Differntial Forms and Integration Beyond Euclidean Space

Daniel Cuneo
Syracuse University
MGO Colloquium
4 November 2016

- Guillem, Victor and Pollack, Alan: *Differential Topology*. AMS publishing, 1974.
- Munkres,James. *Topology, 2nd ed*. Prentice Hall, 2000.
- O'Neill, Barrett. *Elementary Differential Geometry, Revised 2nd ed*. Elsevier, 2006.
- Rudin, Walter. *Principles of Mathematical Analysis, 3rd ed*. McGraw-Hill, 1976.

Change of variables for integrals in $\mathbb{R}^n$:

Let V and U be open subsets of $\mathbb{R}^n$, and g be an integrable function on U. If $f : V \to U$ is a diffeomorphism, then

$$\int_U g(y)dy = \int_V g(f(x))|J_f(x)|dx$$

Definition

Suppose E is an open set of $\mathbb{R}^n$ and D is a compact set in $\mathbb{R}^d$. A d-**surface** in E is a one-to-one C^1-mapping $\Phi : D \to E$.

A d-surface should make us think of a *manifold*- a space $\mathbb{X} \subset \mathbb{R}^N$ locally diffeomorphic to $\mathbb{R}^d$.

Definition

Suppose E is an open set of $\mathbb{R}^n$ and D is a compact set in $\mathbb{R}^d$. A d-**surface** in E is a one-to-one C^1-mapping $\Phi : D \to E$.

A d-surface should make us think of a *manifold*- a space $\mathbb{X} \subset \mathbb{R}^N$ locally diffeomorphic to $\mathbb{R}^d$.

Definition

Let $\mathbb{X}$ be a topological space. If there exists a family $\{V_\alpha\}$ of open sets in $\mathbb{R}^d$ together with continuous maps $\phi_\alpha : V_\alpha \to \mathbb{X}$ such that

- for each $u \in \mathbb{X}$, there exists a neighborhood U_α of u such that $\phi_\alpha : V_\alpha \to U_\alpha$ is a homeomorphism, and
- if $u \in U_\alpha \cap U_\beta$, then the map $\phi_\beta^{-1}\phi_\alpha : \phi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \phi_\beta^{-1}(U_\alpha \cap U_\beta)$ is a diffeomorphism,

then $\mathbb{X}$ is a d-**manifold**.

Example

$\Phi : [0,1] \to \mathbb{R}^3$ is a curve (1-surface) in $\mathbb{R}^3$

Example

$\Phi : D^2 \to \mathbb{R}^3$, $\Phi = (u, v, w)$ is a 2-surface in $\mathbb{R}^3$.

$$u = \frac{2}{1 + x^2 + y^2} x$$

$$v = \frac{2}{1 + x^2 + y^2} y$$

$$w = \frac{1 - (x^2 + y^2)}{1 + x^2 + y^2}$$

Example

$\Phi : [0,1]^2 \to \mathbb{R}^3$, $\Phi = (u, v, w)$ is a 2-surface in $\mathbb{R}^3$.

$$u = \cos(2\pi x)\sin(\pi y)$$
$$v = \sin(2\pi x)\sin(\pi y)$$
$$w = \cos(\pi y)$$

Some other terminology:

- The map $\phi_\alpha : V_\alpha \to U_\alpha$ is called a **parametrization** of the open set U_α.
- The inverse map $\phi_\alpha^{-1} : U_\alpha \to \mathbb{R}^d$ is called a **coordinate system** for U_α.
- The component functions $x_j(u)$ of $\Phi^{-1}(u) = (x_1, x_2, \ldots, x_d)$ are called the **coordinate functions**.
- The **tangent space** of $\mathbb{X}$ at u, $T_u(\mathbb{X})$, is a copy of $\mathbb{R}^d$ that is linearly isomorphic to $\mathbb{R}^d$ via $d\phi_\alpha$, the derivative of ϕ_α.

Definition

A k-**form** on a set $E \subset \mathbb{R}^N$ is a function ω that assigns to each k-surface Φ in E a number $\int_\Phi \omega$

The collection of all k-forms on an open set E is a linear space.

- If ω_1 and ω_2 are k-forms, then

$$\int_\Phi (\omega_1 + \omega_2) = \int_\Phi \omega_1 + \int_\Phi \omega_2$$

- If ω is a k-form and c a constant, then

$$\int_\Phi (c\omega) = c \int_\Phi \omega$$

Let V and W be a real vector spaces, and let $A : V \to W$ be a linear transformation.

- A **functional** on V is a linear transformation $\phi : V \to \mathbb{R}$.
- The set of all functionals on V is a real vector space V^* called the **dual space** of V.
- For $\phi \in W^*$, the functional on V given by $A^*\phi(v) = \phi(Av)$ is called the **pullback** of ϕ via A. The map $A^* : W^* \to V^*$ is a linear transformation.
- A k-**tensor** on V is a multilinear function $T : V^k \to \mathbb{R}$.
- The set of all k-tensors on V is a vector space, denoted $\mathcal{T}^k(V^*)$.
- If T is a k-tensor and S is an l-tensor, then their tensor product $T \otimes S$ is a $(k + l)$-tensor defined by

$$T \otimes S(v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) =$$
$$T(v_1, \ldots, v_k)S(v_{k+l}, \ldots v_{k+l})$$

If $A : V \to W$ is a linear transformation and T is a k-tensor on W, then pullback of T via A is the k-tensor on V given by

$$A^* T(v_1, \ldots, v_k) = T(Av_1, \ldots, Av_k)$$

The map $A^* : \mathcal{T}^k(W^*) \to \mathcal{T}^k(V^*)$ is a linear transformation.

Theorem

If $\{\phi_1, \ldots, \phi_d\}$ is a basis for V^*, then $\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_k} : 1 \leq i_1, \ldots, i_k \leq d\}$ is a basis for $\mathcal{T}^k(V^*)$.

Example

$\mathcal{T}^1(V^*) = V^*$

Example

$\det \in \mathcal{T}^k((\mathbb{R}^k)^*)$

Definition

A k-tensor is **alternating** if transposing two variables switches the sign of the tensor.

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

The set of all alternating k-tensors is a linear subspace of $\mathcal{T}^k(V^*)$, denoted $\Lambda^k(V^*)$.

Let S_k be the permutation group on $\{1, 2, \ldots, k\}$. If T is a k-tensor and $\sigma \in S_k$, we will define a k-tensor by

$$T^\sigma(v_1, \ldots, v_k) = T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

For any k-tensor T, we now define a k-tensor by

$$Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T^\sigma$$

The map $Alt : \mathcal{T}^k(V^*) \to \Lambda^k(V^*)$ is a linear transformation.

Definition

It $T \in \Lambda^k(V^*)$ and $S \in \Lambda^l(V^*)$, then the wedge product $T \wedge S \in \Lambda^{k+l}(V^*)$ is defined by

$$T \wedge S = Alt(T \otimes S)$$

Let $T \in \Lambda^k(V^*)$, $S \in \Lambda^l(V^*)$ and $R \in \Lambda^m(V^*)$. The following are basic properties of the wedge product.

- (Associativity) $(T \wedge S) \wedge R = T \wedge (S \wedge R)$
- (Anticommutivity) $T \wedge S = (-1)^{kl} S \wedge T$
- $T \wedge T = 0$.

Definition

The exterior algebra of V^* is the direct sum

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^d(V^*)$$

together with the wedge product.

Definition

A **differential k-form** is a rule that assigns to each $u \in \mathbb{X}$ an alternating k-tensor on $T_u(\mathbb{X})$. We will denote the linear space of differential k-forms on $\mathbb{X}$ by $\Lambda^k(\mathbb{X})$.

Example

Let $f : \mathbb{X} \to \mathbb{R}$ be a function. Then f is a differential 0-form. At each $u \in \mathbb{X}$, the derivative $df(u)$ of f is a linear map $df(u) : T_u(\mathbb{X}) \to T_{f(u)}(\mathbb{R})$. Thus, $df(u) \in T_u(\mathbb{X})^*$, and df is a differential 1-form, called the **differential** of f.

Example

If α is a differential k-form and β is a differential l-form, then $\alpha \wedge \beta$ is a differential $(k+l)$-form.

If $\mathbb{X}$ is a d-manifold, the differentials of the coordinate functions $dx_1, dx_2, \ldots, dx_d$ form a basis for $\Lambda^1(\mathbb{X})$

Theorem

If $\{\phi_i, \ldots, \phi_d\}$ is a basis for V^*, then $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\}$ is a basis for $\Lambda^k(V^*)$.

Example

If $J = (j_1, j_2, \ldots, j_k)$ with $1 \leq j_1 < \cdots < j_k \leq d$, then any differential k-form can be written uniquely as

$$\sum_J a_J dx_J$$

where each a_J is a function on $\mathbb{X}$ and $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$.

Definition

If $f : \mathbb{X} \to \mathbb{Y}$ is a differentiable function between manifolds, and ω is a differential k-form on $\mathbb{Y}$, then $(df_x)^*\omega(f(x))$ is an alternating k-tensor at x. Thus, $(df_x)^*(\omega \circ f)$ is a differential k-form on $\mathbb{X}$, called the **pullback** of ω via f and denoted $f^*\omega$.

Properties:

- $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
- $f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2)$
- $(f \circ h)^*\omega = h^* f^*\omega$

Example

Let $f : U \to V$ be a diffeomorphism of open sets in $\mathbb{R}^k$, and say $y = f(x)$. If $\omega = dy_1 \wedge \cdots \wedge dy_k$, then

Definition

If $f : \mathbb{X} \to \mathbb{Y}$ is a differentiable function between manifolds, and ω is a differential k-form on $\mathbb{Y}$, then $(df_x)^*\omega(f(x))$ is an alternating k-tensor at x. Thus, $(df_x)^*(\omega \circ f)$ is a differential k-form on $\mathbb{X}$, called the **pullback** of ω via f and denoted $f^*\omega$.

Properties:

- $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
- $f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2)$
- $(f \circ h)^*\omega = h^* f^*\omega$

Example

Let $f : U \to V$ be a diffeomorphism of open sets in $\mathbb{R}^k$, and say $y = f(x)$. If $\omega = dy_1 \wedge \cdots \wedge dy_k$, then

$$f^*\omega(x) = det(df_x) dx_1(x) \wedge \cdots \wedge dx_k(x)$$

If U is an open set in $\mathbb{R}^k$ and $\omega = g(y)dy_1 \wedge \cdots \wedge dy_k$, set

$$\int_U \omega = \int_U g(y)dy.$$

The change of variables formula for open sets U, V in $\mathbb{R}^k$ now reads if ω is an integrable k-form on U and $f : V \to U$ is an orientation-preserving diffeomorphism of open subsets in $\mathbb{R}^k$, then

$$\int_U \omega = \int_V f^*\omega$$

If U is an open set in $\mathbb{R}^k$ and $\omega = g(y)dy_1 \wedge \cdots \wedge dy_k$, set

$$\int_U \omega = \int_U g(y)dy.$$

The change of variables formula for open sets U, V in $\mathbb{R}^k$ now reads if ω is an integrable k-form on U and $f : V \to U$ is an orientation-preserving diffeomorphism of open subsets in $\mathbb{R}^k$, then

$$\int_U \omega = \int_V f^*\omega$$

Definition

The **support** of a k-form on $\mathbb{X}$ is the closure of $\{x : \omega(x) \neq 0\}$.

If ω is a differentiable k-form on a k-manifold $\mathbb{X}$ with support compactly contained in a parametrized open set U_α, then define

$$\int_{\mathbb{X}} \omega = \int_{U_\alpha} \omega = \int_{V_\alpha} \phi_\alpha^* \omega$$

Definition

Let $\{U_\alpha\}$ be a locally finite collection of open cover of a topological space $\mathbb{X}$. Then a **partition of unity** dominated by $\{U_\alpha\}$ is a family of continuous functions $\psi_\alpha : U_\alpha \to [0,1]$ such that

- the support of each ψ_α is contained in the set U_α
- the finite sum $\sum_\alpha \psi_\alpha(x) = 1$ for each $x \in \mathbb{X}$.

If ω is an arbitrary differential k-form on a k-manifold $\mathbb{X}$, we define

$$\int_{\mathbb{X}} \omega = \int_{\mathbb{X}} \sum_\alpha \psi_\alpha \omega = \sum_\alpha \int_{U_\alpha} \psi_\alpha \omega$$

Example

Let f be a function on an open set in $\mathbb{R}^k$. Then its differential df is a differential 1-form given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_k} dx_k$$

Definition

Let $\omega = \sum_I a_I dx_I$ be a differential k-form an open set in $\mathbb{R}^k$. The **exterior derivative** of ω is a differential $(k+1)$-form $d\omega$ given by

$$d\omega = \sum_I da_I \wedge dx_I$$

- $d : \Lambda^k(\mathbb{X}) \to \Lambda^{k+1}(\mathbb{X})$ is a linear homomorphism.
- $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$
- $d(d\omega) = 0$
- $f^*(d\omega) = d(f^*\omega)$

Stokes' Theorem

Let $\mathbb{X}$ be an oriented compact k-manifold with boundary $\partial\mathbb{X}$. If ω is a smooth differential $(k-1)$-form on $\mathbb{X}$, then we have

$$\int_{\partial\mathbb{X}} \omega = \int_{\mathbb{X}} d\omega$$

Example

Fundamental Theorem of Calculus
Green's Theorem
Divergence Theorem

Now, assume $\mathbb{X}$ and $\mathbb{Y}$ are domains in $\mathbb{R}^n$, and the maps $h : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{X} \to \mathbb{Y}$ are diffeomorphisms.

Definition

Suppose $L : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is an integrable function. The n-form Ldx is a free Lagrangian if

$$\int_{\mathbb{X}} L(x, h(x), Dh(x))dx = \int_{\mathbb{X}} L(x, g(x), Dg(x))dx$$

whenever the diffeomorphsims $h : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{X} \to \mathbb{Y}$ are homotopic.

Example

If Φ is an integrable function on $\mathbb{Y}$, then the form $\Phi(h(x))J_h(x)dx$ is a free Lagrangian on $\mathbb{X}$

Let $\mathbb{A} = \{x : r < |x| < R\}$ and $\mathbb{A}^* = \{x : r_* < |x| < R_*\}$.
For $t = |x|$, the differential is $dt = \frac{x_1}{|x|} dx_1 + \cdots + \frac{x_n}{|x|} dx_n$.

The $(n-1)$-form $d\sigma = \sum_{i=1}^{n} \frac{(-1)^i x_i}{|x|^n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

- $\int_{\mathbb{S}_t^{n-1}} d\sigma = \omega_{n-1}$, the surface measure of the unit sphere
- $t^{n-1} d\sigma \wedge dt = dx$
- $d(d\sigma) = 0$

Example

If Φ is a function with integrable derivative, then the n-form $\Phi'(|x|)dt \wedge h^*d\sigma$ is a free Lagrangian on $\mathbb{A}$.

$$\int_{\mathbb{A}} \Phi'(|x|)dt \wedge h^*d\sigma = \int_{\mathbb{A}} (d\Phi(t)) \wedge h^*d\sigma = \int_{\mathbb{A}} d\left(\Phi(t)h^*d\sigma\right)$$
$$= \int_{t=R} \Phi(t)h^*d\sigma - \int_{t=r} \Phi(t)h^*d\sigma$$
$$= \Phi(R) \int_{|x|=R} h^*d\sigma - \Phi(r) \int_{|x|=r} h^*d\sigma$$

$\int_{|x|=t} h^*d\sigma$ is a constant that is invariant under homotopy