Problem 1: The (Alternating) p-series & the Harmonic Series

(a) Recall that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise. Explain why the Integral Test appliesx to the p-series then use WolframAlpha to apply the Integral Test to the p-series. Be sure to indicate what you input into WolframAlpha.

(b) The Alternating p-series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$. Explain why the Alternating Series Test Applies and use this to find values for p that make the series converge. Are these the only possible values for p? If so, *carefully* explain why and if not explain why not not.

(c) Using the previous part, choose a value for p for which the Alternating p-series converges and find how many terms you would need to add to approximate the series to three decimal places. Use WolframAlpha to find this sum. Again, clearly indicate what you inputed into WolframAlpha.

Often with the Alternating p-series, you can add very “few" terms and estimate the sum very accurately. On the other hand, while the Harmonic Series does diverges, looking at the partial sums does make it seem like it might converge as the sums grow very slowly. That is, the sum is not very large even after adding many terms.

(d) Sketch the graph of the values of the Harmonic Sequence, i.e. plot the points $(n, 1/n)$. On the same sketch, plot $1/x$. Use this plot and the Riemann integral to show for $n > 1$ that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

(e) Use the previous part to give the maximum value of the first one million terms of the Harmonic Series. Recently the worlds fastest supercomputer was built in China. This computer can roughly perform 10^{15} additions per second. If this computer were to add terms of the Harmonic Series every second from the moment of your birth to till the moment of your death, what value of the Harmonic Series did it reach?

(f) Calculate at least how many terms you would need to add to get a sum in the Harmonic Series of at least 200. Discuss this answer by comparing this value to the estimated number of particles in the known universe.

Problem 2: An Almost Harmonic Series

(a) Explain why the Integral Test *can* be used to show the divergence of $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ then use the Integral Test to show that the series diverges. You cannot use WolframAlpha; you must show the integration directly.

(b) Does this series diverge faster or slower than the Harmonic Series? Considering Problem 1 is this surprising?

Recall the following "regrouping" that shows the Harmonic Series Diverges

$$\begin{aligned} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{2}+\cdots \end{aligned}$$

[Note: there is actually a shorter way of doing this by assuming the sum converges, say to a, "regrouping" similarly to above, and obtaining $a \geq \frac{1}{2}+a$.]

(c) By regrouping as above, show that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges without using the integral test.

(d) Find the values p so that $\sum_{n=3} \frac{1}{n(\ln n)^p}$ converges.

Problem 3: Alternating Series Test

Of the the following series, determine to which of the series the Alternating Series Test applies. Be sure to briefly explain why or why not.

(i) $\sum_{n=0}^{\infty}(-1)^{n-1} n$

(ii) $\sum_{n=0}^{\infty}(-1)^{n-1} \sin n$

(iii) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(2-\frac{1}{n}\right)$

(iv) $\sum_{n=1}^{\infty}(-1)^{n} \cos\left(\frac{\pi}{n}\right)$

(v) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

Now consider the following series:

$$\sum_{n=0}^{\infty}(-1)^{n} \frac{5n^2}{n^3+9}$$

(b) Because of the $(-1)^n$ term, this series is clearly alternating. Show that the series has decreasing terms for sufficiently large n.

(c) Show that the series converges by the Alternating Series Test.

(d) Find the sum of the first 5 terms of the series. How close is this to the actual sum of the series?

(e) Does this series converge absolutely? If so, show this and if not explain why.

Problem 4: Comparison vs. Limit Comparison

Using the Limit Comparison Test to an appropriate series, determine whether the following series converge or diverge.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}+\sqrt{n+5}}$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n-2}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{2n^4-n}$

Using an appropriate series, use the Comparison Test to show that the above series converge/diverge.

Problem 5: Ratio & Root Test

(a) Use the Root Test to determine the behavior of the following series: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{5^n(n^4+2)}$. Be sure to be as specific as possible.

(b) Use the Ratio Test to determine the behavior of the following series: $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n}{3^{2n+1}(n+2)}$. Be sure to be as specific as possible.

(c) Does the following series converge or diverge: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Explain. What does the Root Test say about its behavior? What does the Ratio Test say about its behavior?

Evaluation

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing" while a 5 is “mind-blowing". For difficulty, 1 is “trivial/routine" while 5 is “brutal." For learning, 1 means “nothing new" while 5 means "profound awakening". Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				

Finally, indicate whether you believe lectures were useful in completing this assignment and whether you believe the problems were useful enough/interesting enough to assign again to future students by checking the appropriate space.

	Lectures		Assign Again	
	Yes	No	Yes	No
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				