MAT 526 – Introduction to Stochastic Processes - Fall 2017

Course Description This is a first course in stochastic processes. Topics to be covered include: random walks, branching processes, Markov chains, the Poisson process and queuing theory.

Prerequisite Solid backgrounds in calculus (MAT 397) and probability (MAT 521), and some familiarity with linear algebra (MAT 331).

Instructor Prof. JT Cox, 311B Carnegie, 443-1488, jtcox@syr.edu

Class Time and Location Tu/Thur 2:00-3:20, Carnegie 115

Office Hours Held in **Carnegie 311B**

- ~~TBA~~ Tu Th 03:45- 05:00pm
- and at other times by appointment

Texts

- A comprehensive calculus book
- Introduction to Stochastic Processes with R, by Robert Dobrow

Course Web page Some use of BlackBoard and/or WebWork may be made.

Calculator policy A calculator is useful for homework problems, but the statistical freeware package "R" is recommended instead. Calculators **are not allowed** on exams.

Cell phone policy

- Cell phones should be turned off and put away during class.
- Cell phones **are not allowed** on exams. Specifically, using or having available for use any calculator, cell phone or other electronic device during any exam will be considered a **violation** of the Academic Integrity Policy. During exams, cell phones and other electronic devices **must be stowed out of reach**, either in a closed backpack or at the front of the room.
- Violations of this policy will be considered Academic Integrity Violations.

Attendance You are expected to attend every class and every exam. You are expected to arrive on time for every class. *Please do not take this course if you cannot arrive on time every day.* If you do miss a class, it is your responsibility to obtain a copy of the lecture notes for that class from another student. You are also responsible for any announcements about changes to the course schedule, the exam schedule or the course requirements made during a missed class.

Homework/Reading There will be daily required reading assignments and weekly homework assignments. You are expected to keep up with both, as both are essential for learning the course material. Homework will be collected weekly.

Exams/Quizzes There will be 2 midterm exams and a final exam. The **tentative** dates are

- Midterm 1: Thur Sep 28
- Midterm 2: Tues Oct 3~~0~~1
- Final: Friday Dec 1~~6~~5, 8:00–10:00am (This date is **NOT** tentative.)

Exams will be based on class notes and examples, text readings and examples, and homework assignments. In addition to problems, definitions and theorem statements, short proofs will be asked on exams. There will be no "make up" exams given. The final exam will be given only at the scheduled time, it **will not be offered at any other time**! Do not make travel plans that conflict with any exam date.

- There **may** be short quizzes.

Grading The course grade weighting scheme is as follows:

- homework/quizz 20%
- each midterm exam 25%
- final exam 30%

Student Learning Outcomes of BS degree mapped to this course

- Demonstrate facility with the techniques of single and multivariable calculus and linear algebra
- Effectively communicate mathematical ideas orally and in writing
- Make accurate calculations by hand and with technological assistance
- Reproduce essential assumptions, definitions, examples, and statements of important theorems
- Describe the logical structure of the standard proof formats, reproduce the underlying ideas of the proofs of basic theorems, and create simple original proofs

Specific Course Goals

- understand the role of stochastic modeling
- gain practice developing and analyzing simple stochastic models
- learn and master some of the basic mathematical tools and techniques of stochastic modeling
- understand the relevant mathematical concepts and methods

Disability-Related Accommodations If you believe that you need accommodations for a disability, please contact the Office of Disability Services (ODS), http://disabilityservices.syr.edu, located in Room 309 of 804 University Avenue, or call (315) 443-4498, TDD: (315) 443-1371 for an appointment to discuss your needs and the process for requesting accommodations. ODS is responsible for coordinating disability-related accommodations and will issue students with documented Disabilities Accommodation Authorization Letters, #as appropriate. Since accommodations may require early planning and generally are not provided retroactively, **please contact ODS as soon as possible.**

Academic Integrity. Syracuse University's academic integrity policy reflects the high value that we, as a university community, place on honesty in academic work. The policy defines our expectations for academic honesty and holds students accountable for the integrity of all work they submit. Students should understand that it is their responsibility to learn about course-specific expectations, as well as about university-wide academic integrity expectations. The university policy governs appropriate citation and use of sources, the integrity of work submitted in exams and assignments, and the veracity of signatures on attendance sheets and other verification of participation in class activities. The policy also prohibits students from submitting the same written work in more than one class without receiving written authorization in advance from both instructors. The presumptive penalty for a first instance of academic dishonesty by an undergraduate student is course failure, accompanied by a transcript notation indicating that the failure resulted from a violation of academic integrity policy. The presumptive penalty for a first instance of academic dishonesty by a graduate student is suspension or expulsion. SU students are required to read an online summary of the university's academic integrity expectations and provide an electronic signature agreeing to abide by them twice a year during pre-term check-in on MySlice. For more information and the complete policy, see http://academicintegrity.syr.edu/. For more precise details, see

- One page guide: AI at SU
- 10 things all students need to know about AI

Religious observances policy SU religious observances policy recognizes the diversity of faiths represented among the campus community and protects the rights of students, faculty, and staff to observe religious holidays according to their tradition. Under the policy, students are provided an opportunity to make up any examination, study, or work requirements that may be missed due to are religious observance provided they notify their instructors before the end of the second week of classes. For fall and spring semesters, an online notification process is available through MySlice (Student Services -> Enrollment -> My Religious Observances) **from the first day of class until the end of the second week of class.**

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MAT 526 – Intro to Stoch. Processes

Stochastic process

Discrete time

- a sequence or ~~finitely~~ family of random variables indexed by a parameter (is usually time), say $X_0, X_1, X_2, \ldots = (X_n)$ ← the whole sequence

  X_n will be the "state" of some "system" at "time" n, $n = 0, 1, 2, \ldots$

Continuous time

$(X_t,\ t \geq 0)$

system at time t, t is a "continuous" variable

Example 1.6 Random walk, Gambler's Ruin

$0 \leq p \leq 1$ is a parameter

A gambler places a sequence of independent bets with win probability p. (given)

- On each bet the gambler's "fortune" ← amount of money the gambler has at a given time

  goes up \$1 with prob. p or down \$1 with probability $q = 1-p$

  after n^{th} gamble

- Notation: S_n = gambler's fortune at "time" n, $n = 0, 1, 2, 3, \ldots$

  S_0 = –"– initial fortune (given)

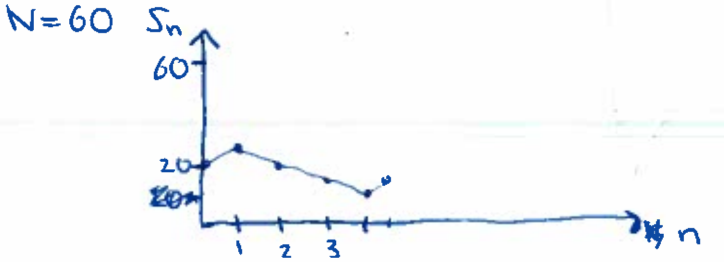
- the gambler is "ruined" if his fortune reaches 0.
- there is a "target" level N (given) and the gambler quits (wins the game) if his fortune reaches N before it reaches 0.

- Think of $p = \frac{2}{3}$, $N = 10$, $S_0 = 2$

Natural problems:

1. Given the initial fortune $S_0 = k$, what is the probability the gambler is ruined (or wins the game)

2. On average, how long does it take to play the game (reach either 0 or N)?



~~Outcome of~~ suppose the first bets are (win, loose, loose, loose, win, –

possible outcomes ($n \leq 1000$) in simulation: ruin, success, not yet resolved

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The answers to these questions cannot be decided by looking at 1, or 2, or any finite nr of random variables.

Given p, N, consider start with $S_0 = k$ for $k = 0, 1, 2, \ldots, N$.

Let $x_k = P(\text{ruin} \mid S_0 = k)$.

We want to find the numbers $x_0, x_1, \ldots, x_N$ or $(x_k)_{0 \le k \le N}$.

(Think of $N=10$, $p = \frac{2}{3}$, we want to find $\cancel{x_0}, x_1, \ldots, \cancel{x_{10}}$.)

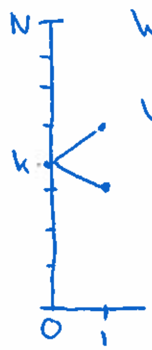
We already know: $x_0 = 1$

$x_N = 0$

We want to find $x_1, x_2, \ldots, x_{N-1}$.

Derive a difference equation (system of equations) for (x_k), $1 \le k \le N-1$.

Say $S_0 = k$, we have k dollars, $x_k = P(\text{ruin} \mid S_0 = k)$



We go up to $k+1$ with probability p, ruin/success have not happened.

Starting at $k+1$, we want x_{k+1}.

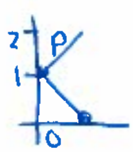
We go down to $k-1$ with prob. q, —"— —"— —"— —"—

Starting at $k-1$, we want x_{k-1}.

So we should have $\boxed{x_k = p\, x_{k+1} + q\, x_{k-1}}$

Is this correct if $k=1$?

Eqn is $x_1 = p x_2 + q x_0 = p x_2 + q \cdot 1 = p x_2 + q$



Prob. we go up to 2 is p, start at 2

Prob. we go down to 0 (hence are ruined) is prob q

$x_1 = p x_2 + q$ (ruined)

Can check for $k=N$ as well

We get $x_0 = 1$, $x_N = 0$

$\underbrace{x_k = p\, x_{k+1} + q x_{k-1}}$ for all $1 \le k \le N-1$ ← has "nothing" to do with probability

$0 = p x_{k+1} - x_k + q x_{k-1}$

general form is $= a x_{k+1} + b x_k + c x_{k-1}$ (a=p, b=-1, c=q)

is a difference equation (system)

Want (x_k), $0 \le k \le N$ with $\mathcal{J}$

DE handout → how to solve such eqns

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1.4 Conditional Probability → MAT 521

Def $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ (if $P(B) \neq 0$)

Multiplication formulas: $P(A\cap B) = P(A|B)\,P(B)$

In some problems, $P(A|B)$ is easy to find and $P(A\cap B)$ is not.

$P(A\cap B) = P(B\cap A) = P(B|A)\,P(A)$

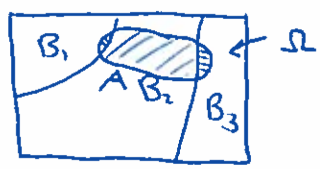
Law of Total Probability (LOTP) $P(A) = ?,\ B_1, B_2, -$

If $B_1, B_2, -$ is a sequence of disjoint$_{①}$ events ~~such~~ (sets) such that

$\bigcup_i B_i \overset{②}{=} \Omega$ ← this is the entire sample space

then $P(A) = \sum_i P(A\cap B_i) = \sum_i P(A|B_i)\,P(B_i)$

The sets $(B_1, B_2, -)$ form a partition of the sample space Ω



B_1, B_2, B_3 form a partition

Reason is: ① $A = (A\cap B_1)\cup(A\cap B_2)\cup(A\cap B_3)$ (property (2))

and these are disjoint (property (1))

② $P(A) = P(A\cap B_1) + P(A\cap B_2) + P(A\cap B_3)$

(additivity ~~of~~ axiom of probab.

Value of this?

Useful when we want $P(A)$ and for some partition (B_i), we know $P(A|B_i)$, $P(B_i)$

Example: We have 2 boxes,
- Box #1 has 4 red chips, 8 green chips
- Box #2 has 9 red chips, 6 —"— —"—

We select Box #1 with probability $\frac{1}{6}$

—"— —"— #2 —"— —"— $\frac{5}{6}$

Then draw a chip from the selected box. Find the probab. the chip is red

Let $A = \{$selected chip is red$\}$. We want $P(A)$.

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Let $B_1 = \{$ select box # 1$\}$
$B_2 = \{$ -"- -"- #2$\}$.

Then B_1, B_2 form a partition. $B_1 \cap B_2 = \emptyset$
$B_1 \cup B_2 = \Omega$.

We know $P(B_1) = \frac{1}{6}$, $P(B_2) = \frac{5}{6}$.

$P(A \mid B_1) = \frac{4}{12}$, $P(A \mid B_2) = \frac{9}{15}$.

Then $P(A) = P(A \mid B_1) P(B_1) + P(A \mid B_2) P(B_2)$
$= \frac{4}{12} \cdot \frac{1}{6} + \frac{9}{15} \cdot \frac{5}{6} \left(= \frac{1}{18} + \frac{3}{6} = \frac{10}{18}\right)$

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Chapter 2 - Markov chains

(think of X_n as the state of some "system" at time n)

Ω = the sample space = the set of all outcomes

Def: Let S be a discrete set, call this the state space.
(either finite or countable)
$\{x_0, x_1, \dots, x_N\}$ or $\{x_0, x_1, x_2, \dots\}$
but not the interval $[a,b]$

A discrete time Markov chain taking values in S is a stochastic process $(X_n) = (X_0, X_1, X_2, \dots)$ with the Markov property:

$$P(\overset{\text{future}}{X_{n+1} = j} \mid \overset{\text{present}}{X_n = i}, \underbrace{X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \dots, X_1 = x_1, X_0 = x_0}_{\text{"past"}})$$
$$= P(X_{n+1} = j \mid X_n = i)$$

$i, j, x_0, x_1, \dots, x_{n-1} \in S$

Remarks
- The time index set is $\{0, 1, 2, \dots\}$
- If we change the (x_k) and keep i, j the same, the probability doesn't change.
- The Gambler's Ruin process is a Markov chain.

Let X_n = gambler's fortune at time n.
The sequence of bets are independent with $P(\text{win}) = p$, $P(\text{loose}) = q$ $(= 1-p)$
What is X_{n+1} equal to given $X_n, X_{n-1}, \dots, X_1, X_0$?

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } q \end{cases}$$

Does not depend on $X_{n-1}, X_{n-2}, \ldots, X_0$.

<u>Def:</u> A Markov-chain is "<u>time-homogeneous</u>" if for all states i, j,

$P(X_{n+1} = j \mid X_n = i)$ does not depend on n. (don't change with time)

(For example, $P(X_5 = j \mid X_4 = i) = P(X_4 = j \mid X_3 = i) = P(X_3 = j \mid X_2 = i)$
$= P(X_2 = j \mid X_1 = i) = P(X_1 = j \mid X_0 = i)$.

We <u>will always have</u> this property.

<u>Ex:</u> Gambler's Ruin process. Time homogeneous?
$P(X_{n+1} = k+1 \mid X_n = k) = p$ for all n.

We can now define the (probability) transition matrix $\underline{\underline{P}}$ of the chain $(P_{ij})_{i,j \in S}$ and $P_{ij} = P(X_1 = j \mid X_0 = i)$ $\left(\overset{\text{also}}{=} P(X_{n+1} = j \mid X_n = i) \right)$

(always square matrix)

transition from i to j

all basic parameters of the model.

<u>Example:</u> $S = \{0, 1, 2\}$. $\underline{\underline{P}}$ is a 3×3 matrix.

$$\underline{\underline{P}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \end{array}$$

$P(X_1 = 1 \mid X_0 = 0)$

transition from 0 to 1

P_{ij} : i is the row number
j is the column -"-

<u>Gambler's Ruin</u> Target level N, win prob. p, (X_n)

What is $\underline{\underline{P}}$?

transition matrix of Markov Chain

$$P_{ij} = \begin{cases} 1 = P_{00} & \text{if } i = j = 0 \\ p \text{ if } j = i+1 & \text{if } 0 < i < N \\ q \text{ if } j = i-1 & \\ 1 = P_{NN} & \text{if } i = j = N \end{cases}$$

and all other P_{ij} equal 0.

$$\underline{\underline{P}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & \cdots & N-1 & N \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 \\ 0 & 0 & q & 0 & p \; 0 \cdots & 0 & 0 \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{array}$$

$q = 1 - p$

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Def: A stochastic matrix is a square matrix with

(1) all entries are non-negative

(2) each row sum equals 1.

Fact: A transition matrix is a stochastic matrix. #

Check Gambler's Ruin $\underline{P}$.

Proof: (P_{ij}), i^{th} row sum

$$\sum_{j\in S} P_{ij} = \sum_{j\in S} P(X_1 = j \mid X_0 = i) \qquad (\text{def of } \underline{P})$$

$$= \sum_{j\in S} \frac{P(X_0=i, \text{ and } X_1=j)}{P(X_0=i)} \qquad (\text{def of cond. prob.})$$

$$= \frac{1}{P(X_0=i)} \underbrace{\sum_{j\in S} P(X_0=i, X_1=j)}_{P(X_0=i)} \qquad (\text{algebra})$$

(LOTP, or additivity axiom)

$$= \frac{1}{P(X_0=i)} P(X_0=i)$$

$$= 1 \checkmark$$

The events $\{X_1=j\}_{j\in S}$ form a partion of Ω

$$\{X_0=i\} = \bigcup_{j\in S} \{X_0=i, X_1=j\}$$
(disjoint union)

$$P(X_0=i) = P(\ \_ \cup \_\ )$$
$$= \sum_{j\in S} P(X_0=i, X_1=j)$$

$\square$

General Goals

(1) Given a "system", (a) determine if (X_n) is a M.C.
(b) find $\underline{P}$.

(2) Given $\underline{P}$, determine various properties of the system.

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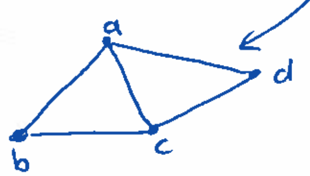
Examples:

2.8 Random walk on graphs

network

A graph has vertices and edges.
$\{a, b, c, d\}$



We say i, j are "neighbours" and write $i \sim j$ if there is an edge joining i and j.

[Large example: World Wide Web]

The **degree** of vertex i, $\deg(i)$ is the number of edges connected to i

[$\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 3$, $\deg(d) = 2$]

A random walk (X_n) jumps from one vertex to another at each time step, it jumps independently "uniformly at random" according to the number of edges at current side.

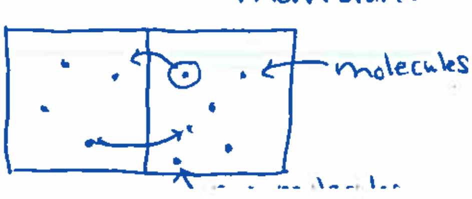
$$\text{If } X_n = b, \quad X_{n+1} = \begin{cases} a & \text{with prob } \frac{1}{2} \\ c & \text{with prob } \frac{1}{2} \end{cases}$$

$$\text{If } X_n = c, \quad X_{n+1} = \begin{cases} a & \text{with prob } \frac{1}{3} \\ b & \text{-''-} \quad \text{-''-} \quad \frac{1}{3} \\ d & \text{-''-} \quad \text{-''-} \quad \frac{1}{3} \end{cases}$$

So in general
$$P_{ij} = \begin{cases} \dfrac{1}{\deg(i)} & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

Example (3.6) Ehrenfest urn model for diffusion of a gas across a membrane (dog-flea model)



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We have 2 urns^{A,B} (containers), N distinct labeled balls) (like $N=10$) in the urns.

"Dynamics": at time step, pick a "ball "uniformly at random" (equal prob for all balls), and move it from the urn it is in to the other urn.

Let X_n = # of balls in urn A after n^{th} move.

Then ① (X_n) is a <u>Markov</u> chain, state space is $S = \{0, 1, 2, \ldots, N\}$.

② If $X_n = i$ then there are $N-i$ balls in urn B.

③

$$P_{ij} = \begin{cases} \frac{N-i}{N} & j = i+1 \\ \frac{i}{N} & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

$\overset{A}{\boxed{i}}$ $\overset{B}{\boxed{N-i}}$ $X_n = i$

$X_{n+1} = i+1$? (Choose from B

$X_{n+1} = i-1$?
↳ Choose from A

<u>Questions</u>

① If $X_0 = N$ (all balls in A), what happens as $n \to \infty$.

② Is there an equilibrium?

③ If $X_0 = N$, how long on average does it take to have all balls in B.

<u>Example</u> Not everything is a Markov chain.

We have 5 quarters (25¢), ~~5 10~~ 5 dimes (10¢), 5 nickels (5¢).

Draw a coin at random, put it on the table.

Then draw again, etc.

X_n = amount of money on the table after n^{th} draw.

$X_0 = 0$ $[X_{15} = 2.00 = X_{16} = X_{17} = \ldots]$

Claim: (X_n) is not a Markov chain.

Why? ① Intuitive?

② Formally, can we violate the Markov property ~~of~~ ~~for~~ any n, any sequence of states.

Claim: Aug 31

$$P(X_5 = .45 \mid X_0 = 0, X_1 = \overset{Q}{.25}, \overset{N}{X_2} = .30, \overset{N}{X_3} = .35, \overset{N}{X_4 = .40})$$

$$\neq P(X_5 = .45 \mid X_0 = 0, X_1 = \overset{D}{.10}, X_2 = \overset{D}{.20}, X_3 = \overset{D}{.30}, \overset{D}{X_4 = .40})$$

Calculate each by def of cnd prob.

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<u>Markov Chain</u> – sequence of rand. var. $(X_n)_{n=0,1,2,\dots}$ taking values in state space S satisfying $P(X_{n+1}=j \mid X_0=x_0, X_1=x_1, \dots, X_{n-1}=x_{n-1}, X_n=i)$

$\overset{(MP)}{=}$ $P(X_{n+1}=j \mid X_n=i) \overset{(TH)}{=} P(X_1=j \mid X_0=i)$

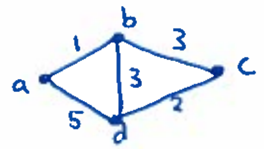
transition matrix $\underline{P}$, $P_{ij} = P(X_1=j \mid X_0=i)$ $\quad i,j \in S$

(the one-step probabilities)

Think of X_n = state of some "system" at time n.

<u>Example 2.11</u> random walk on <u>weighted</u> graphs

(see text directed weighted graphs)



vertices

edges $\quad (i \sim j$ if there is an edge from i to $j)$

weights $\quad w_{ij} \geq 0$ on edges $i \sim j$

Let w_i = total weight of edges containing vertex i $\quad (w_d = 10)$

Let $$P_{ij} \begin{cases} = \dfrac{w_{ij}}{w_i} & \text{if } i \sim j \\ = 0 & \text{if not} \end{cases}$$

Here, $w_d = 10$, $w_a = 6$, $w_b = 7$, $w_c = 5$

$$\underline{P} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & \frac{1}{6} & 0 & \frac{5}{6} \\ b & \frac{1}{7} & 0 & \frac{3}{7} & \frac{3}{7} \\ c & & & 0 & \\ d & & & & 0 \end{array}$$

etc.

<u>The n-step probabilities</u>

are $P(X_n = j \mid X_0 = i)$, $i,j \in S$, $n = 1, 2, 3, \dots$

Can we find these from $\underline{P}$?

$\underline{n=1}$: $P(X_1 = j \mid X_0 = i) = P_{ij}$

$\underline{n=2}$: $P(X_2 = j \mid \underbrace{X_0 = i}) \overset{\text{(LOTP)}}{\underset{\text{(conditional form)}}{=}} \sum_k P(X_2 = j \mid X_1 = k, \underbrace{X_0 = i}) \underbrace{P(X_1 = k \mid \underbrace{X_0 = i})}_{P_{ij}}$

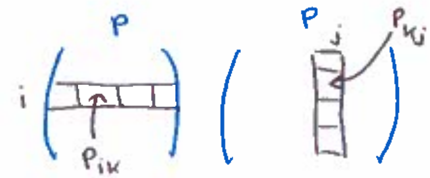
(LOTP)
(conditional form)
HW1 - exer(3)

$\{X_1 = k\}_{k \in S}$ form a partition

$$\overset{=}{\text{MP}} \sum_{k \in S} \underbrace{P(X_2 = j \mid X_1 = k)}_{= P_{kj}} P_{ik}$$

$$= \sum_{k \in S} P_{kj} P_{ik}$$

$$= \sum_k P_{i\underline{k}} P_{\underline{k}j}$$



$$\underset{\substack{\text{matrix} \\ \text{mult.}}}{=} (\underline{P} \times \underline{P})_{ij}$$

$\underline{P} \cdot \underline{P}$ an Stelle (i,j)

$$= (\underline{P}^2)_{ij}$$

$\underline{\text{Notation}}$: We write $(\underline{P}^2)_{ij} = p^2_{ij}$ $\quad (\neq (P_{ij})^2)$

$$\Rightarrow P(X_2 = j \mid X_0 = i) = (\underline{P}^2)_{ij}$$

$\underline{n=3}$: $P(X_3 = j \mid X_0 = i) \overset{\text{LOTP}}{=} \sum_{k \in S} P(X_3 = j \mid X_2 = k, X_0 = i) \cdot P(X_2 = k \mid X_0 = i)$

$\{X_2 = k\}_{k \in S}$ form a partition

$$\overset{\text{MP}}{=} \sum_{k \in S} P(X_3 = j \mid X_2 = k) \cdot p^2_{ik} \quad \swarrow \text{n=2 case}$$

$$\overset{\text{TH}}{=} \sum_{k \in S} P_{kj} \cdot p^2_{ik}$$

$$= \sum_{k \in S} p^2_{ik} P_{kj}$$

$$\underset{\substack{\text{matrix} \\ \text{mult.}}}{=} (\underline{P}^2 \times \underline{P})_{ij}$$

$$= (\underline{P} \times \underline{P}^2)_{ij}$$

$$= (\underline{P}^3)_{ij}$$

Conclusion:

The n-step probabilities are $P(X_n=j \mid X_0=i) = p^n_{ij}$, $n=1,2,3,\ldots$

$$\underline{\underline{p}}^0 = ? = P(X_0=j \mid X_0=i) = \begin{cases} 1 & \text{if } j=1 \\ 0 & \text{if } j \neq 1 \end{cases} = \text{identity matrix}$$

Example

Suppose $S=\{0,1,2\}$,

$$\underline{\underline{P}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} .1 & .1 & .8 \\ .2 & .2 & .6 \\ .3 & .3 & .4 \end{bmatrix} \end{array}$$

Find $P(X_2=0 \mid X_0=0) =$

① Use LOTP

② $P^2_{00} = (\underline{\underline{P}} \times \underline{\underline{P}})_{00} = .27$

Check that

$$\underline{\underline{P}}^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} .27 & 27 & .46 \\ .24 & .24 & .52 \\ .21 & .21 & .58 \end{bmatrix} \end{array} \qquad P^2_{22} = .58 \text{ etc.}$$

Chapman-Kolmogorov equations

By matrix multiplication

$$\underline{\underline{P}}^{n+m} = \underline{\underline{P}}^n \times \underline{\underline{P}}^m = \underline{\underline{P}}^m \times \underline{\underline{P}}^n.$$

$$\underbrace{\underline{\underline{P}} \times \underline{\underline{P}} \times \ldots \times \underline{\underline{P}}}_{n+m}$$

So, this gives us

$$P(X_{n+m}=j \mid X_0=i) = p^{n+m}_{ij} = \sum_{k\in S} p^n_{ik}\, p^m_{kj} \qquad (= (\underline{\underline{P}}^n \times \underline{\underline{P}}^m)_{ij})$$

$$= \sum_{k\in S} P(X_n=k \mid X_0=i)\, \underbrace{P(X_m=j \mid X_0=k)}_{\shortparallel} \text{ m steps from 0}$$

$$= \sum_{k\in S} P(X_n=k \mid X_0=i)\, P(X_{n+m}=j \mid X_n=k) \text{ — m steps from time n}$$

That is,

$$\boxed{P(X_{n+m}=j \mid X_0=i) = \sum_k P(X_n=k \mid X_0=i)\cdot P(X_{n+m}=j \mid X_n=k)}$$

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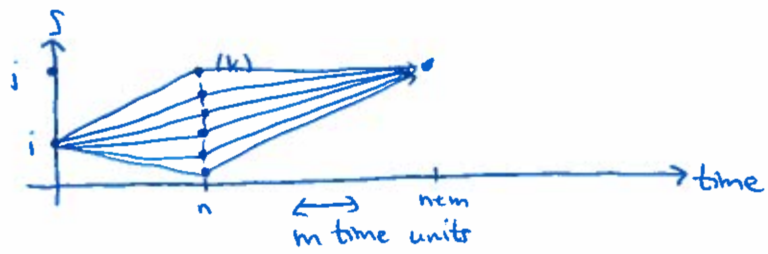

- A probability row vector $\underline{\alpha} = (\alpha_i)_{i \in S}$ satisfies: each $\alpha_i \geq 0$, $\sum_{i \in S} \alpha_i = 1$.

(In effect, $\underline{\alpha}$ is a probability mass fct. of a discrete r.v.)

- A Markov chain ~~has~~ (X_n) has initial distribution $\underline{\alpha}$ (a probability row vector) if $P(X_0 = i) = \alpha_i$, $i \in S$.

↑ time 0

<u>Fact</u> Given initial distribution α, The probability distribution of X_n is given by

$$P(X_n = j) = (\underline{\alpha}\, \underline{P}^n)_j = \sum_{i \in S} \alpha_i \, P^n_{ij}$$

why? LOTP

$\{X_0 = i\}_{i \in S}$ form a partition

$$P(X_n = j) = \sum_{i \in S} P(X_n = j \mid X_0 = i)\, P(X_0 = i)$$

$$= \sum_{i \in S} P^n_{ij} \cdot \alpha_i = \sum_{i \in S} \alpha_i \cdot P^n_{ij} = (\underline{\alpha}\, \underline{P}^n)_j$$

<u>Previous Example:</u>

$$\underline{\mathbb{P}}^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} .27 & .27 & .46 \\ .24 & .24 & .52 \\ .21 & .21 & .58 \end{bmatrix} \end{array}$$

Suppose X_0 has distribution $\underline{\alpha} = (.7 \;\; .2 \;\; .1)$.

Find $P(X_2 = 0) = (\underline{\alpha}\, \underline{P}^2)_0 = \left([.7 \;\; .2 \;\; .1] \begin{bmatrix} .27 & .27 & .46 \\ .24 & .24 & .52 \\ .21 & .21 & .58 \end{bmatrix} \right)_0$

$$= [\overset{0}{(.7)(.27) + (.2)(.24) + (.1)(.21)} \quad \overset{1}{} \quad \overset{2}{}]$$

$$= \left[\tfrac{129}{500} \;\; \tfrac{129}{500} \;\; \tfrac{121}{2500} \right]_0 = \frac{129}{500}.$$

<u>Fact:</u> $P(X_1 = i_1, X_2 = i_2, X_3 = i_3, \ldots, X_n = i_n \mid X_0 = i_0)$

joint dist. of $X_1, \ldots X_n$ given $X_0 = i_0$

$$= P_{i_0 i_1} \cdot P_{i_1 i_2} \, P_{i_2 i_3} \cdots P_{i_{n-1} i_n}$$

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[Given init. distr. α, then

$$P(X_0=i_0, X_1=i_1, \ldots, X_n=i_n) = \alpha_{i_0} P_{i_0 i_1} \cdots P_{i_{n-1} i_n} \;]$$

$$[\; 0 < n_1 < n_2 < \ldots < n_k \Rightarrow P(X_{n_1}=i_1, X_{n_2}=i_{n_2}, \ldots, X_{n_k}=i_{n_k} \mid X_0=i_0) = P^{n_1}_{i_0 i_1} \cdot P^{n_2-n_1}_{i_1 i_2} \cdots P^{n_k-n_{k-1}}_{i_{k-1} i_k} \;]$$

$$P(X_{\underset{n_1}{3}} = b, X_{\underset{n_2}{7}} = c, X_{\underset{n_3}{9}} = d, X_{\underset{n_4}{10}} = f \mid X_0 = a)$$

$$= P^3_{ab} \cdot P^4_{bc} \, P^2_{cd} \cdot P_{df}$$

Mult. Rule(s)

$$P(A_2 \cap A_1) = P(A_2 \mid A_1) P(A_1)$$

$$P(A_3 \cap A_2 \cap A_1) = P(A_3 \mid A_2 \cap A_1) P(A_2 \cap A_1)$$

$$= P(\check{A}_3 \mid A_2 \cap A_1) \, P(\check{A}_2 \mid A_1) P(\check{A}_1)$$

Similar $P(A_4 \cap A_3 \cap A_2 \cap A_1) = P(A_4 \mid A_3 \cap A_2 \cap A_1) \, P(A_3 \mid A_2 \cap A_1) \, P(A_2 \mid A_1) P(A_1)$

... $P(A_n \cap A_{n-1} \cap \ldots \cap A_1) = \ldots$

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Example MC (X_n) with state space $S = \{0,1,2\}$

transition matrix $\underline{P}$

$$\underline{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .1 & .1 & .8 \\ 1 & .2 & .2 & .6 \\ 2 & .3 & .3 & .4 \end{array} \qquad \underline{P}^2 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .22 & .22 & .56 \\ 1 & .23 & .23 & .54 \\ 2 & .24 & .24 & .53 \end{array} \qquad \underline{P}^3 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .219 & .219 & .562 \\ 1 & .228 & .228 & .544 \\ 2 & .237 & .237 & .526 \end{array}$$

ⓐ $P(X_1 = 0, X_2 = 0, X_3 = 2, X_4 = 1 \mid X_0 = 2) = P_{20} P_{00} P_{02} P_{21}$

$= (.3)(.1)(.8)(.3)$

ⓑ $P(X_7 = 2 \mid X_1 = 0, X_2 = 0, X_3 = 2, X_4 = 1) = P(X_7 = 2 \mid X_4 = 1) = P^3_{12} = .544$

$7-4 = 3$ time steps

[formulas (JD1), (JD2), (MP), (JD3) → pp 56-58]

ⓒ $P(X_2 = 1, X_4 = 2, X_5 = 1, X_8 = 0 \mid X_0 = 0) = P^2_{01} P^2_{12} P^1_{21} P^3_{10} = (.22)(.54)(.3)(.228)$

$4-2=2$ steps; 1 step; 3 steps; 2 steps

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Proof of

$m=1$ case of (MP) $\quad n-m=n-1$

$$P(X_{n+1}=j \mid X_0=x_0, X_1=x_1, \ldots, X_{n-1}=i) \overset{\text{LOTP}}{\underset{\text{(cond. form of 1)}}{=}} \sum_{k\in S} P(X_{n+1}=j \mid X_0=x_0, X_1=x_1, \ldots, X_{n-1}=i, X_n=k)$$

1 timestep → (MP)

$$\cdot P(X_n=k \mid X_0=x_0, \ldots, X_{n-1}=i)$$

$$\overset{(MP)}{=} \sum_{k\in S} \cancel{P(X_{n+1})}\, P_{kj} \cdot P_{ik} = \sum_{k\in S} P_{ik} P_{kj} = P^{2=m+1}_{ij}$$

Proof $n=3$ $\quad$ initial dist is α

$$P(X_1=j, X_2=k, X_3=l, X_0=i) = P(X_3=l, X_2=k, X_1=j, X_0=i)$$

$$\overset{\text{mult. rule}}{=} P(X_3=l \mid X_2=k, X_1=j, X_0=i) \cdot P(X_2=k \mid X_1=j, X_0=i)\, P(X_1=j \mid X_0=i) \cdot P(X_0=i)$$

$$\overset{(MP)}{=} P(X_3=l \mid X_2=k) \cdot P(X_2=k \mid X_1=j)\, P(X_1=j \mid X_0=i)\, P(X_0=i)$$

$$= P_{kl}\, P_{jk}\, P_{ij}\, \alpha_i = \alpha_i\, P_{ij}\, P_{jk}\, P_{kl}$$

Ex from book: $\quad$ What happens to P^n_{ij} as $n\to\infty$?

We appear see

(0) 0 entries are absent P^n_{ij} for large n

(1) for each i,j $(0,1,2)$ $\lim_{n\to\infty} P^n_{ij}$ exist

same ↕

(2) There is a limiting matrix. $\quad \lim_{n\to\infty} \underline{P}^n$ exists.

(3) The rows in the limiting matrix are identical.

$$P^{100}_{00} = P^{100}_{10} = P^{100}_{20} \quad \text{(approximately)}$$

That is, $\lim_{n\to\infty} P^n_{ij}$ does not depend on i.

$$\lim_{n\to\infty} P^n_{ij} = P(X_n=j \mid X_i=i)$$

"the chain forgets its starting point"

Another ex: limit does not exist

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Def: A MC $\underbrace{(X_n)}_{\text{with transition matrix } \underline{\underline{P}}}$ has a <u>Limiting</u> <u>distribution</u> λ

if for all i and j, $\underbrace{\lim_{n\to\infty} P_{ij}^n}$ $= \lambda_j$ only depends on j (This is more than just $\lim_{n\to\infty} P_{ij}^n$ exists.)

$$\lim_{n\to\infty} P(X_n = j \mid X_0 = i)$$

<u>Notes:</u> ① A chain cannot have 2 different Limiting distributions.

② It is usually impossible to directly compute $\underline{\underline{P}}^n$ to check if there is a limiting distribution

<u>Example</u> (3.2) The two state M.C.

Let $0 \le p \le 1$, $0 \le q \le 1$ $\quad (q \ne 1-p)$

$$\underline{\underline{P}} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \end{array}$$

<u>Special Cases:</u>

① $p = q = 0$, $\quad \underline{\underline{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \underline{\underline{P}}^n = \underline{\underline{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\forall n$

Note that $\lim_{n\to\infty} P_{ij}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ exists for all i,j, but <u>depends</u> on both i,j, so no <u>limiting</u> <u>distribution</u> for $\underline{\underline{P}}$

② $p = q = 1$, $\quad \underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\underline{\underline{P}}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}, \quad \underline{\underline{P}}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{\underline{P}}^2 \times \underline{\underline{P}} = \underline{\underline{I}} \times \underline{\underline{P}} = \underline{\underline{P}}, \quad \underline{\underline{P}}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{P}}^2 \times \underline{\underline{P}}^2 = \underline{\underline{I}} \times \underline{\underline{I}} = \underline{\underline{I}}$$

$$\Rightarrow \underline{\underline{P}}^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } n \text{ is odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } n \text{ is even} \end{cases}$$

$P_{00}^n = (0, 1, 0, 1, 0, 1, \ldots)$, does not have a limit

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General Case of interest

$$\underline{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \quad \text{where} \quad 0 < p+q < 2$$

Let $r = 1-p-q$, note $-1 < r < 1$

Then $(*)$ $\underline{P}^n = \frac{1}{p+q} \begin{bmatrix} q+pr^n & p-pr^n \\ q-qr^n & p+qr^n \end{bmatrix}$

(The first row sum is $\frac{1}{p+q}\left(q+\cancel{pr^n}+p-\cancel{pr^n}\right) = 1$ ✓)

and $\lim_{n\to\infty} \underline{P}^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix}$ since $\lim_{n\to\infty} r^n = 0$. $(|r|<1)$

$$= \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$$

We get a limiting distribution

$\lambda = (\lambda_1, \lambda_2)$

$\lambda_1 = \lim_{n\to\infty} P^n_{11} = \frac{q}{p+q} = \lim_{n\to\infty} P^n_{21}$

$\lambda_2 = \lim_{n\to\infty} P^n_{12} = \frac{p}{p+q} = \lim_{n\to\infty} P^n_{22}$

Proof of $(*)$

I. Check by induction $(*)$ is correct.

- Check $n=1$
- $\underline{P}^{n+1} = \underline{P} \times \underline{P}^n$

II. See text for a derivation.

Find a formula for P^n_{11} in terms of P^{n-1}_{11}.

III. Use difference eqn's.

What happens if the initial distribution is chosen to be λ the limiting dist.?

ie. $P(X_0=1) = \frac{q}{p+q}$, $P(X_0=2) = \frac{p}{p+q}$.

What is the dist. of X_1.

$$P(X_1=1) \overset{\text{LOTP}}{=} \underbrace{P(X_1=1 \mid X_0=1)}_{=P_{11}} P(X_0=1) + \underbrace{P(X_1=1 \mid X_0=2)}_{=P_{21}} P(X_0=2)$$

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$$=(1-p)\cdot\frac{q}{p+q} + q\cdot\frac{p}{p+q} = \frac{q-pq+pq}{p+q} = \frac{q}{p+q} = P(X_0=1).$$

$P(X_1=1) = P(X_0=1)$
$P(X_1=2) = P(X_0=2)$

→ dist of MC does not change from time 0 to time 1

→ stationary dist

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$$\underline{P} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-p & p \\ 2 & q & 1-q \end{array}$$

Claim: If $0<p,q<1$, and $r=1-p-q$, then

$$\underline{P}^n = \frac{1}{p+q}\begin{bmatrix} q+pr^n & p-pr^n \\ q-qr^n & p+qr^n \end{bmatrix}$$

$$\lim_{n\to\infty} \underline{P}^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} = \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$$

so λ is the limiting distribution.

$\lambda = \left[\frac{q}{p+q} \quad \frac{p}{p+q}\right]$

$$P_{11}^n \stackrel{?}{=} \frac{q+pr^n}{p+q}$$

Proof via DE's

Let $x_n = P_{11}^{\,n} = (\underline{P}\times\underline{P}^{n-1})_{11} = (\underline{P}^{n-1}\times\underline{P})_{11}$

$$\underline{P}^{n-1}\times\underline{P} = \begin{bmatrix} P_{11}^{n-1} & P_{12}^{n-1} \\ P_{21}^{n-1} & P_{22}^{n-1} \end{bmatrix}\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} (1-p)P_{11}^{n-1}+qP_{12}^{n-1} & -- \\ -- & -- \end{bmatrix}$$

$$\Rightarrow x_n = (1-p)P_{11}^{n-1} + q\,\overbrace{P_{12}^{n-1}}^{=1-P_{11}^{n-1}}$$

$$= (1-p)x_{n-1} + q(1-x_{n-1}) = q + x_{n-1}(1-p-q) = q + r x_{n-1}$$

$\Rightarrow \quad x_n - r x_{n-1} = q$ $\qquad$ 1st order difference equation

$\underline{a x_{n+1} + b x_n + c = z_n}$

Characteristic eqn in t is $t-r=0$

Char. eqn $ar^2+br+c=0$

solution is $t=r$

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(1) Solution to $x_n - r x_{n-1} = 0$ is $x_n = Ar^n$

(2) For a particular solution, guess const., so $x_n = c$

[$x_n - r x_{n-1} = q = z_n$ guess sth like $z_n = q$ (const)]

[2nd order linear, const. coeff. differential eqns
$\underbrace{ay'' + by' + cy}_{=0} = f(x)$ $\rightarrow$ guess const. $f(x)$]

Plug in, $x_n - r x_{n-1} = c - rc = q$, solve for c, $c(1-r) = q$, $c = \frac{q}{1-r} = \frac{q}{p+q}$

General sol'n is $x_n = Ar^n + \frac{q}{p+q}$ gen + particular soln (DE handout)

[$\underline{P}^0$ = identity matrix]

$x_0 = P_{11}^0 = 1 = Ar^0 + \frac{q}{p+q} \Rightarrow A = 1 - \frac{q}{p+q} = \frac{p+q-q}{p+q} = \frac{p}{p+q}$

so $x_n = Ar^n + \frac{pq}{p+q} = \frac{p}{p+q} r^n + \frac{pq}{p+q} = \frac{1}{p+q}(q + pr^n)$.

Limiting Distributions ✓

<u>Def:</u> A stationary distribution for a transition matrix $\underline{P}$ is a

① probability row vector $\underline{\lambda}$ such that ② $\underline{\lambda}\,\underline{P} = \underline{\lambda}$ $(= 1 \cdot \underline{\lambda})$

$\rightarrow \underline{\lambda}$ is a <u>left</u> eigenvalue of $\underline{P}$

② = Same as: $(\underline{\lambda}\,\underline{P})_j = \underline{\lambda}_j$ for each j, $\sum_{i \in S} \lambda_i P_{ij} = \lambda_j$ for each j.

We will usually use $\underline{\pi}$ for a stationary distribution.

1) $\underline{\pi}$ is a probability row vector

2) $\pi_j = \sum_i \pi_i P_{ij}$ for each j

<u>Ex:</u> $\underline{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$, we found $\underline{\lambda} = \left[\frac{q}{p+q} \quad \frac{p}{p+q} \right]$

is a stationary distribution. Check!

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Fact: If $\underline{\pi} = \underline{\pi}\,\underline{P}$, then $\underline{\pi} = \underline{\pi}\,\underline{P}^2 = \underline{\pi}\,\underline{P}^3 = \ldots$

In terms of the MC (X_n), if X_0 has distribution $\underline{\pi}$ then X_n has dist. $\underline{\pi}$ for $n = 1, 2, 3, \ldots$ ie, $P(X_n = j) = P(X_0 = j) = \pi_j$ for all n, j

Stationary

Check: $\underline{\pi}\,\underline{P}^2 = \underline{\pi}\,\underline{P}\,\underline{P} = (\underline{\pi}\,\underline{P})\underline{P} = \underline{\pi}\,\underline{P} = \underline{\pi}$ ✓

$\underline{\pi}\,\underline{P}^3 = (\underline{\pi}\,\underline{P}^2)\underline{P} = \underline{\pi}\,\underline{P} = \underline{\pi}$ ✓

If X_0 has distribution $\underline{\pi}$ then $P(X_n = j) \overset{LOTP}{=} \sum_i P(X_n = j \mid X_0 = i)\, P(X_0 = i)$

$= \sum_i P_{ij}^{\,n}\, \pi_i = \sum_i \pi_i\, P_{ij}^{n} = \pi_j = P(X_0 = j)$ ✓

Can we find stationary distributions $\underline{\pi}$ for a given $\underline{P}$?

Yes,

Solve linear eqns for (π_j)

① $\sum_{i \in S} \pi_i = 1$

② $\sum_{i \in S} \pi_i P_{ij} = \pi_j$ for all $j \in S$

Example:

$$\underline{P} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \end{array}$$

① If p and q are not both 0, then there is a single solution to

$$\begin{cases} \pi_1 + \pi_2 = 1 & ① \\ \pi_1 = \pi_1 P_{11} + \pi_2 P_{21} = \pi_1 (1-p) + \pi_2 q & ② \; j=1 \\ \pi_2 = \pi_1 P_{12} + \pi_2 P_{22} = \pi_1 p + \pi_2 (1-q) & ② \; j=2 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_1 + \pi_2 = 1 \\ p\pi_1 - q\pi_2 = 0 \\ p\pi_1 - q\pi_2 = 0 \end{cases}$$

same eqn → one eqn is redundant

is always like this ⇒ have to use ① as well

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$p\pi_1 = q\pi_2$

$\Leftrightarrow \pi_1 = \dfrac{q\pi_2}{p}$ plug in: $\dfrac{q\pi_2}{p} + \pi_2 = 1$

$\Leftrightarrow \pi_2\left(\frac{q}{p}+1\right)=1 \Leftrightarrow \pi_2\left(\frac{q+p}{p}\right)=1 \Leftrightarrow \pi_2 = \dfrac{p}{q+p}$

$\pi_1 = \dfrac{q}{p+q}$

(the limiting distribution)

② If $p=q=0$, $\underline{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{\pi}\,\underline{P} = \underline{\pi}$ $\quad [\pi_1\ \pi_2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\pi_1\ \pi_2]$

$$\left.\begin{array}{ll} ② & \pi_1 = \pi_1 \\ & \pi_2 = \pi_2 \\ ① & \pi_1 + \pi_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \pi_1 = \cancel{\pi_1}\ c \\ \pi_2 = 1 - \cancel{\pi_1}\ c \end{array}$$

For any number c, $0 \le c \le 1$, $\underline{\pi} = [c \quad 1-c]$ is a stationary distribution

This shows there can be more than one stationary distribution.

③ Take $p=q=1$, $\underline{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\underline{\pi} = [\frac{1}{2}\ \frac{1}{2}]$ is the unique stationary distribution.

However, $\lim_{n\to\infty} \underline{P}^n$ does not exist.

Recall: $\underline{P}^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } n \text{ is odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } n \text{ is even} \end{cases}$

<u>Lemma 3.1</u> If (X_n) is a MC with transition matrix $\underline{P}$ ~~and λ~~ which has a limiting distribution $\underline{\lambda}$, then $\underline{\lambda}$ is a stationary distribution. The converse is false by the last example.

<u>Proof:</u> Assume $\lim_{n\to\infty} P^n_{ij} = \lambda_j$ for all i,j.

Consider $P^{n+1}_{ij} = (\underline{P}^n \times \underline{P})_{ij} = \sum_k P^n_{ik} P_{kj}$.

As $n \to \infty$, $P^{n+1}_{ij} \to \lambda_j$, also $P^n_{ik} \to \lambda_k$ Plug this into

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$$P_{ij}^{n+1} = \sum_k P_{ik}^{n} P_{kj}$$

$$\downarrow n\to\infty \qquad \downarrow n\to\infty$$

$$\lambda_j = \sum_k \lambda_k P_{kj} = (\underline{\lambda}\,\underline{\underline{P}})_j \qquad \Rightarrow \lambda \text{ is a stationary distribution.}$$

$\square$

"Regular" transition matrices

A matrix $\underline{\underline{M}}$ is called positive if all entries $M_{ij} > 0$.

Def: A transition matrix $\underline{\underline{P}}$ is called regular if there is some $n \geq 1$ (from book) such that $\underline{\underline{P}}^n$ is a positive matrix.

Examples

① $\underline{\underline{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, not regular

② $\underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, -"- -"-

③ $\underline{\underline{P}} = \begin{bmatrix} .2 & .8 \\ 0 & 1 \end{bmatrix}$ (Check $\underline{\underline{P}}^n$ from our formula, we see $P^n_{21} = 0$ for all n)
not regular

④ $\underline{\underline{P}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ (i) can find $\underline{\underline{P}}^2, \underline{\underline{P}}^3, \underline{\underline{P}}^4$ and see $\underline{\underline{P}}^4$ is positive, so $\underline{\underline{P}}$ is regular.

(ii) Use +/0 method.

$+\cdot+=+$
$0\cdot+=0$

$$\underline{\underline{P}}^2 = \begin{bmatrix} 0 & 0 & + \\ + & 0 & 0 \\ 0 & + & + \end{bmatrix}\begin{bmatrix} 0 & 0 & + \\ + & 0 & 0 \\ 0 & + & + \end{bmatrix} = \begin{bmatrix} 0 & + & + \\ 0 & 0 & + \\ + & + & + \end{bmatrix}$$

$$\underline{\underline{P}}^4 = \underline{\underline{P}}^2 \cdot \underline{\underline{P}}^2 = \begin{bmatrix} 0 & + & + \\ 0 & 0 & + \\ + & + & + \end{bmatrix} \cdot \begin{bmatrix} 0 & + & + \\ 0 & 0 & + \\ + & + & + \end{bmatrix} = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$$

So all entries of $\underline{\underline{P}}^4$ are positive, so $\underline{\underline{P}}$ is regular.

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Fact (p.85): If for some $n>1$, $\underline{\underline{P}}^n$ has at least one zero, and all the zero entries in $\underline{\underline{P}}^{n}$ occur in the same places as all the zero entries in $\underline{\underline{P}}^{n+1}$, then $\underline{\underline{P}}$ is not regular.

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HW solutions: on reserve in the library

Def: $\underline{\underline{P}}$ is regular if for some n, $\underline{\underline{P}}^n$ is positive. (all entries are positive)

Exercise: 3.3.3 If $\underline{\underline{P}}^n$ is positive for some n, then $\underline{\underline{P}}^k$ is positive $\forall k>n$.

Thm 3.2: A MC whose transition matrix is regular has a limiting distribution, and this lim. dist. is the unique stationary distribution

[recall: A prob. row vector $\underline{\lambda}$ is the limiting distribution for $\underline{\underline{P}}$ if

$\lim_{n\to\infty} P^n_{ij} = \lambda_j$ for all states i,j. It is not enough to have $\lim_{n\to\infty} P^n_{ij}$ existing.

That is, there is a unique probability vector π such that

(a) $\lim_{n\to\infty} P^n_{ij} = \pi_j$ for all states i,j.

and (b) $\underline{\pi}\,\underline{\underline{P}}\,\underline{\pi}$.

(a) is the same as $\lim_{n\to\infty} \underline{\underline{P}}^n = \underline{\underline{\Delta}}$, where each row of $\underline{\underline{\Delta}}$ is $\underline{\pi}$

Furthermore, $\underline{\pi}$ is positive ($\pi_i > 0$ for each $i \in S$).

Example: Let $0<p<1$, define

$$\underline{\underline{P}} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1-p & p \\ p & 0 & 1-p \\ 1-p & p & 0 \end{bmatrix} \end{array}$$

← an example of a doubly stochastic matrix

(a) $\underline{\underline{P}}$ is regular because

$$\underline{\underline{P}}^2 = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}.$$

is positive.

(b) Find stationary dist. $\underline{\pi} = [\pi_1\ \pi_2\ \pi_3]$

The Thm says $\underline{\pi}$ is unique

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Def: A doubly stochastic matrix is a stoch. matrix where all the column sums are 1.

Fact: Every finite doubly stochastic matrix $\underline{\underline{P}}$ has a stationary distribution which is uniform over the states. If there are K states, then $\underline{\pi}$, $\pi_i = \frac{1}{K}$ for all i is a stationary distribution.

The Thm says $\underline{\pi}$ is unique. Since $\underline{\underline{P}}$ is doubly stochastic, $\underline{\pi}$ is uniform. $\pi = [\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}]$

(c) Furthermore, $\boxed{\lim_{n\to\infty} P^n_{ij} = \frac{1}{3}}$ for each $i,j \in \{1,2,3\}$.

We did not have to compute powers $\underline{\underline{P}}^n$.

Proof of fact: Let $\pi_i = \frac{1}{k}$

$$(\underline{\pi} \underline{\underline{P}})_j = \sum_{i\in S} \pi_i P_{ij} = \sum_{i \in S} \frac{1}{k} P_{ij} = \frac{1}{k} \sum_{i\in S} P_{ij}$$

sum over entries in column j of $\underline{\underline{P}}$

$$= \frac{1}{k} \cdot 1 = \frac{1}{k}.$$

Note: Unique stationary dist. does not imply existence of a limiting dist.

$$\underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ not regular

Set $\underline{\pi} \stackrel{?}{=} \underline{\pi} \underline{\underline{P}} = [\pi_1 \ \pi_2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [\pi_2 \ \pi_1] \stackrel{?}{=} [\pi_1 \ \pi_2]$

Must have $\pi_1 = \pi_2$. Since $\pi_1 + \pi_2 = 1$, $\pi_1 = \pi_2 = \frac{1}{2}$.

So, $[\frac{1}{2} \ \frac{1}{2}]$ is the unique stationary distribution.

But $\lim_{n\to\infty} P^n_{ij}$ does not exist.

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3.3 Classification of states

Notation: For a transition matrix $\underline{P}$, states i, j, write

- $i \mapsto j$ means $P_{ij} > 0$ (single step)
- $i \to j$ (i leads to j, or j is accessible from i) means $p^n_{ij} > 0$, some $n \geq 0$ (allow $n=0$) $[\underline{P}^0 = I]$
  ($p^n_{ij} = P(X_n = j \mid X_0 = i) > 0$)
- $i \leftrightarrow j$ (i and j communicate) means $i \to j$ and $j \to i$

$$\underline{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix}$$

Remarks

1. If $i \mapsto j$ then $i \to j$
2. $i \to i$ $\quad p^0_{ii} = 1 \ \forall i$
3. If $i \to j$ and $j \to k$ then $i \to k$.

Proof of ③: By assumption, there are m, n such that $p^m_{ij} > 0$ and $p^n_{jk} > 0$.

Consider $L = m+n$, check $p^L_{ik} > 0$?

$\underline{P}^L = \underline{P}^m \times \underline{P}^n$, $\quad p^L_{ik} = \sum_{\alpha \in S} p^m_{i\alpha} p^n_{\alpha k} \geq p^m_{ij} p^n_{jk} > 0.$

each term is ≥ 0
keep $\alpha = j$

Fact: (p. 94)

Communication ($\leftrightarrow$) is an <u>equivalence relation</u> on S (set of states)

That is
(i) $i \leftrightarrow i$ for all $i \in S$ $\qquad (i \to i \Rightarrow i \leftrightarrow i)$
(ii) If $i \leftrightarrow j$ then $j \leftrightarrow i$ $\qquad (i \leftrightarrow j$ means $i \to j$ & $j \to i \Rightarrow j \leftrightarrow i)$
(iii) If $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.

(iii) We need to check that $i \to k$ and $k \to i$.
We know $i \to j$ and $j \to k$ thus $i \to k$. Same arg. gives $k \to i$. $\square$

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~~(iii)~~ As a consequence, we can ~~say~~ break the state space S into disjoint equivalence (communication) classes (sets), say $C_1, C_2, C_3, \ldots$

These have the properties

(i) $S = C_1 \cup C_2 \cup \ldots$ (every state belongs to some class C_i)

(ii) The C_i are disjoint.

(iii) For a given class C, if $i \in C$ and $i \leftrightarrow j$, then $j \in C$.

Also, if i and j belong to different classes, then $i \not\leftrightarrow j$.

<u>Example:</u>

$$\underline{\underline{P}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Find the equivalence classes for $\underline{P}$.

state 0: $0 \not\to j$ for $j \neq 0$, $C_1 = \{0\}$ is a communicating class.

state 3: $3 \not\to j$ for $j \neq 3$, $C_2 = \{3\}$ -"- -"- -"- -"-

states 1, 2: $1 \to j$ for all j so $1 \leftrightarrow 2$
$2 \to j$ for all j

$1 \to 0$ but $0 \not\to 1$ so $0 \not\leftrightarrow 1$, etc. $\Rightarrow C_3 = \{1, 2\}$.

$\rightsquigarrow$ equivalence classes: $\{0\}, \{3\}, \{1, 2\}$.

<u>Why bother?</u>

We will see that certain properties of states are <u>class</u> properties.

That is, if some $i \in C$ has a property, then all states in C have this property.
a communicating class

<u>Proposition:</u> Let i, j be distinct states.

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(i) $i \to j$ iff there is some $k \geq 1$ and sequence of states

$a_0, a_1, a_2, \ldots, a_k$ with $a_0 = i$, $a_k = j$ and $P_{a_0 a_1} \cdot P_{a_1 a_2} \cdots P_{a_{k-1} a_k} > 0$

$\Rightarrow P_{a_0 a_1} > 0,\ P_{a_1 a_2} > 0, \ldots$

Same as $\overset{i}{a_0} \mapsto a_1 \mapsto a_2 \mapsto \cdots \mapsto \overset{j}{a_k}$

or $P(X_1 = a_1, X_2 = a_2, \ldots, X_k = \overset{j}{a_k} \mid X_0 = \overset{i}{a_0}) > 0$

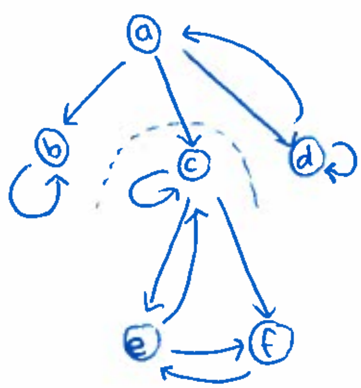
Example:

$$\underline{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Find communicating classes: Determine $i \to j$ (?) for all i, j

$i \mapsto j$

(see p. 95)



$a \to \begin{cases} b \to b \\ c \to \begin{matrix} c \\ e \\ f \end{matrix} \begin{matrix} \to f \\ \to c \\ \to e \end{matrix} \\ d \to \begin{matrix} a \\ d \end{matrix} \end{cases}$ $\Rightarrow$

$a \to b$ ✓	$b \to b$
$a \to c$ ✓	
$a \to d$ ✓	$c \to c$
$a \to e$ ✓	$c \to e$
$a \to f$ ✓	$c \to f$ ...

c, e, f all communicate

$C_1 = \{c, e, f\}$ one class $\quad C_2 = \{b\}$ one class $\quad C_3 = \{a, d\}$ one class.

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Note that $a \to c$, so the class $\{a,d\} \to \{c,e,f\}$

but $\{c,e,f\} \not\to \{a,d\}$.

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i,j states, $\underline{P}$

$i \to j$ if $P^n_{ij} > 0$ for some $\underset{\nwarrow \text{ can depend on } i,j}{n \geq 0}$ "i leads to j"

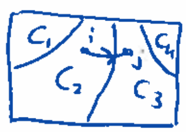
$i \leftrightarrow j$ if $i \to j$ and $j \to i$ " i and j communicate"

$\leftrightarrow$ is an equiv. relation, so there are <u>disjoint</u> sets (classes) $C_1, C_2, \ldots$

s.t.
- every state is in some C_l
- $i \leftrightarrow j$ if $i,j \in$ same class
- $i \not\leftrightarrow j$ if i and j belong to different classes

write $C_k \to C_l$ (C_k leads to C_l)

to mean for some i,j, $i \in C_k$, $j \in C_l$, $i \to j$, and necessarily $j \not\to i$

$S =$ 

<u>Def:</u> A class C is closed if C leads to no other class, and is open if C does lead to a different class.

<u>Example:</u>
$$\underline{P} = \begin{array}{c c} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$
$1 \to 2$, $2 \not\to 1$ so $1 \not\leftrightarrow 2$

But $1 \leftrightarrow 1$, $2 \leftrightarrow 2$, so classes are $\underset{\text{open}}{C_1 = \{1\}}$, $\underset{\text{closed}}{C_2 = \{2\}}$

<u>Def:</u> A MC is called irreducible if it has a single communicating class.

<u>Proposition</u> (p. 95): Let i,j be distinct states. Then

1. $i \to j$ if and only if for some k there are states $a_0, a_1, \ldots, a_k$ with $a_0 = i$, $a_k = j$ and $\underbrace{P_{a_0 a_1} \cdot P_{a_1 a_2} \cdots P_{a_{k-1} a_k} > 0}_{\text{equivalent to } P^k_{ij} > 0}$.

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2. The chain is irreducible iff there is some k, some state a_0, and states $a_1, a_2, \ldots, a_k = a_0$ such that every state appears at least once in this list, and $P_{a_0 a_1} \cdot P_{a_1 a_2} \cdots P_{a_{k-1} a_k} > 0$.

Example:

Is $\underline{P}$ irreducible?

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & & 0 & \frac{1}{8} & 3/4 & \frac{1}{8} & 0 \\ 1 & & 0 & 9/10 & 0 & 0 & 1/10 \\ 2 & & 0 & 0 & 1/3 & & \\ 3 & & & & & & \\ 4 & & & & & & \\ 5 & & & & & & \end{array}$$

$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/8 & 3/4 & 1/8 & 0 & 0 \\ 0 & 9/10 & 0 & 0 & 0 & 1/10 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 3/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Write $i \mapsto j$ if $P_{ij} > 0$

$0 \mapsto 1 \mapsto 5 \mapsto 0 \mapsto 2 \mapsto 4 \mapsto 5 \mapsto 0 \mapsto 3 \mapsto 0$

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
0,	1,	5,	0,	2,	4,	5,	0,	3,	0

All states appear at least once above

$\overset{\text{prop}}{\Longrightarrow}$ $\underline{P}$ irreducible
$\qquad \parallel$
$i \to j$ for every i, j

why: $0 \to$ every state ✓

$1 \to$ every state ✓

$2 \to$ — " — — " — ✓

$3 \to$ — " — — " — ✓

$4 \to$ — " — — " — ✓

$5 \to$ — " — — " — ✓

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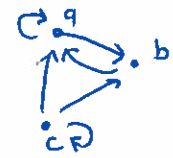
Recurrence & Transcience

Example:

$$\underline{P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1 & 0 & 0 \\ 1/4 & 2/4 & 1/4 \end{bmatrix} \end{array}$$

Sum = 3/4 $\quad \circlearrowleft c$



$a \leftrightarrow b$, $a \not\rightarrow c$, $b \not\rightarrow c$

$c \rightarrow a$, $c \rightarrow b$

Classes are $\{a, b\}$ and $\{c\}$.
closed open

(1) Starting at either a, or b, the chain is certain to return to a or b

recurrent

(2) Starting at c, there is prob. $\frac{3}{4}$ of never returning to c.

transient

Def: Let (X_n) be a MC with state space S.

For each state $a \in S$, define the random variables

$$H_a = \begin{cases} \min\{n \geq 0 : X_n = a\} \\ +\infty \quad \text{if } X_n \neq a \ \forall n \geq 0 \end{cases}$$

hitting time of a

$$R_a = \begin{cases} \min\{n \geq 1 : X_n = a\} \\ +\infty \quad \text{if } X_n \neq a \ \forall n \geq 1 \end{cases}$$

return time if $X_0 = a$

State a is called recurrent if $P(R_a < \infty \mid X_0 = a) = 1$.

-"- -"- -"- transient if $P(R_a < \infty \mid X_0 = a) < 1$.

Ex: $P(R_c < \infty \mid X_0 = c) = \frac{1}{4}$ $\qquad$ c is transient

$P(R_a < \infty \mid X_0 = a) = \frac{1}{3} + \frac{2}{3} \cdot 1 = 1$ $\qquad$ a is recurrent

$[R_a = 1 \text{ or } R_a = 2]$ $\qquad$ (b is too)

Ex: For same MC, we see the "realization" of the experiment

$X_0 = 1,\ X_1 = 1,\ X_2 = 3,\ X_3 = 3,\ X_4 = 2,\ X_5 = 3,\ X_6 = 1,\ X_7 = 2, \ldots$

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For this realization

$R_1 = ?$

$\{n \geq 1: X_n = 1\} = \{1, 6, \ldots\} \Rightarrow R_1 = 1, \quad H_1 = 0$

$\{n \geq 0 \mid X_n = 1\} = \{0, 1, 6, \ldots\}$

$R_2 = \min\{n \geq 1 \mid X_n = 2\} = \min\{4, 7, \ldots\} = 4$

$H_2 = \min\{n \geq 0 \mid X_n = 2\} = \min\{4, 7, \ldots\} = 4, \quad H_2 = R_2$

Def: Let $f_a = P(R_a < \infty \mid X_0 = a)$

a is recurrent if $f_a = 1$ → you come back to a infinitely many times

" " transient if $f_a < 1$. → only finitely many times

Goal: Classify each state in a given chain as recurrent or transient.

How do we do this?

$$f_a = P(\underset{\text{r.v., can be } 1, 2, 3, \ldots, +\infty}{R_a} < \infty \mid X_0 = a)$$

$$= P(R_a = 1 \mid X_0 = a) + P(R_a = 2 \mid X_0 = a) + P(R_a = 3 \mid X_0 = a) + \ldots$$

$$= \sum_{n=1}^{\infty} P(R_a = n \mid X_0 = a) \quad \text{(there is no } n = \infty \text{ term)}$$

(sum does not include $R_a = \infty$ term)

$$\sum_{n=1}^{\infty} P(R_a = n \mid X_0 = a) = \lim_{N \to \infty} \sum_{n=1}^{N} P(R_a = n \mid X_0 = a)$$

If we could find these $P(R_a = n \mid X_0 = a)$, we could add them up and check if the sum is 1 or < 1. (Usually impossible.)

Thm: (p. 98)

State a is ~~reccurrent~~ recurrent iff $\sum_{n=1}^{\infty} P_{aa}^n = +\infty$.

State a is transient iff $\sum_{n=1}^{\infty} P_{aa}^n < \infty$.

Example

$$\underline{P} = \begin{array}{c|cc} & 0 & 1 \\ 0 & .4 & .6 \\ 1 & 0 & 1 \end{array} \qquad \underline{P}^n = ?$$

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$$P^n_{00} = P(X_n=0 \mid X_0=0)$$

$$= P(X_1=0, X_2=0, \_\_, X_n=0 \mid X_0=0)$$

if one goes to 1, one never comes back to 0

$$= \underbrace{P_{00}\, P_{00}\, P_{00} \cdot \_\_ \, P_{00}}_{n \text{ times}} = (0.4)^n \quad \Rightarrow \quad P^n_{01} = 1-(.4)^n$$

$$\underline{P}^n = \begin{bmatrix} (.4)^n & 1-(.4)^n \\ 0 & 1 \end{bmatrix}$$

only possible state after 1 is 1 $(1 \not\to 0)$

$$P_{11}^n = P(X_n=1 \mid X_0=1) = P(X_1=1, X_2=1, \_, X_n=1 \mid X_0=1)$$

$$= P_{11} \cdot P_{11} \cdot \_\_ \cdot P_{11} = 1^n = 1$$

Check $\sum_{n=1}^{\infty} P^n_{aa}$ for $a = 0, 1$.

$a=0$, $\sum_{n=1}^{\infty} (.4)^n < \infty$ geometric series $\Rightarrow$ So, <u>0 is transient.</u>

$a=1$, $\sum_{n=1}^{\infty} 1 = \infty$ $\Rightarrow$ <u>1 is recurrent.</u>

<u>Thm 3.3</u>

If $i \leftrightarrow j$ then either both i and j are recurrent or both i, j are transient.

[Recurrence (transcience) ~~and~~ are class properties.]

[If one state in a communicating class is recurrent then all states are.]

<u>Note:</u> If the chain is irreducible, then either all states are recurrent or are transient.

<u>Ex:</u>

$$\underline{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & .4 & .6 \\ 1 & 0 & 1 \end{array}$$

0 transient

1 recurrent

0, 1 are in different classes. or $\underline{P}$ is <u>not</u> irreducible.

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Proof: Suppose i is recurrent and $i \leftrightarrow j$ (i, j communicate).

We know:

- $\sum_{n=1}^{\infty} p_{ii}^{n} = +\infty$ by Thm (p. 98).
- There is some k such that $\underbrace{p_{ij}^{k}}_{=A} > 0$ $(i \to j)$
- " " " m s.t. $B = p_{ji}^{m} > 0$ $(j \to i)$

We want to show that $\sum_{l=1}^{\infty} p_{jj}^{l} = +\infty$, imply j is recurrent.

Consider $p_{jj}^{m+k+t} = (\underline{P}^{m} \cdot \underline{P}^{t} \cdot \underline{P}^{k})_{jj}$

$$= \sum_{\alpha} P(X_{m+k+t} = j \mid X_0 = j)$$

$$\geq P(X_m = i,\ X_{m+t} = i,\ X_{m+t+k} = j \mid X_0 = j)$$

$$= p_{ji}^{m}\ p_{ii}^{t}\ p_{ij}^{k} = B \cdot p_{ii}^{t} \cdot A = AB\,p_{ii}^{t}$$

$$\sum_{l=0}^{\infty} p_{jj}^{l} \geq \sum_{l=m+n}^{\infty} p_{jj}^{l} = \sum_{\tilde{l}=0}^{\infty} p_{jj}^{\tilde{l}+m+n} = \sum_{\tilde{l}=0}^{\infty} AB\,p_{ii}^{\tilde{l}} = AB \underbrace{\sum_{t=0}^{\infty} p_{ii}^{t}}_{=+\infty} = +\infty$$

$\tilde{l} = l - m - n$, $l = \tilde{l} + m + n$ $\quad [\tilde{l} = t]$

- Exam #1 date moved to Tue $\boxed{\text{Oct } 3}$

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- sol'n$ HW #3 → Library

Prop: (Irreducibility and regularity)

Assume a MC has a finite nr of states.

Then ① If $\underline{P}$ is regular then $\underline{P}$ is irreducible.

② If $\underline{P}$ is irreducible, and there is at least one state i such that $P_{ii} > 0$, then $\underline{P}$ is regular.

$\underline{P}$ regular implies there exists some n with $\underline{P}^{n} > 0$. For this n, $P_{ij}^{n} > 0$, so $i \to j$ for all states i, j.

Example:

$$\underline{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

1) Irreducible? Yes, because $a \to b \to c \to d \to e \to f \to a$

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2) $P_{ff} > 0$

By the prop., $\underline{P}$ is regular.

Note that $\underline{P}^9$ is not positive, but $\underline{P}^{10}$ is positive.

<u>Example:</u> $\underline{P} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$, then $\underline{P}$ is irreducible, but not regular.

<u>Example:</u>

$$\underline{P} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$a \to b \to a$
$\quad\quad\ \searrow c \to a$
$\quad\quad\quad\quad \searrow b$

1) Irreducible ✓
2) $P_{ii} = 0$ for each i

But $\underline{P}$ is regular.

<u>Recall:</u>

$$R_a = \begin{cases} \min\{n \geq 1 : X_n = a\} \\ +\infty \quad \text{if } X_n \neq a \text{ for all } n = 1, 2, \ldots \end{cases}$$

State a is called recurrent if $P(R_a < \infty \mid X_0 = a) = 1$.

(same as $P(R_a = +\infty \mid X_0 = a) = 0$)

State a is called transient if $P(R_a < \infty \mid X_0 = a) < 1$

(same as $P(R_a = +\infty \mid X_0 = a) > 0$).

<u>Thm:</u> • State a is <u>recurrent</u> iff $\sum_{n=1}^{\infty} P_{aa}^n = +\infty$.

equiv. to state a is <u>transient</u> iff $\sum_{n=1}^{\infty} P_{aa}^n < \infty$.

• If state a is transient, and b is any state,
then $\sum_{n=1}^{\infty} P_{ba}^n < \infty$. In particular, $\lim_{n \to \infty} P_{ba}^n = 0$.

Pf like trans $\Rightarrow \Sigma < \infty$

<u>Thm 3.3:</u> If $i \leftrightarrow j$ then either both i and j are recurrent or both i and j are transient.

Proof: Suppose $i \neq j$, $i \leftrightarrow j$, and i is recurrent.

Since $i \to j$ and $j \to i$ there are positive m, n such that

$A = p^n_{ij} > 0$ and $B = p^m_{ji} > 0$.

For any positive k, $p^{k+m+n}_{jj} = P(X_{k+m+n} = j \mid X_0 = j)$

$$\geq P(X_m = i, X_{k+m} = i, X_{n+k+m} = j \mid X_0 = j)$$

$$\overset{MP}{=} \underbrace{p^m_{ji}}_{B} \underbrace{p^k_{ii}}_{p^k_{ii}} \underbrace{p^n_{ij}}_{A} = AB\, p^k_{ii}$$

$$\sum_{L=1} p^L_{jj} \geq \sum_{L=m+n+1}^{\infty} p^L_{jj} = \sum_{k=1}^{\infty} p^{k+m+n}_{jj} \geq \sum_{k=1}^{\infty} AB\, p^k_{ii} = AB \sum_{k=1}^{\infty} p^k_{ii} = +\infty$$

Let $k = L-m-n$, $L = k+m+n$

Since i is recurrent by Thm.

This shows j is recurrent. □

Sketch of Proof: [Thm a rec. $\Leftrightarrow \sum_{..} = \infty$]

① For state a, let V_a be the nr of times the chain "visits" (is at) state a.

Let $I_n = \begin{cases} 1, & \text{if } X_n = a \\ 0 & \text{if } X_n \neq a \end{cases}$ Then $V_a = \sum_{n=1}^{\infty} I_n$, a random variable.

Then $E[V_a \mid X_0 = a] = E\left(\sum_{n=1}^{\infty} I_n \mid X_0 = a\right) = \sum_{n=1}^{\infty} E(I_n \mid X_0 = a) = \sum_{n=1}^{\infty} p^n_{aa}$

$E(I_n \mid X_0 = a) = 1 \cdot P(X_n = a \mid X_0 = a) + 0 \cdot P(X_n \neq a \mid X_0 = a) = p^n_{aa}$

$\Rightarrow E[V_a \mid X_0 = a] = \sum_{n=1}^{\infty} p^n_{aa}$.

② Suppose a is recurrent. Then starting at a, the MC comes back to a w/ prob. 1 at some time. Now, starting at a, the MC comes back to a w/ prob. 1 some further time. Again, starting at a, ... some further time. $\Rightarrow V_a = +\infty$ w/ prob. 1. This implies

$E[V_a \mid X_0 = a] = +\infty$. $\overset{①}{\Rightarrow} \sum_{n=1}^{\infty} p^n_{aa} = +\infty$. See text for the rest. □

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Corollary 3.4:

If a MC has a finite state space and is irreducible, then all states are recurrent.

[This is false in general for MC's with infinite statespace.]

Proof: Let $S=\{1,2,\dots,k\}$.

① For any state i, since $\underline{\underline{P}}^n$ is a stochastic matrix,

$1=\sum_{j=1}^{k} p_{ij}^n$ for all n

② Suppose all states j are transient.

Then, for every state i and j, $\lim_{n\to\infty} p_{ij}^n=0$.

$$1=\lim_{n\to\infty} 1=\lim_{n\to\infty}\sum_{j=1}^{k} p_{ij}^n=\sum_{j=1}^{k}\lim_{n\to\infty} p_{ij}^n=\sum_{j=1}^{k} 0=0.$$ A contradiction

so all states must be recurrent.

Lemma 3.5: Let C be a communicating class for a MC.

Suppose C is ~~closed~~. finite. ① Then C is closed iff all states in C are recurrent.

⇒ All states in a closed communicating class are recurrent.

② Iff C is open, then all states in C are transient.

Example: Suppose a MC has communicating classes $C_1=\{0,1\}$, $C_2=\{2,3\}$, $C_3=\{4,5\}$. Suppose also that $C_1\to C_2$ and $C_2\to C_3$.

What are the recurrent and transient states?

$C_1\to C_2$ means $i\to j$ for some $i\in C_1$, $j\in C_2$. So C_1 is open.

So is C_2 open. How about C_3?

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Suppose C_3 is open. Then either $C_3 \to C_1$ or $\underbrace{C_3 \to C_2}_{\text{contradicts } C_3 \to C_2}$.

(all states in C_2, C_3 would communicate)

(otherwise we have $i \leftrightarrow j$ for all $i \in C_2$, $j \in C_3$: impossible)

Also, $C_3 \not\to C_1$, since $C_1 \to C_3$ b/c $C_1 \to C_2$ and $C_2 \to C_3$.

So, C_3 is closed.

states 0, 1, 2, 3 are transient
4, 5 are recurrent

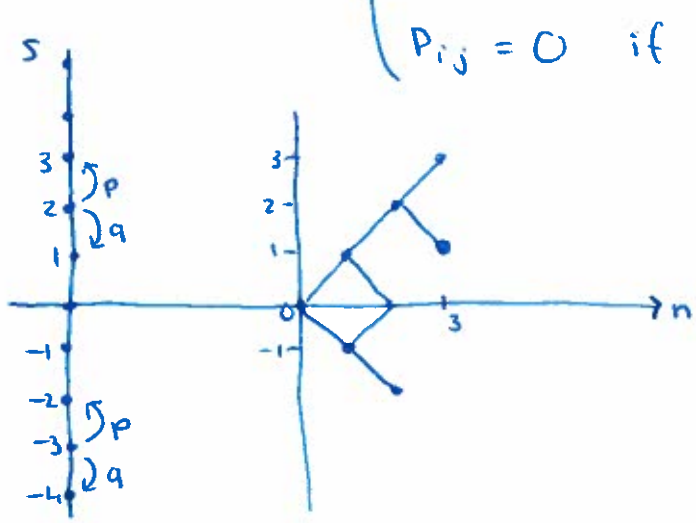
Fact: Every finite ~~state~~ MC must have at least one ~~recurrent~~ closed communicating class.

Example 3.3 Simple one-dimensional random walk
↪ jumps are either +1 or −1

The state space is $\{\text{all integers}\} = \{0, \pm 1, \pm 2, \ldots\}$

$0 < p < 1$ is a fixed parameter, $q = 1 - p$.

$\underline{P}$ is defined by
$$\begin{cases} P_{i,i+1} = p & \forall i \\ P_{i,i-1} = q & \forall i \\ P_{ij} = 0 & \text{if } j \neq i \pm 1 \end{cases}$$



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Thm: This MC is irreducible and

1) if $p=\frac{1}{2}$ then all states are recurrent (check $\sum_{n=1}^{\infty} P_{00}^{n} = +\infty$)

2) if $p \neq \frac{1}{2}$ then all states are transient (check $\sum_{n=1}^{\infty} P_{00}^{n} < \infty$)

Proof sketch: (see text for more details)

(0) $P_{00}^{n} = 0$ if n is odd (starting with $X_0 = 0$ (even), $X_1 =$ odd, $X_2 =$ even, ⋮ $X_n =$ odd if n is odd)

$\quad \| $

$P(X_n = 0 \mid X_0 = 0)$

(1) $P_{00}^{2n} = \binom{2n}{n}(pq)^n$ for $n = 1, 2, 3, \ldots$

(2) $\lim_{n\to\infty} \frac{P_{00}^{n}}{c_n} = 1$, where $c_n = \frac{(4pq)^n}{\sqrt{\pi n}}$

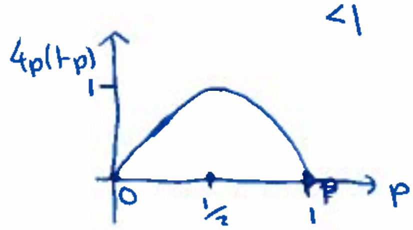
we often write: $P_{00}^{2n} \underset{\substack{\uparrow \\ \text{asymptotically}}}{\sim} \frac{(4pq)^n}{\sqrt{\pi n}}$ as $n\to\infty$

(3) By comparison test for infinite series, $\sum_{n=1}^{\infty} P_{00}^{2n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}}$ converges.

(4) If $p = \frac{1}{2}$, $4pq = 1$, so $\sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = +\infty$

$\Rightarrow$ chain is recurrent

If $p \neq \frac{1}{2}$, $4pq = \underset{<1}{4p(1-p)}$, parabola



(maximize $4p(1-p)$ strict local max at $p = \frac{1}{2}$)

In this case, $\sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}} < \sum_{n=1}^{\infty} (4pq)^n$, geometric series, $4pq < 1$,

converges, so $\sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}} < \infty$ $\Rightarrow$ chain is transient.

$P_{00}^{2n} \overset{?}{=} \binom{2n}{n}(pq)^n$, $P_{00}^{2n} = P(X_{2n} = 0 \mid X_0 = 0)$.

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Fact: $\underline{P}$ is irreducible

Thm: All states are recurrent if $p = \frac{1}{2}$.

All states are transient if $p \neq \frac{1}{2}$.

Intuitive argument for $p = \frac{2}{3}$: there is a drift upwards, so starting at a state the MC may never return.

We will show this using the $\sum_{n=1}^{\infty} P_{ii}^n$ criteria.

It suffices to check if $\sum_{n=1}^{\infty} P_{00}^n$ converges or diverges.

We want a formula for P_{00}^n.

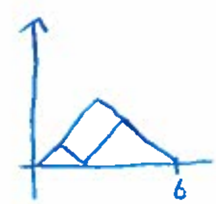
$P_{00}^n = 0$ for all odd n.

We want to find P_{00}^{2n}.

$$P_{00}^{2n} = P(\underbrace{X_{2n} = 0}_{} \mid X_0 = 0) = \binom{2n}{n} p^n q^n.$$

the nr of up $+1$ steps must equal the nr of down -1 steps must equal n (2n steps in total) → n up steps, n down steps

Any such "path" has probability $p^n q^n$
We must multiply by the nr of such paths $= \binom{2n}{n}$



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Read 3.5 ; BB → HW#4 hints

Thursday → problem / review ; next Tuesday – Exam #1

Example 3.3 pp. 99-160

Simple random walk, $S = \{0, \pm 1, \pm 2, \ldots\}$, $0 < p < 1$ fixed, $q = 1-p$

$P_{i,i+1} = p$, $P_{i,i-1} = q$, $P_{ij} = 0$ otherwise

For a path with $X_0=0$, $X_{2n}=0$, must have n +1-steps and n -1-steps, this prob. is $(pq)^n$ $(=p^n q^n)$.

$\Rightarrow P_{00}^{2n} = (pq)^n \cdot$ # "paths" from 0 to 0 with $2n$ steps
↳ sequence of +1's, -1's, n of each.

$= (pq)^n \binom{2n}{n}$
choose n positions for +1's $\to \binom{2n}{n}$ ways to do this
choose pos. for -1's $\to$ 1 way (positions not yet filled)

$= \frac{(2n)!}{n!\,n!}(pq)^n$. (*)

Need Stirling's Formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$ as $n\to\infty$

or $\dfrac{n!}{n^n e^{-n}\sqrt{2\pi n}} \to 1$ as $n\to\infty$.

(For $n=10$, this ratio is 1.0084...)

To get $P_{00}^{2n} \sim \frac{(4pq)^n}{\sqrt{\pi n}}$, plug into (*) with Stirling's formula, simplify. $\left(n! \sim n^n e^{-n}\sqrt{2\pi n},\quad (2n)! \sim (2n)^{2n} e^{-2n}\sqrt{2\pi(2n)}\right)$

- Recall that a finite irreducible MC has all states recurrent.
- This simple random walk example is irreducible but all states are transient if $p \neq \frac{1}{2}$.

<u>Thm 3.6:</u> (Basic limit thm for finite irreducible MC's)

Assume (X_n) is a finite irreducible MC. For every state j define

$\mu_j = E(R_j \mid X_0=j)$.

$R_j \overset{\text{def}}{=} \min\{n\geq 1 \mid X_n=j\}$
= first return time to j (starting at j).

Then

(a) $\mu_j < \infty$ for all j

[Note: random variables can have infinite mean.]
Example: X has pdf $f(x) = \begin{cases} 0 & \text{if } x<1 \\ \frac{1}{x^2} & \text{if } x\geq 1 \end{cases}$

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Check: $\int_1^\infty f(x)\,dx = 1$ (honest pdf)

$$EX = \int_1^\infty x f(x) = \int_1^\infty x \cdot \frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx = +\infty.\ ]$$

(b) There is a unique stationary dist. π given by $\pi_j = \frac{1}{\mu_j}$ for all states j.

(c) For all states i,j, $\frac{1}{n}\sum_{m=1}^{n} P_{ij}^m = \frac{P_{ij}^1 + P_{ij}^2 + \dots + P_{ij}^n}{n}$

= average of P_{ij}^k up to time n

$\xrightarrow[n\to\infty]{} \pi_j$.

Remark:

① Does this apply to regular chains? Yes (reg $\Rightarrow$ irreducl. (MC finite))

② (c) is a weaker statement than $\lim_{n\to\infty} P_{ij}^n = \pi_j$. Also, if $\lim_{n\to\infty} P_{ij}^n = \pi_j$, then (c) holds

Example: Take $\underline{P} = \begin{array}{c|c} & 0 \quad 1 \\ \begin{array}{c}0\\1\end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\end{array}$, $P_{01}^n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

$\underline{P}^2 = \begin{array}{c|c} & 0 \quad 1 \\ \begin{array}{c}0\\1\end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{array}$. $\lim_{n\to\infty} P_{01}^n$ does not exist.

$$\frac{1}{n}\sum_{k=1}^{n} P_{01}^k = \frac{1}{n}(1+0+1+0 \dots +0+1)$$
$$= \frac{1}{n}\left(\frac{n-1}{2}+1\right) = \frac{1}{n}\left(\frac{n-1+2}{2}\right) = \frac{1}{n}\left(\frac{n+1}{2}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right) \to \frac{1}{2} \text{ as } n\to\infty$$

[Thm: If $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} \frac{c_1+c_2+\dots+c_n}{n} = L$]

③ This gives us a way to compute μ_j:

Find π_j, $\mu_j = \frac{1}{\pi_j}$.

Example: If in addition $\underline{P}$ is doubly stochastic, then

$\mu_j = \#$ of states, every state j. $\rightsquigarrow$ uniform stat. distr.

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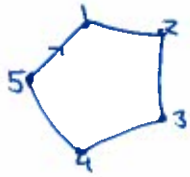

$S=\{1,2,3,4,5\}$

$$\underline{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}\end{array}$$

Doubly stoch. matrix $\Rightarrow \pi_j = \frac{1}{5}$, $\mu_j = 5$.

---

j recurrent $\quad P(R_j < \infty \mid X_0 = j) = 1$

j transient $\quad P(R_j < \infty \mid X_0 = j) < 1$

Def: A recurrent state is positive recurrent if $\mu_j = E(R_j \mid X_0 = j) < \infty$

null recurrent if $\mu_j = E(R_j \mid X_0 = j) = \infty$

Thm 3.6 $\Rightarrow$ a finite irred. MC has all states positive recurrent.

First-step analysis (p. 105) - ask what happens with the first step

Example (3.17)

$$\underline{P} = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\end{array}$$

irreducible, (regular)

Thm 3.6 $\Rightarrow$ $E[R_j \mid X_0 = j] < \infty$ for each j.

Let $e_x = E[R_a \mid X_0 = x]$, $\quad [R_a = \min\{n \geq 1 \mid X_n = a\}]$

$x = a, b, c$.

(e_a, e_b, e_c)

$e_a = 1 + e_b$

additional amount of time depending on 1st step

$e_b = 1 + \underbrace{(P_{ba} \cdot \underset{0}{0} + P_{bb} e_b + P_{bc} e_c)}$ $= 1 + \frac{1}{2} e_c.$

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$e_c = 1 + (P_{ca} \cdot 0 + P_{cb} e_b + P_{cc} e_c) = 1 + \frac{1}{3} e_b + \frac{1}{3} e_c. \quad (= 1 + \sum_{y \neq a} P_{cy} e_y)$

$\Rightarrow \begin{bmatrix} e_a = 1 + e_b, \\ e_b = 1 + \frac{1}{2} e_c \\ e_c = 1 + \frac{1}{3} e_b + \frac{1}{3} e_c \end{bmatrix}$ 3 linear eqns, 3 unknowns, a unique sol'n

$e_a = \frac{10}{3}, \; e_b = \frac{7}{3}, \; e_c = \frac{8}{3}.$

General Formula

For a fixed state a, $e_x = E[R_a \mid X_0 = x]$, then

$$e_x = 1 + \sum_{y \neq a} P_{xy} e_y.$$

Consistency check:

For the example $\pi = \begin{array}{ccc} a & b & c \\ \end{array}$ $\begin{bmatrix} \frac{3}{10} & \frac{2}{5} & \frac{3}{10} \end{bmatrix}$

Thm 3.6. $\mu_a = E[R_a \mid X_0 = a] = \frac{1}{\pi_a} = \frac{1}{3/10} = \frac{10}{3}.$

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1st Step Analysis

$R_a = \min\{n \geq 1 : X_n = a\}$, $H_a = \min\{n \geq 0 \mid X_n = a\}$.

Example:

$$\underline{\underline{P}} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

$e_x = E[R_{\underline{\underline{a}}} \mid X_0 = x]$

= an expected return time value

1st-step analysis:
$e_a = 1 + e_b$
$e_b = 1 + \frac{1}{2} e_c$
$e_c = 1 + \frac{1}{3} e_b + \frac{1}{3} e_c$

$\underline{\underline{e_a}} = 1 + \sum_{j \neq \underline{\underline{a}}} \underline{\underline{P_{aj}}} e_j$

$\underline{\underline{e_b}} = 1 + \sum_{j \neq \underline{\underline{a}}} \underline{\underline{P_{bj}}} e_j$

$\underline{\underline{e_c}} = 1 + \sum_{j \neq \underline{\underline{a}}} \underline{\underline{P_{cj}}} e_j$

Example: Run probability

$N > 0$ is fixed (target for quitting), $S = \{0, 1, \ldots, N\}$

$P_{i,i+1} = p$ $\quad P_{00} = 1, \; P_{NN} = 1$
$P_{i,i-1} = q$

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Ruin event is {hit 0 before N}

$= \{H_0 < H_N\} = \{H_0 < \infty\}$

$[H_N = \infty]$

Let $x_k = P(H_0 < \infty \mid X_0 = k\}$, $k = 0, 1, 2, \ldots, N$.

$x_0 = 1, \quad x_N = 0$

<u>Claim:</u> For $1 \le k \le N-1$, $x_k = p\, x_{k+1} + q\, x_{k-1}$

<u>Sketch:</u> Restrict to $X_0 = k$, $2 \le k \le N-2$ (in this case, X_1 cannot be 0 or N)

$x_k = P(H_0 < \infty \mid X_0 = k) = \sum_{j \in S} P(H_0 < \infty, X_1 = j \mid X_0 = k)$ (LOTP)

$= P(H_0 < \infty, X_1 = k+1 \mid X_0 = k) + P(H_0 < \infty, X_1 = k-1 \mid X_0 = k)$

$= P(H_0 < \infty \mid X_1 = k+1, X_0 = k)\, \underbrace{P(X_1 = k+1 \mid X_0 = k)}_{= p_{k,k+1} = p}$

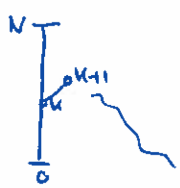
$P(A \cap B) = P(A \mid B) P(B)$

$+ P(H_0 < \infty \mid X_1 = k-1, X_0 = k)\, \underbrace{P(X_1 = k-1, X_0 = k)}_{= p_{k,k-1} = q}$

$= p\, P(H_0 < \infty \mid X_1 = k+1) + q\, P(H_0 < \infty \mid X_1 = k-1)$

time-homog.

$= p\, P(H_0 < \infty \mid X_0 = k+1) + q\, P(H_0 < \infty \mid X_0 = k-1)$ "□"



<u>Fact:</u> In general, $P(H_a < \infty \mid X_0 = x) = u_x$

Then $\begin{cases} u_x = \sum_{y \in S} P_{xy} u_y, & x \ne a \\ u_a = 1 \end{cases}$

$\boxed{\text{Exam}}$ Next Tues

- Topics – lectures, also on topics from 3.4– … in text, text sections 2.1–2.3, 3.1–3.3, difference eqns, HW
- know – defn's, theorems (props, lemmas, facts, …), standard <u>notation</u>

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eg. $p^n_{ij} \neq (p_{ij})^n$; $i \to j$ means for some n, $p^n_{ij} > 0$
can depend on i,j

- closed books, notes, no calc's, cell phones
- Types of question

1) True/false. no justification

T or F if $i \to j$ then $p_{ij} > 0$ [consider $\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & 0 & 0 & 1 \\ c & 1 & 0 & 0 \end{array}$]

$p_{ac} = 0$ but $p^2_{ac} > 0$.

T or F if $\underline{\underline{P}}$ is regular then $\underline{\underline{P}}$ is irred.

regular: there is some (one) n s.t. $p^n_{ij} > 0 \ \forall i,j$

irreducible: $i \leftrightarrow j$ for all i,j or $i \to j$ for all i,j

2) Give an simplest possible example of (no justification)

Give a trans. matrix $\underline{\underline{P}}$ s.t. $\underline{\underline{P}}$ has more than one stationary dist.

$\underline{\underline{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\underline{\pi} = [\, a \quad 1-a \,]$ is a stat. dist. for all $0 \le a \le 1$.

$\underline{\underline{P}}$ always has 0 / 1 / ∞ stat. dist.

3) "standard" calculation problem

For $\underline{\underline{P}} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ find the stationary distribution.

4) Proof

Example

Show: If $\underline{\underline{P}}^N > 0$ then $\underline{\underline{P}}^{N+m} > 0$ for all $m = 1, 2, 3, \ldots$
$\underline{\underline{P}}$ is a stoch. matrix

Sep 28

It suffices to show: if $\underline{\underline{P}}^k > 0$ then $\underline{\underline{P}}^{k+1} > 0$ for all k.

We are given P_{ij}^k for all i,j. Want to show $P_{ij}^{k+1} > 0$ for all i,j.

$$\underline{\underline{P}}^{k+1} = \underline{\underline{P}}^k * \underline{\underline{P}} = \underline{\underline{P}} * \underline{\underline{P}}^k$$

$$P_{ij}^{k+1} = (\underline{\underline{P}} * \underline{\underline{P}}^k)_{ij} = \sum_{l \in S} P_{il} \underbrace{P_{lj}^k}_{\text{all} > 0}$$

Given i, not all P_{il} can be zero because $\sum_{l \in S} P_{il} = 1$

so given i there must be some l_0 s.t. $P_{il_0} > 0$.

Then $P_{ij}^{k+1} \geq \underbrace{P_{il_0}}_{>0} \underbrace{P_{l_0 j}^k}_{>0} > 0.$ ✓

HW 4, 3.28

Show $\{1,2,3\}$ is a closed class and ← all recurrent

$\{4,5,6,7\}$ is an open class ← all transient

$\hookrightarrow \lim_{n\to\infty} P_{ij}^n = 0$ if j is transient, any i

implies $P_{ij}^n = 0$ for all n, all $i \in \{1,2,3\}$, all $j \in \{4,5,6,7\}$.

so $\lim_{n\to\infty} P_{ij}^n = 0$ in this case.

$$\underline{\underline{P}} = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1/3 & 1/3 & 1/3 & & & & \\ 2 & 2/3 & 0 & 1/3 & & & & \\ 3 & 0 & 2/3 & 1/3 & & & & \\ 4 & & & & & & & \\ 5 & & & & & & & \\ 6 & & & & & & & \\ 7 & & & & & & & \end{array}$$

If we let $\underline{\underline{Q}} = [\;]$

1) $P_{12}^n = Q_{12}^n$

2) $\underline{\underline{Q}}$ is doubly stochastic, regular, so $\lim_{n\to\infty} Q_{ij}^n = \pi_j = \frac{1}{3}$

$\underline{\pi} = \left[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right]$

So, $\lim_{n\to\infty} P_{ij}^n = \frac{1}{3}$ for all $i,j \in \{1,2,3\}$.

Sep 28

3) How about $\lim_{n\to\infty} p^n_{ij}$ if $i \in \{4,5,6,7\}$, $j \in \{1,2,3\}$.

Roughly speaking, starting at say $4 = X_0$, eventually, for some random k, $X_k \in \{1,2,3\}$.

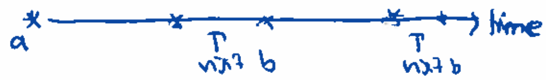
$\lim_{n\to\infty} P^n_{kj} = \pi_j = \frac{1}{3}$ (if first get to 1/2/3)

$\lim_{n\to\infty} p^n_{ij} = \frac{1}{3}$ if $i \in \{4,5,6,7\}$, $j \in \{1,2,3\}$.

Exam $\frac{?}{}$ sol'n → BB | BB: HW 5, Handout 1st step analysis Oct 5

$\boxed{T}$ | F If a, b are distinct states, $a \to b$, a is recurrent then b is recurrent.

We know if $a \leftrightarrow b$ and a is recurrent, then B is recurrent.



$$H_a = \begin{cases} \min\{n \geq 0 : X_n = a\} \\ +\infty \quad \text{if } X_n \neq a \quad \text{all } n = 0, 1, 2, \ldots \end{cases}$$

$$R_a = \begin{cases} \min\{n \geq 1 : X_n = a\} \\ +\infty \quad \text{if } X_n \neq a \quad \text{all } n = 1, 2, \ldots \end{cases}$$

Note: If $X_0 \neq a$, $H_a = R_a$ ✓

Let $u(i) = P(H_a < \infty \mid X_0 = i)$

$v(i) = E(R_a \mid X_0 = i)$

Then
$$\boxed{\begin{aligned} u(i) &= P_{ia} + \sum_{j \neq a} P_{ij}\, u(j) \qquad i \neq a \qquad (*) \\ v(i) &= 1 + \sum_{j \neq a} P_{ij}\, v(j) \qquad \text{all } i \end{aligned}}$$

$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Take $a = 0$, $u(1) = P(H_0 < \infty \mid X_0 = 1)$
< 1
$(\neq 1)$

Gambler's Ruin

Fix $p, q = 1 - p, N$

N, $N-1$, …, 2, 1, 0 ; $\uparrow p$, $\downarrow q$

$P_{00} = P_{NN} = 1$

$P_{i,i+1} = p, \quad 0 < i < N$

$P_{i,i-1} = q, \quad 0 < i < N$

$P_{ij} = 0$ otherwise

1st step analysis equations

$P(\text{ruin} \mid X_0 = i) = P(H_0 < \infty \mid X_0 = i)$
$= u(i)$

↑
get to 0 before N
‖
ever get to 0
‖
$\{H_0 < \infty\}$

Take $2 \le i \le N-2$, then $P_{i0} = 0$

$$\Rightarrow u(i) = \sum_{j \neq 0} P_{ij} u(j) = P_{i,i+1} u(i+1) + P_{i,i-1} u(i-1).$$
$$= p\, u(i+1) + q\, u(i-1).$$

$\Rightarrow u(i) = p u(i+1) + q u(i-1), \quad 2 \le i \le N-2$

Now

If we just put $i=1$ above, using $u(0)=1$, we would get

$$u(1) = pu(2) + q\, u(0) = p\, u(2) + q.$$

~~Check $u(1) = P(H_0 < \infty)$~~

Put $i=1$ in ⊛, we get $u(1) = P_{10} + P_{12}\, u(2) = q + pu(2)$. ✓

We get $u(0)=1, \; u(N)=0$ [Check $i=N-1$]

$$u(i) = pu(i+1) + q u(i-1), \quad 1 \le i \le N.$$

Difference eqn., solved in Problem 9 on DE handout.

~~Set $p=q=\frac{1}{2}$~~, consider $E(\underbrace{\text{duration of game}}_{} \mid X_0 = i)$

$$= \min\{R_0, R_N\} \overset{\text{def}}{=} T, \; i \neq 0 \text{ or } N.$$

Put $w(i) = E(T \mid X_0 = i)$

$w(0) = w(N) = 0.$

By 1st-step analysis $w(i) = 1 + \sum_{j \neq 0, j \neq N} P_{ij} w(j) = 1 + P_{i,i+1} w(i+1) + P_{i,i-1} w(i-1)$

$$= 1 + p w(i+1) + q w(i-1)$$

$\left\{ \begin{array}{l} \text{Set } p = q = \frac{1}{2}. \quad w(i) = 1 + \frac{1}{2} w(i+1) + \frac{1}{2} w(i-1), \; 0 < i < N \\ w(0) = w(N) = 0 \end{array} \right.$

$\rightarrow$ DE, Problem 8 on DE handout

Thm 3.6

Assume (X_n) is a finite state space, irreducible MC, P. For state j define

$\mu_j = E(R_j \mid X_0 = j)$. Then

$\rightarrow R_j \geq 1 \Rightarrow \mu_j \geq 1$

(1) ~~Ex~~ $\mu_j < \infty$ for each $j \in S$

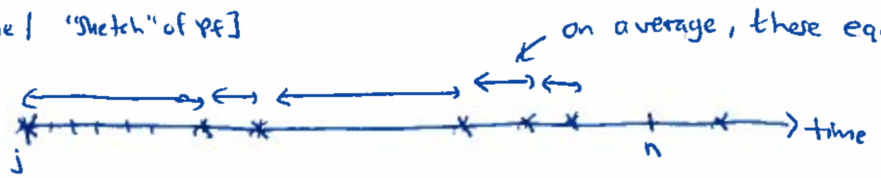
(2) There is a unique stationary distribution π such that $\pi_j = \frac{1}{\mu_j}$ for each j.

(3) For all states i, j $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P_{ij}^{m} = \pi_j$. ~~for every~~

$\left[\sum_{n=1}^{\infty} P_{ii}^{n} < \infty \Rightarrow \text{transient}. \quad \underset{=}{E}\left(\{\text{proportion of } \overbrace{\text{times}}^{\text{up to time } n} \text{ the chain is at state } j\} \right) \right.$

Question: why is $\pi_j = \frac{1}{\mu_j}$?

[Intuitive / "Sketch" of pf]



x = chain is at j

- # of visits to j between times 1 and n = x

  should have $x \cdot \mu_j \approx n$, so $\mu_j \approx \frac{n}{x}$. ← reciprocals

- π_j is proportion of times spent at j up to time n $= \frac{x}{n}$

$\Rightarrow \pi_j = \frac{1}{\mu_j}$.

State i is recurrent if $P(R_i < \infty \mid X_0 = i) = 1$.

i is called positive recurrent if $E(R_i \mid X_0 = i) < \infty$
↑ recurrent

null recurrent if $E(R_i \mid X_0 = i) = +\infty$.

Fact: A finite state space, irred. MC has no null recurrent states.

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Periodicity

The period of state i is defined to be

$$d(i) = \underset{\substack{\uparrow \\ \text{greatest common divisor}}}{g.c.d} \{ n \geq 1 : P_{ii}^n > 0 \}$$

If $d(i) = 1$, then i is called aperiodic

If all states are aperiodic, the MC is called aperiodic.

Simple examples

① Suppose state i satisfies $P_{ii} > 0$. Then $\underline{d(i) = 1}$

② Take

$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Find periods of each state.

$$\underline{P}^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix} \end{array}, \quad \underline{P}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix} = \underline{P}.$$

$$\Rightarrow \underline{P}^n = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix} & n \text{ odd} \\ \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix} & n \text{ even} \end{cases}$$

Check $d(0) = g.c.d. \{ n \geq 1 \; \underline{P_{00}^n} > 0 \} = g.c.d \{ \overset{P_{00}^1 = 0}{\cancel{1}}, 2, \overset{P_{00}^3 = 0}{\cancel{3}}, 4, \cancel{5}, 6, \ldots \}$
$= g.c.d. \{2, 4, 6, \ldots\} = 2.$

$d(1) = g.c.d \{ n \geq 1 : \underline{P_{11}^n} > 0 \} = gcd \{2, 4, 6, \ldots\} = 2.$

Fact: If a state i has period $d(i) > 1$ $(\neq 1)$, then $\lim_{n \to \infty} P_{ii}^n$ does not exist.

Lemma: (see p. 108) Periodicity is a class property.

If $i \leftrightarrow j$, then $d(i) = d(j)$.

Oct 5

3.6 Ergodic MCs

Thm 3.8 (Extended)

Let (X_n) be an irreducible, aperiodic MC. (The state space can be infinite)

Then exactly one of the following is true.

① All states are transient, $\lim_{n\to\infty} P^n_{ij} = 0$ all i,j, there is no stat. distr.

② All states are null recurrent, and $\lim_{n\to\infty} P^n_{ij} = 0$ all i,j, there is no stat. dist.

③ All states are positive recurrent, there is a unique stationary distribution $\underline{\pi}$ s.t.

- $\mu_j = \frac{1}{\pi_j}$ all j
- $\lim_{n\to\infty} P^n_{ij} = \pi_j$ all i,j.

(still true that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k_{ij} = \pi_j$)

Fact: An irreducible MC is positive recurrent iff it has a stationary distr.

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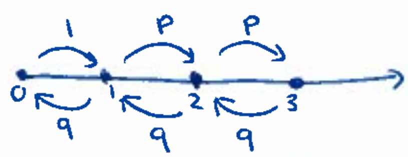
Example – Simple Random Walk on $S = \{0,1,2,\ldots\}$ with reflection at 0.



$0<p<1$
$q = 1-p$

Prop: This chain ~~has~~ is irreducible, period 2, and

1) transient if $p > \frac{1}{2}$

recurrent $\begin{cases} 2) \text{ null recurrent if } p = \frac{1}{2} \\ 3) \text{ positive recurrent if } p < \frac{1}{2}\text{, and} \end{cases}$ $\Rightarrow q > \frac{1}{2}$ and $\frac{p}{q} < 1$

$\underline{\pi}$ is $\pi_0 = \frac{q-p}{2q}$, $\pi_j = \frac{q-p}{2q}\left(\frac{p}{q}\right)^{j-1}$ for $j \geq 1$

If i is recurrent, then
$\begin{cases} \text{pos. recurrent} & E(R_i \mid X_0 = i) < \infty \\ \text{null rec.} & E(R_i \mid X_0 = i) = \infty \end{cases}$

recurrent: $P(R_i < \infty \mid X_0 = i) = 1$

It suffices to consider state 0.

$$P(R_0 < \infty \mid X_0 = 0) = P(H_0 < \infty \mid X_0 = 1)$$

(Starting at 1, $H_0 = R_0$)

R_0: 1st time after time 0

H_0: 1st time including time 0

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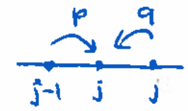
Put $u(i) = P(H_0 < \infty \mid X_0 = i)$, $i \geq 0$ (We really want just $i=1$).

Then, $u(i) = P_{i0} + \sum_{j\neq 0} P_{ij}\, u(j)$



For $i \geq 2$, $P_{i0} = 0$ and $u(i) = p\,u(j+1) + q\,u(\overset{i-1}{\cancel{j+1}})$

or $p\,u(i+1) - u(i) + q\,u(i-1) = 0$

$$u(i) = \begin{cases} A + Bi & p = \frac{1}{2} \\ A + B\left(\frac{q}{p}\right)^i & p \neq \frac{1}{2}. \end{cases}$$ (DE handout HW)

Consider $u(1) = P_{10} + \cancel{\Sigma}\, P_{12}\, u(2)$, with $u(0) = 1$,
$= P_{10}\, u(0) + P_{12}\, u(2) = p\,u(2) + q\,u(0)$

$\boxed{p = \frac{1}{2}}$: $u(i) = A + Bi$

1) If $B \neq 0$, $\lim_{i\to\infty} |u(i)| = +\infty$, impossible, all $u(i)$ are between 0 and 1.

Now, $u(i) = A$ for all $i \geq \cancel{0}1$.

Now, consider $i = 1$, $u(1) = \frac{1}{2} + \frac{1}{2}u(2)$ $\Rightarrow A = \frac{1}{2} + \frac{1}{2}A \Rightarrow A = 1 \Rightarrow u(i) = 1$ for all $i \geq 1$

This means $P(R_0 < \infty \mid X_0 = 0) = u(1) = 1 \Rightarrow$ recurrence.

Take $\boxed{p < \frac{1}{2}}$: $u(i) = A + B\left(\frac{q}{p}\right)^i$, $i \geq 1$

Since $\frac{q}{p} > 1$, $\left(\frac{q}{p}\right)^i \to \infty$ as $i \to \infty$. ~~impo~~ If $B \neq 0$ then $|u(i)| \to \infty$ as $i \to \infty$,

impossible. $\Rightarrow B = 0$. Now, we have $u(i) = A$ for all $i \geq 1$.

Consider $i = 1$. $u(1) = q + p\,u(2)$ $\longrightarrow$ $A(1-p) = q$
$A = q + p\,A$ $\qquad$ $Aq = q$ $\qquad \Rightarrow A = 1$

$\Rightarrow u(i) = 1$ for all $i \geq 1$ $\Rightarrow P(R_0 < \infty \mid X_0 = 0) = u(1) = 1 \Rightarrow$ recurrence.

$\Rightarrow p \leq \frac{1}{2} \Rightarrow$ recurrence

Take $\boxed{p > \frac{1}{2}}$: In analogy with random walk on $\{0, \pm 1, \pm 2, \ldots\}$, which is transient for $p > \frac{1}{2}$, we get transcience.

[We got this by showing $\sum_{n=0}^{\infty} P_{00}^{2n} < \infty$.]

We know $p > \frac{1}{2}$: transient
$p \le \frac{1}{2}$: recurrent case.

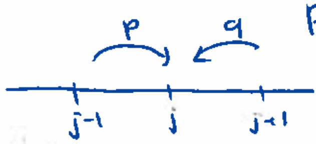
Positive vs. null recurrence?

↑ if there is a stat. dist. $\underline{\pi}$

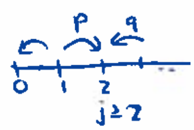
↑ if there is no stat. dist.

$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ ← can be solved or not with $\sum_{j=0}^{\infty} \pi_j = 1$.

$P_{ij} > 0$? $i = j-1$ and $i = j+1$



$\pi_j = \pi_{j-1} p + \pi_{j+1} q$ for $j \ge 2$



$j=0$: $\pi_0 = \sum_{i=0}^{\infty} \pi_i P_{i0} = \pi_1 P_{10} = q\pi_1$

$\pi_1 = \frac{1}{q}\pi_0$

$j=1$: $\pi_1 = \sum_{i=0}^{\infty} \pi_i P_{i1} = \pi_0 P_{01} + \pi_2 P_{21} = \pi_0 + \pi_2 q$

$\Rightarrow q\pi_2 = \pi_1 - \pi_0 = \frac{1}{q}\pi_0 - \pi_0 = \pi_0\left(\frac{1}{q} - 1\right) \Rightarrow \pi_2 = \frac{\frac{1}{q}-1}{q} \cdot \frac{q}{q}\pi_0 = \pi_0 \cdot \frac{1-q}{q^2} = \frac{p}{q^2}\pi_0$

$\Rightarrow \pi_1 = \frac{1}{q}\pi_0$
$\pi_2 = \frac{p}{q^2}\pi_0$

$\pi_j = q\pi_{j+1} + p\pi_{j-1}$, $j \ge 2$

general sol'n ↗ is $\pi_j = \begin{cases} A + Bj & \text{if } p = \frac{1}{2} \\ A + B\left(\frac{p}{q}\right)^j & \text{if } p < \frac{1}{2}. \end{cases}$

$p = \frac{1}{2}$ case: We need $\sum_{j=0}^{\infty} \pi_j = 1 < \infty$

This requires $\lim_{j\to\infty} \pi_j = 0$.

$\pi_j = A + Bj$. In order to have a stat. distr., we need $\lim_{j\to\infty}(A+Bj) = C$

This is false if $B \ne 0$.

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Now, given this, $\pi_j = A,\ j\ge 2$, $\lim_{j\to\infty}\pi_j = A = 0$ iff $A=0$.

$\pi_j = 0,\ j\ge 2$. $[\,0 = \pi_2 = \frac{p}{q}\pi_0$, so $\pi_0 = 0$ and $\pi_1 = 0\,]$

This forces $\pi_j = 0$ for all $j\ge 0$, and $\sum_{j=0}^{\infty}\pi_j = 0$.

$\Rightarrow$ There is no stat. distr. in the case $p=\frac{1}{2}$. $\Rightarrow$ null recurrent case.

Take $p<\frac{1}{2}$. $\pi_j = A + B\left(\frac{p}{q}\right)^j,\ j\ge\frac{1}{2}$.
$(p<q)$

Now $\frac{p}{q}<1$ and $\left(\frac{p}{q}\right)^j \to 0$ as $j\to\infty$. So $\lim_{j\to\infty}\pi_j = \lim_{j\to\infty}\left(A+B\left(\frac{p}{q}\right)^j\right) = A+0 = A \overset{?}{=} 0$

(no matter what B is.)

If $\sum_{j=0}^{\infty}\pi_j < \infty$, this forces $A=0$.

Now have $\pi_j = \left(\frac{p}{q}\right)^j,\ j\ge 2$ $\Rightarrow \sum_{j=0}^{\infty}\pi_j = \pi_0+\pi_1+B\sum_{j=2}^{\infty}\left(\frac{p}{q}\right)^j$

geometric series, $\frac{p}{q}<1$, so it converges.

We now must choose π_0, π_1, B to get $\sum_{j=0}^{\infty}\pi_j = 1$.

3.7 Time reversibility

A transition matrix $\underline{P}$ is called time reversible with respect to a stationary distribution $\underline{\pi}$ if the detailed balance equations are satisfied:
$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for all } i,j$$

This is a fairly "common" property.

Example:

$$\underline{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1/4 & 3/4 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 2 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

regular (since irred, $P_{11}>0$)
$\Rightarrow$ unique stat. distr.

Check: $\underline{\pi} = \left[\ \frac{4}{13}\ \ \frac{6}{13}\ \ \frac{3}{13}\ \right]$

$\pi_0 P_{01} = \pi_1 P_{10}$?
$\frac{4}{13}\cdot\frac{3}{4} = \frac{6}{13}\cdot\frac{1}{2}$ ✓

$\pi_0 P_{02} = \pi_2 P_{20}$ ✓ $(0=0)$
$\pi_1 P_{12} = \pi_2 P_{21}$
$\frac{6}{13}\cdot\frac{1}{4} = \frac{3}{13}\cdot\frac{1}{2}$ ✓

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Prop: Suppose $\underline{\underline{P}}$ is time reversible, stat. dist π, and assume the initial measure is π. Then for all $n \geq 1$, states $i_0, \ldots, i_n$

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_n, X_1 = i_{n-1}, \ldots, X_n = i_0)$$
$$= P(X_n = i_0, X_{n-1} = i_1, \ldots, X_0 = i_n)$$

This means the MC "looks the same" forward and backward in time.

$n=1$: $$P(X_0 = i, X_1 = j) = \underbrace{P(X_1 = j \mid X_0 = i)}_{P_{ij}} \underbrace{P(X_0 = i)}_{\pi_i} = \pi_i P_{ij} \overset{\text{time rev.}}{=} \pi_j P_{ji}$$

$$= P(X_0 = j)\, P(X_1 = i \mid X_0 = j) = P(X_1 = i, X_0 = j) = P(X_0 = j, X_1 = i)$$

Prop: If $\underline{\underline{P}}$ and a prob. vector $\underline{x}$ satisfy $x_i P_{ij} = x_j P_{ji}$ all i,j then $\underline{x}$ must be a stationary distribution.

Sometimes easier to solve than $\pi = \pi \underline{\underline{P}}$

HW: read 4.3 prob. gen. fcts.

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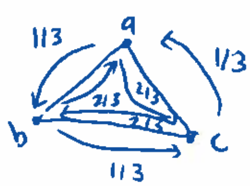
$\underline{\underline{P}}$ is time reversible with respect to π if $\pi_i P_{ij} = \pi_j P_{ji}$ $\forall i,j$.

Intuitively, reversibility means the M.C. "looks the same" forwards and backwards in time. [Simplest case: if X_0 has dist. π then $P(X_0=i, X_1=j) = P(X_0=j, X_1=i)$

Example:

$$\underline{\underline{P}} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{bmatrix} \end{array}$$



- $\underline{\underline{P}}$ is doubly stochastic, $\pi = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$
- Check: $\pi_a P_{ab} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$
  $\neq$ so not time-reversible
  $\pi_b P_{ba} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

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- At each state, the MC has prob. $\frac{1}{3}$ of taking a counterclockwise step
- At each ~~step~~ state, the MC has prob. $2/3$ of taking a clockwise step.
- It is "likely" that a given step will be clockwise.

In a sequence of states $i_0, i_1, \dots, i_n$
$$\overset{i_0}{\underset{X_0}{\|}} \quad \overset{i_1}{\underset{X_1}{\|}} \quad X_n$$

In forwards time, we expect to see $\overset{acba}{\cancel{abc}}$ more often than $\cancel{acb}$.abca.
$\left(\left(\frac{2}{3}\right)^3\right)$ $\qquad \left(\left(\frac{1}{3}\right)^3\right)$

Reversing the sequence, we would see the second more often.

The chain does not look the same forwards and backwards in time.

$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i,j.$

Find $\underline{\pi}$ to check this.

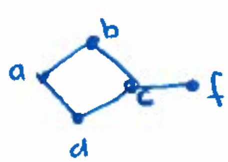
<u>Prop. 3.9:</u> If $\underline{x}$ is a prob. vector and $x_i P_{ij} = x_j P_{ji}$ for all i,j. $(*)$

Then, $\underline{x}$ is a stationary distr., and $\underline{P}$ is reversible with respect to $\underline{x}$.

<u>Proof:</u> Check that $x_j = \sum_{i \in S} x_i P_{ij}$, all i,j:

$$\sum_{i\in S} (x_i P_{ij}) = \sum_{i \in S} x_j P_{ji} = x_j \underbrace{\left(\sum_{i\in S} P_{ji}\right)}_{\text{row sum of row } j} = x_j \cdot 1 = x_j \checkmark$$

<u>Example:</u> Simple random walk on an unweighted graph



$P_{ij} = \frac{1}{\deg(i)}$ provided i,j are "neighbors"

Is $\underline{P}$ reversible ?

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Find $\underline{\pi}$?

Solve $\pi_j = \sum_{i \in S} \pi_i P_{ij}$ all j

<u>Guess:</u>

Try x_j is proportional to $\deg(j)$

(or x_j is -"- to $\frac{1}{\deg(j)}$)

Put $x_j = c \cdot \deg(j)$, some constant

Check detailed balance equations: $x_i P_{ij} = c \cdot \deg(i) \cdot \frac{1}{\deg(i)} = c$ if i,j neighbors

$x_j P_{ji} = c \cdot \deg(j) \frac{1}{\deg(j)} = c$ if j,i neighbors

So we have (*) holds for $x_j = c \cdot \deg(j)$.

Want $1 = \sum_{j \in S} x_j = \sum_{j \in S} c \deg(j) = c \sum_{j \in S} \deg(j)$

Take $c = \frac{1}{\sum_{j \in S} \deg j}$, and $\pi_j = x_j = \frac{\deg(j)}{\sum_{k \in S} \deg(k)}$.

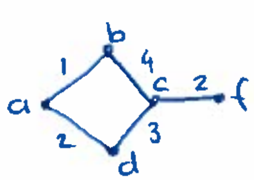
This $\underline{\pi}$ is a stationary distr., $\underline{P}$ is reversible for $\underline{\pi}$,

and $\underline{\pi}$ is unique if $\underline{P}$ is irreducible.

Since $\sum_{j \in S} \deg(j) = 2 \cdot e$, where e is the number of edges,

we get $\pi_j = \frac{\deg(j)}{2e}$.

<u>Example:</u> Random walk on weighted graphs:



$w(i,j)$ = weight of edge between i and j

$w(i) = \sum_{j \in S} w(i,j)$ $\quad P_{ij} = \frac{w(i,j)}{w(i)}$

$w(a,b) = 1$

$w(a,d) = 2$

$w(a) = w(a,b) + w(a,d) = 1 + 2 = 3$ $\quad P_{a,b} = \frac{1}{3}$, $P_{ad} = \frac{2}{3}$

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(If all weights are equal, say 1, then $w(i) = \deg(i)$)

Can we find $\underline{\pi}$, check for time reversibility?

Try: make a guess for π_i, check if $\pi_i P_{ij} = \pi_j P_{ji}$

Recall in unweighted case we guessed $x_i = c \cdot \deg(i)$.

Try $x_j = c \cdot w(j)$. (or $x_j = \frac{c}{w(j)}$).

Check: $x_i P_{ij} = c \cdot w(i) \frac{w(i,j)}{w(i)} = c\, w(i,j)$ (i, j neighbors)

$$\| $$

$x_j P_{ji} = c \cdot w(j) \frac{w(j,i)}{w(j)} = c\, w(j,i)$ ($w(i,j) = w(j,i)$)

Tells us $x_i P_{ij} = x_j P_{ji}$ $\forall i,j$, to make $\underline{x}$ a prob. vector we want

$$1 = \sum_{j \in S} x_j = \sum_{j \in S} c \cdot w(j) = c \sum_{j \in S} w(j) \Rightarrow c = \frac{1}{\sum_{j \in S} w(j)}$$

$\pi_i = x_i = c\, w(i)$

Now we know $\underline{\pi}$: $\pi_i = \dfrac{w(i)}{\sum_{j \in S} w(j)}$ is a stationary dist. and $\underline{P}$ is reversible with respect to $\underline{\pi}$.

~~Example Random walk with reflection at 0.~~

Meaning of null recurrence $\quad P(R_i < \infty \mid X_0 = i) = 1$
$\quad E(R_i \mid X_0 = i) = \infty$

Consider random walk with reflection at 0

$P(i, i+1) = p$, $P(i, i-1) = q = 1-p$ (for $i \geq 1$)

$P(0,1) = 1$.

For $p = \frac{1}{2}$, 0 is null recurrent.

Run simulation: parameter p, $N = \#$ of trials. ($\approx 10{,}000$)

Put $X_0 = 0$, run the random walk until R_0 (1st time rw is at 0 again), record this time, call this T_1

- Repeat, repeat, repeat,... N time in total, the successive return times called $T_1, T_2, \ldots, T_N$.

What can we say about $\frac{T_1+T_2+\cdots+T_N}{N}$ as $N \to \infty$?

$p < \frac{1}{2}$, positive recurrent, $E(R_0 \mid X_0=0) = \frac{1}{\pi_0} = \frac{2(1-p)}{1-2p}$ $\quad q = 1-p$

In this case, for large N we expect, by Law of Large Numbers,

$$\frac{T_1+\cdots+T_N}{N} \approx E(R_0 \mid X_0=0) = \frac{2(1-p)}{1-2p}$$

How about $p=\frac{1}{2}$? As $N\to\infty$, $\frac{T_1+T_2+\cdots+T_N}{N} \to \infty$.

$p = .45$, $E(R \mid X_0=0) = 11$, $\frac{T_1+\cdots+T_{10,000}}{10,000} = 11.01$ $\quad \max\{T_1, \ldots, T_{10,000}\} = 450$

$p = .5$ $\quad N = 20,000 \to$ max return time $= 1,264,924,844$

4.3

Def: Let X be a random variable with range $\{0,1,2,\ldots\}$.

The probability generating fct (pgf) is the fct of a real variable,

$G(s) = \sum_{n=0}^{\infty} s^n P(X=n) = E[s^X]$, s a real number.

We are most interested in $0 \le s \le 1$.

In this case, the series converges.

Example: $P(X=n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$, $n=0,1,2,\ldots$

(Geometric)

The pgf is $G(s) = Es^X = \sum_{n=0}^{\infty} s^n \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}s\right)^n = \frac{1}{3}\,\frac{1}{1-\frac{2}{3}s}$ $\quad$ converges if $\left|\frac{2}{3}s\right|<1$, $|s|<\frac{3}{2}$

$$= \frac{1}{3-2s}.$$

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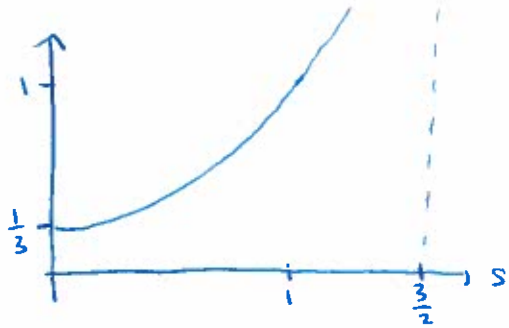

Start with a discrete probability dist. → one fct of a real variable $G(s)$

The pgf is a transform of the prob. dist. of X.

as is $M(t) = Ee^{tX}$, moment gen. fct

Laplace Transform

Fourier Transform

z-trans form

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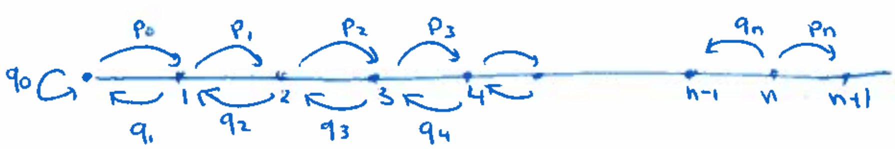
EXAM: Thurs, Nov 2

Example: Birth and death chain

$S = \{0, 1, 2, \dots\}$, $q_n = 1 - p_n$, all p_n satisfy $0 < p_n < 1$ ($0 < q_n < 1$)



Infinitely many parameters $\{p_n\}_{n=0}^{\infty}$

Random walk with reflection is the special case $p_n = p$ for all n

Question: classify the chain

Check for time-reversibility

- find $\underline{x}$ $x_i P_{ij} = x_j P_{ji}$ all i, j

If $\sum_{i=0}^{\infty} x_i < \infty$, $\underline{\pi}$ defined by

$$\boxed{\pi_i = \frac{x_i}{\sum_{i=0}^{\infty} x_i}}$$

is a stat. distr.,

which implies the chain is pos. recurren

* $i = 0, j = 1$ $\quad x_0 P_{01} = x_1 P_{10}$

$x_0 p_0 = x_1 q_1 \quad \Rightarrow x_1 = \frac{p_0}{q_1} x_0$

* $i = 1, j = 2$ $\quad x_1 P_{12} = x_2 P_{21}$

$x_1 p_1 = x_2 q_2 \quad \Rightarrow x_2 = \frac{p_1}{q_2} x_1 = \frac{p_0 p_1}{q_1 q_2} x_0$

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$i=2, j=3 \quad \Rightarrow x_3 = \frac{p_2}{q_3} x_2 = \frac{p_0 p_1 p_2}{q_1 q_2 q_3} x_0$

$$\Rightarrow x_n = \frac{p_0 p_1 p_2 \cdots p_{n-1}}{q_1 q_2 \cdots q_n} x_0$$

Now, put $\gamma_n = \begin{cases} \dfrac{p_0 p_1 \cdots p_{n-1}}{q_1 \cdot q_2 \cdots q_n}, & n \ge 1 \\ 1, & n=0 \end{cases}$

Then $x_n = \gamma_n x_0, \quad n=0,1,2,\ldots$

If $\sum_{n=0}^{\infty} x_n = x_0 \sum_{n=0}^{\infty} \gamma_n < \infty$, then $\pi_i = \dfrac{x_i}{\sum_{n=0}^{\infty} x_n} = \dfrac{\gamma_i x_0}{x_0 \sum_{n=0}^{\infty} \gamma_n} = \dfrac{\gamma_i}{\sum_{n=0}^{\infty} \gamma_n}$

In the random walk with reflection, $\gamma_i = \dfrac{p_0 p_1 \cdots p_{i-1}}{q_1 \cdots q_i} = \dfrac{p^i}{q^i} = \left(\dfrac{p}{q}\right)^i$

$\sum_{i=0}^{\infty} \gamma_i = \sum_{i=0}^{\infty} \left(\frac{p}{q}\right)^i < \infty$ if $p < q$

same as $p < \frac{1}{2}$ ✓ pos. recurrence case.

X is a random var. with range $\{0,1,2,\ldots\}$,

its pgf is $G(s) = \sum_{k=0}^{\infty} \underbrace{P(X=k)}_{p_k} s^k = E(s^X)$, s a real variable

$= \sum_{k=0}^{\infty} p_k s^k$, a power series, radius of convergence is ≥ 1.

Check for absolute convergence,

$$\sum_{k=0}^{\infty} |p_k s^k| = \sum_{k=0}^{\infty} p_k |s|^k \le \sum_{k=0}^{\infty} p_k = 1 < \infty \text{ for } |s| \le 1.$$

Properties of pgf's:

1. $\underline{G(1)} = \sum_{k=0}^{\infty} P(X=k) \cdot 1^k = \underline{1}$, $\underline{G(0)} = \sum_{k=0}^{\infty} P(X=k) s^k \Big|_{s=0} = P(X=0) + P(X=1)s + P(X=2)s^2 + \cdots \Big|_{s=0}$

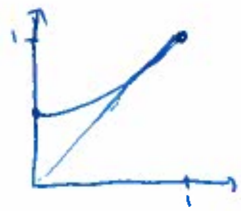
$$= \underline{P(X=0)}$$

2. $G(s) = \sum_{k=0}^{\infty} p_k s^k$

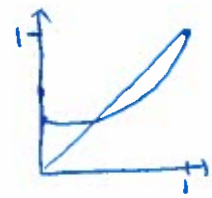
Can differentiate term by term inside the interval of convergence $|s| < 1$

$\Rightarrow G'(s) = \sum_{k=1}^{\infty} p_k \cdot k s^{k-1} \ge 0$, $G''(s) = \sum_{k=2}^{\infty} p_k k(k-1) s^{k-2} \ge 0$ for $0 \le |s| < 1$

$\Rightarrow$ G is increasing and concave up



$P(X=0)$
$\parallel$
$G(0) > 0$
increasing, concave up



stays above diagonal (except at $s=1$)

There is no $s \in [0,1)$ such that $G(s)=s$

crosses diagonal

↳ There is exactly one $s \in [0,1)$ s.t. $G(s)=s$.

But can there be $G(s)$ such that there are two distinct roots of $G(s)=s$ in $[0,1)$, $0 < s_1 < s_2 < 1$ s.t. $G(s_1)=s_1$, $G(s_2)=s_2$?

No.
Prove!

4. $G'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1} \quad 0 \le s < 1$

If the formula is correct at $s=1$, we would get

$G'(1) = \sum_{k=1}^{\infty} k \cdot p_k \cdot 1 = E(X)$

This is always true if $E(X) < \infty$.

If $E(X) = +\infty$, then one can show $G'(1) = +\infty$.

$[G'(1) = \lim_{s \to 1} G'(s)]$

$G''(s) = \sum_{k=2}^{\infty} p_k k(k-1) s^{k-2}, \quad 0 \le s < 1.$

If true for $s=1$, $G''(1) = \sum_{k=2}^{\infty} k(k-1) p_k \cdot 1 = \sum_{k=0}^{\infty} k(k-1) p_k = E(X(X-1))$
$g(x) = x(x-1)$

$E(g(X)) = \sum_{k=0}^{\infty} g(k) p_k.$

5. $E(X) = G'(1)$

$Var(X) = E(X^2) - \underbrace{(E(X))^2}_{(G'(1))^2}$

We know $G''(1) = E(X(X-1)) = E(X^2 - X) = E(X^2) - E(X)$

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$\Rightarrow \quad E(X^2) = G''(1) + E(X) = G''(1) + G'(1)$

$\Rightarrow Var(X) = G''(1) + G'(1) - (G'(1))^2.$

<u>Example:</u> $X \sim Geom(p)$, $P(X=k) = p\cdot(1-p)^k$, $k = 0, 1, 2, \ldots$, $0<p<1$

The pgf is $G(s) = \sum_{k=0}^{\infty} p(1-p)^k \cdot s^k = p \sum_{k=0}^{\infty} ((1-p)s)^k = \dfrac{p}{1-((1-p)s)}$

$= p(1-(1-p)s)^{-1}$

if $|(1-p)s| < 1$, $|s| < \frac{1}{1-p}$ ← larger than 1

$G'(s) = -p(1-(1-p)s)^{-2}(-(1-p)) = \dfrac{p(1-p)}{(1-(1-p)s)^2}$

$G'(1) = \dfrac{p(1-p)}{(1-(1-p))^2} = \dfrac{p(1-p)}{p^2} = \dfrac{1-p}{p} = E(X)$

To find $E(X)$ by the def'n, $E(X) = \sum_{k=0}^{\infty} k\,P(X=k) = \sum_{k=0}^{\infty} k\,p(1-p)^k$

$= p \sum_{k=0}^{\infty} k(1-p)^k$
not a geometric series. more work!

6. $P(X=k) = \dfrac{G^{(k)}(0)}{k!}$, $k = 0, 1, 2, \ldots$

implies: If X and Y have the same generating function, then

$P(X=k) = P(Y=k)$, so X and Y have the same distribution.

<u>Example:</u> Fact: If ~~$X\sim$~~ $Geom(p)$ then $G(s) = \dfrac{p}{1-(1-p)s}$, $|s| < \frac{1}{1-p}$.

Suppose we know a r.v. Y has pgf $H(s) = \dfrac{1}{3-2s}$, then

$Y \sim Geom(\frac{1}{3})$.

Put $p = \frac{1}{3}$ in $G(s) = \dfrac{p}{1-(1-p)s}$, we get $\dfrac{1/3}{1-\frac{2}{3}s} = \dfrac{1}{3-2s}$.

Here, for X taking values in $\{0, 1, 2, \ldots\}$, we have formulas for

$$\text{pmf} \underset{\longleftarrow}{\xrightarrow{G(s)=\sum_k p_k s^k}} \text{pgf} \xrightarrow{} \text{pmf}$$

$p_k = P(X=k) = \dfrac{G^{(k)}(0)}{k!}$

↑
prob. mass fct

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Compare with moment generating fct $\psi(s) = E(e^{sX})$ (any distr. for X),

we have formulas for

in continuous case density fct f $\rightarrow$ mgf $\xrightarrow{?}$ density fct

$$\psi(s) = \int_{-\infty}^{\infty} f(x) e^{sx} dx$$

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$G(s) = E(s^X) = \sum_{k=0}^{\infty} s^k P(X=k)$, $|s| \le 1$.

Thm: If X has pgf $G(s)$ then $P(X=k) = \frac{G^{(k)}(0)}{k!}$, $k = 0, 1, 2, \ldots$

$G^{(k)}$ the k^{th} derivative.

$\Rightarrow$ $G(s)$ determines the distribution of X.

Proof: $G(s)$ is a power series in s, radius of convergence at least one, all its derivatives are power series with same radius of convergence, valid $|s| < 1$

$G(s) = \sum_{k=0}^{\infty} p_k s^k$, $G(0) = p_0$ ✓

(keep $k=1$) $G'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}$, $G'(0) = 1 \cdot p_1$

(keep $k=2$) $G''(s) = \sum_{k=2}^{\infty} k(k-1) p_k s^{k-1}$, $G''(0) = 2(2-1) p_2 = 2! \, p_2$

(keep $k=3$) $G'''(s) = \sum_{k=3}^{\infty} k(k-1)(k-2) p_k s^{k-1}$, $G'''(0) = 3 \cdot 2 \cdot 1 \, p_3 = 3! \, p_3$

$p_1 = \frac{G'(0)}{1!}$

$p_2 = \frac{G''(0)}{2!}$

$p_3 = \frac{G'''(0)}{3!}$

Can do a formal induction argument for the general case.

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$X_1, X_2, \ldots$ independent, pgf's $G_1, G_2, G_3, \ldots$

($\Rightarrow s^{X_1}, s^{X_2}, \ldots$ are independent)

ⓐ $Y = X_1 + \ldots + X_n$

Y has pgf $E(s^Y) = E(s^{X_1 + \ldots + X_n}) = E(s^{X_1} s^{X_2} \cdots s^{X_n})$

$\overset{\text{independent}}{=} E(s^{X_1}) \cdot \ldots \cdot E(s^{X_n}) = G_1(s) \cdot \ldots \cdot G_n(s)$

If the X_i have the same distribution, equivalent to same pgf

$G(s)$, then ~~Ex~~ if $Y = X_1 + \ldots + X_n$, Y has pgf $E(s^Y) = (G(s))^n$.

<u>Ex:</u> If $X \sim \text{Bern}(p)$, $P(X=0) = 1-p$, $P(X=1) = p$ (p is a parameter, $0 \le p \le 1$)

X has pgf $E(s^X) = \sum_{k=0}^{\infty} P(X=k) s^k = (1-p)s^0 + ps^1 = 1-p+ps$

If $X \sim \underset{\text{Binomial}}{\text{Bin}}(n,p)$, then X has pgf $E(s^X) = (1-p+ps)^n$

Since ⓐ by def'n $E(s^X) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k = \sum_{k=0}^{n} \binom{n}{k} (ps)^k (1-p)^{n-k}$

$\overset{\text{Binomial formula}}{=} (ps + (1-p))^n$ ✓

ⓑ We can write $X = X_1 + \ldots + X_n$ where X_i independent, $X_i \sim \text{Bern}(p)$.

Now, by our thm, X has pgf $E(s^X) = (G(s))^n = (1-p+ps)^n$ where $G(s) = \text{pgf of Bern}(p) = 1-p+ps$.

ⓑ r.v.'s $N, X_1, X_2, \ldots$ all independent

N has pgf $H(s)$

the X_i all have pgf $G(s)$

$Y = X_1 + \ldots + X_N$, random sum has pgf $H(G(s))$

[when $N = k$, $Y = X_1 + \ldots + X_k$
If $N = 0$, take $Y = 0$]

<u>Ex:</u> (two-state-experiment)

Toss a biased coin, prob. of heads is $1/5$, tails is $4/5$, repeatedly (10 times).
Assume successive toss are independent.

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~~Roll a fair die, and~~ let N be ($\sim Bin(10, 1/5)$) the number of ~~spots that came~~ up. heads in the 10 tosses

~~and then~~ Now, ~~toss the coin~~ roll a fair die ~~that many times~~. N times, count the nr

of times 2 comes up, call this Y.

Find the pgf of Y, and use it to find $P(Y=2)$.

Let
$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ toss of die comes up } 2 \\ 0 & \text{if not} \end{cases}, \quad i=1,2,\dots$$

$P(X_i = 1) = \frac{1}{6}$

Then $Y = X_1 + \_ + X_N$

① N has pgf $H(t) = (1 - \frac{1}{5} + \frac{1}{5}s)^n = (\frac{4}{5} + \frac{1}{5}t)^n$ b/c $N \sim Bin(10, \frac{1}{5})$.

The X_i have pgf $G(s) = (1 - \frac{1}{6} + \frac{1}{6}s) = (\frac{5}{6} + \frac{1}{6}s)$ b/c $X_i \sim Bern(\frac{1}{6})$.

So, Y has pgf $\varphi(s) = H(G(s)) = H(\underbrace{\tfrac{5}{6} + \tfrac{1}{6}s}_{=t})$

$$= \left(\frac{4}{5} + \frac{1}{5}\left(\frac{5}{6} + \frac{1}{6}s\right)\right)^{10} = \left(\frac{4}{5} + \frac{1}{6} + \frac{1}{30}s\right)^{10} = \left(\frac{29}{30} + \frac{1}{30}s\right)^{10} = \varphi(s)$$

This says Y must be $Bin(10, \frac{1}{30})$, so $P(Y=2) = \binom{10}{2}(\frac{1}{30})^2(\frac{29}{30})^8$.

(or $= \frac{\varphi''(0)}{2!}$)

<u>Claim:</u> $E(Y) = E(X_1 + \_ + X_N) = E(X_1) \cdot E(N)$.

<u>Proof:</u> $E(Y) = \varphi'(1) = \frac{d}{ds}(H(G(s)))\Big|_{s=1} = H'(G(s))\,G'(s)\Big|_{s=1}$

$= H'(G(1))\,G'(1) = EN \cdot EX_1$.

$G'(1) = EX_1$, $H'(G(1)) = H'(1) = EN$

4.1 Intro. to branching processes

individuals, produce "offspring"
(bacteria, ..)

Assume single fixed known <u>offspring distribution</u> $\{p_k\}$, $k = 0, 1, \dots$

Where the prob. any particular indiv. produces k offspring is p_k.

- all indiv. reproduce indep.
- start with a single indiv.

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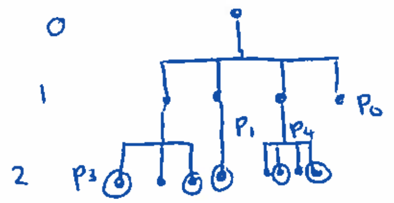

Prob (this event) $= p_4 \cdot (p_3 \cdot p_1 \cdot p_4 \cdot p_0)$

Ex~~tension~~tinction question: Given the sequence $\{p_k\}_{k=0,1,2,\ldots}$, what is the probability of eventual ~~extension~~ extinction?

some generation is empty, has no individuals

Let Z_n = the number of individuals in the n^{th} generation

$Z_0 = 1$ (usually, not always)

Z_1 = the number of offspring of initial individual.

Z_2 = the nr of offspring of all individuals in 1^{st} generation.

In Ex. above $Z_0 = 1,\ Z_1 = 4,\ Z_2 = 8, \ldots$

Claim: $(Z_n)_{n=0,1,2,\ldots}$ is a Markov chain.

Question: Given (p_k), what is the transition matrix $\underline{P} = (P_{ij})$?

$P_{ij} = (Z_1 = j \mid Z_0 = i) = ?$

Let X_k = # of offspring of individual k in generation, $k = 1, 2, \ldots, i$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$ i at gen 0

. . . . j in gen 1

$P_{ij} = P(Z_1 = j \mid Z_0 = i) = P(X_1 + \_ + X_i = j)$

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i$$

Fact: (p. 159) Classification of states

Let (Z_n) be a branching process such that $p_0 > 0$.

In this case state 0 is (always) absorbing, $P_{00} = P(Z_1 = 0 \mid Z_0 = 0) = 1$

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hence is recurrent.

Every other state is transient.

Idea:

Since $P_0 > 0$, there is positive probability that the next generation is empty. $P_{i0} = (P_0)^i > 0$, $i \geq 1$.

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Branching processes

Given an offspring distr. $\{p_k\}_{k=0,1,2,\ldots}$

- Each individual, independently of all other individuals, produces k offspring with probability p_k, $k = 0, 1, 2, \ldots$
- Z_n = # of individuals in generation n, $n = 0, 1, 2, \ldots$
- (Z_n) is a Markov chain, state space $S = \{0, 1, 2, 3, \ldots\}$, trans matrix $\underline{P}$.

Fact:

State 0 is absorbing, $P_{00} = 1$. Assume that $P_0 > 0$.

State i for $i \geq 1$ is transient.

Proof: 1) $P_{i0} = P(Z_1 = 0 \mid Z_0 = i) = (\underline{P})_{i0} = p_0 \cdots p_0 = (p_0)^i > 0$ for any i.

2) We need to check that $P(R_i < \infty \mid Z_0 = i) < 1$.

3) Given $Z_0 = i$. Then $(R_i < \infty$ then $Z_1 \neq 0)$, so $\{R_i < \infty\} \subset \{Z_1 \neq 0\}$

$$P(R_i < \infty \mid Z_0 = i) \leq P(Z_1 \neq 0 \mid Z_0 = i) = 1 - P(Z_1 = 0 \mid Z_0 = i) = 1 - (p_0)^i < 1 \checkmark$$

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<u>Thm:</u> If μ is the mean of the offspring distribution,

$\mu = \sum_{k=0}^{\infty} k p_k$, and $Z_0 = 1$, then $E(Z_n \mid Z_0 = 1) = \mu^n$, $n = 0, 1, 2, \ldots$

<u>Pf:</u> $n=0$, $E(Z_0 \mid Z_0 = 1) = 1 = \mu^0$.

$n=1$, $E(Z_1 \mid Z_0 = 1) = \mu$ ✓

$P(Z_1 = k \mid Z_0 = 1) = p_k$
$\Rightarrow$ The distr. of Z_1 is $(p_k)_{k=0,1,2,\ldots}$
(or the pmf of Z_1)

Now, fix n, and assume

$E(Z_n \mid Z_0 = 1) = \mu^n$, consider Z_{n+1}.

If we let $X_1, X_2, \ldots$ be i.i.d. r.v. with

distr. $(p_k)_{k=0,1,2,\ldots}$, then $Z_{n+1} = X_1 + X_2 + \ldots + X_{Z_n}$, a random sum,

where $X_1, X_2, \ldots$ are independent of Z_n.

By our thm on pgf, $E(Z_{n+1} \mid Z_0 = 1) = E(X_1 + \ldots + X_{Z_n} \mid Z_0 = 1) = E(X_1) \cdot E(Z_n \mid Z_0 = 1) = \mu \cdot \mu^n = \mu^{n+1}$ ✓

□

See text for a formula for $\mathrm{Var}(Z_n \mid Z_0 = 1)$.

What happens to $E(Z_n \mid Z_0 = 1)$ as $n \to \infty$?

$$\lim_{n\to\infty} E(Z_n \mid Z_0 = 1) = \lim_{n\to\infty} \mu^n = \begin{cases} 0, & \mu < 1 \\ 1, & \mu = 1 \\ +\infty, & \mu > 1 \end{cases}$$

What does this suggest about the behavior of the branching process?

- We expect "extinction" if $\mu < 1$ "population dies out"
- $\mu = 1$?
- $\mu > 1$ We expect probability < 1 of extinction.

<u>Def:</u> Extinction means $P(Z_n = 0 \text{ for some } n \geq 1) = 1$. $= P\left(\bigcup_{n=1}^{\infty} \{Z_n = 0\}\right) = 1$

[Assume $p_0 > 0$, $Z_0 \neq 0$]

more formally, let $E_n = \{Z_n = 0\}$, $n = 1, 2, 3, \ldots$

Then (i) $E_1 \subset E_2 \subset E_3 \subset \dots$, and $\{Z_n = 0 \text{ for some } n\} = \underbrace{\bigcup_{n=1}^{\infty} E_n}_{\text{extinction event}}$.

"Survival" means "not extinction" or $P(Z_n = 0 \text{ for some } n \geq 1) < 1$.

<u>Claim:</u> If $\mu < 1$, then there is extinction, or, $P(Z_n = 0 \text{ for some } n) = 1$.

<u>Lemma:</u> For any sequence of events $E_1, E_2, \ldots$ with $E = \bigcup_{n=1}^{\infty} E_n$,

$$P(E) = \lim_{N \to \infty} P\left(\bigcup_{n=1}^{N} E_n\right).$$

<u>Pf.</u> ① Follows from additivity assumption of all probability spaces.

or ② In some sense, $\bigcup_{n=1}^{N} E_n$ "converges" to $\bigcup_{n=1}^{\infty} E_n = E$.

<u>Fact:</u> If e = extinction probability $= P(Z_n = 0 \text{ for some } n \mid Z_0 = 1)$,

then $e = \lim_{N \to \infty} P(Z_N = 0 \mid Z_0 = 1)$.

<u>Pf.</u> Use Lemma with $E_n = \{Z_n = 0\}$ and note that $\bigcup_{n=1}^{N} E_n = E_N$ (since $E_1 \subset E_2 \subset \dots \subset E_N$).

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} P\left(\bigcup_{n=1}^{N} E_n\right) = \lim_{N \to \infty} P(E_N).$$

<u>Back to case $\mu < 1$:</u> Can we find $P(Z_n = 0 \mid Z_0 = 1)$?

1) For any rv X with possible values $0, 1, 2, 3, \ldots$

$$P(X \geq 1) \leq EX \text{ because } EX = \sum_{k=1}^{\infty} k P(X=k) \geq \sum_{k=1}^{\infty} 1\, P(X=k)$$
$$= P(X \geq 1).$$

2) Apply to $X = Z_n$, $P(Z_n \geq 1 \mid Z_0 = 1) \leq E(Z_n \mid Z_0 = 1) = \mu^n \to 0$ as $n \to \infty$ for $\mu < 1$

3) $P(Z_n = 0 \mid Z_0 = 1) = 1 - P(Z_n \neq 0 \mid Z_0 = 1) = 1 - P(Z_n \geq 1 \mid Z_0 = 1)$

$$e = \lim_{n \to \infty} P(Z_n = 0 \mid Z_0 = 1) = 1 - \lim_{n \to \infty} P(Z_n \geq 1 \mid Z_0 = 1) \overset{2)}{=} 1 - 0 = 1. \quad \square$$

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4.4

Goal: find $e = \lim_{n\to\infty} P(Z_n = 0 \mid Z_0 = 1)$ in terms of $(p_k)_{k=0}^{\infty}$.

If we knew the pgf of Z_n, say $H_n(s)$, then

$$P(Z_n = 0 \mid Z_0 = 1) = H_n(0).$$

Thm: Let the offspring dist. have pgf $G(s)$.

Define $(G_n)_{n=0,1,2,\ldots}$ by $G_0(s) = s$, $G_1(s) = G(s)$, and

$$G_{n+1}(s) = G(G_n(s)), \quad n = 0,1,2,\ldots$$
$$= \underbrace{G \circ G \circ G \circ \ldots \circ G}_{(n+1)\text{-times}}(s)$$

Then $G_{n+1}(s) = G_n(G(s))$ and $G_n(s)$ is the pgf of Z_n (given $Z_0 = 1$).

(Note $P(Z_n = 0 \mid Z_0 = 1) = G_n(0)$).

Pf: Fix n, let $X_1, X_2, X_3, \ldots$ be i.i.d. r.v.'s with distr. $(p_k)_{k=0,1,2,\ldots}$

Then $Z_{n+1} = X_1 + X_2 + \ldots + X_{Z_n}$.

By Thm pgf's and random sums, $G_{Z_{n+1}}(s) = (G_{Z_n} \circ G_{X_1})(s) = G_{Z_n}(G_{X_1}(s))$

$= G_n(G(s)) = G_{n+1}(s)$. □

Example: Offspring dist. Geom$(\frac{1}{3})$, $p_k = \frac{1}{3}(\frac{2}{3})^k$, $k = 0,1,2,\ldots$

The pgf is $G(s) = \frac{1}{3-2s} = G_1(s)$.

$$G_2(s) = G(G(s)) = \frac{1}{3 - 2G(s)} = \frac{1}{3 - \frac{2}{3-2s}} \cdot \frac{3-2s}{3-2s} = \frac{3-2s}{9-6s-2} = \frac{3-2s}{7-6s}.$$

Can check $G_3(s) = G_2(G(s)) = \frac{3-2G(s)}{7-6G(s)} = \ldots = \frac{7-6s}{15-14s}$.

We know
$$P(Z_1 = 0 \mid Z_0 = 1) = G(0) = \frac{1}{3}$$
$$P(Z_2 = 0 \mid Z_0 = 1) = G_2(0) = \frac{3}{7}$$
$$P(Z_3 = 0 \mid Z_0 = 1) = G_3(0) = \frac{7}{15}.$$

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$$e = \lim_{n\to\infty} P(Z_n = 0 \mid Z_0 = 1) = \underbrace{\lim_{n\to\infty} G_n(0)}.$$

Can we find this somehow? Yes

Thm 4.2 (Main Thm)

Let (Z_n) be a branching process with offspring dist. (p_k) which has mean μ and pgf $G(s)$. Assume $Z_0 = 1$, and $p_1 \neq 1$ and e = extinction probability $= P(Z_n = 0$ for some $n)$.

omit this from notation

Then ① If $\mu \le 1$ then $e = 1$.

② If $\mu > 1$, then $e < 1$ (survival is possible) and e is the smallest root of $G(s) = s$, $s \in [0,1)$.

Example: $G(s) = \frac{1}{3-2s} = (3-2s)^{-1}$, $G'(s) = -(3-2s)^{-2}(-2) = \frac{2}{(3-2s)^2}$

$\mu = G'(1) = \frac{2}{1} = 2 > 1$, so $e < 1$.

To find e set $G(s) = s$: $\frac{1}{3-2s} = s \Leftrightarrow 1 = 3s - 2s^2 \Leftrightarrow 2s^2 - 3s + 1 = 0$

$(s-1)(2s-1) = 0$

[This has always one solution we know: $s = 1$]
$\Rightarrow s-1$ is always a factor of $G(s) - s = 0$

Roots are $s = 1$, $s = \frac{1}{2}$,

so $\boxed{e = \frac{1}{2}}$.

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Exam: Th, see BB for topics.

Thm (Extinction Thm) (4.2)

hypothesis as above

Then (a) e is the smallest root of $G(s) = s$, $s \in [0$
(b) If $\mu \le 1$ then $e = 1$.
(c) If $\mu > 1$ then $e < 1$.

Proof: Define $e_n = P(Z_n = 0)$, recall $\boxed{\lim_{n\to\infty} e_n = e}$ and also $\lim_{n\to\infty} e_{n+1} = e$.

Recall $G_n(s)$ is the pgf of Z_n, $\underline{G_{n+1}(s) = G(G_n(s))}$, and $e_n = G_n(0)$

<u>I. e solves $G(s) = s$:</u>

$$e_{n+1} = P(Z_{n+1} = 0) = G_{n+1}(0) = G(G_n(0)) = G(e_n)$$

That is, $e_{n+1} = G(e_n)$ for all n.

$\lim_{n\to\infty} e_{n+1} = e$ $\qquad$ G is a continuous fct on $[0,1]$ and $e_n \to e$ as $n\to\infty$, so $G(e_n) \to G(e)$ as $n \to \infty$.

$$\Rightarrow \underbrace{\lim_{n\to\infty} e_{n+1}}_{e} = \lim_{n\to\infty} G(e_n) = G(e) \checkmark$$

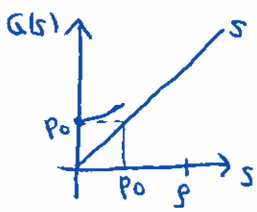
<u>II. Let ρ be the smallest root of $G(s) = s$ in $[0,1]$. Then $e = \rho$.</u>

<u>Case 1:</u> $p_0 = 0$, $G(0) = p_0 = 0$. So 0 is a root of $G(s) = s$, and $\rho = 0$,

so $e = \rho$. $\qquad$ [no 0 offspring $\Rightarrow Z_n > 0 \ \forall n \Rightarrow e = 0$]

<u>Case 2:</u> $p_0 > 0$ $\qquad$ $G(0) < \rho$

Then $\rho > p_0 = G(0)$.



Since G is an increasing fct, $\underbrace{G(G(0))}_{G_2(0)} < G(\rho) = \rho$

Same argument gives $G_3(0) = G(G_2(0)) < G(\rho) = \rho$.

In fact $G_n(0) < \rho$ for every n.

We know $G_n(0) \to e$ as $n \to \infty$, so taking limits we obtain

$e = \lim_{n\to\infty} G_n(0) \le \rho$. We now have $e \le \rho$, but e is a root

of $G(s) = s$, and ρ is the smallest such root, so $e = \rho$. $\Rightarrow$ (a).

(b) <u>If $\mu \le 1$ then there is no root of $G(s) = s$ in $[0,1)$, or $\rho = 1$, $\Rightarrow e = 1$.</u>

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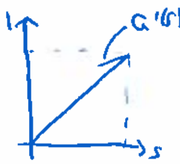
Why? Suppose $\mu = 1$. Then $G'(1) = 1$. Since $G'(s)$ is increasing,

$G'(s) < G'(1) = 1 = \mu$ for $s < 1$.

[If $p_1 = 1$, then $G(s) = s$ $\forall s \in [0,1]$, $G'(s) = 1$ $\forall s$]

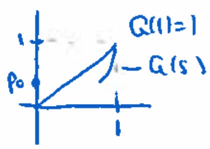
$\overset{\shortparallel}{\sum_{n=0}^{\infty} s^n p_n}$



This tells us that $G(s) > s$ for all $0 \le s < 1$, so no root in $[0,1)$.

(c) Supp. $\mu > 1$. e.g.

There is a unique root in $(0,1)$, so $\rho < 1$, and $e = \rho$.



$\mu = G'(1) > 1$

Calculus shows that $G'(s) > 1$ for s close to 1.
$(G'(s) \to G'(1) = \mu > 1$ as $s \to 1)$

$\Rightarrow G(s) < s$ for s close to 1.

$G(0) > 0$, $G(s) < s$.

Consider $G(s) - s$: $s = 0$, $G(s) - s > 0$

and if s is close to 1, $G(s) - s < 0$.

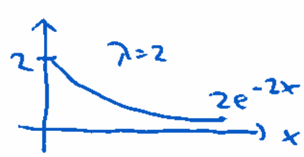
So there must be some s_0 with $G(s_0) - s_0 = 0$, some $0 < s_0 < 1$.

This s_0 is ρ. $\square$

Chapter 6 - Poisson processes

Continuous time

Recall: X is an **exponential** rv with parameter $\lambda > 0$ (we write $X \sim \text{Exp}(\lambda)$) if X has pdf
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \ge 0 \end{cases}$$



$E(X) = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$.

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Fact: $P(X>t)=e^{-\lambda t}$

$$\int_t^\infty \lambda e^{-\lambda x}\,dx = \lim_{A\to\infty}\int_t^A \lambda e^{-\lambda x}\,dx = \lim_{A\to\infty}\left[-e^{-\lambda x}\right]_t^A = \lim_{A\to\infty}\left(-e^{-\lambda A}+e^{-\lambda t}\right)$$

$$=0+e^{-\lambda t}$$

Recall: X is Poisson with parameter $\lambda>0$, ($X\sim \mathrm{Pois}(\lambda)$)

if X has possible values $0,1,2,\ldots$ and $P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$, $k=0,1,2,\ldots$

and $EX=\lambda$, $\mathrm{Var}(X)=\lambda$

Prop: If $X\sim \mathrm{Exp}(\lambda)$ (any $\lambda>0$), then X has the memoryless property, i.e. for all $s,t>0$ $P(X>s+t\mid X>s)=P(X>t)$.

Proof.

$$P(X>s+t\mid X>s)=\frac{P(X>s+t,\,X>s)}{P(X>s)}=\frac{P(X>s+t)}{P(X>s)}=\frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}=e^{-\lambda t}=P(X>t).$$

Counting Processes $(N_t)_{t\geq 0}$

a "continuous" variable

A stochastic process indexed by $[0,\infty)$, each N_t is a r.v, $\underbrace{\text{integer}}_{}$ nonnegative integer-valued

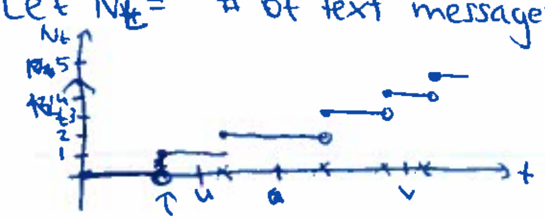
s.t.

1) if $s\leq t$ then $N_s\leq N_t$

2) process can take jumps of only size 1

Example: Turn on cellphone, call this time $t=0$, text messages come in.

Let $N_t=$ # of text messages that have arrived by time t.



times messages arrive.

$N_u=1$, $N_v=4$

The nr of messages that came in during the time interval $(u,v]=N_v-N_u$ an increment in the fct $t\mapsto N_t$.

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Def: (#1) A Poisson process $(N_t)_{t\geq 0}$ with rate λ (>0) is a counting process with the following properties:

1) $N_0 = 0$ (usually).

2) For $t>0$, N_t is a Poisson rv, parameter λt
$\rightsquigarrow E(N_t) = \lambda t$ $\rightsquigarrow$ expect about λt arrivals in t time units
$\rightsquigarrow$ rate of arrivals $= \frac{E(N_t)}{t}$ =

3) (stationary increments)

For each $s,t>0$, the rv $N_{t+s} - N_s$ has the same distribution as $N_t - N_0 = N_t$, that is $N_{t+s} - N_s \sim \text{Poiss}(\lambda t)$

4) (independent increments) If times (t_i) satisfy $0 \le t_0 < t_1 < t_2 < \dots < t_n$,

then the increments

$N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent r.v.'s.

Example: Suppose $(N_t)_{t\geq 0}$ is a PP with rate $\lambda = 0.3$.
N_t = # of text messages that arrive by time t [during $(0,t]$]

Find ⓐ the prob. no messages arrive in 1st 4 minutes

$\rightarrow P(N_4 = 0) \underset{\substack{\nearrow \\ N_4 \sim \text{Pois}(4\lambda)}}{=} \frac{(4\lambda)^0}{0!} e^{-4\lambda} = e^{-4(.3)} = e^{-1.2}$

ⓑ the prob. that 1 message arrived between times 2 and 3 given that 5 messages arrived by time 2.

$P(N_3 - N_2 = 1 \mid \underset{\substack{\| \\ N_2 - N_0}}{N_2 = 5}) = P(N_3 - N_2 = 1)$ (independent increments
$P(A|B) = P(A)$ if A, B are independent)

stat. increments

$\overset{\searrow}{=} P(\underbrace{N_1}_{N_1 - N_0} = 1) = \underset{\substack{\nwarrow \\ N_1 \sim \text{Poiss}(1\cdot\lambda)}}{e^{-\lambda 1}} \cdot \lambda \cdot \frac{1}{1!} = e^{-\lambda}\lambda = e^{-.3}(.3)$

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Exam #2 $\rightarrow$ Thursday, sol'n to hw in library, format same as before (no poisson processes)

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Poisson proc: (3) (stat. increments)

For two increments: $N_t - N_s$ and N_{w-u} have the same (joint) distribution as $N_{t-s} - N_0$ and $N_{w-s} - N_{u-s}$

For 1 increment, say $N_t - N_s$, this has the same dist. as $N_{t-s} - \underbrace{N_0}_{=0}$, so $N_t - N_s$ has Poisson dist., param. $\lambda \cdot (t-s)$

(4) (indep. incr.): $N_t - N_s$ and $N_w - N_u$ are independent

Ex: Cont'd: (c) Find the probability that 6 messages are received in first 10 minutes, and exactly one of these was received in 1st 3 minutes

$$P(N_{10}=6, N_3=1) = P(N_3=1, N_{10}=6) \neq P(N_3=1)\,P(N_{10}=6)$$

$$[\,P(N_s=i, N_t=k) \neq P(N_s=i)\,P(N_t=k)\,]$$

$$P(N_3=1, N_{10}=6) = \underbrace{P(N_3=1, N_{10}-N_3=5)}_{\text{independent increments}}$$

$$= P(N_3=1)\,P(N_{10}-N_3=5) = \underset{\text{Pois}(3\lambda)}{P(N_3=1)}\;\underset{\text{Pois}(7\lambda)}{P(N_7=5)} = \frac{e^{-3\lambda}(3\lambda)^1}{1!}\;\frac{e^{-7\lambda}(7\lambda)^5}{5!}$$

(d) Given that exactly 6 messages were received in 1st ten minutes, what is the prob. that exactly 1 message is received in 1st 3 minutes?

$$P(N_3=1 \mid N_{10}=6) = \frac{P(N_3=1, N_{10}=6)}{\underset{\text{Pois}(10\lambda)}{P(N_{10}=6)}} \overset{(c)}{=} \frac{\frac{e^{-3\lambda}(3\lambda)^1}{1!}\;\frac{e^{-7\lambda}(7\lambda)^5}{5!}}{\frac{e^{-10\lambda}(10\lambda)^6}{6!}}$$

$$= \underbrace{\frac{e^{-3\lambda}e^{-7\lambda}}{e^{-10\lambda}}}_{=1}\;\frac{(3\lambda)^1(7\lambda)^5}{(10\lambda)^6}\cdot\underbrace{\frac{6!}{1!\cdot 5!}}_{=\binom{6}{1}} = 1\cdot\frac{3^1 7^5}{10^6}\cdot\underbrace{\frac{\lambda^1\lambda^5}{\lambda^6}}_{=1}\cdot\binom{6}{1} = \binom{6}{1}\left(\frac{3}{10}\right)^1\left(\frac{7}{10}\right)^5$$

Binomial $(\underset{n}{6}, \underset{p}{3/10})\,(\underset{k=1}{1})$

proportion $3/10$

$P(X=1)$ where $X \sim \text{Bin}(6, \frac{3}{10})$

(e) Find $E(N_3 \cdot N_7)$ $(\neq EN_3 \cdot EN_7)$ [only N_3 and $N_7 - N_3$ are independent!]

$$= E(N_3((N_7-N_3)+N_3)) = E(N_3(N_7-N_3)) + E(N_3^2) = E(N_3)E(N_7-N_3) + EN_3^2$$

$N_7 - N_3$ has same prob. dist. as N_4

$$\overset{*}{=} EN_3 \cdot E(N_4) + EN_3^2$$

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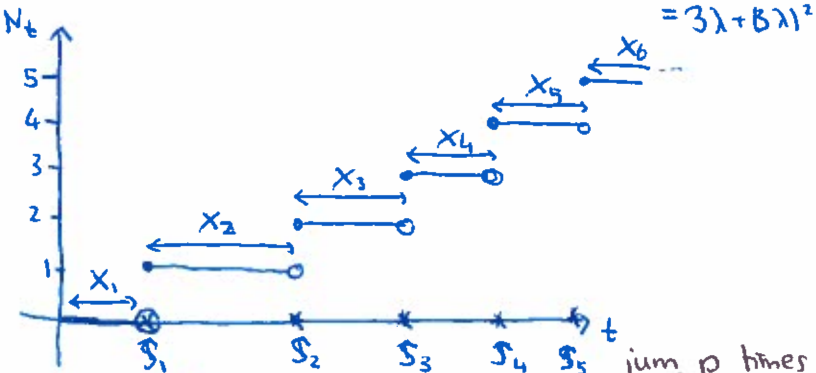
Recall: If $X \sim \text{Pois}(\lambda)$, then $EX = \lambda$, $\text{Var}(X) = \lambda$. $= E(X^2) - (E(X))^2$.
$E(X^2) = \text{Var}(X) + (E(X))^2$

$\Rightarrow E(N_3 \cdot N_7) = E(N_3) \cdot E(N_4) + \underbrace{E(N_3^2)}_{=3\lambda + (3\lambda)^2} = (3\lambda)\cdot(4\lambda) + 3\lambda + (3\lambda)^2$



$*$ = instant a text message arrives

N_t = # of $*$'s during $(0,t]$

Put $S_0 = 0$ and for $k \geq 1$

S_k = time of k^{th} jump of (N_t)

These are random variables, call these the "arrival times".

Define the "inter arrival times" $(X_k)_{k=1,2,\ldots}$ by $X_k = S_k - S_{k-1}$. random variables.

Thm: $X_1, X_2, \ldots$ are indep. identically dist. (i.i.d.)

with the exponential (λ) distribution. [each X_i has pdf $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0, & x < 0 \end{cases}$

$\left[\begin{array}{l} \text{mean} = \frac{1}{\lambda} \\ \text{variance} = \frac{1}{\lambda^2} \end{array}\right]$

$S_k = X_1 + X_2 + \ldots + X_k$ $\quad$ [$X_k = S_{k+1} - S_k$, ... $X_1 = S_1 - S_0 = S_1$, $X_2 = S_2 - S_1 = S_2 - X_1 \Rightarrow X_1 + X_2 = S_2$, ..

As a consequence, the $(S_k)_{k\geq 1}$ are gamma r.v.'s, $S_k \sim \text{Gamma}(k, \lambda)$, and

pdf is $f_k(x) = \begin{cases} \dfrac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ $\quad$ mean is $\frac{k}{\lambda}$ $\quad$ variance $\frac{k}{\lambda^2}$.

Example: Suppose $(N_t)_{t\geq 0}$, Poisson process rate $\lambda = .3$.

(a) Find the expected time of the arrival of the 3^{rd} message

- difficult to consider just using (N_t)

$\rightarrow = E(S_3) = \frac{3}{\lambda} = \frac{3}{.3} = 10.$

(b) Find the probability the 3rd message (S_3) arrived between 10 and 15.

This is $P(10 < S_3 < 15) = \int_{10}^{15} f_3(x)\,dx = \int_{10}^{15} \frac{\lambda^3 x^2}{2!} e^{-\lambda x}\,dx = \ldots$

int. by parts twice..

$N_{15} - N_{10} = 1$, $N_{10} = 2$

$\{10 < S_3 < 15\}$

or  $N_{15} - N_{10} = 3$

or 

$N_{15} - N_{10} = 5$

...

[hard to find just using (N_t) !]

Oct 31

Nov 7

Exam ① $\boxed{F}$ branch. process, $\{p_k\}$, $p_k > 0$ $\forall k$ $\Rightarrow$ irred.

$0 \not\leftrightarrow 1$ b/c $\underline{P_{00} = 1}$, so $P^n_{0j} = 0$ $\forall n, j \neq 0$.

$P^n_{00} = 1$ all n

$\boxed{F}$ finite irred., time-rev. MC has lim. dist: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\pi = [\frac{1}{2}\ \frac{1}{2}]$

$\boxed{F}$ Every MC must have at least one pos. recurrent state.

True if state space is finite.
False for simple symm. rw on $\{0, \pm 1, \pm 2, \ldots\}$, all states null recurrent.

⑥ (Z_n), e, μ, $G(s)$, $Z_0 = 1$, (p_k)
(W_n) branch. process, same (p_k), $W_0 = 3$

W_0

$W_n^{(A)}$ = descendants of (A)

$W_n^{(B)}$ $W_n^{(C)}$

indiv. produce $\boxed{\text{independently}}$

(a) $P(W_n = 0 \text{ for some } n \mid W_0 = 3) = P\big(W_n^{(A)} = 0$ for some n, $W_n^{(B)} = 0$ for some n, $W_n^{(C)} = 0$ for some $n \mid W_0^{(A)} = 1, W_0^{(B)} = 1, W_0^{(C)} = 1\big)$

$\overset{\text{independence}}{=} P(W_n^{(A)} = 0 \text{ for some } n \mid W_0^{(A)} = 1) \cdot P(W_n^{(B)} = 0 \text{ some } n \mid W_0^{(B)} = 1)$

$P(W_n^{(C)} = 0 \text{ some } n \mid W_0^{(C)} = 1) = P(Z_n = 0 \text{ some } n \mid Z_0 = 1)^3 = e^3 \overset{!}{=} 1$

$\Leftrightarrow e = 1$ ← that's the case when $\boxed{\mu \le 1}$

(b) $P(W_n \neq 0 \text{ for all } n \mid W_0 = 3) = 1 - e^3$.

Thm: (X_k) are iid $\text{Expo}(\lambda)$.
$\Rightarrow (S_k)_{k \ge 1}$ are $\text{Gamma}(k, \lambda)$, ... pdf $f_k(t) = \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}$, $t \ge 0$

We will show: the (S_k) are $\text{Gamma}(k, \lambda)$ and the (X_k) are $\text{Expo}(\lambda)$.

Nov 7

I Find the pdf f_k of S_k by finding its cdf F_k and then setting

$f_k = F_k'$.

Fix t.

$$F_k(t) = P(S_k \le t) = P(N_t \ge k) = 1 - P(\underbrace{N_t < k}_{N_t \le k-1}) = 1 - \sum_{j=0}^{k-1} P(N_t = j) = 1 - \sum_{j=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

Exercise: Check $\frac{d}{dt} F_k(t) = \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}$ (telescoping series) ✓

$$\frac{d}{dt} F_k(t) = -\sum_{j=0}^{k-1}\left(\frac{-\lambda e^{-\lambda t}(\lambda t)^j}{j!} + \frac{e^{-\lambda t}\lambda^j t^{j-1} j}{j!}\right) = -\sum_{j=0}^{k-1} \frac{-\lambda^{j+1} e^{-\lambda t} t^j}{j!} - \sum_{j=1}^{k-1} \frac{e^{-\lambda t}\lambda^j t^{j-1}}{(j-1)!}$$

$$= -e^{-\lambda t}\left[\sum_{j=0}^{k-1} \frac{-\lambda^{j+1} t^j}{j!} + \sum_{j=0}^{k-2} \frac{\lambda^{j+1} t^j}{j!}\right] = -e^{-\lambda t}\left(-\frac{\lambda^k t^{k-1}}{(k-1)!}\right) = \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}$$

The dist. of X_{k+1}? Find $P(X_{k+1} \le u)$ and differentiate

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 1 - P(X_{k+1} > u) \leftarrow$ we want

$P(X_{k+1} > u) = P(S_{k+1} - S_k > u)$ $\quad$ [cdf : $F_X(x) = P(X \le x)$, $F_X(x) = \int_{-\infty}^{x} f(t)\,dt$, $\frac{d}{dx} F(x) = f(x)$ ← pdf, $P(X \in A) = \int_A f\,dx$

We can find this if we can find the joint pdf of (S_k, S_{k+1}) $f_{k,k+1}$

If we can find $F_{k,k+1}$ $[F_{k,k+1}(s,t) = P(S_k \le s, S_{k+1} \le t)]$, the joint cdf of (S_k, S_{k+1}), then $f_{k,k+1}(s,t) = \frac{\partial^2}{\partial s \partial t} F_{k,k+1}(s,t)$

<u>Problem:</u> Can we find $F_{k,k+1}(s,t) = P(S_k \le s, S_{k+1} \le t)$ for $s < t$?

Easier to find $G_{k,k+1}(s,t) = P(S_k \le s, S_{k+1} > t)$ ~~$= P(N_s \ge k, N_t - N_s = 0)$~~

$= $ ~~$P(N_s \ge k)$~~ $P(N_s = k, N_t = k) = P(N_s = k, N_t - N_s = 0) = P(N_s = k)\, P(N_t - N_s = 0)$

$$= \frac{e^{-\lambda s}(\lambda s)^k}{k!} \frac{e^{-\lambda(t-s)}(\lambda(t-s))^0}{0!}$$

Now, we have $G_{k,k+1}(s,t) = P(S_k \le s, S_{k+1} > t) =$ ___ specific formula.

We want $F_{k,k+1}(s,t) = P(S_k \le s, S_{k+1} \le t)$.

Can we relate $F_{k,k+1}(s,t)$ and $G_{k,k+1}(s,t)$?

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$\{S_k \le s\} = \{S_k \le s,\ S_{k+1} \le t\} \ \dot\cup\ \{S_k \le s,\ S_{k+1} > t\}$

$P(S_k \le s) = P(\quad " \quad) + P(\quad " \quad)$

$P(S_k \le s) = F_{k,k+1}(s,t) + G_{k,k+1}(s,t)$ or $F_{k,k+1}(s,t) = P(S_k \le s) - G_{k,k+1}(s,t)$

$f_{k,k+1}(s,t) = \frac{\partial^2}{\partial s \partial t}\left(F_{k,k+1}(s,t)\right) = \frac{\partial^2}{\partial s \partial t}\left(0 - G_{k,k+1}(s,t)\right)$

We get : (S_k, S_{k+1}) has joint pdf

$$f_{k,k+1}(s,t) = -\frac{\partial^2}{\partial s \partial t}\left[\frac{e^{-\lambda s}(\lambda s)^k}{k!}\, e^{-\lambda(t-s)} \cdot 1\right],\ s<t$$

Finally, $P(X_{k+1} > u) = P(S_{k+1} - S_k > u) = \dots e^{-\lambda u}$. "□"

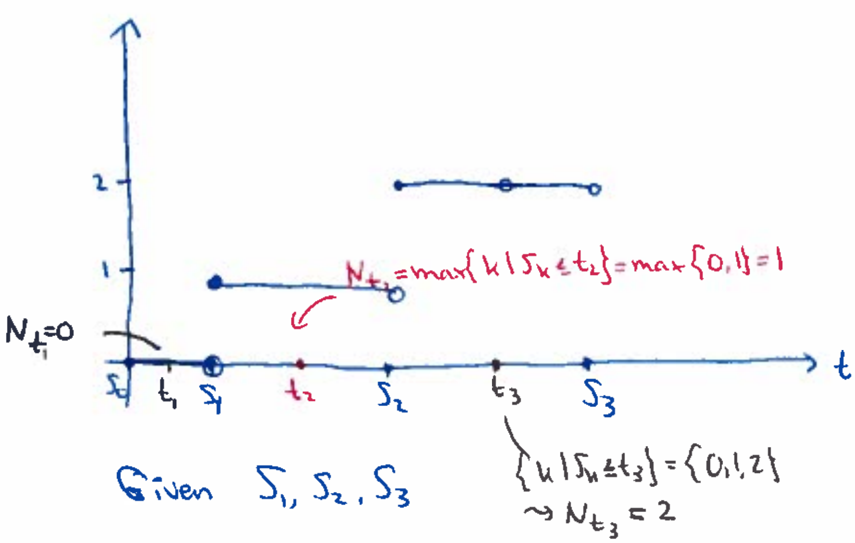
Now, start with iid rv's $\{X_1, X_2, \dots\}$ (positive)

Make a counting process out of these.

1) Define $S_0 = 0$, $S_k = X_1 + \dots + X_k$.

$(S_1 = X_1)$

2) Define $(N_t)_{t \ge 0}$ by $N_t = \max\{k : S_k \le t\}$, $N_0 = 0$



Given S_1, S_2, S_3

<u>Prop:</u> If $(X_1, X_2, \dots$ are iid. Expo(λ), and (S_k) and (N_k) are defined as above, then $(N_t)_{t \ge 0}$ is a Poisson-process, rate λ.

(Def 2 ⇒ Def 1 and Def 1 ⇒ Def 2.

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[Def#2: A Poisson proc. with rate λ is a counting process def'd as above by $X_1, X_2, \ldots$]

Read "little oh" handout on BB. Read 8.2, 8.3.

We say "$f(x) = o(x)$ as $x \to 0$" to mean $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

Intuitive meaning: 1) $f(x)$ goes to 0 "faster" than x goes to 0 [as $x \to 0$]
2) $f(x)$ is negligible compared to x as $x \to 0$.
much more smaller.

Ex: - $h^2 = o(h)$ as $h \to 0$. $\qquad \lim_{h\to 0} \frac{h^{\cancel{2}}}{\cancel{h}} = 0$

- Is $\sqrt{h} = o(h)$ as $h \to 0$? $\qquad \lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h}} \neq 0$.

- $\sin(h) \neq o(h)$ as $h \to 0$ $\qquad$ ($\frac{\sin h}{h} \to 1$ as $h \to 0$)

We say $f(x) = g(x) + o(x)$ to mean $f(x) - g(x) = o(x)$, or
$\lim_{x\to 0} \frac{f(x) - g(x)}{x} = 0$.

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Little oh notation $\qquad f(x) = o(g(x))$ as $x \to 0$ means $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

Ex: ① Is $\cos x = 1 + o(x)$ as $x \to 0$?

$$\lim_{x\to 0} \frac{\cos x - 1}{x} \overset{\text{L'Hopital's Rule}}{\underset{(*)}{=}} \lim_{x\to 0} \frac{-\sin x}{1} = 0 \checkmark$$

② Is $e^{2x} = 1 + 2x + o(x)$ as $x \to 0$?

$$\lim_{x\to 0} \frac{e^{2x} - (1+2x)}{x} \overset{(*)}{=} \lim_{x\to 0} \frac{2e^{2x} - 2}{1} = 0 \checkmark$$

Fact: Let f be a continuous fct. on $[0, \infty)$, $f(0) = 0$.

Then $f(x) = \lambda x + o(x)$ as $x \to 0$ is equivalent to
$f'(0) = \lambda$.

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Pf. (a) If $f(x) = \lambda x + o(x)$, $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)^{0}}{x} = \lim_{x\to 0} \frac{\lambda x + o(x)}{x}$

$= \lim_{x\to 0} \left(\lambda + \frac{o(x)}{x}\right) = \lambda$ ✓

(b) If $f'(0) = \lambda$, then $\lim_{x\to 0} \frac{f(x) - f(0)^{0}}{x} = \lambda$, $\lim_{x\to 0} \frac{f(x)}{x} = \lambda$

Def. $g(x) = \frac{f(x)}{x} - \lambda$, so $\lim_{x\to 0} g(x) = 0$. Solve for $f(x)$,

$f(x) = (\lambda + g(x))x = \lambda x + \boxed{g(x)x}$.

Meaning of $f(x) = \lambda x + o(x)$ is $\lim_{x\to 0} \frac{f(x) - \lambda x}{x} = 0$.

But $\frac{f(x) - \lambda x}{x} = g(x)$, thus $\lim_{x\to 0} \frac{f(x) - \lambda x}{x} = \lim_{x\to 0} g(x) = 0$.

[$h(x) = \underbrace{h(0)}_{0} + \underbrace{h'(0)}_{\lambda} x + \text{error} = \lambda x + \text{error}$.]

Def 3 (p.234)

A counting process $(N_t)_{t\geq 0}$ is a Poisson Process, rate λ, if

1. $N_0 = 0$
2. (N_t) has stationary and independent increments.
3. $P(N_h = 0) = 1 - \lambda h + o(h)$ as $h \to 0$.
   $-o(h) = o(h)$

   or $P(N_h \neq 0) = \lambda h + o(h)$

4. $P(N_h = 1) = \lambda h + o(h)$ as $h \to 0$
5. $P(N_h \geq 2) = o(h)$ as $h \to 0$. [5&4 ⇒ 3]

Fact: $o(h) + o(h) = o(h)$ as $h \to 0$.

Interpretation:

3. Unlikely to have an arrival in a short time period.

   $((0,h])$ $P(N_h) = 0 = P(N_{t+h} - N_t = 0)$

4. The prob. of a single arrival in a short time period is "almost" proportional (λ) to the length of the period.

5. It is very unlikely to have 2 or more arrivals in a short time period.

Note: 1) There is no mention of Poisson distribution.

2) These assumptions are reasonable in modeling a number of situations (not all).

Thm: All three "definitions" of Poisson processes are equivalent.

Note: Def 3 forces the Poisson dist. for N_t.

Thm: (Law of Rare events)

Let $(S_n)_{n=1,2,\dots}$ be a sequence of Binomial r.v.'s s.t. S_n is Binomial with parameter n, p_n where $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} np_n = \lambda > 0$. Then $\lim_{n\to\infty} P(S_n = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0,1,2,\dots$

"$= ES_n$"

[e.g. $p_n = \frac{\lambda}{n}$]

Binom(n, p_n) is approximately same as Poiss(λ) for large n.

If S_n is the nr of successes in n trials with success prob. p_n, where $n\to\infty$, $p_n \to 0$ and $np_n \to \lambda$, then S_n converges in distribution to Pois(λ).

Pf: We need to show $\binom{n}{k} p_n^k (1-p_n)^{n-k} \to \frac{e^{-\lambda}\lambda^k}{k!}$ as $n\to\infty$ for each fixed $k = 0,1,\dots$

Recall $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ any nr x.

This can be improved to: If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \left(1+\frac{x_n}{n}\right)^n = e^x$

Let k be fixed.

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{n!}{(n-k)!\,k!}\, p_n^k \,\frac{(1-p_n)^n}{(1-p_n)^k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\, p_n^k \,\frac{(1-p_n)^n}{(1-p_n)^k}$$

$\lim_{n\to\infty} n p_n = \lambda$

$p_n^k = \frac{(np_n)^k}{n^k}$

$$= \frac{1}{k!}\,\frac{n(n-1)\cdots\overset{\frac{n-(k-1)}{n}}{(n-k+1)}}{n\cdot n\cdots n}\,(np_n)^k\,\frac{\left(1-\frac{p_n n}{n}\right)^n}{(1-p_n)^k}$$

$$= \frac{1}{k!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)(np_n)^k\,\underbrace{\left(1-\frac{p_n n}{n}\right)^n}_{=\left(1+\frac{-np_n}{n}\right)^n \to e^{-\lambda}}\;\underbrace{\frac{1}{(1-p_n)^k}}_{\to \frac{1}{1}}$$

Let $n\to\infty$, get $\quad \frac{1}{k!}\cdot\underbrace{1\cdots 1}_{k \text{ times}}\;\lambda^k e^{-\lambda}\cdot\frac{1}{1^k}$

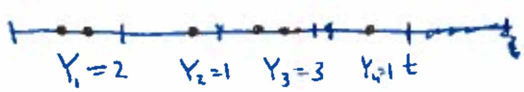
$$= \frac{e^{-\lambda}\lambda^k}{k!}. \qquad \square$$

<u>Sketch of Def 3 implies for fixed t, $N_t \sim \text{Pois}(\lambda t)$.</u>

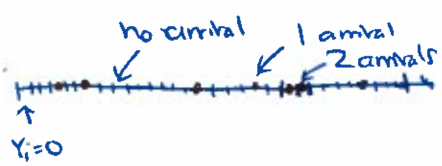
Fix t, we must show $P(N_t = k) = e^{-\lambda}\frac{\lambda^k}{k!}$.

<u>Idea:</u> Show the properties of Def 3 imply N_t is approx. a Binomial rv with parameter n, $p_n \approx \frac{\lambda t}{n}$. Then use Law of rare events.

<u>Step I:</u> Divide $[0,t]$ into n subintervals of length $\frac{t}{n}$.



$n=4$ $\quad \bullet$ = arrival times for our count. process



large n

Let Y_i = # of arrivals in subinterval i.

Let $T_n = Y_1 + \dots + Y_n$ = total nr of arrivals in $[0,t]$

$N_t = T_n$, and $Y_1, Y_2, \dots, Y_n$ are independent rv's.

But T_n is not binomial (some of the Y_i's may be $2, 3, \dots$)

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Step 2: Def. $X_i = \begin{cases} 0 & \text{if } Y_i = 0 \\ 1 & \text{if } Y_i \geq 1 \end{cases}$ and $S_n = X_1 + \dots + X_n$, which [$N_t \neq S_n$ is possible]

is Binomial, parameters n, $p_n = P(X_i = 1) = P(Y_i \geq 1) = P(Y_i = 1) + P(Y_i \geq 2)$

Consider $Y_i = N_{\frac{t}{n}}$. $\Rightarrow p_n = P(N_{\frac{t}{n}} = 1) + P(N_{t/n} \geq 2) \underset{\text{from Def. 3}}{=} \lambda \cdot \frac{t}{n} + o(\frac{t}{n}) + o(\frac{t}{n})$

$\Rightarrow p_n = \frac{\lambda t}{n} + o(\frac{t}{n})$.

Step 3: • $P(S_n \neq T_n) \to 0$ as $n \to \infty$. (check)

$\quad = P(\text{at least one } Y_i \geq 2)$

• $P(N_t = S_n) = P(N_t = T_n, S_n = T_n) + P(N_t = S_n, \underbrace{S_n \neq T_n}_{\to 0})$ b/c

≈ 1 [since $N_t = T_n$]

• $P(N_t = k) \approx P(S_n = k) \to ?$ as $n \to \infty$

S_n is Binomial $(n, p_n = \frac{\lambda t}{n} + o(\frac{t}{n}))$. $np_n = n(\frac{\lambda t}{n} + o(\frac{t}{n}))$

$= \lambda t + n\, o(\frac{t}{n})$

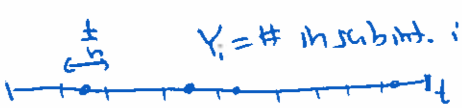
$= \lambda t + t\frac{o(\frac{t}{n})}{\frac{t}{n}}$

$\to \lambda t + 0$

since $h = \frac{t}{n} \to 0$ (as $n \to \infty$)

By law of rare events, $P(S_n = k) \to \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

which shows $P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$



$N_t = \sum_{i=1}^{n} Y_i$ which is approx. $\underbrace{\text{Binom}(n, p_n)}_{\text{Pois}(\lambda t)}$, $p_n = \frac{\lambda t}{n} + o(\frac{t}{n})$

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Def #2: $X_1, X_2, \dots$ are i.i.d. expon. r.v. with param λ, $S_0 = 0$,
$S_k = X_1 + \dots + X_k$, $k = 1, 2, \dots$, (N_t) def'd by $N_0 = 0$, $N_t = \max\{k \mid S_k \leq t\}$, $t > 0$

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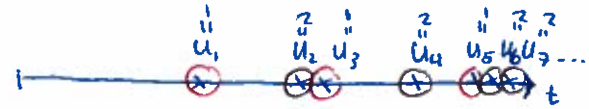
6.4 Thinning and superposition

$(N_t)_{t\geq 0}$ rate λ Poisson process.



arrival times are "x" (S_k's)

Now "mark" the arrival time, with marks of $2^{(k)}$ types O O
1, 2

such that 1) marks are independent

2) mark an arrival as Type 1 with probability p_1

" " " " 2 " " p_2

$(p_1+p_2=1)$

Can let $U_1, U_2, U_3, \ldots$ be i.i.d. with $P(U_i = p_1)$, $P(U_2 = p_2)$ all i,

which are independent of $(N_t)_{t\geq 0}$.

Define $N_t^{(0)}$ = the nr of Type 0 arrivals by time t.

$N_t^{(1)}$ = " " 1 " " "

Note that $N_t = N_t^{(1)} + N_t^{(2)}$.

<u>Thm:</u> (True for all $k\geq 2$) $(N_t)_t$ is Poisson proc., rate λ, $p_0, p_1 = 1-p_0$.

- $(N_t^{(0)})_{t\geq 0}$ is a Poisson process, rate λp_0 (each $N_t^{(0)}$ is a Poisson r.v. (not process) with parameter $\lambda p_1 t$)
- $(N_t^{(1)})_{t\geq 0}$ is a " " " λp_2
- $(N_t^0)_{t\geq 0}$, $(N_t^1)_{t\geq 0}$ are independent

<u>Example:</u> Consider births at a local hospital, assume the prob. a given birth is male to be $p_m = .48$, the prob. " " " is female to be $p_f = .52$.

Assume the nrs over time is a Poisson (rate $\lambda=2$) proc., and successive births

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are independent of one another and the Poisson proc. $(N_t)_{t\geq 0}$.

Def. N_t^m = # of male births by ~~time~~ t (each $t>0$)

N_t^f = # femalle —"— —"—

Then by Thm, $(N_t^m)_{t\geq 0}$ is a Poisson proc., rate $\lambda p_m = 2(.48)$

$(N_t^f)_{t\geq 0}$ rate $\lambda p_f = 2(.52)$

and the two processes are independent.

Find the probability that at least one male and no females are born during a 3 hour period)
$= [0,3]$

We want $P(N_3^M \geq 1, N_3^F = 0) \overset{\text{indep.}}{=} P(N_3^M \geq 1) P(N_3^F = 0)$

$= (1 - P(N_3^M = 0))\ P(N_3^F = 0) = \left(1 - \frac{e^{-3\lambda p_m}(3\lambda p_m)^0}{0!}\right) \frac{e^{-3\lambda p_f}(3\lambda p_f)^0}{0!}$

"Pf of Thm":

Check $(N_t^{(0)})_{t\geq 0}$ is a Poisson proc., rate λp_0.

Use Def 3: $N_0^{(0)} = 0$ ✓ $(N_0 = 0)$

Increments - stationary, independent (for $(N_t^{(0)})$) [inherited from $(N_t)_{t\geq 0}$]

(3) follows from (4), (5)

(5) $0 \leq P(N_h^{(0)} \geq 2) = P(N_h^{(0)} \geq 2, N_h \geq 2) \leq P(N_h \geq 2) = o(h)$ as $h \to 0$ ✓

(4) $\underset{\substack{\uparrow \\ \text{requires } N_h \geq 1}}{P(N_h^{(0)} = 1)} = P(N_h^{(0)} = 1, N_h = 1) + P(N_h^{(0)} = 1, N_h \geq 2)$

$P(N_h^{(0)} = 1, N_h = 1) = P(N_h = 1$, the single mark is type $0)$

$\overset{\text{indep.}}{=} P(N_h = 1)\, p_0 = (\lambda h + o(h))\, p_0 = \lambda p_0 h + o(h)$

$P(N_h^{(0)} = 1, N_h \geq 2) \leq P(N_h \geq 2) = o(h)$ as $h \to 0$.

We get $P(N_h^{(0)} = 1) = \lambda p_0 h + [o(h) + o(h)] = \lambda p_0 h + o(h)$.

Given stat. + indep. increments, this shows $(N_t^{(0)})_{t\geq 0}$ is a Poisson process, rate λp_0. (and $(N_t^{(1)})_{t\geq 0}$ ... rate λp_1).

Check independence of $(N_t^{(0)})$, $(N_t^{(1)})$.

We will show only: for any fixed $t>0$, $P(N_t^{(0)}=i \mid N_t^{(1)}=j) = P(N_t^{(0)}=i)$.

$$= \frac{e^{-\lambda p_0 t}(\lambda p_0 t)^i}{i!}$$

$$P(N_t^{(0)}=i \mid N_t^{(1)}=j) = \frac{P(N_t^{(0)}=i,\ N_t^{(1)}=j)}{P(N_t^{(1)}=j)} \leftarrow \text{know this.}$$

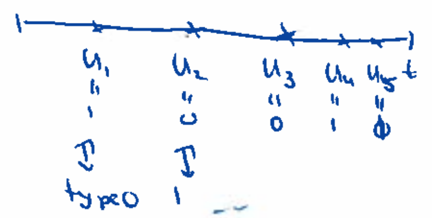
$$P(N_t^{(0)}=i,\ N_t^{(1)}=j) = P(\underbrace{N_t^{(0)}=i,\ N_t=i+j}_{\text{dependent}})$$

Let $U_1, U_2, U_3, \ldots$ be iid Bernoulli r.v.'s, ~~parameter~~ p_0.

and $W_k = U_1+U_2+\dots+U_k$ be the nr of type 0 arrivals in the 1^{st} k arrivals.

Binomial r.v., param. k, p_0

e.g. $N_t^{(0)} = U_1+U_2+\dots+U_5 = 1+0+0+1+1 = 3$



$$\Rightarrow P(N_t^{(0)}=i,\ N_t^{(1)}=j) = P(N_t^{(0)}=i,\ N_t=i+j) = P(W_{i+j}=i,\ N_t=i+j)$$

$$\overset{\text{indep.}}{=} P(W_{i+j}=i)\cdot P(N_t=i+j) = \binom{i+j}{j} p_0^i p_1^j e^{-\lambda t}(\lambda t)^{i+j}\frac{1}{(i+j)!}$$

We now have
$$P(N_t^{(0)}=i \mid N_t^{(1)}=j) = \frac{\binom{i+j}{j} p_0^i p_1^j \frac{e^{-\lambda t}(\lambda t)^{i+j}}{(i+j)!}}{\frac{e^{-\lambda p_1 t}(\lambda p_1 t)^j}{j!}}$$

$$= \text{simplify, get} = \frac{e^{-\lambda p_0 t}(\lambda p_0 t)^i}{i!} = P(N_t^{(0)}=i).$$

Superposition p.240

Thm: If $(N_t^{(1)})_{t\geq 0}, \ldots, (N_t^{(k)})_{t\geq 0}$ are indep. Pois. proc. with rates $\lambda_1, \ldots, \lambda_k$, then $N_t = N_t^{(1)} + \dots + N_t^{(k)}$, $t\geq 0$, (N_t) is a Poisson proc. with rate $\lambda = \lambda_1 + \dots + \lambda_k$.

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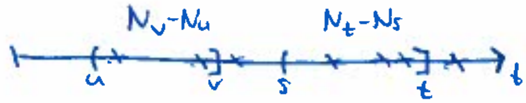
One - dimensional Poisson process:

$(N_t)_{t\geq 0}$

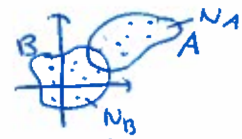


x's are random arrival times.

$N_t - N_s$ = # of arrivals during $(s,t]$ $(=3)$

New notation: N_I = nr of ~~arrivals~~ random points in interval $I=(s,t]$

6.6 Spatial Poisson Processes



N_A, N_B not independent

Def. We have r.v's N_A for each region $A \subseteq \mathbb{R}^2$,

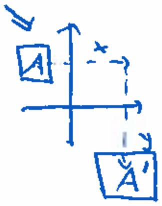
which are Poisson, parameter $\lambda \cdot \underset{\text{area of } A}{|A|}$

(and if $A, B \subseteq \mathbb{R}^2$ are disjoint, then N_A, N_B are independent)

N_A, N_B independent

Stationary increments?

Pois$(\lambda|A|)$



shift it by x in x-direction y in y-dir. to get A'

One dimension: $N_{t+s} - N_s$ is a Poisson r.v. with param λt (and $N_t - N_0$ is Pois., param. λt)

Pois$(\lambda|A'|)$
$=$ Pois$(\lambda|A|)$

$E(N_A) = \lambda|A|$.

Increase λ, expect N_A to increase. "completely random"

Fact Given $N_A = k$, the distribution of the points in A is the same as the distribution of the points $U_1, \dots, U_k$, where $U_1, U_2, \dots, U_k$ are iid, uniform random vectors on A.

6.5 Poisson processes and the uniform distribution

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$(N_t)_{t\geq 0}$ Poisson process, rate λ.

Prop. (p. ~~246~~ 245): Fix $a>0$. For any $k \geq 1$, the conditional distribution of the arrival times given $N_a = k$ is the same as a set of k iid Uniform$(0,a)$ r.v.'s. More precisely, given iid $U_1, U_2, \dots, U_k$,

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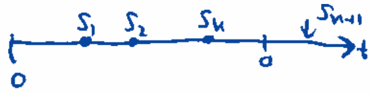
(

Given $N_a = k$
(S_1, S_2, S_k) ← Here $U_{(1)} = U_k$, $U_{(2)} = U_1$, $U_{(3)} = U_2$

(U_1, U_2, U_k))

uniform on $(0, a)$. define the order statistic $U_{(1)}, \dots, U_{(k)}$ to the sequence of points obtained from $U_1 - U_k$ arranged in increasing order.

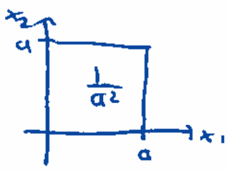
- $U_{(1)} < U_{(2)} < U_{(3)} < \dots < U_{(k)}$
- $\{U_1, \dots, U_k\} = \{U_{(1)}, \dots, U_{(k)}\}$.

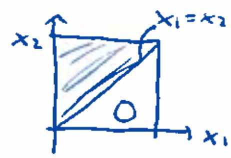
[$k=2$: $(U_{(1)}, U_{(2)})$ is either (U_1, U_2) or (U_2, U_1)]
if $U_1 < U_2$ $\qquad U_2 < U_1$

Then $(S_1, S_2, \dots, S_k)$ given $N_a = k$ has the same distribution as $(U_{(1)}, U_{(2)}, \dots, U_{(k)})$.

Note $(U_1, \dots, U_k)$ has joint pdf $f(x_1, \dots, x_k) = (\frac{1}{a})^k$ (for all $0 < x_i < a$)
$(U_{(1)}, \dots, U_{(k)})$ does not have ↗ as its joint pdf.

$\underline{k=2:}$

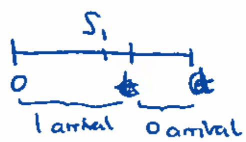




$(\overset{x_1}{U_{(1)}}, \overset{x_2}{U_{(2)}})$ has pdf
$g(x_1, x_2) = \frac{2}{a^2}$ for $0 < x_1 < x_2 < a$

$\underline{\text{Text:}}$ $f_{(U_{(1)}, \dots, U_{(k)})}(x_1, \dots, x_k) = \frac{k!}{a^k}$ if $0 < x_1 < x_2 < \dots < x_k$.

$\underline{\text{Sketch of Pf:}}$ $\underline{k=1}$



$N_a = 1$

Let $F(t) = P(S_1 \le t \mid N_a = 1)$ $\quad 0 < t < a$ (a conditional cdf for S_1 given $N_a = 1$)

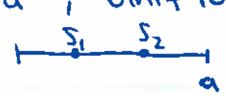
The conditional pdf of S_1 given $N_a = 1$ is $\frac{d}{dt} F(t)$.

$$P(S_1 \le t \mid N_a = 1) = \frac{P(S_1 \le t, N_a = 1)}{P(N_a = 1)} = \frac{P(N_t = 1, N_a - N_t = 0)}{P(N_a = 1)} = \frac{P(N_t = 1)\, P(N_a - N_t = 0)}{P(N_a = 1)}$$

$$= \frac{\frac{e^{-\lambda t}(\lambda t)^1}{1!} \; e^{-\lambda(a-t)} \frac{(\lambda(a-t))^0}{0!}}{e^{-\lambda a}(\lambda a)^1 / 1!} = e^{-\lambda t} e^{-\lambda a + \lambda t} e^{\lambda a} \frac{\lambda t}{\lambda a} \cdot 1 = \frac{t}{a}$$

$\frac{d}{dt}\left(\frac{t}{a}\right) = \frac{1}{a}$, Unif $(0, a)$ pdf

$\underline{k=2}$



Fix $0 < s < t < a$, and $\varepsilon_1, \varepsilon_2 > 0$ such that

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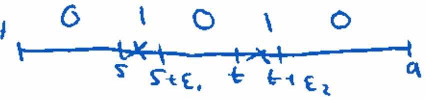
$$f_{(S_1,S_2|N_a=2)}(s,t) = \lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \frac{\{P(s<S_1<s+\varepsilon_1,\ t<S_2<t+\varepsilon_2 \mid N_a=2)\}}{\varepsilon_1\varepsilon_2}$$

$$\text{Num} = \frac{P(s<S_1<s+\varepsilon,\ t<S_2<t+\varepsilon_2,\ N_a=2)}{P(N_a=2)}$$

this event



$$= \frac{P(N_s=0,\ N_{s+\varepsilon_1}-N_s=1,\ N_t-N_{s+\varepsilon_1}=0,\ N_{t+\varepsilon_2}-N_t=1,\ N_a-N_{t+\varepsilon_2}=0)}{P(N_a=2)}$$

factor by indep.

$$= \frac{P(N_s=0)\,P(N_{\varepsilon_1}=1)\,P(N_{t-s-\varepsilon_1}=0)\,P(N_{\varepsilon_2}=1)\,P(N_{a-t-\varepsilon_2}=0)}{P(N_a=2)}$$

put in Pois prob.
=
cancel like terms

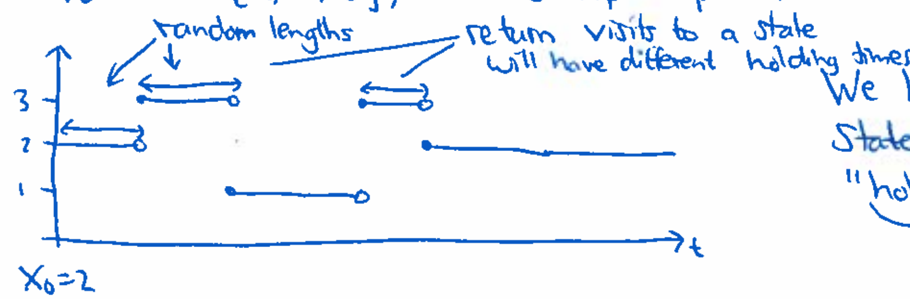
$$\frac{\lambda^2\varepsilon_1\varepsilon_2 e^{-\lambda a}}{e^{-\lambda a}(\lambda a)^2/2} = \frac{2\varepsilon_1\varepsilon_2}{a^2}$$

$$\Rightarrow f_{(S_1,S_2|N_a=2)}(s,t) = \lim_{\varepsilon_1\varepsilon_2\to 0} \frac{2\varepsilon_1\varepsilon_2}{a^2\varepsilon_1\varepsilon_2} = \frac{2}{a^2}.$$ ✓ The pdf of $(U_{(1)}, U_{(2)}$

7.1 Cont. Time MC's

Want a stochastic process $(X_t)_{t\geq 0}$ with state space S which jumps randomly from one state to another at random times.

For $S=\{1,2,3\}$, a sample path



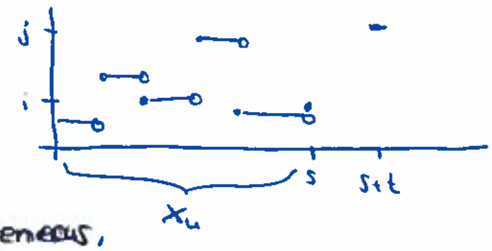
We have random jumps from state to state and random "holding times" in the states

$\underbrace{\text{"holding times"}}_{\text{inter-jump time}}$

Def. A continuous time stochastic process $(X_t)_{t\geq 0}$ with discrete state space is a cont. time MC if it has the Markov property,

$$P(X_{s+t}=j \mid X_s=i,\ X_u=x_u,\ 0\leq u<s)$$
$$= P(X_{s+t}=j \mid X_s=i)$$

where $s,t\geq 0$, and x_u is any given sample (path) of states up to time s $(x_u \in S)$



We assume the MC is time homogeneous,

so $P(X_{s+t} = j \mid X_s = i) = P(X_t = j \mid X_0 = i)$ for all $s, t \geq 0$.

This assumption allows us to define

$P_{ij}(t) = P(X_{s+t} = j \mid X_s = i) = P(X_t = j \mid X_0 = i)$.

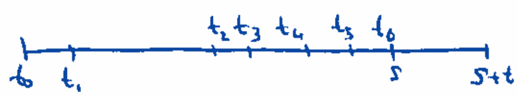
$\underline{\underline{P}}(t) = (P_{ij}(t))_{i,j \in S}$ is called the transition function (fct of t).

[Before $P_{ij} = P(X_1 = j \mid X_0 = i)$]

For every t, $\underline{\underline{P}}(t)$ is a stochastic matrix.

Note : $\underline{\underline{P}}(0) = \underline{\underline{I}}$ (identity matrix as before)

The Markov property is equivalent to:



For any n, any times $0 = t_0 < t_1 < t_2 < \ldots < t_n = s$

$P(X_{s+t} = j \mid X_s = i, X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = P(X_{s+t} = j \mid X_s = i)$.

Basic Problem:

Before, in discrete time, we defined our MC using a trans matrix $\underline{\underline{P}}$.

But now, we have no "single" $\underline{\underline{P}}$ to use.

(Take $\underline{\underline{P}}(1)$ to be our data, how do I find $P(X_{1/2} = j \mid X_0 = i)$? $P(X_{3/2} = j \mid X_0 = i)$?)

Fact: A Poisson process $(N_t)_{t \geq 0}$ (let's allow $N_0 \neq 0$) is a continuous time MC with transition fct (for $j \geq i$) [Use stationary increments]

$$P_{ij}(t) = P(N_t = j \mid N_0 = i) = P(N_t - N_0 = j - i) = \frac{e^{-\lambda t} (\lambda t)^{j-i}}{(j-i)!}$$

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Holding times:

Let T_i be the (first) holding time in state i.

If we start at state i, $T_i = \max\{ t : X_s = i \text{ for all } 0 \le s < t \}$.

$\{ t \mid X_s = 2 \text{ for all } 0 \le s < t \}$

For a rate λ Poisson process, T_i is exponential with parameter λ.

Claim: In general, T_i is an exponential rv with some parameter $q_i > 0$.

[In the discrete case, T_i was geometric
H_i: $P(H_i = k) = a(1-a)^k$]

[midterm #2]

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Continuous Time MC's

Reading - $\boxed{7.1,\ 7.2}$

Memoryless property: $\bullet\ P(X > s+t \mid X > s) = P(X > t)$ for all $s, t > 0$

Further properties of the exponential distribution (p. 230 - 231)
$\text{Expo}(\lambda)$, $P(X \le t) = 1 - e^{-\lambda t}$ or $P(X > t) = e^{-\lambda t}$ $(t > 0)$

(1) The exponential dist. ~~and~~ has the memoryless property and is the only dist. with the memoryless property (Exer. 6.11)

(2) Let $X_1, X_2, \ldots, X_n$ be independent exp. r.v.'s such that X_i has param. $\lambda_i > 0$, and let $M = \min\{X_1, X_2, \ldots, X_n\}$.

Then:

(a) M is an expon. r.v., parameter $\lambda = \lambda_1 + \ldots + \lambda_n$.

Pf: $P(M > t) = P(\min\{X_1, \ldots X_n\} > t) = P(X_1 > t, X_2 > t, \ldots, X_n > t)$

$= P(X_1 > t) P(X_2 > t) \cdots P(X_n > t) = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot \ldots \cdot e^{-\lambda_n t}$

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$= e^{-(\lambda_1 + \dots + \lambda_n)t}$ ✓ expo, param. $\lambda = \lambda_1 + \dots + \lambda_n$

(b) $P(M = X_k) = \dfrac{\lambda_k}{\lambda_1 + \dots + \lambda_n}$

↑

X_k is the minimum of the X_i

Pf: Take $k=1$. $P(M = X_1) = P(X_1 < X_2, X_1 < X_3, \dots, X_1 < X_n)$

[Note: If the rv's were discrete, could write

$$P(X_1 < X_2, X_1 < X_3) = \sum_{k=0}^{\infty} P(X_1 < X_2, X_1 < X_3 \mid X_1 = k) P(X_1 = k)$$

LOTP ← discrete

$$= \sum_{k=0}^{\infty} P(k < X_2, k < X_3 \mid \cancel{X_1 = k}) P(X_1 = k)$$

independence

$$= \sum_{k=0}^{\infty} P(k < X_2) P(k < X_3) P(X_1 = k)$$]

$$P(M = X_1) = P(X_1 < X_2, X_1 < X_3, \dots, X_1 < X_n)$$

$$= \int_0^{\infty} P(X_1 < X_2, \dots, X_1 < X_n \mid X_1 = t) f_1(t)\, dt$$

↑ pdf of X_1

Continuous law of total probability

$$= \int_0^{\infty} P(t < X_2, t < X_3, \dots, t < X_n \mid \cancel{X_1 = t}) f_1(t)\, dt$$

$$= \int_0^{\infty} P(t < X_2, \dots, t < X_n) f_1(t)\, dt = \dots$$

[$f_{Y|X=x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ $\Rightarrow f_{X,Y}(x,y) = f_{Y|X=x}(y) f_X(x)$

$\Rightarrow P((X,Y) \in A) = \iint_A f_{Y|X=x}(y) f_X(x)\, dx.$]

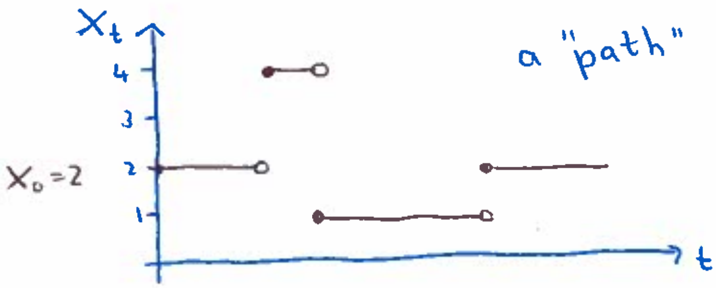
(↑ in $\mathbb{R}^2$)

$$\dots \overset{\text{indep}}{=} \int_0^{\infty} P(X_2 > t) \cdots P(X_n > t) \lambda_1 e^{-\lambda_1 t}\, dt$$

$$= \lambda_1 \int_0^\infty e^{-\lambda_2 t} e^{-\lambda_3 t} \cdot \ldots \cdot e^{-\lambda_n t} e^{-\lambda_1 t} dt = \lambda_1 \int_0^\infty e^{-\underbrace{(\lambda_1 + \ldots + \lambda_n)}_{=:\lambda} t} dt$$

$$= \frac{\lambda_1}{\lambda} \int_0^\infty \lambda e^{-\lambda t} dt = \frac{\lambda_1}{\lambda} \cdot 1 = \frac{\lambda_1}{\lambda_1 + \ldots + \lambda_n} \quad \checkmark$$

Continuous $(X_t)_{t \geq 0}$, finite state space $S = \{1, 2, 3, 4\}$

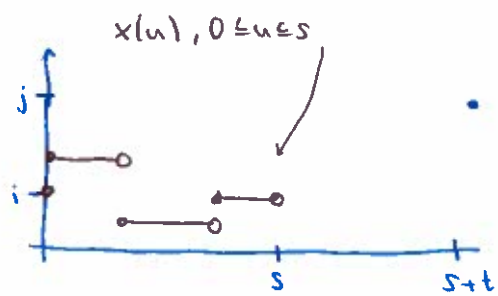


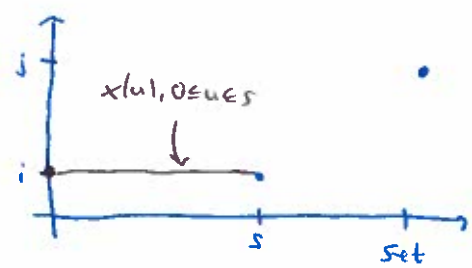
a "path"

<u>Markov Property (I)</u>: Given a path $(x(u))_{0 \leq u \leq s}$,

$$P(X_{s+t} = j \mid X_s = i, \text{ and } X_u = x(u) \text{ for all } 0 \leq u \leq s)$$

$$= P(X_{s+t} = j \mid X_s = i) \underset{\text{time hom.}}{=} P(X_t = j \mid X_0 = i)$$





<u>Markov Property II</u>: Given times $0 = t_0 < t_1 < t_2 < \ldots < t_k < s$ and states $i_1, i_2, \ldots, i_k, i, j$,

$$P(X_{s+t} = j \mid X_{t_j} = i_j, X_s = i) = P(X_{s+t} = j \mid X_s = i) = P(X_t = j \mid X_0 = i)$$

Define $P_{ij}(t) = P(X_t = j \mid X_0 = i)$ all $i, j \in S$, all $t \geq 0$

$\underline{\underline{P}}(t) = (P_{ij}(t))_{i,j \in S}$, all $t \geq 0$.

<u>Chapman–Kolmogorov Equations</u>

For all $s, t \geq 0$, $\underline{\underline{P}}(t+s) = \underline{\underline{P}}(t) \overset{\text{matrix mult.}}{\cdot} \underline{\underline{P}}(s) \quad (= \underline{\underline{P}}(s) \cdot \underline{\underline{P}}(t))$

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<u>Pf:</u> $P_{ij}(s+t) \overset{\text{def}}{=} P(X_{s+t}=j \mid X_0=i) \overset{\text{additivity}}{=} \sum_{k\in S} \underbrace{P(X_{s+t}=j, X_s=k, X_0=i)}_{\substack{= P(X_{s+t}=j \mid X_s=k,\, X_0=i) \\ \cdot\, P(X_s=k \mid X_0=i)}}$

$\cdot\ \cancel{P(X_s=k / X_0=i)}$

$$= \sum_{k\in S} P(X_{t+s}=j / X_s=k, X_0=i)\, P(X_s=k \mid X_0=i)$$

[Mult. form.: $P(A\cap B) = P(A \mid B)\,P(B)$
$P(A\cap B \mid C) = P(A \mid B\cap C)\,P(B \mid C)$]

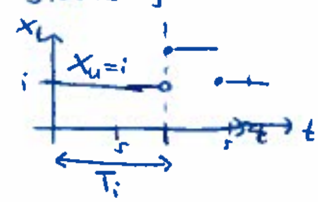
$$\overset{\text{Markov}}{=} \sum_{k\in S} P(X_{t+s}=j \mid X_s=k) \cdot P_{ik}(s)$$

$$= \sum_{k\in S} P_{kj}(t)\cdot P_{ik}(s) = \sum_{k\in S} P_{ik}(s)\,P_{kj}(t) = (\underline{P}(s)\cdot\underline{P}(t))_{ij}.$$

We have shown: $P_{ij}(t+s) = (\underline{P}(s)\cdot\underline{P}(t))_{ij}$
$\quad\quad \|$
$(\underline{P}(t+s))_{ij}$ $\quad\Rightarrow \underline{P}(t+s) = \underline{P}(s)\cdot\underline{P}(t).$ □

<u>Def:</u> The holding time in state i (starting in state i) is

$$T_i = \underset{(\sup)}{\max}\{ s \mid X_u = i \text{ for all } 0 \le u < s \}.$$



or, to say $\quad T_i = s$ means $\begin{cases} X_u = i & \text{for all } 0\le u < s \\ X_s \ne i \end{cases}$

<u>Prop:</u> Each T_i is an exponential r.v., call its parameter q_i.

<u>Pf:</u> Will show T_i has the memoryless property.

For $s,t>0$, $P(\cancel{X_{t+s}}\ T_i > s+t \mid T_i > s, \cancel{\}}\ X_0 = i)$

$$= \frac{P(T_i > s+t, \cancel{T_i} > s \mid X_0=i)}{P(T_i > s \mid X_0 = i)}$$

By the def. of T_i, for any number $\cancel{\omega}\ s$,

$$\{T_i > s\} = \{X_u = i \text{ for all } 0 \le u \le s\}$$

$\Rightarrow P(T_i > s+t, T_i > s \mid X_0=i) = P(T_i > s+t, X_u = i \text{ for all } 0\le u\le s \mid X_0 = i)$

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$= P(T_i > t+s \mid X_u = i \text{ for all } 0 \le u \le s) \cdot P(X_u = i \text{ for all } 0 \le u \le s \mid X_0 = i)$

Markov? $= P(T_i > s+t \mid X_s = i)\, P(X_u = i \text{ for all } 0 \le u \le s \mid X_0 = i)$

$= P(T_i > s+t \mid X_s = i) \cdot P(T_i > s \mid X_0 = i)$

time homog? $= P(T_i > t \mid X_0 = i)\, P(T_i > s \mid X_0 = i)$

$$\Big[\Rightarrow P(T_i > s+t \mid T_i > s, X_0 = i) = \frac{P(T_i > t \mid X_0 = i)\, P(T_i > s \mid X_0 = i)}{P(T_i > s \mid X_0 = i)}$$

$$= P(T_i > t \mid X_0 = i) \Big]$$

[Let $g(s) = P(T_i > s \mid X_0 = i)$
is $g(s+t) = g(t)g(s)$ all $s,t > 0$.]

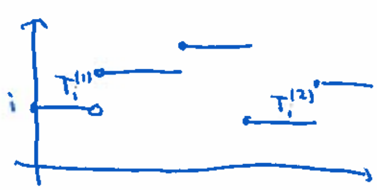
~~This is the~~

We have shown $P(T_i > s+t \mid X_0 = i) = P(T_i > t \mid X_0 = i)\, P(T_i > s \mid X_0 = i)$

This is the (conditional) memoryless property, so T_i must be exponential, same parameter q_i.

$g(t) = e^{\lambda t}$, same λ.

<u>Fact:</u>

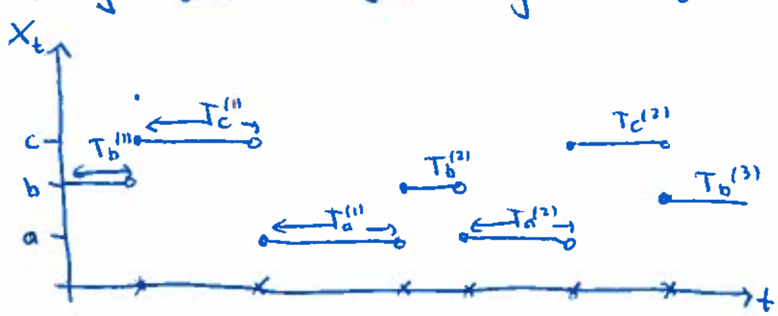


All successive times are exponential.

<u>Nov 30</u>

Continuous Time MC $(X_t)_{t \ge 0}$

T_i = holding time in state i given $X_0 = i$ $= \sup\{t \mid X_s = i \text{ for all } s < t\}$



Let $T_i^{(1)}, T_i^{(2)}, T_i^{(3)}, \ldots$ be the successive holding times in state i

Nov 30

Thm: All holding times are exponential rv's, they are all independent of one another. For each i, there is a $q_i > 0$ such that each $T_i^{(n)}$, $n = 1, 2, \ldots$ are exponential with parameter q_i.

[$T_b^{(1)}$ has ~~They have~~ the memoryless property as a consequence of the Markov property.]

Mathematical Issues

Discrete time chains, $\underline{\underline{P}}$, $v(i) = E(R_i \mid X_0 = i)$, $i \in S = \{1, \ldots, n\}$ where $R_i = \min\{k \geq 1 : X_k = i\}$

the 1st step analysis equations are:

$$v(i) = 1 + \sum_{j \neq i} P_{ij} v_j, \quad \text{each } i \in S: \quad n \text{ linear equations in } n \text{ unknowns } v(1), v(2), \ldots, v(n).$$

Questions: ① How do we know these equations have a solution?
(can compute $v(i)$, fulfill the eqn's)

② How do we know there cannot be two soln's?
(not clear for all)

Given MC $(X_t)_{t \geq 0}$, $\underline{\underline{P}}(t)_{t \geq 0}$, $P_{ij}(t) = P(X_t = j \mid X_0 = i)$.

- Holding Times T_i exp. param. q_i ← Thm
- Embedded discrete time Markov chain $(Y_n, n = 0, 1, 2, \ldots)$

in ex.: $Y_0 = b$, $Y_1 = c$, $Y_2 = a$, $Y_3 = b$, $Y_4 = a$, $Y_5 = c$, $Y_6 = b, \ldots$

with some transition matrix $\underline{\underline{\tilde{P}}}$, $\tilde{P}_{ij} = P(Y_1 = j \mid Y_0 = i)$.

$\tilde{P}$ stoch. with diag. entries 0

Continuous time chain $\rightarrow \begin{cases} q_i \\ \tilde{P}_{ij} \end{cases}$

Example: $S = \{a, b\}$

$$\underline{\underline{\tilde{P}}} = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

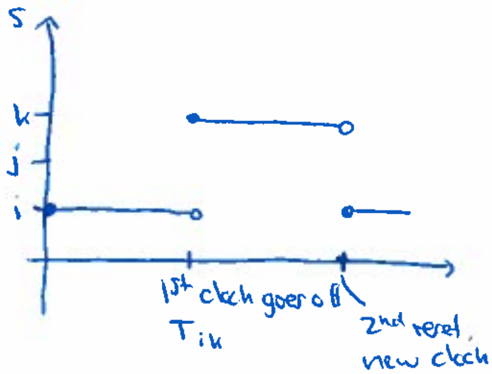
Probl. of Def'n. don't have $\underline{P}(t)$.

Second description:

"build" MC out of given "alarm clock rates" (q_{ij}), $i \neq j$, $i, j \in S$,

each $q_{ij} \geq 0$ if $i \neq j$. def'n $q_{ii} = 0$.

Given (q_{ij}) Fix i, $(T_{ij})_{j \in S}$



1. The first jump.
   - Fix i, put $X_0 = i$ (also $Y_0 = i$)
   - $\forall j$ with q_{ij}, imagine an "alarm clock" which goes off after a random time $T_{ij} \sim Exp(q_{ij})$
   - If the first clock to go off is Clock k ($T_{ik} < T_{ij}$ all $j \neq k$), then the chain jumps to state k (put $Y_1 = k$)
2. ~~Sec~~ 2nd jump
   - 1st jump to state $k \rightsquigarrow$ consider new alarm clocks all set to 0 (indep. $T_{kj}' \sim Exp(q_{kj})$)
   - repeat prev. process
3. repeat again and again...

We call q_{ij} the (instantaneous) rate of jumping from i to j.

the (instantaneous) of making a jump <u>away from i</u> is $\sum_{j \neq i} q_{ij}$.

Back to first jump, fixed i, this time is $\min\{\text{all } T_{ij}, j \in S\}$

$\uparrow$ independent exp. r.v.

By Prop. last time, this is a exponential,

parameter $\sum_{j \neq i} q_{ij}$.

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- Start off at state i, $X_0 = i$, set all clocks to 0
- If clock (i,a) is the 1st clock that rings, and it rings at time t_1, def'n
$$X_t = \begin{cases} i & \text{for } 0 \le t < t_1 \\ a & t = t_1 \end{cases}$$
- (a,b) next clock that rings, at time t_2, def'n $X_t = \begin{cases} a & t_1 \le t_2 < t_2 \\ b, & t = t_2 \end{cases}$

...

Thm: $(X_t)_{t \ge 0}$ has the Markov property
(Relies on Memoryless property!)

Def: Given (q_{ij}), def'n the associated generator to be the matrix $\underline{Q} = (Q_{ij})_{i,j \in S}$

where $Q_{ij} = \begin{cases} q_{ij} & \text{if } j \ne i \\ -\sum_{j \ne i} q_{ij} & \text{if } j = i \end{cases}$ [rows sums of $\underline{Q}$ all 0!]
& neg. diagonal ent's

Ex: $S = \{a, b, c\}$

$$\underline{Q} = \begin{array}{c|ccc} & a & b & c \\ \hline a & -(q_{ab}+q_{ac}) & q_{ab} & q_{ac} \\ b & q_{ba} & -(q_{ba}+q_{bc}) & q_{bc} \\ c & q_{ca} & q_{cb} & -(q_{ca}+q_{cb}) \end{array}$$

Comparison of 2 definitions

(I) Holding times parameters q_i, discrete time jump matrix $\tilde{P}_{ij}$

(II) q_{ij}

What are q_i, $\tilde{P}_{ij}$ for (II)?

1st jump time is exponential, minimum of (i,j) clocks, so $\text{Expo}(\sum_{j \ne i} q_{ij})$
min{all (i,j), $j \in S$, clocks}

$q_i = \sum_{j \ne i} q_{ij}$ (the 1st jump time is the same as the 1st holding time
$$\underbrace{\text{1st holding time}}_{\text{Expo}(q_i)}$$

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How about $\tilde{P}_{ik} = P(Y_1 = k \mid Y_0 = i)$? Given (q_{ij})

$= P((i,k)$ rings ~~first before the~~ first of the (i,j) $j \in S)$

$= P(T_{ik} < T_{im_j},\ j \neq k) = P(T_{ik} = \min\{T_{ij},\ j \in S\})$

$$= \frac{q_{ik}}{\sum_{j \neq i} q_{ij}}$$

That is, given (q_{ij}), $q_i = \sum_{j \neq i} q_{ij}$

$$\tilde{P}_{ik} = \frac{q_{ik}}{\sum_{j \neq i} q_{ij}} = \frac{q_{ik}}{q_i}.$$

$\tilde{P}_{ii} = 0 \ \forall i \quad (q_{ii} = 0)$

Thm: $\underline{\underline{Q}} = \underline{\underline{P}}'(0) = \underbrace{\frac{d}{dt}}_{(P_{ij}'(t))_{i,j \in S}} \underline{\underline{P}}(t)\Big|_{t=0}$ or $q_{ij} = P_{ij}'(0)$, all i,j

Proof: Recall Chapman-Kolmogorov eqn's,

$$\underline{\underline{P}}(t+s) = \underline{\underline{P}}(s)\underline{\underline{P}}(t) = \underline{\underline{P}}(t)\underline{\underline{P}}(s).$$

$$P_{ij}'(0) = \lim_{h \to 0} \frac{P_{ij}(h) - P_{ij}(0)}{h}$$

① Suppose $i \neq j$, then $P_{ij}(0) = 0$ $(=1$ for $i=j)$ $[\underline{\underline{P}}(0) = \underline{\underline{I}}]$

In this case, $P_{ij}'(0) = \lim_{h \to 0} \frac{P_{ij}(h)}{h}$

For small h, $P_{ij}(h) = P(X_h = j \mid X_0 = i) = P(X_h = j$, only one jump in $[0,h] \mid X_0 = i) + P(X_h = j, \geq 2$ jumps in $[0,h] \mid X_0 = i)$

2nd term: $\leq P(\underset{\uparrow}{T_i} + \underbrace{\min\{T_{il}', l \in S\}} \leq h \mid X_0 = i)$

$P(\text{Expo}(q_i) + \text{Expo}(\sum_j q_j) \leq h)$

Fact: If Z_1, Z_2 are independent exponential rv's, parameters λ_1, λ_2
$\Rightarrow P(Z_1 + Z_2 \leq h) = o(h)$.

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Given this, $P(X_h=j, \geq$ two jumps in $[0,h] \mid X_0=i) = o(h)$ as $h \to 0$.

Recall for the Poisson process $(N_t)_{t\geq 0}$ rate λ $\quad P(\underset{\substack{\uparrow \\ \text{two or more jumps}}}{N_h \geq 2}) = o(h)$

Dec 5

CTMC $(X_t)_{t\geq 0}$ $\quad$ Finite state space $|S|=N$

trans. fct $\underline{\underline{P}}(t)$, $P_{ij}(t) = P(X_t=j \mid X_0=i)$, $\underline{\underline{P}}'(t) = (P'_{ij}(t))$

independent:

holding times T_i, exponential, parameter $q_i > 0$

embedded chain $(Y_n)_{n=0,1,\ldots}$ trans. matrix $\underline{\underline{\tilde{P}}}$

Clock description: $(q_{ij})_{i\neq j}$, $\quad (\underline{\underline{Q}})_{ij} = \begin{cases} q_{ij}, & i\neq j \\ -\sum_{j\neq i} q_{ij} & j=i \end{cases}$ $\longrightarrow Q_{ii} = -q_i$

Generator

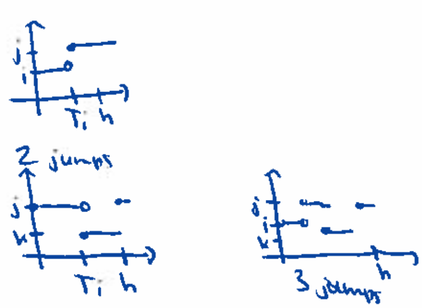
$q_i = \sum_{j\neq i} q_{ij}$, $\quad \tilde{P}_{ij} = \dfrac{q_{ij}}{q_i}$

Thm: $\underline{\underline{P}}'(0) = \underline{\underline{Q}}$.

Sketch of pf: $\quad$ For $i\neq j$, $P'_{ij}(0) = \lim_{h\to 0} \dfrac{P_{ij}(h) - \overset{0}{P_{ij}(0)}}{h}$

$P_{ij}(h) = P(X_h=j \mid X_0=i) = P(X_h=j, \geq 2 \text{ jumps in } [0,h] \mid X_0=i)$

$\quad + P(X_h=j, 1 \text{ jump in } [0,h] \mid X_0=i)$



$= o(h) + P(T_i < h, \text{one } \cancel{\text{jump}}, \tilde{Y}_1 = j \mid X_0=i)$

$= o(h) + [P(T_i<h, \tilde{Y}_1 = j \mid X_0=i) + o(h)]$

$= P(T_i<h, \tilde{Y}_1=j \mid X_0=i) + o(h)$

$= P(T_i<h \mid X_0=i)\, P(\tilde{Y}_1=j \mid X_0=i) + o(h)$

Fact: $1-e^{-\lambda h} = \lambda h + o(h)$ (use l'Hospital's Rule)

$= (1-e^{-q_i h}) \cdot \dfrac{q_{ij}}{q_i} + o(h)$

$= (q_i h + o(h)) \cdot \dfrac{q_{ij}}{q_i} + o(h) = \cancel{q_i h + o(h)} \; q_{ij} h + o(h).$

Dec 5

So, $P_{ij}'(0) = \lim_{h\to 0} \frac{q_{ij} h + o(h)}{h} = \lim_{h\to 0} \left[q_{ij} + \frac{o(h)}{h}\right] = q_{ij} + 0$ ✓

This shows $P_{ij}'(0) = q_{ij}$, $i \neq j$.

The case $j=i$ is easier. "◡̈"

<u>Thm</u> (p. 275): $\underline{\underline{Q}}$ determines $\underline{\underline{P}}(t)$, $t \geq 0$ by either of:

$\underline{\underline{P}}'(t) = \underline{\underline{P}}(t)\, \underline{\underline{Q}}$ (forward eqn's)

$\underline{\underline{P}}'(t) = \underline{\underline{Q}}\, \underline{\underline{P}}(t)$ (backward eqn's)

Each of these is a system of N^2 coupled differential equations.
[$|\mathcal{S}| = N$, $\mathcal{S}$ state space for CTMC]

In component form,
$$P_{ij}'(t) = \sum_k P_{ik}(t)\, Q_{kj}$$
$$\cancel{= P_{ii}(t)}$$
$$= -q_j P_{ij}(t) + \sum_{k\neq j} P_{ik}(t)\, Q_{kj}$$

$k=j$ term: $Q_{jj} = -q_j$

<u>Proof</u>: Use C-K eqns, $\underline{\underline{P}}(s+t) = \underline{\underline{P}}(s)\,\underline{\underline{P}}(t)$, or $\underline{\underline{P}}(t+h) = \underline{\underline{P}}(h)\cdot\underline{\underline{P}}(t)$
$= \underline{\underline{P}}(t)\cdot\underline{\underline{P}}(h)$

$$\Rightarrow \underline{\underline{P}}'(t) = \lim_{h\to 0} \frac{\underline{\underline{P}}(t+h) - \underline{\underline{P}}(t)}{h} = \lim_{h\to 0} \frac{\underline{\underline{P}}(h)\underline{\underline{P}}(t) - \underline{\underline{P}}(t)}{h}$$

$$= \lim_{h\to 0} \frac{(\underline{\underline{P}}(h) - \underline{\underline{I}})\, \underline{\underline{P}}(t)}{h} \qquad \underline{\underline{I}} = \underline{\underline{P}}(0)$$

$$= \lim_{h\to 0} \left(\frac{\underline{\underline{P}}(h) - \underline{\underline{P}}(0)}{h}\right) \underline{\underline{P}}(t) = \underline{\underline{P}}'(0)\, \underline{\underline{P}}(t) \overset{\text{thm}}{=} \underline{\underline{Q}}\, \underline{\underline{P}}(t)$$

This is the system of backward equations.

<u>Example</u>: 2 states $\quad ① \underset{\lambda}{\overset{\mu}{\rightleftarrows}} ② \qquad \begin{array}{c|cc} & 1 & 2 \\ \hline \end{array}$
$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \underline{\underline{Q}} = \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

(FE) Write $\underline{P_{11}'(t)}$ $(i=j=1)$
$= -q_1 P_{11}(t) + P_{12}(t)\, q_{21} = \underline{-\lambda P_{11}(t) + \mu P_{12}(t)}$

Dec 15

$$\underline{P}(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}$$

(BE) $\underline{P'_{11}(t)} = \sum_{k=1}^{2} q_{1k} P_{k1}(t) = q_{12} P_{21}(t) + q_{11} P_{11}(t)$

$$= \underline{-\lambda P_{11}(t) + \lambda P_{21}(t)}$$

In (FE) can use $P_{11}(t) + P_{12}(t) = 1$

$$\rightsquigarrow \underline{P_{11}'(t)} = -\lambda P_{11}(t) + \mu P_{12}(t)$$

$$= -\lambda P_{11}(t) + \mu (1 - P_{11}(t))$$

$$= \underline{P_{11}(t)(-(\lambda+\mu)) + \mu}$$

$$P_{11}(0) = 1$$

Let $y = y(t) = P_{11}(t)$

$$y' = -(\lambda+\mu) y + \mu$$

$$y' + (\lambda+\mu) y = \mu, \qquad y(0) = 1.$$

1st order, linear, constant coeff. diff. eqn.

<u>Solution</u> if $\mu = 0$: $y = C_1 e^{-(\lambda+\mu)t} + C_2$, C_1, C_2 are constants

Check: $y' = -C_1(\lambda+\mu) e^{-(\lambda+\mu)t} + 0$

$$y' + (\lambda+\mu) y = -C_1 \cancel{(\lambda+\mu)} e^{-(\lambda+\mu)t} + (\lambda+\mu)[\cancel{C_1 e^{-(\lambda+\mu)t}} + C_2]$$

$$= (\lambda+\mu) C_2 \stackrel{!}{=} \mu \qquad \Rightarrow C_2 = \frac{\mu}{\lambda+\mu}$$

$$\Rightarrow y = C_1 e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}$$

Set $t = 0$, $y(0) = C_1 \cdot 1 + \frac{\mu}{\lambda+\mu} \stackrel{!}{=} 1 \Rightarrow C_1 = 1 - \frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}$

$$\Rightarrow P_{11}(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}$$

$$\Rightarrow P_{12}(t) = -\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu}. \qquad (P_{11}(t) + P_{12}(t) = 1)$$

Dec 5

$$\underline{\underline{P}}(t) = \frac{1}{\lambda+\mu}\begin{bmatrix} \lambda e^{-(\lambda+\mu)t}+\mu & \lambda-\lambda e^{-(\lambda+\mu)t} \\ \mu-\mu e^{-(\lambda+\mu)t} & \lambda+\mu e^{-(\lambda+\mu)t} \end{bmatrix}$$

Note $\lim_{t\to 0} \underline{\underline{P}}(t) = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$

7.3 Long Term Behaviour

Def: A probability distribution $\underline{\pi}$ is the limiting distribution of $(X_t)_{t\geq 0}$ if $\lim_{t\to\infty} P_{ij}(t) = \pi_j$, all i,j.

A prob. dist. $\underline{\pi}$ is a stationary dist. if $\underline{\pi}\,\underline{\underline{P}}(t) = \underline{\pi}$ for all $t\geq 0$.

Thm 7.2

Let $(X_t)_{t\geq 0}$ be a finite state space, irreducible continuous time MC, with transition fct. $\underline{\underline{P}}(t)$.

Then there exists a unique stationary distribution $\underline{\pi}$ which is also the limiting dist. $\pi_j = \lim_{t\to\infty} P(X_t=j \mid X_0=i)$ for all i,j.

def'n of "irreducible"

discrete time: $\forall i,j$ $P^n_{ij} > 0$ for some n

cont. time: $\forall i,j$ $P_{ij}(t) > 0$ for some t (depending on i,j)

[discrete time: need "aperiodic" to get the thm] counter example: $\underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Lemma: If for some t, $P_{ij}(t) > 0$, then $P_{ij}(s) > 0$ for all $s > 0$.

(Periodicity is impossible [for CTMC]).

Example: $\underline{\underline{Q}} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$, $\pi_1 = \frac{\mu}{\lambda+\mu}$, $\pi_2 = \frac{\lambda}{\lambda+\mu}$. (limiting dist.)

Can check: $\underline{\pi}\,\underline{\underline{P}}(t) = \underline{\pi}$ for all $t > 0$.

Dec 5

Question: How do we find $\underline{\pi}$?

Thm: (p. 286)

A prob. dist. $\underline{\pi}$ is a stationary dist. for a MC with generator $\underline{\underline{Q}}$ iff $\underline{\pi}\,\underline{\underline{Q}} = 0$.

Proof: (a) Suppose $\underline{\pi}$ is stationary, then $\underline{\pi}\,\underline{\underline{P}}(t) = \underline{\pi}$ for all $t>0$.

Then $\underline{\pi}\,\underline{\underline{P}}(t) - \underline{\pi} = \underline{0} \quad \forall t>0$

$$\Leftrightarrow \underline{\pi}\,(\underline{\underline{P}}(t) - \underline{\underline{I}}) = \underline{0} \quad \forall t>0 \qquad [\underline{\underline{I}} = \underline{\underline{P}}(0)]$$

$$\Leftrightarrow \underline{\pi}\,(\underline{\underline{P}}(t) - \underline{\underline{P}}(0)) = \underline{0} \quad \text{for all } t>0$$

$$\Rightarrow \underline{\pi}\left(\frac{\underline{\underline{P}}(t) - \underline{\underline{P}}(0)}{t}\right) = \underline{0} \qquad \text{— " —}$$

Let $t \to 0$, get $\underline{\pi}\,\underline{\underline{P}}'(0) = \underline{0}$ or $\underline{\pi}\,\underline{\underline{Q}} = 0$.

(b) Suppose $\underline{\pi}\,\underline{\underline{Q}} = \underline{0}$. Then.

$$\underline{0} = \underline{\pi}\,\underline{\underline{Q}} \quad \Rightarrow \underline{0} = \underline{0}\,\underline{\underline{P}}(t) = \underline{\pi}\,\underline{\underline{Q}}\,\underline{\underline{P}}(t) \underset{BE}{=} \underline{\pi}\,\underline{\underline{P}}'(t) \quad \forall t>0$$

BE: $\underline{\underline{P}}'(t) = \underline{\underline{Q}}\,\underline{\underline{P}}(t)$

This implies $\underline{\pi}\,\underline{\underline{P}}(t)$ must be constant in t.

Put $t=0$, the constant must be $\underline{\pi}\,\underline{\underline{P}}(0) = \underline{\pi}\,\underline{\underline{I}} = \underline{\pi}$.

$$\Rightarrow \underline{\pi}\,\underline{\underline{P}}(t) = \underline{\pi} \quad \forall t>0, \text{ or } \underline{\pi} \text{ is stationary.} \quad \square$$

Dec 7

Fact: CTMS cannot be periodic.

Thm 7.2: --

π_j represents the long term expected proportion of time spent in state j,

$$\pi_j = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(X_t = j \mid X_0 = i)\,dt.$$

Dec 7

Lemma: If $P(X_t = j \mid X_0 = i) > 0$ for some $t > 0$ then

$P(X_s = j \mid X_0 = i) > 0$ for all $s > 0$.

Partial pf: Assume $P_{ij}(t) > 0$ for some t, fixed. Let $s > t$.

The idea to show $P_{ij}(s) > 0$ is . One way this can happen is to have $X_t = j$ and then have the chain stay at j for $(s-t)$ time.

$$P_{ij}(s) = (\underline{P}(t) \cdot \underline{P}(s-t))_{ij} = \sum_{k \in S} P_{ik}(t) P_{kj}(s-t) \geq P_{ij}(t) P_{jj}(s-t)$$

Take any state a, and any time u,

$$P_{aa}(u) = P(X_u = a \mid X_0 = a) \geq P(X_v = a \text{ for all } 0 \leq v \leq u \mid X_0 = a)$$

$$= P(T_a > u \mid X_0 = a) \underset{T_a \text{ expn. r.v.}}{=} e^{-q_a u}$$

$$\Rightarrow P_{ij}(s) \geq P_{ij}(t) e^{-q_j \cancel{(t-s)}^{(s-t)}} > 0$$

□

Thm: $\underline{\pi}$ is the stationary dist. if and only if $\underline{\pi}\,\underline{Q} = 0$.

($\underline{Q}$ is the generator.)

Example 7.14. A baby has 3 possible states:

e (eating), s (sleeping), p (playing).

The baby eats on average for $\frac{1}{2}$ hr, plays on average for 1 hr, and sleeps on av. 3 hrs. After eating, the baby has a 50% chance of sleeping or playing, after playing, there is a 50% chance of eating or sleeping, after sleeping, the baby always plays.

What proportion of the day does the baby spend sleeping? (π_s)

Let (X_t) be a cont. time MC, X_t = the state of the baby at time t, either e, s, p.

Dec 7

We want $\underline{\pi}$, so we need $\underline{\underline{Q}}$.

We know? q_e, q_p, q_s Assuming times spent in the states are exponential rv's.

q_e = parameter of holding time T_e $\cancel{=2}$ $= \frac{1}{1/2} = 2$. (expected value of an exponential rv. with param. λ is $\frac{1}{\lambda}$)

$q_p = \frac{1}{1} = 1$

$q_s = \frac{1}{3}$

We know $\underline{\underline{\tilde{P}}}$ embedded chain matrix (for Y_n)

$$\underline{\underline{\tilde{P}}} = \begin{array}{c} \\ e \\ p \\ s \end{array} \begin{array}{c} \begin{array}{ccc} e & p & s \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

[always 0 on diagonal]

Recall $\tilde{P}_{ij} = \frac{q_{ij}}{q_i}$ $(i \neq j)$ $\Rightarrow q_{ij} = q_i \tilde{P}_{ij}$

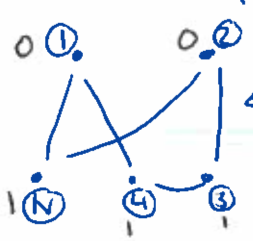
$$\underline{\underline{Q}} = \begin{array}{c} \\ e \\ p \\ s \end{array} \begin{array}{c} \begin{array}{ccc} e & p & s \end{array} \\ \begin{bmatrix} -2 & 1 & 1 \\ 1/2 & -1 & 1/2 \\ 0 & 1/3 & -1/3 \end{bmatrix} \end{array}$$

Solve $[\pi_e \ \pi_p \ \pi_s] \begin{bmatrix} -2 & 1 & 1 \\ 1/2 & -1 & 1/2 \\ 0 & 1/3 & -1/3 \end{bmatrix} = [0 \ 0 \ 0]$

$\Rightarrow \underline{\pi} = \left[\frac{1}{14} \ \frac{4}{14} \ \frac{9}{14}\right]$, $\pi_s = \frac{9}{14}$.

Contact Process

Simple model for "infection" on a graph, N vertices.



← edges joining "neighbors"

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ \underline{x} = (& 0, & 0, & 1, & 1, & 1) \end{array}$$

Dec 7

At each time unit, a vertex is either infected (1) or healthy (0). The state of the system $\underline{x} = (x(1), x(2), \ldots, x(N))$

where
$$x(i) = \begin{cases} 1 & \text{if vertex } i \text{ is infected} \\ 0 & \text{if } \text{-"-} \text{ -"- -"- healthy} \end{cases}$$

(Our state space consists of N-tuples of 1's + 0's)

Let X_t = the state of our system at time t

$= (X_t(1), X_t(2), \ldots, X_t(N))$.

Assume

① an infected vertex "recovers" independently of everything else at exponential rate 1 [Time to recover is an expon. rv, param. 1]

② We have a parameter $\lambda > 0$ such that:

a healthy vertex becomes infected at expon. rate $\lambda \cdot$ (# of infected neighbors)

③ Only a single vertex can change at an instant. (flip from 0 to 1 or 1 to 0)

Current State

$(0,0,1,1,0) \to (0,0,0,1,0)$ rate 1

$\downarrow$

Next state could be $(0,0,1,1,1)$ at rate $\lambda \cdot 1$

Notation: Given $x = (x(1), x(2), \ldots x(N))$

define x^i the same as x except coordinate i is flipped

$$x^i(j) = \begin{cases} x(j) & \text{if } j \neq i \\ 1 - x(i) & \text{if } j = i \end{cases}$$

Define $\underline{Q}$ for (X_t) by:

for $x = (x(1), \ldots, x(N))$, $y = (y(1), \ldots, y(N))$

Dec 7

$$q_{xy} = \begin{cases} 1 & \text{if } y = x^i \text{ and } x(i) = 1 \\ \lambda \cdot \#\text{of infected neighbors of } i \text{ in } x & \text{if } y = x^i \text{ and } x(i) = 0 \end{cases} \quad \text{for any } i$$

This defines $\underline{Q}$, and therefore also $(X_t)_{t \geq 0}$.

<u>Question:</u> What happens as $t \to \infty$?

① nothing if $x_0 = (0,0,\dots,0)$

② Take $X_0 \neq (0,0,\dots,0)$

Take the graph to be $\mathbb{Z}$

~~neares~~ nearest neighbor case.

<u>Thm:</u> Start with a finite nr of infected individuals. There is a critical value λ_c, $0 < \lambda_c < \infty$, such that:

subcritical case: if $\lambda < \lambda_c$, then X_t becomes extinct ($X_t = 0$) eventually

supercritical case: if $\lambda > \lambda_c$, then X_t survives for all time with positive probability

critical case: if $\lambda = \lambda_c$

λ_c unknown, but believed to be ≈ 1.65 (Simulation)

Rigorous: $1 \leq \lambda \leq 2$.

<u>Final Exam:</u>

- Friday Dec 15, 8–10 am, 115 Carnegie
- Approx. $\frac{2}{3}$ will be devoted to last $\frac{1}{3}$ of course
  $\frac{1}{3}$ will approx. uniform over the whole course
- All HW solutions are now in the library
- Graded HW set 10 – will be outside my office starting at noon Monday

BEEZ

Appendix B (B.1, B.2, B.3)

- $F(x) = P(X \le x)$ cumulative distr. fct (cdf) of X

- X random var., discrete r.v. if values in a finite / countably inf. set

  $P(X=x)$ prob. mass fct (pmf) of X — joint pmf of X and Y: $P(X=x, Y=y)$

  $R \subset \mathbb{R} \rightsquigarrow P(X \in R) = \sum_{x \in R} P(X=x)$

  expectation/mean: $E(X) = \sum_x x P(X=x)$

  $Y = g(X) \rightarrow E(Y) = \sum_x g(x) P(X=x) = \sum_y y P(Y=y)$

  $E(aX+b) = aE(X)+b$, $E(X+Y) = E(X) + E(Y)$

  variance: $Var(X) = E((X-E(X))^2) = E(X^2) - (E(X))^2$

  $Var(aX+b) = a^2 Var(X)$ $\quad Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

  standard deviation: $SD(X) = \sqrt{Var(X)}$

  covariance: $Cov(X,Y) = E(XY) - E(X)E(Y)$

  correlation: $Corr(X,Y) = \frac{Cov(X,Y)}{SD(X)SD(Y)}$.

events A, B independent: $P(A \cap B) = P(A)P(B)$

$\Leftrightarrow P(X=x, Y=y) = P(X=x)P(Y=y) \quad \forall x,y$

X, Y indep. $\Rightarrow E(X \cdot Y) = E(X) \cdot E(Y) \Rightarrow Cov(X,Y) = 0$

Cts r.v. $\rightsquigarrow P(X=x) = 0 \quad \forall x \in \mathbb{R} / (0,\infty) / (a,b)$

$\rightsquigarrow$ prob. density fct f: $f \ge 0$, $\int_{-\infty}^{\infty} f = 1$, $P(X \in R) = \int_R f\,dt$, $R \subseteq \mathbb{R}$

cdf: $F(X) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \rightsquigarrow F'(x) = f(x)$

Ex: $f(x) = cx^2$, $0 < x < 3$ $\rightsquigarrow 1 \overset{!}{=} \int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 cx^2 = 9c \Rightarrow c = \frac{1}{9}$

$P(1 < X < 2) = \int_1^2 f(x)\,dx = \frac{7}{27}$

$Y = X^2$ density fct? $F(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y})$

$\rightsquigarrow f_Y(y) = \frac{d}{dy} P(X \le \sqrt{y}) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) = \frac{1}{2\sqrt{y}} \frac{y}{9} = \frac{\sqrt{y}}{18}$, $0 < y < 9$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$, $\quad Var(X) = \int_{-\infty}^{\infty} (x-E(X))^2 f(x)\,dx$

X, Y cts $\rightsquigarrow$ joint density of X, Y: $f(x,y)$ with $P((X,Y) \in R) = \iint_R f(x,y)\,dx\,dy$, $R \subseteq \mathbb{R}^2$

MAT 526
Little oh notation.

Definition
For a function f to say $\boxed{f(x) \text{ is little oh of } x \text{ as } x \text{ goes to } 0}$ means that $\boxed{\lim_{x\to 0} \frac{f(x)}{x} = 0}$.

For this we write

$$f(x) = o(x)) \text{ as } x \to 0$$

More generally, for two functions f, g,

$$f(x) = o(g(x)) \text{ as } x \to a \text{ means that } \lim_{x\to a} \frac{f(x)}{g(x)} = 0$$

and

$$g(x) = f(x) + o(x) \text{ means } g(x) - f(x) = o(x)$$

Intuitively: The informal interpretaton of "$f(x) = o(x)$ as x approaches 0" is:

$f(x)$ is much smaller than x or is negligible compared to x when x is small.

Example (Exercise)
Determine which of the following statements is true. Hint: Use L'Hôpital's Rule.

(a) $x^2 = o(x)$ as $x \to 0$ ✓

(b) $\sqrt{x} = o(x)$ as $x \to 0$ NO

(c) $\sin x = o(x)$ as $x \to 0$ NO

(d) $\cos(x) = 1 + o(x)$ as $x \to 0$ ✓

(e) $e^{2x} = 1 + 2x + o(x)$ as $x \to 0$ ✓

Here is some numerical evidence. For "x small" consider $x = .01$ and $x = .0001$.

$f(x)$	x^2	$\sqrt{x}$	$\sin x$	$1 - \cos x$	$e^{2x} - 1 - 2x$
$\left.\frac{f(x)}{x}\right\|_{x=.01}$	0.0100000	10.0000	0.999983	0.00499996	0.0201340
$\left.\frac{f(x)}{x}\right\|_{x=.0001}$	0.000100000	100.000	0.9999999983	5.00000×10^{-6}	0.000200013

First-Step Analysis Equations

Let (X_n) be a Markov chain with finite state space $\mathcal{S}$ with transition matrix $\underline{\underline{\mathrm{P}}}$. For $a \in \mathcal{S}$ define

$$H_a = \begin{cases} +\infty & \text{if } X_n \neq a \text{ for all } n \geq 0 \\ \min\{n \geq 0 : X_n = a\} & \text{otherwise} \end{cases}$$

$$R_a = \begin{cases} +\infty & \text{if } X_n \neq a \text{ for all } n \geq 1 \\ \min\{n \geq 1 : X_n = a\} & \text{otherwise} \end{cases}$$

Note that if $X_0 = i \neq a$ then $H_a = R_a$.

Theorem (1). *Fix a state a, and for $i \in \mathcal{S}$ define*

$$u(i) = \mathrm{P}(H_a < \infty \mid X_0 = i) \tag{1}$$

$$v(i) = \mathrm{E}(R_a \mid X_0 = i) \tag{2}$$

Then

$$u(i) = P_{ia} + \sum_{j \neq a} P_{ij}\, u(j), \qquad \textit{all } i \neq a \tag{3}$$

$$v(i) = 1 + \sum_{j \neq a} P_{ij}\, v(j), \qquad \textit{all } i \tag{4}$$

Theorem (2). *Fix a set of states B and define*

$$H_B = \min\{H_a, a \in B\} \quad \textit{and} \quad R_B = \min\{R_a, a \in B\},$$

and

$$u_B(i) = \mathrm{P}(H_B < \infty \mid X_0 = i\} \quad \textit{and} \quad v_B(i) = \mathrm{E}(R_B \mid X_0 = i)$$

Then

$$u_B(i) = \sum_{j \in B} P_{ij} + \sum_{j \notin B} P_{ij}\, u_B(j), \qquad \textit{all } i \cancel{\neq a} \notin B$$

$$v_B(i) = 1 + \sum_{j \notin B} P_{ij}\, v_B(j), \qquad \textit{all } i$$

Remark. Theorem (1) is a special case of Theorem (2) (take $B = \{a\}$).

Exercise 1. Prove (3) using equation (5) below, and the fact that

$$u(i) = \mathrm{P}(H_a < \infty \mid X_0 = i) = \lim_{k \to \infty} \mathrm{P}(H_a \leq k \mid X_0 = i)$$

Hint. Apply the "complement rule" in (5).

Exercise 2. Prove (4) using equation (6) below and the fact (see Lemma on Homework Set #5),

$$\mathrm{E}(R_a \mid X_0 = i) = \sum_{k=0}^{\infty} \mathrm{P}(R_a > k \mid X_0 = i) = 1 + \sum_{k=1}^{\infty} \mathrm{P}(R_a > k \mid X_0 = i)$$

Hint: Interchange order of summation appropriately.

Lemma. *For all $k \geq 1$,*

$$P(H_a > k \mid X_0 = i) = \sum_{j \neq a} P_{ij}\, P(H_a > k-1 \mid X_0 = j) \qquad \textit{all } i \neq a \tag{5}$$

$$P(R_a > k \mid X_0 = i) = \sum_{j \neq a} P_{ij}\, P(R_a > k-1 \mid X_0 = j) \qquad \textit{all } i \tag{6}$$

Proof. Consider (5). By the definition of H_a, for $X_0 \neq a$,

$$\{H_a > k\} = \{X_m \neq a \text{ for all } 1 \leq m \leq k\}. \tag{7}$$

Summing in (7) over the possible states for X_m, $1 \leq m \leq k$, and using the "multiplication rule", we get

$$\begin{aligned} &P(H_a > k \mid X_0 = i) \\ &\quad = \sum_{x_1 \neq a} \sum_{x_2 \neq a} \cdots \sum_{x_k \neq a} P(X_1 = x_1, \ldots, X_k = x_k \mid X_0 = i) \\ &\quad = \sum_{x_1 \neq a} \sum_{x_2 \neq a} \cdots \sum_{x_k \neq a} P(X_2 = x_2, \ldots, X_k = x_k \mid X_1 = x_1, X_0 = i) P(X_1 = x_1 \mid X_0 = i) \\ &\quad = \sum_{x_1 \neq a} P_{ix_1} \Big[\sum_{x_2 \neq a} \cdots \sum_{x_k \neq a} P(X_2 = x_2, \ldots, X_k = x_k \mid X_1 = x_1, X_0 = i) \Big] \end{aligned} \tag{8}$$

By the Markov property and then time homogeneity, the probabilities in (8) are

$$\begin{aligned} P(X_2 = x_2, \ldots, X_k = x_k \mid X_1 = x_1, X_0 = i) \\ &= P(X_2 = x_2, \ldots, X_k = x_k \mid X_1 = x_1) \\ &= P(X_1 = x_2, \ldots, X_{k-1} = x_k \mid X_0 = x_1) \end{aligned} \tag{9}$$

By (9), the term in (8) in brackets equals

$$\begin{aligned} \sum_{x_2 \neq a} \cdots \sum_{x_k \neq a} P(X_1 = x_2, \ldots, X_{k-1} = x_k \mid X_0 = x_1) \\ = P(X_1 \neq a, X_2 \neq a, \ldots, X_{k-1} \neq a \mid X_0 = x_1) = P(H_a > k-1 \mid X_0 = x_1) \end{aligned}$$

Plugging this into (8) and changing variables gives

$$P(H_a > k \mid X_0 = i) = \sum_{x_1 \neq a} P_{ix_1}\, P(H_a > k-1 \mid X_0 = x_1) = \sum_{j \neq a} P_{ij}\, P(H_a > k-1 \mid X_0 = j) \tag{10}$$

which is (5).

The proof of (6) is similar. □

Examples:

Gambler's Ruin

- gambler places sequence of independent bets w/ prob. $0 \le p \le 1$.
- on each bet gambler's fortune goes up \$1 or down \$1
- $S_n \overset{def}{=}$ gambler's fort. at "time" n, $n = 0, 1, 2, 3, \dots$ prob p prob $q = 1-p$
- $S_0 \overset{def}{=}$ -"- initial fort. (given)
- g. "ruined" if $S_m = 0$, some m
- "target" level N

given p, N, $S_0 = k$, $k = 0, 1, 2, \dots, N$. $x_k \overset{def}{=} P(\text{ruin} \mid S_0 = k)$. $= ?$

$x_0 = 1$, $x_N = 0$

$x_k = p x_{k+1} + q x_{k-1}$, (up w/ prob. p, down w/ prob. q)

$1 \le k \le N-1$ difference eqn

LOTP

2 boxes, Box #1: 4 red, 8 green chips; Box #2: 9 red, 6 green chips

select #1 w/ prob. $1/6$, #2 w/ prob $5/6$; then draw a chip from selected box. Prob. the chip is red?

$A = \{\text{sel. chip is red}\}$, $B_1 = \{\text{select box \#1}\}$, $B_2 = \{\text{-- \#2}\}$ $\Rightarrow B_1, B_2$ form a partition: $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = \Omega$, $P(B_1) = \frac{1}{6}$, $P(B_2) = 5/6$

$P(A \mid B_1) = 4/12$, $P(A \mid B_2) = 9/15$ $\Rightarrow P(A) = P(A \mid B_1) P(B_1) + P(A \mid B_2) P(B_2) = \_ = \frac{10}{18}$.

Gambler's Ruin

→ Markov chain: $X_n \overset{def}{=}$ gambler's fort. at time n; sequ. of bets are independent w/ $P(\text{win}) = p$, $P(\text{loose}) = q = 1-p$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w/ prob. } p \\ X_n - 1 & \text{w/ prob. } q \end{cases}$$

does not depend on $X_{n-1}, \dots, X_0$.

→ time-homog.: $P(X_{n+1} = k+1 \mid X_n = k) = p \ \forall n$

Target level N, win prob. p, (X_n). What is $\underline{P}$?

$$P_{ij} = \begin{cases} 1 = P_{00} & \text{if } i = j = 0 \\ p \ \text{if } j = i+1 & \text{if } 0 < i < N \\ q \ \text{if } j = i-1 & \\ 1 = P_{NN} & \text{if } i = j = N \end{cases}$$

and all other P_{ij} equal 0

$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & & N-1 & N \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & & & & 0 \\ q & 0 & p & & & & & \vdots \\ 0 & q & 0 & p & & & & \\ & & q & 0 & p & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & q & 0 & p \\ 0 & & & & & & 0 & 1 \end{bmatrix} \end{array}$$

$q = 1-p$

→ stoch. matrix

$\pi_0 = \frac{q-p}{2q}$, $\pi_j = \frac{q-p}{2q}\left(\frac{p}{q}\right)^{j-1}$ for $j \ge 1$.

It suffices to consider state 0.

$P(R_0 < \infty \mid X_0 = 0) = P(H_0 < \infty \mid X_0 = 1)$ (Starting at 1, $H_0 = R_0$)

R_0: 1st time after time 0; H_0: 1st time including time 0

Put $u(i) = P(H_0 < \infty \mid X_0 = i)$, $i \ge 0$ (really want just $i=1$)

Then, $u(i) = P_{i0} + \sum_{j \ne 0} P_{ij}\, u(j)$

$j-1 \xleftarrow{q} j \xrightarrow{p} j+1$

For $i \ge 2$, $P_{i0} = 0$ and $u(i) = p\,u(i+1) + q\,u(i-1)$ or $p\,u(i+1) - u(i) + q\,u(i-1) = 0$

$$u(i) = \begin{cases} A + Bi, & p = \frac{1}{2} \\ A + B\left(\frac{q}{p}\right)^i, & p \ne \frac{1}{2}. \end{cases}$$ (DE HW)

Consider $u(1) = P_{10} + P_{12}\,u(2)$, $u(0) = 1$
$= P_{10}\,u(0) + P_{12}\,u(2) = p\,u(2) + q\,u(0)$

$\boxed{p = \frac{1}{2}}$: $u(i) = A + Bi$

1) If $B \ne 0$, $\lim_{i \to \infty} |u(i)| = +\infty$, impossible, all $u(i)$ are between 0 and 1.

$\Rightarrow u(i) = A$ $\forall i \ge 1$. Consider $i=1$, $u(1) = \frac{1}{2} + \frac{1}{2}u(2)$ $\Rightarrow A = \frac{1}{2} + \frac{1}{2}A$ $\Rightarrow A = 1$ $\Rightarrow u(i) = 1$ all $i \ge 1$

$\Rightarrow P(R_0 < \infty \mid X_0 = 0) = u(1) = 1$ $\Rightarrow$ recurrence.

$\boxed{p < \frac{1}{2}}$: $u(i) = A + B\left(\frac{q}{p}\right)^i$, $i \ge 1$. Since $\frac{q}{p} > 1$, $\left(\frac{q}{p}\right)^i \to \infty$ as $i \to \infty$. If $B \ne 0$, then $|u(i)| \to \infty$ as $i \to \infty$, impossible. $\Rightarrow B = 0$ $\Rightarrow u(i) = A$, all $i \ge 1$.

$i=1$: $u(1) = q + p\,u(2)$ $\Rightarrow A(1-p) = q$
$A = q + pA$ $\quad Aq = q$ $\Rightarrow A = 1$ $\Rightarrow u(i) = 1$, all $i \ge 1$

$\Rightarrow P(R_0 < \infty \mid X_0 = 0) = u(1) = 1 \Rightarrow$ recurrence.

$p \le \frac{1}{2} \Rightarrow$ recurrence.

$\boxed{p > \frac{1}{2}}$: In analogy with random walk on $\{0, \pm 1, \pm 2, \ldots\}$ which is transient for $p > \frac{1}{2}$, we get transcience. [got this by showing $\sum_{n=0}^{\infty} P_{00}^{2n} < \infty$]

$\Rightarrow$ know: $p > \frac{1}{2}$: transient
$p \le \frac{1}{2}$: recurrent case.

Positive vs null recurrence?
(positive:) if there is a stat. dist. — (null:) if there is no stat. dist.

$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ ← can be solved or not with $\sum_{j=0}^{\infty} \pi_j = 1$.

$P_{ij} > 0$? $i = j-1$ or $i = j+1$

$\pi_j = \pi_{j-1}\,p + \pi_{j+1}\,q$ for $j \ge 2$.

$j=0$: $\pi_0 = \sum_{i=0}^{\infty} \pi_i P_{i0} = \pi_1 P_{10} = q\pi_1$ $\quad \pi_1 = \frac{1}{q}\pi_0$

$j=1$: $\pi_1 = \sum_{i=0}^{\infty} \pi_i P_{ij} = \pi_0 P_{01} + \pi_2 P_{21} = \pi_0 + \pi_2 q$ $\Rightarrow q\pi_2 = \pi_1 - \pi_0 = \frac{1}{q}\pi_0 - \pi_0 = \pi_0\left(\frac{1}{q} - 1\right)$

$\Rightarrow \pi_2 = \frac{\frac{1}{q} - 1}{q} \cdot \frac{q}{q}\pi_0 = \frac{1-q}{q^2}\pi_0 = \frac{p}{q^2}\pi_0$

$\Rightarrow \pi_1 = \frac{1}{q}\pi_0$, $\pi_2 = \frac{p}{q^2}\pi_0$, $\pi_j = q\pi_{j+1} + p\pi_{j-1}$, $j \ge 1$

general sol'n: $\pi_j = \begin{cases} A + Bj & p = \frac{1}{2} \\ A + B\left(\frac{p}{q}\right)^j & p < \frac{1}{2} \end{cases}$

Random walk on graphs (network) (Ex. 2.8)

[large ex: world wide web]

graph has vertices $\{a,b,c,d\}$ and edges

i,j are "**neighbors**", write $i \sim j$ if there is an edge joining i and j.

degree of vertex i, $\deg(i)$ = # edges connected to i [$\deg(a)=3$, $\deg(b)=\deg(d)=2$, $\deg(c)=3$]

A random walk (X_n) jumps from one vertex to another at each time step, it jumps independently "uniformly at random" according to the nr of edges at current node.

If $X_n = b$, $X_{n+1} = \begin{cases} a & \text{w/ prob } 1/2 \\ c & \text{w/ prob } 1/2 \end{cases}$ If $X_n = c$, $X_{n+1} = \begin{cases} a & \text{w/ prob. } 1/3 \\ b & \text{w/ prob. } 1/3 \\ c & \text{w/ prob. } 1/3 \end{cases}$

In general, $P_{ij} = \begin{cases} \dfrac{1}{\deg(i)} & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$

$$\underline{P} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1/3 & 1/3 & 1/3 \\ b & 1/2 & 0 & 1/2 & 0 \\ c & 1/3 & 1/3 & 0 & 1/3 \\ d & 1/2 & 0 & 1/2 & 0 \end{array}$$

Example (3.6) Ehrenfest urn model for diffusion of a gas across a membrane (dog-flea model)

- 2 urns A, B (containers), N distinct (labeled) balls (like $N=10$) in the urns
- "Dynamics": at each timestep, pick a ball "uniformly at random" (equal prob. for all balls) and move it from the urn it is in to the other urn.

$\Rightarrow (X_n)$ MC, state space $\{0,1,\dots,N\}$; X_n = # of balls in urn A after n-th move

$X_n = i \Rightarrow N-i$ balls in urn B

$$P_{ij} = \begin{cases} \frac{N-i}{N} & j=i+1 \\ \frac{i}{N} & j=i-1 \\ 0 & \text{otherwise} \end{cases}$$

A: $\boxed{i}$ B: $\boxed{N-i}$

$X_{n+1} = i+1$? choose from B
$X_{n+1} = i-1$? $\rightsquigarrow$ A

Questions: - If $X_0 = N$, what happens as $n \to \infty$?
- equilibrium?
- If $X_0 = N$, how long on average does it take to have all balls in B?

Example: Not everything is a MC

5 quarters (25¢), 5 dimes (10¢), 5 nickels (5¢). draw a coin at random, put it on the table, draw again, etc. X_n = amount of money on the table after nth draw.

$X_0 = 0$ [$X_{15} = 2.00 = X_{16} = X_{17} = \dots$]

$\rightsquigarrow (X_n)$ no MC — intuitive — formally: can violate Markov property for any n, any seq. of states. $P(X_5 = .45 \mid X_0 = 0, X_1 = .25, X_2 = .30, X_3 = .35, X_4 = .40) \neq P(X_5 = .45 \mid X_0 = 0, X_1 = .10, X_2 = .20, X_3 = .30, X_4 = .40)$.

$p=\frac{1}{2}$: need $\sum_{j=0}^{\infty}\pi_j=1<\infty$. This requires $\lim_{j\to\infty}\pi_j=0$. $\pi_j=A+Bj$.

In order to have a stat. distr., we need $\lim_{j\to\infty}(A+Bj)=0$. False if $B\neq0$.

$\Rightarrow \pi_j=A$, $j\geq2$. $\lim_{j\to\infty}\pi_j=A=0$ iff $A=0$

$\Rightarrow \pi_j=0$, $j\geq2$. $[0=\pi_2=\frac{p}{q}\pi_0$, so $\pi_0=0$ and $\pi_1=0]$ $\Rightarrow \pi_j=0$ all j, $\sum_{j=0}^{\infty}\pi_j=0$

$\Rightarrow$ no stat. dist. in the case $p=\frac{1}{2}$. $\Rightarrow$ null recurrent case.

$p<\frac{1}{2}$: $\pi_j=A+B\left(\frac{p}{q}\right)^j$, $j\geq\frac{1}{2}$. $\frac{p}{q}<1 \Rightarrow \left(\frac{p}{q}\right)^j\to0$ as $j\to\infty$.

$\Rightarrow \lim_{j\to\infty}\pi_j=\lim_{j\to\infty}(A+0)=A\overset{?}{=}0$ (no matter what B is)

If $\sum_{j=0}^{\infty}\pi_j<\infty$, this forces $A=0$. $\Rightarrow \pi_j=B\left(\frac{p}{q}\right)^j$, $j\geq2$ $\Rightarrow \sum_{j=0}^{\infty}\pi_j=\pi_0+\pi_1+B\sum_{j=2}^{\infty}\left(\frac{p}{q}\right)^j$

$\Rightarrow$ choose π_0,π_1,B to get $\sum_{j=0}^{\infty}\pi_j=1$. geometric series, $\frac{p}{q}<1$, converges

[$\Rightarrow$ pos. recurrent case.]

Ex: time rev.

$$\underline{P}=\begin{array}{c|ccc} & 0 & 1 & 2\\ \hline 0 & 1/4 & 3/4 & 0\\ 1 & 1/2 & 1/4 & 1/4\\ 2 & 0 & 1/2 & 1/2\end{array}$$

regular (since med. $P_{11}>0$)

$\Rightarrow$ unique stat. dist.

check: $\underline{\pi}=\left[\frac{4}{13}\ \ \frac{6}{13}\ \ \frac{3}{13}\right]$

$\pi_0P_{01}=\pi_1P_{10}$?

$\frac{4}{13}\cdot\frac{3}{4}=\frac{6}{13}\cdot\frac{1}{2}$ ✓

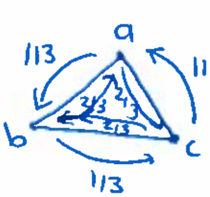
$\pi_0P_{02}=\pi_2P_{20}$ ✓ $(0=0)$

$\pi_1P_{12}=\pi_2P_{21}$

$\frac{1}{6}\cdot\frac{1}{4}=\frac{3}{13}\cdot\frac{1}{2}$ ✓

Ex:

$$\underline{P}=\begin{array}{c|ccc} & a & b & c\\ \hline a & 0 & 1/3 & 2/3\\ b & 2/3 & 0 & 1/3\\ c & 1/3 & 2/3 & 0\end{array}$$



• $\underline{P}$ is doubly stochastic,

$\underline{\pi}=\left[\frac{1}{3},\frac{1}{3},\frac{1}{3}\right]$

Check: $\pi_aP_{ab}=\frac{1}{9}\neq\frac{2}{9}=\pi_bP_{ba}$

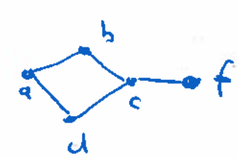
not time-reversible

• At each state, the MC has prob. 1/3 of taking a counterclockwise step
• At each state, the MC has prob 2/3 of taking a clockwise step.
• ~ "likely" that a given step will be clockwise.

In a sequence of states $X_0=i_0, X_1=i_1, \dots, X_n=i_n$ in forwards time, we expect to see acba $\left(\frac{2}{3}\right)^3$ more often than abca $\left(\frac{1}{3}\right)^3$.

Reversing the seq., we would see the second more often. The chain does not look the same forwards and backwards in time.

Ex: Simple random walk on an unweighted graph



$P_{ij}=\frac{1}{\deg(i)}$ provided i,j are "neighbors"

Is $\underline{P}$ reversible? Find $\underline{\pi}$? $\rightarrow$ Solve $\pi_j=\sum\pi_iP_{ij}$ $\forall j$

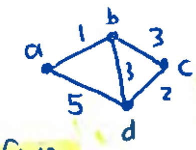
Guess: Try x_j proportional to $\deg(j)$ (or $\frac{1}{\deg(j)}$). Put $x_j=c\deg(j)$, same constant

detailed balance eqn's: $x_iP_{ij}=c\cdot\deg(i)\cdot\frac{1}{\deg(i)}=c$ if i,j are neighbors

$x_jP_{ji}=\ \ =c$ if ...

Example 2.11 random walk on weighted graphs

vertices, edges ($i \sim j$ if $\exists$ edge from i to j),
weights $w_{ij} \geq 0$ on edges $i \sim j$



$w_i :=$ total weight of edges containing vertex i ($w_d = 10$)

$$P_{ij} = \begin{cases} \frac{w_{ij}}{w_i} & \text{if } i \sim j \\ 0 & \text{if not} \end{cases}$$

$w_d = 10,\ w_a = 6,\ w_b = 7,\ w_c = 5$

$$\underline{\underline{P}} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1/6 & 0 & 5/6 \\ b & 1/7 & 0 & 3/7 & 3/7 \\ c & & & 0 & \\ d & & & & 0 \end{array} \quad \text{etc.}$$

Example n-step prob.

Suppose $S = \{0,1,2\}$, $\underline{\underline{P}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .1 & .1 & .8 \\ 1 & .2 & .2 & .6 \\ 2 & .3 & .3 & .4 \end{array}$ Find $P(X_2 = 0 \mid X_0 = 0)$.

① Use LOTP ② $P^2_{00} = (\underline{\underline{P}} \times \underline{\underline{P}})_{00} = .27$, $\underline{\underline{P}}^2 = \ldots$

Supp. X_0 has dist. $\underline{\alpha} = (.7\ \ .2\ \ .1)$. Find $P(X_2 = 0)$.

$$P(X_2 = 0) = (\underline{\alpha}\, \underline{\underline{P}}^2)_0 = \left([.7\ \ .2\ \ .1] \begin{bmatrix} .27 & .27 & .46 \\ .24 & .24 & .52 \\ .21 & .21 & .58 \end{bmatrix} \right)_0 = \ldots = \frac{129}{500}.$$

Example joint dist.

- $P(X_3 = b,\ X_7 = c,\ X_9 = d,\ X_{10} = f \mid X_0 = a) = P^3_{ab} \cdot P^4_{bc} \cdot P^2_{cd} \cdot P_{df}$
- $P(X_1 = 0,\ X_2 = 0,\ X_3 = 2,\ X_4 = 1 \mid X_0 = 2) = P_{20}\, P_{00}\, P_{02}\, P_{21}$
- $P(X_7 = 2 \mid X_1 = 0,\ X_2 = 0,\ X_3 = 2,\ X_4 = 1) = P(X_7 = 2 \mid X_4 = 1) = P^3_{12}$ $7-4=3$ time steps
- $P(X_2 = 1,\ X_4 = 2,\ X_5 = 1,\ X_8 = 0 \mid X_0 = 0) = P^2_{01}\, P^2_{12}\, P_{21}\, P^3_{10}$

Example 3.1 The two-state MC

Let $0 \leq p, q \leq 1$, $(q \neq 1-p)$. $\underline{\underline{P}} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-p & p \\ 2 & q & 1-q \end{array}$

Special cases: ① $p = q = 0$ $\underline{\underline{P}}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\underline{\underline{P}}^n = \underline{\underline{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\forall n$

$\lim_{n \to \infty} P^n_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ exists $\forall i,j$ but depends on both i,j $\Rightarrow$ no lim. dist. for $\underline{\underline{P}}$.

② $p = q = 1$ $\underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\underline{\underline{P}}^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } n \text{ is odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } n \text{ is even.} \end{cases}$ $\rightsquigarrow P^n_{00} = (0, 1, 0, 1, 0, 1, \ldots$ has no limit

General case: ③ $0 < p+q < 2$, let $r = 1-p-q$ $\rightsquigarrow -1 < r < 1$

$$\Rightarrow \underline{\underline{P}}^n = \frac{1}{p+q} \begin{bmatrix} q + pr^n & p - pr^n \\ q - qr^n & p + qr^n \end{bmatrix} \ (*) \text{ and } \lim_{n \to \infty} \underline{\underline{P}}^n = \frac{1}{p+q} \begin{bmatrix} q & p \\ q & p \end{bmatrix} \text{ since } \lim_{n \to \infty} r^n = 0 \ (|r| < 1)$$

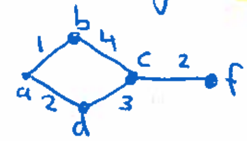
$\rightsquigarrow$ limiting dist. $\underline{\lambda} = (\lambda_1\ \ \lambda_2)$, $\lambda_1 = \lim_{n \to \infty} P^n_{11} = \frac{q}{p+q}$, $\lim_{n \to \infty} P^n_{12} = \frac{p}{p+q}$

Want $1 = \sum_{j\in S} x_j = \sum_{j\in S} c\deg(j) = c\sum_{j\in S}\deg(j)$. Take $c = \frac{1}{\sum_{j\in S}\deg(j)}$, and

$\pi_j = c x_j = \frac{\deg(j)}{\sum_{k\in S}\deg(k)}$. $\Rightarrow \underline{\pi}$ is a stat. dist., $\underline{P}$ is reversible for π, and $\underline{\pi}$ is unique if $\underline{P}$ is irred.

Since $\sum_{j\in S}\deg(j) = 2e$, where e is the nr of edges, we get $\pi_j = \frac{\deg(j)}{2e}$.

Ex: Random walk on weighted graphs:
$w(i,j)$ = weight of edge between i and j
$w(i) = \sum_{j\in S} w(i,j)$, $P_{ij} = \frac{w(i,j)}{w(i)}$



$w(a,b) = 1$
$w(a,d) = 2$
$w(a) = w(a,b) + w(a,d) = 1+2 = 3$
$P_{ab} = \frac{1}{3}$, $P_{ad} = \frac{2}{3}$

(If all weights are equal, say 1, then $w(i) = \deg(i)$)
Can we find $\underline{\pi}$, check for time reversebility?

Try: make a guess for π_i, check if $\pi_i P_{ij} = \pi_j P_{ji}$. Recall: unweighted case $x_i = c\deg(i)$

Try $x_j = c\cdot w(j)$ (or $c/w(j)$)
$\Rightarrow x_i P_{ij} = c\,w(i)\frac{w(i,j)}{w(i)} = c\,w(i,j)$ (i,j neighbors)
$x_j P_{ji} = c\,w(j)\,w(j,i)/w(j) = c\,w(j,i)$ $(w(i,j) = w(j,i))$

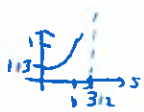
Tells us $x_i P_{ij} = x_j P_{ji}$ $\forall i,j$. To make $\underline{x}$ a prob. vector we want

$1 = \sum_{j\in S} x_j = \sum_{j\in S} c\,w(j) = c\cdot\sum_{j\in S} w(j) \Rightarrow c = \frac{1}{\sum_{j\in S} w(j)}$. $\pi_i = c x_i = c\,w(i)$

$\Rightarrow \underline{\pi}: \pi_i = \frac{w(i)}{\sum_{j\in S} w(j)}$ is a stat. dist. and $\underline{P}$ is reversible w.r.t. $\underline{\pi}$.

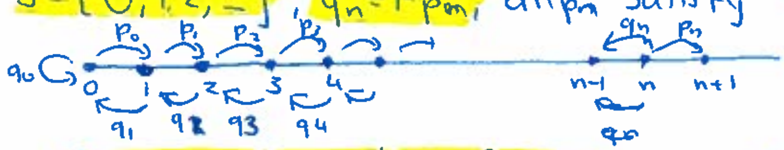
Ex: $P(X=n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$, $n = 0,1,2,\ldots$ (geometric dist.) The pgf is
$G(s) = Es^X = \sum_{n=0}^{\infty} s^n \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}s\right)^n = \frac{1}{3}\frac{1}{1-\frac{2}{3}s} = \frac{1}{3-2s}$

Converges if $|\frac{2}{3}s| < 1 \Leftrightarrow |s| < \frac{3}{2}$.



Ex: Birth and death chain

$S = \{0,1,2,\ldots\}$, $q_n = 1-p_n$, all p_n satisfy $0<p_n<1$ $(0<q_n<1)$
Infinitely many parameters $\{p_n\}_{n=0}^{\infty}$



Random walk with reflection is the special case $p_n = p$ for all n

Q: classify the chain, check for time-reversebility
- find $\underline{x}$ $x_i P_{ij} = x_j P_{ji}$ all i,j

If $\sum_{i=0}^{\infty} x_i < \infty$, $\underline{\pi}$ def'd by $\pi_i = \frac{x_i}{\sum_{i=0}^{\infty} x_i}$ is a stat. dist., which implies the chain is pos. recurrent.

* $i=0, j=1$: $x_0 P_{01} = x_1 P_{10}$, $x_0 p_0 = x_1 q_1 \Rightarrow x_1 = \frac{p_0}{q_1} x_0$
* $i=1, j=2$, $x_1 P_{12} = x_2 P_{21}$, $x_1 p_1 = x_2 q_2 \Rightarrow x_2 = \frac{p_1}{q_2} x_1 = \frac{p_0 p_1}{q_0 q_1} x_0$
* $i=2, j=3$: $\ldots$ $x_3 = \frac{p_0 p_1 p_2}{q_1 q_2 q_3} x_0$ $\ldots \Rightarrow x_n = \frac{p_0 p_1 \cdots p_{n-1}}{q_1 q_2 \cdots q_n} x_0$

Put $r_n = \begin{cases} \frac{p_0\cdots p_{n-1}}{q_1 q_2\cdots q_n}, & n\ge 1 \end{cases}$ $\Rightarrow x_n = r_n x_0$, $n = 0,1,2,\ldots$

Proof of (*): ① induction, $n=1$, $\underline{P}^{n+1} = \underline{P} \times \underline{P}^n$

② textbook derivation. $\rightsquigarrow p^n_{11} = (P^{n-1}P)_{11} = p^{n-1}_{11} p_{11} + p^{n-1}_{12} p_{21} = p^{n-1}_{11}(1-p) + p^{n-1}_{12} q$

$= p^{n-1}_{11}(1-p) + (1-p^{n-1}_{11})q = q + (1-p-q)p^{n-1}_{11}$, $n \geq 1$

$= q + (1-p-q)q + (1-p-q)^2 p^{n-2}_{11} = \dots = \frac{q}{p+q} + \frac{p}{p+q}(1-p-q)^n$, $p^n_{22} = \dots \rightarrow \underline{P}$

③ Use Difference Eqns

What happens if initial distr. is chosen to be $\underline{\lambda}$ the lim. distr.?

i.e. $P(X_0=1) = \frac{q}{p+q}$, $P(X_0=2) = \frac{p}{p+q}$. distn of X_1?

$P(X_1=1) \overset{\text{LOTP}}{=} P(X_1=1 \mid X_0=1)\,P(X_0=1) + P(X_1=1 \mid X_0=2)\,P(X_0=2) = (1-p)\frac{q}{p+q} + q\frac{p}{p+q}$

$= \frac{q}{p+q} = P(X_0=1)$, $P(X_1=2) = P(X_0=2)$ $\rightsquigarrow$ dist of MC does not change from time 0 to time 1 $\rightsquigarrow$ stationary distr.
$[P(X_1=1) = (\lambda P)_1]$

③: Let $x_n = p^n_{11} = (\underline{P} \times \underline{P}^{n-1})_{11} = (\underline{P}^{n-1} \times \underline{P})_{11}$.

$$\underline{P}^{n-1} \times \underline{P} = \begin{bmatrix} p^{n-1}_{11} & p^{n-1}_{12} \\ p^{n-1}_{21} & p^{n-1}_{22} \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} (1-p)p^{n-1}_{11} + q\,p^{n-1}_{12} & \cdot\cdot \\ \cdots & \cdot\cdot \end{bmatrix} \Rightarrow x_n = (1-p)p^{n-1}_{11} + q\,\underbrace{p^{n-1}_{12}}_{=1-p^{n-1}_{11}}$$

$= (1-p)x_{n-1} + q(1-x_{n-1})$

$= q + x_{n-1}(1-p-q) = q + r x_{n-1}$

$\Rightarrow x_n - r x_{n-1} = q$ $\quad$ 1st order diff. eqn. $\quad$ $a x_{n+1} + b x_n + c = z_n$

$\rightsquigarrow$ char. eqn in t is $t - r = 0 \rightarrow$ sol. $t = r$ $\quad$ Char. eqn $ar^2 + br + c = 0$

(1) Soln to $x_n - r x_{n-1} = 0$ is $x_n = A r^n$

(2) For part. soln, guess const., so $x_n = c$ $\quad$ [$x_n - r x_{n-1} = q = z_n$ guess like $z_n = q$ (const)]

Plug in, $x_n - r x_{n-1} = c - rc \overset{!}{=} q \rightsquigarrow c = \frac{q}{1-r} = \frac{q}{p+q}$ $\rightsquigarrow$ gen. soln $x_n = A r^n + \frac{q}{p+q}$

$x_0 = \underset{\text{id. matrix}}{p^0_{11}} = 1 = A r^0 + \frac{q}{p+q} \Rightarrow A = 1 - \frac{q}{p+q} = \frac{p}{p+q}$ $\Rightarrow x_n = \frac{1}{p+q}(q + p r^n)$.

$\rightarrow \underline{\lambda} = \left[\frac{q}{p+q} \quad \frac{p}{p+q}\right]$ (also) stationary distr.

find stat. distr. with linear eqn's:

③ p, q not both zero $\rightarrow$ there is a single soln to
$$\begin{cases} \pi_1 + \pi_2 = 1 \\ \pi_1 = \pi_1 P_{11} + \pi_2 P_{21} = \pi_1(1-p) + \pi_2 q \\ \pi_2 = \pi_1 P_{12} + \pi_2 P_{22} = \pi_1 p + \pi_2(1-q) \end{cases}$$

$\Rightarrow \begin{cases} \pi_1 + \pi_2 = 1 \\ p\pi_1 - q\pi_2 = 0 \\ p\pi_1 - q\pi_2 = 0 \end{cases}$ same eqn $\rightarrow$ one eqn redundant $\Rightarrow \pi_1 = \frac{q\pi_2}{p} \rightsquigarrow \frac{q\pi_2}{p} + \pi_2 = 1$

$\Rightarrow \pi_2 = \frac{p}{q+p}$, $\pi_1 = \frac{q}{p+q}$ (the lim. distn)

① $p = q = 0$, $\underline{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \underline{\pi}\,\underline{P} = \underline{\pi} \Rightarrow \begin{cases} \pi_1 = \pi_1 \\ \pi_2 = \pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{array}{l} \pi_1 = \pi_1 \\ \pi_2 = 1 - \pi_1 \end{array} \rightsquigarrow$ for any $0 \leq c \leq 1$,

$\underline{\pi} = [c \quad 1-c]$ is a stat. distr.

$\rightsquigarrow$ there can be more than one stat. distn

② $p = q = 1$: $\underline{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \underline{\pi} = \left[\frac{1}{2} \quad \frac{1}{2}\right]$ unique stat. dist.

but $\lim_{n\to\infty} \underline{P}^n$ does not exist $\left[\underline{P}^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & n \text{ odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & n \text{ even} \end{cases}\right]$

If $\sum_{n=0}^{\infty} x_n = x_0 \sum_{n=0}^{\infty} \gamma_n < \infty$, then $\pi_i = \frac{x_i}{\sum_{n=0}^{\infty} x_n} = \frac{\gamma_i x_0}{x_0 \sum_{n=0}^{\infty}\gamma_n} = \frac{\gamma_i}{\sum_{n=0}^{\infty}\gamma_n}$.

In the random walk with reflection, $\gamma_i = \frac{p_0 p_1 \cdots p_{i-1}}{q_1 \cdots q_i} = \frac{p^i}{q^i} = \left(\frac{p}{q}\right)^i$

$\sum_{i=0}^{\infty} \gamma_i = \sum_{i=0}^{\infty} \left(\frac{p}{q}\right)^i < \infty$ if $p<q$ same as $p<\frac{1}{2}$ pos. recurrence case.

Ex: Fact: If $X \sim \text{Geom}(p)$, then $G(s) = \frac{p}{1-(1-p)s}$, $|s| < \frac{1}{1-p}$

Supp. we know a r.v. Y has pgf $H(s) = \frac{1}{3-2s}$, then $Y \sim \text{Geom}(\frac{1}{3})$.

Put $p = \frac{1}{3}$ in $G(s) = \frac{p}{1-(1-p)s}$, we get $\frac{1/3}{1-2/3\, s} = \frac{1}{3-2s}$.

Ex: If $X \sim \text{Bern}(p)$, $P(X=0) = 1-p$, $P(X=1) = p$ $(0 \le p \le 1)$, X has pgf

$E(s^X) = \sum_{n=0}^{\infty} P(X=k)s^k = (1-p)s^0 + ps^1 = 1-p+ps$

If $X \sim \text{Bin}(n,p)$, then X has pgf $E(s^X) = (1-p+ps)^n$ since

(a) by def'n $E(s^X) = \sum_{n=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k = \sum_{k=0}^{n} \binom{n}{k} (ps)^k (1-p)^{n-k}$

$\overset{\text{Binomial formula}}{=} (ps + (1-p))^n$ ✓

(b) We can write $X = X_1 + \cdots + X_n$ where X_i indep., $X_i \sim \text{Bern}(p)$.

Now, by our thm, X has pgf $E(s^X) = (G(s))^n = (1-p+ps)^n$ where $G(s) = 1-p+ps$ (pgf of Bern(p))

Ex: (two-state experiment) Toss a biased coin, prob. of heads is 1/5, tails is 4/5, repeatedly 10 times. Assume successive toss are indep. Let $N \sim \text{Bin}(10, 1/5)$ be the nr of heads in the 10 tosses. Now, roll a fair die N times, count the nr of times 2 comes up, call this Y. Find the pgf of Y, and use it to find $P(Y=2)$.

Let $X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ toss of die comes up } 2 \\ 0 & \text{if not} \end{cases}$, $i = 1, 2, \ldots$ $P(X_i = 1) = \frac{1}{6}$

Then $Y = X_1 + \cdots + X_N$. N has pgf $H(t) = (1 - \frac{1}{5} + \frac{1}{5}t)^{10} = (\frac{4}{5} + \frac{1}{5}t)^{10}$ b/c $N \sim \text{Bin}(10, \frac{1}{5})$

The X_i have pgf $G(s) = (1 - \frac{1}{6} + \frac{1}{6}s) = (\frac{5}{6} + \frac{1}{6}s)$ b/c $X_i \sim \text{Bern}(\frac{1}{6})$

So, Y has pgf $\varphi(s) = H(G(s)) = H\underbrace{(\tfrac{5}{6} + \tfrac{1}{6}s)}_{=t} = (\frac{4}{5} + \frac{1}{5}(\frac{5}{6} + \frac{1}{6}s))^{10}$

$= (\frac{4}{5} + \frac{1}{6} + \frac{1}{30}s)^{10} = (\frac{29}{30} + \frac{1}{30}s)^{10} = \varphi(s) \Rightarrow Y \sim \text{Bin}(10, \frac{1}{30})$, so

$P(Y=2) = \binom{10}{2}\left(\frac{1}{30}\right)^2\left(\frac{29}{30}\right)^8$ (or $= \frac{\varphi''(0)}{2!}$)

Ex: Offspring dist. Geom(1/3), $p_k = \frac{1}{3}\left(\frac{2}{3}\right)^k$, $k = 0, 1, 2, \ldots$

The pgf is $G(s) = \frac{1}{3-2s} = G_1(s)$. $G_2(s) = G(G(s)) = \frac{1}{3-2G(s)} = \frac{1}{3 - \frac{2}{3-2s}}$

$\cdot \frac{3-2s}{3-2s} = \frac{3-2s}{9-6s-2} = \frac{3-2s}{7-6s}$. Can check $G_3(s) = G_2(G(s)) = \frac{3-2G(s)}{7-6G(s)} = \cdots = \frac{7-6s}{15-14s}$

We know $P(Z_1 = 0 \mid Z_0 = 1) = G(0) = 1/3$

$P(Z_2 = 0 \mid Z_0 = 1) = G_2(0) = 3/7$

$P(Z_3 = 0 \mid Z_0 = 1) = G_3(0) = 7/15$.

Example: $0<p<1$.
$$\underline{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1-p & p \\ p & 0 & 1-p \\ 1-p & p & 0 \end{bmatrix} \end{array} \leftarrow \text{doubly stoch. matrix}$$

(a) $\underline{P}$ regular b/c $\underline{P}^2 = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$ is positive.

(b) Find stat. distr. $\underline{\pi} = [\pi_1 \ \pi_2 \ \pi_3]$

Thm ($\underline{P}$ regular) $\rightarrow$ $\underline{\pi}$ is unique. $\underline{P}$ doubly stoch. $\rightarrow$ $\underline{\pi}$ uniform, $\underline{\pi} = [\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}]$.

(c) furthermore, $\lim_{n\to\infty} P^n_{ij} = \frac{1}{3}$ $\forall i,j \in \{1,2,3\}$. $\rightarrow$ did not have to compute $\underline{P}^n$ $\forall n$

Example:
$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/2 \\ 1/5 & 2/5 & 1/5 & 1/5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Find the equivalence classes for $\underline{P}$.

State 0: $0 \not\to j$ for $j \neq 0$, $C_1 = \{0\}$ is a communicating class

State 3: $3 \not\to j$ for $j \neq 3$, $C_2 = \{3\}$ – " – – " – – " –

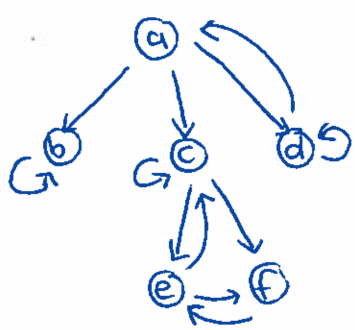
states 1, 2: $1 \to j$ for all j, $2 \to j$ for all j, so $1 \leftrightarrow 2$

$1 \to 0$ but $0 \not\to 1$ so $0 \not\leftrightarrow 1$, etc. $\rightarrow C_3 = \{1,2\}$.

Example:
$$\underline{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1/8 & 3/4 & 1/8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Find comm. classes: determine $i \to j$ (?) for all i,j.

$i \mapsto j$



$$a \to \begin{cases} b \to b \\ c \to \begin{matrix} c \\ e \to f \\ f \to e \end{matrix} \\ d \to \begin{matrix} a \\ d \end{matrix} \end{cases}$$

$\rightarrow$ $a \to b$, $a \to c$, $a \to d$, $a \to e$, $a \to f$
$b \to b$, $c \to c$, $c \to e$, $c \to f$ ...

c, e, f all communicate

$C_1 = \{c,e,f\}$ one class, $C_2 = \{b\}$ one class $C_3 = \{a,d\}$ one class.

Note that $a \to c$, so the class $\{a,d\} \to \{c,e,f\}$, but $\{c,e,f\} \not\to \{a,d\}$.

Example:
$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/8 & 3/4 & 1/8 & 0 & 0 \\ 0 & 9/10 & 0 & 0 & 0 & 1/10 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 3/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$i \mapsto j$ if $P_{ij} > 0$

$0 \mapsto 1 \mapsto 5 \mapsto 0 \mapsto 2 \mapsto 4 \mapsto 5 \mapsto 0$
$\mapsto 3 \mapsto 0$

$\overset{a_0}{0}, \overset{a_1}{1}, \overset{a_2}{5}, 0, 2, 4, 5, 0, 3, \overset{a_9}{0}$ $\rightarrow$ $\underline{P}$ irred.

Ex: $G(s) = \frac{1}{3-2s} = (3-2s)^{-1}$, $G'(s) = -(3-2s)^{-2}(-2) = \frac{2}{(3-2s)^2}$.

$\mu = G'(1) = \frac{2}{1} = 2 > 1$, so $e < 1$. To find e set $G(s) = s$:

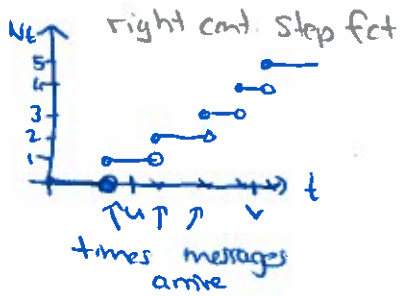
$\frac{1}{3-2s} = s \iff 1 = 3s - 2s^2 \iff 2s^2 - 3s + 1 = 0 \iff (s-1)(2s-1) = 0$

Roots are $s = 1, s = \frac{1}{2}$, so $e = \boxed{\frac{1}{2}}$.

Ex: Turn on cellphone, call this time $t=0$, text messages come in.

Let N_t = # of text messages that have arrived by time t.

[N_t counts events in $[0,t]$ $\Rightarrow$ as t increases, the nr of events N_t increases]



$N_u = 1$, $N_v = 4$

The nr of messages that come in during the time interval $(u,v]$ $= N_v - N_u$ $[0 \le u < v]$

an increment in the fct $t \mapsto N_t$.

Ex: Supp. $(N_t)_{t \ge 0}$ is a PP with rate $\lambda = 0.3$.

Find ⓐ the prob. no messages arrive in 1st 4 minutes

$\rightarrow P(N_4 = 0) = \frac{(4\lambda)^0}{0!} e^{-4\lambda} = e^{-4 \cdot 0.3} = e^{-1.2}$.

$N_4 \sim Pois(4\lambda)$

ⓑ the prob. that 1 message arrived between times 2 and 3 given that 5 mess. arrived by time 2

$\rightarrow P(N_3 - N_2 = 1 \mid \overset{N_2 - N_0}{\cancel{=}} N_2 = 5) = P(N_3 - N_2 = 1)$ (indep. incr. $P(A|B) = P(A)$ if A, B indep)

stat. incr.

$= P(N_1 = 1) = \frac{e^{-\lambda 1} \lambda}{1!} = e^{-\lambda}\lambda = e^{-0.3}(0.3)$.

$N_1 \sim Pois(\lambda \cdot 1)$

ⓒ the prob. that 6 mess. are received in 1st ten minutes and exactly one of these was received in 1st 3 minutes.

$P(N_{10} = 6 \,\&\, N_3 = 1) = P(N_3 = 1, N_{10} = 6)$ $[\ne P(N_3 = 1)\ P(N_{10} = 6)]$

$= P(N_3 = 1, N_{10} - N_3 = 5) \underset{\text{indep incr.}}{=} P(N_3 = 1)\ P(N_{10} - N_3 = 5) = P(N_3 = 1)$

$\cdot P(N_7 = 5) = \frac{e^{-3\lambda}(3\lambda)^1}{1!} \frac{e^{-7\lambda}(7\lambda)^5}{5!}$

ⓓ the prob. that exactly 1 mess. is received in 1st 3 minutes, given that exactly 6 mess. were received in 1st 10 minutes.

Example:

$$\underline{P} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1/3 & 2/3 & 0 \\ b & 1 & 0 & 0 \\ c & 1/4 & 2/4 & 1/4 \end{array}$$

$\underbrace{1/4 \quad 2/4}_{\text{sum}=3/4}$

$a \leftrightarrow b$, $a \nrightarrow b$, $b \nrightarrow c$
$c \to a$, $c \to b$

Classes are $\{a,b\}$ (closed) and $\{c\}$ (open).
- Starting at either a or b, the chain is ~~certain~~ certain to return to a or b → recurrent
- Starting at c, there is prob. 3/4 of never returning to c → transient

$P(R_c < \infty \mid X_0 = c) = \frac{1}{4}$ c is transient

$P(R_a < \infty \mid X_0 = a) = \frac{1}{3} + \frac{2}{3} \cdot 1 = 1$ a is recurrent (b is too)
$[R_a = 1 \text{ or } R_a = 2]$

Example: For same MC, we see the "realization" of the experiment

$X_0 = 1, X_1 = 1, X_2 = 3, X_3 = 3, X_4 = 2, X_5 = 3, X_6 = 1, X_7 = 2, \ldots$ $R_1 = ?$
$\{n \geq 1 : X_n = 1\} = \{1, 6, \ldots\} \Rightarrow R_1 = 1$, $\{n \geq 0 \mid X_n = 1\} = \{0, 1, 6, \ldots\} \Rightarrow H_1 = 0$.
$R_2 = 4$, $H_2 = 4 = R_2$

Example:

$$\underline{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & .4 & .6 \\ 1 & 0 & 1 \end{array} \qquad \underline{P}^n = ?$$

$p^n_{00} = P(X_n = 0 \mid X_0 = 0) = P(X_1 = 0, X_2 = 0, \ldots, X_n = 0 \mid X_0 = 0) = P_{00} \cdot P_{00} \cdots P_{00} = (0.4)^n$

$\Rightarrow p^n_{01} = 1 - (.4)^n$ $\underline{P}^n = \begin{bmatrix} (.4)^n & 1-(.4)^n \\ 0 & 1 \end{bmatrix}$ $p^n_{11} = P(X_n = 1 \mid X_0 = 1) = P(X_1 = 1, X_2 = 1, \ldots, X_n = 1 \mid X_0 = 1)$
$= P_{11} \cdots P_{11} = 1^n = 1$

$a = 0$: $\sum_{n=1}^{\infty} p^n_{aa} = \sum_{n=1}^{\infty} (.4)^n < \infty$ geometric series $\Rightarrow$ 0 is ~~recurrent~~ transient

$a = 1$: $\sum_{n=1}^{\infty} p^n_{aa} = \sum_{n=1}^{\infty} 1 = \infty$ $\Rightarrow$ 1 is recurrent.

Ex: 0 transient, 1 recurrent   0, 1 are in different classes ($\underline{P}$ is not irreducible)

Example:

$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 1 \\ f & .9 & 0 & 0 & 0 & 0 & .1 \end{array}$$

- irreducible b/c $a \to b \to c \to d \to e \to f \to a$
- $P_{ff} > 0 \to \underline{P}$ regular
  P^9 not positive but P^{10} positive

Example:

$$\underline{P} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & \frac{1}{2} & 0 & \frac{1}{2} \\ c & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$a \to b \to a$
$\quad\quad \to c \to a$
$\quad\quad\quad\quad \to b$

irreducible ✓
$P_{ii} = 0 \quad \forall i$
but: $\underline{P}$ regular.

regular $\nRightarrow$ irred $\underline{\&}$ $\exists i: P_{ii} > 0$ (only reg ⇒ irred)

Example: Supp. a MC has comm. classes $C_1 = \{0,1\}$, $C_2 = \{2,3\}$, $C_3 = \{4,5\}$, supp. also that $C_1 \to C_2$, $C_2 \to C_3$. What are the recurrent & transient states?
$C_1 \to C_2$ means $i \to j$ for some $i \in C_1, j \in C_2$ $\Rightarrow C_1$ open, C_2 open. what about C_3?
Supp C_3 open $\Rightarrow C_3 \to C_1$ or $C_3 \to C_2$ but $C_3 \nrightarrow C_1$ since $C_1 \to C_3$ $(C_1 \to C_2, C_2 \to C_3$

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$\rightarrow P(N_3=1 \mid N_{10}=6) = \frac{P(N_3=1, N_{10}=6)}{P(N_{10}=6)} \overset{©}{=} \frac{\frac{e^{-3\lambda}(3\lambda)^1}{1!}\frac{e^{-7\lambda}(7\lambda)^5}{5!}}{\frac{e^{-10\lambda}(10\lambda)^6}{6!}}$

$= \ldots = \binom{6}{1}\left(\frac{3}{10}\right)^1\left(\frac{7}{10}\right)^5$ Binomial $(\underset{n}{6}, \underset{p}{3/10})$ $\underset{k=1}{(1)}$

↳ see HW

$P(X=1)$ where $X \sim Bin(6, \frac{3}{10})$

(e) $E(N_3 \cdot N_7)$ $[\neq EN_3 \cdot EN_7]$ [only N_3 and N_7-N_3 are indep.?]

$= E(N_3((N_7-N_3)+N_3)) = E(N_3(N_7-N_3)) + E(N_3^2) = E(N_3)E(N_7-N_3) + E(N_3^2)$

$= EN_3 \cdot E(N_4) + E(N_3^2) = 3\lambda \cdot 4\lambda + 3\lambda + (3\lambda)^2$

$P(N_7-N_3=\ldots)$
$= P(N_4=\ldots)$

$[X \sim Poi\ \lambda \Rightarrow E(X)=\lambda.\ Var(X)=\lambda = E(X^2)-(E(X))^2$
$\Rightarrow E(X^2) = Var(X) + (E(X))^2]$

Ex: Supp $(N_t)_{t\geq 0}$ PP, rate $\lambda = 0.3$.

(a) Find the expected time of the arrival of the 3rd message

- difficult to consider just using (N_t)

↳ $= E(S_3) = \frac{3}{\lambda} = \frac{3}{.3} = 10$

(b) Find the prob the $\overset{S_3}{\text{3rd mess.}}$ arrived between 10 and 15.

$\rightarrow P(10 < S_3 < 15) = \int_{10}^{15} f_3(x)\,dx = \int_{10}^{15} \frac{\lambda^3 x^2}{2!} e^{-\lambda x}\,dx \overset{\text{int. by parts twice}}{=} \ldots$

Ex: - $h^2 = o(h)$ as $h \to 0$: $\lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$

- Is $\sqrt{h} = o(h)$ as $h\to 0$? $\lim_{h\to 0}\frac{\sqrt{h}}{h} = \lim_{h\to 0}\frac{1}{\sqrt{h}} \neq 0.$

- $\sin h \neq o(h)$ as $h \to 0$: $\frac{\sin h}{h} \to 1$ as $h\to 0$.

- $\cos h = 1 + o(h)$ as $h\to 0$: $\lim_{h\to 0}\frac{\cos h - 1}{h} \underset{\text{l'Hopital's rule}}{=} \lim_{h\to 0}\frac{-\sin h}{1} = 0$ ✓

- $e^{2h} = 1 + 2h + o(h)$ as $h \to 0$: $\lim_{h\to 0}\frac{e^{2h}-(1+2h)}{h} = \lim_{h\to 0}\frac{2e^{2h}-2}{1} = 0$ ✓

Ex: Consider births at a local hospital, assume the prob. a given birth is male to be $p_m = .48$, the prob.
" " is female " $p_f = .52$.

Assume the nrs over time is a rate $\lambda = 2$ PP, and successive births are indep. of one another and the PP $(N_t)_{t\geq 0}$

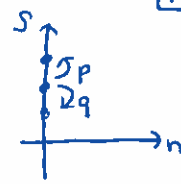
Def $N_t^m = $ # of male births by time t (each $t\geq 0$), $N_t^f = $ # ... female

Thm ⇒ $(N_t^m)_{t\geq 0}$ is PP, rate $\lambda p_m = 2(.48)$, $(N_t^f)_{t\geq 0}$: PP rate $\lambda p_f = 2(.52)$
and the two processes are indep.

Find Prob. at least one male & no females born during a 3 hour period

$P(N_3^M \geq 1, N_3^F = 0) \overset{\text{indep}}{=} P(N_3^M \geq 1)P(N_3^F=0) = (1-P(N_3^M=0))P(N_3^F=0)$
$= (1-e^{-3\lambda p_m}(3\lambda p_m)^0)\, e^{-3\lambda p_f}(3\lambda p_f)^0$

Example 3.13. Simple One-dimensional random walk
↳ jumps are either +1 or −1



State space: {all integers} = $\{0, \pm 1, \pm 2, \dots\}$
$0<p<1$ fixed param., $q = 1-p$.

$$\underline{P} := \begin{cases} P_{i,i+1} = p & \forall i \\ P_{i,i-1} = q & \forall i \\ P_{ij} = 0 & \text{if } j \neq \pm 1 \end{cases}$$

Fact: $\underline{P}$ is irreducible

Thm: All states are recurrent if $p = \frac{1}{2}$. All states are transient if $p \neq \frac{1}{2}$.

- Intuitive argument for $p = \frac{2}{3}$: there is a drift upwards, so starting at a state the MC may never return.
- We will show this using the $\sum_{n=1}^{\infty} p^n_{ii}$ criteria.

It suffices to check if $\sum_{n=1}^{\infty} P^n_{00}$ converges or diverges. Want a formula for P^n_{00}.
$P^n_{00} = 0$ for all odd n. [from 0 the walk can revisit 0 only in an even nr of steps.]
$P^{2n}_{00} = P(\underbrace{X_{2n} = 0}\, | X_0 = 0) = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!\,n!}(pq)^n.$

nr of +1 (up) steps must equal the nr of down (−1) steps must equal n (2n steps in total) ⤳ n up steps, n down steps
any such "path" has prob. $p^n q^n$
nr of such paths = $\binom{2n}{n}$
[choose n positions for +1's ⤳ $\binom{2n}{n}$ ways to do this; pos. for −1's ⤳ 1 way]

Stirling's formula → $P^{2n}_{00} \sim \frac{(4pq)^n}{\sqrt{\pi n}}$

$$\frac{(2n)!}{n!\,n!}(pq)^n \sim \frac{(2n)^{2n} e^{-2n}\sqrt{2\pi (2n)}}{n^n n^n e^{-2n}\sqrt{2\pi n}\sqrt{2\pi n}}(pq)^n = \frac{4^n (pq)^n}{\sqrt{\pi n}}$$

$$\rightsquigarrow \sum_{n=0}^{\infty} P^{2n}_{00} \begin{cases} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty & \text{if } p = \frac{1}{2} \\ \approx \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}} < \infty & \text{if } p \neq \frac{1}{2} \end{cases}$$

- Recall that a finite irred. MC has all states recurrent
- this simple random walk ex. is irred but all states are transient if $p \neq \frac{1}{2}$.

Example: If in addition $\underline{P}$ is doubly stochastic, then μ_j = # of states, $\forall j$.
⤳ uniform stat. distr.

$$\underline{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

$S = \{1, 2, \dots, 5\}$ doubly stoch. matrix $\Rightarrow \pi_j = \frac{1}{5}$, $\mu_j = 5$

Example: Ruin probability

$N > 0$ fixed (target for quitting), $S = \{0, 1, \dots, N\}$ $P_{i,i+1} = p$, $P_{i,i-1} = q$
$P_{00} = 1$, $P_{nn} = 1$
Ruin event is {hit 0 before N} = $\{H_0 < H_N\} = \{H_0 < \infty\}$ $[H_N = \infty]$
$H_a = \min\{n \geq 0 \mid X_n = a\}$

Let $x_k = P(H_0 < \infty \mid X_0 = k)$, $k = 0, 1, 2, \dots, N$. $x_0 = 1$, $x_N = 0$.

Claim: For $1 \leq k \leq N-1$, $x_k = p\,x_{k+1} + q\,x_{k-1}$

Sketch: Restrict to $X_0 = k$, $2 \leq k \leq N-2$ $(X_1 \neq 0/N)$

$x_k = P(H_0 < \infty \mid X_0 = k) \underset{\text{LOTP}}{=} \sum_{j \in S} P(H_0 < \infty, X_1 = j \mid X_0 = k) = P(H_0 < \infty, X_1 = k+1 \mid X_0 = k)$
$+ P(H_0 < \infty, X_1 = k-1 \mid X_0 = k) = P(H_0 < \infty \mid X_1 = k+1, X_0 = k) P(X_1 = k+1 \mid X_0 = k)$
$+ P(H_0 < \infty \mid X_1 = k-1, X_0 = k) P(X_1 = k-1 \mid X_0 = k)$
$= p\,P(H_0 < \infty \mid X_1 = k+1) + q\,P(H_0 < \infty \mid X_1 = k-1) \overset{\text{time-hom.}}{=} p\,P(H_0 < \infty \mid X_0 = k+1) + q\,P(H_0 < \infty \mid X_0 = k \dots$

Ex: $S = \{a, b\}$ $\quad a \rightleftarrows b \quad$ $\underline{P} = \begin{array}{c|cc} & a & b \\ \hline a & 0 & 1 \\ b & 1 & 0 \end{array}$ embedded discrete time MC

Ex: $S = \{a, b, c\}$

generator: $\underline{Q} = \begin{array}{c|ccc} & a & b & c \\ \hline a & -(q_{ab}+q_{ac}) & q_{ab} & q_{ac} \\ b & q_{ba} & -(q_{ba}+q_{bc}) & q_{bc} \\ c & q_{ca} & q_{cb} & -(q_{ca}+q_{cb}) \end{array}$

Ex: 2 states $\quad ① \underset{\lambda}{\overset{\mu}{\rightleftarrows}} ② \quad$ $\underline{Q} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & -\lambda & \lambda \\ 2 & \mu & -\mu \end{array}$

(FE) $\quad P_{11}'(t) = -q_1 P_{11}(t) + P_{12}(t) q_{21} = -\lambda P_{11}(t) + \mu P_{12}(t)$

$$\underline{P}(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}$$

(BE) $\quad P_{11}'(t) = \sum_{k=1}^{2} q_{1k} P_{k1}(t) = q_{12} P_{21}(t) + q_{11} P_{11}(t) = -\lambda P_{11}(t) + \lambda P_{21}(t)$

In (FE) can use $\quad P_{11}(t) + P_{12}(t) = 1$

$\to P_{11}'(t) = -\lambda P_{11}(t) + \mu P_{12}(t) = -\lambda P_{11}(t) + \mu(1 - P_{11}(t))$
$= P_{11}(t)(-(\lambda+\mu)) + \mu$

$P_{11}(0) = 1$

Let $y = y(t) = P_{11}(t) \to \quad y' = -(\lambda+\mu) y + \mu \quad y' + (\lambda+\mu) y = \mu, \; y(0) = 1$

1st order, linear, const. coeff. diff. eqn.

Sol'n: If $\mu = 0$: $\quad y = C_1 e^{-(\lambda+\mu)t} C_2, \; C_1, C_2$ const

$y' = -C_1(\lambda+\mu) e^{-(\lambda+\mu)t} + 0 \quad \Rightarrow y' + (\lambda+\mu) y = -C_1(\lambda+\mu) e^{-(\lambda+\mu)t} + (\lambda+\mu)[C_1 e^{-(\lambda+\mu)t}$
$+ C_2] = (\lambda+\mu) C_2 \overset{!}{=} \mu \Rightarrow C_2 = \frac{\mu}{\lambda+\mu} \quad \Rightarrow y = C_1 e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}.$

Set $t=0$, $y(0) = C_1 \cdot 1 + \frac{\mu}{\lambda+\mu} \overset{!}{=} 1 \quad \Rightarrow C_1 = 1 - \frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}$

$\Rightarrow P_{11}(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} \quad \Rightarrow P_{12} = -\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} \quad (\underbrace{P_{11}(t) + P_{12}(t)}_{=1})$

$$\underline{P}(t) = \frac{1}{\lambda+\mu} \begin{bmatrix} \lambda e^{-(\lambda+\mu)t} + \mu & \lambda - \lambda e^{-(\lambda+\mu)t} \\ \mu - \mu e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{bmatrix}$$

Note $\lim_{t\to\infty} \underline{P}(t) = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$

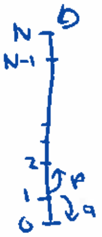
Ex: $\underline{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$, $\pi_1 = \frac{\mu}{\lambda+\mu}$, $\pi_2 = \frac{\lambda}{\lambda+\mu}$ (limiting dist.)

Can check: $\quad \underline{\pi}\, \underline{P}(t) = \underline{\pi} \quad \forall t \geq 0.$

Ex: 7.14 (Eat, play, sleep) Baby 3 states: eat (e), sleep (s), p (play).
eats on average for $\frac{1}{2}$ hr, plays on av. for 1 hr, sleeps on av. 3 hrs.
After eating, 50% chance of s or p. After playing, 50% chance of e or s.
After sleeping, always p. What proportion of the day does the baby spend s? (π_s)
(X_t) CTMC, X_t = state of baby at time t, either e, s, p.
Want $\underline{\pi}$, so we need $\underline{Q}$

Gambler's Ruin

Fix $p, q = 1-p, N$



$P_{00} = P_{NN} = 1$
$P_{i,i+1} = p, \ 0<i<N$
$P_{i,i-1} = q, \ 0<i<N$
$P_{ij} = 0$ otherwise

1st step analysis eqns:

$P(\text{Ruin} \mid X_0 = i) = P(H_0 < \infty \mid X_0 = i) = u(i)$

↑ get to 0 before N $=$ ever get to 0 $= \{H_0 < \infty\}$

Take $2 \le i \le N-2$, then $P_{i0} = 0$.

$\Rightarrow u(i) = \sum_{j \ne 0} P_{ij} u(j) = P_{i,i+1} u(i+1) + P_{i,i-1} u(i-1)$
$= p\, u(i+1) + q\, u(i-1)$

$\Rightarrow u(i) = p u(i+1) + q u(i-1), \quad 2 \le i \le N-2.$

$\underline{i=1}$: $u(1) = P_{10} + P_{12}\, u(2) = q + p u(2) = p u(2) + q u(0)$ ✓

$\underline{i=N-1}$: $u(N-1) = P_{N-1,0} + P_{N-1,N}\, u(N) + P_{N-1,N-2}\, u(N-2) = 0 + p u(N) + q u(N-2)$ ✓

$\Rightarrow$ We get $u(0) = 1, u(N) = 0, \quad u(i) = p u(i+1) + q u(i-1), \ 1 \le i \le N.$
difference eqn, solved in Probl. 9 on DE handout.

Consider $E(\text{duration of game} \mid X_0 = i) = \min\{R_0, R_N\} \stackrel{\text{def}}{=} T$, $i \ne 0$ or N.
Put $w(i) = E(T \mid X_0 = i)$, $w(0) = w(N) = 0$.
By 1st step analysis, $w(i) = 1 + \sum_{j\ne 0, j \ne N} P_{ij} w(j) = 1 + P_{i,i+1} w(i+1) + P_{i,i-1} w(i-1)$
$= 1 + p\, w(i+1) + q\, w(i-1).$

$\begin{cases} \text{Set } p = q = \frac{1}{2}. \quad w(i) = 1 + \frac{1}{2} w(i+1) + \frac{1}{2} w(i-1), \ 0<i<N. \\ w(0) = w(N) = 0. \end{cases}$
↳ DE, Problem 8 on DE handout

Ex: period:

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1/3 & 0 & 2/3 \\ 2 & 0 & 1 & 0 \end{array}$$

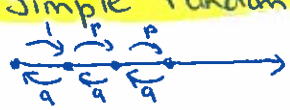
Find periods of each state.

$$P^2 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1/3 & 0 & 2/3 \\ 1 & 0 & 1 & 0 \\ 2 & 1/3 & 0 & 2/3 \end{array}, \quad P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix} = P \Rightarrow P^n = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{bmatrix} & n \text{ odd} \\ \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix} & n \text{ even} \end{cases}$$

$d(0) = \text{g.c.d.}\{n \ge 1 \mid P^n_{00} > 0\} = \gcd\{\cancel{1}, 2, \cancel{3}, 4, \cancel{5}, 6, \dots\}$
$= \gcd\{2, 4, 6, \dots\} = 2$

$d(1) = \gcd\{2, 4, 6, \dots\} = 2.$

Ex: Simple random walk on $S = \{0, 1, 2, \dots\}$ with reflection at 0.



$0 < p < 1$
$q = 1-p$

Prop: This chain is irred., has period 2, and
1) transient if $p > \frac{1}{2}$
recurrent $\begin{cases} 2) \text{ null recurrent if } p = \frac{1}{2} \\ 3) \text{ pos. recurrent if } p < \frac{1}{2}, \text{ and} \end{cases}$

We know q_e, q_p, q_s Assuming times spent in the states are expon. rv's.

q_e = param. of holding time $T_e = \frac{1}{1/2} = 2$ (expected value of an exp. rv. with param λ is $\frac{1}{\lambda}$)

$q_p = \frac{1}{1} = 1$

$q_s = \frac{1}{3}$

Know $\tilde{P}$, embedded chain matrix (for Y_n):
[always 0 on diagonal]

$$\tilde{P} = \begin{array}{c} \\ e \\ p \\ s \end{array} \begin{array}{c} \begin{array}{ccc} e & p & s \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Recall $\tilde{P}_{ij} = \frac{q_{ij}}{q_i}$ $(i \neq j) \Rightarrow q_{ij} = q_i \tilde{P}_{ij}$

$$\to Q = \begin{array}{c} \\ e \\ p \\ s \end{array} \begin{array}{c} \begin{array}{ccc} e & p & s \end{array} \\ \begin{bmatrix} -2 & 1 & 1 \\ 1/2 & -1 & 1/2 \\ 0 & 1/3 & -1/3 \end{bmatrix} \end{array}$$

Solve $\pi Q = 0 = [0\ 0\ 0]$

$\Rightarrow \pi = \left[\frac{1}{14}\ \frac{4}{14}\ \frac{9}{14}\right]$
$(= [\pi_e\ \pi_p\ \pi_s])$

$\Rightarrow \underline{\pi_s = \frac{9}{14}}$ (prop. of time spent sleeping)