Isometric Immersions of a Euclidean Plane Into Hyperbolic Space

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Overview

1. Fundamental Theorem of Surfaces

2. Immersions

3. Isometric Immersion of a Plane into Hyperbolic Three Space
The Second Fundamental Form reflects extrinsic geometry of a surface.

Consider a surface $S = x(D)$ defined by the coordinate patch:

$x: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Consider $(u, v) \in D$

$$x_1 = \frac{\partial x}{\partial u}, x_2 = \frac{\partial x}{\partial v}$$

$$x_{11} = \frac{\partial x}{\partial u^2}, x_{12} = \frac{\partial x}{\partial v \partial u}, x_{21} = \frac{\partial x}{\partial u \partial v}, x_{22} = \frac{\partial x}{\partial v^2}.$$ 

Note that $x_{12} = x_{21}$ by the equality of mixed partials.
Consider a curve $\gamma(t) = x(u(t), v(t))$ on the surface $S$

$$\gamma'(t) = u'x_1 + v'x_2$$  \(1\)

$$\gamma''(t) = u''x_1 + u'(u'x_{11} + v'x_{12}) + v''x_2 + v'(u'x_{21} + v'x_{22})$$

$$= u''x_1 + v''x_2 + u'^2x_{11} + 2u'v'x_{12} + v'^2x_{22}.$$  \(2\)

Goal: Decompose $\gamma''$ into normal and tangential components
The Second Fundamental Form

\[ \Gamma^k_{ij} \] where \( i, j, k = 1, 2 \), Christoffel symbols, denote the coefficients of the tangential component.

(Note that this uses Einstein notation: \( \Gamma^k_{ij} x_k = \sum_k \Gamma^k_{ij} x_k \))

\( L_{ij} \) where \( i, j = 1, 2 \) denote the coefficient of the normal component

\[ x_{ij} = \Gamma^1_{ij} x_1 + \Gamma^2_{ij} x_2 + L_{ij} n = \Gamma^k_{ij} x_k + L_{ij} n. \quad (3) \]

**Figure:** Tangent Plane
The Second Fundamental Form, Christoffel Symbols

\[ l = dS^2 = Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2 = g_{ij}dx_idx_j \quad i, j = 1, 2 \]

\[
[g_{ij}] = \begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22} \\
\end{pmatrix} = \begin{pmatrix} E & F \\
F & G \\
\end{pmatrix} = \begin{pmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 \\
x_2 \cdot x_1 & x_2 \cdot x_2 \\
\end{pmatrix}.
\]

Let \( g = \det[g_{ij}] \).

\[ g^{ij} = [g_{ij}]^{-1} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} \\
-g_{21} & g_{11} \\
\end{pmatrix} \quad (4)\]

Multiplying Equation (3) by \( x_l \) where \( i, j, k, l = 1, 2 \):

\[ x_{ij} \cdot x_l = \Gamma^{k}_{ij}x_k \cdot x_l + L_{ij}n \cdot x_l = \Gamma^{k}_{ij}g_{kl}. \quad (5)\]
Solving Equation (5) for the Christoffel symbol we see

\[ \Gamma^k_{ij} = (x^i_j \cdot x^l) g^{lk}. \]  

(6)

Rewriting this in terms of a metric:

\[ \Gamma^k_{ij} = \frac{1}{2} \left( \frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right) g^{lk}. \]  

(7)
The Second Fundamental Form, Normal Component

Solving for the coefficients of the second fundamental form we will multiply Equation (3) by \( n \)

\[
\mathbf{x}_{ij} \cdot \mathbf{n} = \mathbf{x}_k \Gamma^{k}_{ij} \mathbf{x}_k \cdot \mathbf{n} + L_{ij} \mathbf{n} \cdot \mathbf{n} = L_{ij}.
\]
Referring back to Equation (2) we see that for $i, j, k = 1, 2$:

$$
\gamma'' = (u^i)''x_i + (u^i)'(u^j)'(\Gamma^k_{ij}x_k + L_{ij}n)
$$

$$
= ((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')x_k + (u^i)'(u^j)'L_{ij}n. \tag{8}
$$

The coefficients of the second fundamental form are the coefficients of the normal term, so

$$
II = (u^i)'(u^j)'L_{ij}.
$$
Gauss’ Equation and the Codazzi-Mainardi Equation

Riemann curvature tensor:

\[ R_{ijk}^l = \frac{\partial \Gamma_{ik}^l}{\partial u^j} - \frac{\partial \Gamma_{ij}^l}{\partial u^k} + \Gamma_{ik}^p \Gamma_{pj}^l - \Gamma_{ij}^p \Gamma_{pk}^l \]  \( (9) \)

Gauss’ Equation:

\[ R_{ijk}^l = L_{ik} L_{jp} g^{pl} - L_{ij} L_{kp} g^{pl} \quad (i, j, k, l, p = 1, 2). \]  \( (10) \)

Codazzi-Mainardi Equation:

\[ \frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \Gamma_{ik}^l L_{lj} - \Gamma_{ij}^l L_{lk}. \]  \( (11) \)
(O’Neill) An $n$-dimensional manifold $M$ is a set furnished with a collection $\mathcal{P}$ of abstract patches (smooth, one-to-one functions $x : D \to M$, $D$ and open set in $\mathbb{R}^n$) satisfying:

1. **The covering property:** The images of the patches in the collection $\mathcal{P}$ cover $M$.
2. **The smooth overlap property:** For any patches $x, y$ in $\mathcal{P}$, the composite functions $y^{-1}x$ and $x^{-1}y$ are Euclidean differentiable – and defined on open sets of $\mathbb{R}^n$.
3. **The Hausdorff property:** For any points $p \neq q$ in $M$ there are disjoint patches $x$ and $y$ with $p$ in $x(D)$ and $q$ in $y(E)$. 
(Spivak) Let $U \subset \mathbb{R}^2$ be a convex open set containing the origin.

Let $x, \bar{x} : U \to \mathbb{R}^3$ be two immersions and define:

$$g_{ij} = \langle x_i, x_j \rangle \quad \bar{g}_{ij} = \langle \bar{x}_i, \bar{x}_j \rangle$$

$$N = \frac{x_1 \times x_2}{\sqrt{g_{11}g_{22} - g_{12}^2}} \quad \bar{N} = \frac{\bar{x}_1 \times \bar{x}_2}{\sqrt{\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2}}$$

$$L_{ij} = \langle -N_i, x_j \rangle = \langle N, x_{ij} \rangle \quad \bar{L}_{ij} = \langle -\bar{N}_i, \bar{x}_j \rangle = \langle \bar{N}, \bar{x}_{ij} \rangle$$

Suppose that $g_{ij} = \bar{g}_{ij}$ and $L_{ij} = \bar{L}_{ij}$ on $U$. Then there is a proper Euclidean motion $A$ such that $\bar{x} = A \circ x$. 
Let $g_{ij}$ and $L_{ij}$ ($i, j, = 1, 2$) be functions on $U$ which satisfy:

1. $g_{ij} = g_{ji}$ and $L_{ij} = L_{ji}$ and $(g_{ij})$ is positive definite on $U$,
2. Gauss' Equation:

$$L_{11}L_{22} - (L_{12})^2 = R_{1212} = \sum_{\rho=1}^{2} g_{1\rho} \left( \Gamma_{22,1}^{\rho} - \Gamma_{21,1}^{\rho} + \sum_{h=1}^{2} (\Gamma_{22}^{h} \Gamma_{21}^{\rho} - \Gamma_{21}^{h} \Gamma_{21}^{\rho}) \right)$$

(12)

3. The Codazzi-Mainardi Equations:

$$L_{12,1} - L_{11,2} + \sum_{h=1}^{2} \Gamma_{12}^{h} L_{h1} - \sum_{h=1}^{2} \Gamma_{11}^{h} L_{h2} = 0$$

$$L_{22,1} - L_{21,2} + \sum_{h=1}^{2} \Gamma_{22}^{h} L_{h1} - \sum_{h=1}^{2} \Gamma_{21}^{h} L_{h2} = 0.$$

(13)
Then there is an immersion \( x : U \rightarrow \mathbb{R}^3 \) such that:

\[
g_{ij} = \langle x_i, x_j \rangle,
\]

and

\[
L_{ij} = \langle -N_i, x_j \rangle = \langle N, x_{ji} \rangle, \quad \text{for} \quad N = \frac{x_1 \times x_2}{\sqrt{g_{11}g_{22} - g_{12}^2}}.
\]
Immersion vs. Embedding

Immersion: (O’Neill) if $M$ is an abstract surface and $F : M \to \mathbb{R}^3$ is merely regular, then $F$ is an **immersion** of $M$ into $\mathbb{R}^3$, and the image $F(M)$ is often called an “immersed surface.”

Embedding: (O’Neill) if $M$ is an abstract surface, a proper **embedding** of $M$ into $\mathbb{R}^3$ is a one-to-one regular mapping $F : M \to \mathbb{R}^3$ such that the inverse function $F^{-1} : F(M) \to M$ is continuous.
Let $f : \mathbb{R}^2 \to \mathbb{H}^3$ be a regular isometric immersion of a Euclidean plane ($\mathbb{R}^2$) in three-dimensional hyperbolic space ($\mathbb{H}^3$). Then only one of the following two situations is possible:

1. $f$ is a homeomorphism, and $f(\mathbb{R}^2)$ is a horosphere.
2. $f$ is a locally isometric covering of the surface formed by the rotation of a particular equidistant about its base line (i.e. an equidistant cylinder)
Volkov and Vladamirova: Theorem

\[ \text{Equidistant Cylinder} \]

\[ \text{Horosphere} \]

**Figure:** Poincaré Ball Model Representation of Theorem
The search for isometric immersions of a two-dimensional Riemannian manifold $V^2$ in $\mathbb{H}^3$ reduces to the construction of $V^2$ from the second fundamental form of the sought-after immersion.
In a local coordinate system \((u_1, u_2)\) on \(V^2\) the coefficients \(L_{ij}\) of the second quadratic form satisfy the Gauss and Codazzi-Mainardi equations:

\[
\begin{align*}
\frac{\partial L_{12}}{\partial u_1} - \frac{\partial L_{11}}{\partial u_2} + \tilde{\Gamma}^k_{12} L_{1k} - \tilde{\Gamma}^k_{11} L_{2k} &= -R_{1312} \\
\frac{\partial L_{22}}{\partial u_1} - \frac{\partial L_{12}}{\partial u_2} + \tilde{\Gamma}^k_{22} L_{1k} - \tilde{\Gamma}^k_{12} L_{2k} &= -R_{2312} \\
L_{11} L_{22} - L_{12}^2 &= \tilde{R}_{1212} - R_{1212}.
\end{align*}
\]

\(\tilde{\Gamma}^k_{ij}\), where \((i, j, k = 1, 2)\): the Christoffel symbols of \(V^2\)

\(\tilde{R}\): curvature tensor of \(V^2\)

\(R_{ijkl}\): the components of the curvature tensor of \(H^3\) in any orthogonal coordinate system \((v_1, v_2, v_3)\) such that \(v_1 = u_1, v_2 = u_2\) and \(g_{33} = 1\), where \(g_{ik}\) is the metric tensor of \(H^3\) in the system \((v_1, v_2, v_3)\).
Because $\mathbb{H}^3$ is a space with constant curvature $-1$ its curvature tensor is related to the metric tensor and curvature by the relations:

$$R_{ijkl} = -(-1)(g_{ik}g_{jl} - g_{il}g_{jk}) = g_{ik}g_{jl} - g_{il}g_{jk}$$

In an orthogonal system this yields:

$$R_{ijkl} = 0 \quad (j \neq k)$$

$$R_{ijij} = g_{ii}g_{jj} - g_{ij}^2 \quad (i \neq j).$$
Let the immersible manifold be $\mathbb{R}^2$ with Euclidean metric, and let \((u_1, u_2)\) be Cartesian coordinates. Then:

$$\tilde{\Gamma}^i_{jk} = \tilde{\Gamma}_{ijkl} = 0,$$

and, taking into account Equation (14), Equation (15) assumes the form:

\[
\begin{align*}
\frac{\partial L_{12}}{\partial u_1} - \frac{\partial L_{11}}{\partial u_2} &= 0 \\
\frac{\partial L_{22}}{\partial u_1} - \frac{\partial L_{12}}{\partial u_2} &= 0 \\
L_{11}L_{22} - L_{12}^2 &= 1.
\end{align*}
\]
In hyperbolic space with curvature $k = -1$ a horosphere has principal curvatures $k_1 = k_2 = 1$. Introducing “Cartesian” coordinates $(u_1, u_2)$ on such a horosphere, we find that its first and second fundamental forms are written:

$$I = II = (du_1)^2 + (du_2)^2.$$ 

The first and second fundamental forms of an equidistant cylinder are:

$$I = (du_1)^2 + (du_2)^2$$

$$II = a(du_1)^2 + \frac{1}{a}(du_2)^2 \quad a = \tanh r$$
Every solution of System (16) has the form:

\[ L_{11} = \frac{1}{L_{22}} = a, \quad L_{12} = 0 \]

Thus:

- if \(|a| = 1\), we obtain a horosphere;
- if \(|a| \neq 1\) the surface in question has the same first and second fundamental forms as the surface obtained by the rotation of a certain equidistant, so that by the Fundamental Theorem of Surfaces the surfaces are equal. (Note, we know that this is global because our original manifold, \(\mathbb{R}^2\), is simply connected.)
Figure: Poincaré Ball Model Representation of Theorem


The End