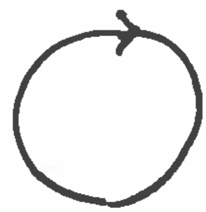
The Alexander Polynomial:

From the twenties to the Teens

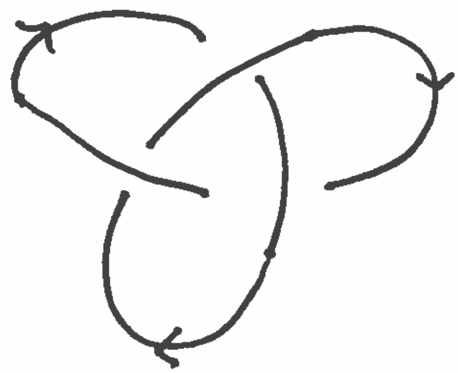
Knots and Diagrams

A knot is a simple closed curve in space. So it is the path travelled by a particle starting at some point in space and moving around until it comes back to its starting point. But it is never allowed to cross its path. We will think of the knot K as being oriented; i.e having a direction of forward motion.

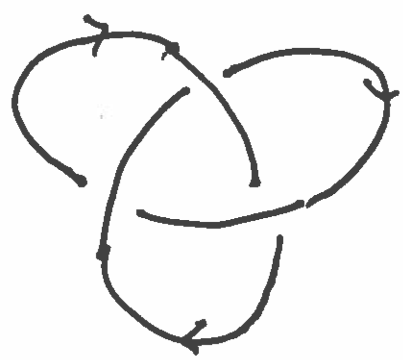
Ex. 1 The unknot



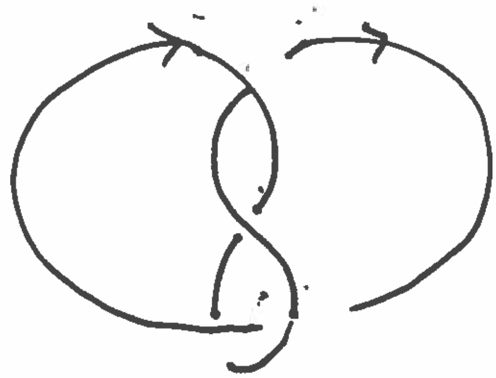
Ex 2. The trefoil

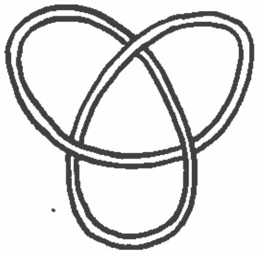


The "other" trefoil



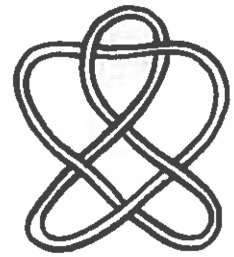
Ex. 3 The Figure 8 Knot





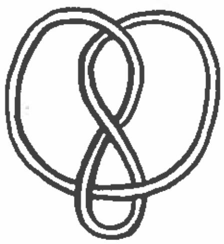
3_1 3

[1-1



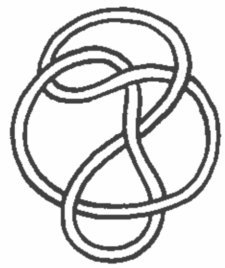
6_2 312

[3-3+1



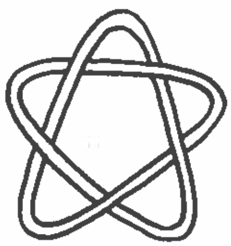
4_1 22

[3-1



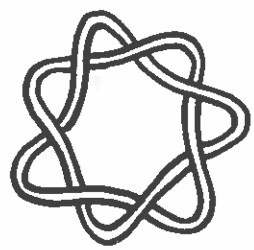
6_3 2112

[5-3+1



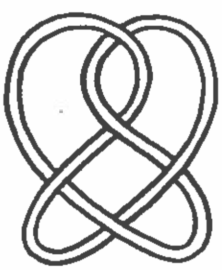
5_1 5

[1-1+1



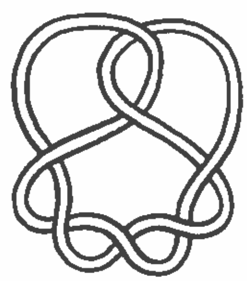
7_1 7

[1-1+1-1



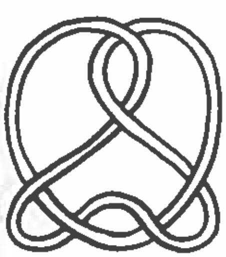
5_2 32

[3-2



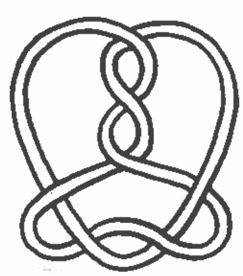
7_2 52

[5-3



6_1 42

[5-2



7_3 43

[3-3+2

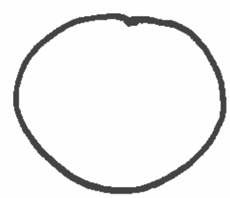
The pictures I've shown are called <u>knot</u> <u>diagrams</u> (or more technically regular projections of the knot).

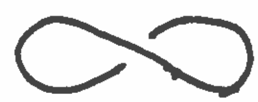
Technically they should be two dimensional (i.e without over/under crossings).

Aside: The work of Peter G. Tait.

Sad Fact: the same knot can be represented by different diagrams

Ex





and

are all diagrams for the un knot.

Question: What does it mean to say two knots are the same?

Intuitive Answer: That you can continuously deform one into the other (but never pulling the knot through itself).

Problem: How can you tell when two knots K_1 and K_2 are different? Or the same?

Basic Idea: Find an invariant of the knot $I(K)$ and show $I(K_1) \neq I(K_2)$.

The Alexander Polynomial

J. W. Alexander, Polynomial Invariants of Knots and Links, Trans. A.M.S., 1928.

The Princeton Duo

Solomon Lefschetz 1884–1972

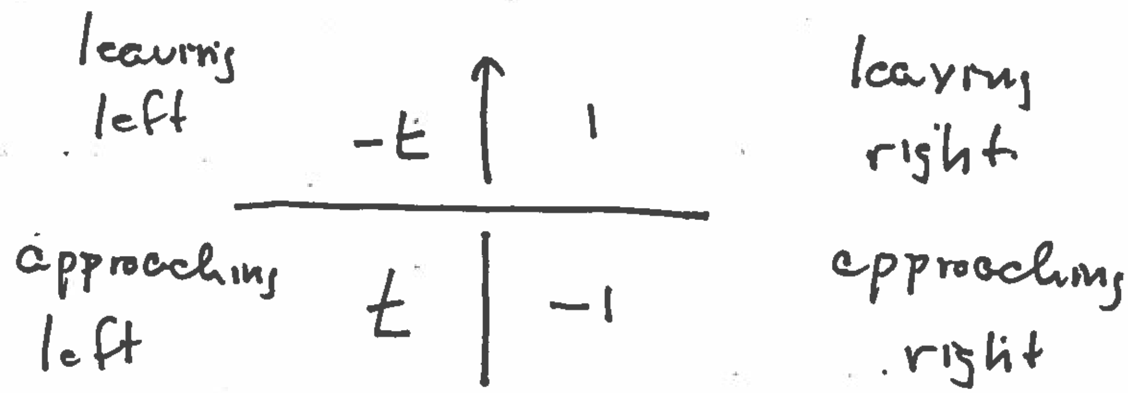
Oswald Veblen 1880–1960

On the Princeton faculty from ~1905 to ~1955

Alexander's Construction

1. Start with a diagram D for the oriented knot K.

2. Assign weights to the 4 regions meeting at each crossing c of D as in the picture



3. Assemble the weights into a $n \times (n+2)$ matrix where $n = \#$ crossings and $n+2 = \#$ regions

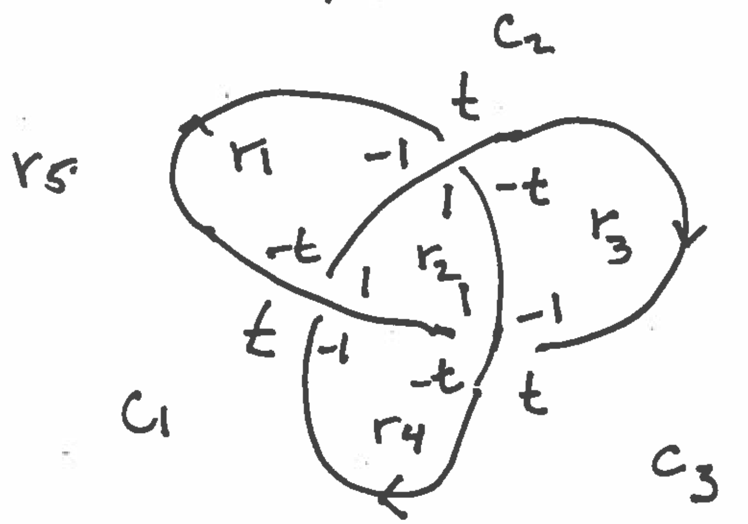
← regions →

↑ crossings ↓

4. Delete two columns corresponding to abutting regions to get an $n \times n$ minor.

5. Then $\Delta_K(t) = \det M$ (almost).

Example: The Trefoil



$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} r_1 & r_2 & r_3 & r_4 & r_5 \end{array} \\ \begin{pmatrix} -t & 1 & 0 & -1 & t \\ -1 & 1 & -t & 0 & t \\ 0 & 1 & -1 & -t & t \end{pmatrix} \end{array}$$

Delete r_1 and r_2

$$M_{12} \begin{pmatrix} 0 & -1 & t \\ -t & 0 & t \\ -1 & -t & t \end{pmatrix}$$

$$\det M_{12} = t^3 - t^2 + t.$$

Delete r_4 r_5

$$M_{45} = \begin{pmatrix} -t & 1 & 0 \\ -1 & 1 & -t \\ 0 & 1 & -1 \end{pmatrix}$$

$$\det M_{45} = -t^2 + t - 1.$$

Upshot / Problem: The determinants of different $n \times n$ minors are different.

Escape / Solution: Allow multiplication by powers of $\pm t$ (or $\pm t^{-1}$).

Technical Thm 1: Let M and M' be obtained from the Alexander matrix by deleting different pairs of columns corresponding to adjacent regions. Then

$$\det M' = \pm t^k \det M$$

New Objection: The construction depends on the choice of diagram representing K.

Technical Thm 2: Up to multiplication by $\pm t^k$, different diagrams give the same polynomial.

Seifert Surfaces - A New Tool

Thm: Let K be an oriented knot in R^3. Then K is the boundary of an oriented surface in R^3.

Frankl-Pontryagin 1930

Seifert 1934

Application

1. Start with an oriented knot K in R^3
2. Pick a Seifert surface S for K.
3. Use "linking numbers" to assign an $n \times n$ matrix V to S. (V is called a Seifert matrix for K.)
4. Thm: $\Delta_K(t) \doteq \det(V^T - tV)$

Cor 1. $\Delta_K(t^{-1}) \doteq \Delta_K(t)$

Pf: Calculate

$$\Delta_K(t^{-1}) = \det(V^T - t^{-1}V)$$
$$= \det(V^T - t^{-1}V)^T$$
$$= \det(V - t^{-1}V^T)$$
$$= \det(-t^{-1}(V^T - tV))$$
$$= (-t^{-1})^n \det(V^T - tV)$$
$$\doteq \Delta_K(t)$$

Cor 2. $\Delta_K(1) = \pm 1$

Thm: Let $p(t)$ be a Laurent polynomial satisfying

1) $p(t) \doteq p(t^{-1})$

2) $p(1) = \pm 1.$

Then there is an oriented knot K in R^3 with

$$\Delta_K(t) \doteq p(t)$$

Recent History

J. H. Conway - 1969

Define

$$\Delta'_K(t) = \det(t^{-1/2}\, V^T - t^{1/2}\, V)$$

Then

$$\Delta'_K(t^{-1}) = \Delta'_K(t)$$

Also

$$\Delta'_K(t) = (t^{-1/2})^n\, \Delta_K(t)$$

But for a knot, $n = 2m$ is even. So

$$\Delta'_K(t) \doteq \Delta_K(t)$$

and $\Delta'_K(t)$ is a (technical) improvement on $\Delta_K(t)$

Thm: $\Delta'_K(t)$ satisfies a "skein relation."

2004 – P. Ozvath and Z. Szebo use "holomorphic disks" to construct "Floer homology" for knots (and links).

2014 – C. Manolescu, P. Ozvath, Z. Szebo, and D. Thurston recast Floer homology in purely combinatorial terms. Now called grid homology.

Thm: There is a homology theory

$$H_{n,i} : \{\text{knots}\} \to \{\mathbb{F}_2 \text{ vector spaces}\}$$

whose Euler characteristic is the Alexander

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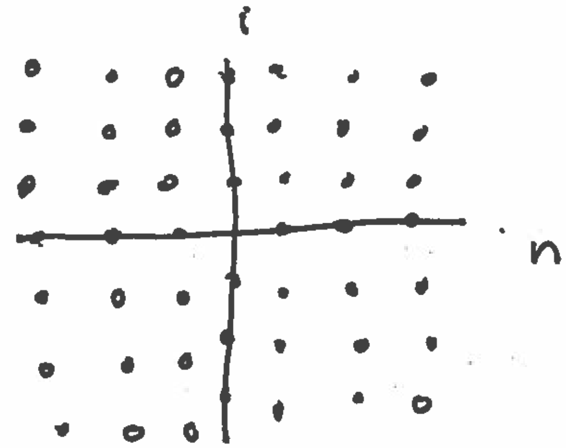
$$H_{n,i} : \{\text{knots}\} \longrightarrow \{\mathbb{F}_2 \text{ vector spaces}\}$$

whose Euler characteristic is the Alexander polynomial.

Remarks: 1) Grid homology is "bigraded" (i.e. indexed on $\mathbb{Z} \times \mathbb{Z}$).



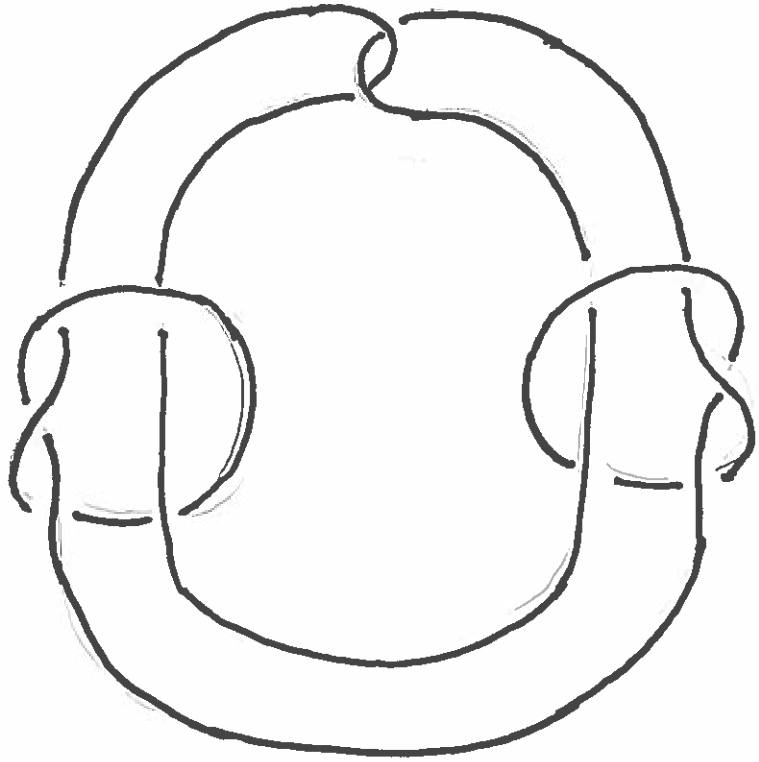
2) The Euler characteristic

$$\chi(H_{**}) = \sum_{n,i} (-1)^n t^i \dim(H_{n,i})$$

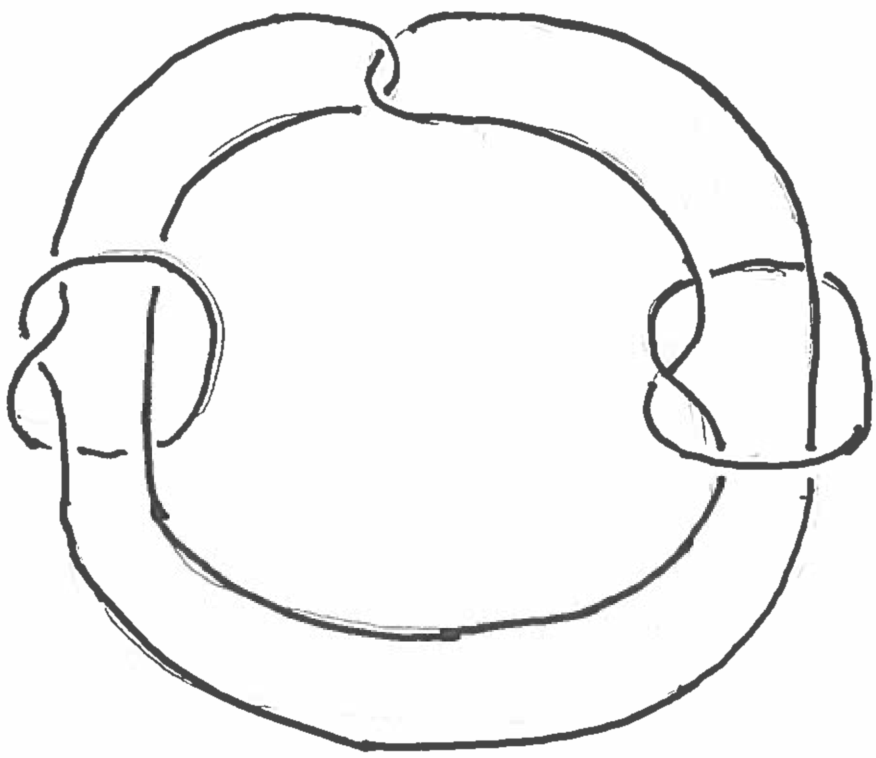
Question: Is Floer/grid homology a stronger invariant than the Alexander polynomial?

E.g. Are there two knots with same Alexander polynomial but with different Floer/grid homology?

Ans: Yes. The Kinoshita-Terasaka knot and its Conway mutant form such a pair!



Kinoshita Terasaka



Conway Mutant