Isometric Immersions of a Euclidean Plane Into Hyperbolic Space

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Overview

The Second Fundamental Form

The Second Fundamental Form reflects extrinsic geometry of a surface.

Consider a surface $S = \mathbf{x}(D)$ defined by the coordinate patch: $\mathbf{x} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. Consider $(u, v) \in D$

$$x_1 = \frac{\partial x}{\partial u} x_2 = \frac{\partial x}{\partial v}$$

$$x_{11} = \frac{\partial x}{\partial u^2} \quad x_{12} = \frac{\partial x}{\partial v \partial u} \quad x_{21} = \frac{\partial x}{\partial u \partial v} \quad x_{22} = \frac{\partial x}{\partial v^2}.$$

Note that $x_{12} = x_{21}$ by the equality of mixed partials.

The Second Fundamental Form

Consider a curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ on the surface S

$$\gamma'(t) = u'\mathbf{x}_1 + v'\mathbf{x}_2 \tag{1}$$

$$\begin{aligned} \gamma''(t) &= u''\mathbf{x}_1 + u'(u'\mathbf{x}_{11} + v'\mathbf{x}_{12}) + v''\mathbf{x}_2 + v'(u'\mathbf{x}_{21} + v'\mathbf{x}_{22}) \\ &= u''\mathbf{x}_1 + v''\mathbf{x}_2 + u'^2\mathbf{x}_{11} + 2u'v'\mathbf{x}_{12} + v'^2\mathbf{x}_{22}. \end{aligned} \tag{2}$$

Goal: Decompose γ'' into normal and tangential components

The Second Fundamental Form

Γ_{ij}^k where $i, j, k = 1, 2$, Christoffel symbols, denote the coefficients of the tangential component.
(Note that this uses Einstein notation: $\Gamma_{ij}^k \mathbf{x}_k = \sum_k \Gamma_{ij}^k \mathbf{x}_k$)
L_{ij} where $i, j = 1, 2$ denote the coefficient of the normal component

$$\mathbf{x}_{ij} = \Gamma_{ij}^1 x_1 + \Gamma_{ij}^2 x_2 + L_{ij}\mathbf{n} = \Gamma_{ij}^k \mathbf{x}_k + L_{ij}\mathbf{n}. \quad (3)$$

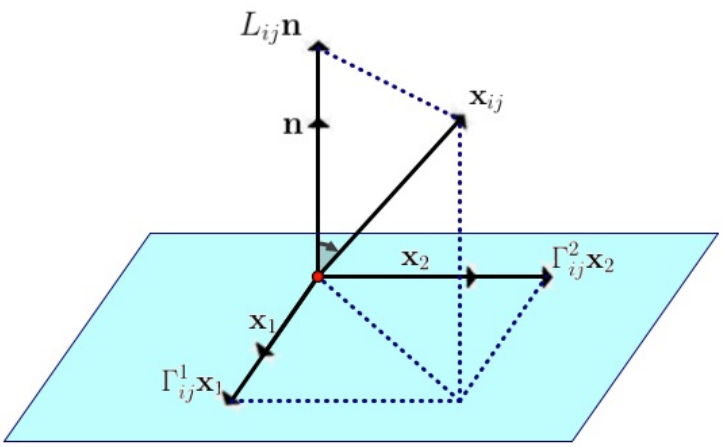

Figure: Tangent Plane

The Second Fundamental Form, Christoffel Symbols

$$I = dS^2 = Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2 = g_{ij}dx_idx_j \quad i,j = 1,2$$

$$[g_{ij}] = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \cdot \mathbf{x}_1 & \mathbf{x}_1 \cdot \mathbf{x}_2 \\ \mathbf{x}_2 \cdot \mathbf{x}_1 & \mathbf{x}_2 \cdot \mathbf{x}_2 \end{pmatrix}.$$

Let $g = det[g_{ij}]$.

$$g^{ij} = [g_{ij}]^{-1} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \tag{4}$$

Multiplying Equation (3) by $\mathbf{x}_l$ where $i, j, k, l = 1, 2$:

$$\mathbf{x}_{ij} \cdot \mathbf{x}_l = \Gamma^k_{ij}\mathbf{x}_k \cdot \mathbf{x}_l + L_{ij}\mathbf{n} \cdot \mathbf{x}_l = \Gamma^k_{ij}g_{kl}. \tag{5}$$

The Second Fundamental Form, Christoffel Symbols

Solving Equation (5) for the Christoffel symbol we see

$$\Gamma_{ij}^{k} = (x_{ij} \cdot x_l) g^{lk}. \tag{6}$$

Rewriting this in terms of a metric:

$$\Gamma_{ij}^{k} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}. \tag{7}$$

The Second Fundamental Form, Normal Component

Solving for the coefficients of the second fundamental form we will multiply Equation (3) by $\mathbf{n}$

$$\mathbf{x}_{ij} \cdot \mathbf{n} = \Gamma_{ij}^{k} \mathbf{x}_k \cdot \mathbf{n} + L_{ij} \mathbf{n} \cdot \mathbf{n} = L_{ij}.$$

The Second Fundamental Form

Referring back to Equation (2) we see that for $i, j, k = 1, 2$:

$$\begin{aligned} \gamma'' &= (u^i)'' x_i + (u^i)'(u^j)'(\Gamma_{ij}^k \mathbf{x}_k + L_{ij}\mathbf{n}) \\ &= ((u^k)'' + \Gamma_{ij}^k (u^i)'(u^j)')\mathbf{x}_k + (u^i)'(u^j)' L_{ij}\mathbf{n}. \end{aligned} \tag{8}$$

The coefficients of the second fundamental form are the coefficients of the normal term, so

$$II = (u^i)'(u^j)' L_{ij}.$$

Gauss' Equation and the Codazzi-Mainardi Equation

Riemann curvature tensor:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk} \tag{9}$$

Gauss' Equation:

$$R^l_{ijk} = L_{ik}L_{jp}g^{pl} - L_{ij}L_{kp}g^{pl} \qquad (i, j, k, l, p = 1, 2). \tag{10}$$

Codazzi-Mainardi Equation:

$$\frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \Gamma^l_{ik}L_{lj} - \Gamma^l_{ij}L_{lk}. \tag{11}$$

Manifold

(O'Neill) An n-dimensional manifold M is a set furnished with a collection $\mathscr{P}$ of abstract patches (smooth, one-to-one functions $\mathbf{x} : D \to M$, D and open set in $\mathbb{R}^n$) satisfying:

1. The covering property: The images of the patches in the collection $\mathscr{P}$ cover M.
2. The smooth overlap property: For any patches $\mathbf{x}, \mathbf{y}$ in $\mathscr{P}$, the composite functions $\mathbf{y}^{-1}\mathbf{x}$ and $\mathbf{x}^{-1}\mathbf{y}$ are Euclidean differentiable – and defined on open sets of $\mathbb{R}^n$.
3. The Hausdorff property: For any points $\mathbf{p} \neq \mathbf{q}$ in M there are disjoint patches $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{p}$ in $\mathbf{x}(D)$ and $\mathbf{q}$ in $\mathbf{y}(E)$.

The Fundamental Theorem of Surfaces

(Spivak) Let $U \subset R^2$ be a convex open set containing the origin.

1. Let $\mathbf{x}, \overline{\mathbf{x}} : U \to R^3$ be two immersions and define:

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle \qquad \overline{g}_{ij} = \langle \overline{\mathbf{x}}_i, \overline{\mathbf{x}}_j \rangle$$

$$N = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g_{11} g_{22} - {g_{12}}^2}} \qquad \overline{N} = \frac{\overline{\mathbf{x}}_1 \times \overline{\mathbf{x}}_2}{\sqrt{\overline{g}_{11} \overline{g}_{22} - {\overline{g}_{12}}^2}}$$

$$L_{ij} = \langle -N_i, \mathbf{x}_j \rangle = \langle N, \mathbf{x}_{ij} \rangle \qquad \overline{L}_{ij} = \langle -\overline{N}_i, \overline{\mathbf{x}}_j \rangle = \langle \overline{N}, \overline{\mathbf{x}}_{ij} \rangle$$

Suppose that $g_{ij} = \overline{g}_{ij}$ and $L_{ij} = \overline{L}_{ij}$ on U. Then there is a proper Euclidean motion A such that $\overline{\mathbf{x}} = A \circ \mathbf{x}$.

The Fundamental Theorem of Surfaces

2. Let g_{ij} and L_{ij} $(i, j, = 1, 2)$ be functions on U which satisfy:
   1. $g_{ij} = g_{ji}$ and $L_{ij} = L_{ji}$ and (g_{ij}) is positive definite on U,
   2. Gauss' Equation:

$$L_{11}L_{22} - (L_{12})^2 = R_{1212} = \sum_{\rho=1}^{2} g_{1\rho} \left(\Gamma^{\rho}_{22,1} - \Gamma^{\rho}_{21,1} + \sum_{h=1}^{2} (\Gamma^{h}_{22}\Gamma^{\rho}_{21} - \Gamma^{h}_{21}\Gamma^{\rho}_{h2}) \right) \tag{12}$$

   3. The Codazzi-Mainardi Equations:

$$\begin{aligned} L_{12,1} - L_{11,2} + \sum_{h=1}^{2} \Gamma^{h}_{12} L_{h1} - \sum_{h=1}^{2} \Gamma^{h}_{11} L_{h2} = 0 \\ L_{22,1} - L_{21,2} + \sum_{h=1}^{2} \Gamma^{h}_{22} L_{h1} - \sum_{h=1}^{2} \Gamma^{h}_{21} L_{h2} = 0. \end{aligned} \tag{13}$$

The Fundamental Theorem of Surfaces

Then there is an immersion $\mathbf{x} : U \to \mathbb{R}^3$ such that:

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle,$$

and

$$L_{ij} = \langle -N_i, \mathbf{x}_j \rangle = \langle N, \mathbf{x}_{ji} \rangle, \text{ for } N = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g_{11}g_{22} - {g_{12}}^2}}.$$

Immersion vs. Embedding

Immersion: (O'Neill) if M is an abstract surface and $F : M \to \mathbb{R}^3$ is merely regular, then F is an **immersion** of M into $\mathbb{R}^3$, and the image $F(M)$ is often called an "immersed surface."

Embedding: (O'Neill) if M is an abstract surface, a proper **embedding** of M into $\mathbb{R}^3$ is a one-to-one regular mapping $F : M \to \mathbb{R}^3$ such that the inverse function $F^{-1} : F(M) \to M$ is continuous.

Volkov and Vladamirova: Theorem

Let $f : \mathbb{R}^2 \to \mathbb{H}^3$ be a regular isometric immersion of a Euclidean plane ($\mathbb{R}^2$) in three-dimensional hyperbolic space ($\mathbb{H}^3$). Then only one of the following two situations is possible:

1. f is a homeomorphism, and $f(\mathbb{R}^2)$ is a horosphere.
2. f is a locally isometric covering of the surface formed by the rotation of a particular equidistant about its base line (i.e. an equidistant cylinder)

Volkov and Vladamirova: Theorem

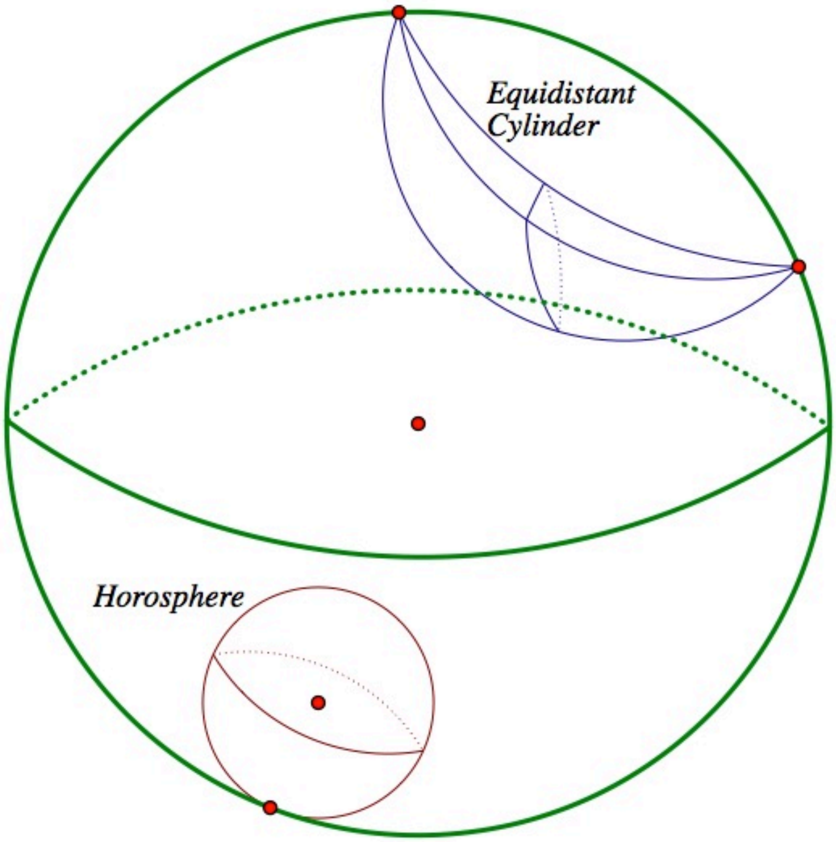

Figure: Poincaré Ball Model Representation of Theorem

Volkov and Vladamirova: Observations

1. The search for isometric immersions of a two-dimensional Riemannian manifold V^2 in $\mathbb{H}^3$ reduces to the construction of V^2 from the second fundamental form of the sought-after immersion.

Volkov and Vladamirova: Observations

2. In a local coordinate system (u_1, u_2) on V^2 the coefficients L_{ij} of the second quadratic form satisfy the Gauss and Codazzi-Mainardi equations:

$$\begin{aligned} \frac{\partial L_{12}}{\partial u_1} - \frac{\partial L_{11}}{\partial u_2} + \tilde{\Gamma}^k_{12} L_{1k} - \tilde{\Gamma}^k_{11} L_{2k} &= -R_{1312} \\ \frac{\partial L_{22}}{\partial u_1} - \frac{\partial L_{12}}{\partial u_2} + \tilde{\Gamma}^k_{22} L_{1k} - \tilde{\Gamma}^k_{12} L_{2k} &= -R_{2312} \\ L_{11} L_{22} - L_{12}^2 &= \tilde{R}_{1212} - R_{1212}. \end{aligned} \tag{14}$$

$\tilde{\Gamma}^k_{ij}$, where $(i, j, k = 1, 2)$: the Christoffel symbols of V^2
$\tilde{R}$: curvature tensor of V^2
R_{ijkl}: the components of the curvature tensor of $\mathbb{H}^3$ in any orthogonal coordinate system $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ such that $\mathbf{v}_1 = u_1, \mathbf{v}_2 = u_2$ and $g_{33} = 1$, where g_{ik} is the metric tensor of $\mathbb{H}^3$ in the system $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Volkov and Vladamirova: Observations

3. Because $\mathbb{H}^3$ is a space with constant curvature -1 its curvature tensor is related to the metric tensor and curvature by the relations:

$$R_{ijkl} = -(-1)(g_{ik}g_{jl} - g_{il}g_{jk}) = g_{ik}g_{jl} - g_{il}g_{jk}$$

In an orthogonal system this yields:

$$\begin{aligned} R_{ijkl} &= 0 \quad (j \neq k) \\ R_{ijij} &= g_{ii}g_{jj} - g_{ij}^2 \quad (i \neq j). \end{aligned} \tag{15}$$

Volkov and Vladamirova: Observations

4. Let the immersible manifold be $\mathbb{R}^2$ with Euclidean metric, and let (u_1, u_2) be Cartesian coordinates. Then:

$$\tilde{\Gamma}^i_{jk} = \tilde{R}_{ijkl} = 0,$$

and, taking into account Equation (14), Equation (15) assumes the form:

$$\begin{aligned} \frac{\partial L_{12}}{\partial u_1} - \frac{\partial L_{11}}{\partial u_2} &= 0 \\ \frac{\partial L_{22}}{\partial u_1} - \frac{\partial L_{12}}{\partial u_2} &= 0 \\ L_{11}L_{22} - L_{12}^2 &= 1. \end{aligned} \tag{16}$$

Volkov and Vladamirova: Observations

5. In hyperbolic space with curvature $k = -1$ a horosphere has principal curvatures $k_1 = k_2 = 1$. Introducing "Cartesian" coordinates (u_1, u_2) on such a horosphere, we find that it's first and second fundamental forms are written:

$$I = II = (du_1)^2 + (du_2)^2.$$

The first and second fundamental forms of an equidistant cylinder are:

$$I = (du_1)^2 + (du_2)^2$$

$$II = a(du_1)^2 + \frac{1}{a}(du_2)^2 \qquad a = \tanh r$$

Volkov and Vladamirova: Result

Every solution of System (16) has the form:

$$L_{11} = \frac{1}{L_{22}} = a, \quad L_{12} = 0$$

Thus:

- if $|a| = 1$, we obtain a horosphere;
- if $|a| \neq 1$ the surface in question has the same first and second fundamental forms as the surface obtained by the rotation of a certain equidistant, so that by the Fundamental Theorem of Surfaces the surfaces are equal. (Note, we know that this is global because our original manifold, $\mathbb{R}^2$, is simply connected.)

Volkov and Vladamirova: Results

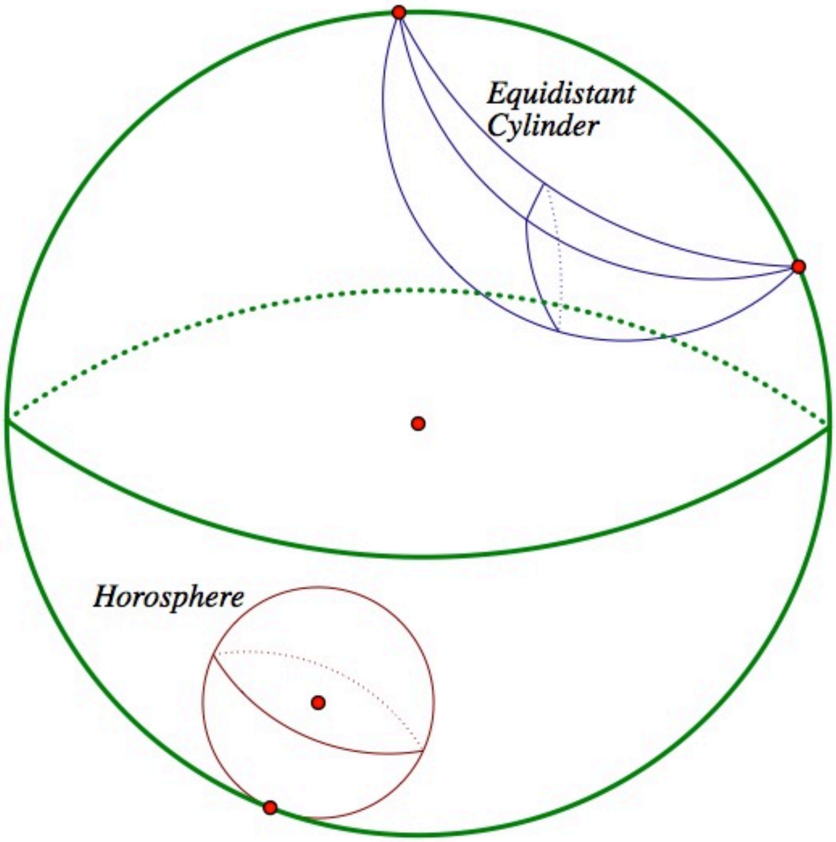

Figure: Poincaré Ball Model Representation of Theorem

References

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