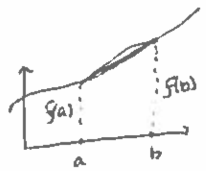
Numerical Analysis

08/29/2017

Discretizations & error:

$$I(f) = \int_a^b f(x)\,dx \cong \underbrace{(b-a)\, f\left(\frac{a+b}{2}\right)}_{\text{Quadrature Midpoint rule}} = Q(f)$$



$$\cong \underbrace{(b-a)\left(\frac{f(a)+f(b)}{2}\right)}_{\text{Trapezoid rule}} = \tilde{Q}(f)$$

$$I(f) - \tilde{Q}(f) = \underbrace{\frac{-(b-a)^3}{12}}_{\text{Error}} f''(a)$$

Notice in each case turn a continuous measurement into a final discrete value.

$$f'(x_0) \cong \frac{f(x_0+h) - f(x_0)}{h}$$

$$\cong \frac{f(x_0) - f(x_0-h)}{h}$$

$$\cong \frac{f(x_0+h/2) - f(x_0-h/2)}{h} = \frac{\delta_h f(x_0)}{h}$$

Notice all these discretizations approximate the same thing but they have different 'costs' in terms of computation & error.

$$\left| f'(x_0) - \frac{\delta_h f(x_0)}{h} \right| \leq \frac{|f'''(\xi)| + |f'''(q)|}{2} \frac{h^2}{24}$$

where $\xi \in [x_0, x_0 + \frac{h}{2}]$, $q \in [x_0 - h/2, x_0]$

Note: $\mathbb{R}$ & $\mathbb{C}$ are not 'computer available'. Only floating point representations to these numbers.

IEEE Standard 754-1985 Binary Floating Point Arithmetic

$$x = \pm \underset{\substack{\text{significance}\\ \text{(mantissa)}}}{s} \cdot \underset{\text{base, radix}}{2}^{E \leftarrow \text{exponent}}$$

$1 \leq s < 2$

$s = 1 + f$ → $0 \leq f < 1$; $f = \sum_{i=1}^{t} f_i 2^{-i}$

$f_i \in \{0,1\}$

binary rep.

$f = (0.f_1 \cdots f_t)_2$

$E = e - b$ ← bias

$e = \sum_{i=0}^{\ell-1} e_i 2^i$

$e_i \in \{0,1\}$

$$x = (1-2p)(1.f_1 \dots f_t)2^E$$

p	$e_{\ell-1}$	$\cdots$	e_0	f_1	$\cdots$	f_{52}
sign	exponent			sign.		

11 e's so $e_0, \dots, e_{10}$

t bits in mantissa	52
ℓ bits in exponent	11
E_{max}	1023
E_{min}	-1022
b bias	1023

$$\underbrace{2}_{\text{sign}} \cdot \underbrace{2^t}_{\text{mant./sig.}} \underbrace{(E_{max} - E_{min} + 1)}_{\text{exp}} + \underbrace{1}_{0} = \#\text{'s in } \mathbb{F}$$

ie floating point numbers

Inc. exp gives larger range. Inc. mant gives finer 'grading' of #'s.

Notice all these numbers are rational.

$$x_{min} = 2^{E_{min}}$$

$$x_{max} = (2 - 2^{-t})\, 2^{E_{max}}$$

$$\Delta_E = 2^{E-t}$$

So the numbers not 'evenly' distributed

Rounding

$fl : \mathbb{R} \to \mathbb{F}$

rounding operation

For $x > 0$,

$$fl(x) = \begin{cases} x_-, & \text{if } x \in [x_-, \mu] \text{ or } x = \mu, a_t = 0 \\ x_+, & \text{if } x \in (\mu, x_+] \text{ or if } x = \mu \text{ \& } b_t = 0 \end{cases}$$

$x_\pm$ left/right neighbor (nearest), ie

$x_+ = \min\{ y \in \mathbb{F} : y \geq x \}$

$x_+ = ((1.f_1 \dots f_t)_2 + (0.0 \dots 1)_2) 2^E$

$\mu = \frac{1}{2}(x_+ + x_-)$

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Floating point arithmetic

$\mathbb{F}(\underbrace{2}_{\text{base}}, \underbrace{t}_{\text{mant.}}, \underbrace{E_{min}, E_{max}}_{\text{exp.}})$

$x \in \mathbb{F}$ then $x = \pm s \cdot 2^E$

$s = 1 + f \; ; \; f = (0.f_1 \dots f_t)_2$

$s = (1.f_1 \dots f_t)_2$

p	e_{10}	...	e_0	f_1	......	f_{52}

$x = \pm s \cdot 2^E = (1 - 2p)(1.f_1 \dots f_t)_2 \, 2^E$

$e = E + b$

$t = 52$

$\ell = 11$

$E_{max} = 1023$

$E_{min} = -1022$

$b = 1023$

Largest $e_{max} = 2^\ell - 1$

Smallest $e_{min} = 0$

We reserve value (max)

$E = e_{max} - b$ and min value

$E = e_{min} - b = -b$

Exponent	frac.	fl #
$E = E_{min} - 1$	$f = 0$	± 0
$E = E_{min}$ ~~with~~	$f \neq 0$	$\pm (0.f)_2 2^{E_{min}}$
$E_{min} < E < E_{max}$	f	$\pm (1.f)_2 2^E$
$E = E_{max} + 1$	$f = 0$	$\pm \infty$
$E = E_{max} + 1$	$f \neq 0$	NaN (not a number)

Subnormals

$x \in [2^E, 2^{E+1}) \cap \mathbb{F}$

$x = s \, 2^E$

Increment between consecutive fl-#'s in $[2^E, 2^{E+1})$ is $\Delta_E = 2^{E-t}$

The increment doubles from $[2^E, 2^{E+1})$ to $[2^{E+1}, 2^{E+2})$

There are 2^t fl-numbers in each $[2^E, 2^{E+1})$

subnormals (not of form $(1-2p)(1.f_1 \dots f_t)_2 2^E$)

$x = \pm f \, 2^{E_{min}}$

The gap between subnormals is

$\Delta = 2^{E_{min} - t}$

Rounding

$fl: \mathbb{R} \to \mathbb{F}$

$fl(x)$

Certainly want $fl(fl(x)) = fl(x)$

and also $x < y \to fl(x) < fl(y)$

and $fl(x) = x$ for $x \in \mathbb{F}$

and $fl(-x) = -fl(x)$

Suppose $x \in [2^E, 2^{E+1})$

$x = (1.\underbrace{f_1 \dots f_{\cdot}}_{\text{possibly infinite}})_2 \boxed{2^E}$. So $x \in \mathbb{R}$

$x_- = \max\{y \in \mathbb{F} : y \le x\} = (1.a_1 \dots a_t)2^E$

$x_+ = \min\{y \in \mathbb{F} : x \le y\} = (1.b_1 \dots b_t)_2 2^E$

If $x \notin \mathbb{F}$ then one of $f_{t+1} \dots$ is not zero.

$x_+ = ((1.f_1 \dots f_t)_2 + (0.0 \dots 1)_2)2^E$

$x_- = x_+ - 2^{E-t}$

$\mu = \frac{1}{2}(x_+ + x_-)$

$$fl(x) = \begin{cases} x_- & \text{if } x \in [x, \mu) \text{ or } x = \mu \text{ and } a_t = 0 \\ x_+ & \text{if } x \in (\mu, x_+] \text{ or } x = \mu \text{ and } b_t = 0 \end{cases}$$

Relative Error: $x \in [2^E, 2^{E+1})$; $\Delta_E = 2^{E-t}$ } gap

$$\frac{|fl(x) - x|}{|x|} \le \frac{\frac{1}{2} 2^{E-t}}{2^E} \;\Big\}\; \tfrac{1}{2}\Delta_E$$

$$= \frac{1}{2}(2^{-t}) \text{ epsilon, eps}$$

But $(-)_2$ rep. do not have to be 'nice', even for 'nice' numbers:

$\frac{1}{10} = 0.10 = (0.0\ 0011\ 0011 \dots)_2$

$1, 2^{-53} \in \mathbb{F}$ but $1 + 2^{-53} \notin \mathbb{F}$

So floating point operations are very complicated.

$x, y \in \mathbb{F}$; $\odot \in \{+, -, \cdot, \div\}$

denote $\boxdot$ ~~to~~ be $\odot$ but in $\mathbb{F}$.

$x \boxdot y = fl(x \odot y)$

So the operations in $\mathbb{F}$ – though ugly – work as well as one would hope.

$fl(x \odot y) = (x \odot y)(1 + \delta)$

$|\delta| \le eps$

Add + mult. in fl is not associative

Distributivity does not hold.

Mult. and div. are not inverse operations.

Even worse, let h be 'very small'

$b \boxplus h = 1$ "absorption"

Then $f'(b) := \dfrac{f(b \boxplus h) - f(b)}{h} = 0$

Lem: (Absorption) $x, y \in \mathbb{F}$, $0 < y < x$

If $y < \frac{1}{4} 2^{-t} x$ then $fl(x+y) = x$

Pf: $x = s 2^E$; $1 \le s < 2$

the next largest fl number larger than x is $x + \Delta_E = s2^E + 2^{E-t}$

$\mu = x + \frac{1}{2} 2^{E-t}$

$x + y < x + \frac{1}{4} 2^{-t} < \mu$ so

$fl(x+y) = x.$

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Computational consequences of using fl arithmetic.

Ex: Consider two mathematically equivalent algorithms.

$A_1(a,b) = a^2 - b^2$

$A_2(a,b) = (a-b)(a+b)$

$$fl\,(a^2 - b^2) = \left(a^2(1+\epsilon_1) - b^2(1+\epsilon_2)\right)(1+\epsilon_3)$$

$$= (a^2-b^2)\left(\frac{a^2(1+\epsilon_1) - b^2(1+\epsilon_2)}{a^2-b^2}\right)(1+\epsilon_3)$$

$$= (a^2-b^2)\left(1 + \frac{a^2\epsilon_1 - b^2\epsilon_2}{a^2-b^2}\right)(1+\epsilon_3)$$

$|\epsilon_i| \leq$ eps. Say $a \approx b$ and sign $\epsilon_1 = -$sign ϵ_2

Then the result is 'bad'.

$$fl\left((a-b)(a+b)\right) = \left((a-b)(1+\epsilon_1)(a+b)(1+\epsilon_2)\right)(1+\epsilon_3)$$

$$= (a^2-b^2)\,\underbrace{(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)}_{1+\delta}$$

So $1+\delta = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3$

$$|\delta| = 3\,\text{eps} + \mathcal{O}(\text{eps}^2)$$

So the second choice is clearly better. So in an algorithm one may see A_2 but not A_1 despite being the same algebraically.

Ex: Preventing overflow.

$$x = \sqrt{a^2 + b^2} \quad ; \quad a = 10^{70}, b = 1$$

$$x = s\sqrt{(a/s)^2 + (b/s)^2}$$

with $s = \max(|a|, |b|)$

$$x = 10^{70}\sqrt{1^2 + (1/10^{70})^2}$$

This version will result in a number whereas the first will not.

Ex:

$$1 - \sqrt{1-x} = \frac{x}{1+\sqrt{1-x}}$$

Ex: $1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$

Polynomial Computations

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

It takes $1, 2, \ldots, n-1$ mult. to compute $x^2, x^3, \ldots, x^n$. Then $n-1$ mult. for $a_i x^i$, then $n+1$ tot additions at least computing traditionally.

Horner's Scheme:

$$p(x) = \left(\cdots\left(\left(a_n x - a_{n-1}\right)x + a_{n-2}\right)x + \cdots\right) + a_0$$

This involves far less multiplication

$b_n = a_n$

$b_{n-1} = a_n x + a_{n-1}$

$\vdots$

$b_i = b_{i+1} x + a_i$

$\vdots$

$b_0 = b_1 x + a_0$

The b_i are poly. in x. n mult. & n additions

$$p(y) = a_n y^n + a_{n-1} y^{n-1} + \dots + a_1 y = (b_n y^{n-1} + \dots + b_2 y + b_1)(y - x) + b_0$$

Comparing coefficients

$$a_n = b_n$$
$$a_{n-1} = -b_n x + b_{n-1}$$
$$\vdots$$
$$a_1 = -b_2 x + b_1$$
$$a_0 = -b_1 x + b_0$$

This gives division.

Notice also...

$$b'_{n-1}(x) = a_n \qquad (b_n' = 0)$$
$$\vdots$$
$$b_i'(x) = \frac{d}{dx}\left(\underbrace{b_{i+1}}_{x \text{ dep.}} x + a_i\right)$$
$$= b'_{i+1}(x) \cdot x + b_{i+1} \quad ; i = n-1, \dots, 0$$
$$b_0'(x) = p'(x)$$

So this gives derivatives. Back to division: div. by $x - \alpha$

$$p(x) = Q_\alpha(x)(x - \alpha) + r$$

	a_n	a_{n-1}	....	a_1	a_0
α	b_n	b_{n-1}	....		

Number in second line is sum of the number above it and number on left mult. by α.

Ex: $p(x) = x^3 - 4x^2 + 3x + 2$

$p(3) = ?$

	1	-4	3	2
3		3	-3	0
	1	-1	0	2

algorithmic rep. of process at top left of this page

So $p(3) = 2$ and $p(x) = (x^2 - x)(x - 3) + 2$

See p. 112

Now say we want to rep. $p(x)$ as...

$$p(x) = \sum_{i=1}^{n} a_i(\alpha)(x - \alpha)^i$$

Given $p(x)$, find $a_i(\alpha)$.

$$p(x) = Q_{n-1}(x)(x - \alpha) + p(\alpha)$$

Denote $Q_n(\alpha) = p(\alpha)$.

$$Q_{n-1}(x) = Q_{n-2}(x)(x - \alpha) + Q_{n-1}(\alpha)$$

Then...

$$Q_n(x) = \underset{Q_0(x)}{\underset{\parallel}{Q_0(\alpha)}}(x-\alpha)^n + Q_1(\alpha)(x-\alpha)^{n-1} + \dots + Q_{n-1}(\alpha)(x-\alpha) + Q_n(\alpha)$$

Hence, $a_i(\alpha) = Q_{n-i}(\alpha)$.

$$p(x) = x^3 - 4x^2 + 3x + 2$$

	1	-4	3	2
3		3	-3	0
	1	-1	0	$\boxed{2}$
3		3	-6	
	1	-2	$\boxed{-6}$	
3		3		
	$\boxed{1}$	$\boxed{1}$		

Lagrange Interpolation

Thm: Given distinct $x_0, \dots, x_n$ $(n \geq 0)$ and arbitrary numbers $f_0, \dots, f_n$, there exists a unique polynomial L of degree $\leq n$ such that

$$L(x_i) = f_i \; ; \; i = 0, \dots, n$$

if $f_i = f(x_i)$.

Cardinal Lag. intp. poly. l_i

$$l_i(x_j) = \delta_{i,j} \text{ , the Kronecker function.}$$

$$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Then let $L(x) = \sum_{i=0}^{n} f_i \, l_i(x)$

Now $L(x_K) = f_K \underbrace{l_K(x_K)}_{=1} = f_K$

This is existence. For uniqueness, suppose $\tilde{L}(x)$ is another such polynomial.

Then $L(x) - \tilde{L}(x)$ is a poly. with at least $n+1$ roots, namely the x_i so it is identically 0 and uniqueness is satisfied.

x_i = nodes

Notice the polynomial 'depends' on the nodes, in that for numerical accuracy, if the nodes are all very close, this could be an issue!

Suppose

$$L(x) = a_0 + a_1 x + \dots + a_n x^n$$

$$L(x_i) = f_i$$

Find $a_0, \dots, a_n$. Well

$$a_0 + a_1 x_i + \dots + a_n x_i^n = f_i.$$

So we can write:

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}}_{\text{Vandermonde matrix, } V} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

This matrix is often ill conditioned. So finding the a_i with this are often inaccurate.

So we may want another way of determining the polynomial.

Newton basis:

$\pi_0(x) = 1$ deg 0

$\pi_1(x) = x - x_0$ deg 1

$\pi_2(x) = (x-x_0)(x-x_1)$ deg 2

$\vdots$

$\pi_n(x) = (x-x_0)\cdots(x-x_{n-1})$ deg n

We want to express L in Newton's basis.

$L_{0,1,\dots,K-1} \in \mathbb{P}_{K-1}$ } poly at most deg $K-1$

$L_{0,\dots,K} \in \mathbb{P}_K$

$L_{0,\dots,K-1}(x_i) = f_i \; ; \; i = 0, \dots, K-1$

$L_{0,\dots,K}(x_i) = f_i \; ; \; i = 0, \dots, K-1, K$

$L_{0,\dots,k} - L_{0,\dots,k-1}$ vanishes at $x_0, \dots, x_{k-1}$. So in particular, $x - x_i$ is a factor of the diff for $i = 0, \dots, k-1$. But then it is a multiple of $\pi_k(x)$.

$$L_{0,\dots,k} - L_{0,\dots,k-1} = b_k \pi_k(x)$$

Clearly, b_k = leading coefficient of $b_k \pi_k(x)$ = leading coefficient of $L_{0,\dots,k} - L_{0,\dots,k-1}$ = leading coefficient of $L_{0,\dots,k}$.

$$L(x) = \sum_0^k f_i \ell_k(x) = \sum_0^k f_i \frac{\prod_0^k (x - x_j)}{\prod_{\substack{0 \\ j \neq i}}^k (x_i - x_j)}$$

So...

$$b_k = \sum_0^k \frac{f_i}{\prod_{\substack{0 \\ j \neq i}}^{k} (x_i - x_j)}$$

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Newton's Form

$\pi_0(x) = 1$

$\pi_1(x) = x - x_0$

$\vdots$

$\pi_n(x) = (x - x_0) \cdots (x - x_{n-1})$

We would like to express $L_{0,\dots,n}(x)$ in terms of π_n.

$L_{0,\dots,k-1}(x_i) = f_i \; ; \; i = 0, \dots, k-1$

$L_{0,\dots,k}(x_i) = f_i \; ; \; i = 0, \dots, k-1, k$

$(L_{0,\dots,k} - L_{0,\dots,k-1})(x_i) = 0 \; ; \; i = 0, \dots, k-1$

$\therefore (L_{0,\dots,k} - L_{0,\dots,k-1})(x) = b_k \pi_k(x)$

$$L_{0,\dots,k}(x) = \sum_0^k f_i \prod_{\substack{j=0 \\ j \neq i}}^{k} \frac{x - x_j}{x_i - x_j}$$

$$b_k = \sum_0^k \frac{f_i}{\prod_{\substack{j \neq i \\ j > 0}}^{k} (x_i - x_j)} := f[x_0, \dots, x_k]$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$L_0(x) = f(x_0)$

$$L_{0\dots k}(x) = L_{0\dots k-1}(x) + f[x_0, \dots, x_k]\, \pi_k(x)$$

$$L_{0,\dots,n}(x) = f(x_0) + f[x_0, x_1](x - x_0) + \dots + f[x_0, \dots, x_n](x - x_0) \cdots (x - x_n)$$

$$L_{0,\dots,n}(x_i) = \sum b_k \pi_k(x_i) = f_i$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \pi_1(x_1) & & 0 \\ \vdots & & \ddots & \vdots \\ 1 & \pi_n(x_1) & \cdots & \pi_n(x_n) \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Properties of divided differences

a) Linearity: $f(x) = \alpha g(x) + \beta h(x)$

$f[x_0, \dots, x_k] = \alpha g[x_0, \dots, x_k] + \beta h[x_0, \dots, x_k]$

b) Commutativity

$f[x_0, \dots, x_k] = f[x_{\sigma(1)}, \dots, x_{\sigma(n)}]$

c) Recurrence $f[x_0, \dots, x_k] =$

$$\frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

Define $q \in \mathbb{P}_k$

$$q(x) = \frac{(x-x_0)L_{1,\dots,k}(x) - (x-x_k)L_{0,\dots,k-1}(x)}{x_k - x_0}$$

q interpolates at $x_0,\dots,x_k$ then

$q \equiv L_{0,\dots,k}$

Compare leading coefficients.

$$\begin{array}{ccccc}
x_0 & f_0 & & & \\
x_1 & f_1 & f[x_0,x_1] & & \\
x_2 & f_2 & f[x_1,x_2] & f[x_0,x_1,x_2] & \\
\vdots & \vdots & \vdots & \vdots \ \ddots & \\
x_n & f_n & f[x_{n-1},x_n] & \cdots & f[x_0,\dots,x_n]
\end{array}$$

The following identity holds:

$$f(x) = \sum_{0}^{n} f[x_0,\dots,x_k]\,\pi_k(x) + f[x_0,\dots,x_n,x]\,\pi_{n+1}(x)$$

Interpolation Remainder:

$f \in C^{n+1}([a,b])$; $\{x_0,\dots,x_n\} \subset [a,b]$

$L_n = L_{0,\dots,n}$

$$f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

for $\xi \in (a,b)$

Let $q \in \mathbb{P}_{n+1}$ which interpolates f at $x_0,\dots,x_n,x$

$$q(t) = L(t) + \frac{f(x)-L_n(x)}{\pi_{n+1}(x)}\pi_{n+1}(t)$$

$E(t) = f(t) - q(t)$

$E(y) = 0$ for $y \in \{x_0,\dots,x_n,x\}$

$$0 = E^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{f(x)-L_n(x)}{\pi_{n+1}(x)}(n+1)!$$

Remainder Est.

$$x_m = \frac{1}{2}\left((b-a)\cos\left(\frac{(2m+1)\pi}{2(n+1)}\right) + (b+a)\right)$$

$m = 0,\dots,n$

If $(a,b) = (-1,1)$ such nodes of intp. in $(-1,1)$ would be roots of

$$T_{n+1}(x) = \cos((n+1)\arccos x)$$

$$x_m = \cos\left(\frac{(2m+1)\pi}{2(n+1)}\right) \ ; \ m = 0,1,\dots,n$$

$$\pi_{n+1}(x) = (x-x_0)\cdots(x-x_n) = 2^{-n}T_{n+1}(x)$$

$$\max_{-1<x<1}|(x-x_1)\cdots(x-x_n)| = 2^{-n}$$

$M_{n+1} = \sup_{0<x<1}|f^{(n+1)}(x)|$ / $|E(x)| \le \dfrac{M_{n+1}}{(n+1)!\,2^n}$

Hermite Interpolation

$x_0 \neq x_1$. Want $h_{0,1} \in \Pi_3$ with...

$h_{0,1}(x_0) = 0 \quad ; \quad h'_{0,1}(x_0) = 1$

$h_{0,1}(x_1) = 0 \quad ; \quad h'_{0,1}(x_1) = 0$

$p(x) = (x-x_0)(x-x_1)^2$

$p'(x) = (x-x_1)^2 + \cancel{2(x-x_0)(x-x_1)}$

$p'(x_0) = (x_0 - x_1)^2$

$\therefore \dfrac{p(x)}{p'(x_0)} = h_{0,1}(x)$

$= (x-x_0)\left(\dfrac{x-x_1}{x_0-x_1}\right)^2$

Find $h_{0,0}(x)$

$h_{0,0}(x_0) = 1 \quad ; \quad h'_{0,0}(x_0) = 0$

$h_{0,0}(x_1) = 0 \quad ; \quad h'_{0,0}(x_1) = 0$

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$$L_{0,0}(x) = \left(\frac{x-x_1}{x_0-x_1}\right)^2$$

$$h_{0,0}(x) = L_{0,0}(x) - L'_{0,0}(x_0)h_{0,1}(x)$$

$$= \frac{(x-x_1)(2x+x_1-3x_0)}{(x_1-x_0)^3}$$

We can construct $h_{1,1}$ & $h_{1,0}$

$$h_{1,0}(x) = -\frac{(x-x_0)^2(2x+x_0-3x_1)}{(x_1-x_0)^3}$$

$$h_{1,1}(x) = (x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$

We may solve the 2pt Hermite interp problem. $H \in \Pi_3$

$H^0(x_0) = f_0 \quad ; \quad H'(f_0) = f'(x_0)$

$H^0(x_1) = f_1 \quad ; \quad H'(f_1) = f'(x_1)$

$H(x) = f(x_0)h_{0,0}(x) + f'(x_0)h_{0,1}(x) + f(x_1) + f(x_1)h_{1,0}(x) + f'(x_1)h_{1,1}(x)$

Newton's form:

$H(x) = f_0^{(x_0)} + f[x_0,2](x-x_0) +$

$f[x_0,2,x_1](x-x_0)^2 +$

$f[x_0,2,x_1,2](x-x_0)^2(x-x_1)$

x_0	$f(x_0)$				
x_0	$f(x_0)$	$f'(x_0)$			
x_1	$f(x_1)$	$f[x_0,x_1]$	$f[x_0,2;x_1]$		
x_1	$f(x_1)$	$f'(x_1)$	$f[x_0;x_1 2]$	$f[x_0,2,x_1,2]$	

Thm: Let $x_0,\dots,x_K$ be distinct & $m_0,\dots,m_K$ be integers ≥ 1 such that $\sum_{i=0}^{K} m_i = m+1$. Let f be $\ni f^{(m_i-1)}(x_i)$ exists for $0 \leq i \leq K$ exists, then $\exists!$ poly $h \in \Pi_m$ with $h^{(\ell)}(x_i) = f^{(\ell)}(x_i)$

$i = 0,\dots,K \; ; \; 0 \leq \ell \leq m_i - 1$

Our system has a unique solution if and only if a homog. system has a unique solution.

$$h^{(\ell)}(x_i) = 0 \; ; \quad \begin{array}{l} i=0,\dots,K \\ \ell = 0,\dots,m_i-1 \end{array}$$

$$h(x) = h(x_i) + h'(x_i)(x-x_i) + \cdots + \frac{1}{(m_i-1)!} h^{(m_i-1)}(x_i)(x-x_i)^{m_i-1} + \frac{1}{m_i!} h^{(m_i)}(x_i + \theta(x-x_i))(x-x_i)^{m_i}$$

$$h(x) = (x-x_i)^{m_i} g(x)$$

$$h(x) = \prod_{i=0}^{K} (x-x_i)^{m_i} g(x)$$

$$h(x) \equiv 0 \quad \} \; n+1 \text{ - zeros}$$

from counting zeros. [so $g(x) = 0$ Then $h \equiv 0$, as 'desired'].

Thm: $h(x) = \sum_{i=0}^{K} \sum_{\ell=0}^{m_i-1} f^{(\ell)}(x_i) h_{i,\ell}(x)$

where $h_{i,\ell} \in \mathbb{P}_m$ satisfy...

$$h^{(m)}_{i,\ell}(x_j) = \begin{cases} 1, & i=j \text{ and } m=\ell \\ 0, & i\neq j \text{ or } m \neq \ell \end{cases}$$

Define poly:

$$L_{i,j}(x) = \frac{(x-x_i)^\ell}{\ell!} \prod_{\substack{j=0 \\ j\neq i}}^{K} \left(\frac{x-x_j}{x_i - x_j} \right)^{m_j}$$

$$i=0,\dots,K \; ; \; \ell = 0,\dots,m_i-1$$

then $h_{i,m_i-1}(x) = L_{i,m_i-1} \; ; \; i=0,\dots,K$

$$h_{i,m}(x) = L_{i,m}(x) - \sum_{v=m+1}^{m_i-1} L^{(v)}_{i,m}(x_i) h_{i,v}(x)$$

$$m = m_i-2, \dots, 0$$

Consider $L_{i,\ell}(x)$

$$L^{(0)}_{i,\ell}(x_i) = \cdots = L^{(\ell-1)}_{i,\ell}(x_i) = 0$$

$$L^{(\ell)}_{i,\ell}(x_i) = 1$$

and at $x = x_j$

$$L^{(0)}_{i,\ell}(x_j) = \cdots = L^{(m_j-1)}_{i,\ell}(x_j) = 0$$

$$j \neq i$$

Formal definition of divided diff with multiple nodes:

a) $f[x_0, i] = \dfrac{f^{(i-1)}(x_0)}{(i-1)!} \; ; \; i \geq 1$

b) $f[x_0, m_0; \dots; x_K, m_K] =$

$$\frac{f[x_0, m_0-1; \dots; x_K, m_K] - f[x_0, m_0; \dots; x_K, m_K-1]}{x_K - x_0}$$

For any i, $0 \leq i \leq K$, define...

$$s(i) = \begin{cases} 0, & i=0 \\ m_0 + \cdots + m_{i-1} \; ; & 0 < i \leq K \end{cases}$$

Now every integer p; $0 \le p \le m$ can be represented by

$$p = s(i) + j \ ; \quad \begin{matrix} 0 \le i \le K \\ 0 \le j \le m_i - 1 \end{matrix}$$

$$\pi_0(x) = 1$$

$$\pi_1(x) = \pi_{s(0)+1}(x) = (x - x_0)$$

$$\vdots$$

$$\pi_{m_0-1}(x) = \pi_{s(0)+m_0-1}(x) = (x - x_0)^{m_0 - 1}$$

$$\pi_{m_0}(x) = \pi_{s(1)+0}(x) = (x - x_0)^{m_0}$$

$$\pi_{s(i)+j}(x) = (x - x_0)^{m_0} \cdots (x - x_{i-1})^{m_{i-1}} (x - x_i)^j$$

$$0 \le j \le m_i - 1$$

$$h_m(x) = \sum_{p=0}^{m} b_p \pi_p(x) = \sum_{i=0}^{K} \sum_{j=0}^{m_i - 1} b_{s(i)+j} \, \pi_{s(i)+j}(x)$$

$$b_{s(i)+j} = f[x_0, m_0; \ldots; x_{i-1}, m_{i-1}; x_i, j+1]$$

09/14/2017

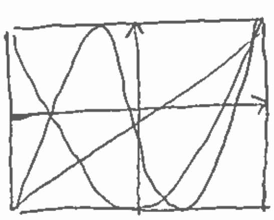
Chebyshev Polynomials

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$\vdots$$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$



This has many expressions

$$* \ T_n(x) = \cos(n \cos^{-1} x) \ ; \quad n \ge 0, \quad -1 < x < 1$$

OR

$$* \ T_n(x) = \frac{1}{2}(z^n + z^{-n});$$

$$|z| = 1$$

$$x = \operatorname{Re} z$$

$$T_n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)\right)^n + \left(x - \sqrt{x^2-1}\right)^n ; \ x \in \mathbb{R}$$

$$T_n(x) = \begin{cases} \cosh(n \operatorname{arccosh} x), & x \ge 1 \\ (-1)^n \cosh(n \operatorname{arccosh}(-x)), & x \le -1 \end{cases}$$

For the first, use

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Choose $n+1$ & $n-1$. Add & rearrange then let $\theta = \cos^{-1} x$ & $x = \cos\theta$

Show it satisfies same recurrence relations.

zeros

$$T_n(t_{n,k}) = 0$$

$$t_{n,k} = \cos\left(\frac{2k-1}{2n}\pi\right) \ ; \ k = 1, \ldots, n$$

$$T_n(s_{n,k}) = (-1)^k$$

$$s_{n,k} = \cos\left(\frac{k\pi}{n}\right) \ ; \ k = 0, \ldots, n$$

Chebyshev points

Complex Analytic Def:

$$T_n(x) = \cos(n\theta)$$

$$\theta = \arccos x$$

$$z = e^{i\theta}$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

$$\cos(n\theta) = Re(z^n) = \frac{z^n + z^{-n}}{2} = \frac{z^n + \overline{z^n}}{2}$$

Consider points $\{z_j\}$, $(n+1)$ total

$$z_j = e^{\frac{j\pi}{n} i} \; ; \; j = 0, \dots, n$$

The proj. to x-axis is no longer uniform

$$Re\, z_k = \cos\left(\frac{k\pi}{n}\right) \; ; \; k = 0, \dots, n$$

Chebyshev points

Minimal Property of T_n

Thm: Let $p \in \Pi_n$ be monic

then $\|p\|_\infty = \max_{-1 \le x \le 1} |p(x)| \ge 2^{1-n}$

&

$$2^{1-n} = \max_{-1 \le x \le 1} |\underbrace{2^{1-n} T_n(x)}_{\text{monic poly}}| \le \max_{-1 \le x \le 1} |p(x)|$$

Orthogonality of T_n

$T_n(x)$; $-1 < x < 1$

Consider $L^2_w([-1,1])$

$$w(x) = \frac{1}{\sqrt{1-x^2}} > 0$$

$$\|f\| = \int_{-1}^{1} |f(x)|^2 w(x)\, dx$$

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) w(x)\, dx$$

$$\langle T_n, T_m \rangle = \begin{cases} 0, & n \ne m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \ne 0 \end{cases}$$

09/19/2017

Neville's algorithm of poly. interpolation

$L_{0,1,\dots,n}(x)$

$L_{0,1,\dots,n}(x_i) = f_i$; $i = 0, 1, \dots, n$

$L_{m,\dots,m+k}(x_{m+j}) = f_{m+j}$; $j = 0, \dots, k$

$$L_{0,1}(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

Observe:

$$\frac{x - x_1}{x_0 - x_1} + \frac{x - x_0}{x_1 - x_0} = 1$$

$$L_{1,2}(x) = \frac{x - x_2}{x_1 - x_2} f_1 + \frac{x - x_1}{x_2 - x_1} f_2$$

$$L_{0,1,2} = \frac{x - x_2}{x_0 - x_2} L_{0,1}(x) + \frac{x - x_0}{x_2 - x_0} L_{1,2}(x)$$

$$\frac{x_2 - x + x - x_0}{x_2 - x_0} = 1$$

$$L_{0,1,2}(x) = \frac{(x - x_0) L_{1,2}(x) - (x - x_2) L_{0,1}(x)}{x_2 - x_0}$$

$$L_{m,\dots,m+k}(x) = \frac{x - x_{m+k}}{x_m - x_{m+k}} L_{m,\dots,m+k-1}(x) + \frac{x - x_m}{x_{m+k} - x_k} L_{m+1,\dots,m+k}(x)$$

for $m \geq 0$ and $n \geq m+k$

$x_0 :\ f_0 = L_0(x)$
$x_1 :\ f_1 = L_1(x) - L_{0,1}(x)$
$x_2 :\ f_2 = L_2(x) - L_{1,2}(x) - L_{0,1,2}(x)$
$x_3 :\ f_3 = L_3(x) - L_{2,3}(x) - L_{1,2,3}(x) - L_{0,1,2,3}(x)$

Bernstein Polynomials

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) g_{n,k}(x)$$

where $g_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$

We can also write

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$n=0$: $B_{0,0}(x) = 1$

$n=1$: $B_{1,0}(x) = 1-x$
$B_{1,1}(x) = x$

$n=2$: $B_{2,0}(x) = (1-x)^2$
$B_{2,1}(x) = 2x(1-x)$
$B_{2,2}(x) = x^2$

1) $B_{n,k}$ has a zero of mult k @ $x=0$ and one of mult. $n-k$ @ $x=1$

2) $B_{n,k} \in \mathbb{P}_n$ are nonnegative on $[0,1]$

3) $B_{n,k}(x) \geq 0$ for $x \in [0,1]$

4) $\underbrace{\sum_{k=0}^{n} B_{n,k}(x) = 1}_{\text{Partition of unity}}$ for $x \in \mathbb{R}$

5) Symmetry:
$B_{n,k}(x) = B_{n,n-k}(1-x)$; $k=0,\dots,n$

6) $k=0,\dots,n$
$B_{n,k}(x) = x B_{n-1,k-1}(x) + (1-x) B_{n-1,k}(x)$

Pf:

4) $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1$

5) $\binom{n}{k} = \binom{n}{n-k}$

6) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Max of $B_{n,k}$ occurs at $x = k/n$

$B_{n,0}, \dots, B_{n,n}$ are lin. independent:

$$\sum_{k=0}^{n} b_k B_{n,k}(x) = 0$$

$$0 = \sum_{k=0}^{n} b_k \underbrace{B_{n,k}(1)}_{\substack{=0 \text{ except for } B_{n,n}(x)=1 \\ \text{ie when } k=n}} = b_n B_{n,n}(1) \rightarrow b_n = 0$$

$B_{n,k}(x)$ div. by $1-x$. Divide by $1-x$ then $x=1 \rightsquigarrow b_{n-1} = 0$ and continue $b_0 = \dots = b_n = 0$

$$B_{n,k}(a,b) = \frac{1}{(b-a)^n} \binom{n}{k} (x-a)^k (b-x)^{n-k}$$

Bernstein Poly for f

$f \in C[0,1]$.

Consider $B_n f$.

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

Goal $B_n f \xrightarrow{unf} f$ on $[0,1]$

That is, $\|f - B_n(f)\|_\infty \to 0$

$\|f\|_\infty = \sup_{0 \le x \le 1} |f(x)|$

Think of an operator:

$B_n : C[0,1] \to \mathbb{P}_n \subset C[0,1]$

Bernstein op. are monotone:

$f \le g \Rightarrow B_n f \le B_n g$

Thm: (Bohman-Korovkin) If L_n seq. monotone lin op. on $C[a,b]$

If $\|L_n f - f\|_\infty \to 0$ for $f = 1, x, x^2$

then true for all $f \in C[a,b]$.

OR

$H_n : C[a,b] \to C[a,b]$, monotone

$\|H_n(t^r)(x) - x^r\| \to 0$

for $r = 0,1,2 \to$

$\|H_n(f) - f\| \to 0$

$$B_n(1)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

Modify partition of unity formula replace n with $n-1$, mult by nx

$$nx = \sum_{k=0}^{n-1} n\binom{n-1}{k} x^{k+1} (1-x)^{n-(k+1)}$$

$$= \sum_{k=0}^{n-1} (k+1)\binom{n}{k+1} x^{k+1} (1-x)^{n-(k+1)}$$

$$= \sum_{s=1}^{n} s \binom{n}{s} x^s (1-x)^{n-s}$$

$$= \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k}$$

Then $nx =$ So

$$x = \sum_{k=0}^{n} \frac{k}{n} \cdot \binom{n}{k} x^k (1-x)^{n-k}$$

Then $B_n(t)(x)$

So B_n invariant on constants & lin. pieces. Using above calculation twice

$$B_n(t^2)(x) = \sum_{0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \left(\frac{k}{n}\right) \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \left(\frac{n-1}{n} \frac{k-1}{n-1} + \frac{1}{n}\right) \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= \frac{n-1}{n} x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{n-k} + \frac{x}{n}$$

$$= \frac{n-1}{n} x^2 + \frac{x}{n} \xrightarrow{unf} x^2$$

That is, use above with $n \to n-1$, mult. by nx and manipulate.

09/21/2017

Bernstein to Bézier Curves

$p_0, \dots, p_n \in \mathbb{R}^d$ control points

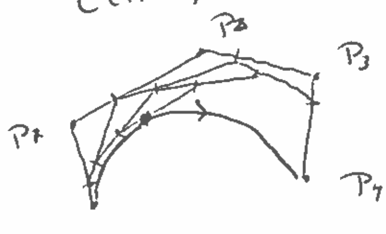
Bézier curve is...

$$C(t) = \sum_{k=0}^{n} p_k B_{n,k}(t)$$

$$t \in [0,1]$$

$C(0) = p_0$

$C(1) = p_2$



$n=2$: $B_{0,2}(t) = (1-t)^2$

$B_{2,1}(t) = 2(1-t)t$

$B_{2,2}(t) = t^2$

$$C(t) = (1-t)^2 p_0 + 2(1-t)t\, p_1 + t^2 p_2$$

$C(0) = p_1$; $C(1) = p_2$

$$C(\tfrac{1}{2}) = \frac{1}{2}\left(\frac{p_0+p_1}{2} + \frac{p_1+p_2}{2}\right) = \frac{1}{2}\left(\frac{p_0+p_2}{2} + p_1\right)$$

$$C(t) = (1-t)\big((1-t)p_0 + t p_1\big) + t\big((1-t)p_1 + t p_2\big)$$

$$0 \le t \le 1$$

Convex Comb. of Convex Comb

$\Downarrow$

Convex Comb.

This recursion is called Casteljau alg.

09/26/2017

Spline Functions

$S_1^0(\Delta)$

$\Delta = \{x_0, \dots, x_n\} \subset [a,b]$

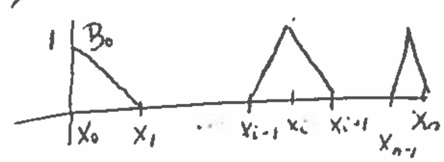
Consider $f \in S_1^0(\Delta)$ such that

$f \in C^0([a,b])$

$f\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_1$

$\dim S_1^0(\Delta) = n+1$

Define basis in $S_1^0(\Delta)$



$$B_i(x_j) = \underbrace{\delta(i-j)}_{\text{Kronecker function}}$$

$$B_0(x) = \begin{cases} \dfrac{x-x_1}{x_0-x_1}, & [x_0, x_1] \\ 0, & \text{otherwise} \end{cases}$$

$$B_i(x) = \begin{cases} \dfrac{x-x_{i-1}}{x_i-x_{i-1}}, & [x_{i-1}, x_i] \\ \dfrac{x_{i+1}-x}{x_{i+1}-x_i} & [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

$$B_n(x) = \begin{cases} \dfrac{x-x_{n-1}}{x_n-x_{n-1}}, & [x_n, x_n] \\ 0, & \text{otherwise} \end{cases}$$

x_i

Clearly, these are lin. independent

Given $f_0, \dots, f_n$; $f_i = f(x_i)$

Find $s \in S_1^0(\Delta)$ such that

$$s(x_i) = f_i$$

$$s\big|_{[x_{i-1}, x_i]} = f_{i-1} + (x - x_{i-1}) f[x_{i-1}, x_i]$$

$$s\big|_{[x_i, x_{i+1}]} = f_i + (x - x_i) f[x_i, x_{i+1}]$$

Let $\mathcal{L}(f)(x) = \sum_{i=0}^{n} f(x_i)\, B_i(x)$

$$\| f - \mathcal{L}(f) \|_\infty = ?$$

$$\mathcal{L}(f)\big|_{[x_i, x_{i+1}]} = f(x_i) + (x - x_i) f[x_i, x_{i+1}]$$

$$|f(x) - \mathcal{L}(f)(x)| = |(x - x_i)(x - x_{i+1}) f[x_i, x_{i+1}, x]|$$

$$\leq \frac{h^2}{2} \sup_{a \leq \xi \leq b} \left| \frac{f''(\xi)}{2} \right|$$

where $h = \max_i h_i$, $h_i = x_i - x_{i-1}$

Notice then that as # mesh points $\to \infty$,

$\mathcal{L}(f)(x) \to f$

$$\|\mathcal{L}(f)\| = \max_{0 \leq i \leq n-1} \sup_{x_{i-1} \leq x \leq x_{i+1}} |\mathcal{L}(f)(x)|$$

$$= \max_{0 \leq i \leq n-1} |f(x_i)| \leq \|f\|$$

Let $g \in S_1^0(\Delta)$.

$$\mathcal{L}(g) = g$$

$$\|f - \mathcal{L}(f)\| \leq \|f - g - \mathcal{L}(f) + \mathcal{L}(g)\|$$

$$\leq \|f - g\| + \|\mathcal{L}(g - f)\|$$

$$\leq \|f - g\| + \|f - g\|$$

$$= 2\|f - g\|$$

$$\inf_{g \in S_1^0(\Delta)} \|f - g\| \leq \|f - \mathcal{L}(f)\| \leq 2\|f - g\|$$

Piecewise Cubic Splines

$$S_3^1 = \{ f : f \in C^1[a,b],\ f\big|_{[x_i, x_{i+1}]} \in \Pi_3 \}$$

$h \in S_3^1(\Delta)$. On $[x_i, x_{i+1}]$

Poly given by 4 coefficients so

$4n$ coeff. in total. But we need

C^1, so at nodes the derivatives

agree so

$$4N - 2(N-1)$$

$$= \underbrace{2N + 2}_{\#\text{ parameters}}$$

If $h^{(j)}(x_i) = f^{(j)}(x_i)$

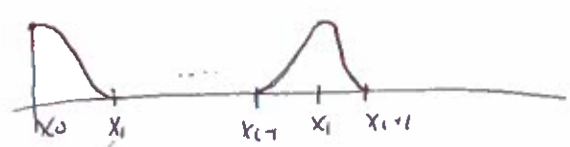
$i = 0, \dots, n$; $j = 0, 1$

$h_{i,0}(x_j) = \delta_{ij}$ $\quad i,j = 0, \dots, n$

$h_{i,0}'(x_j) = 0$

$h_{i,1}(x_j) = 0$

$h_{i,1}'(x_j) = \delta_{ij}$



$$h_{i,0}(x) = \begin{cases} \frac{2}{h_{i-1}^3} (x - x_{i-1})^2 \left(x - x_i - \frac{h_{i-1}}{2}\right), & [x_{i-1}, x_i] \\ \frac{2}{h_i^3} (x - x_{i+1})^2 \left(x - x_i + \frac{h_i}{2}\right), & [x_i, x_{i+1}] \end{cases}$$

$$|f - H(f)(x)| = |(x-x_i)^2 (x-x_{i+1})^2 f[x_i,2; x_{i+1},2]|$$

$$\le \frac{h_i^2}{4} \max_{a \le \xi \le b} |f^{(4)}(\xi)|$$

$$\Delta = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

$$S_m^K(\Delta) = \left\{ s : \begin{array}{l} s|_{[x_i, x_{i+1}]} \in \mathbb{P}_m \\ s \in C^K[a,b] \end{array} \right\}$$

$$\dim S_m^K(\Delta) = \underbrace{(m+1)(n)}_{\text{coeff on int.}} - \underbrace{(K+1)(n-1)}_{\text{cont.}}$$

$$= mn + n - Kn + K - n + 1$$

$$= mn - Kn + K + 1$$

$$= n(m-K) + (K+1)$$

$S(x_j) = f(x_j)$; $j = 0, \ldots, n$

Take $K = m-1$

$S_m^{m-1}(\Delta)$

$\dim S_m^{m-1}(\Delta) = m+n$

periodic spline:

$$S^{(l)}(a) = S^{(l)}(b) \; ; \; l = 1, \ldots, m-1$$

Natural Spline

$m = 2l-1$; $l \ge 2$

$$S^{(l+j)}(a) = S^{(l+j)}(b) = 0$$

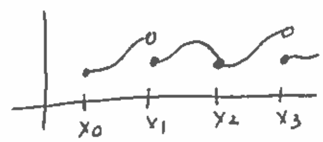
$$j = 0, 1, \ldots, l-2$$

09/28/2017

Splines

$$\Delta = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

$f : [a,b] \to \mathbb{R}$ is piecewise poly if $f|_{[x_i, x_{i+1}]}$ is a polynomial.



Def: f spline degree m if

1) $f|_{[x_i, x_{i+1}]} \in \mathbb{P}_m$

2) $f \in C^{m-1}([a,b])$

$$S_m^{m-1}(\Delta) = S_m(\Delta)$$

$$\dim S_m(\Delta) = n(m+1) - m(n-1)$$
$$= m+n$$

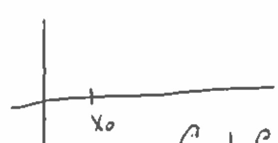
Want simplest spline. $\Delta = \{x_0\}$. Let $S \in S_m(\Delta)$. Let $S \equiv 0$ for $x < x_0$. Then $S \in \mathbb{P}_m$ for $x \ge x_0$

$$S^0(x_0) = S'(x_0) = \cdots = S^{m-1}(x_0) = 0$$

So $S(x) = C(x - x_0)^m$

Fitting a spline: Consider $S_2(\Delta) \ni S$
$\Delta = \{0, 1, 2\}$

$$S(x) = \begin{cases} a_{0,2}x^2 + a_{0,1}x + a_{00}, & [0,1] \\ a_{12}x^2 + a_{11}x + a_{10} & [1,2] \end{cases}$$



Also $S \in C^1(0,2)$.

Cont. @ $x = 1 \to$

$$a_{0,2} + a_{0,1} + a_{00} = a_{12} + a_{11} + a_{10}$$

Cont. S' @ $x = 1 \to$

$$2a_{02} + a_{01} = 2a_{12} + a_{11}$$

Need more constraints:

$S(0) = f_0$, $S(1) = f_1$

$S(2) = f_2$; $S'(0) = f_0'$

$a_{00} = f_0$

$a_{12} + a_{11} + a_{10} = f_1$

$4a_{12} + 2a_{12} + a_{10} = f_2$

$a_{01} = f_0'$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 2 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{inv.}} \begin{pmatrix} a_{02} \\ a_{01} \\ a_{00} \\ a_{12} \\ a_{11} \\ a_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \\ f_0' \end{pmatrix}$$

Second possibility: Match value at $\frac{1}{4}$, $\frac{1}{2}$, $\frac{5}{4}$, $3/2$, two on each int. in Δ

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 2 & 1 & 0 & -2 & -1 & 0 \\ 1/16 & 1/4 & 1 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25/16 & 5/4 & 1 \\ 0 & 0 & 0 & 9/4 & 3/2 & 1 \end{pmatrix} \begin{pmatrix} a_{02} \\ a_{01} \\ a_{00} \\ a_{12} \\ a_{11} \\ a_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_{1/4} \\ f_{1/2} \\ f_{5/4} \\ f_{3/2} \end{pmatrix}$$

Third possibility: Match values at $0, 1/8, 1/4, 1/2$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 2 & 1 & 0 & -2 & -1 & 0 \\ 1/16 & 1/4 & 1 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/64 & 1/4 & 1 & 0 & 0 & 0 \end{pmatrix}}_{\text{non-inv.}} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Determination of Cubic Spline

$S''(a) = 0$, $S''(b) = 0$

$S \in S_3(\Delta)$

$\Delta = \{x_0, \dots, x_n\}$

$S|_{[x_i, x_{i+1}]} \in \mathbb{P}_3$

$4n - 3(n-1) = 4n - 3n + 3 = n + 3$

We impose

$S(x_i) = f_i$; $i = 0, \dots, n$

$h_i = x_{i+1} - x_i$

Call $S''(x_j) = M_j$ moments of S

We show S can be determined in terms of moments. Moments can be computed from interpolating data by solving a tridiag system

1) $S''(x) = M_i \dfrac{x_{i+1} - x}{h_{i+1}} + M_{i+1} \dfrac{x - x_i}{h_{i+1}}$

Integrate twice & det. int. constants from f_i

2) $S(x) = M_i \dfrac{(x_{i+1} - x)^3}{6h_{i+1}} + M_{i+1} \dfrac{(x - x_i)^3}{6h_{i+1}} + A_i (x - x_i) + B_i$

Set $S(x_i) = f_i$

$S(x_{i+1}) = f_{i+1}$

4) $f_i = S(x_i) = \dfrac{M_i h_{i+1}^2}{6} + B_i$

5) $f_{i+1} = S(x_{i+1}) = M_{i+1}\dfrac{h_{i+1}^2}{6} + A_i h_{i+1} + B_i$

$B_i = f_i - M_i \dfrac{h_{i+1}^2}{6}$

6) $A_i = \dfrac{f_{i+1}-f_i}{h_{i+1}} - \dfrac{h_{i+1}}{6}(M_{i+1}-M_i)$

$$S(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

$a_i = f_i$

$c_i = M_i/2$

$b_i = \dfrac{f_{i+1}-f_i}{h_{i+1}} - \dfrac{2M_i + M_{i+1}}{6} h_{i+1}$

Show how to compute M_i; $i=0,\dots,n$ from $f_0,\dots,f_n$

$M_0 = M_n = 0$

S' is cont. at int. nodes

$S'(x_i) = S'(x_i^-) = S'(x_i^+)$

$$S'(x) = -M_i\frac{(x_{i+1}-x)^2}{2h_{i+1}} + M_{i+1}\frac{(x-x_i)^2}{2h_{i+1}} + \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{h_{i+1}}{6}(M_{i+1}-M_i)$$

$$S'(x_i^+) = -M_i\frac{h_{i+1}}{2} + \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{h_{i+1}}{6}(M_{i+1}-M_i)$$

$$= \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{h_{i+1}}{3}M_i - \frac{h_{i+1}}{6}M_{i+1}$$

$$S'(x_i^-) = \frac{f_i - f_{i-1}}{h_i} + \frac{h_i}{3}M_i + \frac{h_i}{6}M_{i-1}$$

Now...

$$\frac{h_i}{6}M_{i-1} + \frac{h_i+h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}$$

$i = 1,\dots,n-1$

$\mu_i := \dfrac{h_i}{h_i+h_{i+1}}$

$\lambda_i := \dfrac{h_{i+1}}{h_i+h_{i+1}}$; $\mu_i + \lambda_i = 1$

$\delta_i := \dfrac{6}{h_i+h_{i+1}}\left(\dfrac{f_{i+1}-f_i}{h_{i+1}} - \dfrac{f_i - f_{i-1}}{h_i}\right)$

Then...

$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = \delta_i$

If $M_0 = M_n = 0$

$\lambda_0 = \mu_n = \delta_0 = \delta_n = 0$

$$\begin{pmatrix} 2 & \lambda_0 & & & 0 \\ \mu_1 & 2 & \lambda_1 & & \\ & \mu_2 & 2 & \ddots & \\ & & \ddots & \ddots & \lambda_{n-1} \\ 0 & & & \mu_n & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_n \end{pmatrix} = \begin{pmatrix} \delta_0 \\ \vdots \\ \delta_n \end{pmatrix}$$

Try to think or prove that the above matrix is strictly diagonally dominant.

10/03/2017

Cubic Splines & B-splines (absent)

10/05/2017

B-splines

Our def. based on divided diff.

$$f[t_0,\dots,t_n] = \sum_{j=0}^{n} f(t_j) \prod_{\substack{s=0 \\ s\neq j}}^{n} (t_j - t_s)^{-1}$$

$$\delta^n(t_0,\dots,t_n)f = f[t_0,\dots,t_n]$$

$f(t,y)$

$t_+ = \max\{t,0\}$ $\qquad t_+^0(t) = \begin{cases} 0, & t\le 0 \\ 1, & 0<t \end{cases}$

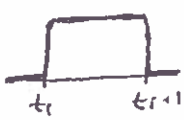
Nodes:

$\cdots < t_{-1} < t_0 < t_1 < t_2 < \cdots$

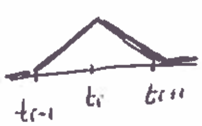
$$\lim_{i\to\pm\infty} t_i = \pm\infty$$

$$B_i^r(x) = (t_{i+r+1} - t_i)\,\delta_t^{r+1}(t_i,\dots,t_{i+r+1})(t-x)_+^r$$

$B_i^0(x)$



$B_i^1(x)$



Lemma: (Leibniz)

$f = gh \rightarrow$

$$f[t_i,\dots,t_{i+k}] = \sum_{r=i}^{i+k} g[t_i,\dots,t_r]\,h[t_r,\dots,t_{i+k}]$$

Compare to: $f^{(n)} = \sum \binom{n}{k} f^{(n-k)} g^k$

Pf:

$$G(t) = g[t_i] + \sum_{r=i+1}^{i+k} g[t_i,\dots,t_r](t-t_i)\cdots(t-t_{r-1})$$

$$H(t) = h[t_{i+k}] + \sum_{s=i}^{i+k-1} h[t_s,\dots,t_{i+k}](t-t_{s+1})\cdots(t-t_{i+k})$$

Consider $F(t) = G(t)H(t)$

$$F(t) = \underbrace{\left(\sum_{r=i}^{i+k} a_r b_s\right)}_{G(t)}\underbrace{\left(\sum_{s=i}^{i+k} a_r b_s\right)}_{H(t)}$$

$$= \underbrace{\sum_{r\le s} a_r b_s}_{P_1(t)} + \underbrace{\sum_{r>s} a_r b_s}_{P_2(t)}$$

Examine $P_2(t)$

$P_2(t_j) = 0 \quad j = i, i+1,\dots, i+k \rightarrow$

$\delta^k(t_i,\dots,t_{i+k})P_2 = 0$

a_r $r = i,\dots,i+k$ contains $(t-t_i)\cdots(t-t_{r-1})$ of deg $r+i$

b_s $s\ge r$ contains $(t-t_{s+1})\cdots(t-t_{i+k})$ of deg $r+k-s$

$a_r b_s$ is of deg $r+k-s$

when $r>s$, $r-1\ge s$ then each term a_r contains $(t-t_i)\cdots(t-t_s)$

$P_1(t)$ must interpolate f @ $t_i,\dots,t_{i+k}$

$\underbrace{\delta^k(t_i,\dots,t_{i+k})}_{\delta^k}$

$\delta^k F = \delta^k P_1 + \cancel{\delta^k P_2}$

$\delta^k f = \delta^k P_1$

Leading coeff. of $P_i(t)$

Leading coefficient of P_i is a sum of leading coeff. in terms of deg K in

$$\sum_{r=i}^{i+K} a_r b_r = \sum_{r=i}^{i+K} g[t_i,\dots,t_r]\, h[t_r,\dots,t_{i+K}]$$

$$(t-t_i)\widehat{(t-t_r)}(t-t_{i+K})$$

Compact Support

$B_i^r(x) = 0$ for $x \notin (t_i, t_{i+r+1})$ $r \geq 0$

$$\delta_t^{r+1}(t_i,\dots,t_{i+r+1})(t-x)_+^r$$

$$t_i \leq t \leq t_{i+r+1} < x$$

$$x < t_i \leq t \leq t_{i+r+1}$$

Recurrence (de Boor, Cox)

$$B_i^r(x) = \left(\frac{x-t_i}{t_{i+r}-t_i}\right) B_i^{r-1}(x) + \left(\frac{t_{i+r+1}-x}{t_{i+r+1}-t_{i+1}}\right) B_{i+1}^{K-1}(x)$$

Positivity in r+1 conjec. int.

$B_i^r(x) > 0$; $x \in (x_i,\dots,x_{i+r+1})$, $r \geq 0$

$$(t-x)_t^r = (t-x)(t-x)_t^{r-1}$$

$$g(t) = t-x$$

$$g[t_i] = g(t_i) = t_i - x$$

$$g(t_i, t_{i+1}) = 1$$

$$g[t_i,\dots,t_j] = 0,\ j > i+1$$

$$\delta_t^{r+1}(t_i,\dots,t_{i+r+1})(t-x)_t^r =$$

$$\delta^{r+1}\left[(t-x)(t-x)_+^{r-1}\right]$$

$$= g[t_i]\,\delta^{r+1}(t-x)_+^{r-1} + g[t_i,t_{i+1}]\,\delta_t^r (t-x)_+^{r-1}$$

$$= (t_i - x)\quad \frac{\delta_t^r(t_{i+1},\dots t_{i+r+1})(t-x)_+^{r-1} - \delta_t^r(t_i,\dots,t_{i+r})(t-x)_+^{r-1}}{t_{i+r+1}-t_i} + \delta_t^r (t-x)_+^{r-1}$$

$\vdots$

$$= \frac{x-t_i}{t_{i+r+1}-t_i}\,\delta_t^r(t_i,\dots,t_{i+r})(t-x)_+^{r-1} + \frac{t_{i+r+1}-x}{t_{i+r+1}-t_i}\,\delta_t^r(t_{i+1},\dots,t_{i+r+1})(t-x)_+^{r-1}$$

Compact Support:

$B_i^r(x) = 0$ for $x \notin (t_i, t_{i+r+1})$ $r \geq 0$

$$\delta_t^{r+1}(t_i,\dots,t_{i+r+1})(t-x)_+^r$$

10/10/2017

Numerical Integration (Quadrature)

$$I(f) := \int_a^b f(x)\,dx \approx \lambda_1 f_1 + \cdots + \lambda_n f_n =: Q(f)$$

$$E(f) = I(f) - Q(f) \quad \text{\} want small}$$

Interpolatory Quadratures

$$I(f) = \int_a^b f(x)\,dx \qquad a, b \text{ finite}$$

$$x_i := a + ih, \quad h = \frac{b-a}{n} \;;\; i = 0, 1, \ldots, n$$

$$Q(f) = \int_a^b L_n(x)\,dx \;;\; L_n(x) = \sum_{i=0}^{n} l_i(x) f_i$$

Lag. int. of f @ x_i

$$Q(f) = \int_a^b \sum_{i=0}^{n} l_i(x) f_i \,dx = \sum_{i=0}^{n} \underbrace{\int_a^b l_i(x)\,dx}_{\lambda_i \text{ weights}} f_i$$

Can determine weights by undet. coefficients

$$Q(x^i) = I(x^i) \;;\; i = 0, 1, \ldots, n$$

Find weights λ_j of such poly.

$$Q(f) = \sum \lambda_j f_j$$

$$\sum \lambda_j x_j^i = \int_a^b x^i\,dx$$

$[x_j^i]$ Vandermonde matrix

We say Q is exact of deg $\le n$ if $Q(f) = I(f) \quad \forall f \in \mathbb{P}_n$

$$E(f) = I(f) - Q(f)$$

$$= \int_a^b f - L_n \,dx$$

$$= \int_a^b \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)\cdots(x - x_n)\,dx$$

$$\xi_x \in [a, b]$$

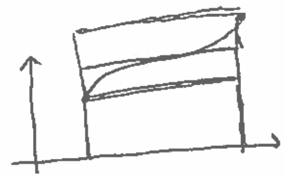
Other (constant) interpolants of f

$$Q(f) = (b-a) f(a) = h f(a)$$

$$Q(f) = (b-a) f(b) = h f(b)$$

$$Q(f) = (b-a) f\left(\frac{a+b}{2}\right) = h f(x_{1/2})$$

$$h = x_1 - x_0 = b - a$$



We will prove:

$$\int_a^b f(x)\,dx = \underbrace{h f(x_{1/2})}_{Q(f)} + \frac{1}{24} h^3 \overset{\text{2nd deriv.}}{f''}(\mu)$$

$$L_0(x) = f(x_{1/2})$$

$$f(x) - L_0(x) = f'(\xi_x)(x - x_{1/2})$$

$$E(f) = \int_a^b f'(\xi_x)(x - x_{1/2})\,dx$$

$$|E(f)| \le \overset{\text{bound}}{M_1} \int_a^b |x - x_{1/2}|\,dx$$

$$= M_1 \frac{(b-a)^2}{2}$$

$$M_1 = \sup_{a \le x \le b} |f'(x)|$$

But cannot infer degree of exactness from this.

$$H(x) = f(x_{1/2}) + f'(x_{1/2})(x - x_{1/2})$$

Hermite interpolant at $x = x_{1/2}$

$H(x_{1/2}) = f(x_{1/2})$

$H'(x_{1/2}) = f'(x_{1/2})$

$$Q(f) = I(H) = I(\underbrace{f(x_{1/2})}_{I(L_0)}) + I(f'(x_{1/2})(x - x_{1/2}))$$

$$Q(f) = I(L_0)$$

$$f(x) - H(x) = \frac{1}{2!}(x - x_{1/2})^2 f^{(2)}(\xi_x)$$

$$E(f) = I(f - Q) = \int_a^b \frac{1}{2}(x - x_{1/2}) f''(\xi_x)\, dx$$

MVTI:

ϕ, ψ cont. on $[a,b]$. if ψ doesn't change

$\exists\ \mu$

$$\int_a^b \phi(x)\psi(x)\, dx = \phi(\mu)\int_a^b \psi(x)\, dx$$

$$E(f) = \frac{1}{2} f''(\mu) \int_a^b (x - x_{1/2})^2\, dx$$

$$= \frac{h^3}{24} f''(\mu)$$

Lin. interpolant of f. Trap. rule

$x_0 = a \quad x_1 = b$

$$L(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$$

$$Q(f) = I(L) = \frac{f(a)}{b-a}\int_a^b b - x\, dx + \frac{f(b)}{b-a}\int_a^b x - a\, dx$$

$$= \frac{f(a)}{b-a}\frac{1}{2}(b-a)^2 + \frac{f(b)}{b-a}\frac{1}{2}(b-a)^2$$

$$Q(f) = \frac{b-a}{2}\left(f(a) + f(b) \right)$$

$$I(f) - Q(f) =$$

$$\int_a^b \frac{f''(\xi_x)}{2!}\underbrace{(x-a)(x-b)}_{\text{always neg.}}\, dx$$

$$= \frac{(b-a)^3}{8}\int_{-1}^{1}(t^2 - 1)\, dt = -\frac{(b-a)^3}{6}$$

$$x = \frac{b-a}{2}t + \frac{a+b}{2} \quad ; -1 \le t \le 1$$

$$E(f) = \frac{(b-a)^3}{12} f''(\xi)$$

Trap rule exact for $\mathbb{P}_1$.

Newton - Cotes

Quad. Interp.: Simpson's Rule

$$x_0 = a,\quad x_1 = \frac{a+b}{2},\quad x_2 = b$$

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_1-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

Integrate $L_2(x)$

$x - x_1 = s$

$x - x_0 = s + h$

$x - x_2 = s - h$

$$\int_{x_0}^{x_2} (x-x_1)(x-x_2)\,dx =$$

$$\int_{-h}^{h} s(s-h)\,ds = \frac{2}{3}h^3$$

x

$$\int_{x_0}^{x_2} (x-x_0)(x-x_2)\,dx =$$

$$\int_{-h}^{h} (s-h)(s+h)\,ds =$$

$$\int_{-h}^{h} (s^2-h^2)\,ds =$$

$$\frac{4}{3}h^3$$

x

$$\int_{x_0}^{x_2} (x-x_0)(x-x_1)\,dx =$$

$$\int_{-h}^{h} s(s+h)\,ds = \frac{2}{3}h^3$$

$$Q(f) = h\left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right)$$

$$= \frac{b-a}{6}\left(f_0 + 4f_1 + f_2\right)$$

$$E(f) = I(f - L_2)$$

Define $H_3 \in \mathbb{P}_3$

$$\begin{cases} H_3(x_i) = f_i \;;\; i = 0,1,2 \\ H_3'(x_1) = f'(x_1) \end{cases}$$

$$H_3(x) = L_2(x) + K(x-x_0)(x-x_1)(x-x_2)$$

$$H_3'(x_1) = f'(x_1)$$

$$H_3'(x) = L_2'(x)$$

10/12/2017

Simpson's Rule:

$[a,b]$ $x_0 = a$, $x_1 = \frac{a+b}{2}$ $x_2 = b$

$$Q(f) = \frac{b-a}{6}\left(f_0 + 4f_1 + f_2\right)$$

Exact for poly. of deg. ≤ 2

Actually exact for $p \in \mathbb{P}_3$

Smooth function f. Define Hermite $H_3 \in \mathbb{P}_3$

$$H_3(x_i) = f(x_i) \quad i = 0,1,2$$

$$H^{(1)}(x_1) = f^{(1)}(x_1)$$

Newton:

$$H_3(x) = L_2(x) + K(x-x_0)(x-x_1)(x-x_2)$$

Find K

$$L_2'(x_1) = \frac{f_2 - f_0}{2h}$$

$$H_3'(x_1) = \frac{f_2 - f_0}{2h} - Kh^2$$

$$K = \frac{1}{h^2}\left(\frac{f_2 - f_0}{2h} - f'(x_1)\right)$$

$$I(H_3) = I(L_2) + K\int_a^b (x-x_0)(x-x_1)(x-x_2)\,dx$$

$$= I(L_2) + K\int_{-h}^{h} \underbrace{\underbrace{(s+h)s(s-h)}_{s(s^2-h^2)}}_{\text{odd}}\,ds$$

$$= I(L_2)$$

$$E(f) = I(f - H_3)$$

$$f(x) = H_3(x) + (x-x_0)(x-x_1)^2(x-x_2)\frac{f^{(4)}(\xi)}{4!}$$

$$E(f) = \int_{x_0}^{x_2} (x-x_0)(x-x_1)^2(x-x_2)\frac{f^{(4)}(\xi)}{4!}\,dx$$

$$= \frac{1}{4!} f^{(4)}(\mu)\underbrace{\int_{-1}^{h}(s+h)(s-h)s^2\,ds}_{-\frac{4}{15}h^5}$$

$$= \frac{-1}{90}h^5 f^{(4)}(\mu)$$

We need a 'good' set of nodes for quadrature $\to$ Orth. poly.

$$\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx$$

$\underbrace{w(x)}_{\text{weight}} > 0$ & $x^n w \in L^1(a,b)$ $n \ge 1$

$$\|f\|_{L^2(a,b)} = \int_a^b |f(x)|^2 w(x)\,dx$$

Use Graham-Schmidt to produce orth. poly.

$\{1, x, x^2, \dots\}$ $\{p_0, p_1, \dots\}$

$p_0(x) = 1$; $p_1(x) = x - \frac{\langle 1,x\rangle}{\langle 1,1\rangle}1$

$$p_n(x) = x^n - \cancel{\langle x^n, p_i\rangle} \sum_{i=0}^{n-1} \lambda_{i,n}\, p_i(x)$$

monic

$$\lambda_{i,n} = \frac{\langle x^n, p_i\rangle}{\langle p_i, p_i\rangle}$$

Thm (Triple Recursion) $\exists!$ seq. of poly. $\{p_n\}_{n=0}^{\infty}$ with p_n monic $\ni$

$$\langle p_n, q\rangle = 0 \quad \forall q \in \mathbb{P}_{n-1}$$

satisfying...

$$p_n(x) = (x - \lambda_n)p_{n-1}(x) - \mu_n p_{n-2}(x)$$

$$\lambda_n = \frac{\langle x p_{n-1}, p_{n-1}\rangle}{\|p_{n-1}\|^2}$$

$$\mu_n = \frac{\|p_{n-1}\|^2}{\|p_{n-2}\|^2}$$

$$\langle p_n, q\rangle = 0 \quad \forall q \in \mathbb{P}_{n-1}$$

$$p_n(x) - x p_{n-1}(x) \in \mathbb{P}_{n-1}$$

$$= \sum_{i=0}^{n-1} a_i p_i(x)$$

For $i \le n-3$

$$x p_i(x) \in \mathbb{P}_{n-2}$$

$$a_i \langle p_i, p_i\rangle = \langle p_n, p_i\rangle - \langle x p_{n-1}, p_i\rangle$$

$$= -\langle x p_{n-1}, p_i\rangle$$

$$= -\langle p_{n-1}, x p_i\rangle$$

$$= 0$$

$i=n-1$; $a_{n-1} = -\frac{\langle xp_{n-1}, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} = -\lambda_n$

$i=n-2$: $a_{n-2} = \frac{\langle xp_{n-1}, p_{n-2} \rangle}{\langle p_{n-2}, p_{n-2} \rangle}$

$\langle xp_{n-1}, p_{n-2} \rangle = \langle p_{n-1}, p_{n-1} - p_{n-1} + xp_{n-2} \rangle$

$= \langle p_{n-1}, p_{n-1} \rangle + \langle p_{n-1} \underbrace{- p_{n-1} + xp_{n-2}}_{=0} \rangle$

Extended prop:

$\|p_n\| \leq \|S\|$ S monic in $\mathbb{P}_n$

$S = p_n - \sum_{i=0}^{n-1} c_i p_i$

$\|S\|^2 = \|p_n\|^2 + \sum_{i=0}^{n-1} |c_i|^2 \|p_i\|^2 \geq \|p_n\|^2$

p_n has n-real roots in (a,b)

Let $x_1, \dots, x_k$ real roots of p_n of odd multiples.

If $k=n \rightsquigarrow$ true

If $k<n \rightsquigarrow q(x) = \prod^{k} x - x_i$

$q \equiv 1$ $k=0$

$\int_a^b \underbrace{p_n(x) q(x)}_{\text{only roots mult.}} w(x)\, dx = 0$

Gauss Quadrature

Want quad

$Q(f) = \sum^{n} \lambda_i f(x_i)$

Weights computed from

$\int_a^b l_i(x) w(x)\, dx = \lambda_i$

By choosing Gauss nodes, exactness of Q could be $2k+1$ instead of k.

If $k=0$ set $l_0(x) \equiv 1$

A quad. Q with $k+1$ nodes and exact for $2k+1$ poly is unique called Gauss quad

10/19/2017

Gaussian Quad.

$I(f) = \int_a^b f(x) w(x)\, dx$

$Q(f) = \sum_{i=0}^{k} \lambda_i f(x_i)$

Idea: Choose $x_0, \dots, x_k$ so that as to raise exactness of Q to $2k+1 = 2\,\#\text{nodes} - 1$

Lem: A quad rule Q with nodes $x_0, \dots, x_K$ equal to roots of p_{K+1}. and weights $\lambda_0, \dots, \lambda_K$ given

$$\lambda_i = \int_a^b \ell_i(x) w(x) dx \quad i = 0, \dots, K$$

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{K} \frac{x - x_j}{x_i - x_j}$$

Claim exact for $P \in \mathbb{P}_{2K+1}$

$p_{K+1} \in \mathbb{P}_{K+1}$ orthogonal in $\mathcal{L}^2_w(a,b)$ monic

First show exact for $P \in \mathbb{P}_K$.

$$Q(f) = \sum_{i=0}^{K} \lambda_i f(x_i) = \sum_{i=0}^{K} \int_a^b \ell_i(x) w(x) dx \, f(x_i)$$

$$= \int_a^b \underbrace{\sum_{i=0}^{K} \ell_i(x) f(x_i)}_{\mathcal{L}_K(f)(x)} w(x) dx$$

$$= \int_a^b f(x) \omega(x) dx = I(f)$$

There is a poly. in $\mathbb{P}_{K+1}$ for which Q is also exact $\mathbb{P}_{K+1}$

$$Q(p_{K+1}) = 0$$
$$= \langle 1, p_{K+1} \rangle$$
$$= \int_a^b p_{K+1}(x) w(x) dx$$
$$= I(p_{K+1})$$

$f \in \mathbb{P}_{2K+1}$

$$f(x) = q(x) p_{K+1} + r(x)$$

$$r \in \mathbb{P}_K$$

$$p_{K+1}(x) = (x - x_0) \cdots (x - x_K)$$

$$f(x_i) = r(x_i) \; ; \; i = 0, \dots, K$$

$$\int_a^b f(x) w(x) dx =$$

$$\underbrace{\int_a^b q(x) p_{K+1}(x) w(x) dx}_{\langle q, p_{K+1} \rangle}$$

$$+ \int_a^b r(x) w(x) dx$$

$$= Q(r) \sum_{i=0} \lambda_i r(x_i)$$

$$= \sum_i \lambda_i f(x_i) = Q(f)$$

Uniqueness

Suppose Q exact for $\mathbb{P}_{2K+1}$ so in particular for $q\pi$

$$\pi(x) = \prod x - x_i$$

$$q \in \mathbb{P}_K$$

$$0 = Q(q\pi) = I(q\pi)$$

$$0 = \int_a^b \pi(x) q(x) w(x) dx$$

$\langle \pi, q \rangle = 0$ and π monic $\rightarrow \pi = p_{K+1}$

$$Q(\ell_i) = \sum_{j=0}^{K} \lambda_i \ell_i(x_j) = \int_a^b —$$

The weights of Q are positive

Q is exact for $\ell_i^2(x) \in \mathbb{P}_{2K}$

$$\lambda_i = \sum_{j=0}^{K} \lambda_j \ell_i^2(x_j)$$

$$= Q(\ell_i^2)$$

$$= I(\ell_i)^2$$

$$= \int_a^b \ell_i^2(x) w(x)\,dx$$

Error Gauss quad.

Let $H \in \mathbb{P}_{2K+1}$

$$H^{(\ell)}(x_i) = f^{(\ell)}(x_i)$$

$i = 0, \cdots, K$

$\ell = 0, 1$

$$f(x) = H(x) + f[x_0, 2; \cdots; x_{K+1}, 2, x]\,\pi^2(x)$$

$$I(f) = I(H) + \int_a^b f[x_1, 2; \cdots; x_K, 2, x]\,\pi^2(x) w(x)\,dx$$

$$I(H) = Q(H) = Q(f)$$

$$E(f) = I(f) - Q(f)$$

$$= I(f) - I(H)$$

$$= I(f - H)$$

$$= \int_a^b f[x_0, 2; \cdots; x_K, 2, x]\,\pi^2(x) w(x)\,dx$$

$$= f[x_0, 2; \cdots; x_K, 2, \xi] \int_a^b \pi^2(x) w(x)\,dx$$

$$= \frac{1}{(2K+2)!} f^{(2K+2)}(\xi) \int_a^b \pi^2(x) w(x)\,dx$$

~~Otra~~ The weights of Q are positive. Q exact for $\ell_i^2(x) \in \mathbb{P}_{2K}$

$$0 < \lambda_i = \sum_{j=0}^{K} \lambda_j \ell_i^2(x_j)$$

$$= Q(\ell_i^2)$$

$$= I(\ell_i^2) = \int_a^b \ell_i^2(x) w(x)\,dx$$

Gauss - Lobatto

Degree exactness $2K-1$

$x_0 = a \quad x_K = b$

$x_1, \cdots, x_{K-1}$ are roots of P_{K-1}

Orth. poly of deg $K-1$ on (a,b) with weight

$$\tilde{w}(x) = w(x)(x-a)(b-x)$$

$(-1, 1)$

$$w(x) = (1 - x^2)^{-1/2}$$

Gauss - Chebyshev quad.

$(0,1) \quad w(x) \equiv 1$

Gauss - Legendre

Ex: Gauss - Hermite quad nodes

x_0, x_1

$$Q(f) = \sum_{i=0}^{1} A_i f(x_i)$$

exact for poly. of deg. $2K+1 = 3$

$$Q(f) \approx I(f) = \int_{-\infty}^{\infty} f(x) \underbrace{e^{-x^2}}_{w(x)}\,dx$$

$Q(x^i) = I(x^i)$; $i=0,1,2,3$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$Q(1) = I(1)$

$A_0 + A_1 = \sqrt{\pi}$

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = 0$$

$$\int_{-\infty}^{\infty} x^3 e^{-x^2} dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx =$$

$$-\frac{1}{2}\int_{-\infty}^{\infty} x(e^{-x^2})' dx$$

$$\int_{-\infty}^{\infty} x(e^{-x^2})' dx = \left[x e^{-x^2}\right]_{-\infty}^{\infty}$$

$$(\;) - \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= \sqrt{\pi}$$

$$\begin{cases} A_0 + A_1 = \sqrt{\pi} \\ A_0 x_0 + A_1 x_1 = 0 \\ A_0 x_0^2 + A_1 x_1^2 = \frac{\sqrt{\pi}}{2} \\ A_0 x_0^3 + A_1 x_1^3 = 0 \end{cases}$$

Let $\pi(x) = (x-x_0)(x-x_1)$

Find x_0, x_1

$= x^2 + px + q$

Want to find p, q not x_0, x_1.

$\overset{eq}{q(1)}$ $\overset{eq}{p(2)}$ $\overset{eq}{1(3)}$

$q\sqrt{\pi} = A_0 q + A_1 q$

$0 = A_0 x_0 p + A_1 x_1 p$

$\frac{\sqrt{\pi}}{2} = A_0 x_0^2 + A_1 x_1^2$

$$\sqrt{\pi}\left(q+\tfrac{1}{2}\right) = A_0(x_0^2 + \cancel{p x_0}^{\,0} + q) + A_1(x_1^2 + \cancel{p x_1}^{\,0} + q)$$

$q = -\frac{1}{2}$

$0 = x^2 - \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$

$A_0 = A_1 = \frac{\sqrt{\pi}}{2}$

$\downarrow$

$$Q(f) = \frac{\sqrt{\pi}}{2} f\left(-\tfrac{1}{\sqrt{2}}\right) + \frac{\sqrt{\pi}}{2} f\left(\tfrac{1}{\sqrt{2}}\right)$$

10/24/2017

Convergence of Quad. Formulas

$$Q_k(f) = \sum_{i=0}^{k} \lambda_{i,k} f(x_{i,k}) \cong \int_a^b f(x) w(x)\, dx$$

Formula with $k+1$ nodes

$$E_k(f) = \int_a^b f(x) w(x)\, dx - Q_k(f)$$

$$\lim_{k\to\infty} E_k(f) \overset{?}{=} 0$$

Thm: (Polya) $\lim_{k\to\infty} E_k(f)=0$ for any cont. function f iff

a) $\exists M \ni \forall k \sum_{i=0}^{k} |\lambda_{ik}| \le M$

b) $\forall n \lim_{k\to\infty} E_k(x^n)=0$

Sufficiency: Assume (a)&(b)

$\exists p$

$$\sup_{x\in[a,b]} |(f-p)(x)| = \|f-p\|_\infty \le \epsilon$$

$$|E_k(f)| - |E_k(p)| \le |E_k(f-p)|$$

$$= \left| \int_a^b (f-p)(x)w(x)dx - \sum_{i=0}^{k} \lambda_{ik}(f-p)(x_{ik}) \right|$$

$$\le \epsilon \left[\int_a^b w(x)dx + \sum_{i=0}^{k} |\lambda_{ik}| \right]$$

$$\le \epsilon \left[M + \int_a^b w(x)\,dx \right]$$

$$|E_k(f)| \le \epsilon \left[M + \int_a^b w(x)\,dx \right] + \underbrace{|E_k(p)|}_{\to 0}$$

Euler-MacLaurin Formula:

Study of quad. error in comp. trap.

Trap. Rule:

$$\int_{x_0}^{x_0+h} f(x)\,dx \approx \frac{h}{2}\left(f(x_0)+f(x_0+h)\right)$$

Comp Trap.

$$\int_{x_0}^{x_k} f(x)\,dx \approx T_k(f) = h\left(\tfrac{1}{2}f(x_0)+f(x_1)+\cdots+f(x_{k-1})+\tfrac{1}{2}f(x_k)\right)$$

$$x_i = x_0 + ih \;;\; i=0,\dots,k$$

Develop the formula for error on $[0,1]$ interpolating by parts

$$\int_0^1 f(x)\,dx = \int_0^1 (x-1/2)'\, f(x)dx$$

$$= (x-\tfrac{1}{2})f(x)\Big|_0^1 - \int_0^1 (x-1/2)f'(x)dx$$

$$= \tfrac{1}{2}\left(f(0)+f(1)\right) - \int_0^1 (x-1/2)f'(x)\,dx$$

$$B_1(x) = x - 1/2 \qquad \int_0^1 B_1(x)\,dx = 0$$

$$B_1'(x) = 1$$

$$B_1(x) = \int_0^t dt + \overbrace{B_1(0)}^{-1/2 = b_1}$$

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}\left(f(0)+f(1)\right) - \underbrace{\int_0^1 B_1(x)f'(x)\,dx}_{\text{Error in trap.}}$$

$$f(0) = \int_0^1 f(x)\,dx - \tfrac{1}{2}\left(f(1)-f(0)\right) + \int_0^1 B_1(x)f'(x)\,dx$$

$$B_2(x) : \tfrac{1}{2}B_2'(x) = B_1(x) \;;\; \int_0^1 B_2(x)dx = 0$$

$$B_2(x) = 2\int_0^x B(t)\,dt + \underbrace{B_2(0)}_{b_2} \qquad b_2 = B_2(0) = B_1(0)$$

$$B_2(x) = x^2 - x + 1/6$$

$$\int_0^1 B_1(x)f'(x)dx = \int_0^1 \tfrac{1}{2}B_2'(x)f'(x)\,dx$$

$$= \tfrac{1}{2}B_2(x)f'(x)\Big|_0^1 - \int_0^1 \tfrac{1}{2}B_2(x)f''(x)\,dx$$

$$= \frac{b_2}{2}\left(f'(1)-f'(0)\right) - \tfrac{1}{2}\int_0^1 B_2(x)f''(x)dx$$

2-term Euler - Maclaurin

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left(f(0)+f(1)\right) - \frac{b_2}{2}\left(f'(1) - f'(0)\right) + \frac{1}{2}\int_0^1 B_2(x) f''(x)\,dx$$

$$\begin{cases} B_0(x) \equiv 1 \\ B_{n+1}'(x) = (n+1) B_n(x) \ ; \ n \ge 0 \\ \int_0^1 B_n(x)\,dx = 0 \end{cases}$$

$$B_{n+1}(x) = (n+1)\int_0^x B_n(t)\,dt + B_{n+1}(0)$$

$$b_{n+1} = -(n+1)\int_0^1 \int_0^x B_n(t)\,dt\,dx$$

$$b_{n+1} = B_{n+1}(0) = B_{n+1}(1)$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k \quad \} \text{Taylor formula}$$

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n$$

$$B_n(x+1) - B_n(x) = n x^{n-1} \ ; \ n \ge 2$$

$$B_n(1-x) = (-1)^n B_n(x)$$

$$\sum_{k=0}^{n} \binom{n}{k} b_k = 0 \ ; \ n \ge 2$$

$$b_{2j+1} = 0$$

$$B_n''(x) = n(n-1) B_{n-2}(x)$$

$$\vdots$$

$$B_n^{(k)}(x) = n(n-1)\cdots(n-k+1) B_{n-k}(x)$$

$$B_0(t) = 1$$

$$B_1(t) = t - 1/2$$

$$B_2(t) = t^2 - t + 1/6$$

$$B_3(t) = t^3 - 3/2\,t^2 + 1/2\,t$$

$$\int_0^1 f(x) = \frac{1}{2}\left(f(0)+f(1)\right) - \sum_{j=1}^{n} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(1) - f^{(2j-1)}(0)\right) + R_{2n}$$

$$R_{2n} = \frac{1}{(2n)!}\int_0^1 B_{2n+1}(x) f^{(2n+1)}(x)\,dx$$

$$T(h) = \tau_0 + \tau_1 h^\gamma + O(h^{2\gamma})$$

General Scheme for Rich extrapolation

computable

$\tau_0 = T(0)$ unknown

error

$$0 < b < 1$$

$$T(h) = \tau_0 + \tau_1 h^\gamma + O(h^{2\gamma})$$

$$T(bh) = \tau_0 + \tau_1 b^\gamma h^\gamma + O((bh)^{2\gamma})$$

$$= \tau_0 + \tau_1 b^\gamma h^\gamma + O(h^{2\gamma})$$

$$\tau_1 h^\gamma = \frac{T(h) - T(bh)}{1 - b^\gamma} + O(h^{2\gamma})$$

$$\tau_0 = T(h) + \frac{T(bh) - T(h)}{1 - b^\gamma} + O(h^{2\gamma})$$

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$$\int_0^1 f(x)\,dx = \frac{1}{2}\left(f(0) - f(1)\right) - \sum_{1}^{n} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(1) - f^{(2j-1)}(0)\right) + R_{2n}$$

$$R_{2n} = \frac{1}{(2n)!}\int_0^1 B_{2n}(x) f^{(2n)}(x)\,dx$$

$$\int_0^N f(x)\,dx = \frac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2}(f(N)) + R_{2n}$$

$$\phi(t) = f(a+th)$$

$$\int_a^b f(x)\,dx = h\int_0^N \phi(t)\,dt$$

$$\int_a^b f(x)\,dx = T_n(f) - h\sum \frac{b_{2j}}{(2j)!}\left(\phi^{(2j-1)}(N) - \phi^{(2j-1)}(0)\right)$$

$$- \frac{h}{(2n+1)!}\int_0^N B_{2n+1}(\{t\})\,\phi^{(2n+1)}(t)\,dt$$

$$- \frac{h^{2n+1}}{(2n+1)!}\int_a^b B_{2n+1}\left(\left\{\frac{x-a}{h}\right\}\right) f^{(2n+1)}(x)\,dx$$

$$\int_a^b f(x)\,dx = T_n(f) - \sum_{j=1}^{n} \frac{b_{2j}}{(2j)!} h^{2j}\left(f^{(2j-1)}(b) - f^{(2j-1)}(a)\right) + R_{2n+1}$$

$$R_{2n+1} = \frac{-h^{2n+1}}{(2n+1)!}\int_a^b B_{n+1}\left(\left\{\frac{x-a}{h}\right\}\right) f^{(2n+1)}(x)\,dx$$

If $a=0,\ b=N,\ f(x) = x^{m-1}$

$$\sum_{0\le k<N} f(k) = \int_0^N x^{m-1}\,dx + \sum_{k=1}^{n} \frac{b_k}{k!}\left(f^{(k-1)}(N) - f^{(k-1)}(0)\right)$$

$$= \frac{1}{n}\sum_{k=0}^{m} \binom{m}{k} b_k N^{m-k}$$

$m=3$

$$\sum_{0\le k<N} k^2 = \frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6}$$

[Early Fourier Analysis - Montgomery]

Approximation

Let V be a normed vector space over $\mathbb{R}$ or over $\mathbb{C}$.

Let $W \subseteq V$ be a finite dim. vector subspace.

$f \in V$. Find $h^* \in W$ such that

$$\|f - h^*\| \leq \|f - h\| \quad \forall h \in W$$

Ex:

$V = C[a,b]$, $\|f\| = \sup_{a \leq x \leq b} |f(x)|$

$$\underbrace{W = \mathbb{P}_n}_{\text{uniform approximation}}$$

Ex:

$$\underbrace{V = L^2(a,b), \quad \|f\| = \langle f, f\rangle^{1/2} = \left(\int_a^b |f(x)|^2\, dx\right)^{1/2}}_{\text{least square approximation}}$$

$V = L_p^2[(0,a)]$, where $L_p^2[(0,a)] = \{ f : (0,a) \to \mathbb{R} \}$

$f(x) = f(x+a)$ 'a-periodic'

$\|f\| = \int_0^a |f(x)|^2\, dx$

$W = \{\text{trig. poly.}\}$

$$p(x) = \sum_{n=-N}^{N} c_n e^{i 2\pi n x/a}$$

Orthogonal Projection

$\langle u, w\rangle = 0$ orth. vectors

Lem: $v \in V$, $w \neq 0$. $\exists!$ vect.

$v_\parallel$ & $v_\perp$ $\ni$

$v = v_\parallel + v_\perp$

$v_\parallel = cw$ for some $c \in \mathbb{C}$

$\langle v_\perp, v_\parallel\rangle = 0$

Unique:

$\langle v, w\rangle = \langle v_\parallel, w\rangle + \langle v_\perp, w\rangle$

$= \langle cw, w\rangle + \langle v_\perp, w\rangle$

$= c\langle w, w\rangle$

$= c\|w\|^2$ (why?)

$c = \dfrac{\langle v, w\rangle}{\|w\|^2}$

Exist: $\dfrac{\langle v, w\rangle}{\|w\|^2}$; $v_\parallel = cw$, $v_\perp = v - cw$

$v_\perp + v_\parallel = v$

$\langle v_\perp, w\rangle = \langle v, w\rangle - c\langle w, w\rangle$

$= \langle v, w\rangle - \dfrac{\langle v, w\rangle}{\|w\|^2}\langle w, w\rangle$

$= 0$

Cauchy-Sch.

$|\langle v, w\rangle| \leq \|v\| + \|w\|$

$w = 0$ eq.

$w \neq 0 \rightarrow v = v_\parallel + v_\perp$; $v_\parallel = cw$

$c = \dfrac{\langle v, w\rangle}{\|w\|}$

$\|v\|^2 = \|v_\parallel\|^2 + \|v_\perp\|^2$

$\geq \|v_\parallel\|^2 = |c|^2 \|w\|^2$

$= \frac{|\langle v,w\rangle|^2}{\|w\|^2}$

$\|v\|^2 \|w\|^2 \geq |\langle v,w\rangle|^2$

Lem: $\|u+v\| \leq \|u\| + \|v\|$

Thm: $W \subset V$, $(V, \|\cdot\|)$, $\dim W < \infty$
for any $f \in V$, $\exists$ an 'optimal' element h^*
$\|f-h^*\| \leq \|f-h\| \quad \forall h \in W$

Pf: $\|f-h^*\| \leq \|f\| \quad (h=0)$

$K = \{h \in W : \|f-h\| \leq \|f\|\}$

K closed in f.d. $\rightarrow$ K compact

Want to obtain min. $\underbrace{\|f-h\|}_{\text{cont. in } h}$; $h \in K$

$\big| \|f-(h+g)\| - \|f-h\| \big| \leq \|g\|$

$\|g\| = \|f-(h+g) - (f-h)\| \geq \|f-(h+g)\| - \|f-h\|$

$\|g\| = \|f-h - (f-(h+g))\|$

$\|f-h\|$ attains min on K

$\|f-h^*\|$ min.

h^* may not be unique.

10/31/2017

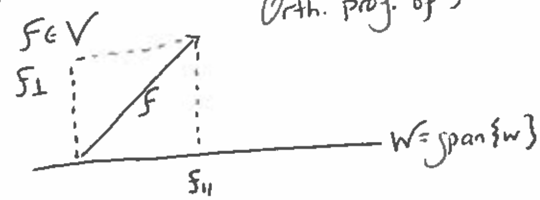
$W \subset V$, $f \in V$. Find $h^* \in W \ni \|f-h^*\| \leq \|f-h\|$ for all $h \in W$.

$\underbrace{E_W}_{}(f) := \|f-h^*\|$

Error approx f by fun. in W.

Know W fin dim. $\rightarrow \exists h^*$

$\text{span}\{w\} = W \subset V$ Orth. proj. of f onto W



$f = f_\parallel + f_\perp$; $f_\parallel = cw$

$c = \frac{\langle f,w\rangle}{\|w\|^2}$

$f_\parallel$ will be our approx.

$\langle \underbrace{f - f_\parallel}_{= f_\perp}, w \rangle = 0 \quad \forall w \in W$

Then $\langle f - f_\parallel, f_\parallel - w\rangle = 0$

By Pythag.

$\|f - f_\parallel\|^2 + \|f_\parallel - w\|^2 = \|f-w\|^2$

Generally, $\|f - f_\parallel\| \leq \|f-w\|$

Let $S \subset W$ be set of best approx to $f \in V$. Then S is convex: $h_1, h_2 \in S$ distinct

$\|f-h_1\| = E_W(f) = \|f-h_2\|$

Consider $\alpha h_1 + \beta h_2$; $\alpha\beta \neq 0$; $\alpha + \beta = 1$

$\|f - (\alpha h_1 + \beta h_2)\| = \|\alpha(f-h_2) + \beta(f-h_2)\|$

$* \; f = (\alpha+\beta) f$

$\leq \alpha \|f-h_1\| + \beta\|f-h_2\|$

$= (\alpha+\beta) E_W(f) = E_W(f)$

We say that normed space V is strictly convex if $\|x\| \le r$, $x \ne y \Rightarrow \|x+y\| < 2r$

OR

$\|f\| = \|g\|$, $f \ne g \Rightarrow \|f+g\| < 2$

Approx in strictly convex V gives a unique solution.

$g_1 \ne g_2$ $\quad \|f-g_1\| = \|f-g_2\| = E_W(f) = \inf_{h \in W} \|f-h\|$

$\|f - \frac{1}{2}(g_1+g_2)\| = \|\frac{1}{2}(f-g_1) + \frac{1}{2}(f-g_2)\|$

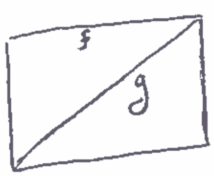
$< E_W(f)$

Contradicts $g_1, g_2 \in E_W(f)$

$V = C[0,1]$, $\|f\| = \sup_{0 \le x \le 1} |f(x)|$

$(V, \|\cdot\|)$ is not strictly convex.

$f(t) = 1$; $g(t) = t$



$\|f\| = \|g\| = 1$ as sup 1

$(f+g)(t) = t+1$

$\sup_x |t+1| = 2$

$W \subset V$. V inner product space

Thm: Given $f \in V$ there exists a unique $f^* \in W$.

$\|f - f^*\| \le \inf_{w \in W} \|f - w\|$

$E_W(f) = \inf_{w \in W} \|f - w\| = d > 0$

$D_n = \{w \in W : \|f - w\| \le d + 1/n\}$

When $n \to \infty$, diameter $D_n \to 0$

$w_1, w_2 \in D_n$

$\|w_1 + w_2 - 2f\|^2 + \|w_1 - w_2\|^2$

$= 2(\|w_1 - f\|^2 + \|w_2 - f\|^2)$

$\le 4(d^2 + \frac{2d}{n} + \frac{1}{n^2})$

$= 4d^2 + \frac{8d}{n} + \frac{4}{n^2}$

$\|\frac{w_1+w_2}{2} - f\| \ge d$

$\|w_1 - w_2\|^2 \le \frac{8d}{n} + \frac{4}{n^2}$

Choose $\tilde{w}_n \in D_n$; $n \ge 1$ then

$\lim \tilde{w}_n = f^*$

Orthogonal Proj. Thm: V inner product $W \subset V$. W complete vector subspace $f \in V \Rightarrow f^* \in W$ is the best approx of f iff $\langle f - f^*, w \rangle = 0$ $\forall w \in W$

Pf:

Suff: $\langle f - f^*, \tilde{w} \rangle = 0$ $\forall \tilde{w} \in W$

Set $\tilde{w} = \underbrace{f^* - w}_{\in W}$ for $w \in W$. Gives

$\langle f - f^*, f^* - w \rangle = 0$

$\|f - f^*\|^2 + \|f^* - w\|^2 = \|f - w\|^2$

$\|f - f^*\| \le \|f - w\|$

$\|f - f^*\| \le \|f - w\|$ $\forall w \in W$

Nec: Show $\langle f - f^*, \tilde{w} \rangle = 0$ $\forall \tilde{w}$

Let $w_\alpha = f^* + \alpha(w - f^*)$

$\alpha \in \mathbb{C}$, $w \in W$ arbitrary

$$\|f - f^*\|^2 \le \|f - w_\alpha\|^2$$
$$= \|f - f^* - \alpha(w - f^*)\|^2$$
$$= \|f - f^*\|^2 + \|w - f^*\|^2 - \alpha\langle w - f^*, f - f^*\rangle - \bar{\alpha}\overline{\langle w - f^*, f - f^*\rangle}$$

But then
$$|\alpha|^2 \|w - f^*\|^2 \ge \alpha\langle w - f^*, f - f^*\rangle + \bar{\alpha}\overline{\langle w - f^*, f - f^*\rangle}$$

Let $\alpha = |\alpha| e^{i\theta}$. Divide by $|\alpha|$,

Charge $\alpha \to 0$. This gives

$0 \ge e^{i\theta} z + \bar{e}^{i\theta} \bar{z}$; $z = \langle w - f^*, f - f^*\rangle$

Set $\theta = 0$, $\operatorname{Re} z \le 0$

If $\theta = \pi$ $\operatorname{Re} z \ge 0$ $\Big\}$ $\operatorname{Re} z = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\operatorname{Im} z = 0$

If $f^* = \sum_{j=1}^{n} c_j \phi_j$; $f \in V$

$$\langle f - f^*, \phi_j\rangle = 0$$

$$\left\langle f - \sum_1^n c_j \phi_j, \phi_i \right\rangle$$

$$\sum c_j \langle \phi_j, \phi_i\rangle = \langle f_i, \phi_i\rangle$$

$$\underbrace{\begin{pmatrix} \langle \phi_1, \phi_1\rangle & \cdots & \langle \phi_n, \phi_1\rangle \\ \vdots & & \vdots \\ \langle \phi_1, \phi_n\rangle & \cdots & \langle \phi_n, \phi_n\rangle \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{c} = \underbrace{\begin{pmatrix} \langle f, \phi_1\rangle \\ \langle f, \phi_2\rangle \\ \vdots \\ \langle f, \phi_n\rangle \end{pmatrix}}_{F}$$

G ← Gramm matrix

$$Gc = F$$

$$G_{i,j} = \langle \phi_i, \phi_j\rangle$$

(j: column, i: row)

If $\{\phi\}$ orthogonal

$$c_i = \frac{\langle f, \phi_i\rangle}{\langle \phi_i, \phi_i\rangle}$$

$$f^* = \sum \frac{\langle f, \phi_i\rangle}{\langle \phi_i, \phi_i\rangle} \phi_j$$

G symmetric or hermitian

symmetric positive definite:

$\langle Gx, x\rangle > 0$ unless $x = 0$

11/02/2017

Thm (Best L^2 approx)

V inner product space

$W \subset V$ complete subspace

$f \in V$, $f^* \in W$ is the B.A. element in W ($\leftarrow$ best approx)

$$\|f - f^*\| \le \|f - w\| \quad \forall w \in W$$
$$\Updownarrow$$
$$\langle f - f^*, w\rangle = 0 \quad \forall w \in W$$

If $\dim W = n$ then

$$f^* = \sum_{i=1}^{n} \frac{\langle f, \phi_i\rangle}{\langle \phi_i, \phi_i\rangle} \phi_i$$

$\{\phi_i\}_{i=1}^{n}$ orth. system in W (basis)

If $\dim W = \infty$, then need $\{\phi_i\}$ orthogonal set which is dense in W.
$\{\phi_i\}$ then a Hilbert basis in W.

$\{\cos nx\}$ on $\mathcal{L}^2(-\pi,\pi)$

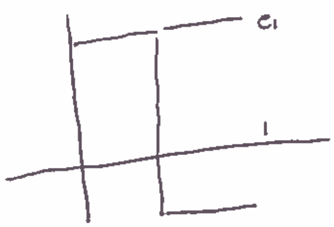
$V = \mathcal{L}^2[(0,1)]$

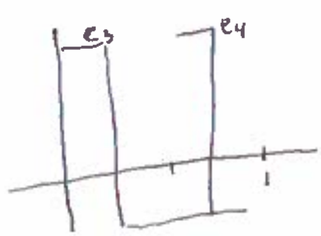
$f \in V \qquad f(x) = \begin{cases} x & 0 < x < 1/2 \\ 1-x & \frac{1}{2} \le x < 1 \end{cases}$

$\phi(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$\psi(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

$e_1(x) = \phi(x)$
$e_2(x) = \psi(x)$
$e_3(x) = \psi(2x)$
$e_4(x) = \psi(2x-1)$





$\{e_1, \dots, e_4\}$ orth. system in $\mathcal{L}^2(0,1)$

$\langle e_1, e_1 \rangle = 1$
$\langle e_2, e_2 \rangle = 1$
$\langle e_3, e_3 \rangle = 1/2$
$\langle e_4, e_4 \rangle = 1/2$

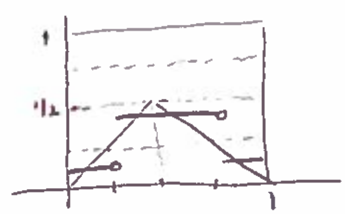
$$f^* = \sum_{i=1}^{4} \frac{\langle f, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

$\langle f, e_1 \rangle = 1/2$
$\langle f, e_2 \rangle = 0$
$\langle f, e_3 \rangle = -1/16$
$\langle f, e_4 \rangle = 1/16$

$$f^* = \frac{1}{4} e_1 - \frac{1}{8} e_3 + \frac{1}{8} e_4$$

$$= \frac{2e_1 - e_3 + e_4}{8}$$

$$f^* = \begin{cases} \frac{1}{8} & 0 - 1/4 \\ 3/8 & 1/4 \le - 1/2 \\ 3/8 & \frac{1}{2} \le x < 3/4 \\ 1/8 & 3/4 \le - < 1 \\ 0 & \text{otherwise} \end{cases}$$



$W = \text{span}\{e_1, \dots, e_4\}$

Finite dim. spaces complete

Example above Haar functions

Linear Least Square Fitting

A set of data points

$(t_1, b_1), \ldots, (t_m, b_m)$; $t_i \in \mathbb{R}^K$, $b_i \in \mathbb{R}$

We want to approx f by a lin. comb. of functions $\phi_1, \ldots, \phi_n$ (model functions)

$$\sum_{i=1}^{m} c_i \phi_i \quad ; \text{ model of } f.$$

$$f^* = \sum^{m} c_i \phi_i$$

where $\phi_i = \begin{pmatrix} \phi_i(t_1) \\ \vdots \\ \phi_i(t_m) \end{pmatrix}$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Want

$$\langle f - f^*, \phi_i \rangle = 0$$

$$\langle b, \phi_i \rangle = \langle f^*, \phi_i \rangle$$

$$\sum_{k=1}^{m} \left(\sum_{j=1}^{n} c_j \phi_j(t_k) \phi_i(t_k) \right)$$

$$\sum b_k \phi_i(t_k)$$

$$A = \begin{pmatrix} \phi_1(t_1) & \cdots & \phi_n(t_1) \\ \vdots & & \vdots \\ \phi_1(t_m) & \cdots & \phi_n(t_m) \end{pmatrix}$$

$$A^T A c = A^T b$$

A sym. pos. def. $\rightsquigarrow$ inv.

By solving for c

$$r = b - Ac$$

When c satisfies normal system.

Obtain c^* ∋

$$\| r(c^*) \|_2 \leq \| r(c) \|_2$$

11/07/2017

$f \in V$; $W \subset V$ closed subspace. (V inner product space)

Want $f^* \in W$ ∋

$$\| f - f^* \| \leq \| f - w \| \quad \forall w \in W$$

$$\langle f - f^*, w \rangle = 0 \quad \forall w \in W$$

Thm: $\{\phi_i\}_{i=1}^{\infty}$ countable orthonormal family in V ∋ $S_N f := \sum_{n=1}^{N} \langle \phi_n, f \rangle \phi_n$

The following are equiv:

1) Fin. lin. comb. $\sum a_i \phi_i$ are dense in V

2. For any $f \in V$

$$\lim_{N \to \infty} \| f - S_N f \| = 0$$

3. For any $f \in V$

$$\| f \|^2 = \sum_{n=1}^{\infty} | \langle f, \phi_i \rangle |^2$$

$\geq$ Bessel Inequality

4. $\forall f \in V$ ∋ $\langle f, \phi_i \rangle = 0 \ \forall i$, $f \equiv 0$

Pf:

$1 \to 2$: $\| f - \sum^{N} a_i \phi_i \| \leq \epsilon$

$\leq$

$\| f - S_{N_0}(f) \|$

$2 \to 3$: $\| f \|^2 = \| f - S_N f + S_N f \|^2$

$= \langle f - S_N f, f - S_N f \rangle$

$= \| S_N(f) \|^2 + \| f - S_N f \|^2 + 2 \langle S_N f, f - S_N f \rangle$

$\langle S_N f, f - S_N f \rangle = 0$

$\| f - \sum \langle f, \phi_k \rangle \phi_k \|^2 \to 0$ as $N \to \infty$

$= \|f\|^2 - \sum^{N} |\langle f, \phi_k \rangle|^2$

$3 \to 4$: Trivial

$4 \to 1$: $\tilde{V}$ set of all fin. lin. comb $\{\phi_i\}$

WTS $\tilde{V}$ dense in V.

$\langle f, \phi_i \rangle = 0 \rightsquigarrow f = 0 \overset{\text{WTS}}{\Rightarrow} \tilde{V}$ dense

Suppose $\bar{\tilde{V}} \neq V$. $\exists\ \psi \in V \setminus \bar{\tilde{V}}$

Uniform Approx of Cont. Functions

$f \in C[a,b]$. Find $p \in \mathbb{P}_n$.

$E_{\mathbb{P}_n}(f) = \| f - p^* \| \le \| f - q \| \quad \forall q \in \mathbb{P}_n$

$\|f\| = \sup_{a \le x \le b} |f(x)|$

Characterization of $p^* \in \mathbb{P}_n$ given by Chebyshev Thm

Def: Function e equioscillates on $x_0 < x_1 < \dots < x_k$ iff

a) $|e(x_i)| = \|e\|$

b) $e(x_i) = -e(x_{i+1})$; $i = 0, \dots, k-1$

Points $x_0, \dots, x_k$ an alternating set for e (alternans)

Thm (Chebyshev) $f \in C[a,b]$. $p \in \mathbb{P}_n$ is B.A.P. iff $f - p$ equioscillates at $n+2$ points in $[a,b]$.

$\sup \max(f,g) = \max(\sup f, \sup g)$

Lem 1: $\exists$ at least 2 distinct points $x_1, x_2 \in [a,b] \ni$

$|f(x_0) - p(x_0)|$

$= |f(x_2) - p(x_1)| = E$

$f(x_0) - p(x_1) = -(f(x_1) - p(x_1))$

Pf: $e(x) = (f - p)(x)$ cont.

$|e|$ cont. takes value ~~[illegible]~~

$y = E$ or $y = -E$. WTS both

If not $e(x) > -E \quad \forall x \in [a,b]$

$\min_{x \in [a,b]} e(x) = m > -E$

$c = \frac{E+m}{2} > 0$

$q = p + c$; $f - q = f - p - c = e - c$

$-(E-c) = m - c \le e(x) - c \le E - c$

$\|f - q\| = \sup_{x \in [a,b]} |e(x) - c| \le E - c < E$

$\times$

$f \in C[a,b]$ Find $p_0 \in \mathbb{P}_0$ constant

$\ni \|f - p_0\| \le \|f - c\|$; $c \in \mathbb{R}$

$M = \sup_x f(x) = f(x_2)$

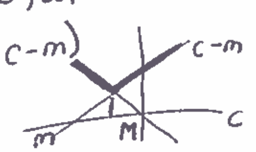
$E(c) = \sup_{x \in [a,b]} |f(x) - c|$

$|z| = \max(z, -z)$

$|f(x) - c| = \max(f(x) - c,\ c - f(x))$

$\sup |f(x) - c| = \max(\sup f - c, \sup c - f)$

$= \max(M - c, c - m)$



$c = p_0 = \frac{m+M}{2}$

$E(p_0) = \|f - p_0\|$ func.

$f - p_0$ equioscillates $x_1 < x_2$

$\frac{M-m}{2} = |f(x_i) - \frac{m+M}{2}| = \|f - \frac{m+M}{2}\|$; $f(x_1) = \frac{m-M}{2}$

$f(x_2) = \frac{M-m}{2}$

11|09|2017

Thm: (Int. Value Theorem) f cont. function on $[a,b]$ with $f(a) < f(b)$, then $\forall y$ with $f(a) < y < f(b)$, $\exists\ y_0 \in [a,b]$ with $f(x_0) = y$.

Pf: (Divide & Conquer) Consider sequence of int. $[a_1,b_1] := [a,b]$, $[a_2,b_2]$, ...

with $f(a_k) < y < f(b_k)$

$[a_2,b_2]$ 1/2 of $[a_1,b_1]$

$f(\frac{a+b}{2}) < y \longrightarrow [a_2,b_2] := [\frac{a+b}{2}, b]$

otherwise $[a, \frac{a+b}{2}]$

$$b_k - a_k = 2^{1-k}(b_1 - a_1)$$

$y \in \bigcap [a_k, b_k]$. Seq. nested closed intv. with $\underbrace{\text{diam}\,[a_k,b_k]}_{b_k - a_k} \to 0$. $a_k, b_k \to x_0$ limit

and $f(x_0) = y$. ∎

Thm: Let $f: [a,b] \to \mathbb{R}$ cont. $\ni$ $f(a)f(b) \leq 0 \longrightarrow \exists\ \xi \in [a,b]$ $\ni$ $f(\xi) = 0$

Pf: $f(a) = 0$ or $f(b) = 0$. Done. Then $f(a) > 0$ & $f(b) < 0$ or $f(a) < 0$, $f(b) > 0$

Now use Int. Value Thm. ∎

Thm (Brouwer Fixed Pt. Thm)

$g: [a,b] \to [a,b]$ cont.

then $\exists\ \xi \in [a,b]$ $\ni$

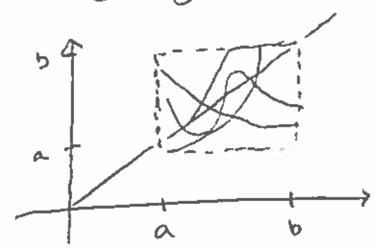
$g(\xi) = \xi$. ← Fixed point of g.

Pf: $f(x) := x - g(x)$

$f(a) = a - g(a) \leq 0$

$f(b) = b - g(b) \geq 0$

Opposite signs. $f(a)f(b) \leq 0$. ∎



Bisection

$[a,b] = [a_0, b_0]$

✱ $x_0 := a + \frac{1}{2}(b_0 - a_0)$

[num. better than $x_0 = \frac{b_0 + a_0}{2}$]

$f(f(x_0)) = 0$ if not

$$(a_1,b_1) = \begin{cases} (x_0, b_0), & f(x_0)f(a_0) > 0 \\ (a_0, x_0), & f(x_0)f(a_0) < 0 \end{cases}$$

Interpret bisection in interpolation terms:

$L(x)$ affine function $\ni$ $L(a_k) = -1$ $L(b_k) = 1$

↑ just need opp. signs

$$L(x) = -1 + (x - a_k)\frac{2}{b_k - a_k}$$

Solve $L(x) = 0$ to find x_k (middle of int.)

$$x_k = \frac{1}{2}(a_k + b_k)$$

False position

$L(a_k) = f(a_k)$

$L(b_k) = f(b_k)$

$L(x) = f(a_k) + (x - a_k)\dfrac{f(b_k) - f(a_k)}{b_k - a_k}$

$L(x) = 0$

$x_k = a_k - \dfrac{f(a_k)}{f(a_k) - f(b_k)}(b_k - a_k)$

Simple Iteration of $f(x) = 0$ (finding solution)

Converting into fixed pt. eqn

$f(x) = 0$ $\qquad \overline{|\, g(x) = x \,|}$

$\alpha f(x) = 0$ $\qquad \underbrace{g(x) = x + \alpha f(x)}_{\text{fixed point} \to f(x) = 0}$

x_0 arb.

$g(x_k) = x_{k+1}$

Sufficient condition for conv. of simple interation that $g: [a,b] \to [a,b]$

a $\underbrace{\text{contraction mapping}}$

$\exists\ 0 < L < 1 \ni |g(x) - g(y)| < L|x-y|$

ie Lipschitz function with constant < 1.

$|g(x)| - |g(y)| < |g(x) - g(y)| < L|x-y|$

$|g(x)| < |g(y)| + L(x-y)$

"cone bounding g"

Thm (Contraction Mapping Thm)

$g: [a,b] \to [a,b]$ cont.

g assumed to be a contraction

Then g has unique fixed point in $[a,b]$

Seq $\{x_k\}$ conv. to fixed pt for any starting point $x_0 \in [a,b]$.

Pf: ξ exists by Brouwer's. If 2

$|\xi - \eta| = |g(\xi) - g(\eta)| < L|\xi - \eta|$

$x_k \underset{k}{\to} \xi$

$|\xi - x_k| = |g(\xi) - g(x_{k-1})|$
$< L|\xi - x_{k-1}|$

$\forall k \geq 1$ Generally,

$0 \leq |\xi - x_k| \leq L^k |x_0 - \xi|$

Squeeze & done. ☒

If $|g'(x)| < L$ in (a,b) then

g is contraction:

$\dfrac{g(x) - g(y)}{x - y} = g'(\alpha)$

& shown.

11/14/2017

Thm: (Local Contraction Thm)

$g: [a,b] \to [a,b]$ cont.
$\xi \in [a,b]$; $\xi = g(\xi)$
($\exists$ by Brauwer). Assume g' cont. in some neigh of ξ and $g'(\xi) < 1$
The seq $\{x_k\}$; $g(x_k) = x_{k+1}$ conv. to ξ provided x_0 'close' to ξ

Pf: g' cont. $[\xi - h, \xi + h]$
$|g'(\xi)| < 1$. Let $I_\delta = [\xi - \delta, \xi + \delta]$ such that $|g'| \le L < 1$ in I_δ.
Take $L = \frac{1}{2}(1 + |g'(\xi)|) < 1$ *

Choose $\delta \le h \ni$
$$|g'(y) - g'(\xi)| \le \frac{1}{2}(1 - |g'(\xi)|)$$
for $x \in I_\delta$. g' cont @ ξ. For all $x \in I_\delta$,
$$|g'(x)| \le |g'(x) - g'(\xi)| + |g'(\xi)| \le \frac{1}{2}(1 - |g'(\xi)|) + |g'(\xi)| = \frac{1}{2}(1 + |g'(\xi)|) < 1$$

Suppose $x_k \in I_\delta$.
$$x_{k+1} - \xi = g(x_k) - g(\xi) = (x_k - \xi)\, g'(\underbrace{\eta_k}_{\text{between } x_k,\ \xi})$$

$|x_{k+1} - \xi| \le L|x_k - \xi| \le L\delta < \delta$
$x_{k+1} \in I_\delta$, if $x_0 \in I_\delta \to x_k \in I_\delta$

$|x_k - \xi| \le L^k |x_0 - \xi|$ ∎

$f: \mathbb{R} \to \mathbb{R}$. f cont. diff
$f(\alpha) = 0$. $f(x) = 0$? Suppose $f'(\alpha) \ne 0$.

Construct $H(x)$, Hermite int. poly of f. If $x_n \approx \alpha$.

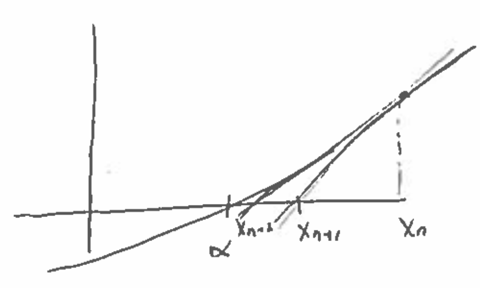
$H(x_n) = f(x_n)$
$H'(f_n) = f'(x_n)$
$$H(x) = \underbrace{f(x_n) + f'(x_n)(x - x_n)}_{\text{lin. of } f}$$

$H(x) = 0$
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ Newton's

$$g(x) := x - \frac{f(x)}{f'(x)}$$

$x_{n+1} = g(x_n)$ finds fixed point of g. $g(\alpha) = \alpha$.
$$g'(x) = 1 - \frac{f'^2 - f f''}{f'^2}$$
$$g'(\alpha) = 1 - \frac{f'(\alpha)^2 - 0}{f'(\alpha)^2} = 0$$

In neigh α, $|g'(x)| \le L < 1$
Newton's converges.



$f: \mathbb{R}^n \to \mathbb{R}^n$

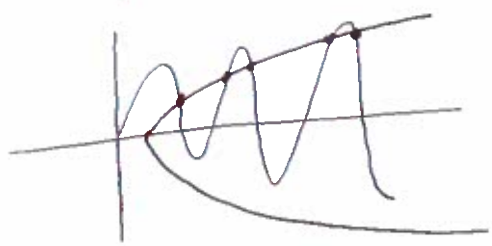
$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$f(x) = 0$

$$\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} = \vec{0}$$

$f_1(x_1, x_2) = \frac{1}{2} x_1 \sin\left(\frac{1}{2}\pi x_1\right) - x_2 = 0$

$f_2(x_1, x_2) = x_2^2 - x_1 + 1$



$$0 = f(\alpha) \approx \underbrace{f(x_n) + f'(x_n)(\alpha - x_n)}_{H(\alpha)}$$

$F(x_n) + F'(x_n)(x - x_n) = 0$

$F'(x_n)(x - x_n) = -F(x_n)$

$$F'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \quad \text{Jacobian}$$

$x_{n+1} = x_n - f'^{-1} f$

Secant Method

$f(x) = 0 \qquad f(x) \approx L(x)$

$$L(x) = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n) = 0$$

$L(x_{n-1}) = f(x_{n-1})$

$L(x_n) = f(x_n)$

$$(x - x_n) \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = -f(x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{\underbrace{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}_{\approx f'(x_n)}}$$

$$x_{n+1} = x_n - \frac{(x - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

$f = x^2 - 6 \qquad (\sqrt{6})^2 - 6 = 0$

$$x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n}$$

$$= x_n - \frac{1}{2}x_n + \frac{6}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{6}{x_n}\right)$$

Let 'a' be a first guess at $a > \sqrt{b}$

$$\frac{b}{a} < \frac{b}{\sqrt{b}} = \sqrt{b}$$

$$\tilde{x} = \frac{1}{2}\left(a + \frac{b}{a}\right)$$

$\uparrow \; > \sqrt{b} \qquad \uparrow \; < \sqrt{b}$

$\underbrace{\qquad\qquad}_{\text{avg.}}$

$b < x_1^2 < x_0^2$

$\underbrace{\qquad}_{\text{Monotone Conv. Thm}}$

11|16|2017

Kantorovich Thm, Global Conv. of Newton's Method:

$$x_{n+1} = x_n + \underbrace{[DF(x_n)]}_{\text{Jac. matrix}}^{-1} F(x_n)$$

where $F: \mathbb{R}^n \to \mathbb{R}^n$; $n = 0, 1, 2, \dots$

Thm: $F: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be diff mapping. $x_0 \in U$ & $DF(x_0)$ inv. Define $h_0 = -[DF(x_0)]^{-1} F(x_0)$.

(1) Define
$$\begin{cases} h_0 = -[DF(x_0)]^{-1} F(x_0) \\ x_1 = x_0 + h_0 \\ U_1 = B_{|h_0|}(x_1) \end{cases}$$

If $\bar{U}_1 \subset U$ & $DF(x)$ satisfy the Lipschitz cond.

(2) $$|DF(u_1) - DF(u_2)| \le M |u_1 - u_2|$$

and if the inequality

(3) $$|F(x_0)| \, |[DF(x_0)]^{-1}|^2 M \le 1/2$$

is satisfied, the equation $F(x) = 0$ has a unique solution $\bar{u}_1$ & Newton's Method e x_0 converges to this solution.

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$; $|u| = \sqrt{u_1^2 + \dots + u_n^2}$

$M = [m_{ij}]$; $|M| = \sqrt{m_{ij}^2}$

Units:

u unit u

r unit $F(u)$

r unit $|F(x_0)|$

r/u unit $DF(x_0)$

$\underbrace{u^2/r^2}_{u^2/r^2}$ unit $([DF(x_0)]^{-1})^2$

Then M unit r/u^2

———— x ————

Finding Lipschitz Constant M

Ex: $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{pmatrix} x_1 - x_2^2 \\ x_1^2 + x_2 \end{pmatrix}$$

$$DF = \begin{pmatrix} 1 & -2x_2 \\ 2x_1 & 1 \end{pmatrix}$$

$$DF(x) - DF(y) = \begin{pmatrix} 0 & -2(x_2 - y_2) \\ 2(x_1 - y_1) & 0 \end{pmatrix}$$

$$|DFx - DFy| = \sqrt{4(x_1 - y_1)^2 + 4(x_2 - y_2)^2}$$

$$2\sqrt{(x_1 - y_1)^2 + (x_2 - y_1)^2}$$

$$2|x - y|$$

$$|DFx - DFy| \le \underbrace{\boxed{2}}_{M}|x - y|$$

Ex:
$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^5 - y^2 + xy - a \\ y^4 + x^2 y - b \end{pmatrix}$$

$u_1 = x^2$ $\quad u_2 = y^2$ $\quad u_3 = x^4 = u_1^2$

$$\tilde{F}\left(\begin{pmatrix} x \\ y \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}\right) = \begin{pmatrix} xu_3 - u_2 + xy - a \\ u_2^2 + u_1 y - b \\ u_1 - x^2 \\ u_2 - y^2 \\ u_3 - u_1^2 \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D\vec{F} = \begin{pmatrix} u_3+y & x & 0 & -1 & x \\ 0 & u_1 & y & 2y_2 & 0 \\ -2x & 0 & 1 & 0 & 0 \\ 0 & -2y & 0 & 1 & 0 \\ 0 & 0 & -2y_1 & 0 & 1 \end{pmatrix}$$

$D\vec{F}p - D\vec{F}p' =$

$$\begin{pmatrix} u_3-u_3'+y-y' & x-x' & 0 & 0 & x-x' \\ 0 & u_1-u_1' & y-y' & 2(u_2-u_2') & 0 \\ -2(x-x') & 0 & 0 & 0 & 0 \\ 0 & 0 & -2(u_1-u_1') & 0 & 0 \\ 0 & 0 & -2(y-y') & 0 & 0 \end{pmatrix}$$

$(a+b)^2 \le 2(a^2+b^2)$

as $(a-b)^2 \ge 0$

$|D\vec{F}p - D\vec{F}p'| \le$

$$\left(6|x-x'|^2 + 7|y-y'|^2 + 5|u_1-u_1'|^2 + 4|u_2-u_2'|^2 + 2(u_3-u_3')^2\right)^{1/2}$$

$$\le \sqrt{7}\left|\begin{pmatrix} x-x' \\ y-y' \\ \vdots \end{pmatrix}\right| = \sqrt{7}\,|p-p'|$$

Want to prove 4 statements

1) $[DF]$ @ x_1 inv. so $h_1 = -[DF x_1]^{-1} f(x_1)$ well def

2) $|h_1| \le |h_0|/2$

3) $|f(x_1)|\,|[DF x_1]^{-1}|^2 \le |f x_0|\,|[DF(x_0)]^{-1}|^2$

4) $|f(x_1)| \le \frac{M}{2}|h_0|^2$

If 1, 2, 3 hold $\rightarrow$ we can define h_i, x_i, U_i given by

$x_{i+1} = x_i + h_i$

$U_{i+1} = \{x : |x - x_{i+1}| \le |h_i|\}$

so that each step i, hypotheses of thm hold.

From (2), $\sum_i |h_i|$ conv.

(Compare with geo.)

Then $\{x_i\}$ conv. say to α.

From (4)

$$|f(x_i)| \le \frac{M}{2}|h_{i-1}|^2 \le \underbrace{\frac{M}{2^i}|h_0|^2}_{\to 0 \text{ as } i \to \infty}$$

By cont., $|f(\alpha)| = 0$

11/28/2017

Kanturovich Thm on Global Conv. of Newton's Method

$f: \mathbb{R}^n \to \mathbb{R}^n$

$x_{n+1} = x_n - [Df(x_n)]^{-1} f(x_n)$

Thm: $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ diff mapping. $x_0 \in U$ & $[Df(x_0)]$ inv.

Define $h_0 = -[Df(x_0)]^{-1} f(x_0)$

$x_1 = x_0 + h_0$

$U_1 = B_{|h_0|}(x_1)$

If $\bar{U}_1 \subset U$ if $[Df(x)]$ satisfies Lipschitz & $|[Df(u_1)] - [Df(u_2)]| \le M|u_1 - u_2|$

$u_1, u_2 \in \bar{U}_1$ & if

$|F(x_0)| \cdot |[Df(x_0)]^{-1}|^2 M \le 1/2$

is satisfied then $f(x) = 0$ has unique solution in $\bar{U}_1$ & Newton's method with starting point x_0 converges to this solution.

Ex:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} x^2 - y - 2 \\ y^2 - x - 6 \end{pmatrix}$$

Find Lipschitz constant

$$Df\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 2x & -1 \\ -1 & 2y \end{pmatrix}$$

$$\left[Df\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)\right] - \left[Df\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)\right]$$

$$\begin{pmatrix} 2(x_1 - x_2) & 0 \\ 0 & 2(y_1 - y_2) \end{pmatrix}$$

So in norm....

$$\sqrt{4(x_1 - x_2)^2 + 4(y_1 - y_2)^2}$$

$$= \underbrace{2}_{M} \left| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right|$$

$x_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $Df(x_0) = \begin{pmatrix} 4 & -1 \\ -1 & 6 \end{pmatrix}$

$[Df(x_0)]^{-1} = \frac{1}{23}\begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$

$f(x_0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$h_0 = \frac{1}{23}\begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{23}\begin{pmatrix} -5 \\ 3 \end{pmatrix}$

$U_1 = B_{|h_0|}(x_1) =$; $x_1 = x_0 + h_0$

Kant. assumption?

$\sqrt{2} \; \frac{1}{23^2}\sqrt{36 + 2 + 16}^{\,2} \cdot 2 = \frac{\sqrt{108}}{23}^2 = 0.288 < \frac{1}{2}$

Want to prove the 4 statements from before.

Lem 1: $f: \mathbb{R}^n \to \mathbb{R}^n$ diff w/ deriv satisfying

$|Df(x) - Df(y)| \le M|x - y|$; $x, y \in \bar{U}$

Then for $x, y \in \bar{U}$

$|f(y) - f(x) - Df(x)(y - x)| \le \frac{M}{2}|y - x|^2$

Pf: $x, y \in \bar{U}$, $h = y - x$. $g(t) = f(x + th)$

$g(1) - g(0) = \int_0^1 g'(t)$; $g'(t) = Df(x + th)h$

$f(y) - f(x) = \int_0^1 Df(x + th)h \, dt$

$g'(t) = Df(x)h + (Df(x + th)h - Df(x)h)$

$$f(x+h) - f(x) = \int_0^1 Df(x)h\,dt + \int_0^1 Df(x+th)h - Df(x)\,dt$$

$$|f(x+h) - f(x) - Df(x)h| \leq \left|\int_0^1 Df(x+th) - Df(x)\,dt\right|$$

$$\leq \int_0^1 |Df(x+th) - Df(x)|\,dt$$

$$\leq \int_0^1 Mt|h|^2\,dt = \frac{M}{2}|h|^2 \checkmark$$

$\blacksquare$

Lem 2: $\underline{Df(x_1)}$ inv.

$$|[Df(x_1)]|^{-1} \leq 2|[Df(x_0)]^{-1}|$$

$$[Df(x_0)]^{-1} Df(x_1) \approx I$$

$$A = I - [Df(x_0)]^{-1}[Df(x_1)]$$

$$= [Df(x_0)]^{-1}[Df(x_0)] - [Df(x_0)]^{-1}[Df(x_1)]$$

$$= [Df(x_0)]^{-1}([Df(x_0) - Df(x_1)])$$

$$|[Df(x_0)] - [Df(x_1)]| \leq M|x_0 - x_1|$$

$$\leq M|h_0|$$

$$|A| \leq |[Df(x_0)]^{-1}|\,|h_0| M \cdot$$

$$h_0 = -[Df(x_0)]^{-1} f(x_0)$$

$$|h_0| \leq |[Df(x_0)]^{-1}| \cdot |f(x_0)|$$

Then $|A| \leq |[Df(x_0)]^{-1}|^2\,|f(x_0)|\,M \leq \frac{1}{2}$

$I - A$ inv. $\longrightarrow$ $[Df]$ inv.

$\uparrow$

$[Df(x_0)]^{-1}$ inv.

$$B = (I - A)^{-1} = [Df(x_1)]^{-1}[Df(x_0)]$$

$$[Df(x_1)]^{-1} = B[Df(x_0)]^{-1}$$

$$= (I + A + A^2 + \cdots)[Df(x_0)]^{-1}$$

$$|[Df(x_1)]^{-1}| \leq |I + A + A^2 \cdots|\,|[Df(x_0)]^{-1}|$$

$$\leq (1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots)|[Df(x_0)]^{-1}|$$

$$2|[Df(x_0)]^{-1}|$$

$\blacksquare$

Lem 3 $\quad |f(x_1)| \leq \frac{M}{2}|h_0|^2$

Lem 1 gives

$$|f(x_1) - f(x_0) - [Df(x_0)]h_0| \leq \frac{M}{2}|h_0|^2$$

$$f(x_0) + [Df(x_0)](x_1 - x_0) = 0$$

$$h_0 = x_1 - x_0 = -[Df(x_0)]^{-1} f(x_0)$$

- - - - - - - - -

$$|h_1| < \frac{|h_0|}{2}$$

$$|h_1| \leq |[Df(x_1)]^{-1}|\,|f(x_1)|$$

$$\xrightarrow{\text{Lem2}} \leq |f(x_1)| \cdot 2|Df(x_0)^{-1}|$$

$$\leq \frac{M}{2}|h_0|\,2\,|Df(x_0)^{-1}|\,|h_0|$$

$$\leq M|h_0|\,|Df(x_0)^{-1}|^2\,|f(x_0)|$$

Wed Dec 6 1pm – 2pm pick up 'final exam'.

11/30/2017

- Why Newton's method is important in diff eq.
- How to use Newton to factor real quad (find quad factors).

Nonlinear alg system resulting from a nonlin. system b.v.p.

$$u''(x) = \underbrace{f(x,u)}_{\sin(u(x))} \quad ; \quad 0 \le x \le 1 \qquad u(0) = u(1) = 0$$

$$x_j = jh \; ; \; h = \frac{1}{n+1} \quad j = 0, \cdots, n+1$$

0 x_j 1

$$u''(x_j) \cong \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2}$$

$$\frac{1}{h^2}\left(u(x_{j+1}) - 2u(x_j) + u(x_{j-1})\right) =$$

$$f(x_j, u(x_j)) + \underbrace{r(x_j, h)}_{\substack{\text{goes to } 0 \\ \text{as } h \to 0}}$$

$$u_j \equiv u(x_j)$$

$$u_{j+1} - 2u_j + u_{j-1} = h^2 f(x_j, u_j) \; ; \; 1 \le j \le n$$

$$0 = \underbrace{\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} + h^2 \begin{pmatrix} f(x_1, u_1) \\ \vdots \\ f(x_n, u_n) \end{pmatrix}}_{F(u)}$$

$$F: \mathbb{R}^n \to \mathbb{R}^n$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$0 = Au + \tilde{F}(u) = F(u)$$

Suppose want $f(x) = 0$,
Poly with real coefficients.

Want to find quad factors (will always have if deg > 1 but not nec. linear and can then use quad form).

$$f(x) = Q^*(x) m^*(x)$$

$$m^*(x) = m^*(x, p^*, r^*) = x^2 - p^* x - r^*$$

$$m(x) = m(x, p, r) = x^2 - px - r$$

$$* \; f(x) = m(x,p,r) Q(x,p,r) + q_1(p,r) x + q_0(p,r)$$

m^* quad. factor of f if $q_1(p^*, r^*) = 0$ and $q_0(p^*, r^*) = 0$ to determine we need p^*, r^* such that

$$\begin{cases} q_0(p,r) = 0 \\ q_1(p,r) = 0 \end{cases}$$

Start at $\begin{pmatrix} p_0 \\ r_0 \end{pmatrix}$

$$\begin{pmatrix} p_{k+1} \\ r_{k+1} \end{pmatrix} = \begin{pmatrix} p_k \\ r_k \end{pmatrix} - \begin{pmatrix} \frac{\partial q_0}{\partial p}(p_k, r_k) & \frac{\partial q_0}{\partial q}(p_k, r_k) \\ \frac{\partial q_1}{\partial p}(p_k, r_k) & \frac{\partial q_1}{\partial q}(p_k, r_k) \end{pmatrix}^{-1} \begin{pmatrix} q_0(p_k, r_k) \\ q_1(p_k, r_k) \end{pmatrix}$$

How to compute Jacobian? Diff * wrt p & r

$$\begin{cases} 0 = \frac{\partial f}{\partial p} = \frac{\partial m}{\partial p} Q + m \frac{\partial Q}{\partial p} + \frac{\partial q_1}{\partial p} x + \frac{\partial q_0}{\partial p} \\ \quad = -xQ + m \frac{\partial Q}{\partial p} + \frac{\partial q_1}{\partial p} x + \frac{\partial q_0}{\partial p} \\ 0 = \frac{\partial f}{\partial r} = \overset{-1}{\frac{\partial m}{\partial r}} Q + m \frac{\partial Q}{\partial r} + \frac{\partial q_1}{\partial r} x + \frac{\partial q_0}{\partial r} \end{cases}$$

$$x\,Q(x,p,r) = m(x,p,r) \frac{\partial Q(x,p,r)}{\partial p} + \frac{\partial q_1(p,r)}{\partial p} x + \frac{\partial q_0}{\partial p}(p,r)$$

$$Q(x,p,r) = m(x,p,r) \cdot \frac{\partial Q}{\partial r}(x,p,r) + \frac{\partial q_1(p,r)}{\partial r} + \frac{\partial q_0}{\partial r}(p,r)$$

We obtain $\frac{\partial q_1}{\partial p}$, $\frac{\partial q_1}{\partial r}$, $\frac{\partial q_0}{\partial p}$, $\frac{\partial q_0}{\partial r}$ by dividing xQ and Q by m.

Div. by quad. factor

$$f(x) = \sum_{0}^{n} a_i x^i$$

$$Q(x) = \sum_{i=0}^{n-2} q_{i+2} x^i$$

$$m = x^2 - px - r$$

$$q_n = a_n$$

$$q_{n-1} = p q_n + a_{n-1}$$

$$q_i = p q_{i+1} + r q_{i+2} a_i \; ; \; i = n-2, n-3, \ldots, 0$$

Thm (Superconv. of Newton's method)

Let $K = |f(x_0)| \, |[Df(x_0)]^{-1}|^2 M < 1/2$

Let $C = \frac{1-K}{1-2K} |[Df(x_0)]^{-1}| \frac{M}{2}$

If $|h_n| \le \frac{1}{2c} \rightarrow |h_{n+m}| \le \frac{1}{c} \left(\frac{1}{2}\right)^{2^m}$

Lem: If $K < 1/2$, if $|h_i| \le 1/2c$ then $|h_{i+1}| \le c|h_i|^2$

Pf (Thm using Lem): Denote $y_i = c|h_i|$

$$y_{i+1} = c|h_{i+1}| \le c^2 |h_i|^2 = y_i^2$$

If $|h_n| \le 1/2c$, $y_n = c|h_n| \le c \, 1/2c = 1/2$

$$y_{n+1} \le y_n^2 \le \frac{1}{4} = \left(\frac{1}{2}\right)^{2^1}$$

$$y_{n+2} \le y_{n+1}^2 \le (y_n^2)^2 \le \left(\frac{1}{4}\right)^2 = \left(\frac{1}{2}\right)^{2^2}$$

$$y_{n+m} \le y_n^{2^m} \le \left(\frac{1}{2}\right)^{2^m} \quad \blacksquare$$

Pf (Lem): From prev lem,

$$|f(x_i)| \le \frac{M}{2} |h_{i-1}|^2$$

$$h_i = -[Df(x_i)]^{-1} f(x_i)$$

$$|h_i| \le |[Df(x_i)]^{-1}| \, |f(x_i)|$$

$$\le \frac{M}{2} |[Df(x_i)]^{-1}| \, |h_{i-1}|^2$$

Need bound for $|[Df(x_i)]^{-1}|$

Lem 5: If $K < 1/2$ then $[Df(x_i)]^{-1}$ exists and satisfies

$$|Df(x_i)^{-1}| < |Df(x_0)^{-1}| \frac{1-K}{1-2K}.$$

12/05/2017

Uniqueness in Newton's Method

$f: \mathbb{R}^n \to \mathbb{R}^n$, $U \subset$

$x_0 \in U$

$Df(x_0)$ invertible

$$h_0 = -[Df(x_0)]^{-1} f(x_0)$$

$$x_1 = x_0 + h_0$$

$$U_1 = B_{|h_0|}(x_1)$$

Assume Df Lipschitz (with M)

$$|f(x_0)| \, |Df(x_0)^{-1}|^2 M \le 1/2$$

Then $f(x) = 0$ has unique solution in U_1.

If $y \in \overline{U}_1$ & $f(y) = 0 \rightarrow$

$*$ $|y - x_{i+1}| \le \frac{1}{2} |y - x_i|$. Know $x_i \to \alpha$ with $f(\alpha) = 0$. $x_i \to \alpha$

$$|y - \alpha| \le \frac{1}{2} |y - \alpha|$$

$\frac{1}{2}|y-\alpha| \le 0$

$y = \alpha.$

$*$PF: $f(y) = f(x_i) + [Df(x_i)](y-x_i) + r_i$

$-f(x_i) = [Df(x_i)](y-x_i) + r_i$

$y - x_i = -[Df(x_i)]^{-1} f(x_i) - [Df(x_i)]^{-1} r_i$

$y - x_{i+1} = -[Df(x_i)]^{-1} r_i$

Lem 1: $|r_i| = |f(y) - f(x_i) - Df(x_i)(y-x_i)|$

$\le \frac{M}{2}|y-x_i|^2$

$|y-x_i| \le |Df(x_i)^{-1}||r_i|$

$\le |Df(x_i)^{-1}| \frac{M}{2}|y-x_i|^2$

Now...

$|y-x_1| \le |[Df(x_0)^{-1}]| \frac{M}{2}|y-x_0|^2$

$= |Df(x_0)^{-1}| \frac{M}{2}|y-x_0||y-x_0|$

$\le |Df(x_0)^{-1}| \frac{M}{2} \cdot 2|h_0| \cdot |y-x_0|$

(y in ball radius that centered at x_1 so most $2|h_0|$ from x_0).

$\le |Df(x_0)^{-1}| M |h_0| |y-x_0|$

$\le \underbrace{|Df(x_0)^{-1}|^2 M |f(x_0)|}_{\le 1/2} |y-x_0|$

induction

Assume induction up to $|y-x_j| \le \frac{1}{2}|y-x_{j-1}|$

$\frac{|y-x_{i+1}|}{|y-x_i|} \le |Df(x_i)^{-1}| \frac{M}{2}|y-x_i|$

$\le 2|Df(x_{i-1})^{-1}| \frac{M}{2}|y-x_{i-1}|$

$\le \frac{1}{2}M|y-x_0||Df(x_0)^{-1}|$

$\le \frac{1}{2}M\, 2|h_0|\, |Df(x_0)^{-1}|$

$\le M|Df(x_0)^{-1}|^2 f(x_0) \le 1/2$

If $K < \frac{1}{2}$ & if $C = \frac{1-K}{1-2K}|Df(x_0)^{-1}|\frac{M}{2}$

If $|h_n| \le \frac{1}{2C} \to |h_{n+m}| \le \frac{1}{C}\left(\frac{1}{2}\right)^{2^m}$

Holds if
$|h_{i+1}| \le C|h_i|^2$

Lem 3 $|f(x_i)| \le \frac{M}{2}|h_0|^2$

From they

$|f(x_i)| \le \frac{M}{2}|h_{i-1}|^2$

$h_i = -[Df(x_i)^{-1}] f(x_i)$

$|h_i| \le |Df(x_i)^{-1}| |f(x_i)|$

$\le |Df(x_i)^{-1}| \frac{M}{2}|h_{i-1}|^2$

Show $\le C$

WTS $|Df(x_i)^{-1}| \le |Df(x_0)^{-1}| \frac{1-K}{1-2K}$

$|h_1| \le |f(x_1)||Df(x_1)^{-1}| \le \frac{M}{2}|h_0|^2 2|Df(x_0)^{-1}|$

$$= \frac{M}{2} \underbrace{|h_0|}_{\le |f(x_0)|\,|Df(x_0)^{-1}|} |h_0| \, |Df(x_0)^{-1}| = |h_0| M \, |Df(x_0)^{-1}|^2 \, |f(x_0)|$$

$$= K|h_0|$$

$$|h_1| \le K|h_0|$$

$$K = |f(x_0)|\,|Df(x_0)^{-1}|^2 M$$

Assume $|h_i| \le K|h_{i-1}|$

$$|x_n - x_0| = \left|\sum_0^{n-1} h_i\right| \le \sum_{i=0}^{n-1} |h_i|$$

$$= |h_0| + |h_1| + \cdots + |h_{n-1}|$$

$$\le \cancel{K}|h_0| + K|h_0| + \cdots + K^{n-1}|h_0|$$

$$= |h_0|\left(1 + K + \cdots + K^{n-1}\right)$$

$$|h_0| \frac{1-K^n}{1-K} \underset{\text{hence } \le}{\to} |h_0| \frac{1}{1-K}$$

$$A_n = I - Df(x_0)^{-1} Df(x_n) = Df(x_0)^{-1}\left(Df(x_0) - Df(x_n)\right)$$

$$|A_n| \le |Df(x_0)^{-1}| \, M \, |x_0 - x_n|$$

$$\le |Df(x_0)^{-1}| M \frac{|h_0|}{1-K}$$

$$\le |Df(x_0)^{-1}| \, M \, \frac{|Df(x_0)^{-1}|\,|f(x_0)|}{1-K}$$

$$= \frac{K}{1-K} < 1 \text{ if } K < 1/2$$

$|A_n| < 1$. Hence, $I - A_n$ invertible.

$$(I - A_n)^{-1} = I + A_n + A_n^2 + \cdots$$

$$(I - A_n)^{-1} = [Df(x_n)]^{-1}[Df(x_0)]$$

$$[Df(x_n)^{-1}] = (I + A_n + A_n^2 + \cdots)[Df(x_0)]^{-1}$$

$$=$$

$$|Df(x_n)^{-1}| \le |Df(x_0)^{-1}| \left(1 + |A_n| + |A_n|^2 + \cdots\right)$$

$$= |Df(x_0)^{-1}| \frac{1}{1 - |A_n|} \le |Df(x_0)^{-1}| \frac{1}{1 - \frac{K}{1-K}} = |Df(x_0)^{-1}| \frac{1-K}{1-2K}$$