Determinantal Rings

MGO Conference Talk 4-8-17

What is a determinantal ring?

We assume $m \le n$ are positive integers, ~~and~~ that X is the $m \times n$ matrix

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$$

For $t \in \mathbb{Z}^+$ with $1 \le t \le m$ and any field k (can consider k a ring, but we are considering k a field), form the <u>determinantal ring</u> $R = \dfrac{k[X]}{I_t(X)}$,

where $k[X] = k[x_{ij}] = k[x_{11}, \ldots, x_{mn}]$

and $I_t(X) = \{$ideal generated by all $t \times t$ minors of X (determinants)$\}$

⌈Note: in $(m\times n)$ matrices of rank $<t$, it is the $t\times t$ minors that vanish, so that $R=\frac{k[X]}{I_t(X)}$ is the ring of polynomial functions on the set of $m\times n$ matrices with rank $<t$⌋

Some properties of Determinantal Rings:

- Cohen-Macaulay
- sometimes Gorenstein
- easily described singular locus
- rings of Grassmannians
- complex enough to vary greatly in their behavior, so good to use for checking conjectures

say, don't write

Finding systems of parameters for determinantal rings when $t=2$

What is a system of parameters (s.o.p)?

R commutative, noetherian, (local)

Def: $\dim R = \max\{n \mid p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$ is a chain of prime ideals in $R\}$

Def: A s.o.p is a collection of elts. $x_1, \ldots, x_{\dim R} \in R$ s.t. $\dim \left(R/(x_1, \ldots, x_{\dim R}) \right) = 0$

For determinantal rings,

Fact 1: $\dim k[X] = mn$

$\dim R = (m+n-t+1)(t-1)$

"dim" for ideals $\rightarrow$ $\operatorname{ht} I_t = (m-t+1)(n-t+1)$

$$\left[CM: \dim \underbrace{\frac{k[X]}{I_t(X)}}_{= R} + \operatorname{ht} I_t = \dim k[X] \right]$$

Fact 2: If $\alpha \in R$ is nilpotent, then $\dim R = \dim (R/\alpha)$

Fact 3: For $t = 2$, R has a linear s.o.p given by the sums of variables along "diagonals," that is, along lines of slope -1 in the matrix X.

<u>Example</u>: $m=2, n=3, t=2$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \Longrightarrow$$

~~According to~~ Fact 1, $\dim R = (m+n-t+1)(t-1)$
$= 4$,

so a(n) s.o.p. should have four elements.

~~According to~~ Fact 3 $\Rightarrow$ $\underbrace{x_{21}, x_{11}+x_{22}, x_{12}+x_{23}, x_{13}}_{*}$

is a(n) s.o.p. for R

<u>Check</u>: Need $\dim\left(R/(*)\right) = 0$ for $*$ to be a(n) s.o.p.

$$R = \frac{k[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]}{(\underbrace{x_{11}x_{22}}_{-x_{11}^2} - x_{12}x_{21}, \underbrace{x_{11}x_{23}}_{-x_{11}x_{12}} - x_{13}x_{21}, \underbrace{x_{12}x_{23}}_{-x_{12}^2} - x_{13}x_{22})}$$

$\Big\downarrow$ mod $x_{21}, x_{13}, x_{11}+x_{22}, x_{12}+x_{23}$

$$R/(*) = \left.\frac{k[x_{11}, x_{12}]}{(x_{11}^2, x_{11}x_{12}, x_{12}^2)}\right\} S$$

Fact 2 $\Rightarrow$ (since x_{11} and x_{12} are now nilpotent)

$$\dim R/(*) = \dim S/(x_{11}, x_{12}) = \dim k = 0 \checkmark$$

$\therefore *$ is a(n) s.o.p.