Lights Out for Related Graphs

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Abstract

The Lights Out problem on graphs, in which each vertex of the graph is in one of two states ('on' and 'off'), has been investigated from a variety of perspectives over the last several decades, including parity domination, cellular automata, and harmonic functions on graphs. In this paper, we consider a variant of the Lights Out problem in which the possible states for each vertex are indexed by the integers modulo k. We examine the space of "null patterns" (i.e. harmonic functions) on graphs, and use this as a way to prove theorems about Lights Out on various related graphs.

1 Introduction

In the classical version of the Lights Out puzzle, each vertex of a finite graph is either 'on' or 'off.' By 'pressing' a vertex, the player toggles the state of that vertex and all adjacent vertices. The goal is to turn off the lights by pressing the correct sequence of vertices. While any particular example is solvable by means of ordinary linear algebra over $\mathbb{Z}_2$ (see [3]), this puzzle has deep connections to various areas of combinatorics, including parity domination (see [1] and [2]), cellular automata (see [8]), and harmonic functions on graphs (see [9]).

The generalized Lights Out puzzle can be described as follows. Throughout this paper, the term **graph** will mean a finite graph without multiple edges or loops. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let k be a prime number. A **state** on G is a function $\mathbf{s} : V \to \mathbb{Z}_k$. By fixing an ordering on V, we may regard a state $\mathbf{s}$ as a column vector in $\mathbb{Z}_k^n$ where $n = |V|$. We will denote the zero state by $\vec{0}$. For any vertex $v \in V$, we define the **closed neighborhood** of v as

$$X(v) = \{v\} \cup \{u \in V \,:\, (u, v) \in E\}.$$

Given $v \in V$, there is an associated state $\mathbf{m}_v$ defined by

$$\mathbf{m}_v(u) = \begin{cases} 1 & u \in X(v) \\ 0 & u \notin X(v) \end{cases}$$

We think of the states $\mathbf{m}_v$ as "moves" in the Lights Out puzzle. Adding state $\mathbf{m}_v$ to state $\mathbf{s}$ in $\mathbb{Z}_k^n$ corresponds to the action of "pressing" vertex v. That is, pressing vertex v increments the state of v and every vertex adjacent to v by 1 in $\mathbb{Z}_k$. The goal in the puzzle is to convert an initial state $\mathbf{s}$ into the zero state by adding a sequence of states of the form $\mathbf{m}_v$. For other recent work on this generalized Lights Out problem, see [4, 5, 6, 7, 10].

It is immediately apparent that the ordering of the vertices in the solution sequence is unimportant; we only need to keep track of the number of times each vertex is pressed. Therefore, a **pattern** on G is a function $\mathbf{p} : V \to \mathbb{Z}_k$ where we interpret $\mathbf{p}(v)$ as the number of times vertex v is pressed. Let $V = \{v_1, \dots, v_n\}$. Given an initial state $\mathbf{s}$, $\mathbf{p}$ will be called a **winning pattern** for $\mathbf{s}$ if

$$\mathbf{s} + \sum_{i=1}^{n} \mathbf{p}(v_i)\mathbf{m}_{v_i} = \vec{0}.$$

One goal is to determine which initial states on G have associated winning patterns.

Much of our study involves rephrasing the Lights Out puzzle in terms of linear algebra, which is introduced for the basic $\mathbb{Z}_2$ puzzle on grids in [3]. Let $A = A(G)$ be the adjacency matrix of G based on the ordering $V = \{v_1, \dots, v_n\}$. The matrix $N = N(G) = A(G) + I_n$ is called the **neighborhood matrix** of G. We use $\mathrm{CS}_k(N)$, $\mathrm{RS}_k(N)$, and $\mathrm{NS}_k(N)$ to denote the column space, row space, and null space of N over $\mathbb{Z}_k$, respectively. For any pattern $\mathbf{p}$ on G,

$$\mathbf{s} + \sum_{i=1}^{n} \mathbf{p}(v_i)\mathbf{m}_{v_i} = \mathbf{s} + N\mathbf{p},$$

where $\mathbf{s}$ and $\mathbf{p}$ are considered as a column vectors in $\mathbb{Z}_k^n$. Thus, $\mathbf{p}$ is a winning pattern for $\mathbf{s}$ if and only if $N\mathbf{p} = -\mathbf{s}$, and $\mathbf{s}$ has a winning pattern if and only if $\mathbf{s} \in \mathrm{CS}_k(N)$. Since N is a symmetric matrix, it follows that $\mathrm{CS}_k(N)$ can be identified with $\mathrm{RS}_k(N)$, the orthogonal complement of $\mathrm{NS}_k(N)$. A state $\mathbf{s}$ on G will be called **winnable** if and only if $\mathbf{s} \in \mathrm{CS}_k(N)$, and we will refer to elements of $\mathrm{NS}_k(N)$ as **null patterns** on G. We have

$$\dim \mathrm{CS}_k(N) + \dim \mathrm{NS}_k(N) = n.$$

A graph G will be called **always winnable** over $\mathbb{Z}_k$ if $\dim \mathrm{CS}_k(N) = n$. A graph is always winnable over $\mathbb{Z}_k$ if and only if $\det(N) \not\equiv 0 \pmod{k}$.

Our overall goal is to study winnable states and null patterns for various families of graphs. To do this, we develop tools which tell us what happens to these spaces when graphs are combined with one another in various ways. In Section 2, we introduce some basic notions, prove our main result, and explore some consequences. The main result, Theorem 2.10, gives $\dim \mathrm{NS}_k(N(H))$, where H is formed from a finite number of graphs G_i and chosen vertices $v_i \in V(G_i)$ by identifying all of the v_i with one another inside the disjoint union of the graphs G_i. Section 3 explores some basic applications of these results.

2 Lights Out for Related Graphs

We describe how the sets of winnable states (or, equivalently, null patterns) are related for graphs formed from others via various constructions. Given a graph G, one can consider the effect of deleting one vertex of G. We will denote the graph formed by deleting vertex $v \in V(G)$ (along with all incident edges) from G by $G - v$.

Definition 2.1 The **label** of a vertex v in a graph G will be defined by

$$l_G(v) = \dim \mathrm{NS}_k(N(G-v)) - \dim \mathrm{NS}_k(N(G)).$$

The label of a vertex v may depend on the prime k. Indeed, we will show in Section 3 that this is the case for cycles. Since k is considered to be a fixed prime, we will use the notation $l_G(v)$ without reference to k.

Proposition 2.2 Let G be a graph. For all $v \in V(G)$, we have $l_G(v) \in \{-1, 0, 1\}$. If G is always winnable over $\mathbb{Z}_k$, then for all $v \in V(G)$, we have $l_G(v) \in \{0, 1\}$.

Proof. The matrix $N(G-v)$ is formed by deleting exactly one row and exactly one column from $N(G)$. For any matrix, deleting a column will either decrease the null space dimension by 1 or leave it the same. Similarly, for any matrix, deleting a row will either increase the null space dimension by 1 or leave it the same. Therefore, deleting a column then a row from a matrix will change the null space dimension at most by 1 either up or down. Thus, $l_G(v) \in \{-1, 0, 1\}$.

For an always winnable graph G, we have $\dim \mathrm{NS}_k(N(G)) = 0$, and therefore for all $v \in V(G)$, we have $\dim \mathrm{NS}_k(N(G-v)) \geq \dim \mathrm{NS}_k(N(G))$. □

Let G be a graph with $v \in V(G)$. Let $\mathbf{e}_v$ be the state on G such that $\mathbf{e}_v(v) = 1$ and $\mathbf{e}_v(w) = 0$ if $w \neq v$. Let f_v be the $\mathbb{Z}_k$-linear transformation which extends a pattern on $G-v$ to a pattern on G that is zero at v. Let r_v be the $\mathbb{Z}_k$-linear transformation that restricts a pattern on G to a pattern on $G-v$.

Proposition 2.3 Let G be a graph, and let $v \in V(G)$. The following are equivalent:

1. $l_G(v) = -1$,
2. The state $\mathbf{e}_v$ is not winnable on G.
3. There exists $\mathbf{p} \in \mathrm{NS}_k(N(G))$ with $\mathbf{p}(v) \neq 0$.
4. The function f_v restricts to an injective linear transformation from $\mathrm{NS}_k(N(G-v))$ to $\mathrm{NS}_k(N(G))$, and the restriction of $f_v : \mathrm{NS}_k(N(G-v)) \to \mathrm{NS}_k(N(G))$ has 1-dimensional cokernel.

Proof. (1) $\Rightarrow$ (3): We prove the contrapositive. If $\mathbf{p}(v) = 0$ for all $\mathbf{p} \in \mathrm{NS}_k(N(G))$ then every null pattern on G restricts to a null pattern on $G-v$. This would imply

$$\dim \mathrm{NS}_k(N(G-v)) \geq \dim \mathrm{NS}_k(N(G)),$$

implying $l_G(v) \in \{0, 1\}$.

(3) $\Leftrightarrow$ (2): This equivalence follows immediately from the facts that the winnable states on G are precisely the elements of $\mathrm{CS}_k(N(G))$ and that $\mathrm{CS}_k(N(G))$ is the orthogonal complement of $\mathrm{NS}_k(N(G))$.

(2), (3) $\Rightarrow$ (4): Suppose that $\mathbf{e}_v$ is not winnable on G, and let $f_v : \mathbb{Z}_k^{n-1} \to \mathbb{Z}_k^n$ be as above. A pattern on G that is null on $G-v$ is also null at v; otherwise that pattern would win $\lambda\mathbf{e}_v$

for some $\lambda \in \mathbb{Z}_k^*$, contradicting (2). In particular, a null pattern on $G - v$ extended by 0 at v is null on G, ensuring that the restriction $f_v : \mathrm{NS}_k(N(G-v)) \to \mathrm{NS}_k(N(G))$ is well-defined. Clearly, f_v is injective. Now by (3), there exists a null pattern $\mathbf{p}$ on G such that $\mathbf{p}(v) \neq 0$, so f_v cannot be surjective. Hence, by Proposition 2.2, the restriction of f_v has 1-dimensional cokernel.

(4) $\Rightarrow$ (1): This is immediate from the definition of $l_G(v)$. □

Corollary 2.4 Let G be a graph, and let $v \in V(G)$. Then $l_G(v) \in \{0, 1\}$ if and only if $\mathbf{p}(v) = 0$ for every $\mathbf{p} \in \mathrm{NS}_k(N(G))$.

Proof. This follows directly from the equivalence of (1) and (3) in Proposition 2.3. □

It also follows from Proposition 2.3 that the second statement in Proposition 2.2 can be made biconditional.

Corollary 2.5 For any graph G, G is always winnable over $\mathbb{Z}_k$ if and only if $l_G(v) \in \{0, 1\}$ for all $v \in V(G)$.

Proof. The 'only if' statement is proven as part of Proposition 2.2. If $l_G(v) \in \{0, 1\}$ for all $v \in V(G)$, then by Proposition 2.3, the state $\mathbf{e}_v$ is winnable on G for all $v \in V(G)$. This implies that G is always winnable. □

Proposition 2.6 Let G be a graph and let $v \in V(G)$. The following are equivalent:

1. $l_G(v) = 0$,
2. For all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda\mathbf{e}_v$ is winnable on G, and any winning pattern $\mathbf{p}$ for $\lambda\mathbf{e}_v$ satisfies $\mathbf{p}(v) \neq 0$.
3. The functions r_v and f_v restrict to give bijective linear transformations between $\mathrm{NS}_k(N(G))$ and $\mathrm{NS}_k(N(G-v))$, and these restrictions are inverses of one another.

Proof.
(1) $\Rightarrow$ (2): Suppose that $l_G(v) = 0$. By Corollary 2.4, $\mathbf{q}(v) = 0$ for every $\mathbf{q} \in \mathrm{NS}_k(N(G))$. The space of winnable states is the orthogonal complement to the space of null patterns, and therefore, for all $\lambda \in \mathbb{Z}_k^*$, $\lambda\mathbf{e}_v$ is winnable because $\lambda\mathbf{e}_v \perp \mathbf{q}$ for all $\mathbf{q} \in \mathrm{NS}_k(N(G))$.

Now suppose that $\mathbf{p}$ is a winning pattern for $\lambda\mathbf{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$ with $\mathbf{p}(v) = 0$. It follows that $\mathbf{p}|_{G-v}$ is null, but $\mathbf{p}$ is not null on G. For every $\mathbf{q} \in \mathrm{NS}_k(N(G))$, $\mathbf{q}(v) = 0$ and thus $\mathbf{q}|_{G-v}$ is always a null pattern on $G - v$. Notice that $\mathbf{p}|_{G-v}$ must be distinct from $\mathbf{q}|_{G-v}$ for all $\mathbf{q} \in \mathrm{NS}_k(N(G))$, since the effect of extending these patterns by 0 at v is different. This implies that $\dim_k \mathrm{NS}(N(G-v)) > \dim_k \mathrm{NS}(N(G))$, contradicting (1). Thus $\mathbf{p}(v) \neq 0$.

(2) $\Rightarrow$ (3): Suppose (2) is true. Since a pattern $\mathbf{q} \in \mathrm{NS}_k(N(G))$ must be orthogonal to every winnable pattern, $\mathbf{q}(v) = 0$ in all null patterns on G. If any $\mathbf{q} \in \mathrm{NS}_k(N(G))$ is restricted to $G - v$, the result is a null pattern on $G - v$, and therefore, r_v restricts to give a well-defined function from $\mathrm{NS}_k(N(G))$ to $\mathrm{NS}_k(N(G-v))$.

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Clearly, r_v is injective. Let $\mathbf{r} \in \mathrm{NS}_k(N(G-v))$. Extending $\mathbf{r}$ to G by setting $\mathbf{r}(v) = 0$ yields either a null pattern on G or a pattern that wins $\lambda \mathbf{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$. The latter is impossible by (2), since $\mathbf{r}(v) = 0$. Thus, the extension of $\mathbf{r}$ by 0 is a null pattern on G. This implies that the restriction of r_v is an isomorphism from $\mathrm{NS}_k(N(G))$ to $\mathrm{NS}_k(N(G-v))$ with inverse given by f_v.

(3) ⇒ (1): This is immediate from the definition of $l_G(v)$. □

Proposition 2.7 Let G be a graph, and let $v \in V(G)$. The following are equivalent:

1. $l_G(v) = 1$,
2. For all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\mathbf{p}$ for $\lambda \mathbf{e}_v$ satisfies $\mathbf{p}(v) = 0$.
3. The function r_v induces an injective linear transformation from $\mathrm{NS}_k(N(G))$ to $\mathrm{NS}_k(N(G-v))$, and the restriction $r_v : \mathrm{NS}_k(N(G)) \to \mathrm{NS}_k(N(G-v))$ has 1-dimensional cokernel.

Proof. (1) ⇒ (2): Suppose that $l_G(v) = 1$. Again by Corollary 2.4, every null pattern on G is zero at v. Therefore, as in the first part of the proof of Proposition 2.6, the state $\lambda \mathbf{e}_v$ is winnable on G for all $\lambda \in \mathbb{Z}_k^*$. Suppose there exists $\mu \in \mathbb{Z}_k^*$ and a winning pattern $\mathbf{p}$ for $\mu \mathbf{e}_v$ with $\mathbf{p}(v) \neq 0$. Since every winning pattern for $\mu \mathbf{e}_v$ is in the set $\mathbf{p} + \mathrm{NS}_k(N(G))$ and every element of $\mathrm{NS}_k(N(G))$ is zero at v by Proposition 2.4, every winning pattern for $\mu \mathbf{e}_v$ is nonzero at v. Moreover, for any nonzero $\lambda \in \mathbb{Z}_k^*$, every winning pattern for $\lambda \mathbf{e}_v$ is in the set $\lambda\mu^{-1}\mathbf{p} + \mathrm{NS}_k(N(G))$, and this implies that every winning pattern for $\lambda \mathbf{e}_v$ is also nonzero at v. By Proposition 2.6, this would imply $l_G(v) = 0$, a contradiction to (1). Therefore, any winning pattern $\mathbf{p}$ for $\mu \mathbf{e}_v$ satisfies $\mathbf{p}(v) = 0$.

(2) ⇒ (3): Assume (2) is true. Since a null pattern on G has to be perpendicular to every winnable pattern, $\mathbf{q}(v) = 0$ for all null patterns $\mathbf{q}$ on G. Therefore, a null pattern on G restricted to $G - v$ is still null. Thus, the restriction of r_v gives a well-defined linear transformation $\mathrm{NS}_k(N(G)) \to \mathrm{NS}_k(N(G-v))$.

Clearly, the restriction of r_v is injective. If the restriction of r_v were also surjective, then $l_G(v) = 0$, and this contradicts (2) by Proposition 2.6. The cokernel of the restriction of r_v to $\mathrm{NS}_k(N(G))$ is therefore 1-dimensional by Proposition 2.2.

(3) ⇒ (1): This is immediate from the definition of $l_G(v)$. □

We will determine what happens to the dimension of the space of null patterns as graphs are connected together in various ways. A convenient way to describe this is to determine $l_H(v)$ where H is the newly formed graph and v represents the vertex (or one of the vertices) where the connection is taking place.

For vertices in a graph with label 0, we will need to record more information.

Definition 2.8 1. Let G be a graph and suppose $v \in V(G)$ with $l_G(v) = 0$. By Proposition 2.6, the state $\mathbf{e}_v$ has a winning pattern $\mathbf{q}$, and $\mathbf{q}(v) \in \mathbb{Z}_k^*$. Let

$$\lambda_G(v) = -\mathbf{q}(v)^{-1} \in \mathbb{Z}_k^*.$$

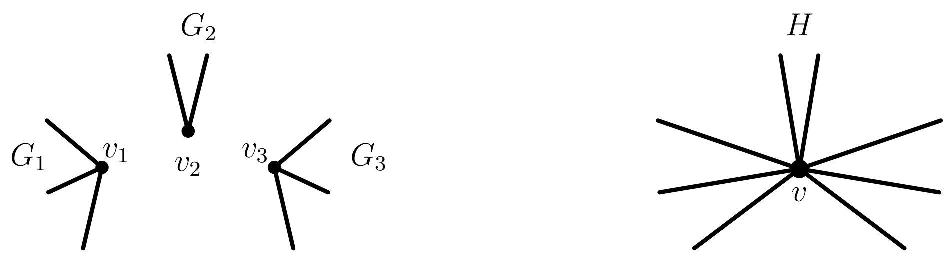
In this situation, all null patterns $\mathbf{p}$ on G have $\mathbf{p}(v) = 0$, and therefore $\lambda_G(v)$ is independent of the winning pattern $\mathbf{q}$ chosen.

2. Let G be a graph and suppose $v \in V(G)$ with $l_G(v) = -1$. We define $\lambda_G(v) = 0$. (This secondary label on the $l_G(v) = -1$ vertices carries no information, but it is convenient for summation notation later.)

In either case, the number $\lambda_G(v) \in \mathbb{Z}_k$ will be called the **secondary label** of v. The label of a vertex v with $l_G(v) = 0$ and secondary label λ will typically be written as $0(\lambda)$.

Our main result explores the process of joining two or more graphs together by vertex identification. One chooses a vertex in each graph and creates a new graph H by identifying the chosen vertices to a single vertex v inside the disjoint union of the original graphs. If the labels and secondary labels of the chosen vertices are known, then the following theorem gives $\dim \mathrm{NS}_k(N(H))$ by computing $l_H(v)$.

Definition 2.9 Let $m \in \mathbb{Z}$ with $m \geq 2$. For $1 \leq i \leq m$, let G_i be a graph with $v_i \in V(G_i)$. The graph $H = \mathrm{VJ}(\{G_i, v_i\})$ is defined by starting with the disjoint union $\bigcup G_i$ and identifying $\{v_1, v_2, \ldots, v_m\}$ to a single vertex v.



Theorem 2.10 Let G_i be graphs for $1 \leq i \leq m$, and let $v_i \in V(G_i)$. Let $H = \mathrm{VJ}(\{G_i, v_i\})$.

1. $l_H(v) = 1$ if and only if $l_{G_i}(v_i) = 1$ for at least one i.

2. $l_H(v) \in \{0, -1\}$ if and only if $l_{G_i} \in \{0, -1\}$ for all i. Moreover, in this case, $l_H(v) = -1$ if and only if $\displaystyle\sum_{i=1}^{m} \lambda_{G_i}(v_i) = m - 1 \pmod{k}$.

3. In the case that $l_H(v) = 0$, the secondary label of v in H is given by

$$\lambda_H(v) = 1 - m + \sum_{i=1}^{m} \lambda_{G_i}(v_i) \pmod{k}.$$

Once $l_H(v)$ is known, $\dim \mathrm{NS}_k(N(H))$ can be computed from the definitions of the labels as

$$\dim \mathrm{NS}_k(N(H)) \quad = -l_H(v) + \sum_{i=1}^{m} (\dim \mathrm{NS}_k(N(G_i)) + l_{G_i}(v_i))$$

Proof. For the main statements in (1) and (2), we prove only the "if" implication. The converses will be clear from Proposition 2.2 because $l_H(v) = 1$ and $l_H(v) \in \{0, -1\}$ are mutually exclusive and exhaustive outcomes.

(1) Suppose that $l_{G_j}(v_j) = 1$ for some j. By Proposition 2.7, there is a pattern $\mathbf{p}$ on G_j that wins $\mathbf{e}_{v_j}$ with $\mathbf{p}(v_j) = 0$. Extend $\mathbf{p}$ to H such that $\mathbf{p}(w) = 0$ for all vertices $w \in V(H)$ not originally coming from G_j. Then $\mathbf{p}$ is a winning pattern on H for $\mathbf{e}_v$ with $\mathbf{p}(v) = 0$. It now follows from Proposition 2.3 that $l_H(v) \neq -1$, and then Corollary 2.4 implies that every null pattern on H is zero at v. It follows that every winning pattern $\mathbf{q}$ on H for $\lambda\mathbf{e}_v$ with $\lambda \in \mathbb{Z}_k^*$ also satisfies $\mathbf{q}(v) = 0$. By Proposition 2.7, we have $l_H(v) = 1$.

(2) Suppose that $l_{G_i}(v_i) \in \{0, -1\}$ for all i. Also, for purposes of contradiction, assume that $l_H(v) = 1$. Proposition 2.7 implies that $\mathbf{e}_v$ is winnable on H, and winning patterns $\mathbf{p}$ for $\mathbf{e}_v$ on H satisfy $\mathbf{p}(v) = 0$. If $\mathbf{p}$ is any such pattern, then for each i, $\mathbf{p}$ restricts to a null pattern on $G_i - v_i$. For any i such that $l_{G_i}(v_i) = 0$, $\mathbf{p}$ induces a null pattern on G_i by restriction as well. (This is true because this pattern restricted to G_i would either have to be null or be a winning pattern for $\lambda\mathbf{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$, but could not do the latter with both $\mathbf{p}(v_i) = 0$ and $l_{G_i}(v_i) = 0$; see Proposition 2.6.) In addition, for any i such that $l_{G_i}(v_i) = -1$, $\mathbf{p}$ induces a null pattern on G_i since nonzero multiples of $\mathbf{e}_{v_i}$ are not winnable on G_i. Putting this information together gives $\mathbf{p} \in \mathrm{NS}_k(N(H))$, contradicting the fact that $\mathbf{p}$ is a winning pattern for $\mathbf{e}_v$ on H. Therefore, $l_H(v) \in \{0, -1\}$.

We now prove the second biconditional statement in (2).

($\Rightarrow$) Suppose that $l_H(v) = -1$. Choose $\mathbf{p} \in \mathrm{NS}_k(N(H))$ such that $\mathbf{p}(v) = 1$, which exists by Proposition 2.3. Let $\mathbf{p}_i$ be the pattern on G_i given by the restriction of $\mathbf{p}$. For all G_i such that $l_{G_i}(v_i) = -1$, $\mathbf{p}_i$ is null because $\mathbf{e}_{v_i}$ is not winnable on G_i. However, for all i such that $l_{G_i}(v_i) = 0$, the fact that every null pattern on G_i is zero at v_i (Corollary 2.4) shows that $\mathbf{p}_i$ is a winning pattern for $\lambda\mathbf{e}_{v_i}$ for some $\lambda \in \mathbb{Z}_k^*$. In both cases, by definition of the secondary labels, $\mathbf{p}_i$ is a winning pattern on G_i for $-\lambda_{G_i}(v_i)\mathbf{e}_{v_i}$. Since $\mathbf{p}_i(v_i) = 1$ for all i, the contribution of $\mathbf{p}$ from all vertices in $G_i - v_i$ to the state of v_i must be $\lambda_{G_i}(v_i) - 1$. Because $\mathbf{p} \in \mathrm{NS}_k(N(H))$, we must have

$$1 + \sum_{i=1}^{m} (\lambda_{G_i}(v_i) - 1) = 0 \pmod{k},$$

which implies

$$\sum_{i=1}^{m} \lambda_{G_i}(v_i) = m - 1 \pmod{k}.$$

($\Leftarrow$) Suppose that $\sum_{i=1}^{m} \lambda_{G_i}(v_i) = m - 1 \pmod{k}$. To show that $l_H(v) = -1$, we construct a null pattern $\mathbf{p}$ on H that has $\mathbf{p}(v) = 1$. For each i such that $l_{G_i}(v_i) = -1$, Proposition 2.3 implies that there exists a null pattern $\mathbf{p}_i$ on G_i with $\mathbf{p}_i(v_i) = 1$. For each i such that $l_{G_i}(v_i) = 0$, the definition of the secondary labels implies that a winning pattern $\mathbf{p}_i$ for $-\lambda_{G_i}(v_i)\mathbf{e}_{v_i}$ has $\mathbf{p}_i(v_i) = 1$. Since all of the patterns $\mathbf{p}_i$ have $\mathbf{p}_i(v_i) = 1$, they glue together to form a pattern $\mathbf{p}$ on H, which we will show is null. By construction, $\mathbf{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ on H for some $\lambda \in \mathbb{Z}_k$. We need to show that $\lambda = 0$. By adding up all of the contributions from the different G_i, we see that

$$-\lambda = 1 + \sum_{i=1}^{m} (\lambda_{G_i}(v_i) - 1) = 1 + (m-1) - m = 0 \pmod{k}.$$

Therefore, $\mathbf{p} \in \mathrm{NS}_k(N(H))$. Because $\mathbf{p}(v) \neq 0$, we have $l_H(v) = -1$ by Proposition 2.3.

(3) Suppose that $l_H(v) = 0$. By (2), we must have $l_{G_i}(v_i) \in \{0, -1\}$ for all i and

$$\sum_{i=1}^{m} \lambda_{G_i}(v_i) \neq m - 1 \pmod{k}.$$

If $l_{G_i}(v_i) = -1$, then by Proposition 2.3 there is a null pattern $\mathbf{p}_i$ on G_i with $\mathbf{p}_i(v_i) = 1$. If $l_{G_i}(v_i) = 0$, then by Proposition 2.6 there is a pattern $\mathbf{p}_i$ on G_i such that $\mathbf{p}_i(v_i) = 1$ and $\mathbf{p}_i$ wins $-\lambda_{G_i}(v_i)\mathbf{e}_{v_i}$. Gluing these patterns together, we form a pattern $\mathbf{p}$ on H with $\mathbf{p}(v) = 1$. By the definition of the secondary labels, $\mathbf{p}$ is a winning pattern on H for $-\lambda_H(v)\mathbf{e}_v$. Adding up the contributions from pressing v once and all of the patterns $\mathbf{p}_i$ being glued together gives

$$\lambda_H(v) = 1 + \sum_{i=1}^{m} (\lambda_{G_i}(v_i) - 1) = 1 - m + \sum_{i=1}^{m} \lambda_{G_i}(v_i).$$

□

Corollary 2.11 Consider two always winnable graphs G_1 and G_2 over $\mathbb{Z}_k$, and let

$$H = \mathrm{VJ}(\{G_1, v_2\}, \{G_2, v_2\}).$$

Then H is always winnable if and only if $l_H(v) = l_{G_1}(v_1) + l_{G_2}(v_2)$. This can happen in only the following two ways:

1. One of the $l_{G_i}(v_i)$ is 1 and the other is 0, in which case $l_H(v) = 1$.
2. $l_{G_1}(v_1) = 0(\lambda)$ and $l_{G_2}(v_2) = 0(\mu)$ with $\lambda + \mu \neq 1 \pmod{k}$, in which case $l_H(v) = 0(\lambda + \mu - 1)$.

Proof. This is immediate from the $m = 2$ case of Theorem 2.10. □

One application of Theorem 2.10 is to determine the dimension of the space of null patterns (and hence, the space of winnable patterns) when P_2 is attached to a graph by identifying one of the vertices of P_2 with a chosen vertex of the graph.

Corollary 2.12 Let G_1 be a graph and let $v \in V(G_1)$. Let P_2 be a path with 2 vertices v' and w'. Let

$$G_1' = \mathrm{VJ}(\{G_1, v\}, \{P_2, v'\}).$$

Let $d = \dim \mathrm{NS}_k(N(G_1))$. Then $\dim \mathrm{NS}_k(N(G_1'))$ is given by the following table.

$l_{G_1}(v)$	$\dim \mathrm{NS}_k(N(G_1'))$	$l_{G_1'}(v)$	$l_{G_1'}(w')$
1	d	1	$0(1)$
-1	$d-1$	$0(-1)$	1
$0(\lambda) \quad (\lambda \neq 1)$	d	$0(\lambda - 1)$	$0(1-\lambda^{-1})$
$0(1)$	$d+1$	-1	-1

Proof. For all k we have

$$l_{P_2}(v') = l_{P_2}(w') = -1.$$

In forming G_1', there are three main cases to consider depending on whether $l_{G_1}(v)$ is 1, $0(\lambda)$, or -1. For ease of notation, we will refer to the vertex of G_1' corresponding to identifying $v' = v$ as v.

Case 1: $[l_{G_1}(v) = 1 \Rightarrow l_{G_1'}(v) = 1$ and $l_{G_1'}(w') = 0(1)]$
Suppose $l_{G_1}(v) = 1$. Then by Theorem 2.10, $l_{G_1'}(v) = 1$, showing that

$$\begin{aligned} \dim \mathrm{NS}_k(N(G_1')) &= \dim \mathrm{NS}_k(N(G_1' - v)) - 1 \\ &= \dim \mathrm{NS}_k(N(G_1 - v)) - 1 \\ &= \dim \mathrm{NS}_k(N(G_1)) \qquad = d. \end{aligned}$$

Then $l_{G_1'}(w') = 0$. To win $\mathbf{e}_{w'}$ on G_1', we press w' exactly $k-1$ times, relying on the fact that the pattern $(k-1)\mathbf{e}_v$ can be won on G_1 without pressing v (by Proposition 2.7). Therefore, the secondary label on w' in G_1 is $-(k-1)^{-1} = 1$, showing that $l_{G_1'}(w') = 0(1)$.

Case 2: $[l_{G_1}(v) = -1 \Rightarrow l_{G_1'}(v) = 0(-1)$ and $l_{G_1'}(w') = 1]$
Suppose $l_{G_1}(v) = -1$. Then by Theorem 2.10, $l_{G_1'}(v) = 0(-1)$, showing that

$$\begin{aligned} \dim \mathrm{NS}_k(N(G_1')) &= \dim \mathrm{NS}_k(N(G_1' - v)) \\ &= \dim \mathrm{NS}_k(N(G_1 - v)) \\ &= \dim \mathrm{NS}_k(N(G_1)) - 1 \quad = \quad d-1. \end{aligned}$$

Then $l_{G_1'}(w') = 1$.

Case 3a: $[l_{G_1}(v) = 0(\lambda)$ where $(\lambda \neq 1) \Rightarrow l_{G_1'}(v) = 0(\lambda - 1)$ and $l_{G_1'}(w') = 0(1-\lambda^{-1})]$
Suppose $l_{G_1}(v) = 0(\lambda)$ where $\lambda \neq 1$. Theorem 2.10, $l_{G_1'}(v) = 0(\lambda - 1)$, showing that

$$\begin{aligned} \dim \mathrm{NS}_k(N(G_1')) &= \dim \mathrm{NS}_k(N(G_1' - v)) \\ &= \dim \mathrm{NS}_k(N(G_1 - v)) \\ &= \dim \mathrm{NS}_k(N(G_1)) \qquad = d. \end{aligned}$$

Then $l_{G_1'}(w') = 0$. We know from Proposition 2.6 that $\mathbf{e}_{w'}$ is winnable on G_1'. Let $\mathbf{p}$ be a pattern on G_1' that wins $\mathbf{e}_{w'}$, and suppose $\mathbf{p}(w') = t$. Then $\mathbf{p}$, when restricted to G_1, gives a pattern on G_1 that wins $t\mathbf{e}_v$ with $\mathbf{p}(v) = -t-1$. Since v is pressed $-\lambda^{-1}$ times in winning $\mathbf{e}_v$ on G_1, it follows that v is pressed $-t\lambda^{-1}$ times in winning $t\mathbf{e}_v$ on G_1. Thus $-t\lambda^{-1} = -t-1$. Solving for t gives $t = (\lambda^{-1} - 1)^{-1}$. The secondary label of w' in G_1' is $-t^{-1}$, and therefore this secondary label is $1 - \lambda^{-1}$. Thus $l_{G_1'}(w') = 0(1-\lambda^{-1})$.

Case 3b: $[l_{G_1}(v) = 0(1) \Rightarrow l_{G_1'}(v) = -1$ and $l_{G_1'}(w') = -1]$
Suppose $l_{G_1}(v) = 0(1)$. Then by Theorem 2.10, $l_{G_1'}(v) = -1$, showing that

$$\begin{aligned} \dim \mathrm{NS}_k(N(G_1')) &= \dim \mathrm{NS}_k(N(G_1' - v)) + 1 \\ &= \dim \mathrm{NS}_k(N(G_1 - v)) + 1 \\ &= \dim \mathrm{NS}_k(N(G_1)) + 1 \qquad = d+1. \end{aligned}$$

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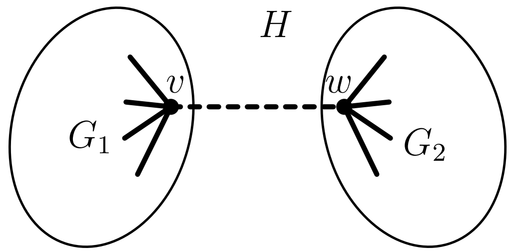
Then $l_{G_1'}(w') = -1$. □

As another application of these results, we determine the dimension of the space of null patterns for joining two graphs via a new edge.

Definition 2.13 Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $H = \mathrm{EJ}(\{G_1, v\}, \{G_2, w\})$ be the graph with $V(H) = V(G_1) \cup V(G_2)$ and $E(H) = E(G_1) \cup E(G_2) \cup \{(v, w)\}$.



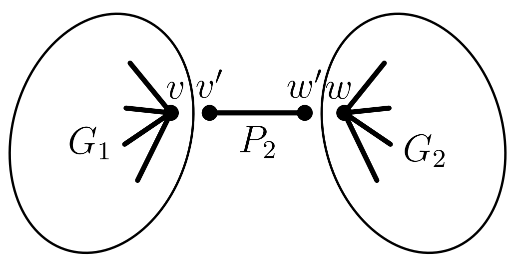
Theorem 2.14 Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $d_i = \dim \mathrm{NS}_k(N(G_i))$, and let $H = \mathrm{EJ}(\{G_1, v\}, \{G_2, w\})$. Then $\dim \mathrm{NS}_k(N(H))$ is given by the following table.

$l_{G_1}(v)$	$l_{G_2}(w)$	$\dim \mathrm{NS}_k(N(H))$
1	any	$d_1 + d_2$
-1	any	$d_1 + d_2 + l_{G_2}(w) - 1$
$0(\lambda)$	$0(\mu)$	$d_1 + d_2 \quad (\mu \neq \lambda^{-1})$
$0(\lambda)$	$0(\mu)$	$d_1 + d_2 + 1 \quad (\mu = \lambda^{-1})$

Cases not covered by the chart can be handled by symmetry.

Proof. Let P_2 be a path with two vertices, v' and w'. We construct H in two steps. Let

$$G_1' = \mathrm{VJ}(\{G_1, v\}, \{P_2, v'\}) \quad \text{and} \quad H = \mathrm{VJ}(\{G_1', w'\}, \{G_2, w\}).$$



We use Corollary 2.12 and then Theorem 2.10 to find $\dim \mathrm{NS}_k(N(H))$. To ease notation, we will refer to the identified vertex $v = v'$ in G_1' as simply v, and similarly we will refer to the vertices $v = v'$ and $w = w'$ of H as v and w, respectively.

Case 1: Suppose $l_{G_1}(v) = 1$. Corollary 2.12 implies that $l_{G_1'}(w') = 0(1)$. If $l_{G_2}(w) = 1$, Theorem 2.10 shows that $l_H(w) = 1$. If $l_{G_2}(w) = -1$, Theorem 2.10 shows that $l_H(w) = -1$. If $l_{G_2}(w) = 0(\lambda)$, Theorem 2.10 shows that $l_H(w) = 0$, since $\lambda + 1 \neq 1$. Therefore, since $l_H(w) = l_{G_2}(w)$ in every case,

$$\begin{aligned} \dim \mathrm{NS}_k(N(H)) &= \dim \mathrm{NS}_k(N(H - w)) - l_H(w) \\ &= \dim \mathrm{NS}_k(N(G_1)) + \dim \mathrm{NS}_k(N(G_2 - w)) - l_H(w) \\ &= \dim \mathrm{NS}_k(N(G_1)) + \dim \mathrm{NS}_k(N(G_2)) + l_{G_2}(w) - l_H(w) \\ &= d_1 + d_2. \end{aligned}$$

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Case 2: Suppose $l_{G_1}(v) = -1$ and $l_{G_2}(w) \in \{0, -1\}$. Corollary 2.12 implies that $l_{G_1'}(w') = 1$. Theorem 2.10 implies that $l_H(w) = 1$. Therefore,

$$\begin{aligned}
\dim \mathrm{NS}_k(N(H)) &= \dim \mathrm{NS}_k(N(H-w)) - 1 \\
&= \dim \mathrm{NS}_k(N(G_1)) + \dim \mathrm{NS}_k(N(G_2 - w)) - 1 \\
&= \dim \mathrm{NS}_k(N(G_1)) + \dim \mathrm{NS}_k(N(G_2)) + l_{G_2}(w) - 1 \\
&= d_1 + d_2 + l_{G_2}(w) - 1.
\end{aligned}$$

Case 3: Suppose $l_{G_1}(v) = 0(\lambda)$ and $l_{G_2}(w) = 0(\mu)$. Using Corollary 2.12, we find that $l_{G_1'}(w') = -1$ if $\lambda = 1$ and $l_{G_1'}(w') = 0(1 - \lambda^{-1})$ if $\lambda \neq 1$. In the case that $\lambda = 1$, Theorem 2.10 gives $l_H(w) = 0$ when $\mu \neq 1$ and $l_H(w) = -1$ when $\mu = 1$. In the case that $\lambda \neq 1$, Theorem 2.10 gives $l_H(w) = 0$ when $\mu \neq \lambda^{-1}$ and $l_H(w) = -1$ when $\mu = \lambda^{-1}$. In terms of computing dimensions, we then have two possibilities: either $\mu = \lambda^{-1}$, in which case $\dim \mathrm{NS}_k(N(H)) = d_1 + d_2 + 1$, or $\mu \neq \lambda^{-1}$, in which case $\dim \mathrm{NS}_k(N(H)) = d_1 + d_2$. ☐

Corollary 2.15 Consider two always winnable graphs G_1 and G_2 over $\mathbb{Z}_k$, and let $H = \mathrm{EJ}(\{G_1, v\}, \{G_2, w\})$. Then H is always winnable if and only if one of the following occurs:

1. Either $l_{G_1}(v) = 1$ or $l_{G_2}(w) = 1$, or both.
2. $l_{G_1}(v) = 0(\lambda)$ and $l_{G_2}(w) = 0(\mu)$ with $\mu \neq \lambda^{-1} \pmod{k}$.

Proof. This is immediate from Theorem 2.14. We note that part 1 gives a different proof of [4, Corollary 2.11]. ☐

One useful application of Theorem 2.10 is the idea of graph reduction, i.e. removing a set of vertices, along with all incident edges, from a graph without changing the dimension of the null space of the neighborhood matrix.

Corollary 2.16

1. Let G_1 and G_2 be graphs with $v_i \in V(G_i)$. Let $H = \mathrm{VJ}(\{G_1, v_1\}, \{G_2, v_2\})$. Suppose that G_2 is always winnable and that $l_{G_2}(v_2) = 1$. Then

$$\dim \mathrm{NS}_k(N(H)) = \dim \mathrm{NS}_k(N(G_1 - v_1)).$$

2. Let H be a graph that has a degree 1 vertex x adjacent to a degree 2 vertex w. Let v be the vertex of H other than x that is adjacent to w. Then

$$\dim \mathrm{NS}_k(N(H)) = \dim \mathrm{NS}_k(N(H - \{v, w, x\})).$$

Proof. (1) Let v be the vertex of H corresponding to the identification $v_1 = v_2$. By Theorem 2.10, $l_H(v) = 1$. Therefore,

$$\begin{aligned}
\dim \mathrm{NS}_k(N(H)) &= \dim \mathrm{NS}_k(N(H-v)) - l_H(v) \\
&= \dim \mathrm{NS}_k(N(G_1 - v_1)) + \dim \mathrm{NS}_k(N(G_2 - v_2)) - l_H(v) \\
&= \dim \mathrm{NS}_k(N(G_1 - v_1)) + \dim \mathrm{NS}_k(N(G_2)) + l_{G_2}(v_2) - l_H(v) \\
&= \dim \mathrm{NS}_k(N(G_1 - v_1)),
\end{aligned}$$

where the last inequality is true since $\dim \mathrm{NS}_k(N(G_2)) = 0$ and $l_{G_2}(v_2) = l_H(v) = 1$.

(2) This comes from part (1) applied to $G_1 = H - \{w, x\}$ and $G_2 = P_3$, a path with 3 vertices $\{v', w, x\}$ where $\deg(w) = 2$. For all k, P_3 is always winnable, and $l_{P_3}(v') = 1$. ☐

Corollary 2.17 Let H be a graph and $\{v_1, \ldots, v_k\} \in V(H)$ such that $\deg(v_i) = 1$ for all i and each v_i is adjacent to the same vertex $x \in V(H)$. Then

$$\dim \mathrm{NS}_k(N(H)) = \dim \mathrm{NS}_k(N(H - \{v_i : i = 1, \ldots, k\})).$$

Proof. We apply Theorem 2.10 to the graphs $\{G = H - \{v_i : i = 1 \ldots k\}, x\}$ and $\{E_i = P_2, x\}$ for $i = 1, \ldots, k$, where $V(E_i) = \{x, v_i\}$. We have $l_{E_i}(x) = -1$ for all i. By Theorem 2.10, we have:

$$l_H(x) = \begin{cases} 1 & \text{if } l_G(x) = 1 \\ -1 & \text{if } l_G(x) = -1 \\ 0 & \text{if } l_G(x) = 0(\lambda) \end{cases}$$

where the last two equalities are true since there are $m = k+1$ graphs involved. Since $l_H(x) = l_G(x)$ in every case and $\dim \mathrm{NS}_k(N(E_i - \{x\})) = 0$ for all i, we have $\dim \mathrm{NS}_k(N(H)) = \dim \mathrm{NS}_k(N(G))$. □

Proposition 2.18 Let H be a graph with distinct vertices $v, p_1, p_2, w \in V(H)$ such that $\deg(p_1) = \deg(p_2) = 2$, $\{(v, p_1), (p_1, p_2), (p_2, w)\} \subseteq E(H)$, and $(v, w) \notin E(H)$. Let H' be the graph defined by identifying the vertices v and w inside $H - \{p_1, p_2\}$.

1. For every $\mathbf{p} \in \mathrm{NS}_k(N(H))$, we have $\mathbf{p}(v) = \mathbf{p}(w)$.
2. The induced mapping $\mathrm{NS}_k(N(H)) \to \mathrm{NS}_k(N(H'))$ is an isomorphism, and therefore

$$\dim \mathrm{NS}_k(N(H)) = \dim \mathrm{NS}_k(N(H')).$$

Proof. (1) Let $\mathbf{p} \in \mathrm{NS}_k(N(H))$. Then

$$\begin{aligned} \mathbf{p}(v) + \mathbf{p}(p_1) + \mathbf{p}(p_2) &= 0 \quad \pmod k \\ \mathbf{p}(p_1) + \mathbf{p}(p_2) + \mathbf{p}(w) &= 0 \quad \pmod k \end{aligned}$$

This shows that $\mathbf{p}(v) = \mathbf{p}(w)$.

(2) Let $\mathbf{p} \in \mathrm{NS}_k(N(H))$. Since $\mathbf{p}(v) = \mathbf{p}(w)$, $\mathbf{p}$ naturally induces a pattern on H' which we will denote by $\mathbf{p}'$. We will now show that $\mathbf{p}' \in \mathrm{NS}_k(N(H'))$. Let $t = \sum_{u \in X(v) \setminus \{v, p_1\}} \mathbf{p}(u)$ and $s = \sum_{u \in X(w) \setminus \{w, p_2\}} \mathbf{p}(u)$.

Since $\mathbf{p}$ is null at v and w, we have

$$\begin{aligned} \mathbf{p}(v) + \mathbf{p}(p_1) + t &= 0 \quad \pmod k \\ \mathbf{p}(p_2) + \mathbf{p}(w) + s &= 0 \quad \pmod k \end{aligned}$$

When combined with the equations in part (1), this implies

$$\begin{aligned} t &= \mathbf{p}(p_2) \quad \pmod k \\ s &= \mathbf{p}(p_1) \quad \pmod k \end{aligned}$$

Clearly $\mathbf{p}'$ is null on H' except possibly at v', the vertex created by the identification of v with w. We have

$$\begin{aligned} \textstyle\sum_{u \in X(v')} \mathbf{p}'(u) &= \mathbf{p}'(v') + s + t \\ &= \mathbf{p}(v) + \mathbf{p}(p_1) + \mathbf{p}(p_2) \\ &= 0 \qquad \pmod k \end{aligned}$$

Hence, $\mathbf{p}' \in \mathrm{NS}_k(N(H'))$, and $\mathbf{p} \mapsto \mathbf{p}'$ gives a linear transformation from $\mathrm{NS}_k(N(H))$ to $\mathrm{NS}_k(N(H'))$. To see that this linear transformation is bijective, notice that any null pattern $\mathbf{q}'$ on H' can be extended uniquely to a null pattern $\mathbf{q}$ on H as follows:

- $\mathbf{q}$ is identical to $\mathbf{q}'$ away from $\{v, p_1, p_2, w\}$,
- $\mathbf{q}(v) = \mathbf{q}(w) = \mathbf{q}'(v')$,
- $\mathbf{q}(p_1) = -\sum_{u \in X(v)\setminus\{p_1\}} \mathbf{q}'(u)$,
- $\mathbf{q}(p_2) = -\sum_{u \in X(w)\setminus\{p_2\}} \mathbf{q}'(u)$

The pattern $\mathbf{q}$ is null by construction on vertices of H not in $\{p_1, p_2\}$. To see that $\mathbf{q}$ is null at p_1, note that

$$\begin{aligned}\mathbf{q}(v) + \mathbf{q}(p_1) + \mathbf{q}(p_2) &= \mathbf{q}'(v') - \textstyle\sum_{u \in X(v)\setminus\{p_1\}} \mathbf{q}(u) - \sum_{u \in X(w)\setminus\{p_2\}} \mathbf{q}(u) \\ &= \mathbf{q}'(v') - \mathbf{q}'(v') - \textstyle\sum_{u \in X(v')} \mathbf{q}'(u) \\ &= 0 \pmod{k}\end{aligned}$$

where the last equality is true because $\mathbf{q}'$ is null on H'. Now $\mathbf{q}$ is also null at p_2 since $\mathbf{q}(v) = \mathbf{q}(w)$. □

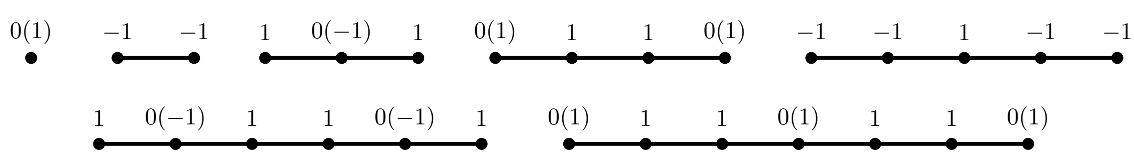
3 Some Examples and Applications

The following information for paths, cycles, complete graphs, and complete bipartite graphs can be obtained directly, but also follows from [5, Theorem 4.4], which gives the result in terms of winnable states.

Paths: For P_n, a path with n vertices, we have

$$\dim \mathrm{NS}_k(N(P_n)) = \begin{cases} 0 & \text{if } n \neq 2 \pmod{3} \\ 1 & \text{if } n = 2 \pmod{3} \end{cases}$$

This shows that the labels of the vertices of P_n follow the following pattern:



When $n = 2 \pmod{3}$, a basis for $\mathrm{NS}_k(N(P_n))$ is given by a pattern of the form

$$(1, -1, 0, 1, -1, 0, \dots, 1, -1),$$

where the vertices are listed in the order that they are connected along the path.

Cycles: Let C_n be the n-cycle. Then

$$\dim \mathrm{NS}_k(N(C_n)) = \begin{cases} 0 & \text{if } 3 \nmid n \text{ and } k \neq 3 \\ 1 & \text{if } 3 \nmid n \text{ and } k = 3 \\ 2 & \text{if } 3 | n \end{cases}$$

If $v \in V(C_n)$, the label on v is given by

$$l_{C_n}(v) = \begin{cases} 0(3) & \text{if } n = 1 \pmod 3 \text{ and } k \neq 3 \\ 0(-3) & \text{if } n = 2 \pmod 3 \text{ and } k \neq 3 \\ -1 & \text{if } 3|n \text{ or } k = 3 \end{cases}$$

If 3 is not a divisor of n and $k = 3$, a basis for $\mathrm{NS}_k(N(C_n))$ is given by the pattern $\mathbf{p}$ such that $\mathbf{p}(v) = 1$ for every vertex $v \in V(C_n)$. If $3|n$, a basis for $\mathrm{NS}_k(N(C_n))$ is given by

$$\{(1, -1, 0, 1, -1, 0, \ldots, 1, -1, 0), (0, 1, -1, 0, 1, -1, \ldots, 0, 1, -1)\},$$

where the vertices are listed in the order given by proceeding around the cycle.

Complete Graphs: We have $\dim \mathrm{NS}_k(N(K_n)) = n-1$, and $l_{K_n}(v) = -1$ for all $v \in V(K_n)$. Choose a vertex $v \in V(K_n)$. A basis of $\mathrm{NS}_k(N(K_n))$ is given by the set of patterns of the form $\mathbf{p}_w$ where $w \in V(K_n) \setminus \{v\}$, $\mathbf{p}_w(v) = 1$, $\mathbf{p}_w(w) = -1$, and $\mathbf{p}_w(u) = 0$ if $u \in V(K_n) \setminus \{v, w\}$.

Complete Bipartite Graphs: Let $K_{m,n}$ be the complete bipartite graph on m and n vertices. We will refer to the set of m vertices as the 'left-hand' vertices and the set of n vertices as the 'right-hand' vertices. We have

$$\dim \mathrm{NS}_k(N(K_{m,n})) = \begin{cases} 0 & \text{if } k \nmid (mn-1) \\ 1 & \text{if } k|(mn-1) \end{cases}$$

When $k|(mn-1)$, a basis of $\mathrm{NS}_k(N(K_{m,n}))$ is given by the pattern $\mathbf{p}$ which has value n at all left-hand vertices and value -1 at all right-hand vertices.

If $k|(mn-1)$, then k can divide neither $(m-1)n-1$ nor $m(n-1)-1$, and therefore, $l_{K_{m,n}}(v) = -1$ for all $v \in V(K_{m,n})$. If $k \nmid (mn-1)$, then k may divide neither, one or both of $(m-1)n-1$ and $m(n-1)-1$. We summarize the possibilities in the following chart. Here, 'LH label' means the label on the left-hand vertices, and 'RH label' means the label on the right-hand vertices, and $\lambda_L = (mn-1)(mn-n-1)^{-1}$ and $\lambda_R = (mn-1)(mn-m-1)^{-1}$.

$k\|(mn-1)$	$k\|((m-1)n-1)$	$k\|(m(n-1)-1)$	LH label	RH label
Yes	No	No	-1	-1
No	No	No	$0(\lambda_L)$	$0(\lambda_R)$
No	No	Yes	$0(\lambda_L)$	1
No	Yes	No	1	$0(\lambda_R)$
No	Yes	Yes	1	1

We give an application to generalized star graphs (called 'spider graphs' in [4]). A **generalized star** is a connected graph of the form $G = \mathrm{VJ}(\{P_{n_i}, v_i\})$, where $n_i \geq 2$ are integers, P_{n_i} is a path with n_i vertices, and v_i is a degree 1 vertex of P_{n_i}. The vertex $v \in V(G)$ is called the **center** of G, and $G - v$ is a disjoint union of the paths $P_{n_i - 1}$. To avoid trivial cases, we assume $\deg(v) > 2$. Every other vertex of G has degree 1 or 2.

The following result gives a more general version of [4, Theorem 3.4].

Proposition 3.1 Let $G = \mathrm{VJ}(\{P_{n_i}, v_i\})$ be a generalized star as defined above, where $\{n_1, n_2, \ldots, n_m\}$ is a set of integers with $n_i \geq 2$ and $m \geq 3$. For $j \in \{0, 1, 2\}$, let p_j be the number of n_i such that $n_i = j \pmod 3$. Then

$$l_G(v) = \begin{cases} -1 & \text{if } p_0 = 0 \text{ and } k|(p_2-1) \\ 0(1-p_2) & \text{if } p_0 = 0 \text{ and } k \not| (p_2-1) \\ 1 & \text{if } p_0 \neq 0 \end{cases}$$

This implies that

$$\dim \mathrm{NS}_k(N(G)) = \begin{cases} 1 & \text{if } p_0 = 0 \text{ and } k|(p_2-1) \\ 0 & \text{if } p_0 = 0 \text{ and } k \not| (p_2-1) \\ p_0 - 1 & \text{if } p_0 \neq 0 \end{cases}$$

In particular, G is always winnable over $\mathbb{Z}_k$ if and only if either $p_0 = 1$ or both $p_0 = 0$ and $k \not| (p_2 - 1)$.

Proof. This follows immediately from Theorem 2.10 and the characterization of paths given above. □

Let S_n be the star with $n \geq 3$ edges. By Proposition 3.1, $\dim \mathrm{NS}_k(N(S_n)) = 1$ if $k|(n-1)$ and $\dim \mathrm{NS}_k(N(S_n)) = 0$ otherwise. This implies that for $v \in V(S_n)$ such that $\deg(v) = 1$, we have

$$l_{S_n}(v) = \begin{cases} -1 & \text{if } k|(n-1) \\ 1 & \text{if } k|(n-2) \\ 0((n-1)(n-2)^{-1}) & \text{otherwise} \end{cases}$$

Proposition 3.2 Let $G = \mathrm{VJ}(\{S_{n_i}, v_i\})$, where $\{n_1, n_2, \ldots, n_m\}$ is a set of integers with $n_i \geq 2$, $m \geq 2$ and $\deg_{S_{n_i}}(v_i) = 1$. Let v be the vertex of G created by the identification of the vertices v_i. Let p_2 be the number of the n_i such that $n_i = 2 \pmod k$. Then

$$l_G(v) = \begin{cases} -1 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} = m-1 \pmod k \\ 0 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} \neq m-1 \pmod k \\ 1 & \text{if } p_2 \neq 0 \end{cases}$$

which implies that

$$\dim \mathrm{NS}_k(N(G)) = \begin{cases} 1 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} = m-1 \pmod k \\ 0 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} \neq m-1 \pmod k \\ p_2 - 1 & \text{if } p_2 \neq 0 \end{cases}$$

In particular, G is always winnable over $\mathbb{Z}_k$ if and only if either $p_2 = 1$ or both $p_2 = 0$ and

$$\sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} \neq m-1 \pmod k.$$

Proof. This follows immediately from Theorem 2.10 using the properties of stars given after Proposition 3.1 and the properties of P_2 and P_3 (to handle the case that n_i might be equal to 2 for some values of i). □

Proposition 3.3 Let $G = \mathrm{VJ}(\{C_{n_i}, v_i\})$, where $\{n_1, n_2, \ldots, n_m\}$ is a set of integers with $n_i \geq 3$, $m \geq 2$, and $v_i \in V(C_{n_i})$. Let v be the vertex of G created by the identification of the vertices v_i. For $j \in \{0, 1, 2\}$, let p_j be the number of n_i that are congruent to j modulo 3. Then

$$l_G(v) = \begin{cases} -1 & \text{if } 3(p_1 - p_2) = m - 1 \pmod{k} \\ 0 & \text{if } 3(p_1 - p_2) \neq m - 1 \pmod{k} \end{cases}$$

which implies that

$$\dim \mathrm{NS}_k(N(G)) = \begin{cases} p_0 + 1 & \text{if } 3(p_1 - p_2) = m - 1 \pmod{k} \\ p_0 & \text{if } 3(p_1 - p_2) \neq m - 1 \pmod{k} \end{cases}$$

In particular, G is always winnable over $\mathbb{Z}_k$ if and only if $p_0 = 0$ and $3(p_1 - p_2) \neq m - 1 \pmod{k}$.

Proof. This follows immediately from Theorem 2.10 and the characterization of cycles and paths given above. □

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