Lights Out on Related Graphs

Laura Ballard

Hope College REU with
Erica Budge and Dr. Darin Stephenson
Summer 2012

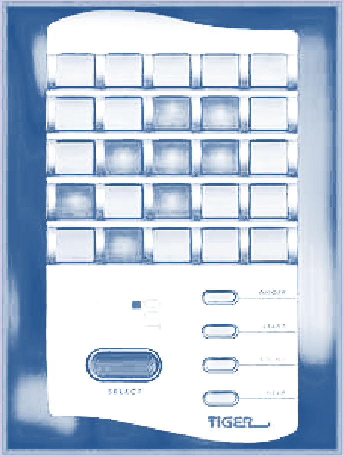
October 16, 2015

What is “Lights Out”?

The original Tiger Electronics game:

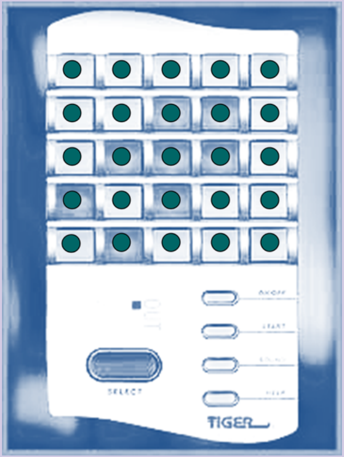
What is "Lights Out"?

The original Tiger Electronics game:



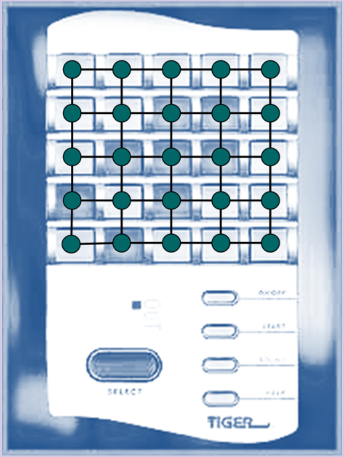
What is “Lights Out”?

Each light is a vertex



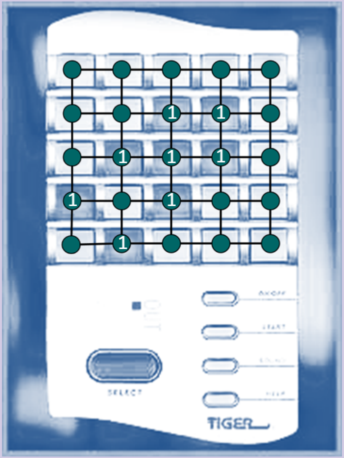
What is "Lights Out"?

Connections are edges



What is "Lights Out"?

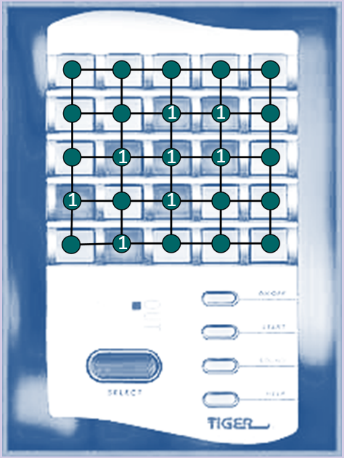
In the $\mathbb{Z}_2$ case, 1 is "on" and 0 is "off"



What is "Lights Out"?

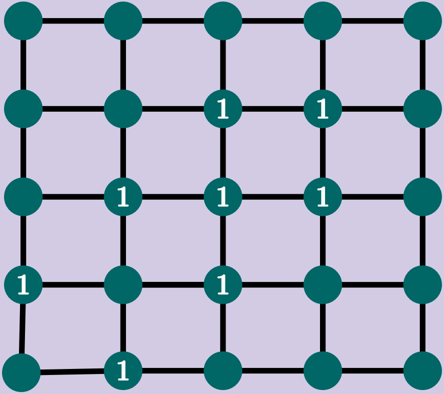
In the $\mathbb{Z}_2$ case, 1 is "on" and 0 is "off"

Blank vertices are in state 0.



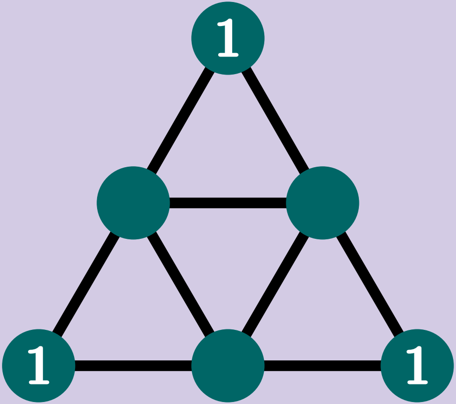
What is "Lights Out"?

The finished product:



What is "Lights Out"?

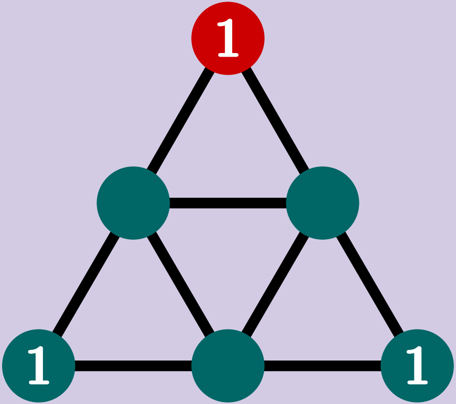
Another graph:



What is "Lights Out"?

Definition

A **press** on a vertex v increases the state of v and all adjacent vertices by 1.

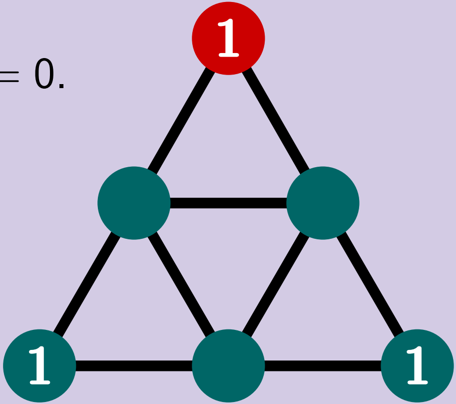


What is "Lights Out"?

Definition

A **press** on a vertex v increases the state of v and all adjacent vertices by 1.

In $\mathbb{Z}_2$,
$1 + 1 = 0$.

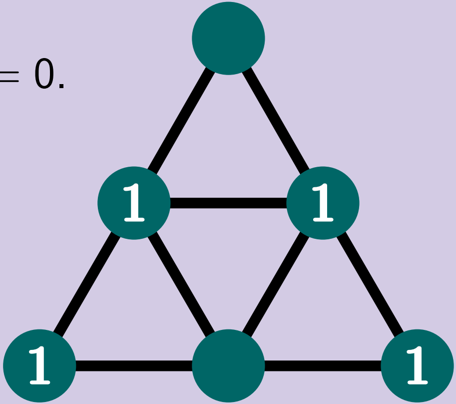


What is “Lights Out”?

Definition

A **press** on a vertex v increases the state of v and all adjacent vertices by 1.

In $\mathbb{Z}_2$,
$1 + 1 = 0.$

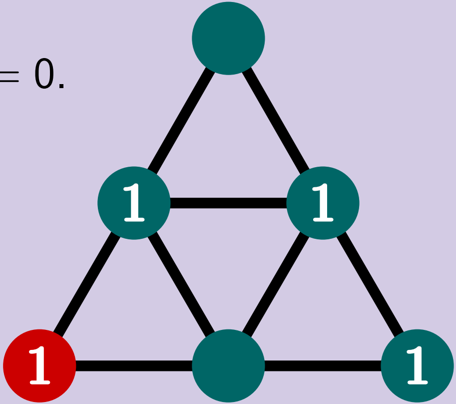


What is "Lights Out"?

Definition

A **press** on a vertex v increases the state of v and all adjacent vertices by 1.

In $\mathbb{Z}_2$,
$1 + 1 = 0.$

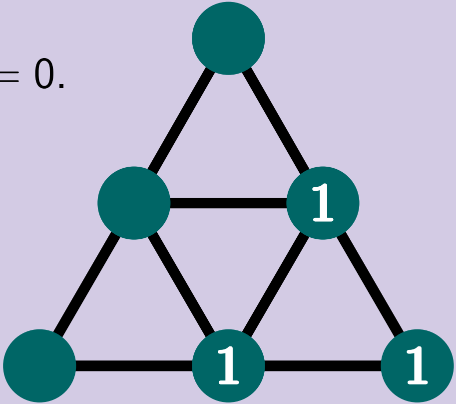


What is “Lights Out”?

Definition

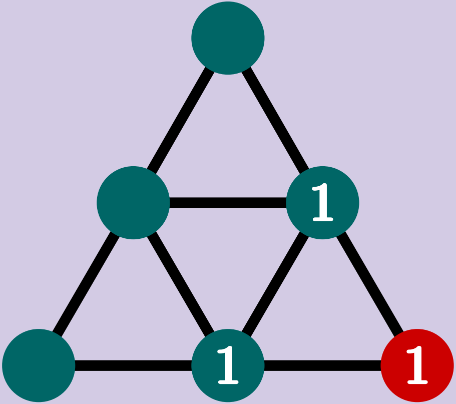
A **press** on a vertex v increases the state of v and all adjacent vertices by 1.

In $\mathbb{Z}_2$,
$1 + 1 = 0.$



What is “Lights Out”?

Remember, the objective is to turn all of the lights out.

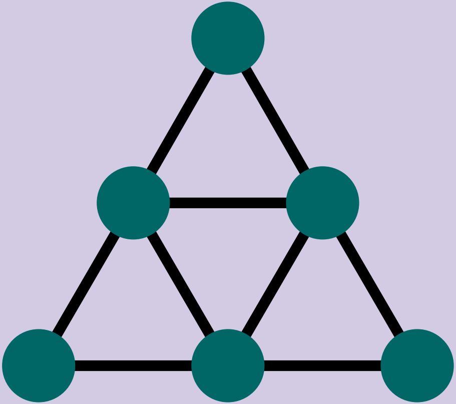


What is “Lights Out”?

Lights Out on Related Graphs

L. Ballard

Lights out!!



It gets better!

But we don't just work in $\mathbb{Z}_2$...

It gets better!

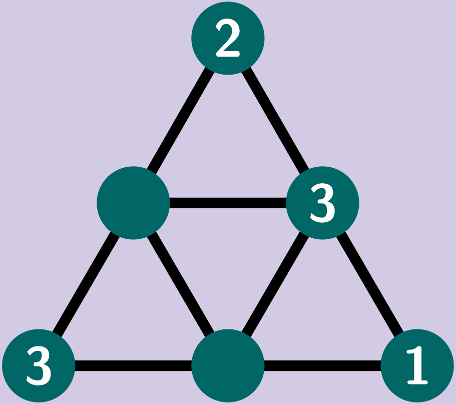
But we don't just work in $\mathbb{Z}_2$...

- In $\mathbb{Z}_k$, possible states are between 0 and $k-1$.

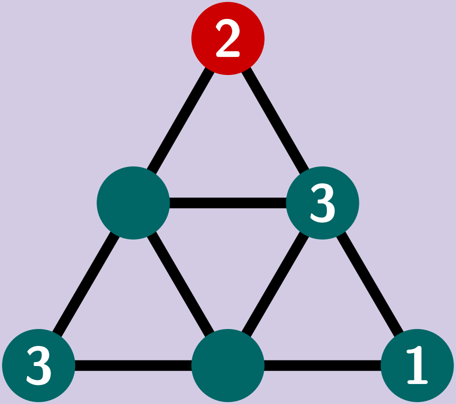
In $\mathbb{Z}_4$:

But we don't just work in $\mathbb{Z}_2$...

- In $\mathbb{Z}_k$, possible states are between 0 and $k-1$.

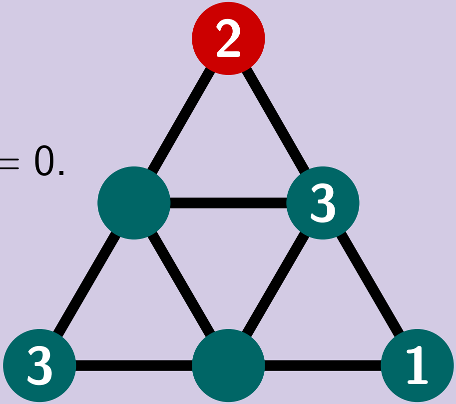


In $\mathbb{Z}_4$:



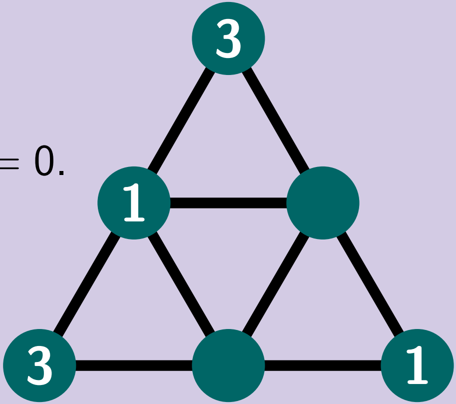
In $\mathbb{Z}_4$:

In $\mathbb{Z}_4$,
$3 + 1 = 0.$



In $\mathbb{Z}_4$:

In $\mathbb{Z}_4$,
$$3 + 1 = 0.$$



For multiple presses:

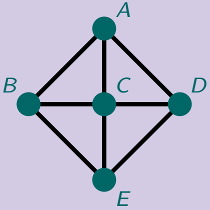
For multiple presses:

A **pattern** is a vector that represents multiple presses on a graph G.

For multiple presses:

A **pattern** is a vector that represents multiple presses on a graph G.

- For example, pattern $\vec{y}$ would press A twice and every other vertex once:

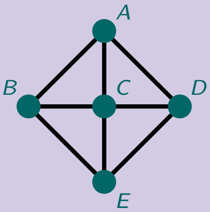


$$\vec{y} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For multiple presses:

A **pattern** is a vector that represents multiple presses on a graph G.

- For example, pattern $\vec{y}$ would press A twice and every other vertex once:



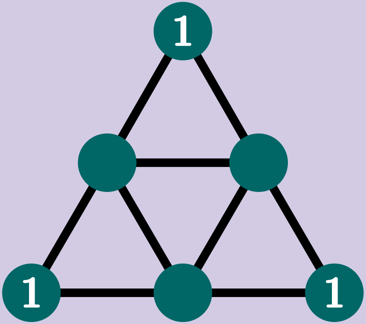
$$\vec{y} = \begin{array}{cc} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} \end{array}$$

Definition

If a state can be turned all off by applying a pattern, that state is a **winnable** state.

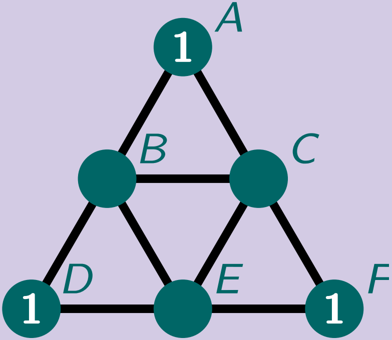
Definition

If a state can be turned all off by applying a pattern, that state is a **winnable** state.



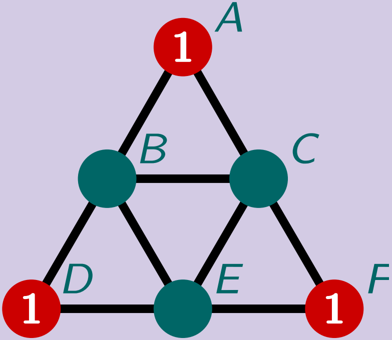
Definition

If a state can be turned all off by applying a pattern, that state is a **winnable** state.



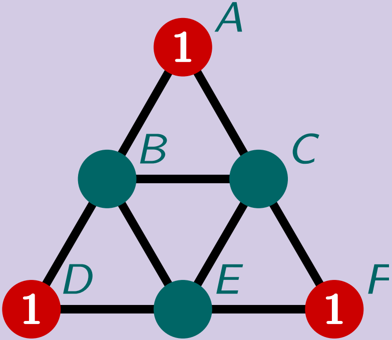
Definition

If a state can be turned all off by applying a pattern, that state is a **winnable** state.



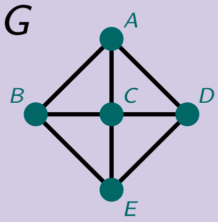
Definition

If a state can be turned all off by applying a pattern, that state is a **winnable** state.

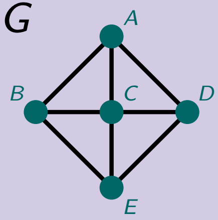


$$\begin{array}{cc} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \\ \text{F} \end{array} \end{array}$$

Linear Algebra



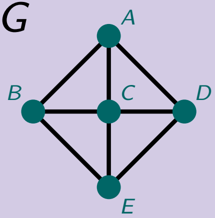
Linear Algebra



Neighborhood Matrix of G
(denoted $N(G)$)

Linear Algebra

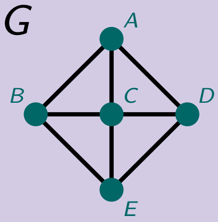
G



Neighborhood Matrix of G (denoted $N(G)$)

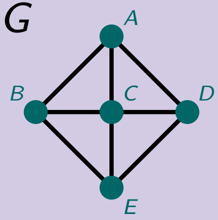
	A	B	C	D	E
A					
B					
C					
D					
E					

Linear Algebra



Neighborhood Matrix of G
(denoted $N(G)$)

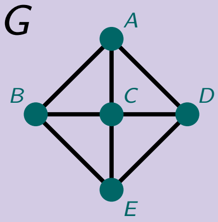
	A	B	C	D	E
A		1	1	1	
B	1		1		1
C	1	1		1	1
D	1		1		1
E		1	1	1	



Neighborhood Matrix of G (denoted $N(G)$)

	A	B	C	D	E
A		1	1	1	0
B	1		1	0	1
C	1	1		1	1
D	1	0	1		1
E	0	1	1	1	

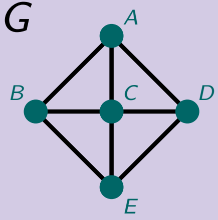
Linear Algebra



Neighborhood Matrix of G (denoted $N(G)$)

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ A & 1 & 1 & 1 & 1 & 0 \\ B & 1 & 1 & 1 & 0 & 1 \\ C & 1 & 1 & 1 & 1 & 1 \\ D & 1 & 0 & 1 & 1 & 1 \\ E & 0 & 1 & 1 & 1 & 1 \end{array}$$

Linear Algebra



Neighborhood Matrix of G
(denoted $N(G)$)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

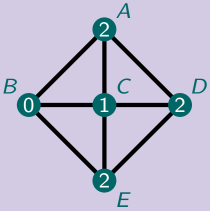
Linear Algebra

The "initial state" of a graph can also be represented as a vector $\vec{x}$.

Linear Algebra

The “initial state” of a graph can also be represented as a vector $\vec{x}$.

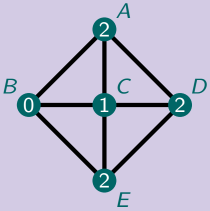
In $\mathbb{Z}_3$:



Linear Algebra

The "initial state" of a graph can also be represented as a vector $\vec{x}$.

In $\mathbb{Z}_3$:



$$\vec{x} = \begin{array}{c} \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \end{array} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}$$

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

$$N\vec{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

$$N\vec{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

$$N\vec{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$\underbrace{2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\text{Pressing A}} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

$$N\vec{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

Pressing vertices in a pattern is commutative.

$$\underbrace{2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\text{Pressing A}} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

$$N\vec{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

Pressing vertices in a pattern is commutative.

$$\underbrace{2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For every pattern $\vec{y}$, $N\vec{y} \in \mathrm{CS}_k(G)$

Applying Patterns

- $N\vec{y}$ gives the effect of applying pattern $\vec{y}$ to graph G.

$$N\vec{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

Pressing vertices in a pattern is commutative.

$$\underbrace{2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

For every pattern $\vec{y}$, $N\vec{y} \in \mathrm{CS}_k(G)$

Applying Patterns

- The new state of G after pattern $\vec{y}$ is complete is $N\vec{y} + \vec{x}$.

Applying Patterns

- The new state of G after pattern $\vec{y}$ is complete is $N\vec{y} + \vec{x}$.

$$\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} =$$

Applying Patterns

- The new state of G after pattern $\vec{y}$ is complete is $N\vec{y} + \vec{x}$.

$$\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \\ 3 \\ 3 \end{pmatrix} \bmod 3$$

Applying Patterns

- The new state of G after pattern $\vec{y}$ is complete is $N\vec{y} + \vec{x}$.

$$\begin{pmatrix}2\\1\\2\\1\\1\end{pmatrix} + \begin{pmatrix}2\\0\\1\\2\\2\end{pmatrix} = \begin{pmatrix}4\\1\\3\\3\\3\end{pmatrix} \bmod 3 = \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}$$

When we apply patterns, the goal is to change each vertex to state 0:

When we apply patterns, the goal is to change each vertex to state 0:

$$N\vec{y} + \vec{x} = \vec{0}$$

When we apply patterns, the goal is to change each vertex to state 0:

$$N\vec{y} + \vec{x} = \vec{0}$$

$$N\vec{y} = -\vec{x}$$

When we apply patterns, the goal is to change each vertex to state 0:

$$N\vec{y} + \vec{x} = \vec{0}$$

$$N\vec{y} = -\vec{x}$$

$$\vec{y} = N^{-1} * (-\vec{x})$$

When we apply patterns, the goal is to change each vertex to state 0:

$$N\vec{y} + \vec{x} = \vec{0}$$

$$N\vec{y} = -\vec{x}$$

$$\vec{y} = N^{-1} * (-\vec{x})$$

If N is invertible mod k, every initial state $\vec{x}$ is winnable, and $N^{-1} * (-\vec{x})$ is the winning pattern.

When we apply patterns, the goal is to change each vertex to state 0:

$$N\vec{y} + \vec{x} = \vec{0}$$

$$N\vec{y} = -\vec{x}$$

$$\vec{y} = N^{-1} * (-\vec{x})$$

If N is invertible mod k, every initial state $\vec{x}$ is winnable, and $N^{-1} * (-\vec{x})$ is the winning pattern.

Graphs for which N is invertible mod k are called **always winnable** mod k.

- Remember that $N\vec{y} \in \mathrm{CS}_k(G)$.

- Remember that $N\vec{y} \in \mathrm{CS}_k(G)$.
- $N\vec{y} = -\vec{x}$ is solvable if $\vec{x}$ is winnable.

- Remember that $N\vec{y} \in \mathrm{CS}_k(G)$.
- $N\vec{y} = -\vec{x}$ is solvable if $\vec{x}$ is winnable.

Thus, $\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

- Remember that $N\vec{y} \in \mathrm{CS}_k(G)$.
- $N\vec{y} = -\vec{x}$ is solvable if $\vec{x}$ is winnable.

Thus, $\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

(Null Space of G - $\mathrm{NS}_k(G)$)

Since N is a symmetric matrix, $\mathrm{NS}_k(G)$ is the orthogonal complement of $\mathrm{CS}_k(G)$.

- Remember that $N\vec{y} \in \mathrm{CS}_k(G)$.
- $N\vec{y} = -\vec{x}$ is solvable if $\vec{x}$ is winnable.

Thus, $\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

(Null Space of G - $\mathrm{NS}_k(G)$)

Since N is a symmetric matrix, $\mathrm{NS}_k(G)$ is the orthogonal complement of $\mathrm{CS}_k(G)$.

A graph is always winnable mod k if and only if $\dim \mathrm{NS}_k(G) = 0$.

Our Research Focus

$\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

Our Research Focus

$\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

A graph is always winnable mod k if and only if $\dim \mathrm{NS}_k(G) = 0$.

Our Research Focus

$\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

A graph is always winnable mod k if and only if $\dim \mathrm{NS}_k(G) = 0$.

$\mathrm{NS}_k(G)$ is the orthogonal complement of $\mathrm{CS}_k(G)$.

Our Research Focus

$\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

A graph is always winnable mod k if and only if $\dim \mathrm{NS}_k(G) = 0$.

$\mathrm{NS}_k(G)$ is the orthogonal complement of $\mathrm{CS}_k(G)$.

- Characterize the way the null space changes when certain subgraphs are removed, or when two graphs are joined together.

Our Research Focus

$\vec{x}$ is winnable if and only if $\vec{x} \in \mathrm{CS}_k(G)$.

A graph is always winnable mod k if and only if $\dim \mathrm{NS}_k(G) = 0$.

$\mathrm{NS}_k(G)$ is the orthogonal complement of $\mathrm{CS}_k(G)$.

- Characterize the way the null space changes when certain subgraphs are removed, or when two graphs are joined together.
- Look for ways to connect graphs or remove subgraphs that do not change the null space.

The Null Space

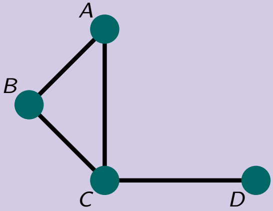
- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$

The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

The Null Space

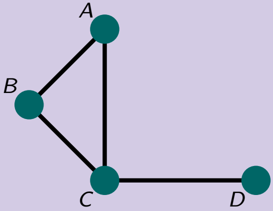
- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.



The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

In $\mathbb{Z}_2$

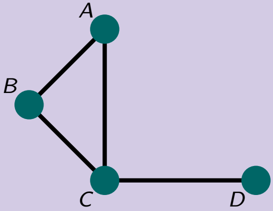


$$\vec{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

In $\mathbb{Z}_2$

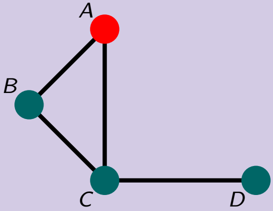


$$\vec{z} = \begin{array}{c} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array}$$

The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

In $\mathbb{Z}_2$

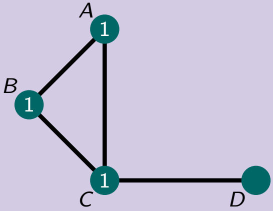


$$\vec{z} = \begin{array}{cc} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \end{array}$$

The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

In $\mathbb{Z}_2$

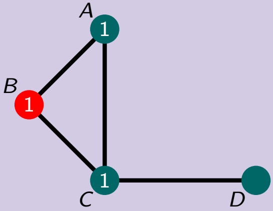


$$\vec{z} = \begin{array}{cc} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array} \end{array}$$

The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

In $\mathbb{Z}_2$

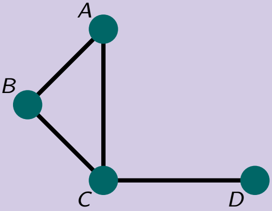


$$\vec{z} = \begin{array}{cc} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \end{array}$$

The Null Space

- By definition, if $\vec{z} \in \mathrm{NS}_k(G)$, $N\vec{z} = \vec{0}$
- Then $\vec{z}$ is a **null pattern** that will not change the state of graph G, since $N\vec{z} + \vec{x} = \vec{x}$.

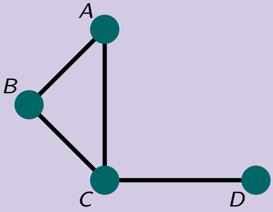
In $\mathbb{Z}_2$



$$\vec{z} = \begin{array}{c} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}$$

The Null Space

This null pattern can be generalized to $\mathbb{Z}_k$.



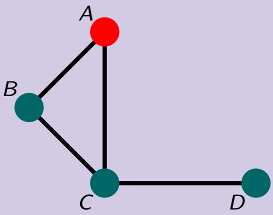
$$\vec{z} = \begin{array}{c} \begin{pmatrix} t \\ -t \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}$$

The Null Space

Lights Out on Related Graphs

L. Ballard

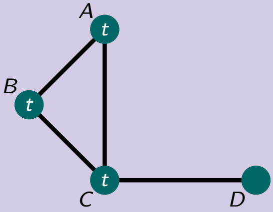
This null pattern can be generalized to $\mathbb{Z}_k$.



$$\vec{z} = \begin{array}{c} \begin{pmatrix} t \\ -t \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}$$

The Null Space

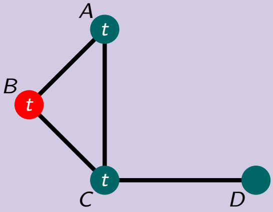
This null pattern can be generalized to $\mathbb{Z}_k$.



$$\vec{z} = \begin{array}{cc} \begin{pmatrix} t \\ -t \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \end{array}$$

The Null Space

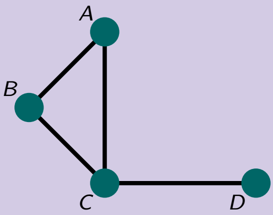
This null pattern can be generalized to $\mathbb{Z}_k$.



$$\vec{z} = \begin{array}{cc} \begin{pmatrix} t \\ -t \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \end{array}$$

The Null Space

This null pattern can be generalized to $\mathbb{Z}_k$.



$$\vec{z} = \begin{array}{cc} \begin{pmatrix} t \\ -t \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \end{array}$$

A Sampling of Our Results

Definition

The **label** of a vertex v in a graph G will be defined by

$$l_G(v) = \dim \mathrm{NS}_k(G - v) - \dim \mathrm{NS}_k(G).$$

In other words, $\dim \mathrm{NS}_k(G - v) = \dim \mathrm{NS}_k(G) + l_G(v)$

A Sampling of Our Results

Proposition

Let G be a graph. For all $v \in V(G)$, we have $l_G(v) \in \{-1, 0, 1\}$. If G is always winnable over $\mathbb{Z}_k$, then for all $v \in V(G)$, we have $l_G(v) \in \{0, 1\}$.

A Sampling of Our Results

Proposition

Let G be a graph. For all $v \in V(G)$, we have $l_G(v) \in \{-1, 0, 1\}$. If G is always winnable over $\mathbb{Z}_k$, then for all $v \in V(G)$, we have $l_G(v) \in \{0, 1\}$.

Proof. The neighborhood matrix of $G - v$ is formed by deleting exactly one row and one column from the neighborhood matrix of G.

A Sampling of Our Results

Proposition

Let G be a graph. For all $v \in V(G)$, we have $l_G(v) \in \{-1, 0, 1\}$. If G is always winnable over $\mathbb{Z}_k$, then for all $v \in V(G)$, we have $l_G(v) \in \{0, 1\}$.

Proof. The neighborhood matrix of $G - v$ is formed by deleting exactly one row and one column from the neighborhood matrix of G. Thus $l_G(v) \in \{-1, 0, 1\}$.

A Sampling of Our Results

Proposition

Let G be a graph. For all $v \in V(G)$, we have $l_G(v) \in \{-1, 0, 1\}$. If G is always winnable over $\mathbb{Z}_k$, then for all $v \in V(G)$, we have $l_G(v) \in \{0, 1\}$.

Proof. The neighborhood matrix of $G - v$ is formed by deleting exactly one row and one column from the neighborhood matrix of G. Thus $l_G(v) \in \{-1, 0, 1\}$. For an always winnable graph G, $\dim \mathrm{NS}_k(G) = 0$, and therefore for all $v \in V(G)$, we have $\dim \mathrm{NS}_k(G - v) \geq \dim \mathrm{NS}_k(G)$, so $l_G(v) \in \{0, 1\}$.

A Sampling of Our Results

Proposition

If there exists $\vec{p} \in \mathrm{NS}_k(G)$ *with* $\vec{p}(v) \neq 0$*, then* $l_G(v) = -1$.

A Sampling of Our Results

Proposition

If there exists $\vec{p} \in \mathrm{NS}_k(G)$ with $\vec{p}(v) \neq 0$, then $l_G(v) = -1$.

Suppose $\vec{p} \in \mathrm{NS}_k(G)$ such that $\vec{p}(v) \neq 0$.

A Sampling of Our Results

Proposition

If there exists $\vec{p} \in \mathrm{NS}_k(G)$ with $\vec{p}(v) \neq 0$, then $l_G(v) = -1$.

Suppose $\vec{p} \in \mathrm{NS}_k(G)$ such that $\vec{p}(v) \neq 0$. Then $\mathbf{e}_v$ is not winnable on G, since $\mathbf{e}_v$ is not in the orthogonal complement of $\mathrm{NS}_k(G)$.

A Sampling of Our Results

Proposition

If there exists $\vec{p} \in \mathrm{NS}_k(G)$ with $\vec{p}(v) \neq 0$, then $l_G(v) = -1$.

Suppose $\vec{p} \in \mathrm{NS}_k(G)$ such that $\vec{p}(v) \neq 0$. Then $\mathbf{e}_v$ is not winnable on G, since $\mathbf{e}_v$ is not in the orthogonal complement of $\mathrm{NS}_k(G)$. It follows that a pattern on G that is null on $G - v$ is also null at v; otherwise that pattern would win $\lambda \mathbf{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$.

A Sampling of Our Results

Proposition

If there exists $\vec{p} \in \mathrm{NS}_k(G)$ with $\vec{p}(v) \neq 0$, then $l_G(v) = -1$.

Suppose $\vec{p} \in \mathrm{NS}_k(G)$ such that $\vec{p}(v) \neq 0$. Then $\mathbf{e}_v$ is not winnable on G, since $\mathbf{e}_v$ is not in the orthogonal complement of $\mathrm{NS}_k(G)$. It follows that a pattern on G that is null on $G - v$ is also null at v; otherwise that pattern would win $\lambda \mathbf{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$. In particular, a null pattern on $G - v$ extended by 0 at v is null on G, so $\dim \mathrm{NS}_k(G) \geq \dim \mathrm{NS}_k(G - v)$.

A Sampling of Our Results

Proposition

If there exists $\vec{p} \in \mathrm{NS}_k(G)$ with $\vec{p}(v) \neq 0$, then $l_G(v) = -1$.

Suppose $\vec{p} \in \mathrm{NS}_k(G)$ such that $\vec{p}(v) \neq 0$. Then $\mathbf{e}_v$ is not winnable on G, since $\mathbf{e}_v$ is not in the orthogonal complement of $\mathrm{NS}_k(G)$. It follows that a pattern on G that is null on $G - v$ is also null at v; otherwise that pattern would win $\lambda \mathbf{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$. In particular, a null pattern on $G - v$ extended by 0 at v is null on G, so $\dim \mathrm{NS}_k(G) \geq \dim \mathrm{NS}_k(G - v)$. The fact that there exists a null pattern $\vec{p} \in \mathrm{NS}_k(G)$ not of this type gives us a strict inequality, or in other words gives us $l_G(v) = -1$.

A Sampling of Our Results

Corollary

Let G be a graph, and let $v \in V(G)$. Then if $l_G(v) = 0$, $\vec{p}(v) = 0$ for every $\vec{p} \in \mathrm{NS}_k(G)$.

A Sampling of Our Results

Corollary

Let G be a graph, and let $v \in V(G)$. Then if $l_G(v) = 0$, $\vec{p}(v) = 0$ for every $\vec{p} \in \mathrm{NS}_k(G)$.

Proof. If $l_G(v) = 0$, then $l_G(v) \neq -1$, so by the contrapositive of the previous proposition, $\vec{p}(v) = 0$ for every $\vec{p} \in \mathrm{NS}_k(G)$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. If $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. If $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Proof. Suppose that $l_G(v) = 0$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. If $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Proof. Suppose that $l_G(v) = 0$. Then by the previous corollary, $\vec{q}(v) = 0$ for every $\vec{q} \in \mathrm{NS}_k(G)$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. If $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Proof. Suppose that $l_G(v) = 0$. Then by the previous corollary, $\vec{q}(v) = 0$ for every $\vec{q} \in \mathrm{NS}_k(G)$. The space of winnable states is the orthogonal complement of the space of null patterns, and therefore, for all $\lambda \in \mathbb{Z}_k^*$, $\lambda \mathbf{e}_v$ is winnable because $\lambda \mathbf{e}_v \perp \vec{q}$ for all $\vec{q} \in \mathrm{NS}_k(G)$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Assume for the sake of contradiction that $\vec{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ such that $\vec{p}(v) = 0$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Assume for the sake of contradiction that $\vec{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ such that $\vec{p}(v) = 0$. It follows that $\vec{p}|_{G-v}$ is null, but $\vec{p}$ is not null on G.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Assume for the sake of contradiction that $\vec{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ such that $\vec{p}(v) = 0$. It follows that $\vec{p}|_{G-v}$ is null, but $\vec{p}$ is not null on G. We also know that for all $\vec{q} \in \mathrm{NS}_k(G)$, $\vec{q}(v) = 0$ and thus $\vec{q}|_{G-v}$ is always a null pattern on $G - v$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $I_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Assume for the sake of contradiction that $\vec{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ such that $\vec{p}(v) = 0$. It follows that $\vec{p}|_{G-v}$ is null, but $\vec{p}$ is not null on G. We also know that for all $\vec{q} \in \mathrm{NS}_k(G)$, $\vec{q}(v) = 0$ and thus $\vec{q}|_{G-v}$ is always a null pattern on $G - v$. We note that $\vec{p}|_{G-v}$ must be distinct from $\vec{q}|_{G-v}$ for all $\vec{q} \in \mathrm{NS}_k(G)$, since the effect of extending these patterns by 0 at v is different.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Assume for the sake of contradiction that $\vec{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ such that $\vec{p}(v) = 0$. It follows that $\vec{p}|_{G-v}$ is null, but $\vec{p}$ is not null on G. We also know that for all $\vec{q} \in \mathrm{NS}_k(G)$, $\vec{q}(v) = 0$ and thus $\vec{q}|_{G-v}$ is always a null pattern on $G - v$. We note that $\vec{p}|_{G-v}$ must be distinct from $\vec{q}|_{G-v}$ for all $\vec{q} \in \mathrm{NS}_k(G)$, since the effect of extending these patterns by 0 at v is different. This implies that $\dim \mathrm{NS}_k(G - v) > \dim \mathrm{NS}_k(G)$, contradicting the fact that $l_G(v) = 0$.

A Sampling of Our Results

Proposition

Let G be a graph and let $v \in V(G)$. Then if $l_G(v) = 0$, then for all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \mathbf{e}_v$ is winnable on G, and any winning pattern $\vec{p}$ for $\lambda \mathbf{e}_v$ satisfies $p(v) \neq 0$.

Assume for the sake of contradiction that $\vec{p}$ is a winning pattern for $\lambda \mathbf{e}_v$ such that $\vec{p}(v) = 0$. It follows that $\vec{p}|_{G-v}$ is null, but $\vec{p}$ is not null on G. We also know that for all $\vec{q} \in \mathrm{NS}_k(G)$, $\vec{q}(v) = 0$ and thus $\vec{q}|_{G-v}$ is always a null pattern on $G - v$. We note that $\vec{p}|_{G-v}$ must be distinct from $\vec{q}|_{G-v}$ for all $\vec{q} \in \mathrm{NS}_k(G)$, since the effect of extending these patterns by 0 at v is different. This implies that $\dim \mathrm{NS}_k(G - v) > \dim \mathrm{NS}_k(G)$, contradicting the fact that $l_G(v) = 0$. Thus $\vec{p}(v) \neq 0$.

A Sampling of Our Results

Definition (The Secondary Label of v in G)

A Sampling of Our Results

Definition (The Secondary Label of v in G)

1. Let G be a graph and suppose $v \in V(G)$ with $l_G(v) = 0$. By the previous proposition, the state $\mathbf{e}_v$ has a winning pattern $\vec{q}$, and $\vec{q}(v) \in \mathbb{Z}_k^*$. Let

$$\lambda_G(v) = -\vec{q}(v)^{-1} \in \mathbb{Z}_k^*.$$

In this situation, all null patterns $\vec{p}$ on G have $\vec{p}(v) = 0$, and therefore $\lambda_G(v)$ is independent of the winning pattern $\vec{q}$ chosen.

A Sampling of Our Results

Definition

In either case, $\lambda_G(v) \in \mathbb{Z}_k^*$ will be called the **secondary label** of v. The label of a vertex v with $l_G(v) = 0$ and secondary label λ will typically be written as $0(\lambda)$.

A Sampling of Our Results

Definition

2. Let G be a graph and suppose $v \in V(G)$ with $l_G(v) = -1$. We define $\lambda_G(v) = 0$. (This secondary label on the $l_G(v) = -1$ vertices carries no information, but is convenient for summation notation later.)

In either case, $\lambda_G(v) \in \mathbb{Z}_k^*$ will be called the **secondary label** of v. The label of a vertex v with $l_G(v) = 0$ and secondary label λ will typically be written as $0(\lambda)$.

A Sampling of Our Results

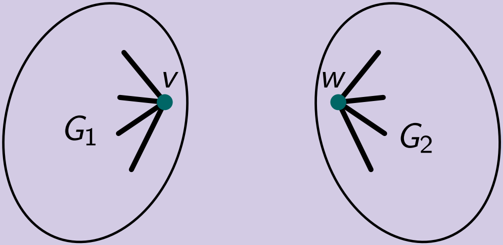
Definition (Edge Join of G_1 and G_2 at v and w, respectively)

Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $H = \mathbf{EJ}(\{G_1, v\}\{G_2, w\})$ be the graph with $V(H) = V(G_1) \cup V(G_2)$ and $E(H) = E(G_1) \cup E(G_2) \cup (v, w)$.

A Sampling of Our Results

Definition (Edge Join of G_1 and G_2 at v and w, respectively)

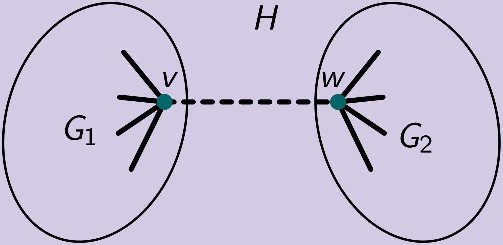
Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $H = \mathbf{EJ}(\{G_1, v\}\{G_2, w\})$ be the graph with $V(H) = V(G_1) \cup V(G_2)$ and $E(H) = E(G_1) \cup E(G_2) \cup (v, w)$.



A Sampling of Our Results

Definition (Edge Join of G_1 and G_2 at v and w, respectively)

Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $H = \mathbf{EJ}(\{G_1, v\}\{G_2, w\})$ be the graph with $V(H) = V(G_1) \cup V(G_2)$ and $E(H) = E(G_1) \cup E(G_2) \cup (v, w)$.



A Sampling of Our Results

Theorem

Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $d_i = \dim \mathrm{NS}_k(G_i)$, and let $H = \mathsf{EJ}(\{G_1, v\}\{G_2, w\})$. Then $\dim \mathrm{NS}_k(H)$ is given by the following table.

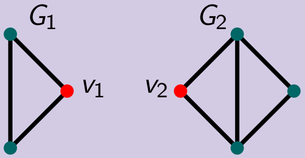
$l_{G_1}(v)$	$l_{G_2}(w)$	$\dim \mathrm{NS}_k(H)$
1	*any*	$d_1 + d_2$
-1	*any*	$d_1 + d_2 + l_{G_2}(w) - 1$
$0(\lambda)$	$0(\mu)$	$d_1 + d_2 \ \ (\mu \neq \lambda^{-1})$
$0(\lambda)$	$0(\mu)$	$d_1 + d_2 + 1 \ \ (\mu = \lambda^{-1})$

In particular, if $l_{G_1}(v) = -1$, then
$\dim \mathrm{NS}_k(H) = \dim \mathrm{NS}_k(G_1 - v) + \dim \mathrm{NS}_k(G_2 - w)$.

A Sampling of Our Results

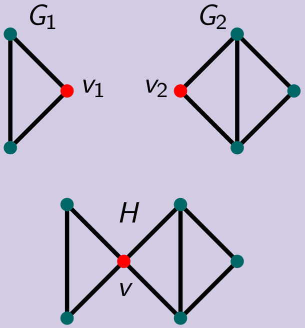
Vertex joining:

Vertex joining:



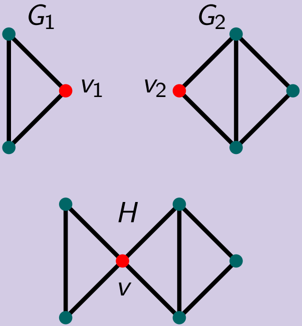
A Sampling of Our Results

Vertex joining:



A Sampling of Our Results

Vertex joining:



The following theorem will show how $\mathrm{NS}_k(H)$ depends on $I_{G_i}(v_i)$ for $i \in \{1, 2\}$.

A Sampling of Our Results

Theorem

For $1 \leq i \leq m$, let G_i be a finite graph with $v_i \in V(G_i)$. The graph $H = \mathrm{VJ}(\{G_i, v_i : 1 \leq i \leq m\})$ is defined by

$$V(H) = \bigcup_{i=1}^{m} V(G_i - v_i) \cup \{v\} \text{ and}$$

$$E(H) = \bigcup_{i=1}^{m} E(G_i - v_i) \cup \left\{ (wv) \ : \ (wv_i) \in \bigcup_{i=1}^{m} E(G_i) \right\}.$$

A Sampling of Our Results

Theorem

For $1 \leq i \leq m$, let G_i be a finite graph with $v_i \in V(G_i)$. The graph $H = \mathrm{VJ}(\{G_i, v_i : 1 \leq i \leq m\})$ is defined by

$$V(H) = \bigcup_{i=1}^{m} V(G_i - v_i) \cup \{v\} \text{ and}$$

$$E(H) = \bigcup_{i=1}^{m} E(G_i - v_i) \cup \left\{ (wv) \ : \ (wv_i) \in \bigcup_{i=1}^{m} E(G_i) \right\}.$$

1. *If $l_{G_i}(v_i) = 1$ for at least one i, then $l_H(v) = 1$.*

A Sampling of Our Results

Theorem

For $1 \leq i \leq m$, let G_i be a finite graph with $v_i \in V(G_i)$. The graph $H = \mathrm{VJ}(\{G_i, v_i : 1 \leq i \leq m\})$ is defined by

$$V(H) = \bigcup_{i=1}^{m} V(G_i - v_i) \cup \{v\} \text{ and}$$

$$E(H) = \bigcup_{i=1}^{m} E(G_i - v_i) \cup \left\{ (wv) \,:\, (wv_i) \in \bigcup_{i=1}^{m} E(G_i) \right\}.$$

1. *If $l_{G_i}(v_i) = 1$ for at least one i, then $l_H(v) = 1$.*
2. *$l_H(v) \in \{0, -1\}$ if and only if $l_{G_i}(v_i) \in \{0, -1\}$ for all i. Moreover, $l_H(v) = -1$ if and only if* $\sum_{i=1}^{m} \lambda_{G_i}(v_i) = m - 1 (\mod k)$.

Acknowledgments

I would like to acknowledge:

- Syracuse University MGO
- Dr. Darin Stephenson
- Hope College
- The National Science Foundation
- Drs. Kristin Camenga and Rebekah Yates

References

Anderson, M., & Feil, T. (1998). Turning lights out with linear algebra. *Mathematics Magazine, 71*(4), 300-303.

Edwards, S., Elandt, V., James, N., Johnson, K., Mitchell, Z., & Stephenson, D. (2008). Lights out on finite graphs. 1-13.

Giffen, A., & Parker, D. (2009). On generalizing the "lights out" game and a generalization of parity domination. 1-14.

Stephenson, D. R. (2012). Lights out, σ^+-automata, and domination in graphs. 1-10.

Theorem

Let G be a finite graph with $v \in V(G)$, and let f_v be the function which extends a pattern on $G - v$ to a pattern on G that is zero at v. The following are equivalent:

Theorem

Let G be a finite graph with $v \in V(G)$, and let f_v be the function which extends a pattern on $G - v$ to a pattern on G that is zero at v. The following are equivalent:

1. $l_G(v) = -1$,

Theorem

Let G be a finite graph with $v \in V(G)$, and let f_v be the function which extends a pattern on $G - v$ to a pattern on G that is zero at v. The following are equivalent:

1. $l_G(v) = -1$,
2. The state $\mathbf{e}_v$ is not winnable on G.

Theorem

Let G be a finite graph with $v \in V(G)$, and let f_v be the function which extends a pattern on $G - v$ to a pattern on G that is zero at v. The following are equivalent:

1. $I_G(v) = -1$,
2. The state $\mathbf{e}_v$ is not winnable on G.
3. There exists $\mathbf{p} \in \mathrm{Null}(N(G))$ with $\mathbf{p}(v) \neq 0$.

Theorem

Let G be a finite graph with $v \in V(G)$, and let f_v be the function which extends a pattern on $G - v$ to a pattern on G that is zero at v. The following are equivalent:

1. $l_G(v) = -1$,
2. The state $\mathbf{e}_v$ is not winnable on G.
3. There exists $\mathbf{p} \in \mathrm{Null}(N(G))$ with $\mathbf{p}(v) \neq 0$.
4. The function f_v induces an injective linear transformation from $\mathrm{Null}(N(G - v))$ to $\mathrm{Null}(N(G))$, and $\dim \operatorname{coker}(f_v) = 1$.