HW 1 Notes

$E, F \subseteq \mathbb{R}$, $E - F = \{x - y : x \in E, y \in F\}$

Show $\sup(E-F) = \sup E - \inf F$

Let $M = -F = \{z : z = -y, \text{ for some } y \in F\}$

So prove: $\sup(E+M) = \sup E + \sup M$ $\quad E+M = \{x+z \mid x \in E, z \in M\}$

Step 1:

WTS $\sup(E+M) \leq \sup E + \sup M$

Consider any $x \in E$ and $z \in M$, so $x+z \in E+M$

It follows from def of sup. that $x \leq \sup E$ and $z \leq \sup M$

$\Rightarrow x + z \leq \sup E + \sup M$

We see that $w = x+z$ can be any number in $E+M$.

$\therefore w \leq \sup E + \sup M \Rightarrow \sup E + \sup M$ is an upper bound of $E+M$

By def. of $\sup(E+M)$ being LUP, $\sup E + \sup M \geq \sup(E+M)$

Step 2:

WTS $\sup(E+M) \geq \sup E + \sup M$

$\forall \varepsilon > 0$, we have $\sup(E+M) \geq \sup E + \sup M - \varepsilon$,

which would imply $\sup(E+M) \geq \sup E + \sup M$.

By def. of sup., $\exists x_1 \in E$ s.t. $x_1 \geq \sup E - \frac{1}{2}\varepsilon$

$\exists z_1 \in M$ s.t. $z_1 \geq \sup M - \frac{1}{2}\varepsilon$

Adding these inequalities we obtain $x_1 + z_1 \geq \sup E + \sup M - \varepsilon$

Hence $\sup(E+M) \underbrace{\geq}_{\text{By def. of sup.}} x_1 + z_1 \geq \sup E + \sup M - \varepsilon$, since $x_1 + z_1 \in E+M$

$f : X \to Y$ $\quad f(A \cap B) = f(A) \cap f(B)$ show that f is $1 \to 1$

Let $f(x_1) = f(x_2)$. WTS $x_1 = x_2$

Suppose, to the contrary, that $x_1 \neq x_2$

$\{x_1\} \cap \{x_2\} = \emptyset$

$f(\{x_1\}) \cap f(\{x_2\}) = f(x_1) = f(x_2) \neq \emptyset$ ... solved already

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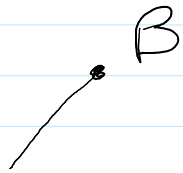
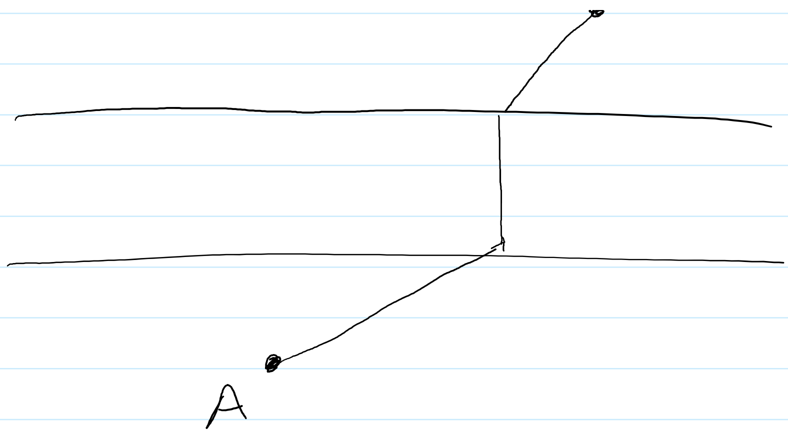
Exercise



Build bridge



Build bridge s.t. distance is shortest possible?

HW 2

1.3.9) $a \leq x_n \leq b$

WTS $\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n > N \ |x_n - x| < \varepsilon$

Su ose to the contrary it fails.

$\Rightarrow \exists \varepsilon_0 > 0 \ \forall N$ s.t. $\exists n > N \ |x_n - x| \geq \varepsilon_0$

Choose and fix ε_0.

Take $N = 1$: $\exists n_1 > 1$ s.t. $|x_{n_1} - x| \geq \varepsilon_0$

Take $N = n_1$: $\exists n_2 > n_1$ s.t $|x_{n_2} - x| \geq \varepsilon_0$

$N = n_2$: $\exists n_3 > n_2$ s.t. $|x_{n_3} - x| \geq \varepsilon_0$

$\vdots$

$N = n_k$: $\exists n_k > n_{k-1}$ s.t $|x_{n_k} - x| \geq \varepsilon_0$

$\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and thus bounded.

$\Rightarrow \exists$ some subsequence of $\{x_{n_k}\}$ call it $\{a_{n_k}\}$ that converges

$a_{n_k} \to a$ for some $a \neq x$ since $|a_{n_k} - x| \geq \varepsilon_0 \Rightarrow\Leftarrow$

since $\{a_{n_k}\}$ is also a subsequence of $\{x_n\}$ and must converge to x if it does

1.3.12) $\dfrac{x_1 + \dots + x_n}{n} \to 0$ if $x_n \to 0$

$$\left| \frac{x_1 + \dots + x_n}{n} \right| < \varepsilon \text{ for } n > N$$

$$\leq \frac{\overbrace{|x_1| + \dots}^{\text{Fixed}} + \overbrace{|x_n|}^{\leq \varepsilon/2}}{n}$$

$$\leq \frac{M + |x_k| + |x_{k+1}| + \dots + |x_n|}{n} \leq \frac{M + (n-k)\varepsilon}{n}$$

TM

$$\le \frac{\cdots}{n} \le \frac{\cdots}{n}$$

$$\le \frac{M}{n} + \frac{(n-k)\varepsilon}{n}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

for $n > 2M/\varepsilon$

HW #3

Problem #5

Induction ...

a) $a_{n+1} \le \frac{b_{n+1}}{b_n} \cdot a_n \le \frac{b_{n+1}}{\cancel{b_n}} \cdot \frac{\cancel{b_n}}{b_{n-1}} \cdots \frac{\cancel{b_2}}{b_1} \le \frac{b_{n+1}}{b_1}$

1.1

Thursday, August 23, 2018 4:01 PM

8/28/18

Section 1.1: Sets and Functions

$\mathbb{X}$ - a collection of things called elements or points

$x \in \mathbb{X}$, $x \notin \mathbb{X}$ $\quad \mathbb{X} = \{a, I, \Delta, 5\}$

$x, y \in \mathbb{X}$, $x, y, z \in \mathbb{X}$

Cantor: Let $\mathbb{F}$ be a collection of sets which are not members of themselves.

Question: $\underbrace{\mathbb{F} \in \mathbb{F}}$ or $\mathbb{F} \notin \mathbb{F}$?

Can't be by definition. $\quad$ Then $\mathbb{F} \in \mathbb{F}$ by definition $\otimes$

$\boxed{\text{The sentence on the other side is false.}}$ $\quad$ $\boxed{\text{The sentence on the other side is false}}$ $\quad$ Opposite sides of the same paper.

Subset

$A \subseteq \mathbb{X}$ $\quad$ A is contained in $\mathbb{X} \Rightarrow \forall a \in A,\ a \in \mathbb{X}$

$\mathbb{X} \supseteq A$ $\quad$ $\mathbb{X}$ contains A

$(A \subset X)$, $A = \mathbb{X}$

Proper Subset: $A \subseteq X$ and $A \neq X$ (Can use $A \subsetneq X$)

Empty Set: $\emptyset$

Natural Numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$

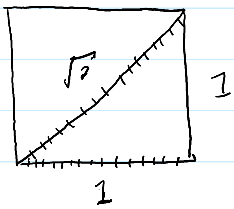
Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} = \{n - m \mid n, m \in \mathbb{N}\}$

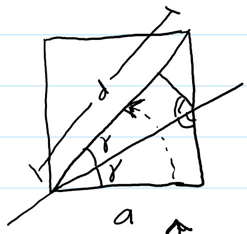
Rational Numbers: $\mathbb{Q} = \{p/q \mid q \in \mathbb{N},\ p \in \mathbb{Z}\}$

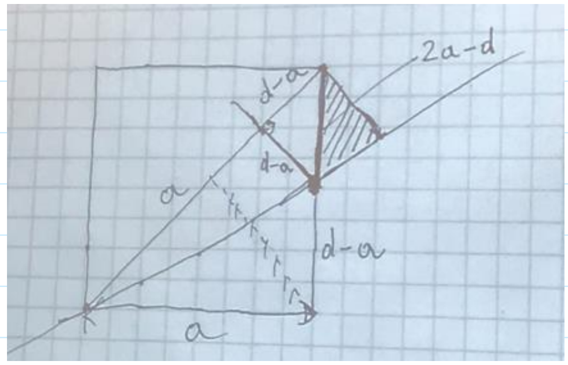
Example: "$\sqrt{2}$ is not rational"

Pf:

WTS $\nexists\ p/q \in \mathbb{Q}$ s.t. $p^2/q^2 = 2$







Assume smallest possible square.

⨂ from finding smaller square.

Intersection: $A, B \subseteq X$

$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$

disjoint if $A \cap B = \emptyset$

Union: $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$

Examples 1.1.2, 1.1.3, 1.1.4

Prop 1.1.5 (Distributive Laws)

$A, B, C \subseteq X$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Pf:

WTS $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$.

$\Rightarrow x \in A$ and $x \in B \cup C$

$x \in A$ and $x \in B$ or $x \in C$

$\left.\begin{array}{l} \text{If } x \in B \Rightarrow x \in A \cap B. \\ \text{If } x \in C \Rightarrow x \in A \cap C. \end{array}\right\} \Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (B \cap C)$

Let $y \in (A \cap B) \cup (A \cap C)$.

$\Rightarrow y \in A \cap B$ or $y \in A \cap C$

$\Rightarrow y \in A$ and $y \in B$ or $y \in A$ and $y \in C$

In either case, $y \in A$.

$\Rightarrow y \in B$ or $y \in C \Rightarrow y \in B \cup C$

$\Rightarrow y \in A \cap (B \cup C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$\Rightarrow$ They are equal #

Difference of Sets: $A \setminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}$

Example 1.1.6 is hw. *

Complement: If the "universe" X is fixed.

$A \subseteq X$, $X \setminus A$ is the compliment.

Note that $X \setminus X \setminus A = A$

Prop 1.1.7 De Morgan's Law

1) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

2) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Infine Collection of Sets: $A_1, A_2, \dots$

$$\bigcap_{i=1}^{\infty} A_n = \{x \in X \mid x \in A_n \ \forall n \in \mathbb{N}\}$$

$$\bigcup_{i=1}^{\infty} A_n = \{x \in X \mid x \in A_n \text{ for some } n \in \mathbb{N}\}$$

Remark: Uncountable collection of sets $\{A_\alpha\}_{\alpha \in F}$

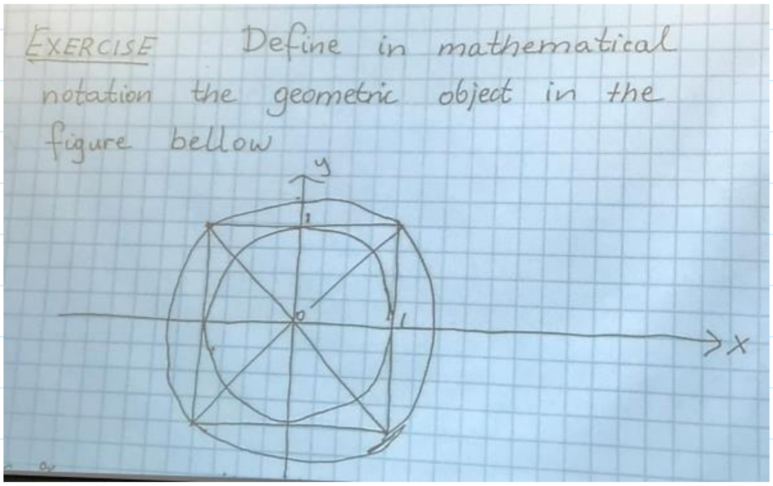
$$\bigcap_{\alpha \in F} A_\alpha = \{x \in X \mid x \in A_\alpha \ \forall \alpha \in F\}$$

$$\bigcup_{\alpha \in F} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in F\}$$

$$X \setminus \bigcup_{\alpha \in F} A_\alpha = \bigcap_{\alpha \in F} (X \setminus A_\alpha)$$

$$X \setminus \bigcap_{\alpha \in F} A_\alpha = \bigcup_{\alpha \in F} (X \setminus A_\alpha)$$

Exercise:



$Q = \{(x,y) \mid -1 \le x \le 1, -1 \le y \le 1\}$

$D = \{(x,y) \mid x = y, -1 \le x \le 1\}$

$C = \{(x,y) \mid x^2 + y^2 = 1\}$

$\text{Disk} = \{(x,y) \mid x^2 + y^2 \le 1\}$

Function: X, Y - sets $f : X \to Y$

Synonyms: map, mapping, transformation, ...

X is domain of f

Y is range of f (target)

$f(X) = \{f(x) \in Y \mid x \in X\}$ is called the image

Example 1.1.9

- $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = x^2$

- sign function $\begin{cases} f(x) = +1 & \text{if } x \ge 0 \\ f(x) = -1 & \text{if } x < 0 \end{cases}$

Day 2 8/30/18 ↓

- $A \subseteq X$, Characteristic (Indicator) Function

  $\chi_A : X \to \mathbb{R} \quad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

  Note: $- \chi_\emptyset(x) = 0 \quad \forall x \in X$

  $\chi_X(x) = 1 \quad \forall x \in X$

- Constant function $f : X \to Y$

  $f(x) = y_0 \quad \forall x \in X$ and some $y_0 \in Y$

  That is $f(X) = \{y_0\}$

<u>Composition</u>: $f: X \to Y$, $g: Y \to Z$
then
$$g \circ f: X \to Z \text{ and } g \circ f = g(f(x))$$
Note the image of f must be subset of the domain of g

<u>Ex:</u>
$f(x) = x^2$, $g(y) = y^3$ on $\mathbb{R}$
$g \circ f = [f(x)]^3 = (x^2)^3 = x^6$
Incidentally, $f \circ g = f(y)^2 = (y^3)^2 = y^6$

<u>Def</u>: 1.1.10
A) $f: X \to Y$ is <u>surjective</u> or <u>onto</u> if $\forall y \in Y \; \exists x \in X$ s.t. $y = f(x)$.

B) $f: X \xrightarrow{1-1} Y$ is <u>injective</u> or <u>one-to-one</u> if $\forall x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$

c) $f: X \to Y$ is <u>bijective</u> if f is onto and one-to-one.

<u>Ex.</u> (1) $f: \mathbb{Q} \to \mathbb{Q}$; $f(x) = x+1$ is bijective
(2) $f: \mathbb{N} \to \mathbb{N}$; $f(x) = x^2$ is injective but not bijective/surjective
(3) $f: \mathbb{Z} \setminus \{0\} \to \{n^2 \mid n \in \mathbb{N}\}$, $f(x) = x^2$ is surjective but not bijective

(1) Let $y \in \mathbb{Q}$.
$y = f(y-1)$ and $y-1 \in \mathbb{Q}$ since $y, 1 \in \mathbb{Q}$
$\Rightarrow$ surjective.
Let $x_1, x_2 \in \mathbb{Q}$ s.t. $f(x_1) = f(x_2)$
$\Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2 \Rightarrow$ injective
$\Rightarrow$ bijective #

(2) Let $x_1, x_2 \in \mathbb{N}$ s.t. $f(x_1) = f(x_2)$.
$\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ since $x_1, x_2 \in \mathbb{N}$
$\Rightarrow$ injective
Note, $2 \in \mathbb{N}$ but $\sqrt{2} \notin \mathbb{N} \Rightarrow$ not surjective

(3) Let $y \in \{n^2 \mid n \in \mathbb{N}\} \Rightarrow y = m^2$ for some $m \in \mathbb{N}$ by def.
$m \in \mathbb{Z} \setminus \{0\}$ since $\mathbb{N} \subseteq \mathbb{Z} \setminus \{0\}$
Note: $-1, 1 \in \mathbb{Z} \setminus \{0\}$ and $f(-1) = f(1) = 1$ but $1 \neq -1$. #

1.2

Thursday, August 30, 2018 3:32 PM

Real Numbers

Axiom (Density Property)

- $a<b$ rational numbers. Then $\exists$ an irrational number x, s.t. $a<x<b$
- $a<b$ irrational numbers. Then $\exists$ rational number x, s.t. $a<x<b$

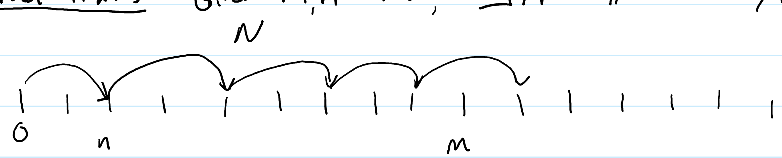
Def: $E \subseteq \mathbb{R}$ is bounded above if $\exists\, a \in \mathbb{R}$ s.t. $x \leq a \;\forall x \in E$.
Such a is called an upper bound of E.
Similarly: E is bounded below if $\exists\, b \in \mathbb{R}$ s.t. $b \geq x \;\forall x \in E$.
Such b is called a lower bound.
E is bounded if it is bounded below and bounded above.

Def: (Least Upper Bound or Supremum)
$E \subseteq \mathbb{R}$ bounded above. The least upper bound (LUB) is a number α that satisfies: Denoted: $\alpha = \sup E$
(i) α is an upper bound of E
(ii) $\alpha \leq a$ for any other upper bound a.

Def: (Greatest Lower Bound or Infimum)
$E \subseteq \mathbb{R}$ bounded below. The greatest lower bound (GLB) is a number β that satisfies Denoted: $\beta = \inf E$
(i) β is a lower bound
(ii) $b \leq \beta$ for any other lower bound b

Axiom of Completeness: If a set has a UP it has a supremum.
If a set has a LB it has an infimum.

Lemma 1.2.5 Given $m, n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ s.t. $Nn > m$.



Prop 1.2.6 $x, \varepsilon \in \mathbb{Q}$, $x, \varepsilon > 0$ then $\exists N \in \mathbb{N}$ s.t. $N\varepsilon > x$

Pf:

Let $x = \frac{a}{b}$, $a,b \in \mathbb{N}$ and $\varepsilon = \frac{c}{d}$, $c,d \in \mathbb{N}$

$\varepsilon - \frac{x}{n} = \frac{c}{d} - \frac{a}{nb} = \frac{nbc - ad}{nbd} > 0$ since $nbc > ad$ for large n by Lemma 1.2.5

$\Rightarrow x < N\varepsilon$ #

Lemma 1.2.7 There is no natural number x s.t. $x^2 = 2$.

Pf: If there were such $x = n/m$ w/ n,m having no common divisor other than ± 1.

$2 = x^2 = m^2/n^2$

$\Rightarrow 2m^2 = n^2 \Rightarrow n^2$ is even $\Rightarrow n$ is even

$\Rightarrow \exists k \in \mathbb{Z}$ s.t. $n = 2k$

$\Rightarrow 2m^2 = (2k)^2 = 4k^2 \Rightarrow 2m^2 = 4k^2 \Rightarrow m = 2k^2$

$\Rightarrow m$ is even $\Rightarrow$ m and n have a common divisor $\Rightarrow \Leftarrow$

$\Rightarrow$ no such x exists #

Lemma 1.2.8 If $a,b \in \mathbb{Q}^+$ s.t. $a^2 < b^2$, then $a < b$.

Pf: $0 < b^2 - a^2 = (b-a)(b+a) \Rightarrow b - a > 0 \Rightarrow b > a$

Prop 1.2.9 The set $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$ is bounded above, but has no supremum in $\mathbb{Q}$.

Pf:

(1) A is bounded $a < 2$.

$a^2 < 2 < 4 = 2^2 \Rightarrow a < 2$ by Lemma 1.2.8

$\Rightarrow A$ is bounded above.

(2) $\sup A \in \mathbb{R}$ does exist by completeness

(3) Suppose to the contrary

Define $x = \sup A \in \mathbb{Q}$ $\quad (x > 1$ since $1 \in A)$

Case 1: Consider $x^2 > 2$ (We shall see that $x \neq \sup A$)

Note $(x - \frac{1}{n})^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2} > 2$ for large n, because

$x^2 - 2 > \frac{2x - 1/n}{n} \Rightarrow n(x^2 - 2) > 2x - 1/n$

Case 2: $x^2 < 2$

Indeed $(x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2$ for large n, because

$(2x + \frac{1}{n}) < (2 - x^2)n$

$2x + 1 < (2 - x^2)n$ ⇑

$\Rightarrow x^2 = 2$ but no such x exists #

⋮

<u>Prop 1.2.15</u>

(a) ?

(b) If $\alpha, \beta \in \mathbb{R}$, then either $\alpha \leq \beta$ or $\beta \leq \alpha$

(c) (Trichotomy) If $\gamma \in \mathbb{R}$, then either $\gamma < 0$, $\gamma > 0$, or $\gamma = 0$

<u>Thm 1.2.16</u> $E \subseteq \mathbb{R}$ nonempty, bounded above, then $\sup E$ exists.

↓

Note: We say $\mathbb{R}$ has the <u>completeness property</u>.

? {

<u>Prop 1.2.19</u> If $\alpha, \beta, \gamma \in \mathbb{R}$, the following holds

(i) $\alpha + \beta = \beta + \alpha$

(ii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma = \alpha + \beta + \gamma$

(iii) $\alpha + 0 = \alpha$

<u>Prop 1.2.20</u> $\forall \alpha \in \mathbb{R}\ \exists \beta \in \mathbb{R}$ s.t. $\alpha + \beta = 0$ where β is unique.

We denote $-\alpha = \beta$.

- Then $-(-\alpha) = \alpha$.
- If $\alpha \geq 0$ then $-\alpha \leq 0$

<u>Pf:</u> (That β' is unique)

Suppose $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$.

$\alpha + \beta_1 + \beta_2 = 0 + \beta_2 = \beta_2$

$\alpha + \beta_1 + \beta_2 = \alpha + \beta_2 + \beta_1 = 0 + \beta_1 = \beta_1 \Rightarrow \beta_1 = \beta_2$ #

<u>Notation/Def:</u> $\mathbb{R}_+ = \{\alpha \in \mathbb{R} \mid \alpha > 0\}$

? {

<u>Prop 1.2.25</u> If $\alpha, \beta, \gamma \in \mathbb{R}^+$, then

(a) $\alpha\beta = \beta\alpha$

(b) $(\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha\beta\gamma$

(c) $\alpha = 1 \cdot \alpha$ where 1 is unique.

(d) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

(e) $\alpha^{-1}\alpha^{1} = 1$

<u>Pf:</u> (That 1 is unique).

$1_2 1_1 = 1_2$ and $1_1 1_2 = 1_1$ (by (c)) $\Rightarrow 1_1 = 1_2 \Rightarrow$ unique

Def: (Absolute Value, Modulus)

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \geq 0 \\ -\alpha & \text{if } \alpha < 0 \end{cases}$$

Prop 1.2.27

(a) $|\alpha| = |-\alpha|$

(b) $|\alpha\beta| = |\alpha||\beta|$

(c) $|\alpha+\beta| \leq |\alpha| + |\beta|$ (Triangle Inequality)

(d) $\big||\alpha| - |\beta|\big| \leq |\alpha - \beta|$

Pf: of (b)

Case 1) $\alpha \geq 0$ and $\beta \geq 0$

$|\alpha\beta| = \alpha\beta$ and $|\alpha||\beta| = \alpha\beta$

Case 2) $\alpha \geq 0$ and $\beta \leq 0$

$|\alpha\beta| = -\alpha\beta = |\alpha||\beta|$

Case 3) same as case 2

Case 4) $\alpha \leq 0$ and $\beta \leq 0$

$|\alpha\beta| = \alpha\beta = (-\alpha)(-\beta) = |\alpha||\beta|$

Exercise: Solve $|3x-6| + x - |2x-8| = 3$

2 4

$-(3x-6) + x - (-(2x-8)) = 3$

$6 - 3x + x + 2x - 8 = 3$

$3 = -2 \Rightarrow \Leftarrow$

no solution.

(If like $-2 = -2$, then all $x < 2$ sol.)

$3x - 6 + x - (-(2x-8)) = 3$

$3x - 6 + x + 2x - 8 = 3$

$6x = 17$

$x = 17/6 \in (2,4)$

$\Rightarrow$ is sol.

$3x - 6 + x - (2x-8) = 3$

$2x + 2 = 3$

$2x = 1$

$x = 1/2 \notin (4, \infty)$

so no solution

Thm: 1.2.8

(a) If $\alpha, \beta \in \mathbb{Q}$ where $\alpha < \beta$ then $\exists$ an irrational number γ s.t. $\alpha < \gamma < \beta$

(b) If $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha < \beta$, then $\exists x \in \mathbb{Q}$ s.t. $\alpha < x < \beta$

Pf: Already given.

1.3 Convergence

Thursday, September 6, 2018 3:19 PM

Def: Open Interval $I = \{x \in \mathbb{R} \mid c < x < d\} = (c, d)$
where $-\infty \le c < d \le \infty$

Note we could have: $(-\infty, d)$, (c, ∞), $(-\infty, \infty)$

Def: Closed Interval $I = \{x \in \mathbb{R} \mid c \le x \le d\} = [c, d]$
where $-\infty < c \le d < \infty$

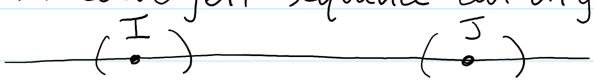
Def: Half-Open Interval: $[c, d)$ or $(c, d]$

Def: A sequence in $\mathbb{R}$ is an enumeration of some real numbers: $a_1, a_2, \ldots$
$\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$

Notice a sequence is a function $f: \mathbb{N} \to \mathbb{R}$

Def: $\{a_n\}$ converges to a if every open interval containing a, contains all but a finite number of terms of the sequence.
$a_n \to a$ or $a = \lim_n a_n = \lim_{n \to \infty} a_n = \lim a_n$

Prop 1.3.2: A convergent sequence can only converge to one point.



Prop 1.3.3

(a) $a_n \to a$ iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon\ \ \forall n \ge N$
(N need not be an integer)

(b) $a_n \to a$ iff $(a_n - a) \to 0$

Pf:

(a) Suppose $a_n \to a$ and $\varepsilon > 0$.
$I = (a-\varepsilon, a+\varepsilon)$ contains all but finite number of terms of a_n
$\Rightarrow a - \varepsilon < a_n < a + \varepsilon$ for $n \ge N$.
Conversely, let $I = (c, d)$ contain a. $\exists \varepsilon > 0$ s.t.
$c < a - \varepsilon < a < a + \varepsilon < d$.
There is an N s.t. $|a_n - a| < \varepsilon$ for $n \ge N$.

Prop 1.3.6

$a_n \to a$, $b_n \to b$ then

(i) $a_n + b_n \to a + b$

(ii) $a_n b_n \to ab$

(iii) If $b \neq 0$ then $b_n \neq 0$ for large n, then $\frac{b_n}{a_n} \to \frac{b}{a}$

Pf:

(ii) $a_n b_n - ab = (a_n - a) b_n + a(b_n - b)$

$$|a_n b_n - ab| \leq |a_n - a||b_n| + |a||b_n - b| \leq \underbrace{|a_n - a|}_{\text{small b/c}} M + |a| \underbrace{|b_n - b|}_{\text{convergent}} \leq \varepsilon$$

Convergent sequence bounded (haven't stated)

Def: If $\{a_n\}$ is a sequence and $n_1 < n_2 < \dots$ then $\{a_{n_k}\}_{k=1,2,\dots}$ is called a subsequence.

Prop 1.3.10 If $a_n \to a$ then $a_{n_k} \to a$

$\lim_{k \to \infty} a_{n_k} = \infty$, $|a_{n_k} - a| < \varepsilon$ $\forall k$ but finite number

Thm: Bolzano-Weierstrass

A bounded sequence has a convergent subsequence.

Pf:

Assume that the sequence $\{x_n\}$ contains infinitely many distinct points.

If not, there are infinitely equal points that form a convergent subsequence

☆Insert Image

J_1 is a half of J_0 which contains an infinite number of distinct points

J_2 is a half of J_1 which contains an infinite number of distinct points.

$\vdots$

J_k is a half of J_{k-1} which contains an infinite number of distinct points.

We have $J_0 \supseteq J_1 \supseteq J_2 \supseteq \dots \supseteq J_{k-1} \supseteq J_k \supseteq \dots$

The left endpoints of $J_1, \dots$ denoted $l_1, l_2, \dots$ form an increasing sequence, thus converging to $l \geq x$.

The right endpoints $r_1, r_2, \dots$ form a decreasing sequence, thus converging to $r \le y$.

Note that $0 \le r_k - l_k = \frac{y-x}{2^k}$. Hence $0 \le r - l \le r_k - l_k \to 0$

Define $r = l \overset{\text{Def}}{=} a$

Now choose $a_{n_1} \in J_1$

$a_{n_2} \in J_2 \quad n_2 > n_1 \quad$ ($n_2 > n_1$ b/c ∞ly many points in J_2)

$a_{n_3} \in J_3 \quad n_3 > n_2$

$\vdots$

$a_{n_k} \in J_k \quad n_k > n_{k-1}$

We conclude $\underbrace{l_{n_k} - a}_{\downarrow\, 0} \le a_{n_k} - a \le \underbrace{r_{n_k} - a}_{\downarrow\, 0}$

so $a_n \to 0$ $\blacksquare$

<u>Def:</u> (i) $a_n \to +\infty$ if $\forall R\ \exists N$ s.t. $\forall n > N$, $a_n > R$

(ii) $a_n \to -\infty$ if $\forall R\ \exists N$ s.t. $\forall n > N$, $a_n < R$

<u>Def:</u> (Limit Superior or Upper Limit)

Sequnce $\{a_n\}$

$$\lim_n \sup a_n = \limsup a_n = \lim_n \sup\{a_n, a_{n+1}, \dots\} \quad (= \overline{\lim}\, a_n)$$

- $x_n = \sup\{a_n, a_{n+1}, \dots\}$ is a decreasing sequence.
  - ↳ Thus it has a limit.

<u>Def:</u> (Limit Inferior or Lower Limit)

$$\lim_n \inf a_n = \liminf a_n = \lim \inf\{a_n, a_{n+1}, \dots\}$$

- Increasing sequence, thus it has a limit.

<u>Exercise</u> 1.3.17)

Show that $\exists$ $\{a_n\}$, $\{b_n\}$ of real numbers s.t.

$\limsup (a_n + b_n) \ne \limsup a_n + \limsup b_n$

<u>Solution:</u>

Note: $\limsup (a_n + b_n) = \lim \sup\{a_n + b_n, a_{n+1} + b_{n+1}, \dots\}$

$\le \lim \left[\sup\{a_n, a_{n+1}, \dots\} + \sup\{b_n, b_{n+1}, \dots\}\right]$

$= \limsup a_n + \limsup b_n$

Consider $a_n = (-1)^n$ and $b_n = -(-1)^n$

$a_n + b_n = 0 \quad \limsup (a_n + b_n) = 0$

$\limsup a_n = 1 \quad \limsup b_n = 1$

Note: Any counter example must be nonconverging sequences otherwise $\limsup x_n = \lim x_n = x$ if $x_n \to x$.

<u>Def:</u> (Cauchy Sequence)

$\{a_n\}$ is a Cauchy Sequence if $\forall \varepsilon > 0$

<u>Thm:</u> Cauchy $\Leftrightarrow$ convergent

⋮

1.4 Infinite Series

Tuesday, September 11, 2018 4:42 PM

$\sum_{n=1}^{\infty} a_n$ converges if ...

Prop 1.4.2 $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow a_n \to 0$

Ex: Harmonic Series

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots = \sum \frac{1}{n}$

Lets show we can't add this sum.

Suppose $\cancel{1 + 1/2 + 1/3 + \dots} = S$

$\cancel{1/2 + 1/4 + 1/6 + \dots} = 1/2\, S$

$\Rightarrow \frac{1}{1} + 1/3 + 1/5 + 1/7 + \dots = S - 1/2\, S = 1/2\, S$

$\Rightarrow \frac{1}{2} + 1/4 + \frac{1}{6} + \dots = 1/2\, S$

$\Rightarrow \frac{1}{1} + 1/3 + 1/5 + \dots = 1/2\, S$ $\Leftarrow$ Denomnators larger $\Rightarrow$ $1/2\, S > 1/2\, S$ $\otimes$

Cauchy Condition: $S_n = \sum_{k=1}^{\infty} a_k$

$|S_n - S_m| < \varepsilon \quad n, m > \varepsilon$

$|S_n - S_{n+1}| < \varepsilon \quad n > N$

necessary but not sufficient for $a_n \to 0$

Used to show harmonic diverges

On board ??

Ex: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots = 1$

via telescoping decomposition.

Pf:

Ex: $1 + \frac{1}{} + 1 + \dots + 1 \quad -$

Ex: $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \dots + \frac{1}{(k-1)k(k+1)} =$

Why?

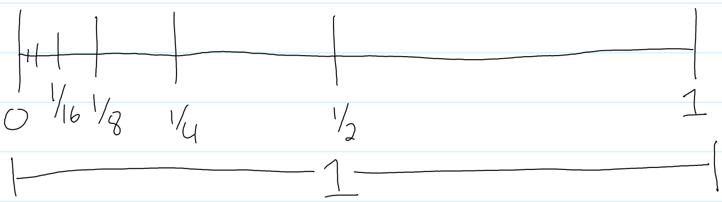
Note: $\frac{1}{(k-1)k(k+1)} = \frac{1}{k-1} - \frac{2}{k} + \frac{1}{k+1}$

$$\Rightarrow \text{Sum} = \left(\frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \frac{1}{5}\right)$$
$$+ \left(\cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}\right) + \dots + \left(\cancel{\frac{1}{k}} - \frac{2}{k} + \frac{1}{k+1}\right)$$

$$= \frac{1}{2} - \frac{1}{k+1} + \frac{1}{k+2} \longrightarrow \frac{1}{2}$$

Exercise: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$

why?



Adding them all we must get 1.

Exercise 11 on pg 24

If $|x|<1$, show that $\lim_n x^n = 0$.

Proof:

Note $\{x_n\}$ is decreasing. Thus it has a limit.

Let $\lim |x|^n = a$.

Hence $\lim |x|^{n+1} = a|x|$

$a = \lim |x|^n = \lim |x|^{n+1} = a|x|$

Hence $a=0$.

Proposition 1.4.4 Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x|<1$$

and diverges for $|x|>1$.

Note: (On board)

$1 + x + x^2 + x^3 + \dots + x^{n-1} + x^n$.

$+(1-x)(1+x+x^2+\dots+x^{n-1}+x^n)$

$$1+\cancel{x}+\cancel{x^2}+\dots+\cancel{x^{n-1}}+\cancel{x^n}$$
$$-\cancel{x}-\cancel{x^2}-\dots-\cancel{x^{n-1}}-\cancel{x^n}-x^{n+1}$$
$$=1-x^{n-1}\to 1$$

<u>Proposition 1.4.5</u> Cauchy Criterion

$\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon>0\ \exists N$ s.t. $\forall n,m\geq N$
$|a_m+a_{m+1}+\dots+a_n|<\varepsilon$

<u>Proof:</u> Text

<u>Proposition 1.4.6</u>
If $a_n\geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges
iff
the partial sums are bounded

<u>Comparison Test:</u>
Let $|a_n|\leq b_n$ for $n\geq N$.
If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
Moreover $\left|\sum_{n=1}^{\infty} a_n\right|\leq\sum_{n=1}^{\infty}|a_n|\leq\sum_{n=1}^{\infty} b_n$

<u>Pf:</u>
$|a_n+a_{n+1}+\dots+a_m|\leq|a_n|+|a_{n+1}|+\dots+|a_m|\leq b_n+b_{n+1}+\dots+b_m<\varepsilon$
b/c of Cauchy Criterion on b_n

<u>Corollary 1.4.8</u> If $0\leq a_n\leq b_n$ for $n\geq N$ and $\sum a_n$ diverges
then $\sum b_n$ diverges

<u>Def:</u> $\sum a_n$ <u>converges absolutely</u> if $\sum|a_n|$ converges

<u>Proof 1:</u> Verify the Cauchy Condition
$|a_n+a_{n+1}+\dots+a_m|\leq|a_n|+|a_{n+1}|+\dots+|a_m|<\varepsilon$ for $m,n\geq N$

<u>Example 1.4.11</u> $\sum(-1)^n\frac{1}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots$
converges but not absolutely

Why?
$$\left|\sum_{k=m}^{n}(-1)^k\frac{1}{k}\right|=\frac{1}{m}+\left(-\frac{1}{m+1}+\frac{1}{m+2}\right)+\left(\frac{1}{m+3}+\frac{1}{m+4}\right)+\dots+\begin{cases}\text{either} \\ -\frac{1}{n-1}+\frac{1}{n} \\ \text{or} \\ -\frac{1}{n}\end{cases}<\frac{1}{m}$$

On board:

On board:

$$1 + x + x^2 + \dots = \frac{1}{1-x}$$

$\Rightarrow$ by integration $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\log(1-x)$

If $x = -1$, then $= -\log(2)$

Similarly, $\sum (-1)^n a_n$, for $a_n \geq a_{n+1} \geq a_{n+2} \geq \dots \to 0$ — Alternating Series Test

then the same proof gives $\left| \sum_{k=m}^{n} (-1)^k a_k \right| < a_m$

Root Test: (Thm 1.4.12)

$\{a_n\}$ a sequence in $\mathbb{R}$

$r = \limsup |a_n|^{\frac{1}{n}}$

(a) If $r < 1$, then $\sum a_n$ converges absolutely

(b) If $r > 1$, then $\sum a_n$ diverges ($a_n \not\to 0$)

Proof:

(a) If $r < 1$, then $\underbrace{|a_n|^{\frac{1}{n}} < x}_{\text{otherwise } r > x} < 1$, for $n \geq N$

$\Rightarrow |a_n| < x^n$ and we compare with geometric series

(b) If $r > 1$, then $|a_n|^{\frac{1}{n}} > 1$ for infinitely many n and hence $a_n \not\to 0$

Remark: If $r = 1$, there is no conclusion

Ratio Test: (Thm 1.4.14)

Assume $R = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists.

(a) If $R < 1$, then the series $\sum a_n$ converges absolutely.

(b) If $R > 1$, then the series $\sum a_n$ diverges

Proof:

Let $R < 1$, choose and fix $R < x < 1$.

$$\left| \frac{a_{n+1}}{a_n} \right| \leq x \quad \text{for } n \geq N$$

$$a_{n+k} \leq x^k |a_N| \quad \text{for } n > N$$

Comparison Test works

Remark:

Under the assumption of the Ratio Test,
Show that
$$r = \limsup |a_n|^{\frac{1}{n}} \le \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = R < 1$$

Proof:
$$a_n = \underbrace{\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{k+1}}{a_k}}_{(n-k \text{ terms})} \cdot \underbrace{\left(\frac{a_{k+2}}{a_k} \cdots \frac{a_2}{a_1}\, a_1\right)}_{A_k\text{-product}}$$

Given any $\varepsilon > 0$, $\exists K$ s.t. $\left|\frac{a_{K+1}}{a_K}\right|, \left|\frac{a_{K+2}}{a_{K+1}}\right|, \ldots < R + \varepsilon$

since the limit is R.

Now consider $n > K$ We have
$$|a_n| \le (R+\varepsilon)^{n-K} |A_K| \quad (K \text{ is fixed})$$
Let $n \to \infty$ to obtain
$$|a_n|^{\frac{1}{n}} \le (R+\varepsilon)^{1-K/n} |A_K|^{\frac{1}{n}} \to (R+\varepsilon)$$
Since ε is arbitrary we have that $r = \limsup |a^{\frac{1}{n}}| \le R$ #

It is well known that:

For any sequence $\{a_n\}$ of positive numbers,
$$\liminf \frac{a_{n+1}}{a_n} \le \liminf (a_n)^{1/n} \le \limsup (a_n)^{1/n} \le \limsup \left(\frac{a_{n+1}}{a_n}\right)$$

In particular, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists then $r = R$.

Homework: 5, 6, 8, 9 on pages 28 and 29

Ex: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0 < r$ for every $r > 0$

$\Rightarrow$ converges for any x

$f(0) = 1$ so $f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$
$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

Hence $f'(x) = f(x)$!!

$$\Rightarrow \frac{f'(x)}{f(x)} = 1 \Rightarrow \log f(x) = x + C$$
$$f(x) = Ce^x \text{ and } f(0) = 1$$

thus $f(x) = e^x$

So we have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

1.5 Countable and Uncountable Sets

Tuesday, September 18, 2018 3:35 PM

Def: A set A is countable if it is finite or $\exists$ a bijection $f: A \to \mathbb{N}$
$f^{-1}: \mathbb{N} \to A$

- Countably infinite sets

Def: A set A is said to be uncountable if it fails to be countable.

Intuition: The countable infinite sets can be elements of a sequence.

Ex: $f: \mathbb{N} \to \mathbb{Z}$ bijection

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, \dots\}$
$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$

Why?
$m \in \mathbb{Z}$ show $m = \frac{n}{2}$ or $m = -\frac{n-1}{2}$
Case 1: $m > 0$ then $n = 2m$ even
Case 2: $m \leq 0$ then $n = 2m+1$ odd

Prop 1.5.3
a) If $\exists A \subseteq \mathbb{N}$ and a surjeetive function $f: A \xrightarrow{\text{onto}} X$ then X is countable.
(We can cover X by a subset of $\mathbb{N}$.)
b If X is countable and $\exists$ a surjective $f: X \xrightarrow{\text{onto}} Y$ then Y is countable.

Pf:
(a) $\forall x \in X$ Let $n = n_x$ be the first integer in A with $f(n) = x$.
So $B = \{n_x ; x \in X\}$ is another subset on $\mathbb{N}$ and the

function $g(n_x) = x$ is a bijection.
$\Rightarrow X$ is countable.
(b) Immediate from part (a) #

Def: Cartesian Product
$X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$
$X_1 \times X_2 \times \dots \times X_n = \{(x_1, x_2, \dots, x_n\} \mid x_i \in X_i \quad i = 1, 2, \dots, n\}$
$\bigtimes_{n=1}^{\infty} X_n = X_1 \times X_2 \times \dots \times X_k \times \dots$

Prop: If X and Y are countable sets, then so is $X \times Y$.
Pf:

	1	2	3	4	...
1	(1,1)	(2,1)	(3,1)	(4,1)	
2	(1,2)	(2,2)	(3,2)	(4,2)	
3	(1,3)	(2,3)	(3,3)	(4,3)	

$X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$
WOLG we can assume $X = \mathbb{N} = Y$ by indicies.
Define sequence as above

Cor: $\mathbb{Q}$ is countable.
Pf:
$r \in \mathbb{Q}$, $r = \frac{a}{b}$ (reduced terms) uniquely
Thus $\exists$ a bijection b/w $\mathbb{Q}$ and a subset of $\mathbb{Z} \times \mathbb{Z}$.

Cor 1.5.6
$X_1, \dots, X_n$ countable, then $X_1 \times X_2 \times \dots \times X_n$ is countable (by induction)
Pf:
X_1, X_2, X_3
$X_1 \times X_2 \times X_3 \neq (X_1 \times X_2) \times X_3$ but there is a 1-1 correspondence.
Proof follows by induction

Prop: 1.5.7
If $X = \bigcup_{n=1}^{\infty} X_n$ and each X_n is countable, then X is countable.

Pf:
Write $X_n = \{X_n^1, X_n^2, X_n^3, \dots\}$ where $X_n^i \neq \emptyset$ for any i.
(Note we can write finite set $X_m = \{x_1, x_2, \dots, x_m\} = \{x_1, \dots, x_m, x_m, x_m, \dots\}$)

$f: \mathbb{N} \times \mathbb{N} \to X$, $f(n,k) = x_n^k$ is surjective

Cor 1.5.8: The set of all finite subsets of $\mathbb{N}$ is countable.
Pf:
Let F be the family of all finite subsets of $\mathbb{N}$, then
$F = \bigcup_{n=1}^{\infty} S_n$, where S_n is the set of all subsets of $\{1, 2, \dots, n\}$
$\# S_n = 2^n$ (by induction)
So F is countable.

Notation: 2^X is the family of subsets of X.

Prop 1.5.9
The set of all sequences of zeros and ones is not countable.
Pf:
Suppose
$$\begin{cases} 0, 0, 1, 1, 0, 1, \dots \\ 1, 0, 0, 0, 1, 0, \dots \\ 0, 1, 0, 0, \dots \\ 1, 0, 0, 1, \dots \end{cases}$$

We can manufacture a sequence of 0s and 1s not in the list.
$1, 1, 1, 0, \dots$

Cor: The collection of all subsets on $\mathbb{N}$, denoted $2^{\mathbb{N}}$, is not countable.
Pf:
Characteristic functions of subsets of $\mathbb{N}$ are the sequences $(0, 1, 0, 0, 1, \dots)$ and are the subsets of $\mathbb{N}$, so we have

a bijection. $\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Cantor: There is not bijection (or surjection) of X onto 2^X.

Pf:

Suppose $f: X \xrightarrow{\text{onto}} 2^X$.

Consider $N = \{x \mid x \notin f(x)\}$ $\quad (x \in X$ and $f(x) \subsetneq X)$

Since f is onto, $N = f(z)$ for some $z \in X$.

Case 1: $z \in N$

$z \notin f(z) = N \Rightarrow\Leftarrow$

Case 2: $z \notin N$

$z \in f(z) = N \Rightarrow\Leftarrow$

Thus no such f exists.

Continuum Hypothesis:

$\mathbb{N}$ $\qquad\qquad X \qquad\qquad \mathbb{R}$

$\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$

countable $\qquad\qquad\qquad\qquad$ uncountable

Does there exist X?

1.6 Open and Closed Sets

Thursday, September 20, 2018 3:56 PM

<u>Def:</u> (1) $F \subseteq \mathbb{R}$ is <u>closed</u> iff every limit point of F lies F

$x_n \to x$, $x_n \in F$, implies $x \in F$

(2) $G \subseteq \mathbb{R}$ is open if $\underbrace{\mathbb{R} \setminus G}_{\text{compliment of } G}$ is closed.

<u>Ex:</u> (1) $\emptyset \subseteq \mathbb{R}$ is both open and closed.

(2) $\mathbb{R}$ is open and closed

(3) $(-\infty, a]$ and $[b, \infty)$ are closed

(4) Finite subsets of $\mathbb{R}$ are closed

(5) $[a,b]$ is closed but not open b/c $(-\infty, a) \cup (b, \infty)$ is not closed.

(6) (a,b) is open $= \mathbb{R} \setminus \{(-\infty, a] \cup [b, \infty)\}$

<u>Prop 1.6.3</u>

a) $F_1, F_2, \ldots, F_n$ closed sets, then $F_1 \cup F_2 \cup \ldots \cup F_n$ is closed

<u>Pf:</u> Textbook

<u>Note:</u> Doesn't work for countable. The set of points $\frac{1}{n}$ for $n \in \mathbb{N}$ is closed but 0 is a limit point and is not in the set.

b) $F_1, F_2, \ldots, F_n, \ldots$ closed sets, then $F_1 \cap F_2 \cap \ldots \cap F_n \cap \ldots$ is closed.

More generally, consider a family $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ of closed sets then $\cap \mathcal{F} \overset{def}{=} \bigcap_{\alpha \in A} F_\alpha$ is closed.

<u>Proof:</u> Textbook

<u>Note:</u> This does not need to be countable.

c) $G_1, \ldots G_n$ open sets, then $G_1 \cap G_2 \cap \ldots \cap G_n$ is open.

d) Consider a family $\mathcal{G} = \{G_\alpha\}_{\alpha \in A}$ of open sets.

$$\cup \mathcal{G} \overset{def}{=} \bigcup_{\alpha \in A} G_\alpha \text{ is open.}$$

(Use DeMorgan's Law)

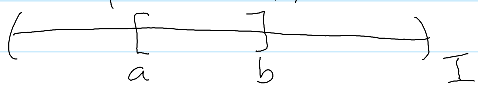
Prop 1.6.4 $G \subseteq \mathbb{R}$ is open iff $\forall x \in G\ \exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq G$.

Pf:

Skip/Exercise

Ex: $(a,b) \cap \mathbb{Q}$ is neither open or closed.

Lemma 1.6.6: Let $I \subseteq \mathbb{R}$. The I is an interval iff whenever $a, b \in I$ it follows that $[a,b] \subseteq I$.



(Connected Sets)

Prop 1.6.7 $G \subseteq \mathbb{R}$ is open iff $G = I_1 \cup I_2 \cup I_3 \cup \dots$ for countable pairwise disjoint intervals

Pf: (Idea) (must be since each interval contains an element of $\mathbb{Q}$)

To every $x \in G$ there corresponds the largest open interval containing x, say J_x.

(union of open intervals containing x in G)

Observation either $J_x = J_y$ or $J_x \cap J_y = \phi$.

Def: The diameter of $E \subseteq \mathbb{R}$ is

$$\operatorname{diam} E = \sup \{ |x-y| : x, y \in E \}$$

Note: E is bounded iff $\operatorname{diam} E < \infty$.

HW Exercise: n points in the plane. Show that $\exists$ at most n diameters.

Thm: Cantor's Theorem

For $F_1 \supseteq F_2 \supseteq F_3 \supseteq$ are nonempty closed subsets of $\mathbb{R}$ (usually bounded, so compact)

(a) If one subset is bounded, then

$$F \overset{def}{=} \bigcap_{n=1}^{\infty} F_n \neq \phi$$

(b) If $\operatorname{diam} F_n \to 0$, then

$$F \overset{def}{=} \bigcap_{n=1}^{\infty} F_n = \{x\} \text{ a singleton}$$

Pf:

- F is closed by Prop 1.6.3 b
- If F_n is bounded, then F_k are bounded $\forall k \geq n$.

For each $k \geq n$ pick a point $x_k \in F_k$

$\{x_k\}_{k=n}^{\infty}$ is a bounded sequence.

$\Rightarrow$ By Bolzano-Waistrass

$\exists$ subsequence $x_{k_i} \to x$ $\quad k_1 < k_2 < \dots \to \infty$

$\forall m \geq n$ we have

$x \in F_m$ - closed set containing $x_{k_m}, x_{k_{m+1}}, \dots$

$\Rightarrow x \in \bigcap_{m \geq n}^{\infty} F_m \Rightarrow F = \bigcap_{n=1}^{\infty} F_m \neq \emptyset$

b) All sets F_m, $m \geq n$ are bounded

$F \subseteq F_m$

$\operatorname{diam} F \leq \operatorname{diam} F_m \to 0$

so $F = \{x\}$.

<u>Thm 1.6.9</u> ("closed bounded sets are compact")

$X \subseteq \mathbb{R}$ closed and bounded

$X \subseteq \bigcup_{n=1}^{\infty} G_n$, cover of X by open sets G_n.

then $\exists$ a finite subcover $X \subseteq \bigcup_{n=1}^{N} G_n$.

(This is often the definition of a compact set.)

<u>Pf:</u>

May assume $G_1 \subseteq G_2 \subseteq \dots$

Simply, replace $G_1' = G_1$

$G_2' = G_1 \cup G_2$

$\vdots$

Thus $X \subseteq \bigcup_{n=1}^{\infty} G_n'$

The sets $F_1 \supseteq F_2 \supseteq \dots$, $F_n = X \setminus G_n'$ are closed and bounded.

Therefore $F_1 \cap F_2 \cap \dots = \underbrace{X \setminus \bigcup_{n=1}^{\infty} G_n' = \emptyset}_{\text{since } \bigcup G_n' \text{ covers } X}$

$\Rightarrow$ By Cantor's Thm, one of the sets $F_N = \emptyset$

Hence $X \subseteq G_N' = G_1 \cup G_2 \cup \dots G_N$ #

Remark: (about Lindelöf property of $\mathbb{R}$)

Every open cover of a set $X \subseteq \mathbb{R}$, if $X = \bigcup_{\alpha \in A} G_\alpha$

admits a countable subcover $X = \bigcup_{n=1}^{\infty} G_{\alpha_n}$, $\alpha_n \in A$

Pf:

X is open since union of open sets.

$x \in X \Rightarrow x \in G_\alpha$ for some $\alpha \in A$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq G_\alpha$

Choose $(x-\varepsilon_0, x+\varepsilon_0) \subseteq (x-\varepsilon, x+\varepsilon)$ s.t. endpts. are rational.

There are a countable number of these sets.

Choose only G_α that contain these subintervals.

Hence countable subcover.

Cor: Bounded closed subsets of $\mathbb{R}$ are compact, meaning that every open cover admits a finite open subcover.

(In Topology, this property is used as a definition of compactness.)

Thm: (Lindelöf Space) Optional.

Let (M,d) be a metric space and suppose that M has a countable dense set D. Then every open cover of M admits a countable subcover.

Polish Space, by definition, is a space homomorphic to a metric space that has a dense countable subset

Def: $E \subseteq \mathbb{R}$, $E \neq \emptyset$, $x \in \mathbb{R}$

$$\mathrm{dist}(x, E) = \inf\{|x-y| : y \in E\}$$

Def: The closure of $E \neq \emptyset$

$$\mathrm{cl}\, E = \{x \in \mathbb{R} \mid \mathrm{dist}(x,E) = 0\} = \overline{E}$$

Prop 1.6.11 $E \subseteq \mathbb{R}$, $E \neq \emptyset$

a) $\overline{E}$ is a closed set.

b) $E \subseteq F \Rightarrow \overline{E} \subseteq \overline{F}$

c) If E is closed then $E = \overline{E}$.

Pf:

a) $x_n \in \overline{E}$, $x_n \to x$

$\mathrm{dist}(x, E) \le \mathrm{dist}(x_n, E) + \mathrm{dist}(x_n, x)$

$= 0 + |x_n - x| \to 0$

$\Rightarrow x \in \overline{E}$

b) Let $x \in \overline{E}$ show $\mathrm{dist}(x, E) = 0$

$\mathrm{dist}(x, F) \le \mathrm{dist}(x, E) = 0$

$\mathrm{dist}(x, F) \le |x - y|$, $y \in E \subset F$

c)?

Note: This prop tells us that the $\overline{E}$ is the smallest closed set that contains E:

$$\overline{E} = \cap F_E$$

where F_E are the family of closed sets containing E.

1.7 Continuous Functions

Tuesday, September 25, 2018 4:45 PM

Def: $X \subseteq \mathbb{R}$, $a \in X$
$f: X \to \mathbb{R}$ is continuous at a if
$x_n \to a$ implies $f(x_n) \to f(a)$

Def: Let X be any abstract set.
$f, g: X \to \mathbb{R}$ We define
$f \pm g: X \to \mathbb{R}$
$fg: X \to \mathbb{R}$
$f/g: X \to \mathbb{R}$ for $g \neq 0$ in X.

Def: $X \subseteq \mathbb{R}$ $f, g: X \to \mathbb{R}$
$f \pm g$, fg, f/g $(g \neq 0)$ are continuous at a.
Polynomials
Rational Functions

Prop 1.7.4 $X, Y \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$, $f(X) \to Y$, $g: Y \to \mathbb{R}$, then
then $g \circ f: X \to \mathbb{R}$
f-continuous at a and g-continuous at $\alpha = f(a)$
then $g \circ f$ is continuous at a.

Thm: 1.7.5 $f: X \to \mathbb{R}$ is continuous at $a \in X$
iff $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ $x \in X$

Extreme Value Theorem: $[a, b]$ bounded closed interval
and $f: [a,b] \to \mathbb{R}$ continuous then $\exists$ $x_0, y_0 \in [a,b]$ s.t.
$f(x_0) \leq f(x) \leq f(y_0)$ $\forall x \in [a,b]$
(min) (max)

Intermediate Value Theorem
$I \subseteq \mathbb{R}$ an interval, $f: I \to \mathbb{R}$ continuous.
Suppose $a, b \in I$ and $f(a) < \tau < f(b)$. Then
$\exists t$ s.t. $a < t < b$ and $f(t) = \tau$.
Pf:
Let $A = \{x \in [a,b] \mid f(x) < \tau\} \neq \emptyset$
Let $t = \sup A$

$t = \lim a_n$, $a_n \in A$

$f(t) = \lim f(a_n)$ $\quad f(a_n) < \tau$

Hence $f(t) \leq \tau$.

Suppose to the contrary, that $f(t) < \tau$. Thus $\exists f(t) + \varepsilon < \tau$ but $\exists \delta \; 0$ s.t. $|f(x) - f(t)| < \varepsilon$ whenever $|x - t| < \delta$. In part., $t < x < t + \delta$, $f(x) < f(t) + \varepsilon < \tau$ so $x \in A \Rightarrow x \leq \sup A = t$, a contradition.

Cor 1.7.8 $f : [a,b] \to \mathbb{R}$ continuous then $f([a,b])$ is a closed and bounded interval.

Pf:

Let $\alpha = \inf f([a,b])$

$\beta = \sup f([a,b])$

Thus $f([a,b]) \subseteq [\alpha, \beta]$

$$\left.\begin{array}{l} \alpha \in f([a,b]) \\ \beta \in f([a,b]) \end{array}\right\} \text{by M.V.T.}$$

Given any $\tau \in (a,b)$ we find $\tau = f(t)$ for some $t \in [a,b]$

Thus $f([a,b]) \supseteq [\alpha, \beta]$

$\Rightarrow f([a,b]) = [\alpha, \beta]$ #

Definition: $f, g : X \to \mathbb{R}$

(X: Arbitrary set)

Define:

$f \vee g = \max(f,g) : X \to \mathbb{R}$

$f \wedge g = \min(f,g) : X \to \mathbb{R}$

$f \vee g = \frac{1}{2}[f + g + |f - g|]$

$f \wedge g = \frac{1}{2}[f + g - |f - g|]$

Prop 1.7.11 $f, g : X \to \mathbb{R}$ continuous ($X \subseteq \mathbb{R}$) then both $f \vee g$ and $f \wedge g$ are continuous.

Def: If $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0 \; \exists \delta \; \forall x, y \in X \; |x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

Negation: $\exists \varepsilon > 0 \; \forall \delta \; |x - y| < \delta$ then $|f(x) - f(y)| > \varepsilon$.

Example: 1.7.13 $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$

is not u.c.

Pf:

Suppose to the contrary it is u.c.

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta$ s.t. $|x^2 - y^2| < \varepsilon$ whenever $|x-y| < \delta$

??

Ex: $f(x) = \sin\frac{1}{x}$ is not uc $0 < x < \frac{1}{\pi}$

why?

$x = \frac{1}{2\pi n}$ $y = \frac{1}{2\pi n + \pi/2}$ then $|f(x) - f(y)| = 1$

$x - y = \frac{\pi/2}{(2\pi n)(2\pi n + \pi/2)} \leq \frac{\pi}{2\pi n}$

So $x-y$ is arbitrarily small and $|f(x) - f(y)|$ is not

Def:

$E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$. f is a Lipshitz Function if $\exists M$ s.t. $\forall x, y \in E$ $|f(x) - f(y)| \leq M|x-y|$

Prop: A Lipshitz function is uniformly continuous.

Remark: Hölder continuous functions of exponent α are continuous.

Same as Lipshitz but $|f(x) - f(y)| \leq M|x-y|^\alpha$, $0 < \alpha \leq 1$

★ ~~A~~: If $|f(x) - f(y)| \leq |x-y|^2$

★ $X = [0,1]$ $X \subseteq \mathbb{R}$

$|f(x) - f(y)| \leq |f(x) - f(x_1)| + |f(x_1) + f(x_2)| + \ldots + |f(x_n) - f(y)|$

$\leq |x - x_1|^2 + |x_1 - x_2|^2 + \ldots + |x_n - y|^2$

Dividing into subintervals of length ε.

$\leq \varepsilon|x - x_1| + \varepsilon|x_1 - x_2| + \ldots + \varepsilon|x_n - y|$

$= \varepsilon|x-y|$ So $f(x)$ is constant

Ex: $f(x) = \log x$, $x \geq 1$ is a Lipshitz function.

why?

$\log x - \log y$

Note: Any fcn for which $|f'(x)| \le M$ is Lipshitz!

Ex: $|x|-|y| \le |x-y|$ so $f(x)=|x|$ is Lipshitz

Prop 1.4.15: If $E \subseteq \mathbb{R}$, $E \ne \emptyset$ then
$|dist(x,E) - dist(y,E)| \le |x-y|$

Pf:
Let $e \in E$, then $|x-e| \le |x-y| + |y-e|$
Taking the infimum over all $e \in E$
$dist(x,E) \le \inf_{e \in E} \{|x-y| + |y-e\}$

$$\le |x+y| + \inf_{e\in E} |y-e| = x+y+dist(y,E)$$

$\Rightarrow dist(x,E) - dist(y,E) \le |x-y|$
Reversing the roles of x and y, we get
$dist(y,E) - dist(x,E) \le |x-y|$
Thus we have our result. #

Thm 1.7.16 (Important)
X - bounded closed subset of $\mathbb{R}$
$f: X \to \mathbb{R}$ continuous
then f is u.c. that is
$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x,y \quad |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$
Pf:
Suppose to the contrary
$\exists \varepsilon_0 > 0 \ \forall \delta > 0 \ \exists x,y \quad |x-y| < \delta$ and $|f(x)-f(y)| \ge \varepsilon_0$

Take $\delta = \frac{1}{n}$, $\exists x_n, y_n \quad |x_n - y_n| < \frac{1}{n}$
Choose a sequence $x_n \to a \in X$
so $y_n \to a \in X$
$f(a) = \lim f(x_n)$
$\underbrace{f(a) = \lim f(y_n)}_{\text{since f continuous}}$ but $|f(x_n) - f(y_n)| \ge \varepsilon_0$ $\otimes$.

$\Rightarrow f$ is u.c. ■

Thm 1.7.17 (Tietze Extension Thm.)
X closed subset of [a,b]

X - closed subset of $[a,b]$
$f: X \to [m,M]$ continuous then $\exists$ a continuous fcn
$\tilde{f}: [a,b] \to [m,M]$ s.t. $\tilde{f}(x) = f(x)$ for $x \in X$.

Illustration:

Pf:
First suppose $X \subseteq (a,b)$.
Then $(a,b) \setminus X$ is open and as such

$$(a,b) \setminus X = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

Define:

$$\tilde{f}(x) = \begin{cases} f(x), & x \in X \\ \text{straight line connecting } (a_n, f(a_n)) \text{ with } (b_n, f(b_n)) & n = 1, 2, \dots \end{cases}$$

$$= \begin{cases} f(x) & x \in X \\ f(a_n) + \dfrac{f(b_n) - f(a_n)}{b_n - a_n}(x - a_n) & x \notin X \end{cases}$$

Equivalently: $\tilde{f}(x) = f(b_n) + \dfrac{f(a_n) - f(b_n)}{b_n - a_n}(b_n - x) \quad x \notin X$

Note 1) $\sum_{n=1}^{\infty} (b_n - a_n) \le b - a$
$\lim_{n \to \infty} (b_n - a_n) = 0$

2) $f(b_n) - f(a_n) \to 0$ b/c $a_n, b_n \in X$ and f is u.c.

Let $x_k \to x$, $x \in X$ (the case of infinitely many $x_k \in (a,b) \setminus X$)
Let $x_{n_k} \in (a_{n_k}, b_{n_k})$ $x_{n_k} \to x \in X$
Then $f(b_{n_k}) \to f(x)$, b/c, $b_{n_k} \in X$

$$f(a_{n_k}) \to f(x) \quad b/c \quad a_{n_k} \in X$$

Thus,

$$\underbrace{|b_{n_k} - x|}_{\substack{\uparrow \quad \uparrow \\ x \quad x}} \le \underbrace{|b_{n_k} - x_{n_k}|}_{\to 0} + \underbrace{|x_{n_k} - x|}_{\to 0}$$

$$|b_{n_k} - x_{n_k}| \le |b_{n_k} - a_{n_k}| \to 0$$

Similarly, $|a_{n_k} - x| \to 0$

By u.c. $f(b_{n_k}) - f(x) \to 0$

$f(a_{n_k}) - f(x) \to 0$

$$\tilde{f}(x) - f(x) = f(a_{n_k}) + \frac{f(b_{n_k}) - f(a_{n_k})}{b_{n_k} - a_{n_k}} \cdot (x_{n_k} - a_{n_k}) - f(x)$$

$$0 \le \frac{(x_{n_k} - a_{n_k})}{b_{n_k} - a_{n_k}} \le 1 \le \tilde{f}(x) - f(x) \le f(b_{n_k}) \to f(x)$$

HW Problem: $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/b & \text{if } x = a/b,\ a \in \mathbb{Z},\ b \in \mathbb{N},\ \gcd(a,b) = 1 \end{cases}$

Case 1: x-rational, $x = \frac{a}{b}$, $f(x) = \frac{1}{b} \neq 0$

$$\frac{a}{b} + \frac{1}{n}\sqrt{2} \to \frac{a}{b}$$

$$f\left(\frac{a}{b} + \frac{1}{n}\sqrt{2}\right) = 0 \to 0 \neq \frac{a}{b}$$

Hence f is not continuous at rational numbers.

Case 2: x - irrational. $f(x) = 0$

Consider $\{x_n\}$ s.t. $x_n \to x$.

Suppose to the contrary $f(x_n) \not\to 0$. Hence $f(x_n)$ contains an infinite number of rationals say $x_{n_k} = \frac{a_{n_k}}{b_{n_k}}$ $\gcd(a_{n_k}, b_{n_k}) = 1$

Claim $b_{n_k} \to \infty$ and in particular $f(x_{n_k}) = \frac{1}{b_{n_k}} \to 0$.

Suppose $b_{n_k} \not\to \infty$

$\Rightarrow \exists$ a subsequence $\{b_{n_{k_i}}\}$ which is bounded say $0 < b_{n_{k_i}} < M$

$\Rightarrow x_{n_{k_i}} = \frac{a_{n_{k_i}}}{b_{n_{k_i}}} \to x \Rightarrow \frac{|a_{n_{k_i}}|}{|b_{n_{k_i}}|} = |x| + 1 \Rightarrow |a_{n_{k_i}}| < M(|x|+1)$ for large i

$\Rightarrow a_{n_{k_i}} < N \quad \forall i$

Hence, the number of ratios $\frac{a_{n_{k_i}}}{b_{n_{k_i}}}$ is finite.

$\Rightarrow x$ is rational $\emptyset$

Insert trig illistration

$|\sin\theta| < |\theta|$

$\lim_{\theta \to 0} \sin\theta = 0 = \sin 0 \Rightarrow \sin\theta$ continuous at 0

$\cos\theta = \sqrt{1-\sin^2\theta}$ is continuous

$\sin(\theta \pm \varepsilon) = \sin\theta\cos\varepsilon \pm \cos\theta\sin\varepsilon$

Def: 2.1.1
$X \subseteq \mathbb{R}$, a is a limit point of X iff
$\forall \varepsilon > 0 \ \exists x \in X$ s.t. $0 < |x-a| < \varepsilon$

$f: X \to \mathbb{R}$
$\lim_{x \to a} f(x) = L$, $f(x) \to L$ as $x \to a$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $x \in X$ and $0 < |x-a| < \delta$

Ex: $f(x) = \sin \frac{1}{x}$, $x \in \mathbb{R}^+$
$\nexists \lim_{x \to 0} f(x)$.

Why?
Show $\exists \varepsilon_0 > 0 \ \forall \delta > 0 \ |f(x) - L| > \varepsilon_0 \quad 0 < |x-a| < \delta$
Consider $\varepsilon_0 = 1/2$

Ex: $f(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$

$\lim_{x \to 0^+} f(x) = 1 \neq \lim_{x \to 0^-} f(x) = -1$

Def: ① $\lim_{x \to a^+} f(x) = L \overset{def}{=} f(a^+)$
② $\lim_{x \to a^-} f(x) = L \overset{def}{=} f(a^-)$

Prop 2.1.2 $X \subseteq \mathbb{R}$ a is a limit pt.
$f: X \to \mathbb{R}$
$\lim_{x \to a} f(x) = L$ iff for every sequence converging $x_n \to a$ then $f(x_n) \to f(a)$

Cor 2.1.3
$f: X \to \mathbb{R}$ is continuous at a iff $\lim_{x \to a} f(x) = f(a)$.

Prop 2.1.4 $f: (a,b) \to \mathbb{R}$ is continuous at $c \in (a,b)$ iff $f(c^-) = f(c^+) = f(c)$

Prop 2.1.5 $X \subseteq \mathbb{R}$, $a \in \overline{X}$, $f, g: X \to \mathbb{R}$
If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to g} g(x) = K$ then
1) $\lim_{x \to a} [f(x) + g(x)] = L + K$
2) $\lim f(x) g(x) = L \cdot K$
3) $\lim f(x)/g(x) = L/K$ assuming $K \neq 0$.

Def 2.1.6 (Jump Continuity, Simple Continuity)
$f: (a,b) \to \mathbb{R}$ has a jump discontinuity at a point $c \in (a,b)$ if $f(c^+)$ and $f(c^-)$ exists, but $f(c^+) \neq f(c^-)$

Definition: Montonic Functions

$X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$

- increasing if $f(x) \leq f(y)$ whenever $x \leq y$
- decreasing if $f(x) \geq f(y)$ whenever $x \leq y$

(strictly)

- monotonic means either inc. or dec.

Prop 2.1.8

$f: (a,b) \to \mathbb{R}$ bounded increasing, $x \in (a,b)$ then

- $\sup_{y<x} f(y) = f(x^-) \leq f(x) \leq f(x^+) = \inf_{y>x} f(x)$
- every discontinuity is simple
- $f(a^+) = \inf \{ f(y) \mid a < y \}$
- $f(a^-) = \sup \{ f(y) \mid y > a \}$

Prop 2.1.9 If $f: (a,b) \to \mathbb{R}$ is monotonic, then the set of discontinuities is countable.

(Insert Picture)

Def 2.1.10 $X \subseteq \mathbb{R}$, a is a limit point $f: X \to \mathbb{R}$

1) $\forall P > 0 \ \exists \delta > 0 \ x \in X \ 0 < |a-x| < \delta$ then $f(x) > P$

then $\lim_{x \to a} f(x) = \infty$, $f(x) \to \infty$ as $x \to a$

2) $\forall N < 0 \ \exists \delta > 0 \ x \in X \ 0 < |a-x| < \delta$ then $f(x) < N$

then $\lim_{x \to a} f(x) = -\infty$, $f(x) \to -\infty$ as $x \to a$

2.2 Differentiation

Tuesday, October 9, 2018 3:31 PM

Def 2.2.1 (a,b), $x \in (a,b)$

$f:(a,b) \to \mathbb{R}$ is differentiable if

$\lim_{t \to 0} \frac{f(x+t)-f(x)}{t}$ exists $\underset{=}{\text{def}}$ $f'(x) = \frac{df}{dx}$

$$f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$$

Def 2.2.3 $f:[a,b] \to \mathbb{R}$ is said to be differentiable at a if $\lim_{t \to 0^+} \frac{f(x+t)-f(x)}{t} = f'(a)$

Similarly, is defined $f'(b)$.

Prop 2.2.4 $f:[a,b] \to \mathbb{R}$ differentiable at $x \in [a,b]$ then f is continuous at x.

Pf: Trivial

Prop 2.2.5 f, g diff. at $x \in [a,b]$ then

(a) $f+g$ is diff. and $(f+g)' = f' + g'$

(b) $(fg)' = f'g + fg'$

(c) $g \neq 0$ in $[a,b]$ then $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Pf: From limit def.

Ex: $f(x) = x^n$

$f(x) - f(a) = x^n - a^n = (x-a)(x^{n-1} + \dots + a_1)$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = na^{n-1}$$

Lemma: 2.2.7

a) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

b) $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

Pf:

$\sin\theta < \theta < \tan\theta, \quad \theta > 0, \quad \theta \approx 0$

Insert Image

area AOB $\quad$ area of sector $\quad$ area AOC

$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \to 1$

$$\frac{\cos\theta - 1}{\theta} = \frac{(\cos\theta - 1)(\cos\theta + 1)}{\theta(\cos\theta + 1)}$$

$$= \frac{-\sin^2\theta}{\theta} \cdot \frac{1}{\cos\theta + 1} \to 0 \cdot \frac{1}{2} = 0$$

Prop 2.2.8

$(\sin x)' = \cos x$ and $(\cos x)' = \sin x$

Pf:

$$\frac{\sin(x+\theta) - \sin x}{\theta} = \frac{\sin x \cos\theta + \sin\theta \cos x - \sin x}{\theta}$$

$$= \sin x\left(\frac{\cos\theta - 1}{\theta}\right) + \cos x\left(\frac{\sin\theta}{\theta}\right) \to \sin x \cdot 0 + \cos x \cdot 1$$

$$= \cos x$$

$$\cos x = \sin\left(\tfrac{\pi}{2} - x\right) \Rightarrow (\cos x)' = \cos\left(\tfrac{\pi}{2} - x\right)\cdot(-1) = -\sin x \quad \#$$

<u>Remark:</u> $u'' = -u$ then $u = c_1 \cos x + c_2 \sin x$

$\underbrace{\text{Optional Homework:}}_{\text{Just Kidding.}}$ Prove that there are no other solutions.

<u>Proof:</u> $\downarrow$

$u''(x) = u(x), \quad 0 \le x \le T$

$u(0) = c_1$ and $u'(0) = c_2$

$u(x) = u(0)\cos x + u'(0)\sin x + v(x)$

$v(x)$ must also be a solution with $v(0) = 0$, $v'(0) = 0$

We will show $v(x) = 0$

$$v'(x) = \int_0^x v''(t)\,dt = \int_0^x v(t)\,dt \Rightarrow v(z) = \int_0^z v'(x)\,dx = \int_0^z\int_0^x v''(t)\,dt\,dx$$

Consider, $|v(z)| \le \int_0^z\int_0^x |v(t)|\,dt\,dx$

$$= \int_0^z\int_0^x \left(|v(t)|e^{-Nt}\right)e^{Nt}\,dt\,dx$$

$$\le \max_{0\le t\le x} |v(t)|e^{-Nt} \int_0^z \underbrace{\int_0^x e^{Nt}\,dt}_{\frac{e^N-1}{N} \le \frac{e^N}{N}}\,dx$$

$$\le \max |v(t)|e^{-Nt} \int_0^z \frac{e^{Nx}}{N}\,dx$$

$$\le \max |v(t)|e^{-Nt} \cdot \frac{e^{Nz}-1}{N^2} \le \max |v(t)|e^{-Nt}\,\frac{e^{Nz}}{N^2}$$

So $|v(t)|e^{-Nz} \le \frac{1}{N^2}\max_{0\le t\le x} |v(t)|e^{-Nt} \le \frac{1}{N^2}\max_{0\le t\le T} |v(t)|e^{-Nt}$

Thus $\max_{0\le z\le T} |v(t)|e^{-Nt} \le \frac{1}{N^2}\max_{0\le t\le T} |v(t)|e^{-Nt}$

$\downarrow$

$0 \quad \#$

$\breve{o}$ $\nexists$

Optional H/W: $u'' = u$, solutions are e^{x} and e^{-x}

$\Rightarrow u = c_1 e^{x} + c_2 e^{-x}$.

Show there are no other solutions.

Thm 2.2.10 Chain Rule

$$(a,b) \xrightarrow{f} (c,d) \xrightarrow{g} \mathbb{R}$$

$$x \in (a,b), \quad y = f(x) \in (c,d)$$

$h = g \circ f : (a,b) \to \mathbb{R}$ then

$$h'(x) = g'(y) f'(x)$$

Prop 2.2.9

$f: X \to \mathbb{R}$ is differentiable at $x \in X$ iff

$\exists D \in \mathbb{R}$ s.t.

$$f(y) = f(x) + D(y-x) + F(y)(y-x) \quad \forall y \in X.$$

Here $F: X \to \mathbb{R}$ has limit 0 at x and

$$F(y) \overset{def}{=} \frac{f(y) - f(x)}{y - x} - D$$

Proof of Chain Rule

$$f(y) - f(x) = (y-x) \overbrace{[f'(x) + F(y)]}^{\text{Taylor Exp.}}$$

$$g(\xi) - g(f(x)) = (\xi - f(x))[g'(f(x)) - G(\xi))$$

Hence for $h = g(f(x))$

$$h(y) - h(x) = g(f(y)) - g(f(x))$$

$$= [f(y) - f(x)][g'(f(x)) - G(f(y))]$$

$$= (y-x)[f'(x) + F(y)][g'(f(x)) + G(f(y))]$$

$$= (y-x)[g'(f(x)) f'(x) + T(y)]$$

$$T(y) = f'(x)\, G(\underbrace{f(y)}_{0}) + \underbrace{F(y)}_{0}\, g' f(x) + \underbrace{F(y)}_{0}\, \underbrace{G(f(y))}_{0}$$

Local Maximum &	Local Minimum
f has a local max at $x \in (a,b)$ if $f(x) \geq f(y)$ $\forall y$ in neighborhood of x; that is for all $x-\delta < y < x+\delta$ where δ is small enough so $(x-\delta, x-\delta) \subset (a,b)$	f has a local min $f(x) \leq f(y)$

Thm 2.3.2

$f: (a,b) \to \mathbb{R}$ differentiable

If f has a local max or local min at $c \in (a,b)$ then $f'(c) = 0$.

Pf.

Suppose f has local max at x; then for $t \approx 0$, $t > 0$

$$\frac{f(x+t) - f(x)}{t} \leq 0$$

$$\Rightarrow f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t} \leq 0$$

Similarly for $t < 0$ $f'(x) \geq 0 \Rightarrow f'(x) = 0$ #

Thm 2.3.4 Mean Value Theorem.

$f: [a,b] \to \mathbb{R}$ continuous

$f: (a,b) \to \mathbb{R}$ differentiable

then $\frac{f(b) - f(a)}{b-a} = f'(x)$ for some $x \in (a,b)$.

then $\frac{f(b)-f(a)}{b-a} = f'(x)$ for some $x \in (a,b)$.

Pf:

Consider function on $[a,b]$

$h(t) = [f(b)-f(a)][t] - (b-a) f(t)$

$h(a) = h(b)$.

Thus h assumes $\begin{cases} \text{maximum at } x \in (a,b) \\ \text{or} \quad \text{minimum at } x \in (a,b) \end{cases}$

$\Rightarrow h'(x) = 0$

Cor 2.35 (Generalized MVT)

$f, g : (a,b) \to \mathbb{R}$ differentiable

$f, g : [a,b] \to \mathbb{R}$ continuous

then $\exists x \in (a,b)$ s.t.

$[f(b)-f(a)] g'(x) = [g(b)-g(a)] f'(x)$

Equivalently,

$$\frac{f(b)-f(a)}{b-a} \cdot g'(x) = \frac{g(b)-g(a)}{b-a} \cdot f'(x)$$

Pf:

Consider

$$F(t) = \frac{f(b)-f(a)}{b-a} g(t) - \frac{g(b)-g(a)}{b-a} f(t)$$

$\frac{F(b)-F(a)}{b-a} = 0$ so $F'(x) = 0$ for some $x \in (a,b)$

Cor 2.3.6

$f: (a,b) \to \mathbb{R}$ $f'(x) = 0$ then $f =$ constant

Cor 2.3.6

$f:(a,b)\to\mathbb{R}$, $f'(x)=0$, then $f=$ constant

Prop 2.3.7

$f:(a,b)\to\mathbb{R}$ differentiable, then

(a) f is increasing iff $f'(x)\geq 0$ in (a,b)

(b) If $f'(x)>0$ $\forall x$, then f is strictly increasing

(c) f is decreasing iff $f'(x)\leq 0$ in (a,b)

(d) If $f'(x)<0$ $\forall x$, then f is strictly decreasing.

Prop 2.3.8

Let I be an interval;

Let $f:I\to\mathbb{R}$ be continuous and injective, then $f(I)$ is an interval and f is strictly monotonic.

Proof:

- $f(I)$ is an interval by IVT

  Indeed, every two points in $f(I)$ can be connected

It suffices to show that f is monotonic on every closed interval $[\alpha,\beta]\subset I$.

Consider any closed interval $[\alpha,\beta]\subset I$.

We may assume WOLG, $f(\alpha)\leq f(\beta)$.

(Otherwise consider the function $-f$.)

We aim to prove f is increasing.

Claim: $f(\alpha)\leq \min\limits_{x\in[\alpha,\beta)} f(x)\leq \max\limits_{x\in[\alpha,\beta)} f(x)\leq f(\beta)$

If $\max f(x)>f(\beta)$ $\exists c\in(\alpha,\beta)$ s.t. $f(c)=f(\beta)$ by IVT $\not\otimes$ since f is injective. $\Rightarrow \max f(x)\leq f(\beta)$

Similarly, $f(\alpha)\leq \min f(x)$.

by I.V.T $\otimes$ since f is injective. $\Rightarrow \max f(x) \le f(\beta)$

Similarly, $f(\alpha) \le \min f(x)$.

Now suppose f is not increasing.

That is $\exists x_1, x_2$ s.t. $x_1 < x_2$ and $f(x_1) < f(x_2)$.

This contradicts by I.V.T since $\exists y_1 \in (\alpha, x_1)$, $y_2 \in (x_1, x_2)$ s.t. $f(y_1) = f(y_2)$. #

Prop 2.3.9

I - an interval.

$f: I \xrightarrow{\text{onto}} J$ injective continuous $(y = f(x))$

Then $f^{-1}: J \xrightarrow{\text{onto}} I$ is continuous

Proof:

Certainly J is an interval and $f^{-1}: J \to I$ is monotonic.

Choose and fix a point $y_0 \in J$.

Case 1: y_0 is not an endpoint of J.

Denote by L^+ and L^- the right limit and left limit of f^{-1}, respeetively.

$$L^+ = \lim_{J \ni y \searrow y_0} f^{-1}(y) \,;\; f^{-1}(y) \to L^+ \,;\; f(\underbrace{f^{-1}(y)}_{y_0}) \to f(L^+)$$

$$L^- = \lim_{J \ni y \nearrow y_0} f^{-1}(y) \,;\; f^{-1}(y) \to L^- \,;\; f(f^{-1}(y)) \to f(L^{-1})$$

b/c injective

so $f(L^+) = f(L^-) = y_0$, $L^+ = L^-$ $\boxtimes$

Prop 2.3.10

$f: (a,b) \to (\alpha, \beta)$ differentiable bijection, then $f^{-1}: (\alpha, \beta) \to (a,b)$ is a diff. bijection.

Moreover,

$$0 \ne (f^{-1})'(\xi) = \frac{1}{f'(f^{-1}(\xi))} \qquad \xi \in (\alpha, \beta)$$

Pf:

Pf:

Follows by chain rule; proof on pg 61 is repeat of proof of chain rule.

2.4-Critical Points

Thursday, October 11, 2018 4:26 PM

Defi: $f:(a,b)\to\mathbb{R}$ differentiable, $x\in(a,b)$ is said to be a critical point if $f'(x)=0$.

Prop 2.4.2

$f:(a,b)\to\mathbb{R}$ differentiable (thus continuous) and x_0 is a C.P.

a) $f'(x)<0$ for $x<x_0$ and $f'(x)>0$ for $x>x_0$ then f has a local max. at x_0

b) $f'(x)>0$ for $x<x_0$ and $f'(x)<0$ for $x>x_0$ then f has a local min at x_0.

Thm: 2.4.6 (Darboux's Theorem)

If $f:(a,b)\to\mathbb{R}$ is differentiable.

(f' need not be continuous)

Then $f':(a,b)\to\mathbb{R}$ has IV Property.

Pf: Text?

Thm: (L'Hôpital's Rule)

f,g - differentiable on $(a,b)\setminus\{c\}$

$g'(x)\neq 0$ in all x

$\lim_{x\to c} f(x)=0=\lim_{x\to c} g(x)$, $\exists \lim_{x\to c}\frac{f'(x)}{g'(x)}=L$

then $\lim_{x\to c}\frac{f(x)}{g(x)}=L$.

Under such conditions, $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}=L$.

Error

Tuesday, October 23, 2018 4:23 PM

Prop 2.3.10 Is false; $f(x) = x^3 \Rightarrow f^{-1} = \sqrt[3]{y}$ is a counter example.
Requires contition that $f'(x) > 0$ or $f'(x) < 0$ everywhere.
In particular, pg. 60.

2.5-

Tuesday, October 23, 2018 4:28 PM

Thm: 2.5.1 (L'Hopital's Rule)

Assume $c \in (a,b)$

f, g - differentiable on $(a,b) \setminus \{c\}$

$g'(x) \neq 0 \quad \forall x \in (a,b) \setminus \{c\}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \quad \text{and} \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$$

Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L \in \mathbb{R}$$

Pf:

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = L \quad \text{(To be proven)}$$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $c < z < c + \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$ since $\frac{f'}{g'} \to L$

By Corollary 2.3.5, $\exists z \in (y, x)$ s.t.

$$[f(x) - f(y)] g'(z) = [g(x) - g(y)] f'(z)$$

Note $g(x) - g(y) \neq 0$ otherwise g' would vanish at some point on (y, x)

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)} \quad (z \text{ depends on } x \text{ and } y)$$

Letting $y \searrow c$, we find

$$\left| \frac{f(x) - 0}{g(x) - 0} - L \right| < \varepsilon$$

$\vdots$

Remark: Thm 2.5.1 holds if $L = \infty$.

Pf:

$$\frac{f'(x)}{g'(x)} \to \infty \quad \text{as } x \to c$$

This implies that close to c, $f'(x) = 0$.

This implies that close to c, $f'(x)=0$.

Hence, $\dfrac{g'(x)}{f'(x)} \to 0$

Hence, $\dfrac{g(x)}{f(x)} \to 0$ by L'H.

Hence, $\dfrac{f(x)}{g(x)} \to \infty$ $\blacksquare$

<u>Thm 2.5.2</u> (L'H)

Assume $c \in (a,b)$

f, g - differentiable on $(a,b)\setminus\{c\}$, $g'(x) \neq 0$ $\forall x \in (a,b)\setminus\{c\}$

$\lim\limits_{x\to c} g(x) = \infty$ and $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$

Then, $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = L$

<u>Why?</u>

Insert Image

<u>Pf:</u> May assume that $L=0$. (To simplify)

Consider if $L \neq 0$, we can replace f with $f-Lg$

$\dfrac{f-Lg}{g}$ $\quad \dfrac{f'-Lg'}{g'} \to 0$

<u>Thm 2.5.6</u> Wrong in Textbook (Taylor's Theorem)

$f \in \mathcal{C}^{n-1}(a,b)$, $n \geq 1$

$f^{(n-1)}$ is differentiable at $c \in (a,b)$

$$P(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + \ldots$$

Then,

$$f(x) = P(x) + h(x)(x-c)^n, \quad h(x) \to \frac{f^{(n)}(c)}{n!} \text{ as } x \to c$$

Moreover, if f is n-times differentiable $\forall x \in (a,b)$

$h(x) = \frac{f^{(n)}(d)}{n!}$, $d = d(x)$ for some d b/w x and c.

Ex: $P_0(x) = f(c)$

$P_1(x) = f(c) + f'(c)(x-c)$

$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!}$

Proof:

Define $h(x) = \begin{cases} \frac{f(x) - P(x)}{(x-c)^n} & x \neq c \\ \frac{f^{(n)}c}{n!} & x = c \end{cases}$

L'H applied $(n-1)$ times

$\lim_{x \to c} f(x) = \frac{[f(x) - P(x)]'}{[(x-c)^n]'} = \cdots = \frac{1}{n!} \lim_{x \to c} \frac{f^{(n-1)}(x) - f^{(n-1)}(c)}{x-c}$

$= \frac{1}{n!} f^{(n)}(c)$

b/c $f^{(n-1)}$ cont and diff at $x=c$.

Moreover: Fix x

Consider $f(y) - p(y) - \frac{f(x) - P(x)}{(x-c)^n}(y-c)^n$

$\begin{cases} g(x) = 0 \\ g(c) = 0 \end{cases}$, $g'(c) = g''(c) = \ldots g^{n-1}(c) = 0$

Hence by MVT/Darboux

$g'(y_1) = 0$, $y_1 \sim (x, c)$

$g''(y_2) = 0$, $y_2 \sim (y_1, c) \subset (x, c)$

$\vdots$ $\vdots$

$g^{(n-1)}(y_{n-1}) = 0$, $y_{n-1} \sim (y_{n-2}, c)$

$g^{(n)}(y_n) = 0$, $y_n \sim (y_{n-1}, c)$

$\Rightarrow g^{(n)}(y) = f^{(n)}(y) - 0 - n! \cdot \frac{f(x) - P(x)}{(x-c)^n} = 0$

$\Rightarrow g^{(n)}(y) = f^{(n)}(y) - 0 - n! \cdot \frac{f(x) - P(x)}{(x-c)^n} = 0$

$$\Rightarrow \frac{f(x) - P(x)}{(x-c)^n} = \frac{f^{(n)}(y)}{n!} \quad \#$$

Ex: $f(x) = a_0 + a_1 x + \ldots + a_n x^n$

$P(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$

3.1-The Riemann Integral

Thursday, October 25, 2018 4:08 PM

Def:

$[a,b]$

$P = \{a = x_0 < x_1 < \dots < x_n = b\}$ a partition

Q a refinement of a partition P,

adds points to P. Q is a partition with more pts.

$f : [a,b] \to \mathbb{R}$ a bounded function

$M_j = M_j^P = \sup\{f(x) \mid x_{j-1} \le x \le x_j\}$

$m_j = m_j^P = \inf\{f(x) \mid x_{j-1} \le x \le x_j\}$

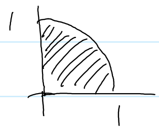
Upper Sum $U(f,P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1})$

Lower Sum $L(f,P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$

Ex. Show $\int_0^1 \sqrt[p]{1-x^q}\,dx = \int_0^1 \sqrt[q]{1-x^p}\,dx$

Sol.

Consider $x^p + y^q = 1$



Same Area.

Prop:

$m \le f(x) \le M$ in $[a,b]$

then

a) $m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$

b) If Q is a refinement of P, then

$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$

c) $0 \le U(f,Q) - L(f,Q) \le U(f,P) - L(f,P)$

$\sup\{L(f,P);\ P \text{ partitions}\} \le \inf\{U(f,Q) \mid Q \text{ partitions}\}$

(not necessarily refinements)

If equality occurs then

$$\int_a^b f = \int_a^b f(x)\,dx = \sup_P L(f,P) = \inf_Q U(f,Q)$$

Note: Trick to get a common refinement of P, Q is to "add" them together.

Prop 3.1.4 $f: [a,b] \longrightarrow \mathbb{R}$ bounded

Then f is Riemann integrable iff

$\forall \varepsilon > 0\ \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Moreover,

$\int_a^b f$ is the unique number s.t.

$L(f,Q) \leq \int_a^b f \leq U(f,Q)$

for any Q a refinement of P.

Pf:

We always have

$L \overset{def}{=} \sup_P \{L(f,P)\} \leq U \overset{def}{=} \inf_Q \{U(f,Q)\}$

Suppose for $\varepsilon > 0$ we can find a partition P s.t.

$U(f,P) - L(f,P) > \varepsilon$

but

$U - L \leq U(f,P) - L(f,P) < \varepsilon$

Hence $U = L$, $f \in \mathcal{R}[a,b]$ #

Cor 3.1.5 $f: [a,b] \to \mathbb{R}$ bounded

then $f \in \mathcal{R}[a,b]$ iff $\exists$ a sequence $\{P_k\}$ s.t.

Insert

Prop 3.1.6 $f: [a,b] \to \mathbb{R}$ bounded,

$f \in \mathcal{R}[x_{j-1}, x_j]$, $a = x_0 < x_1 < \dots < x_n = b$ for $1 \leq j \leq n$ then

f is Riemann Int. on $[a,b]$ and

$f \in R[x_{j-1}, x_j]$, $a = x_0 < x_1 < \dots < x_n = b$ for $1 \le j \le n$ then f is Riemann Int. on $[a,b]$ and

$$\int_a^b f = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f$$

Pf:

Let P_j be a partition of $[x_{j-1}, x_j]$ s.t. $U(f, P_j) - L(f, P_j) < \varepsilon/n$

If $P = \bigcup_{j=1}^{n} P_j$ then P is a partition on $[a,b]$.

$\vdots$

Thm 3.1.7 If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded and continuous except at finitely many points, then $f \in \mathcal{R}$

Pf:

i) If f is continuous

Observe f is u.c.

$\forall \varepsilon > 0 \ \exists \delta \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Consider any partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ s.t.

$0 < x_j - x_{j-1} < \delta$, $j = 1, 2, \dots, n$

So $M_j - m_j \overset{def}{=} \max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) < \frac{\varepsilon}{b-a}$

Therefore, $U(f, P) - L(f, P) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \frac{\varepsilon}{b-a} \cdot \sum_{j=1}^{n} (x_j - x_{j-1}) = \varepsilon$

Hence $f \in \mathcal{R}[a,b]$

ii) f is continuous on $[a,b] \setminus \{c\}$ $c \in [a,b]$

Let $m \le f \le M$ since bdd.

Remove a small neighborhood of c of the type

$[a, a+\varepsilon)$, $(c-\varepsilon, c+\varepsilon)$, $(b-\varepsilon, b]$

$c = a$ $\quad c \in (a,b)$ $\quad c = b$

Let P_1 = partition of $(a+\varepsilon, c-\varepsilon)$

P_2 = partition of $(c+\varepsilon, b-\varepsilon)$ s.t.

$U(f,P_1) - L(f,P_1) < \varepsilon$ and $U(f,P_2) - L(f,P_2) < \varepsilon$

$$U(f,P_1) - L(f,P_1) \le (M-m)\varepsilon + U(f,P_1) - L(f,P_1) + (M-m)2\varepsilon + U(f,P_2) - L(f,P_2) + (M-m)\varepsilon$$
$$\le (M-m)\varepsilon + (M-m)2\varepsilon + \varepsilon + (M-m)\varepsilon = (3M - 3m + 2)\varepsilon$$

iii) f is cont. except at a finite number of pts $c_1 < c_2 < \dots < c_m$

Pick points $a = x_0 < x_1 < \dots < x_m = b$ s.t.

each subinterval $[x_{k-1}, x_k]$ contains at most 1 point of discontinuity

$\Rightarrow f \in \mathcal{R}[x_{k-1}, x_k]$, $1 \le k \le m$ and consequently

$f \in \mathcal{R}(\bigcup [x_{k-1}, x_k])$ and $\displaystyle\int_a^b f(x)dx = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} f(x)dx$ $\boxtimes$

Prop 3.1.10

$f, g \in \mathcal{R}[a,b]$

a) If $f(x) \le g(x)$ $\forall x \in [a,b]$ then $\int_a^b f \le \int_a^b g$

Pf.

Since $f, g \in \mathcal{R}[a,b]$ $\exists$ sequences $\{P_k\}$ and $\{Q_k\}$ of partitions of $[a,b]$ s.t.

$$\int_a^b f = \lim_{k\to\infty} U(f,P_k) = \lim_{k\to\infty} L(f,P_k) \quad \text{and}$$

$$\int_a^b g = \lim_{k\to\infty} U(f,Q_k) = \lim_{k\to\infty} L(f,Q_k)$$

Consider common refinements $\{R_\ell\}$ of $\{P_k\}$ and $\{Q_k\}$ then

$$\int_a^b f$$

$\vdots$

b) $f \in \mathcal{R}[a,b]$, $|f(x)| \le M$ then $\left|\int_a^b f\right| \le M(b-a)$

Pf:

$-M \le f(x) \le M$ $\square$

Prop: 3.1.11

$f, g \in \mathcal{R}[a,b]$ $\alpha, \beta \in \mathbb{R}$ then

$\alpha f + \beta g \in \mathcal{R}[a,b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$

Convention: $\int_a^b f = -\int_b^a f$, $\int_a^a f = 0$

3.2-The Fundamental Theorem of Calculus

Tuesday, November 6, 2018 3:29 PM

Thm 3.2.1 $f \in \mathcal{R}[a,b]$

$F(x) \stackrel{def}{=} \int_a^x f(t)\,dt$ – indefinite integral of f or anti-derivative of f

then $F \in \mathcal{C}[a,b]$

If f is cont. at a point $c \in [a,b]$ then f is diff. at c and $F'(c) = f(c)$.

Pf:

F is Lipshitz cont. so u.c.

$|F(x)-F(y)| = \left|\int_a^x f - \int_a^y f\right| = \left|\int_x^y f\right| \le M|x-y|$ where $|f(x)| \le M$

$$\left|\frac{F(c+t)-F(c)}{t} - f(c)\right| = \left|\frac{1}{t}\int_c^{c+t} f - f(c)\right|$$

$$= \left|\frac{1}{t}\int_c^{c+t}[f(t)-f(c)]\,dt\right| \le \frac{\varepsilon|t|}{|t|} = \varepsilon$$

Thm: (MVT)

$f:[a,b] \to \mathbb{R}$ cont. then $\frac{1}{b-a}\int_a^b f(t)\,dt = c$ for some $c \in [a,b]$

Equivalently, $F'(c) = \frac{F(b)-F(a)}{b-a}$

Thm 3.2.5 ((Change of Variables) (Int. by Substitution))

$\phi:[a,b] \to I$ (interval)

continuously differentiable

$f: I \to \mathbb{R}$ continuous then

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\,\phi'(t)\,dt$$

Substitution $x = \phi(t)$, $dx = \phi'(t)\,dt$

$a \le t \le b$

Pf:

Let $F: I_u \to \mathbb{R}$, indef. int. of $\mathbb{R}$.

$F(u) = \int_{\phi(a)}^{u} f(x)\,dx$ so $F'(u) = f(u)$.

The Chain Rule Implies

$$[F(\phi(t))]' = F'(\phi(t))\,\phi'(t) = f(\phi(t))\,\phi'(t)$$

$$\int_a^b [F(\phi(t))]'\,dt = \int_a^b f(\phi(t))\,\phi'(t)\,dt$$

$$\begin{aligned} &\parallel \\ &F(\phi(t))\Big|_a^b \\ &\parallel \\ &F(\phi(t)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(x)\,dx - 0 \end{aligned}$$

$\boxtimes$

Thm 3.2.7 (Integration by Parts)

f, g are continuously differentiable. Then

$$\int_a^b fg' = \underbrace{f(b)g(b) - f(a)f(b)}_{\parallel \text{def} \; fg|_a^b} - \int_a^b f'g$$

Untitled page

Thursday, November 8, 2018 4:22 PM

<u>Def:</u> $x > 0$ $\log x = \int_0^x \frac{1}{t}\, dt$

<u>Prop:</u> (1) $\log(ab) = \log a + \log b$

(2) $\log(a/b) = \log a - \log b$

(3) $\log : (0, \infty) \to \mathbb{R}$ bijection

<u>Def:</u> The inverse of log is called the exponential function.

$\exp : \mathbb{R} \to (0, \infty)$ bijection

<u>Def:</u> $x > 0$ $a \in \mathbb{R}$

$x^a = \exp(a \log x)$

$x^a x^b = x^{a+b}$

$x^a y^a = (xy)^a$

$(x^a)^b = x^{ab}$

$x^{a-b} = x^a / x^b$

<u>Prop:</u> $e^a = \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n$

<u>Pf.</u>

Equivalently, $a = \lim \log \left(1 + \frac{a}{n}\right)^n$

Consider $f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$

$$\lim_{n\to\infty} \log\left(1 + \frac{a}{n}\right)^n = \lim_{n\to\infty} n \log\left(1 + \frac{a}{n}\right)$$

$$= a \lim_{n\to\infty} \frac{\log\left(1 + \frac{a}{n}\right) - \log 1}{a/n}$$

$$= a \lim \frac{\log(1+t) - \log 1}{}$$

$$= a \lim_{t \to 0} \frac{\log(1+t) - \log 1}{t}$$

$$= a f'(1)$$

Prop: For any $n \geq 1$

$$\lim_{x \to \infty} x^n e^{-x} = 0 \quad \text{and} \quad \lim_{x \to 0^+} \frac{e^{-1/x}}{x^n} = 0$$

Pf:
Use L'H

Ex: $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

$f \in C^\infty(\mathbb{R})$, $f^{(n)}(0) = 0$

HW 3.3.3 & 3.3.12

3.4

Tuesday, November 13, 2018 3:38 PM

__Def:__ $f:(a,b)\to\mathbb{R}$ $-\infty\le a<b\le\infty$

is said to be __locally integrable__ if it integrable on every bnd. closed subinterval $[a',b']\subset(a,b)$ $a<a'<b'<b$

We say f is integrable on (a,b) (or __improperly integrable__) if it is locally integrable and

$$\int_a^b f \overset{\text{def}}{=} \lim_{a'\searrow a}\left(\lim_{b'\nearrow b}\int_{a'}^{b'} f\right) \text{ exists and is finite.}$$

__Prop 3.4.2__ $\lim\limits_{a'\searrow a}\left(\lim\limits_{b'\nearrow b}\int_{a'}^{b'} f\right)=\lim\limits_{b'\nearrow b}\left(\lim\limits_{a'\searrow a}\int_{a'}^{b'} f\right)$

__Pf:__

Pick a pt $x\in(a,b)$

$$\lim_{c\searrow a^+}\left[\lim_{b'\nearrow b}\int_c^{b'} f\right]=\lim_{c\searrow a^+}\left[\int_c^x f+\lim_{b'\nearrow b}\int_x^{b'} f\right]$$

$$=\lim_{c\searrow a^+}\int_c^x f+\lim_{b'\nearrow b}\int_x^{b'} f$$

$$=\lim_{b'\nearrow b}\left[\lim_{c\searrow a^+}\int_c^x f+\int_x^{b'} f\right]=\lim_{b'\nearrow b}\left[\lim_{c\searrow a^+}\int_c^{b'} f\right] \quad \boxtimes$$

__Prop:__ If $f:(a,b)\to\mathbb{R}$ is absolutely integrable, meaning that $|f|$ is integrable, then $\left|\int_a^b f\right|\le\int_a^b |f|$

(and $\int_a^b f$ exists)

__Example:__ Show $\displaystyle\int_0^\infty \frac{\cos x}{1+x}\,dx=\int_0^\infty\frac{\sin x}{(1+x)^2}\,dx$

Sol:

$$\lim_{b\to\infty}\int_0^b \frac{\cos x}{1+x}\,dx = \lim_{b\to\infty}\int_0^b \frac{(\sin x)'}{1+x}\,dx$$

$$= \lim_{b\to\infty} \frac{\sin x}{1+x}\Big|_0^b + \lim_{b\to\infty}\int_0^b \frac{\sin x}{(1+x)^2}\,dx = 0 + \lim_{b\to\infty}\int_0^b \frac{\sin x}{(1+x)^2}\,dx$$

$$a_n = \int_0^n \frac{\sin x}{(1+x)^2}\,dx$$

$$|a_m - a_n| = \left|\int_n^m \frac{\sin x}{(1+x)^2}\,dx\right| \le \int_n^m \left|\frac{\sin x}{(1+x)^2}\right| dx = \int_n^m \frac{1}{(1+x)^2}\,dx$$

$$= -\frac{1}{1+x}\Big|_n^m = \frac{1}{1+n} - \frac{1}{1+m} \to 0$$

so $a_n \to a$ for some a

$$\Rightarrow \int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx = a$$

Ex: $\displaystyle\int_0^1 \frac{dx}{x^p} \Rightarrow \frac{x^{-p+1}}{-p+1}\Big|_a^1 = \frac{1}{-p+1} - \frac{a^{-p+1}}{-p+1} = \frac{1}{1-p}$

for $-p+1 > 0 \Rightarrow p < 1$

Def: 3.5.1
Consider open intervals $I=(a,b)$ $a<b$ and its length, $I=b-a$,
a set $E\subset\mathbb{R}$ has measure zero if every $\varepsilon>0$ $\exists$ a sequence $\{I_n\}$ of open intervals s.t.

$$E\subset\bigcup_{n=1}^{\infty} I_n,\quad \sum_{n=1}^{\infty}|I_n|<\varepsilon$$

We write $|E|=0$

Example 1: Any countable set has finite measure.

Prop 3.5.3
If $|E_n|=0$ $n=1,2,\ldots$ then $|\bigcup E_n|=0$.

4.2-Power Series

Tuesday, December 4, 2018 3:56 PM

Def: $X \subseteq \mathbb{R}$ $\quad f_n : X \to \mathbb{R}$

$\sum_{n=1}^{\infty} f_n$ converges uniformly if the partial sums converge uniformly.

Thm 4.2.2 (Weierstrass M-Test)

$X \subseteq \mathbb{R}$ $\quad f_n : X \to \mathbb{R}$

$\exists M_n$ $\quad |f_n(x)| \leq M_n \quad \forall x \in X$

$\sum M_n < \infty$

then $\sum_{n=1}^{\infty} f_n$ converges uniformly on X (and absolutely)

$\sum_{n=1}^{\infty} |f_n|$ converges uniformly

$\ast$ M_n can depend on n.

Def: If $c \in \mathbb{R}$ a power series about c is defined as $\sum_{n=0}^{\infty} a_n (x-c)^n$, $\quad a_n \in \mathbb{R}$

Thm 4.2.4

Given $\sum_{n=0}^{\infty} a_n (x-c)^n$ we define $R \in [0, \infty]$ by rule

$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$ called the radius of convergence

Then

(a) The series converges absolutely when $|x-c| < R$

(b) The series diverges when $|x-c| > R$

(c) If $0 < r < R$, the series converges uniformly on $|x-c| \leq r < R$

(d) R is unique.

Pf:

Wlog assume $c = 0$

May assume $c=0$

(a) and (b) by root test

$\limsup \sqrt[n]{|a_n x^n|} = |x| \limsup \sqrt[n]{|a_n|} = xR^{-1} \overset{?}{<} 1$

(c) If $|x| \le r$, then $|a_n x^n| \le a_n r^n \overset{def}{=} M_n$

where $\sum M_n < \infty$ by (a)

so by Weiestrauss M-Test conv. unif.

(d) Suppose $\exists$ another R_* say $0 \le R_* < R$

Then by (b) the series diverges when $R_* < |x| < R$

a contradiction with (a).

Examples:

- $\sum x_n$, $R=1$
- $\sum \frac{x_n}{n!}$, $R=\infty$
- $\sum \frac{(-1)^n}{n 2^n} x^{2n}$, no ratio test applies b/c gaps

$y = x^2$ $\quad \sum \frac{(-1)^n}{n2^n} y^n$, $R=2$

So for x, $R=\sqrt{2}$

Thm 4.2.6

Let $f = \sum_{n=0}^{\infty} a_n x^n$, $-R<x<R$ then

(a) $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n x^{n-k}$

(b) The radius of conv. is still R.

(c) $a_n = \frac{1}{n!} f^{(n)}(0)$

Pf:

(c) follows from (a)

(a) and (b) by induction.

$k=1 \quad f'(x) = \sum_{k=1}^{\infty} n a_n x^{n-1} \overset{\text{def}}{=} g(x)$

$\limsup \sqrt[n]{n|a_n||x|^{k-1}} = \limsup \sqrt[n]{|a_n|}\sqrt[n]{|x|^{n-1}} = \frac{|x|}{R}$

We need to show $g(y) = f'(x)$

Consider partial sums

$$g_N(y) = \sum_{n=1}^{N} n a_n y^{n-1}$$

$$\downarrow\downarrow$$

$$g(y) = \sum_{n=1}^{\infty} n a_n y^{n-1}$$

$$\int_0^x g_N(y)\,dy = \sum_{n=1}^{N} a_n x^n$$

$$\downarrow$$

$$\int_0^x g(y)\,dy = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

Hence by FTC, $g(x) = f'(x)$ $\blacksquare$

<u>Ex:</u> $e^x = \sum \frac{x^n}{n!} \quad (e^x)' = e^x$

<u>Thm 4.2.9</u>

$f \in C^\infty(-r,r) \quad |f^{(n)}(x)| \leq M^n$ then

$$f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$$

<u>Pf.</u> By Taylor's Formula

$$|f(x) - P_n(x)| = \left|\frac{f^{(n)}(d_n(x))}{n!} x^n\right| \leq \frac{M^n}{n!} r^n \to 0$$

<u>Ex:</u>

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad R = \infty$
- $\cos x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad R = \infty$
- $\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = f$ but not if $0 < x < r < 1$

$\cdot \cos x = \sum_{n=1} \frac{(-1)}{(2n)!} x^{2n} \qquad R = \infty$

$\cdot \log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = f$ but not if $0 < x < r < 1$

or

$$\log x = \int_1^x \frac{dt}{t} = \int_1^x \frac{dt}{1-(1-t)}$$

$$= \int_1^x \sum_{n=1}^{\infty} (1-t)^n dt$$

$\Downarrow$ u.c.

fix $x < 1$,

$$\log x = \sum_{n=0}^{\infty} \int_1^x (1-t)^n dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \qquad \square$$

$\cdot f = \frac{1}{x} = \sum \frac{f^{(n)}(c)}{n!} (x-c)^n = \sum \frac{(-1)^n n!}{c^{n+1}} (x-c)$ not banded

Problem #8) Wrong

5.1-Metric and Euclidean Space

Tuesday, January 15, 2019 3:31 PM

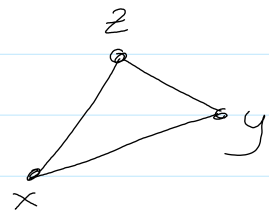
<u>Definitions:</u> (X, d), $d: X \times X \to [0, \infty)$

<u>Metric</u>

a) $d(x,y) = d(y,x)$

b) $d(x,y) = 0 \iff x = y$

c) $d(x,y) \leq d(x,z) + d(z,y)$



<u>Example</u>

1) Trivial discrete metric on X

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

c) If $d(x,z) = 1$ or $d(z,y) = 1$ we are done.

Assume not

$\Rightarrow z = x$ and $z = y$

$\Rightarrow x = y$ $\square$

2) Euclidean (Standard) Metric

$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

$\|x\| = (|x_1|^2 + |x_2|^2 + \dots + |x_n|^2)^{1/2}$

$d(x,y) = \|x - y\|$

<u>Exercise:</u>

If $d(x,y)$ is a metric then $2d$ is a metric, $\lambda d(x,y)$

3) $D(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$ is a bounded metric.

<u>Pf:</u>

a & b are obvious.

c) $D(x,y) \leq \dfrac{d(x,z)}{\dots} + \dfrac{d(y,z)}{\dots}$ is to be shown.

c) $D(x,y) \le \dfrac{d(x,z)}{1+d(x,z)} + \dfrac{d(y,z)}{1+d(y,z)}$ is to be shown.

$$D(x,y) \le \frac{d(x,z)+d(y,z)}{1+d(x,z)+d(y,z)}$$

So we are to show $\dfrac{a+b}{1+a+b} \le \dfrac{a}{1+a} + \dfrac{b}{1+b}$

$\iff (a+b)(1+a)(1+b) \le (1+a+b)[a(1+b)+b(1+a)]$

$\iff (a+b)(1+a+b+ab) \le (1+a+b)(a+b+2ab)$

$\iff a+a^2+a^2b+b+ab+b^2+ab^2 \le a+b+2ab+a^2+ab+2a^2b+ab+b^2+2ab^2$

$\iff 0 \le 2ab+a^2b+ab^2$ true since a,b are metrics. ∎

<u>Remark</u>: This is important for discussing convergence.

4) A lin. combo. of metrics is a metric

5)

6)

7) Exercise

5.2-Sequences and Completeness

Thursday, January 17, 2019 3:58 PM

(X,d) a metric space.

<u>Def 5.2.1</u> $x_n \to x$ if
$\forall \varepsilon > 0 \ \exists N$ s.t. $d(x, x_n) < \varepsilon \quad \forall n \geq N$

<u>Ex 5.2.2</u>
In $\mathbb{R}^p$,
$x_n = (x_n^1, x_n^2, \ldots, x_n^p) \to (x^1, x^2, \ldots, x^p)$ iff
$x_n^1 \to x^1, \ldots, x_n^p \to x^p$

<u>Pf:</u>
$\Rightarrow$ $d(x, x_n) = (|x_n^1 - x^1|^2 + |x_n^2 - x^2| + \ldots + |x_n^p - x^p|)^{1/2} \longrightarrow 0$
$\Rightarrow |x_n^k - x^k| \longrightarrow 0$
$\Leftarrow$ $|x_n^1 - x^1| < \varepsilon/\sqrt{p}, \ |x_n^2 - x^2| < \varepsilon/\sqrt{p}, \ldots, |x_n^p - x^p| < \varepsilon/\sqrt{p}$

$\Rightarrow d(x, x_n) < \left(p\left(\frac{\varepsilon}{\sqrt{p}}\right)^2\right)^{1/2} = \varepsilon$

<u>Ex:</u> (X,d) - discrete metric space
$x_n \to x$ iff $\exists N$ s.t. $x_n = x \quad \forall n \geq N$
Note: If $x_n \to x$ then every subsequence converges to x.

<u>Def:</u> <u>Cauchy Sequence</u>

$\{x_n\} \subset (X,d)$
$\forall \varepsilon > 0, \ \exists N$ s.t. $d(x_m, x_n) < \varepsilon \quad \forall m, n \geq N$.

<u>Prop:</u> Every convergent sequence is a Cauchy Sequence.

<u>Def:</u> <u>Complete Space</u>
(X,d) is complete if every Cauchy Sequence is convergent.

<u>Ex:</u> • $\mathbb{R}, \mathbb{R}^p$ are complete
(X,d) - discrete metric space

$\mathbb{C} \simeq \mathbb{R}^2$

- Open disk in $\mathbb{R}^2$ is not complete.
- $(0,1)$ is not complete since $\frac{1}{n}$ is Cauchy, $\frac{1}{n} \to 0 \notin (0,1)$.
- Any "closed" subset of $\mathbb{R}^2$ is complete

5.3

Thursday, January 17, 2019 4:31 PM

Def:

① (X,d) - metric space

$F \subset X$ is <u>closed</u> if $(x_n) \in F$, $x_n \to x$ then $x \in F$.

Note: $\emptyset$ and X are closed.

② $G \subset X$ is <u>open</u> iff $X \setminus G$ is closed. ($\emptyset, X$ are open)

Prop 5.3.2

a) $F_1, \dots, F_n$ are closed then $\bigcup_{k=1}^{n} F_k$ is closed.

Pf:

<u>One</u> of the F_k contains infinitely many terms of any sequence.

This is a subsequence so it must verge w/i F_k.

b) $G_1, G_2, \dots, G_n$ are open then $\bigcap G_k$ is open.

c) $\{F_i\}_{i \in I}$ are closed then $\bigcap_{i \in I} F_i$ is closed.

d) $\{G_i\}_{i \in I}$ are open then $\bigcup_{i \in I} G_i$ is open

Def:

Balls in (X,d) centered at x.

$B(x,r) = \{y \in X \mid d(x,y) < r\} \to$ open

$\overline{B(x,r)} = \{y \in X \mid d(x,y) \leq r\} \to$ closed

Prop 5.3.3

G is open $\iff$ G is the union of open balls.

Pf: (Sketch)

Let $x \in G$

Suppose $B(x, \frac{1}{n}) \not\subset G$

So $x_n \in B(x, \frac{1}{n})$ $d(x_n, x) < 1/n$

$x_n \in G^c$ - closed, $x_n \to x \Rightarrow x \in F \Rightarrow F = X \setminus G$

<u>Relatively Open</u> and <u>Relatively Closed</u> sets in $Y \subset X$

Prop 5.3.5

(X,d) - a metric space

$Y \subset X$

a) $G \subset Y$ is said to be relatively open in Y if
$G = U \cap Y$, for some U in X

b) $F \subset Y$ is said to be relatively closed in Y if
$F = D \cap Y$, for som D in X.

Remark: This is the same as the open/closed sets in (Y, d).
To see this, restricting a ball in X gives a ball in Y.

Def:
(X, d)

① The interior of a set:

$$\text{int } A = \bigcup_{\substack{G\text{-open} \\ G \subset A}} G$$

(is the largest open set contained in A.)

② The closure of a set:

$$\text{cl} A = \bar{A} = \bigcap_{\substack{F\text{-open} \\ F \subset A}} F$$

(is the smallest closed set in A.)

③ The boundry of $A \subset X$.

$$\partial A = (\text{cl} A) \cap \text{cl}(X \setminus A)$$

Prop 5.3.7
Let $A \subset X$

a) $x \in \text{int } A \iff B(x, \varepsilon) \subset A$ for some $\varepsilon > 0$.

b) $x \in \bar{A} \iff B(x, \varepsilon) \cap A \neq \emptyset \quad \forall \varepsilon > 0$.

c) $x \in \partial A \iff B(x, \varepsilon)$ intersects A and $X \setminus A \quad \forall \varepsilon > 0$.

Ex: $\mathbb{Q} \subset \mathbb{R}$ - metric space

- $\text{int } \mathbb{Q} = \emptyset$
  $\nexists I \subset \mathbb{Q}$, I open int.
- $\text{cl } \mathbb{Q} = \mathbb{R}$
- $\partial \mathbb{Q} = \mathbb{R}$

- $\text{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$
- $\text{cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$
- $\partial(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$

Ex:

$X = \odot \subset \mathbb{R}^2$

In this space X we have

$B(0,1) = \{0\}$ so $\mathrm{cl}\, B(0,1) = \{0\} \neq \overline{B(0,1)}$

Prop 5.3.9 $A \subset X$

a) A is closed $\iff$ $A = \overline{A}$

b) A is open $\iff$ $A = \mathrm{int}\, A$

c) $\overline{A} = X \setminus \mathrm{int}(X \setminus A)$

$\mathrm{int}\, A = X \setminus \mathrm{cl}(X \setminus A)$

$\partial A = \mathrm{cl}\, A \setminus \mathrm{int}\, A$

d) $\mathrm{cl}(A_1 \cup A_2 \cup \dots \cup A_k) = \mathrm{cl}\, A_1 \cup \mathrm{cl}\, A_2 \cup \dots \cup \mathrm{cl}\, A_k$

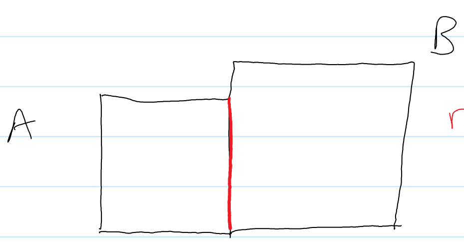
- $\mathrm{int}(A \cup B) \supset \mathrm{int}\, A \cup \mathrm{int}\, B$

Why?

A' open $\subset A$, B' open $\subset B$

Then $\underbrace{A' \cup B'}_{\text{open}} \subset A \cup B$

Ex:



red line $\notin \mathrm{int}\, A \cup \mathrm{int}\, B$

HW pg 127 11, 12, 13 Due Jan 29.

Def: 5.3.10

$E \subset X$ is *dense* if $\mathrm{cl}\, E = X$.

Example:

$\mathbb{Q} \subset \mathbb{R}$ is dense

(X,d) is seperable if it has a countable dense subset.

Ex:
- $\mathbb{R}^p$ is seperable.
- Let (X,d) be a discrete uncountable metric space. Then (X,d) is not seperable.

Prop 5.3.12
$E \subset X$ is dense in (X,d) iff every ball intersects E.

Def: 5.3.13
$A \subset X$. A point $x \in X$ is called a limit point of A iff $\forall B(x,\varepsilon)$ $\exists a \in B(x,\varepsilon)$ s.t. $a \neq x$

Def:
$x \in A$, but is not a limit point is called an isolated point.

Ex:
$\mathbb{Q} \subset \mathbb{R}$ every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

Prop 5.3.15 $A \subset X$
a) $x \in X$ is limit point if

$x = \lim x_i$
$x_i \in A$
$x_i \neq x_n$ (non-constant sequence)

b) A set is closed iff it contains all its limit points.

c) $\operatorname{cl} A = A \cup \{\text{set of all limit points}\}$

Def: 5.3.16
$A \subset X$, $x \in X$
$\operatorname{dist}(x, A) = \inf \{ d(x,a) \mid a \in A \}$

Prop: 5.3.17
$A \subset X$
Then $\operatorname{cl} A = \{ x \in X \mid \operatorname{dist}(x, A) = 0 \}$

Def:

$E \subset X$, the diameter of E,

$$\operatorname{diam} E = \sup\{d(x,y) \mid x,y \in E\}$$

Cantor's Theorem

The space (X,d) is complete iff it satisfies the following:

Whenever $F_1 \supset F_2 \supset \dots$ (nonempty closed sets)
$\operatorname{diam} F_n \to 0$,

we have $\bigcap_{n=1}^{\infty} F_n$ is a single point.

Prop 5.3.19

(X,d) complete, $Y \subset X$.
Then (Y,d) is complete iff Y is closed in X.

Def:

A subset of (X,d) is bounded if $\operatorname{diam} A < \infty$.

Thm:

a) A is bounded iff $\forall x \in X$ $A \subset B(x,R)$ for some $R > 0$.

Better:

Balls are bounded, so are their subsets.
Every bounded set is contained in a ball.

Indeed pick up a point $a \in A$ and ball ??

b) The union of a finite number of bounded sets is bounded.

c) A Cauchy Sequence in (X,d) is a bounded set.

5.4-Continuity

Thursday, January 24, 2019 3:39 PM

<u>Def:</u> Continuity

(X,d), (Z,ρ) $f: X \xrightarrow{\text{into}} Z$

is continuous at a point $a \in X$, iff

$$\begin{pmatrix} x_n \to a \\ \cap \quad\; \cap \\ X \quad\; X \end{pmatrix} \Rightarrow \begin{pmatrix} f(x_n) \to f(a) \\ \cap \qquad\quad \cap \\ Z \qquad\quad Z \end{pmatrix}$$

Equivalently, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d(a,x) < \delta \Rightarrow \rho(f(a), f(x)) < \varepsilon$

<u>Def:</u>

$f: X \to Z$ is continuous if f is continuous at every point.

<u>Thm 5.4.3</u> (a) f continuous

(b) $\forall\, U \in Z$ open the set $f^{-1}(U)$ is open in X.

(c) $\forall\, D \in Z$ closed the set $f^{-1}(D)$ is closed in X.

<u>Prop: 5.4.4, 5.4.5</u>

$f \circ g$ is continuous if f and g are continuous.

If $f, g \in \mathcal{C}(X, \mathbb{R}^p)$ then

$f + g \in \mathcal{C}(X, \mathbb{R}^p)$

$f \cdot g \in \mathcal{C}(X, \mathbb{R}^p)$

$\alpha f \in \mathcal{C}(X, \mathbb{R}^p)$ $\alpha \in \mathbb{R}$

<u>Prop 5.4.7</u>

(X,d), $A \subset X$

$|\operatorname{dist}(x,A) - \operatorname{dist}(y,A)| \le d(x,y)$

Is Lipshitz continuous.

Dist from x to A.

Cor: $f(x) = \text{dist}(x, A)$ is Lipshitz cont.

Thm: Urysohn's Lemma

A, B disjoint closed subsets of X.

$$f(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}.$$

f is cont. ($\text{dist}(x, A)$, $\text{dist}(x, B)$ can't both be 0.)

(a) $0 \le f(x) \le 1$

(b) $f(x) = 0 \quad \forall x \in A$

(c) $f(x) = 1 \quad \forall x \in B$

Cor 5.4.10 Cutoff Function

(X, d)

G - open

F closed subset of G.

$\exists f \in \mathcal{C}(X, \mathbb{R})$

$$\begin{cases} 0 \le f(x) \le 1 & \text{in } X \\ f(x) = 1 & \text{on } F \\ f(x) = 0 & \text{outside } G. \end{cases}$$

Def: 5.4.11

$f : (X, d) \to (Z, \rho)$ is uniformly cont. if

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$.

Example 5.4.12

Lipshitz functions.

$f : (X, d) \to (Z, \rho)$

$\rho(f(x), f(y)) \le M d(x, y)$

5.5-Compactness

Tuesday, January 29, 2019 4:39 PM

Def: Cover and Subcover

$E \subset X$, (X, d)

A collection of subsets of X $\mathcal{G} \subset 2^X$ is a cover of E if $E \subset \bigcup_{G \in \mathcal{G}} G$.

A subcover $\mathcal{G}_1 \subset \mathcal{G}$ which is a cover of E

Open cover if $\mathcal{G}$ consists of open subsets of X

Def: 5.5.1

(X, d), $K \subset X$

K is compact if every open cover of K has a finite open subcover.

Exercise 2 p 139

K is compact iff every cover K by relatively open subsets of K has a finite open subcover.

Ex:

① $(0,1) \subset \mathbb{R}$ is not compact

$G_n = (\frac{1}{n}, 1)$ $\bigcup G_n = (0,1)$

② $\ell^\infty = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{R},\ x \text{ bounded}\}$

$$E \begin{cases} e_1 = (1, 0, 0, \ldots) \\ e_2 = (0, 1, 0, \ldots) \\ \vdots \\ e_n = (0, 0, \ldots, 0, 1, 0, \ldots) \end{cases}$$

$\|x - y\| = \sup_n |x_n - y_n|$

E not compact

③ Every finite set is compact.

④ $G = \{x_i\}$, $x_i \to x$, $x \in G$ then G is compact.

Prop:

(a) K-compact $\Rightarrow$ K is closed and bounded.
(b) K-compact, F closed set $F \subset K$, then F is compact.
(c) Continuous image of K (compact) is compact.

Thm: Extreme Value Theorem
(X, d) compact
$f: X \to \mathbb{R}$ continuous
Then $\exists a, b \in X$ s.t. $f(a) \leq f(x) \leq f(b)$ $\forall x \in X$.
(f obtains its min/max)

Thm 5.5.5
Let K be a closed subset of X.
TFAE
a) K is compact.
b) Every $\mathcal{F}$-collection of closed subsets of K has FIP (Defined below)
c) Every sequence in K has a convergent subsequence.
d) Every infinite subset of K has a limit point.
e) (K, d) is a complete metric space, that is every Cauchy sequence is convergent.

Def: Finite Intersection Property (FIP)
$\mathcal{F}$ a collection of subsets of K s.t. $F_1, F_2, \ldots, F_n \in \mathcal{F}$ implies $F_1 \cap F_2 \cap \ldots \cap F_n \neq \emptyset$.

5.6-Connectedness

Thursday, February 7, 2019 3:40 PM

Def 5.6.1

(X,d) is connected if it cannot be written as

$X = A \cup B$, A, B nonempty open sets and $A \cap B = \emptyset$.

X is disconnected if

$X = A \cup B$, A, B nonempty open sets, $A \cap B = \emptyset$.

Def:

Prop 5.6.2

TFAE

(a) (X,d) is connected

(b) If $X = A \cup B$ and $A \cap B = \emptyset$, A, B are open (or both closed)

then either $A = \emptyset$ or $B = \emptyset$.

(c) If $A \subset X$, $A \neq \emptyset$

A both open and closed then $A = X$.

Prop 5.6.3

A subset of $\mathbb{R}$ is connected $\Leftrightarrow$ it is an interval.

(a,b), $[a,b)$, $[a,b]$, $(a,b]$

Thm 5.6.4

The cont. image of a connected set is connected.

Cor 5.6.5 (IVT)

$f: (X,d) \to \mathbb{R}$, (X,d) connected, f continuous

$a, b \in f(X) \subseteq \mathbb{R}$,

then $[a,b] \subset f(X)$.

Ex:

(a) $x, y \in \mathbb{R}^p$ straight line segment

$[x,y] = \{ty + (1-t)x \,,\; 0 \le t \le 1\}$ is connected

(b) $B(a,r) \subset \mathbb{R}^p$ is connected

(c) Circle in $\mathbb{R}^2$ is connected.

Pf: $f[0, 2\pi] = [\cos\theta, \sin\theta : 0 \le \theta \le 2\pi]$

Def:

(i) A continuum in $\mathbb{R}^n$ is any compact connected subset.

(ii) A domain in $\mathbb{R}^n$ is any open connected subset.

Prop 5.6.7

(a) E_i, $i \in I$ are connected; $E_i \cap E_j \neq \emptyset$

then $\cup E_i$ connected

(b) $E_1 \cap E_2 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, ...

then $\cup E_i$ connected.

Remark:

E connected $\Leftrightarrow$ E does not contain a subset that is relatively open and closed.

Prop 5.6.11

$C \subset X$ connected

Then every set Y b/w C and $\mathrm{cl}\,C$ is connected.

$C \subset Y \subset \mathrm{cl}\,C$, Y is connected.

Def 5.6.9 (Component)

Def 5.6.9 (Component)

A component of (X, d) is a maximal connected subset of X.

5.7-Space of Continuous Functions

Tuesday, February 12, 2019 4:24 PM

Def: Uniformly convergent

(X,d) a fixed metric space.

$\{f_n\}$ $f_n : X \to \mathbb{R}$

$f_n \rightrightarrows f$ uniformly if

$\forall \varepsilon > 0 \ \exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon \ \forall x \in X$ and $n \geq N$.

Prop 5.7.2

$$g_n(x) \leq f_n(x) \leq h_n(x)$$

$$\downarrow\downarrow \qquad\qquad \downarrow\downarrow$$

$$f(x) \qquad\qquad f(x)$$

then $f_n(x) \rightrightarrows f(x)$

Def/Notation:

$C(X)$ is the space of continuous fcns on X

$C_b(X)$ is " " " bounded " " " "

- These are algebras.

Prop 5.7.3

$f_n : X \to \mathbb{R}$ bnded fcns

$f_n \rightrightarrows f : X \to \mathbb{R}$

Then f is bounded and the sequence is uniformly bnded.

Uniformly bounded:

$\exists M$ s.t. $|f_n(x) \leq M \ \forall x \in X$ and $\forall n \geq 1$.

Thm 5.7.4

$f_n \in C_b(X)$

$f_n \rightrightarrows f$

then $f \in C_b(X)$

Thm: 5.7.5

$X \subset \mathbb{R}$

$\{f_n\}$ bnd unif. cont, $f_n : X \to \mathbb{R}$, $f_n \rightrightarrows f$

then $f : X \to \mathbb{R}$ is bounded and uniformily continuous.

Def 5.7.6

Def 5.7.6

Norm in $C_b(X)$, $\|f\| = \sup\{|f(x)| : x \in X\} < \infty$

Prop 5.7.7 (X, d)

(a) $C(X)$ and $C_b(X)$ are algebras

(b) $d(f,g) \stackrel{def}{=} \|f-g\|$, $f,g \in C_b(X)$

(c) $f_n \longrightarrow f$ in C_b $\iff$ $f_n \rightrightarrows f$

(d) $C_b(X)$ is a complete metric space.

Lemma 5.7.9 Stone-Weierstrass

Polynomials:

$$p_1(x) = 0, \quad p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x))$$

HW 4) 5.7.8 and

8' let $f \in \mathcal{C}[-1,1]$ s.t.

$$\int_{-1}^{1} f(x) x^{2n}\,dx = 0 \quad \forall n \geq 0.$$

Show that $f(-x) = -f(x)$.

Lemma 5.7.9 Stone-Weierstrass

Polynomials:

$$p_1(x) = 0, \quad p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x))$$

(α) For $x \in [0,1]$ we have

$p_n(x) \leq \sqrt{x} \leq 1$ by induction (Note $0 \leq p_n(x)$)

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}(\sqrt{x} - p_n(x))(\sqrt{x} + p_n(x))$$
$$\leq p_n(x) + (\sqrt{x} - p_n(x)) = \sqrt{x}$$

(β) $p_n(x) \leq p_{n+1}(x)$ obvious

(γ) $\lim p_n(x) = t \leq \sqrt{x}$, $\quad t = t + \frac{1}{2}(x - t^2)$

$t = \sqrt{x}$

(γ) $p_n \rightrightarrows \sqrt{x}$

Consider closed sets $F_n = \{x \in [0,1] : f(x) \geq p_n(x) + \varepsilon\}$, $F_1 \supset F_2 \supset \dots$

$\cap F_i = \emptyset$ so $\exists N$ s.t. $F_N = \emptyset$ $\square$

Subalgebras of $\mathcal{C}_b(x)$

Lemma:

$\mathcal{A} \subset \mathcal{C}_b(x)$ closed algebra, $1 \in \mathcal{A}$.

Then given $f, g \in \mathcal{A}$

then $\max\{f g\} \in \mathcal{A}$ and

$\min\{f g\} \in \mathcal{A}$

5.7.1 (Stone-Weierstrauss)

(X, d) compact

$\mathcal{A} \subset \mathcal{C}(X)$ closed subalgebra

$1 \in \mathcal{A}$ so $\mathcal{A}$ contains constant fcns.

A-seperates the points in X given $x_1 \neq x_2$, then

$\exists f \in \mathcal{A}$ s.t. $f(x_1) \neq f(x_2)$

Then $\mathcal{A} = \mathcal{C}(X)$.

HW ① $f \in \mathcal{R}(a,b)$ s.t. $\int_a^b f x^n = 0 \quad \forall n$

Show $f = 0$ a.e.

② $f, f' \in \mathcal{C}[a,b]$

Show $\exists p_n$ polynomials s.t. $p_n \rightrightarrows f$ and $p_n' \rightrightarrows f'$.

Hint for 1)

f is bounded $\Rightarrow m \leq f \leq M$.

$f_n \in \mathcal{C}[a,b]$ s.t. $\int_a^b |f(x) - f_n(x)| dx \to 0$

f is continuous except on a set of zero-measure say E

Cover w/ open intervals of length as small as you wish.

G = good sets.

Have function on G

Extend $f|_G$ to entire interval (pg. 42) for continuous fcn.

f_n coinsides w/ f on G, dont care on E.

Approx. f_n by $p_n \rightrightarrows f_n$

So $\int |f - p_n| \longrightarrow 0$

Now same as for cont. fcns.

$$\int f^2 = \int (f - p_n) f + \int \underbrace{f p_n}$$

$$\int f = \int (f - p_n) f + \underbrace{\int f p_n}_{=0}$$

$$\leq M \int |f - p_n| \to 0$$

Suppose $f(c) \neq 0$ at some c. $\Rightarrow f^2(c) \neq 0$ continuous at c

$\Rightarrow f^2(c) > 0$ on some $(\alpha, b) \ni c$

$$\Rightarrow 0 = \int f > \int_\alpha^\beta f > c \int_\alpha^\beta$$

<u>Def: 5.7.14</u>

(X, d) - metric space.

$\mathcal{F} \subset \mathcal{C}(X)$ is <u>equicontinuous</u> if

$\forall \varepsilon > 0 \quad \forall x_0 \in X \quad \exists \mathcal{U}$ a neighborhood of x_0 s.t.

$|f(x) - f(x_0)| < \varepsilon \quad \forall x \in \mathcal{U}$ and

$\forall f \in \mathcal{F}$.

(Continuous at every point not depending on f.)

<u>Ex:</u>

- $\mathcal{F} = \{f\}$
- $\mathcal{F} = \{f_1, f_2, \dots, f_N\}$
- $\mathcal{F} = \{f_1, \dots\} \quad f_n \rightrightarrows f$

<u>Thm:</u> (Arzela - Ascoli)

(X, d) - compact

$\mathcal{F} \subset \mathcal{C}(X)$ is ~~totally~~ bounded iff

$\mathcal{F}$ is bounded and equicontinuous

(not necessarily closed family)

<u>Cor:</u> (Should be thm)

$\mathcal{F} \subset \mathcal{C}(X)$ is compact $\iff$ $\mathcal{F}$ is closed and equicontinuous.

6.1

Tuesday, February 12, 2019 4:05 PM

Def: 6.1.1

$\gamma . (a,b) \longrightarrow \mathbb{R}^p$

$$\gamma'(x) = \lim_{t \to 0} \frac{\gamma(x+t) - \gamma(x)}{t}$$

Def 6.1.2

A curve in $\mathbb{R}^p$

$\gamma : [a,b] \longrightarrow \mathbb{R}^p$ continuous

$\gamma([a,b]) \subset \mathbb{R}$ the trace of the curve

$\gamma'(t)$ is tangent vector

Prop 6.1.3 (differentiability)

$x, y \in (a,b)$

$$\gamma(x) - \gamma(y) = \int_0^1 \frac{d}{dt} \gamma(tx + (1-t)y)\, dt$$

$$= \int_0^1 [\gamma'(tx + (1-t)y) \mid x - y]\, dt$$

$$= \langle V \mid x - y \rangle$$

$$V = \int_0^1 \gamma'(tx + (1-t)y)\, dt$$

Letting $y \to x$ we see

$$\lim_{y \to x} [\gamma(y) - \gamma(x)] = \int_0^1 \gamma'(x)\, dt = \gamma'(x)$$

So

$$\gamma(y) - \gamma(x) = \gamma'(x)(y - x) + \mathcal{E}(y) \qquad \lim_{y \to x} \mathcal{E}(y) = 0$$

Prop 6.1.4 Prod. Rule

$f(t) = \langle \gamma(t) \mid \tau(t) \rangle$

$f'(t) = \langle \gamma'(t) \mid \tau(t) \rangle + \langle \gamma(t) \mid \tau'(t) \rangle$

Cor from Prop 6.1.3

$|\gamma(x) - \gamma(y)| \leq M|x-y|$ where $M = \sup_{a \leq s \leq b} |\gamma'(s)|$

Remark:

No MVT here.

Ex:

$\gamma(t) = (t^2, t^3) \quad 0 \leq t \leq 1$

$\gamma(1) - \gamma(0) \neq \gamma'(c)$ for any $c \in (0,1)$

$(1,1) \neq (2c, 3c^2)$

6.2

Tuesday, February 26, 2019 4:03 PM

Def: 6.2.1 Partial Derivative

$G \subset \mathbb{R}^p$ open subset of $\mathbb{R}^p$

$f : G \to \mathbb{R}$, $x \in G$, $1 \le j \le p$,

j-th partial derivative

$$\lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t} = \frac{\partial f}{\partial x_j}(x) = \partial_j f(x) = f_{x_j}(x)$$

Ex:

- $\partial_{ji} f = \partial_i(\partial_j f)$

- $$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\partial_{21} f(0,0) = 1 \quad , \quad \partial_{12} f(0,0) = -1$$

Prop

If $\partial_i \partial_j f$ and $\partial_j \partial_i f$ exist everywhere in G and are continuous at $x \in G$, then they are equal (at x)

Def: 6.2.6 Linear functionals in $\mathbb{R}^p$

$L : \mathbb{R}^p \to \mathbb{R}$ $\quad (\mathbb{R}^p)^* \approx \mathbb{R}^{p*}$

Let $e_1, \dots, e_p$ form a basis for $\mathbb{R}^p$ (usually standard binary basis)

Let $\varepsilon_j = L(e_j)$

$x = (x_1, \dots, x_p) = x_1 e_1 + x_2 e_2 + \dots + x_p e_p$

Then $L(x) = \sum_{j=1}^{p} \varepsilon_j x_j$

Prop 6.2.7

The functional $E_j(x) = x_j$, $j = 1, \dots, p$ form a basis for $(\mathbb{R}^p)^*$

Prop 6.2.8

If $L \in (\mathbb{R}^p)^*$ then $\exists!$ vector $a \in \mathbb{R}^p$ s.t.
$L(x) = \langle x \mid a \rangle$.

In fact $a = (L(e_1), \dots, L(e_p))$
$= (\varepsilon_1, \dots, \varepsilon_p)$

Prop 6.2.9 The norm of L

$\|L\| = \left(\sum_{j=1}^{p} |L(e_j)|^2 \right)^{1/2}$

$= \sup \{ |L(x)| : \|x\| \le 1 \}$
$\hookrightarrow x_1^2 + x_2^2 + \dots + x_p^2 \le 1$

Note: There is an x s.t. supremum is obtained.

Def: 6.2.11 (Differentiability)

$f: G \to \mathbb{R}$, G is open in $\mathbb{R}^p$

f is differentiable at x_0 if $\qquad \frac{o(s)}{s} \to 0$

$f(y) - f(x_0) = L(y - x_0) + o(\|y - x_0\|)$ (Nonlinear part is small)
$= \langle a \mid y - x_0 \rangle$, $a = \nabla f(x_0)$

By definition: $L = Df(x) = f'(x)$ — differential at x.
$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$

Note that L is unique.

Thm 6.2.12

$f: G \to \mathbb{R}$ differentiable at x_0 then
(a) f is continuous at x_0
(b) the partial derivative of f at x_0 exists.
(c) $\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \dots, \frac{\partial}{\partial x_n} f(x) \right)$, the gradient
$[Df(x)](y) = \langle y, \nabla f(x) \rangle$ directional derivative.

Thm 6.2.13

If $f: G \to \mathbb{R}$ has partial derivatives in G which are continuous at $x_0 \in G$, then f is differentiable at x_0.

Moreover,

$L = Df(x)h = \langle \nabla f \mid h \rangle$ for $h \in \mathbb{R}^p$.

<u>Thm 6.2.14</u> (Chain Rule)

$\gamma : (a,b) \longrightarrow \underset{\text{open}}{G} \subset \mathbb{R}^p$, $f : G \longrightarrow \mathbb{R}$

$f \circ \gamma : (a,b) \longrightarrow \mathbb{R}$ is differentiable at x_0

$$(f \circ \gamma)'(x_0) = \langle \nabla f(\gamma(x_0)) \mid \gamma'(x_0) \rangle$$

Directional Derivatives

⋮

<u>Prop 6.2.18</u>

G - domain, $f : G \to \mathbb{R}$, $f'(x) = 0$ $\forall x \in G$, then f is constant.

<u>Directional Derivatives</u>

G - open in $\mathbb{R}^p$

$f : G \to \mathbb{R}$

$x \in G$

d - a vector in $\mathbb{R}^p$, usually a unit vector

$t \rightsquigarrow f(x+td)$ differentiable at $t = 0$

$$\nabla_d f(x) \overset{def}{=} \lim_{t \to 0} \frac{f(x+td) - f(x)}{t} \in \mathbb{R}$$

<u>Prop 6.2.17</u>

$$\nabla_d f(x) = \langle \nabla f(x), d \rangle$$

Remark:

$$\nabla_{\alpha a+\beta b} f(x) = \alpha \nabla_a f(x) + \beta \nabla_b f(x)$$

whenever f is differentiable at x;

$$f(x+h) - f(x) - \langle \nabla f(x) \mid h \rangle = o(|h|)$$
$h \in \mathbb{R}^n$ as $h \to 0$

Local Critical Points

$G \subset \mathbb{R}^p$ open $f: G \to \mathbb{R}$ diff. at G. Then

- f has a critical point at a if $\nabla f(a) = 0$.
- f has a local maximum at a if $\exists \delta > 0$ s.t. $f(a) \geq f(x)$, whenever $|a-x| < \delta$
- Local minimum $f(a) \leq f(x)$
- Local extremum is either local max/min.
- Saddle Point. not needed

Thm 6.2.21

$a \in G$, $f: G \to \mathbb{R}$ diff. on G.

- If $\forall d \in \mathbb{R}^p$ $\exists \delta > 0$ s.t.
  $\nabla_d f(a-td) > 0$ and $\nabla_d f(a+td) < 0$ for $0 < t < \delta$
  Then
  f has a local maximum at a
- Switch inequalities for local minimum.

Prop 6.2.22

$$|\nabla_d f(a)| \leq |\nabla f(a)| = \left(\sum_{i=1}^{p} \left| \frac{\partial f}{\partial x_i}(a) \right|^2 \right)^{1/2}$$

Equality occurs when $d = \dfrac{\nabla f(a)}{|\nabla f(a)|}$

HW 6

Thursday, February 28, 2019 3:35 PM

6) Suppose $F = A \cup B$, closed sets s.t. $A \neq \emptyset$, $B \neq \emptyset$
(So suppose not connected) $A \cap B = \emptyset$

Define: $f(x) = \text{dist}(x, A) - \text{dist}(x, B)$ is cont. in X.

If $x \in A$, $f(x) \geq 0$.
If $x \in B$, $f(x) \leq 0$
So by MVT, $\exists\, x_1 \in X_1$ s.t. $f(x_1) = 0$
since F_i connected. $x_2 \in X_2$ s.t. $f(x_2) = 0$

$\vdots$

$x_n \in X_n$ s.t. $f(x_n) = 0$

Consider $\{x_1, x_2, \ldots, \}$ (take subset $x_n \to x$ since compact)
Claim $x \in F$.
Why? $x \in F_1$ since $x_1, x_2, \ldots \in F_1$
$x \in F_2$ since $x_2, x_3, \ldots \in F_2$

$\vdots$

f continuous so $f(x_n) \to f(x)$
$\overset{\parallel}{0} \longrightarrow \overset{\parallel}{0}$

$\Rightarrow \text{dist}(x, A) = \text{dist}(x, B)$ but $x \in A$ or $x \in B$ so $= 0$ $\otimes$

cl of connected set is connected

6.3

Tuesday, March 19, 2019 3:23 PM

2) Take $x \in \Gamma_\varepsilon \setminus \Gamma$

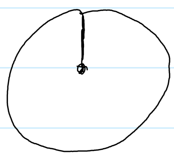
a) Take closest point $y \in \Gamma$ (exists since Γ compact)

dist is not in Γ otherwise not min. dist

So connected.

These segments share Γ.

c)



1) Δ inequality

Lemma 1: Let $a \in A$

$$\operatorname{dist}(a,B) \le \sup_{x \in A} \operatorname{dist}(x,C) + \sup_{z \in C} \operatorname{dist}(z,B)$$

pf.

$\forall\, b \in B,\ c \in C$

$$\operatorname{dist}(a,B) \le d(a,b) \le d(a,c) + d(c,b)$$

Take inf. WRT b to obtain

$$\operatorname{dist}(a,B) \le d(a,c) + d(z,B)$$
$$\le d(a,c) + \sup_{z \in C} \operatorname{dist}(z,B)$$

Take inf. WRT c to obtain

$$\operatorname{dist}(a,B) \le \operatorname{dist}(a,C) + \sup_{z \in C} \operatorname{dist}(z,B)$$
$$\le \sup_{x \in A} \operatorname{dist}(x,C) + \sup_{z \in C} \operatorname{dist}(z,B)$$

Lemma 2:

Let $b \in B$. Then

$$\operatorname{dist}(b,A) \le \sup \operatorname{dist}(u, C) + \sup (z, A)$$

Let $b \in B$. Then

$$\text{dist}(b, A) \le \sup_{y \in B} \text{dist}(y, C) + \sup_{z \in C} (z, A)$$

- Add them up.

- Take supremum WRT a and b to get inequality.

6.4

Tuesday, March 26, 2019 4:06 PM

<u>Def 6.4.11</u>

$A \in \mathcal{L}(\mathbb{R}^p)$ is <u>self-adjoint</u> or hermitian if

$A = A^*$

<u>Def: 6.4.13</u> (Orth. Projection)

$M \subset \mathbb{R}^p$, $x \in \mathbb{R}^p$

then $\exists!\ y_0 \in M$ s.t. $x - y_0 \perp M$

Denote $y_0 = P(x)$ - call P the <u>orthogonal projection</u>

$P: \mathbb{R}^p \to X$

<u>Prop; 6.4.14</u>

$M \subset \mathbb{R}^p$ linear subspace $P: \mathbb{R}^p \to M$ orth. proj - then

(a) $P: \mathbb{R}^p \to \mathbb{R}$ is a hermition transformation
(linear)

Choose a fix an orthonormial basis for M

$\{y_1, y_2, \dots, y_m\}$

Then,

$$P(x) = y_0 = \sum_{j=1}^{m} \langle x, y_j \rangle y_j$$

$$\langle Px, w \rangle = \sum \langle x, y_j \rangle \langle y_j, w \rangle$$

$$= \sum \langle x, y_j \langle y_j, w \rangle \rangle$$

$$= \langle x, Pw \rangle \overset{def}{=\!=} \langle P^* x, w \rangle$$

$Px = P^* x$

(b) $\|Px\| \leq \|x\|$

$Px \perp x - Px$, $\|x\|^2 = \|x - Px + Px\|^2$

$$= \|x - Px\|^2 + \|Px\| \geq \|Px\|^2$$

equality iff $x = Px$

(c) P is <u>idempotent</u>:

$$P^2 = P$$

$$P^2_x = P(\underset{\uparrow M}{P_x}) = P_x$$

(d) $\ker P = M^\perp$

$\operatorname{ran} P = M = \{x : Px = x\}$

<u>Remark:</u>

We need not specify M. In particular, we have the following definition.

<u>Def 6.4.15</u>

An <u>orth. projection</u> (we do not specify M) is an idempotent $P : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ s.t.

$$(x - Px) \perp \operatorname{ran} P$$

<u>Exercise:</u>

Compute

$$\det \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -2 & 0 & -3 & 1 & 6 \\ 5 & -4 & 0 & 2 & 0 \\ 0 & 3 & 0 & -1 & 4 \\ -9 & 8 & 0 & 0 & 0 \end{pmatrix} = 24$$

<u>Remark:</u>

① For matrix A, $\|A\|$ is the largest root of $\det(A^*A - \lambda^2) = 0$.

② For diagonal matrix $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $\|A\| = \max_{\forall i} \{\lambda_i\}$

6.5-Differentiable Mappings

Tuesday, April 2, 2019 4:31 PM

Def:

G - open subset of $\mathbb{R}^p$, $x \in G$

$$f : G \longrightarrow \mathbb{R}^q$$

is differentiable at x if there is a linear transformation

$A : \mathbb{R}^p \longrightarrow \mathbb{R}^q$ s.t.

$f(y) = f(x) + A(y-x) + \|y-x\| F(y)$ where

$F : G \longrightarrow \mathbb{R}^q$ satisfies

$\lim_{y \to x} F(y) = 0$, $\|y-x\| F(y) = o(\|y-x\|)$ at $y \to x$

Equivalently,

$$\lim_{y \to x} \overbrace{\frac{f(y) - f(x) - A(y-x)}{\|y-x\|}}^{F(y)} = 0$$

$$f = (f_1, f_2, \ldots, f_q)$$

Prop 6.5.2 (Uniqueness of $A : \mathbb{R}^p \to \mathbb{R}^q$)

A is unique.

Thm 5.6.7

$G \subset \mathbb{R}^p$ open, $x \in \mathbb{R}^p$, $f : G \longrightarrow \mathbb{R}^q$ differentiable at x

$$f = (f_1, f_2, \ldots, f_q)$$

Then

(a) f is continuous at x

(b) Partial derivatives

$\frac{\partial f_i}{\partial x_j}(x)$ exists at x

$i = 1, 2, \ldots, q$

$j = 1, 2, \ldots, p$

(c) The matrix representation of $Df(x)$ is

$Df(x)v = \Big[\frac{\partial f_1}{\partial x_1}(x) \quad \frac{\partial f_1}{\partial x_2}(x) \quad \cdots \quad \frac{\partial f_1}{\partial x_p}(x) \Big] \Big[v_1$

$$Df(x)v = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x), & \frac{\partial f_1}{\partial x_2}(x), & \dots, & \frac{\partial f_1}{\partial x_p}(x) \\ \vdots & & & \\ \frac{\partial f_q}{\partial x_1}(x), & \frac{\partial f_q}{\partial x_2}(x), & \dots, & \frac{\partial f_q}{\partial x_p}(x) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{bmatrix}$$

$$= \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}$$

Remark: Partial derivatives exist $\not\Rightarrow$ differentiable

Thm 6.5.9
G-open in $\mathbb{R}^p$, $f: \mathbb{R}^p \to \mathbb{R}^q$,
If

$\left[\frac{\partial f_i}{\partial x_j}\right]: G \longrightarrow \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ exists at every $x \in G$ and is continuous, then f is differentiable at every $x \in G$ and

$$\left[\frac{\partial f_i}{\partial x_j}\right] = Df(x)$$

Prop 6.5.10
If G is connected and $f: G \to \mathbb{R}^q$ is differentiable and $Df(x) = 0$ $\forall x \in G$, then f is constant.

Thm 6.5.11
We say $f: G \to \mathbb{R}^p$ is continuously differentiable if f is differentiable everywhere in G

$Df: G \longrightarrow \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ is cont.

Equivalently, f is differentiable and

$\frac{\partial f_i}{\partial x_j}: X \longrightarrow \mathbb{R}$ are continuous

<u>Thm 6.5.13</u> (Chain Rule)

$\mathbb{R}^p \qquad\qquad \mathbb{R}^q \qquad\qquad \mathbb{R}^d$

$\cup$

G

<u>Chain Rule</u>:

$X \subset \mathbb{R}^p$ open, $Y \subset \mathbb{R}^q$ open

$f: X \longrightarrow Y$ differentiable at $x_0 \in X$.

$g: Y \longrightarrow \mathbb{R}^d$ differentiable at $y_0 = f(x_0) \in Y$

Then,

$g \circ f: X \to \mathbb{R}^d$ is differentiable at x_0 and

$$\underbrace{D(g\circ f)(x_0)}_{\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)} = \underbrace{D_g(y_0)}_{\mathcal{L}(\mathbb{R}^q, \mathbb{R}^d)} \underbrace{D_f(x_0)}_{\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)}$$

6.6-Critical Points

Thursday, April 4, 2019 4:21 PM

$a \in G$, $f: G \to \mathbb{R}$ diff. at a.
Then f has a critical point at a if $\nabla f(a) = 0$.

<u>Defi</u>
Let $f: G \to \mathbb{R}$ be twice differentiable

$$D^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right] = \left[\qquad\right] = A$$

Hessian Matrix $= \mathcal{H} f(x)$

<u>Def 6.6.3</u>

- A is <u>positive</u> if all eigenvalues are positive (or non-negative)
- A is <u>negative</u> if all eigenvalues are negative (or non-positive).
- A is <u>positive definite</u> if all eigenvalues are strictly positive.
- A is <u>negative definite</u> if all eigenvalues are strictly negative.

<u>Lemma</u>

① $f: G \to \mathbb{R}$ - twice continuously diff.
then,

$$f(x+h) = f(x) + \langle \nabla f(x) | h \rangle + \frac{1}{2} \langle \mathcal{H} f(x) h | h \rangle + o(|h|^2) = \eta(h)|h|^2$$

② Let a be a critical point.

Then,

$$f(a+h)-f(a)=\frac{1}{2}\langle Ah|h\rangle+\eta(h)|h|^2$$

$$\lim_{h\to 0}\eta(h)=0$$

③ A is pos. def then

$\langle Ah|h\rangle \geq c|h|^2 \quad \forall h\in\mathbb{R}^p$ where $C>0$.

④ • Suppose now that A is pos. def at C.P. a

$$f(a+h)-f(a)=\frac{1}{2}\langle Ah,h\rangle+\eta(h)|h|^2$$

$$\geq\left[\frac{c}{2}+\eta(h)\right]|h|^2>0$$

so a is a local minimum.

• Similarly, if A is neg. def. at C.P. then a is a local maximum.

• Saddle point: $\lambda_i>0$ and $\lambda_j<0$

Remark: The eigenvalues tell us the nature of C.P.

6.7 ??

Def:

Tangent Hyperplane:

$f: G\to\mathbb{R}$, $G\subseteq\mathbb{R}^p$ differentiable (say at point $c\in G$)

Thus,

$f(x)=f(c)+\langle\nabla f(x)|x-c\rangle+o(|x-c|)$ for x sufficiently close to c.

Consider the linear function in $\mathbb{R}^p$

$u(x)=f(c)+\langle\nabla f(c)|x-c\rangle$

Let $\mathbb{F}$ denote the graph of f, $\mathbb{F}=\{(x,f(x))\in\mathbb{R}^{p+1}: x\in G\}$

Let $\mathbb{U}$ denote the graph of u, $\mathbb{U} = \{(x, u(x)) \in \mathbb{R}^{p+1} : x \in G\}$
Note that $(c, u(c)) = (c, f(c)) \in \mathbb{F} \cup \mathbb{U}$.
Then,
we define $\mathbb{U}$ to be the tangent affine hyperplane to the graph $\mathbb{F}$ of f at the point $(c, f(c))$.

<u>Def:</u>

Now consider any point $(x, u(x)) \in \mathbb{U}$ and the vector
$v = (x, u(x)) - (c, u(c)) = (x-c, \langle \nabla f(c) | x-c \rangle)$
We see that v is orthogonal to the vector
$(\nabla f(c), -1) \in \mathbb{R}^{p+1}$.
Thus, $N = (\nabla f(c), -1) \in \mathbb{R}^{p+1}$ is the <u>normal vector</u>.

6.8-Inverse Function Theorem

Tuesday, April 9, 2019 4:03 PM

Thm: Contraction Principle

X - complete metric space

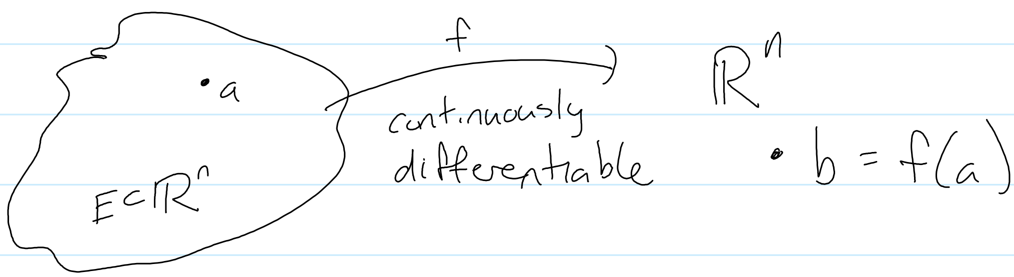
$\varphi : X \to X$ a contraction

i.e. $d[\varphi(x), \varphi(y)] \le c\, d(x,y)$, $0 \le c < 1$, $x, y \in X$

Then,

$\exists !$ fixed point of φ s.t. $\varphi(x) = x$.

The Inverse Function Theorem



$Df(a) = f'(a) \in L(\mathbb{R}^n, \mathbb{R}^n)$ invertible

Then, $\exists$ open sets $U, V \in \mathbb{R}^n$ s.t.

$a \in U \subset E$, $b \in V$

$f : U \xrightarrow{\text{bijection}} V$

The inverse map $g = f^{-1} : V \xrightarrow{\text{bijection}} U$ is continuously differentiable. $g(f(x)) = x$ $x \in U$ so

$g'(y) = [f'(x)]^{-1}$ $y = f(x)$

Define $V = f(U)$

Brouwer's Theorem on the Invariance of Domain:

V is open and f is a homeomorphism b/w U and V.

In particular, $f^{-1} : V \xrightarrow{\text{onto}} V$ is continuous.

Not used

Cor:

f is an open map. (Takes open sets to open sets.)

6.9-Implicit Function Theorem

Thursday, April 11, 2019 4:37 PM

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$

$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m} \cong \mathbb{R}^n \times \mathbb{R}^m$

$A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$, $A(h,k) = A(h,0) + A(0,k)$

$A(h,k) = A_x h + A_y k$

$A_x \in L(\mathbb{R}^n, \mathbb{R}^m)$ $\quad$ $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$

$A_x h = A(h, 0)$ $\quad$ $A_y k = A(0, k)$

Consider an $f \in C'(E, \mathbb{R}^n)$, $\underset{\text{open}}{E} \subset \mathbb{R}^{n+m}$

Thm 6.92

Let $f \in C'(E, \mathbb{R})$, $f(a,b) = 0$,

$A = f'(a,b)$, A_x is invertible

Then

- $\exists\, U \in \mathbb{R}^m$ open containing (a,b)

Problem 7

Ex: $f(x,y) = e^{xy} + \sin y + y^2 - 1$

Find h - defined near 2, $h = h(x)$ s.t. $f(x, h(x)) = 0$.

Sol:

$$e^{xh(x)} + \sin(h(x)) + h(x)^2 - 1 = 0$$

For $x=2$, we have solution when $y=0$.

So we want $h(2)=0$.

Problem 6

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases}$$

near the point

$$(x,y,u,v) = (2,-1,2,1)$$

Solve the equations for u and v in terms x and y. (Prove existence and uniqueness.)

Sol:

$$\begin{cases} u^3 - v^2 = x^2 - y^2 + 4 \\ 2u^2 + 3v^4 = 2xy + y^2 + 8 \end{cases}$$

Jacobian matrix of LHS (at (2 1))

$$J = \begin{bmatrix} 3u^2 & -2v \\ 4u & 12v^3 \end{bmatrix} = \begin{bmatrix} 12 & -2 \\ 8 & 12 \end{bmatrix}$$

$\det J \neq 0$ so unique x, y solutions

Find $\frac{\partial ①}{\partial x}$ $\quad 3u^2 u_x - 2v v_x = 2x \quad (*)$

$\frac{\partial ②}{\partial y}$ $\quad 3u^2 u_y - 2v v_y = -2y \quad (**)$

$\frac{\partial ①}{\partial x}$ $\quad 4u u_x + 12v^3 v_x = 2y \quad (*)$

$\frac{\partial ②}{\partial y}$ $\quad 4u u_y + 12v^3 v_y = 2x + 2y \quad (**)$

Solve for u_x, v_x by $(*)$'s at $(2,-1,2,1)$

Solve for u_y, v_y by $(**)$'s at $(2,-1,2,1)$

Ex:

$f(x,y) = 0 \quad x \in \mathbb{R}, \; y \in \mathbb{R}$

$f(x_0, y_0) = 0$

Find x

$x = g(y) \rightarrow f(g(y), y) = 0$

$\Rightarrow x_0 = g(y_0)$

$\frac{d}{dy} \Rightarrow f(g(y), y) = 0$

$$f_x(g(y),y)\cdot g'(y) + f_y(g(y),y) = 0$$

$$g'(y) = -(f_x)^{-1} f_y$$

Ex:

$$f_1 = 2e^{x_1} + x_2 y_1 - 4y_2 + 3 = 0$$

$$f_2 = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3 = 0$$

Solve for x_1, x_2

Sol:

$$x_1 = g_1(y_1, y_2, y_3) \qquad (x_1^0, x_2^0) = (0,1)$$

$$x_2 = g_2(y_1, y_2, y_3 \qquad (y_1^0, y_2^0, y_3^0) = (3,2,7)$$

$$f_x = \begin{bmatrix} 2e^{x_1} & y_1 \\ -x_2 \sin x_1 - 6 & \cos x_1 \end{bmatrix}$$

$$\det f_x = 2e^{x_1}\cos x_1 + y_1 x_2 \sin x_1 + 6y_1$$

at $(x_0, y_0) = 2 + 0 + 6\cdot 3 = 20 \neq 0$

So we have solution.

$$D_y = -(f_x)^{-1} f_y = \begin{bmatrix} * & * \\ * & * \end{bmatrix}\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} = \overbrace{\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}}^{g'(3,2,7)}$$