Decision Making in Mathematics: Prison Rules

The Riddle: There a 100 prisoners serving lifetime sentences. A cruel yet playful warden gives them a few hours to strategize for the following game:

- All prisoners kept in solitary confinement.
- Each day ~~the~~ one selected randomly and brought to a room with one lightbulb.
- She may choose to turn the light on/off and it stays that way until the next day.
- Any day the visitor can guess that all 100 prisoners have been selected. If correct → freedom! If wrong, all DED.

Goal: Find the "best" strategy.

Optimal Strategies

① The 1st person turns the light on. No one turns it off on their first visit to the room. Anyone turns it off on their second visit. Noone besides 1st person turns it on again. If in 100 days light on: freedom!

If not, start over.
How long will this take?

let $p =$ prob of success in a 100 day block. The expected # of blocks is $1/p$.

But $p = \frac{n!}{n^n}$ where $n =$ # of prisoners. So expected number of blocks is $\frac{n^n}{n!}$ for $n = 100$ # of days $\frac{100^{100}}{100!}$. This is about 10^{41} years. Seems long...

Strategy Two:

One prisoner is determined to be the "counter."
Any time a prisoner goes to the room their first time, they turn the light on if it is off. The only person allowed to turn it off is the counter, who takes note of how many times they've done this. Once they turn the light off 100 times, freedom!

So, how long will it take?

set $T_i =$ 1^{st} day the ith prisoner is recorded for having turned light on.

$X_i =$ # of days from T_i to T_{i+1}

$x_i = Y_i + Z_i$ Y_i = amount of time for i^{th} prisoner to get there.

Z_i = time for counter to get there after.

Total time $= \sum_i' x_i = \sum_i' Y_i + Z_i$

Y_i is geometric ~~of the~~ random variable of parameter $\frac{n-i}{n}$, Z_i is geometric with parameter $\frac{1}{n}$

Thus $E[X] = \sum E[Y_i] + E[Z_i]$

$$= \sum \frac{n}{n-i} + \cancel{\frac{n(n-i)}{n-i}} = n^2 + n + \underbrace{n H_{n-1}}_{\sum \frac{1}{i}}$$

$$\cancel{\sum_{i=1} \frac{n + n^2 - ni}{n-i}}$$

Thus $E[x] \sim O(n^2)$.

For $n = 100$ $E[x] = 10417$ days ~ 29 years.

You can make more complicated algorithms, to get to ~11 years.

What if we stop caring about optimality, but go only for practicality.

<u>The Sit and Wait Method</u>: Decide on a day to just guess that everyone has been there.

How long would we expect until chances were 99% that it is true? How long until 99.99%?

<u>This is a graph theory question</u>:

You have a complete graph with 100 vertices.

You are taking a random walk around it.

Start at any vertex v. The prob of going to any particular other vertex is $\frac{1}{n-1}$. The prob of not hitting some vertex is $1-\frac{1}{n-1}$.

The prob of not hitting a particular vertex after m days is $(1-\frac{1}{n-1})^m$. Thus prob of hitting a part. vertex after m moves is $1-(1-\frac{1}{n-1})^m$, thus prob of reaching all vertices after m moves is $(1-(1-\frac{1}{n-1})^m)^n$.

For us $n = 100$.

For $m = 730$ (two years) prob $= 94\%$

$m = 1461$ (4 years) prob $= 99.996\%$

Seems better.

Cellular Sheaf Theory
Curry 2014