Adic and Perfectoid Spaces

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6.) (Complex–Analytic/Rigid–Analytic Spaces)

Definition (Huber Ring)

A Huber ring is a topological ring A containing an open subring A_0 carrying the linear topology induced by a finitely generated ideal $I \subseteq A_0$.

Remark

The data (A_0, I) are *not* included along with A.

Definition (Bounded)

We say that $S \subseteq A$ is bounded if for all open $U \ni 0$, there is an open neighborhood $V \ni 0$ such that $VS \subset U$.

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Definition (Pseudo–Uniformizer, ϖ)

A pseudo–uniformizer is a topological nilpotent unit.

Definition (Tate)

A Huber ring A is called Tate if it contains a topological nilpotent unit. Such an element is called a pseudo–uniformizer.

Definition (Uniform)

A Huber ring A is uniform if $A^\circ \subset A$ is bounded.

A	A_0	I_0	Tate	Uniform
A	A	0	✗	✓
K	K^0	(ϖ)	✓	✓
$K\langle T_1, \cdots, T_n\rangle$	$K^0\langle T_1, \cdots, T_n\rangle$	(ϖ)	✓	✓
$R[\![T_1, \cdots, T_n]\!]$	$R[\![T_1, \cdots, T_n]\!]$	$(T_1, \cdots, T_n)$	✗	✓
$\mathbb{Q}_p[T]/T^2$	$\mathbb{Z}_p + \mathbb{Q}_p T$	(T)	?	✗

Definition (Continuous Valuation)

If A is a topological ring, a continuous valuation on A is a map $|\cdot| : A \to \Gamma \cup \{0\}$ such that $|\cdot|$ is a valuation on A and for all $\gamma \in \Gamma$, $\{a \in A \mid |a| < \gamma\}$ is open in A.

Note: $\ker|\cdot| \lhd A$ is prime and only depends on its equivalence class.

Definition (Cont(A))

The set of equivalence classes of continuous valuations on A.

If $x \in \mathrm{Cont}(A)$, we write $f \mapsto |f(x)|$ to denote a continuous valuation representing x.

Definition (Integral Elements)

Let A be a Huber ring. A subring $A^+ \subset A$ is a *ring of integral elements* if it is open and integrally closed and $A^+ \subset A^\circ$.

Definition (Huber Pair)

A *Huber pair* is a pair (A, A^+), where A is Huber and $A^+ \subset A$ is a ring of integral elements.

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Definition (Spa)

Given a Huber pair (A, A^+), we let $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ be the subset of continuous valuations x for which $|f(x)| \leq 1$ for all $f \in A^+$.

Definition

Let $s_1, \cdots, s_n \in A$ and $T_1, \cdots, T_n \subset A$ be finite subsets such that for each i, $T_i A \subset A$ is open. Define a subset

$$U\left(\left\{\frac{T_i}{s_i}\right\}\right) = U\left(\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right)$$
$$= \{x \in X : |t_i(x)| \leq |s_i(x)| \neq 0 \text{ for all } t_i \in T_i\}$$

Subsets of this form are called *rational subsets*.

Theorem

Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ factors over U and is universal for such maps. Moreover, this map is a homeomorphism onto U. In particular, U is quasicompact.

Definition (Structure Presheaf)

Define a presheaf $\mathscr{O}_X$ of topological rings on $\mathrm{Spa}(A, A^+)$ as follows: if $U \subset X$ is rational, $\mathscr{O}_X(U)$ is as in the theorem. On a general open $W \subset X$, define

$$\mathscr{O}_X(W) = \varprojlim_{U \subset W \text{ rational}} \mathscr{O}_X(U)$$

Definition (Adic Space)

An *adic space* consists of a topological space X, a sheaf of rings $\mathscr{O}_X$, and the data of a continuous valuation on $\mathscr{O}_{X,x}$ for each $x \in X$. We require that X be covered by open subsets of the form $\mathrm{Spa}(A, A^+)$, where each (A, A^+) is a sheafy Huber pair.

Definition (Perfectoid Field)

Let K be a nonarchimedean field of residue characteristic p. Then K is said to be a perfectoid field if...

- $|K^\times|$ is nondiscrete.
- $K^\circ/p \to K^\circ/p$ is surjective.

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Example

Both the completions of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ are perfectoid fields. In fact, the completion of any arithmetically profinite extension is a perfectoid field.

Definition (Tilt)

Let K be a perfectoid field with absolute value $|\cdot|$. Let $K^\circ = \{|x| \leq 1\}$ be the ring of integers. Define

$$K^\flat = \varprojlim_{x \mapsto x^p} K$$

$$c_n = \lim_{m \to \infty} (a_{m+n} + b_{m+n})^{p^m}$$

Remark

Note that the perfectoid field $K^\flat$ contains a pseudo–uniformizer ϖ with $|\varpi| = |p|$ and

$$K^{\flat\circ} \cong \varprojlim_{x \mapsto x^p} K^\circ/p \text{ and } K^{\flat\circ}/\varpi \cong K^\flat/p$$

Example

Let $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$. Then $K^\flat$ contains $t = (p, p^{1/p}, \cdots)$ with $|t| = |p|$. Therefore, t is a pseudo–uniformizer of $K^\flat$ and since $K^\flat$ is perfectoid, $K^\flat$ contains $\mathbb{F}_p((t^{1/p^\infty}))$. In fact, $K^\flat = \mathbb{F}_p((t^{1/p^\infty}))$.

Theorem (Tilting Equivalence)

Let K be a perfectoid field. Then for any finite extension L/K (necessarily separable), L is a perfectoid field and $L^\flat/K^\flat$ is a finite extension of the same degree as L/K. The categories of finite extensions of K and $K^\flat$ are equivalent via $L \mapsto L^\flat$. Therefore, there is an isomorphism $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K}^\flat, K^\flat)$.

Definition (Untilt)

An untilt K is a pair $(K^\sharp, \iota)$, where $K^\sharp$ is a perfectoid field and $\iota : K \xrightarrow{\sim} K^{\sharp\flat}$ is an isomorphism.

Definition (Perfectoid Ring)

If A is a Huber ring, then A is a perfectoid ring if...

1. A is Tate.
2. A is uniform.
3. $\varpi \in A$ with $\varpi^p \mid p$ and $A^0/\varpi \to A^0/\varpi^p$ is an isomorphism.

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A perfectoid space is an adic space which is covered by affinoids of the form $\mathrm{Spa}(A, A^+)$, where A is perfectoid.

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Example

If K is a perfectoid field and $K^+ \subset K$ is a ring of integral elements, then $\mathrm{Spa}(K, K^+)$ is a perfectoid space.

Example

Let K be a perfectoid field. Let $A = K\langle T^{1/p^\infty}\rangle$. Then A is a perfectoid ring and $\mathrm{Spa}(A, A^\circ)$ is a perfectoid space.