MAT 705, Calculus on Manifolds, Fall 2017

TR 9:30-10:50, Carnegie 109

Instructor: Loredana Lanzani, 313G Carnegie, phone 443-1496, e-mail: llanzani@syr.edu
Office hours: W 1:00-2:00pm, R 4:00-5:00pm or by appointment.

Topics: differentiable manifolds, differential forms, exterior calculus, integration over manifolds, Stokes' theorem, other topics.

Prerequisites: MAT 602, MAT 632, MAT 661.

Textbook: Instructor's notes based upon material taken from (but not limited to): *Analysis on Manifolds*, by James R. Munkres, Westview Press 1991; *An introduction to differentiable manifolds and Riemannian Geoometry*, by William M. Boothby, Academic Press 1986; *Holomorphic functions and integral representations in several complex variables* by R. Michael Range, Springer 1998.

Grading: Students will be expected to make short presentations on topics assigned by the instructor.

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MAT705 Fall 2017 Final assignments and presentations schedule

- *Instructions*:

- Each enrolled student will write a paper (up to 10 pages for a single paper or up to 20 pages for a joint paper by two persons) and make a 35-40min presentation to the class and/or to the professor.
- **All papers are due Monday Nov. ~~28~~** 27 (e-mail an e-scan to (pdf) professor who will post on BlackBoard for everyone to access)

- Topics and assignments listed below. Study material for Topics 1, 2 and 3 from J. R. Munkres *Analysis on Manifolds* (sections and pages indicated below).
- You may borrow a copy of the book from Prof. Lanzani and make photocopies of relevant pages.

- Feel free to exchange your assignment with another student's (who will agree to do so) based on interest and availability. **Let me know asap if you have switched assignment with another student's.**

- **Presentations calendar:**
- Tuesday Nov. 28 (2 persons; 35-40min each);
- Thursday Nov. 30 (2 persons; 35-40min each)

Thursday Dec. 7 (2 persons; 35-40min each);

Tuesday Dec. 12 1:00pm in CARN 109 (1 person; 35-40min).

Note: Class cancelled Tuesday Dec. 5 (we meet Dec. 12 instead).

- ***Presentation topics and assignments*:**

- **Topic 1:** *Wedge product*. **Assigned to: Erin and Stephen** (write joint paper, up to 20pp total; split presentation 35-40min each).

  Content:

  Thm on existence; uniqueness and properties of wedge product: Munkres Thm. 28.1 pp. 237-243, plus exercises #1, 2, 5 (include pb. 5 in the proof of Thm 28.1) and 6, pp. 243-244.

  Presentation schedule: Tuesday Nov. 28.

- **Topic 2:** *Exterior derivative and pull-back.* **Assigned to: Fabian and Paula** (write joint paper, up to 20pp total; split presentation 35-40min each).

  Content:

  Theorem on exterior derivative (existence; uniqueness and properties): Munkres Thm. 30.4 pp. 256-259 and exercise #2 p. 260, plus correlation with div-curl-grad (Thm. 31.1 and Thm. 31.2 pp. 263-265: fill-in missing details).

  Theorem on pull-back representation: Munkres Thm. 32.2 pp. 269-270 and exercise #4 p. 273. pp. 256-273

  Theorem on invariance of exterior derivative: Munkres Thm. 32.3 pp.270-272 and exercises #5.

  If you have time, also try exercise #6, p. 273.

  Presentation schedule: Thursday Nov. 30.

  Paper due: Monday Nov. 27

- **Topic 3**: *The classical theorems of vector integral calculus re-interpreted via differential forms.*

Assigned to: Erin and Felix (write joint paper, up to 20pp total; split presentation 35-40min each).

Content:

Gradient thm for 1-manifolds: Munkres Lemma 38.1 pp. 310-311 (proof 2 only). Them 38.2 p. 312.

Divergence thm for (n-1) manifolds: Munkres pp. 312-319 (in Lemma 38.6 do proof 2 only)

Stokes' the for 2-manifolds in 3D: Thm 38.9 pp.319-320.

Presentation schedule: Thursday Dec. 7.

- **Topic 4:** *Applications of manifolds to Physics.* **Assigned to Arthur** (write a paper up to 10pp; give presentation 35-40min)

  Presentation schedule: Thursday Dec. 12.

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MAT 705 - Calculus on Manifolds

HW: send e-mail with available time-slots on Mondays

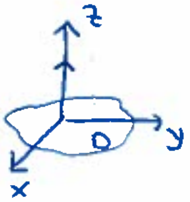
A quick review of Multivariable Calc. (Cal III)

$D \subset \mathbb{R}^3$ domain (open & connected), $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ a given vector field (assume as much differentiability as we need)

- $\operatorname{div}\vec{F} := P_x + Q_y + R_z$ (a scalar fct)
- $\operatorname{Curl}\vec{F} :=$ (= rot $\vec{F}$) $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \vec{i}\ \cancel{\partial_y}(R_y - Q_z) - \vec{j}\,(R_x - P_z) + \vec{k}\,(Q_x - P_y)$
- Special Case: $D \subset \mathbb{R}^2$ & $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$. Then:

  - We may think of $\vec{F}$ as a vector field in $\mathbb{R}^3$ by setting $R(x,y,z) \equiv 0$.
  - $\operatorname{div}\vec{F} = P_x + Q_y$
  - $\operatorname{Curl}\vec{F} = (Q_x - P_y)\,\vec{k}$

$\left(\begin{array}{ll} R_y, R_x = 0 & (R=0) \\ Q_z, P_z = 0 & (P, Q \text{ indep of } z) \end{array} \right)$



(A) Divergence Thm

$n = k = 3$ (see (0) later)

Ⓗ Hypothis $D \subset \mathbb{R}^3$ domain; $\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

Ⓒ Conclusion

$$\oiint_{bD} \vec{F}\cdot\underbrace{\vec{n}}_{\substack{\text{outer unit} \\ \text{normal } \underbrace{\text{vector}}_{\omega}}}\ \underbrace{dS}_{\substack{\text{surface} \\ \text{area element}}} = \iiint_D \underbrace{\operatorname{div}\vec{F}}_{d\omega}\ \underbrace{dV}_{\substack{\text{vol. el.} \\ dx\,dy\,dz}}$$

Ex.: $D = \mathbb{B}_1(0) = \{x^2+y^2+z^2 < 1\}$

$bD = \mathbb{S}^2_1(0) = \{x^2+y^2+z^2 = 1\}$

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(B) Stokes' Thm

$n=3;\ k=2$

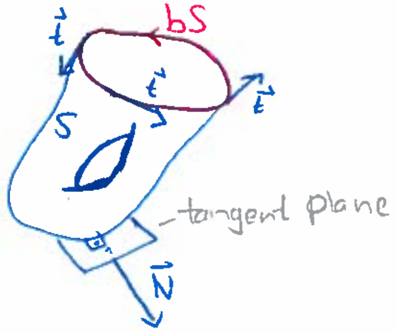
Ⓗ $S \subset \mathbb{R}^3$ a given surface with boundary bS, $\vec F = P\vec i + Q\vec j + R\vec k$

Ⓒ $$\oint_{bS} \vec F \cdot \overbrace{\vec t}^{\text{unit tangent vector to } bS}\ \underbrace{ds}_{\substack{\text{arclength el.}\\ \text{for } bS}} = \iint_S (\mathrm{Curl}\ \vec F)\cdot \underbrace{\vec N}_{\substack{\text{outer unit}\\ \text{normal vector}\\ \text{to } S}}\ \underbrace{dS}_{\substack{\text{surface area}\\ \text{el. for } S}}$$

ω

$d\omega$

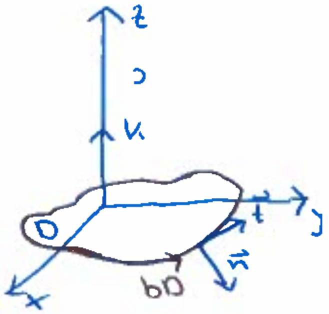
Ex.:



(C) Green's Thm

Ⓗ $D \subset \mathbb{R}^2$ a given domain, $\vec F = P(x,y)\vec i + Q(x,y)\vec j$

Ⓒ $$\oint_{bD} P(x,y)\,dx + Q(x,y)\,dy = \iint_D (Q_x - P_y)\underbrace{dA}_{\substack{\text{area element}\\ dxdy}}$$



Alternate formulations of Ⓒ

(a) $$\oint_{bD} \vec F \cdot \underbrace{\vec n}_{\substack{\text{outer}\\ \text{unit normal}\\ \text{vector to } bD}}\ \underbrace{dS}_{\substack{\text{arc length el.}\\ \text{for } bD}} = \iint_D \mathrm{div}\ \vec F\ \underbrace{dA}_{\text{area element}}$$

(b) $$\oint_{bD} \vec F \cdot \underbrace{\vec t}_{\substack{\text{unit tangent}\\ \text{vector to } bD}}\ \overbrace{ds} = \iint_D (\mathrm{Curl}\ \vec F)\cdot \vec k\ \underbrace{dA}_{\text{area el. } dxdy}$$

Connections between (A), (B) and (C)

- Green (formulation (b)) is a special case of Stokes with $S = D$

  so that $\vec N = \vec k$ ("flat" case)

- Green (formulation (a)) is a 2-dim analog of Divergence Thm:
  - dV in $\mathbb{R}^3$ is analog to dA in $\mathbb{R}^2$ ("Lebesgue measure")
  - dS for surface in $\mathbb{R}^3$ is analog of ds for curve in $\mathbb{R}^2$ ("Induced Leb. measure")

As we shall see sometime ~ Nov, each of (A), (B), (C) is a

special case of:

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(D) Generalized Stokes' Thm:

Ⓗ $M \subset \mathbb{R}^n$ is a k-dim. manifold (any $n \geq 2$, $1 \leq k \leq n$; integers)

with boundary bM

- given a differential form ω in $\mathbb{R}^n$ of degree $k-1$ (a "k-1-form", for short)

Ⓒ $$?\int_{bM} \omega = ?\iint_{M} d\omega$$, where $d\omega$ = the differential of ω.

Manifolds in $\mathbb{R}^n$

- Manifolds without boundary

Assumptions: (Given)
- $n, k \in \mathbb{Z}^+$; $1 \leq k \leq n$; $r \in \mathbb{Z}^+$
- $M \subset \mathbb{R}^n$ (a subset)

Def.: We say that M is a (differentiable) k-manifold of Class C^r without boundary in $\mathbb{R}^n$ if $\forall p \in M$:

- $\exists$ set $p \in V_p \subset M$ that is open in M (i.e. $V_p = \tilde{V} \cap M$, $\exists \tilde{V}$ open set in $\mathbb{R}^n$)
- $\exists$ open set $U_p \subset \mathbb{R}^k$
- $\exists$ map $\alpha : U_p \to V_p$

$$x = (x_1, \ldots, x_k) \longmapsto \alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x))$$ s.t.

(0) α is one-to-one & onto : $U_p \to V_p$

(1) α is of class C^r (i.e. $\frac{\partial^L \alpha_j}{\partial x_1^{L_1} \cdots \partial x_k^{L_k}}$ $\left(\begin{matrix} 0 \leq L \leq r \\ L_1 + \cdots + L_k = L \end{matrix} \right)$ are all continuous) $\forall j = 1, \ldots, n$

(2) $\alpha^{-1} : V_p \to U_p$ is continuous

(3) $$D\alpha(x) = \begin{pmatrix} \frac{\partial \alpha_1}{\partial x_1}, & \cdots & \frac{\partial \alpha_1}{\partial x_k} \\ \frac{\partial \alpha_2}{\partial x_1}, & \cdots, & \frac{\partial \alpha_2}{\partial x_k} \\ \vdots & & \\ \frac{\partial \alpha_n}{\partial x_1}, & \cdots, & \frac{\partial \alpha_n}{\partial x_k} \end{pmatrix}$$ $n \times k$ matrix

has maximal rank $\forall x \in U_p$. max. rank = k

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i.e. $\underset{\forall x}{\wedge}$ $\exists$ k×k minor of $D\alpha(x)$ whose determinant is $\neq 0$.

We call such map α a <u>coordinate chart</u> for M.

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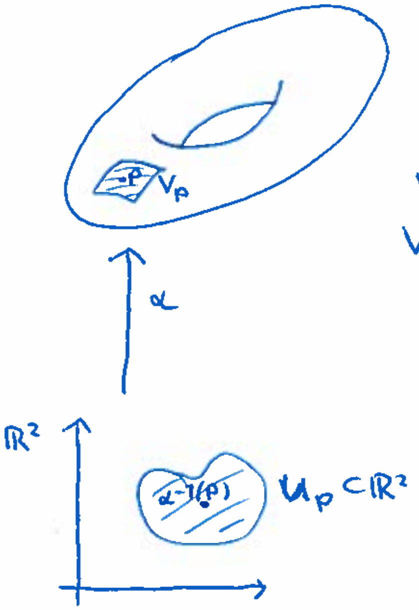
HW: reading assignement (next week: no classes) → BB

α is a <u>coordinate patch</u>.

<u>Ex:</u> M = torus in $\mathbb{R}^3$



$V = \cancel{E}^2\ \mathbb{T}^2 \cap \underbrace{B_r(p)}_{\text{Ball in } \mathbb{R}}$

$V = V(p, \alpha)$

$\{(x_1-p_1)^2+(x_2-p_2)^2 + (x_3-p_3)^2 < r\}$

<u>Significance of "Maximal Rank" condition for $D\alpha$</u>

Example

(i) $k=1$; any n Then $U_p = (a,b) \subset \mathbb{R} \ni t$

$$\alpha(t) = (\alpha_1(t), \_, \alpha_n(t))$$

$$D\alpha(t) = \begin{pmatrix} \alpha_1'(t) \\ \vdots \\ \alpha_n'(t) \end{pmatrix} \quad n\times 1$$

maximal rank = rank = 1 $\quad \forall t\ \exists j \in \{1, \to, n\}$ s.t. $\alpha_j'(t) \neq 0$

<u>Ex:</u> $k=1$, $n=2$. $\alpha: \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (t^3, t^2)$; $M = \alpha(\mathbb{R})$.

- α class C^∞ (polynomial)
- 1-1 ; onto (check!) onto of map $\mathbb{R} \to \alpha(\mathbb{R}) = M$
- α^{-1} is continuous (check!)

But $D\alpha(t)$ does not have rank = 1 at every $t \in \mathbb{R}$

$D\alpha(t) = (3t^2, 2t)$ so $D\alpha(0) = (0,0)$ ☹

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it turns out that M has a cusp

<u>Ex:</u> $k=1, n=2;\ M=\beta(\mathbb{R})$

$$\beta(t) = (t^3, |t^3|)$$

- β of class C^2 (check!)
- β^{-1} is continuous (check!)
- $D\beta(0) = \binom{0}{0}$ again, rank <u>not</u> maximal.

$\frac{D\beta(t)}{|D\beta(t)|} = \vec{\tau}(t)$ so if rank not maximal at t_0, $\vec{\tau}(t_0)$ "discontinuous"

Comparing "discontinuity" of $\vec{\tau}(0)$: <u>Cusp</u> vs <u>Corner</u>:

- <u>Cusp:</u> $D\alpha(t) = (3t^2, 2t) = t(3t, 2)$

  $|D\alpha(t)| = |t|(9t^2+4)^{1/2}$ <u>note:</u> $(9t^2+4)^{1/2} \approx 1$

  $\tau(t) \approx \underbrace{\frac{t}{|t|}}_{\text{sign}(t)}(3t, 2) \quad \rightarrow \tau(t) \approx (\underbrace{3t\,\text{sign}(t)}_{\substack{\text{continuous}\\ \text{at } t=0,\\ \text{not of class } C^1}}, \underbrace{2\,\text{sign}(t)}_{\text{discontinuous at } t=0})$

- <u>Corner:</u> $\beta(t) = (t^3, |t^3|)$

  $D\beta(t) = (3t^2, 3t^2\,\text{sign}(t)) = 3t^2(1, \text{sign}(t))$

  $|D\beta(t)| = 3t^2\sqrt{2}\ ;\ \vec{\tau}(t) = \frac{1}{\sqrt{2}}(\underbrace{1}_{C^\infty!!}, \underbrace{\text{sign}(t)}_{\text{discont. at } 0})$

Failure of maximal rank

$k=2$, any n

$$D\alpha(x) = \begin{pmatrix} \frac{\partial\alpha_1}{\partial x_1} & \frac{\partial\alpha_1}{\partial x_2} \\ \vdots & \vdots \\ \frac{\partial\alpha_n}{\partial x_1} & \frac{\partial\alpha_n}{\partial x_2} \end{pmatrix} \quad n \times 2 \text{ matrix}$$

Maximal rank = $k=2$

<u>Note:</u> $\vec{v}_i(y) := \left(\frac{\partial\alpha_1}{\partial x_i}(y), \dots, \frac{\partial\alpha_n}{\partial}(y)\right)$ = <u>velocity</u> ~~vector~~ vector of

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Curve:

$$t \longmapsto (\alpha_1(y_1+t, y_2), \dots, \alpha_n(y_1+t, y_2)) = f_1(t) \subset M$$

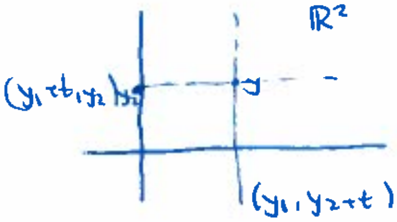
$\Rightarrow$ $\vec{v}_1(y)$ is <u>tangent</u> to $f_1(t)$ $\Rightarrow$ tangent to M at pt $\alpha(y)$

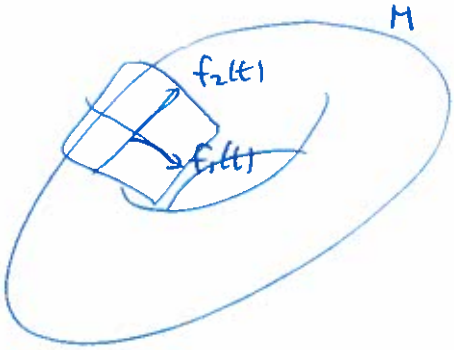
Similarly: $\vec{v}_2(y) = \left(\frac{\partial \alpha_1}{\partial x_2}(y), \dots, \frac{\partial \alpha_n}{\partial x_2}(y) \right)$ tangent to

$$t \mapsto f_2(t) = (\alpha_1(y_1, y_2+t), \dots \alpha_n(y_1, y_2+t)) \subset M$$

$\boxed{\text{Rank}=2}$ $\rightsquigarrow$ $\vec{v}_1$ & $\vec{v}_2$ are $\left\{ \begin{array}{l} \text{linearly indep.} \\ \text{tangent} \end{array} \right\}$ $\Rightarrow$ rank $D\alpha(t) \neq 0$

then $\frac{\partial \alpha}{\partial x_1}(y)$ & $\frac{\partial \alpha}{\partial x_2}(y)$ span the tangent plane





What happens if $D\alpha$ fails max rank at some pt x_0?

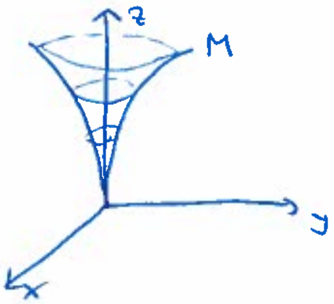
<u>Ex:</u> $k=2$; $n=3$; $\alpha: \mathbb{R}^2 \to \mathbb{R}^3$

$$(x,y) \longrightarrow (\underbrace{x(x^2+y^2)}_{\alpha_1}, \underbrace{y(x^2+y^2)}_{\alpha_2}, \underbrace{x^2+y^2}_{\alpha_3})$$

<u>Check:</u> $D\alpha$ (3×2-matrix)

$D\alpha(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ (check!) failure of max rank

$M = \alpha(\mathbb{R}^2)$ has a cusp at $(0,0)$



<u>Ex:</u> try on your own to study: M = cone

find: $\beta: (x,y) \to (\beta_1(x,y), \beta_2(x,y), \beta_3(x,y))$

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Significance of condition that α^{-1} be continuous

Ex: $n=2$; $k=1$; $\alpha : \overset{\subset\mathbb{R}}{(0,\pi)} \to \mathbb{R}^2$

$t \mapsto \alpha(t) := \sin(2t)\,(|\cos t|, \sin t)$

$M = \alpha(0,\pi)$ is "figure 8 in $\mathbb{R}^2$":

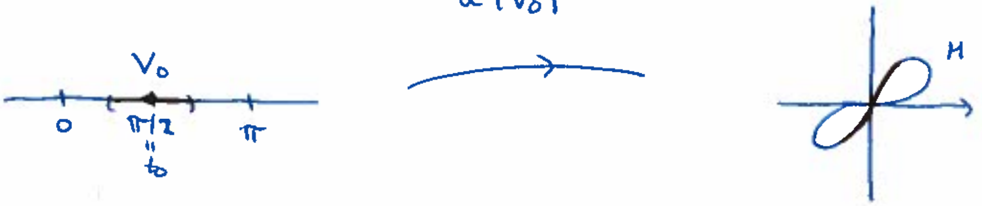
α class C^1 (check!)

$\alpha : (0,\pi) \to M$ one-to-one (check!)

$\underset{2\times 1}{D\alpha}$ has rank 1   Check!!

$\underbrace{\alpha^{-1} \text{ not continuous}}$ at $\frac{\pi}{2} = t_0$.

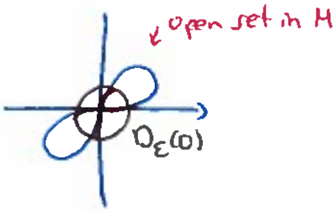
for continuity: $\underbrace{(\alpha^{-1})^{-1}(V_0)}_{\alpha(V_0)}$ open $\forall$ V_0 open in ~~M~~ $\mathbb{R}^3$



$t_0 = \frac{\pi}{2}$, $V_0 = (\frac{\pi}{4}, \frac{3}{4}\pi)$ but

$\alpha(V_0)$ not open

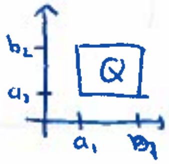
[α^{-1} not cts ⇝ double-crossing]



[every nbhd of (0,0) contains a "cross"]

Partitions of unity

Lemma: Ⓗ $Q = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ "rectangle" in $\mathbb{R}^n$



Ⓒ $\exists\ \varphi : \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$ such that

$\varphi(x) > 0$ if $x \in \operatorname{Int} Q = (a_1, b_1) \times \_\_ \times (a_n, b_n)$

and $\varphi(x) = 0$ if $x \in \mathbb{R}^n \setminus \operatorname{Int} Q$.

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Proof: read notes. → BB

Lemma 2: Ⓗ $\mathcal{A} = \{A_\lambda\}_\lambda$ collection of open sets in $\mathbb{R}^n$

$$A = \bigcup_\lambda A_\lambda$$

Ⓒ $\exists$ countable collection of rectangles $\{Q_i\}_{i \in \mathbb{N}}$ s.t. $Q_i \subset A$ $\forall i = 1, \dots \infty$

(1) $A \subset \bigcup_{i=1}^{\infty} Q_i$

(2) $\forall i$ $\exists$ $\lambda = \lambda(i)$ s.t. $Q_i \subset A_{\lambda(i)}$

(3) $\forall x \in A$ $\exists$ $U(x)$ open nbhd of x st. $U(x) \cap Q_i \neq \emptyset$ for only finitely many i's ("local finiteness cond.")

Partitions of unity

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Thm ($\exists$ of partitions of unity)

Ⓗ $\mathcal{A} = \{A_\alpha\}_\alpha$ collection of open sets in $\mathbb{R}^n$, $A = \bigcup_\alpha A_\alpha$

Ⓒ $\exists$ $\{\varphi_i\}_{i \in \mathbb{N}}$ a sequence of functs : $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ s.t.

(1) $\varphi_i(x) \geq 0$ $\forall x \in \mathbb{R}^n$, $i = 1, 2, 3, \dots$

(2) $\operatorname{Supp} \varphi_i \subseteq A_{\alpha(i)}$ $\exists$ $\alpha(i)$ (supp $\varphi_i := \overline{\{x \in \mathbb{R}^n \mid \varphi_i(x) \neq 0\}}$, so $y \notin \operatorname{supp} \varphi_i$ then $\exists$ $B_r(y)$ s.t $\varphi_i(x) = 0$ $\forall x \in B_r(y)$)

(3) $\forall x \in A$, $\exists$ $U(x)$ open nbhd s.t. $U(x) \cap \operatorname{Supp} \varphi_i \neq \emptyset$ for only finitely many i's

(4) $\underbrace{\sum_{i \in \mathbb{N}}}_{} \varphi_i(x) = 1$ $\forall x \in A$

↑ finite for each $x \in A$

(5) $\varphi_i \in C_0^\infty(A)$

Def: $\{\varphi_i\}_{i \in \mathbb{N}}$ as above are called a <u>partition of unity</u> for A dominated by $\{A_\alpha\}_\alpha$.

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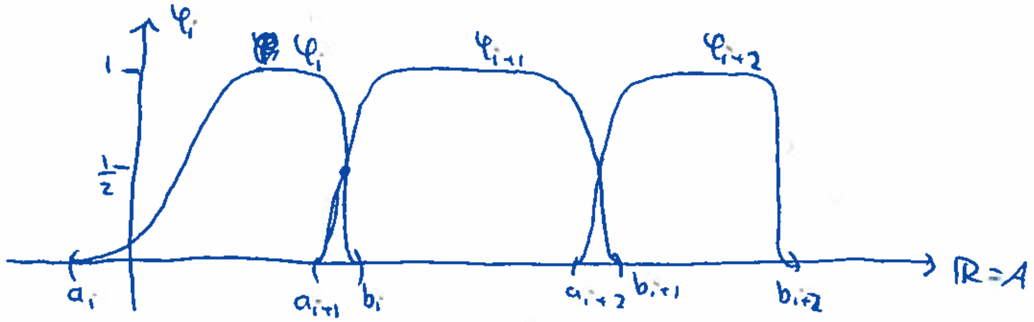
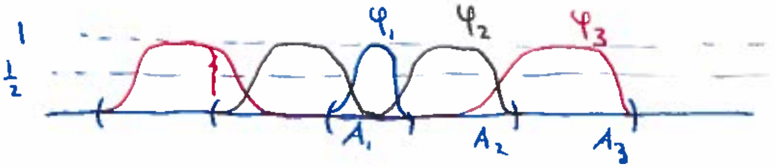
Ex: a partition of unity for $A=\mathbb{R}$ dominated by $\{A_i\}$



$A_i=(a_i,b_i)$ s.t. $a_i<b_i$ & $a_{i+1}<b_i<a_{i+2}$

In this ex, $\sum \varphi_i(x)$ consists of at most two terms $\forall x\in\mathbb{R}$

More interesting configurations $(A=\mathbb{R})$: $A_{i_1} \subset A_{i_1+1} \subset A_{i+2} \cdots$



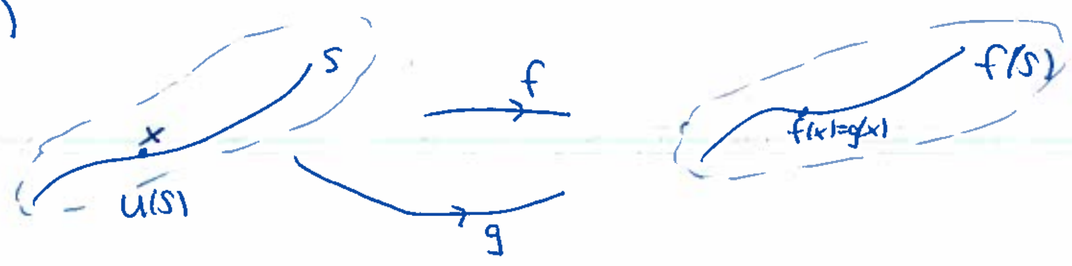
etc. ...

Question: How about

Def: (Differentiability on arbitrary sets in $\mathbb{R}^k$)

Let $S\subseteq\mathbb{R}^k$ be (any) subset; let $f: S\to\mathbb{R}^n$. We say that "f is of class C^r on S" $(\exists r\in\mathbb{N})$ if $\exists$ $U(S)$ open set in $\mathbb{R}^k$ that contains S, $\exists g: U(S)\to\mathbb{R}^n$ of class C^r s.t. $g|_S=f$ i.e. $g(x)=f(x)$ $\forall x\in S$ (g is "a C^r-extension of f").

i.e. $(n=k=2)$



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Fact: given $f_1: S\subset\mathbb{R}^k \to \mathbb{R}^n$, $f_1(S)\subset T\subset\mathbb{R}^n$
-"- $f_2: T\subset\mathbb{R}^n\to\mathbb{R}^p$ $\Big\}$ f_1, f_2 of class C^r

then: $f := f_2\circ f_1 : S\subset\mathbb{R}^k\to\mathbb{R}^p$ is also of class C^r.
check!

Lemma (being of class C^r is a local property)

Ⓗ ~~Suppose~~ $S\subset\mathbb{R}^k$, a subset, $f: S\to\mathbb{R}^n$

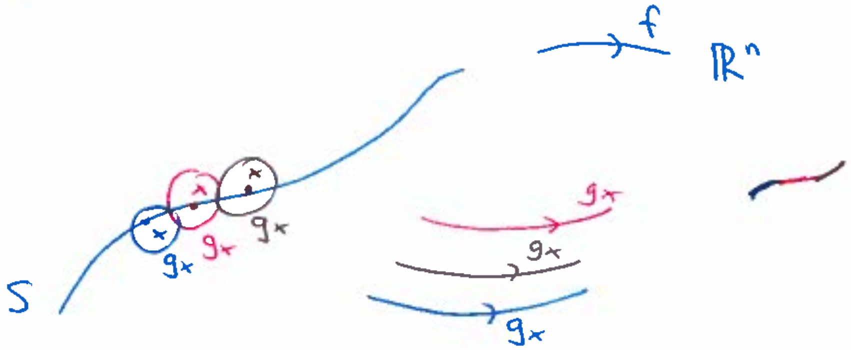
$\forall x\in S$ $\exists U_x$ an open set in $\mathbb{R}^k$ containing x & $\exists g_x: U_x\to\mathbb{R}^n$ of class C^r

s.t. $g_x(y)=f(y)$ $\forall y\in S\cap U_x$.

Ⓒ f is of class C^r.

Proof: Need to find $U=U(S)$ and $g: U\to\mathbb{R}^n$ of class C^r s.t.

$g(y)=f(y)$ $\forall y\in S$.



Let $U=\bigcup_x U_x$; let $\mathcal{A}=\{A_x\}=\{U_x\}_{x\in S}$. Let $\{\varphi_i\}_{i\in}$ be a $\underbrace{\text{partition of unity}}_{\text{of class } C^\infty}$ dominated by $\{U_x\}_{x\in S}$.

$\forall i=1,2,3,\ldots$ pick U_{x_i} s.t. $\operatorname{supp}\varphi_i\subset U_{x_i}$, and call: $g_i := g_{x_i}$.

Note that: $\underbrace{\varphi_i\cdot g_i}_{\text{ptwise product}} : U_{x_i}\to\mathbb{R}^n$ is zero outside $\operatorname{Supp}\varphi_i$ = a cpct subset of U_{x_i}.

Thus we may extend $\varphi_i g_i$ to a fct $\underbrace{h_i}_{\text{of class } C^r}$ on U by setting it $=0$ on $U\setminus U_{x_i}$.

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Def: $g(x) := \sum_{i=1}^{\infty} h_i(x) = \sum_{i=1}^{\infty} \varphi_i(x)\, g_i(x).$

Note: $\forall x \in U \quad \exists U_x \quad \exists \quad N(x) \in \mathbb{Z}^+$ s.t. $g(y) = \sum_{i=1}^{N(x)} \varphi_i(y)\, g_i(y) \quad \forall y \in U_x$

So g of class C^r on U_x $\forall x \in U$ $\Rightarrow g$ of class C^r

(g = finite sum of C^r-fcts)

Finally, now let $x \in S$, $g(x) = \sum \varphi_i(x) \underbrace{g_i(x)}_{=f(x)}$ Ⓗ $= \underbrace{\left(\sum_i \varphi_i(x)\right)}_{=1 \ \forall x \in U} f(x) = f(x)$

So, $g|_S = f$, indeed. □

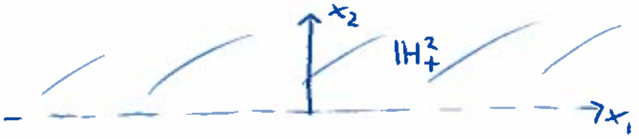
Def: • $\mathbb{H}^k := \{x \in \mathbb{R}^k,\ x = (x_1, \dots, x_k) \text{ s.t. } x_k \geq 0\}$ i.e. the upper half space in $\mathbb{R}^k$.

ex: $(k=2)$

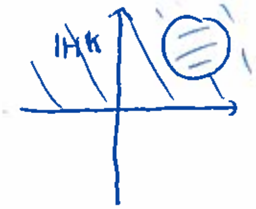
• $\mathbb{H}^k_+ := \{x \in \mathbb{R}^k \mid x = (x_1, \dots, x_k),\ x_k > 0\}$

i.e. the open upper half space in $\mathbb{R}^k$

ex: $(k=2)$

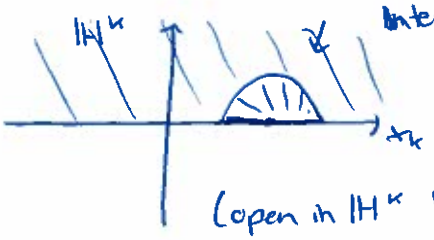


Open sets in $\mathbb{H}^k$:



(same as open sets in $\mathbb{R}^k$)

or



We are especially interested in those

(open in $\mathbb{H}^k$ but not open in $\mathbb{R}^k$)

Lemma: Ⓗ Let U be open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$. Let $\alpha: U \to \mathbb{R}^n$ be of class C^r. Let $U' \subset \mathbb{R}^k$ be open in $\mathbb{R}^k$ that contains U & let $\beta: U' \to \mathbb{R}^n$ be a C^r-extension of α.

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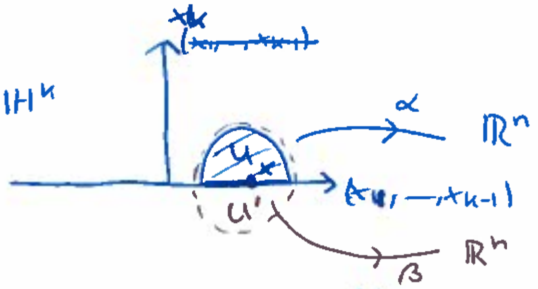


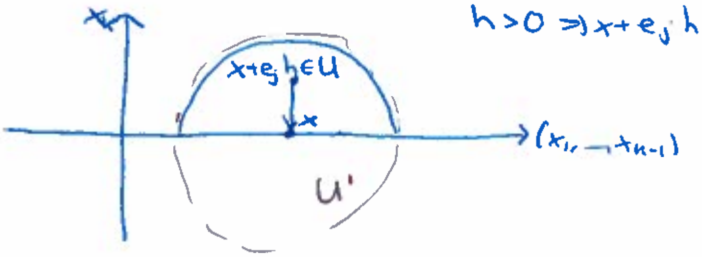
© $\forall x \in U$, we have that $D_\beta(x) = \left(\frac{\partial \beta_i(x)}{\partial x_j}\right)_{\substack{i=1,-n \\ j=1,-k}}$

depends only on α and is independent of the choice of extension β.

It follows that we may write $D\alpha(x)$ without ambiguity, $x \in U$.

Proof: Fix $i \in \{1, -, n\}$. For ease of notation, write $\frac{\partial \beta(x)}{\partial x_j}$ for $\frac{\partial \beta_i(x)}{\partial x_j}$

Recall: $\frac{\partial \beta(x)}{\partial x_j} = \lim_{h \to 0} \frac{\beta(x+he_j) - \beta(x)}{h}$, $e_j = (0, -, 0, \underset{\overset{\uparrow}{j}}{1}, 0, -, 0)$



$h > 0 \Rightarrow x + e_j h \in U$

Now, we do now that $\beta \in C^r(U')$ & this means that above limit exists $\forall j = 1, -, k$ $\forall x \in U'$, (so in particular for $x \in U$) & takes same value no matter how you let $h \to 0$. Pick $h \geq 0$: for such h, we have that $x + he_j \in U$. But β is an extension of α, so: $x \in U \Rightarrow \beta(x) = \alpha(x)$

& $x + he_j \in U \Rightarrow \beta(x+he_j) = \alpha(x+he_j)$

Thus for $0 < h < d$ $\frac{\beta(x+he_j) - \beta(x)}{h} = \frac{\alpha(x+he_j) - \alpha(x)}{h}$

i.e. diff. quotient of α is independent of choice of extension β.

□

Def: Let $k, n \in \mathbb{Z}^+$, $k \leq n$. A "k-manifold in $\mathbb{R}^n$ of class C^r" is a subset $M \subseteq \mathbb{R}^n$ with the following properties:

$\forall p \in M \quad \exists V(p) \subset \mathbb{R}^n$ (open nbhd of p in $\mathbb{R}^n$);

$\exists U_p$ open either in $\mathbb{R}^k$ or in $\mathbb{H}^k$

$\exists$ continuous, 1-1, onto map $\alpha : U_p \to \overset{\alpha(U_p)=}{V(p)}$ s.t.

(1) α is of class C^r (i.e. $\exists U_p' \subset \mathbb{R}^k$ open $\exists g_\alpha : U_p' \to \mathbb{R}^n$ of class C^r s.t. $g_\alpha|_{U_p} = \alpha$)

(2) $\alpha^{-1} : V(p) \to U_p$ is continuous

(3) $D\alpha(x)$ has (maximal) rank k $\forall x \in U_p$.

The map α is called a coordinate patch on M about p.

<u>Def:</u> A discrete collection of points in $\mathbb{R}^n$ is called a zero manifold in $\mathbb{R}^n$.

<u>ex:</u> $n=3$, $k=2$

a 2-mfd in $\mathbb{R}^3$, with two coord. patches

(U_q open in $\mathbb{H}^2$, but not in $\mathbb{R}^2$)

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<u>Lemma:</u> (H) M is a k-mfd in $\mathbb{R}^n$, $\alpha : U \to V$ is a coord. patch on M.

$U_0 \subset U$ <u>open in U</u> $\quad (U_0 = U \cap \underbrace{B_r(z_0)}_{\text{ball in } \mathbb{R}^k})$

(C) $\alpha|_{U_0} : U_0 \to \alpha(U_0) = V_0$ is also a coord. patch

Pf: • U_0 open in U means U_0 is open in $\mathbb{R}^k$ or $\mathbb{H}^k$

• α^{-1} continuous $\Rightarrow \alpha(U_0) =: V_0$ is open in $\mathbb{R}^n$

$\left((\alpha^{-1})^{-1}(U_0) \text{ open} \quad \text{but: } (\alpha^{-1})^{-1}(U_0) = \alpha(U_0) \right)$

It follows that $\alpha|_{U_0}$ is a coord. patch.

- 1-1, onto
- class C^r
- inverse continuous
- $D(\alpha|_{U_0})$ maximal

} immediate from properties of α & V_0 open

The boundary of a manifold

Fact: coordinates patches of a k-mfd "overlap differentiability"

Thm: (H) M is a k-mfd in $\mathbb{R}^n$ of class C^r

$\alpha_0 : U_0 \subseteq \mathbb{R}^k$ (or $\mathbb{H}^k$) $\to V_0 \subset M$

$\alpha_1 : U_1 \subset \mathbb{R}^k$ (or $\mathbb{H}^k$) $\to V_1 \subset M$

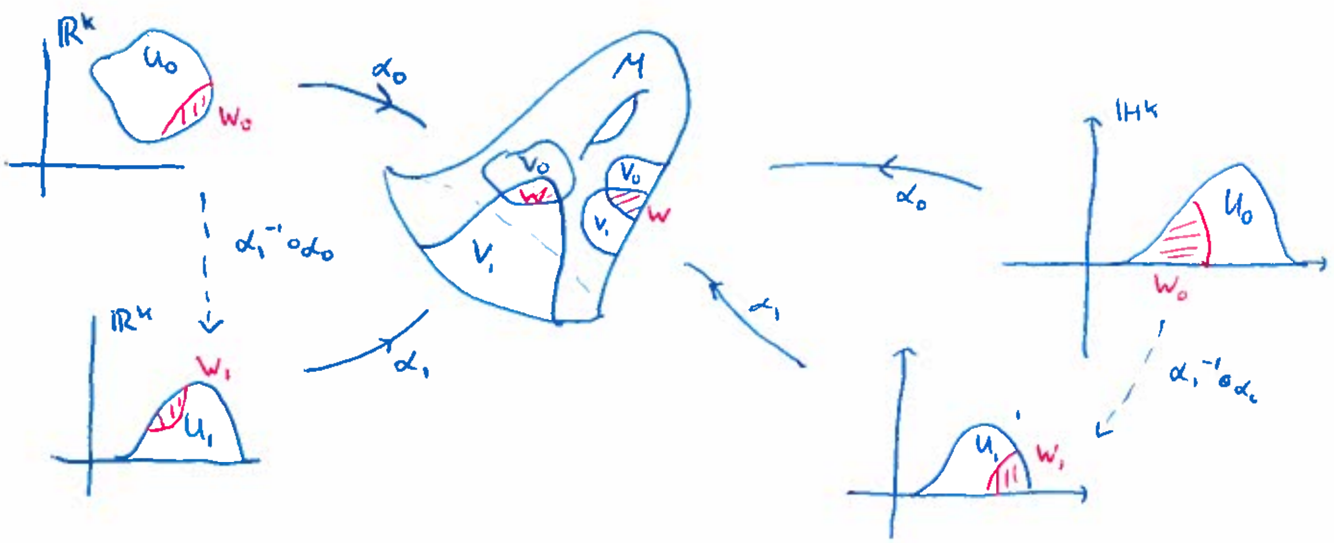
are two coordinate patches such that $V_0 \cap V_1 = W \neq \emptyset$

Let $W_i = \alpha_i^{-1}(W) \subset U_0$, $i = 0, 1$.

(C) $\alpha_1^{-1} \circ \alpha_0 : W_0 \to W_1$ is of class C^t and the matrix

$D(\alpha_1^{-1} \circ \alpha_0)(x)$ is non-singular $\forall x \in W_0$

e.g.



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<u>So:</u> $D(\alpha_1^{-1}\circ\alpha_0)$ is a $k\times k$ - matrix $\forall\ x\in W_0 = \alpha^{-1}(\underset{V_0\cap V_1}{\underset{\shortparallel}{W}})$

& thm claims: $\alpha_1^{-1}\circ\alpha_0$ is of class C^r and

$$\det\left(D(\alpha_1^{-1}\circ\alpha_0)\right)(x)\neq 0 \quad \forall x\in W_0.$$

We call $\alpha_1^{-1}\circ\alpha_0$ a transition fct for the patches α_0, α_1.

<u>Pf.</u> <u>Claim:</u> enough to show that if $\alpha: U\to V$ is a coord. patch on M, then $\alpha^{-1}: V\to U$ is of class C^r.

Assuming that <u>claim</u> is true, let's see why © follows:

if α_1^{-1} is of class C^r then so is $\alpha_1^{-1}\circ\alpha_0$ ✓

Also,

$$Id = (\alpha_0^{-1}\circ\alpha_1)\circ(\alpha_1^{-1}\circ\alpha_0) : W_0 \to W_0$$

$$\det D(Id)(x) = \underbrace{\det\Big(\underbrace{D(\alpha_0^{-1}\circ\alpha_1)((\alpha_1^{-1}\circ\alpha_0)(x))}_{B(x)} \underset{\substack{\uparrow\\ \text{matrix product}}}{\cdot} D(\alpha_1^{-1}\circ\alpha_0)(x)\Big)}_{}, \ x\in W_0$$

chain rule

$\alpha_0^{-1}\circ\alpha_1$ & $\alpha_1^{-1}\circ\alpha_0$ are of class C^r

$$\underset{0\neq}{1} = \det B(x)\cdot \det(D(\alpha_1^{-1}\circ\alpha_0)(x))$$

$$\Rightarrow \quad \det B(x)\neq 0 \text{ and } \det(D(\alpha_1^{-1}\circ\alpha_0)(x))\neq 0$$

$$\Rightarrow \quad D(\alpha_1^{-1}\circ\alpha_0)(x) \text{ non-singular!}$$

We are left to prove the claim, that is to show that if $\alpha: U\to V$ is a coord. patch, <u>then α^{-1} is of class C^r</u>.

$\alpha^{-1}: V\to U$ & we would need to show that $\exists\ \tilde V\subset\mathbb{R}^n$ open in $\mathbb{R}^n$ $\exists\ G:\tilde V\to\mathbb{R}^k$ of class C^r s.t. $V\subset\tilde V$ & $G(p) = \alpha^{-1}(p)$ $\forall p\in V\cap\tilde V$. But we know that the property of being of class C^r is <u>local</u> (lemma we proved last time), so, enough to show that

α^{-1} is <u>locally of class C^r</u> that is: $\forall p_0\in V$ $\exists\ V'(p_0)$ open in $\mathbb{R}^n$ (containing p_0) and $\exists g: V'(p_0)\to\mathbb{R}^k$ of class C^r s.t.

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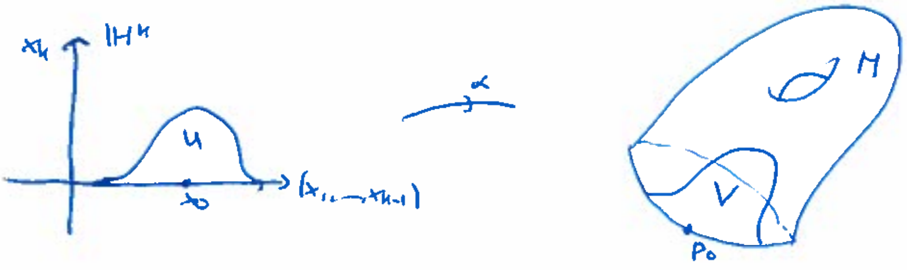
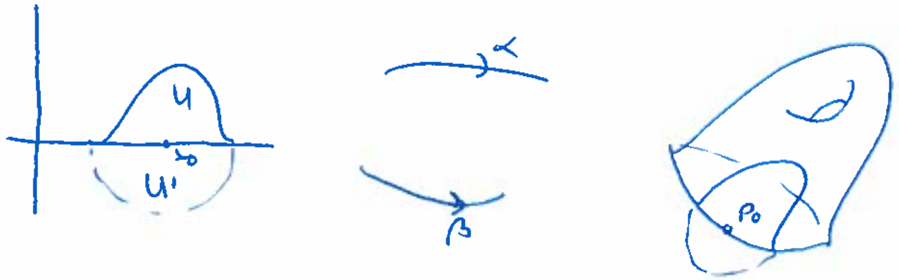
$g(p) = \alpha^{-1}(p) \quad \forall p \in V^{-1}(p_0) \cap V.$

recall: $\alpha^{-1} : V \to U$ where U is either open in $\mathbb{H}^k$ (but not in $\mathbb{R}^k$), or open in $\mathbb{R}^k$. Set: $x_0 := \alpha^{-1}(p_0) \in U$.

Case I: U open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$.



Ⓗ $\Rightarrow$ we may extend α to a C^r-map β on an open set $U' \subseteq \mathbb{R}^n$.



Also Ⓗ $\Rightarrow$ $D\alpha(x_0)$ has rank k, so there are k-many lin. independent rows in $D\alpha(x_0)$ & we may assume that these are the first k rows.

let $\pi : \mathbb{R}^n \to \mathbb{R}^k$, $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Then $h := \pi \circ \beta : U' \to \mathbb{R}^k$ is of class C^r (check!)

& $\det(Dh)(x_0) \neq 0$ (check!!)

Aside: Recall from Advanced Calculus

Inverse Function Thm:

Ⓗ $U' \subseteq \mathbb{R}^k$ open set, $h : U' \to \mathbb{R}^k$ of class C^r

$\det Dh(x_0) \neq 0 \quad \exists x_0 \in U'$

Ⓒ $\exists W = W(x_0)$ open nbhd of x_0 in $\mathbb{R}^k$ s.t. $h(W)$ is open in $\mathbb{R}^k$

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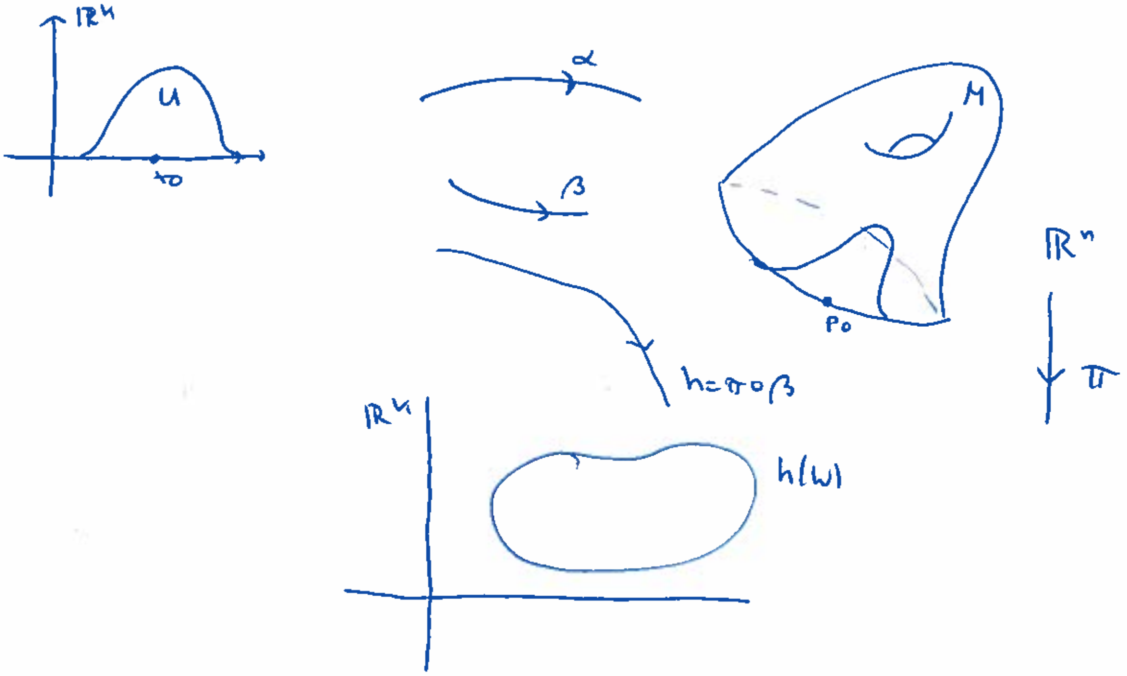
& $h: W \to h(W)$ 1-1 & onto. Furthermore, h^{-1} is of class C^r.

We call such fct a diffeomorphism of class C^r
also "change of variables of class C^r"

By Inverse fct Thm, $\exists\ W = W(t_0) \subset \mathbb{R}^k$ open set s.t. $h: W \to h(W) \subset \mathbb{R}^k$ is diffeom. of class C^r, i.e. h invertible and h^{-1} of class C^r, & $h(W)$ open set in $\mathbb{R}^k$:



Def: $g = h^{-1} \circ \pi$. Note that g is of class C^r.

<u>Claim:</u> $\exists\ V' = V'(p_0)$ open in $\mathbb{R}^n$ such that $g(p) = \alpha^{-1}(p)\ \forall p \in V'$.

<u>Pf of claim:</u> first note that $U_0 := U \cap W$ is open in U. (b/c W is open in $\mathbb{R}^k$). Thus $\alpha(U_0) =: V_0$ is open in V (b/c α^{-1} continuous since α is coord. chart)

So $\exists\ V'$ open in $\mathbb{R}^n$ s.t. $V_0 = V' \cap V$ & we may choose V' so that $V' \subset$ domain of g (intersect V' with $\pi^{-1}(h(W))$)

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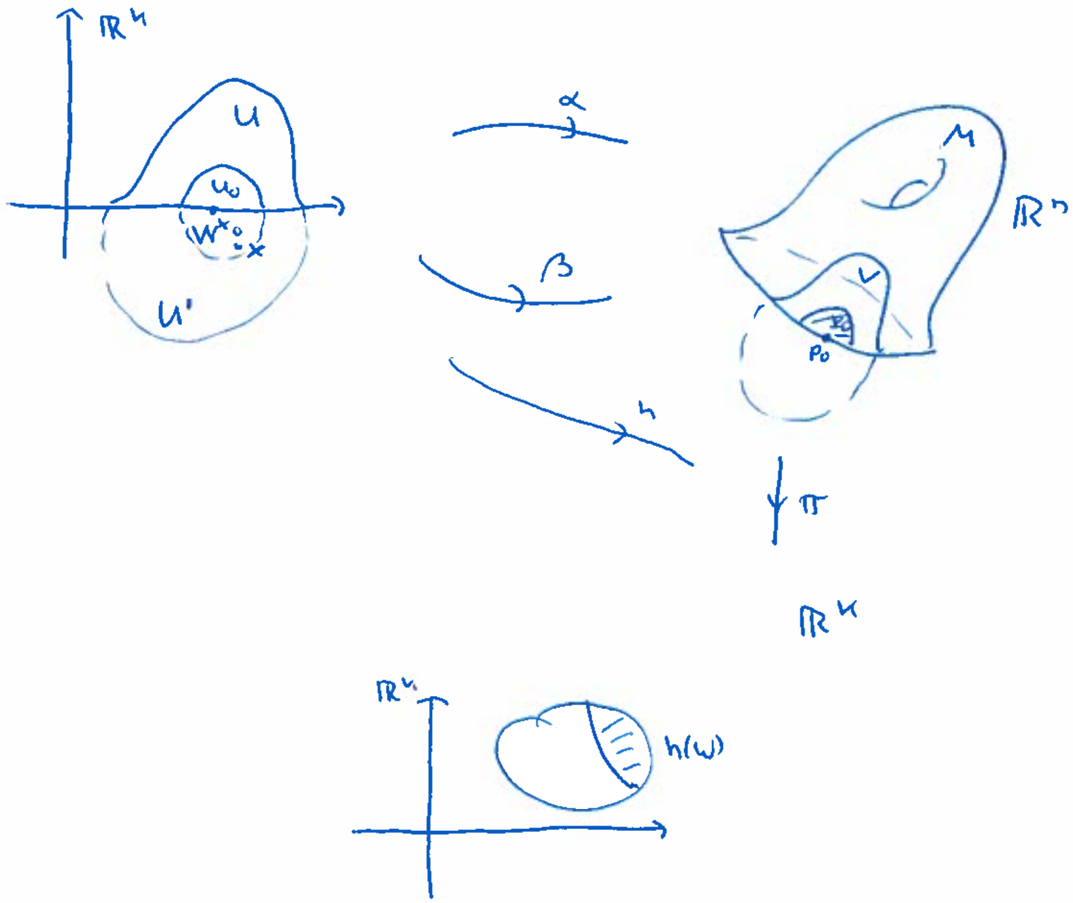

$\forall\, p \in V_0 = V \cap V'$ let $x = \alpha^{-1}(p) \in U_0$ & compute

$$g(p) = g(\alpha(x)) \underset{\substack{x \in U_0 \subset U' \,\&\\ \beta \text{ is an extension of } \alpha}}{=} h^{-1}(\pi(\beta(x))) = h^{-1}(h(x)) = x$$

So: $g(p) = x = \alpha^{-1}(p)$.

□

<u>Case 2:</u> U (domain of chart α) is open in $\mathbb{R}^k$.

Proof is similar (same steps) but much easier.

We may pick $U' = U$ & $\beta = \alpha$. □

<u>Def:</u> (interior points & boundary pts of k-mfd in $\mathbb{R}^n$).

Let M be a k-mfd in $\mathbb{R}^n$, class C^r. Let $p \in M$.

- If there is a coord. patch $\alpha: U \to V$ on M about p such that <u>U is open in $\mathbb{R}^n$</u>, then we say that p is an interior point of M.

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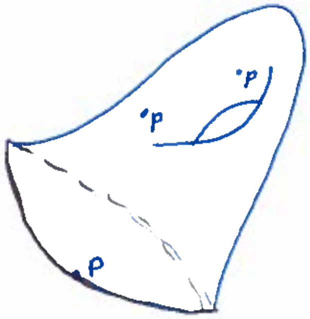
- Otherwise (i.e. if all coord. patches about p are defined on sets that are open in $\mathbb{H}^k$ but not open in $\mathbb{R}^k$) we say that p is a boundary pt of M [computationally hard to check]

Notation: $\{\text{boundary pts of } M\} = bM$ "boundary of M"

$M \setminus bM = \operatorname{Int} M$ "interior of M"



Lemma 2: (Criterion to identify boundary pts on a k-mfd in $\mathbb{R}^n$)

Ⓗ $M \subset \mathbb{R}^n$ is a k-mfd in $\mathbb{R}^n$, $p \in M$, $\alpha: U \to V$ chart about p.

Ⓒ (a) if U is open in $\mathbb{R}^k$ then $p \in \operatorname{Int} M$

(b) if U is open in $\mathbb{H}^k$ (not in $\mathbb{R}^k$) and $p = \alpha(x_0)$ $\exists x_0 \in \mathbb{H}^k_+$

then $p \in \operatorname{Int} M$

$\exists\, \cancel{\forall}\, \alpha_0: U_0 \to V_0$, U_0 open in $\mathbb{H}^k$, not in $\mathbb{R}^k$

(c) if ~~U is open in $\mathbb{H}^k$~~ and $p = \alpha_0(x_0)$ $\exists x_0 \in \mathbb{R}^{k-1} \times \{0\}$

then $p \in bM$

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Lemma3: Ⓗ $p \in bM$

Ⓒ $\forall$ charts $\alpha: U \to V$ about p, $\exists\, x_\alpha \in U \cap (\mathbb{R}^{k-1} \times \{0\})$ s.t. $p = \alpha(x_\alpha)$

Pf: By def. we know that $U \cap (\mathbb{R}^{k-1} \times \{0\}) \neq \emptyset$.

By contradiction: $\exists$ $\beta: U \to V$ a chart s.t. $p = \beta(x_1)$ $\exists x_1 \in U \setminus (\mathbb{R}^{k-1} \times \{0\})$

This means that $x_1 \in \mathbb{H}^k_+$ which is open in $\mathbb{R}^k \Rightarrow \exists$ $B^k_\varepsilon(x_1) \subset U \setminus (\mathbb{R}^{k-1} \times \{0\})$,

$\Rightarrow \beta|_{B^k_\varepsilon(x_1)}$ is also a chart about p $\rightarrow\!\leftarrow$ (b/c $p \in bM$: every chart has domain not open in $\mathbb{R}^k$, but $B^k_\varepsilon(x_1)$ is!)

So $p \in bM \Rightarrow \forall$ chart about p, $p = \alpha(x_0) \Rightarrow x_0 \in \mathbb{R}^{k-1} \times \{0\}$. Sep 19 $\square$

Proof of Lemma 2:

(i) trivial (def of Int M!)

(ii) semi-trivial: $x_1 \in \mathbb{H}^k_+ \Rightarrow \exists\, B^k_\varepsilon(x_1)$ open in $\mathbb{R}^k$ & $\alpha|_{B^k_\varepsilon(x_1)}$ is also a chart.

(iii) By contradiction: Say $p \notin bM$. Then $p \in \text{Int } M$ i.e. $\exists\, \alpha_1 : U_1 \to V_1$ about p with U_1 open in $\mathbb{R}^k$.

Set $V := V_0 \cap V_1$ open in M & $p \in V$. Set $W_0 := \alpha_0^{-1}(V)$: oppen in $\mathbb{H}^k$ and $x_0 \in W_0$. Set $W_1 := \alpha_1^{-1}(V)$: open in $\mathbb{R}^n$

Consider transition fct: $\alpha_0^{-1} \circ \alpha_1 : W_1 \to W_0$. By previous Thm,

$\alpha_0^{-1} \circ \alpha_1$ is Class C^r; 1-1; onto; $\det(D(\alpha_0^{-1} \circ \alpha_1)(x)) \neq 0 \quad \forall x \in W_1$

By global inverse fct thm it follows that ~~$W_0 = \alpha_0^{-1} \circ$~~

$W_0 = (\alpha_0^{-1} \circ \alpha_1)(W_1)$ is open in $\mathbb{R}^k$: impossible b/c $x_0 \in W_0 \cap (\mathbb{R}^{k-1} \times \{0\})$

& $W_0 \subseteq \mathbb{H}^k$ so $\nexists\, B^k_\varepsilon(x_0) \subset W_0$, so W_0 not open in $\mathbb{R}^k$. $\square$

Summary:

Lemma 1: $p \in bM \Rightarrow \forall$ chart about p $\exists\, x_\alpha \in \mathbb{R}^{k-1} \times \{0\}$ s.t. $p = \alpha(x_\alpha)$

Lemma 2: if $p = \alpha(x_0)$ $\exists\, x_0 \in \mathbb{R}^{k-1} \times \{0\}$ $\exists$ chart $\rightarrow p \in bM$.

Ex: $M = \mathbb{H}^k$ is a k-mfd in $\mathbb{R}^k$ of class C^∞, & $b\mathbb{H}^k = \mathbb{R}^{k-1} \times \{0\}$ check!!

Thm: Ⓗ M is a k-manifold in $\mathbb{R}^n$ of class C^r.

Ⓒ bM is a $(k-1)$-mfd in $\mathbb{R}^n$ of class C^r, and $b(bM) = \emptyset$.

Pf: To show: $\forall p \in bM$ $\exists\, \alpha' : U' \to V'$ chart about p with U' open in $\mathbb{R}^{k-1}$

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Pick $p \in bM$. Ⓗ $\Rightarrow$ $\exists\, \alpha: U \to V$ chart about p of class C^r with U open in $\mathbb{H}^k$ (not in $\mathbb{R}^k$) & $p = \alpha(x_0)$ $\exists\, x_0 \in U \cap (\mathbb{R}^{k-1} \times \{0\})$

Notation: $\mathbb{R}^k = \{(x', x_k) \mid x' \in \mathbb{R}^{k-1}, x_k \in \mathbb{R}\}$.

Claim: $U \cap (\mathbb{R}^{k-1} \times \{0\}) = U' \times \{0\}$ $\exists$ open set U' in $\mathbb{R}^{k-1}$

Pf of Claim: WLOG : $U = B^k_\varepsilon(x_0) \cap \mathbb{H}^k = \{(x', x_k) \mid |x' - x_0'|^2 + \overset{x_k^2}{\cancel{(x_k - x_{0k})}}^2 < \varepsilon\}$

$\Rightarrow U \cap (\mathbb{R}^{k-1} \times \{0\}) = \{(x', x_k) \mid |x' - x_0'|^2 + 0 < \varepsilon^2, x_k = 0\} \cap \{x_k \geq 0\}$

$= B^{k-1}_\varepsilon(x_0') \times \{0\}$

$\Rightarrow U' = B^{k-1}_\varepsilon(x_0')$. Let $x' \in U'$

Def. $\alpha'(x') := \alpha(x', 0)$: • class C^r b/c α is so

• $D\alpha'(x')$ has rank $k-1$ ✓ ($(n \times (k-1))$ submatrix of $n \times k$: $\underbrace{D\alpha(x', 0)}_{\text{rank } k}$)

check!

Also: Lemmas 1 & 2 : $\alpha: \underbrace{U \cap \{x_k = 0\}}_{U' \times \{0\}} \to \underbrace{V \cap bM}_{(\tilde{V} \cap M) \cap bM}$
$= \underbrace{\tilde{V} \cap bM}_{\text{open in } bM}$

$\alpha': U' \to U' \times \{0\} \xrightarrow{\alpha} V \cap bM =: V'$ open set in bM

$\mathbb{R}^{k-1} \xleftarrow[\pi]{} \mathbb{R}^k$ $\qquad$ q

and $\forall\, q \in V \cap bM$, $(\alpha')^{-1}(q) = \underbrace{(\pi \circ \alpha^{-1})}_{\text{continuous}}(q)$ b/c α^{-1} is so

All together: $\forall\, p \in bM$ $\exists\, U'$ open in $\mathbb{R}^{k-1}$ & C^r-chart

$\alpha': U' \to V'$ about p.
$\quad V'$ open in bM.

$\square$

Recall: $\alpha'(x') = \alpha(x', 0)$

α': chart for $p \in bM$ $\qquad$ α: chart for $p \in bM \subset M$.

We call α' the restriction of the chart.

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A procedure for constructing n-mfd in $\mathbb{R}^n$ of class C^r and $(n-1)$-mfds in $\mathbb{R}^n$ (class C^r).

<u>Thm:</u> Ⓗ $A \subseteq \mathbb{R}^n$ open, $f: A \to \mathbb{R}$ of class C^r, $f \neq$ const.

Def. $N := f^{-1}([0, +\infty)) = \{y \in A \mid f(y) \geq 0\}$

Def. $M := f^{-1}(0) = \{x \in A \mid f(x) = 0\}$.

Suppose that $M \neq \emptyset$ & $Df(x) \in \mathbb{R}^{n \times 1}$ has rank 1 $\forall x \in M$.

Ⓒ N is an n-mfd class C^r & $bN = M$.

(<u>Note:</u> if $N = M$ i.e. $f(y) \leq 0$ $\forall y \in A$: replace f with $g := -f$).

<u>Proof:</u> $p \in N$. Two cases: $f(p) > 0$ or $f(p) = 0$ (def N)

<u>Case 1:</u> $f(p) > 0$. Def. $U := f^{-1}(0, \infty)$ open in $\mathbb{R}^n$ (b/c f continuous)

& $p \in U$ & $\alpha : U \to U$, $\alpha(x) := x$ is trivially a coord.

chart about p, & since U open in $\mathbb{R}^n$, $p \in \operatorname{Int} N$.

<u>Case 2:</u> $f(p) = 0 \Rightarrow p \in M$ (def M) & Ⓗ $\to Df(p)$ has rank 1

$\Rightarrow \frac{\partial f(p)}{\partial x_j} \neq 0 \quad \exists j = 1, \ldots, n.$

W.L.O.G.: $j = n$ i.e. $\frac{\partial f}{\partial x_n}(p) \neq 0$

Def. $F: A \to \mathbb{R}^n$

$$x = (x', x_n) \mapsto F(x) = (x', f(x))$$

$$DF(x) := \begin{pmatrix} I^{(n-1)\times(n-1)} & \overset{\frac{\partial x_1}{\partial x_n}}{\overset{\shortparallel}{0}} \\ \vdots & \\ \underset{\frac{\partial f}{\partial x_j}}{\underbrace{*}} & \underset{\neq 0}{\frac{\partial f}{\partial x_n}(x)} \end{pmatrix} \qquad \det DF(p) = 1 \cdot D_n f(p) \neq 0$$

$DF(p)$ non-singular. By <u>Inverse Fct Thm</u> $\exists V(p)$; $\exists U(F(p))$

Sep 19

Open sets in $\mathbb{R}^n$ s.t.

$F: V(p) \to U(F(p))$ is diffeo of class C^r

& $F: \underbrace{V(p) \cap N}_{\text{open in } N} \to \underbrace{U \cap \mathbb{H}^n}_{\text{open in } \mathbb{H}^n}$ (b/c $x \in N \Leftrightarrow f(x) \geq 0$)

Thus: $\alpha := F^{-1} : \underbrace{U \cap \mathbb{H}^n}_{U'} \to \underbrace{V(p) \cap N}_{V'}$

is the required chart.

Also: $F: V(p) \cap M \to U(p) \cap b\,\mathbb{H}^n$

so $bN = M$. □

Thm. Ⓗ $A \subset \mathbb{R}^n$ open set, $f: A \to \mathbb{R}$ class C^r, $f \neq$ const.

Ⓒ $f^{-1}([0,\infty))$ is n-mfd class C^r, call it N, $f^{-1}(0) = \partial N$.

Ex: $x \in \mathbb{R}^n$, $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$

let $g: \underset{\text{open}}{A'} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ class C^r

Def. $f(x) := g(x') - x_n$. $Df(x) = (Dg(x'), \underset{\neq 0}{-1})$

has max'l rank.

Thm $\Rightarrow$ $N := \{(x', x_n) \mid x_n < g(x')\}$ & $M = \{(x', x_n) \mid x_n = g(x')\}$

n mfd ↑ epigraph of g

n-1 mfd ↑ graph of g

are class C^r-mfds

So epigraphs & graphs of fcts of class C^r always have mfd structure.

& can describe N & M with a single chart

$\alpha: \mathbb{R}^n \to N \cap \mathbb{R}^n$

$x \mapsto \alpha(x) = (x', g(x') - x_n)$.

$\alpha': \underbrace{\mathbb{R}^{n-1}}_{U'} \to M \cap \mathbb{R}^n$

$x' \mapsto \alpha'(x') := \alpha(x', 0)$

$= (x', g(x'))$

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Ex: $\overline{\mathbb{B}^n_r(p)} = \{ |x-p|^2 \le r^2 \}$ is mfd with $b\,\mathbb{B}^n_r(p) = S^{n-1}_r(p)$

with $f(x) := r^2 - |x-p|^2$ Check!

Def: For a k-mfd $M \subset \mathbb{R}^n$ of class C^r, an atlas for M is $\{\alpha_i, U_i, V_i\}_{i \in J}$, a collection of coord. charts s.t. $M = \bigcup_{i \in J} V_i$. Sep 21

Next goal: "integrals of scalar fcts over mfds" i.e. $\int_M f\, dv = ?$

But first: quick detour of linear algebra

A quick review of (some) linear algebra & integration over $\mathbb{R}^k$

Def: Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation

$(h(ax+by) = ah(x) + bh(y) \ \forall a,b \in \mathbb{R}, \forall x,y \in \mathbb{R}^n)$

Facts:

- $h(\vec{x}) = A \cdot \vec{x}$ $\exists A \in \mathbb{R}^{n \times n}$ (matrix), $\vec{x} \in \mathbb{R}^n$ ("vector")
- h is orthogonal if A is an orthogonal matrix i.e. $A^{tr} \cdot A = I^{n \times n}$

  where if $A = (a_{ij}) \Rightarrow A^{tr} := (b_{ij})$, $b_{ij} := a_{ji}$.
- h is isometry if $\|h(\vec{x}) - h(\vec{y})\|_{\mathbb{R}^n} = \|x-y\|_{\mathbb{R}^n}$ (distance preserving)

Def: Linear subspace of $\mathbb{R}^n$ is $W \subseteq \mathbb{R}^n$ s.t. $\forall \vec{u}, \vec{v} \in W \Rightarrow a\vec{u} + b\vec{v} \in W$ $\forall a,b \in \mathbb{R}$

(ex: line through origin is linear subspace ☺)

Lemma: Ⓗ W is linear subspace of $\mathbb{R}^n$ with dimension $n \le k$.

Ⓒ
- $\exists$ orthonormal basis for $\mathbb{R}^n$ whose first k elements are a basis for W
- $\exists$ $h : \mathbb{R}^n \to \mathbb{R}^n$ orthogonal transf. s.t. $h(W) = \mathbb{R}^k \times \{\underbrace{(0,0,\ldots,0)}_{(n-k)\text{-times}}\} \subset \mathbb{R}^n$

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Thm 1 ("Volume fct in $\mathbb{R}^n$")

Ⓗ $k, n \in \mathbb{Z}^+$, $k \leq n$

Ⓒ There ~~exists~~ a unique fct: $V: \mathbb{R}^{n\times k} \to [0,\infty) \subset \mathbb{R}$

$$(\underset{\in\mathbb{R}^n}{\vec{x}_1}, \underset{,\ldots,}{\ldots}, \underset{\in\mathbb{R}^n}{\vec{x}_k}) \to V(\vec{x}_1, \ldots, \vec{x}_k) \in [0,\infty) \text{ s.t.}$$

(1) if $h: \mathbb{R}^n \to \mathbb{R}^n$ is orthog. transf. then $V(h(\vec{x}_1), \ldots, h(\vec{x}_k)) = V(\vec{x}_1, \ldots, \vec{x}_k)$

(2) if $\vec{y}_j \in \mathbb{R}^k \times \{\overset{\in\mathbb{R}^{n-k}}{\vec{0}}\} \in \mathbb{R}^n$ $\forall j = 1, \ldots, k$, so: $\vec{y}_j = \begin{bmatrix} z_{1j} \\ \vdots \\ z_{kj} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{z}_j \\ 0 \end{bmatrix}$

then $V(\vec{y}_1, \ldots, \vec{y}_k) = |\det Z|$, $Z = (z_{ij}) \in \mathbb{R}^{n\times k}$

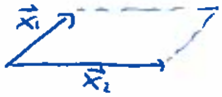
(3) In general: $V(\vec{x}_1, \ldots, \vec{x}_k) = (\det(X^{tr} \cdot X))^{1/2}$ where $X := [\vec{x}_1, \ldots, \vec{x}_k] \in \mathbb{R}^{n\times k}$

(so $X^{tr} \cdot X \in \mathbb{R}^{k\times k}$)

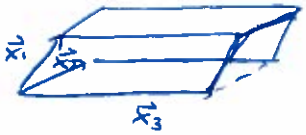
It follows $V(\vec{x}_1, \ldots, \vec{x}_k) = 0 \Leftrightarrow \{\vec{x}_1, \ldots, \vec{x}_k\}$ are linearly dependent vectors in $\mathbb{R}^n$.

<u>Notation:</u> $V(\vec{x}_1, \ldots, \vec{x}_k) =: V(X)$ volume function

<u>Ex:</u>
- $k = 2$; $n = 3$ then $V(\vec{x}_1, \vec{x}_2)$ = area of parallelogram in $\mathbb{R}^n$ with edges $\vec{x}_1$ & $\vec{x}_2$:



- $k = 3$; $n = 3$ then $V(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ = volume of parallelepiped with edges $\vec{x}_1, \vec{x}_2, \vec{x}_3$



<u>Back:</u> $k = 2$; $n = 3$: Recall from Calc III that Area(parallelogram) = magnitude of $\vec{x}_1 \times \vec{x}_2$ (cross pr.)

So, two ways for computing area of a parallelogram:

via <u>Thm 1</u> (formula 3) <u>or</u> via cross product.

Similarly, for $X \in \mathbb{R}^{n\times k}$ we have:

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Thm 2 Ⓗ $X \in \mathbb{R}^{n \times k}$, $k \leq n$

Ⓒ $V(X) = \left(\sum_{[I]} (\det X_I)^2 \right)^{1/2}$,

where • $[I]$ means that above summation is taken over all "ascending k-tuples of $\{1,\dots,n\}$" i.e. over all $I = (i_1, i_2, \dots, i_k) \subset \{1,\dots,n\}$ s.t. $i_1 < i_2 < \dots < i_k$

(ex: "ascending 2-ples of $\{1,2,3\} = \{(1,2),(1,3),(2,3)\}$)

• X_I is $k \times k$ submatrix of X consisting of rows $i_1, i_2, \dots, i_k \in I$.

Ex: $k=2$; $n=3$

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \begin{matrix} \left.\vphantom{x}\right] x_{12} \\ \left.\vphantom{x}\right] x_{23} \end{matrix} \; \rangle\, x_{13}$$

$$\begin{matrix} \uparrow & \uparrow \\ \vec{x}_1 & \vec{x}_2 \end{matrix}$$

So in this ex. by Thm 2, $V(X) = \left(\det^2 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \det^2 \begin{pmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{pmatrix} + \det^2 \begin{pmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \right)^{1/2}$

Thm 1: $\Rightarrow X^{tr} \cdot X = \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} x_{11}^2 + x_{21}^2 + x_{31}^2 ; & x_{11}x_{12} + x_{21}x_{22} + x_{31}x \\ x_{12}x_{11} + x_{22}x_{21} + x_{32}x_{31} ; & x_{12}^2 + x_{22}^2 + x_{32}^2 \end{pmatrix}$

$V(X) = \det(X^{tr} \cdot X)^{1/2} = \dots$

depending on relative size of k and n, formula from Thm 1 may be easier or harder than formula from Thm 2.

Def: Given a set $\{\vec{x}_1, \dots, \vec{x}_k\}$ of linearly independent vectors in $\mathbb{R}^n$, the k-parallelepiped $P(\vec{x}_1, \dots, \vec{x}_k)$ in $\mathbb{R}^n$ is

$$\left\{ \vec{y} \in \mathbb{R}^n \mid \vec{y} = \sum_{j=1}^{k} a_j \vec{x}_j ,\ a_j \in [0,1] \right\}$$

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<u>So:</u> • 2-parallelepiped in $\mathbb{R}^n$ = parallelogram

• 3- –"– –"– in $\mathbb{R}^3$ = "parallelepiped"

Def. Given $X \in \mathbb{R}^{n\times k}$, Volume of $\{\vec{x}_1, \dots, \vec{x}_k\} := V(X)$

Note: if $\{\vec{x}_1, \dots, \vec{x}_k\}$ are linearly indep. then $V(X)$ = volume of $P(\vec{x}_1, \dots, \vec{x}_k)$.

□

Integrating a continuous, scalar fct over a k-mfd in $\mathbb{R}^n$

Just consider special case when M (mfd) is a cpct subset of $\mathbb{R}^n$ (eg. M = ball in $\mathbb{R}^n$ or M = sphere in $\mathbb{R}^n$)

Extension to general case (M unbounded) is analog to ~~dialog~~ dealing with "improper integrals".

Preliminaries: Let M be a cpct k-mfd in $\mathbb{R}^n$, class C^r.

Let $f: M \to \mathbb{R}^n$ be a continuous funct.

Let $K = \text{supp} f$. Note that K is a cpct set in $\mathbb{R}^n$.

To begin with, suppose that Supp f is contained in a single chart: $\alpha: U \to V$, so supp $f \subset V$.

Claim: $\alpha^{-1}(\text{Supp} f)$ is cpct subset of $\mathbb{H}^k$. (Check!)

Thus, wlog we may assume that U is bounded.

So, wlog we may assume ~~for~~

- M cpct k-mfd in $\mathbb{R}^n$, class C^r, that
- $f: M \to \mathbb{R}$ continuous & let $K = \text{Supp} f$ (cpct)

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- $\alpha : U \to V$ a coord. chart s.t. U is <u>bounded</u> & $\text{Supp} f \subset V$.

<u>Def:</u> $\int_M f\,dV := \int_{\text{Int}\,U} (f\circ\alpha)\; V(D\alpha)(x)\, dx_1 \dots dx_k$ ↖ Volume fct

an integral over an <u>open</u> & <u>bounded</u> subset of $\mathbb{R}^k$.

So: $V^2(D\alpha)(x) = \sum_{[I]} \det{}^2 \underline{(D\alpha)}_I$

$$\det{}^2 \begin{pmatrix} \frac{\partial\alpha_{i_1}}{\partial x_1}, & \dots, & \frac{\partial\alpha_{i_1}}{\partial x_n} \\ \vdots & & \\ \frac{\partial\alpha_{i_n}}{\partial x_1}, & \dots, & \frac{\partial\alpha_{i_n}}{\partial x_n} \end{pmatrix}$$

<u>Remark:</u> $\int_M f\,dV$ exists as an ordinary integral:

- $\int_{\text{Int}\,U}$ — bounded set

- $F(x_1, \dots, x_k) = f(\alpha(x_1, \dots, x_n))\; V(D\alpha)(x_1, \dots, x_n) \in C_0(\mathbb{R}^k)$

<u>Lemma:</u> ("$\int_M f\,dV$ is well-defd")

Ⓗ M is cpct k-mfd M of class C^r

$f : M \to \mathbb{R}$ continuous ; $\text{Supp} f \subset V$ for a single chart $\alpha : U \to V$.

Ⓒ $\int_M f\,dV$ does not depend on choice of coord. chart.

<u>Pf:</u> <u>Step 1</u>: (independence of range of α):

let W be open set in U s.t. $\text{Supp} f \subset \alpha(W)$

$$\int_{\text{Int}\,U} (f\circ\alpha)(x)\; V(D\alpha)(x)\, dx = \int_{\text{Int}\,W} \text{same} + \underbrace{\int_{\text{Int}\,U \setminus \text{Int}\,W} \text{same}}_{f=0} = \int_{\text{Int}\,W} (f\circ\alpha)(x)\, V(D\alpha)(x)\, dx$$

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Step 2: (Independence of choice of α):

let $\alpha_0 : U_0 \to V_0$
$\alpha_1 : U_1 \to V_1$ s.t.

$\mathrm{Supp} f \cap V_0 \cap V_1 \neq \emptyset$.

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Recall from last time:

- M cpct k-mfd in $\mathbb{R}^n$, class C^r
- $f: M \to \mathbb{R}$, continuous
- Supp f contained in single chart $\alpha: U \to V$ & w.l.o.g. U bounded open set in $\mathbb{H}^k$ & sufficiently regular to support a notion of Riemann integral

Def:
$$\int_M f\,dV = \int_{\text{Int } U} (f\circ\alpha)\, \overset{\text{volume fct}}{V(D\alpha)}\, dx_1 \ldots dx_k$$
Riemann Integral over a bdd open set in $\mathbb{R}^k$

Lemma: def. above does not depend on choice of coord. chart.

Pf: **Step 1:** $\int_{\text{Int } U} f\,dV = \int_{\text{Int } W} f\,dV$ for any two open sets in $\mathbb{R}^k$ w/ Supp $f \subset U, W$

Step 2: (Independence of α): Let $\alpha_0 : U_0 \to V_0$ & $\alpha_1 : U_1 \to V_1$ s.t.

Supp $f \subset V_0 \cap V_1$.

Claim:
$$\int_{\text{Int } U_0} (f\circ\alpha_0)\, V(D\alpha_0)\, dx_1 \ldots dx_k = \int_{\text{Int } U_1} (f\circ\alpha_1)\, V(D\alpha_1)\, dy_1 \ldots dy_k$$

Pf of claim: Set $V := V_0 \cap V_1$, $W_i := \alpha_i^{-1}(V) \subset U_i$, $i = 0,1$

By **Step 1**: Enough to show that

$$\underbrace{\int_{\text{Int } U_0} f\circ\alpha_0}_{\int f\,dV} \overset{\downarrow}{=} \int_{\text{Int } \alpha_0^{-1}(V)} (f\circ\alpha_0)\, V(D\alpha_0)\, dx \overset{?}{=} \int_{\text{Int } \alpha_1^{-1}(V)} (f\circ\alpha_1)\, V(D\alpha_1)\, dy \overset{\text{Step 1}}{=\!=} \int_{\text{Int } U_1} f\circ\alpha_1 = \int f\,dV$$

Pf is by Change of Var. Formula for integrals in $\mathbb{R}^k$, b/c we know

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$g := \alpha_1^{-1} \circ \alpha_0 : \operatorname{Int} W_0 \to \operatorname{Int} W_1$ is a diffeom., so:

$$\int_{\operatorname{Int}\alpha_1^{-1}(V)} (f\circ\alpha_1)(y)\, V(D\alpha_1)(y)\, dy \underset{\substack{\uparrow \\ y=g(x)=\alpha_1^{-1}\circ\alpha_0(x)}}{=} \int_{\operatorname{Int}(\alpha_0^{-1}(V))} (f\circ \overbrace{\alpha_1)(g(x))}^{\alpha_0(x)}\ \underbrace{V(D\alpha_1)(g(x))\, |\det Dg(x)|}_{\substack{= V(D\alpha_0)(x) \\ \text{check!!}}}\, dx$$

□

<u>Def.</u> of $\int_M f\,dV$ for cpct k-mfd M in $\mathbb{R}^n$ in case when

Supp f not contained in <u>single chart</u>

Need notion of <u>partition of unity</u> fitted to M:

<u>Lemma:</u> Ⓗ M cpct, k-mfd in $\mathbb{R}^n$, class C^r

$M = V_1 \cup \dots \cup V_N$ with $\alpha_i : U_i \to V_i$ coord. charts

Ⓒ $\exists \{\varphi_1, \dots, \varphi_N\} \subset C_0^\infty(\mathbb{R}^n, \mathbb{R})$ i.e. $\varphi_i : \mathbb{R}^n \to \mathbb{R}$, s.t.

(1) $\varphi_i(x) \geq 0 \quad \forall x \in \mathbb{R}^n \quad \forall i \in \{1, \dots, N\} = I$

(2) $\forall i \in I \quad \exists \alpha_i : U_i \to V_i$ s.t. Supp $\varphi_i \cap M \subset V_i$

(3) $\sum_i \varphi_i(x) = 1 \quad \forall x \in M$

<u>Pf.</u> φ_i = partition of unity dominated by $\tilde{V}_i$ where $\tilde{V}_i$

open in $\mathbb{R}^n$ s.t. $V_i = \tilde{V}_i \cap M$. □

<u>Def.</u> Let M be a cpct k-mfd in $\mathbb{R}^n$, class C^r, let $f : M \to \mathbb{R}$

continuous.

$$(*) \quad \int_M f\, dV := \sum_{i=1}^N \int_M (f \overset{\leftarrow \text{pointwise mult.}}{\bullet} \varphi_i)\, dV \quad \text{"new defn"}$$

where $\{\varphi_1, \dots, \varphi_N\}$ is a part. of unity subordinated

to an (some; any) atlas for M.

<u>Remarks:</u> (1) If Supp f lies indeed in <u>single chart</u> $\alpha : U \to V$

then "new def" agrees with "original def":

Call $A = \operatorname{Int} U$ (open in $\mathbb{R}^k$)

$$\sum_{i=1}^{N} \int_M (\varphi_i f) dV = \sum_{i=1}^{N} \int_{\operatorname{Int} U} (\varphi_i \circ \alpha)(x) (f \circ \alpha)(x) V(D\alpha) dx$$

$$= \int_{\operatorname{Int} U} \underbrace{\left(\sum_{i=1}^{N} (\varphi_i \circ \alpha)(x) \right)}_{=1 \quad (\varphi_i \text{ are part. of unity})} (f \circ \alpha)(x) V(D\alpha) dx$$

$$= \underbrace{\int_{\operatorname{Int} U} (f \circ \alpha)(x) V(D\alpha)(x) dx}_{\text{"old def"}}$$

□

(2) New def $(*)$ is indep. of choice of part. of unity.

Let $\{\varphi_i\}$, $\{\psi_j\}$ be two part. of unity.

By Remark (1) for $\psi_j f$ (fix j & ψ_j)

$$\sum_i \int_M \varphi_i (\psi_j f) dV = \int_M \psi_j f dV \quad \underline{\underline{\forall j.}}$$

$$\Rightarrow \sum_{i,j} \int_M (\varphi_i \psi_j f) dV = \sum_j \int_M \psi_j f dV \qquad (a)$$

By same token (switch ψ_j & φ_i):

$$\sum_j \int_M \psi_j (\varphi_i f) dV = \int_M \varphi_i f dV \quad \underline{\underline{\forall i}}$$

$$\sum_{i,j} \int_M (\psi_j \varphi_i f) dV = \sum_i \int_M \varphi_i f dV \qquad (b)$$

Comparing (a) and (b) (which have <u>same</u> left hand sides) get

$$\sum_i \int_M \varphi_i f dV = \sum_j \int_M \psi_j f dV.$$

<u>Thm:</u> (linearity of $\int_M$):

$$\int_M (af + bg) dV = a \int_M f dV + b \int_M g dV \quad \forall a,b \in \overset{\mathbb{R}?}{\mathbb{C}} \quad \forall f,g : M \to \mathbb{R} \text{ cont.}$$

□

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Note: Our current def of $\int_M f\,dV$, although rigorous & robust, is difficult to implement

(need to compute part. of unity, φ_i; need to integrate product of φ_i & f)

Need a more practical algorithm (method), which requires:

Def: M cpct mfd in $\mathbb{R}^n$, class C^r.

$D \subset M$ (a given subset of M).

We say that D has zero measure in M if $\exists$ atlas $\{\alpha_i, U_i, V_i\}$ s.t. D covered by (countably many) charts $\alpha_i : U_i \to V_i$, i.e. $D \subset \bigcup_i V_i$.

s.t. $D_i := \alpha_i^{-1}(D \cap V_i) \subset \mathbb{R}^k$ has zero-Euclidean measure in $\mathbb{R}^k$, $\forall i$.

Equivalently: $\forall$ atlas, $\forall$ chart in atlas, $\alpha : U \to V$ $\alpha^{-1}(D \cap V)$ has zero meas. in $\mathbb{R}^k$. Check! □

Remark: if M has boundary bM, then bM has zero-measure in M b/c we know that $\forall V$ s.t. $bM \cap V \neq \emptyset$, we have:

$$\alpha^{-1}(bM \cap V) \subset \underbrace{\mathbb{R}^{k-1} \times \{0\}}_{\text{a zero-meas. subset of } \mathbb{R}^k}.$$

Computationally friendly tool:

Thm: Ⓗ M cpct k-mfd in $\mathbb{R}^n$; $f : M \to \mathbb{R}$ cont.

$\alpha_i : A_i \to M_i$ are charts $(i = 1, \dots, N)$ s.t.

- A_i is open in $\mathbb{R}^k$, &
- $M = M_1 \dot{\cup} \dots \dot{\cup} M_N \dot{\cup} K$ with $K \subset M$ a set of measure zero in M.

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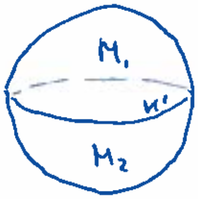
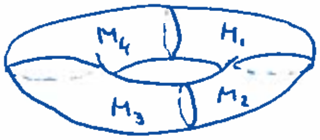
Ⓒ $\int_M f\,dV = \sum_{i=1}^{N} \int_{A_i} (f\circ\alpha_i)\, V(D\alpha_i)$ (**)

ex: $M = S^2 \subseteq \mathbb{R}^3$



M_i = upper & lower hemispheres
K = equator
$A_1 = A_2 = \mathbb{D}_1(0) \subset \mathbb{R}^2$.



K = union of four circles.

Def. M_i called a parametrized mfd (atlas is single chart)

Thm says that $\int_M f\,dV$ can be computed by cutting up M into parametrized mfds ("pieces") and integrating on each piece separately.

Think of $\{M_j\}_{j=1,\ldots}$ as a "tiling" of M & think of K as the "grout" between your tiles.

Note: if $bM \neq \emptyset$ then $bM \subset K$ (b/c the A_i are open in $\mathbb{R}^k$).

Pf of Thm: since both sides of (**) are linear in f, wlog may assume that Supp f is contained in single chart:
(use part. of unity)

$\beta: U \to V$ & U is bounded. Thus:

$$\int_M f\,dV = \int_{\text{Int}\, U} (f\circ\beta)\, V(D\beta).$$

Step 1: Set $W_i := \beta^{-1}(M_i \cap V)$ & $L := \beta^{-1}(K \cap V)$.

Then W_i open in $\mathbb{R}^k$ (b/c $W_i \subset A_i$ open in $\mathbb{R}^k$ by Ⓗ)
& L has zero measure in $\mathbb{R}^k$ (by Ⓗ)

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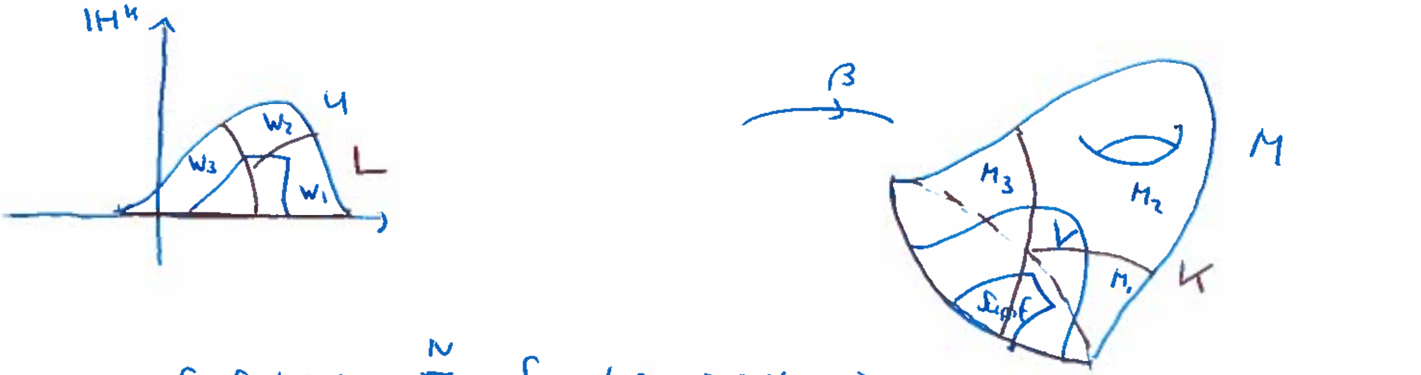


Claim: $\int_M f\,dV = \sum_{i=1}^{N} \int_{W_i} \underbrace{(f\circ\beta)\,V(D\beta)}_{=:F}$.

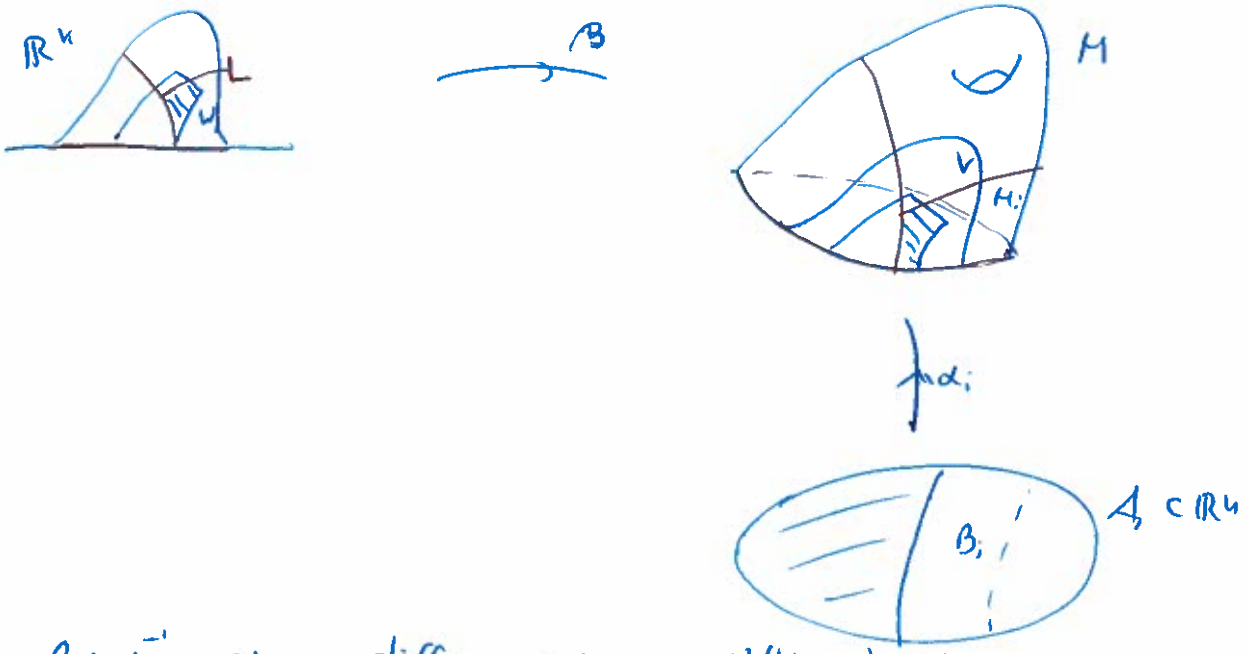
Then: $\sum_{i=1}^{N} \int_{W_i} F = \int_{\text{Int}\,U\setminus L} F$ (def of W_i; def L; additivity of integral)

$= \int_{\text{Int}\,U} F$ (L has 0-meas. in $\mathbb{R}^k$ & F cont. on U)

$= \int_M f\,dV$ (def of $\int_M$ & ⊕ f)

Step 2: Claim: $\int_{W_i} F = \int_{A_i} F_i$, $F_i := (f\circ\alpha_i)\,V(D\alpha_i)$

Consider: $\alpha_i^{-1}\circ\beta$ (e.g. focus on W_1)



Then: $\beta\circ\alpha_i^{-1}$ is a diffeom. : $W_i \to \alpha_i^{-1}(M\cap V) =: B_i$

1-1; onto

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Recall from last time:

M cpct k-mfd in $\mathbb{R}^n$ of class C^r. $f: M \to \mathbb{R}$ continuous

$\{\beta_i, U_i, V_i\}$ atlas for M, $\{\psi_i\}_{i=1,\dots,N}$ partition of unity on M

dominated by atlas.

$$\int_M f\,dV := \begin{cases} \int_U (f\circ\beta)\, V(D\beta)\, dx_1 \dots dx_k, & \text{if supp } f \subset V \text{ (one chart)} \\ \sum_{i=1}^N \int (f\circ\beta_i)(x)\,(\psi_i \circ \beta_i)(x)\, V(D\beta_i)\, dx_1 \dots dx_k, & \text{otherwise} \end{cases}$$

Thm: Ⓗ M cpct k-mfd in $\mathbb{R}^n$, class C^r, $f: M \to \mathbb{R}$ cont.

$\alpha_i: A_i \to M_i$ charts st. • A_i open in $\mathbb{R}^k$
• $M = M_1 \,\dot\cup \dots \dot\cup\, M_k \,\dot\cup\, K$, K zero meas. in M

Ⓒ $\int_M f\,dV = \sum_{i=1}^N \int_{A_i} (f\circ\alpha_i)\, V(D\alpha_i)$

Pf of Thm: to show: $\int_M f\,dV = \sum_i \int_{A_i} F_i$ $(*)$

wlog: assume $f:\beta: U \to V$ (single chart; then invoke linearity of lhs & of rhs of $(*)$).

to show: $$\int_{\text{int } U} \underbrace{(f\circ\beta)\, V(D\beta)}_{\substack{=:F \\ \uparrow \\ \text{given (def.)}}}\, dx = \sum_i \int_{A_i} F_i$$

Step 1: Show: $\int_{\text{int } U} F = \sum_i \int_{W_i} F\,dx$, where:

$W_i := \beta^{-1}(M_i \cap V)$ (is open in $\mathbb{R}^k$ b/c $\subset A_i$ open by Ⓗ).

Step 2: $\int_{W_i} F\,dx = \int_{A_i} F_i$, $F_i = (f\circ\alpha_i)\, V(D\alpha_i)$

Consider: $g := \alpha_i^{-1} \circ \beta : \underset{\beta^{-1}(M_i \cap V)}{\overset{\shortparallel}{W_i}} \to \alpha_i^{-1}(M \cap V) =: B_i$ diffeom. (see last picture from last time)

Then $\int_{B_i} F_i(x)\,dx \underset{\substack{y = g(x) = \alpha_i^{-1}(\beta(x)) \\ \text{CVF}}}{=} \int_{W_i} (f\circ\beta)(x)\, \underbrace{V(D\beta)(x)}_{\text{check!}}\,dx = \int_{W_i} F.$

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Thus: $\int_{W_i} F = \int_{B_i} F$

Moreover, since Supp f <u>closed</u> in M, then $\alpha_i^{-1}(\text{Supp} f)$ <u>closed</u> in A_i, so: $O_i := A_i \setminus \alpha_i^{-1}(\text{Supp} f)$ <u>open in A_i</u> ($\therefore$ in $\mathbb{R}^k$)

<u>So</u>: $$\int_{A_i} F_i = \int_{B_i} F_i + \underbrace{\int_{O_i} F_i - \int_{B_i \cap O_i} F_i}_{=0-0 \text{ since } \text{Supp} F \subset B_i}$$

$A_i = B_i \cup O_i \setminus (B_i \cap O_i)$

Summary: $$\int_M f\, dV \underset{\substack{\uparrow \\ \text{Step 1}}}{\equiv} \sum_{i=1}^{N} \int_{W_i} F \underset{\substack{\uparrow \\ \text{Step 2}}}{=} \sum_i \int_{A_i} F_i$$ $\square$

<u>Ex:</u> Compute surface area of $M = \mathbb{S}^2(a) = \{(x,y,z) \mid x^2+y^2+z^2 = a^2\}$

$$\text{Area}(\mathbb{S}^2(a)) = \int_{\mathbb{S}^2(a)} 1\, dV$$ ($\mathbb{S}^2$ = cpct 2-mfd in $\mathbb{R}^3$)
f=1

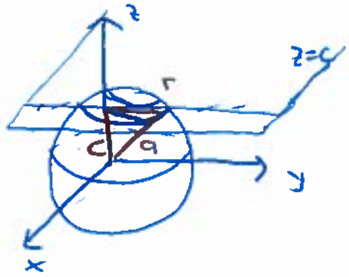
We may implement "Thm" in two different ways:

<u>Method 1:</u> use 2-tiling of $\mathbb{S}^2$ from last time (A_1 & A_2 : hemispheres, K= equator)

<u>Method 2:</u> use another tiling of $\mathbb{S}^2(a)$, as follows:

<u>note:</u> $\mathbb{S}^2(a) \cap \{z=c\}$, $|c|<a$:
= circle, radius $\rho^2 = a^2-c^2$

$$= \begin{cases} x^2+y^2 = a^2-c^2 \\ z = c \end{cases}$$



"Inspired" by this picture, we realize that $\mathbb{S}^2(a)$ may be tiled as follows: $$\alpha(t,z) := \left((a^2-z^2)^{1/2} \cos t,\ (a^2-z^2)^{1/2} \sin t,\ z \right),$$

$(t,z) \in A := \{(t,z) \in \mathbb{R}^2 \mid 0<t<2\pi \ ;\ |z|<a\}$ open in $\mathbb{R}^2$

$$\underset{t=0 \text{ \& } t=2\pi}{\alpha =} \{ ((a^2-z^2)^{1/2}, 0, z) \ ,\ |z|<a \}$$

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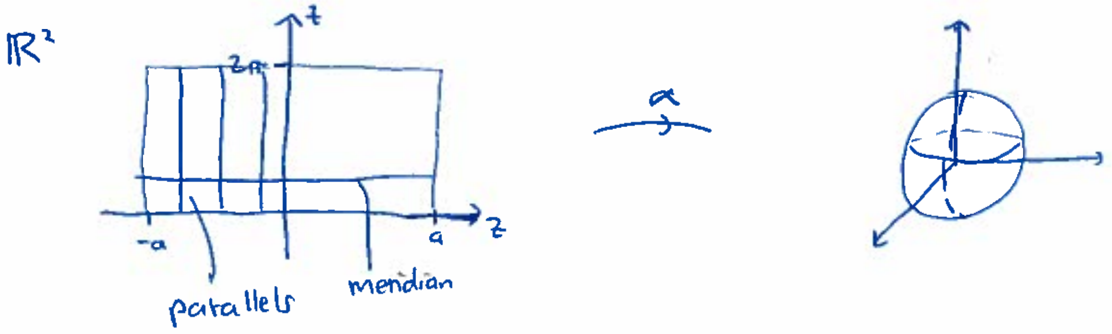

$\mathrm{Surf}\,(\,S^2(a)\,) = \int_A V(D\alpha)\,dx$

- Compute $D\alpha$
- Compute $V(D\alpha)$ check: $V(D\alpha) = a$

get: $\mathrm{Surf}(S^2(a)) = \int_{(0,2\pi)\times(-a,a)} a\,dz\,dt = a\cdot 2\pi\cdot(2a) = 4\pi^2 a^2$:)

Check: that 2-tile thing (hemispheres) gives same answer.

[Check: torus, area]

Differential forms

Goal: extend "vector integral calculus" [Green's thm
div thm
Stokes thm]

from: Surfaces in $\mathbb{R}^3$; to: cpct k-mfds in $\mathbb{R}^n$

Tool for vector calculus in $\mathbb{R}^3$: linear algebra: (vector fields acting on fcts ; ∇, Curl)

Tool for vector calculus on k-mfd in $\mathbb{R}^n$:

Multilinear Algebra : k-tensors acting on differential forms

Multilinear Algebra: a survey

Given any two positive integers, k & n:

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Def: (k-tensor) : Let V be an n-dim. v.s. over $\mathbb{R}$.

Let $V^k = \underbrace{V \times \ldots \times V}_{k\text{-many}} = \{ (\vec{v}_1, \ldots \vec{v}_k) \mid \vec{v}_j \in V, j = 1, \ldots k \}$

Given : $f : V^k \to \mathbb{R}$ we say that f is a k-tensor on V

if $\forall i \in \{1, \ldots, k\}$, $\forall \vec{a}_j \in V$, $j = 1, \ldots, i-1, i+1, \ldots, k$,

we have that

$$\vec{v} \in V \to f(\vec{a}_1, \ldots, \vec{a}_{i-1}, \vec{v}, \vec{a}_{i+1}, \ldots, \vec{a}_k)$$

is a linear transformation : $V \to \mathbb{R}$ $\forall i, \forall \vec{a}_j$

(k-tensors also called multilinear transf.)

Notation: $\mathcal{L}^k(V) = \{$ all k-tensors on $V \}$

Remarks:

- $k=1 \Rightarrow \mathcal{L}^1(V) = \{$ all linear transf. $V \to \mathbb{R} \} = V^*$ (dual space of V)
- $k=2 \Rightarrow \mathcal{L}^2(V) = \{$ bilinear transformations : $V \times V \to \mathbb{R} \}$
- check: k-tensor on $V = \mathbb{R}$ = 1-tensor on $V = \mathbb{R}^k$

Ex: $\forall i \in \{1, \ldots k\}$; $\forall j \in \{1 \ldots n\}$ $\forall V$ (n-dim.)

$f_{ij}(\vec{v}_1, \ldots, \vec{v}_k) := v_{ij}$ (j-th component of $\vec{v}_i$)

Thm: $\mathcal{L}^k(V)$ is a vector space over $\mathbb{R}$ via:

$(f+g)(\vec{v}_1, \ldots, \vec{v}_k) := f(\vec{v}_1, \ldots, \vec{v}_k) + g(\vec{v}_1, \ldots, \vec{v}_k)$

tensor sum "+"

$\underset{\text{prod. by scalar}}{(cf)}(\vec{v}_1, \ldots, \vec{v}_k) = \underset{\text{product in } \mathbb{R}}{c\, f}(\vec{v}_1, \ldots, \vec{v}_k)$

Pf: Check! □

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Lemma: Ⓗ $\{\vec{a}_1, - \vec{a}_n\}$ is a basis for V.

$f, g \in \mathcal{L}^k(V)$ s.t. $\forall\ I := \{i_1, -, i_k\} \subset \{1, -, n\}$ we have

$f(\vec{a}_{i_1}, -, \vec{a}_{i_n}) = g(\vec{a}_{i_1}, -, \vec{a}_{i_n})$

Ⓒ $f(\vec{v}_1, -, \vec{v}_k) = g(\vec{v}_1, -, \vec{v}_k) \quad \forall \vec{v}_j \in V.$

Note: no preassigned ordering of elts of I

no requirements that elts of I be distinct.

Pf: Check!! $\square$

Thm: (basis for $\mathcal{L}^k(V)$):

Ⓗ $\{\vec{a}_1, -, \vec{a}_n\}$ basis for V

$I = \{i_1, - i_k\} \subset \{1, -, n\}$ (no preassigned order, repititions allowed within I)

Ⓒ $\exists!\ \varphi_I \in \mathcal{L}^k(V)$ s.t.

$\forall\ J = \{j_1, j_k\} \subset \{1, -, n\}$ we have:

$$(*) \quad \varphi_I(\vec{a}_{j_1}, -, \vec{a}_{j_k}) = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

Furthermore, $\{\varphi_I\}_I$ are a <u>basis</u> for $\mathcal{L}^k(V)$

The φ_I's are called: elementary k-tensors on V corresponding to basis $\{\vec{a}_1, -, \vec{a}_n\}$ for V.

Note: $|\{\text{distinct } k\text{-tuples from } \{1, -, n\}| = n^k$ (check!!)

eg. $k = 2$; $n = 3$

12 21
13 31
23 32
Also 11, 22, 33

$\}\ 9 = 3^2$

$k = 3$; $n = 2$

112, 121, 211
122, 212, 221
222, 111

$\}\ 8 = 2^3.$

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<u>Pf.</u> <u>Step 1:</u> Prove uniqueness : follows from preceding lemma.

<u>Step 2:</u> prove $\exists$ existence :

<u>Case 1:</u> $k=1$ $\mathcal{L}^1(V) = V^*$; $I = \{1\}; \{2\}; \dots; \{n\}$

Linear Algebra says we can determine any linear transf.

$f: V \to \mathbb{R}$ by specifying its values on (some) basis of V.

So we define $\underset{\uparrow}{\varphi_i}(\vec{a}_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ $\forall i,j = 1, \dots, n.$ (**)

these are the desired 1-tensors.

<u>General case:</u> $k \geq 2$. $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ (repetition allowed etc.)

<u>Def.</u> $\varphi_I(\vec{v}_1, \dots, \vec{v}_k) := \overbrace{\varphi_{i_1}(\vec{v}_1)}^{\in \mathbb{R}} \overbrace{\varphi_{i_2}(\vec{v}_2)}^{\in \mathbb{R}} \dots \overbrace{\varphi_{i_k}(\vec{v}_k)}^{\in \mathbb{R}}$ ← product in $\mathbb{R}$

def. in Case 1 $(k=1)$

<u>Check.</u> (1) φ_I satisfies (*) immediate from (**)

(2) $\{\varphi_I\}_I$ are a <u>basis</u> for $\mathcal{L}^k(V)$ < linearly indep, generate $\mathcal{L}^k(V)$

use lemma. □

<u>Corollary:</u> Ⓗ $\{\vec{a}_1, \dots, \vec{a}_n\}$ basis for V

© $\forall I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ (reps. allowed etc.)

$\forall d_I \in \mathbb{R}$ $\exists!$ $f_{d_I} \in \mathcal{L}^k(V)$ s.t. $f_I(\vec{a}_{i_1}, \dots, \vec{a}_{i_k}) = d_I$.

<u>Thus:</u> a k-tensor is determined by specifying its values on all k-tuples of elts from (any) basis of V.

<u>Oct 3</u>

Recall from last time:

- $k, n \in \mathbb{Z}^+$, V: = n-dim vector space over $\mathbb{R}$
- $\mathcal{L}^k(V) = \{$k-tensors on $V\} = \{$tensors of <u>order k</u> on $V\}$ (a vs. over $\mathbb{R}$)

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k-tensors: multilin. transf.: $\underbrace{V\times\ldots\times V}_{k\text{-linear}} \to \mathbb{R}$

- $k=1 \to \mathcal{L}^1(V) = V^*$
- $k=2 \to \mathcal{L}^2(V) = \{$bilinear transf. on $V\}$
- Basis for $\underline{\mathcal{L}^k(V)}$ (given by basis for $V = \{\vec{a}_1, \ldots, \vec{a}_n\}$) is the set

$$\{\underbrace{\varphi_I}_{\substack{\text{elementary}\\ k\text{-tensor, where:}}} \mid I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}\}$$

↪ • repts allowed
• all possible orderings allowed

$$\forall J = (j_1, \ldots, j_k) : \varphi_I(\vec{a}_{j_1}, \ldots, \vec{a}_{j_k}) = \begin{cases} 1, & J=I \\ 0, & \text{otherwise} \end{cases}$$

$$\dim \mathcal{L}^k(V) = n^k$$

- fact: $\Phi_I(\vec{v}_1, \ldots, \vec{v}_k) = \Phi_{i_1}(\vec{v}_1) \cdot \ldots \cdot \Phi_{i_k}(\vec{v}_k)$, where $\Phi_{i_l} \in \mathcal{L}^1(V)$, $\Phi_{i_l}(\vec{a}_j) = \begin{cases} 1, & i_l = j \\ 0, & \text{otherwise} \end{cases}$

<u>Corollary:</u> Ⓗ $\{\vec{a}_1, \ldots, \vec{a}_n\}$ = basis for V

Ⓒ $\forall$ I = k-subset of $\{1, \ldots, n\}$ $\forall d_I \in \mathbb{R}$ $\exists!$ $f \in \mathcal{L}^k(V)$ s.t.

$$f(\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}) = d_I \qquad (f := d_I \varphi_I)$$

<u>Ex:</u> $V = \mathbb{R}^n$ with canonical basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$

- $k=1$ let $\{\varphi_1, \ldots, \varphi_n\}$ be dual basis of for $\mathcal{L}^1(V) = V^*$

(ie. $\varphi_{E,i}(\vec{e}_j) = \delta_{ij}$). Writing $\vec{x} = x_1\vec{e}_1 + \ldots + x_n\vec{e}_n$ get: $\varphi_i(\vec{x}) = x_i$

(φ_i = projection onto i-th coord.)

- $k \geq 2$: $I = \{i_1, \ldots, i_k\}$ Then: $\varphi_I(\vec{x}_1, \ldots, \vec{x}_k) = \underbrace{\varphi_{i_1}(\vec{x}_1)}_{x_{i_1,1}} \cdot \underbrace{\varphi_{i_2}(\vec{x}_2)}_{x_{i_2,2}} \cdot \ldots \cdot \underbrace{\varphi_{i_k}(\vec{x}_k)}_{x_{i_k,k}}$

where: $\vec{x}_1 = (x_{11}, x_{21}, \ldots, x_{n1})$ etc.

Thus: elementary k-tensor φ_I is <u>monomial of degree k</u>.

& general k-tensor on $\mathbb{R}^n$ is linear comb. of such monomials.

eg. • 1-tensors in $\mathbb{R}^n$: $f(\vec{x}) = d_1x_1 + \ldots + d_nx_n$, $d_j \in \mathbb{R}$

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- 2-tensors on $\mathbb{R}^n$: $f(\vec{x},\vec{y}) = \sum_{i,j=1}^{n} d_{ij}\, x_i x_j$, $d_{ij} \in \mathbb{R}$
- k-tensor on $\mathbb{R}^n$: $f(\vec{x}_1, \dots, \vec{x}_k) = \sum_{i_1,\dots,i_k=1}^{n} \underbrace{d_{i_1 \dots i_k}}_{\in \mathbb{R}} x_{i_1} \cdots x_{i_k}$

<u>Ex:</u> $k=2;\ n=4$: 2 tensors in $\mathbb{R}^4$

11	21
12	22
13	23
14	24

etc.

<u>Ex:</u> $\varphi_{11}(\vec{x},\vec{y}) = x_1 y_1$; $\varphi_{34}(\vec{x},\vec{y}) = x_3 y_4$ etc.

- is $h(\vec{x},\vec{y}) = x_1 y_1 - 7 x_3 y_3 \in \mathcal{L}^2(\mathbb{R}^4)$?

$$= \varphi_{11} - 7\varphi_{33} \quad ☺$$

is $f(\vec{x},\vec{y}) := \underset{\varphi_{12}}{3 x_1 y_2} - 5 \underset{\uparrow\ ☹}{x_3 x_3} \in \mathcal{L}^2(\mathbb{R}^4)$?

not elementary 2-tensor!

Tensor Product

<u>Def:</u> Let $f \in \mathcal{L}^k(V)$ & $g \in \mathcal{L}^l(V)$

$f \otimes g \in \mathcal{L}^{k+l}$ defined as follows:

$$\underset{\uparrow}{f \otimes g}(\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_{k+l}) = f(\vec{v}_1, \dots, \vec{v}_k) \underset{\text{product in } \mathbb{R}}{\cdot} g(\vec{v}_{k+1}, \dots, \vec{v}_{k+l})$$

Tensor product of f & g

<u>Thm:</u> (properties of $\otimes$): Ⓗ f, g, h given tensors on V

Ⓒ (1) (Associativity) $f \otimes (g \otimes h) = (f \otimes g) \otimes h$

(2) (Homogeneity) $(cf) \otimes g = f \otimes cg \quad \forall c \in \mathbb{R}$

(3) (Distributivity) Assume that f, g have <u>same order</u>

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$$\underset{\substack{\uparrow \\ \text{Sum of} \\ k\text{-tensors}}}{(f+g)} \otimes h = f\otimes h \underset{\substack{\uparrow \\ \text{Sum of } k+l\text{-tensors}}}{+} g\otimes h \qquad (\text{if order of } h = l)$$

$$h\otimes \overset{\substack{\text{Sum of} \\ k\text{-tensors} \\ \nwarrow}}{(f+g)} = h\otimes f + h\otimes g$$

(4) Let $\{\vec{a}_1, \dots, \vec{a}_n\}$ be a basis for V & let φ_I elem. tensor.

$I = \{i_1, \dots, i_k\}$, then $\varphi_I = \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}$.

Addition of linear transformations

Let V, W be two vector spaces on $\mathbb{R}$; $k \in \mathbb{Z}^+$.

Let $T: V \to W$ be a linear transf.

Def: dual transformation: $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$
$f \longmapsto T^* f$

where $(T^* f)(\vec{v}_1, \dots, \vec{v}_k) = f(T\vec{v}_1, \dots, T\vec{v}_k)$ i.e.

$$\begin{tikzcd}
V\times\dots\times V \arrow[r, "T\times\dots\times T"] \arrow[dr, dashed, "T^*f"'] & W\times\dots\times W \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}$$

Remarks:

- $\forall f \in \mathcal{L}^k(W)$, $T^* f$ is indeed multilin. : $V\times\dots\times V \to \mathbb{R}$

  thus $T^* f \in \mathcal{L}^k(V)$

- $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ is ~~not~~ linear, more precisely:

Thm (linearity of T^*):

Ⓗ V, W any two vector spaces on $\mathbb{R}$, $T: V \to W$ lin. transf., dual: $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$.

Ⓒ T^* is linear: $T^*(af \underset{\substack{\uparrow \\ \text{lin. comb. of} \\ k\text{-tensors on } W}}{+} bg) = aT^*(f) \underset{\substack{\uparrow \\ \text{Sum of } k\text{-tensors on } V}}{+} b\,T^*(g)$ $\forall a, b \in \mathbb{R}, f, g \in \mathcal{L}^k(W)$

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(2) $T^*(\underbrace{f \otimes g}_{2k\text{-tensor on } W}) = \underbrace{T^* f \otimes T^*(g)}_{2k\text{-tensor on } V}$

(3) Let X be a v.s. on $\mathbb{R}$ & $S: W \to X$ be a linear transf. Then:

$(S \circ T)^* = T^* \circ S^*$ i.e. $\quad (S \circ T)^*(\underset{\in \mathcal{L}^k(X)}{h})\, e = \underbrace{T^*(\underbrace{S^* h}_{\in \mathcal{L}^k(W)})}_{\in \mathcal{L}^k(V)}$ $\quad$ i.e.

That is:
if :

$$\begin{tikzcd} & W \arrow[dr, "S"] & \\ V \arrow[ur] \arrow[rr, "S \circ T"'] & & X \end{tikzcd}$$

then

$$\begin{tikzcd} & \mathcal{L}^k(W) \arrow[dl, "T^*"'] & \\ \mathcal{L}^k(V) & & \mathcal{L}^k(X) \arrow[ul, "S^*"'] \arrow[ll, "(S \circ T)^*"] \end{tikzcd}$$

Pf: check !! $\square$

A quick review of permutations

Let $k \in \mathbb{Z}^+, k \geq 2$. Permutation of $\{1, \ldots, k\}$ is a 1-1, onto fct :

$\sigma : \{1, \ldots, k\} \circlearrowleft$ (a rearrangement of the set)

$S_k = \{$all permutations of $\{1, \ldots, k\}\}$ is a group under composition of functions ($Id: 1 \to 1, \ldots, k \to k$)

Note: σ is 1-1 $\Rightarrow$ repetitions not allowed in output

$i_i \neq i_j \Rightarrow \sigma(i_i) \neq \sigma(i_j) \quad \to \quad |S_k| = k!$

Def: $1 \leq i < k$. Def. $e_i \in S_k$ as follows:

$$e_i(j) = \begin{cases} j, & j \neq i, i+1 \\ i+1, & j = i \\ i, & j = i+1 \end{cases}$$

elementary (i-th) permutation

Remarks:
- Identity $\neq$ elementary!
- $k = 2 \rightsquigarrow S_2 = \{12, 21\}$, e_1 note: $12 \xrightarrow{e_1} 21 \xrightarrow{e_1} 12$
  So : $Id = e_1 \circ e_1$.

Lemma: $\forall \sigma \in S_k$ is a composition of elementary permutations.

Pf: Check! (Note : $Id = e_i \circ e_i \quad \forall i \in \{1, \ldots, k\}$).

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Def. (sign } of permutation):

Let $\sigma \in S_k$.

- Inversion in $\sigma := \{\text{all pairs } i<j \in \{1,\dots,k\} \text{ s.t. } \sigma(i) > \sigma(j)\}$.
- Sign of $\sigma :=$ $\begin{cases} -1, & \text{if total \# of inversion of } \sigma \text{ is odd} \\ +1, & \text{---''--- \quad ---''--- \quad even} \end{cases}$

Ex: $k=2$: $Id : \{1,2\} \to 12$ even $\text{sign}(Id) = +1$

$\sigma_1 : \{1,2\} \to 21$ odd $(\sigma(1) > \sigma(2) \therefore \text{sign } \sigma_1 = -1)$

Ex: $k=3$: $\sigma : \{1,2,3\} \to 213$

$1,2 \mapsto 21$ ← inversion
$1,3 \to 31$ ← no inv.
$2,3 \to 13$ ← ---''--- ''

$\}$ $\text{sign}(\sigma) = -1$
note: σ is elementary

τ: $\{1,2,3\} \to 312$

$1,2 \to 31$ ← inv.
$1,3 \to 32$ ← inv.
$2,3 \to 12$ ← not inv.

$\}$ $\text{sign}(\tau) = +1$
note: τ not elem.

but: $123 \xrightarrow{\sigma_3} 131 \xrightarrow{\sigma_1} 312$: $\tau = \sigma_1 \circ \sigma_2$

Ex: $\sigma \in S_k$ is any elementary permutation check: $\text{sign}\,\sigma = -1$

Lemma: Ⓗ $\sigma, \tau \in S_k$

Ⓒ (1) If τ is a composition of m elementary permutations,

then $\text{sign}(\tau) = (-1)^m$.

(2) $\underbrace{\text{sign}(\sigma \circ \tau) = \text{sign}(\sigma) \cdot \text{sign}(\tau)}_{\text{product of signs.}}$

(3) if $p \neq q$ & $\tau \in S_k$ s.t. $\tau_j = \begin{cases} q & \text{if } j = p \\ p & \text{if } j = q \\ j & \text{if } j \neq p,q \end{cases}$

(e.g. $k = \tau$: $1,2,3 \to 3,2,1$)

then $\text{sign } \tau = -1$.

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Pf: Check! □

Back to $\mathcal{L}^k(V)$

Alternating k-tensors

Def: $f\in\mathcal{L}^k(V)$. We say that

- f is an alternating k-tensor ("f is alternating") if

$$f(\vec{v}_1, \dots, \underline{\vec{v}_{i+1}, \vec{v}_i}, \dots, \vec{v}_k) = -f(\vec{v}_1, \dots, \vec{v}_k) \quad \forall \vec{v}_j \in V, \ \forall i=1,\dots,k.$$

Alt. notation: $\underbrace{f(\sigma_i(\vec{v}_1, \dots, \vec{v}_k))} = -f(\vec{v}_1, \dots, \vec{v}_k)$

$$f^{\sigma_i}(\vec{v}_1, \dots, \vec{v}_k) = -f(\vec{v}_1, \dots, \vec{v}_k)$$ (note: here $\sigma_i := e_i$)

- f is a symmetric k-tensor ("f is symmetric") if

$$f^{\sigma_i} = f \quad \forall i = 1, \dots, k-1$$

Focus on alternating k-~~vectors~~ tensors

Ex: $k=2$. Then $S_2 = \{12, \underset{\overset{\uparrow}{e_1}}{21}\}$ the only elem. perm.

So $f\in\mathcal{L}^2(V)$ is alternating $\Leftrightarrow$ $f(\vec{v}_2, \vec{v}_1) = -f(\vec{v}_1, v_2)$ $\forall \vec{v}_i \in V$.

Def: V = v.s. over $\mathbb{R}$, $k\in\mathbb{Z}^+$, $k\geq 2$.

$A^k(V) = \{$Alternating k-tensors on $V\}$

Fact: $A^k(V)$ is a vector subspace of $\mathcal{L}^k(V)$.

Def: V v.s. over $\mathbb{R}$, $k=1$. ($S_1=\{1\}$ identity so no elem. perm.)

Define $A^1(V) = \mathcal{L}^1(V) = V^*$.

Ex: $k\geq 2$ & $f = \varphi_I$ (elem. k-tensor)

Show that $\varphi_I \notin A^k(V)$: check!

Ex: if $k\geq 2$: certain linear ~~transf.~~ combinations of elem. k-tensors

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are in $A^k(V)$!

For instance:

- $k=2$, $V=\mathbb{R}^n$, $f := \underbrace{\varphi_{ij}}_{I} - \underbrace{\varphi_{ji}}_{J} \in A^2(V)$ $\forall i,j = 1 \ldots n$

and: $f(\vec{x},\vec{y}) = x_i y_j - x_j y_i = \det\begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix}$:

from this interpretation of $f(\vec{x},\vec{y})$ it follows right away that $f(\vec{y},\vec{x}) = -f(\vec{x},\vec{y})$

- $k=3$, $V=\mathbb{R}^n$, $g(\vec{x},\vec{y},\vec{z}) = \det\begin{pmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{pmatrix} \in A^3(\mathbb{R}^n)$

$$g = \varphi_{ijk} + \varphi_{jki} + \varphi_{kij} - \varphi_{jik} - \varphi_{ikj} - \varphi_{kji} .$$ Check!! □

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Recall from last time

$k\in\mathbb{Z}^+$, $k\geq 2$

$V = n$ -dim v.s. over $\mathbb{R}$

$A^k(V) = \{$alternating k-tensors on $V\}$ ∴ elements in $A^k(V)$

- multilinear : $\underbrace{V\times\_\times V}_{k}\to\mathbb{R}$

(Def: $f^\sigma(\vec{x}_1, - \vec{x}_k) = f(\vec{x}_{\sigma(1)}, -, \vec{x}_{\sigma(k)})$, $f\in\mathcal{L}^k(V), \sigma\in S_k$)

- $f^{e_i} = -f$ $\quad\forall e_i =$ elementary in S_k $\quad i\leftrightarrow i+1$

$A^1(V) := V^* = \mathcal{L}^1(V)$.

Ex: $V=\mathbb{R}^3$, $k=3$

$f(\vec{x}_1,\vec{x}_2,\vec{x}_3) = \det[\vec{x}_1,\vec{x}_2,\vec{x}_3] \in A^3(\mathbb{R}^3)$

Lemma 1: Ⓗ $V = n$-dim. v.s. over $\mathbb{R}$, $f\in\mathcal{L}^k(V)$, $\sigma,\tau\in S_k$

© (1) The mapping : $f\to f^\sigma$ is a linear transf.: $\mathcal{L}^k(V)\to\mathcal{L}^k(V)$

$(af^{\cancel{\sigma}}+bg)^\sigma = af^\sigma+bg^\sigma \quad \forall f,g\in\mathcal{L}^k(V), \forall a,b\in\mathbb{R}$

also: $(f^\sigma)^\tau = f^{\tau\circ\sigma}$

(2) $f\in A^k(V) \Leftrightarrow f^\sigma = (\text{sign}\,\sigma)f \quad \forall\sigma\in S_k$

Moreover, if $f\in A^k(V)$ & $(\vec{v}_1, -, \vec{v}_k)$ is s.t. $\vec{v}_p=\vec{v}_q$ $\exists p\neq q$

(*) then $f(\vec{v}_1, \to \vec{v}_k)=0$

Proof of (*): (only) Let $\sigma_{pq} : p\leftrightarrow q$ (fixes everything else)

then: sign $\sigma_{pq} = -1 \Rightarrow \underline{f^{\sigma_{pq}}(\vec{v}) \overset{(2)'}{=} -f(\vec{v})}$ by (2)

But $\sigma_{pq}(\vec{v}_1, -, \vec{v}_k) = (\vec{v}_{\sigma_{pq}(1)}, - \vec{v}_{\sigma_{pq}(k)}) \underset{\vec{v}_p=\vec{v}_q}{=} (\vec{v}_1, -, \vec{v}_k)$

so $\underline{f^{\sigma_{pq}}(\vec{v}_1, -, \vec{v}_k) = f(\vec{v}_1, -, \vec{v}_k)} \Rightarrow f(\vec{v}) = -f(\vec{v}) \Rightarrow f(\vec{v})=0$

Corollary: Ⓗ $V = n$-dim v.s. over $\mathbb{R}$, $k \in \mathbb{Q}^+$, $k \geq 2$, $k > n$.

Ⓒ $A^k(V) = \{0\}$

Pf: We know any $f \in \mathcal{L}^k(V)$ is ! determined by its values on all k-subsets of vectors from $\{\vec{a}_1, -, \vec{a}_n\}$ (basis for V)

(by specifying $f(\vec{a}_{j_1}, -, \vec{a}_{j_k})$ $\forall J = \{j_1, -, j_k\} \subset (1, -, n)$)

Now: $k > n \Rightarrow J$ must contain repeated labels i.e. $\exists\, j_i = j_l$ for $i \neq l$.

$\Rightarrow$ by (*) previous lemma : $f \in A^k(V) \Rightarrow f(\vec{a}_J) = 0$ $\therefore f \equiv 0$ □

Notation: ascending k-tuple is $J = \{j_1, -, j_k\} \subset \{1, -, n\}$ s.t.

$j_i \neq j_l$ $\forall i \neq l$ & $j_1 < j_2 < \ldots < j_k$

Next: $f \in A^k(V)$ is completely determined by its values on ascending k-tuples of $\{1, -, n\}$.

Lemma: Ⓗ $2 \leq k \leq n$; $V = n$-dim. v.s. over $\mathbb{R}$, Let $\{\vec{a}_1, -, \vec{a}_n\}$ be a basis for

$f, g \in A^k(V)$ s.t. $f(\vec{a}_I) = g(\vec{a}_I)$ for any ascending k-tuple I in $\{1, -, n\}$

Ⓒ $f = g$

Pf: enough to show that $f(\vec{a}_J) = g(\vec{a}_J)$ $\forall J$ = any k-tuple of basis element of V.

Case 1: $j_l = j_m$ $\exists\, l \neq m$ $\underset{\text{Coroll.}}{\Rightarrow}$ $f(\overset{J = \{j_1, - j_k\}}{\vec{a}_J}) = 0 = g(\vec{a}_J)$

Case 2: $j_l \neq j_m$ $\forall l \neq m$. Let $\sigma \in S_k$ be the permutation of $\{j_1, - j_k\}$ that rearranges the labels in increasing order. Call $\sigma(J) =: I = (i_1 < \ldots < i_k$

$f(\vec{a}_I) = f^\sigma(\vec{a}_J)$ (def σ, def f^σ)

$= (\text{sign}\,\sigma)\, f(\vec{a}_J)$ (b/c $f \in A^k$)

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Likewise, $g(\vec{a}_I) = (\text{sign}\,\sigma)\, g(\vec{a}_J)$, by Ⓗ $g(\vec{a}_I) = f(\vec{a}_I)$

thus $f(\vec{a}_J) = g(\vec{a}_J) \Rightarrow f = g$. □

Thm: (Basis for $A^k(V)$.)

Ⓗ $2 \le k \le n$, V n-dim. v.s. over $\mathbb{R}$, $\{\vec{a}_1, \dots, \vec{a}_n\}$ basis for V

I = ascending k-tuple in $(1, \dots, n)$

Ⓒ $\exists!\ \psi_I \in A^k$ s.t. $\forall$ ascending k-tuple J:

(*) $\psi_I(\vec{a}_J) = \delta_{IJ}$ where in fact

(**) $\psi_I := \sum_{\sigma \in S_k} (\text{sign}\,\sigma)\, \varphi_I^\sigma$ where φ_I is the elemantary k-tensor for $\mathcal{L}^k(V)$ corresponding to I.

Furthermore, $\{\psi_I\}_{[I]}$ ← all ascending k-tubles, only is a basis for $A^k(V)$.

Def: ψ_I given by (**) is called elementary alternating k-tensor on V corresponding to basis $\{\vec{a}_1, \dots, \vec{a}_n\}$.

Proof: ! immediate from Lemma 2 (check!)

· $\exists$ define $\psi_I := \sum_{\sigma \in S_k} \text{sign}\,\sigma\, (\Phi_I^\sigma)$

Show $\psi_I \in A^k(V)$ & (*) $\psi_I(\vec{a}_J) = \delta_{IJ}$ $\forall$ ascending k-tuple J

(i) $\psi_I \in A^k(V)$: $\overset{\text{Lemma 1}}{\iff}$ $\psi_I^\tau \overset{?}{=} (\text{sign}\,\tau)\, \psi_I \quad \forall \tau \in S_k$.

$$\psi_I^\tau = \sum_\sigma (\text{sign}\,\sigma)\, (\varphi_I^\sigma)^\tau \quad \text{(linearity)}$$

$$= \sum_\sigma (\text{sign}\,\sigma)\, \varphi_I^{\tau\circ\sigma}$$

(sign $(\tau\circ\sigma)$ = sign $\tau \cdot$ sign σ)

$$= \text{sign}\,\tau \underbrace{\sum_\sigma \text{sign}(\tau\circ\sigma)\, \varphi_I^{\tau\circ\sigma}}$$

$$= \text{sign}\,\tau\ \psi_I \quad \text{(def } \psi_I \text{ b/c } \sigma\circ\tau \text{ also spans } S_k\text{).}$$

So : $\psi_I^\tau = \text{sign}\,\tau\ \psi_I$, thus $\psi_I \in A^k(V)$.

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(ii) Show $\psi_I(\vec{a}_J) = \delta_{IJ}$ ∀ ascending k-tuples I & J.

$$\psi_I(\vec{a}_J) = \sum_\sigma \text{sign}\,\sigma \, \underbrace{\varphi_I(\vec{a}_{\sigma(J)})}_{=1} \quad (\text{def } \psi_I \text{ ; def } \varphi_I^0)$$

$\neq 0 \iff \exists \sigma \in S_k$ s.t. $\sigma(J) = I$.

$\iff I = J$ & $\sigma = \text{Identity}$

sign $\sigma = (-1)^0 = 1$

$= 1$ (def $\varphi_I(\vec{a}_I)$)

$\therefore \psi_I(\vec{a}_J) = \delta_{IJ}$

∎

(iii) Show : $\{\psi_I\}_{[I]}$ are <u>basis</u> of A^k.

i.e. show: any $f \in A^k(V)$ can be ! expressed as a lin. combination of $\{\psi_I\}_{[I]}$:

Given $f \in A^k$. Def $d_I := f(\vec{a}_I) \in \mathbb{R}$ ∀ I ascending.

<u>def:</u> $g := (\sum_{[J]} d_J \psi_J$

<u>Then:</u> $g(\vec{a}_I) = d_I = f(\vec{a}_I)$ ∀ ascending I (def ψ_J; def d_J).

So: $g = f$ (they agree on all ascending k-tuples of basis vectors.)

Uniqueness follows from lemma. □

<u>Thus:</u> $\forall f \in A^k(V)$ $\exists! \{d_I\}_{[I]} \subset \mathbb{R}$ s.t. $f = \sum_{[I]} d_I \psi_I$.

<u>Def:</u> d_I = components of f relative to $\{\vec{a}_1, \dots, \vec{a}_n\}$.

<u>Dimension of $A^k(V)$:</u>

- $k = 1$ ⇝ $A^1(V) = V^*$ ⇒ $\dim A^1 = n$
- $2 \le k \le n$: $\dim A^k = |\{\text{all ascending k-tuples from } \{1, \dots, n\}\}|$

∀ k-subset of $\{1, \dots, n\}$ ∃! k-ascending set with same content as this subset

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$|\{\text{all ascending } k\text{-tuples in } \{1,\dots,n\}\}| = |\{k\text{-subsets of } \{1,\dots,n\}\}|$

$= \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\dim A^k(V) = \binom{n}{k}$ if $\dim V = n$.

Ex: Basis for $\mathcal{L}^2(\mathbb{R}^4)$

$$\underbrace{\begin{array}{ccc} 11 & 21 & 31 \\ \textcircled{12} & 22 & 32 \\ \textcircled{13} & \textcircled{23} & 33 \\ \textcircled{14} & \textcircled{24} & \textcircled{34} \end{array}}_{\text{Basis for } A^2(\mathbb{R}^4)} \begin{array}{c} 41 \\ 42 \\ 43 \\ 44 \end{array}$$

Most triking situation: $k=n$

$\dim \mathcal{L}^n(\mathbb{R}^n) = n^n$

vs. $\dim A^n(\mathbb{R}^n) = \binom{n}{n} = 1$

Thm (Stability of A^k)

Ⓗ $T: V \to W$ Linear transf. between v.s. over $\mathbb{R}$

$f \in A^k(W)$

Ⓒ $T^*f \in A^k(V)$

$$\begin{tikzcd} \underbrace{V \times \dots \times V}_{k} \arrow[r, "T\times\dots\times T"] \arrow[dr, "T^*f"'] & \underbrace{W \times \dots \times W}_{k} \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

So: $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$
and $A^k(W) \to A^k(V)$.

Determinants (def. for arbitrarily sized matrices)

Def. $V = \mathbb{R}^n$, basis $\{\vec{e}_1, \dots, \vec{e}_n\}$, canonical basis

$\{\varphi_1, \dots, \varphi_n\}$ = dual basis (for $\mathcal{L}^1(\mathbb{R}^n) = A^1(\mathbb{R}^n)$)

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$k=n$: $\dim A^n(\mathbb{R}^n)^* = 1 = \binom{n}{n}$

basis: a single alternating elementary n-tensor: $\psi_{(1,\ldots,n)}$

Let $X=[\vec{x}_1,\ldots,\vec{x}_n]$ be any $n\times n$ matrix.

Def: $\det X := \psi_{(1,\ldots,n)}(\vec{x}_1,\ldots,\vec{x}_n)$ $(*)$

Fact: $(*)$ satisfies all axioms of determinant fct as we know them from lin. alg.

(1) linearity on rows / columns $\Leftrightarrow$ $\psi_{(1,\ldots,n)} \in \mathcal{L}^n(V)$.

(2) changing sign by switching columns (rows) $\Leftrightarrow$ $\psi_{(1,\ldots,n)} \in A^n(V)$.

(3)
$$\det X = \sum_{\sigma\in S_n} (\operatorname{sign}\sigma)\, \underbrace{\varphi_{(1,\ldots,n)}(x_{\sigma(1)},\ldots,x_{\sigma(n)})}_{\varphi^{\sigma}_{(1,\ldots,n)}} \qquad (**)$$

$$= \underbrace{\sum_{\sigma\in S_n} (\operatorname{sign}\sigma)\, x_{1,\sigma(1)}\cdot x_{2,\sigma(2)}\cdots x_{n,\sigma(n)}}_{\text{"determinant formula" from lin. algebra}} \qquad (\text{def } \varphi_{(1,\ldots,n)})$$

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Recall from last week

$V = n$-dim v.s. over $\mathbb{R}$. $\{\vec{a}_1,\ldots,\vec{a}_n\}$ = basis for V

$A^k(V) = \{$alternating k-tensors$\}$, $k\geq 2$

$A^1(V) = \mathcal{L}^1(V) = V^*$

$A^k(V) = \{0\}$, $k \geq n$

$\dim A^k(V) = \binom{n}{k}$

Basis for $A^k(V) = \{\psi_I\}_{(I)} = \{$elem. alternating k-tensors$\}$

ascending k-tuples in $\{1,\ldots,n\}$.

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$$\psi_I(\vec v_{(1,\dots,k)}) = \sum_{\sigma\in S_k} (\text{sign}\ \sigma)\ \Phi_I^\sigma(\vec v_{(1,\dots,k)})$$

<u>notation:</u> $\vec v_{(1,\dots,k)} = (\vec v_1,\dots,\vec v_k)$. $\vec v_j \in \mathbb R^n$

where Φ_I = elementary k-tensor (i.e. $\Phi_I(\vec a_J) = \delta_{IJ}$)

& $\Phi_I^\sigma(\vec v_{(1,\dots,k)}) := \Phi_I(\vec v_{(\sigma(1),\dots,\sigma(k))})$

<u>Recall that:</u> $\Phi_I = \Phi_{i_1}\otimes\dots\otimes\Phi_{i_k}$, $\Phi_{i_j}\in\mathcal L^1(V)$. $\Phi_I(\vec v_{(1,\dots,k)}) = \Phi_{i_1}(\vec v_1)\dots\Phi_{i_k}(\vec v_{i_k}$

<u>Determinants</u>

$V=\mathbb R^n$, basis $\{\vec e_1,\dots,\vec e_n\}$; $k=n$; dim $A^n(\mathbb R^n) = 1 = \langle\psi_{(1,\dots,n)}\rangle$

$$\forall\ \underset{\vec x_{(1,\dots,n)}}{\overset{\parallel}{X}} \in\mathbb R^{n\times n},\quad \det X :\overset{\text{def}}{=} \psi_{(1,\dots,n)}(\vec x_{(1,\dots,n)}) = \sum_{\sigma\in S_n} \text{sign}\ \sigma\ \varphi_{(1,\dots,n)}(\vec x_{(\sigma(1),\dots,\sigma(n))})$$

$$= \sum_{\sigma\in S_n} \text{sign}\ \sigma \cdot \underbrace{x_{1,\sigma(1)}}_{\varphi_1(\vec x_{\sigma(1)})} \cdot \underbrace{x_{2,\sigma(2)}}_{\varphi_2(\vec x_{\sigma(2)})} \cdot\dots\cdot \underbrace{x_{n,\sigma(n)}}_{\varphi_n(\vec x_{\sigma(n)})}$$

$\uparrow$ used as "def'n" of det X in linear algebra

Now we can say that the $n\times n$ determinant fct is the elementary alternating n-tensor on $\mathbb R^n$ relative to canonical basis in $\mathbb R^n$.

<u>"Conversely":</u>

<u>Thm:</u> Ⓗ $V=\mathbb R^n$; $\{\vec e_1,\dots,\vec e_n\}$ canonical basis , ψ_I = elementary alternating k-tensor on $\mathbb R^n$.

Ⓒ $\psi_I(\vec x_{(1,\dots,k)}) = \det X_I$ where X_I is the <u>$k\times k$ minor</u> of $n\times k$-matrix $X = (\vec x_1,\dots,\vec x_k) = \vec x_{(1,\dots,k)}$ obtained by selecting rows $i_1, i_2,\dots,i_k$ (in given order) from all n-rows of X.

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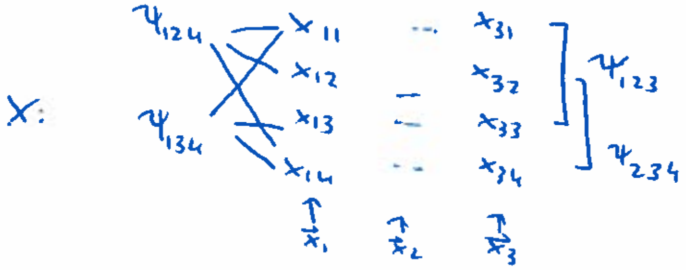
Ex: $\dim \mathcal{A}^3(\mathbb{R}^4) = \binom{4}{3} = 4$

$$\psi_I \in \{\psi_{123}, \psi_{124}, \psi_{134}, \psi_{234}\}$$



"Every elem. alt. k-tensor in $\mathbb{R}^n$ relative to canonical basis is a $k\times k$ determinant fct"

The same can be said for elementary alternating k-tensors ψ_I on an arbitrary n-dim. v.s. V in the sense that $T^*\psi_I \in \mathcal{A}^k(\mathbb{R}^n)$ is a $k\times k$-determinant fct

where $T: \mathbb{R}^n \to V$
$\{\vec{e}_1, .., \vec{e}_n\} \to \{\vec{a}_1, .., \vec{a}_n\}$ (extended by linearity)

The Wedge Product

Recall: if $T: V \to W$ then

$T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and

$T^*: \mathcal{A}^k(W) \to \mathcal{A}^k(V)$

Recall: $\otimes \quad \mathcal{L}^k(V) \times \mathcal{L}^l(V) \to \mathcal{L}^{k+l}(V)$

$$(f,g) \mapsto f\otimes g(\vec{v}_{1,..,k+l}) = f(\vec{v}_{1,..,k}) \cdot g(\vec{v}_{k+1,..,k+l})$$

However, in general: $\otimes \; \mathcal{A}^k(V) \times \mathcal{A}^l(V) \not\to \mathcal{A}^{k+l}(V)$!!!

Need: new "product" that preserves alternating tensors

& this is the notion of "wedge product"

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We have the following:

Thm: Ⓗ V = n-dim v.s. V over $\mathbb{R}$

Ⓒ $\exists : \wedge : A^k(V) \times A^l(V) \to A^{k+l}(V)$ s.t. $\forall f \in A^k(V), g \in A^l(V), h \in A^s(V)$ the following holds:

① Associativity: $f \wedge (g \wedge h) = (f \wedge g) \wedge h$

② Homogeneity: $(cf) \wedge g = f \wedge (cg)$ $\forall c \in \mathbb{R}$

③ Distributivity: (w.r.t. tensor sum) if $k = l$ then $(f+g) \wedge h = f \wedge h + g \wedge h$
$h \wedge (f+g) = h \wedge f + h \wedge g$

④ Anti-commutativity: $g \wedge f = (-1)^{kl} f \wedge g$

⑤ Let $\{\vec{a}_1, \ldots, \vec{a}_n\}$ basis for V; $\{\varphi_1, \ldots, \varphi_n\}$ dual basis for V^*

Let $\{\psi_I\}_{[I]}$ = elementary k-tensors on V

then for any ascending k-tuple $I = (i_1, i_2, \ldots, i_k)$ we have:

$\psi_I = \varphi_{i_1} \wedge \varphi_{i_2} \wedge \ldots \wedge \varphi_{i_k}$ [vs. $\varphi_I = \varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k}$]

Properties (1)-(5) characterize $\wedge$ uniquely

Moreover, if: $T: V \to W$ (linear transformation of vector spaces)

$\forall f \in A^k(W), g \in A^l(W)$ we have that

$$A^{k+l}(W) \ni T^*(f \wedge g) = T^* f \wedge T^* g$$ □

Corollary: Ⓗ $f \in A^{2k+1}(V)$ (odd order)

Ⓒ $f \wedge f = 0$

(Pf $f \wedge f \overset{④}{=} \underbrace{(-1)^{(2k+1)^2}}_{=-1} f \wedge f \Rightarrow 2(f \wedge f) = 0$ □)

Def: $A : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$, $f \mapsto Af := \sum_{\sigma \in S_k} (\operatorname{sign} \sigma) f^\sigma$

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- Then, in fact: $A: \mathcal{L}^k(V) \to \mathcal{A}^k(V)$

  i.e. $(Af)^{e_i} = -Af$ all $f \in \mathcal{L}^k(V)$

  check!

- if $f \in \mathcal{A}^k(V)$ then $Af = k!f$
- if $f \in \mathcal{A}^k(V)$ & $g \in \mathcal{A}^l(V)$ then $f \wedge g := \frac{1}{k!\,l!} A(f \otimes g)$

Pf of Thm: a final exam "assignment"

$\downarrow$

- last two weeks of classes (4 classes)
- 4 assignements: work in pairs; each member talks 40 minutes
- read, understand proof + understandable presentation

Tangent vectors & differential forms

Recall from multivariable calculus in $\mathbb{R}^3$:

"Vector algebra in $\mathbb{R}^3$":

- vector sum
- dot product
- cross product
- scalar fields ($f: \mathbb{R}^3 \to \mathbb{R}$)
- vector fields ($F: \mathbb{R}^3 \to \mathbb{R}^3$)
- $\vec{\nabla}$: {scalar field} $\to$ {vector fields}
- $\overrightarrow{\text{Curl}} = \vec{\nabla} \times \vec{F}$: {vector fields} $\circlearrowleft$
- div $= \vec{\nabla} \cdot \vec{F}$: {vector fields} $\to$ {scalar fields}

Now: "Tensor algebra in $\mathbb{R}^n$":

- Tensor sum
- alternating tensors
- wedge product of alternating tensors
- "Alternating tensor field": $\vec{v} \in V \to \omega(\vec{v}) \in \mathcal{A}^k$

(in fact: $p \in \underset{\substack{\uparrow \\ m\text{-mfd in } \mathbb{R}^n}}{M} \longrightarrow \omega(p) \in \mathcal{A}^k(\underbrace{T_pM}_{\substack{\text{tangent space} \\ \text{of } M \text{ at } p)}})$

Alternating tensor fields are also called differential forms

Later: "exterior derivative": d: {differential forms} $\circlearrowleft$ ← generalize div, Curl, $\vec{\nabla}$

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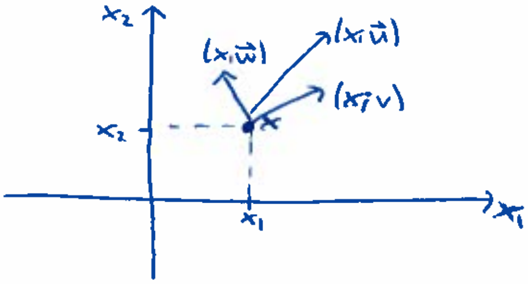
Tangent vectors & tangent vector fields

Def: Given $x \in \mathbb{R}^n$ (a point in the metric space $\mathbb{R}^n$) a "tangent vector to $\mathbb{R}^n$ at x" is a pair $(x;\vec{v})$ where $v \in \mathbb{R}^n$ (a vector in the vector space $\mathbb{R}^n$)

ex: $n=3$



We represent $(x;\vec{v})$ as an arrow with same direction as $\vec{v}$ and initial pt at x

Def. Given $x \in \mathbb{R}^n$ (a pt in metric space $\mathbb{R}^n$), $\mathcal{T}_x(\mathbb{R}^n) = \{(x;\vec{v}) \mid \vec{v}$ any vector in v.s. $\mathbb{R}^n\}$ is the tangent space to $\mathbb{R}^n$ at x

Facts: $\mathcal{T}_x(\mathbb{R}^n)$ is a v.s. W over $\mathbb{R}$ via: $(x;\vec{v})+(x;\vec{u}) := (x;\vec{v}+\vec{u})$

$c(x;\vec{v}) = (x;c\vec{v})$, $c \in \mathbb{R}$

Note: $T_x : \mathcal{T}_x(\mathbb{R}^n) \to \{x\} \times \mathbb{R}^n$
$(x;\vec{e}_j) \mapsto (x,\vec{e}_j)$ is 1-1, onto, linear

$\dim \mathcal{T}_x(\mathbb{R}^n) = n.$

Def: Given $(a,b) \subset \mathbb{R}$ (open interval) & $\gamma : (a,b) \to \mathbb{R}^n$ (map of class C^r), $t \mapsto \gamma(t)$ (a point in $\mathbb{R}^n$)

Velocity vector of γ: $(\gamma(t), D\gamma(t))$

$\vec{v} = \left(\frac{\partial \gamma_1}{\partial t}(t), \ldots, \frac{\partial \gamma_n}{\partial t}(t)\right)$

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More generally:

Def: given U open set in $\mathbb{R}^k$ or $\mathbb{H}^k$

given $\alpha : U \to \mathbb{R}^n$ class C^r

given $x \in U$

def: $\alpha_* : \mathcal{T}_x(\mathbb{R}^k) \to \mathcal{T}_{\alpha(x)}(\mathbb{R}^n)$ via

$$\alpha_*(x; \vec{v}) := (\alpha(x); \underbrace{\underbrace{D\alpha(x)}_{n\times k} \cdot \underbrace{\vec{v}}_{k\times 1}}_{n\times 1})$$

α_* is called transformation of tangent spaces induced by α

Tangent:

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$$\left.\begin{array}{l}\text{vectors}\\ \text{spaces}\\ \text{vector fields}\\ m\text{-tensor fields}\\ m\text{-forms}\end{array}\right\} \text{for } \mathbb{R}^n \text{ \& for a } k\text{-mfd in } \mathbb{R}^n$$

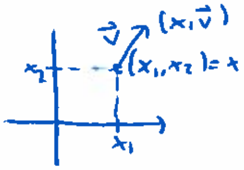
For $\mathbb{R}^n$:

Def: Given $x \in \mathbb{R}^n$ (pt in metric space $\mathbb{R}^n$)

"tangent vector to $\mathbb{R}^n$ at x" is $(x; \vec{v})$, any $\vec{v} \in \mathbb{R}^n$

(a vector in v.s. $\mathbb{R}^n$)

Ex: $\mathbb{R}^2$



$(x,\vec{v})$ represented as an arrow
- initial pt is at x, "base pt"
- direction = same as $\vec{v}$

Def $\mathcal{T}_x(\mathbb{R}^n) = \{(x,\vec{v}) \mid \vec{v} \in \mathbb{R}^n\}$

tangent space to $\mathbb{R}^n$ at x

Facts:
- $\mathcal{T}_x(\mathbb{R}^n)$ v.s. (operations† inherited from $\mathbb{R}^n$ as v.s. e.g. $(x,\vec{v}) + (x,\vec{u}) = (x, \vec{v}+\vec{u})$, etc.
- $\mathcal{T}_x(\mathbb{R}^n) \cong \{x\} \times \mathbb{R}^n$, via $T_x : \{x\} \times \mathbb{R}^n$
  $(x, \vec{e}_j) \mapsto (x, \vec{e}_j)$

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dim $\mathcal{T}_x(\mathbb{R}^n) = n \quad \forall x$.

- $(x;\vec{v}) + (y;\vec{u})$ is undefined

- Interpretation of $\mathcal{T}_x(\mathbb{R}^n)$ as a space of "tangent vectors"

Given $(a,b) \subset \mathbb{R}$ & $\gamma : (a,b) \to \mathbb{R}^n$ (class C^r)

Let $x = \gamma(t)$ $\exists\, t \in (a,b)$. Then: $(\gamma(t), (D\gamma)(t))$

$= \vec{v} = (\dot{\gamma}_1(t), \dots, \dot{\gamma}_n(t))$

is tangent vector at $\gamma(t)$

to curve $\{\gamma(t)\}_{t\in\mathbb{R}} \subset \mathbb{R}^n$

More generally:

- transformation of tangent spaces:

Given $U \subset \mathbb{R}^k$ on $\mathbb{H}^k$, open set

$\alpha : U \to \mathbb{R}^n$, class C^r (before: $\alpha = \gamma$, $k=1$)

$x \in U$

Def. $\alpha_* : \mathcal{T}_x(\mathbb{R}^k) \to \mathcal{T}_x(\mathbb{R}^n)$

$$(x;\vec{v}) \mapsto \alpha_*(x;\vec{v}) = (\alpha(x), \underbrace{\underset{\mathbb{R}^{n\times k}\cdot\mathbb{R}^{k\times 1}}{D\alpha(x)\cdot\vec{v}}}_{\in\mathbb{R}^n})$$

...

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Pf. Check ! (chain rule) □

Lemma: Ⓗ U open in $\mathbb{R}^k$ or $\mathbb{H}^k$; $x \in U$

$\alpha : U \to \mathbb{R}^n$, class C^r

V open in $\mathbb{R}^n$, $\alpha(U) \subset V$

$\beta : V \to \mathbb{R}^n$ (class C^r)

Ⓒ $(\beta \circ \alpha)_* = \beta_* \circ \alpha_* : T_x(\mathbb{R}^n) \to T_{(\beta\circ\alpha)(x)}(\mathbb{R}^n)$

Pf. Check! $(\beta \circ \alpha)_*(x, \vec{v}) = \beta_*(\alpha_*(x, \vec{v}))$ i.e.

$$\begin{tikzcd} U \arrow[rr, "\beta\circ\alpha"] \arrow[dr, "\alpha"'] & & \mathbb{R}^n \\ & V \arrow[ur, "\beta"'] & \end{tikzcd} \Rightarrow \begin{tikzcd} T_x(\mathbb{R}^k) \arrow[rr, "(\beta\circ\alpha)_*"] \arrow[dr, "\alpha_*"'] & & T_{\beta(\alpha(x))}(\mathbb{R}^n) \\ & T_{\alpha(x)}(\mathbb{R}^n) \arrow[ur, "\beta_*"'] & \end{tikzcd}$$

Tangent vector fields in $\mathbb{R}^n$ (or $U \subset \mathbb{R}^n$, open)

Def: ↳ continuous fct $F: U \to \mathbb{R}^n \times \mathbb{R}^n$ s.t.
$x \mapsto F(x) \in T_x(\mathbb{R}^n)$ $\forall x \in U$

ie, $F(x) \in \{x\} \times \mathbb{R}^n$ $\forall x$ & $F(x) = (x, \vec{f}(x))$ $\exists \vec{f} : U \to \mathbb{R}^n$

We say that F is of class C^r iff $\vec{f}$ so is.

Def: Let $U \subseteq \mathbb{R}^n$; open ; $1 \le m \le n$.

m-tensor field on U is $w : U \to \bigcup_{y \in U} \mathcal{L}^m(T_y(\mathbb{R}^n))$

s.t. $\forall x \in U$, $w(x) \in \mathcal{L}^m(T_x(\mathbb{R}^n))$ ~~& we require w to be~~

~~continuous: $U \to \bigcup_{y\in U} \mathcal{L}$~~

Def: differential form order m (m-form; m co-vector) on U

is an m-tensor field such that, in fact:

$\forall x \in U$, $w(x) \in \mathcal{A}^m(T_x(\mathbb{R}^n))$ which we require to be

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continuous (recall $A^m(T_x(\mathbb{R}^n)) = \mathbb{R}^{\binom{n}{m}}$)

So: $\forall x$, ~~ω_x~~ $\omega(x) : T_x(\mathbb{R}^n) \times \ldots \times T_x(\mathbb{R}^n) \to \mathbb{R}$ with:

$$((x;\vec{v}_1), \ldots, (x;\vec{v}_m)) \mapsto \omega(x)((x;\vec{v}_1), \ldots, (x;\vec{v}_m)) \in \mathbb{R}$$

s.t. $(\omega(x))^\sigma((x;\vec{v}_1), \ldots, (x;\vec{v}_m)) = \operatorname{sign}\sigma\, (\omega(x))((x;\vec{v}_{\sigma(1)}), \ldots, (x;\vec{v}_{\sigma(m)})) \in \mathbb{R}$ ~~$\in\mathbb{R}$ $\in$~~

$\forall \sigma \in S_m$

Definitions for M = kmfd in $\mathbb{R}^n$

Given: M = a k-mfd in $\mathbb{R}^n$

$p \in M$

$\alpha : U \to V$ coord chart about p, (U open in $\mathbb{R}^k$ or $\mathbb{H}^k$)

$p = \alpha(x)$ $\exists x \in U$

Def: Tangent vector for M at p: it's $\alpha_*(x;\vec{v})$, any $v \in \mathbb{R}^k$.

Def: Tangent space to M at p is $T_p(M) := \alpha_*(T_x(\mathbb{R}^k))$

$$= \{(\underbrace{p}_{\alpha(x)}; D\alpha(x)\cdot\vec{v}) \mid \vec{v} \in \mathbb{R}^k\}$$

Notes (check!)
- Def of $T_p(M)$ independent of choice of coord. chart about p (up to isomorphism)
- $T_p(M) \subset T_p(\mathbb{R}^n)$
- $T_p(M)$ is a v.s. & a subspace of $T_p(\mathbb{R}^n)$
- If we choose $\{\vec{e}_1, \ldots, \vec{e}_k\}$, canonical basis for $\mathbb{R}^k$, then:

  $T_p(M)$ is spanned by $\{(p; \underbrace{D\alpha(x)\cdot\vec{e}_j}_{\vec{u}_j(x) = \frac{\partial\alpha}{\partial x_j}(x) = \left(\frac{\partial\alpha_1}{\partial x_j}(x), \ldots, \frac{\partial\alpha_n}{\partial x_j}(x)\right)})\}_{j=1,\ldots,k}$
- $\{(x;\vec{u}_1(x)), \ldots, (x;\vec{u}_k(x))\}$ linearly indep. b/c the $n\times k$ matrix $D\alpha(x)$ has max rank k.

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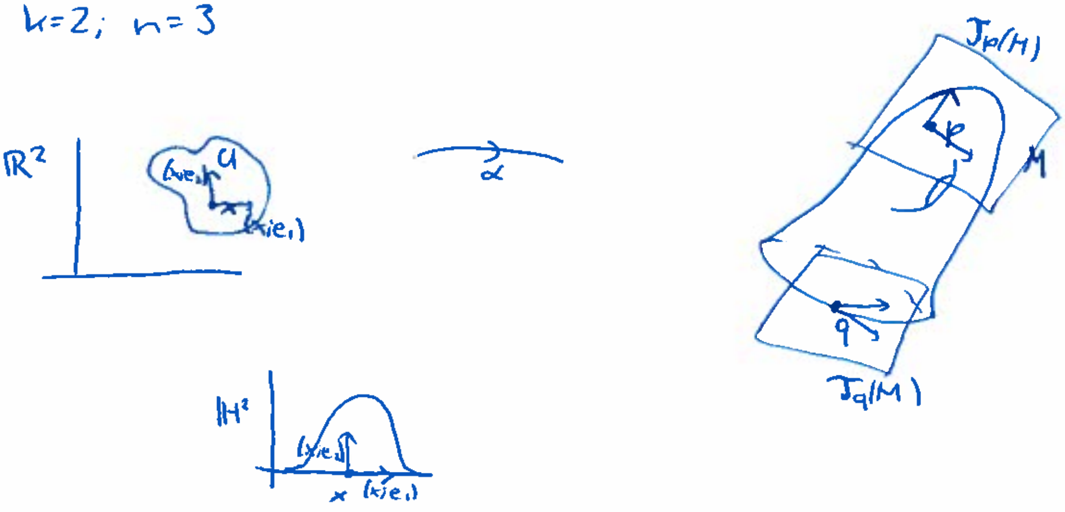
So: $\left\{ \left(x; \frac{\partial \alpha_1}{\partial x_j}(x)\right), \ldots, \left(x; \frac{\partial \alpha_n}{\partial x_j}(x)\right) \right\}_{j=1,\ldots,k}$ basis for $T_p(M)$

e.g. $k=2$; $n=3$



Def: Tangent bundle of M is: $T(M) = \bigcup_{q \in M} T_q(M)$.

Note: $T_q(M) \cong \{q\} \times \mathbb{R}^k$ then: $T(M) \cong M \times \mathbb{R}^k$.

& endow with natural topology

Def: A tangent vector field to M is a continuous fct

$F: M \to T(M)$
$p \mapsto F(p) \in T_p(M)$, $\forall p \in M$.

Def: $\forall\ 1 \le m \le k$, M = k-mfd in $\mathbb{R}^n$, an m-tensor field on M

is a map $\omega: M \to \bigcup_{q \in M} \mathcal{L}^m(T_q(M))$ s.t.

$\forall p \in M$, $\omega(p) \in \mathcal{L}^m(T_q(M))$.

- differential form of order m (m-form; m covector) on M:

is an m-tensor field on M s.t.

$\forall p \in M$, $\omega(p) \in \underset{\nwarrow \text{alternating } m\text{-tensors}}{A^m}(T_p(\mathbb{R}^n))$.

s.t. ω: continuous: $M \to \bigcup_{q \in M} A^m(T_q(M)) \cong M \times \mathbb{R}^{\binom{k}{m}}$

$\uparrow$ endowed w/ natural topology from $\nearrow$

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Note: $\omega(p): T_p(M)\times\dots\times T_p(M)\to\mathbb{R}$

Fact: if Ω is an m-tensor field on U open in $\mathbb{R}^n$

since $T_p(M)\subset T_p(\mathbb{R}^n)$ (subspace) it follows that

Ω determines ω an m-tensor field on M by setting

$$\omega(p)((p;\vec u_1),\dots,(p;\vec u_m)) := \Omega(p)((p;\vec u_1)\dots(p;\vec u_m))$$

ω is "restriction of Ω to M".

Converse also true (but non trivial!):

every m-tensor field ω on M extends to m-tensor field on $U\subset\mathbb{R}^n$, $U\supset M$.

From now on: we will only work with m-forms defined on open sets in $\mathbb{R}^n$. More precisely: we will only work with m-forms ω over M that are restrictions to M of m-forms Ω on open sets of $\mathbb{R}^n$ (that contain M).

Recall from last week   Oct 17

$$T_x(\mathbb{R}^k) = \{(x;\vec v) \mid \underset{pt}{x\in\mathbb{R}^k},\ \vec v\in\mathbb{R}^k\}$$

$$T_p(M) = \alpha_*(T_{\alpha^{-1}(p)}(\mathbb{R}^k)) = \{(p; D\alpha(x)\cdot\vec v) \mid \vec v\in\mathbb{R}^k\}$$

$x=\alpha^{-1}(p)$, α = any coord. chart at p

$T_p(M)$ is a vector space with basis:

$$\left\{\underset{\in\mathbb{R}^n}{\frac{\partial\vec\alpha}{\partial x_1}(\alpha^{-1}(p))},\dots,\underset{\in\mathbb{R}^k}{\frac{\partial\vec\alpha}{\partial x_k}(\alpha^{-1}(p))}\right\}$$

$1\le m\le k\le n$, M = k-mfd in $\mathbb{R}^n$

- m-tensor field: $\omega: M\to\bigcup_{q\in M}\mathcal{L}^m(T_q(M)) \cong M\times\mathbb{R}$

$$p\mapsto\omega(p)\in\mathcal{L}^m(T_p(M))$$

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- m - differential form (m - form (m - covector)

$$\omega: M \to \bigcup_{q\in M} A^m(T_q(M)) \qquad \cong M \times \mathbb{R}^{\binom{n}{m}}$$

$$M \ni p \mapsto \omega(p) \in A^m(T_p(M))$$

WLOG: assume ω = restriction to M of Ω = m-form on $V \in \mathbb{R}^n$

with $V \supset M$: $\Omega: V \subset \mathbb{R}^n \to \bigcup_{x\in V} A^m(T_x(\mathbb{R}^n))$

Notation: $\Lambda^m(V)$ or $\Lambda^m(V) = \{m\text{-forms on } V\}$

is a v.s. over $\mathbb{R}$: $\forall x \in V$, $(a\omega(x) + b\eta(x))\underbrace{(x;\vec{v})}_{\in T_x(\mathbb{R}^n)} := a\omega(x)(x;\vec{v}) + b\eta(x)(x;\vec{v})$

Basis for $\Lambda^m(V)$:

- $\{\vec{e}_1, \dots, \vec{e}_n\}$ = canonical basis for $\mathbb{R}^n$; $x \in \mathbb{R}^n$ $(x \in V)$

Canonical 1-forms (m=): $\underbrace{\varphi_j(x)}_{\in A^m(T_x(\mathbb{R}^n))}(x;\vec{e}_i) = \delta_{ij}$

Canonical m-forms: $(m \geq 1)$

(I = ascending m-tuple in $(1, \dots, n)$)

$\psi_I(x) = \varphi_{i_1}(x) \wedge \varphi_{i_2}(x) \wedge \dots \wedge \varphi_{i_m}(x)$ if $I = (i_1, \dots, i_m)$

any ascending m-tuple in $(1, \dots, n)$.

recall: $\psi_I(x)(x; \underbrace{[\vec{v}_1, \dots, \vec{v}_m] \in \mathbb{R}^{n\times m}}_{X}) = \det \underbrace{X_I}_{\substack{m\times m \text{ minor of } X \\ \text{with rows } (i_1,\dots,i_m)}}$

General m-form $\omega(x) = \sum_{[I]} b_I(x)\psi_I(x)$

↖all ascending m-tuples in $(1, \dots, n)$

where $b_I : V \to \mathbb{R}$ is a scalar function.

ω class C^r $\Leftrightarrow$ b_I class C^r, $r = 0, 1, \dots, \infty$

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We call $\{b_I\}_{[I]}$ the components of ω relative to canonical basis for $\mathbb{R}^n$.

Def: 0-forms: $\Lambda^0(V)$: scalar functions : $V \to \mathbb{R}$

$f, g \in \Lambda^0(V)$, $(f \wedge g)(x) := \underset{\text{ptwise product}}{f(x)g(x)} \in \Lambda^0(V)$

$$\underset{\in \Lambda^0}{(f} \wedge \underset{\in \Lambda^m}{\omega)}(x) := \underbrace{f(x)\omega(x)}_{\sum_{[I]} f(x) b_I(x) \psi_I(x)}$$

Consistent w/ all the axiomatic prop's of $\wedge$

Exterior Derivative (aka differential operator)

We are going to define a linear operator:

$$d : \Lambda^m(V) \to \Lambda^{m+1}(V) \qquad \forall\, 0 \le m \le n$$

- $m = 0$, $f \in \Lambda^0(V)$, $df \in \Lambda^1(V)$ & is defined as follows:

$$(df)(x)\,(x; \vec{v}) = \underset{(1\times n)}{(Df)(x)} \cdot \underset{(n\times 1)}{\vec{v}} \in \mathbb{R}$$

Lemma: $d : \Lambda^0(V) \to \Lambda^1(V)$ is linear:

$$d(af+bg)(x) = a\,df(x) + b\,dg(x) \qquad \text{check!!}$$

□

Representation of canonical 1-forms via d:

Lemma: Ⓗ $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$, $x = (x_1, \dots, x_n)$

φ_i = canonical 1-form, i.e. $\varphi_i(x; \vec{v}) = v_i$

Ⓒ $\varphi_i = d\pi_i$

Pf: Check $d\pi_i(x; \vec{v}) = \underset{= \varphi_i(x;\vec{v})}{v_i}$ $\forall\, v, \forall\, x$ ✓

$$\Big[D\pi_i(x) = (0, \dots, 0, \underset{\overset{\uparrow}{i}}{1}, 0, \dots, 0) \;.\; D\pi_i(x)(x; \vec{v}) = (0, \dots 0, 1, 0, \dots 0)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_i\Big]$$

□

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It follows that (since: $\psi_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_m}$)

$$\psi_I = d\pi_{i_1} \wedge \dots \wedge d\pi_{i_m}$$

Notation: $x_i = \pi_i$ & $\varphi_i = dx_i$ & : $\psi_I = dx_{i_1} \wedge \dots \wedge dx_{i_m}$ ∀ canonical m-forms

So: $\forall \omega \in \Lambda^m(V)$: $\omega(x) = \sum_{[I]} b_I(x) \underbrace{dx_{i_1} \wedge \dots \wedge dx_{i_m}}$

notation: $= dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_m}$

↗

note: dx_I is not a differential

In particular: $f \in \Lambda^0(V)$, $df = (D_1 f) dx_1 + \dots + (D_n f) dx_n$.

- $d : \Lambda^m(V) \to \Lambda^{m+1}(V)$ $\forall m \geq 0$

Thm: ($\exists!$ of exterior derivative op.)

Ⓗ $V \subseteq \mathbb{R}^n$ open

Ⓒ $\forall\, 0 \leq m \leq n$ $\exists!$ linear transf. $d : \Lambda^m(V) \to \Lambda^{m+1}(V)$ s.t.

(1) If $f \in \Lambda^0(V)$, $(df)(x)(x; \vec{v}) = (Df)(x) \cdot \vec{v}$ $\forall x \in V, \forall \vec{v} \in \mathbb{R}^n$.

(2) If $\omega \in \Lambda^m(V)$, $\eta \in \Lambda^l(V)$ then

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^m \omega \wedge (d\eta).$$

("Leibniz rule for ∧")

(3) $\forall \omega \in \Lambda^m(V)$, $d(d\omega) = 0$ ("$d^2 = 0$")

Proof: "final exam assignement". □

Fact: $\omega(x) = \sum_{[I]} \underbrace{b_I(x)}_{\in \Lambda^0(V)} dx_I$

$$d\omega(x) = \sum_{[I]} (db_I)(x) \wedge dx_I = \sum_{[I]} \sum_{j=1}^{n} (D_j b_J) \underbrace{dx_j \wedge dx_I}_{\substack{J = (j, I) \\ (m+1)\text{-tuple}}}$$

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Def: Let $0 \le m \le n$, $w \in \Lambda^m(V)$

- w is closed if $dw \equiv 0$
- w is exact if $\exists\, \eta \in \cancel{V}\Lambda^{m-1}(V)$ s.t. $w = d\eta$

(note: w exact $\Rightarrow dw = 0$ b/c $d^2 \equiv 0$)

so: exact $\Rightarrow$ closed
but $\not\Leftarrow$ in general, unless V is "star-shaped"

Connection with multivariable calculus:

~~Def.~~ $V \subseteq \mathbb{R}^3$ open $(n=3)$

Def: $\vec{\nabla}: \{\text{functions}: V \to \mathbb{R}\} \to \{\text{vector fields}: V \to \mathbb{R}^3\}$

$$f \longmapsto \vec{\nabla} f(x) = (x\,;\,((D_1 f)(x)\vec{e}_1 + \_\_ + (D_n f)(x)\vec{e}_n))$$

div : $\{\text{vector fields}: V \to \mathbb{R}^3\} \to \{\text{functions}: V \to \mathbb{R}\}$

$$\underset{(g_1, g_2, g_3)}{G} \longmapsto \operatorname{div} \vec{G} = (D_1 g)(x) + (D_2 g)(x) + (D_3 g_3)(x)$$

$\overrightarrow{\text{Curl}}$: $\{\text{vector fields}: V \to \mathbb{R}^3\} \circlearrowleft$

$$\vec{F} = (f_1, f_2, f_3) \longmapsto \overrightarrow{\text{Curl}}\, \vec{F}(x) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ D_1 & D_2 & D_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Thm: Ⓗ $V \subseteq \mathbb{R}^3$, open set.

Ⓒ $\exists$ v.s. isomorphism $\alpha_j\,;\,\beta_j$:

$$\begin{tikzcd}
\{\text{scalar fields}: V \to \mathbb{R}\} \arrow[r, "\alpha_0"] \arrow[d, "\vec{\nabla}"] & \Lambda^0(V) \arrow[d, "d"] \\
\{\text{vector fields}: V \to \mathbb{R}^3\} \arrow[r, "\alpha_1"] \arrow[d, "\overrightarrow{\text{Curl}}"] & \Lambda^1(V) \arrow[d, "d"] \\
\{\text{vector fields}: V \to \mathbb{R}^3\} \arrow[r, "\beta_2"] \arrow[d, "\text{div}"] & \Lambda^2(V) \arrow[d, "d"] \\
\{\text{scalar fields}: V \to \mathbb{R}\} \arrow[r, "\beta_3"] & \Lambda^3(V).
\end{tikzcd}$$

s.t. above diagrams commute, i.e.
$d \circ \alpha_0 = \alpha_1 \circ \vec{\nabla}$; $d \circ \alpha_1 = \beta_1 \circ \text{Curl}$, $d \circ \beta_2 = \beta_2 \circ \text{div}$.

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Pf: part of "final exam assignation". $\square$

Action of a differential map

Recall, given:

- $U \subset \mathbb{R}^k$ open set ; $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$, class C^r

$$\alpha_* : T_x(\mathbb{R}^k) \to T_{\alpha(x)}(\mathbb{R}^n) \qquad \forall x \in U$$

$$(x;\vec{v}) \mapsto \alpha_*(x;\vec{v}) = (\alpha(x), \underbrace{(D\alpha)(x)\cdot\vec{v}}_{\vec{u}\in\mathbb{R}^n})$$

- $T : V \to W$, lin. transf. of v.s. $(V \& W)$

$$T^* : \mathcal{A}^l(W) \to \mathcal{A}^l(V)$$

$$\uparrow \qquad \overset{\rotatebox{90}{$\in$}}{w} \qquad \overset{\rotatebox{90}{$\in$}}{(T^* w)(\vec{v})} := w(T\vec{v}_1, \_, T\vec{v}_l)$$

dual transf. of T

Def: **Given:**

- $U \subset \mathbb{R}^k$ open
- $\alpha : U \to \mathbb{R}^n$, class C^r
- $V \subset \mathbb{R}^n$ open s.t. $\alpha(U) \subset V$.

$0 \le l \le \min\{k, n\}$

Pullback by α: $\alpha^* : \Lambda^l(V) \to \Lambda^l(U)$:

- $l = 0$, $f \in \Lambda^0(V)$, $(\alpha^* f)(x) = (f \circ \alpha)(x)$, $x \in U$
- $l \ge 1$, $w \in \Lambda^l(V)$, $(\alpha^* w)(x)((x;\vec{v}_1), \_, (x;\vec{v}_l))$

$$:= w(\alpha(x))(\alpha_*(x;\vec{v}_1), \_; \alpha_*(x;\vec{v}_l)).$$

In practice:

Thm: (1) $U \subset \mathbb{R}^k$ open ; $k \le n$

$\alpha : U \to \mathbb{R}^n$, class C^r

$x \mapsto \alpha(x) = y$, ie. $\alpha(x) = (y_1, \_, y_n)$

dx_i = canonical 1-form in $\mathbb{R}^k$

dy_i = " " in $\mathbb{R}^n$

Oct. 19, 2017

Recall from last time:

$U \subset \mathbb{R}^k$ open; $m \in \{0, 1, \dots, k\}$.

- $\exists !\ d : \Lambda^m(U) \longrightarrow \Lambda^{m+1}(U)$ s.t.
  - $\omega \in \Lambda^m$, $\eta \in \Lambda^l$:

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^m \omega \wedge d\eta$$

  - $d^2 \equiv 0$ $\quad (d(d\omega) = 0 \ \forall \omega)$

in practice: $\omega(x) := \sum_{[I]_m} b_I(x)\, dx_I$

$$\Rightarrow \quad d\omega(x) = \sum_{[I]_m} (db_I)(x) \wedge dx_I = \sum_{\substack{[I]_m \\ j=1,\dots,n}} \frac{\partial b_I}{\partial x_j}(x)\, dx_j \wedge dx_I$$

Note: ω class C^r (i.e. b_I class C^r all I)
$d\omega$ class C^{r-1}

For sake of simplicity, we assume: class C^∞

- ω <u>closed</u> $:=$ $d\omega = 0$.
- ω <u>exact</u> $:=$ $\omega = d\eta \ \exists\, \eta \in \Lambda^{m-1}$

<u>Note</u>: exact $\Longrightarrow$ closed (b/c $d^2 = 0$)

in general $\not\Longleftarrow$

<u>Poincaré's lemma</u>: U convex or more generally, $\star$-shaped then "$\Longleftarrow$" true.

- given $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ class C^∞

$$U \longrightarrow \mathbb{R}^n$$

$$x \longrightarrow y = \alpha(x) = (\alpha_1(x), \dots, \alpha_n(x))$$

$$\alpha^* : \Lambda^m(\mathbb{R}^n) \longrightarrow \Lambda^m(\underset{U}{\mathbb{R}^k}) \quad \forall\ m = 0, 1, \dots, \min\{k, n\}$$

$$\alpha^*(dy_I)(x) = \sum_{\substack{[J]_m \\ \subset (1,\dots,k)}} \left(\det \frac{\partial \alpha_I}{\partial x_J}(x)\right) dx_J$$

$$D\alpha(x) = \underbrace{\begin{matrix} \frac{\partial \alpha_1}{\partial x_1} & \cdots & \frac{\partial \alpha_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial \alpha_1}{\partial x_k} & & \frac{\partial \alpha_n}{\partial x_k} \end{matrix}}_{I \text{ columns}} \Bigg\rangle \text{ pick ``J'' rows}$$

<u>Note:</u> $m = 1 \Rightarrow dy_I = dy_i$, $i = 1, \dots, n$ & $J = j$, $j = 1, \dots, k$

$$\frac{\partial \alpha_I}{\partial x_J} = \frac{\partial \alpha_i}{\partial x_j}(x) \Rightarrow \alpha^*(dy_i) = \sum_{j=1}^{k} \frac{\partial \alpha_i}{\partial x_j}(x)\, dx_j = d\alpha_i$$

by linearity :

$$\text{if}\quad \omega(y) = \sum_{\substack{[I]_m \\ \subseteq (1,\dots,n)}} \underbrace{b_I(y)}_{\Lambda^0}\; dy_I = \sum_{[I]_m} \boxed{b_I(y)\wedge dy_I}$$

$$\Rightarrow \alpha^*\omega(x) = \sum_{[I]_m} \underbrace{\alpha^* b_I(y)}_{\parallel} \wedge \underbrace{\alpha^* dy_I}$$

$$(b_I\circ\alpha)(x) \quad (m=0) \qquad\qquad y=\alpha(x)$$

$$\boxed{\alpha^*(\mu\wedge\eta) = \alpha^*\mu\wedge\alpha^*\eta}$$

$$= \sum_{[I]_m} b_I(\alpha(x))\left(\alpha^* dy_I\right)$$

$$\alpha^*\omega(x) = \sum_{\substack{[I]_m \\ \subset (1,\dots,n) \\ [J]_m \\ \subset (1,\dots,k)}} \left(b_I(\alpha(x))\det\frac{\partial\alpha_I(x)}{\partial x_J}\right) dx_J$$

$$\parallel$$

$$\sum_{[J]_m} c_J(x)\,dx_J$$

$$= \sum_{[J]_m} \underbrace{\left(\sum_{[I]_m} b_I(\alpha(x))\det\frac{\partial\alpha_I(x)}{\partial x_J}\right)}_{c_J(x)} dx_J$$

Thm (properties of pull-back)

Ⓗ $U \subset \mathbb{R}^k$ open

$\alpha : U \longrightarrow \mathbb{R}^m$, class C^∞

$V \subset \mathbb{R}^m$ open ; $\alpha(U) \subset V$

$\beta : V \longrightarrow \mathbb{R}^n$, class C^∞.

ω; η be l-forms (same order)

Ⓒ (1) $\beta^*(a\omega + b\eta) = a\beta^*\omega + b\beta^*\eta$

(2) $\beta^*(\omega \wedge \theta) = \beta^*\omega \wedge \beta^*\theta \quad \forall\, \theta$

(3) $(\beta \circ \alpha)^*\omega = \alpha^*(\beta^*\omega)$.

Pf argument.

Thm (d commutes with pull-back):

Ⓗ $U \subset \mathbb{R}^k$ open; $\alpha : U \to \mathbb{R}^n$ class C^∞
$x \mapsto y = \alpha(x)$
$V \subset \mathbb{R}^n$ open, $\alpha(U) \subset V$
$\omega \in \Lambda^\ell(V)$

Ⓒ $d(\alpha^*\omega) = \alpha^*(d\omega)$ i.e.

$$\begin{tikzcd}
\Lambda^\ell(U) \arrow[r, "d"] & \Lambda^{\ell+1}(U) \\
\Lambda^\ell(V) \arrow[u, "\alpha^*"] \arrow[r, "d"'] \arrow[ur, phantom, "\circlearrowright"] & \Lambda^{\ell+1}(V) \arrow[u, "\alpha^*"']
\end{tikzcd}$$

pf: final assignation!

Importance of last thm ($d\alpha^* = \alpha^* d$)

- $\omega \in \Lambda^m(\mathbb{R}^n)$ ~~closed ($d\omega = 0$)~~

  ω closed $\iff$ $\alpha^*\omega$ closed

  $$0 = d\omega \Rightarrow 0 = \alpha^*(d\omega) \Rightarrow d(\alpha^*\omega) = 0 \Rightarrow \alpha^*\omega \text{ closed.}$$

  $\Leftarrow$ true (same proof).

- $\omega \in \Lambda^m(\mathbb{R}^n)$ <u>exact</u> i.e. $\omega = d\eta$ $\exists\, \eta \in \Lambda^{m-1}(\mathbb{R}^n)$

  $\Rightarrow$ $\alpha^*\omega$ <u>also</u> <u>exact</u>:

  $$\omega = d\eta \Rightarrow \underbrace{\alpha^*\omega} = \alpha^* d\eta \overset{\text{Thm}}{=} d(\underbrace{\alpha^*\eta})$$

- if α diffeomorphism $: \mathbb{R}^n \to \mathbb{R}^n$ then:

  ω exact $\iff$ $\alpha^*\omega$ exact.

pf: $\Rightarrow$ ✓

$\Leftarrow$ suppose $\alpha^*\omega = d\mu$ ($\alpha^*\omega$ exact)

$$\Rightarrow \underbrace{(\alpha^{-1})^*\alpha^*\omega} = (\alpha^{-1})^* d\mu$$

// (3) in thm on properties of pull-back

$$\underbrace{(\alpha\circ\alpha^{-1})^*\omega}$$

$$\underbrace{(Id)^*\omega}$$

$$\boxed{\omega} = (\alpha^{-1})^* d\mu \overset{\text{Thm}}{=} d(\underbrace{(\alpha^{-1})^*\mu})$$

~~Upa~~ Aside: $U \subset \mathbb{C}^n = \mathbb{R}^{2n}$

$$\boxed{\bar\partial : \Lambda^m \to \Lambda^{m+1}}$$

$$(\bar\partial)^2 = 0$$

~~dz_I~~ dz_I ; $d\bar z_J$

$\parallel$

$dz_{i_1}\wedge dz_{i_2}\wedge\dots\wedge dz_{i_k}$

$d\bar z_{j_1}\wedge d\bar z_{j_2}\wedge\dots\wedge d\bar z_{j_k}$

$$\bar\partial\left(f\, dz_I\wedge d\bar z_J\right)$$

$$\bar\partial f \wedge \left(dz_I\wedge d\bar z_J\right)$$

$$\sum_{j=1}^{n}\frac{\partial f}{\partial \bar z_j}\, dz_j\wedge dz_I\wedge d\bar z_J$$

∂

$$\alpha^*\bar\partial \neq \bar\partial\,\alpha^*$$

ex/ $n=3$; $k=2$ $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$U = \mathbb{R}^2$; $(u,v) \longrightarrow (x, y, z) := (\underset{\alpha_1}{uv}, \underset{\alpha_2}{u^2}, \underset{\alpha_3}{3u+v})$

$$\alpha^* : \Lambda^m(\mathbb{R}^3) \longrightarrow \Lambda^m(\mathbb{R}^2), \quad m = 0, 1, 2.$$

$m=1$, $\omega = xy\,dx + 2z\,dy - y\,dz \in \Lambda^1(\mathbb{R}^3)$

Compute directly: $d\omega$; $\alpha^*\omega$; $\alpha^*(d\omega)$; $d(\alpha^*\omega)$.

$$\cdot\, d\omega = \underbrace{d(xy)}_{} \wedge dx + \underbrace{2dz \wedge dy}_{-2dy \wedge dz} - dy \wedge dz$$

$$0 + x\,dy \wedge dx$$

$$\boxed{d\omega = -\{x\,dx \wedge dy + 3\,dy \wedge dz\}}$$

~~$\cdot\, \alpha^*\omega =$~~

use: $\alpha^* dy_i = d\alpha_i$

$$,\ \omega = \underset{b_1}{xy}\,dx + \underset{b_2}{2z}\,dy - \underset{b_3}{y}\,dz$$

$$\alpha^*\omega(u,v) = \sum_j (b_j \circ \alpha)\, d\alpha_j = \underbrace{(uv)u^2}_{b_1(\alpha)} d\underset{\alpha_1}{(uv)} +$$

$$+ \underbrace{2(3u+v)}_{b_2 \circ \alpha} d\underset{\alpha_2}{(u^2)} - \underbrace{u^2}_{b_3 \circ \alpha} d(3u+v) =$$

$$\alpha^+ w(u,v) = u^3 v\left(v\,\underset{\frown}{du} + u\,\underset{\smile}{dv}\right) + 4u(3u+v)\,\underset{\frown}{du} \; +$$

$$-u^2\left(3\,\underset{\frown}{du} + \underset{\smile}{dv}\right) =$$

$$= \left(v^2u^3 + 12u^2 - 3u^2 \underset{4uv}{+}\right)du + \left(u^4 v - u^2\right)\underset{\frown}{dv}$$

$$\boxed{\alpha^* w(u,v) = \left(v^2u^3 + 9u^2 + 4uv\right)du + \left(u^4 v - u^2\right)dv}$$

$$d(\alpha^* w) = d\left(v^2u^3 + 9u^2 + 4uv\right)\wedge du + d\left(u^4 v - u^2\right)\wedge dv$$

$$= \left(2vu^3 + 4u\right)dv\wedge du + \left(4u^3 v - 2u\right)du\wedge dv$$

$$= \left(4u^3 v - 2u - 4u - 2vu^3\right)du\wedge dv$$

$$\boxed{d(\alpha^* w) = \left(2u^3 v - 6u\right)du\wedge dv}$$

• $\alpha^+(dw)$. recall: $dw = -\{x\,dx\wedge dy - 3\,dy\wedge dz\}$

$$\alpha(u,v) = (\underset{\alpha_1}{uv},\ \underset{\alpha_2}{u^2},\ \underset{\alpha_3}{3u+v})$$

$$D\alpha(u,v) = \begin{pmatrix} \frac{\partial\alpha_1}{\partial u} & \text{etc.} \\ \frac{\partial\alpha_1}{\partial v} & \end{pmatrix} = \begin{pmatrix} v & 2u & 3 \\ u & 0 & 1 \end{pmatrix}$$

$$\underbrace{u \quad 0}_{(1,2)} \qquad \underbrace{0 \quad 1}_{(1,3)}$$

$\cdot\ \alpha^*(\underset{I=12}{dx\wedge dy}) = -2u^2\, du\wedge dv$

$\cdot\ \alpha^*(\underset{I=23}{dy\wedge dz}) = 2u\, du\wedge dv$

$$\alpha^* d\omega = (-uv)(-2u^2\, du\wedge dv) - 3(2u\, du\wedge dv)$$

$$\boxed{\alpha^* d\omega = (2u^3v - 6u)\, du\wedge dv \qquad ☺}$$

Stokes' Thm

Recall from last time:

- m-forms in $\mathbb{R}^n$ are a generalization to $\mathbb{R}^n$ of $\left.\begin{matrix}\text{scalar fields}\\ \text{vector fields}\end{matrix}\right\}$ in $\mathbb{R}^3$
- exterior derivative d is a generalization to $\mathbb{R}^n$ of $\left.\vec{\nabla}, \vec{\text{curl}}, \text{div}\right\}$ in $\mathbb{R}^3$

Next:

- integral of an m-form over a (cpct) k-mfld in $\mathbb{R}^n$:

  this will be a generalization ~~of~~ to $\mathbb{R}^n$ of

  $\left.\begin{matrix}\cdot\ \text{line integrals}\\ \cdot\ \text{surface integrals}\end{matrix}\right\}$ in $\mathbb{R}^3$.

Standing assumptions:

- k-mfld in $\mathbb{R}^n$, M, is cpct & C^∞.
- m-form ω is defined on V open set in $\mathbb{R}^n$ with $M \subset V$
- m-form ω class C^∞

$$\omega(x) = \sum_{\substack{[I]m \\ \subset \{1,\dots,n\}}} b_I(x)\, dx_I \rightsquigarrow \begin{cases} b_I \text{ class } C^\infty \\ \text{~~de~~ } b_I \text{ defined on open set } V \supset M. \end{cases}$$

- integral of $\eta \in \Lambda^k(U)$, $U \subset \mathbb{R}^k$ open, over $\overline{U}$:

Then $[I]_k = (1, 2, \ldots, k)$ a single ascending k-labels in $\mathbb{R}^k$

$$\eta(x) = \underbrace{a(x)}_{a : U \to \mathbb{R}} dx_1 \wedge \cdots \wedge dx_k$$

Def : $$\int_U \eta = \int_U a(x) \underbrace{dV(x)}_{\text{Euclidean vol. meas. in } \mathbb{R}^k}$$

provided the latter exists.

- Integral over a parametrized k-mfld, M, in $\mathbb{R}^n$ of $\omega \in \Lambda^k(V)$, with $V \subset \mathbb{R}^n$ open, $M \subset V$

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Ⓒ (a) $l=1$, $\alpha^*(dy_i) = d\alpha_i$

(b) $l=k$ $\forall\, I=(i_1,\dots,i_k)$ = ascending k-tuple in $(1,\dots,n)$

$$\alpha^*(dy_I) = \left(\det \frac{\partial \alpha_I}{\partial x}\right) dx_1\wedge\dots\wedge dx_k.$$

$(D\alpha(x) \in \mathbb{R}^{n\times k}$, $D\alpha_I \in \mathbb{R}^{k\times k}$

(c) General l: $I=(i_1,\dots,i_l)$ ascending in $(1,\dots,n)$

$$\alpha^*(dy_I) = \sum_{[I]} \left(\det \frac{\partial \alpha_I}{\partial x_J}\right) dx_J$$

$J=(j_1,\dots,j_l)$ = ascending l-tuples in $(1,\dots,k)$

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<u>Oct 19</u>: see printed sheets

Recall from last time: Stoke's thm

<u>Next:</u> • integral of k-form ω over (cpct) k-mfd in $\mathbb{R}^n$:

- generalization of <u>line integrals</u> in $\mathbb{R}^3$; surface integrals in $\mathbb{R}^3$

$$\uparrow$$

orientation

$$\underbrace{\oint_C Pdx+Qdy \qquad \oint_S dxdy}$$

need "orientation" for C or S.

Orientable mfds

<u>Def:</u> Let M be a k-mfd in $\mathbb{R}^n$, let $\alpha_j: \underset{\subset\mathbb{R}^k}{U_j} \to \underset{\subset\mathbb{R}^n}{V_j}$, $j=1,2$ be two coord. charts for M s.t. $V_1\cap V_2\neq\emptyset$ ("overlapping charts") Assume $V_1\subset\alpha_2(U_2)$

- We say that the charts α_1 & α_2 overlap positively if transition funct. $\alpha_2^{-1}\circ\alpha_1: \underset{\subset\mathbb{R}^k}{U_1} \to \underset{\subset\mathbb{R}^k}{U_2}$ has: $\det\{D(\alpha_2^{-1}\circ\alpha_1)(x) > 0$ $\forall x\in U_1$
- We say that M is orientable if M has an atlas $A=\{(U_j,\alpha_j)\}_j$. s.t. any two overlapping coord charts in A overlap positively.

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Otherwise, M is not orientable

- If M is orientable, then an orientation for M is a collection of charts that cover M & overlap positively.

Orientable k-mfds in $\mathbb{R}^n$: three special cases:

$k=1$; $k=n-1$; $k=n$ (class C^∞)

- <u>$k=1$:</u> M orientable 1-mfd in $\mathbb{R}^n$ $\Rightarrow$ M admits a <u>unit</u> tangent field of class C^∞:

$T: M \to T(M)$ class C^∞:

given $p \in M$ let (U_1, α) be any coord. chart about p in the orientation of M s.t. $p = \alpha(t_1)$ $\exists t_1 \in U_1$ def. $T(p) := \left(p, \frac{D\alpha(t_1)}{\|D\alpha(t_1)\|}\right)$

To show: this def indep. of choice of α.

Let (U_2, β) be another coord. chart about p in orientation for M s.t. $p = \beta(t_2)$ $\exists t_2 \in U_2$ then

$g: \beta^{-1} \circ \alpha: U_1 \to U_2$ & $Dg(t)$ is 1×1 matrix

with $\det Dg(t) > 0$ (b/c α_0 and β overlap positively $\forall t \in U_1$)

$$D\alpha(t_1) \underset{\beta\circ g=\alpha}{=} D(\beta\circ g)(t_1) \underset{\substack{\text{chain}\\ \text{rule}\\ g(t_1)=\beta^{-1}(p)=t_2}}{=} D\beta(t_2)\underbrace{Dg(t_1)}_{\mu_1>0}$$

$$\Rightarrow D\alpha(t_1) = \mu_1 D\beta(t_2)$$

$$\Rightarrow \frac{D\alpha(t_1)}{\|D\alpha(t_1)\|} \overset{(*)}{=} \frac{\mu D\beta(t_2)}{\|\mu D\beta(t_2)\|} = \frac{\mu D\beta(t_2)}{|\mu| \|D\beta(t_2)\|} \overset{\mu=|\mu|}{=} \frac{D\beta(t_2)}{\|D\beta(t_2)\|}$$

- <u>$k=n-1$:</u> $\forall p \in M$, $T_p(M)$ has dim $= n-1$ & a basis $T_p(M)$ is $\left\{\left(p; \frac{\partial\alpha}{\partial x_1}(x_0)\right); \dots; \left(p, \frac{\partial\alpha}{\partial x_{n-1}}(x_0)\right)\right\}$ if (U, α) is a coord chart

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about p in orientation of M with $p = \alpha(x_0)$.

Complete to a basis for $T_p(\mathbb{R}^n)$.

$$\left\{ \left(p; \underset{\substack{\uparrow \\ \text{unit vector s.t.}}}{\vec{n}(p)}\right); \left(p, \frac{\partial \alpha}{\partial x_1}(x_0)\right); \dots; \left(p, \frac{\partial \alpha}{\partial x_n}(x_0)\right) \right\}$$

$$\det \underbrace{[n, D\alpha(x_0)]}_{n \times n \text{ matrix}} > 0$$

This defines a unit vector field & one may show that:

M orientable & class C^∞ $\Rightarrow$ $p \to (p, \vec{n}(p))$ is well-def'd

(indep. of choice of α chart) & class C^∞.

"The unit normal field to M corresponding to orientation of M".

$n = 3$
$k = 2$ Möbius band

• **k=n:** M n-mfd in $\mathbb{R}^n$. is always orientable:

$\forall$ atlas $A = \{(U_j, \alpha_j)\}$ $\quad \alpha: \underset{\subset \mathbb{R}^n}{U} \to \underset{\subset \mathbb{R}^n}{V}$ has $\det \underbrace{D\alpha(x)}_{n \times n} \neq 0$ $\forall x \in U$

Def. Natural orientation of $M = \{(U, \alpha) \mid \det \alpha(x) > 0 \ \forall x \in U\}$

Let (U_j, α_j) be an atlas for M.

Assume w.l.o.g. that each U_j is connected (either $U_j = \mathbb{B}_{\varepsilon_j}(p)$

$\exists p \in M$, or $U_j = \mathbb{B}_{\varepsilon_j}(p) \cap \mathbb{H}^n$)

then: either : $\left. \begin{array}{l} \det D\alpha_j(x) > 0 \\ \det D\alpha_j(x) < 0 \end{array} \right\} \forall x \in U_j$

if $\det D\alpha_j(x) > 0 \ \forall x \in U_j$, save (U_j, α_j)

if $\det D\alpha_j(x) < 0 \ \forall x \in U_j$, replace (U_j, α_j) with $(\tilde{U}_j, \tilde{\alpha}_j)$, $\tilde{\alpha}_j := \alpha_j \circ r$

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where $r : \mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \dots, x_n) \to r(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$

then: $\det(D\tilde{\alpha}_j)(x) > 0$; $\tilde{U}_j = \{(-x_1, x_2, \dots, x_n) \mid (x_1, \dots, x_n) \in U_j\}$

So: $\tilde{A}$: $\{ \quad \det D\alpha_j > 0\} \cup \{\alpha_j \circ r, \det D\alpha_j > 0\}$ is an atlas

for M made of positively overlapping charts: M orientable.

Reversing orientation of an orientable k-mfd in $\mathbb{R}^n$

Suppose M is orientable k-mfd in $\mathbb{R}^n$.

Def: $r : \mathbb{R}^k \to \mathbb{R}^k$

$(x_1, \dots, x_k) \to (-x_1, x_2, \dots, x_n) = r(x_1, \dots, x_n)$

Suppose (U, α) is coord. patch belonging to orientation of M.

Def: $(\tilde{U}, \tilde{\alpha})$, with $\tilde{\alpha}(\tilde{x}) := \alpha \circ r(\tilde{x})$, $\tilde{x} \in \tilde{U}$, $\tilde{U} = \{(-x_1, \dots, x_k) \mid (x_1, \dots, x_n) \in U\}$.

Then: (U, α) & $(\tilde{U}, \tilde{\alpha})$ overlap negatively

But: $(\tilde{U}_1, \tilde{\alpha}_1)$ & $(\tilde{U}_2, \tilde{\alpha}_2)$ overlap positively (if (U_1, α_1) & (U_2, α_2) so do)

So, $\{(\tilde{U}, \tilde{\alpha}) \mid (U, \alpha) \in \text{orientation}\}$ is another orientation of M.

So: every orientable mfd has at least two orientations

(& if M connected, then M has exactly two orientations)

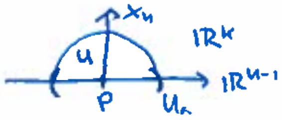
Induced orientation on boundary of orientable mfd (with boundary!)

Thm: (H) M orientable k-mfd in $\mathbb{R}^n$ with $bM \neq \emptyset$ $(k \geq 1)$

(C) bM is orientable.

Pf: Let $p \in bM$. Let (U, α) be a coord. chart for p in M.

Then $(p \in bM)$ U open in $\mathbb{H}^k$. set: $U_0 := U \cap \mathbb{R}^{k-1}$.



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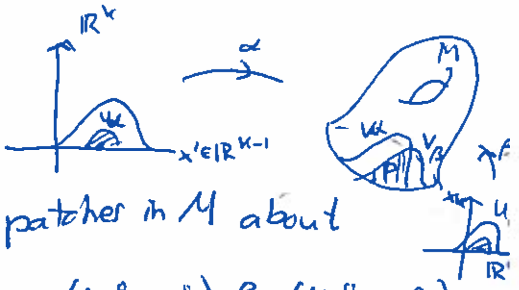
def $\alpha_0^\circ \underbrace{(x_1, \ldots, x_{k-1})}_{\in U_0} = \alpha(x_1, \ldots, x_{k-1}, 0)$. is a coord. patch

for p in bM. ("restriction of α")



<u>Fact:</u> if (U_i, α_i) & (U_j, α_j) are two coord. patches in M about $p \in bM$ that overlap positively then so do: $(U_i^\circ, \alpha_i^\circ)$ & $(U_j^\circ, \alpha_j^\circ)$

So, if $\{(U_i, \alpha_i, p) \mid p \in M\}$ is an orientation for M, then $\{(U_i^\circ, \alpha_i^\circ, p) \mid p \in bM\}$ is an orientation for bM. $\square$

<u>Oct 26</u>

Recall from last time:

M = k-mfd in $\mathbb{R}^n$.

M <u>orientable</u> if M admits an atlas $\{U_\alpha, \alpha, V_\alpha\}_\alpha$ s.t. any two overlapping charts in atlas overlap <u>positively</u>, i.e.

if $V_\alpha \cap V_\beta \neq \emptyset$ then $g := \beta^{-1} \circ \alpha : \alpha^{-1}(V_\alpha \cap V_\beta) \to \beta^{-1}(V_\alpha \cap V_\beta)$ has $\det Dg(x) > 0 \quad \forall x \in \alpha^{-1}(V_\alpha \cap V_\beta)$.

We call such an atlas <u>"an orientation for M"</u>.

M orientable $\Rightarrow$ M has (at least) two orientations: $\to (\tilde{U}_\alpha, \tilde{\alpha}, \tilde{V}_\alpha)_\alpha$

~~Let f U~~

- Any n-mfd M in $\mathbb{R}^n$ $(k=n)$ <u>is</u> orientable with a "natural" orientation induced by $\mathbb{R}^n$: (U_α, α) s.t. $\det D\alpha(x) > 0 \ \forall x \in U_\alpha$.
- Orientable 1-mfds & (n-1)-mfds $(k=1, n-1)$ have "intrinsic" orientations (via tangent vector fields $(k=1)$ or normal vector field $(k=n-1)$).

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Induced orientation on bM

Thm: Ⓗ M is orientable k-mfd in $\mathbb{R}^n$, $bM \neq \emptyset$

Ⓒ bM is orientable $(k-1)$-mfd in $\mathbb{R}^n$)

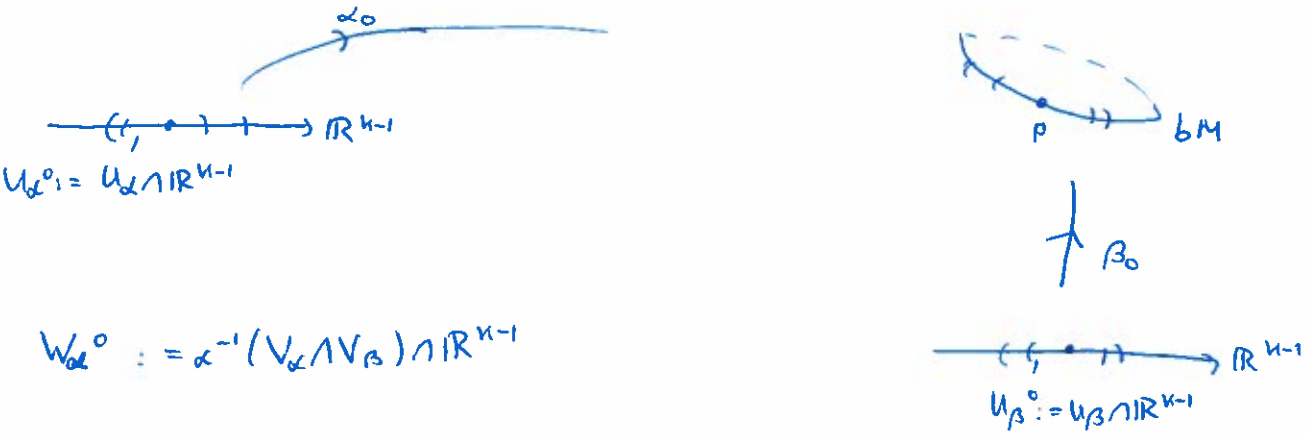
Pf. $p \in bM$, α, β two coord charts for M about p that overlap positively.

$$g: \beta^{-1} \circ \alpha : \alpha^{-1}(V_\alpha \cap V_\beta) \to \beta^{-1}(V_\alpha \cap V_\beta) \text{ has } \det Dg(t) > 0$$
$$\forall t \in \alpha^{-1}(V_\alpha \cap V_\beta). (1)$$

Let α_0 & β_0 be the coord charts for bM about p obtained by

restricting α & β to bM:

$$\alpha_0(x') := \alpha(x', 0) \; ; \; \beta_0(x') = \beta(x', 0)$$
$$(x', 0) \in U_\alpha \qquad (x', 0) \in U_\beta$$



$U_\alpha^0 := U_\alpha \cap \mathbb{R}^{k-1}$

$W_\alpha^0 := \alpha^{-1}(V_\alpha \cap V_\beta) \cap \mathbb{R}^{k-1}$

$U_\beta^0 := U_\beta \cap \mathbb{R}^{k-1}$

def. $g^0 := \beta_0^{-1} \circ \alpha_0 : W_\alpha^0 \to W_\beta^0$

$W_\beta^0 := \beta^{-1}(V_\alpha \cap V_\beta) \cap \mathbb{R}^{k-1}$

Claim: $\det Dg^0(x') > 0 \;\; \forall x' \in W_\alpha^0$

This will show that α_0 & β_0 overlap positively.

$$Dg^T(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_{k-1}}{\partial x_1} & \frac{\partial g_k}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_1}{\partial x_{k-1}} & \cdots & \frac{\partial g_{k-1}}{\partial x_{k-1}} & \frac{\partial g_k}{\partial x_{k-1}} \\ \frac{\partial g_1}{\partial x_k} & \cdots & \frac{\partial g_{k-1}}{\partial x_k} & \frac{\partial g_k}{\partial x_k} \end{pmatrix}, \quad x \in W_\alpha$$

Focus on W_α^0:

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We know : $g : (x';0) \mapsto (y';0) \quad \forall x' \in W_k^\circ$ (*)

Claim: $\frac{\partial g_k}{\partial x_j}(x';0) = 0 \quad \forall j = 1, \ldots, k-1, \ \forall x' \in W_k^\circ$

Pf: ie. $j=1$: $\frac{\partial g_k}{\partial x_j}(x';0) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{g_k(x_1+h, x_2, \ldots, x_{k-1}, 0) - g_k(x';0)}{h}$

$\overset{(*)}{=} \lim_{k \to 0} \frac{0-0}{h} = 0$. Same all $j = 1, \ldots, k-1$.

Claim: $\frac{\partial g_k}{\partial x_k}(x';0) \geq 0 \quad \forall x' \in W_k^\circ$.

≥ 0 b/c $g(x';h) \in \mathbb{H}^k$

$$\frac{\partial g_k}{\partial x_k}(x';0) = \lim_{h \downarrow 0} \frac{g_k(x';h) - g_k(x';0)}{h} \overset{(*)}{=} \lim_{h \downarrow 0} \frac{g_k(x';h) - 0}{h} \geq 0$$

$$Dg^T(x';0) = \left(\begin{array}{ccc|c} \frac{\partial g_1}{\partial x_1}(x';0) & \cdots & \frac{\partial g_{k-1}}{\partial x_1}(x';0) & 0 \\ \vdots & & & \vdots \\ \frac{\partial g_1}{\partial x_{k-1}}(x';0) & \cdots & \frac{\partial g_{k-1}}{\partial x_{k-1}}(x';0) & 0 \\ \hline \cdots & \cdots & \cdots & \frac{\partial g_k}{\partial x_k}(x';0) \end{array} \right)$$

$(Dg)_I \rightarrow$

$$\det Dg(x';0) = \frac{\partial g_k}{\partial x_k}(x';0) \cdot \det (Dg)_I (x';0)$$

But: $(Dg)_I(x';0) = (Dg^\circ)(x')$ check!

Thus: $\underbrace{\det Dg(x';0)}_{>0} = \underbrace{\frac{\partial g_k}{\partial x_k}(x';0)}_{>0} \cdot \det (Dg^\circ)(x')$

$\Rightarrow \det (Dg^\circ)(x') > 0$ all $x' \in W_k^\circ$. ∎

Q: Does that work for any $(k-1)$-submfd? Probably not.

Integrations of k-forms over k-mfds in $\mathbb{R}^n$

Case I: $k=n$ & M = open subset, U, of $\mathbb{R}^n$; $\omega \in \Lambda^n(U)$.

- $\omega(x) = a(x)\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, $\exists a : U \to \mathbb{R}$
- $\int_M \omega \underset{\text{def}}{:=} \int_U a(x)\, dx_1\, dx_2 \ldots dx_n$, provided this exists as a Riemann integral in $\mathbb{R}^n$

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Case 2: $k \leq n$ & $M =$ "parametrized n-mfd in $\mathbb{R}^n$" ie.

M covered by a single chart $\alpha: U \subset \mathbb{R}^n \to \alpha(U) = V \supset M$.
U open, connected

Let $\omega \in \Lambda^n(V)$. Def.: $\displaystyle\int_M \omega = \int_U \underbrace{\alpha^* \omega}_{\in \Lambda^n(U)\ \leadsto\ \text{Case }1}$ provided latter exists

Recall that: $\omega \in \Lambda^n(V) \to \alpha^*\omega \in \Lambda^n(U)$

$\alpha: \underset{\mathbb{R}^n}{U} \to \underset{\mathbb{R}^n}{V}$ $\qquad \alpha^*: \Lambda^n(V) \to \Lambda^n(U)$

$$\alpha^*\omega(x) = \sum_{\substack{[I]\\ \in\{1,\dots,n\}}} (b_I \circ \alpha)(x) \underbrace{\alpha^* dy_I}_{= \det\left(\frac{\partial \alpha_I}{\partial x}\right)(x)\, dx_1 \wedge \dots \wedge dx_n} \overset{k=n}{=} (b \circ \alpha)(x) (\det D(\alpha)(x))\, dx_1 \wedge \dots \wedge dx_n$$

$$\alpha^*\omega(x) = b(\alpha(x)) \det D\alpha(x)\, dx_1 \wedge \dots \wedge dx_n$$

Lemma: ~~Thm:~~ [(independence of choice of parametrization)]

Ⓗ M parametrized ~~k~~-mfd in $\mathbb{R}^n$

$M = \alpha(U) = \beta(\tilde{U})$, $\quad U; \tilde{U} \subset \mathbb{R}^n$ open

$\alpha: U \to \alpha(U) = M$, U open in $\mathbb{R}^k$
$\beta: \tilde{U} \to \beta(\tilde{U}) = M$, $\tilde{U}$ " " " " are two param. of M, ie.
α: 1-1, $D\alpha(x)$ has rank k $\forall x \in U$
β: " " $D\beta(y)$ " " $\forall y \in \tilde{U}$

$\omega \in \Lambda^n(M)$
$g := \beta^{-1} \circ \alpha$

Ⓒ $\displaystyle\int_U \alpha^*\omega = \varepsilon \int_{\tilde{U}} \beta^*\omega$, where $\varepsilon = \operatorname{sign} \det Dg(x)$ $\forall x \in U$

$k=n$: Check! (CVF for integrals in $\mathbb{R}^n$)
$\Rightarrow$ (Thm) indep. of choice of param. $\qquad \varepsilon = 1$

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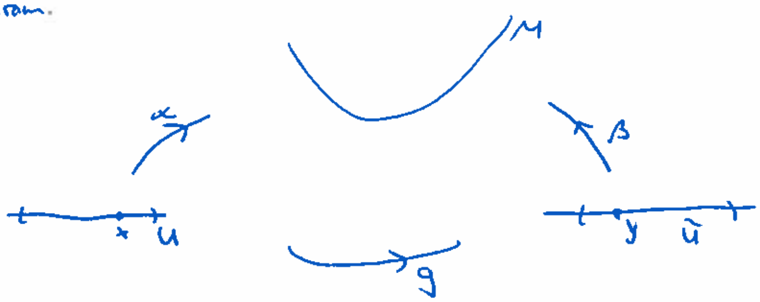
Pf.



Ⓗ $\Rightarrow \det Dg(x) \neq 0$ $\forall x \in U$ $\therefore$ sign ~~θ~~ $\det Dg(x) = +1$ or -1 all $x \in U$ "ε".

Now $g = \beta^{-1} \circ \alpha \Leftrightarrow \alpha = \beta \circ g$. Set $\eta := \beta^*\omega$.

Then $g^*\eta = g^*\beta^*\omega = (\beta^{-1}\circ\alpha)^*\beta^*\omega = [\beta \circ (\beta^{-1} \circ \alpha)]^*\omega = \alpha^*\omega$

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~~⇐~~ **So:** $\eta = \beta^* \omega \Rightarrow g^* \eta = \alpha^* \omega$.

Thus: $\int_{\mathcal{U}} \alpha^* \omega \overset{?}{=} \varepsilon \int_{\tilde{\mathcal{U}}} \beta^* \omega \Leftrightarrow \int_{\mathcal{U}} g^* \eta \overset{?}{=} \varepsilon \cdot \int_{\tilde{\mathcal{U}}} \eta$

Now: $\eta = f dy_1 \wedge \dots \wedge dy_k \Rightarrow g^* \eta = (f \circ g) g^* (dy_1 \wedge \dots \wedge dy_k)$

$= (f \circ g) \det Dg(x)\, dx_1 \wedge \dots \wedge dx_k$.

Thus: $\int_{\mathcal{U}} g^* \eta \overset{?}{=} \varepsilon \int_{\tilde{\mathcal{U}}} \eta \Leftrightarrow \int_{\mathcal{U}} (f \circ g)(x) \det Dg(x)\, dx_1 \dots dx_k$

$\overset{?}{=} \varepsilon \int_{\tilde{\mathcal{U}}} f(y)\, dy_1 \dots dy_k$

$$\Leftrightarrow \int_{\mathcal{U}} (f \circ g)(x) \underbrace{\varepsilon \det Dg(x)}_{= |\det Dg(x)|} dx_1 \dots dx_k \overset{?}{=} \int_{\tilde{\mathcal{U}}} f(y)\, dy_1 \dots dy_k$$

↑ which is true by CVF (change of var. form.) for Riemann integrals in $\mathbb{R}^k$ □

Case 3:

- $\mathcal{M}$ cpct oriented k-mfd in $\mathbb{R}^n$
- $\omega \in \Lambda^k(W)$ $\exists W$ open in $\mathbb{R}^n$, s.t. $\mathcal{M} \subset W$ i.e.

$$\omega(y) = \sum_{\substack{[I]_k \\ \subset \{1, \dots, n\}}} a_I(y)\, dy_I \quad \exists a_I : W \to \mathbb{R}$$

Def.: $\operatorname{supp} \omega := \bigcup_{[I]_k} \operatorname{supp} a_I$, $C = \mathcal{M} \cap \operatorname{supp} \omega$ (cpct subset of $\mathbb{R}^n$)

- Suppose $\exists$ single coord. chart (connected) : $(\mathcal{U}, \alpha, V)$ belonging to the orientation of $\mathcal{M}$ s.t. $C \subset V$

Def.: $\int_{\mathcal{M}} \omega := \int_{\operatorname{Int} \mathcal{U}} \alpha^* \omega$ provided latter exists as Riemannian integral in $\mathbb{R}^k$

(recall: $\mathcal{U}$ is open in $\mathbb{R}^k$ or in $\mathbb{H}^k$)

Remarks:

- Def of $\int_{\mathcal{M}} \omega$ is indep. of choice of coord. chart in orientation of $\mathcal{M}$ (previous lemma with $\varepsilon = +1$)
- Def of $\int_{\mathcal{M}} \omega$ is linear: if ω & η are supported in single coord. chart

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(U, α, V), then $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$ (linearity of α^* linearity of Riem. integral)

- if $-M := M$ with reversed orient., then $\int_{-M} \omega = -\int_M \omega$.

Case 4: • M cpct oriented k-mfd in $\mathbb{R}^n$

• $\omega \in \Lambda^k(W)$ $\exists W \subset \mathbb{R}^n$ open, $M \subset W$

• Cover M by coord. charts $\{U_\alpha, \alpha, V_\alpha\}$, $V_\alpha = \alpha(U_\alpha) = W_\alpha \cap M$,

$\exists W_\alpha$ open in $\mathbb{R}^n$

ie. $\det D(\beta^{-1} \circ \alpha)(x) > 0$ $\forall x \in U_\alpha$

• choose partition of unity $\varphi_1, \ldots, \varphi_L$ on M that is dominated by $\{U_\alpha\}$

(Recall: $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ s.t.
(1) $\varphi_i(x) \geq 0$ $\forall x \in \mathbb{R}^n$
(2) $(\operatorname{Supp} \varphi_i) \cap M \subset V_i$
(3) $\sum_{i=1}^{L} \varphi_i(x) = 1$ $\forall x \in M$)

Def: $\int_M \omega := \sum_{i=1}^{L} \int_M (\varphi_i \omega)$. $(*)$

Remarks: • $\omega_i := \varphi_i \omega$ is supported in single coord. chart: (U_i, α_i, V_i),

so rhs of $(*)$ is defined as in Case 3.

• def of $\int_M \omega$ is indep. of choice of partition of unity

• $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$

• $\int_{-M} \omega = -\int_M \omega$

Thm: (Computation of $\int_M \omega$ via tiling)

Ⓗ M cpct oriented k-mfd in $\mathbb{R}^n$ $\{V_1, \ldots, V_N, K\}$ is a tiling of M by coord. charts that belong to orientation for M, ie.

• $\{U_i, \alpha_i, V_i\}_{i=1,\ldots,N}$ are coord. charts belong. to orient. of M

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- $V_i \cap V_j = \emptyset \quad \forall i \neq j$

- $M = V_1 \dot{\cup} \ldots \dot{\cup} V_N \cup K$

- K has <u>zero meas.</u> in M [ie, $\alpha^{-1}(K)$ has zero meas. in $\mathbb{R}^k$ $\forall$ coord charts α]

Ⓒ $\displaystyle \int_M \omega = \sum_{i=1}^{N} \underbrace{\int_{U_i} \alpha_i^* \omega}_{\text{Case 2}}$

Stokes Thm

- a thm about integrals of k-forms over cpct, orientable k-mfds M in $\mathbb{R}^n$, that includes all thms of vector calculus ($n=2$ or $n=3$) as special case.

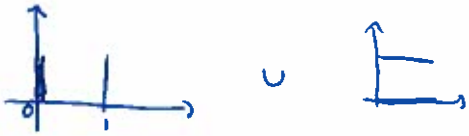
- We begin with a <u>special case:</u> $M = \underbrace{[0,1]^k}_{= I^k}$ in $\mathbb{R}^k$

$I^k = \underbrace{[0,1]}_{I_1} \times \ldots \times \underbrace{[0,1]}_{I_k}$ k times

$\text{Int}(I^k) = (0,1)^k = (0,1) \times \ldots \times (0,1)$

$b(I^k) = I^k \setminus \text{Int}(I^k) = \bigcup_{j=1}^{k} I_1 \times \ldots \times I_{j-1} \times \cancel{b} \underbrace{b I_j}_{\{0,1\}} \times I_{j+1} \times \ldots \times I_k$

e.g. $k=2$:



<u>Lemma:</u> (Stokes' for $M = I^k$)

Ⓗ $\eta \in \Lambda^{\overset{k-1}{\cancel{k}}}(W)$ $\exists$ W open subset of $\mathbb{R}^k$ s.t. $I^k \subset W$ s.t. $\eta(x) = 0$ $\forall x \in b(I^k)$ except possibly on $(0,1)^{k-1} \times \{0\}$

eg. $k=2$

Ⓒ $\displaystyle \int_{(0,1)^k} d\eta = (-1)^k \int_{(0,1)^{k-1}} i^* \eta$, where i is the inclusion map: $i: [0,1]^{k-1} \to [0,1]^{k-1}$

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via $i(u_1, \dots, u_{k-1}) = (u_1, \dots, u_{k-1}, 0)$.

<u>Notation:</u> • u is general pt in $\mathbb{R}^{k-1}$

• x " " " $\mathbb{R}^k$

• $\forall j \in \{1, \dots, k\}$ let $j' = (1, 2, \dots, j-1, j+1, \dots, k)$
$= (1, 2, \dots, \hat{j}, \dots, k)$

• elementary $(k-1)$-forms in $\mathbb{R}^k$: $dx_{j'} = dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_k$

<u>Pf</u> by linearity of $\cdot \int_M \omega$; $\cdot\, d$; i^*: <u>enough</u> to consider case

when $\eta = f\, dx_{j'}$, $j \in \{1, \dots, k\}$, $f: W \to \mathbb{R}$.

<u>Step 1a:</u> • Compute $d\eta$:

$\sum_{l=1}^{k} \frac{\partial f}{\partial x_l} dx_l \wedge dx_{j'}$ $dx_l \wedge dx_l = 0$

$$d\eta = d(f\, dx_{j'}) = df \wedge dx_{j'} \underset{+f \cdot 0\ (d^2=0)}{} = \frac{\partial f}{\partial x_j} dx_j \wedge dx_{j'}$$

$$= \frac{\partial f}{\partial x_j} \underbrace{dx_j \wedge dx_1 \wedge dx_2 \wedge \dots \wedge dx_{j-1}}_{(j-1)\text{-many "jumps"}} \wedge dx_{j+1} \wedge \dots \wedge dx_k$$

$$\Rightarrow d(f\, dx_j) = (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \wedge \dots \wedge dx_k.$$

<u>Step 1b:</u> Compute $\int_{(0,1)^k} d\eta = (-1)^{j-1} \int_{(0,1)^k} \frac{\partial f}{\partial x_j}(x) dx_1 \wedge \dots \wedge dx_k$

$$\overset{\text{Case 1 in def} \int_M \omega}{=} (-1)^{j-1} \int_{(0,1)^k} \frac{\partial f}{\partial x_j}(x)\, dx_1 \dots dx_k \overset{\text{Fubini for Riem. int. in } \mathbb{R}^k}{=} (-1)^{j-1} \int_{(0,1)^{k-1}} \underbrace{\left(\int_0^1 \frac{\partial f}{\partial x_j}(x)\, dx_j \right)}_{\text{// FTC fund. thm. calc}} dx_j$$

$$0 \overset{Ⓗ}{=} \underbrace{\{ f(x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_k) - f(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_k)}_{Ⓗ \neq 0 \Leftarrow j = k}$$

<u>Thus:</u> for $f = \eta\, dx_j$ $\forall j = 1, \dots, k$ we have:

$$\int_{(0,1)^k} d\eta = \begin{cases} 0 & \text{if } j < k \\ (-1)^k \int_{(0,1)^{k-1}} (f \circ i) & \text{if } j = k \end{cases} \quad (1)$$

<u>Step 2a:</u> <u>Compute</u> $i^*(f\, dx_{j'}) = (f \circ i)\, i^*(dx_{j'})$

recall. $i: \mathbb{R}^{k-1} \to \mathbb{R}^k$, $(u_1, \dots, u_{k-1}) \to (u_1, \dots, u_{k-1}, 0)$ has

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Jacobian $\underbrace{D_i(u)}_{\in \mathbb{R}^{k\times(k-1)}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$ So: $i^*(dx_{j'})$ $= \det(D_i)_{j'}\, du_1\wedge\dots\wedge du_{k-1}$

$$\& \quad \det(D_i)_{j'} = \begin{cases} 0 & j\neq k \\ 1 & j=k \end{cases}$$ (picks $(k-1)$-minors that have a zero row)

So: $$i^*(\eta) = i^*(f\,dx_{j'}) = \begin{cases} 0 & j\neq k \\ (f\circ i)(u)\,du_1\wedge\dots\wedge du_k, & j=k \end{cases}$$

$$\int_{[0,1]^{k-1}} i^*\eta = \begin{cases} 0 \text{ if } j\neq k \\ \int_{[0,1]^{k-1}} (f\circ i), & \text{if } j \cancel{\neq} = k. \end{cases} \qquad (2)$$

Conclusion: Comparing (1) & (2) we obtain for $\eta = f\,dx_{j'}$. $\forall j = 1,\dots,k$:

$$\int_{[0,1]^k} d\eta \overset{(1)}{=} 0 \overset{(2)}{=} (-1)^k \int_{[0,1]^{k-1}} i^*\eta \quad \text{if } j\neq k$$

$$\int_{[0,1]^k} d\eta \overset{(1)}{=} (-1)^k \int_{[0,1]^{k-1}} (f\circ i) \overset{(2)}{=} (-1)^k \int_{[0,1]^{k-1}} i^*\eta \quad \text{if } j=k.$$

Thus: $$\int_{[0,1]} d\eta = (-1)^k \int_{[0,1]^{k-1}} i^*\eta. \qquad \square$$

Nov 2

Stokes Thm

Ⓗ $k\geq 2$, M is a cpct oriented k-mfd in $\mathbb{R}^n$. If $bM\neq\emptyset$, give bM orientation induced by M (induced orient.)

$\omega\in\Lambda^{k-1}(\overset{W}{\cancel{A}})$, W open set in $\mathbb{R}^n$ s.t. $M\subset W$.

Ⓒ $$\int_M d\omega = \begin{cases} \int_{bM} \omega & \text{if } bM\neq\emptyset \\ 0 & \text{if } bM=0 \end{cases}$$

Pf: Let $\{U_\alpha, V_\alpha\}$ be an atlas for M in orient. of M

Step 1: Augment atlas by including suitably built coord. charts, which overlap positively.

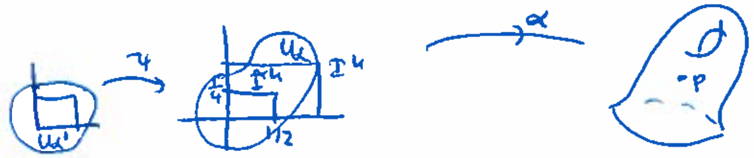
Nov 2

Let $p \in M = (M \setminus bM) \cup bM$.

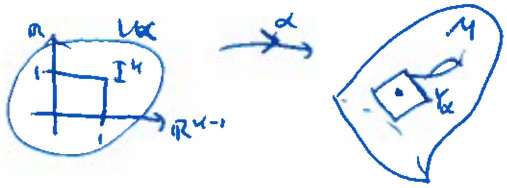
Case 1: $p \in M \setminus bM$ (this will be only case if $bM = \emptyset$)

Choose $(U_\alpha, \alpha, V_\alpha)$ coord chart about p s.t. U_α open in $\mathbb{R}^k$, and contains $I^k = [0,1]^k$ & s.t. $\alpha^{-1}(p) \in (0,1)^k = \text{Int}(I^k)$ (always achievable via translation & reescaling in $\mathbb{R}^k$)
See picture

k=2:



Def $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ rescaling fitted to $\tilde{I}^2$
$(v_1, v_2) \to (\frac{1}{2} v_1, \frac{1}{4} v_2)$ ("re-scaling")
$U_\alpha' = \psi^{-1}(U_\alpha)$
& consider $\tilde{\alpha} := \alpha \circ \psi$ instead.



Set: $Y_\alpha := \alpha((0,1)^k) = \alpha(\text{int}(I^k))$.

Then: restriction of α to $(0,1)^k$, that is

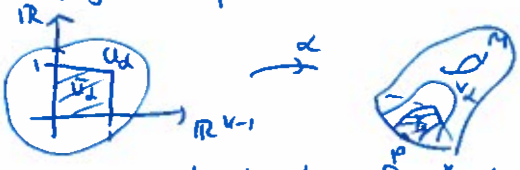
$\{(0,1)^k, \alpha, Y_\alpha\}$ is also a coordchart in the orient. of M.

with $\tilde{U}_\alpha := (0,1)^k$ open in $\mathbb{R}^k$, & by construction α extends to an open nbhd of $(0,1)^k$ (namely U_α).

Case 2: $p \in bM$. Choose $\{U_\alpha, \alpha, V_\alpha\}$ from atlas belonging to orientation of M s.t. U_α is open in $\mathbb{H}^k$ ($\therefore U_\alpha = \mathbb{H}^k \cap (\text{open set } \tilde{U} \text{ in } \mathbb{R}^k)$) that contains $I^k = [0,1]^k$ (again: achievable via transl. & rescalling)

& s.t. $\alpha^{-1}(p) \in (0,1)^{k-1} \times \{0\}$.

Def: $\tilde{U}_\alpha = (0,1)^k \cup ((0,1)^{k-1} \times \{0\})$ open in $\mathbb{H}^k$ (not in $\mathbb{R}^k$), $Y_\alpha = \alpha(\tilde{U}_\alpha)$
See picture:



Then restriction of α to $\tilde{U}_\alpha$ is a coord chart for bM about $p \in bM$ that belongs to orient. of M with $\tilde{U}_\alpha$ open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$.

Nov 2

In the sequel, we will $\overset{\text{work}}{\cancel{\text{look}}}$ exclusively with these special charts

$$\{(0,1)^k, \alpha, Y_\alpha\}, \ p\in \operatorname{Int} M \quad \& \quad \{(0,1)^k \times \{0\}, \alpha, Y_\alpha\} \text{ if } p \in bM.$$

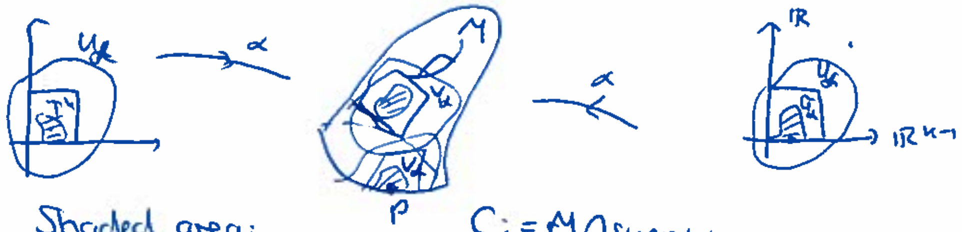
<u>Step 2:</u> by linearity of "d" & of $\int_M d\omega$ enough to prove © in case when $C := M \cap \operatorname{Supp}\omega \subset Y_\alpha = \begin{cases} \alpha(I_k) &, p\in \operatorname{Int} M \\ \alpha((0,1)^k \cup ((0,1)^{k-1}\times\{0\})) &, p \in bM \end{cases}$

(image of a single chart α)

(Say not, $C \subset Y_\alpha \overset{\cup}{\cancel{A}} Y_\beta$, say. $\underset{Y_\alpha\cap Y_\beta \neq 0}{}$ Then $\omega = \underbrace{\varphi_\alpha\omega}_{\substack{\omega_\alpha \\ C_\alpha\subset Y_\alpha}} + \underbrace{\varphi_\beta\omega}_{\substack{\omega_\beta \\ C_\beta\subset Y_\beta}}$ for a part. of unity $\{\varphi_\alpha, \varphi_\beta\}$ dominated by $\{Y_\alpha, Y_\beta\}$

$C_\alpha := M \cap \operatorname{Supp}\omega_\alpha$

$C_\beta := M \cap \operatorname{Supp}\omega_\beta$ & $d\omega = d\omega_\alpha + d\omega_\beta$ so $\int_M d\omega = \int_M d\omega_\alpha + \int_M d\omega_\beta$)



Shaded area: $C := M \cap \operatorname{supp}\omega$

<u>Def:</u> $\eta := \alpha^*\omega \in \Lambda^{k-1}(\mathbb{R}^k)$ Then:

- η extends to U_α open in $\mathbb{R}^k \supseteq I^k = [0,1]$
- $\eta(x) = 0 \ \forall x \in b(I^k)$ except possibly for points in $(0,1)^{k-1}\times\{0\}$.

Thus η satisfies all hypotheses of lemma.

<u>Step 3:</u> Let ω & η as in Step 2. We prove ©

<u>Case 1:</u> $bM = \emptyset$. Then © needs $\int_M d\omega = 0$.

Note that if $bM = \emptyset$, M is covered by charts of type <u>Case 1</u> in <u>Step 1</u>.

Then: $$\int_M d\omega = \int_{(0,1)^k} \alpha^*(d\omega) \underset{\alpha^* d = d\alpha^*}{=} \int_{(0,1)^k} d(\alpha^*\omega) \underset{\eta = \alpha^*\omega}{=} \int_{(0,1)^k} d(\eta)$$

$$\overset{\text{Lemma}}{=} (-1)^k \int_{(0,1)^{k-1}} i^*\eta \underset{\uparrow}{=} 0$$

<u>Case 1:</u> $i^*\eta = 0$ on $(0,1)^{k-1}$. <u>Thus:</u> $\int_M d\omega = 0$ done.

<u>Case 2:</u> $bM \neq \emptyset$. Then © needs: $\int_M d\omega = \int_{bM} \omega$.

<u>Case 2a:</u> ω supported in a chart α s.t. $Y_\alpha \subset \operatorname{Int} M$.

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ie. 

Then, proceeding as in Case 1, we get $\int_M d\omega = 0$. But also $\int_{bM}$ ~~deo~~ $\omega = 0$ as well, b/c in Case 2a: Supp $\omega \cap bM = \emptyset$. $\therefore\ \omega(p) = 0$ $\forall p \in bM$

Case 2b: ω supported in a chart α as in Case 2 of Step 1

Recall: $\tilde{U}_\alpha = (0,1)^k \cup ((0,1)^{k-1} \times \{0\})$ open in $\mathbb{H}^k$, but not open in $\mathbb{R}^k$

Computing as before, $\int_M d\omega = \int_{(0,1)^k} \alpha^* d\omega = \int_{(0,1)^k} d(\alpha^*\omega) = \int_{(0,1)^k} d\eta$

$\underset{=}{\text{Lemma}}\ (-1)^k \int_{(0,1)^{k-1}} i^*\eta$

Thus: $\int_M d\omega = (-1)^k \int_{(0,1)^{k-1}} i^*\eta$, $\eta = \alpha^*\omega$ (1)

Next, compute $\int_{bM} \omega$. Now: Supp $\omega \cap bM$ is covered by the following chart for bM, $\beta := \alpha \circ i : (0,1)^{k-1} \to Y_\alpha \cap bM$

& recall that if k even then β belongs to induced orient. of bM, or else (if k odd), β belongs to opposite of induced orient. of M.

Thus:

$$\int_{bM} \omega ~~\alpha~~ = \begin{cases} \int_{(0,1)^{k-1}} \beta^*\omega & \text{if } k \text{ is even} \\ -\int_{(0,1)^{k-1}} \beta^*\omega & \text{if } k \text{ odd} \end{cases}$$

That is: $\int_{bM} \omega = (-1)^k \int_{(0,1)^{k-1}} \beta^*\omega$ any k

But: $\beta^*\omega \underset{\text{def } \beta}{=} (\alpha \circ i)^*\omega \overset{\text{prop. of pullbacks}}{=} i^*\alpha^*\omega = i^*\eta$. So:

$\int_{bM} \omega = (-1)^k \int_{(0,1)^{k-1}} i^*\eta$ all k (2) & recall: $\int_M d\omega = (-1)^k \int_{(0,1)^{k-1}} i^*\eta$ (1)

Comparing (1) & (2) get: $\int_M d\omega = \int_{bM} \omega$. □

Complex Manifolds

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Let ~~$N \geq 2k$~~. $n \geq k$.

Def M is a complex k-mfd in $\mathbb{C}^n$ if it is a

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2k-mfd in $\mathbb{R}^{2n}$ s.t. ∀ atlas $\{U_i, \alpha_i, V_i\}$, then coord. charts "overlap holomorphically" ie.:

if $\alpha_1 : U_1 \subset \mathbb{R}^{2k} \to V_1$ & $\alpha_2 : U_2 \subset \mathbb{R}^{2k} \to V_2$ are two charts with $V_1 \cap V_2 \neq \emptyset$

then the transition map : $\varphi : \alpha_2^{-1} \circ \alpha_1 : \underset{\overset{!!}{D}}{\alpha_1^{-1}(V_1 \cap V_2)} \to \underbrace{\alpha_2^{-1}(V_1 \cap V_2)}_{\overset{!!}{W}}$

D open set in $U_1 \subset \mathbb{R}^{2k}$ W open in $U_2 \subset \mathbb{R}^{2}$

is a holomorphic map of open sets $\mathbb{C}^k$ under the natural identification of $\mathbb{R}^{2k}$ with $\mathbb{C}^k$:

$$(x_1, y_1, x_2, y_2, \ldots x_k y_k) \to (z_1, \ldots, z_k),\ z_j = x_j + i y_j$$

$$\varphi : D \subset \mathbb{C}^k \to W \subset \mathbb{C}^k$$

$$(z_1, \ldots, z_k) \mapsto (w_1, \ldots, w_k),\ w_j = \varphi_j(z_1, \ldots, z_n)$$

holomorphic map means: $\forall j = 1, \ldots, k$ φ_j holomorphic fct $: D \to \mathbb{C}$

That is, $\varphi_j = u_j + i v_j$ $\forall j = 1, \ldots, k$ & we require that u_j & v_j satisfy the Cauchy-Riemann eqn's:

$$(CR) \begin{cases} \dfrac{\partial u_j}{\partial x_l} = \dfrac{\partial v_j}{\partial y_l} \\ \dfrac{\partial u_j}{\partial y_l} = -\dfrac{\partial v_j}{\partial x_l} \end{cases} \quad \forall l = 1, \ldots, k,\ \forall j = 1, \ldots, k$$

Alternate formulation of CR:

Def:
$$\frac{\partial}{\partial z_l} := \frac{1}{2}\left(\frac{\partial}{\partial x_l} + \frac{1}{i}\frac{\partial}{\partial y_l}\right)$$
$$\frac{\partial}{\partial \bar{z}_l} := \frac{1}{2}\left(\frac{\partial}{\partial x_l} - \frac{1}{i}\frac{\partial}{\partial y_l}\right)$$

Lemma 0: Ⓗ $\varphi_j = u_j + i v_j$

Ⓒ φ_j satisfies (CR) $\Leftrightarrow \dfrac{\partial \varphi_j}{\partial \bar{z}_l} = 0$ $\forall l = 1, \ldots, k$.

Pf:
$$\frac{\partial \varphi_j}{\partial \bar{z}_l} = \frac{\partial u_j}{\partial \bar{z}_l} + i \frac{\partial v_j}{\partial \bar{z}_l} = \frac{1}{2}\left\{\left(\frac{\partial u_j}{\partial x_l} - \frac{1}{i}\frac{\partial u_j}{\partial y_l}\right) + i\left(\frac{\partial v_j}{\partial x_l} - \frac{1}{i}\frac{\partial v_j}{\partial y_l}\right)\right\}$$

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$$= \frac{1}{2}\left\{ \left(\frac{\partial u_j}{\partial x_l} - \frac{\partial v_j}{\partial y_l} \right) + i \left(\frac{\partial v_j}{\partial x_l} + \frac{\partial u_j}{\partial y_l} \right) \right\} \rightarrow \frac{\partial \varphi_j}{\partial \bar{z}_l}$$

So: $\frac{\partial \varphi_j}{\partial \bar{z}_l} \equiv 0 \iff \begin{matrix} \mathrm{Re}\left(\frac{\partial \varphi_j}{\partial \bar{z}_l} \right) = 0 \\ \mathrm{Im}\left(\frac{\partial \varphi_j}{\partial \bar{z}_l} \right) = 0 \end{matrix} \iff$ (CR). □

Lemma 1: $\frac{\partial \varphi_j}{\partial \bar{z}_l} \equiv 0 \iff \frac{\partial \varphi_j}{\partial z_l} = \frac{\partial \varphi_j}{\partial x_l}$.

Pf:

$$\begin{array}{rl} & \frac{\partial \varphi_j}{\partial \bar{z}_l} = \frac{1}{2}\left(\frac{\partial \varphi_j}{\partial x_l} - \frac{1}{i} \frac{\partial \varphi_j}{\partial y_l} \right) \\ + & \frac{\partial \varphi_j}{\partial z_l} = \frac{1}{2}\left(\frac{\partial \varphi_j}{\partial x_l} + \frac{1}{i} \frac{\partial \varphi_j}{\partial y_l} \right) \\ \hline \end{array}$$

$$\frac{\partial \varphi_j}{\partial \bar{z}_l} + \frac{\partial \varphi_j}{\partial z_l} = \frac{\partial \varphi_j}{\partial x_l}, \text{ so } \frac{\partial \varphi_j}{\partial \bar{z}_l} = 0 \iff \frac{\partial \varphi_j}{\partial z_l} = \frac{\partial \varphi_j}{\partial x_l}. \quad □$$

Lemma 2: For any $\psi_j = u_j + i v_j$,

(a) $\overline{\frac{\partial \psi_j}{\partial z_l}} = \frac{\partial \bar{\psi}_j}{\partial \bar{z}_l}$; (b) $\frac{\partial \psi_j}{\partial \bar{z}_l} = \overline{\left(\frac{\partial \bar{\psi}_j}{\partial z_l} \right)}$

Pf: ~~(a)~~ Note that (b) is (a) applied to $\bar{\psi}_j$.

(a) $$\frac{\partial \psi_j}{\partial z_l} = \frac{\partial u_j}{\partial z_l} + i \frac{\partial v_j}{\partial z_l} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_l} + \frac{1}{i} \frac{\partial u_j}{\partial y_l} \right] + \frac{i}{2}\left[\frac{\partial v_j}{\partial x_l} + \frac{1}{i} \frac{\partial v_j}{\partial y_l} \right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial v_j}{\partial y_l} \right) - i \left(\frac{\partial u_j}{\partial y_l} - \frac{\partial v_j}{\partial x_l} \right) \right].$$

Thus: $$\overline{\frac{\partial \psi_j}{\partial z_l}} = \frac{1}{2}\left[\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial v_j}{\partial y_l} \right) + i \left(\frac{\partial u_j}{\partial y_l} - \frac{\partial u_j}{\partial x_l} \right) \right]. \quad (*)$$

Next: $$\frac{\partial \bar{\psi}_j}{\partial \bar{z}_l} = \frac{\partial u_j}{\partial \bar{z}_l} - i \frac{\partial v_j}{\partial \bar{z}_l} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_l} - \frac{1}{i} \frac{\partial u_j}{\partial v_j} \right] - \frac{i}{2}\left[\frac{\partial v_j}{\partial x_l} - \frac{1}{i} \frac{\partial v_j}{\partial y_l} \right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial v_j}{\partial y_l} \right) + i \left(\frac{\partial u_j}{\partial y_l} - \frac{\partial v_j}{\partial x_l} \right) \right] \quad (**)$$

Compare (*) and (**). □

Lemma 3:

Ⓗ $\frac{\partial \varphi_j}{\partial \bar{z}_l} = 0$

Ⓒ $\frac{\partial \bar{\varphi}_j}{\partial x_l} = \overline{\frac{\partial \varphi_j}{\partial \bar{z}_l}}$

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Pf: $\overline{\frac{\partial \varphi_j}{\partial x_l}} \overset{\text{Lemma 1}}{=} \overline{\frac{\partial \varphi_j}{\partial z_l}}$ (*)

But $\overline{\frac{\partial \varphi_j}{\partial x_l}} = \overline{\frac{\partial u_j}{\partial x_l} + i\frac{\partial v_j}{\partial x_l}} = \frac{\partial u_j}{\partial x_l} - i\frac{\partial v_j}{\partial x_l} = \frac{\partial \overline{\varphi_j}}{\partial x_l}$ (* *)

All together: $\frac{\partial \overline{\varphi_j}}{\partial x_l} = \overline{\frac{\partial \varphi_j}{\partial x_l}} = \overline{\frac{\partial \varphi_j}{\partial z_l}}$ ∎

Def: Complex Jacobian matrix of φ at $a \in D$:

$$D^{\mathbb{C}}\varphi(a) := \begin{bmatrix} \frac{\partial \varphi_1}{\partial z_1}(a), & \dots, & \frac{\partial \varphi_1}{\partial z_k}(a) \\ \vdots & & \\ \frac{\partial \varphi_k}{\partial z_1}(a), & \dots, & \frac{\partial \varphi_k}{\partial z_k}(a) \end{bmatrix} \in \mathbb{C}^{k\times k}$$

Lemma 4: Ⓗ $\varphi = (\varphi_1, \dots, \varphi_k) : D \subseteq \mathbb{C}^k \to W \subseteq \mathbb{C}^k$ is a holom. map

Let $D\varphi(a)$ be the (real) Jacobian matrix of φ at $a \in D$.

Ⓒ $\det D\varphi(a) = |\det D^{\mathbb{C}}\varphi(a)|^2$!!

Pf: Writing $\varphi_j = u_j + i v_j$ (& omit (a)):

$$D\varphi = \begin{bmatrix} \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial y_1}, & \dots, & \frac{\partial u_1}{\partial y_k} \\ \frac{\partial v_1}{\partial x_1}, & \frac{\partial v_1}{\partial y_1}, & \dots, & \frac{\partial v_1}{\partial y_k} \\ \vdots & & & \\ \frac{\partial v_k}{\partial x_1}, & \frac{\partial v_k}{\partial y_1}, & \dots, & \frac{\partial v_k}{\partial y_k} \end{bmatrix}$$

After a permutation of rows & columns we may re-write:

$$\det D\varphi = \det \left[\begin{array}{c:c} \frac{\partial u_j}{\partial x_l} & \frac{\partial u_j}{\partial y_l} \\ \hdashline \frac{\partial v_j}{\partial x_l} & \frac{\partial v_j}{\partial y_l} \end{array}\right] \underset{\text{add } i(\text{bottom rows}) \text{ to top rows}}{\overset{\cancel{CR}}{=}}$$

$$\det \left[\begin{array}{c:c} \frac{\partial u_j}{\partial x_l} + i\frac{\partial v_j}{\partial x_l} & \frac{\partial u_j}{\partial y_l} + i\frac{\partial v_j}{\partial y_l} \\ \hdashline \frac{\partial v_j}{\partial x_l} & \frac{\partial v_j}{\partial y_l} \end{array}\right] \underset{}{\overset{\frac{\partial u_j}{\partial x_l} = \frac{\partial v_j}{\partial y_l},\ \frac{\partial u_j}{\partial y_l} = -\frac{\partial v_j}{\partial x_l}}{\overset{CR}{=}}} \det \left[\begin{array}{c:c} \frac{\partial u_j}{\partial x_l} + i\frac{\partial v_j}{\partial x_l} & -\frac{\partial v_j}{\partial x_l} + i\frac{\partial u_j}{\partial x_l} \\ \hdashline \frac{\partial v_j}{\partial x_l} & \frac{\partial u_j}{\partial x_l} \end{array}\right]$$

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subtract
i(left rows)
from corresp.
right
columns

$$\det\left[\begin{array}{c|c} \frac{\partial u_j}{\partial x_l}+i\frac{\partial v_j}{\partial x_l} & 0 \\ \hline \frac{\partial v_j}{\partial x_l} & \frac{\partial u_j}{\partial x_l}-i\frac{\partial v_j}{\partial x_l} \end{array}\right] = \det\left[\begin{array}{c|c} \frac{\partial \varphi_j}{\partial x_l}) & 0 \\ \hline \frac{\partial v_j}{\partial x_l} & \frac{\partial \overline{\varphi_j}}{\partial x_l} \end{array}\right]$$

$$\overset{\text{Lemma 1\& 3}}{=} \det\left[\begin{array}{c|c} \frac{\partial \varphi_j}{\partial z_l} & 0 \\ \hline \frac{\partial v_j}{\partial x_l} & \overline{\frac{\partial \varphi_j}{\partial z_l}} \end{array}\right]$$

So: $\det D\varphi = \det\left[\begin{array}{c|c} D^c\varphi & 0 \\ \hline \frac{\partial v_j}{\partial x_l} & \overline{D^c\varphi} \end{array}\right] = \det(D^c\varphi)\det(\overline{D^c\varphi})$

$$= \det(D^c\varphi)\,\overline{\det(D^c\varphi)} = |\det D^c\varphi|^2 \qquad \square$$

We have proved that $\det D\varphi = |\det D^c\varphi|^2$ (if φ has holom. components)

Coroll. 1: Ⓗ M is a complex k-mfld in $\mathbb{C}^n$

Ⓒ M is orientable.

Coroll. 2: Ⓗ M is a cplx k-mfd in $\mathbb{C}^n$

Ⓒ $\forall$ transition map φ we have:

$\det D^c\varphi(a) \neq 0 \quad \forall a \in$ domain of φ.

Corollary 3: Ⓗ M is a complex k-mfd in $\mathbb{C}^n$

Ⓒ any trans. map φ is a biholomorphism of open sets in $\mathbb{C}^n$.

Pf: Coroll. 2 + Complex implicit funct. thm

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Stokes' Thm in 1 cplx variable

Recall: $f(x+iy) = \underset{\text{Re}f}{u(x,y)} + i\underset{\text{Im}f}{v(x,y)}$, $(x,y)\in \underset{\text{domain}}{D} \subseteq \mathbb{C}$

Def: f analytic at $z\in D$ if following Lim exists:

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f'(z)$$

"$f\in A(D)$"

Thm 1: $f\in A(D) \Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $\forall (x,y)\in D$ & $f'(z) = u_x + iv_x$

Def: $z := x+iy$, $\bar{z} := x-iy$. then : $T:(x,y)\to(z,\bar z)$ is a coord. change

in sense that T invertible & inverse is:

$$U: (z,\bar z)\to\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right) =: (x,y)$$

By Chain Rule:

$$\frac{\partial f}{\partial z} = f_x\cdot\frac{\partial x}{\partial z} + f_y\frac{\partial y}{\partial z} = \frac{1}{2}\left(f_x + \frac{1}{i}f_y\right)$$

$$\frac{\partial f}{\partial \bar z} = f_x\frac{\partial x}{\partial \bar z} + f_y\frac{\partial y}{\partial \bar z} = \frac{1}{2}\left(f_x - \frac{1}{i}f_y\right)$$

Thm 2: $f\in A(D) \Leftrightarrow \frac{\partial f}{\partial \bar z}(z,\bar z) = 0$ in D.

Pf: $\frac{\partial f}{\partial \bar z} = \frac{\partial u}{\partial \bar z} + i\frac{\partial v}{\partial \bar z} = \frac{1}{2}\{[u_x + iu_y] + i[v_x + iv_y]\} = \frac{1}{2}\{(u_x - v_y) + i(u_y + v_x)\}$

$$\left.\begin{aligned} \text{So: Re}\frac{\partial f}{\partial \bar z} = 0 &\Leftrightarrow \text{CR I} = 0 \\ \text{Im}\frac{\partial f}{\partial \bar z} = 0 &\Leftrightarrow \text{CR II} = 0 \end{aligned}\right\} \overset{\text{Thm 1}}{\Leftrightarrow} f\in A(D)$$

Def: $\frac{\partial f}{\partial \bar z} = 0$ "CR eqn's in complex form"

Thm 3:
Ⓗ $f\in A(D)$
Ⓒ $f'(z) = \frac{\partial f}{\partial z}(z)$ in D.

Pf: $\frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z} = \frac{1}{2}[(u_x - iu_y) + i(v_x - iv_y)] \overset{CR}{=} \frac{1}{2}[(u_x + iv_x) + \underset{u_x + iv_x}{(v_y - i\overset{v_x}{\overset{\|}{u_y}})}]$

$= u_x + iv_x \overset{\text{Thm 1}}{=} f'(z)$

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Thm: (properties of $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial \bar z}$):

- linearity: $\frac{\partial}{\partial z}(af+bg) = a\frac{\partial f}{\partial z} + b\frac{\partial g}{\partial z}$

  $\frac{\partial}{\partial \bar z}(\qquad) = \ldots \qquad -$

- Leibniz: $\frac{\partial}{\partial z}(fg) = g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}$

  $\frac{\partial}{\partial \bar z}(\ldots) = \ldots$

- $\frac{\partial f}{\partial \bar z} = \overline{\frac{\partial \bar f}{\partial z}}$; $\frac{\partial f}{\partial z} = \overline{\frac{\partial \bar f}{\partial \bar z}}$

Differential forms in $\mathbb{C}$, cplx notation

$\mathbb{R}^2$: Elementary 1-forms : $dx; dy$

-||- 2-forms : $dx \wedge dy$

$\mathbb{C}$: Elementary 1-forms : $dz = dx + i dy$ =: "type (1,0)"

$d\bar z = dx - i dy$ =: "type (0,1)"

2-forms: $d z \wedge d\bar z = (dx+idy)\wedge(dx-idy)$
$= i\, dy \wedge dx - i\, dx \wedge dy$
$= -2i\, dx \wedge dy$

Notes: $dz \wedge dz = 0$ (check!)
$d\bar z \wedge d\bar z = 0$ -||-
$d\bar z \wedge dz = -dz \wedge d\bar z$ (check!)

Forms of type (0,0) = forms of degree 0 = $f(z,\bar z)$.

type (1,0): $\omega = f(z,\bar z)\, dz$
(0,1): $\omega = f(z,\bar z)\, d\bar z$
(1,1): $\omega = f(z,\bar z)\, dz \wedge d\bar z$.

Exterior derivative in cplx form

Recall: (d for $\mathbb{R}^2$): $\Lambda^0(D) \xrightarrow{d} \Lambda^1(D) \xrightarrow{d} \Lambda^2(D) \xrightarrow{d} \{0\}$.

Def: ∂: $\Lambda^{(0,0)}(D) \xrightarrow{\partial} \Lambda^{(1,0)}(D) \xrightarrow{\partial} \{0\}$; $\Lambda^{(0,1)}(D) \xrightarrow{\partial} \Lambda^{(1,1)}(D)$

$\bar\partial$: $\Lambda^{(0,0)}(D) \xrightarrow{\bar\partial} \Lambda^{(0,1)}(D) \xrightarrow{\bar\partial} \{0\}$; $\Lambda^{(1,0)}(D) \xrightarrow{\bar\partial} \Lambda^{(1,1)}(D)$.

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via: (0,0): $\partial f = \frac{\partial f}{\partial z}\, dz$

(1,0): $\partial(f\, dz) := \partial f \wedge dz = \frac{\partial f}{\partial z} \underbrace{dz \wedge dz}_{=0} = 0$

(0,1): $\partial(f\, d\bar z) := \partial f \wedge d\bar z = \frac{\partial f}{\partial z}\, dz \wedge d\bar z$

(1,1): $\partial(f\, dz \wedge d\bar z) := \partial f \wedge dz \wedge d\bar z = 0 \wedge d\bar z = 0$

- (0,0): $\bar\partial f := \frac{\partial f}{\partial \bar z}\, d\bar z$

(1,0): $\bar\partial(f\, dz) := \bar\partial f \wedge dz = \frac{\partial f}{\partial \bar z}\, d\bar z \wedge dz$

(0,1): $\bar\partial(f\, d\bar z) := \bar\partial f \wedge d\bar z = 0$

(1,1):

Corollary: $f \in A(D) \Leftrightarrow \bar\partial f = 0.$

Coroll: $\partial^2 = 0,\ \bar\partial^2 = 0$

Pf. (0,0): f $\quad \partial f = \frac{\partial f}{\partial z}\, dz, \quad \partial(\partial f) = (\partial(\frac{\partial f}{\partial z})) \wedge dz$

$= f_{zz} \underbrace{dz \wedge dz}_{=0}$

So: $\partial^2 f = 0$ if $f \in \Lambda^{0,0}$

(1,0): $f\, dz, \quad \partial(f\, dz) = 0, \quad \partial^2(f\, dz) = \partial(0) = 0.$

etc. . . . □

Thm: $d = \partial + \bar\partial$ (any degree / type)

Aside: $f \in \Lambda^{(0,0)}$ $\quad \frac{\partial f}{\partial z} = f_z = \frac{1}{2}(f_x - i f_y)$

$\frac{\partial f}{\partial \bar z} = f_{\bar z} = \frac{1}{2}(f_x + i f_y)$

Thus: $f_x = f_z + f_{\bar z}$ (1)

$-i f_y = f_z - f_{\bar z} \rightsquigarrow f_y = i(f_z - f_{\bar z})$ (2)

Pf of Thm: degree zero: $df = f_x\, dx + f_y\, dy \overset{(1),(2)}{=} (f_z + f_{\bar z}) \frac{1}{2}(dz + d\bar z)$
$+ i(f_z - f_{\bar z}) \frac{1}{2i}(dz - d\bar z)$

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$$= \frac{1}{2}\left\{ \left(f_z\, dz + f_{\bar z}\, d\bar z + \cancel{f_z\, d\bar z} + \cancel{f_{\bar z}\, dz} \right) \right.$$

$$\left. + \left(f_z\, dz + f_{\bar z}\, d\bar z - \cancel{f_{\bar z}\, dz} - f_z\, \cancel{d\bar z} \right) \right\}$$

$$= \partial f + \bar\partial f.$$

- degree 1: $\omega = f\,dx + g\,dy$

$$d\omega = df \wedge dx + dg \wedge dy \overset{f,g \text{ have degree } 0}{=} (\partial f + \bar\partial f) \wedge \tfrac{1}{2}(dz + d\bar z) + (\partial g + \bar\partial g) \wedge \tfrac{1}{2i}(dz - d\bar z)$$

$$(1) \quad d\omega = \tfrac{1}{2}\{ f_z + i g_z \}\, dz \wedge d\bar z + \tfrac{1}{2}\{ -f_{\bar z} + i g_{\bar z} \}\, dz \wedge d\bar z$$

On the other hand:

$$\omega = f\,dx + g\,dy = f \cdot \tfrac{1}{2}(dz + d\bar z) + g\, \tfrac{1}{2i}(dz - d\bar z)$$

$$\Rightarrow \partial\omega = \tfrac{1}{2}\{ \partial f \wedge (dz + d\bar z) - i\, \partial g \wedge (dz - d\bar z) \}$$

$$\overset{\text{degree } 0}{=} \tfrac{1}{2}\{ f_z\, dz \wedge d\bar z + i g_z\, dz \wedge d\bar z \} = \boxed{\tfrac{1}{2}\{ \underset{= \partial f}{f_z + i g_z} \}\, dz \wedge d\bar z. \quad (2)}$$

Similarly: $\bar\partial\omega = \ldots \Rightarrow \boxed{\bar\partial\omega = \tfrac{1}{2}\{ -f_z + i g_{\bar z} \}\, dz \wedge d\bar z \quad (3)}$

Notice that: $\underset{d\omega}{(1)} = \underset{\partial\omega}{(2)} + \underset{\bar\partial\omega}{(3)}$ $\therefore$ $d\omega = \partial\omega + \bar\partial\omega.$ ∎

- degree 2: ...

∎

<u>Stokes in one cplx vble</u>

$(*)$ $\displaystyle\oint_{bD} \omega = \int_D d\omega$ $\forall \omega \in \Lambda^1$ with coeff. of class $C^1(D) \cap C(\bar D)$

& $\forall$ sufficiently regular bounded open set D.

(a) type $(1,0)$: $\omega = f\,dz$ ~~for~~ $f \in C^1(D) \cap C(\bar D)$

(*) reads: $\displaystyle\int_{bD} f\,dz \underset{\substack{\uparrow \\ d = \partial + \bar\partial,\ \bar\partial(f\,dz) = 0}}{=} \int_D \bar\partial f \wedge dz = \int_D f_{\bar z}\, d\bar z \wedge dz.$

(b) type $(0,1)$: $\omega = f\,d\bar z$, $f \in C^1(D) \cap C(\bar D)$ &

$(*) \Rightarrow \displaystyle\int_{bB} f\,d\bar z = \int_D \partial f \wedge d\bar z = -\int_D f_z\, d\bar z \wedge dz.$

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Corollary: **Cauchy-Thm:** Ⓗ $f \in A(D) \cap C(\overline{D})$

Ⓒ $\oint_{bD} f\, dz = 0$

Pf: Stokes formulation (a): $\omega = f\,dz$ [(1,0)-form]

& $d\omega = \underbrace{\overline{\partial} f}_{=0} \wedge dz$ b/c $f \in A(D)$.

Corollary: Cauchy formula for analytic functs:

Ⓗ $g \in A(D) \cap C(\overline{D})$

Ⓒ $\forall w \in D,\ g(w) = \frac{1}{2\pi i} \oint_{bD} \frac{g(z)}{z-w}\, dz$

(not nec. simply connct

Pf: Let $\varepsilon_0 > 0$ s.t. $\overline{\mathbb{D}_\varepsilon(w)} \subset D\ \forall\ 0 < \varepsilon < \varepsilon_0$

Def: $D_\varepsilon := D \setminus \overline{\mathbb{D}_\varepsilon(w)}$

def: $f(z) := \frac{1}{2\pi i} \frac{g(z)}{z-w} \in A(D_\varepsilon)\quad \forall \varepsilon > 0$

By Cauchy Thm (applied to f on D_ε) get:
$\Downarrow$

$$0 = \frac{1}{2\pi i} \int_{bD_\varepsilon} \frac{g(z)}{z-w}\, dz = \frac{1}{2\pi i} \oint_{bD} \frac{g(z)}{z-w}\, dz - \frac{1}{2\pi i} \oint_{b\mathbb{D}_\varepsilon(w)} \frac{g(z)}{z-w}\, dz$$

So: $\frac{1}{2\pi i} \int_{bD} \frac{g(z)}{z-w}\, dz = \underbrace{\frac{1}{2\pi i} \oint_{b\mathbb{D}_\varepsilon(w)} \frac{g(z)}{z-w}\, dz}_{I_\varepsilon(w)}\quad \forall \varepsilon > 0.$

Now: $I_\varepsilon(w) \underset{\substack{\uparrow \\ z = w + \varepsilon e^{i\theta} \\ dz = i\varepsilon e^{i\theta} d\theta}}{=} \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(w+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta} d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(w+\varepsilon e^{i\theta})\, d\theta \overset{\pm g(w)}{=}$$

$$= g(w) + \frac{1}{2\pi} \int_0^{2\pi} \left(g(w+\varepsilon e^{i\theta}) - g(w)\right) d\theta$$

But $g \in A(D) \Rightarrow g \in C(\overline{\mathbb{D}_\varepsilon(w)})$ & we conclude that

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$$\lim_{\varepsilon\to 0} I_\varepsilon(w) = g(w) + \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\lim_{\varepsilon\to 0}\left(g(w)+\varepsilon e^{i\theta} - g(w)\right)}_{=0} dw$$

All together: $\frac{1}{2\pi i}\int_{bD} \frac{g(z)}{z-w}\, dz = g(w).$ $\square$

Remark: in proving that $\lim_{\varepsilon\to 0} I_\varepsilon = g(w)$ we only used the fact that $g \in C(\overline{D_\varepsilon(w)})$ (we did not use $g \in A(D)$).

Corollary: Pompeiu (also generalized Cauchy) formula for $C^1(D)\cap C(\overline{D})$.

Ⓗ $g \in C^1(D) \cap C(\overline{D})$

Ⓒ $\forall w \in D,\ g(w) = \frac{1}{2\pi i}\oint_{bD}\frac{g(z)}{z-w}\,dz - \frac{1}{\pi^2}\int_D \frac{g_{\bar z}(z,\bar z)}{z-w}\,dx\,dy$

Pf:

$$\omega = \frac{1}{2\pi i}\frac{g(z)}{z-w}\,dz \in \Lambda^1(D_\varepsilon) \quad \forall \varepsilon > 0$$

Stokes $\Rightarrow$ $\frac{1}{2\pi i}\int_{bD_\varepsilon}\frac{g(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_{D_\varepsilon} dz\left(\frac{g(z)}{z-w}\right)\wedge dz$

$\overset{d=\partial+\bar\partial \text{ etc..}}{=} \frac{1}{2\pi i}\int_{D_\varepsilon} \bar\partial_z\left(\frac{g(z)}{z-w}\right)\wedge dz \overset{\text{Leibniz } \& \ \frac{1}{z-w}\in A(D_\varepsilon)}{=}$

$$= \frac{1}{2\pi i}\int_{D_\varepsilon}\frac{1}{z-w}\,\partial_{\bar z} g(z)\wedge dz = \frac{1}{2\pi i}\int_{D_\varepsilon}\frac{1}{z-w}\, g_{\bar z}(z,\bar z)\underbrace{d\bar z\wedge dz}_{2i\,dx\wedge dy}$$

So: $\frac{1}{2\pi i}\int_{bD_\varepsilon}\frac{g(z)}{z-w}\,dz = \frac{1}{\pi}\int_{D_\varepsilon}\frac{1}{z-w}\,g_{\bar z}(z,\bar z)\,dx\,dy$

But $\frac{1}{z-w}$ is absolutely integrable at w in D

$$\int_{D,(w)}\frac{1}{|z-w|}\,dx\,dy \underset{\text{Polar coord's}}{=} \int_0^{2\pi}\int_0^1 \frac{1}{r}\, r\,dr\,d\theta < \infty$$

So: $\int_{D_\varepsilon}\frac{g_{\bar z}}{z-w}\,dx\,dy \xrightarrow[\varepsilon\to 0]{} \int_{z\in D}\frac{g_{\bar z}}{z-w}\,dx\,dy$

All together: $\int_{bD_\varepsilon}\frac{g(z)}{z-w}\,dz = \int_{D_\varepsilon}\frac{g_{\bar z}}{z-w}\, d$

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$$\int_{bD} \quad // \qquad \underbrace{-\int_{bD_\varepsilon(w)} \frac{g(z)}{z-w}\,dz}_{I_\varepsilon(w)} = \Bigg) \downarrow$$

$$\downarrow$$

$$\int_{bD} \frac{g(z)}{z-w}\,dz \quad -g(w) = \int_D \frac{g_{\bar z}}{z-w}\,dx\,dy$$

$\square$

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Stokes' Thm in SCV

several cplx variables

Preliminaries:

$\mathbb{R}^{2n} = \{ (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \} = \{ (x_1, x_2, \dots, x_{2n}) \}$

$\mathbb{C}^n$: $z_j := x_j + i y_j \ , j = 1, \dots, n$

alt.: $z_j := x_{2j-1} + i x_{2j} \ , j = 1, \dots, n$

$z_j := x_j + i x_{j+n} \ , j = 1, \dots, n$

Consequently, $x_j := \frac{1}{2}(z_j + \bar z_j)$ $\quad$ (alt: $x_{2j-1} = \frac{1}{2}(z_j + \bar z_j)$

$y_j := \frac{1}{2i}(z_j - \bar z_j)$ $\quad$ $x_{2j} = \frac{1}{2i}(z_j - \bar z_j)$)

Given $f: D \to \mathbb{C}$,

def: $\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right)$

$\frac{\partial f}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right)$

Def. f is holomorphic in D ("analytic": $n=1$) iff $\frac{\partial f}{\partial \bar z_j} \equiv 0$ in D $\forall j = 1, \dots, n$

Notation: $f \in \mathcal{O}(D)$

- Differential forms in cplx notation:
- Elementary 1-forms ("degree 1") in $\mathbb{R}^{2n}$, dx_j, dy_j, $j = 1, \dots, n$

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in $\mathbb{C}^{2n}$: $dz_j = dx_j + i\,dy_j$; $d\bar{z}_j = dx_j - i\,dy_j$, $j = 1 \to n$

$\uparrow$ type 10 $\qquad\qquad$ $\uparrow$ type (0,1)

- Elementary 2-forms in $\mathbb{R}^{2n}$: $dx_j \wedge dx_l$; $dx_j \wedge dy_l$; $dy_j \wedge dy_l$

in $\mathbb{C}^{2n}$: $dz_j \wedge dz_l$, $dz_j \wedge d\bar{z}_l$, $d\bar{z}_j \wedge d\bar{z}_l$

type $\quad$ 2,0 $\quad$ 1,1 $\quad$ 0,2

- Given $p, q \in \{0,1,2,\dots,n\}$: $\left.\begin{array}{l} I = (i_1, i_2, \dots, i_p) \\ J = (j_1, j_2, \dots, j_q) \end{array}\right\}$ ascending labels in $(1, \to n)$

Elem. $(p,0)$-forms: $dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_p}$

Fact: $dz_j \wedge dz_j = 0$

Elem. $(0,q)$-forms : $d\bar{z}_J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \dots \wedge d\bar{z}_{j_q}$

(again : $d\bar{z}_j \wedge d\bar{z}_j = 0$ all j)

Elem. (p,q)-forms: $dz_I \wedge d\bar{z}_J = dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$

degree $r = p+q$

General form type (p,q): $\omega = \sum_{\substack{[I]_p \\ [J]_q}} a_{IJ}\, dz_I\, d\bar{z}_J$

$a_{IJ} : O \to \mathbb{C}$

Notation: $\overset{\mathbb{C}^n}{\Lambda^{p,q}(O)}$; $\overset{\mathbb{R}^{2n}}{\Lambda^{r}(O)}$, $\quad p, q \in \{0, \dots, n\}$, $r \in \{0, \dots, 2n\}$

Fact: $\Lambda^r(D) = \bigoplus_{p+q=r} \Lambda^{p,q}(D)$

Λ^r: $\uparrow$ $\mathbb{R}^{2n}$: $z_j = x_j + i y_j$ $\uparrow$ etc

$\Lambda^{p,q}$: $\uparrow$ $\mathbb{C}^n$

Functions: $f : D \to \mathbb{C}$: $\Lambda^{0,0}(D)$

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Def: $\omega \in \Lambda^{p,q} \rightsquigarrow \bar{\omega} \in \Lambda^{p,q}$.

$$\omega = \sum_{\substack{[I]_p \\ [J]_q}} a_{IJ}\, dz_I \wedge d\bar{z}_J \qquad \rightsquigarrow \bar{\omega} = \sum_{\substack{[I]_p \\ [J]_q}} \bar{a}_{IJ}\, d\bar{z}_I \wedge dz_J$$

Exterior derivative in cplx form:

- $r=0$, $f \in \Lambda^{0,0}$

$$\partial f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j \in \Lambda^{1,0}$$

$$\bar{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j \in \Lambda^{0,1}$$

Note: $f \in \mathcal{O}(D) \Leftrightarrow \bar{\partial} f = 0$.

- $r \in \{1, \dots, 2n\}$
- $q \in \{0, \dots, n\}$

$$\partial : \{0\} \hookrightarrow \Lambda^{0,q}(D) \to \Lambda^{1,q}(D) \to \dots \to \Lambda^{p,q}(D) \to \Lambda^{p+1,q} \to \dots \to \Lambda^{n,q} \to \{0\}$$

$$\partial\Big(\sum a_{IJ}\, dz_I \wedge d\bar{z}_J\Big) = \sum \underset{\substack{\uparrow \\ \text{def for } r=0}}{(\partial a_{IJ})} \wedge dz_I \wedge d\bar{z}_J$$

- $p \in \{0, \dots, n\}$

$$\bar{\partial} : \{0\} \hookrightarrow \Lambda^{p,0}(D) \xrightarrow{\bar{\partial}} \Lambda^{p,1} \to \dots \to \Lambda^{p,q} \to \Lambda^{p,q+1} \to \dots \to \Lambda^{p,n} \to \{0\}$$

$$\bar{\partial}\Big(\sum a_{IJ}\, dz_I \wedge d\bar{z}_J\Big) = \sum (\bar{\partial} a_{IJ}) \wedge dz_I \wedge d\bar{z}_J$$

Thm: Ⓗ d = exterior derivative for $\mathbb{R}^{2n}$

Ⓒ $d = \partial + \bar{\partial}$ $\quad \Big(d\Big(\sum a_{IJ}\, dz_I \wedge dz_J\Big) = \sum (\partial a_{IJ} + \bar{\partial} a_{IJ}) \wedge dz_I \wedge d\bar{z}_J\Big)$

$$\partial \circ \partial = 0; \quad \bar{\partial} \circ \bar{\partial} = 0; \quad (\partial \circ \bar{\partial} + \bar{\partial} \circ \partial) = 0$$

Furthermore, ∂ & $\bar{\partial}$ commute with pull-back under holomorphic maps $F: \mathbb{C}^n \to \mathbb{C}^n$.

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- Volume form for $\mathbb{R}^{2n}$

$$dV = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$$

$$= \left(\tfrac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$$

$$= \frac{1}{n!} \left(\frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \right)^n$$

$$= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2i)^n} \quad d\bar{z}_1 \wedge \cancel{d} \ldots \wedge d\bar{z}_n \wedge dz_1 \wedge \ldots \wedge dz_n$$

- Inner product of forms: $\omega = \sum a_{IJ}\, dz_I \wedge d\bar{z}_J$ (same type (p,q))

$$\eta = \sum b_{IJ}\, dz_I \wedge d\bar{z}_J$$

$$\langle \omega, \eta \rangle_{\cancel{p}z} = \sum_{\substack{[I]_p \\ [J]_q}} a_{IJ}(z\overline{b}_{IJ}(z) \ , \ z \in D$$

$$(\omega, \eta)_D = \int_{z \in D} \langle \omega, \eta \rangle_z \, dV$$ provided this integral makes sense

<u>Thm (Hodge $*$ operator for $\mathbb{R}^{2n}$ & $\mathbb{C}^n$)</u>: There is a ! $\mathbb{C}$-linear map $*$, s.t. $\forall r \in \{0, \ldots, 2n\}$,

$$*: \Lambda^r(D) \to \Lambda^{2n-r}(D)$$

$$*(\bar{\varphi}) = \overline{*\varphi}$$

$$**\varphi = (-1)^{2n-r} \varphi$$

$$*1 = dV \ , \quad *dV = 1$$

$$(\psi \wedge *\bar{\varphi})_z = \langle \psi, \varphi \rangle dV \quad \forall\ \varphi, \psi \in \Lambda^r$$

Furthermore, in $\mathbb{C}^n = \mathbb{R}^{2n}$:

$$*: \Lambda^{p,q} \to \Lambda^{n-q, n-p}$$

$$**\varphi = (-1)^{p+q} \varphi \quad \forall \varphi \in \Lambda^{p,q}$$

$$*\underbrace{dz_J}_{\in \Lambda^{q,0}} = \frac{(-1)^{\frac{q(q-1)}{2}}}{2^{n-q}\, i^n} dz_J \wedge \left(\bigwedge_{\nu \in J'} d\bar{z}_\nu \wedge dz_\nu \right)$$

$J' = (1, \ldots, n) \setminus J$

$\forall [I]_q$ □

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e.g. in $\mathbb{C}^3$, $q=2$, $J=(1,3)$, $J'=(2)$

$q=1$, $J=(2)$, $J'=(1,3)$

Stokes' Thm for $D\subseteq\mathbb{C}^n$

Recall from last time (Stokes' Thm for $D\subseteq\mathbb{C}$):

Stokes' Thm: Ⓗ $D\subset\mathbb{C}$ (bdd, smooth)
$j: bD\hookrightarrow\mathbb{C}$
$f\in C^1(D)\cap C(\bar{D})$

Ⓒ (a) $\displaystyle\oint_{bD} j^*(f\,dz) = \int_D \bar{\partial} f\wedge dz$

(b) $\displaystyle\oint_{bD} j^*(f\,d\bar{z}) = \int_D \partial f\wedge d\bar{z}$

Corollary: (Cauchy-Pompeiu formula):

Ⓗ $g\in C^1(D)\cap C(\bar{D})$

Ⓒ $$g(w)=\frac{1}{2\pi i}\oint_{bD} j^*g(z)\, j^*\left(\frac{dz}{z-w}\right)-\frac{1}{2\pi i}\int_D\frac{\bar{\partial}g\wedge dz}{z-w}$$

In particular, if $g\in A(D)\cap C(\bar{D})$, then:

$$g(w)=\frac{1}{2\pi i}\oint_{z\in bD} j^*g(z)\quad j^*\left(\frac{dz}{z-w}\right)\qquad\forall w\in D$$

Def: $\frac{1}{2\pi i}\, j^*\left(\frac{dz}{z-w}\right)$ =: Cauchy kernel for D at $w\in D$

Q: Is there an analog of Cauchy-Pompeiu & Cauchy formula for $D\subseteq\mathbb{C}^n$, $n\geq 1$?

Stokes in $\mathbb{R}^{2n}$: $\displaystyle\oint_{bD} j^*\omega=\int_D d\omega\quad\forall\ \omega\in\Lambda^{2n-1}(\mathbb{R}^{2n})$

Now: $$\Lambda^{2n-1}=\bigoplus_{\substack{p+q=2n-1\\ p,q\in\{0,1,\dots,n\}}}\Lambda^{p,q}=\Lambda^{n,n-1}\oplus\Lambda^{n-1,n}$$

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Short hand: $dz_N = dz_1 \wedge - \wedge dz_n$

$d\bar z_N = d\bar z_1 \wedge - \wedge d\bar z_n$

$$\underset{(a)}{\omega = \sum_{j=1}^{n} a_j\, dz_N \wedge d\bar z_j}\,; \quad \underset{(b)}{\omega = \sum_{j=1}^{n} b_j\, dz_j \wedge d\bar z_N}$$

$a_j, b_j : \mathbb{R}^{2n}$ (or $\mathbb{C}^n$) $\to \mathbb{C}$

(a) $d\omega = (\partial + \bar\partial)(\omega) = \bar\partial\omega = \sum_j \bar\partial a_j \wedge dz_N \wedge d\bar z_j$

(b) $d\omega = (\partial + \bar\partial)(\omega) = \partial\omega = \sum_j \partial b_j \wedge dz_j \wedge d\bar z_N$

Stokes in cplx form for $D \subseteq \mathbb{C}^n$:

Ⓗ $\omega \in \Lambda^{2n-1}(D)$

(a) $(\omega \in \Lambda^{n,n-1})$: $$\int_{\partial D} j^*\omega \underset{\substack{\uparrow \\ \omega = \sum_j a_j dz_N \wedge d\bar z_j}}{=} \sum_j \int_D \bar\partial a_j \wedge dz_N \wedge d\bar z_j,$$

(b) $(\omega \in \Lambda^{n-1,n})$: $$\int j^*\omega \underset{\substack{\uparrow \\ \omega = \sum b_j dz_j \wedge d\bar z_N}}{=} \sum_j \int_D \partial b_j \wedge dz_j \wedge d\bar z_N$$

Q. Cauchy-Pompeiu for $D \subseteq \mathbb{C}^n$?

Back to $n=1$: Cauchy-Kernel : $j^*\left(\frac{dz}{z-w}\right)$

Now, if $z \in \mathbb{C}^n$ then $\frac{1}{z} = ??$

However, note that if, $z, w \in \mathbb{C}\,(n=1)$: rewrit: $\frac{1}{z-w} = \frac{\bar z - \bar w}{|w-z|^2}$ ↗

meaningful in any dimension!

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$n \geq 2$: **Def:** $\beta(z,w) := |w-z|^2 = |z_1-w_1|^2 + \dots + |z_n-w_n|^2$

$= (z_1-w_1)(\overline{z_1-w_1}) + \dots + (z_n-w_n)(\overline{z_n-w_n})$.

Def: Bochner - Martinelli kernel $K_0(z,w)$

1943, Ann. Math

1938, Mem. R. Acad. Italia

$$K_0(z,w) := \frac{(n-2)!}{2\pi^n} \underbrace{\left(- \ast \underbrace{\partial_z \beta^{1-n}}_{\Lambda^{1,0}} \right)}_{\Lambda^{n,n-1}} \qquad \forall w \in \mathbb{C}^n,\ z \in \mathbb{C}^n \setminus \{w\}.$$

Lemma:

$$K_0(z,w) = \frac{1}{(2\pi i)^n} \underbrace{\frac{\partial\beta}{\beta}}_{\Lambda^{1,0}} \underset{\wedge}{\wedge} \Big[\underbrace{\bar\partial \left(\frac{\partial\beta}{\beta}\right)}_{(\Lambda^{1,1})^{n-1} = \Lambda^{n-1,n-1}} \Big]^{(n-1)}$$

Pf: omitted.

Remark: $n=1$:

$$K_0(z,w) = \frac{1}{2\pi i} \frac{\partial\beta}{\beta} = \frac{1}{2\pi i} \frac{\partial(|w-z|^2)}{|w-z|^2}$$

$$= \frac{1}{2\pi i} \frac{\partial\left(\overset{z-w}{\cancel{(w-z)}}(\overline{z-w})\right)}{|w-z|^2} = \frac{1}{2\pi i} \frac{dz\,(\overline{z-w}) + (z-w)\cdot 0}{|z-w|^2}$$

$$= \frac{1}{2\pi i} \frac{(\overline{z-w})}{|z-w|^2} dz \underset{n=1}{=} \frac{1}{2\pi i} \frac{dz}{z-w} \quad ☺$$

So: $n=1$ BM = Cauchy ☺

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too lazy and late to write sth down :P