MAT 601 HW 1.3-4 SOLUTION: FIELD OF REAL NUMBERS

Problem 1. Let A be a nonempty set of real numbers which is bounded above. Let $B = \{-3x : x \in A\}$. Prove that $\inf B = -3 \sup A$.

Proof. Let $a = \sup A$. For every $x \in A$ we have $x \leq a$, hence $-3x \geq -3a$. This shows that $-3a$ is a lower bound for B.

It remains to show that a number $b > -3a$ cannot be a lower bound for B. Indeed, $b > -3a$ implies $-b/3 < a$. Hence, there exists $x \in A$ such that $x > -b/3$. This implies $-3x < b$, and since $-3x \in B$, we conclude that b is not a lower bound for B. □

Problem 2. Let A be a nonempty bounded set of real numbers. Let $B = \{x - y : x, y \in A\}$. Prove that $\sup B = \sup A - \inf A$.

Proof. Let $s = \sup A$ and $t = \inf A$; both exist because A is bounded. For every $x, y \in A$ we have $x \leq s$ and $y \geq t$, hence $x - y \leq s - t$. This shows that $s - t$ is an upper bound for B.

It remains to show that a number $c < s - t$ cannot be an upper bound for B. Let $\epsilon = (s - t - c)/2$; this is a positive number chosen so that

$$s - \epsilon - (t + \epsilon) = s - t - 2\epsilon = c.$$

Since $s - \epsilon < \sup A$, there exists $x \in A$ such that $x > s - \epsilon$. Since $t + \epsilon > \inf A$, there exists $y \in A$ such that $y < t + \epsilon$. Since $x - y \in B$ and

$$x - y > s - \epsilon - (t + \epsilon) = c$$

it follows that c is not an upper bound for B. □

MAT 601 HW 1.7 SOLUTION: EUCLIDEAN SPACES

Problem 1. Let $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (0, 1)$ be the standard basis vectors in $\mathbb{R}^2$. **Prove** that for any $\mathbf{x} \in \mathbb{R}^2$

$$|\mathbf{x}|^2 + |\mathbf{x} - \mathbf{a}|^2 + |\mathbf{x} - \mathbf{b}|^2 \geq \frac{4}{3}$$

and find all vectors $\mathbf{x}$ for which equality is attained.

Solution. Expand in terms of coordinates $\mathbf{x} = (x_1, x_2)$:

$$|\mathbf{x}|^2 + |\mathbf{x} - \mathbf{a}|^2 + |\mathbf{x} - \mathbf{b}|^2 = x_1^2 + x_2^2 + (x_1 - 1)^2 + x_2^2 + x_1^2 + (x_2 - 1)^2 \quad \geq 0$$

then rearrange by combining terms and completing squares:

$$3(x_1 - 1/3)^2 + 3(x_2 - 1/3)^2 + \frac{4}{3} \geq \frac{4}{3}$$

Equality is attained if and only if both squares are zero, i.e., $\mathbf{x} = (1/3, 1/3)$.

Generalization (optional, not for grade): find the best lower bound for the sum $\sum_{j=1}^{m} |\mathbf{x} - \mathbf{a}_j|^2$ where $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ are fixed and $\mathbf{x} \in \mathbb{R}^n$ can vary.

Solution. The example above suggests that the minimum may be attained at the mean $\mathbf{z} = \frac{1}{m} \sum_{j=1}^{m} \mathbf{a}_j$. To show this, expand the sum as

$$\begin{aligned}\sum_{j=1}^{m} |\mathbf{x} - \mathbf{a}_j|^2 &= \sum_{j=1}^{m} |(\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{a}_j)|^2 \\ &= m|\mathbf{x} - \mathbf{z}|^2 + \sum_{j=1}^{m} |\mathbf{z} - \mathbf{a}_j|^2 + 2\sum_{j=1}^{m} (\mathbf{x} - \mathbf{z}) \cdot (\mathbf{z} - \mathbf{a}_j)\end{aligned}$$

Here the last sum is zero, because it can be written as

$$(\mathbf{x} - \mathbf{z}) \cdot \sum_{j=1}^{m} (\mathbf{z} - \mathbf{a}_j) = (\mathbf{x} - \mathbf{z})(m\mathbf{z} - m\mathbf{z}) = 0$$

The minimum of

$$m|\mathbf{x}-\mathbf{z}|^2 + \sum_{j=1}^{m} |\mathbf{z}-\mathbf{a}_j|^2$$

is attained when the first term vanishes, i.e., $\mathbf{x} = \mathbf{z}$. The minimal value is the second term.

Problem 2. Prove that for $n \in \mathbb{N}$ and for every vector $\mathbf{x} \in \mathbb{R}^n$

$$\sum_{k=1}^{n} |x_k| \le \sqrt{\sum_{k=1}^{n} 2^k x_k^2}$$

Also find all vectors $\mathbf{x}$ for which equality is attained. → $|x\cdot y| \le |x|\cdot|y|$

Solution. Use the Cauchy-Schwarz inequality: and $\sum|x_k| = |x^{?}|$?

$$\sum_{k=1}^{n} |x_k| = \sum_{k=1}^{n} 2^{-k/2}(2^{k/2}|x_k|) \le \sqrt{\sum_{k=1}^{n} 2^{-k}}\sqrt{\sum_{k=1}^{n} 2^k x_k^2} \tag{1}$$

By the geometric sum formula, the first factor on the right is

$$\sqrt{\sum_{k=1}^{n} 2^{-k}} = \sqrt{1-2^{-n}} < 1$$

Thus, the expression (1) does not exceed the second factor, $\sqrt{\sum_{k=1}^{n} 2^k x_k^2}$, and equality is possible only when this factor is zero, i.e., when $\mathbf{x} = 0$.

MAT 601 HW 2.2A: METRIC SPACES, PART 1

Due Wednesday 09/14

Problem 1. For $x, y \in \mathbb{R}$ define

$$d(x,y) = \begin{cases} |x| + |y|, & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Prove that d is a metric on $\mathbb{R}$.

The properties $d(x,x) = 0$, $d(x,y) \geq 0$, and $d(x,y) = d(y,x)$ are evident from the definition. Also, if $x \neq y$, then either x or y is different from 0, resulting in $d(x,y) = |x| + |y| > 0$. It remains to prove the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z), \qquad x, y, z \in \mathbb{R} \tag{1}$$

Observe (and this applies to any metric) that if some of the points x, y, z coincide, (1) follows from the properties already established: for example, if $x = y$, then it becomes $d(x,z) \leq d(x,z)$, and if $x = z$, it becomes $0 \leq d(x,y) + d(y,x)$. So we only need to consider the case of distinct x, y, z. In this case, (1) can be written as

$$|x| + |z| \leq |x| + |y| + |y| + |z|$$

which is true because $|y| \geq 0$.

Problem 2. For the metric in #1, describe the following sets: (a) neighborhoods $N_r(p)$; (b) open sets; (c) closed sets; (d) sets that are dense in $\mathbb{R}$.

(a) We have $x \in N_r(p)$ if either $x = p$ or $|x| + |p| < r$. The latter can only happen if $|p| < r$. Thus,

$$N_r(p) = \begin{cases} \{p\} & \text{if } 0 < r \leq |p| \\ \{p\} \cup (|p| - r, r - |p|) & \text{if } r > |p| \end{cases}$$

(b) By (a), every nonzero point has a neighborhood that is only that point. Hence, any subset not containing 0 is open. A set containing 0 must also contain an interval of the form $(-r, r)$ because such are neighborhoods of 0. In conclusion, a set is open iff it either contains an interval of the form $(-r, r)$, or does not contain 0.

(c) A nonzero point cannot be a limit point of any set, since it has a neighborhood that contains no other points. Since the only possible limit point is 0, any set containing 0 is closed. A set not containing 0 must avoid having it as a limit point, which means being disjoint from some interval $(-r, r)$. Conclusion: a set is closed iff it either contains 0 or is disjoint from some interval of the form $(-r, r)$.

(d) A dense set must contain every nonzero point, since those points are their own neighborhoods. This leaves us with just two candidates for dense sets: $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$. And since they intersect every neighborhood of 0, they are indeed dense.

MAT 601 HW 2.2B: METRIC SPACES, PART 2

Problem 1. Let $E_1, \ldots, E_n$ be subsets of a metric space X. Prove that

allowed here if only ⊃ ∞

$$\overline{\bigcup_{j=1}^{n} E_j} = \bigcup_{j=1}^{n} \overline{E_j}$$

Remark: Definition 2.26 in the book ($\overline{E} = E \cup E'$) is not the best way to work with the closures, because it leads to considering multiple cases (either $x \in E$ or $x \in E'$). Theorem 2.27 offers a better way to think about the closure of E: it's the smallest closed sets containing E. This leads to a short proof, as follows:

Proof. [⊃ part.] Let $A = \overline{\bigcup_{j=1}^{n} E_j}$. This is a closed set, and it contains E_j for every j. Therefore, it also contains $\overline{E_j}$ for every j (since $\overline{E_j}$ is the **smallest** closed set containing E_j). This proves that $A \supset \bigcup_{j=1}^{n} \overline{E_j}$.

[⊂ part.] Being a finite union of closed sets, $\bigcup_{j=1}^{n} \overline{E_j}$ is closed. Since it's a closed set that contains $\bigcup_{j=1}^{n} E_j$, it must also contain $\overline{\bigcup_{j=1}^{n} E_j}$, again by the minimality of the closure. □

Remark: The preceding proof is slick and does not deal directly with neighborhoods. But sometimes we have to work with closures in terms of neighborhoods. In this case it's convenient to use the following fact:

$$x \in \overline{E} \iff \forall r > 0 \ N_r(x) \cap E \neq \emptyset \tag{1}$$

Indeed, if $x \in E$, then it satisfies both sides of (1). And if $x \notin E$, then the right hand side of (1) says precisely that $x \in E'$.

Here is another proof of #1, using (1).

Alternative proof. [$\supset$ part.] Suppose $x \in \bigcup_{j=1}^n \overline{E_j}$. Then there is j such that $N_r(x) \cap E_j \neq \emptyset$ for every $r > 0$. This implies that $N_r(x)$ intersects $\bigcup_{j=1}^n E_j$ for all $r > 0$, hence $x \in \overline{\bigcup_{j=1}^n E_j}$.

[$\subset$ part, by contrapositive.] Suppose $x \notin \bigcup_{j=1}^n \overline{E_j}$. Then for each j there exists r_j such that $N_{r_j}(x) \cap E_j = \emptyset$. Let $r = \min(r_1, \dots, r_n)$. Then $N_r(x)$ is disjoint from $\bigcup_{j=1}^n E_j$, proving that $x \notin \overline{\bigcup_{j=1}^n E_j}$. □

Remark: The $\supset$ part of either proof works for arbitrary unions. The $\subset$ part requires a **finite** union. For a counterexample, consider $\mathbb{Q}$ as the countable union of one-point sets.

Problem 2. Prove that every open set $G \subset \mathbb{R}$ can be written as a countable union of closed subsets of $\mathbb{R}$. That is, there exist closed sets $E_n \subset \mathbb{R}$ such that $G = \bigcup\limits_{n=1}^{\infty} E_n$.

Hint: By definition of an open set, every point of G has a neighborhood contained in G. Define the sets E_n using the size of such a neighborhood.

Remark: My hint goes along the lines of $E_n = \{x \colon N_{1/n}(x) \subset G\}$; it's clear that $\bigcup_{n=1}^{\infty} E_n$, but one needs some work to prove each E_n is closed. The hint has some merit in that it also helps in $\mathbb{R}^n$ and many other spaces. But for $\mathbb{R}$, there is an easier proof without using the hint. It also does not use the fact that I proved in class, about G being a disjoint union of open intervals.

Proof. For each $x \in G$ there is $r > 0$ such that $(x - r, x + r) \subset G$. By the density of rationals, there exist $p, q \in \mathbb{Q}$ such that $x - r < p < x < q < x + r$. Let $I_x = [p, q]$; by construction, $x \in [p, q] \subset G$. Thus, the union of the collection of intervals $\{I_x \colon x \in G\}$ is precisely G. And this collection is at most countable, since there are countably many rational numbers. □

MAT 601 HW 2.3A: COMPACT SETS IN GENERAL METRIC SPACES

Due Monday 09/19

Problem 1. Let A and B be compact subsets of a metric space X. Prove that the closure of the set $A \setminus B$ is compact.

Proof. Being compact, A is closed. Since $A \setminus B \subset A$, it follows that

$$\overline{A \setminus B} \subset \overline{A} = A$$

The closure of any set is closed. Being a closed subset of a compact set (A), the set $\overline{A \setminus B}$ is compact. □

Remark: the assumption that B is compact is extraneous.

Problem 2. Let X be a compact metric space. Suppose that $f\colon X \to \mathbb{R}$ is a function with the following property: for any $x \in X$ there exists $r > 0$ such that the image set $f(N_r(x))$ has an upper bound. Prove that there exists a real number M such that $f(x) \le M$ for all $x \in X$.

Proof. Consider the following open cover $\{G_x\colon x \in X\}$: for each $x \in X$, let G_x be a neighborhood of x whose image under f has an upper bound. Since $x \in G_x$, this is indeed a cover of X. By compactness of X, it has a finite subcover, say $G_{x_1}, \dots, G_{x_n}$. Let

$$M = \max_{i=1,\dots,n} \sup f(G_{x_i})$$

then $f \le M$ holds in every set G_{x_i}, and therefore in all of X. □

MAT 601 HW 2.3B: COMPACT SETS IN $\mathbb{R}$ AND $\mathbb{R}^n$

Problem 1. Give an example of an open cover of the set $[0,1] \setminus \{3/4\}$ which has no finite subcover. (The set is a closed interval minus a point; it's considered a subset of $\mathbb{R}$ with the standard metric.)

Example: $G_n = \mathbb{R} \setminus (3/4 - 1/n, 3/4 + 1/n)$, $n = 1, 2, \ldots$. This is an open cover because $\bigcup_{n=1}^{\infty} = \mathbb{R} \setminus \{3/4\}$, but any finite union of these sets is just the largest of them, $\mathbb{R} \setminus (3/4 - 1/N, 3/4 + 1/N)$, which omits points such ~~that~~ as $3/4 - 1/(2N)$.

Problem 2. Define another metric on $\mathbb{R}$ by $d_1(x,y) = \min(|x-y|, 1)$. (You don't need to prove that d_1 is a metric). The standard metric is $d(x,y) = |x-y|$.

(i) Prove that a set $A \subset \mathbb{R}$ is open with respect to d_1 if and only if it is open with respect to d.

(ii) Prove that a set $A \subset \mathbb{R}$ is compact with respect to d_1 if and only if it is compact with respect to d.

(iii) Conclude that the metric space $(\mathbb{R}, d_1)$ has subsets that are closed and bounded, but not compact.

(i) Let $N_r^1(x)$ denote the neighborhoods with respect to d_1, to distinguish them from neighborhoods with respect to d. Note that $N_r(x) = N_r^1(x)$ for $r < 1$ because $d = d_1$ when either metric is less than 1. Also note that any neighborhood contains a neighborhood of radius less than 1, with the same center. Thus,

$$(1) \qquad U \text{ open wrt } d \iff \forall x \in U\ \exists r \in (0,1) \text{ such that } N_r(x) \subset U$$

Similarly,

$$(2) \qquad U \text{ open wrt } d_1 \iff \forall x \in U\ \exists r \in (0,1) \text{ such that } N_r^1(x) \subset U$$

The right hand sides of (1) and (2) are equivalent, hence

$$U \text{ open wrt } d \iff U \text{ open wrt } d_1$$

(ii) Compactness is defined in terms of open sets. An open cover with respect to d is also an open cover with respect to d_1, and vice versa. So, compactness (being the existence of finite subcovers of open covers) is the same for either metric.

(iii) The set $\mathbb{R}$ itself gives such an example. It is closed and bounded wrt d_1 (since $\mathbb{R} \subset N_2^1(0)$). If it was compact wrt d_1, it would also be compact wrt d. But it is not bounded with respect to d, so cannot be compact.

Remark 1: The same logic applies to $[0, \infty)$, $[1.\infty)$, or $\mathbb{N}$ or $\mathbb{Z}$.

Remark 2: the fact that $\bigcup_{n=1}^{\infty} [n, \infty) = \emptyset$ shows that the intersection of closed bounded nested sets can be empty in a general metric space.

MAT 601 REMARKS ON 2.4-5: PERFECT SETS AND CONNECTED SETS

Bonus Theorem 1 from 9/26. I overstated the result, claiming it's true for an arbitrary set $E \subset \mathbb{R}$ (it can't be for many reasons). The correct statement is: every closed set $E \subset \mathbb{R}$ is the union of a perfect set and an at most countable set.

Proof. Let $\mathcal{J}$ be the set of all intervals I with rational endpoints such that $E \cap I$ is at most countable. Let $C = \bigcup_{I \in \mathcal{J}} (E \cap I)$; this is an at most countable set. It is also open in E.

If $x \in E \setminus C$, then $E \cap N_r(x)$ is uncountable for every $r > 0$, for otherwise x would be contained in some interval $I \in \mathcal{J}$. Therefore, $(E \setminus C) \cap N_r(x)$ is also uncountable. This shows that x is a limit point of $E \setminus C$. Finally, $E \setminus C$ is closed in E and since E is closed in $\mathbb{R}$, it follows that $E \setminus C$ is closed in $\mathbb{R}$.

Summary: $E \setminus C$ is perfect and C is at most countable. $\square$

(Note that the assumption that E is closed is used only to show that $E \setminus C$ is closed.)

Bonus Theorem 2 from 9/26. Suppose that $K_1 \supset K_2 \supset \cdots$ are nonempty compact connected sets in a metric space X. Then the set $K = \bigcap_{n=1}^{\infty} K_n$ is also connected.

Proof. Suppose to the contrary that $K = A \cup B$ where A and B are nonempty, disjoint and open in K. We have $A = U \cap K$ where U is open in X. Let $V = X \setminus \overline{U}$; this set is also open in X. We have $V \cap K = B$ because on one hand, V is disjoint from A, and on the other, $\overline{U}$ is disjoint from B.

The sets $E_n = K_n \backslash (U \cup V)$ are compact and nested. Since $K \subset U \cup V$, the intersection of E_n is empty. Hence, there exists n such that $E_n = \emptyset$, meaning that $K_n \subset U \cup V$. But the sets $U \cap K_n$ and $V \cap K_n$ are nonempty, disjoint, and open in K_n, so K_n being covered by them contradicts the assumption that K_n is connected. ☐

Hint for homework problem 2. The key step is to prove that after $K_1, \ldots, K_n$ have been constructed, the set $I_{n+1} \setminus (K_1 \cup \cdots \cup K_n)$ is nonempty. Here's a hint for this step.

Pick any $x \in I_{n+1}$. If it's none in $K_1, \ldots, K_n$, done. Otherwise it's in exactly one of them, say K_j. Then there is a neighborhood $N_r(x)$ that is contained in I_{n+1} and is disjoint from K_i for $i \in \{1, 2, \ldots, n\} \setminus \{j\}$. (Why?) Once you have this $N_r(x)$, the conclusion follows since K_i does not contan any interval.

MAT 601 HW 2.4-5 SOLUTION: PERFECT SETS AND CONNECTED SETS

Problem 1. Let E_α ($\alpha \in I$) be some collection of connected subsets of a metric space X. Suppose that $\bigcap_{\alpha \in I} E_\alpha$ is nonempty. Prove that $\bigcup_{\alpha \in I} E_\alpha$ is a connected set.

Proof. Suppose $\bigcup_{\alpha \in I} E_\alpha = A \cup B$ where A, B are nonempty and separated. Pick $p \in \bigcap_{\alpha \in I} E_\alpha$. Then $p \in A$ or $p \in B$; without loss of generality, $p \in A$. This implies that the set $E_\alpha \cap A$ is nonempty for every $\alpha \in I$. If $E_\alpha \cap B$ was also nonempty, we'd have nonempty separated sets $E_\alpha \cap A$ and $\alpha \in I$ covering the connected set E_α, which is impossible. (Note: if two sets are separated, then their subsets are also separated, since neither contains the limit points of the other one.) Hence, $E_\alpha \cap B = \emptyset$ for every α. But then $B = \bigcup_{\alpha \in I} E_\alpha \cap B$ is empty, a contradiction. □

Problem 2. Let's say that K is a Cantor-type set if it is the image of the standard Cantor set (example 2.44) under some transformation $y = ax + b$, $a \neq 0$. Suppose I_n, $n \in \mathbb{N}$, are nonempty open intervals in $\mathbb{R}$. Prove that there exist disjoint Cantor-type sets K_n, $n \in \mathbb{N}$, such that $K_n \subset I_n$ for every n.

Proof. For any open interval J we can pick two numbers $c < d$ in it and map the interval $[0, 1]$ onto $[c, d]$ by a linear map. This will map the standard Cantor set into a Cantor-type set contained in J.

Do this to produce $K_1 \subset I_1$. Then proceed inductively: after $K_1, \dots, K_n$ are constructed, consider the set $G = I_{n+1} \setminus (K_1 \cup \cdots \cup K_n)$. It is open, being a finite intersection of open sets.

E_n is the union of 2^n intervals each of length 3^{-n}
Cantor set $= \bigcap_{n=1}^{\infty} E_n$

Claim: G is nonempty. To prove this, pick $x \in I_{n+1}$. If it is in G, the claim is proved. Otherwise, there is $i \in \{1, \dots, n\}$ such that $x \in G$. The set $E = \bigcup_{j \neq i} K_j$ is closed and does not contain x. Thus, $x \in I_{n+1} \setminus E$, which is an open set. This implies that some neighborhood $N_r(x)$ is contained in $I_{n+1} \setminus E$. Since K_i cannot contain $N_r(x)$, there is a point $y \in N_r(x) \setminus K_i$. This point is in G, proving the claim.

Being nonempty and open, G contains an open interval J. We construct $K_{n+1} \subset J$ as described above, and the process continues indefinitely. □

Application (not for grade): use #2 to prove that there exists a function $f \colon \mathbb{R} \to \mathbb{R}$ such that for every nonempty open interval I we have $f(I) = \mathbb{R}$.

Proof. Consider the set of all open intervals with rational endpoints. Since it's countable, the intervals can be put in bijection with $\mathbb{N}$, i.e., enumerated as I_1, I_2, ... Construct Cantor-type sets K_n, $n \in \mathbb{N}$, as in #2. For each n, pick a surjective map $f_n \colon K_n \to \mathbb{R}$. (For example, recall that an element of the standard Cantor set is $2\sum_{k=1}^{\infty} d_k/3^k$ with $d_k \in \{0,1\}$, so we can send this element to $\sum_{k=1}^{\infty} d_k/2^k$, thus obtaining a map of K_n onto $[0,1]$. Then compose with any map of $[0,1]$ onto $\mathbb{R}$.) Finally, define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} f_n(x), & x \in K_n \\ 0, & \text{otherwise} \end{cases}$$

Since every nonempty interval I contains some I_n, we have $f(I) \supset f(I_n) \supset f(K_n) = \mathbb{R}$ as desired. □

MAT 601 HW 3.1 SOLUTION: CONVERGENCE OF SEQUENCES

Problem 1. Let $\{p_n\}$ be a convergent sequence in a metric space X. Prove that the set

$$T = \bigcap_{m=1}^{\infty} \overline{\{p_n \colon n \geq m\}} \tag{1}$$

consists of one point.

Proof. Let $p = \lim_{n\to\infty} p_n$.

Step 1: $p \in T$. Recall that a point is in the closure of a set if and only if the set intersects every neighborhood of that point. So, we must show that for every $r > 0$ the intersection $N_r(p) \cap \{p_n \colon n \geq m\}$ is nonempty. But this follows from the definition of the limit: there exists N such that $d(p_n, p) < r$ for all $n \geq N$; this means $p_n \in N_r(p)$ for all $n \geq N$.

Step 2: if $q \neq p$, then $q \notin T$. Let $\epsilon = d(p,q)/2$. Since $p_n \to p$, there exists N such that $d(p_n, p) < \epsilon$ for all $n \geq N$. The triangle inequality $d(p,q) \leq d(p_n,p) + d(p_n,q)$ implies $d(p_n, q) > \epsilon$ for all $n \geq N$. This says precisely that $N_\epsilon(q) \cap \{p_n \colon n \geq N\} = \emptyset$, which in turn implies $q \notin \overline{\{p_n \colon n \geq N\}}$. Thus $q \notin T$ as claimed. □

Problem 2. Give an example of a sequence in $\mathbb{R}$ (with the standard metric) for which the set T in (1) consists of one point, but the sequence does not converge.

Let $x_n = (1 + (-1)^n)n$, so that $x_n = 2n$ when n is even, and $x_n = 0$ when n is odd. The sequence does not converge, since it's unbounded. On the other hand,

$$\overline{\{p_n \colon n \geq m\}} = \overline{\{0\} \cup \{2n \colon n \geq m,\ n \text{ even}\}}$$

The set $\{0\} \cup \{2n\colon n \geq m,\ n \text{ even}\}$ is closed, since any two of its elements are at distance at least 2 from each other, making it impossible for any point to be its limit point. Hence,

$$T = \bigcap_{m=1}^{\infty} (\{0\} \cup \{2n\colon n \geq m,\ n \text{ even}\}) = \{0\}$$

MAT 601 HW 3.2-3 SOLUTION: SUBSEQUENCES, CAUCHY SEQUENCES

Problem 1. Suppose that $\{p_n\}$ is a sequence in a metric space X, and there is a point $p \in X$ such that every subsequence of $\{p_n\}$ has a subsubsequence converging to p. Prove that $\{p_n\}$ converges to p.

Proof. Suppose to the contrary that $\{p_n\}$ does not converge to p. Negating the definition of the limit yields the following: there exists $\epsilon > 0$ such that for every N there is $n \geq N$ such that $d(p_n, p) \geq \epsilon$. Construct an increasing sequence of integers $\{n_k\}$ as follows: use the above with $N = 1$ to get n_1, then use $N = n_1 + 1$ to get n_2, then use $N = n_2 + 1$ to get n_3, etc. This results in a subsequence $\{p_{n_k}\}$ such that $d(p_{n_k}, p) \geq \epsilon$ for all k. By the hypothesis, some subsubsequence $\{p_{n_{k_j}}\}$ converges to p; but this implies $d(p_{n_{k_j}}, p) < \epsilon$ for all sufficiently large j, a contradiction. □

Problem 2. Give an example of a sequence $\{x_n\}$ in $\mathbb{R}$ such that for every $k \in \mathbb{N}$ we have $\lim_{n\to\infty}(x_{n+k} - x_n) = 0$ but $\{x_n\}$ is not a Cauchy sequence.

Proof. Let $x_n = \sqrt{n}$. This is not a Cauchy sequence since it's unbounded. On the other hand, for every k we have

$$x_{n+k} - x_k = \sqrt{n+k} - \sqrt{n} = \frac{k}{\sqrt{n+k} + \sqrt{n}} \to 0$$

since the numerator stays fixed while the denominator grows indefinitely. □

MAT 601 HW 3.3B SOLUTION: COMPLETENESS

Problem 1. Let X be $\mathbb{R}$ with the metric $d_1(x,y) = \min(|x-y|, 1)$. Prove that X is a complete metric space, using the fact that $\mathbb{R}$ is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence with respect to d_1. The goal is that to prove that it converges with respect to d_1.

Claim 1: $\{x_n\}$ is a Cauchy sequence with respect to the standard metric $d(x,y) = |x-y|$. Indeed, given $\epsilon > 0$, we can let $\epsilon' = \min(\epsilon, 1)$ and pick N such that $d_1(x_n, x_m) < \epsilon'$ for all $n, m \geq N$. Since both metrics agree when the value of one of them is less than 1, we have $d(x_n, x_m) = d_1(x_n, x_m) < \epsilon' \leq \epsilon$ for all $n, m \geq N$, proving the claim.

Since $\mathbb{R}$ is complete, Claim 1 implies that x_n converges to some point $x \in \mathbb{R}$ with respect to d.

Claim 2: $x_n \to x$ with respect to d_1. Indeed, given $\epsilon > 0$, we can pick N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. Since $d_1 \leq d$, it follows that $d_1(x_n, x) < \epsilon$ for all $n \geq N$, proving the claim.

We've proved that every Cauchy sequence with respect to d_1 converges with respect to d_1, meaning X is complete. □

Problem 2. Suppose that $\{p_n\}$ is a Cauchy sequence in a metric space (X, d), which does not converge. Consider the set $Y = X \cup \{p\}$ where p is an abstract point we add to the set X. Define a metric d_Y on Y as follows:

- $d_Y(a,b) = d_X(a,b)$ if $a, b \in X$;
- $d_Y(a,p) = d_Y(p,a) = \lim_{n\to\infty} d(p_n, a)$ if $a \in X$;
- $d(p,p) = 0$.

Prove that d_Y is indeed a metric (this includes showing that the limit in the definition of d_Y exists). Also prove that $p_n \to p$ in the space Y.

Proof. General remark: if all terms of sequences $\{x_n\}, \{y_n\} \subset \mathbb{R}$ satisfy $x_n \leq y_n$, and the limits $\lim x_n = x$, $\lim y_n = y$ exist, then $x \leq y$. Indeed, if $x > y$, then taking $\epsilon = (x-y)/2$ we'd find that $x_n > x - \epsilon = y + \epsilon > y_n$ for sufficiently large n, contradicting the assumption.

One can summarize the above by saying that we can pass to the limit in a non-strict inequality.

Before showing that d is a metric, it must be shown that it is well-defined, that is the limit $\lim_{n\to\infty} d(p_n, a)$ exists. By the triangle inequality,

$$|d(p_n, a) - d(p_m, a)| \leq d(p_n, p_m) \tag{1}$$

Since $\{p_n\}$ is a Cauchy sequence, (1) implies that $\{d(p_n, a)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus it converges.

The metric properties only need to be proved when one of the points involved is p — otherwise, they follow from d being a metric on X. Also, the triangle inequality (property 3) only needs to be proved for three *distinct* points, because when some of them coincide, it reduces to the properties 1-2.

Property 1: $d_Y(x, y) \geq 0$ for all $x, y \in Y$, with equality iff $x = y$. The only case of interest is when one of two points is p, so let's say $x = p$ and $y \neq p$. Then $d_Y(p, y) \geq 0$ because it's the limit of nonnegative numbers $d(p_n, y)$. Also, this limit cannot be zero, for otherwise p_n would converge to y, contradicting the assumption that $\{p_n\}$ does not converge.

Property 2: $d_Y(x, y) = d_Y(y, x)$. This is by the definition, bullet item 2.

Property 3: $d_Y(x, y) \leq d_Y(x, z) + d_Y(y, z)$. First consider the case $x = p$. Since $d(p_n, y) \leq d(p_n, z) + d(y, z)$ for every n, passing to the limit $n \to \infty$ yields $d_Y(p, y) \le d_Y(p, z) + d_Y(y, z)$, as desired. The case $y = p$ is the same up to swapping x and y. Consider the case $z = p$. Since $d(x, y) \le d(x, p_n) + d(y, p_n)$, passing to the limit $n \to \infty$ we get $d(x, y) \le d(x, p) + d(y, p)$ as desired.

Finally, to prove that $p_n \to p$ in Y, we must show that for every $\epsilon > 0$ there exists N such that $d_Y(p_n, p) < \epsilon$ for all $n \ge N$. Using the Cauchy property of $\{p_n\}$, pick N such that $d(p_n, p_m) < \epsilon/2$ for all $n, m \ge N$. Keeping n fixed, pass to the limit $m \to \infty$, getting $d_Y(p_n, p) \le \epsilon/2$ (only *non-strict* inequalities are preserved under taking limits). Thus, $d_Y(p_n, p) < \epsilon$ for all $n \ge N$, as required. □

MAT 601 HW 3.4 SOLUTION: UPPER/LOWER AND INFINITE LIMITS

Problem 1. Define a sequence by $x_1 = 1/3$ and $x_{n+1} = x_n(1 - x_n)$ for $n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} x_n$ exists.

Proof. The sequence is decreasing because $x_{n+1} = x_n - x_n^2 \le x_n$ (the square of any number is nonnegative). Hence $x_n \le x_1 < 1$ for all n.

Claim: $x_n \ge 0$ for all n. The base case is $x_1 = 1/3 \ge 0$. The step of induction is $x_n \ge 0 \implies x_n(1 - x_n) \ge 0 \implies x_{n+1} \ge 0$ which works because $1 - x_n > 0$. The claim is proved.

Since $\{x_n\}$ is decreasing and bounded below, it converges. $\square$

Problem 2. Define a sequence by $x_1 = 1/3$ and $x_{n+1} = 4x_n(1 - x_n)$ for $n = 1, 2, \ldots$. Prove that

(a) $0 \le x_n \le 1$ for all n;

(b) $\liminf_{n\to\infty} x_n \le 3/4$.

Proof. (a) will be proved by induction. Base: $0 \le 1/3 \le 1$. Step of induction: if $0 \le x_n \le 1$, then on one hand $x_{n+1} = 4x_n(1 - x_n) \ge 0$, and on another

$$x_{n+1} = 4x_n(1 - x_n) = 1 - (2x_n - 1)^2 \le 1 \tag{1}$$

(b) If $x_n > 3/4$ then $2x_n - 1 > 1/2$, which in view of (1) yields $x_{n+1} < 1 - (1/2)^2 = 3/4$. So,

$$\min(x_n, x_{n+1}) \le \frac{3}{4} \quad \text{for every } n \tag{2}$$

This implies $\inf\{x_n : n \ge m\} \le \min(x_m, x_{m+1}) \le 3/4$. Hence

$$\liminf_{n\to\infty} x_n = \sup_m \inf\{x_n : n \ge m\} \le 3/4$$

Alternative end of proof, using the definition of lim inf via the set S of subsequential limits: Select a subsequence x_{n_k} by picking the smaller of the elements x_{2k-1}, x_{2k} for every k. By (a) and (2) we have $0 \leq x_{n_k} \leq 3/4$ for all k. Hence, there is a convergent subsubsequence $x_{n_{k_j}}$ with limit contained in $[0, 3/4]$. Hence $\inf S \leq 3/4$. $\square$

MAT 601 HW 3.4 SOLUTION: UPPER/LOWER AND INFINITE LIMITS

Problem 1. Define a sequence by $x_1 = 1/3$ and $x_{n+1} = x_n(1 - x_n)$ for $n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} x_n$ exists.

Proof. The sequence is decreasing because $x_{n+1} = x_n - x_n^2 \le x_n$ (the square of any number is nonnegative). Hence $x_n \le x_1 < 1$ for all n.

Claim: $x_n \ge 0$ for all n. The base case is $x_1 = 1/3 \ge 0$. The step of induction is $x_n \ge 0 \implies x_n(1 - x_n) \ge 0 \implies x_{n+1} \ge 0$ which works because $1 - x_n > 0$. The claim is proved.

Since $\{x_n\}$ is decreasing and bounded below, it converges. □

Problem 2. Define a sequence by $x_1 = 1/3$ and $x_{n+1} = 4x_n(1 - x_n)$ for $n = 1, 2, \ldots$. Prove that

(a) $0 \le x_n \le 1$ for all n;

(b) $\liminf_{n\to\infty} x_n \le 3/4$.

Proof. (a) will be proved by induction. Base: $0 \le 1/3 \le 1$. Step of induction: if $0 \le x_n \le 1$, then on one hand $x_{n+1} = 4x_n(1 - x_n) \ge 0$, and on another

$$x_{n+1} = 4x_n(1 - x_n) = 1 - (2x_n - 1)^2 \le 1 \tag{1}$$

(b) If $x_n > 3/4$ then $2x_n - 1 > 1/2$, which in view of (1) yields $x_{n+1} < 1 - (1/2)^2 = 3/4$. So,

$$\min(x_n, x_{n+1}) \le \frac{3}{4} \quad \text{for every } n \tag{2}$$

This implies $\inf\{x_n : n \ge m\} \le \min(x_m, x_{m+1}) \le 3/4$. Hence

$$\liminf_{n\to\infty} x_n = \sup_m \inf\{x_n : n \ge m\} \le 3/4$$

Alternative end of proof, using the definition of $\liminf$ via the set S of subsequential limits: Select a subsequence x_{n_k} by picking the smaller of the elements x_{2k-1}, x_{2k} for every k. By (a) and (2) we have $0 \leq x_{n_k} \leq 3/4$ for all k. Hence, there is a convergent subsubsequence $x_{n_{k_j}}$ with limit contained in $[0, 3/4]$. Hence $\inf S \leq 3/4$. □

MAT 601 HW 3.5 SOLUTION: EVALUATING LIMITS

Problem 1. Suppose that $\{x_n\}$ is a sequence of real numbers that converges to $L \in \mathbb{R}$. Let $y_n = (x_1 + x_2 + \cdots + x_n)/n$ for $n \in \mathbb{N}$. Prove that $y_n \to L$.

Proof. Let $z_n = x_n - L$, so that $z_n \to 0$. Observe that

$$y_n - L = \frac{x_1 + x_2 + \cdots + x_n - nL}{n} = \frac{z_1 + \cdots + z_n}{n} \tag{1}$$

Since convergent sequences are bounded, there exists M such that $|z_n| \leq M$ for all $n \in \mathbb{N}$.

Given $\epsilon > 0$, let K be such that $|z_n| < \epsilon/2$ when $n \geq K$, and choose an integer N such that

$$N > \max\left(K, \frac{2MK}{\epsilon}\right) \tag{2}$$

I claim that $|y_n - L| < \epsilon$ whenever $n \geq N$. Indeed, for such n we have, using (1),

$$|y_n - L| \leq \frac{|z_1| + \cdots + |z_K|}{n} + \frac{|z_{K+1}| + \cdots + |z_n|}{n} < \frac{KM}{n} + \frac{n\epsilon/2}{n} < \epsilon$$

since $KM/n \leq KM/N < \epsilon/2$ according to (2). □

Problem 2. Suppose that $\{x_n\}$ is a sequence of real numbers that converges to $L \in \mathbb{R}$. Let $y_n = (x_1 + 2x_2 + 3x_3 + \cdots + nx_n)/n^2$ for $n \in \mathbb{N}$. Prove that $y_n \to L/2$.

Proof. Let $z_n = x_n - L$, so that $z_n \to 0$. Since $1 + \cdots + n = n(n+1)/2$, it follows that

$$y_n - \frac{L}{2} = \frac{z_1 + 2z_2 + \cdots + nz_n}{n^2} + \left(\frac{n(n+1)}{2n^2}L - \frac{L}{2}\right) \tag{3}$$

The term in parentheses simplifies to $L/(2n)$ and therefore tends to 0 as $n \to \infty$. It remains to show that

$$\frac{z_1 + 2z_2 + \cdots + nz_n}{n^2} \to 0 \tag{4}$$

Given $\epsilon > 0$, let K be such that $|z_n| < \epsilon/2$ when $n \geq K$, and choose an integer N such that

$$N > \max\left(K, \frac{2MK^2}{\epsilon}\right) \tag{5}$$

I claim that $|z_1 + 2z_2 + \cdots + nz_n|/n^2 < \epsilon$ whenever $n \geq N$. Indeed, for such n we have

$$\begin{aligned}\frac{|z_1 + 2z_2 + \cdots + nz_n|}{n^2} &< M\frac{1+2+\cdots+K}{n^2} + \frac{\epsilon}{2}\frac{(K+1)+\cdots+n}{n^2} \\ &< M\frac{K^2}{n^2} + \frac{\epsilon}{2}\frac{n^2}{n^2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

completing the proof. □

MAT 601 HW 5.1: THE NOTION OF DERIVATIVE

`Due Friday 11/18`

Problem 2. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $|f|$ is differentiable, then f is also differentiable.

Lemma 1. *If* $\lim\limits_{t \to p} F(t) > 0$, *then there is* $\delta > 0$ *such that* $F(t) > 0$ *whenever* $0 < d(t,p) < \delta$. *(This works on any metric space, and works the same for "< 0").*

Proof. Let $L = \lim_{t \to p} F(x)$ and choose $\epsilon = L$. By the definition of limit there exists $\delta > 0$ such that $|F(t) - L| < \epsilon$ when $0 < d(t,p) < \delta$. This implies

$$-\epsilon < F(t) - L < \epsilon$$

hence $F(t) > L - \epsilon = 0$ as claimed. □

Proof. Consider a point $x \in \mathbb{R}$. There are three cases.

Case 1: $f(x) > 0$. By the lemma there exists $\delta > 0$ such that $f(t) > 0$ when $0 < |t - x| < \delta$. The existence of a limit as $t \to x$ is determined by values of t close to x, and for such t we have $|f(t)| = f(t)$, hence

$$\frac{f(t) - f(x)}{t - x} = \frac{|f(t)| - |f(x)|}{t - x} \to |f|'(x)$$

as $t \to x$, proving $f'(x)$ exists.

Case 2: $f(x) < 0$. Let $g = -f$. This is also a continuous function. Since $|g| = |f|$ everywhere, $|g|$ is continuous. Applying Case 1 to g, we get that g is differentiable at x. Hence f is differentiable there too (the product of differentiable function and the constant -1.)

Case 3: $f(x) = 0$. If $|f|'(x) > 0$ then by applying the lemma to $F(t) = |f(t)|/(t-x)$ we get $\delta > 0$ such that

$$\frac{|f(t)|}{t-x} > 0$$

when $0 < |t-x| < \delta$; but this is impossible when $t < x$. Similarly, the assumption $|f|'(x) < 0$ leads to $\frac{|f(t)|-0}{t-x} < 0$ around x, which can't happen when $t > x$. We conclude that $|f|'(x) = 0$. By definition of the limit this means that for every $\epsilon > 0$ there exists δ such that

$$\left|\frac{|f(t)|}{t-x}\right| < \epsilon \qquad \text{whenever} \qquad 0 < |t-x| < \delta.$$

But the above is the same as

$$\left|\frac{f(t)}{t-x}\right| < \epsilon \qquad \text{whenever} \qquad 0 < |t-x| < \delta.$$

which means $\lim_{t\to x} \frac{f(t)}{t-x} = 0$. □

Remark: The continuity of f is not needed in Case 3.

MAT 601 HW 5.2 SOLUTION: MEAN VALUE THEOREMS

Problem 2. Suppose that $f\colon (0,1) \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \le 1/\sqrt{x}$ for $x \in (0,1)$. Prove that $\lim_{x\to 0+} f(x)$ exists.

Proof 1. Let $g(x) = f(x) + 2\sqrt{x}$ and $h(x) = f(x) - 2\sqrt{x}$. Then $g'(x) \ge -1/\sqrt{x} + 1/\sqrt{x} = 0$ and $h'(x) \le 1/\sqrt{x} - 1/\sqrt{x} = 0$. Hence g is increasing and h is decreasing on the interval $(0,1)$. Also, $g(x) > h(x)$ for all $x \in (0,1)$.

For every $x \in (0,1/2)$ we have $g(x) \le g(1/2)$ and $g(x) \ge h(x) \ge h(1/2)$. Thus, g is bounded between $h(1/2)$ and $g(1/2)$ on this interval. As in the proof of Theorem 4.29, this leads to the conclusion that $g(0+) = \inf\{g(x)\colon 0 < x < 1/2\}$. Since $2\sqrt{x} \to 0$ as $x \to 0+$, it follows that $f(0+) = g(0+)$. ◻

Proof 2. Let $g(t) = f(t^2)$. By the chain rule, $g'(t) = 2tf'(t^2)$. Hence,

$$|g'(t)| = 2|t||f'(t^2)| \le 2|t|/\sqrt{t^2} = 2$$

for all $t \in (0,1)$. By the mean value theorem, $|g(t) - g(s)| \le 2|t-s|$ for all $s, t \in (0,1)$. This implies that g is uniformly continuous (as proved in class; specifically, one can take $\delta = \epsilon/2$). A uniformly continuous function on an open interval has limits at both ends; this is a special case of Problem 2 of Homework 4.3. So, there is $L \in \mathbb{R}$ such that for every $\epsilon > 0$ there is $\delta > 0$ with the property that $|g(t) - L| < \epsilon$ whenever $0 < t < \delta$. Returning to f, we see that $|f(x) - L| < \epsilon$ whenever $0 < x < \delta^2$. Thus, $f(0+) = L$. ◻

MAT 601 HW 4.5-6 SOLUTION: DISCONTINUITIES, MONOTONE FUNCTIONS

Problem 1. Let $f\colon X \to Y$ be a function, where X and Y are metric spaces. Let C be the set of all points of X at which f is continuous. Prove that C can be written as a countable intersection of open sets.

Proof. For $n = 1, 2, \dots$ let A_n the set of all points x such that $\operatorname{diam} f(N_r(x)) < 1/n$ for some $r > 0$.

Claim 1: A_n is open. Indeed, for every $x \in A_n$ we have $r > 0$ such that $\operatorname{diam} f(N_r(x)) < 1/n$. For $y \in N_r(x)$ let $\rho = r - d_X(x, y)$; then the triangle inequality implies $N_\rho(y) \subset N_r(x)$, hence $\operatorname{diam} f(N_\rho(y)) < 1/n$. This shows $y \in A_n$, hence x is an interior point of A_n.

Claim 2: $C \subset A_n$ for every n. Suppose $p \in C$; then by definition of continuity there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < 1/(3n)$ whenever $d_X(x, p) < \delta$. Therefore, for every two points $x, y \in N_\delta(p)$ we have

$$d_Y(f(x), f(y)) \le d_Y(f(x), f(p)) + d_Y(f(y), f(p)) < \frac{2}{3n}$$

This implies $\operatorname{diam} f(N_\delta(p)) \le \frac{2}{3n} < 1/n$, hence $p \in A_n$ as claimed.

Claim 3: If $p \in A_n$ for all n, then $p \in C$. Given $\epsilon > 0$, pick n such that $1/n < \epsilon$. Since $p \in A_n$, there exists r such that $\operatorname{diam} f(N_r(x)) < 1/n$. This implies $d_Y(f(x), f(p)) < 1/n < \epsilon$ whenever $d_X(x, p) < r$; thus, the definition of continuity is satisfied by taking $\delta = r$.

The combination of three claims shows that C is the intersection of open sets A_n. □

Remark: Not every set can be written as a countable intersection of open sets. For example, $\mathbb{Q}$ cannot be written in such a way. Indeed, suppose $\mathbb{Q} = \bigcap_{n=1}^{\infty} A_n$ and consider also the sets $B_n = \mathbb{R} \setminus \{r_n\}$ where $\{r_1, r_2, \ldots, \}$ is an enumeration of all rational numbers. Note that every A_n and every B_n is a dense open subset of $\mathbb{R}$. Since $\mathbb{R}$ is complete, Baire's category theorem implies that the intersection $A_1 \cap B_1 \cap A_2 \cap B_2 \cap \cdots$ must be dense. On the other hand, this intersection is $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ which is empty.

As a consequence of this remark and #1, we obtain that there does not exist a function $f \colon \mathbb{R} \to \mathbb{R}$ that is continuous at all rational points and discontinuous at all irrational points.

Problem 2. Define $f \colon \mathbb{R} \to \mathbb{R}$ so that $f(x) = 1$ when x is rational and $f(x) = 0$ when x is irrational. Prove that it is impossible to find two monotone functions $g, h \colon \mathbb{R} \to \mathbb{R}$ such that $f = g + h$.

Proof. We know that the set of discontinuities of a monotone function is at most countable. Since the sum of two functions continuous at p is also continuous at p, it follows that the sum of two monotone functions also has an at-most-countable set of discontinuities.

On the other hand, the given function f is discontinuous at every point, since every neighborhood of any point p contains both rational and irrational points, making it impossible to have $|f(x) - f(p)| < \epsilon$ with $\epsilon < 1$. □

MAT 601 REVIEW FOR EXAM 3

Exam 3 Monday covers 3.11-4.7.

1. Let X be a metric space. Suppose that $f\colon X \to \mathbb{R}$ is a continuous surjective function. Prove that the set $\{x \in X\colon f(x) \neq 0\}$ is not connected.

2. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that the set $\{f(x)\colon 0 < x < 1\}$ can be written as a countable union of closed sets.

3. Suppose that $f\colon (0,1) \to \mathbb{R}$ is an increasing bounded continuous function. Prove that f is uniformly continuous.

4. If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, does it follow that $g(x) = \dfrac{1}{(f(x))^2+1}$ is uniformly continuous?

5. Let a and b be distinct points of a connected metric space X. Let r be a number such that $0 < r < d(a,b)$. Prove that there exists $x \in X$ such that $d(x,a) = r$.

6. Suppose that $f\colon (0,+\infty) \to \mathbb{R}$ is a uniformly continuous function such that $f(2x) = f(x)$ for all $x \in (0,\infty)$. Prove that f is a constant function.

7. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that f is uniformly continuous.

8. If $\sum a_n$ converges absolutely, then $\sum |a_n - a_{n+1}|$ converges. Prove this is false without "absolutely".

MAT 601 HW 4.3 SOLUTION: CONTINUITY AND COMPACTNESS

Problem 1. Suppose that A and B are disjoint nonempty subsets of a metric space X. Let $d_B(x) = \inf\{d(x,b) : b \in B\}$ for $x \in X$.

(a) Prove that d_B is a continuous function on X.

(b) Suppose in addition that A is compact and B is closed. Prove that there exists $\epsilon > 0$ such that $d_B(x) \geq \epsilon$ for all $x \in A$.

(c) Show that the conclusion of the previous item fails if A is assumed to be closed instead of compact. (*Hint*: an example can be found with $X = \mathbb{R}$ and $B = \mathbb{N}$.)

Proof.

(a) The function $d_B(x)$ is well-defined since $\{d(x,b) : b \in B\}$ is bounded below by 0. I claim that

$$|d_B(x) - d_B(y)| \leq d(x,y) \quad \text{for all } x, y \in X, \tag{1}$$

from where the uniform continuity of d_B follows by taking $\delta = \epsilon$. To prove (1), note that for every $b \in B$ the triangle inequality implies

$$d_B(x) \leq d(x,b) \leq d(y,b) + d(x,y), \quad \text{hence } d_B(x) - d(x,y) \leq d(y,b)$$

So, $d_B(x) - d(x,y)$ is a lower bound for $\{d(y,b) : b \in B\}$, which by the definition of infimum yields $d_B(y) \geq d_B(x) - d(x,y)$. Thus $d_B(x) - d_B(y) \leq d(x,y)$, and the inequality $d_B(y) - d_B(x) \leq d(x,y)$ follows by interchanging x and y above. This completes the proof of (1).

(b) For every $x \in A$ we have $x \notin B$, and since B is closed, there exists $r > 0$ such that $N_r(x) \cap B = \emptyset$. So, r is a lower bound for $\{d(x,b) : b \in B\}$, which implies $d_B(x) \geq r > 0$.

Since d_B is continuous and A is compact, by the Extreme Value theorem there exists $a \in A$ such that $d_B(a) = \inf\{d_B(x) \colon x \in A\}$. Choose $\epsilon = d_B(a)$, which is positive; then $d_B(x) \geq \epsilon$ for all $x \in A$ as claimed.

(c) Let $B = \mathbb{N}$ and $A = \{n + 2^{-n} \colon n \in \mathbb{N}\}$. Both A and B are closed because they have no limit points: indeed, since $\lim(n + 2^{-n}) = \infty$, every neighborhood $N_r(x)$ of any point contains only finitely many points from A (and from B). Also, they are disjoint because $n + 2^{-n}$ has nonzero fractional part, namely 2^{-n}.

The fact that $d_{\mathbb{R}}(n + 2^{-n}, n) = 2^{-n}$ shows that $\inf_{x \in A} d_B(x) \leq 2^{-n}$. And since n is arbitrarily large, $\inf_{x \in A} d_B(x) = 0$.

Problem 2. Let E be a subset of a metric space X such that $E' = X$. Suppose that $f \colon E \to Y$ is a uniformly continuous function, where Y is a complete metric space. Prove that for every $p \in X$ there exists a limit $\lim\limits_{x \to p} f(x)$.

Proof. Given $p \in X$, consider any sequence $\{x_n\}$ of elements of $E \setminus \{p\}$ that converges to p. Being convergent, this sequence is Cauchy. Being uniformly continuous, f maps Cauchy sequences to Cauchy sequences (stated in class; proved below for completeness). Since $\{f(x_n)\}$ is a Cauchy sequence in a complete metric space, it has a limit $q \in Y$.

In order to use the sequential characterization of $\lim_{x \to p} f(x)$, we must show that q is the same for every sequence $\{x_n\}$ as above. One way to do this is the *interlacing trick*: given another sequence $\{y_n\}$ of elements of $E \setminus \{p\}$ that converges to p, consider the combined sequence $\{x_1, y_1, x_2, y_2, x_3, y_3, \dots\}$. It also converges to p, which by the above implies that $\{f(x_1), f(y_1), f(x_2), f(y_2), \dots\}$ has a limit. Since all subsequences of a convergent sequence converge to the same limit, we have $\lim f(x_n) = \lim f(y_n)$. $\square$

Lemma If $\{x_n\}$ is a Cauchy sequence in X and $f\colon X \to Y$ is uniformly continuous, then $\{f(x_n)\}$ is Cauchy.

Proof. Given $\epsilon > 0$, let $\delta > 0$ be as in the definition of uniform continuity, so that $d_X(a,b) < \delta \implies d_Y(f(a), f(b)) < \epsilon$. Pick N such that $d_X(x_n, x_m) < \delta$ whenever $m, n \geq N$. Then for $m, n \geq N$ we have $d_Y(f(x_n), f(x_m)) < \epsilon$ as required. □

MULTIPLIERS PRESERVING SERIES CONVERGENCE

Let's say that a sequence $\{b_n\}$ *preserves convergence of series* if for every convergent series $\sum a_n$, the series $\sum a_n b_n$ also converges.

Similarly, a sequence $\{b_n\}$ *preserves absolute convergence of series* if for every absolutely convergent series $\sum a_n$, the series $\sum a_n b_n$ also converges absolutely.

How to describe such convergence-preserving sequences?

Theorem 1. *A sequence $\{b_n\}$ preserves convergence of series if and only if it has **bounded variation**, meaning $\sum_{n=1}^{\infty} |b_n - b_{n+1}|$ converges. (One can say that $\{b_n\}$ is BV for brevity).*

The situation with absolute convergence is much simpler.

Theorem 2. *A sequence $\{b_n\}$ preserves absolute convergence of series if and only if it is **bounded**.*

Remark 3. By the triangle inequality, every BV sequence is Cauchy:

$$|b_n - b_m| \le \sum_{k=m}^{n-1} |b_k - b_{k-1}|$$

where the sum on the right is small when m, n are large. Thus, a BV sequence has a limit. But the converse is false: for example, the sequence $\{(-1)^n/n\}$ has a limit but is not BV since

$$\sum_{n=1}^{\infty} |b_n - b_{n+1}| = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n+1}\right)$$

diverges by comparison to harmonic series.

We'll prove Theorem 2 first, as its proof is much simpler.

Proof of Theorem 2. If there is M such that $|b_n| \le M$ for all n, then $|a_n b_n| \le M|a_n|$, so the series $\sum |a_n b_n|$ converges by comparison to $M \sum |a_n|$.

Conversely, suppose b_n is unbounded. Then for every k there is n_k such that $|b_{n_k}| \geq 2^k$. We can ensure $n_k > n_{k-1}$ since there are infinitely many candidates for n_k. Define

$$a_n = \begin{cases} 2^{-k} & \text{if } n = n_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum a_n$ converges, since its partial sums are bounded by $\sum_{k=1}^{\infty} 2^{-k} = 1$. But $\sum a_n b_n$ diverges because its terms do not approach 0: we have $|a_n b_n| \geq 1$ for infinitely many values of n. □

The proof of Theorem 1 requires two lemmas.

Lemma 4. *A sequence $\{b_n\}$ is BV if and only if there are two increasing bounded sequences $\{c_n\}$ and $\{d_n\}$ such that $b_n = c_n - d_n$ for all n.*

Proof. If such c_n, d_n exist, then by the triangle inequality

$$\begin{aligned} \sum_{n=1}^{N} |b_n - b_{n+1}| &= \sum_{n=1}^{N} (|c_n - c_{n+1}| + |d_{n+1} - d_n|) \\ &= \sum_{n=1}^{N} (c_{n+1} - c_n) + \sum_{n=1}^{N} (d_{n+1} - d_n) \end{aligned}$$

and the latter sums telescope to $c_{N+1} - c_1 + d_{N+1} - d_1$ which has a limit as $N \to \infty$ since bounded monotone sequences converge.

Conversely, suppose $\{b_n\}$ is BV. Let $c_n = \sum_{k=1}^{n-1} |b_k - b_{k+1}|$, understanding that $c_1 = 0$. By construction, the sequence $\{c_n\}$ is increasing and bounded. Also let $d_n = c_n - b_n$; as a difference of bounded sequences, this is bounded too. Finally,

$$d_{n+1} - d_n = c_{n+1} - c_n + b_n - b_{n+1} = |b_n - b_{n+1}| + b_n - b_{n+1} \geq 0$$

which shows that $\{d_n\}$ is increasing. □

Lemma 5. *If a series of nonnegative terms $\sum A_n$ diverges, then there is a sequence $c_n \to 0$ such that the series $\sum c_n A_n$ still diverges.*

Proof. This was proved in class. (The idea was to make it so that partial sums of $\sum c_n A_n$ are square roots of the partial sums of $\sum A_n$.) □

Proof of Theorem 1. Suppose $\{b_n\}$ is BV. Using Lemma 4, write $b_n = c_n - d_n$. Since $a_n b_n = a_n c_n - a_n d_n$, it suffices to prove that $\sum a_n c_n$ and $\sum a_n d_n$ converge. Consider the first one; the proof for the other is the same. Let $L = \lim c_n$ and write $a_n c_n = La_n - a_n(L - c_n)$. Here $\sum La_n$ converges as a constant multiple of $\sum a_n$. Also, $\sum a_n(L - c_n)$ converges by the Dirichlet test: the partial sums of $\sum a_n$ are bounded, and $L - c_n$ decreases to zero.

For the converse, suppose $\{b_n\}$ is not BV. The goal is to find a convergent series $\sum a_n$ such that $\sum a_n b_n$ diverges. If $\{b_n\}$ is not bounded, then the proof of Theorem 2 applies here as well, delivering an absolutely convergent series $\sum a_n$ such that $a_n b_n \not\to 0$. It remains to consider the case when $\{b_n\}$ is a bounded sequence.

Since $\sum_{n=1}^{\infty} |b_n - b_{n+1}|$ diverges, by Lemma 5 there exists $\{c_n\}$ such that $c_n \to 0$ and $\sum_{n=1}^{\infty} c_n |b_n - b_{n+1}|$ diverges. Let d_n be such that $d_n(b_n - b_{n+1}) = c_n|b_n - b_{n+1}|$; that is, d_n differs from c_n only by sign. In particular, $d_n \to 0$. Summation by parts yields

$$\sum_{n=1}^{N} d_n(b_n - b_{n+1}) = \sum_{n=2}^{N} (d_n - d_{n-1})b_n + d_1 b_1 - d_N b_{N+1} \tag{1}$$

As $N \to \infty$, the left hand side of (1) does not have a limit since $\sum d_n(b_n - b_{n+1})$ diverges. On the other hand, $d_1 b_1 - d_N b_{N+1} \to d_1 b_1$ since $d_N \to 0$ while b_{N+1} stays bounded. Therefore,

$$\lim_{N\to\infty} \sum_{n=2}^{N} (d_n - d_{n-1})b_n \quad \text{does not exist} \tag{2}$$

Let $a_n = d_n - d_{n-1}$. The series $\sum a_n$ converges (by telescoping, since $\lim_{n\to\infty} d_n$ exists) but $\sum a_n b_n$ diverges, as shown by (2). □

MAT 601 HW 3.9-10 SOLUTION: ROOT AND RATIO TESTS; POWER SERIES

Problem 1. Determine, with a proof, the set of all complex numbers z such that the following power series converges.

$$\sum_{n=1}^{\infty}(2+(-1)^n)^n z^n$$

Claim: the set of convergence is $\{z \in \mathbb{C} \colon |z| < 1/3\}$.

Proof. If $|z| < 1/3$, then

$$|(2+(-1)^n)^n z^n|^{1/n} \le (3^n|z|^n)^{1/n} = 3|z|$$

hence $\limsup_{n\to\infty} |(2+(-1)^n)^n z^n|^{1/n} \le 3|z| < 1$, which implies the series converges by the Root Test.

If $|z| \ge 1/3$, then for even n we have

$$|(2+(-1)^n)^n z^n| \ge 3^n(1/3)^n = 1$$

Since the terms of the series do not converge to 0, the series diverges. □

Problem 2. Determine, with a proof, the set of all complex numbers z such that the following power series converges.

$$\sum_{n=1}^{\infty}\frac{n!}{3^n}z^{2^n}$$

Claim: the set of convergence is $\{z \in \mathbb{C} \colon |z| < 1\}$.

Proof. The series clearly converges when $z = 0$. For $z \ne 0$ we can use the Ratio test, which seems practical because of $n!$:

$$\left|\frac{(n+1)!}{3^{n+1}}z^{2^{n+1}}\right| \Big/ \left|\frac{n!}{3^n}|z|^{2^n}\right| = \frac{n+1}{3}|z|^{2^n}$$

where $|z|^{2^{n+1}}/|z|^{2^n} = |z|^{2^{n+1}-2^n} = |z|^{2^n}$.

When $|z| < 1$, we have $(n+1)|z|^{2^n} \leq (n+1)|z|^n \to 0$, because the exponential function beats any power of n (considered in section 3.5). Thus, the series converges by the Ratio Test.

When $|z| \geq 1$, the ratio $\dfrac{n+1}{3}|z|^{2^n} \geq \dfrac{n+1}{3}$ tends to ∞. The series converges by the Ratio Test. $\square$

MAT 601 HW 3.7 SOLUTION: SERIES WITH POSITIVE TERMS

Problem 1. Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ such that $a_n > 0$ for all n, and the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ diverges.

Let $P = \{2^j : j = 0, 1, 2, 3, \dots\}$ and define

$$a_n = \begin{cases} 1/n, & n \in P \\ 1/n^2, & n \notin P \end{cases}$$

To prove $\sum a_n$ converges, it suffices to show the partial sums are bounded above. We have

$$\sum_{k=1}^{n} a_k \le \sum_{k \in P,\ k \le n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k^2}$$

The first sum on the right is bounded by $\sum_{j=0}^{\infty} 1/2^j = 2$. The second one is bounded because $\sum 1/k^2$ converges. Thus, $\sum a_k$ converges.

On the other hand, $2^k a_{2^k} = 2^k 1/2^k = 1$ for every k, so $\sum_{k=0}^{\infty} 2^k a_{2^k}$ diverges (the terms do not approach 0.)

Problem 2. Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ such that $a_n > 0$ for all n, and the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Let $P = \{2^j : j = 0, 1, 2, 3, \dots\}$ and define

$$a_n = \begin{cases} 1/n^2, & n \in P \\ 1, & n \notin P \end{cases}$$

Since the terms do not approach 0, the series $\sum a_n$ diverges. On the other hand, $2^k a_{2^k} = 2^k \cdot 1/(2^k)^2 = 1/2^k$ for every k, so $\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 1/2^k$ converges as a geometric series.

MAT 601 HW 3.6 SOLUTION: SERIES

Problem 1. Define a sequence by $a_1 = 1/3$ and $a_{n+1} = a_n^2$ for $n = 1, 2, \ldots$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. The geometric series $\sum_{n=1}^{\infty}(1/3)^n$ is known to converge. Since $a_n \geq 0$ for all n it it suffices to show that $a_n \leq (1/3)^n$ to prove that $\sum a_n$ converges by comparison.

Base case: $a_1 = (1/3)^1$. Assuming $a_n \leq (1/3)^n$, we get $a_{n+1} = a_n^2 \leq (1/3)^{2n} \leq (1/3)^{n+1}$, completing the proof by induction. □

Problem 2. Define a sequence by $a_1 = 1/3$ and $a_{n+1} = a_n(1 - a_n)$ for $n = 1, 2, \ldots$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. The series $\sum \frac{1}{3n}$ diverges since its partial sums are $1/3$ of the partial sums of the harmonic series $\sum 1/n$, which was shown to be divergent. Thus, it suffices to prove that $a_n \geq 1/(3n)$ for all n.

Base case: $a_1 = 1/3$.

The inductive step requires some preparation. From homework 3.4, the sequence a_n decreases to zero; thus $0 < a_n \leq 1/3$ for all n. Observe that when $0 < x \leq y < 1/2$, we have $1 - 2x \geq 1 - 2y > 0$, hence $(1 - 2x)^2 \geq (1 - 2y)^2$. This leads to

$$x(1 - x) = 1 - (1 - 2x)^2 \leq 1 - (1 - 2y)^2 = y(1 - y) \tag{1}$$

Now, assuming $a_n \geq 1/(3n)$ and using (1) we get

$$a_{n+1} = a_n(1 - a_n) \geq \frac{1}{3n}\left(1 - \frac{1}{3n}\right) = \frac{3n - 1}{9n^2}$$

It remains to show that the right hand side is $\geq 1/(3(n+1))$. To this end, notice that

$$\frac{3n-1}{9n^2} - \frac{1}{3(n+1)} = \frac{(3n-1)(n+1)-3n^2}{9n^2(n+1)} = \frac{2n-1}{9n^2(n+1)} > 0$$

which completes the inductive proof. □

LIMSUP "EXERCISE"

Define the sequence $\{x_n\}$ as follows: x_1 is some number in $(0,1)$ and $x_{n+1} = 4x_n(1-x_n)$ for $n \in \mathbb{N}$.

Theorem 1. $\limsup x_n = 1$ *if and only if*
(A) the number $\alpha = \frac{1}{\pi}\cos^{-1}(1-2x_1)$ *is irrational, and*
(B) the binary expansion of α *has arbitrarily long runs of the same digit.*

Remark 2. Condition (B) means that for any k, the binary expansion of α has either k consecutive 0s or k consecutive 1s.

Remark 3. Condition (A) holds when $x_1 = 1/3$ (proof will be given at the end) but I don't know whether condition (B) holds.

Remark 4. $\limsup x_n = 1$ implies that $\liminf x_n = 0$ because when x_n is close to 1, its successor $x_{n+1} = 4x_n(1-x_n)$ is close to 0.

Proof of Theorem 1. The quadratic polynomial $f(x) = 4x(1-x)$ maps the interval $[0,1]$ onto itself. Observe that the linear function $g(x) = 1-2x$ maps $[0,1]$ onto $[-1,1]$. It follows that the composition $h = g \circ f \circ g^{-1}$ maps $[-1,1]$ onto $[-1,1]$. This composition is easy to compute:

$$h(x) = 1 - 2f((1-x)/2) = 1 - 4(1-x)(1+x)/2 = 2x^2 - 1$$

We want to know whether the iteration of the map f, starting from x_1, produces numbers arbitrarily close to 1. Since

$$f \circ f \circ \cdots \circ f = g^{-1} \circ h \circ h \circ \cdots \circ h \circ g$$

the goal is equivalent to finding whether the iteration of h, starting from $g(x_1)$, produces numbers arbitrarily close to $g(1) = -1$. To shorten formulas, let's write $h^{(n)}$ for the nth iterate of h, for example, $h^{(3)} = h \circ h \circ h$.

So far we traded one quadratic polynomial f for another, h. But h satisfies a nice identity: $h(\cos t) = 2\cos^2 t - 1 = \cos(2t)$, hence

$$h^{(n)}(\cos t) = \cos(2^n t), \qquad n \in \mathbb{N}$$

Recalling the definition of α from Theorem 1, we see that

$$h^{(n)}(g(x_1)) = h^{(n)}(\cos 2\pi\alpha) = \cos(2^n \cdot 2\pi\alpha)$$

The problem becomes to determine whether the numbers $2^n \cdot 2\pi\alpha$ come arbitrarily close to π, modulo an integer multiple of 2π. Dividing by 2π rephrases this as: does the fractional part of $2^n\alpha$ come arbitrarily close to $1/2$?

A number that is close to $1/2$ has binary expansion beginning either with $0.01111111\ldots$ or with $0.10000000\ldots$. Since the binary expansion of $2^n\alpha$ is just the binary expansion of α shifted n digits to the left, we conclude that the property $\limsup x_n = 1$ is equivalent to the following: for every $k \in \mathbb{N}$ the binary expansion of α has infinitely many groups of the form "1 followed by k 0s" or "0 followed by k 1s".

A periodic expansion cannot have the above property; this, α must be irrational. The property described above can then be simplified to "irrational and has arbitrarily long runs of the same digit", since a long run of 0s will be preceded by a 1, and vice versa. ☐

To prove Remark 3, we need a lemma.

Lemma 5. *For every $n \in \mathbb{N}$ there exists a monic polynomial P_n with integer coefficients such that $P_n(2\cos t) = 2\cos nt$ for all t.*

Proof. Induction, the base case $n = 1$ being $P_1(x) = x$. Assuming the result for integers $\leq n$, we have

$$\begin{aligned} 2\cos(n+1)t &= e^{i(n+1)t} + e^{-i(n+1)t} \\ &= (e^{int} + e^{-int})(e^{it} + e^{-it}) - (e^{i(n-1)t} + e^{-i(n-1)t}) \\ &= P_n(2\cos t)(2\cos t) - P_{n-1}(2\cos t) \end{aligned}$$

which is a monic polynomial of $2\cos t$. ☐

Remark 6. P_n are related to the Chebyshev polynomials of the 1st kind: `https://en.wikipedia.org/wiki/Chebyshev_polynomials` Namely, $P_n(x) = 2T_n(x/2)$ where T_n is a Chebyshev polynomial.

Proof of Remark 3. Let $\alpha = \frac{1}{2\pi}\cos^{-1}(1/3)$. Suppose to the contrary that there exists n such that $n\alpha \in \mathbb{Z}$. Then $2\cos(2\pi n\alpha) = 2$. By Lemma 5 this means $P_n(2\cos(2\pi\alpha)) = 2$, that is $P_n(2/3) = 2$.

Since $2/3$ is a root of a polynomial with integer coefficients, the Rational Root Theorem implies that 3 divides the leading coefficient of P_n. This contradicts the fact that P_n is a monic polynomial (the leading coefficient is 1). □

Remark 7. I learned the proof of Remark 3 from a newsgroup post by Robert Israel, `http://mathforum.org/kb/message.jspa?messageID=1675813`

Final Exam (Friday 12/16, 3-5 pm) is cumulative. There will be 8 problems; do any 7 of them. Sample problem set:

1. Let X be a metric space. Suppose that (x_n) is a sequence of elements of X such that $d(x_n, x_m) \geqslant 1$ whenever $n \neq m$. Prove that X is not compact. BW: compact ⇒ ∀ seq. has conv subseq

2. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $f(0) = f'(0) = 0$ and $f''(x) \geqslant 10$ for all $x \in \mathbb{R}$. Prove that $f(2) \geqslant 20$. Taylor

3. Suppose that (x_n) is a Cauchy sequence in $\mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that $(f(x_n))$ is a Cauchy sequence.

4. Suppose $\{x_n : n = 1, 2, \dots\}$ is a sequence in a complete metric space X such that the series $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges. Prove that $\lim_{n\to\infty} x_n$ exists.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two differentiable functions. You are given that $f'(x) = g'(x)$ for all $x \neq 0$. Prove that $f'(0) = g'(0)$. • intermediate values attained by deriv's

6. Let a and b be distinct points of a connected metric space X. Let r be a number such that $0 < r < d(a, b)$. Prove that there exists $x \in X$ such that $d(x, a) = r$.

7. Let E be a closed subset of a metric space X. Prove that there exists a sequence of open sets $G_n \subset X$ such that (i) $G_{n+1} \subset G_n$ for all n; (ii) $\bigcap_{n=1}^{\infty} G_n = E$.

8. Suppose that $f\colon (0, +\infty) \to \mathbb{R}$ is a uniformly continuous function such that $f(2x) = f(x)$ for all $x \in (0, \infty)$. Prove that f is a constant function.

9. Given two power series: $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R_1, and $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence R_2. Suppose that $0 < R_1 < R_2 < \infty$. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ is equal to R_1.

10. The power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 3. The radius of convergence of $\sum_{n=0}^{\infty} b_n z^n$ is equal to 4. Prove that the radius of convergence of $\sum_{n=0}^{\infty} (a_n b_n) z^n$ is at least 12.

11. Let Y be a metric space. Suppose that $f\colon \mathbb{R} \to Y$ is a function such that for any $x \in \mathbb{R}$

$$\lim_{t\to x} \frac{d(f(t), f(x))}{|t - x|} = 0$$

$g(x) = d_Y(f(x), f(0))$
get $|g(t) - g(x)| \leq d(f(t), f(x)) \to 0$

Prove that f is a constant function; that is, $f(x) = f(0)$ for all $x \in \mathbb{R}$.

12. Prove or disprove: "If $f\colon \mathbb{R} \to \mathbb{R}$ is a function differentiable at 0, then $\lim_{x\to 0} \frac{f(x) - f(0)}{\sqrt{|x|}} = 0$."

MATH 601 EXAM 1 SOLUTION

1. Suppose that $f\colon \mathbb{R} \to \mathbb{Z}$ is a function, and define a relation $\prec$ on $\mathbb{R}$ by

$$x \prec y \quad \text{if and only if} \quad f(x) < f(y)$$

Prove that $\prec$ is **not** an order.

Proof. Suppose $\prec$ is an order. By the Trichotomy property, whenever $x \neq y$ we have either $f(x) < f(y)$ or $f(y) < f(x)$. Thus, f is injective. It follows that $f(\mathbb{R})$ is equivalent to $\mathbb{R}$ and therefore uncountable. But $f(\mathbb{R}) \subset \mathbb{Z}$, and every subset of a countable set is at most countable. This contradiction shows that $\prec$ is not an order. ☐

2. Suppose that a and b are complex numbers such that $|a+b| \geqslant 5$ and $|a-b| < 1$. Prove that $|a| > 2$.

Proof. Suppose to the contrary that $|a| \leqslant 2$. By the triangle inequality,

$$|b| = |a + (b-a)| \leqslant |a| + |b-a| < 2 + 1 = 3$$

hence

$$|a+b| \leqslant |a| + |b| < 2 + 3 = 5$$

contradicting the assumption $|a+b| \geqslant 5$. ☐

3. For $x, y \in \mathbb{R}$ let $d(x,y) = \sqrt{|x-y|}$. Prove that d is a metric.

Proof. The square root is nonnegative. It is zero iff $|x-y| = 0$, that is $x = y$. The symmetry follows from $|x-y| = |y-x|$. To prove the triangle inequality, note that

$$d(x,y)^2 = |x-y| \leqslant |x-z| + |y-z| \leqslant |x-z| + |y-z| + 2\sqrt{|x-z||y-z|} = (d(x,z) + d(z,x))^2$$

for any $x, y, z \in \mathbb{R}$. Taking square roots yields $d(x,y) \leqslant d(x,z) + d(z,x)$. ☐

4. Suppose that A is a nonempty, open and bounded subset of $\mathbb{R}$. Prove that $\sup A \notin A$.

Proof. Let $s = \sup A$ and suppose that $s \in A$. Then there exists $r > 0$ such that $N_r(s) \subset A$. In particular $s + r/2 \in A$. But $s + r/2 > s$, contradicting that s is the supremum of A. ☐

5. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that for every $t \in \mathbb{R}$ the set $\{x\colon |f(x)| \leqslant t\}$ is compact. Prove that there exists $a \in \mathbb{R}$ such that $|f(x)| \geqslant |f(a)|$ for all $x \in \mathbb{R}$.

Proof. The set $E = \{|f(x)|\colon x \in \mathbb{R}\}$ is nonempty and bounded below by 0, hence it has an infimum. Let $u = \inf E$. For $n \in \mathbb{N}$ let $K_n = \{x\colon |f(x)| \leqslant u + 1/n\}$. By assumption, each K_n is compact. Also, these sets are nested: $K_1 \supset K_2 \supset \dots$ and nonempty (since $u + 1/n$ is not a lower bound for E). By the theorem on nested compact sets, the intersection $K = \bigcap_{n=1}^{\infty} K_n$ is nonempty. Pick $a \in K$. Since $u \leqslant |f(a)| \leqslant u + 1/n$ for every n, it follows that $|f(a)| = u$. Recalling that u is a lower bound for E, we conclude that $|f(x)| \geqslant |f(a)|$ for every $x \in \mathbb{R}$. ☐

1. Suppose that A, B are disjoint closed sets in a metric space X such that $A \cup B$ is perfect. Prove that both A and B are perfect.

Proof. Suppose the conclusion is false. Without loss of generality, A is not perfect. Since it's closed, this means there is a point $x \in A$ and a neighborhood $N_r(x)$ such that $N_r(x) \cap A = \{x\}$. Since A and B are disjoint, we have $x \notin B$. As B is closed, there is a neighborhood $N_s(x)$ that is disjoint from B. Let $\rho = \min(r, s)$. Then $N_\rho(x) \cap (A \cup B) = \{x\}$ which shows that x is an isolated point of $A \cup B$, contradicting the assumption that $A \cap B$ is perfect. □

2. Suppose that $\{x_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} x_n = L \in \mathbb{R}$. Prove that $\limsup_{n\to\infty}(-1)^n x_n = |L|$.

Proof. Let $y_n = (-1)^n x_n$. We have $y_{2k} = x_{2k} \to L$ and $y_{2k-1} = -x_{2k-1} \to -L$, so both L and $-L$ are subsequential limits. It remains to prove that there are no other subsequential limits, because then the statement $\limsup_{n\to\infty} y_n = \sup\{L, -L\} = |L|$ will follow.

Suppose y_{n_k} is a convergent subsequence. If infinitely many of the indices n_k are even, these terms form a subsubsequence that converges to L. Hence $y_{n_k} \to L$ (the limit of a convergent sequence is equal to the limit of any of its subsequences).

If only finitely many of the indices n_k are even, then infinitely many of them are odd. They form a subsubsequence that converges to $-L$. Hence $y_{n_k} \to -L$. This proves that $\pm L$ are the only subsequential limits. □

Alternative proof, using tail supremum

Proof. For every $\epsilon > 0$ there exists N such that $|x_n - L| < \epsilon$ for all $n \geqslant N$. This implies

$$(-1)^n x_n \leqslant |x_n| \leqslant |L| + \epsilon$$

hence $\limsup_{n\to\infty}(-1)^n x_n \leqslant |L| + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have

$$\limsup_{n\to\infty}(-1)^n x_n \leqslant |L|$$

On the other hand, the subsequence $\{(-1)^{2n} x_{2n}\}$ converges to L while the subsequence the subsequence $\{(-1)^{2n-1} x_{2n}\}$ converges to $-L$. Since both L and $-L$ are subsequential limits of $\{(-1)^n x_n\}$, it follows that

$$\limsup_{n\to\infty}(-1)^n x_n \geqslant \max(L, -L) = |L|$$

completing the proof. □

3. Suppose that $\{x_n\}$ is a Cauchy sequence in a metric space (X, d), and that this sequence does not converge. Prove that the set $E = \{x_n : n \in \mathbb{N}\}$ is closed in X.

Proof. Suppose to the contrary that there is a limit point x of E that does not belong to E. Then each neighborhood of x contains infinitely many points of E. Using this fact, for each $k \in \mathbb{N}$ we can pick an index n_k such that $x_{n_k} \in N_{1/k}(x)$ and $n_k > n_{k-1}$ (the latter is not applicable when $k = 1$). This creates a subsequence $\{x_{n_k}\}$ that converges to x, since $d(x_{n_k}, x) < 1/k$. But a Cauchy sequence with a convergent subsequence converges (proved in class), contradicting the assumption. □

4. Let $a_1 = 1$ and $a_{n+1} = (a_n/2)^2$ for $n = 1, 2, \ldots$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. The terms are nonnegative by definition. Claim: $a_n \leqslant 1/4^{n-1}$ for all n. The base case is $a_1 = 1 = 1/4^0$. Inductive step: if $a_n \leqslant 1/4^{n-1}$, then

$$a_{n+1} = a_n^2/4 \leqslant a_n/4 \leqslant 1/4^n$$

Since the geometric series $\sum 1/4^{n-1}$ converges, the series $\sum a_n$ converges by comparison. □

5. Suppose that a power series $\sum_{n=0}^{\infty} c_n z^n$ converges at $z = a$ and $z = b$ (here a and b are complex numbers). Prove that it also converges at $z = \dfrac{a+b}{2}$.

Proof. If $a = b$, then $(a+b)/2 = a$, so the series converges there. Suppose $a \neq b$. Let R be the radius of convergence of the series. Since it diverges for $|z| > R$, it follows that $|a| \leqslant R$ and $|b| \leqslant R$. In order to prove convergence at $(a+b)/2$, it suffices to show that $|(a+b)/2| < R$.

This can be done by canceling cross-products as follows:

$$|a+b|^2 + |a-b|^2 = (a+b)(\bar{a}+\bar{b}) + (a-b)(\bar{a}-\bar{b}) = 2|a|^2 + 2|b|^2 \leqslant 4R^2$$

hence $|a+b|^2 \leqslant 4R^2 - |a-b|^2 < 4R^2$, and $|(a+b)/2| < R$. □

MATH 601 EXAM 3 SOLUTION

1. Suppose $f: \mathbb{R} \to \mathbb{R}$ is a bounded monotone function. Let $a_n = f(n+1) - f(n)$ for $n \in \mathbb{N}$. Prove that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Proof. If f is decreasing, then $-f$ is increasing. Since $|a_n|$ is the same for $-f$ as for f, it suffices to consider the case of increasing f.

Since $f(n+1) \geqslant f(n)$, we have $|a_n| = f(n+1) - f(n)$. Hence, a partial sum $\sum_{n=1}^{N} |a_n|$ simplifies to

$$f(2) - f(1) + f(3) - f(2) + \cdots + f(N+1) - f(N) = f(N+1) - f(1)$$

There exists M such that $|f(x)| \leqslant M$ for all x. Hence,

$$\sum_{n=1}^{N} |a_n| \leqslant 2M$$

for all N. A series with nonnegative terms converges iff its partial sums are bounded. □

2. Let X, Y be metric spaces. Suppose $f: X \to Y$ is a surjection such that

$$d_Y(f(a), f(b)) \geqslant d_X(a,b)^2 \quad \text{for all } a, b \in X.$$

Prove that for every open set $A \subset X$ the set $f(A)$ is open in Y.

Proof. If a, b are distinct points of X, then $d_Y(f(a), f(b)) \geqslant d_X(a,b)^2 > 0$, hence f is injective. Since it's also surjective, it has an inverse $g = f^{-1}$. We get

$$d_X(g(p), g(q)) \leqslant \sqrt{d_Y(p,q)} \text{ for all } p, q \in Y.$$

by using the assumption about f with $a = g(p)$, $b = g(q)$. Therefore, g is uniformly continuous: given $\epsilon > 0$, let $\delta = \epsilon^2$ so that

$$d_Y(p,q) < \delta \implies d_X(g(p), g(q)) < \epsilon$$

The inverse image of an open set under a continuous map is open. Thus $g^{-1}(A)$ is open in Y. But $g^{-1} = f$, so $f(A)$ is open. □

3. Let X be a compact metric space. For $x \in X$ define $f(x) = \sup\{d_X(x,y) : y \in X\}$. Prove that there exists $a \in X$ such that $f(a) = \inf\{f(x) : x \in X\}$.

Proof. The existence of a follows from the Extreme Value Theorem once we prove that f is continuous. To this end, consider any $x, z \in X$. Since the distance function $d(y) = d(x,y)$ is continuous, it attains its supremum on X at some point y. That is, $f(x) = d(x,y)$. By the triangle inequality

$$f(z) \geqslant d(z,y) \geqslant d(x,y) - d(x,z) = f(x) - d(x,z)$$

We have proved $f(x) - f(z) \leqslant d(x, z)$. Applying the same to x, z in opposite order yields $f(z) - f(x) \leqslant d(x, z)$. So, $|f(x) - f(z)| \leqslant d(x, z)$, which implies that f is uniformly continuous: given $\epsilon > 0$ we can let $\delta = \epsilon$, and the definition holds. □

4. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $\lim_{x \to a} f(x) = 0$ for every $a \in \mathbb{R}$. Prove that the set $\{x \in \mathbb{R}\colon f(x) \neq 0\}$ is at most countable.

Proof. By assumption, for every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists a neighborhood N of x such that $|f| < \epsilon$ at every point of $N \setminus \{x\}$. For each $n \in \mathbb{N}$ the set $[-n, n]$ can be covered by finitely many such neighborhoods (since it's compact). Taking the union over n, conclude that $\mathbb{R}$ can be covered by countably many such neighborhoods. Therefore, the set $\{x\colon |f(x)| \geqslant \epsilon\}$ is at most countable (it can only contain the centers of the neighborhoods we used to cover $\mathbb{R}$).

Since

$$\{x \in \mathbb{R}\colon f(x) \neq 0\} = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}\colon |f(x)| \geqslant 1/k\}$$

and each set on the right is at most countable, it follows that the set on the left is at most countable. □