Complex Analysis

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<u>multiplication on $\mathbb{R}^2$</u> $(x_1,y_1)\cdot(x_2,y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$

$(\mathbb{R}^2,+,\cdot)$ comm. field $\quad \mathbb{R} \hookrightarrow \mathbb{R}^2 \quad x \mapsto (x,0)$ subfield

$1 = (1,0)$

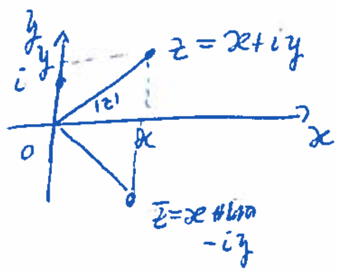
$i = (0,1) \qquad i^2 = (0,1)(0,1) = (-1,0) = -1.$

$(x,y) = (x,0) + (0,y) = (x,0) + (0,1)(y,0) = x + iy.$

$\mathbb{C} = \{ z = x+iy : x,y \in \mathbb{R} \}$, $\quad x = \operatorname{Re} z$ real part of z

$y = \operatorname{Im} z$ imaginary part of z

Complex plane



modulus $|z| = \sqrt{x^2+y^2}$

conjugate $\bar{z} = x - iy$

$z = x+iy \qquad x = \operatorname{Re} z = \dfrac{z+\bar z}{2}, \quad y = \operatorname{Im} z = \dfrac{z-\bar z}{2i}$

$\bar z = x - iy$

<u>Properties</u> 1) triangle inequality $\quad |z+w| \le |z| + |w|$

$\big|\,|z| - |w|\,\big| \le |z-w|$

2) $|zw| = |z|\cdot|w|$

$|z| = |\bar z|$

3) $\overline{z+w} = \bar z + \bar w$

$\overline{zw} = \bar z \cdot \bar w$.

$z\bar z = |z|^2$

If $z \ne 0 \quad \dfrac{1}{z} = \dfrac{\bar z}{|z|^2} = \dfrac{x - iy}{x^2+y^2}$

Topology on $\mathbb{C}$ is that of $\mathbb{R}^2$

$\Delta(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}$

domain = open and connected

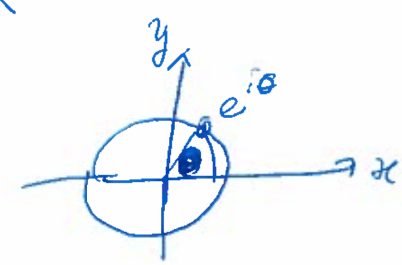
$E \subseteq \mathbb{C}$, $\overline{E} = \{ z \in \mathbb{C} \mid \Delta(z, r) \cap E \neq \phi \ \forall r > 0 \}$ closure

boundary $\partial E = \{ z \in \mathbb{C} \mid \forall r > 0 \ \Delta(z,r) \cap E \neq \phi$ and $\Delta(z,r) \setminus E \neq \phi \}$

1.2 Polar representation

Def: $e^{i\theta} := \cos\theta + i \sin\theta, \ \theta \in \mathbb{R}$

$|e^{i\theta}| = 1$

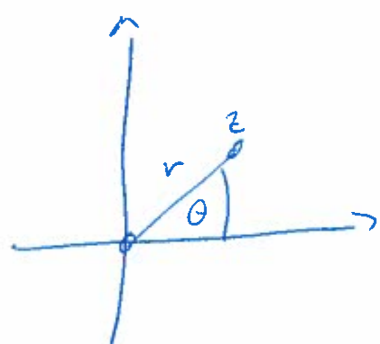


Have: $e^{i(\theta + \varphi)} = e^{i\theta} e^{i\varphi}$

$z = x + iy$ $(x, y) \longmapsto (r, \theta)$ polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$z = r(\cos\theta + i \sin\theta) = re^{i\theta}$.



$r = \sqrt{x^2 + y^2} = |z|$, $\theta =$ an argument of $z \neq 0$

$\arg z = \{ \theta + 2k\pi \mid k \in \mathbb{Z} \}$ Set of all arguments

$\operatorname{Arg} z = \theta \in \arg z \cap (-\pi, \pi]$ principal value of argument

Ex $\operatorname{Arg}(-1) = \pi$, $\arg(-i) = \left\{ -\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z} \right\}$

$\operatorname{Arg}(-i) = -\pi/2$.

Properties $e^{-i\theta} = \cos(\theta) - i\sin\theta = \overline{e^{i\theta}}$.

$$\frac{1}{e^{i\theta}} = \frac{e^{-i\theta}}{1} = e^{-i\theta}.$$

$z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, $\frac{1}{z} = \frac{1}{r} e^{-i\theta}$

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2} \Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$, $z_1, z_2 \neq 0$
as sets.

$\arg(\frac{1}{z}) = \arg(\bar{z}) = -\arg(z)$

$z = re^{i\theta}$, $z^n = r^n e^{in\theta}$. $n \in \mathbb{Z}$

De Moivre's formula

$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

$$\cos(n\theta) = \operatorname{Re}\left[\sum_{j=0}^{n} \binom{n}{j} (\cos\theta)^{n-j} (i\sin\theta)^j\right]$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (\cos\theta)^{n-2k} (-1)^k (\sin\theta)^{2k}$$

$\sin(n\theta) = \dots$

n-th root of $w \neq 0$: $z^n = w$ $n \in \mathbb{N}$.

$w = \rho e^{i\varphi} = \rho e^{i(\varphi + 2k\pi)}$ $k \in \mathbb{Z}$. look for $z = re^{i\theta}$

$$r^n e^{in\theta} = \varrho e^{i(\varphi + 2k\pi)}$$

$$r^n = \varrho > 0 \Rightarrow r = \varrho^{1/n}$$

$$n\theta = \varphi + 2k\pi$$

$$\theta = \frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, \dots, n-1 \rightsquigarrow n \text{ roots of } w$$

$$w^{1/n} = \{ \underbrace{\varrho^{1/n}}_{=|w|^{1/n}} e^{i\frac{\varphi + 2k\pi}{n}}, \ k \in \{0, \dots, n-1\} \}$$

roots of unity $z^n = 1$ $\quad z_k = e^{\frac{2k\pi i}{n}}, \ k = 0, \dots, n-1$

vertices of regular n-gon prescribed in unit circle

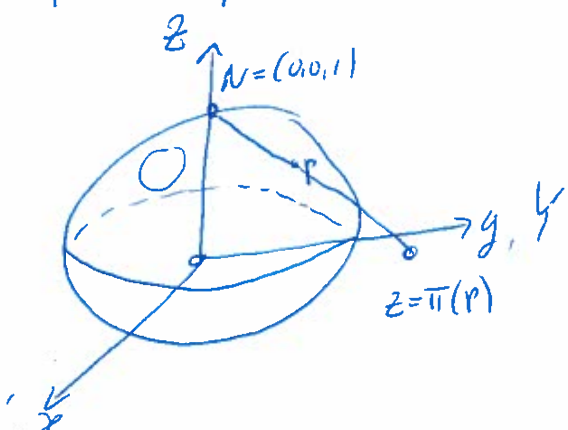
1.3 Stereographic projection

$$\mathbb{C}^* := \mathbb{C} \cup \{\infty\} \quad (\mathbb{C}_\infty)$$

nbd of ∞ $\quad U \supset (\{z : |z| > R\} \cup \{\infty\})$

$$z \to \infty \iff |z| \to +\infty$$

$\mathbb{C}^*$ compact, homeo to $S^2 \subseteq \mathbb{R}^3$



$\pi : S^2 \to \mathbb{C}^*$

$\pi(N) = \infty$

$P \to N, \quad z = \pi(P) \to \infty$

circle on S^2 mapped to circles in $\mathbb{C}$

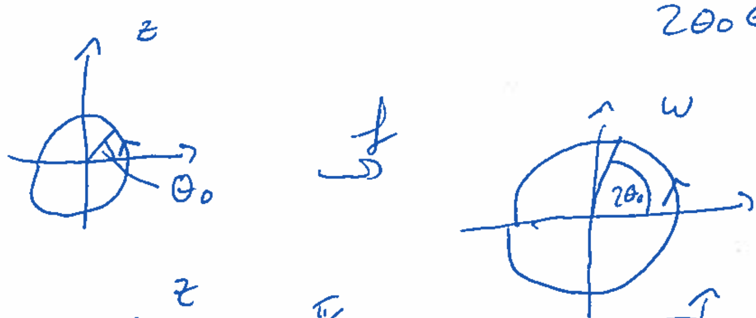
lines if ∞ contained inside circle.
$\quad \overset{\shortparallel}{N}$

1.4 Square & square-root functions

1. $f(z) = z^2$, $z = re^{i\theta}$

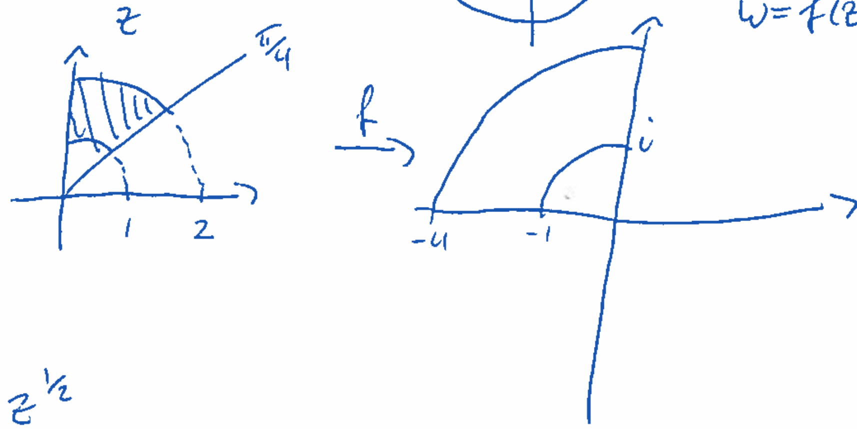
$f(z) = r^2 e^{2i\theta}$, $|z| = r \Rightarrow |f(z)| = r^2$ covers circle twice

ray $\theta = \theta_0$ mapped to the ray $\underbrace{\arg w = 2\theta_0}_{2\theta_0 \in \arg w.}$ $(w = z^2)$



$w = f(z) = z^2$.

Ex:



2. $w = z^{1/2}$

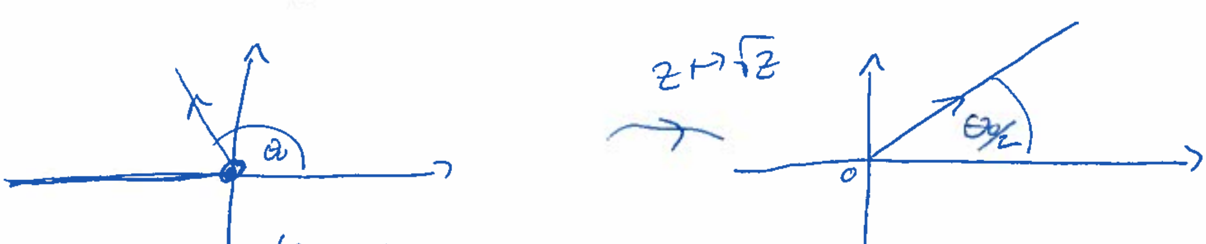
$w^2 = z = re^{i\theta}$

$w_{0,1} = \sqrt{r}\, e^{i\frac{\theta + 2k\pi}{2}}$, $k = 0, 1$

$w_0 = \sqrt{r}\, e^{i\frac{\theta}{2}}$, $w_1 = \sqrt{r}\, e^{i\frac{\theta}{2} + \pi i} = \sqrt{r}\, e^{i\frac{\theta}{2}} \underbrace{e^{i\pi}}_{=-1} = -w$

principal value $w = \sqrt{z} = |z|^{1/2} e^{i\frac{\operatorname{Arg} z}{2}}$ $-\pi < \operatorname{Arg} z \le \pi$

Continuous on $\mathbb{C} \setminus (-\infty, 0]$ (slit plane)



[illegible] $\mathbb{C} \setminus (-\infty, 0] \xrightarrow{bij} \{\operatorname{Re} w > 0\}$

Cannot be extended to $\mathbb{C}$! (Cont.)

1.5 Exponential Function

$z = x+iy$, $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$

$|e^z| = e^x = e^{\operatorname{Re} z} > 0$, $e^z \neq 0$

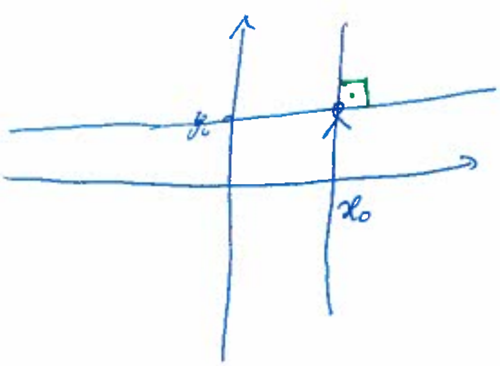
$\arg e^z = y$, $e^{z+w} = e^z e^w$

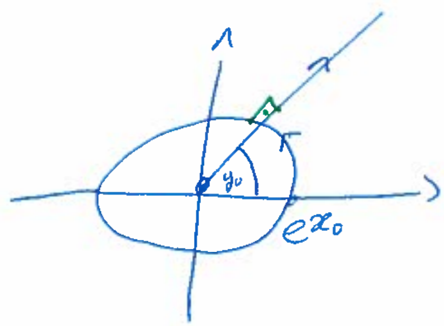
Periodicity $e^{z+2k\pi i} = e^z$

Mapping properties $w = e^z$

Vertical lines $\operatorname{Re} z = x_0 \longmapsto$ Circle $|w| = e^{x_0}$

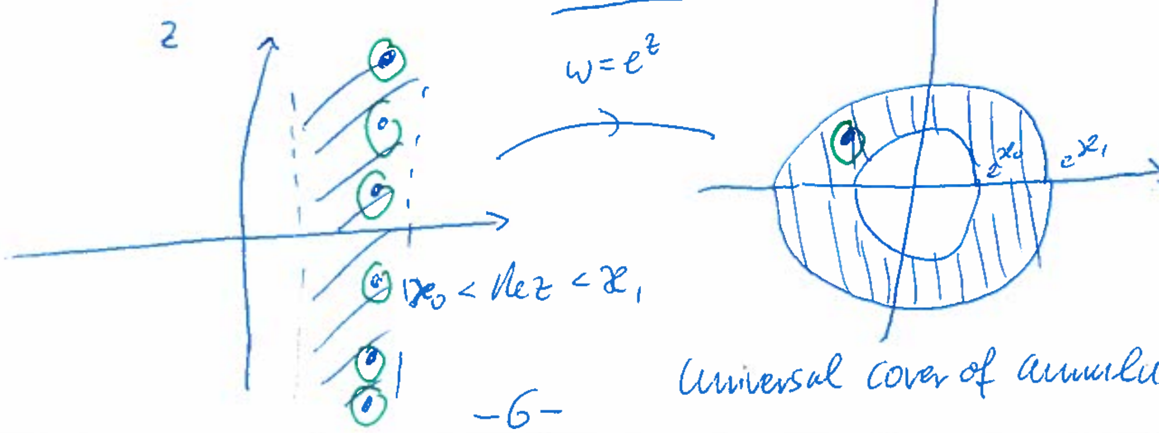
horizontal line $\operatorname{Im} z = y_0 \overset{\text{inj.}}{\longmapsto}$ ray $\arg w = y_0$





Ex:

Covering map



Universal cover of annulus.

1.6 Logarithm

$w \overset{?}{=} \log z$, $e^w = z = re^{i\theta}$, $r>0$

$w = x+iy$ $\quad e^x e^{iy} = re^{i\theta}$ $\quad e^x = r$ so $x = \ln r = \log r$

$y = \theta + 2k\pi$, $k \in \mathbb{Z}$.

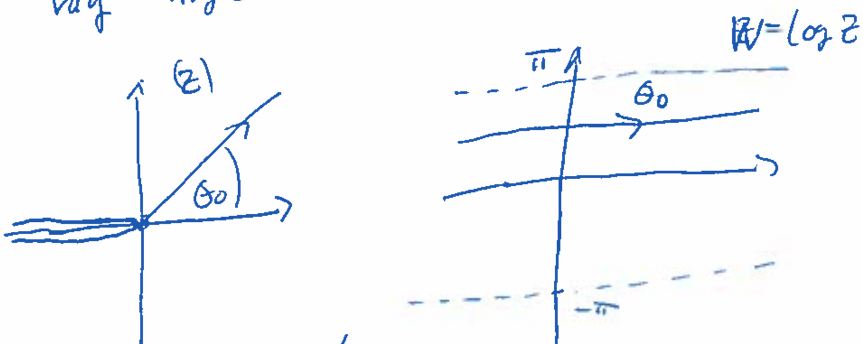
$w = \log z = \log|z| + i(\operatorname{Arg} z + 2k\pi)$, $k \in \mathbb{Z}$

$\log z = \log|z| + i \arg z$, $z \in \mathbb{C} \setminus \{0\}$ (multivalued function)

principal value of log $\quad \operatorname{Log} z = \log|z| + i \operatorname{Arg} z$.

$w = \operatorname{Log} z$

ray $\operatorname{Arg} z = \theta_0 \in (-\pi, \pi] \longrightarrow \operatorname{Im} w = \theta_0$ horizontal line



$$\mathbb{C} \setminus (-\infty, 0] \underset{\text{bij}}{\xrightarrow{\operatorname{Log}}} \{-\pi < \operatorname{Im} w < \pi\}$$

$\xleftarrow{\exp}$

1.7 Power functions

$\alpha \in \mathbb{C}$, $z \neq 0$

$$z^\alpha := e^{\alpha \log z} = e^{\alpha(\log|z| + i \operatorname{Arg} z) + 2k\pi\alpha i}$$

$$= e^{\alpha(\log|z| + i\operatorname{Arg} z)} e^{2k\pi\alpha i} \quad k \in \mathbb{Z} \quad \text{multivalued}$$

principal value: $z^\alpha = e^{\alpha(\log|z| + i\operatorname{Arg} z)}$

$\alpha = n \in \mathbb{N}$. $z^n = e^{n(\log|z| + i\operatorname{Arg} z)} \underbrace{e^{2kn\pi i}}_{=1}$

$$= \left(e^{\log|z|}\right)^n e^{in\operatorname{Arg} z} = \underbrace{z \cdot \ldots \cdot z}_{n\text{-times}} \quad \text{similarly } n \in \mathbb{Z}_{<0}$$

So $\alpha \in \mathbb{Z}$ $\quad z^n = \begin{cases} z \cdot z \cdots z, & n > 0 \\ \frac{1}{z} \cdot \frac{1}{z} \cdots \frac{1}{z}, & n < 0 \end{cases}$ Single valued

But $\alpha \notin \mathbb{Z} \leadsto$ multivalued.

$z^{\alpha+\beta} \quad \underset{\neq}{=} \quad z^\alpha \cdot z^\beta$ Check!

$$i^{i-i} = i^0 = e^{0 \log i} = 1$$

$$i^i = e^{i \log i} = e^{i(\log 1 + i(\frac{\pi}{2} + 2k\pi))} = e^{-\pi/2 + 2k\pi}, \quad k \in \mathbb{Z}$$

$$i^{-i} = e^{-i \log i} = e^{-i(\log 1 + i(\frac{\pi}{2} + 2\ell\pi))} = e^{\frac{\pi}{2} + 2\ell\pi}, \quad \ell \in \mathbb{Z}$$

So $i^i i^{-i} = \{e^{2(k+l)\pi} : k, l \in \mathbb{Z}\}$

$= \{e^{2k\pi}, k \in \mathbb{Z}\}$. $\quad i^{i-i} \neq i^i \cdot i^{-i}$

Similar for $\log(zw) \overset{?}{=} \log z + \log w$

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1.8. Trigonometric and Hyperbolic functions

1) $e^{i\theta} = \cos\theta + i\sin\theta, \ \theta \in \mathbb{R}$

$\cos\theta = \mathrm{Re}(e^{i\theta}) = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \mathrm{Im}(e^{i\theta}) = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Def: ~~$\cos z = e^{iz}$~~ $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ $\quad \forall z \in \mathbb{C}$

$\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

Prop $\cos(-z) = \cos z, \ \sin(-z) = -\sin z$

$\cos(z+2k\pi) = \cos(z), \ \sin(z) = \sin(z+2k\pi)$ (Since $e^{2k\pi i} = 1 \ \forall k \in \mathbb{Z}$)

$\cos^2 z + \sin^2 z = 1$

Pf: $\dfrac{e^{2iz}+e^{-2iz}+2}{4} + \dfrac{e^{2iz}+e^{-2iz}-2}{-4} = 1$

Similarly: $\cos(\frac{\pi}{2}-z) = \sin(z)$

$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ $\quad z, w \in \mathbb{C}$.

$\sin(z+w) = \sin z\cos w + \sin w \cos z$

$e^{iz} = \cos z + i\sin z$

2) $\cosh z = \dfrac{e^{z}+e^{-z}}{2}$, $\sinh = \dfrac{e^{z}-e^{-z}}{2}$

$\cosh(\cancel{z}) = \cosh(z)$; $\sinh(-z) = -\sinh(z)$

$\cosh^2(z) - \sinh^2(z) = 1$

$\cosh(iz) = \cos(z)$, $\cos(iz) = \cosh(z)$

$\sinh(iz) = i\sin(z)$, $\cancel{\sin(z)}$ $\cancel{-\sinh z = i \sin z}$

$\sin(iz) = i\sinh(z)$

$-\sinh(z) = +i(\sin iz)$,

$z = x+iy$, $\sin(z) = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \sinh y \cos x$

$\operatorname{Re} \sin z = \cancel{\sin} \sin x \cosh y$, $\operatorname{Im} \sin z = \cos x \sinh y$.

$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x (1 + \sinh y) + (1 - \sin^2 x)\sinh^2 y$

$= \sin^2 x + \sinh^2 y$

$\sin z = 0 \iff \begin{cases} \sin x = 0 \\ \sinh y = 0 \end{cases} \iff z = k\pi$, $k \in \mathbb{Z}$ (only zeros are the real ones)

Similar $\cos z = 0 \iff z = \frac{\pi}{2} + k\pi$

Inverse trig functions:

$W = \arccos z = \cos^{-1} z$

$\Leftrightarrow \quad \cos w = z$

$$e^{iw} + e^{-iw} = 2z \quad \Leftrightarrow \quad e^{2iw} - 2z e^{iw} + 1 = 0$$

$$0 \neq e^{iw} = \frac{z \pm \sqrt{z^2-1}}{1} \quad \text{so} \quad iw = \log\left(z \pm \sqrt{z^2-1}\right)$$

$$w = \cos^{-1} z = -i \log\left(z \pm \sqrt{z^2-1}\right)$$ <u>multivalued</u>

<u>Chapter 2: Analytic functions</u>

<u>Def:</u> $f: U \to \mathbb{C}$, $z_0 \in U$ open. f is (complex) differentiable at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists.}$$

Then $f'(z_0)$ is called <u>derivative of f at z_0.</u>

<u>Ex:</u> 1) $f(z) = z^n$, $n \in \mathbb{N}$, $f'(z) = nz^{n-1}$

$$f'(z) = \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h \to 0} \frac{h\left((z+h)^{n-1} z + (z+h)^{n-2} z + \dots + z^{n-1}\right)}{h} = nz$$

2) $f(z) = \bar{z}$

$$\lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h \to 0} \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta} \quad \text{(radially)}$$

$h = re^{i\theta}, r \to 0$

depends on direction of approach

does not exist.

$z = x+iy$, $f(z) = x - iy$. $u(x,y) = x$, $v(x,y) = -y$ 01/22

real polynomial mapping, nowhere complex differentiable.

1. f is diff. at $z_0 \Rightarrow f$ cont. at z_0

$z \neq z_0$ $f(z) - f(z_0) = \dfrac{f(z)-f(z_0)}{z-z_0} \cdot (z-z_0)$

$$\lim_{z\to z_0} \left(f(z) - f(z_0)\right) = f'(z_0)\cdot 0 = 0$$

2. f, g differentiable, $c \in \mathbb{C}$ $(f+g)' = f'+g'$, $(cf)' = cf'$

$$(fg)' = f'g + fg'$$

If $g(z_0) \neq 0$ then $\left(\dfrac{f}{g}\right)'(z_0) = \dfrac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}$

$$\frac{f(z)g(z) - f(z_0)g(z_0)}{z-z_0} = \frac{(f(z)-f(z_0))\, g(z)}{z-z_0} + \frac{f(z_0)\,(g(z)-g(z_0))}{z-z_0}$$

$$\to f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Chain Rule If f diff'ble at z_0, g differentiable at $w_0 = f(z_0)$. Then

of $g\circ f$ diff'ble at z_0 with ~~$(f\circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$~~

$$(g\circ f)'(z_0) = g'(f(z_0))\cdot f'(z_0)$$

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Pf 1) $f'(z_0) \neq 0 \Rightarrow \exists \varepsilon > 0$ if $0 < |z - z_0| < \varepsilon$ then $f(z) \neq f(z_0)$

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \cdot \frac{f(z) - f(z_0)}{z - z_0}$$

$$\xrightarrow[z \to z_0 \Rightarrow f(z) \to f(z_0)]{} g'(f(z_0)) \cdot f'(z_0)$$

2) $f'(z_0) = 0$, $\dfrac{g(w) - g(w_0)}{w - w_0}$ is bounded near w_0

$$\Rightarrow \left| \frac{g(w) - g(w_0)}{w - w_0} \right| \leq C, \quad 0 < |w - w_0| < \delta$$

$$\Rightarrow |g(w) - g(w_0)| \leq C|w - w_0| \quad \forall\ |w - w_0| < \delta$$

$$|g(f(z)) - g(f(z_0))| \leq C|f(z) - f(z_0)| \quad \text{for } |z - z_0| < \text{some } \varepsilon$$

$$\left| \frac{g(f(z)) - g(f(z_0))}{z - z_0} \right| \leq C \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \quad 0 < |z - z_0| < \varepsilon$$

$$\to 0 = |f'(z_0)|. \qquad \square$$

Def: $f : U_{open} \to \mathbb{C}$ is called analytic on U (holomorphic) if f is differentiable at any $z \in U$ and f' is continuous on U.

2.3 The Cauchy Riemann equations

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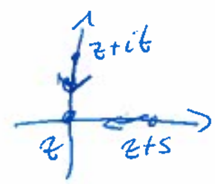
Theorem $f = u + iv : D \underset{\text{open}}{\subseteq} \mathbb{C} \to \mathbb{C}$ is holomorphic on D iff. $u, v \in \mathcal{C}^1(D)$ and ~~stati~~ satisfy the CR-equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Pf: "$\Rightarrow$" f' exists on D and is cont. on D. $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

$z = x + iy$, $h = s + it$. If $h = s \to 0$



$$\frac{f(z+s) - f(z)}{s} = \frac{u(x+s, y) - u(x,y)}{s} + i \frac{v(x+s,y) - v(x,y)}{s}$$

$\downarrow$

$f'(z) \Rightarrow \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist and $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$h = it \to 0$

$$\frac{f(z+it) - f(z)}{it} = -i \left(\frac{u(x, y+t) - u(x,y)}{t} + i \frac{v(x, y+t) - v(x,y)}{t} \right)$$

$$f'(z) = \frac{v(x, y+t) - v(x,y)}{t} - i \frac{u(x, y+t) - u(x,y)}{t}$$

$\Rightarrow \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}$ exist and $f' = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \operatorname{Re} f'$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \operatorname{Im} f'$.

Moreover $u, v \in \mathcal{C}^1(D)$.

"$\Leftarrow$" $u, v \in \mathcal{C}^1(D) \Rightarrow u, v$ real differentiable

$z \in D$, $h = s + it \ (\to 0)$

$$u(z+h) = u(z) + \frac{\partial u}{\partial x}(z)\cdot s + \frac{\partial u}{\partial y}(z)\cdot t + R(h)$$

with $\left|\frac{R(h)}{h}\right| \to 0$ as $h \to$

and $v(z+h) = v(z) + \frac{\partial v}{\partial x}(z)\cdot s + \frac{\partial v}{\partial y}(z)t + S(h)$, $\left|\frac{S(h)}{h}\right| \to 0$.

$$f(z+h) - f(z) = \frac{\partial u}{\partial x}(z)s + \underbrace{\frac{\partial u}{\partial y}(z)}_{= -\frac{\partial v}{\partial x}(z)}t + i\frac{\partial v}{\partial x}(z)t + i\underbrace{\frac{\partial v}{\partial y}(z)}_{\frac{\partial u}{\partial x}(z)}t + R(h) + iS(h)$$

$$= \frac{\partial u}{\partial x}(z)\underbrace{(s+it)}_{h} + i\frac{\partial v}{\partial x}(z)\underbrace{(it+s)}_{h} + R(h) + iS(h)$$

$$\frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z) + \underbrace{\frac{R(h)+iS(h)}{h}}_{\to 0}$$

$\downarrow$

exists and $f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$

Moreover $f' \in \mathcal{C}(D)$ as $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are. $\square$

<u>Ex:</u> $f(z) = e^z = e^x(\cos y + i\sin y)$, $u(x,y) = e^x\cos y$, $v(x,y) = e^x\sin y$

$u_x = e^x\cos y = v_y$

$u_y = -e^x\sin y = -v_x$.

So $f(z) = e^z$ is holomorphic on $\mathbb{C}$ (entire function)

and $f'(z) = \underbrace{e^x\cos y}_{u_x} + i\underbrace{e^x\sin y}_{v_x} = e^z$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z$, $\sinh z$, $\cosh z$ are all entire

and $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$, $(\sinh z)' = \cosh z$

$(\cosh z)' = \sinh z$

Thm f holom. on a domain D and $f' \equiv 0$ on D. Then f is constant.

Pf: $f = u + iv$, $f' = u_x + iv_x = v_y - iu_y = 0 \Rightarrow u_x = v_x = 0$

$u_y = v_y = 0$

$\square$

real analysis
$\Rightarrow$ $u = \text{const}$, $v = \text{const}$.

Thm If $f : D \to \mathbb{C}$, D domain is holom. and real ~~f~~ valued ($f(D) \subseteq \mathbb{R}$) then it is constant.

Pf: $f = u + iv$, $v = 0 \Rightarrow v_x = 0$, $v_y = 0 \overset{CR}{\Rightarrow} u_y = 0$, $u_x = 0 \Rightarrow f' = 0$

So $\underset{u}{\overset{\shortparallel}{f}}$ is constant.

2.4 Inverse function theorem

Remarks

1 $f : \underset{\mathbb{C}\text{ open}}{\overset{}{D}} \to \mathbb{C}$, $f = u + iv$ holom.

$$\det J f = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = u_x v_y - u_y v_x = u_x^2 + v_x^2$$

$$= |f'(z)|^2 \geq 0$$

2) $f : \underset{\text{domain}}{D_{\subseteq \mathbb{C}}} \longrightarrow f(D)$ is injective and holom. 01/2

So $f(D)$ is a domain. Assume $D, f(D)$ bounded. (inj. + cont. $\Rightarrow$

Let h continuous on $\overline{f(D)}$

$$\iint_{f(D)} h \, d\lambda \underset{w=f(z)}{=} \iint_D (h \circ f) |f'|^2 \, d\lambda \quad \text{(holom. change of variables)}$$

$h \equiv 1$ $\quad \operatorname{area}(f(D)) = \iint_D |f'|^2 \, d\lambda$.

Inverse function theorem Assume $f : D_{\text{open}} \to \mathbb{C}$ holomorphic and $f'(z_0) \neq 0$ for some $z_0 \in D$. Then, there exists $\mathcal{U} \subseteq D$ $\overset{\text{open}}{\text{disk}}$ centered at z_0 so that $f'(z) \neq 0$ on $\forall z \in \mathcal{U}$, $V = f(\mathcal{U})$ open. $f|_{\mathcal{U}} : \mathcal{U} \to V$ is bijective and $f^{-1} : V \to \mathcal{U}$ is holomorphic. and if $w = f(z) \in V$, $z \in \mathcal{U}$ then $(f^{-1})'(w) = \dfrac{1}{f'(z)}$.

Pf: $f \in \mathcal{C}^1(D)$, $\det(\partial f(z_0)) = |f'(z_0)|^2 > 0$. $\exists\, \mathcal{U} \subseteq D$ $\overset{\text{open}}{\text{disk}}$ about z_0 with $f'(z) \neq 0$ $\forall z \in \mathcal{U}$. $\overset{\text{Real}}{\underset{\text{result}}{\Rightarrow}}$ $V = f(\mathcal{U})$ open, $f : \mathcal{U} \to V$ bijective

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$$(f^{-1})'(w_1) = \lim_{w\to w_1} \frac{f^{-1}(w) - f^{-1}(w_1)}{w - w_1} \qquad (f^{-1} \text{ cont.})$$

$$= \lim_{z\to z_1} \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{f'(z_1)} \quad \text{So } f^{-1} \text{ is holom.}$$

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Ex: $\mathrm{Log}: \mathbb{C} \setminus (-\infty, 0] \to \{-\pi < \mathrm{Im}\, z < \pi\}$ holomorphic.

$e^{\log z} = z \Rightarrow e^{\log z}(\log z)' = 1$, so $(\log z)' = \frac{1}{z}$

Ex: $\pm\sqrt{z}$ holom. on $\mathbb{C} \setminus (-\infty, 0]$

Derivative $(\sqrt{z})' = \frac{1}{2\sqrt{z}}$ (same branch) $\quad \sqrt{z} = \pm |z|^{\frac{1}{2}} e^{i\frac{\mathrm{Arg}\, z}{2}}$

$f(z)^2 = z \Rightarrow 2f(z)f'(z) = 1 \Rightarrow f'(z) = \frac{1}{2f(z)}$

2.5 Harmonic functions

$u: D_{\text{open}} \subseteq \mathbb{C} \to \mathbb{R}$ is harmonic if $u \in C^2(D)$ and $\Delta u = u_{xx} + u_{yy} = 0$ on D.

$f = u + iv : D \to \mathbb{C}$ is harmonic if $\Delta f = \Delta u + i\Delta v = 0$ on D

equivalently u and v are harmonic.

Theorem: Assume f is holom. on D, $f = u + iv$ and $f \in C^2(D)$

Then u, v are harmonic on D.

Proof $\Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$ 01/26

Since $v \in \mathcal{C}^2(D)$. Thus, $v = Re(\underbrace{-if}_{\text{holom.}})$ is harmonic as well $\square$

Def: If $u: D \to \mathbb{R}$ is harmonic, then a <u>harmonic conjugate of u in D</u> is a function $v \in \mathcal{C}^2(D)$ such that $f = u + iv$ is holom. in D.

Remarks If D is connected and v_1, v_2 are harmonic conjugates of u, then

$v_1 = v_2 + \text{constant}$.

Ex $u(x,y) = x^2 - y^2$ is harmonic on $\mathbb{C}$. $\Delta u = 2 - 2 = 0$.

harmonic conjugate? $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x$

$\Rightarrow v = 2yx + h(y)$, $\frac{\partial v}{\partial y} = 2x + h'(y) = 2x$, so $h'(y) = 0$, $h(y) = c$

$v(x,y) = 2xy + c$, then $f = u + iv = z^2 + ic$.

Theorem Let $P, Q \in \mathcal{C}^1(\Delta(z_0, r))$ ← Disk. There exists a function $v \in \mathcal{C}^2(\Delta(z_0, r))$ with $\frac{\partial v}{\partial x} = P$, $\frac{\partial v}{\partial y} = Q$ if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } \Delta(z_0, r)$$

Corollary If u is harmonic in $\Delta(z_0, r)$ then u has a harmonic conjugate in $\Delta(z_0, r)$.

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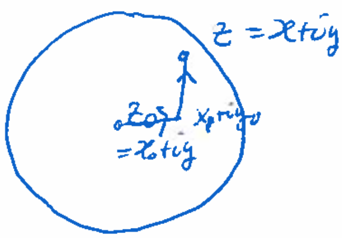
<u>Proof of corollary</u> Need V: $\underbrace{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}_{P}$, $\underbrace{\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}}_{Q}$

$$\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial y} \text{ since } \Delta u = 0 \quad \square$$

<u>Pf of Theorem</u> <u>"$\Rightarrow$"</u> $\overset{\text{if } v \text{ exists}}{\frac{\partial P}{\partial y}} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial Q}{\partial x}$.

<u>"$\Leftarrow$"</u>



<u>Define</u> $$v(x,y) = \int_{x_0}^{x} P(s,y_0)\,ds + \int_{y_0}^{y} Q(x,t)\,dt$$

$$\frac{\partial v}{\partial x}(x,y) = P(x,y_0) + \int_{y_0}^{y} \frac{\partial Q}{\partial x}(x,t)\,dt = P(x,y_0) + \int_{y_0}^{y} \frac{\partial P}{\partial t}(x,t)\,dt$$

$$= P(x,y) + P(x,y_0) - P(x,y_0) = P(x,y).$$

$$\frac{\partial v}{\partial y}(x,y) = Q(x,y)$$

2.6 Conformal Mappings

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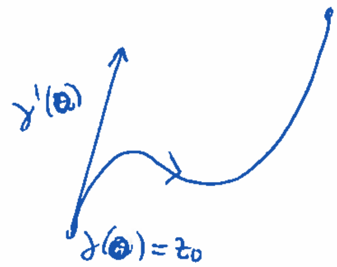
$\gamma : [0,1] \to \mathbb{C}$ smooth. $\gamma(t) = x(t) + i y(t)$

$\gamma(0) = z_0$ tangent at z_0 is $\gamma'(0) = x'(0) + i y'(0)$



Suppose f holom. in a nbhd of $\gamma([0,1])$

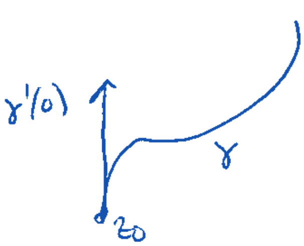
$f \circ \gamma(t)$ is another curve. Then

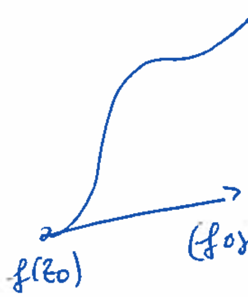
$$(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t) \quad \text{(Chain rule)}$$

$$\boxed{\begin{array}{l} CR \quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \qquad\quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \\ \hline \text{equivalently} \quad \dfrac{\partial f}{\partial y} = i \dfrac{\partial f}{\partial x} \end{array}}$$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))\, x'(t) + \underbrace{\frac{\partial f}{\partial y}}_{= i \frac{\partial f}{\partial x}}(x(t), y(t))\, y'(t)$$

$$= \underbrace{\frac{\partial f}{\partial x}}_{= f'(\gamma(t))}(\gamma(t)) \underbrace{\left(x'(t) + i y'(t) \right)}_{= \gamma'(t)} = f'(\gamma(t))\, \gamma'(t).$$





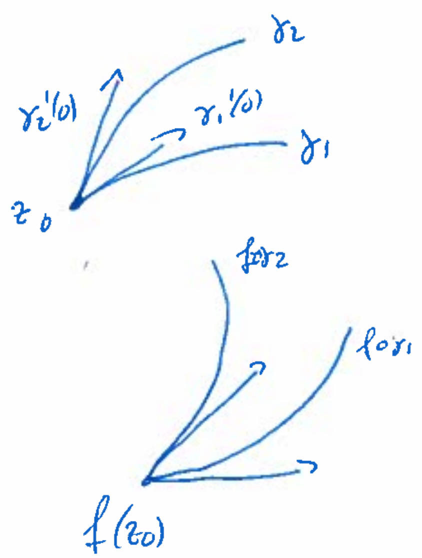
$$(f \circ \gamma)'(0) = \underbrace{f'(z_0)}_{\text{transforms tangent}} \gamma'(0)$$

$|(f \circ \gamma)'(0)| = |f'(z_0)|\, |\gamma'(0)|$ dilation by $|f'(z_0)|$ and $\arg((f \circ \gamma)'(0))$

$= \arg(\gamma'(0)) + \arg(f'($

if $f'(z_0) \neq 0$.

rotation by $\arg(f'(z_0))$. f holom. near z_0, $f'(z_0) \neq 0$.



$$\frac{(f\circ\gamma_2)'(0)}{(f\circ\gamma_1)'(0)} = \frac{f'(z_0)\gamma_2'(0)}{f'(z_0)\gamma_1'(0)} = \frac{\gamma_2'(0)}{\gamma_1'(0)}$$

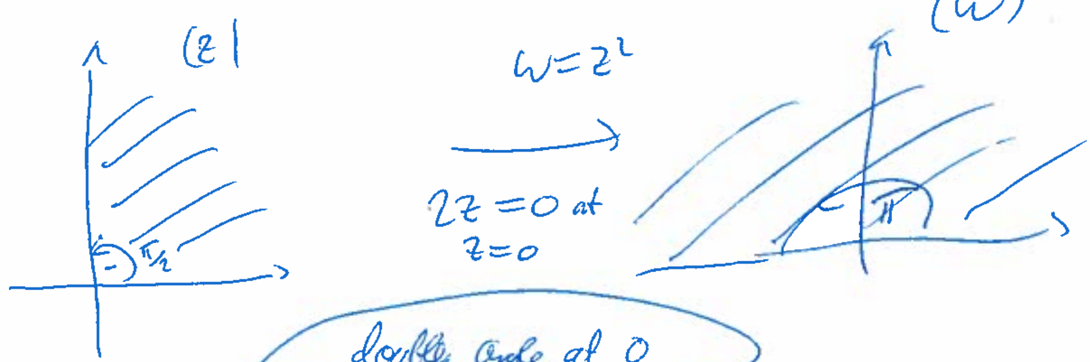
$$\arg\left(\frac{f\circ\gamma_2'(0)}{f\circ\gamma_1'(0)}\right) = \measuredangle\left((f\circ\gamma_1)'(0), (f\circ\gamma_2)'(0)\right)$$
$$= \measuredangle\left(\gamma_1'(0), \gamma_2'(0)\right).$$

$\therefore$ f preserves angles at z_0

Def: f is conformal at z_0 if it preserves angles at z_0

f is conformal on D if $f \in \mathcal{C}^1(D)$ and injective and preserves angles at every point $z_0 \in D$.

f hol. near z_0, $f'(z_0) \neq 0 \Rightarrow f$ is conformal at z_0
$\Leftarrow$ Chapter 4



$2z = 0$ at $z = 0$

double angle at 0

2.7. Fractional linear maps (Moebius transformations)

$$w = f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \ ad-bc \neq 0$$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0 \quad f \text{ nonconstant.}$$

1) $c=0$ $\quad w = a'z+b', \quad a' = \frac{a}{d} \neq 0 \quad f : \mathbb{C} \to \mathbb{C}$ bijection

$f(\infty) = \infty$ continuous extension to $\mathbb{C}^*$ (Riemann sphere) (Check!)

$f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ homeomorphism.

2) $c \neq 0$, $z_0 = -\frac{d}{c}$ pole of f $\quad \lim_{z \to z_0} f(z) = \infty$ (in $\mathbb{C}^*$)

$f : \mathbb{C} \setminus \{-\frac{d}{c}\} \longrightarrow \mathbb{C} \setminus \{\frac{a}{c}\}$ bijection with.

Inverse $z = \frac{-dw+b}{cw-a}$. $\quad$ Set $f(-\frac{d}{c}) = \infty, \ f(\infty) = \frac{a}{c}$

$\therefore f : \mathbb{C}^* \to \mathbb{C}^*$ homeomorphism

$$GL_2(\mathbb{C}) = \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C} \ \det A = ad-bc \neq 0 \}$$

$$\pi : (GL_2(\mathbb{C}), \cdot) \longrightarrow (\underset{\substack{\uparrow \\ \text{set of moebius maps}}}{H}, \circ)$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\pi}{\longmapsto} \left(w = \frac{az+b}{cz+d} \right)$$ is a group homomorphism

$$\pi(AB) = \pi(A) \circ \pi(B).$$

<u>Elementary Mobius maps</u> 1) translation $f(z) = z+b$

2) dilation; $f(z) = az$, $a \neq 0$ (dilation by $|a|$ + rotation by $\arg(a)$)

3) inversion: $f(z) = \frac{1}{z} = w$ $\quad |w| = \frac{1}{|z|}$, $\arg w = -\arg z$

Any Mob. is a composition of 1)–3). Let ~~ez~~ $c \neq 0$

$$f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}} \quad \text{So}$$

$$\underset{f_1 \text{ translation}}{z \mapsto z+\frac{d}{c}} \underset{f_2 \text{ inversion}}{\longmapsto} \frac{1}{z+d/c} \underset{f_3 \text{ dilation}}{\longmapsto} \frac{bc-ad}{c^2} \cdot \frac{1}{z+d/c} \underset{f_4 \text{ translation}}{\longmapsto} \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{z+d/c} = f(z).$$

So $f = f_4 \circ f_3 \circ f_2 \circ f_1$

<u>Def:</u> Let z_0, z_1, z_2, z_3 <u>distinct</u> complex numbers

cross ratio $(z_0, z_1, z_2, z_3) = \frac{z_0 - z_1}{z_0 - z_3} : \frac{z_2 - z_1}{z_2 - z_3} = \underbrace{\frac{z_0 - z_1}{z_0 - z_3}}_{\to 1 \, (z_0 \to \infty)} \frac{z_2 - z_3}{z_2 - z_1}$

extends continuously on $\mathbb{C}^*$

e.g. $(\infty, z_1, z_2, z_3) = \frac{z_2 - z_3}{z_2 - z_1}$

Lemma If $f(z)$ is a Möb. map, $z_0, \to z_4 \in \mathbb{C}^*$ are distinct, then $(f(z_0), f(z_1), f(z_2), f(z_3)) = (z_0, z_1, z_2, z_3)$ $\square$ 01/2

Proof Calculate the LHS.

Corollary Suppose $z_1, z_2, z_3 \in \mathbb{C}^*$ distinct and $(w_1, w_2, w_3) \in \mathbb{C}^*$ distinct then $\exists!$ Mobius map f s.t. $f(z_j) = w_j$, $j = 1, 2, 3$.

Proof $z \in \mathbb{C}$ arb. $(f(z), w_1, w_2, w_3) = (z, z_1, z_2, z_3)$

i.e. $$\frac{f(z)-w_1}{f(z)-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$ solve for $f(z$

Special case $f(z_1)=0$, $f(z_0)=\infty$

$$f(z) = \lambda \frac{z-z_0}{z-z_1}, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Equation of a circle in $\mathbb{C}^*$ (circle or line)

$$A(x^2+y^2) + bx + cy + D = 0 \quad (A=0 \text{ line}, A \neq 0 \Rightarrow \text{circle})$$

A, b, c, D real

$z = x+iy$, $x^2+y^2 = |z|^2 = z\bar{z}$, $x = \operatorname{Re} z = \frac{z+\bar{z}}{2}$, $y = \operatorname{Im} z = \frac{z-\bar{z}}{2i}$

$$A z\bar{z} + b\frac{z+\bar{z}}{2} - ic\frac{z-\bar{z}}{2} + D = 0$$

$$Az\bar z + \frac{b-ic}{2}z + \frac{b+ic}{2}\bar z + D = 0.$$ 01/29

$$\Rightarrow \boxed{Az\bar z + Bz + \bar B\bar z + D = 0 \qquad A, D \in \mathbb{R},\ B, \cancel{A} \in \mathbb{C}}$$

Theorem $f(z) = \dfrac{az+b}{cz+d}$ maps circles in $\mathbb{C}^*$ onto circles in $\mathbb{C}^*$.

Pf: given $Az\bar z + Bz + \bar B\bar z + D = 0$ $\quad A, D \in \mathbb{R},\ B, \cancel{A} \in \mathbb{C}$

$$w = \frac{az+b}{cz+d}, \quad z = \frac{dw-b}{-cw+a}$$

$$A\,\frac{dw-b}{-cw+a}\cdot\frac{\overline{dw-b}}{\overline{-cw+a}} + B\,\frac{dw-b}{-cw+a} + \bar B\,\frac{\overline{dw-b}}{\overline{-cw+a}} + D = 0$$

$$A\,(dw-b)(\bar d\bar w - \bar b) + B\,(dw-b)(-\bar c\bar w + \bar a)$$
$$+ \bar B\,(\bar d\bar w - \bar b)(-cw+a) + D(-cw+a)(-\bar c\bar w + \bar a) = 0$$

$$w\bar w\,(Ad\bar d - Bd\bar c - \bar B\bar d c + Dc\bar c) +$$
$$A|d|^2 - 2\,\mathrm{Re}(Bd\bar c) + D|c|^2 \in \mathbb{R}$$

$$+\ w\,\underbrace{(-Ad\bar b + Bd\bar a + \bar B\cancel{a}\,\bar b c - Dc\bar a)}_{=B'}$$

$$\bar w\,\underbrace{(-Ab\bar d + Bb\bar c + \cancel{B}\,\bar B\bar d a - Da\bar c)}_{=\bar B'} \ *$$

$+ (Ab\bar{b}\ A - \bar{B}b\bar{a} - \bar{B}\bar{b}a + Da\bar{a}) = 0$ is a circle. $\square$

$A|b|^2 - 2\,\mathrm{Re}(\bar{B}b\bar{a}) + D|a|^2 \in \mathbb{R}$

<u>Cor</u> $(z_0, z_1, z_2, z_3) \in \mathbb{R} \iff z_0, z_1, z_2, z_3$ lie on a circle in $\mathbb{C}^*$.

<u>Pf:</u> $w_0 = \infty, w_1 = 0, w_2 = 1$ $\exists!$ Moebius map f s.t. $f(z_j) = w_j$, $j = 0$

Set $w_3 = f(z_3)$. Now $(z_0, z_1, z_2, z_3) = (\infty, 0, 1, w_3) = \dfrac{\infty - 0}{\infty - w_3} : \dfrac{1-0}{1-w_3}$

$= 1 - w_3$

<u>„$\Rightarrow$"</u> $(z_0, z_1, z_2, z_3) \in \mathbb{R} \Rightarrow w_3 \in \mathbb{R}$ So $w_0, \dots, w_3$ on real axis.

$\Rightarrow z_0, \dots, z_3 \in f^{-1}(\mathbb{R})$ circle in $\mathbb{C}^*$.

<u>"$\Leftarrow$"</u> $w_0, \dots, w_3$ are on a circle in $\mathbb{C}^*$ hence on the real axis $\Rightarrow w_3 \in \mathbb{R}$

So $(z_0, z_1, z_2, z_3) \in \mathbb{R}$. $\square$

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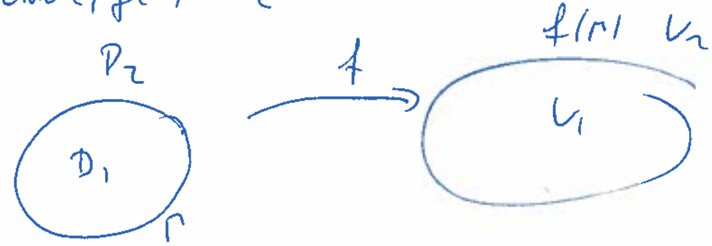
z_0 pole of f, Γ circle in $\mathbb{C}^*$ $z_0 \in \Gamma \Rightarrow f(\Gamma)$ line

$z_0 \notin \Gamma \Rightarrow f(\Gamma)$ circle

Since f homeom.

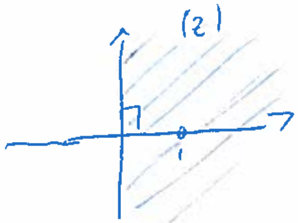
If Γ circle, $f(\Gamma)$ circle



then $f(D_1) = U_1$ or
$f(D_2) = U_2$

Ex: f Moeb, $f(-1)=0$, $f(1)=\infty$, $f(0)=1$ Find the image of real, imaginary axis, right half plane, left unit circle / disk — 01/31

$f(z)=\lambda\dfrac{z+1}{z-1}=-1\dfrac{z+1}{z-1}=\dfrac{1+z}{1-z}$ (real coefficients) $\Rightarrow f(\mathbb{R})=\mathbb{R}$



$\xrightarrow{w=\frac{1+z}{1-z}}$

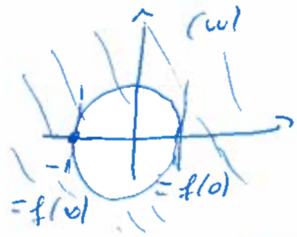
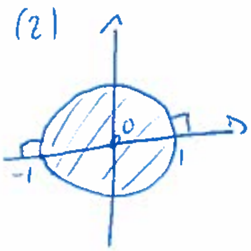


image circle orthogonal to the real axis

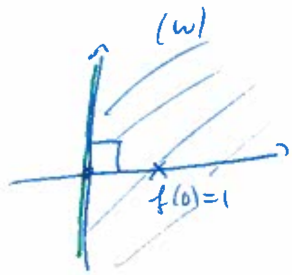
$f(i\mathbb{R})=\{|w|=1\}$

$f(\{\operatorname{Re} z>0\})=\{|w|>1\}$

$f(\infty)=-1$



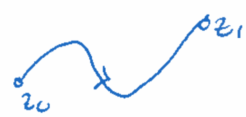
$\xrightarrow{w=\frac{1+z}{1-z}}$



$f(\{|z|=1\})=\{\operatorname{Re} w=0\}$

line through $f(-1)=0$ $\perp f(\mathbb{R})=\mathbb{R}$

$f(\{|z|<1\})=\{\operatorname{Re} w>0\}$

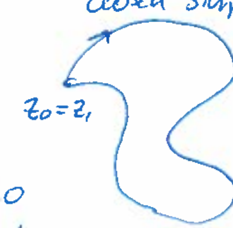


Chapter 3: Line Integrals and Green's Thm

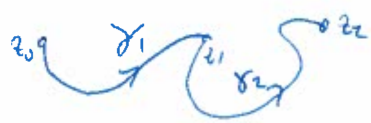
Paths of $\mathbb{C}$: 1) path: from z_0 to z_1 $\gamma:[0,1]\to\mathbb{C}$ continuous $\gamma(0)=z_0$, $\gamma(1)=z_1$

2) Simple path: γ is injective 3) closed path $\gamma(0)=\gamma(1)$

4) closed simple path γ closed path s.t. $\gamma(s)\neq\gamma(t)$ if $0\le s<t<1$



5) reparametrization of γ $\gamma:[0,1]\to\mathbb{C}$, $\varphi:[0,1]\to[0,1]$ cont. and strictly increasing $\varphi(0)=0$, $\varphi(1)=1$



$\gamma\circ\varphi$ is a reparametrization of γ.

7) trace of γ is $\gamma([0,1])$ trace of $\gamma\circ\varphi$ = trace of γ.

Concatenation of paths: $\left.\begin{array}{l}\gamma_1:[0,1]\to\mathbb{C}\\ \gamma_2:[0,1]\to\mathbb{C}\end{array}\right\}$ $\gamma_1+\gamma_2(t):=\begin{cases}\gamma_1(2t) & 0\le t\le\frac12\\ \gamma_2(2t-1), & \frac12\le t\le 1\end{cases}$

$\gamma_1(1)=\gamma_2(0)$

$-\gamma(t)=\gamma(1-t)$ opposite path.

<u>length of γ</u>

$\Pi = \{t_0 = 0 < t_1 < \dots < t_n = 1\}$ $01/31$

$$L(\gamma) := \sup_{\Pi} \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \in [0, +\infty], \quad \gamma \text{ rectifiable if } L(\gamma) < \infty$$
(γ is BV)

$\gamma(t) = x(t) + iy(t)$. Then x, y are BV.

<u>Ex:</u> $[z_0, z_1]$, $\gamma(t) = (1-t)z_0 + tz_1$ segment

S^1 $\gamma(t) = e^{2\pi i t}$

$\gamma'(t) = x'(t) + iy'(t)$. γ smooth $\Leftrightarrow$ $x, y \in C^1$, $\gamma' \neq 0$.

<u>Line integrals</u> $\gamma : [0,1] \to \mathbb{C}$ rectifiable, $\gamma(t) = x(t) + iy(t)$, $P, Q : \gamma([0,1]) \to \mathbb{R}$ (
continuous. Then $\displaystyle\int_\gamma P\,dx + Q\,dy = \int_0^1 P(x(t), y(t))\,dx(t) + \int_0^1 Q(x(t), y(t))\,dy(t)$

as <u>Riemann-Stieltjes integrals</u>.

<u>Recall</u> $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = 1\}$, $t_j' \in [t_{j-1}, t_j]$. Then

$$\int_\gamma P\,dx + Q\,dy = \lim_{\|\Pi\| \to 0} \sum_{j=1}^{n} \left[P(x(t_j'), y(t_j'))\,(x(t_j) - x(t_{j-1})) + Q(x(t_j'), y(t_j'))\,(y(t_j) - \dots\right.$$

<u>Piecewise smooth γ</u>

γ' exist everywhere but finitely many interior points.

$$\int_\gamma P\,dx + Q\,dy = \int_0^1 \left[P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)\right] dt$$

<u>Properties</u> 1) $\gamma = x + iy$ $\quad \Gamma = \gamma \circ \varphi = u + iv$, $\varphi : [0,1] \to [0,1]$ C^1 and strictly increasing, $\varphi(0) = 0$, $\varphi(1) =$

$$\Rightarrow \int_\Gamma \cancel{P\,dx + Q\,dy}\; P\,du + Q\,dv = \int_\gamma P\,dx + Q\,dy$$

2) $$\int_{\gamma_1 + \gamma_2} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy + \int_{\gamma_2} P\,dx + Q\,dy$$

and $\int_{-\gamma} Pdx+Qdy = -\int_{\gamma} Pdx+Qdy$ 01/31

3) $\gamma_n,\gamma : [0,1]\to D$ rectifiable, $\gamma_n \to \gamma$ uniformly on $[0,1]$, $\exists M\geq 0$ s.t.

$L(\gamma_n)\leq M$ $\forall n$. P,Q cont. on D.

$\Rightarrow \lim_{n\to\infty} \int_{\gamma_n} Pdx+Qdy = \int_{\gamma} Pdx+Qdy.$

02/02

D domain piecewise smooth boundary

positive orientation of ∂D

(Domain to the left)

Green's Thm $P,Q \in C^1(\bar{D})$, D bdd

$$\Rightarrow \int_{\partial D} Pdx+Qdy = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

EX: $\gamma=\gamma_1+\gamma_2+\gamma_3$

$\gamma_1(t)=t \quad 0\leq t\leq 1$

$\gamma_2(t)=e^{it} \quad 0\leq t\leq \frac{\pi}{2}$

$\gamma_3(t)=(1-t)i \quad 0\leq t\leq 1$

$$\int_\gamma xy\,dx = \int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3} = \underset{y=0}{0}+\int_0^{\pi/2}\cos t\sin t(-\sin t)dt+\underset{x=0}{0}$$

$$=-\frac{1}{3}\sin^3 t\Big|_0^{\frac{\pi}{2}} = \int_0^{\pi/2}\int_0^1 -r\cos\theta\, r\,dr\,dt = \iint_D -x\,dxdy.$$

3.2 Independence of paths

Differential 1-forms If $h : D \subset \mathbb{C} \to \mathbb{R}$ ($\mathbb{C}$) is $\mathcal{C}^1$

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy.$$ Then ~~the~~ $dh(x_0, y_0) : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$

is a $\mathbb{R}$ linear map.

Diff. 1-form on D $\quad \omega = P dx + Q dy, \; P, Q \in \mathcal{C}(D)$

$(x_0, y_0) \in D \mapsto \omega(x_0, y_0) = P(x_0, y_0) dx + Q(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}$ $\quad \mathbb{R}$-linear.

ω is ~~an~~ exact on D if $\exists\, h \in \mathcal{C}^1(D)$ s.t. $\omega = dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$

$\omega = P dx + Q dy$ is closed $\left(P, Q \in \mathcal{C}^1(D)\right)$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

Then $\int_\gamma P dx + Q dy$ is path independent on D if one of the following ~~con~~ equivalent conditions hold: i) If γ_1, γ_2 rectifiable paths in D s.t.

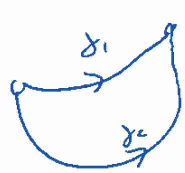
$\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$

Then $\int_{\gamma_1} P dx + Q dy = \int_{\gamma_2} P dx + Q dy$.

ii) $\int_\gamma \overbrace{P dx + Q dy}^{\omega} = 0 \quad \forall \gamma$ rectifiable closed path in D.

i) $\Rightarrow$ ii) $\gamma(0) = \gamma(1) = z_0$, $\gamma_1(t) = z_0$ $\quad$ i) $\Rightarrow \int_{\gamma_2} \omega = \int_{\gamma_1} \omega = 0$

ii) $\Rightarrow$ i)



Then $\gamma = \gamma_1 - ~~\gamma~~\gamma_2$ is rectifiable and closed

$$0 = \int_\gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega$$

Theorem $\mathbb{C} \supseteq D$ domain, $P, Q : D \to \mathbb{R}(\mathbb{C})$ cont. $\omega = P\,dx + Q\,dy$

a) If $P, Q \in \mathcal{C}^1(D)$ and ω is exact, then it is closed.

b) ω is exact in D iff. $\int_\gamma \omega$ is path independent in D.

If $\omega = dh$ and $\gamma(0) = z_0$, $\gamma(1) = z_1$ then $\int_\gamma \omega = \int_\gamma dh = h(z_1) - h(z_0)$
$= h(\gamma(1)) - h(\gamma(0))$.

Pf: a) let $\omega = dh$ $P = \frac{\partial h}{\partial x}$, $Q = \frac{\partial h}{\partial y}$, $h \in \mathcal{C}^2(D)$.

$$\frac{\partial P}{\partial y} = \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial Q}{\partial x} \Rightarrow \omega \text{ closed.}$$

b) "exact $\Rightarrow \int_\gamma \omega$ path indep."

γ piecewise smooth, $\omega = dh$
$\gamma = x + iy$

$$\int_\gamma \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy = \int_0^1 \frac{\partial h}{\partial x}(x(t), y(t))x'(t) + \frac{\partial h}{\partial y}(x(t), y(t))y'(t)\,dt$$

$$= \int_0^1 \frac{d}{dt} h(x(t), y(t))\,dt = h(x(1), y(1)) - h(x(0), y(0)) = h(\gamma(1)) - h(\gamma(0)).$$

γ_n polygonal path inscribed in γ, joining $\gamma(\frac{j}{n})$, $\gamma(\frac{j+1}{n})$, $0 \le j \le n-1$

$\gamma_n \xrightarrow{\text{unif}} \gamma$, $L(\gamma_n) \le L(\gamma) < +\infty$

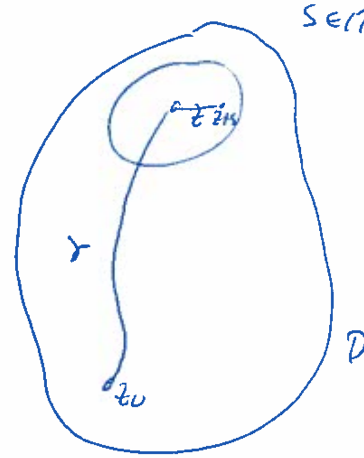
$\int_{\gamma_n} dh = \gamma(1) - \gamma(0)$ and $\int_{\gamma_n} dh \to \int_\gamma dh$.

"$\Leftarrow$" Fix $z_0 \in D$, D path connected

$h(z) = \int_{z_0}^{z} P\,dx + Q\,dy$ along any rectifiable path in D, joining z_0 to z, This is well defined since $\int_\gamma w$ is path independent in D.

Let $s \in \mathbb{R}$ $\quad \tilde{\gamma} = \gamma + [z, z+s]$

$S \in \mathbb{R}$



$$\frac{h(z+s) - h(z)}{s} = \frac{1}{s}\left(\int_{\gamma + [z, z+s]} P\,dx + Q\,dy - \int_\gamma P\,dx + Q\,dy\right)$$

$$= \frac{1}{s}\int_0^s P(x_1 + t, y_1)\,dt$$

$z = \cancel{x_1 + i} (x_1, y_1)$

on segment $\begin{cases} x = x_1 + t & 0 \le t \le s \\ y = x_1 \end{cases}$

MVT integral

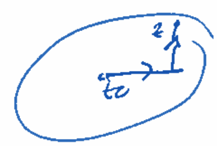
$$= \frac{1}{s}\cdot s\, P(x_1 + \xi, y_1) \quad \text{with } \xi \in [0, t]$$

$$\xrightarrow[s \to 0]{} P(x_1, y_1). \Rightarrow \frac{\partial h}{\partial x} = P. \quad \left(\text{similarly } \frac{\partial h}{\partial y} = Q\right).$$

P cont.

<u>Remarks</u> 1) If D ~~is closed~~ is a disc then w closed $\Rightarrow$ w exact.

$$w = dh \text{ for } h(z) = \int_{\gamma_z} w$$



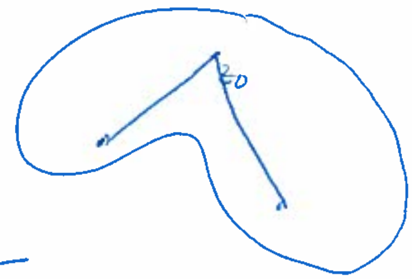
2) HW#2: $\exists$ form in $\mathbb{C}\setminus\{0\}$ closed but not exact.

Check $\int_C w \neq 0$.

<u>Remark</u> w closed in D, γ simple closed curve piecewise smooth. $\gamma = \partial U$

and $U \subset D$, then $\displaystyle\int_\gamma P\,dx + Q\,dy \overset{\text{Green}}{=} \iint_U \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dx\,dy$

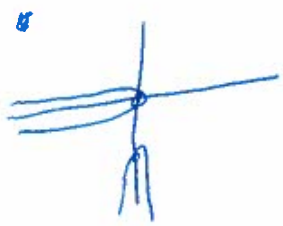
Domain D is called <u>star shaped</u> if $\exists z_0 \in D$ (called star center) s.t. $\forall z \in D$ $[z_0, z] \subseteq D$. 02/05



<u>Examples</u> 1) D convex $\Rightarrow$ D star shaped

2) $\mathbb{C} \setminus (-\infty, 0]$ star shaped

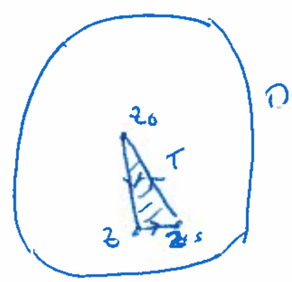
3) $\mathbb{C}$ with 2 cuts, not star shaped but simply connected



<u>Theorem</u> If $\omega = P\,dx + Q\,dy$, $P, Q \in \mathcal{C}^1(D)$ is closed on a star shaped domain D, then ω is exact.

Pf:

$$h(z) := \int_{[z_0, z]} \omega =: \int_{z_0}^{z} \omega \,, \qquad s \in \mathbb{R} \qquad \frac{h(z+s) - h(z)}{s} = \boxed{\frac{1}{s} \int_{z}^{z+s} \omega \to \underset{\text{same arg}}{P(z)}}$$



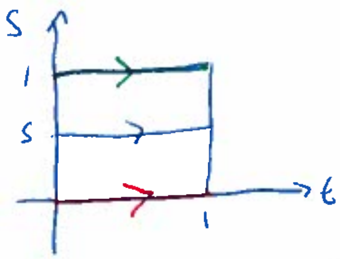
$$T = \Delta(z_0, z, z+s) \subseteq D.$$

$$\left(\int_{z_0}^{z} \omega + \int_{z}^{z+s} \omega + \int_{z+s}^{z_0} \omega \right) = \iint_T \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)}_{=0} dx\,dy = 0$$

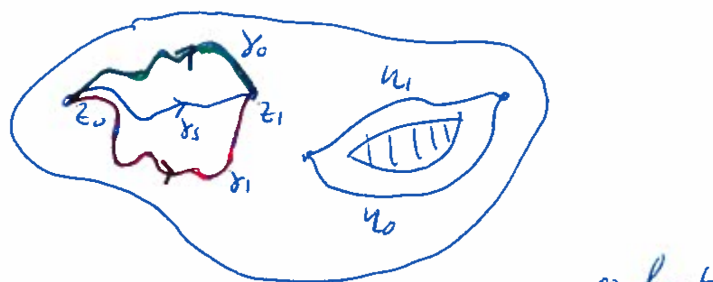
$$h(z) - h(z+s) + \int_{z}^{z+s} \omega = 0$$

Def: 1) $I=[0,1]\subseteq\mathbb{R}$, D domain in $\mathbb{C}$.

Two paths $\gamma_0,\gamma_1 : I\to D$ cont. with $\gamma_0(0)=\gamma_1(0)=z_0$ and $\gamma_0(1)=\gamma_1(1)=z_1$ are path homotopic in D. if $\exists\ \Gamma: I\times I\to D$ continuous s.t. $\Gamma(\cdot,0)=\gamma_0$, $\Gamma(\cdot,1)=\gamma_1$, $\Gamma(0,s)=z_0$, $\Gamma(1,s)=z_1$.



$\gamma_s=\Gamma(\cdot,s)$



η_0,η_1 not path homot[opic] in D.

2) Two closed paths $\gamma_0,\gamma_1 : I\to D$ are path homotopic in D if $\exists\ \Gamma: I\times I\to D$ cont. such that $\Gamma(t,0)=\gamma_0(t)$, $\Gamma(t,1)=\gamma_1(t)$ $t\in I$ and $\Gamma(0,s)=\Gamma(1,s)$ $\forall s\in I$. (each $\gamma_s(t)=\Gamma(t,s)$ is closed)

If $\gamma:[0,1]\to D$ closed path is homotopic to a constant path then we say γ is homotopic to 0 in D.

A domain $D\subseteq\mathbb{C}$ is simply connected if every closed path in D is homotop[ic] to 0 in D. ("no holes").

Ex: Star shaped domains simply connected

$\gamma: I\to D$, $\gamma(0)=\gamma(1)$ cont.

$$\Gamma(t,s)=(1-s)\gamma(t)+s z_0\in D.$$

continuous on $I\times I$. $\Gamma(0,s)=(1-s)\gamma(0)+s z_0=\Gamma(1,s)$.

Theorem Let $w = Pdx + Qdy$ be a closed in D. 02/05

1) If $\gamma_0, \gamma_1 : I \to D$ are (rectifiable) paths, $\gamma_0(0) = \gamma_1(0) = z_0$, $\gamma_0(1) = \gamma_1(1) = z_1$ are path homotopic in D, then $\int_{\gamma_0} w = \int_{\gamma_1} w$.

2) If $\gamma_0, \gamma_1 : I \to D$ are closed rectifiable paths in D and path homotopic in D, then $\int_{\gamma_0} w = \int_{\gamma_1} w$.

Corollary If w is closed on a simply connected domain D, then w is exact in D.

Pf of ~~Theorem~~ Cor γ closed rectifiable path $\Rightarrow$ γ homotopic to 0 $\Rightarrow \int_\gamma w = 0$.

So path independent, hence exact.

Proof of Theorem 1) $\Gamma : I \times I \to D$ cont., $\Gamma(t,0) = \gamma_0(t)$, $\Gamma(t,1) = \gamma_1(t)$, $t \in I$.
$\Gamma(0,s) = z_0$, $\Gamma(1,s) = z_1$, $\gamma_s(t) = \Gamma(t,s)$, $K := \Gamma(I \times I) \subseteq D$ compact.

$\varepsilon := dist(K, \partial D) = \inf\{|z-w| \mid z \in K, w \in \partial D\} > 0$.

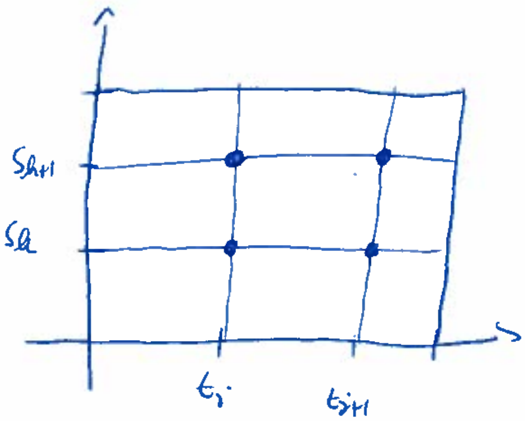
$\Delta(z, \varepsilon) \subset D$ $\forall z \in K$, Γ unif. cont. $\Rightarrow \exists\, \delta > 0$ s.t. if $(s-s')^2 + (t-t')^2 < \delta^2$

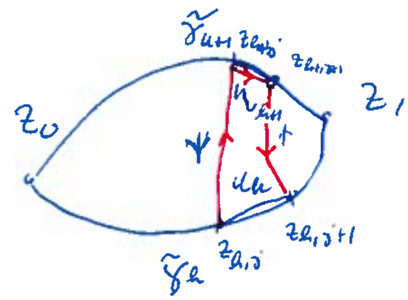
then $|\Gamma(s,t) - \Gamma(s',t')| < \varepsilon$. Fix n s.t. $\frac{\sqrt{2}}{n} < \delta$.

$t_j = \frac{j}{n}$, $s_k = \frac{k}{n}$ $j, k = 0, \dots, n$. and let $z_{kj} = \Gamma(t_j, s_k)$

$\tilde{\gamma}_k(t) = \Gamma(t, \frac{k}{n})$, $\eta_k =$ polygonal path inscribed in $\tilde{\gamma}_k$ joining z_{kj} to $z_{k(j+1)}$

let





02.

$z_{kj},\ z_{k+1,j},\ z_{k,j+1},\ z_{k+1,j+1} \in \Delta(z_{kj}, \varepsilon) \subset D$.

ω closed in $\Delta(z_{kj}, \varepsilon) \subset D \Rightarrow \displaystyle\int_{z_{kj}}^{z_{k,j+1}} \omega = \int \omega = \int_{z_{kj}}^{z_{k+1,j}} \omega + \int_{z_{k+1,j}}^{z_{k+1,j+1}} \omega + \int_{z_{k+1,j+1}}^{z_{k,j+1}} \omega$

Sum $\displaystyle\sum_{j=0}^{n-1}$ and get $\displaystyle\int_{\eta_k} \omega = \int_{\eta_{k+1}} \omega \Rightarrow \int_{\eta_0} \omega = \int_{\eta_1} \omega$

by defn!

let $n \to \infty$: $\eta_0 \xrightarrow{\text{unif}} \gamma_0$, $\eta_1 \xrightarrow{\text{unif}} \gamma_1$ $L(\eta_0) \le L(\gamma_0)$

$L(\eta_1) \le L(\gamma_1)$

$\Rightarrow \displaystyle\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$

3.3. Harmonic conjugates D domain, $u \in C^2(D)$, $\Delta u = 0$

v harmonic conjugate of u in D if v is harmonic and $u + iv$ is holom. in D.

Lemma $u \in C^2(D)$, $\omega = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \in C^1(D)$. 02/07

1) u harmonic in D $\Leftrightarrow$ ω closed in D.

2) u has harmonic conjugate v in D $\Leftrightarrow$ $\omega = dv$ ($\Leftrightarrow \omega$ is exact)

Pf: 1) ω closed $\Leftrightarrow$ $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \Leftrightarrow \Delta u = 0$

2) $\exists\, v$ s.t. $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Leftrightarrow \omega = dv$. $\square$

Theorem If u is harmonic on D simply connected, then u has a harmonic conjugate v in D, given by

$$v(z) = \int_{z_0}^{z} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

where the integral is over any rectifiable path in D joining z_0 to z.

Pf: u harmonic $\Rightarrow -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ is closed $\overset{D \text{ simply connected}}{\Rightarrow}$ $\underbrace{-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy}_{= dv}$ is exact.

Lemma $\Rightarrow$ v is harmonic conjugate of u.

$\int_\gamma -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ path independent in D, $v(z)$ as is in statement $\blacksquare$

Application (HW#4) $D = \{a < |z| < b\}$

$\gamma_r = re^{it}$, $0 \le t \le 2\pi$, $a < r < b$

u has harmonic conjugate in D if and only if $\int_{\gamma_r} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 0$

Pf of Application: u has harmonic conjugate in D $\Leftrightarrow$

the form $-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ is exact in D

$(\Leftrightarrow)$ $\displaystyle\int_\gamma \underbrace{-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy}_{w} = 0$ $\forall$ closed rectifiable path γ in D.

alg. top $\Rightarrow$ γ is homotopic in D to $n\gamma_r$, where $n \in \mathbb{Z}$

so $\displaystyle\int_\gamma w = n\int_{\gamma_r} w$ $\square$

3.4. Mean value Property (MVP)

$h: \Delta(z_0, \rho) \to \mathbb{R}$ ($\mathbb{C}$) continuous. Integral average on the circle of radius r

$$A(r) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{it})\, dt, \quad 0 \le r < \rho$$

Theorem

1) $\lim_{r\to 0} A(r) = h(z_0)$

2) If $h \in C^2(\Delta(z_0,\rho))$ then $r\frac{d}{dr}A(r) = \frac{1}{2\pi}\iint_{|z-z_0|\le r} \Delta h \, dx dy$

Corollary If h is harmonic on $\Delta(z_0,\rho)$, then $h(z_0) = A(r)$ $\forall\ 0 \le r < \rho$.

Pf: $\frac{d}{dr}A(r) = 0$, $A(r) \equiv c$ $\forall_{0 \le r < \rho} \Rightarrow \lim_{r\to 0} A(r) = h(z_0) = c\ \forall\ 0 \le r < \rho$.

Proof of Theorem 1) $\left|A(r)-h(z_0)\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left| h(z_0+re^{it}) - h(z_0)\right| dt$

2) $r\frac{d}{dr}A(r) = r\frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dr} h(\underbrace{z_0+re^{it}}_{=(x_0+r\cos t,\, y_0+r\sin t)})\, dt$

$$= \frac{r}{2\pi}\int_0^{2\pi} \frac{\partial h}{\partial x}(\underbrace{x_0+r\cos t}_{x}, \underbrace{y_0+r\sin t}_{y})\cos t + \frac{\partial h}{\partial y}(x_0+r\cos t, y_0+r\sin t)\sin t\, dt$$

$$= \frac{1}{2\pi}\int_{|z-z_0|=r} -\frac{\partial h}{\partial y}dx + \frac{\partial h}{\partial x}dy \overset{\text{Green}}{=} \frac{1}{2\pi}\int_{|z-z_0|\le r}\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)dx\,dy$$

Def $h: D\to\mathbb{R}\ (\mathbb{C})$ continuous on D has the MVP on D if $\forall z_0\in D$ $\exists\ \Delta(z_0,\rho)\subset D$ s.t. $h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0+re^{it})dt$, $0\le r<\rho$

Cor: h harmonic on $D \Rightarrow h$ has MVP on D.

f holom and C^2 on $D \Rightarrow f$ has MVP on D.

3.5 Maximum Principle

Strict maximum principle D domain in $\mathbb{C}$

1) If $u: D\to\mathbb{R}$ is harmonic in D, $u\le M$ and $u(z_0)=M$ for some $z_0\in D$. Then $u(z)\equiv M$ $\forall z\in D$.

2) If $h : D \to \mathbb{C}$ is harmonic in D, $|h| \le M$ on D and

$|h(z_0)| = M \Rightarrow h(z) \equiv c$ is constant ($|c| = M$).

(Maximum modulus principle)

<u>Proof.</u> 1) $S = \{z \in D : u(z) = M\} \ni z_0$, S open and closed $\Rightarrow S = D$.

$S = u^{-1}(\{M\})$ closed in D ✓.

Suppose $z_1 \in S$, let $\Delta(z_1, \rho) \subset D$ (any)

$$u(z_1) = M = \frac{1}{2\pi}\int_0^{2\pi} u(z_1 + re^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} M\,dt = M.$$

$\Rightarrow u(z_1 + re^{it}) = M \ \forall\ 0 \le t \le 2\pi,\ 0 \le r < \rho \Rightarrow \Delta(z_1, \rho) \subset S$, so S is open ☐

2) $h(z_0) = Me^{i\theta}$ some θ. , $\tilde{h} = e^{-i\theta}h = u + iv$, u, v harmonic.

$u(z_0) = \tilde{h}(z_0) = M.$, $u \le |\tilde{h}| = |h| \le M$.

$|\tilde{h}| = \sqrt{M^2 + v^2} \le M \Rightarrow v \equiv 0$, so $\tilde{h} \equiv M \Rightarrow h \equiv e^{i\theta}M$

$\overset{1)}{\Rightarrow} u \equiv M$ on D.

<u>Max Principle</u> D <u>bounded domain</u>, $h : D \to \mathbb{C}\ (\mathbb{R})$ <u>harmonic on D</u>, cont. on $\overline{D}$. If $|h| \le M$ on ∂D, then $|h| \le M$ on D. , i.e.

$$\max_{\overline{D}} |h| = \max_{\partial D} |h|.$$

Pf: $M' = \max_{z \in \bar{D}} |h(z)| = h(z_0)$. Some $z_0 \in \bar{D}$. 02/09

If $z_0 \in D \Rightarrow h \equiv M' e^{i\theta}$ on D, hence on $\bar{D} \Rightarrow M' \leq M$.

If $z_0 \in \partial D \Rightarrow M' \leq M$.

Remark! Both theorems apply to holom. functions of class C^2 (they are harmonic

Cor. D bounded domain, $u: \bar{D} \to \mathbb{R}$ cont. on $\bar{D}$, u verifies MVP on D.

If $u \leq M$ on ∂D then $u \leq M$ on D.

Ch. 4 4.1 Complex line integrals

$\gamma: [0,1] \to \mathbb{C}$ rectifiable, $f: \gamma([0,1]) \to \mathbb{C}$ continuous.

$$\int_\gamma f(z)dz = \int_\gamma f\,dx + i f\,dy.$$ If γ piecewise smooth, then

$$dz = \gamma'(t)dt = (x'(t) + i y'(t))dt, \quad \gamma = x + iy$$

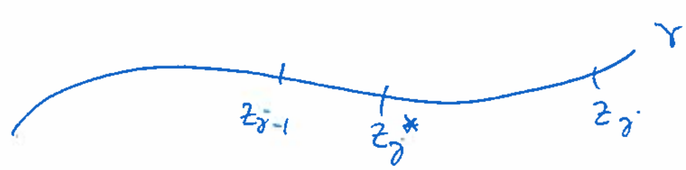
$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

Let $\pi = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ be a partition of $[0,1]$ and

$t_j^* \in [t_{j-1}, t_j]$. $$\int_\gamma f(z)dz = \lim_{|\pi| \to 0} \sum_{j=1}^n f(z_j^*)(z_j - z_{j-1})$$

$z_j = \gamma(t_j)$

$z_j^* = \gamma(t_j^*)$



d2/0

__Ex 1:__ $\gamma(t) = (1+i)t \qquad 0 \le t \le 1$

$$\int_\gamma z^2 dz = \int_0^1 (1+i)^2 t^2 (1+i) dt = (1+i)^3 \frac{t^3}{3}\Big|_0^1 = \frac{(1+i)^3}{3}.$$

__Ex 2:__ $m \in \mathbb{Z}$

$$\int_{\substack{|z-z_0|=R \\ \text{(counter clockwise)}}} (z-z_0)^m dz = \begin{cases} 0, & m \neq -1 \\ 2\pi i, & m = -1 \end{cases}$$

$z = z_0 + Re^{it}, \; 0 \le t \le 2\pi \qquad dz = iRe^{it} dt$

$$\int_{|z-z_0|=R} (z-z_0)^m dz = \int_0^{2\pi} R^m e^{mit} \, iRe^{it} dt = iR^{m+1} \int_0^{2\pi} e^{(m+1)it} dt$$

$$= \begin{cases} iR^{m+1} \frac{1}{m+1} e^{(m+1)it}\Big|_0^{2\pi}, & m \neq -1 \\ 2\pi i, & m = -1 \end{cases}$$

~~(is [illegible])~~

__Integral wrt. arc length__ $\gamma : [0,1] \to \mathbb{C}$ rectifiable, $f : \gamma([0,1]) \to \mathbb{C}$ cont.

$$\int_\gamma f(z)\, ds \overset{\text{notation}}{=:} \int_\gamma f(z) |dz| = \lim_{|\pi| \to 0} \sum_{j=1}^n f(z_j^*) |z_j - z_{j-1}|$$

In particular $\int_\gamma |dz| = L(\gamma)$.

If $\gamma(t) = x(t) + i\,y(t)$ piecewise smooth

$\Rightarrow$ $ds = |dz| = \sqrt{x'(t)^2 + y'(t)^2}\,dt = |\gamma'(t)|\,dt$ justifies $|dz|$

$$\int_\gamma f(z)\,|dz| = \int_0^1 f(\gamma(t))\,|\gamma'(t)|\,dt$$

Theorem Suppose $\gamma : [0,1] \to \mathbb{C}$ rectifiable, $f : \gamma([0,1]) \to \mathbb{C}$ cont., $|f| \le M$ on $\gamma([0,1])$. Then:

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \underset{\text{trivial}}{\le} M \cdot L(\gamma)$$

Pf:

$$\left|\sum_{j=1}^{n} f(z_j^*)\,(z_j - z_{j-1})\right| \le \sum_{j=1}^{n} |f(z_j^*)| \cdot |z_j - z_{j-1}| \text{ and } \lim_{|\pi| \to 0}. \quad \square$$

Equality 1) $\gamma(t) = (1+i)\,t$, $0 \le t \le 1$

$f(z) = z^2$ $\quad \left|\int_\gamma f(z)\,dz\right| = \left|\frac{(1+i)^3}{3}\right| = \frac{2\sqrt{2}}{3}$

and $\quad \int_\gamma |f(z)|\,|dz| = \int_0^1 t^2 (\sqrt{2})^2 \sqrt{2}\,dt = \frac{2\sqrt{2}}{3}$.

$z = (1+i)t$

$|z| = \sqrt{2}\,t$

$|dz| = \sqrt{2}\,dt$

$M = \max\limits_{0 \le t \le 1} |\gamma(t)|^2 = 2$, $L(\gamma) = \sqrt{2}$

2) $\displaystyle\int_{|z-z_0|=R} \frac{1}{z-z_0}\,dz = 2\pi i$, $\quad M = \frac{1}{R} = \frac{1}{|z-z_0|}$, $\quad L(\gamma) = 2\pi R$

$$\left|\int_{|z-z_0|=R} \frac{1}{z-z_0}\,dz\right| = 2\pi = \frac{1}{R}\,2\pi R.$$

harmonic conjugate $u \to \int_\gamma -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ 02/09

$$f \longrightarrow \int_\gamma f\,dx + i f\,dy = \int_\gamma f(z)\,dz.$$

4.2 Fundamental Theorem of Calc. for holom. functions

Def: A continuous $f: D \to \mathbb{C}$ has ~~a primitive in D~~
domain

a primitive in D if $\exists$ F holom. in D and $F' = f$.

(If $G' = F' = f \Rightarrow G = F + \text{const}$ (D connected)).

Theorem 1) If f has a primitive F on D, then $\omega = f\,dx + i f\,dy$ (02/11)
is exact and $\omega = dF$. If γ is a rectifiable path
joining z_0 to z_1 in D, then $\int_\gamma f(z)\,dz = F(z_1) - F(z_0)$.

(fundamental thm. of calculus)

2) If $\omega = dF$ is exact, then F is a primitive of f.

We have $F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$ along any rectifiable path from z_0 to z

Pf: 1) F holom, $F' = f \Rightarrow \frac{\partial F}{\partial y} = i\frac{\partial F}{\partial x}$ (complex CR-equations)

$f = F' = \frac{\partial F}{\partial x}$, $\omega = f\,dx + i f\,dy = \frac{\partial F}{\partial x}dx + i\frac{\partial F}{\partial x}dy$

$= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF.$

$$\int_\gamma f(z)\,dz = \int_\gamma dF = F(z_1)-F(z_0).$$

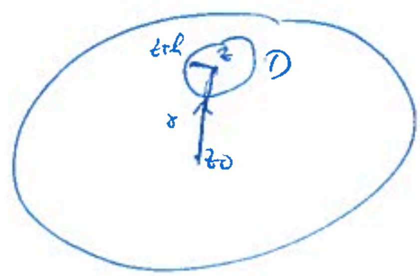
(If γ piecewise C^1: $\int_\gamma f(z)dz = \int_0^1 F'(\gamma(t))\gamma'(t)dt$

$\overset{\text{chain rule}}{=} \int_0^1 \frac{d}{dt}(F\circ\gamma)(t)\,dt \overset{\text{FTC}}{=} F(\gamma(1)) - F(\gamma(0)) = F(z_1)-F(z_0)$)

2) $w = f dx + i f dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \Rightarrow f = \frac{\partial F}{\partial x},\ if = \frac{\partial F}{\partial y}$

$\Rightarrow \frac{\partial F}{\partial y} = i\frac{\partial F}{\partial x}$, F holom. and $F' = \frac{\partial F}{\partial x} = f$. so F is a primitive

[Direct: $\frac{d}{dz}\int_{z_0}^{z} \underbrace{f(\zeta)d\zeta}_{\text{exact}} = f(z)$]



$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} f(\zeta)d\zeta - f(z) \to 0.$

Cor. $f: D\to\mathbb{C}$ has a primitive on D if and only if

$\int_\gamma f(z)dz = 0$ along any closed path (rectifiable!) in D.

Ex: $f(z) = \frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$

$\int_{|z-0|=1} f(z)dz = 2\pi i \neq 0$. does not have a primitive

But $\log z$ is a primitive of f on $D = \mathbb{C}\setminus(-\infty,0]$.

<u>Theorem</u> If $f \in \mathcal{C}^1(D)$, then $\omega = f(z)\,dz$ is closed in D if and only if f is holomorphic.

<u>Pf:</u> $\omega = f\,dx + i f\,dy$ closed $\Leftrightarrow i \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ (complex CR) $\Leftrightarrow f$ holom. in D. $\square$

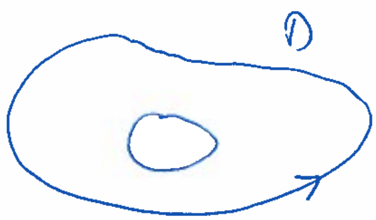
<u>Theorem</u> If f is holom. in a simply connected domain D, then f has a primitive F in D given by $F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$. $\square$

<u>Pf:</u> Follows from previous 2 Theorems. $\omega = f\,dz$ closed $\xrightarrow[\text{connected}]{\text{Simply}}$ exact.

4.3. Cauchy's Theorem

" f holom., γ closed path $\int_\gamma f(z)\,dz = 0$ with hypothesis".

<u>Cauchy's Theorem</u> If D is a bounded domain with piecewise smooth boundary and f is holom. on D, and of class $\mathcal{C}^1$ in a nbhd of the closure $\bar{D}$.

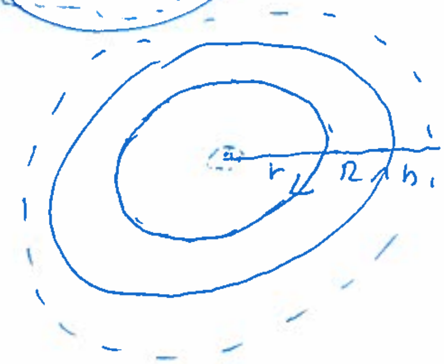


$$\int_{\partial D} f(z)\,dz = 0.$$

<u>Pf</u> $$\int_{\partial D} \overset{P}{f\,dx} + \overset{Q}{i f\,dy} \overset{\text{Green}}{=} \iint_D \Big(\underbrace{i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}}_{=0 \text{ CR}}\Big)\,dx\,dy = 0.$$

Ex: Suppose f holom. in $A = \{a < |z| < b\}$ and $a < r < R < b$ 02/12

Then $\int_{|z|=r} f(z)dz = \int_{|z|=R} f(z)dz$



$D = \{r < |z| < R\}$

Cauchy Thm

$$0 = \int_{\partial D} f(z)dz = \int_{|z|=R} f(z)dz - \int_{|z|=r} f(z)dz$$

Cauchy Thm In simply connected domains

If f holom on a simply connected domain D, then $\int_\gamma f(z)dz = 0$ for any closed rectifiable path γ in D.

Pf: f holom on $D \Rightarrow \exists F$ holom on D, $F' = f$

$$\Rightarrow \int_\gamma f(z)dz = \int_\gamma F'(z)dz = F(\gamma(1)) - F(\gamma(0)) = 0.$$

Homotopic version of Cauchy's Theorem

1) If f hol. on D (domain) and γ_0, γ_1 are homotopic (rectifiable) paths in D joining $z_0 \in D$ and $z_1 \in D$. Then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

2) If f holom. in D and γ_0, γ_1 are $\underline{\underline{\text{closed}}}$ rectifiable homotopic paths in D joining $z_0 \in D$ and $z_1 \in D$. Then $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.

In particular if γ is in addition homotopic to 0 in D, then $\int_\gamma f(z)dz =$
("homological" to 0 is enough)

$\square$

<u>Pf:</u> f holom. $\rightarrow$ $\omega = f(z)dz$ closed + Chapter 3.

4.4. Cauchy's Integral Formula

<u>Theorem</u> $\gamma : [0,1] \to \mathbb{C}$ rectifiable, $K = \gamma([0,1])$

$h : D \times K \to \mathbb{C}$, D open s.t.

1) $h(z, \cdot)$ is cont. in K, $\forall z \in D$

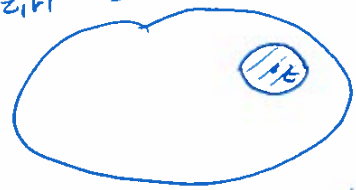
2) $h(\cdot, \zeta)$ is holom. on D $\forall \zeta$ in K.

3) $\dfrac{\partial h}{\partial z}(z, \zeta)$ is cont. on $D \times K$.

Then: $H(z) = \int_\gamma h(z,\zeta)\,d\zeta$ is holom. on D and

$$H'(z) = \int_\gamma \frac{\partial h}{\partial z}(z,\zeta)\,d\zeta$$

Proof: $z \in D$, $\overline{\Delta(z,r)} \subset D$ $\qquad \frac{\partial h}{\partial z}$ unif. cont. on $\overline{\Delta(z,r)} \times K$



Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $|w| < \delta$ then

$$\left| \frac{\partial h}{\partial z}(z+w, \zeta) - \frac{\partial h}{\partial z}(z, \zeta) \right| < \varepsilon \quad \forall \zeta \in K$$

$$\left| \frac{H(z+w) - H(z)}{w} - \int_\gamma \frac{\partial h}{\partial z}(z,\zeta)\, d\zeta \right| = \left| \int_\gamma \frac{1}{w}\Big(h(z+w,\zeta) - h(z,\zeta)\Big) - \frac{\partial h}{\partial z}(z,\zeta)\, d\zeta \right|$$

$$\leq \int_\gamma \left| \frac{1}{w}\Big(h(z+w,\zeta) - h(z,\zeta)\Big) - \frac{\partial h}{\partial z}(z,\zeta) \right| d\zeta$$

FTC

$$= \int_\gamma \left| \frac{1}{w} \int_{[z,z+w]} \frac{\partial h}{\partial z}(\xi,\zeta)\, d\xi - \frac{\partial h}{\partial z}(z,\zeta) \right| d\zeta$$

$$= \int_\gamma \left| \frac{1}{w} \int_0^1 \frac{\partial h}{\partial z}(z+tw,\zeta)\, w\, dt - \frac{\partial h}{\partial z}(z,\zeta) \right| d\zeta$$

$$\leq \int_\gamma \int_0^1 \underbrace{\left| \frac{\partial h}{\partial z}(z+tw,\zeta) - \frac{\partial h}{\partial z}(z,\zeta) \right|}_{<\varepsilon} dt\, d\zeta \leq \varepsilon L(\gamma) \quad \text{if } |w| < \delta \ \forall \zeta \in K.$$

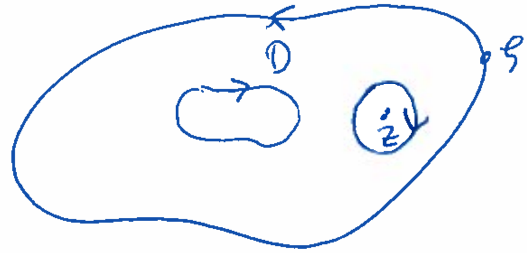
$\Rightarrow \exists\, H'(z) = \int_\gamma \underbrace{\frac{\partial h}{\partial z}(z,\zeta)}_{\text{cont on } D\times K} d\zeta$ is continuous on D $\qquad \cancel{\square}$

$\Rightarrow H$ holom. $\qquad \square$

Theorem (Cauchy Integral Formula) Let D bounded domain with piecewise smooth boundary, f holom. in D and of class $\mathcal{C}^1$ in a nbd of $\overline{D}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \forall z \in D$$

$\uparrow$ pos. oriented



Proof: Pick $\varepsilon > 0$ small s.t. $\overline{\Delta(z,\varepsilon)} \subset D$

$D_\varepsilon = D \setminus \overline{\Delta(z,\varepsilon)}$ has piecewise smooth boundary

Now $\zeta \mapsto \dfrac{f(\zeta)}{\zeta - z}$ holom. on D_ε and of class $\mathcal{C}^1(\overline{D_\varepsilon})$

Cauchy's Theorem $\quad 0 = \displaystyle\int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{|\zeta - z| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{|\zeta - z| = \varepsilon} \frac{f(\zeta)}{z - \zeta}\, d\zeta \qquad \zeta = z + \varepsilon e^{it},\ 0 \le t \le 2\pi$$

$$d\zeta = \varepsilon i e^{it} dt$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}}\, \varepsilon i e^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{it})\, dt \longrightarrow f(z), \quad \varepsilon \to 0$$

Cor: D, f as above. Then f has derivatives of all orders and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \forall n \ge 0.$$

Pf: Induction on m: $m=0$ ✓

Assume $f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta-z)^{m+1}} d\zeta$

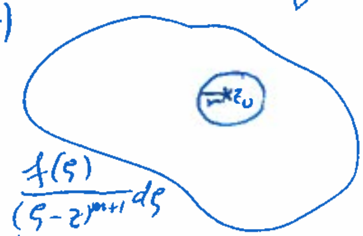
$h(z,\zeta) = \frac{f(\zeta)}{2\pi (\zeta-z)^{m+1}}$ on $\underset{z}{\underset{\Psi}{D}} \times \underset{\zeta}{\underset{\Psi}{\partial D}}$

$\frac{\partial h}{\partial z}(z,\zeta) = (m+1) \frac{f(\zeta)}{(\zeta-z)^{m+2}}$ cont. on $D \times \partial D$.

$\Rightarrow f^{(m)}$ holom. $\quad f^{(m+1)}(z) = \frac{(m+1)!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta-z)^{m+2}} d\zeta$

Cor: If f is holom. on an open set D, then f has complex derivatives of all orders.

D



Pf: Fix any $\overline{\Delta}(z_0,r) \subset D$, $z \in \Delta(z_0,r)$

$\overset{CIF}{\Rightarrow} f(z) = \frac{1}{2\pi i} \int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta-z} d\zeta \Rightarrow f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta-z)^{m+1}} d\zeta$

for all $m \geq 0$ by the previous Corollary.

Cor: If $f = u+iv$ is holom. in an open set D, then u,v are harmonic on D.

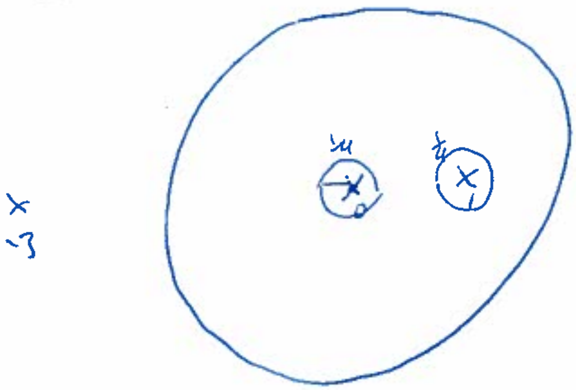
Pf: $f' = u_x + iv_x = v_y - iu_y$ is holom on D

$\Rightarrow f' \in \mathcal{C}^1(D) \Rightarrow u,v \in \mathcal{C}^2(D)$ so they are harmonic

□

Thus mean value property, maximum modulus property holds for any holomorphic function. 02/14

$$I = \int_{|z|=2} \frac{1}{z^3(z-1)(z-3)}\, dz$$



$$\overset{CT}{=} \int_{|z|=\frac14} \frac{dz}{z^3(z-1)(z+3)} dz + \int_{|z-1|=\frac14} \frac{dz}{z^3(z-1)(z+3)}$$

$$= \int_{|z|=\frac14} \frac{\frac{1}{(z-1)(z+3)}}{z^3}\, dz + \int_{|z-1|=\frac14} \frac{\frac{1}{z^3(z+3)}}{z-1}$$

$$= \frac{2\pi i}{2!} \underbrace{\left(\frac{d}{dz}\right)^2 \frac{1}{(z-1)(z+3)}\Big|_{z=0}}_{=-\frac{14}{27}} + 2\pi i \cdot \underbrace{\frac{1}{z^3(z+3)}\Big|_{z=1}}_{=\frac14}.$$

$= \cancel{2\pi i}$

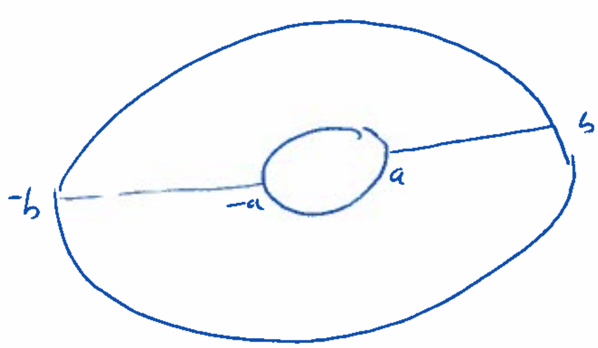
02/16

Comment on HW #3 III.3.#4

Hint from book

$A_- = A \setminus [-b,-a]$ simply connected
$\Rightarrow v_1$ harmonic conj. of u on A_-

$A_+ = A \setminus [a,b]$ simply connected.
$\Rightarrow v_2$ harmonic conj of u in A_+



$v_1 = v_2 + C_1$ in $A \cap \{\mathrm{Im}\, z > 0\}$

$v_1 = v_2 + C_2$ in $A \cap \{\mathrm{Im}\, z < 0\}$

$$\lim_{\substack{z \to x \\ \mathrm{Im}\, z > 0}} v_1(z) = v_2(x) + C_1 \qquad \lim_{\substack{z \to x \\ \mathrm{Im}\, z < 0}} v_1(z) = v_2(x) + C_2$$

$$\lim_{\substack{z \to x \\ \mathrm{Im}\, z > 0}} \left(v_1(z) - C\,\mathrm{Arg}\, z\right) = v_2(x) + C_1 - C\pi$$

$$\lim_{\substack{z \to x \\ \mathrm{Im}\, z < 0}} \left(v_1(z) - C\,\mathrm{Arg}\, z\right) = v_2(x) + C_2 + C\pi$$

$C_1 - C\pi = C_2 + C\pi$, $C = \frac{C_1 - C_2}{2\pi} \in \mathbb{R}$ ($C_i \in \mathbb{R}$)

$\exists C \in \mathbb{R}$ s.t. $v(z) := v_1(z) - C\,\mathrm{Arg}\, z$ is harmonic in $A \setminus [-b,-a]$

and v is cont. in A.

$\mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg}\, z$ hol. on $\mathbb{C} \setminus (-\infty, 0]$

$f(z) = u(z) - C\log|z| + i v(z)$

$= \underbrace{u(z) + i v_1(z)}_{\text{holom.}} - C\,\mathrm{Log}\, z$ hol. on $A \setminus [-b,-a]$ is cont. on A.

Moreover $\Rightarrow$ f is holom. on A.

$u+iv$ ~~[illegible]~~ holom. 02/16

$\Rightarrow$ CR in polar coords. $\quad r\frac{\partial}{\partial r}\left(u(z)-\underbrace{C\log|z|}_{=\log r}\right) = \frac{\partial}{\partial\theta}v(z), \quad z = re^{i\theta}$

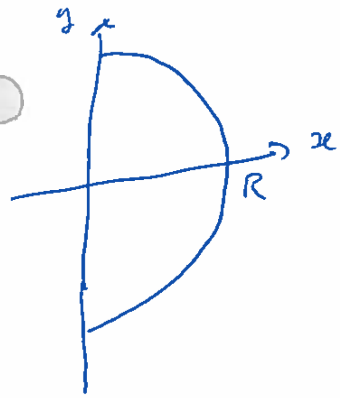
integrate $\Rightarrow$ $\int_0^{2\pi}\left(r\frac{\partial u}{\partial r}(re^{i\theta}) - C\right) d\theta = \int_0^{2\pi}\frac{\partial}{\partial\theta}v(re^{i\theta})\,d\theta = v(re^{i\theta})\Big|_{\theta=0}^{\theta=2\pi}$

$\Rightarrow C\,2\pi = \int_0^{2\pi} r\frac{\partial u}{\partial r}(re^{i\theta})\,d\theta$

III. ~~5~~ #5 f holom on $D=\{\operatorname{Re} z>0\}$ cont. on $\overline{D} = \{\operatorname{Re} z \geq 0\}$

If $|f(z)|\leq N$, $z\in D$, $|f(iy)|\leq \underline{\underline{M}}$, $y\in\mathbb{R}$ $\Rightarrow$ $|f(z)|\leq M$ on D.

$f_\varepsilon(z) = \frac{1}{1+\varepsilon z}, \ \varepsilon>0 \quad \left(\text{or } \frac{1}{(1+z)^\varepsilon}\right)$



hol. in D, cont on $\overline{D}$

$|f_\varepsilon(iy)| = \frac{1}{|1+\varepsilon iy|} = \frac{1}{\sqrt{1+\varepsilon^2y^2}} \leq 1$

$|f_\varepsilon(z)| \leq \frac{1}{\varepsilon|z|-1} = \frac{1}{\varepsilon R-1}, \quad R>\frac{1}{\varepsilon}$

$f_\varepsilon(z)\to 1$ as $\varepsilon\to 0$ for all $z\in D$.

Let $\varepsilon>0$, any $R>\frac{1}{\varepsilon}$. $|(f_\varepsilon f)(iy)|\leq M \quad \forall y\in\mathbb{R}$.

$\underset{|z|=R}{|(f_\varepsilon f)(z)|} \leq \frac{N}{\varepsilon R-1} < M$ if $R>R_\varepsilon$. $\overset{\text{max principle}}{\Rightarrow}$ $|f_\varepsilon f|\leq M$ if $z\in D, |z|$

$\overset{R\to\infty}{\Rightarrow}$ $|f_\varepsilon f|\leq M$ on D $\overset{\varepsilon\to 0}{\Rightarrow}$ $|f|\leq M$ on D.

4.5 Liouville's Theorem

Cauchy's Estimates If f is holom. in a nbd. of $\bar{\Delta}(z_0, \rho)$ and $|f(z)| \le M$ for $|z-z_0| = \rho$ then

$$\left| f^{(m)}(z_0) \right| \le \frac{m!\, M}{\rho^m}, \quad m \ge 0.$$

$$\left| f^{(m)}(z_0) \right| = \left| \frac{m!}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{m+1}} dz \right| = \frac{m!}{2\pi} \frac{M}{\rho^{m+1}} \cdot 2\pi\rho. \quad \square$$

Liouville's Thm If f is ~~entire~~ an entire function (holom on $\mathbb{C}$) and it is bounded, then f is constant.

Pf: Let $|f(z)| \le M$ $\forall z \in \mathbb{C}$. Fix $R > 0$ ~~arbitrary~~

Cauchy's estimate on $\bar{\Delta}(z, \rho)$, $m = 1$

$$|f'(z)| \le \frac{1!\, M}{\rho} \to 0, \ \rho \to \infty. \Rightarrow f' \equiv 0. \quad \square$$

Fundamental Theorem of Algebra If $P(z)$ is a polynomial of degree n, $a_i \in \mathbb{C}$

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $n \ge 1$, $a_n \ne 0$. Then $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$. (hence n roots)

Pf: $|P(z)| \ge |a_n| |z|^n - |a_{n-1} z^{n-1} + \dots + a_0|$

$$\ge |a_n| |z|^n - \left(|a_{n-1}| |z^{n-1}| + \dots + |a_0| \right)$$

$$= |z|^n \left(|a_n| - \underbrace{\left(\frac{|a_{n-1}|}{|z|} + \frac{|a_{n-1}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n} \right)}_{< \frac{|a_n|}{2} \text{ if } |z| \geq R} \right)$$

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So $|P(z)| \geq \frac{|a_n|}{2} |z|^n$, $|z| \geq R$.

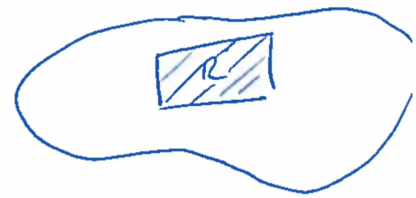
Suppose P has no zeroes, then $\frac{1}{P(z)}$ is an entire function. Let $f(z) = \frac{1}{P(z)}$ $z \in \mathbb{C}$

and $|f(z)| \leq \frac{2}{|a_n| R^n}$ if $|z| \geq R$. and f is cont. on $\{|z| \leq R$

thus bounded. Liouville $\Rightarrow f(z) \equiv C \Rightarrow P(z) \equiv \frac{1}{C}$, a contradiction

02/14

4.6 Morera's Theorem



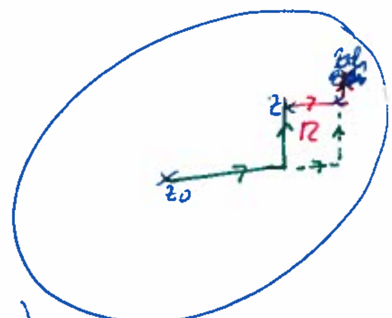
Recall f cont. on D, $\int_\gamma f\,dz = 0$ $\forall$ closed rectifiable path γ in D.

$\Rightarrow \exists F$ holom on D : $F' = f$. $\overset{F\in C^\infty}{\Rightarrow}$ f holom. on D.

Morera's Thm If f cont. on D and $\int_{\partial R} f\,dz = 0$ $\forall$ rectangle $R \subseteq D$ parallel to the coordinate axes. Then f is holom.

Pf: Pick any disk $\Delta(z_0, r) \subset D$.

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta \quad \text{on the path shown}$$



$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \cancel{\oint \frac{1}{h}\int}$$

$$= \left|\frac{1}{h}\left(\int_{[z, z+\operatorname{Re} h]} f(\zeta)\,d\zeta + \int_{[z+\operatorname{Re} h,\, z+h]} f(\zeta)\,d\zeta + F(z) - F(z)\right) - f(z)\right|$$

$= \gamma_h$

$$= \left|\frac{1}{h}\int_{\gamma_h} f(\zeta)\,d\zeta - f(z)\right| = \left|\frac{1}{h}\int_{\gamma_h}(f(\zeta)-f(z))\,d\zeta\right| \le \frac{1}{|h|} \varepsilon L(\gamma_h) \le$$

For $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\overline{\Delta}(z,\delta) \subset \Delta(z_0, r)$: If $|\zeta - z| < \delta$ then $|f(\zeta) - f(z)| <$

If $|h| < \delta \Rightarrow |f(\zeta) - f(z)| < \varepsilon$ $\forall \zeta$ on γ_h.

$\Rightarrow F' = f$ on $\Delta(z_0, r)$ f holom. on $\Delta(z_0, r)$, hence on D.

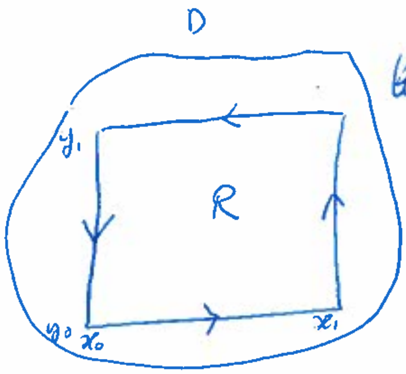
Cor: Let $h : D \times [a,b] \to \mathbb{C}$ be continuous and $\subset \Omega$ 02/19

$h(\cdot, t)$ holom. on D $\forall t \in [a,b]$. Then

$$H(z) = \int_a^b h(z,t)\,dt \text{ is holom. on } D.$$

H cont. on D. ✓

Pf: ~~...~~ Let $R \subset D$



$$\int_{\partial R} H(z)\,dz = \int_{x_0}^{x_1} \int_a^b \cancel{h(z,t)}\, h(x+iy_0, t)\,dt\,dx$$

$$+ \int_{y_0}^{y_1} \int_a^b h(x_1+iy, t)\,dt\,dy + \int_{x_1}^{x_0} \cancel{\ast}\_\_\_ + \int$$

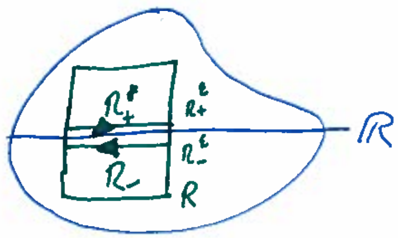
Fubini

$$= \int_a^b \int_{x_0}^{x_1} h(x+iy_0, t)\,dx\,dt + \int_a^b \int_{y_0}^{y_1} h(x_1+iy, t)\,dy\,dt + \int + \int.$$

$$= \int_a^b \underbrace{\int_{\partial R} h(z,t)\,dz}_{=0\ \forall t.} dt = 0$$

as $h(\cdot, t)$ holom.

Cor. f is cont on D, holom. on $D \setminus \mathbb{R} \Rightarrow f$ holom. on D. 02/19



$R \subset D$ rectangle parallel to coord. axes.

$R^+ = R \cap \{\operatorname{Im} z \geq 0\}$

$R_- = R \cap \{\operatorname{Im} z \leq 0\}$

$\varepsilon > 0$: $R_+^\varepsilon = R \cap \{\operatorname{Im} z \geq \varepsilon\}$, $R_-^\varepsilon = R \cap \{\operatorname{Im} z \leq -\varepsilon\}$

C.T: $\displaystyle\int_{\partial R_+^\varepsilon} f\,dz = 0 \overset{\varepsilon \to 0}{\Longrightarrow} \int_{\partial R_+} f(z)\,dz = \lim_{\varepsilon \to 0} \int_{\partial R_+^\varepsilon} f(z)\,dz = 0$

$\displaystyle\int_{\partial R_-^\varepsilon} f\,dz = 0 \Rightarrow \int_{\partial R_-} f(z)\,dz = 0.$

$\displaystyle\int_{\partial R} f(z)\,dz = \int_{\partial R_+} f(z)\,dz + \int_{\partial R_-} f(z)\,dz = 0. \overset{\text{Morera}}{\Longrightarrow} f$ holom. on D. □

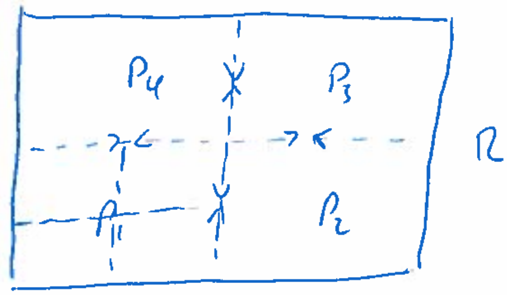
4.8. **Goursat's Theorem** If f is (complex) diff'ble on D, i.e.

$\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z)$ exists $\forall z_0 \in D \Rightarrow f$ is holomorphic on D.

Pf: Note that f is continuous on D. Take a rectangle $R \subset D$, parallel to axes.

$$\Big| \underbrace{\int_{\partial R} f(z)\,dz}_{I} \Big| \overset{?}{=} 0.$$

Let $\ell = L(\partial R)$, $d = \operatorname{diam} R$



$$I = \left| \int_{\partial P_1} f\,dz + \int_{\partial P_2} f\,dz + \int_{\partial P_3} f\,dz + \int_{\partial P_4} f\,dz \right|$$

$$\le \sum_{j=1}^{4} \left| \int_{\partial P_j} f\,dz \right| \qquad \exists j \text{ s.t. } \left| \int_{\partial P_j} f\,dz \right| \ge \frac{1}{4} I.$$

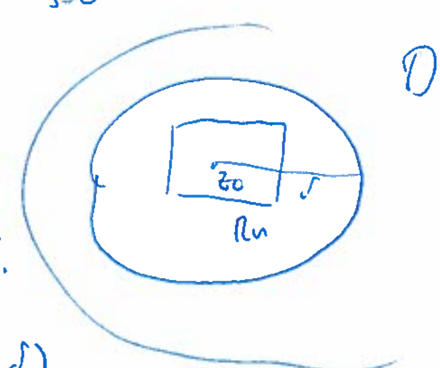
Let $R_1 = P_j$ s.t. $\left| \int_{\partial R_1} f\,dz \right| \ge \frac{1}{4} I$, $\underset{\ell_1 =}{\ell(\partial R_1)} = \frac{\ell}{2}$, $\underset{d_1 =}{\operatorname{diam}(R_1)} = \frac{d}{2}$

Induction: Construct a sequence of rectangles

$$R_n \subset R_{n-1}, \quad \ell(\partial R_n) = \ell_n = \frac{\ell}{2^n}, \quad \operatorname{diam}(R_n) = d_n = \frac{d}{2^n}.$$

and $\left| \int_{\partial R_n} f(z)\,dz \right| \ge \frac{1}{4} \left| \int_{\partial R_{n-1}} f(z)\,dz \right| \ge \frac{I}{4^n}$, $R_0 = R$.

Let $\{z_0\} = \bigcap_{n=0}^{\infty} R_n$ (unique point). Let $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|z - z_0| < \delta$, $z \neq z_0$



then $\left| \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$.

$\Rightarrow |f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon |z - z_0| \quad \forall |z - z_0| < \delta$.

Fix n s.t. $\frac{d}{2^n} = \operatorname{diam} R_n < \delta$. $\Rightarrow R_n \subset \Delta(z_0, \delta)$

02/19

$$\frac{1}{4^n} I \leq \left| \int_{\partial R_n} f\, dz \right| = \left| \int_{\partial R_n} (f(z) - \underbrace{f(z_0) - f'(z_0)(z-z_0)}_{\text{is holom.} \Rightarrow \int_{\partial R_n} = 0}) \, dz \right|$$

$$\leq \ell(\partial R_n) \cdot \varepsilon \cdot \operatorname{diam}(R_n) = \frac{\ell}{2^n}\, \varepsilon \cdot \frac{d}{2^n} = \frac{\varepsilon \ell d}{4^n} \Rightarrow \forall \varepsilon \;\; I \leq \varepsilon \ell d.$$

$\square$

$\overset{\varepsilon \to 0}{\Rightarrow} I = 0$ so f holom. by Morera's Theorem.

02/2

4.8 Complex notation, Pompeiu's Formula

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

Theorem $f \in \mathcal{C}^1(D)$, f holom on $D \iff \frac{\partial f}{\partial \bar z} = 0$ on D

f holom $\Rightarrow f' = \frac{\partial f}{\partial z}$.

Pf: f holom. on $D \underset{\text{complex CR.}}{\iff} \frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x} \iff \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) =$

$$\iff i \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x} \iff -\frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial x}$$

If f holom. $\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \underbrace{\frac{\partial f}{\partial y}}_{= i \frac{\partial f}{\partial x}}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\right) = f'(z).$

Differentiation rules
$f, g \in \mathcal{C}^1$

1) Linearity
2) Product and Quotient rules $\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z} g + f \frac{\partial g}{\partial z}$

and $\frac{\partial}{\partial z}\left(\frac{f}{g}\right) = \frac{\frac{\partial f}{\partial z} g - f \frac{\partial g}{\partial z}}{g^2}.$

3) Chain Rule $\xi = h(w)$, $w = g(z)$.

$$\frac{\partial}{\partial z}(h \circ g) = \frac{\partial h}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial h}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial z}$$

$$\frac{\partial}{\partial \bar{z}}(h \circ g) = \frac{\partial h}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}}$$

Like $z, \bar{z}$ are independent variables.

$$\left[\begin{array}{l} \frac{\partial}{\partial z} z = 1, \ \frac{\partial}{\partial \bar{z}} z = 0 \\ \frac{\partial}{\partial z} \bar{z} = 0, \ \frac{\partial}{\partial \bar{z}} \bar{z} = 1 \end{array}\right]$$

4) $$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}$$

$f = u + iv$. $$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)$$

So $$\overline{\frac{\partial f}{\partial z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} + i\underbrace{\left(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\right)}_{= \frac{\partial \bar{f}}{\partial y}}\right) = \frac{\partial \bar{f}}{\partial \bar{z}}.$$

Ex: $f(z) = 4ixy + e^{x^2+y^2}$ $\qquad x = \frac{z+\bar{z}}{2}, \ y = \frac{z-\bar{z}}{2i}$

$f(z) = z^2 - \bar{z}^2 + e^{z\bar{z}}$ $\qquad w = z\bar{z}$

$$\frac{\partial f}{\partial z} = 2z - 0 + e^{z\bar{z}} \cdot \bar{z} + 0 \cdot z \qquad \frac{\partial (e^w)}{\partial \bar{w}} = 0 \quad \text{holom}$$

$$\frac{\partial f}{\partial \bar{z}} = 0 - 2\bar{z} + e^{z\bar{z}} \cdot z$$

$f \in \mathcal{C}^1(D)$: holom $\Leftrightarrow \frac{\partial f}{\partial \bar z} = 0$ 02/21

Call $f \in \mathcal{C}^1(D)$ antiholom. if $\frac{\partial f}{\partial z} = 0$.

<u>Remark:</u> f antiholom. $\Leftrightarrow \bar f$ holom.

$$\left[\frac{\partial \bar f}{\partial \bar z} = \overline{\frac{\partial f}{\partial z}} = 0. \right].$$

Differential of $f \in \mathcal{C}^1(D)$ at $z_0 = 0$ ($z = x + iy$.

$$f(z) = f(0) + \frac{\partial f}{\partial x}(0)x + \frac{\partial f}{\partial y}(0)y + g(z), \quad \frac{g(z)}{z} \to 0 \text{ as } z \to 0.$$

$$df(0)(x,y) = \frac{\partial f}{\partial x}(0)x + \frac{\partial f}{\partial y}(0)y. \quad \mathbb{R} \text{ linear} \qquad \Leftrightarrow |g(z)| = o(|z|) \text{ as } z \to 0.$$

$$df(0) : \mathbb{C} \to \mathbb{C}, \quad df(0)(x,y) = \frac{\partial f}{\partial x}(0)\, \frac{z+\bar z}{2} + \frac{\partial f}{\partial y}(0)\, \frac{z - \bar z}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x}(0) - i \frac{\partial f}{\partial y}(0) \right) z + \frac{1}{2}\left(\frac{\partial f}{\partial x}(0) + i \frac{\partial f}{\partial y}(0) \right) \bar z$$

$$df(0)(z) = \frac{\partial f}{\partial z}(0) z + \frac{\partial f}{\partial \bar z}(0) \bar z, \quad df(0)(\lambda z) = \lambda\, df(0)(z) \quad \forall \lambda \in \mathbb{R}.$$

$df(0) : \mathbb{C} \to \mathbb{C}$ $\mathbb{C}$-linear $\Leftrightarrow df(0)(i) = i\, df(0)(1)$

$$= \frac{\partial f}{\partial z}(0)\, i + \frac{\partial f}{\partial \bar z}(0)(-i) = i \left(\frac{\partial f}{\partial z}(0) + \frac{\partial f}{\partial \bar z}(0) \right)$$

$$\Leftrightarrow 2i \frac{\partial f}{\partial \bar z}(0) = 0 \Leftrightarrow \frac{\partial f}{\partial \bar z}(0) = 0.$$

<u>Remark</u> $f \in \mathcal{C}^1(D)$ f complex diff'ble at $z_0 \in D$ iff.

$df(z_0): \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$ linear.

<u>Chain rule</u>: $f: D \to \mathbb{C}$ is $\mathcal{C}^1$, $\gamma: [0,1] \to D$ smooth, $\gamma(0) = z_0 \in D$.

$$(f \circ \gamma)'(0) = \frac{\partial f}{\partial z}(z_0)\gamma'(0) + \frac{\partial f}{\partial \bar z}(z_0)\overline{\gamma'(0)}.$$

(already did it if f holom.)

$$f(\gamma(t)) - f(\gamma(0)) = \frac{\partial f}{\partial z}(z_0)\left(\gamma(t) - \gamma(0)\right)$$
$$+ \frac{\partial f}{\partial \bar z}(z_0)\left(\overline{\gamma(t) - \gamma(0)}\right) + g\left(\gamma(t) - \gamma(0)\right)$$

$\in \mathbb{R}$

$$\Rightarrow \frac{f(\gamma(t)) - f(\gamma(0))}{t} = \frac{\partial f}{\partial z}(z_0)\left(\frac{\gamma(t) - \gamma(0)}{t}\right) + \frac{\partial f}{\partial \bar z}(z_0)\overline{\left(\frac{\gamma(t) - \gamma(0)}{t}\right)} + \frac{g(\gamma(t) - \gamma(0))}{t}$$

$$t \to 0 \quad \left|\frac{g(\gamma(t) - \gamma(0))}{t}\right| = \underbrace{\left|\frac{g(\gamma(t) - \gamma(0))}{\gamma(t) - \gamma(0)}\right|}_{\downarrow \; 0} \cdot \underbrace{\left|\frac{\gamma(t) - \gamma(0)}{t}\right|}_{\downarrow \; |\gamma'(0)|} \to 0.$$

$$\Rightarrow \exists \; (f \circ \gamma)'(0) = \frac{\partial f}{\partial z}(z_0)\gamma'(0) + \frac{\partial f}{\partial \bar z}(z_0)\overline{\gamma'(0)}.$$

$02/12$

<u>Prop.</u> If $f \in C^1(D)$ is conformal at $z_0 \in D$, then f is complex diff'ble (at z_0) and $f'(z_0) \neq 0$.

<u>Pf:</u> If $\gamma_1, \gamma_2 : [0,1] \to D$ smooth and $\begin{cases} \gamma_i(0) = z_0 \\ \gamma_i'(0) \neq 0 \end{cases}$ $i=1,2$

then ~~$f(\gamma_1$~~ $(f\circ\gamma_1)'(0) \neq 0, (f\circ\gamma_2)'(0) \neq 0$ and $\measuredangle(\gamma_1'(0), \gamma_2'(0))$

$$= \measuredangle\left((f\circ\gamma_1)'(0), (f\circ\gamma_2)'(0)\right)$$

<u>Have:</u> $\arg \gamma_2'(0) - \arg \gamma_1'(0) = \arg(~~f~~((f\circ\gamma_1)'(0)) - \arg((f\circ\gamma_2)'(0))$

$\Rightarrow \arg((f\circ\gamma_2)'(0)) - \arg(\gamma_2'(0)) = \arg((f\circ\gamma_1)'(0)) - \arg(\gamma_1'(0))$

<u>conf:</u> $\Leftrightarrow \arg\left(\dfrac{(f\circ\gamma)'(0)}{\gamma'(0)}\right)$ is independent of γ.

Take $\gamma(t) = z_0 + te^{i\theta}$, θ fixed, $\gamma'(0) = e^{i\theta} \neq 0$.

Then: $0 \neq (f\circ\gamma)'(0)$ and $\arg\left(\dfrac{(f\circ\gamma)'(0)}{\gamma'(0)}\right) = \arg\left(\dfrac{\frac{\partial f}{\partial z}(z_0)e^{i\theta} + \frac{\partial f}{\partial \bar z}(z_0)e^{-i\theta}}{e^{i\theta}}\right)$

$$= \arg\left(\underbrace{\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar z}(z_0)e^{-2i\theta}}_{\text{circle centered at } \frac{\partial f}{\partial z}(z_0) \text{ with radius } \left|\frac{\partial f}{\partial \bar z}(z_0)\right|}\right) \text{ independent of } \theta.$$

$\Rightarrow \dfrac{\partial f}{\partial \bar z}(z_0) = 0$ complex diff'ble, $(f\circ\gamma)'(0) = \dfrac{\partial f}{\partial z}(z_0)e^{i\theta} \neq 0$

$\Rightarrow f'(z) = \dfrac{\partial f}{\partial z}(z_0) \neq 0.$

02/23

Green's Theorem D bounded domain, piecewise smooth boundary, f of class $\mathcal{C}^1$ in a nbd of $\bar{D}$ $(f \in \mathcal{C}^1(\bar{D}))$

$$\int_{\partial D} f(z)\,dz = 2i \iint_D \frac{\partial f}{\partial \bar{z}}\,dx\,dy$$

Proof: $$\int_{\partial D} f\,dx + if\,dy = \iint_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = 2i \iint_D \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$$

$$= 2i \iint_D \frac{\partial f}{\partial \bar{z}}\,dx\,dy.$$

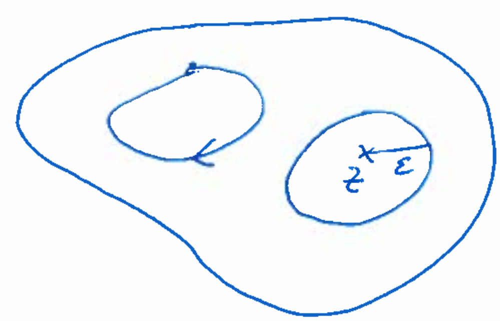
Pompeiu's Formula D, f as before.

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}\,d\xi - \frac{1}{\pi}\iint_D \frac{\partial f}{\partial \bar{z}}(\xi)\,\frac{1}{\xi - z}\,dx\,dy$$

$(\xi = x + iy)$

(If f holom, $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow$ Cauchy formula)

Pf: Pick $\bar{\Delta}(z,\varepsilon) \subseteq D$, $D_\varepsilon = D \setminus \bar{\Delta}(z,\varepsilon)$



$\xi \mapsto \dfrac{f(\xi)}{\xi - z}$ is $\mathcal{C}^1(D_\varepsilon)$

Green: $$\Rightarrow \int_{\partial D} \frac{f(\xi)}{\xi - z}\,d\xi - \int_{|\xi - z| = \varepsilon} \frac{f(\xi)}{\xi - z}\,d\xi = 2i \iint_{D_\varepsilon} \frac{\partial f}{\partial \bar{z}}(\xi)\,\underbrace{\frac{1}{\xi - z}}_{\text{holom in } \xi}\,dx\,dy$$

$\varepsilon \to 0$: as before (f cont.)

02/2

$$\int_{|z-\xi|=\varepsilon} \frac{f(\xi)}{\xi - z}\, d\xi \to f(z)\cdot 2\pi i.$$

Note $\xi \mapsto \frac{1}{\xi - z} \in L^1_{loc}(\mathbb{C})$

$$\iint_{|\xi - z|\le R} \frac{1}{|\xi - z|}\,dx\,dy = \int_0^{2\pi}\int_0^R \frac{1}{r}\, r\,dr\,d\theta = 2\pi R < \infty$$

So $\left(\xi \mapsto \frac{\partial f}{\partial \bar z}(\xi)\frac{1}{\xi - z}\right) \in L^1(D)$. Thus, $\iint_{D_\varepsilon} \frac{\partial f}{\partial \bar z}(\xi)\frac{1}{\xi - z}\,dx\,dy$ ~~as~~ $\varepsilon \to 0$

$$\to \iint_D \frac{\partial f}{\partial \bar z}(\xi)\frac{1}{\xi - z}\,dx\,dy$$

(L.D.C).

$\square$

So $\int_{\partial D} \frac{f(\xi)}{\xi - z} - 2\pi i f(z) = 2i \iint_D \frac{\partial f}{\partial \bar z}(\xi)\frac{1}{\xi - z}\,dx\,dy$

Chapter 5: Power series

5.2. Sequences and series of functions

Def: $f_j, f: E \to \mathbb{C}$, $E \subseteq \mathbb{C}$

1) f_j ~~and~~ converge pointwise to f on E if $f_j(z) \to f(z)\ \forall z \in E$

2) f_j converge uniformly to f on E.
if $\forall \varepsilon > 0\ \exists j_0 = j_0(\varepsilon)$ s.t. $j \ge j_0, z \in E \Rightarrow |f_j(z) - f(z)| < \varepsilon$.

Equivalently $\|f_j - f\|_\infty = \sup_{z\in E} |f_j(z) - f(z)| \longrightarrow 0$, $j \to \infty$ 02/23

3) $\sum_{j=1}^{\infty} f_j$ converges pointwise / uniformly on E, if the sequence of partial sums does.

Weierstraß M-test: $\sum_{j=1}^{\infty} f_j$, $f_j : E \to \mathbb{C}$. $M_j \geq 0$, $\sum_{j=1}^{\infty} M_j < \infty$ s.t. $|f_j(z)| \leq M_j \; \forall z \in E$. Then $\sum_{j=1}^{\infty} f_j$ converges absolutely and uniformly on E

Theorem 1) If $f_j : E \to \mathbb{C}$ are continuous, $f_j \to f$ uniformly on E, then f is continuous.

2) $\gamma : [0,1] \to \mathbb{C}$ rectifiable, f_j are cont. on $K = \gamma([0,1])$ and $f_j \to f$ uniformly on K.

$\Rightarrow \int_\gamma f_j(z)\,dz \longrightarrow \int_\gamma f(z)\,dz$.

Proof: 2) $\left| \int_\gamma f_j(z)\,dz - \int_\gamma f(z)\,dz \right| = \left| \int_\gamma (f_j - f)\,dz \right| \leq \ell(\gamma) \, \|f_j - f\|_\infty \to 0$, $j \to \infty$.

Def: $f_j, f : D_{\text{open}} \subseteq \mathbb{C} \to \mathbb{C}$. $\{f_j\}$ converges normally on D to f or locally uniformly if $f_j \to f$ uniformly on each compact subset of D.

0

<u>Theorem</u> $f_j \to f$ normally on D ($f_j \rightrightarrows f$)

$\Leftrightarrow \forall z \in D\ \exists\ \overline{\Delta}(z,r) \subseteq D,\ r = r(z) > 0$ s.t. $f_j \to f$ uniformly on $\overline{\Delta}(z,r)$

<u>Pf:</u> "$\Rightarrow$" ✓

<u>"$\Leftarrow$"</u> Let $K \subseteq D$, K compact. $\forall z \in D\ \exists \overline{\Delta}(z, r_z) \subset D$, $r_z > 0$ s.t.

$f_j \to f$ uniformly on $\overline{\Delta}(z, r_z)$. $K \subseteq \bigcup_{z \in K} \Delta(z, r_z)$. Open cover.

$\Rightarrow \exists\ z_1, \dots, z_n \in D$ s.t. $K \subseteq \bigcup_{j=1}^{n} \Delta(z_j, r_{z_j})$.

$$\| f_j - f \|_{L^\infty(K)} \le \max_{k=1,\dots,n} \| f_j - f \|_{L^\infty(\overline{\Delta}(z_k, r_{z_k}))} \to 0.$$

<u>Theorem</u> Suppose f_j holom. on D and $f_j \rightrightarrows f$ on D. Then f is holom. ~~on~~ on D and $f_j^{(m)} \rightrightarrows f^{(m)}$ on D. $\forall m \in \mathbb{N}_0$.

<u>Pf:</u> Morera: R rectangle $\subseteq D$

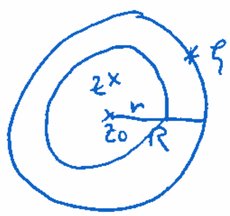
$$0 = \int_{\partial R} f_j(z)\,dz \to \int_{\partial R} f(z)\,dz. \Rightarrow f \text{ holomorphic.}$$

Pick $z_0 \in D$, $\overline{\Delta}(z_0, R) \subseteq D$. f_j holom.

$$f_j^{(m)}(z) = \frac{m!}{2\pi i} \int_{|\zeta - z_0| = R} \frac{f_j(\zeta)}{(\zeta - z)^{m+1}}\,d\zeta \quad \forall m \ge 0, \quad z \in \Delta(z_0, R).$$

$m = 0$, $j \to \infty$: $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z}\,d\zeta$ so f holom. on $\Delta(z_0, R)$

and moreover
$$f^{(m)}(z) = \frac{1}{2\pi i} \int_{|\xi - z_0| = R} \frac{m!\, f(\xi)}{(\xi - z)^{m+1}}\, d\xi .$$



Fix $0<r<R$, $|z-z_0|\le r$

$$|\xi - z| \ge \underbrace{|\xi - z_0|}_{=R} - \underbrace{|z - z_0|}_{\le r} \ge R - r > 0 .$$

$$\left| f_j^{(m)}(z) - f^{(m)}(z) \right| \le \frac{m!}{2\pi} \left| \int_{|\xi - z_0| = R} \frac{f_j(\xi) - f(\xi)}{(\xi - z)^{m+1}}\, d\xi \right|$$

$$\le \frac{m!}{2\pi}\, 2\pi R\, \frac{1}{(R-r)^{m+1}}\, \underbrace{\|f_j - f\|_{L^\infty(\partial\Delta(z_0,R))}}_{\to 0,\ j\to\infty} \quad \text{if } |z - z_0| < r$$

$\Rightarrow f_j^{(m)} \rightrightarrows f^{(m)}$ on $\bar{\Delta}(z_0,r)$ $\Rightarrow$ ~~$f_j^{(m)}$~~ $f_j^{(m)} \rightrightarrows f^{(m)}$, $\forall m \ge 0$ $\square$

5.3 Power series

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $a_n \in \mathbb{C}$ coefficients power series, centered at z_0

$w = z - z_0 \to \sum_{n=0}^{\infty} a_n w^n$, $x_n \in \mathbb{R}$ $\boxed{\ell = \limsup_{n\to\infty} x_n \in [-\infty, +\infty]}$

1) $\ell \in \mathbb{R}$: a) $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $x_n < \ell + \varepsilon$ $\forall n \ge N$

b) $\exists$ subsequence $x_{n_k} \to \ell$, $k \to \infty$

2) $\ell = +\infty$ $\exists\, x_{n_k} \to +\infty$

3) $\ell = -\infty$ $x_n \to -\infty$.

<u>Hadamard's Theorem</u>: Let $R = \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} \in [0, +\infty]$.

Then $\sum\limits_{n=0}^{\infty} a_n (z-z_0)^n$ converges absolutely and normally on $\Delta(z_0, R)$.

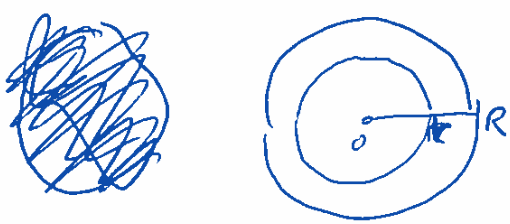
and diverges on $\{|z-z_0| > R\}$.

<u>Pf</u>: $z_0 = 0$, 1) $|z| > R = \frac{1}{\ell}$, $\ell = \limsup \sqrt[n]{|a_n|}$

$\Rightarrow \ell > \frac{1}{|z|}$, $\exists$ subsequence n_k s.t. $\sqrt[n_k]{|a_{n_k}|} > \frac{1}{|z|}$

$\Rightarrow |a_{n_k} z^{n_k}| > 1 \Rightarrow \sum\limits_{n=0}^{\infty} a_n z^n$ diverges.

2)



Fix $0 < r < R$. Want to show: abs. unif. convergence on $\{|z| \le r\}$

Fix $\varepsilon > 0$, $r + \varepsilon < R = \frac{1}{\ell}$, so $\ell < \frac{1}{r+\varepsilon}$

$\Rightarrow \sqrt[n]{|a_n|} < \frac{1}{r+\varepsilon}$ for $n \ge N(\varepsilon)$ suff. large.

$|a_n z^n| \le \frac{1}{(r+\varepsilon)^n} r^n = M_n := \left(\frac{r}{r+\varepsilon}\right)^n$, $n \ge N$
$(|z| \le r)$

$\sum\limits_{n=0}^{\infty} M_n < +\infty$. (geometric) $\square$

Remarks 1) If $\ell = \lim_{n\to\infty} \sqrt[n]{|a_n|} \in [0,+\infty]$ exists, then $R = \frac{1}{\ell}$. 02/26

2) If $a_n \neq 0$ for all n suff. large and $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \in [0,+\infty]$

then $\exists\ \ell = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, so $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

Examples 1) $\sum_{n=0}^{\infty} z^n = 1 + z^2 + z^3 + \dots = \frac{1}{1-z}$, $|z|<1$.

diverges for $|z|\geq 1$. ($z^n \not\to 0$)

2) $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R = 1$. If $|z|=1$. $z=1 \Rightarrow \sum_{n=0} \frac{1}{n}$ diverges

If $|z|=1$, $z\neq 1$, $\sum_{n=0}^{\infty} \frac{z^n}{n}$ converges.

(Abel - Dirichlet: $\sum_{n=0}^{\infty} a_n b_n$, $a_n, b_n \in \mathbb{C}$ ① $\left|\sum_{k=0}^{n} a_k\right| \leq M$ $\forall n$ ② $b_n \searrow 0$

then $\sum_{n=0}^{\infty} a_n b_n$ converges).

$b_n = \frac{1}{n} \searrow 0$, $a_n = z^n$, $|1+z+\dots+z^n| = \frac{|1-z^{n+1}|}{|1-z|} \leq \frac{2}{|1-z|}$, $z\neq 1$

3) $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ Conv. absolute and uniformly on $\{|z|\leq 1\}$. (M-test.)

4) $\sum_{n=0}^{\infty} n^n z^n$ $R=0$ converges only at $z=0$

5) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $R=+\infty$.

Theorem Assume $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence $R>0$. 02/26

Then $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is holom. on $\Delta(z_0,R)$ and

$$f'(z)=\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \text{ in } \Delta(z_0,R).$$

and $\quad a_n=\dfrac{f^{(n)}(z_0)}{n!}$, $n\in\mathbb{N}_0$.

A primitive of $f(z)$ in $\Delta(z_0,R)$ is $F(z)=\sum_{n=0}^{\infty}\dfrac{a_n}{n+1}(z-z_0)^{n+1}$, $z\in\Delta$

(term-by-term integration). $\square$

Proof: Follows from Thm. of unif. convergence of holom. functions.

Ex: 1) $\sum_{n=0}^{\infty} z^n = 1+z+z^2+\dots = \dfrac{1}{1-z}$, $|z|<1$.

2) $\sum_{n=0}^{\infty} n z^{n-1} = 1+2z+3z^2+\dots = \dfrac{1}{(1-z)^2}$, $|z|<1$.

$\sum_{n=1}^{\infty} n z^n = z+2z^2+3z^3+\dots = \dfrac{z}{(1-z)^2}$, $|z|<1$.

3) ~~$\int_0^z \frac{d\xi}{1-\xi}$~~ $\quad -\mathrm{Log}(1-z)=\int_0^z \dfrac{d\xi}{1-\xi}$ $\qquad |z|<1 \Rightarrow 1-z\in\Delta(1,1)$

$$=\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{z^n}{n},\quad |z|<1.$$

$\Rightarrow \mathrm{Log}(1-z) = -\sum_{n=1}^{\infty}\dfrac{z^n}{n}$, $z\to -z$ $\quad \mathrm{Log}(1+z)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}z^n}{n}$.

02/26

5.4 Power series expansion of holom. functions.

Def: f is analytic on D (open): If $\forall z_0 \in D$ $\exists \Delta(z_0, r) \subset D$, $r>0$ and $a_n = a_n(z_0) \in \mathbb{C}$, s.t. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in $\Delta(z_0, r)$.

Theorem f holom. on $D \Leftrightarrow f$ analytic on D.

Pf: "$\Leftarrow$" Done.

"$\Rightarrow$" Next Thm. $\square$

Theorem If f is holom. in $\Delta(z_0, \rho)$, then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ (*)

where power series has radius of convergence $R \geq \rho$. and $a_n = \frac{f^{(n)}(z_0)}{n!}$.

$$= \frac{1}{2\pi i} \int_{|\rho - z_0| = r} \frac{f(\rho)}{(\rho - z_0)^{n+1}} d\rho, \quad n \geq 0.$$

If $|f(z)| \leq M$ on $\{|z| = r\}$, then $|a_n| \leq \frac{M}{r^n}$, $n \geq 0$.

Pf: Will show (*) converges pointwise $\forall z \in \Delta(z_0, \rho)$ to f.

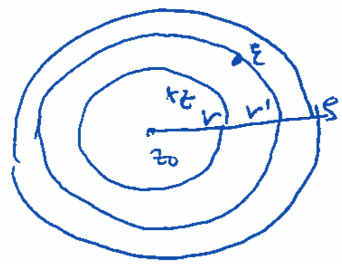
Then the power series will converge normally and absolutely in $\Delta(z_0, \rho)$.

$\Rightarrow R \geq \rho$.

Cauchy estimate for derivatives.

Cauchy estimates $|a_n| = \left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{1}{n!} n! \frac{M}{r^n}$

Pf: Pick $z \in \Delta(z_0, \rho)$, $|z-z_0| \le r < r' < \rho$ for some r, r' 02/2



$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r'} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}$$

$\left|\frac{z-z_0}{\zeta - z_0}\right| \le \frac{r}{r'} < 1$ $\qquad = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\zeta - z_0}\right)^n$ uniformly in ζ.

$\Rightarrow \frac{f(\zeta)}{\zeta - z} = \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n$. conv. uniformly in ζ on $|\zeta - z_0| = r'$!

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r'} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta - z_0| = r'} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \right) (z - z_0)^n$$

02/28

Cor: f, g holom in $\Delta(z_0, \rho)$ and $f^{(n)}(z_0) = g^{(n)}(z_0)$ $\forall n \ge 0$

$\Rightarrow f = g$.

Cor: If f holom. in a nbd. of z_0 (so $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$) $|z - z_0| < \varepsilon$

Then the radius of convergence R of the power series is the radius of the largest disc centered at z_0 to which f extends holomorphically.

Ex: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$

$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$, $z \in \mathbb{C}$

$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$, $z \in \mathbb{C}$

$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$, $z \in \mathbb{C}$

$\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$; $|z| < 1$

Ex: 1) $f(z) = \frac{z}{\sin z}$ holom. in nbd. of 0.

$\lim_{z \to 0} \frac{z}{\sin z} = \lim_{z \to 0} \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = 1$

f holom on $B_\varepsilon \setminus \{0\}$ cont. on $B_\varepsilon \Rightarrow$ f holom

(cont on D + holom on $D \setminus \mathbb{R} \Rightarrow$ holom on D)

f holom in some $0 < |z| < \varepsilon$, $\lim_{z \to 0} f(z) = 1 \Rightarrow$ f holom in $|z| < \varepsilon$.

Radius of conv. of p.s. at 0 of $f = \pi$

f holom on $\mathbb{C} \setminus \{\pm\pi, \pm 2\pi, \ldots\}$ and $\lim_{z \to n\pi} |f(z)| = \infty$, $n \neq 0$

Largest disk centered at 0 on which f holom is $\Delta(0, \pi)$.

2) $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$

02/28

$$= \sum_{n=0}^{\infty} (-1) x^{2n}, \quad |x|<1 \quad (x \in (-1,1))$$

$f(z) = \frac{1}{1+z^2}$, $z = \pm i$ so radius of convergence has to be $=1$.

5.6 Power Series at infinity

Def: If ~~f~~ f is holom on some $\{|z|>R\}$, then f is called holomorphic at ∞ if $\lim_{z\to\infty} f(z) = \ell \in \mathbb{C}$. $z = \frac{1}{w}$

$g(w) = f(\frac{1}{w})$ if $0<|w|<\frac{1}{R}$ holom. and $\lim_{w\to 0} g(w) = \lim_{z\to\infty} f(z)$

Define $g(0) = \ell \Rightarrow g$ cont. on $\Delta(0,\frac{1}{R})$ and holom on $\Delta(0,\frac{1}{R}) \setminus \{0\}$.

$\Rightarrow g$ holom. on $\Delta(0,\frac{1}{R})$. $g(w) = \ell + a_1 w + a_2 w^2 + \cdots$

converging uniformly in $\{|w| < \frac{1}{r} < \frac{1}{R}\}$ if $r > R$.

Now $w = \frac{1}{z} \Rightarrow f(z) = g(\frac{1}{z}) = \ell + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$

(Power series expansion of f at ∞)

converges uniformly on $|z| > r \ \forall r > R$.

Ex: $f(z) = \frac{1}{1+z^2}$ holom. at ∞: hol. on $|z|>1$ and

$\lim_{z\to\infty} f(z) = 0$.

02/29

$$g(w) = f\left(\frac{1}{w}\right) = \frac{1}{1+\frac{1}{w^2}} = \frac{w^2}{1+w^2} \quad \text{holom.} \quad |w|<1$$

$$= w^2 \sum_{n=0}^{\infty} (-1)^n w^{2n} = \sum_{n=0}^{\infty} (-1)^n w^{2(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} w^{2n}, \quad w = \frac{1}{z}$$

$$\Rightarrow \quad f(z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^{-2n}$$ Converges uniformly on $|z| \geq 1+\varepsilon \ \forall \varepsilon > 0$.

Another way to get this. $|z|>1$

$$f(z) = \frac{1}{1+z^2} = \frac{1}{z^2}\left(\frac{1}{1+\underbrace{\frac{1}{z^2}}_{|\cdot|<1}}\right) = \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z^2}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} z^{-2n}$$

5.7 Manipulation of Power Series

Theorem If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$, ~~let~~ $|z-z_0|<\rho$

Then $(f+g)(z) = \sum_{n=0}^{\infty} (a_n+b_n)(z-z_0)^n, \quad |z-z_0|<\rho$

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

Cauchy product

for $|z-z_0|<\rho$

<u>Pf:</u> fg holom. in $|z-z_0|<\rho$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$f(z)g(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(z_0)}{n!}(z-z_0)^n, \quad |z-z_0|<\rho$$

$$\frac{(fg)^{(n)}(z_0)}{n!} = \frac{1}{\cancel{n!}} \sum_{k=0}^{n} \frac{\cancel{n!}}{k!(n-k)!} f^{(k)}(z_0)\, g^{(n-k)}(z_0) = \sum_{k=0}^{n} a_k b_{n-k} = c$$

<u>Dividing:</u> g holom. near 0, on $\{|z|<r\}$, $g(0)=a_0 \neq 0$.

$\Rightarrow g(z) \neq 0$ for $|z|<\varepsilon<r$. (e.g. $g(z)=\cos(z)$)

$g(z) = a_0 + a_1 z + a_2 z^2 + \ldots, \quad |z|<r$.

$f(z) = \frac{1}{g(z)}$ holom on $\{|z|<\varepsilon\}$.

$$\frac{1}{g(z)} = \frac{1}{a_0 + a_1 z + a_2 z^2 + \ldots} = \frac{1}{a_0} \frac{1}{1 + \underbrace{\left(\cancel{\tfrac{a_0}{a_0}} + \frac{a_1}{a_0} z + \frac{a_2}{a_0} z^2 + \ldots\right)}_{|\cdot|<1 \text{ if } |z| \text{ small}}}$$

$$= \frac{1}{a_0}\left(1 - \left(\frac{a_1}{a_0} z + \frac{a_2}{a_0} z^2 + \ldots\right) + \left(\frac{a_1}{a_0} z + \frac{a_2}{a_0} z^2 + \ldots\right)^2 - \ldots \right)$$

$$= b_0 + b_1 z + b_2 z^2 + \ldots$$

$$b_0 = \frac{1}{a_0}, \quad b_1 = \frac{1}{a_0}\left(-\frac{a_1}{a_0}\right) = -\frac{a_1}{a_0^2}, \quad b_2 = \frac{1}{a_0}\left(\left(-\frac{a_2}{a_0}\right) + \frac{a_1^2}{a_0^2}\right)$$

$$b_3 = \frac{1}{a_0}\left(\left(-\frac{a_3}{a_0}\right) + 2\frac{a_1}{a_0}\frac{a_2}{a_0} - \frac{a_1^3}{a_0^3}\right)$$

Comments on HW 4

03/05

I.5.4 f entire, $|f(z)| \le M|z|^n \; \forall z \in \mathbb{C}$, $f(z) = a_0 + a_1 z + \dots + a_n z^n$

$f(0) = 0 \Rightarrow a_0 = 0 \Rightarrow |a_1 + a_2 z + \dots + a_n z^{n-1}| \le M|z|^{n-1}, z \in \mathbb{C}$

$\Rightarrow a_1 = 0 \quad \dots \quad \Rightarrow f(z) = a_n z^n, \; z \in \mathbb{C}.$

IV.8.2 $\frac{\partial}{\partial \bar{z}}(az^2 + bz\bar{z} + c\bar{z}^2) = bz + 2c\bar{z} = 0$ $\quad b = b_1 + ib_2$, $c = c_1 + ic_2$

$\rightsquigarrow \begin{cases} Ax + By = 0 \\ Cx + Dy = 0 \end{cases}$ Three possibilities

1) $z = 0$ only solution

2) line of solution (1-D)

3) all z are solutions

$bz = -2c\bar{z} \Rightarrow \quad |b||z| = 2|c||z|$

1) $|b| \neq 2|c| \Rightarrow$ only sol.'n $z = 0$

2) $|b| = 2|c| > 0$

$c = re^{i\varphi}, \; b = 2re^{i\theta}$

$e^{i\theta} z + e^{i\varphi}\bar{z} = 0 \quad | \; e^{-i/2(\theta+\varphi)}$

$e^{i\frac{\theta-\varphi}{2}} z + e^{i\frac{\varphi-\theta}{2}}\bar{z} = 0 \Rightarrow \operatorname{Re}\left(e^{i\frac{\theta-\varphi}{2}} z\right) = 0$

$\Rightarrow e^{i\frac{\theta-\varphi}{2}} z = it$

$\Rightarrow z = i e^{i\frac{\varphi-\theta}{2}} t$ is a line. 03/05

5.7 Zeros of Analytic Functions

f holom. on open D; Say f has a zero of order N at $z_0 \in D$

if $f(z_0) = 0 = f'(z_0) = \dots = f^{(N-1)}(z_0)$ and $f^{(N)}(z_0) \neq 0$.

$N=1$ zero of order $1 \Leftrightarrow$ simple zero $\Leftrightarrow f(z_0)=0, f'(z_0) \neq 0$

$N=2$ double zero

$N=3$ triple zero $f(z_0) = f'(z_0) = f''(z_0) = 0$, $f'''(z_0) \neq 0$.

Lemma f has a zero of order N at z_0 in D

$\Leftrightarrow f(z) = (z-z_0)^N h(z)$, h holom in D, & $h(z_0) \neq 0$.

Pf: "$\Leftarrow$" ✓

"$\Rightarrow$" $\Delta(z_0, R) \subset D$ $f(z) = \cancel{a_N(z-z_0)^N + a_{N+1}z^{N+1} + \dots}$

$$\underbrace{a_N}_{\neq 0}(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \dots$$

$$f(z) = (z-z_0)^N \big(\underbrace{a_N + a_{N+1}(z-z_0) + \dots}_{h(z) \text{ in } \Delta(z_0, R)}\big)$$

03/05

$$h(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^N}, & z \in D \setminus \{z_0\} \\ a_N, & z = z_0 \end{cases}$$ cont. on D, holom. on $D \setminus \{z_0\}$

$\Rightarrow$ holom. on D. $\square$

Exe: $\sin z, \cos z \Rightarrow$ all zeros are simple

$f(z) = \cos z - 1 = -\dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots$ zero of order 2 / double zero at $z_0 = 0$.

Def: order of f at z_0 $\quad \operatorname{ord}(f, z_0) = \begin{cases} 0, & f(z_0) \neq 0 \\ \text{order of zero of } f \text{ at } z_0 & \text{if } f(z_0) = 0 \end{cases}$

Lemma $\operatorname{ord}(fg, z_0) = \operatorname{ord}(f, z_0) + \operatorname{ord}(g, z_0)$

Pf:

$f(z) = a_N (z-z_0)^N + \text{—}, \quad a_N \neq 0$

$g(z) = b_M (z-z_0)^M + \text{—}, \quad b_M \neq 0$

$f(z)g(z) = a_N b_M (z-z_0)^{M+N} + \text{—}, \quad a_N b_M \neq 0 \Rightarrow \operatorname{ord}(fg, z_0) = N+M.$

Def. f holom. on $\{|z| > R\}$ has a zero of order N at infinity if $g(w) = f(1/w), |w| < \frac{1}{R}$ is holom. at 0 and has a zero of order N at 0.

$f(z) = \dfrac{a_N}{z^N} + \dfrac{a_{N+1}}{z^{N+1}} + \text{—}, \quad a_N \neq 0$

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Ex (recall) $f(z) = \frac{1}{1+z^2} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \dots \quad |z|>1$

has a double zero at ∞.

Def: $z_0 \in \mathbb{C}$ is a <u>limit point</u> of E if $\forall r>0\ \exists z \in E \setminus \{z_0\}$ s.t. $|z-z_0|<r$

(equivalently $\exists z_k \in E \setminus \{z_0\}$, $z_k \to z_0\ (k\to\infty)$). E' = set of limit points

$z_0 \in E \setminus E'$ is called an <u>isolated point of E</u>.

$\iff \exists r>0$ s.t. $\Delta(z_0, r) \cap E = \{z_0\}$.

Theorem: Suppose f holom. on domain D. TFAE

1) $E = \{z \in D \mid f(z)=0\}$ has a limit point in D

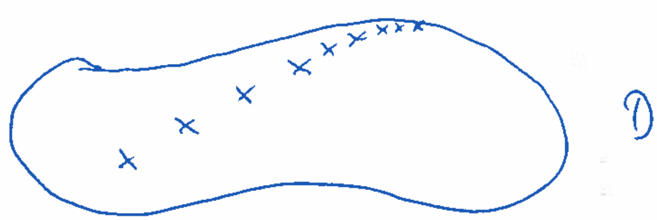
2) f has a zero of infinite order.

3) $f \equiv 0$ on D.

So if $f \not\equiv 0$, then all zeros have finite order and are isolated.

E can be infinite but has to be countable. limit points of E lie on ∂

Picture:



<u>Ex:</u> $x \in \mathbb{R}$, $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then $f^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$.

and $f \in \mathcal{C}^\infty(\mathbb{R})$.

<u>in $\mathbb{R}^n$:</u> $f(x) = \begin{cases} e^{-\frac{1}{|x|^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, $f^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$.

<u>Proof</u>

<u>1) ⇒ 2):</u> let $z_0 \in E' \cap D \Rightarrow \exists z_n \in E, z_n \neq z_0, z_n \to z_0$.

$\Rightarrow f(z_n) = 0$ so $f(z_0) = 0$. Assume z_0 has order N

$\Rightarrow f(z) = (z - z_0)^N h(z)$, $h(z_0) \neq 0 \Rightarrow h(z) \neq 0$ in $\Delta(z_0, r)$

$f(z_n) = 0 \Rightarrow h(z_n) = 0$ and $z_n \to z_0$. ↯.

<u>2) ⇒ 3)</u> $\emptyset \neq \mathcal{U} = \{z \in D \mid f^{(n)}(z) = 0 \ \forall n \in \mathbb{N}_0\}$ set of zeros of infinite order.

$\mathcal{U} = \bigcap_{n=0}^{\infty} (f^{(n)})^{-1}(\{0\})$ closed in D.

$z_0 \in \mathcal{U} \Rightarrow f(z) \equiv 0$ in any $\Delta(z_0, R) \subset D \Rightarrow \Delta(z_0, R) \subset \mathcal{U} \Rightarrow \mathcal{U}$ open

$\Rightarrow \mathcal{U} = D$, so $f \equiv 0$. 3) ⇒ 1) ✓. □

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Corollary (Identity Theorem) If f, g are holom. in a domain D and $S = \{z \mid f(z) = g(z)\}$ has a limit point in $D \Rightarrow f \equiv g$.

Apply Thm. to $f - g$.

Exe: Find all holom. fcts. f, g in $\Delta(0,1)$ s.t.

1) $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$, $n \geq 2$

2) $g\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$, $n \geq 2$

1) If $h(z) = z^2 \Rightarrow f\left(\frac{1}{n}\right) = h\left(\frac{1}{n}\right)$ but $\{\frac{1}{n}, n \geq 2\}$ has a limit point in Δ

$\Rightarrow f \equiv h$.

2) $\nexists$ such a g: n even $g\left(\frac{1}{2k}\right) = \frac{1}{2k} \Rightarrow g(z) = z$ on Δ

$n = 2k+1$ $g\left(\frac{1}{2k+1}\right) = \frac{-1}{2k+1} \Rightarrow g(z) = -z$ on Δ

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Theorem Suppose $D \subseteq \mathbb{C}$ domain, $E \subset D$ has a limit point in D. $F : D \times D \to \mathbb{C}$ holom. in each variable separately, i.e.

$\forall z \in D$ $F(z, \cdot)$ holom on D, $\forall w \in D$ $F(\cdot, w)$ holom. on D. If $F = 0$ on $E \times E$ then $F \equiv 0$ on $D \times D$.

~~E~~: Pf: Fix $z \in E$ $g(w) = F(z,w)$ holom in D, $g = 0$ on E 03/07

$\overset{\text{Identity}}{\Rightarrow}$ $g = 0 \Rightarrow F(z,w) = 0 \ \forall z \in E, w \in D$.
Theorem

$\Rightarrow F = 0$ on $E \times D$. Fix $w \in D$, $h(z) = F(z,w)$, so $h = 0$ on E

$\Rightarrow h = 0$ on D. $\Rightarrow F = 0$ on $D \times D$ $\square$

Appl: $F(z,w) = \sin(z+w) - \sin z \cos w - \cos z \sin w$ holom. in each variable on $\mathbb{C} \times \mathbb{C}$.

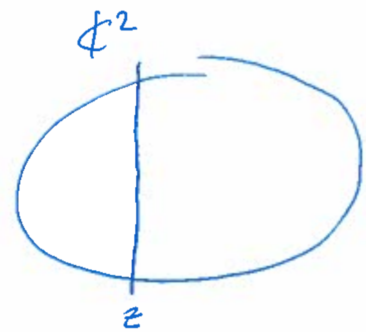
$F = 0$ if $z, w \in \mathbb{R}$. $\Rightarrow F = 0$ on $\mathbb{C} \times \mathbb{C}$.

$F: G_{\text{open}} \subset \mathbb{C}^n \to \mathbb{C}$ holom. if ~~F~~ $F \in C^1(G)$ and
$z = (z_1, \dots, z_n) \in G$ $\quad \dfrac{\partial F}{\partial \bar{z}_1} = 0, \dots, \dfrac{\partial F}{\partial \bar{z}_n} = 0$ on G.

$\left(dF = \dfrac{\partial F}{\partial z_1} dz_1 + \dots + \dfrac{\partial F}{\partial z_n} dz_n + \dfrac{\partial F}{\partial \bar{z}_1} d\bar{z}_1 + \dots + \dfrac{\partial F}{\partial \bar{z}_n} d\bar{z}_n\right)$

$: \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{C}$ is $\mathbb{R}$-linear. dF $\mathbb{C}$ linear $\Leftrightarrow$ F holom.

Clearly such F is holom. in each variable separately.



Hartogs Theorem If $F: G \subset \mathbb{C}^n \to \mathbb{C}$ is holom. in each variable separately, then F is holom.

Ch. 6: Laurent Series

03/5

6.1 Laurent decomposition.

A Laurent series centered at $z_0 \in \mathbb{C}$ is ~~seen~~ a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}, \quad a_n \in \mathbb{C}, n \in$$

L.S. is convergent at z if both series converge at z.

let $\sigma := \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} \in [0, +\infty]$

$$\rho := \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|} \in [0, +\infty]$$

$\sum\limits_{n=0}^{\infty} a_n (z-z_0)^n$ conv. abs. and ~~unif.~~ normally in $\{|z-z_0| < \sigma\}$ to a holom. function f_0.

$\sum\limits_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$ conv. abs. ~~and unif.~~ in $\{|z-z_0| > \rho\}$ and unif. in $\{|z-z_0| \geq \rho+\varepsilon\}$ $\forall \varepsilon > 0$, to a holom. function f_1. f_1 holom in $\{|z-z_0| > \rho\}$.

If $\rho < \sigma$ then $\sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$ conv. absolutely in

$\{\rho < |z-z_0| < \sigma\}$ and unif. on in any annulus $\{r \le |z-z_0| \le s\}$

for any $\rho < r \le s < \sigma$, and the sum is the holom. function $f_0 + f_1$.

in $\{\rho < |z-z_0| < \sigma\}$.

<u>Theorem</u>: (L.S. expansion) If f holom in $A = \{\rho < |z-z_0| < \sigma\}$

$0 \le \rho < \sigma \le +\infty$. Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ where L.S.

Converges normally and absolutely in A. and a_n are uniquely determined

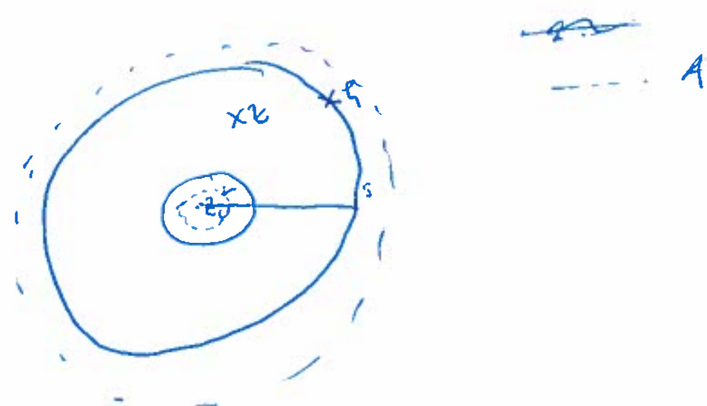
by f and $a_n = \frac{1}{2\pi i} \int_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} d\zeta$ for any $n \in \mathbb{Z}$

does not depend on r.

and $\rho < r < \sigma$.

<u>Pf</u>: Fix $z \in A$ show $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ with a_n given by the formula.

Fix $\rho < r \le |z-z_0| \le s < \sigma$



$$f(z) = \frac{1}{2\pi i} \int_{|\zeta-z_0|=s} \frac{f(\zeta)}{\zeta-z} d\zeta$$

$$- \frac{1}{2\pi i} \int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta-z} d\zeta$$

$$\frac{1}{\xi-z}=\frac{1}{\xi-z_0-(z-z_0)}=\frac{1}{\xi-z_0}\cdot\frac{1}{1-\underbrace{\frac{z-z_0}{\xi-z_0}}_{|\cdot|=\frac{|z-z_0|}{s}<1\ \forall\xi}}$$

$$=\frac{1}{\xi-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\xi-z_0)^n}\quad \text{unif. in } \xi \text{ on } |\xi-z_0|=s.$$

So
$$\frac{f(\xi)}{\xi-z}=\sum_{n=0}^{\infty}\frac{f(\xi)}{(\xi-z_0)^{n+1}}(z-z_0)^n\quad \text{unif. in } \xi \text{ on } |\xi-z_0|=s.$$

$$\frac{1}{2\pi i}\int_{|\xi-z_0|=s}\frac{f(\xi)}{\xi-z}\,d\xi=\sum_{n=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\int_{|\xi-z_0|=s}\frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi\right]}_{=a_n}(z-z_0)^n=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$-\frac{1}{\xi-z}=\frac{1}{z-z_0-(\xi-z_0)}=\frac{1}{z-z_0}\cdot\frac{1}{1-\underbrace{\frac{\xi-z_0}{z-z_0}}_{|\cdot|=\frac{r}{|z-z_0|}<1\ \forall\xi}}$$

$n+1=m$

$$=\sum_{n=0}^{\infty}\frac{(\xi-z_0)^n}{(z-z_0)^{n+1}}=\sum_{m=1}^{\infty}\frac{(\xi-z_0)^{m-1}}{(z-z_0)^m}\quad \text{unif. in } \xi \text{ on } |\xi-z_0|=r$$

$$-\frac{1}{2\pi i}\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z}\,d\xi=\sum_{m=1}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\int_{|\xi-z_0|=r}\frac{f(\xi)}{(\xi-z_0)^{1-m}}\,d\xi\right]}_{=a_{-m}}(z-z_0)^{-m}$$

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$$f(\xi) = \sum_{m=-\infty}^{+\infty} a_m (\xi - z_0)^m \quad \text{unif. in } \xi \text{ on } |\xi - z_0| = r$$

$$\int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi = \sum_{m=-\infty}^{\infty} a_m \underbrace{\int_{|\xi - z_0| = r} (\xi - z_0)^{m-n-1} d\xi}_{\begin{cases} = 0 \text{ if } m-n-1 \neq -1 \\ = 2\pi i \text{ if } m-n-1 = -1,\ m=n \end{cases}} = 2\pi i\, a_n. \quad \square$$

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Recall f holom in $\{ \rho < |z - z_0| < \sigma \}$ $0 \leq \rho < \sigma \leq +\infty$

$\Rightarrow$ $f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$ (abs. norm. conv.)

$$a_n = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \quad n \in \mathbb{Z}, \ \rho < r < \sigma$$

Corollary (Laurent decomp.) If f holom. in $A = \{ \rho < |z - z_0| < \sigma \}$ then $f = f_0 + f_1$ with f_0 holom. in $\{ |z - z_0| < \sigma \}$, f_1 is holom on $\{ |z - z_0| > \rho \}$. $f_1(\infty) = 0$; and f_0, f_1 are unique with these properties.

Pf: $f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$, let $f_0(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

$f_1(z) = \frac{a_1}{z - z_0} + \frac{a_2}{(z - z_0)^2} + \dots$ have the properties.

Uniqueness: $f = f_0 + f_1$, $g = g_0 + g_1$ g_0, g_1 same properties

$\Rightarrow f_0 - g_0 = g_1 - f_1$ in A

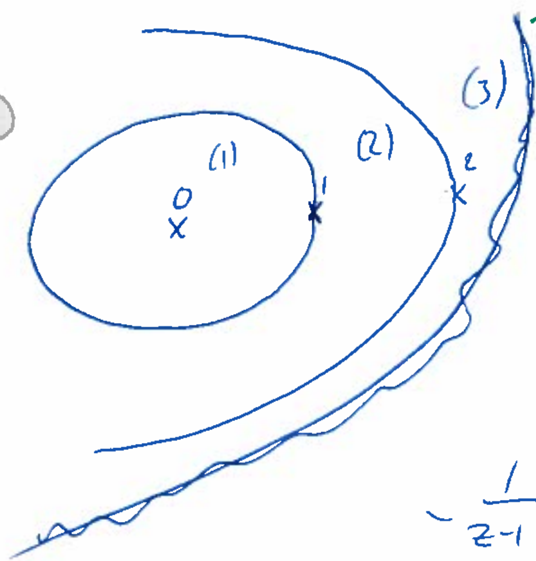
Let $F = \begin{cases} f_0 - g_0 & |z-z_0| < \sigma \\ g_1 - f_1 & |z-z_0| > \rho \end{cases}$ F entire, $F(\infty) = 0$
$= g_1(\infty) - f_1(\infty)$

Liouville $\Rightarrow$ $F \equiv 0$.

Laurent series: i) Find all Laurent series expansions centred at 0 for

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Three annuli.



(1) $|z| < 1$

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \quad |z| <$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

So $f(z) = \sum_{n=0}^{\infty} (1 - 2^{n+1}) z^n$ (Taylor series)

(2) $1 < |z| < 2$

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$ as above in (1)

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$$-\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-\underbrace{\frac{1}{z}}_{|\cdot|<1}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{z^{n}} \quad \forall |z|>1.$$

$$f(z) = -\sum_{n=0}^{\infty}\frac{z^{n}}{2^{n+1}} - \sum_{n=1}^{\infty}\frac{1}{z^{n}}$$

(3) $|z|>2$

$$-\frac{1}{z-1} = -\sum_{n=1}^{\infty}\frac{1}{z^{n}}$$

$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\underbrace{\frac{2}{z}}_{|\cdot|<1}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n} = \sum_{n=0}^{\infty}\frac{2^{n}}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^{n}}$$

$$f(z) = \sum_{n=1}^{\infty}\frac{2^{n-1}-1}{z^{n}} = \sum_{n=2}^{\infty}\frac{2^{n-1}-1}{z^{n}}$$

2) All Laurent series expansions of

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$ centered at $z_0 = 1$

(1) $0<|z-1|<1$

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-\underbrace{(z-1)}_{|\cdot|<1}} = -\sum_{n=0}^{\infty}(z-1)^{n}.$$

$$\Rightarrow f(z) = -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^{n}.$$

(2) $|z-1|>1$

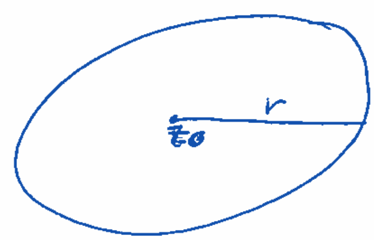
$$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{1}{z-1}\cdot\frac{1}{1-\underbrace{\frac{1}{z-1}}_{|\cdot|<1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{(z-1)^n}$$

$$f(z) = -\frac{1}{z-1} + \sum_{n=1}^{\infty}\frac{1}{(z-1)^n} = \sum_{n=2}^{\infty}\frac{1}{(z-1)^n}.$$

6.2 Isolated Singularities

Call $z_0 \in \mathbb{C}$ an isolated singularity of f if f is defined and holom. in $0<|z-z_0|<r$ for some $r>0$.



Ex: 1) $f(z) = \frac{1}{\sin z}$ holom. on $\mathbb{C}\setminus\{n\pi \mid n\in\mathbb{Z}\}$

all $n\pi$ are isolated singularities.

2) $g(z) = \frac{1}{\sin(\frac{1}{z})}$ $\quad z\neq 0,\ \sin(\frac{1}{z})\neq 0 \quad z\neq \frac{1}{n\pi},\ n\in\mathbb{Z}\setminus\{0\}$

$E = \left\{\frac{1}{n\pi} \mid n\in\mathbb{Z}\setminus\{0\}\right\}\cup\{0\}$ closed, g holom. on $\mathbb{C}\setminus E$.

$\frac{1}{n\pi}$ are isolated singularities, but $z_0=0$ is not!

z_0 isol. singularity $f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$, $0<|z-z_0|<r$

1) all $a_n = 0$, $n<0$

2) all but finitely many $a_n = 0$ for $n<0$

3) for infinitely many $n<0$, $a_n \neq 0$

Case 1) $a_n = 0 \ \forall n<0 \rightsquigarrow$ Call z_0 removable singularity

$f(z) = a_0 + a_1(z-z_0) + \text{—}$ Define $f(z_0) = a_0$, then get holom. function in $|z-z_0|<r$.

Ex: $f(z) = \frac{\sin z}{z}$ 0 isolated singularity

$$= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \text{—}}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

removable singularity at 0, $f(0) = 1$

Riemann Ext. Thm f holom. and bounded in $0<|z-z_0|<r$

then z_0 removable singularity.

Pf: let $|f(z)| \leq M \ \forall \ 0<|z-z_0|<r$.

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$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{1}{2\pi} \frac{M}{\varepsilon^{n+1}} 2\pi\varepsilon = \frac{M}{\varepsilon^n} = M\varepsilon^{-n}$$

$\square$

$n<0$, $\varepsilon \to 0 \Rightarrow a_n = 0$. $\Rightarrow$ z_0 is removable.

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Case 2 $\exists N > 0$ s.t. $a_{-N} \neq 0$ but $a_n = 0$ $\forall n < -N$.

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$a_{-N} \neq 0$. Call z_0 pole of order n for f, $\frac{a_{-1}}{z-z_0} + \cdots + \frac{a_{-N}}{(z-z_0)^N}$ is called principal part of the Laurent series.

Theorem TFAE:

1) z_0 pole of order N of f.

2) $f(z) = \frac{g(z)}{(z-z_0)^N}$, g holom. near z_0, $g(z_0) \neq 0$.

3) $\frac{1}{f}$ extends holom. at z_0 and has a zero of order N at z_0.

<u>Proof</u> <u>1) ⇒ 2)</u> $f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \cdots$

$g(z) = (z-z_0)^N f(z) = a_{-N} + a_{-N+1}(z-z_0) + \cdots$

extends holom. to $\{|z-z_0|<r\}$, and $g(z_0) = a_{-N}$

$\therefore\ f(z) = \frac{g(z)}{(z-z_0)^N}$

<u>2) ⇒ 3)</u> $\frac{1}{f(z)} = \frac{1}{g(z)}(z-z_0)^N$, $\frac{1}{g}$ holom. near z_0, $\frac{1}{g(z_0)} \neq 0$.

$\Rightarrow$ $\frac{1}{f}$ has a zero of order N at z_0.

<u>3) ⇒ 1)</u> $\frac{1}{f(z)} = h(z)(z-z_0)^N$, h holom. near z_0 $(|z-z_0|<r')$

$h(z_0) \neq 0$.

So $h(z) \neq 0$ in $|z-z_0|<r''$ $\Rightarrow g = \frac{1}{h}$ holom. in $|z-z_0|<r''$

$\Rightarrow f(z) = \frac{g(z)}{(z-z_0)^N} = \frac{1}{(z-z_0)^N}(b_0 + b_1(z-z_0) + \cdots)$, $b_0 \neq 0$

$= \frac{b_0}{(z-z_0)^N} + \frac{b_1}{(z-z_0)^{N-1}} + \cdots$, $a_{-N} = b_0 \neq 0$.

$\Rightarrow f$ has a pole of order N at z_0 $\square$

Theorem z_0 isol. sing.

f has a pole at $z_0 \iff \lim_{z \to z_0} f(z) = \infty$

Pf: "$\Rightarrow$" ✓ by 2).

"$\Leftarrow$" $\lim_{z\to z_0} \frac{1}{f(z)} = 0 \Rightarrow \frac{1}{f}$ extends holom. at z_0 and has a zero at z_0

has isol. sing. at z_0.

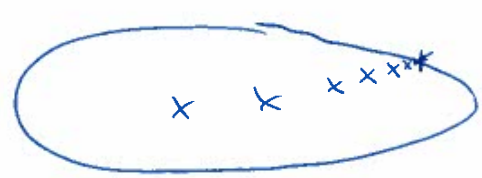
$\overset{3)}{\Rightarrow}$ f has a pole (of order ≥ 1)

Example $f(z) = \dfrac{1}{\sin z - z} = \dfrac{1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) - z}$

$$= \frac{\underbrace{\dfrac{1}{-\frac{1}{3!} + \frac{z^2}{5!} - \dots}}_{\neq 0 \text{ at } 0}}{z^3} \Rightarrow f \text{ has pole of order 3 at } 0.$$

Def: A meromorphic function on $D \underset{\text{open}}{\subseteq} \mathbb{C}$ is a function f holom. on D except at isolated singularities which are poles.

Remarks 1) f can have at most countably many poles. If infinitely many, they accumulate at ∂D



2) D domain $\mathcal{M}(D)$ set of meromorphic functions on D.

$\Rightarrow \mathcal{M}(D)$ is a field

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<u>Theorem</u> If f is meromorphic on D (domain), then $\exists\, g, h$ holom.

$h \not\equiv 0$, s.t. $f = \frac{g}{h}$.

"$\Leftarrow$" exercise "$\Rightarrow$" harder $\quad \square \quad$ (Will prove when $D = \mathbb{C}^*$)

<u>Case 3</u> $a_n \neq 0$ for infinitely many $n < 0$. Call z_0 an <u>isolated essential singularity</u>.

<u>Ex:</u> $e^{\frac{1}{z}} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$

0 isolated essential singularity.

<u>Theorem</u> (Casorati-Weierstraß) If f has an isolated essential singularity at z_0, then $\forall\, w_0 \in \mathbb{C}\ \exists\, z_n \to z_0$ s.t. $f(z_n) \to w_0$.

<u>Pf:</u> Assume $\exists\, \varepsilon > 0$ s.t. $|f(z) - w_0| > \varepsilon$ for $|z - z_0| < \varepsilon$.

$g(z) = \frac{1}{f(z) - w_0}$ holom. in $0 < |z - z_0| < \varepsilon$.

and $|g(z)| < \frac{1}{\varepsilon}\ \forall |z - z_0| < \varepsilon \overset{\text{Riemann}}{\Rightarrow} z_0$ removable singularity of g.

$g(z_0) = 0$ (z_0 zero of g)

or

$g(z_0) \neq 0$

$f(z) = w_0 + \frac{1}{g(z)}$, $g(z_0) \neq 0 \Rightarrow z_0$ removable for f

$g(z_0) = 0 \Rightarrow z_0$ pole of f. In either case ~~ɣ~~ to f essential

Singularity at z_0.

$\Rightarrow \varepsilon = \frac{1}{n}$ $\exists z_n$ $0 < |z_n - z_0| < \frac{1}{n}$, $|f(z_n) - w_0| \leq \frac{1}{n} \Rightarrow z_n \to z_0$, $f(z_n) \to w_0$ $\square$

"Big" Picard Theorem If f has isol. essential singularity at z_0, then for all except at most one $w_0 \in \mathbb{C}$ $\exists z_n \to z_0$ s.t. $f(z_n) = w_0$.

"Little" Picard Theorem If f is entire, (not poly) then for all except at most one $w_0 \in$ $\exists z_n \to +\infty$ with $f(z_n) = w_0$.

6.3 Isolated Singularities at ∞ We say f has isol. Singularity at ∞ if f holom. on some $\{|z| > R\}$

$$f(z) = \sum_{n=-\infty}^{+\infty} b_n z^n, \quad |z| > R, \quad g(w) = f\left(\frac{1}{w}\right), \quad 0 < |w| < \frac{1}{R}$$

g has isolated singularity at 0 and

$$g(w) = \sum_{n=-\infty}^{+\infty} b_n \left(\frac{1}{w}\right)^n, \quad 0 < |w| < \frac{1}{R}.$$

1) ∞ is removable for f if 0 is removable for g. 03/21

$$f(z) = b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \dots \quad (f \text{ holom. at } \infty)$$

2) ∞ is a pole of order N of f, if 0 is a pole of order N of g:

$$f(z) = \underbrace{b_N z^N + b_{N-1} z^{N-1} + \_\_ + b_1 z}_{\text{principal part is polynomial}} + b_0 + \frac{b_{-1}}{z} + \dots$$

$b_N \neq 0$

3) ∞ is isolated (essential) sing. of f, if 0 ess. singularity of g, i.e. $b_n \neq 0$ for infinitely many $n > 0$.

Ex: $e^z = 1 + z + z^2 + \dots$ has essential singularity at ∞.

Def: f transzendental entire function has essential sing. at ∞.
$\hookrightarrow$ not polynomial

Ex: ∞ for $f(z) = \frac{1}{\sin(\frac{1}{z})}$?

$z_0 = 0$, $z_n = \frac{1}{n\pi}$, $n \in \mathbb{Z} \setminus \{0\}$ singularities, $|z_n| \le \frac{1}{\pi}$

f holom. in $|z| > \frac{1}{\pi}$ $\Rightarrow$ ~~is not~~ isolated sing. at ∞.

$$g(w) = f\left(\frac{1}{w}\right) = \frac{1}{\sin w} = \frac{1}{w - \frac{w^3}{3!} + \dots} = \frac{1}{w} \cdot \underbrace{\frac{1}{1 - \frac{w^2}{3!} + \dots}}_{\text{holom}, \neq 0 \text{ at } 0}$$

$\Rightarrow 0$ pole of order 1 for g

$\Rightarrow \infty$ is a simple pole for f. 03/21

03/2[illegible]

6.4. Partial Fractions

$D \subseteq \mathbb{C}^*$ open, f merom. on D if it is holom. except for isolated singularities which are poles.

Ex: rational function $f = \frac{P}{Q}$ merom on $\mathbb{C}^*$.

Theorem If f merom. on $\mathbb{C}^*$, then f is rational.

Pf: $\mathbb{C}^*$ cpt $\Rightarrow$ only finitely many isol. sing.

∞ is a pole: $P_\infty(z) = b_N z^N + b_{N-1} z^{N-1} + \_\_ + b_0$ principal part of f.

f hol. at $\infty \rightarrow P_\infty(z) = f(\infty)$.

P_∞ polyn. $f(z) - P_\infty(z) \to 0$ as $z \to \infty$. $z_1, \ldots, z_n$ poles of f in $\mathbb{C}$.

$$P_k(z) = \frac{a_{N_k}}{(z-z_k)^{N_k}} + \cdots \frac{a_{-1}}{(z-z_k)}$$ principal part of f at z_k (pole of order N_k)

$1 \le k \le n$. P_k holom. on $\mathbb{C} \setminus \{z_k\}$, $P_k(z) \to 0$, $z \to \infty$.

Let $g(z) = f(z) - P_\infty(z) - \sum_{k=1}^{N} P_k(z)$ holom. on $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$

$f - P_k$ holom. on z_k, P_j holom. at z_k, $k \neq j$. $\Rightarrow g$ is entire func

and $g(z) \to 0$ as $z \to \infty$ $\xrightarrow{\text{Liouville}} g \equiv 0$.

$\Rightarrow f(z) = P_\infty(z) + \sum_{k=1}^{n} P_k(z)$ is rational.

Remark If f is rational, the Theorem shows the partial fraction decomposition of f.

Ch. 7 Residues

7.1 Residue Theorem

: If f has an isolated singularity at $z_0 \in \mathbb{C}$

$$f(z) = \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots, \quad 0 < |z-z_0| < r$$

$\mathrm{Res}(f, z_0) := a_{-1}$ (Residue of f at z_0)

$$= \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z)\,dz, \quad 0 < \varepsilon < r.$$

Residue Theorem D bounded domain in $\mathbb{C}$, with piecewise smooth boundary, f holom. in a nbd. of $\bar{D}$, except at finitely many singular points $z_1, \dots, z_n \in D$.

Then $$\boxed{\int_{\partial D} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)}$$

$\underline{Pf}$: Fix $\varepsilon > 0$ s.t. $\overline{\Delta}(z_j;\varepsilon) \subset D$ and $\overline{\Delta}(z_j;\varepsilon) \cap \overline{\Delta}(z_k,\varepsilon) = \phi$ if $k \neq j$.

$$D_\varepsilon = D \setminus \bigcup_{j=1}^{n} \overline{\Delta}(z_j;\varepsilon) \rightarrow f \text{ holom. in } D_\varepsilon.$$

$$\Rightarrow \int_{\partial D_\varepsilon} f(z)\,dz = 0 \Rightarrow \int_{\partial D} f(z)\,dz = \sum_{j=1}^{n} \int_{|z-z_j|=\varepsilon} f(z)\,dz$$

$$= \sum_{j=1}^{n} 2\pi i \operatorname{Res}(f, z_j). \qquad \square$$

$\underline{Ex}$: $\operatorname{Res}(\frac{1}{z}, 0) = 1$, $\operatorname{Res}(\frac{1}{z^2}, 0) = 0$, $\operatorname{Res}(\sin\frac{1}{z}, 0) = 1$

$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \dots$

Residues at poles:

1) z_0 simple pole of $f \Rightarrow f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots$

$$a_{-1} = \operatorname{Res}(f, z_0) = \lim_{z \to z_0} f(z)(z-z_0).$$

$$\operatorname{Res}\left(\frac{1}{z^3+1}, -1\right) = \lim_{z \to -1} \frac{z+1}{z^3+1} = \lim_{z \to -1} \frac{1}{z^2-z+1} = \frac{1}{3}$$

Special case: $f(z) = \frac{g(z)}{h(z)}$, $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$

$\rightarrow$ simple pole for f.

$$\Rightarrow \operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)}$$

$$= \lim_{z \to z_0} \frac{g(z)}{\frac{h(z) - h(z_0)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}. \Rightarrow \operatorname{Res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}$$

2.) z_0 double pole. $f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \dots$$

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1} + 2a_0(z-z_0) + \dots$$

$$\Rightarrow \operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right].$$

$$\operatorname{Res}\left(\frac{1}{(z^2-1)^2}, 1\right) = \lim_{z \to 1} \left[\frac{d}{dz} (z-1)^2 \frac{1}{(z^2-1)^2}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \frac{1}{(z+1)^2} = \lim_{z \to 1} -2(z+1)^{-3}\Big|_{z=1} = -\frac{2}{8} = -\frac{1}{4}$$

3) z_0 pole of order n of f.

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + \dots$$

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$\rightarrow (z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + \cdots$

$\rightarrow \operatorname{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \left[(z-z_0)^n f(z)\right].$

7.2 Integrals of rational functions

General case $\quad f(x) = \frac{P(x)}{Q(x)}, \ x \in \mathbb{R}, \ Q(x) \neq 0 \ \forall x \in \mathbb{R}.$

$q = \deg Q \geq p + 2 = \deg P + 2.$

$\left| \frac{P(x)}{Q(x)} \right| \leq \frac{C}{x^2}, \ |x| \geq 1 \Rightarrow \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ is abs. convergent.

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{\operatorname{Im} z_j > 0} \operatorname{Res}\left(\frac{P(z)}{Q(z)}, z_j\right) \quad (z_j : \text{roots of } Q)$$

Ex $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, i\right) = 2\pi i \left.\frac{1}{2z}\right|_{z=i} = \pi.$

Pf:

Pf:

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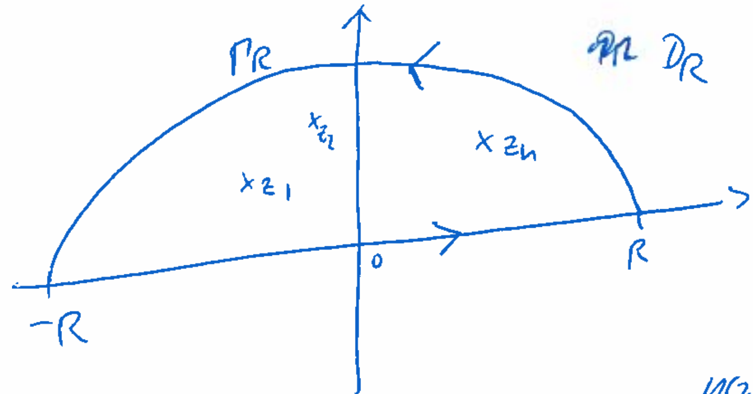

$R>0$ large, D_R contains all sing. points of $\frac{P(z)}{Q(z)}$ on $\{\operatorname{Im} z>0\}$

Residue Theorem on D_R and $\frac{P}{Q}$.

$$\Rightarrow \int_{-R}^{R} \frac{P(x)}{Q(x)}dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)}dz = 2\pi i \sum_{\operatorname{Im} z_j>0} \operatorname{Res}\left(\frac{P}{Q}, z_j\right)$$

$R\to\infty$: the first integral $\longrightarrow \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}dx$, the second $\xrightarrow{?} 0$

$|P(z)| \le C|z|^p, \quad |z|\ge 1$

$|Q(z)| \ge \varepsilon |z|^q \quad |z|\ge R_0$

If $R\ge R_0+1$, $|z|=R$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{C}{\varepsilon}\frac{R^p}{R^q} \le \frac{C}{\varepsilon}\frac{1}{R^2}$$

So $\left|\int_{\Gamma_R} \frac{P(z)}{Q(z)}dz\right| \le \pi R \frac{C}{\varepsilon}\frac{1}{R^2} \to 0, \; R\to\infty.$ $\square$

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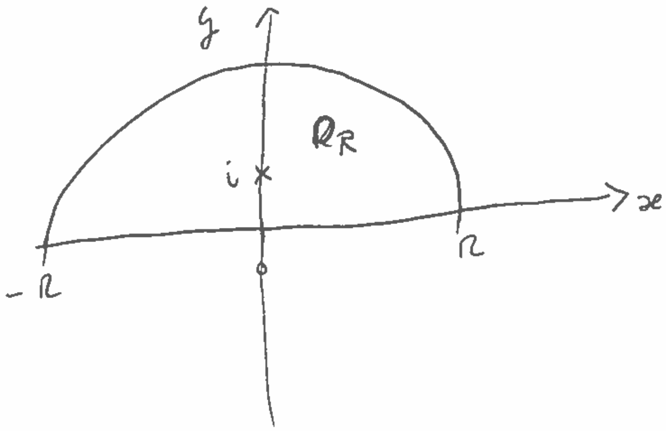
Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx = \pi e^{-a}, \quad a>0$

$f(z) = \dfrac{e^{iaz}}{1+z^2}$, $z=i$ simple pole



$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}\,dx + \int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\,dz = 2\pi i \underbrace{\frac{e^{aiz}}{2z}\Big|_{z=i}}_{=\operatorname{Res}(f,i)}$$

$|e^{aiz}| = e^{-ay}$

$$\left| \int_{\Gamma_R} \frac{e^{aiz}}{z^2+1}\,dz \right| \le \pi R \, \frac{\cancel{e}\;1}{R^2-1} \to 0, \quad R\to\infty$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\cos ax}{x^2+1}\,dx + i\int_{-\infty}^{+\infty} \frac{\sin ax}{x^2+1}\,dx = \pi e^{-a}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \quad \& \quad \int_{-\infty}^{+\infty} \frac{\sin ax}{x^2+1}\,dx = 0 \quad \text{(odd function)}$$

7.3 Integrals of trig. Functions

R rational

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$$

Poisson-kernel $\quad P(r,\theta) = \dfrac{1-r^2}{1+r^2-2r\cos\theta}, \quad 0 \le r < 1$

$$\int_0^{2\pi} P(r,\theta)\,d\theta = 2\pi \quad \forall\, 0 \le r < 1.$$

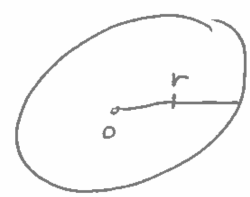
$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = \int_{|z|=1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}$$

$$z = e^{i\theta}, \quad dz = i\underbrace{e^{i\theta}}_{z}\,d\theta$$

$$\cos\theta = \operatorname{Re} z = \frac{z+\bar{z}}{2} = \frac{z+\frac{1}{z}}{2} \quad (|z|=1)$$

$$\sin\theta = \operatorname{Im} z = \frac{z-\bar{z}}{2i} = \frac{z-\frac{1}{z}}{2i}$$

$$\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta}\,d\theta = \int_{|z|=1} \frac{1-r^2}{1+r^2-2r\cdot\frac{z+\frac{1}{z}}{2}}\cdot\frac{dz}{iz}$$



$$= \frac{1-r^2}{i}\int_{|z|=1} \frac{dz}{(1+r^2)z - rz^2 - r} = \frac{1-r^2}{i}\int_{|z|=1} \frac{dz}{(z-r)(1-rz)}$$

zeros $z = r < 1$

$z = \frac{1}{r} > 1$

$$= \frac{1-r^2}{i}\, 2\pi i \operatorname{Res}\left(\frac{1}{(z-r)(1-rz)}, r\right) = \frac{1-r^2}{i}\, 2\pi i \,\frac{1}{(1-r^2)} = \underline{\underline{2\pi}}.$$

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7.4 Integrals with branch points

Contains $\log x$, x^a, $a \in \mathbb{R}$, $x > 0$

$$\log z = \log|z| + i \arg z, \quad \arg z \in (0, 2\pi).$$

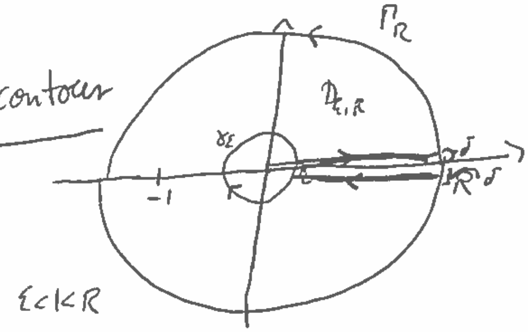
$$z^a = e^{a \log z} = |z|^a e^{ia \arg z}$$

Ex: $$\int_0^\infty \frac{x^a}{(1+x)^2}\, dx = \frac{\pi a}{\sin(\pi a)}, \quad -1 < a < 1$$

$$\frac{x^a}{(1+x)^2} \sim \frac{1}{x^{2-a}}, \quad x > 1 \quad \leadsto a < 1$$

$$\frac{x^a}{(1+x)^2} \sim x^a, \quad |x| < 1 \quad \leadsto a > -1.$$

Keyhole contour



$$D_{\varepsilon,R} = \{\varepsilon < |z| < R\} \setminus [\varepsilon, R]$$

$f(z) = \dfrac{z^a}{(1+z)^2}$ holom. in $D_{\varepsilon,R}$ except $z = -1$ double pole

$$\int_{\partial D_{\varepsilon,R}} f(z)\,dz = 2\pi i \operatorname{Res}(f, -1) = 2\pi i \left. \frac{d}{dz} e^{a \log z} \right|_{z=-1}$$

$$= 2\pi i \left. e^{a \log z} \cdot \frac{a}{z} \right|_{z=-1} = 2\pi i\, e^{a i \pi} \frac{a}{-1}$$

$$\int_\varepsilon^R \frac{x^a \boxed{1}}{(1+x)^2}dx - \int_\varepsilon^R \frac{x^a \boxed{e^{2\pi i a}}}{(1+x)^2}dx +$$

(Using the jump in the argument)

$$+\int_{|z|=R} \frac{z^a}{(1+z)^2}dz - \int_{|z|=\varepsilon}\frac{z^a}{(1+z)^2}dz = -2\pi i e^{\pi i a}$$

$R \to \infty$: $$\left|\int_{|z|=R}\frac{z^a}{(1+z)^2}dz\right| \le 2\pi R \frac{R^a}{(R-1)^2} = 2\pi\frac{R^{a+1}}{(R-1)^2} \to 0 \quad R\to\infty,\ a<1$$

$\varepsilon \to 0$: $$\left|\int_{|z|=\varepsilon}\frac{z^a}{(1+z)^2}dz\right| \le 2\pi\varepsilon\frac{\varepsilon^a}{\cancel{2r}(1-\varepsilon)^2} = \frac{2\pi\,\varepsilon^{a+1}}{(1-\varepsilon)^2} \to 0,\ \varepsilon\to 0,\ \cancel{a<}\ a>-1$$

$$\Rightarrow (1-e^{2\pi i a})\int_0^\infty \frac{x^a}{(1+x)^2}dx = -2\pi i a e^{\pi i a}$$

$$\Rightarrow \int_0^\infty \frac{x^a}{(1+x)^2}dx = \frac{-2\pi i a e^{\pi i a}}{1-e^{2\pi i a}} = \frac{-2\pi i a}{e^{-\pi i a}-e^{\pi i a}} = \frac{-2\pi i a}{-2i\sin(\pi a)} = \frac{\pi a}{\sin(\pi a)}$$

$$\int_0^{+\infty}\frac{x^w}{(1+x)^2}dx \text{ abs. conv. if } -1<\operatorname{Re} w<1.$$

$$\left|\frac{x^w}{(1+x)^2}\right| = \frac{1}{(1+x)^2}\left|e^{w\log x}\right| = \frac{e^{\operatorname{Re}(w\log x)}}{(1+x)^2} = \frac{\cancel{x^w}\ x^{\operatorname{Re} w}}{(1+x)^2}$$

$\Rightarrow$ Int. converges absolutely

Can show $g(w) = \int_0^\infty \frac{x^w}{(1+x)^2}\,dx$ is holom. in $-1 < \operatorname{Re} w < 1$

$$\Rightarrow \quad g(w) = \int_0^{+\infty} \frac{x^w}{(1+x)^2}\,dx = \frac{\pi w}{\sin(\pi w)}, \quad -1 < \operatorname{Re} w < 1 \quad \text{by Identity Theorem}$$

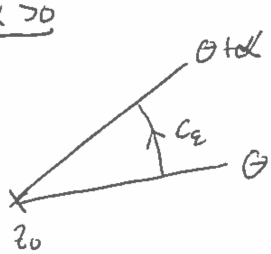
7.5 Fractional residues

Theorem z_0 simple pole of f, C_ε an arc of the circle $\{|z - z_0| = \varepsilon\}$ of opening α

then $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,dz = \alpha i \operatorname{Res}(f, z_0)$ $\qquad (\alpha = 2\pi$ Residue Thm$)$

$(\alpha < 0, \ C_\varepsilon$ clockwise$)$

$\alpha > 0$



$f(z) = \frac{a}{z - z_0} + g(z)$, g holom. near z_0.

$$\left| \int_{C_\varepsilon} \underset{\substack{\uparrow \\ \text{bounded}}}{g(z)}\,dz \right| \le M \varepsilon \alpha \to 0, \ \varepsilon \to 0$$

$$\int_{C_\varepsilon} \frac{a}{z - z_0}\,dz = \int_\theta^{\theta + \alpha} \frac{a}{\varepsilon e^{it}} \varepsilon i e^{it}\,dt = a i \alpha.$$

$z = z_0 + \varepsilon e^{it}$, $\theta \le t \le \theta + \alpha$

$$\Rightarrow \int_{C_\varepsilon} f(z)\,dz = \alpha i \operatorname{Res}(f, z_0) + \underbrace{\int_{C_\varepsilon} g(z)\,dz}_{\to\, 0,\ \varepsilon \to 0} \qquad \square$$

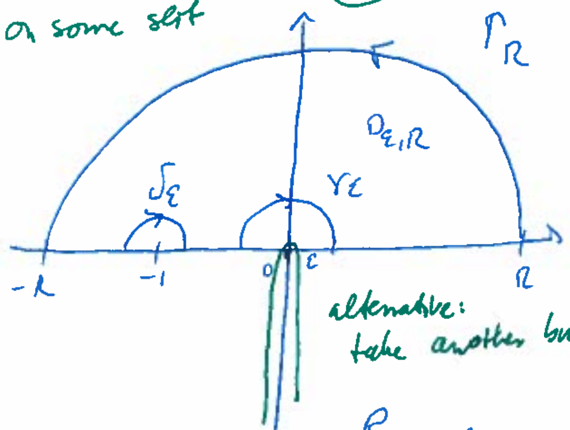
Application: $\displaystyle\int_0^\infty \frac{\log x}{x^2-1}\,dx = \frac{\pi^2}{4}$

$f(x) = \dfrac{\log x}{x^2-1}$, $f(z) = \dfrac{\operatorname{Log} z}{z^2-1}$, $f(1) = \frac{1}{2}$ holom. extension

use $f(z) = \dfrac{\log z}{z^2-1}$, $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$ hol. on $\mathbb{C}\setminus\{\operatorname{Im} z \le 0\}$

holom b/c is inverse f exp on some slit plane

f holom. in $\overline{D_{\varepsilon,R}}$



alternative: take another branch of the log.

$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$

$$0 = \int_{\partial D_{\varepsilon,R}} f(z)\,dz = \int_\varepsilon^R \frac{\log x}{x^2-1}\,dx + \left(\int_{-R}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{-\varepsilon}\right) \frac{\log|x| + i\pi}{x^2-1}\,dx$$

$$+ \left(\int_{\Gamma_R} + \int_{\gamma_\varepsilon} + \int_{\delta_\varepsilon}\right) f(z)\,dz$$

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \pi R \frac{\log R + \pi}{R^2-1} \to 0, \quad R\to\infty$$

$\boxed{R>1}$

$$\left|\int_{\gamma_\varepsilon} f(z)\,dz\right| \le \pi\varepsilon \frac{-\overbrace{\log\varepsilon}^{<0,\ \varepsilon<1} + \pi}{1-\varepsilon^2} \to 0 \quad \varepsilon\to 0$$

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$$\lim_{\varepsilon\to 0} \int_{\delta_\varepsilon} \frac{\operatorname{Log} z}{z^2-1}\,dz = -\pi i \operatorname{Res}\left(\frac{\operatorname{Log} z}{z^2-1}, -1\right) = -\pi i \left.\frac{\operatorname{Log} z}{2z}\right|_{z=-1}$$

$$= -\pi i \frac{i\pi}{2} = -\frac{\pi^2}{2}, \quad -1 \text{ simple pole}$$

$\tilde{f}(z) = \frac{\log z}{z^2-1}$, $\log z = \log$ $+ i \arg z$

$\pi - \delta < \arg z < \pi + \delta$

$$\lim_{\varepsilon\to 0} \operatorname{Re} \int_{\delta_\varepsilon} \frac{\operatorname{Log} z}{z^2-1}\,dz = -\frac{\pi^2}{2}$$

Take Real parts

$$0 = \int_\varepsilon^R \frac{\log x}{x^2-1}\,dx + \int_{[-R,-1-\varepsilon]\cup[-1+\varepsilon,-\varepsilon]} \frac{\log|x|}{x^2-1}\,dx + \operatorname{Re}\left[\left(\int_{\Gamma_R} + \int_{\gamma_\varepsilon} + \int_{\delta_\varepsilon}\right) f(z)\,dz\right]$$

$\underline{\varepsilon \downarrow 0,\ R\uparrow\infty:}$
$$0 = \int_0^{+\infty} \frac{\log x}{x^2-1}\,dx + \int_{-\infty}^{0} \frac{\log|x|}{x^2-1}\,dx - \frac{\pi^2}{2}.$$

cont. at -1

$$\Rightarrow 2\int_0^\infty \frac{\log x}{x^2-1}\,dx = \frac{\pi^2}{2}.$$

$\square$

7.6. Principal values

f cont. on $[a,b] \setminus \{x_0\}$, $a < x_0 < b$.

$$\text{p.v.} \int_a^b f(x)\,dx := \lim_{\varepsilon \to 0} \left(\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx \right) \quad \underline{\text{Principal value}}$$

provided the limit exists.

$\int_a^b f(x)\,dx$ is conv. if $\int_a^{x_0} f(x)\,dx$ & $\int_{x_0}^b f(x)\,dx$ <u>both</u> converge.

$$\text{So} \quad \int_a^b f(x)\,dx = \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \left[\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\delta}^b f(x)\,dx \right].$$

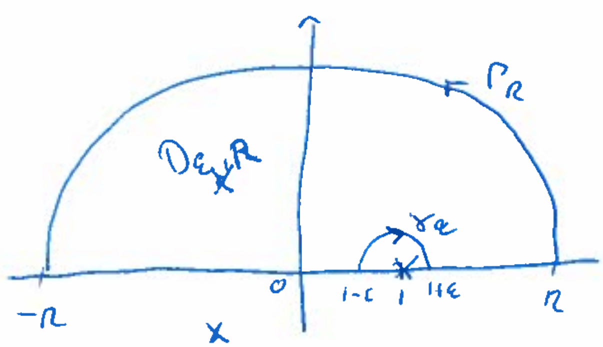
$\int_{-1}^1 \frac{1}{x}\,dx$ div. b/c $\int_0^1 \frac{1}{x}\,dx = +\infty$ <u>but</u> $\text{p.v.} \int_{-1}^1 \frac{1}{x}\,dx$

$$= \lim_{\varepsilon \to 0} \left(\int_{-1}^{\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^1 \frac{1}{x}\,dx \right)$$

$$= \lim_{\varepsilon \to 0} (\log \varepsilon - \log \varepsilon) = 0.$$

<u>Ex:</u> $\text{p.v.} \int_{-\infty}^{+\infty} \frac{dx}{x^3-1}$ (div. b/c. $\frac{1}{x^3-1} \sim \frac{1}{x-1}$ near 1)

$$= -\frac{\pi}{\sqrt{3}}$$



$z_0 = e^{\frac{2\pi i}{3}}$ simple pole of f

Resid. Thm: $\displaystyle\int_{\partial D_{\varepsilon,R}} \frac{dz}{z^3-1} = 2\pi i \left.\frac{1}{3z^2}\right|_{z=e^{\frac{2\pi i}{3}}} = \frac{2\pi i}{3} \cdot \frac{e^{\frac{2\pi i}{3}}}{1}$

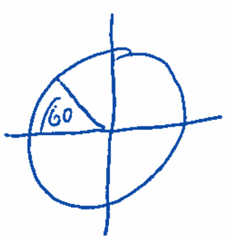
$$\int_{-R}^{R} \quad \underbrace{\int_{[-R,1-\varepsilon]\cup[1+\varepsilon,R]} \frac{dx}{x^3-1}} + \underbrace{\int_{\Gamma_R} \frac{dz}{z^3-1}}_{\to 0,\ R\to\infty} + \int_{\gamma_\varepsilon} \frac{dz}{z^3-1} = \frac{2\pi i}{3} e^{\frac{2\pi i}{3}}$$

$$\int_{\gamma_\varepsilon} \frac{dz}{z^3-1} \to -\pi i \frac{1}{3}, \quad \mathrm{Res}\left(\frac{1}{z^3-1}, 1\right)$$

$R \to \infty$; $\varepsilon \to 0$

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{dx}{x^3-1} = \cancel{\frac{\pi i}{3} e^{\frac{2\pi i}{3}} + \frac{2\pi i}{3} e^{\frac{2\pi i}{3}}}$$

$$= \frac{2\pi i}{3} e^{\frac{2\pi i}{3}} + \frac{\pi i}{3}$$



$$= \frac{\pi i}{3} + \frac{2\pi i}{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \qquad \cos\left(\tfrac{2\pi}{3}\right) = -\tfrac{1}{2}$$

$$= \frac{-\pi}{\sqrt{3}}$$

7.7. Jordan's lemma

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \sin(x)\, dx, \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x)\, dx, \quad Q \neq 0 \text{ on } \mathbb{R}$$

$\deg P = n$, $\deg Q = n+1$ ($\deg Q \geq n+2 \Rightarrow$ integrals are abs. conv.)

For $x > R$: $\left.\begin{array}{l} |P(x)| \geq \varepsilon x^n \\ |Q(x)| \leq M x^{n+1} \end{array}\right\} x$ large

$$\int_R^{+\infty} \cancel{Abs} \left|\frac{P(x)}{Q(x)}\right| |\sin(x)|\, dx \geq \frac{\varepsilon}{M} \int_R^{+\infty} \frac{|\sin x|}{x} dx = +\infty.$$

not abs. conv., but still conv.

$\int_0^\infty f(x) g(x)\, dx$ if: 1) $f(x) \searrow 0$ as $x \to +\infty$.

2) $\exists\, G$ bounded ($|G(x)| \leq M$), $G' = g$.

then $\int_0^\infty f(x) g(x)\, dx$ converges

Related to:

Dirichlet'um $\sum_{n=1}^{\infty} a_n b_n$, 1) $a_n \searrow 0$ 2) $\sum_{n=1}^{N} b_n$ bounded $\forall N$.

$\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ is convergent.

03/30

Dirichlet Thm:

If $f(x) \downarrow 0$ $\mathcal{C}^1$

and $\left| \int_0^x g(t)\,dt \right| \le M$ $g \in \mathcal{C}$. $\Rightarrow \int_0^\infty fg$ converges

<u>Pf:</u> $\exists\, G,\ G' = g,\ 0 \le G \le N$ on $[0, +\infty)$

$$\rightarrow \int_0^x f(t)\,g(t)\,dt = f\,G\Big|_0^x - \int_0^x f'G$$

$$= f(x)\,G(x) - f(0)\,G(0) - \int_0^x f'(t)\,G(t)\,dt$$

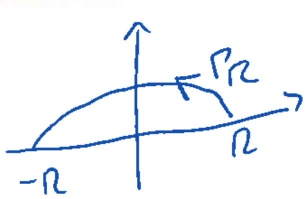
<u>Note</u> $\displaystyle\int_0^x |f'G|\,dt \underset{f'\le 0}{=} \int_0^x -f'G\,dt \le N(f(0) - f(x)) \le N(f(0))$

<u>abs. conv.</u>

$$\therefore \lim_{x \to \infty} \int_0^x fg = 0 - f(0)\,G(0) - \int_0^\infty f'G\,dt. \qquad \square$$

Jordan's Lemma

$$\left| \int_{\Gamma_R} e^{iz}\,dz \right| \le \int_{\Gamma_R} |e^{iz}|\,|dz| < \pi, \quad \forall R > 0$$



$$|e^{iz}| = e^{-\operatorname{Im} z} = e^{-R\sin t}$$

$z = Re^{it}$

03/30

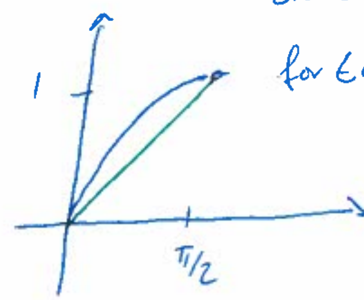
$$\therefore \int_{\Gamma_R} |e^{iz}|\,|dz| = \int_0^{\pi} e^{-R\sin t} R\,dt$$

$$= R\left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right) e^{-R\sin t} dt = 2R\int_0^{\pi/2} e^{-R\sin t}\, dt$$

$\theta = \pi - t$

$\sin(\pi - t) = \sin(t)$

$\sin t \geq \frac{2t}{\pi}$ for $t \in [0, \pi/2]$



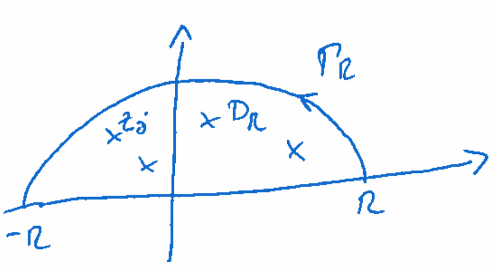
$$\leq 2R\int_0^{\pi/2} e^{-R\frac{2t}{\pi}} dt = -\pi e^{-\frac{2R}{\pi}t}\Big|_0^{\pi/2}$$

$$= \pi - \pi e^{-R} < \pi.$$

<u>Prop</u> P, Q polys, $Q(x) \neq 0\ \forall x \in \mathbb{R}$ $\deg Q \geq \deg P + 1$

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cos(x)\,dx + i\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \sin x\,dx$$

$$= 2\pi i \sum_{\operatorname{Im} z_j > 0} \operatorname{Res}\left(\frac{P(z)}{Q(z)} e^{iz}, z_j\right)$$



$$\int_{-R}^{R} \frac{P(x)}{Q(x)} e^{ix}\,dx + \underbrace{\int_{\Gamma_R} \frac{P(z)}{Q(z)} e^{iz}\,dz}_{\longrightarrow 0,\ R\to\infty} \underset{\text{Thm.}}{\overset{\text{Res.}}{=}} 2\pi i \sum_{\Im z_j>0} \text{Res} \dots$$ 03/3(

Let $\deg P = n$, $\deg Q \geq n+1$

So ~~At~~ $|P(z)| \leq C|z|^n$, $|z|\geq 1$, $|Q(z)| \geq \varepsilon |z|^{n+1}$, $|z| \geq R_0 \geq 1$

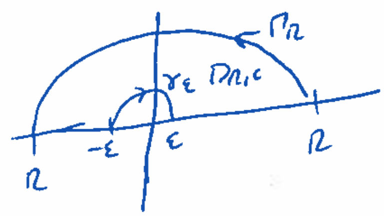
$$\left|\int_{\Gamma_R} \frac{P(z)}{Q(z)} \cancel{dz}\, e^{iz}\,dz\right| \leq \int_{\Gamma_R} \left|\frac{P(z)}{Q(z)}\right| |e^{iz}|\,|dz| \leq \frac{C}{\varepsilon R} \underbrace{\int_{\Gamma_R} |e^{iz}|\,|dz|}_{\leq \pi}$$

$\longrightarrow 0$, $R \to \infty$.

$$\Rightarrow \lim_{R\to+\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{ix}\,dx = RHS.$$ Moreover, the integral converg[...] t

~~If~~ by Dirichlet's Thm.

Ex: $\displaystyle\int_0^{+\infty} \frac{\sin x}{x} = \frac{\pi}{2}$ $\qquad f(z) = \dfrac{e^{iz}}{z}$



$$0 \overset{\text{Cauchy}}{=} \int_{\partial D_{\varepsilon,R}} \frac{e^{iz}}{z}\,dz = \underbrace{\int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\cos x}{x} + i\frac{\sin x}{x}\,dx}_{= 2i\int_{\varepsilon}^{R} \frac{\sin x}{\cancel{dx}\, x}\,dx} + \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz + \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,d[...]$$

$$\left|\int_{\Gamma_R}\frac{e^{iz}}{z}dz\right| \le \frac{1}{R}\int_{\Gamma_R}|e^{iz}|\,|dz| < \frac{\pi}{R} \to 0,\ R\to\infty$$

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{e^{iz}}{z}dz = -\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{z},0\right) = -\pi i$$

$$\Rightarrow R\to\infty,\ \varepsilon\to 0\ \exists\quad 2i\lim_{R\to\infty}\int_0^R\frac{\sin x}{x}dx = -\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{e^{iz}}{z}dz = \pi i. \quad \square$$

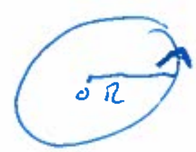
7.8. Residue at ∞

Recall $\quad \mathrm{Res}(g, z_0) = a_{-1} = \dfrac{1}{2\pi i}\displaystyle\int_{|z-z_0|=\varepsilon} g(z)\,dz$

Suppose f holom. in $\{|z|>R\}$. (∞ isolated singularity)

Laurent Series $\quad f(z) = \cdots + \dfrac{a_{-1}}{z} + a_0 + a_1 z + \cdots$

$$\mathrm{Res}(f,\infty) = -a_{-1} = -\frac{1}{2\pi i}\int_{|z|=R} f(z)\,dz.$$



$\times\ \infty$

Ex: $f(z) = \dfrac{b_1}{z} + \dfrac{b_2}{z^2} + \cdots$ for $|z|>R_0$

$f(\infty)=0 \quad$ holom. at infinity, but $\mathrm{Res}(f,\infty) = -b_1$.

03/30

<u>Ex:</u> $f(z) = \dfrac{z^4}{z^3-1}$, $|z|>1$

<u>∞:</u> $g(w) = f\left(\frac{1}{w}\right) = \dfrac{\frac{1}{w^4}}{\frac{1}{w^3}-1} = \dfrac{1}{w-w^4} = \dfrac{1}{w(1-w^3)}$

$$= \cancel{\ldots} \overset{|w|<1}{=} \frac{1}{w}\left(1+w^3+w^6+\ldots\right) = \frac{1}{w} + w^2 + w^5 + \ldots$$

$w=\frac{1}{z}$ $\quad f(z) = z + \dfrac{1}{z^2} + \dfrac{1}{z^5} + \ldots$, $|z|>1 \rightarrow \operatorname{Res}(f,\infty)=0$

<u>Lemma</u> $\operatorname{Res}(f(z),\infty) = \operatorname{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right)$ if ∞ <u>isol. sing. for.</u>

<u>Pf:</u> $\operatorname{Res}(f(z),\infty) = -\dfrac{1}{2\pi i}\displaystyle\int_{|z|=R} f(z)\,dz$

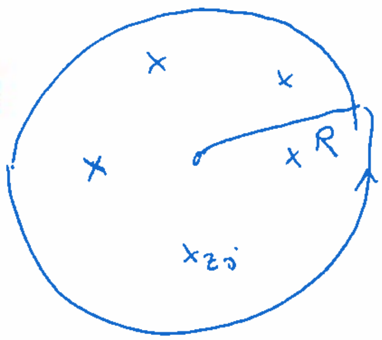
$$\overset{w=\frac{1}{z}}{=} -\frac{1}{2\pi i}\int_{\substack{|w|=\frac{1}{R} \\ \text{clockwise}}} f\left(\frac{1}{w}\right)\left(-\frac{dw}{w^2}\right) = -\frac{1}{2\pi i}\int_{|w|=\frac{1}{R}} f\left(\frac{1}{w}\right)\frac{1}{w^2}$$

$$= \operatorname{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right).$$

<u>Prop</u> Suppose f holom. in $\mathbb{C}$, ~~exp~~ except at $z_1, \dots, z_n$.

Then $$\sum_{j=1}^{n} \mathrm{Res}(f_j, z_j) + \mathrm{Res}(f, \infty) = 0$$

<u>Pf:</u>



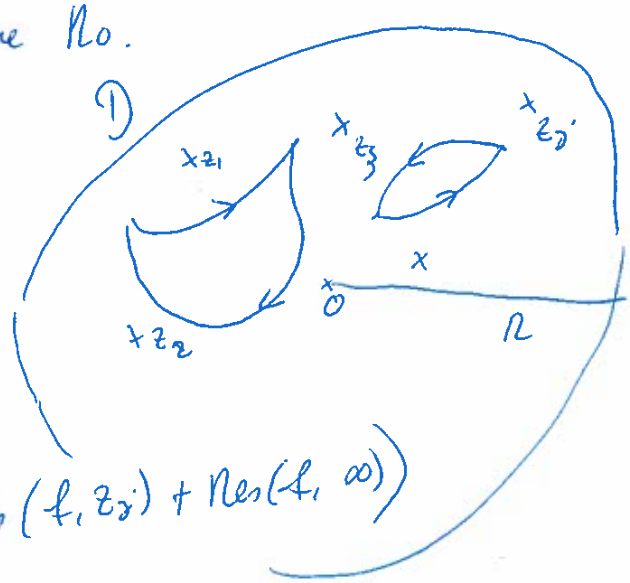
f holom. in $\{|z|>R\}$

$$\int_{|z|=R} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

$$= -2\pi i \,\mathrm{Res}(f(z), \infty).$$

$\square$

<u>Residue Thm. for exterior domains</u> $D \subseteq \mathbb{C}$ domain, piecewise smooth boundary. $D \supseteq \{|z| > R_0\}$ some R_0.



f holom. in a nbd. of $\bar{D}$.

except at $z_1, \dots, z_n \in D$.

$$\Rightarrow \int_{\partial D} f(z)\,dz = 2\pi i \left(\sum_{j=1}^{n} \mathrm{Res}(f, z_j) + \mathrm{Res}(f, \infty) \right)$$

∂D (pos. orient)

Fix $R > R_0$, ∂D, $z_1, \dots, z_n$ in $\{|z| < R\}$.

Apply Res. Thm. on $D \cap \{|z|<R\}$ 03/30

$$\underbrace{\int_{|z|=R} f(z)dz}_{= -2\pi i \,\mathrm{Res}(f,\infty)} + \int_{\partial D} f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

$= \cancel{2\pi i}$ $-2\pi i \,\mathrm{Res}(f,\infty)$

$\square$

04/02

8.1 The Argument Principle

Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ closed $\gamma(0)=\gamma(1)$ rectifiable

$w(\gamma,0)$

~~wt(γ)~~ $w(\gamma,0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$

$w(\gamma,0)$

<u>Lemma</u> $w(\gamma,0) \in \mathbb{Z}$. If $\arg(\gamma(t))$ continuous branch of $\arg(\gamma(t))$ ($t \mapsto \arg(\gamma(t))$ cont., $\arg(\gamma(t)) \in \arg(\gamma(t))$)

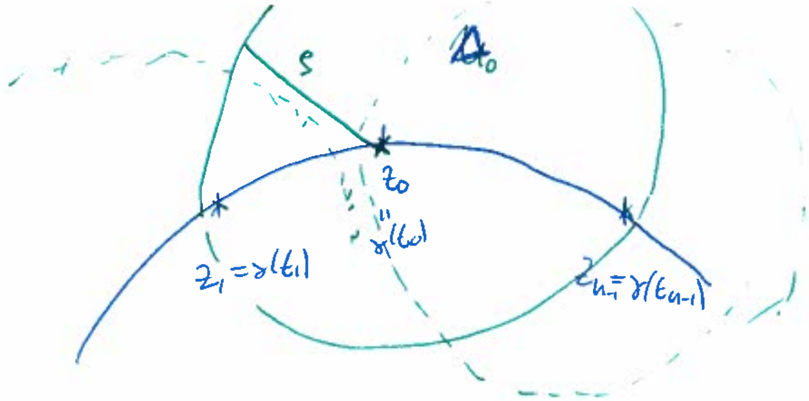
then $w(\gamma,0) = \frac{1}{2\pi}\left(\arg(\gamma(1)) - \arg(\gamma(0))\right)$.

<u>Pf:</u> $\varepsilon = \mathrm{dist}(0, \mathrm{tr}\,\gamma) > 0$, $\mathrm{tr}\,\gamma = \{\gamma(t) \mid t \in [0,1]\}$

$\exists\, \delta > 0$ s.t. $|t-t'| < \delta \Rightarrow |\gamma(t) - \gamma(t')| < \varepsilon$.

Pick $\frac{1}{n} < \delta$, $t_0 = 0 < t_1 = \frac{1}{n} < \dots < t_{n-1} = \frac{n-1}{n} < t_n = 1$.

$t_j = \frac{j}{n}$, $z_j = \gamma(t_j)$, $\Delta_j = \Delta(z_j, \varepsilon) \not\ni 0$.



$\gamma = \gamma_0 + \cdots + \gamma_{n-1}$, $\gamma_j(t) = \gamma(t)$, $t_j \le t \le t_{j+1}$.

Δ_0: $\frac{1}{z}$ holom. on Δ_0; $\exists f_0$ holom. $f_0' = \frac{1}{z}$, $f_0(z_0) \in \log(z_0)$

$\Rightarrow$ $\boxed{e^{f_0(z)} = z}$, so $f_0(z) \in \log z$, so $f_0(z) = \log|z| + i\,\mathrm{Arg}_0(z)$

$f_0(\gamma(t)) = \log \gamma(t) + i\,\mathrm{Arg}_0(\gamma(t))$ $0 = t_0 \le t \le t_1$, $t_{n-1} \le t \le t_n$

Check $(z e^{-f_0(z)})' = e^{-f_0(z)} + z e^{-f_0(z)}(-\frac{1}{z}) = 0$, $\therefore z e^{-f_0(z)} = z_0 e^{-f_0(z_0)} = 1$

Δ_1: $\exists f_1$ holom. $f_1'(z) = \frac{1}{z}$, $f_1(z_1) = f_0(z_1) \in \log z_1$

$\Rightarrow e^{f_1(z)} = z$. $\therefore f_1(z) = \log|z| + i\,\mathrm{Arg}_1(z)$

$\vdots$

Δ_j: $\exists f_j$ holom $f_j'(z) = \frac{1}{z}$, $f_j(z_j) = f_{j-1}(z_j) \in \log z_j$.

$\Rightarrow e^{f_j(z)} = z$, so $f_j(z) = \log|z| + i\,\mathrm{Arg}_j(z)$.

$\vdots$

Δ_{n-1} $\cdots\cdots$

$$\int_{\gamma_0} \frac{dz}{z} = f_0(z_1) - f_0(z_0)$$

$$\int_{\gamma_1} \frac{dz}{z} = f_1(z_2) - f_1(z_1)$$

$$\vdots$$

$$\int_{\gamma_{n-1}} \frac{dz}{z} = f_{n-1}(\underset{z_0}{\overset{\parallel}{z_n}}) - f_{n-1}(z_{n-1})$$

So $$W(\gamma, 0) = \frac{1}{2\pi i}\sum_{j=0}^{n-1}\int_{\gamma_j}\frac{dz}{z} = \frac{1}{2\pi i}\left(f_{n-1}(z_0) - f_0(z_0)\right)$$

$$= \frac{1}{2\pi i}\left(\arg_{n-1}(z_0) - \arg_0(z_0)\right).$$

Remark If $z_0 \notin \operatorname{tr}\gamma$, $W(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = W(\gamma - z_0, 0)$

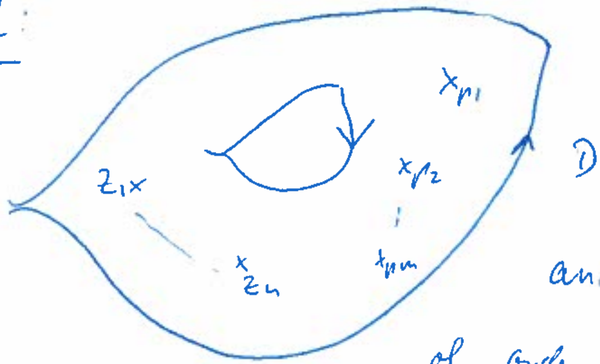
Theorem (Avg. Principle) 1) D bounded domain, piecew. smooth bdry. 2) f holom. in nbd of $\overline{D}$ except at finitely many poles, all of which are in D. 3) $f(z) \neq 0$ if $z \in \partial D$.

Then $$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty.$$

where $N_0 = \#$ zeros of f in D counting multiplicity

$N_\infty = \#$ poles of f in D counting order.

Pf:



let $z_1, \ldots, z_n$ denote the zeros in D of order $N_1, \ldots, N_n$

and $p_1, \ldots, p_m$ denote the poles in D of order $M_1, \ldots, M_m$.

$\frac{f'}{f}$ holom. in a nbd. of $\overline{D}$ except at $z_1, \ldots, z_n$ and $p_1, \ldots, p_m$.

z_1 : $f(z) = (z-z_1)^{N_1} h(z)$, h holom., $h(z_1) \neq 0$

$$\frac{f'(z_1)}{f(z_1)} = \frac{N_1}{z-z_1} + \underbrace{\frac{h'(z)}{h(z)}}_{\text{holom.}}$$

log. diff., z_1 simple pole of $\frac{f'}{f}$ and

order. $\operatorname{Res}\left(\frac{f'}{f}, z_1\right) = N_1$.

p_1 : $f(z) = (z-p_1)^{-M_1} h(z)$ h holom., $h(p_1) \neq 0$.

$$\frac{f'(z)}{f(z)} = -\frac{M_1}{z-p_1} + \underbrace{\frac{h'(z)}{h(z)}}_{\text{holom.}} \Rightarrow \operatorname{Res}\left(\frac{f'(z)}{f(z)}, p_1\right) = -M_1. \quad p_1 \text{ simple pole}$$

$\underline{\text{Res. Thm}}$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{n} \operatorname{Res}\left(\frac{f'}{f}, z_j\right) + \sum_{j=1}^{m} \operatorname{Res}\left(\frac{f'}{f}, p_j\right)$$

$$= \sum_{j=1}^{n} N_j - \sum_{j=1}^{m} M_j = N_0 - N_\infty.$$

$\square$

Remark f holom., nonvanishing in a nbd. of ∂D. 02/06

$\partial D = \gamma_1 \cup \cdots \cup \gamma_k$ with orientation given by bdry.

γ_j closed piecewise smooth.

$\Gamma_j = f \circ \gamma_j$ closed piecewise smooth., $0 \notin$ tr Γ_j. $\forall j$.

$$N_0 - N_\infty = \frac{1}{2\pi i} \sum_{j=1}^{k} \int_{\gamma_j} \frac{f'(z)}{f(z)} dz \qquad w = f(z)$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{k} \int_{\Gamma_j} \frac{dw}{w} = \sum_{j=1}^{k} W(\Gamma_j, 0)$$

$$= \sum_{j=1}^{k} \frac{1}{2\pi} \arg_j f \circ \gamma_j(t) \Big|_{t=0}^{t=1}$$

04/06

8.2 Rouché's Theorem

Theorem D bounded, piecewise smooth bdry. f, h holom. in a nbd of $\bar{D}$ s.t. $|h(z)| < |f(z)|$ $\forall z \in \partial D$. Then f, $f+h$ have the same number of zeros in D counting multiplicity.

04/64

Pf $f(z) \neq 0$, $|f(z)+h(z)| \geq |f(z)| - |h(z)| > 0 \quad \forall z \in \partial D$.

(*) $\Rightarrow$ arg. principle, $\partial D = \gamma_1 \cup \dots \cup \gamma_k$ piecewise smooth closed curves

$$N_0(f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)}\, dz$$

$$N_0(f+h) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z) + h'(z)}{f(z)+h(z)}\, dz$$

$$f+h = f\left(1 + \frac{h}{f}\right) \Rightarrow \frac{(f+h)'}{f+h} = \frac{f'}{f} + \frac{\left(1+\frac{h}{f}\right)'}{\left(1+\frac{h}{f}\right)}$$

$$N_0(f+h) = N_0(f) + \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{\gamma_j} \frac{\left(1+\frac{h}{f}\right)'}{\left(1+\frac{h}{f}\right)}\, dz,$$

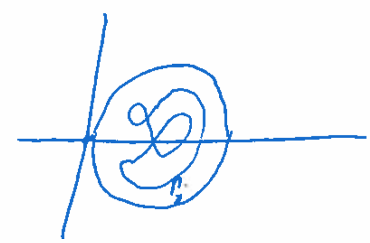
$w = 1 + \frac{h(z)}{f(z)}$ ~~near~~ near ∂D

$\Gamma_j = \left(1+\frac{h}{f}\right)(\gamma_j)$

$|\Gamma_j(t) - 1| < \left|\frac{h(\gamma_j(t))}{f(\gamma_j(t))}\right| < 1.$

$$= N_0(f) + \sum_{j=1}^{k} \frac{1}{2\pi i} \underbrace{\int_{\Gamma_j} \frac{dw}{w}}_{=0}$$

Γ_j closed, $\frac{1}{w}$ has primitive



Ex: How many roots does $p(z) = z^9 + z^5 - 8z^3 + 2z + 1$ have in $A = \{ 1 < |z| < 2 \}$.

1) Apply Rouché on $\{|z| < 2\}$, $f(z) = z^9$, $h(z) = z^5 - 8z^3 + 2z + 1$

$|z| = 2$, $|h(z)| \le 32 + 64 + 4 + 1 < 512 = |z|^9$

$p = f + h$ has 9 roots counting multipl. in $|z| < 2$.

2) Rouché on $\{|z| \le 1\}$ $f(z) = -8z^3$, $h(z) = z^9 + z^5 + 2z + 1$

If $|z| = 1$, $|h(z)| \le 1 + 1 + 2 + 1 = 5 < 8 = |f(z)|$.

$\Rightarrow$ $p = f + h$ has 3 roots counting multiplicity in $\{|z| < 1\}$. and none on $\{|z| = 1\}$. (by the inequality for Rouché)

$\therefore$ P has 6 roots in $\{ 1 < |z| < 2 \} = A$.

8.3 Hurwitz's Theorem

Theorem ~~If~~ Suppose f_k, f holom. in D, D domain. and $f_k \to f$ normally. If f has a zero of order N at $z_0 \in D$. (so $f \not\equiv 0$

Then $\exists\ \delta > 0, k_0 \in \mathbb{N}$ so that $\overline{\Delta}(z_0, \delta) \subseteq D$. and $\forall k \ge k_0$

f_k has exactly N zeros counting multipl. in $\Delta(z_0, \delta)$ counting multiplicity.

The zeros converge to z_0 as $k \to \infty$.

Ex: $f_k(x) = x^2 + \frac{1}{k}$, $f(x) = x^2$

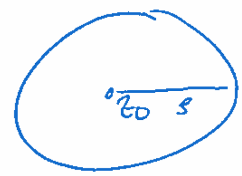
$f_k > 0$ on $\mathbb{R}$, $f_k \to f$ unif. on $\mathbb{R}$, $f(0) = 0$.

Hurwitz does not work for functions on $\mathbb{R}$.

Pf: Fix any $\rho > 0$ s.t.

- $\overline{\Delta}(z_0, \rho) \subseteq \mathcal{O}$.
- $f(z) \neq 0 \quad \forall z \in \overline{\Delta}(z_0, \rho) \setminus \{z_0\}$.

$$\delta := \min\{|f(z)| : |z - z_0| = \rho\} > 0.$$



want to use Rouché for: $\begin{cases} f(z) = 0 \\ f_k(z) = f(z) + (f_k(z) - f(z)) = 0 \end{cases}$

$\exists\, k_0 : \forall k \geq k_0$ s.t. $|f_k(z) - f(z)| < \delta$ for $|z - z_0| \leq \rho$.

Then $|f_k(z) - f(z)| < \delta < |f(z)| \overset{\text{Rouché}}{\Rightarrow} f_k$ has exactly N zeros

counting multiplicity in $\Delta(z_0, \rho)$ by Rouché.

$\forall \rho' < \rho \ \exists\, k_0(\rho')$ s.t. if $k \geq k_0(\rho')$ f_k has N zeros in $\Delta(z_0, \rho')$

$\therefore$ zeros of $f_k \to z_0$. $\square$

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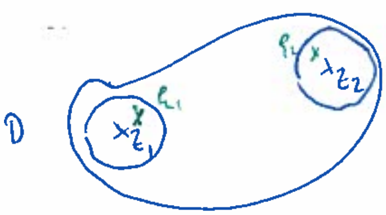
Corollary If $f_k \to f$ normally on a domain D, with $f_k(z) \neq 0$ $\forall z \in D$, $k \geq 1$. Then either $f \equiv 0$ or else $f(z) \neq 0$ $\forall z \in D$.

Def: f univalent on D if f holom. and injective on D.

Corollary If f_k are univalent on a domain D and $f_k \to f$ normally on D, then either f is constant or else it is univalent in D.

Pf. Assume f non-constant, not injective on D. $\exists$ $z_1, z_2 \in D$ where $f(z_1) = f(z_2) = w$. Apply ~~previous Corollary~~ Hurwitz Thm. to $f_k - w$ and $f - w \not\equiv 0$.



for some k large enough $\exists$ ξ_1, ξ_2 $f_k(\xi_j) = w$

$\Rightarrow$ f_k not injective. $\square$

§4 Open Mapping Theorem

Def: Say f ~~assumes~~ ~~attains~~ assumes $w \in \mathbb{C}$ m-times at $z_0 \in \mathbb{C}$ if $f - w$ has a zero of order m at z_0. $(m \geq 1)$

$$\left(f(z) = w + a_m (z - z_0)^m + \dots \right)$$

$a_m \neq 0$ near z_0.

Ex: $f(z) = z^3 - z^2 + 1$, $w = 1$

$z_0 = 1$ $\quad f(z) - 1 = z^3 - z^2 = z^2(z-1)$, so $w = 1$ is assumed once by f at $z_0 = 1$

$z_1 = 0$ $\quad$ Then $w = 1$ is assumed twice by f at $z_1 = 0$

Def: Say f assumes a value $w \in \mathbb{C}$ m-times at ~~the~~ ∞ if $f(\frac{1}{z}) - w$ has a zero of order m at 0.

Say f assumes the value ∞ m-times at $z_0 \in \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ if z_0 is a pole of order m.

Ex: $f(z) = z^3 - z^2 + 1$ assumes every value $w \in \mathbb{C}^*$ 3 times in $\mathbb{C}^*$.

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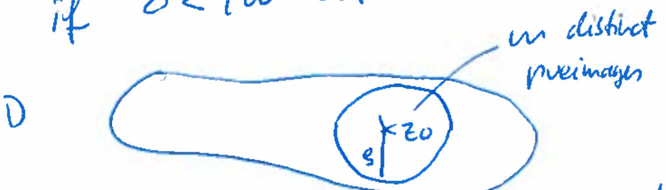
Def: f assumes value w_0 at z_0 $m \geq 1$-times ($f - w_0$ has ~~an~~ zero of order m at z_0)

$\Rightarrow$ f' has zero of order $m-1$ at z_0.

Call z_0 critical point of order $m-1$ of f., $w_0 = f(z_0)$ is a critical value.

Theorem Suppose f holom. on D and value $w_0 = f(z_0)$ is assumed $m \geq 1$ times at z_0. Then $\exists\ \rho, \delta > 0$ s.t. $\bar{\Delta}(z_0, \rho) \subset D$ and if $0 < |w - w_0| < \delta$ then $f(z) = w$ has m distinct roots in $\Delta(z_0, \rho)$



Example $f(z) = z^m$.

Proof: Pick $\rho > 0$ s.t. 1) $\overline{\Delta}(z_0,\rho) \subset D$.

2) If $z \in \overline{\Delta}(z_0,\rho) \setminus \{z_0\}$ then $f(z) \neq w_0$

($f - w_0 \not\equiv 0 \to$ zeros isolated)

3) if $z \in \overline{\Delta}(z_0,\rho) \setminus \{z_0\}$, then $f'(z) \neq 0$.

Let $\delta := \min\{|f(z) - w_0| \mid |z - z_0| = \rho\} > 0$

$f(z) - w_0 = 0$ has m roots in $\Delta(z_0,\rho)$ (all at z_0)

$f(z) - w = f(z) - w_0 + (w_0 - w)$

If $|w_0 - w| < \delta \leq |f(z) - w_0|$ if $|z - z_0| = \rho \Rightarrow$ Claim by Rouché.

(Roots are distinct by 3)). $\square$

Open Mapping Theorem If f is holom, not const. on D domain, then

f is open: $f(U)$ open $\forall U \subseteq D$ open.

Pf: Let $w_0 \in f(U)$, $z_0 \in U$, $f(z_0) = w_0$. Apply Theorem to $f|U$

get $\Delta(w_0, \delta) \subseteq f(U)$. ($w_0$ is assume m-times $1 \leq m < \infty$ b/c f not const.)

Theorem If f univalent in D, then $f'(z) \neq 0$. $\forall z \in D$.

Pf: Assume $\exists z_0 \in D$ $f'(z_0) = 0$, so $w_0 = f(z_0)$ is assumed $m \geq 2$-times. Contradiction: w near w_0 are assumed at (at least) 2 distinct points near z_0. $\square$

Ex: $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x^2, y^2)$ not open
$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ inj. on $\mathbb{R}$, $f'(0) = 0$.

Inverse fct Theorem

Let f holom. in a nbd. of $\bar{\Delta}(z_0, \rho)$, $w_0 = f(z_0)$, $f'(z_0) \neq 0$ and $f(z) \neq z_0$ $\forall z \in \bar{\Delta}(z_0,\rho) \setminus \{z_0\}$. Let $\delta = \min\{|f(z) - w_0|, |z - w_0| = \rho\} > 0$.

Then for each $w \in \Delta(w_0, \delta)$, $\exists ! z \in \Delta(z_0, \rho)$; $z =: f^{-1}(w)$ with $f(z) = w$.) We have

up to here follows from thm.

$$z = f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta \quad \text{for } w \in \Delta(w_0, \delta).$$

Pf: Only need to prove formula

$f(\zeta) = w$ has a simple root at $z = f^{-1}(w)$ in $\Delta(z_0, \rho)$

Res. Thm $\Rightarrow$ $$\frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta = \operatorname{Res}\left(\frac{\zeta f'(\zeta)}{f(\zeta) - w}, \zeta = z\right)$$

$\square$ 04/06

$$= \frac{\xi f'(\xi)}{\underbrace{f'(\xi)}_{\neq 0 \text{ at } z}} \Big|_{\xi = z} = z = f^{-1}(w)$$

by since $f|_{\Delta(z_0,\delta)}$ univalent.

9.1 Schwarz Lemma

If f is holom. in $D = \Delta(0,1) = \{|z|<1\}$, $f(0)=0$, $|f(z)| \leq 1$ for $z \in$

Then 1) $|f(z)| \leq |z|$ $\forall z \in D$. Have $|f(z_0)| = |z_0|$ some $z_0 \neq 0$.

iff. $f(z) = \lambda z$, $|\lambda| = 1$.

2) $|f'(0)| \leq 1$. Have $|f'(0)| = 1$ iff. $f(z) = \lambda z$, $|\lambda| = 1$.

Pf: $f(z) = z\, g(z)$, g holom. in D.

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

Pick $|z| < r < 1$. Apply the maximum principle on $\{|z| \leq r\}$

$\Rightarrow |g(z)| \leq \max\limits_{|\xi| = r} |g(\xi)| \leq \frac{1}{r}$. let $r \nearrow 1$ so, then

$|g(z)| \leq 1$ $\forall z \in D$. $\Rightarrow |f(z)| = |z|\,|g(z)| \leq |z|$ and

$|g(0)| = |f'(0)| \leq 1$.

If $|f(z_0)| = |z_0|$, $z_0 \neq 0$ or $|f'(0)| = 1$

$\Rightarrow |g(z_0)| = 1$ or $|g(0)| = 1 \overset{\text{Strong}}{\underset{\text{max. principle}}{\Rightarrow}}$ g const., $g(z) = \lambda$, $|\lambda| = 1$ $\square$

Schwarz lemma for arbitrary disks (by rescaling)

If f holom. on $\Delta(z_0, R)$, $|f(z)| \leq M$, $\forall z$, and $f(z_0) = 0$

$g : \mathbb{D} \to \mathbb{D}$, $g(\xi) = \dfrac{f(z_0 + R\xi)}{M}$, $g(0) = \dfrac{f(z_0)}{M} = 0$.

$\overset{\text{Schw.}}{\Rightarrow} |g(\xi)| \leq |\xi| \Rightarrow |f(\underbrace{z_0 + R\xi}_{z})| \leq M|\xi|$ $\quad \xi = \dfrac{z - z_0}{R}$

$\square$

$\Rightarrow \boxed{|f(z)| \leq \dfrac{M}{R} |z - z_0|}$

$|g'(0)| \leq 1 \Rightarrow \dfrac{R}{M} |f'(z_0)| \leq 1$; $\quad |f'(z_0)| \leq \dfrac{M}{R}$

Equality $|f(z)| = \dfrac{M}{R}|z - z_0|$, some $z \neq z_0$ or $|f'(z_0)| = \dfrac{M}{R}$

iff. $g(\xi) = \lambda \xi$, $|\lambda| = 1$, so

$$f(z) = M\lambda \cdot \frac{z - z_0}{R} \iff f(z) = k(z - z_0)$$
$$|k| = \frac{M}{R}$$

9.2. Conformal self maps of $\mathbb{D}$

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$f : \mathbb{D} \to \mathbb{D}$ holom. bijective called an automorphism of $\mathbb{D}$

$(\operatorname{Aut} \mathbb{D}, \circ)$ group

Ex: $\varphi(z) = e^{i\varphi} \dfrac{z-a}{1-\bar{a}z}$, $|a|<1$, $\varphi \in \mathbb{R}$

$\varphi \in \operatorname{Aut}(\mathbb{D})$: $\varphi(a) = 0$ Moebius map $z = e^{i\theta}$

$$|\varphi(e^{i\theta})| = \left| \frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}} \right| = \left| \frac{e^{i\theta}-a}{e^{-i\theta}-\bar{a}} \right| = 1 \quad (\text{max. principle} \Rightarrow \varphi : \mathbb{D} \to \underline{\mathbb{D}})$$

$$\varphi'(z) = e^{i\varphi} \frac{1-\bar{a}z-(z-a)(-\bar{a})}{(1-\bar{a}z)^2} = e^{i\varphi} \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

Inverse $w = \dfrac{z-a}{1-\bar{a}z}$, $w(1-\bar{a}z) = z-a \Rightarrow w+a = z+\bar{a}wz$

$$\Rightarrow \quad z = \frac{w+a}{1+\bar{a}w}$$

Theorem $\operatorname{Aut} \mathbb{D} = \left\{ \varphi(z) = e^{i\varphi} \cdot \dfrac{z-a}{1-\bar{a}z} : |a|<1, \varphi \in \mathbb{R} \right\}$

($\exists$ bijection $\operatorname{Aut} \mathbb{D}$ to $\mathbb{D} \times \partial\mathbb{D}$)

$$\varphi \longmapsto \left(\underset{a}{\text{zero of } \varphi}, \frac{\varphi'(a)}{|\varphi'(a)|} \right)$$

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Pf: 1) $F \in \operatorname{Aut}\mathbb{D}$, $F(0)=0$

Schwarz lemma: $|F(z)| \le |z|$, $z \in \mathbb{D}$

Schwarz lemma to F^{-1}: $|F^{-1}(w)| \le |w|$, $w \in \mathbb{D}$.

if $w = F(z)$: $|z| \le |F(z)| \Rightarrow |F(z)| = |z| \Rightarrow F(z) = e^{i\varphi} z$

2) $F \in \operatorname{Aut}\mathbb{D}$. $\exists\, a \in \mathbb{D}$, $F(a)=0$

$z = \psi(\xi) = \dfrac{\xi + a}{1 + \bar{a}\xi} \Rightarrow \therefore F \circ \psi \in \operatorname{Aut}\mathbb{D}$, $F(0)=0$

1) $\Rightarrow F(\underbrace{\psi(\xi)}_{z}) = e^{i\varphi}\xi = e^{i\varphi}\dfrac{z-a}{1-\bar{a}z}$.

Remark: $\operatorname{Aut}\mathbb{D}$ acts transitively on $\mathbb{D}$.

$\forall\, z_0, w_0 \in \mathbb{D}\ \exists\, \psi \in \operatorname{Aut}\mathbb{D}$: $\psi(z_0) = w_0$.

Pick's lemma (Invariant form of Schwarz lemma)

Let $f : \mathbb{D} \to \mathbb{D}$ be holom., Then:

1) $$\left| \frac{f(z) - f(\xi)}{1 - \overline{f(\xi)} f(z)} \right| \le \left| \frac{z - \xi}{1 - \bar{\xi} z} \right|, \quad \forall\, \xi, z \in \mathbb{D}.$$

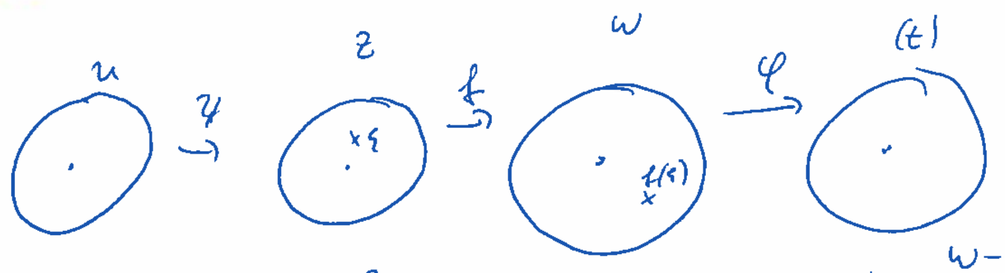
↑ characterises isometries of hyperbolic disk.

Equality holds for $\xi \neq z$ iff. $f \in \operatorname{Aut}\mathbb{D}$.

2) $$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2} \quad \forall z \in \mathbb{D}$$

Equality holds at $z \in \mathbb{D}$ iff. $f \in \operatorname{Aut}\mathbb{D}$.

Pf: 1) $\xi \in \mathbb{D}$ fixed.



$$z = \psi(u) = \frac{u+\xi}{1+\bar{\xi}u} \qquad t = \varphi(w) = \frac{w-f(\xi)}{1-\overline{f(\xi)}\,w}$$

$$\psi(0) = \xi$$

Now $F = \varphi \circ f \circ \psi : \mathbb{D} \to \mathbb{D}$, $F(0) = 0$.

Schwarz: $|F(u)| \le |u| \quad \forall u \in \mathbb{D}$ (**)

$$u = \psi^{-1}(z) = \frac{z-\xi}{1-\bar{\xi}z} \Rightarrow |\varphi \circ f(z)| \le |u| = \left|\frac{z-\xi}{1-\bar{\xi}z}\right|$$

$$\Downarrow$$

(*) $$\left|\frac{f(z)-f(\xi)}{1-\overline{f(\xi)}f(z)}\right| \le \left|\frac{z-\xi}{1-\bar{\xi}z}\right|$$

Equality (*) at $z \neq \xi$ equivalent to $=$ in (**) at $u = \psi^{-1}(z) \neq 0$

$\Leftrightarrow F(u) = \lambda u$, $|\lambda| = 1$

$\Rightarrow \quad f = \varphi^{-1} \circ F \circ \psi^{-1}$, so $f \in \mathrm{Aut}\,\mathbb{D}$.

If $f \in \mathrm{Aut}\,\mathbb{D} \Rightarrow F \in \mathrm{Aut}\,\mathbb{D}$ w. $F(0)=0 \Rightarrow F(u) = \lambda u \Rightarrow$ "$=$" in (**)

thus in (*)

$\Rightarrow$ 1).

2) $|F'(0)| \le 1$ by Schwarz.

$$F'(0) = \varphi'(f(\xi)) \cdot f'(\xi) \cdot \psi'(0)$$

$$= \frac{1-|f(\xi)|^2}{(1-\overline{f(\xi)}f(\xi))^2} \cdot f'(\xi) \cdot \frac{1-|\xi|^2}{(1+\bar{\xi}\cdot 0)^2} = \frac{1-|f(\xi)|^2}{(1-|f(\xi)|^2)^2} f'(\xi) \cdot (1-|\xi|^2)$$

$$\Rightarrow |F'(0)| = \frac{1-|\xi|^2}{1-|f(\xi)|^2} |f'(\xi)| \le 1 \iff |f'(\xi)| \le \frac{1-|f(\xi)|^2}{1-|\xi|^2} \quad (*)$$

"$=$" in (*) at some $\xi \iff |F'(0)| = 1 \iff F(u) = \lambda u,\ |\lambda| = 1$

$\square$

$\overset{\text{see}}{\underset{\text{before}}{\iff}} f \in \mathrm{Aut}\,\mathbb{D}$.

Alternative Proof of 2) If $z \neq \xi$

$$\left| \frac{f(z) - f(\xi)}{z - \xi} \right| \le \left| \frac{1 - \overline{f(\xi)} f(z)}{1 - \bar{\xi} z} \right| \qquad \text{Now } z \to \xi.$$

$$\Rightarrow \quad |f'(\xi)| \le \frac{1-|f(\xi)|^2}{1-|\xi|^2}$$

10.1 Poisson Integral Formula

Poisson kernel in $\mathbb{D}$, $z = re^{i\theta}$, $0 \le r < 1$

$$P(r,\theta) = P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta} = \frac{1-|z|^2}{|1-z|^2} = \operatorname{Re}\left(\frac{1+z}{1-z}\right)$$

imaginary

$$\frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-|z|^2 + z-\bar{z}}{|1-z|^2} \Rightarrow \operatorname{Re}\left(\frac{1+z}{1-z}\right) = \frac{1-|z|^2}{|1-z|^2}$$

and $|1-z|^2 = (1-z)(1-\bar{z}) = 1+|z|^2 - \underbrace{(z+\bar{z})}_{=2\operatorname{Re} z} = 1+|z|^2 - 2\operatorname{Re} z.$

Properties

1) $P(r,\cdot)$ is 2π-periodic, $P(r,\theta) = P(r,-\theta)$

2) $\int_0^{2\pi} P(r,\theta)\,d\theta = 2\pi$, $0 \le r < 1$ (Ch. 7)

3) $P(r,\cdot)$ decreases on $[0,\pi]$ and increasing on $[-\pi,0]$

Max at $\theta = 0$: $P(r,\theta) = \frac{1+r}{1-r} \nearrow +\infty$ as $r \nearrow 1$.

$\forall\, \delta > 0$ $\quad \max\{P(r,\theta) : \theta \in [-\pi,-\delta] \cup [\delta,\pi]\} = P(r,\delta)$

$$= \frac{1-r^2}{1+r^2-2r\cos\delta} \to 0$$

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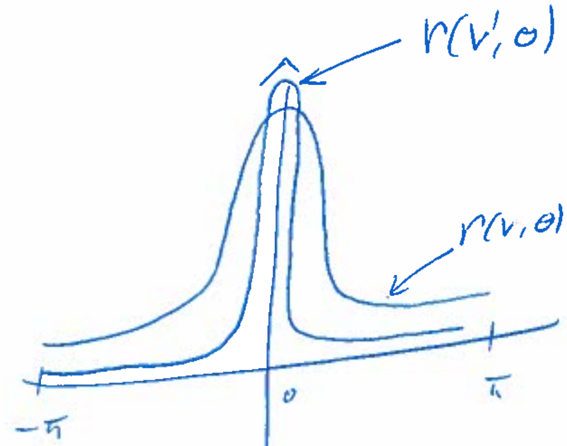

with $1>r'>r$.

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Let $h(e^{i\varphi})$ be cont. on $\mathbb{R}$

Poisson integral of h: $\underline{z\in\mathbb{D}}$.

$$\tilde{u}(z) = \hat{h}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\varphi})\, P(r,\theta-\varphi)\, d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\varphi})\ \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}\, d\varphi$$

<u>Lemma</u> $\hat{h}$ harmonic in $\mathbb{D}$.

<u>Pf:</u> Let $h = u+iv \Rightarrow \hat{h} = \tilde{u} + i\tilde{v}$

$$\tilde{u}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})\ \cancel{\mathrm{Re}}\ \mathrm{Re}\left(\frac{1+re^{i(\theta-\varphi)}}{1-re^{i(\theta-\varphi)}}\right) d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{u(e^{i\varphi})}_{\in\mathbb{R}}\, \mathrm{Re}\left(\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right) d\varphi = \mathrm{Re}\underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \overbrace{u(e^{i\varphi})\, \frac{e^{i\varphi}+z}{\underbrace{e^{i\varphi}-z}_{\neq 0,\ z\in\mathbb{D}}}}^{\text{holom in } z\text{, cont.}} d\varphi\right)}_{\text{holom in } \mathbb{D}.}$$

$\Rightarrow \tilde{u}$ is harmonic, similarly for $\tilde{v}$. □

<u>Theorem</u> If $\xi \in \partial\mathbb{D}$, then $\displaystyle\lim_{\substack{z\to\xi \\ z\in\mathbb{D}}} \hat{h}(z) = h(\xi)$. Thus

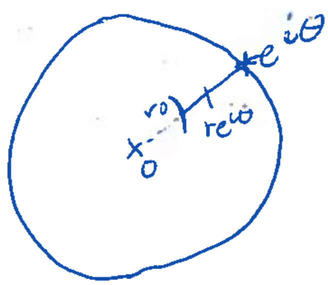
$h(e^{i\theta})$ extends to a cont. function on $\overline{\mathbb{D}}$, which is harmonic in $\mathbb{D}$.

<u>Remark</u> Dirichlet Problem for Δ, h given cont. function on $S^1 = \partial\mathbb{D}$.

Find $\check{h}$ on $\overline{\mathbb{D}}$ s.t. $\check{h}$ is cont. on $\overline{\mathbb{D}}$ and

$$\begin{cases} \Delta \check{h} = 0 & \text{on } \mathbb{D} \\ \check{h} = h & \text{on } \partial\mathbb{D}. \end{cases}$$

<u>Proof of Thm:</u>



$\forall \varepsilon > 0$, $\exists r_0(\varepsilon) < 1$ s.t. if $r_0 < r < 1$, $\theta \in [0, 2\pi]$ then $|\hat{h}(re^{i\theta}) - h(e^{i\theta})| < \varepsilon$

Fix M s.t. $|h(e^{i\theta})| \le M\ \forall \theta$, unif. cont $\Rightarrow \exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$|\varphi - \theta| < \delta \Rightarrow |h(e^{i\varphi}) - h(e^{i\theta})| < \varepsilon$.

$\max\{ P(r,\theta)\ \ \delta \le |\theta| \le \pi\} = P(r,\delta) < \varepsilon$ if $r_0 < r < 1$ for $r_0 = r_0(\delta) = r_0($

For $r_0<r<1$, $\theta\in[0,2\pi]$ 04/11

$$\left|\hat{h}(re^{i\theta})-h(e^{i\theta})\right| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\varphi})\, P(r,\underbrace{\theta-\varphi}_{=t})\,d\varphi - h(e^{i\theta})\right|$$

$\varphi=\theta-t$
$d\varphi=-dt$

$$= \left| \frac{1}{2\pi}\underbrace{\int_{\theta-\pi}^{\theta+\pi} h(e^{i(\theta-t)})\, P(r,t)\,dt}_{2\pi \text{ periodic in } t} - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\theta})\, P(r,\varphi)\,d\varphi\right|$$

$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{P(r,\varphi)}_{\text{pos.}} \left(h(e^{i(\theta-\varphi)})-h(e^{i\theta})\right) d\varphi\right|$$

$$\le \frac{1}{2\pi}\left(\int_{-\delta}^{\delta} + \int_{[-\pi,-\delta]\cup[\delta,\pi]}\right) P(r,\varphi)\, \left|h(e^{i(\theta-\varphi)})-h(e^{i\theta})\right| d\varphi$$

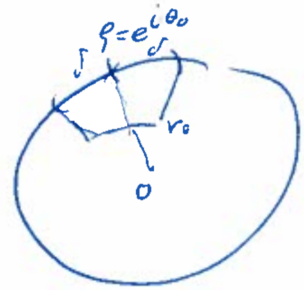
$$\le \frac{1}{2\pi}\left(\varepsilon \underbrace{\int_{-\delta}^{\delta} P(r,\varphi)\,d\varphi}_{\le \int_{-\pi}^{\pi} P(r,\varphi)\,d\varphi = 2\pi} + 2M\int_{[-\pi,\delta]\cup[\delta,\pi]} \overbrace{P(r,\varphi)}^{<\varepsilon \text{ if } r_0<r<1}\,d\varphi\right)$$

$$\le \frac{1}{2\pi}\left(\varepsilon\cdot 2\pi + 2M\varepsilon\cdot 2\pi\right) = \varepsilon + 2M\varepsilon = (2M+1)\cdot\varepsilon.$$

Now, suppose $\xi = e^{i\theta_0}\in\partial\mathbb{D}$. Let $\varepsilon>0$. $z=re^{i\theta}$, $r_0<r<1$, and

$|\theta-\theta_0|<\delta$

$$|\hat{h}(z) - h(e^{i\theta_0})| \le \underbrace{|\hat{h}(re^{i\theta}) - h(e^{i\theta})|}_{< 2(M+1)\varepsilon} + \underbrace{|h(e^{i\theta}) - h(e^{i\theta_0})|}_{< \varepsilon \text{ unif. cont.}}$$

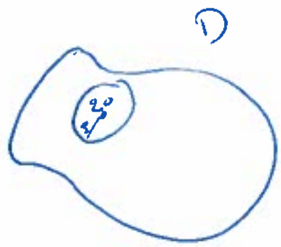


$$\le 2(M+1)\varepsilon.$$

10.2. Characterisation of harmonic functions

Recall: $h : D \to \mathbb{R}$ (or $\mathbb{C}$) cont. has MVP on D if $\forall z_0 \in D \ \exists \ \rho > 0$ $\Delta(z, \rho) \subseteq D$ s.t.

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{it})\, dt \quad \forall\, 0 < r < \rho.$$



Lemma Suppose $h : \overline{D} \to \mathbb{R}$ continuous, D bounded domain, and h has MVP in D. If $a \le h \le b$ on ∂D, $a, b \in \mathbb{R}$ then $a \le h(z) \le b$ for any $z \in D$.

Pf: let $M = \max\{h(z) : z \in \overline{D}\}$. Assume $M > b \Rightarrow \exists z_0 \in \underline{D}$, $h(z_0) = M$

$E := \{z \in D \mid h(z) = M\} \neq \emptyset$, E is closed in D. (h cont.)

and E is open by MVP: If $z \in E$, $\Delta(z, \rho)$ where MVP holds

then $M = h(z) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{h(z + re^{it})}_{\le M}\, dt \le M.$

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$\Rightarrow h(z+re^{it}) = M \quad \forall\, 0<r<\rho \Rightarrow \Delta(z,\rho) \subset E. \Rightarrow E = D.$

□

cont.
$\Rightarrow h = M$ on $\overline{D}$ but $M > b$ ↯.

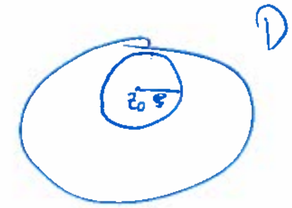
Theorem D open $\subset \mathbb{C}$, $h: D \to \mathbb{C}$ cont. Then

h has the MVP in D iff. h is harmonic.

Pf: "$\Leftarrow$" Ch. 3

"$\Rightarrow$" Let $h = u + iv$, then u, v have MVP on D (and are cont.)

Suffices to assume $h: D \to \mathbb{R}$. Fix $\overline{\Delta}(z_0,\rho) \subset D$.



$\exists\, \tilde{h}$ cont. on $\overline{\Delta}(z_0,\rho)$, harmonic in $\Delta(z_0,\rho)$

and $\tilde{h} = h$ on $\{|z-z_0| = \rho\}$

$\tilde{h} - h$ cont. on $\overline{\Delta}(z_0,\rho)$ and has the MVP on $\Delta(z_0,\rho)$

and $\tilde{h} - h|_{\partial\Delta(z_0,\rho)} = 0 \overset{\text{lemma}}{\Rightarrow} \tilde{h} = h$ on $\overline{\Delta}(z_0,\rho) \Rightarrow h$ is harmonic in $\Delta(z_0,\rho)$. □

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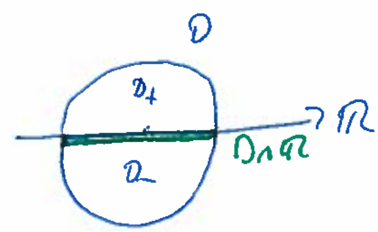
Cor. If u_n harmonic on D open, $u_n \to u$ normally, then u is harmonic.

Pf: u is continuous and has MVP. $\square$

10.3 Schwarz Reflection Principle

Let D symmetric domain wrt. real axis: $z \in D \Longleftrightarrow \bar{z} \in D$.

$D_+ = D \cap \{\operatorname{Im} z > 0\}$, $D_- = D \cap \{\operatorname{Im} z < 0\}$



Reflection for harmonic functions

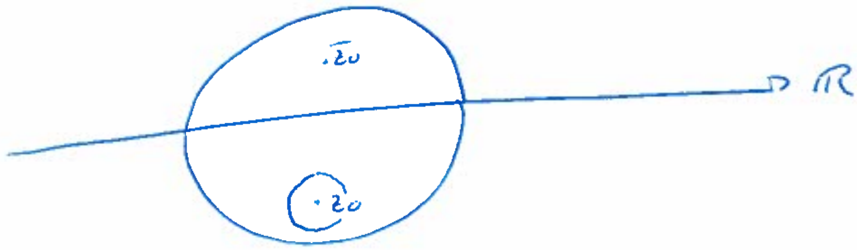
If $u : D_+ \to \mathbb{R}$ harmonic and $\forall \xi \in D \cap \mathbb{R}$ the limit $\lim\limits_{D_+ \ni z \to \xi} u(z) = 0$. Then u extends to a harmonic function on D which verifies $u(\bar{z}) = -u(z)$ $\forall z \in D$. $(*)$

Pf: $u(z) = \begin{cases} u(z), & z \in D^+ \\ 0, & z \in D \cap \mathbb{R} \\ -u(\bar{z}), & z \in D^- \end{cases}$ on D

$\Rightarrow$ u cont. on D and $u(\bar{z}) = -u(z)$ $\forall z \in D$.

u is harmonic on D since it has MVP on D:

1) $z_0 \in D_+$ ~~[illegible]~~



2) $z_0 \in D_-$, $\bar\Delta(z_0,\varrho) \subset D_-$, $r \le \varrho$.

$$\frac{1}{\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt = \frac{-1}{2\pi}\int_0^{2\pi} u(\bar z_0 + re^{-it})\,dt$$

$$\underset{d\theta=-dt}{\overset{\theta=2\pi-t}{=}} -\frac{1}{2\pi}\int_0^{2\pi} u(\bar z_0 + re^{i\theta})\,d\theta \overset{u\ \text{MVP}}{=} -u(\bar z_0) = u(z_0).$$

3) $z_0 \in D\cap\mathbb{R}$, $\bar\Delta(z_0,\varrho)\subset D$, $r\le\varrho$

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt = \frac{1}{2\pi}\left(\int_0^{\pi} u(z_0+re^{it})\,dt + \int_\pi^{2\pi} -u(\underbrace{\bar z_0}_{=z_0}+re^{-it})\,dt\right)$$

$$\underset{d\theta=-dt}{\overset{\theta=2\pi-t}{=}} \frac{1}{2\pi}\left(\int_0^{\pi} u(z_0+re^{it})\,dt - \int_0^{\pi} u(z_0+re^{i\theta})\,d\theta\right) = 0 = u(z_0). \quad \square$$

Reflection for holom. functions

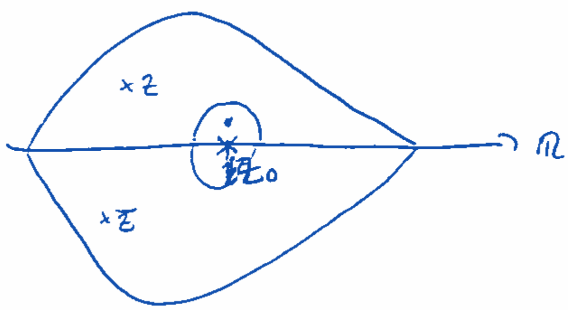
Let $f = u + iv$ holom on D_+ s.t. $\forall \xi \in D \cap \mathbb{R}$ $\lim_{D_+ \ni z \to \xi} v(z) = 0$.

Then f extends to a holom. function f on D ~~s.t.~~ which verifies

$$f(\bar{z}) = \overline{f(z)}, \quad z \in D \quad (*).$$

Pf: Step 1 f extends cont. to $D \cap \mathbb{R}$ and $f(D \cap \mathbb{R}) \subseteq \mathbb{R}$.

(i.e. if $z_0 \in \mathbb{R} \cap D$, then $\lim_{D \ni z \to z_0} f(z) \in \mathbb{R}$)



Pick $z_0 \in D \cap \mathbb{R}$, $\bar{\Delta}(z_0, \delta) \subset D$. Let v be the harmonic extension of v to $\Delta(z_0, \delta)$. $\exists$ $\hat{u}$ harmonic function on $\Delta(z_0, \delta)$ s.t. $g = \hat{u} + iv$ is holom in $\Delta(z_0, \delta)$ and $\hat{u}(z_0 + i\frac{\delta}{2}) = u(z_0 + i\,\delta/2)$

$\Rightarrow g = f$ on $\Delta(z_0, \delta) \cap \{\operatorname{Im} z > 0\}$. (agree at one point & have same imag. part)

$\Rightarrow \exists \lim_{D_+ \ni z \to z_0} f(z) = g(z_0) \in \mathbb{R}$ as $v(z_0) = 0$.

Step 2 Define for $z \in D$

$$f(z) = \begin{cases} f(z), & z \in D_+ \cup (D \cap \mathbb{R}) \\ \overline{f(\bar{z})}, & z \in D_- \end{cases}$$

verifies $(*)$

and is cont. on D.

Cont. at. $z_0 \in D \cap \mathbb{R}$:

$$\lim_{D_- \ni z \to z_0} f(z) = \lim_{D_- \ni z \to z_0} \overline{f(\bar{z})} = \lim_{D_+ \ni \bar{z} \to z_0} \overline{f(\bar{z})} = \underbrace{\overline{f(z_0)}}_{\in \mathbb{R}} = f(z_0).$$

□

f holom. on $D_+ \cup D_-$ and cont. on $D \overset{\text{Morera}}{\Rightarrow} f$ holom. on D.

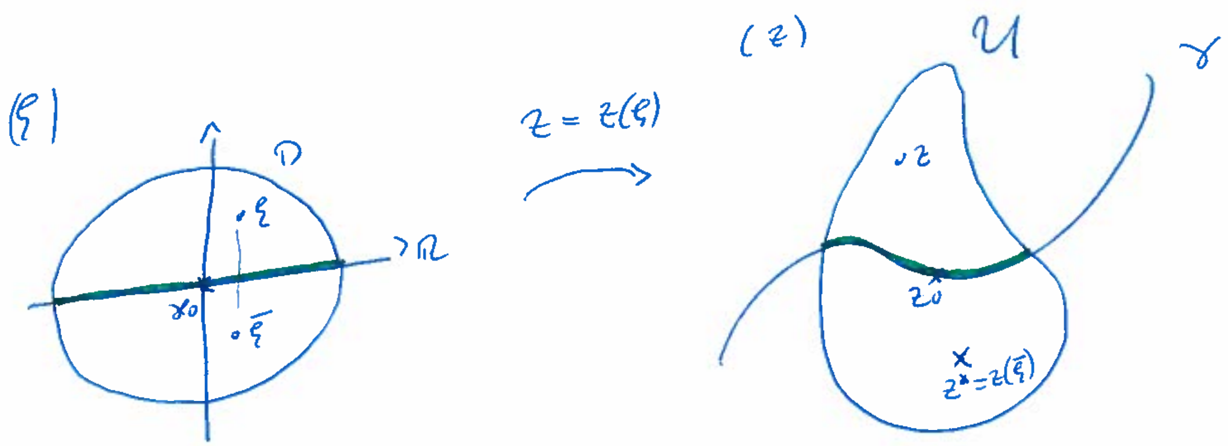
Reflection across analytic arcs

$\gamma : [0,1] \to \mathbb{C}$ curve, $\gamma = \gamma([0,1]) \subseteq \mathbb{C}$.

Def: curve $\gamma \subset \mathbb{C}$ is an analytic arc. if $\forall z_0 \in \gamma$ $\exists$ open nbd. $\mathcal{U} \ni z_0$ s.t. with the following properties:

$\exists$ $D = \Delta(x_0, r)$, $x_0 \in \mathbb{R}$ and an injective holom. map $z = z(\xi)$

$\xi \in D$, mapping D onto $\mathcal{U}$. and $D \cap \mathbb{R}$ onto $\mathcal{U} \cap \gamma$.



Reflection across γ is defined locally near each $z_0 \in \gamma$ (on $\mathcal{U}$):

$z^* = R(z)$ is given by $z^* = z(\bar{\xi})$ where $z = z(\xi) \in \mathcal{U}$.

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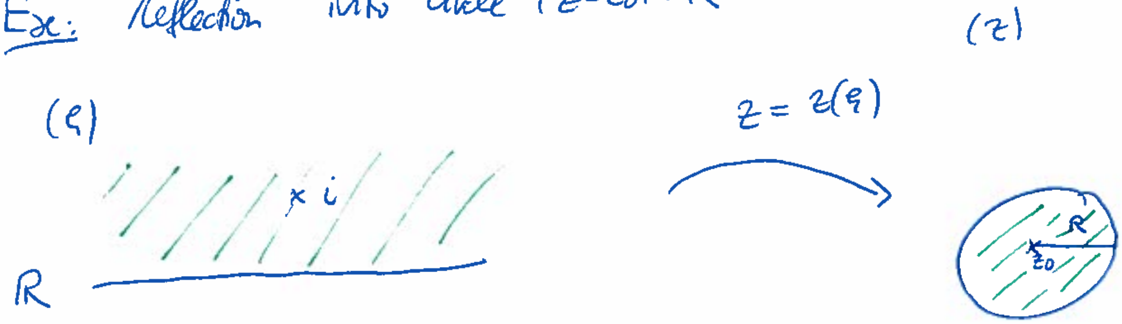
Note $\mathcal{R}$ is antiholom:

$$\overline{\mathcal{R}(z)} = \overline{z(\bar{\varsigma})}$$

Ex: Reflection into circle $|z-z_0|=R$



$$z = z_0 + R\cdot\frac{\varsigma-i}{\varsigma+i}$$ maps $\{\operatorname{Im}\varsigma>0\}$ into $\{|z-z_0|<R\}$

and $\mathbb{R}\cup\{\infty\}$ onto $\{|z-z_0|=R\}$

$\varsigma\in\mathbb{R}$: $|z-z_0| = R\left|\frac{\varsigma-i}{\varsigma+i}\right| = R$, $\quad i \mapsto z_0$.

$$z^* = z_0 + R\,\frac{\bar\varsigma-i}{\bar\varsigma+i} \quad \text{if } z = z_0 + R\,\frac{\varsigma-i}{\varsigma+i}$$

$$= z_0 + R\,\overline{\left(\frac{\varsigma+i}{\varsigma-i}\right)} = z_0 + R\cdot\frac{R}{\overline{z-z_0}} = z_0 + \frac{R^2}{\bar z - \bar z_0}$$

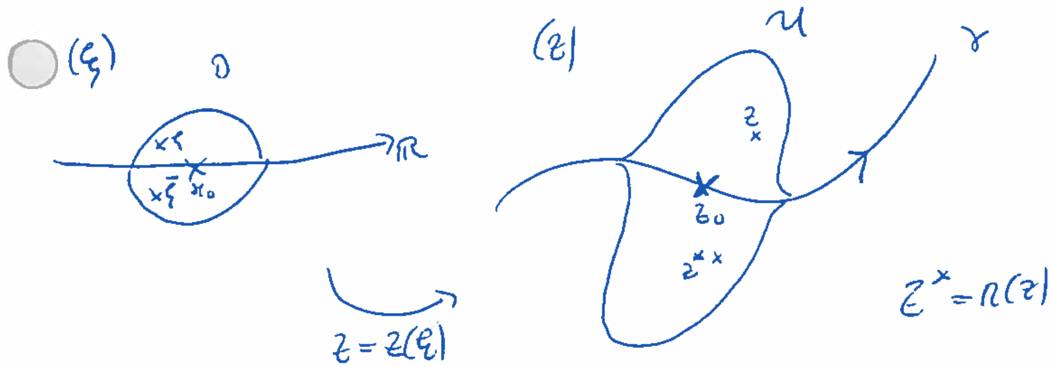
$$|z^*-z_0| = \frac{R^2}{|z-z_0|}, \quad \arg(z^*-z_0) = \arg(z-z_0).$$

"Inversion" In particular if $z_0=0$, $R=1$ (unit circle), then

$$z^* = \frac{1}{\bar z}$$ antiholom. (depends holom on $\bar z$, $\frac{\partial}{\partial z}\cdot = 0$)

R maps circles in $\mathbb{C}^*$ to circles in $\mathbb{C}^*$.

Analytic arc



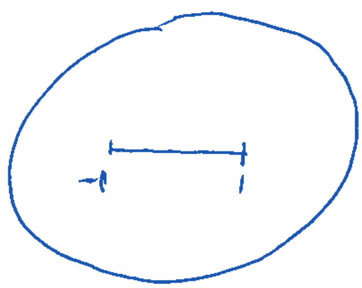
$z = z(\xi)$
biholom. $D \to U$
$D \cap \mathbb{R} \to U \cap \gamma$.

Circle: reflection into $|z| = R$, $z^* = \dfrac{R^2}{\bar{z}}$

Def: D domain, $\gamma \subseteq \partial D$ is called free analytic boundary arc if γ is an analytic arc $\forall z_0 \in \gamma$ $\exists$ nbd U s.t. $U \setminus \gamma$ has 2 connected components, one in D and the other in $\mathbb{C} \setminus \bar{D}$.

5-free analytic boundary arcs.

$D = \Delta(0,2) \setminus [-1,1]$



$(-1,1)$ analytic boundary arc, not free.

Lemma If f is holom. on a simply connected domain D, and $f(z) \neq 0 \ \forall z \in D$. 04/16

Then $\exists\, g$ holom. on D s.t. $f = e^{g}$.

(So $\log f$ has a holomorphic branch on D.)

Pf: $\frac{f'}{f}$ holom. in D. Let g be holom. on D s.t. $g' = \frac{f'}{f}$ (D simply connected)

and $g(z_0) \in \log f(z_0)$ for a fixed $z_0 \in D$.

Now $h = f e^{-g}$. Then $h(z_0) = 1$ and $h' = f' e^{-g} - f e^{-g} g' = 0$

$\Rightarrow h \equiv 1$, so $f \equiv e^{g}$.

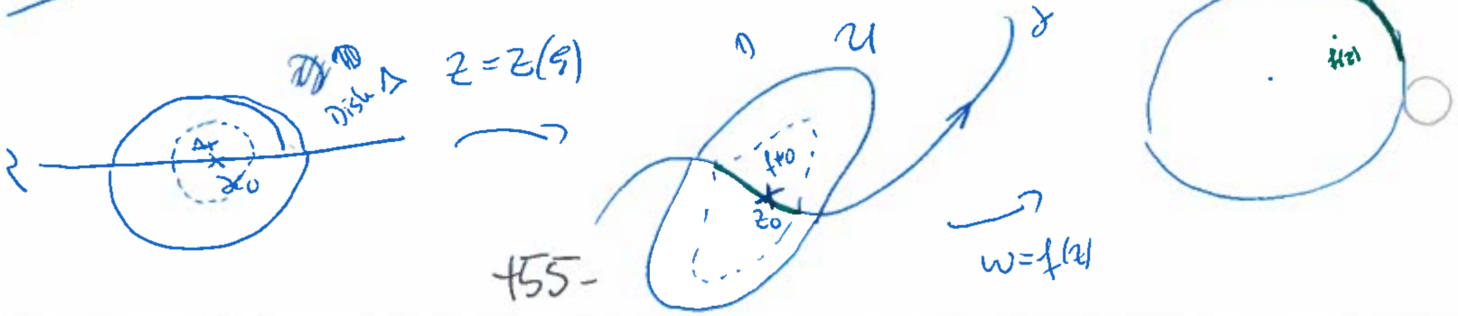
$\Rightarrow$ we can take n-th roots. ($f = e^{g} \rightsquigarrow \sqrt[n]{f} := e^{g/n}$)

Theorem Let D domain, γ free analytic boundary arc. Let f be holom. on D such that $|f(z)| \to R > 0$ as $z \in D \to \gamma$. Then f extends holomorphically to a nbhd. of γ and the extension satisfies

$$f(z^*) = \frac{R^2}{\overline{f(z)}}$$

for z near γ, z^* is the reflection of z across γ.

Pf: Pick $z_0 \in \gamma$, U nbhd. of z_0



Similarly ~~if~~ Disk Δ we may assume $\mathcal{U} \setminus \gamma$ has 2 components, one is D one is $\mathbb{C} \setminus \overline{D}$, $f(z) \neq 0$ if $z \in \mathcal{U} \cap D$ 04/16

<u>Wlog</u> $R = 1$. Assume true for $R = 1$, then $h = \frac{f}{R}$ extends and verifies

$$h(z^*) = \frac{1}{\overline{h(z)}} \Rightarrow f = R \cdot h \text{ extends}, \; f(z^*) = R \, h(z^*) = \frac{R}{\overline{h(z)}} = \frac{R^2}{\overline{f(z)}}.$$

$f \neq 0$ in $D \cap \mathcal{U}$ simply connected ($\exists$ $z(\xi)$ homeom.)

$\Rightarrow \exists \, g$ holom. in $D \cap \mathcal{U}$ s.t. $e^g = f$, $\operatorname{Re} g = \log |f|$

$g(z(\xi))$ holom. in Δ_+ and $\operatorname{Re} g(z(\xi)) \to 0 = \log |f|$, $\xi \to \mathbb{R} \cap \Delta$

$\therefore$ $g(z(\xi))$ extends holom. to Δ by classical Schwarz reflection principle.

$\therefore$ $f = e^g$ extends holom. to $\mathcal{U}$, $f(z) \neq 0 \; \forall z \in \mathcal{U}$.

For the identity let $F(z) := \dfrac{1}{\overline{f(z^*)}}$ holom. on $\mathcal{U}$ (z^* antiholom.)

$F = f$ on $\mathcal{U} \cap \gamma$. $z \in \mathcal{U} \cap \gamma \Rightarrow z = z^*$ and $|f(z)| = 1$.

then $F(z) = \dfrac{1}{\overline{f(z)}} = \dfrac{|f(z)|}{\overline{f(z)}} = f(z)$. Identity Theorem $\Rightarrow F \equiv f$ on $\mathcal{U}$,

So $\dfrac{1}{\overline{f(z^*)}} = f(z)$. $\square$

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Ch. 11 Conformal Mappings

11.1 Mappings to $\mathbb{D}$ and $\mathbb{H}$

Def: 1) $\mathbb{D} = \{ |z| < 1 \}$, $\mathbb{H} = \{ \operatorname{Im} z > 0 \}$

2) D, V domains, $f : D \to V$ conformal if it is holom. and bijective.

(biholomorphic map)

Problems 1) If D is domain, is there a conformal map $f : D \to \mathbb{D}$?

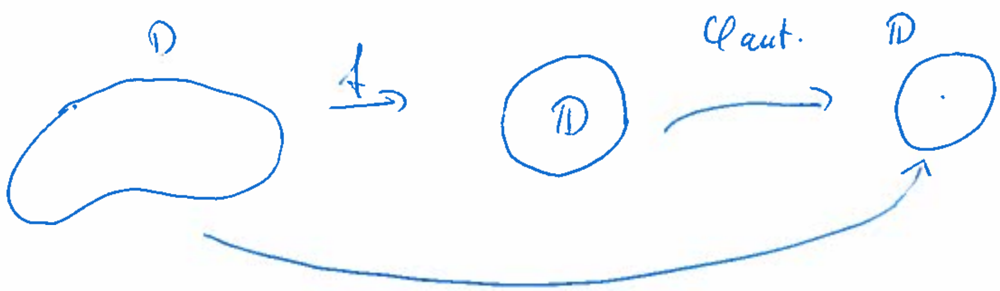
Necessary condition: $\mathbb{D}$ simply connected. Riemann mapping theorem: this is sufficient.

2) If f exists, what are all the conformal maps $D \to \mathbb{D}$?

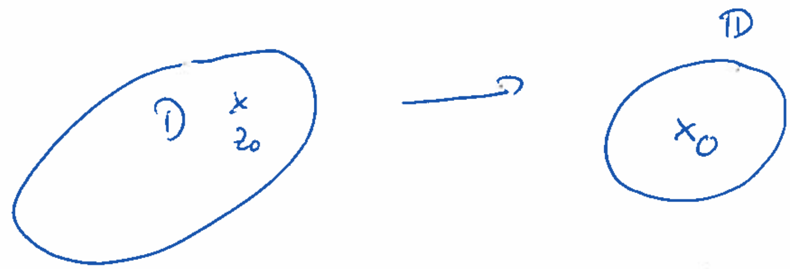
2): Let $f : D \to \mathbb{D}$ conformal, $g : \overset{z}{D} \to \overset{\xi}{\mathbb{D}}$ conformal.

then $g \circ f^{-1} : \mathbb{D} \to \mathbb{D}$ conf. $\Rightarrow$ $g \circ f^{-1}(\xi) = e^{i\theta} \dfrac{\xi - a}{1 - \bar{a}\xi}$, $\xi \in \mathbb{D}$ some $|a| < 1$, $\theta \in \mathbb{R}$

$$\xi = f(z) \Rightarrow g(z) = e^{i\theta} \frac{f(z) - a}{1 - \bar{a} f(z)}.$$



So: $\{ g : D \to \mathbb{D} \text{ conformal} \}$ depends on 3 real parameters.



If $z_0 \in D$ $\exists!\ g : D \to \mathbb{D}$ conf. s.t. $g(z_0) = 0$ and $g'(z_0) > 0$.

$\hookrightarrow \arg g'(z_0) = 0$

$$g(z) = e^{i\theta} \cdot \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}\, f(z)}, \quad g'(z) = e^{i\theta} \frac{1 - |f(z_0)|^2}{(1 - \overline{f(z_0)}\, f(z))^2} f'(z)$$

$$g'(z_0) = e^{i\theta} \frac{f'(z_0)}{1 - |f(z_0)|^2} \qquad \theta = -\arg f'(z_0).$$

$$g(z) = \frac{|f'(z_0)|}{f'(z_0)} \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}\, f(z)}, \quad g'(z_0) = \frac{|f'(z_0)|}{1 - |f(z_0)|^2}$$

---

1) Maps $\mathbb{H} \to \mathbb{D}$

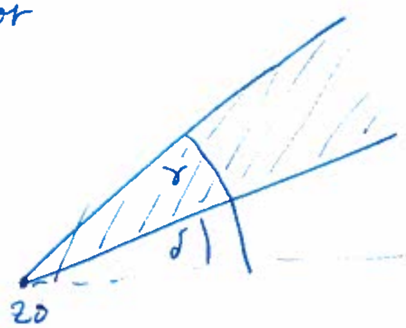
$$f(z) = e^{i\theta} \frac{z - z_0}{z - \overline{z_0}}, \quad z_0 \in \mathbb{H}, \theta \in \mathbb{R}.$$

$f(z_0) = 0$

$z \in \mathbb{R}$: $|f(z)| = \dfrac{|z - z_0|}{|z - \overline{z_0}|} = 1$

Ex: $w = \dfrac{z-i}{z+i}$

2) $D = \{\delta < \arg(z - z_0) < \gamma\}$ sector

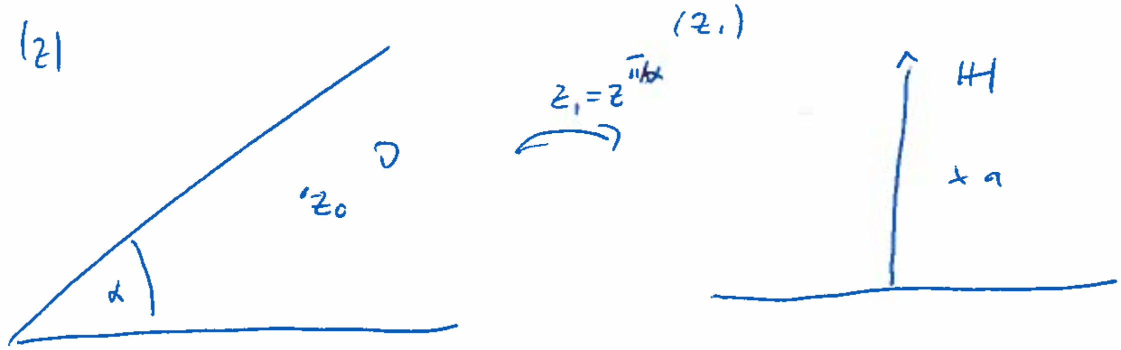


$z' = e^{-i\delta}(z - z_0)$

maps D onto standard sector D'

$D' = \{0 < \arg z' < \underbrace{\gamma - \delta}_{\alpha}\}$

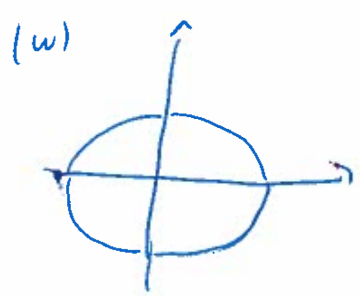
Assume $D = \{a \;\; 0 < \arg z < \alpha\}$, $0 < \alpha \le 2\pi$



$z_1^{\pi/\alpha} = |z|^{\pi/\alpha} e^{i\frac{\pi}{\alpha}\theta}$

$z = |z| e^{i\theta}, \; 0 < \theta < \alpha$

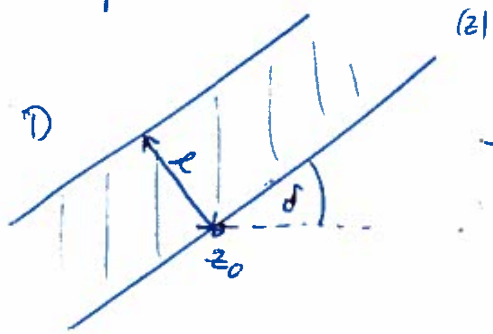
$\rightarrow$



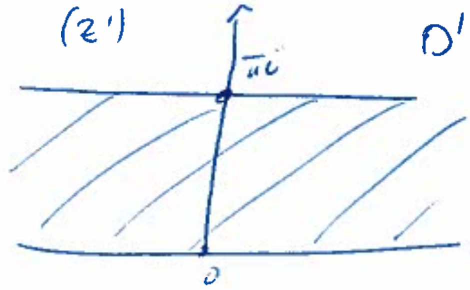
$w = e^{i\theta} \dfrac{z_1 - a}{z_1 - \bar{a}}$

$w = e^{i\theta} \dfrac{z^{\pi/\alpha} - a}{z^{\pi/\alpha} - \bar{a}}$

3) D = strip



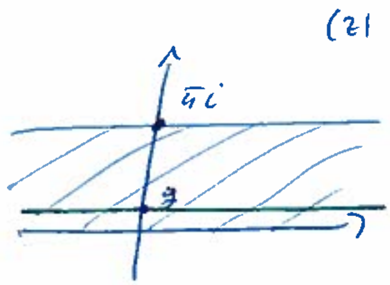
$\longrightarrow$



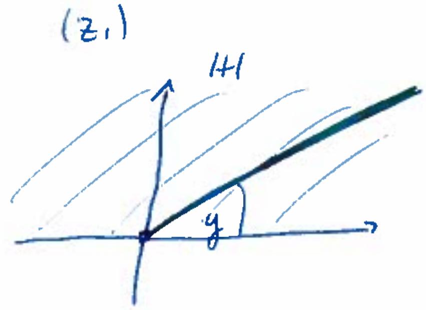
Standard strip

$$z' = \frac{\pi}{\ell} e^{-i\delta}(z - z_0)$$

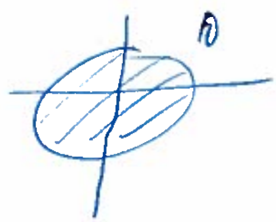
So WLOG. $D = \{0 < \operatorname{Im} z < \pi\}$



$$\xrightarrow{\quad} \quad z_1 = e^z = e^x e^{iy}$$



$\longrightarrow$ (w)



$$w = \frac{z_1 - i}{z_1 + i}, \text{ so } w = \frac{e^z - i}{e^z + i}.$$

4. Lunar Domains

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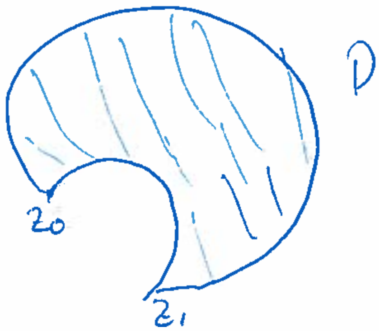
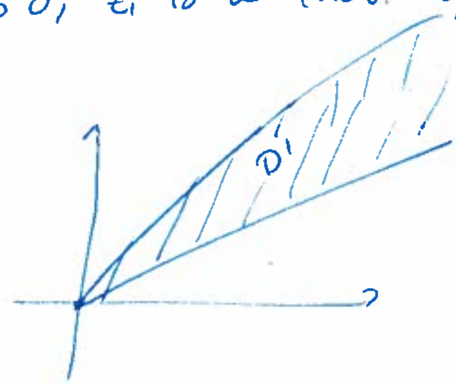
D enclosed by 2 circles (in $\mathbb{C}^*$) intersecting at z_0, z_1



(or )

$w = \dfrac{z - z_0}{z - z_1}$ maps z_0 to 0, z_1 to ∞ (Moeb. map)

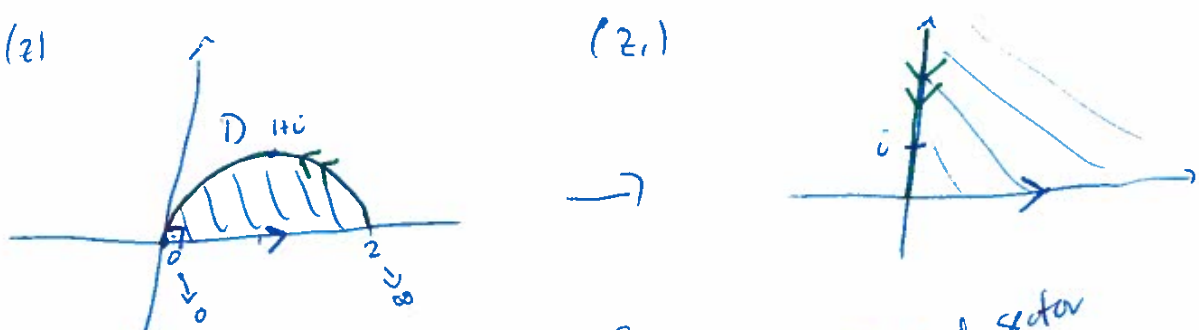
So map D to D'



⤳ Case 2.

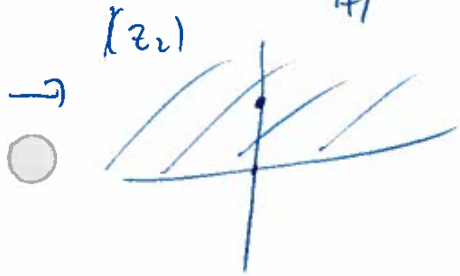
Ex: Find conformal map from

$D = \{ |z-1| < 1, \operatorname{Im} z > 0 \}$ onto $\mathbb{D}$.

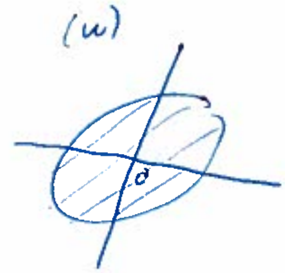


$z_1 = \dfrac{z}{2-z}$

Standard sector $\alpha = \pi/2$



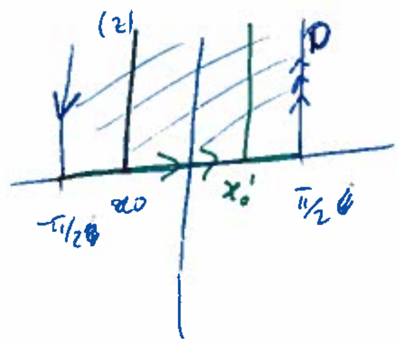
$z_2 = z_1^2$



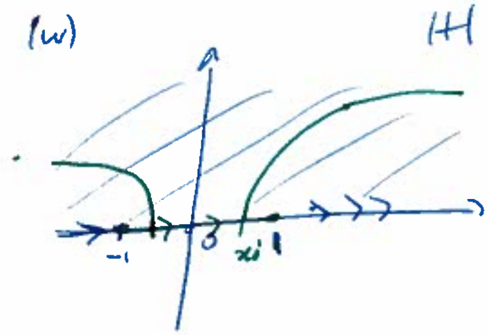
$w = \dfrac{z_2 - i}{z_2 + i}$

$$w = \frac{z_1^2 - i}{z_1^2 + i} = \frac{\left(\frac{z}{2-z}\right)^2 - i}{\left(\frac{z}{2-z}\right)^2 + i}$$

Standard half strip



$w = \sin z$



$D = \{ -\pi/2 < \operatorname{Re} z < \pi/2,\ \operatorname{Im} z > 0 \}$

$z = x + iy$, $w = \sin(z) = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

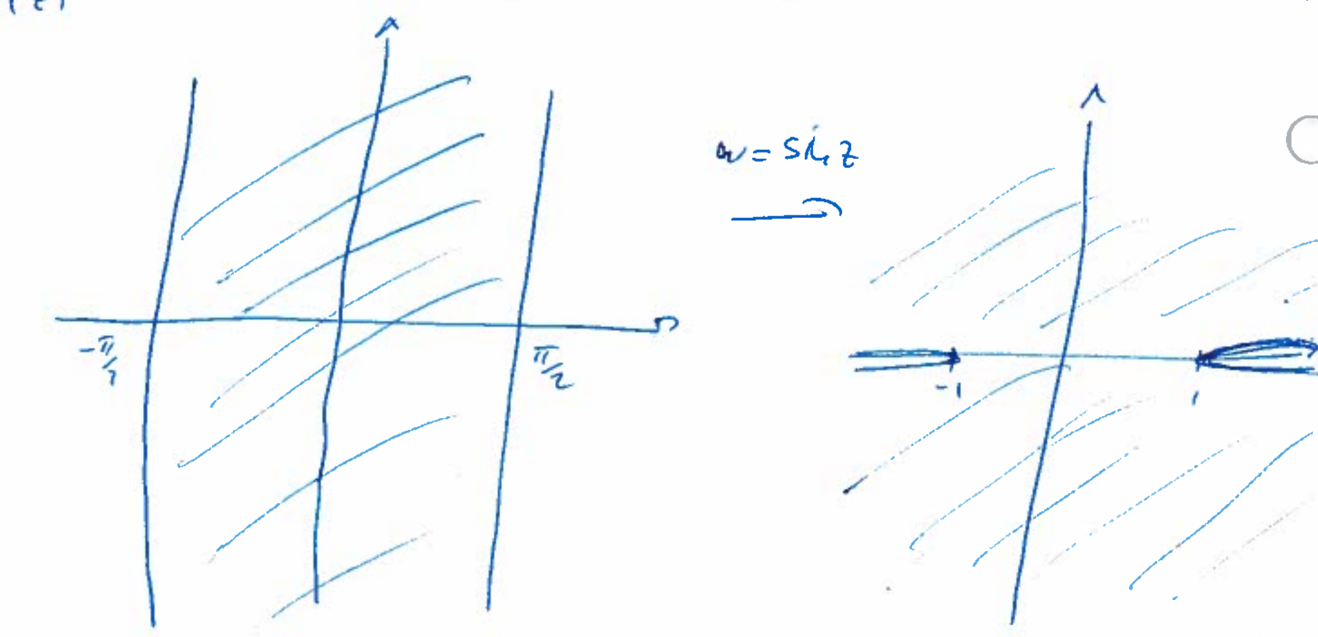
$= u + iv.$

$u = \sin(x_0)\cosh(y)$ $\quad (y > 0)$

$v = \cos(x_0)\sinh(y)$ $\quad \dfrac{u}{\sin(x_0)}$

$$\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = \cosh^2 y - \sinh^2 y = 1$$

(2)



$\mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty))$.

11.2 Riemann Mapping Theorem

$D \subseteq \mathbb{C}$ simply connected domain $\overset{def}{\iff}$ every closed path in D homotopic to 0 in D.

Fact: If $D \subseteq \mathbb{C}$ simply connected $\iff \mathbb{C}^* \setminus D$ is connected $\subseteq \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$

$\iff \partial D \subseteq \mathbb{C}^*$ is connected. (doesn't work if the complement taken in $\mathbb{C}$

)

Not true in $\mathbb{R}^n$, $n \geq 3$.

$\{ 1 \leq \|x\| \leq 2,\ x \in \mathbb{R}^3 \}$ simply connected

09
1

Thm If $\emptyset \neq D \subsetneq \mathbb{C}$ is a domain. TFAE

1) D simply connected

2) every closed form in D is exact $\left(\omega = P\,dx + Q\,dy,\ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \ldots\right.$

$\omega = df,\ f \in \mathcal{C}^2(D)$

3) $\forall a \in \mathbb{C} \setminus D,\ \exists f$ holom. on D s.t. $e^{f(z)} = z - a$

$(f(z) \in \log(z-a))$. holom. branch.

4) $\exists\ \varphi : D \to \mathbb{D}$ conformal.

Pf: 1) $\Rightarrow$ 2) chapter 3.

2) $\Rightarrow$ 3) $\quad \omega = \underbrace{\frac{1}{z-a}}_{\text{holom.}}\,dz$ is closed. $\overset{2)}{\Rightarrow}$ ω is exact, so

$\exists\ f$ holom. in D : $f'(z) = \frac{1}{z-a}$ choose f s.t. at $z_0 \in D$

$f(z_0) \in \log(z-a)$. $\left(e^{f(z_0)} = z-a\right) \Rightarrow e^{f(z)} = z-a$

$\left(\frac{d}{dz}\,\frac{e^{f(z)}}{z-a} = 0\right)$

3) $\Rightarrow$ 4) (Riemann mapping theorem) Later.

4) $\Rightarrow$ 1) φ homeom., $\mathbb{D}$ simply connected $\Rightarrow D = \varphi^{-1}(\mathbb{D})$ simply connected. $\square$

Riemann Mapping Theorem

If $D \subseteq \mathbb{C}$, $D \neq \mathbb{C}$ is simply connected domain then

$\exists\, \varphi : D \to \mathbb{D}$ conformal (holom., bijective) (φ called a Riemann map: $D \to \mathbb{D}$)

Remark $\varphi : \mathbb{C} \to \mathbb{D}$ holom., then φ is constant by Liouville.

$\therefore$ $\mathbb{C}$ not conf. equivalent to $\mathbb{D}$.

Corollary $D \subseteq \mathbb{C}^*$ simply connected domain, then either $D = \mathbb{C}^*$, or

D is conformally equivalent to $\mathbb{D}$ or $\mathbb{C}$.

Pf of Corollary: Assume $D \neq \mathbb{C}^*$

Case 1 $D = \mathbb{C}^* \setminus \{a\}$. If $a = \infty$ done $D = \mathbb{C}$. Assume $a \in \mathbb{C}$.

$$\varphi : \mathbb{C}^* \setminus \{a\} \to \mathbb{C},\ \varphi(z) = \begin{cases} \frac{1}{z-a}, & z \neq \infty \\ 0, & z = \infty \end{cases}$$ is conformal.

holom. at ∞.

Case 2 $D \subseteq \mathbb{C}^* \setminus \{a, b\}$. Wlog $a \neq \infty$

$\varphi : \mathbb{C}^* \setminus \{a\} \to \mathbb{C}$. $\varphi(z) = \frac{1}{z-a}$ maps D into $\mathbb{C} \setminus \{\varphi(b)\}$

$\therefore$ $\varphi(D) \subseteq \mathbb{C} \setminus \{\varphi(b)\}$ simply connected. Riemann mapping thm

$\Rightarrow$ $\varphi(D)$ conf. equivalent to $\mathbb{D}$. $\square$

Lemma $\varphi: G_1 \to G_2$ homeo, $G_i \subseteq \mathbb{C}$ open $\forall i$.

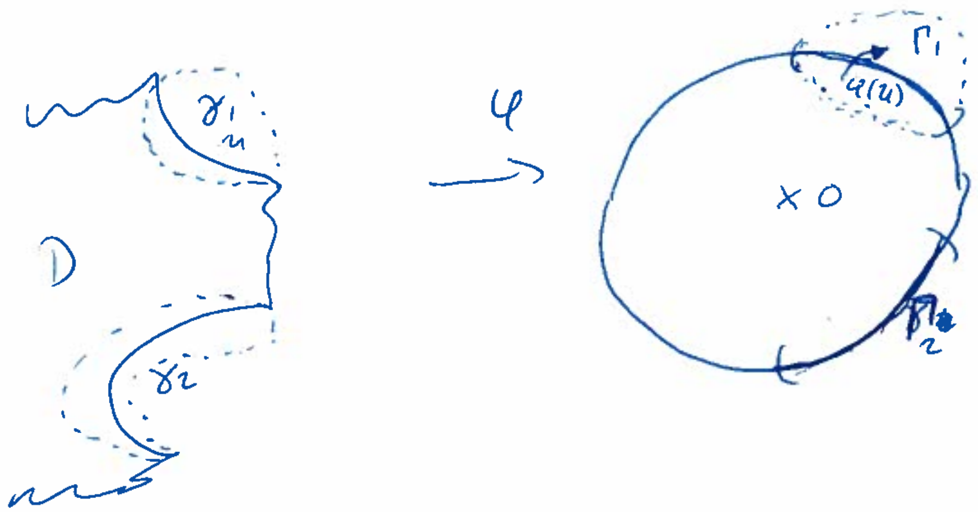
Then $\varphi^{-1}(K)$ is compact for $K \subseteq G_2$ compact.

Pf: φ^{-1} is cont.

$\Rightarrow$ If $z_n \in G_1$, $z_n \to \partial G_1$, then $\varphi(z_n) \to \partial G_2$.

leaves every compact. $\Leftrightarrow$ dist$(z_n, \partial G_1) \to 0$ (if G_1 bounded

Boundary behaviour of $\varphi: D \to \mathbb{D}$ conformal. Suppose γ_1, γ_2 are disjoint free analytic boundary arcs in ∂D. Then φ extends analytically across γ_1 and γ_2 and maps γ_1, γ_2 1-1 and onto arcs Γ_1, Γ_2 in $\partial\mathbb{D}$ which are disjoint. The extended function satisfies $\varphi'(z) \neq 0$ on $\gamma_1 \cup \gamma_2$.



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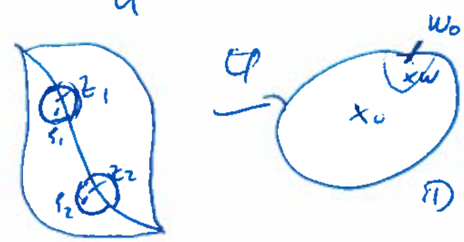
Lemma $\Rightarrow$ $|\varphi(z)| \to 1$ as $z \to \partial D$. Can use reflection

principle $\exists\, \mathcal{U}$ nbd of γ_1, φ holom. on $\mathcal{U}$ verifying $\varphi(z^*) = \frac{1}{\overline{\varphi(z)}}$

$\Rightarrow$ $|\varphi(z)| > 1$ on $\mathcal{U} \setminus \overline{D}$. , $|\varphi(z)| = 1$ on γ_1. φ injective? on $\mathcal{U} \setminus \gamma_1$

yes, since φ injective on $\mathcal{U} \cap D$ (and by reflection formula). Assume $z_1, z_2 \in \gamma_1$

s.t. $\varphi(z_1) = \varphi(z_2) = w_0 \in \partial\mathbb{D}$.



If w is close to w_0, $|w| < 1$ $\exists\, \xi_1 \in D$

near z_1 and $\exists\, \xi_2 \in D$ near z_2, $\xi_1 \neq \xi_2$ but $\varphi(\xi_1) = w = \varphi(\xi_2)$

contradiction. $\to$ φ injective on $\mathcal{U}$ $\Rightarrow$ $\varphi'(z) \neq 0$ on $\mathcal{U}$.

Same argument ~~for~~ $\Rightarrow$ $\Gamma_1 \cap \Gamma_2 = \emptyset$ as $\gamma_1 \cap \gamma_2 = \emptyset$.

Cont $\gamma : [0,1] \to \mathbb{C}$ simple closed curve (Jordan curve) if

$\gamma(0) = \gamma(1)$, $\gamma(s) \neq \gamma(t)$ if $0 \leq s < t < 1$.

get $\tilde{\gamma} : S^1 \to \mathbb{C}$ cont. injective **Note**: $\tilde{\gamma}^{-1} : \tilde{\gamma}(S^1) \to S^1$ is cont, too.

cpt.

Jordan curve is homeomorphic image of S^1.

04

<u>Jordan curve Theorem</u> If $\Gamma \subseteq \mathbb{C}$ is a Jordan curve.

Then $\mathbb{C} \setminus \Gamma$ has exactly two connected components, one bounded $\mathcal{U}$, and one unbounded V, with $\partial \mathcal{U} = \partial V = \Gamma$, and $\mathcal{U}$ is simply connected in $\mathbb{C}$ and $V \cup \{\infty\}$ is simply connected in $\mathbb{C}^*$.

<u>Carathéodory Extension Theorem</u> If D is a Jordan Domain (i.e. ∂D is a Jordan curve $\leadsto D$ bounded) and $\varphi: D \to \mathbb{D}$ is conformal. Then φ extends cont. to ∂D, $\varphi: \overline{D} \to \overline{\mathbb{D}}$ is a homeomorphism, and $\varphi(\partial D) = \partial \mathbb{D}$.

<u>Remark:</u> If such an extension of φ exists, then D must be a Jordan domain
$\varphi: \partial D \to \partial \mathbb{D} = S^1$ homeom. $\Rightarrow$ ∂D is a Jordan curve and D is a ~~Jo~~ Jordan domain.

11.5 Compact families of functions (Montel's Thm)

1) Equicontinuity $(E,d),(K,d')$ metric spaces

$\mathcal{F}$ = family of fcts $f: E \to K$.

i) $\mathcal{F}$ is equicontinuous at $z_0 \in E$ if $\forall \varepsilon>0\ \exists \delta=\delta(\varepsilon)>0$ such that if $d(z,z_0)<\delta$ then $d'(f(z), f(z_0))<\varepsilon\ \forall f \in \mathcal{F}$.

ii) $\mathcal{F}$ is equicontinuous on E if $\forall \varepsilon>0\ \exists \delta=\delta(\varepsilon)>0$ s.t. if ~~$d(z,z_0)$~~ $z,z' \in E$, $d(z,z')<\delta \Rightarrow d'(f(z), f(z'))<\varepsilon$ $\forall f \in \mathcal{F}$.

Lemma: If E is compact, then $\mathcal{F}$ is equicontinuous on E iff. $\mathcal{F}$ is equicontinuous at each $z_0 \in E$.

2) Assume $K = \mathbb{C}$.

i) $\mathcal{F}$ is pointwise bounded if $\forall z \in E\ \exists M_z>0$ s.t. $|f(z)| \le M_z \forall f \in$

ii) $\mathcal{F}$ is uniformly bounded on E if $\exists M>0$ s.t. $|f(z)|\le M\ \forall f \forall z$

Lemma If E is compact and $\mathcal{F}$ pointwise bounded and equicontinuous on E, then $\mathcal{F}$ is pointwise bounded on E.

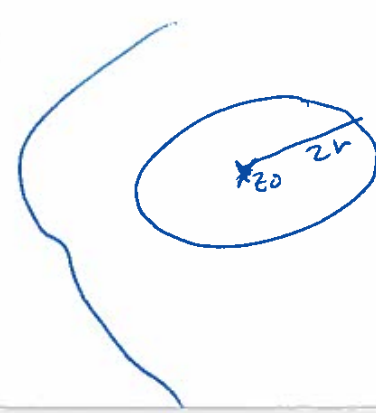
Arzela-Ascoli Thm

1) (E,d) compact metric space, $\mathcal{F}$ family of functions from: $E \to \mathbb{C}$ $(\mathbb{R}^n)$ and $\mathcal{F}$ pointwise bounded and $\mathcal{F}$ equicontinuous. Then every sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence ~~which~~ $\{f_{n_k}\}$ that converges uniformly on E.

2) ~~If~~ Let (E,d), (K,d') compact metric spaces, $\mathcal{F}$ family of functions: $E \to K$ which is equicontinuous. Then the same conclusion holds.

Montel's Theorem Let $\mathcal{F}$ be a family of holom. functions on a domain $D \subseteq \mathbb{C}$ s.t. $\mathcal{F}$ is uniformly bounded on each compact $K \subseteq$ (locally uniformly bounded) $\forall K \subseteq D$ cpt $\exists\, M_K > 0$ s.t. $\|f\|_{L^\infty(K)} \leq$ ... Then every sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges normally on D to a holom. function.

Pf: 1) $\mathcal{F}$ equicont. at each $z_0 \in D$. Fix $\overline{\Delta}(z_0, 2r) \subset D$. $\exists\, M >$ s.t. $|f(z)| \leq M$ $\forall f \in \mathcal{F}$ $\forall z \in \overline{\Delta}(z_0, 2r)$.



~~$z \in \overline{\Delta}(z_0, r)$~~

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If $z \in \overline{\Delta}(z_0, r)$ (then $\overline{\Delta}(z, r) \subseteq \overline{\Delta}(z_0, 2r)$).

Then $|f'(z)| \le \frac{M}{r}$ (Cauchy's estimate)

$$|f(z) - f(z_0)| = \left| \int_{[z_0, z]} f'(\xi) d\xi \right| \le \frac{M}{r} |z - z_0| \Rightarrow \mathcal{F} \text{ equicont.}$$

at z_0.

2) Let $K_n = \{ z \in D \mid \operatorname{dist}(z, \partial D) \ge \frac{1}{n}, |z| \le n \}$

Then $K_{n+1} \supseteq K_n$, $D = \bigcup_{n=1}^{\infty} K_n$ (compact exhaustion of D)

Pick $\{f_n\} \subseteq \mathcal{F}$. Arzela-Ascoli on $K_1 \Rightarrow \exists\, f_{11}, f_{12}, f_{13}, —$

Subsequence of $\{f_n\}$ uniformly convergent on K_1.

Arzela-Ascoli on $K_2 \Rightarrow \exists\, f_{21}, f_{22}, f_{23}, —$ subsequence of $\{f_{1k}\}_k$

conv. uniformly on K_2 .... cont. by induction

$\exists\, f_{k1}, f_{k2}, — f_{kk}, —$ subsequence of $\{f_{k-1, j}\}_j$ conv. unif.

on K_k.

Then $\{f_{11}, f_{22}, f_{33}, —, f_{kk}, —\}$ conv. uniformly on

each K_n. (subsequence of n-th subsequence eventually). This converges

normally in D (each compact $K \subset$ some K_n) $\square$

~~Theorem~~ Theorem (Ahlfors fcn. on D)

Let $D \subseteq \mathbb{C}$ be a domain such that there exists a bounded, nonconstant holom. function h on D. Fix $z_0 \in D$. Then there exist a holom. function $G : D \to \mathbb{D}$ s.t. $G(z_0) = 0$, $G'(z_0) \neq 0$ and $|f'(z_0)| \leq |G'(z_0)|$ for every ~~all~~ holom. f on D with $|f(z)| \leq 1$ on D

(If $D = \mathbb{C} \setminus S$, S discrete then every bounded holom fctn on D is constant

Pf: $\mathcal{F} := \{ f : D \to \mathbb{C}$ holom, $|f(z)| \leq 1 \; \forall z \in D \}$.

Step 1 $\mathcal{F} \neq \phi$ $\quad h(z) = h(z_0) + a(z-z_0)^N + \ldots \quad a \neq 0$, near z_0

$N = \text{ord}(h - h(z_0), z_0)$. $\quad g(z) = \underbrace{\varepsilon \dfrac{h(z) - h(z_0)}{(z-z_0)^{N-1}}}_{\text{holom. on } D} = \varepsilon a (z - z_0) + \cdots$ near z_0.

Then $g'(z_0) = \varepsilon a \neq 0$. Fix $\overline{\Delta}(z_0, r) \subset D$. , $|g(z)| \leq \varepsilon C$ on for $z \in \overline{\Delta}(z_0, r)$

(where $\left| \dfrac{h(z) - h(z_0)}{(z-z_0)^{N-1}} \right| \leq C$

on $D \setminus \overline{\Delta}(z_0, r)$

$$|g(z)| \leq \varepsilon \frac{2 \cdot \|h\|_{L^\infty(D)}}{r^{N-1}}$$

Take $\varepsilon > 0$ small s.t. $|g(z)| \leq 1 \; \forall z \in D$.

$\Rightarrow g \in \mathcal{F}$, $g'(z_0) \neq 0$.

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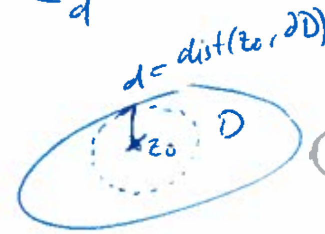
Step 2 $A := \sup\{ |f'(z_0)| : f \in \mathcal{F} \}$, $0 < A < \frac{1}{d}$



$\Rightarrow \exists f_n \in \mathcal{F}, |f_n'(z_0)| \to A.$

$\Rightarrow \exists f_{n_k} \to G$ normally on D. $|G'(z_0)| = A$ G nonconstant

$\overset{\text{open}}{\Longrightarrow} G : D \to \mathbb{D}$. To show $G(z_0) = 0$

$$\tilde{G}(z) = \frac{G(z) - G(z_0)}{1 - \overline{G(z_0)}\, G(z)} : D \to \mathbb{D}, \quad \tilde{G} \in \mathcal{F}, \text{ so}$$

$$|\tilde{G}'(z_0)| = \frac{1}{1 - |G(z_0)|^2} |G'(z_0)| = \frac{A}{1 - |G(z_0)|^2} \overset{!}{\le} A$$

$\Rightarrow G(z_0) = 0.$ □

04/25

Riemann Mapping Theorem

If $D \subsetneq \mathbb{C}$ simply connected. Then $\exists f_0 : D \to \mathbb{D}$ conformal

(f_0 univalent, onto).

(Koebe)

04

Proof: $\mathcal{F} = \{ f : D \to \mathbb{D} \text{ holom.}, f \text{ univalent}, f(z_0) = 0 \}$

$z_0 \in D$. Fixed

Step 1: $\mathcal{F} \neq \phi$.

Step 2: $M = \sup\{ |f'(z_0)| : f \in \mathcal{F} \}$ $\exists f_0 \in \mathcal{F}$ s.t. $M = |f_0'(z_0)|$

Step 3: f_0 is onto, $f_0(D) = \mathbb{D}$.

Step 1 Fix $a \in \mathbb{C} \setminus D$. $\exists h$ holom on D s.t. $e^{h(z)} = z - a$ ($\neq 0$)

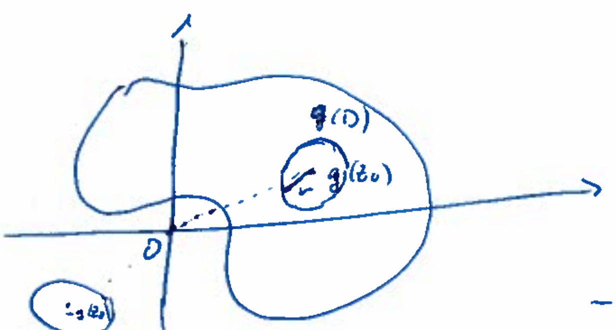
(b/c D simply connected)

$g := e^{h/2} \Rightarrow g^2(z) = z - a$. 1) g is injective: $g(z) = g(z')$ $\overset{2}{\Rightarrow} z - a = z' - a$.

2) $\Rightarrow g(D)$ is open, connected, $0 \notin g(D)$.

3) $\forall z, z' \in D$ $g(z) \neq -g(z')$. [If $g(z) = -g(z') \overset{2}{\Rightarrow} z - a = z' - a$

$\Rightarrow g(z) = -g(z) = 0$ contrad.]



$\exists \Delta(g(z_0), r) \subset g(D)$ open.

Then $\Delta(-g(z_0), r) \cap g(D) = \phi$. by 3).

If $w^{\in}$, then $w = g(z)$, $z \in D$ and

$-w = g(z')$ $\overset{z' \in D}{\Rightarrow}$ $g(z) = -g(z')$ contrad. to 3)

So $|g(z)+g(z_0)| \ge r$, $\forall z \in D$

Then $h(z) = \dfrac{r}{g(z)+g(z_0)}$ holom., inj. on D, $|h| \le 1$.

As h is open, $h : D \to \mathbb{D}$. $\varphi \in \mathrm{Aut}(\mathbb{D})$, $\varphi(h(z_0)) = 0$

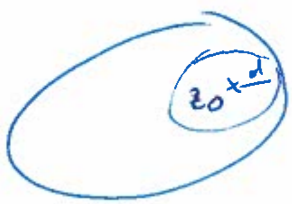
$h(D)$ open in $\mathbb{C}$

And $f := \varphi \circ h = \dfrac{h - h(z_0)}{1 - \overline{h(z_0)}\, h} : D \to \mathbb{D}$, $f(z_0) = 0$, injective.

$\Rightarrow f \in \mathcal{F} \ne \phi$. , $|f'(z_0)| \ne 0$. since f is injective.

Step 2 $0 < M \le +\infty$

$d = \mathrm{dist}(z_0, \partial D)$, $\Delta(z_0, d) \subseteq D$. $\forall f \in \mathcal{F}$, $|f| < 1$



$\Rightarrow |f'(z_0)| \le \frac{1}{d}$. (Cauchy estimate on $\Delta(z_0, d-\varepsilon)$ & $\varepsilon \to 0$)

$\therefore\ 0 < M \le \frac{1}{d} \Rightarrow \forall n\ \exists f_n$ ~~st~~ ~~$M - \frac{1}{n} < |f_n'(z_0)| \le \frac{1}{d}$~~

s.t. $M - \frac{1}{n} < |f_n'(z_0)| \le M$. so $|f_n'(z_0)| \to M$. Also $|f_n| < 1$ on D

for all n. $\Rightarrow \exists f_{n_k} \to f_0$ normally on D.

$f_0 : D \to \overline{\mathbb{D}}$ is holom., $f_0(z_0) = 0$ (as $f_n(z_0) = 0\ \forall n$)

to $|f_0'(z_0)| = M > 0 \Rightarrow f_0$ not constant. Then

f_0 open map, so $f_0 : D \to \mathbb{D}$ and f_0 injective (Hurwitz) as f_0 not constant.

$\Rightarrow f_0 \in \mathcal{F}$, $|f_0'(z_0)| = M$

Step 3 Assume $\exists\ \alpha \in \mathbb{D} \setminus f_0(\mathbb{D})$ $(\alpha \neq 0)$

$$\psi(z) = \frac{z-\alpha}{1-\bar{\alpha} z}, \quad \psi \circ f_0(z) = \frac{f_0(z) - \alpha}{1 - \cancel{f_0} \bar{\alpha} f_0(z)} \neq 0 \ \forall z \in D$$

$\psi \circ f_0 : D \to \mathbb{D}$ univalent, never zero.

So $\exists\ F : D \to \mathbb{D}$ s.t. $F^2 = \psi \circ f_0$ (D simply connected)

F injective since F^2 is. Let $f = \dfrac{F - F(z_0)}{1 - \overline{F(z_0)} \cdot F} \in \mathcal{F} : D \to \mathbb{D}$

$(f = \varphi \circ f)$

f injective, $f(z_0) = 0$. $\rightsquigarrow f \in \mathcal{F}$. So $|f'(z_0)| \leq M$.

Calculate $f'(z_0)$:

$F^2(z_0) = \psi \circ f_0(z_0) = \psi(0) = -\alpha$, $|F(z_0)| = \sqrt{|\alpha|}$

$$2F\,F' = (\psi' \circ f_0) \cdot f_0' = \frac{1-|\alpha|^2}{(1-\bar{\alpha} f_0)^2} \cdot f_0'$$

~~$f_0(z_0)$~~

$$2F(z_0)\,F'(z_0) = (1-|\alpha|^2)\, f_0'(z_0)$$

$$\Rightarrow |F'(z_0)| = \frac{(1-|\alpha|^2)\,M}{2F(z_0)} = \frac{1-|\alpha|^2}{2\sqrt{|\alpha|}} M.$$

$f = \varphi \circ F$

$\varphi(w) = \dfrac{w - F(z_0)}{1 - \overline{F(z_0)} w}$

$$f' = (\varphi' \circ F) \cdot F' = \frac{1 - |F(z_0)|^2}{(1 - \overline{F(z_0)}\, F)^2} \cdot F'$$

$$f'(z_0) = \frac{F'(z_0)}{1 - |F(z_0)|^2}$$

Thus $|f'(z_0)| = \dfrac{|F'(z_0)|}{1-|F(z_0)|^2} = \dfrac{(1-|\alpha|^2)\; M}{2\sqrt{|\alpha|}\;(1-|\alpha|)}$

$= \dfrac{1+|\alpha|}{2\sqrt{|\alpha|}} \cdot M > M. \qquad \left((1-2\sqrt{|\alpha|})^2 > 0 \right)$

$\to f \in \mathcal{F}$ but $|f'(z_0)| > M$, a contradiction. Thus f_0 is onto $\square$

Comments on HW

$|f(z)| \leq 1$ in $|z| < 3$, $f(\pm i) = f(\pm 1) = 0$

$g(z) = f(3z)$, $|z| < 1$, $|g(z)| \leq 1$, $g(\pm i/3) = g(\pm 1/3) = 0$.

Blaschke product: $z_1, \dots, z_n \in \mathbb{D}$. $B(z) = e^{i\theta} \left(\dfrac{z - z_1}{1-\bar{z}_1 z}\right) \cdot \left(\dfrac{z - z_2}{1-\bar{z}_2 \cdot z}\right) \cdots \left(\dfrac{z - z_n}{1-\bar{z}_n z}\right)$

zeros at $z_1, \dots, z_n$, $|B(z)| = 1$ for $|z| = 1$ hol. in bigger disks than $\overline{\mathbb{D}}$.

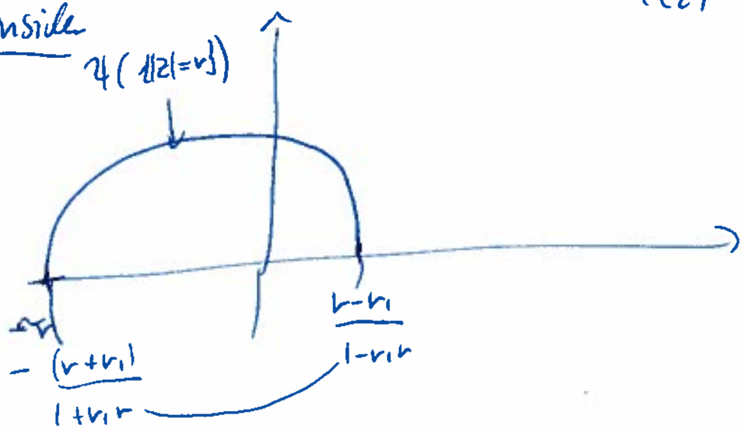
If $|z| \in (|z_1|, 1)$, then $(|z| = r, |z_1| = r_1)$

$$\frac{r - r_1}{1 - r_1 r} \leq \left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| \leq \frac{r + r_1}{1 + r_1 r}$$

$$\left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| = \left| \frac{re^{i\theta} - r_1 e^{i\theta_1}}{1 - r_1 e^{-i\theta_1} re^{i\theta}} \right| = \left| \frac{re^{i(\theta-\theta_1)} - r_1}{1 - r_1 r e^{i(\theta - \theta_1)}} \right|$$

$$\varphi(z) = \frac{z - r_1}{1 - r_1 z}$$

Consider



$f : \mathbb{D} \to \overline{\mathbb{D}}$ zeros at $z_1, \ldots, z_n$.

$$g(z) = \frac{f(z)}{B(z)} \quad \text{holom. on } \mathbb{D}.$$

$r < 1$ close to 1 $\quad |g(z)| \le \dfrac{1}{\min\limits_{|z|=r} |B(z)|}$, $\quad |z| < r \overset{r \to 1}{\Rightarrow} |g(z)| \le 1$

$\Rightarrow |f(0)| \le |B(0)| = |z_1| \cdot \ldots \cdot |z_n|$. equality $\Leftrightarrow g = e^{i\theta}$ const.

$\Leftrightarrow f = e^{i\theta} B$.

Comment on HW f entire, $|f(z)|=1$ for $|z|=1$

$\Rightarrow f(z) = e^{i\theta} z^n$, $n \geq 0$

Lemma f hol. on $\mathbb{D}$, cont. on $\overline{\mathbb{D}}$, $|f(e^{i\theta})|=1$ $\forall \theta$, then

$f(z) = e^{i\theta} \frac{z-z_1}{1-\bar{z}_1 z} \cdot \ldots \cdot \frac{z-z_n}{1-\bar{z}_n z}$ is a finite Blaschke product

for some $z_j \in \mathbb{D}$.

Case 1 $f(z) \neq 0$ $\forall z \in \mathbb{D}$, $|z|<r<1$ $\left|\frac{1}{f(z)}\right| \leq \frac{1}{\min_{|z|=r} |f(z)|} \to 1$ as $r \to 1$.

$\Rightarrow |f(z)| \geq 1$ and $|f(z)| \leq 1$ $\Rightarrow f$ is constant, $f = e^{i\theta}$.

Pf Case 2 $z_1, \ldots, z_n$ zeros of f. $B(z) = \frac{z-z_1}{1-\bar{z}_1 z} \cdot \ldots \cdot \frac{z-z_n}{1-\bar{z}_n z}$.

$g = \frac{f}{B}$, $|g(e^{i\theta})| = 1$, $g(z) \neq 0$ on $\mathbb{D}$.

Case 1 $\Rightarrow$ g is constant.

f entire, $f = B \to f(z) = e^{i\theta} z^n$.

Sol: $g(z) = f(z) \overline{f(\frac{1}{\bar{z}})}$ holom. on $\mathbb{C} \setminus \{0\}$, $g(e^{i\theta}) = f(e^{i\theta}) \overline{f(e^{i\theta})}$
$= |f(e^{i\theta})|^2 = 1$.

identity
$\Rightarrow$ $g(z) = f(z) \overline{f(\frac{1}{\bar{z}})} = 1$ $\forall z \in \mathbb{C} \setminus \{0\}$.
Theorem

$\Rightarrow f(z) \neq 0$ $\forall z \neq 0$. $N = \text{Ord}(f, 0)$.

$h(z) = \frac{f(z)}{z^N}$, $|h(e^{i\theta})| = 1$, $h(z) \neq 0$ on D $\Rightarrow$ previous result

implies $h(z) = e^{i\theta}$ $\square$

<u>Qual Problems</u>

1. $f(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$, $|z| \leq r < 1$ $\quad |f(z)| \leq \sum_{n=1}^{\infty} \frac{1}{n} r^{n!} \leq \sum_{n=1}^{\infty} \frac{r^n}{n} <$

$\Rightarrow$ normal & abs. conv. in D.

$z = e^{i \frac{k}{e} 2\pi}$, $\frac{k}{e} \in \mathbb{Q} \cap [0,1]$.

$n \geq e \qquad z^{n!} = e^{i \frac{k}{e} n! \, 2\pi} = 1$ So $f(e^{i \frac{k}{e} 2\pi}) = \sum_{n=0}^{e-1} \cdots$

$+ \underbrace{\sum_{n=e}^{\infty} \frac{1}{n}}_{\to \infty}$ di

2. f holom. on $\mathbb{C}$ except poles, ∞ isolated, removable or a pole.

a) f has finitely many poles in $\mathbb{C} \cup \{\infty\}$. f holom in $\Delta \{|z| > R\}$
$\Rightarrow$ all poles in $\mathbb{C}$ are in $\{|z| \leq R\}$ so finitely many.

b) let $z_1, \ldots, z_N$ be the poles in $\mathbb{C} \cup \{\infty\}$
$P_j(z)$ principal part of f z_j. To show $f - \sum_{j=1}^{N} P_j = \text{const.}$

$z_j \in \mathbb{C}$: $P_j(z) = \dfrac{b^j_{m_j}}{(z-z_j)^{m_j}} + \ldots + \dfrac{b^j_1}{z-z_j}$ holom on $\mathbb{C} \setminus \{z_j\}$

$P_j(z) \to 0$, $z \to \infty$.

$z_N = \infty$: $P_N(z) = a_m z^m + \ldots + a_1 z$. polynomial.

$g = f - \sum_{j=1}^{N} P_j$ is entire and $\lim_{z\to\infty} g(z) = a \in \mathbb{C}$

L.S. at ∞

$f(z) = P_N(z) + a + \underbrace{\frac{c_1}{z} + \ldots}_{\to 0.}$

$\Rightarrow$ bounded, so const. by Liouville.

3. f cont. on $\mathbb{C}$, holom. on $\mathbb{C} \setminus \{|z|=1\}$.

g entire, $f = g$ on $|z|=1$. $\Rightarrow$ f entire

$f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in \mathbb{C}$.

$r<1$

$a_n = \dfrac{1}{2\pi i} \displaystyle\int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz$ $\qquad b_n = \dfrac{1}{2\pi i} \displaystyle\int_{|z|=1} \frac{g(z)}{z^{n+1}}\, dz$

$r \to 1$ unif cont. of integrand on $\frac{1}{2} \leq |z| \leq 1$

$\dfrac{1}{2\pi i} \displaystyle\int_{|z|=1} \frac{\overbrace{f(z)}^{=g(z)}}{z^{n+1}}\, dz \Rightarrow a_n = b_n \ \forall n.$, so $f \equiv g$.

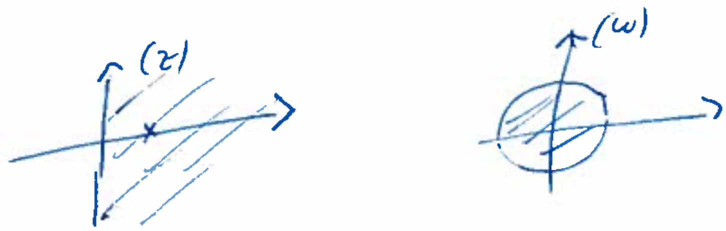
Alternatively use f composed Mobius map.

4. $f_n : \mathbb{D} \to P = \{w : \operatorname{Re} w > 0\}$, f_n holom. 04 $\frac{1}{2}$

$f_n \to f$ ptwise, $\operatorname{Re} f \leq 0$. $\Rightarrow$ f const.

If f was holom. $\operatorname{Re} f \leq 0$ so $\operatorname{Re} f = 0$ taking ptwise limit $\xRightarrow{\text{OMT}}$ f const.

$\exists \varphi : P \to \mathbb{D}$



Now $\qquad w = \dfrac{z-1}{z+1} = \varphi(z)$

$g_n = \varphi \circ f_n$ holom. unif. bounded.

$g_n \to \cancel{f}\, g$ normally, g holom. $g : \mathbb{D} \to \overline{\mathbb{D}}$.

$\Rightarrow$ $f_n \to \varphi^{-1} \circ g. = f$ holom.

---

f, g $\overset{\text{loc bdd}}{(\text{holom})}$ on D open $\bigcup_{n=1}^{\infty} K_n = D$, $K_n \subseteq K_{n+1}$ compact $\frac{@}{3}$

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{L^\infty(K_n)}}{1 + \|f-g\|_{L^\infty(K_n)}} \quad \text{metric.} \quad d(f_n, f) \to 0$$

$\Leftrightarrow$ $f_n \to f$ normally.

A. 2017 / #3 $f: D \to D$ holom., D bounded domain

$f(z_0) = z_0 \Rightarrow |f'(z_0)| \leq 1$

$f^{(n)} = f \circ \dots \circ f$, $(f^{(n)})'(z_0) = (f'(z_0))^n$

D is bounded $|w| \leq M\ \forall w \in D$. $\to |f^{(n)}(z)| \leq M$. th, z.

$\overline{\Delta}(z_0, R) \subset D$ $\quad |(f^{(n)})'(z_0)| \leq \frac{M}{R}$ (Cauchy Estimate)

$$|f'(z_0)| \leq \sqrt[n]{\frac{M}{R}} \to 1, \quad n \to \infty$$

Comment on HW

$\varphi: \underset{s.c}{D} \to \mathbb{D}$, $\varphi(z_0) = 0$, $\varphi'(z_0) > 0$ Riemann map

$f: D \to \mathbb{D}$ holom. $\Rightarrow |f'(z_0)| \leq \varphi'(z_0)$ & "=" $\Leftrightarrow f = e^{i\theta}\varphi$.

$g = f \circ \varphi^{-1}: \mathbb{D} \to \mathbb{D}$

Pick's $|g'(\zeta)| \leq \frac{1 - |g(\zeta)|^2}{1 - |\zeta|^2}$, $|g'(0)| = \frac{|f'(z_0)|}{\varphi'(z_0)} \leq 1 - |g(0)|^2$

$\Rightarrow |f'(z_0)| \leq \varphi'(z_0)(1 - |g(0)|^2) \leq \varphi'(z_0)$

$\Rightarrow |f'(z_0)| = \varphi'(z_0) \Rightarrow g(0) = 0 \overset{\text{Schw}}{\Rightarrow} g(\zeta) = e^{i\theta}\zeta$.

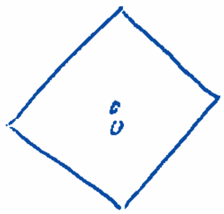
$\Rightarrow f = e^{i\theta}\varphi$.

A 2016/#3

S square centered at 0.

$F : \Delta \to S$ holom. 1-1, onto, $F(0)=0$.

(Riemann map)



to show: $\Rightarrow$ $F(iz) = i F(z)$ $\forall z \in D$.

$G(z) = \frac{1}{i} F(iz) : \Delta \to S$ Riemann map.

$G(0) = 0$, $G'(0) = \frac{1}{i} F'(0) \cdot i = F'(0) \Rightarrow F = G$.

A 2016/#4

$$|a_j| \le \frac{\max_{|z|=R} |p(z)|}{R^j} \quad \text{(Cauchy estimates)}$$

$R = 1 \Rightarrow |a_j| \le \frac{\max_{|z|=1} |p(z)|}{1^j} < 1$, $j \ge 0$ but $a_n = 1$ contradiction.

A 2017/#4

$f_n : \overline{D} \to \mathbb{C}$ cont., holom. on D, $f_n(0) = 0$

$u_n = \operatorname{Re} f_n \to u$ unif. on $\{|z|=1\}$.

To show $f_n \xrightarrow[\text{on } D]{\text{normally}} f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta})\, d\theta$

$$f_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u_n(e^{i\theta})\, d\theta + i \underbrace{\operatorname{Im} f_n(0)}_{=0} \quad \text{(Schwarz Represent. formula)}$$

04/30

$\Rightarrow \quad f_n(z) \to f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(e^{i\theta})\, d\theta \quad \forall z \in \mathbb{D}$

(Integrand conv. uniformly in θ, $n\to\infty$, for each $z \in \mathbb{D}$).

To show $f_n \to f$ uniformly on $\{|z| \le r\}$, $r<1$.

If $|z| \le r < 1$

$$|f_n(z) - f(z)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| \, |u_n(e^{i\theta}) - u(e^{i\theta})|\, d\theta$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+r}{1-r} \|u_n - u\|_{L^\infty(\partial\mathbb{D})}\, d\theta$$

$= \frac{1+r}{1-r}\|u_n - u\|_{L^\infty(\partial\mathbb{D})} \to 0$ as $n\to\infty$ uniformly in z.

A 2017 #2 How many zeros does $f(z) = e^z - 4z^2 + 3z + 1$ have on $\{|z|<2\}$?

$|e^z| = e^{\operatorname{Re} z} \le e^2 < 9$

for $|z|=2$: $|e^z + 3z + 1|$
$\le e^2 + 6 + 1 < 9+6+1 = 16$
$= |-4z^2|$

$g(z) = -4z^2$
$f(z) = g(z) + e^z + 3z + 1$

Rouché
$\Rightarrow$ g & f have same # of zeros

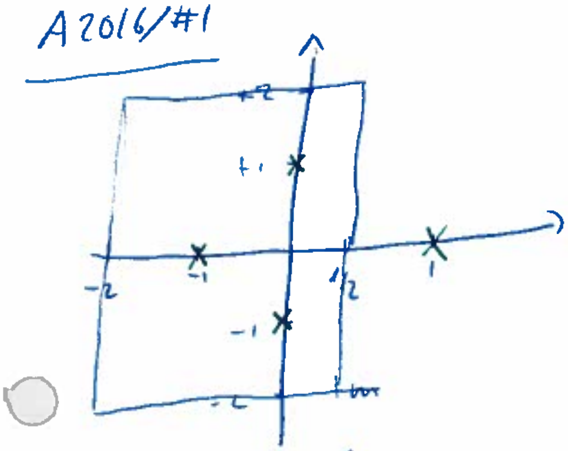
$\Rightarrow$ f has 2 zeros in $\{|z|<2\}$.

Remark: The roots are distinct. If double zero

$$\left.\begin{aligned} f(z) &= e^z - 4z^2 + 3z + 1 = 0 \\ f'(z) &= e^z - 8z + 3. = 0 \end{aligned}\right\} \quad -4z^2 + 3z + 1 = -8z + 3$$

$$\Rightarrow 4z^2 - 11z + 2 = 0$$

A2016/#1



$$I = \int_C \frac{z^n}{z^4 - 1}\, dz, \qquad n \geq 0$$

$$I = 2\pi i \left(\operatorname{Res}\left(\frac{z^n}{z^4-1}, -1\right) + \operatorname{Res}\left(\frac{z^n}{z^4-1}, -i\right) + \operatorname{Res}\left(\frac{z^n}{z^4-1}, i\right) \right)$$

$$\operatorname{Res}\left(\frac{z^n}{z^4-1}, \xi \right) \overset{\text{simple pole}}{=} \frac{z^n}{4z^3}\Big|_{z=\xi} = \frac{\xi^{n+1}}{4} \qquad (\xi^4 = 1)$$

$$I = \frac{2\pi i}{4} \left((-i)^{n+1} + i^{n+1} + (-1)^{n+1} \right)$$

$$= \begin{cases} -\frac{\pi i}{2}, & n = 2k \\ \ldots, & n = 4k+1 \\ \ldots, & n = 4k+3 \end{cases}$$

A 2017 / #1 $\qquad\qquad\qquad$ 04/30

$$D = \{ z \in \mathbb{C},\ \operatorname{Arg} z \in (0, \pi/4) \}$$