MAT 731 RINGS AND MODULES Aug 28

Recall:

① Ring $(R,+,\cdot)$, 2 binary operations $+,\cdot : R\times R \to R$

$(a,b) \mapsto a+b$

$(a,b) \mapsto a\cdot b = ab$

s.t. ⓐ $(R,+)$ abel. group $0 = 0_R$

$(R,\cdot)$ associative monoid $1 = 1_R$

\set, op., neutral el., assoc.

ⓑ distr properties

$$a(b+c) = ab+ac$$
$$(a+b)c = ac+bc$$

not commutative $ab \overset{?}{=} ba$

Ex: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, k any field, $k[x]$, $C[0,1]$ ring of continuou fcts on $[0,1]$

$(A,+)$ abelian group $\Rightarrow$ $\mathrm{End}_{\mathbb{Z}}(A)$ is a ring

② A left R-module M (R ring) (denoted ${}_RM$ ~~RM~~) is an abelian group $(M,+)$ (or $(M,+_M)$) with a map

$R\times M \to M$ [scalar mult]

$(a,m) \mapsto a\cdot m = am$

s.t. ⓐ $\underbrace{(ab)}_{\text{in } R}\cdot m = \underbrace{a\cdot(b\cdot m)}_{\text{no product in } R!}$

ⓑ $(a+b)\cdot m = am+bm$

$a\cdot(m+m') = am+am'$ $\forall a,b\in R,\ m,m'\in M$

ⓒ $1_R\cdot m = m$, $\forall m\in M$. (unitary property)

Ex: $R = k$ field

- $k\times M \to M$ scalar mult. $\rightsquigarrow$ M v.s.
- v.s. is also a module over $k = R$ ✓

module $\equiv$ v.s. (vector space)

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Ex.: $R = \mathbb{Z}$, an R-module is simply an abel. group $(M,+)$

$R=\mathbb{Z} \rightsquigarrow \mathbb{Z} \times M \to M$
$(n,m) \to nm$
and (n_1+n_2, m)
$= n_1 m + n_2 m$
and $1m = m$
(no prop. added through "module"

Remark:
ab. group $M \longrightarrow$ module over $\mathbb{Z}$
$[n \cdot x := x + \dots + x]$

$$2 \cdot m = (1+1)m = 1 \cdot m + 1 \cdot m = m + m.$$

Ex.: $M_n(R)$ $n \times n$ matrices $/ R$.

$R^n = \mathrm{Col}_n(R) = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : a_i \in R \right\}$ is a left $M_n(R)$-module with matrix mult.

$\mathrm{Col}_n(R)$: $\underline{M_n(R)\text{-module}}$

$\Big[R$ module: $R \times \mathrm{Col}_n(R) \to \mathrm{Col}_n(R)$
$\left(k, \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right) \longmapsto \begin{pmatrix} k a_1 \\ \vdots \\ k a_n \end{pmatrix} \Big]$

Ex.: ${}_R R$ $R \times R \to R$ mult.

Similarly we have right R-modules $(M,+)$ $\rightsquigarrow M_R$

$$M \times R \to M$$

Def.: Given ${}_R M, {}_R N$. a fct (map) $\Phi : M \to N$ s.t.

ⓐ Φ is a ~~graph~~ group homomorphism

ⓑ $\Phi(rm) = r\Phi(m)$ $\forall r \in R, m \in M$

$\Rightarrow \Phi(rm + sn)$
$= r\Phi(m) + s\Phi(n$

is an $\underline{R\text{-hom.}}$ [homomorphism of R-modules]

Ex.: $R = \mathbb{Z}$, ${}_{\mathbb{Z}}A, {}_{\mathbb{Z}}B$ R-hom is an group hom. (not more)

Ex.: $R = k$ field, R-hom. $\equiv$ linear transformations
$\rightsquigarrow$ R-mod $=$ v.s. (linear map)

Write $\mathrm{Hom}_R(M,N)$ for set off all R-homs $\Phi : M \to N$.

This is an abelian group $(\Phi + \Psi)(m) \overset{\text{def}}{=} \Phi(m) + \Psi(m)$
(M, N ab. groups $\to$ abelian; group ✓)

Write $\mathrm{End}_R(M)$ for $\mathrm{Hom}_R(M,M)$.

Prop.: Given R, ${}_R M$, ${}_R N$, and ${}_R P$ with R-homomorphisms,

$f, g : M \to N$, $h, l : N \to P$ then

ⓐ $h \circ f \in \mathrm{Hom}_R(M,P)$ $M \xrightarrow{f} N \xrightarrow{h} P$

ⓑ $h \circ (f+g) = h \circ f + h \circ g$ ⓓ $\circ$ is associative

ⓒ $(h+l) \circ f = h \circ f + l \circ f.$

Pf.: Exercise.

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Prop.: Given R, ${}_RM$. $\mathrm{End}_R(M)$ is a ring.
Pf.: Exercise!

~~Prop.: Let R be a ring, (M,+) an abelian group.~~

Def.: R, S rings. A fct $\Phi: R \to S$ st.

ⓐ $\Phi(r+r') = \Phi(r) + \Phi(r')$

ⓑ $\Phi(rr') = \Phi(r)\,\Phi(r')$

ⓒ $\Phi(1_R) = 1_S$ is an $\overset{\text{ring}}{\cancel{R}}$-hom.

Recall: group G. X is G-set if we have a group hom.
$\Phi: G \to \mathrm{Sym}_X = S_X$ $\quad \Phi(g)(x) = g\cdot x$

Prop.: Given a ring R and an abelian group $(M,+)$.
M is a left R-module iff we have a ring homom.
$\Phi: R \to \mathrm{End}_{\mathbb{Z}}(M)$.
[representation of R]

Proof: (sketch)

Given $\Phi: R \to \mathrm{End}_{\mathbb{Z}}(M)$. define $R \times M \to M$
$(a,m) \to \Phi(a)(m)$.

Check this makes M an R-module.

[$(a+b,m) \to \Phi(a+b)(m) = [\Phi(a)+\Phi(b)](m) = \Phi(a)(m) + \Phi(b)(m)$
$= a\cdot m + b\cdot m$]

Conversely if ${}_RM$ define $\Phi: R \to \mathrm{End}_R(M)$ by
$\Phi(r): M \to M$
$m \mapsto r\cdot m$. **Check** Φ is a ring hom.

Note. $\mathrm{End}_R(M) \subset \mathrm{End}_{\mathbb{Z}}($
is a subring

Finally these two processes are inverse (check). □

Submodules and factor modules

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Def Given ${}_RM$, $N \subseteq M$ s.t.

(a) $N \neq \emptyset$

(b) If $a \in R$, $n \in N$ then $a\cdot n \in N$

(c) If $n, n' \in N$ then $n + n' \in N$.

is a **submodule**.

[add. subgroup closed under scalar mult.]

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Note: N inherits structure of an R-module.

$[n, n' \in N \Rightarrow n + (-1)\cdot n' = n - n' \in N.]$

$0 \cdot m = 0$

Ex: $(-1)m = -m$.

Why? $0\cdot m = (0+0)\cdot m = 0\cdot m + 0\cdot m$. Add $-(0\cdot m)$ to both sides. $\Rightarrow 0\cdot m = 0$

$0 = 0\cdot m = (1+(-1))m = 1\cdot m + (-1)\cdot m = m + (-1)m \Rightarrow (-1)m = -m$.

$\Rightarrow$ Submodule is an additive subgroup $[0 \in N$, closed under $+$, $m \in N \Rightarrow -m \in N]$

Ex: $R = M_2(k)$, k field.

$M = {}_R R$ Then $\begin{bmatrix} k & 0 \\ k & 0 \end{bmatrix} \subseteq M$ submodule

$P = \left\{ \begin{bmatrix} \alpha & \alpha \\ \beta & \beta \end{bmatrix} \mid \alpha, \beta \in k \right\} \subseteq M$ submodule

Def: $I \subseteq {}_R R$ a submodule is a Left ideal.

Exercise: Check $P \subseteq M_2(\mathbb{R})$ is not an ideal.

Ex: R comm. Left ideal $\Leftrightarrow$ ideal.

Ex: R PID. all (left) ideals are 0, $Ra, a \in R \setminus 0$. [all ideals are generated by $a \in R$ that is they look like Ra, $a \in R$.

Prop: Let $f : {}_R M \to {}_R N$ be an R-mod hom. Then

(1) $\operatorname{Ker} f \subseteq M$ and $\operatorname{Ker} f = \{ m \in M .\ f(m) = 0_N \}$

(2) $\operatorname{Im} f = f(M) \subseteq N$ $\operatorname{Im} f = \{ n \in N .\ \exists m \in M .\ f(m) = n \}$

are both submodules.

Why? (1) $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0_M) = 0_N \Rightarrow 0 \in \operatorname{Ker}(f) \neq \emptyset$

If $a \in R$, $n \in \operatorname{Ker}(f)$. $f(an) = af(n) = a0 = 0 \Rightarrow an \in \operatorname{Ker}(f)$. (2) Exer. $\square$

If $n, n' \in \operatorname{Ker}(f)$ $f(n+n') = f(n) + f(n') = 0+0 = 0 \Rightarrow n + n' \in \operatorname{Ker}(f)$

Def: If $N \subset_R M$ is a submodule. The group $\dfrac{M}{N}$ with

$a \cdot (m + N) = am + N$ becomes an R-module called the factor module.

[Check: ① $\dfrac{M}{N}$ is a group ② Is mult. well defined?

remember: M ab. group, N normal subgroup $\rightarrow$ M/N quot. group

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$m+N = m'+N \Rightarrow m-m' \in N \Rightarrow a(m-m') = am-am' \in N$

$\Rightarrow a(m+N) = am+N = am'+N = a(m'+N) \quad \forall a \in R.$

③ other properties are "obvious". $\square$]

Theorem: Given $N \subseteq_R M$ submodule. $\exists$ a bijective correspondence between $\{P \mid N \subseteq P \subseteq M,\ P \text{ submod}\}$ and submods of M/N where P corresponds to $\frac{P}{N} \subseteq \frac{M}{N}$. (Correspondence thm)

why? Correspondence thm for groups give subgroups of M/N. Check that $\frac{P}{N}$ Submodule $\Leftrightarrow$ $P \subseteq M$ submod.

Note: $N \subseteq_R M$ submod. $\pi: M \to \frac{M}{N}$ (projection)

$m \mapsto m+N$

is an R-hom with $\operatorname{Ker}(\pi) = N$.

Theorem: (1^{st} isom thm for modules)

Let $f: {}_RM \to {}_RN$ be an R-hom. Then $\bar{f}: \frac{M}{K} \to f(M)$

$m+K \to f(m)$

is an isom of R-modules, where $K = \operatorname{Ker}(f)$.

Proof: Recall $\bar{f}$ is an isom of groups. from group theory

$\bar{f}(rm+K) = f(rm) = rf(m) = r\bar{f}(m+K)$. $\to$ isom of R-mods $\square$

Def: If $N, K \subseteq_R M$ are submods then $N+K = \{n+k \mid n \in N, k \in K\}$ is again a submodule of M. ${}_R M$

$\neq \emptyset$ ✓

$a \in R, n+k \in N+K \Rightarrow a(n+k) = an+ak \in N+K$ ✓

$(n+k), (n'+k') \in N+K \Rightarrow \ldots + \_ \in N+K$ ✓ ${}_R N_R K$

Theorem: (2^{nd} isomorphism thm for mods)

$N, K \subseteq_R M$ submods. Then $N \cap K \subseteq_R M$ is a submod and

(✓)

$0 \in N \cap K$

$a \in R, m \in N \cap K \Rightarrow a \cdot m \in N, a \cdot m \in K$

$\Rightarrow a \cdot m \in N \cap K$

$n, n' \in N \cap K \Rightarrow n+n' \in N \cap K$ ✓

$$\frac{N}{N \cap K} \cong \frac{N+K}{K}.$$

Proof: Let $f: N \overset{\text{(inclusion)}}{\hookrightarrow} N+K \xrightarrow{\text{(proj)}} \frac{N+K}{K}$.

$n \longmapsto n+0 \longmapsto [n+0]$

$K \trianglelefteq (N+K)$ normal subgroup

$K = \{0+k \mid k \in K\} \subseteq (N+K)$ subgr.

normal: $N+K \subseteq M$ abelian

f is onto: $(n+k)+K = n+K = f(n)$.

$f: N \hookrightarrow N+K \to \frac{N+K}{K}$ Ker(f) $\doteq N\cap K$ "⊆" $m \in N, f(m)=0$ Aug 30

$\Rightarrow m \in K \Rightarrow m \in N\cap K$

"⊇" $m \in N\cap K \Rightarrow m \in K, \Rightarrow f(m)=0$

Ker f is $N \cap K$. $\Rightarrow$ $N\cap K$ is a submodule! (Prop Ker f is submod)

1st isom thm $\frac{N}{N\cap K} \cong f(N) \underset{f \text{ onto}}{=} \frac{N+K}{K}$. □

Thm (3rd isom. thm)

If $K \subseteq N \subseteq_R M$ are submodules. Then $\frac{N}{K} \subseteq \frac{M}{K}$ is a submodule and

$$\frac{\frac{M}{K}}{N/K} \cong M/N.$$

$n+K \in M/K$ ($N \subset M$)

$N/K \neq \emptyset$ ✓

$a \in R$, $n+K \in N/K \Rightarrow a\cdot(n+K) = an+K \in N/K$

... $\Rightarrow N/K$ Submod ✓

Proof: $f: \frac{M}{K} \to \frac{M}{N}$

$m+K \longmapsto m+N$

$m+K = m'+K \Rightarrow m-m' \in K \subseteq N \Rightarrow m+N = m'+N$.

$\Rightarrow f$ is well-def.

Clearly $\operatorname{Im}(f) = \frac{M}{N}$, $\operatorname{Ker}(f) = \frac{N}{K}$. □

$m+N = f(m+K)$. $f(n+K) = n+N = N$, $f(m+K) = m+N = 0 \iff m \in N$

$\Rightarrow m \in K \in N/K$

Def: ${}_RS$ is simple if $S \neq 0$ & only submodules are 0 and S.

Theorem: ${}_RS$ is simple iff $S \cong \frac{R}{I}$ where I is a maximal left ideal of R.

Proof: "⇐": If $I \subseteq R$ is a maximal left ideal. Then $\frac{R}{I} \neq 0$ since $I \neq R$.

By the correspondence thm: no submods of R/I ~~expe~~ except $0 = \frac{I}{I}$ and $\frac{R}{I}$. $\Rightarrow R/I$ is a simple R-module.

submods of R/I are bij to submods = left ideals $I \subseteq J \subseteq R$

$\rightsquigarrow$ only $J = I$ and $J = R$

$\rightsquigarrow R/I$ and $I/I = 0$

"⇒" Conversely Let S be a simple R-mod. Choose $x \in S$, $x \neq 0$.

Define $f: R \to S$. f is an R-hom. (S, R R-modules) group hom ✓

$r \mapsto r\cdot x$

$f(r'\cdot r) = (r'\cdot r)\cdot x = r'rx = r'f(r)$

Let $I = \operatorname{Ker}(f)$. Then $x \in f(R) \subseteq S$ is non zero $\overset{S \text{ simple}}{\Rightarrow} f(R) = S$.

$\Rightarrow \bar{f}: \frac{R}{I} \to \underset{f(R)}{S}$ is an isom (1st isom thm). ■

Note: I is a maximal left ideal since $\frac{R}{I}$ simple. □

Sep 01

Direct sums

R-mods $M_1, - M_t$

$\bigoplus_i M_i := M_1 \oplus \ldots \oplus M_t = \{(m_1, m_2, -, m_t) \mid m_i \in M_i\}$

$+$ is componentwise

$r(m_1, -, m_t) = (rm_1, -, rm_t)$

This makes $M_1 \oplus - \oplus M_t$ an R-mod.

(external) direct sum of $M_1, -, M_t$.

$\bigoplus_i M_i$ is also written $\coprod_i M_i$

$\alpha_j : M_j \to \bigoplus_i M_i$

$m_j \mapsto (0, -, 0, \underset{\substack{\uparrow \\ j\text{-pos.}}}{m_j}, 0, -, 0)$

α_j R-hom: group hom ✓

$\alpha_j(rm_j) = (0, -, 0, rm_j, 0, -, 0)$

$= r(0, -, 0, m_j, 0, -, 0)$

$= r\alpha_j(m_j)$ ✓

$P_j : \bigoplus_i M_i \to M_j$

$(m_1, -, m_t) \mapsto m_j$

P_j R-hom: group hom ✓

$P_j(r(m_1, -, m_t)) = P_j((rm_1, -, rm_t))$

$= rm_j = r P_j((m_1, -, m_t))$ ✓

α_j, P_j R-homs.

Prop: Using notation above.

(1) $P_j \circ \alpha_j = \overset{\swarrow \text{identity}}{I_{M_j}}$ ✓ $: M_j \to M_j$ R-hom as composition

(2) $P_l \circ \alpha_j = 0$ ✓ $: M_j \to M_l$ R-hom ✓

(3) $\sum_{i=1}^{t} \underbrace{\alpha_i \circ P_i}_{\bigoplus_i M_i \to \bigoplus_i M_i} = I_{\bigoplus_i M_i}$ R-hom

Check! → Extra Sheet

Def: ① If $A, B \leq_R M$ R-mods. $A + B = \{a + b \mid a \in A, b \in B\} \leq_R M$ is again a submodule. ✓ (earlier N+K submod)

② If $M_1, - M_t \leq_R M$. $M_1 + M_2 + - + M_t = \{m_1 + - + m_t \mid m_i \in M_i\}$ ← submod: $\neq \emptyset$ ✓

$a \in R$, $m_1 + - + m_t \in$

$\Rightarrow a(m_1 + - + m_t)$

$= (am_1 + - + am_t)$

$\in M_1 + - + M_t$ ✓

$m, m' \in M_1 + - + M_t$

$\Rightarrow m + m' \in M_1 + - + M_t$ ✓

Def: M is the (internal) direct sum of $M_1, -, M_t \leq_R M$ if

(1) $M_1 + - + M_t = M$

(2) every element of M can be written $\overset{(!)}{\text{~~can~~}}$ [uniquely] in form $m_1 + - + m_t$, $m_i \in M_i$.

Ex: $R = \mathbb{R}$, $M = \mathbb{R}^3$

$A = \mathbb{R} \oplus \mathbb{R} \oplus 0$ (xy-plane)
$B = 0 \oplus \mathbb{R} \oplus \mathbb{R}$ (yz-plane)

$M = A + B$

$$(\alpha, \beta, \gamma) = \underset{\in A}{(\alpha, \beta, 0)} + \underset{\in B}{(0, 0, \gamma)}$$

But: $(1,1,1) = \underset{\in A}{(1,1,0)} + \underset{\in B}{(0,0,1)}$
$= (1,0,0) + (0,1,1)$ $\Rightarrow \mathbb{R}^3$ is not the internal direct sum of A, B.

Prop: If M is the internal direct sum of $M_1, \dots, M_t \subseteq M$ then

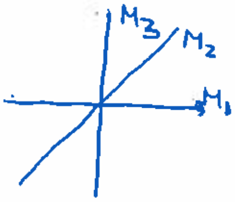
$$\bigoplus_i M_i \to M$$
$$(m_1, \dots, m_t) \mapsto m_1 + \dots + m_t$$

is ~~the~~ an isomorphism of R-modules.

Exercise: $M_1, \dots, M_t \leq_R M$ R-modules with $M_1 + \dots + M_t = M$
If $M_{i+1} \cap [M_1 + \dots + M_i] = 0$, $i = 1,2, \dots, t-1$ then $M = M_1 \oplus \dots \oplus M_t$ (internal dir. sum).

Def:

Ex: $R = \mathbb{R}$, $M = \mathbb{R}^2$
$M_1 = \mathbb{R} \cdot (1,0)$, $M_2 = \mathbb{R}(1,1)$, $M_3 = \mathbb{R} \cdot (0,1)$



$\rightsquigarrow$ pairwise "linear indep" ($M_i \cap M_j = \{0\}$) is not enough

Def: $A \leq_R M$ R-mods. A is a direct summand of ${}_R M$ if $\exists B \leq_R M$ R-submodule s.t. $A \oplus B = M$. Notation: $A \mid_R M$

Ex: $R = \mathbb{Z}$, $M = {}_{\mathbb{Z}}\mathbb{Z}$, $A = 2\mathbb{Z}$. If $0 \neq B \overset{\text{submod}}{\subseteq} \mathbb{Z}$ then $A \cap B \neq 0$.
$0 \neq b \in B \Rightarrow 2b \in B \cap A$
$\Rightarrow A$ is not a direct summand of ${}_R M$.

Note: If $M = M_1 \oplus \dots \oplus M_t$ (internal direct sum)

$\alpha_i : M_i \to M$
$P_i : M \to M_i$
$m_1 + \dots + m_t \mapsto m_i$ are R-homs & satisfy same properties as before.

Lemma: If $A \oplus B = M$. Then $M/A \cong B$.

A direct summand $\Rightarrow A \subseteq_R M$ submod
$\Rightarrow (A, +)$ normal subgroup
$(M,+)$ ab. group

Proof: $P_2 : M \to B$
$a + b \mapsto b$ R-hom, see above

P_2 is onto, $0 + b \mapsto b \ \forall b \in B$

$\ker P_2 = A$.
$P_2(a+b) = 0 \Leftrightarrow b = 0 \Leftrightarrow a + b = a \in A$

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1st Isom, $\frac{M}{A} \cong B$.

[also $\frac{M}{B} \cong A$.]

$$\begin{tikzcd} M/\ker p_2 \arrow[d, equal] \arrow[r, phantom, "\cong"] & p_2(M) = B \\ M/A & \end{tikzcd}$$

<u>Def:</u> If ${}_R M$ R-module, $X \subseteq M$ subset. The submodule of M generated by X is $\bigcap_{\substack{N \subseteq M \\ X \subseteq N \\ N \text{ submod}}} N$.

<u>Note</u>: $X \subseteq M$ so not trivial intersection. $(X \subset \bigcap_{\substack{N \subseteq M \\ X \subseteq N}} N)$

This is the unique smallest submodule of M containing X, check ;-) and equals the set $\{ \sum_{i=1}^{n} r_i x_i \mid n \geq 0, x_i \in X \}$.

<u>Def.</u> ${}_R M$ is finitely generated (fin. gen.) if <u>$\exists$ finite set X</u> that <u>generates M</u>.

$\rightsquigarrow {}_R M = \bigcap_{\substack{N \subseteq M \\ X \subseteq N \\ N \text{ submod}}} N$, $X \overset{\subseteq M}{}$ finite set

ZORN'S LEMMA

<u>Def:</u> Set S, a relation on S is a subset $R \subseteq S \times S$.

<u>Ex:</u> equivalence relation

(i) $(x,x) \in R \quad \forall x \in S$ (reflexive)

(ii) $(x,y) \in R \Rightarrow (y,x) \in R$ (symmetric)

(iii) $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$ (transitive)

[write $x \sim y$ for $(x,y) \in R$.]

<u>Def:</u> R a relation on S is a partial order if

(i) $(x,x) \in R \quad \forall x \in S$ (reflexive)

(ii) $(x,y), (y,x) \in R \Rightarrow x = y$ (anti-symmetric)

(iii) $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$ (transitive)

We ~~call~~ write $x \preceq y$ for $(x,y) \in R$.

We call $(S, \preceq)$ a poset. (partially ordered set)

<u>Ex:</u> $(\mathbb{R}, \leq)$

$(\mathbb{Z}, \leq)$

$(\mathbb{N}, \leq)$

<u>Ex:</u> X any set $S = \mathcal{P}(X)$ set of subsets of X. $A \preceq B$ if $A \subseteq B$.

Sep 1

Ex: $X = \{a, b, c\}$

$A = \{a, b\}$, $B = \{b, c\}$, $C = \{b\}$

$C \le A$, $C \le B$ $A \nleq B$ $B \nleq A$

Def: poset $(S, \le)$ is linearly ordered (or totally ordered) if given $a, b \in S$ either $a \le b$ or $b \le a$.

Ex: $(\mathbb{R}, \le)$, $(\mathbb{N}, \le)$ are linearly ordered

Def: If $C \subseteq S$, $(S, \le)$ poset then C inherits "posetness." (✓)

C is a chain in S if C is linearly ordered.

Def: If $B \subseteq S$, $(S, \le)$ poset. An upper bound for B is $m \in S$ s.t. $b \le m \ \forall b \in B$

Ex: $(\mathbb{R}, \le)$ $(0,1)$ has $1, 17, \sqrt{\pi}$ as upper bounds.

$[0,1]$ has upper bounds $1, 17, \sqrt{\pi}$.

Theorem (ZORN'S)

Let $(S, \le)$ be a nonempty poset. If every chain in S has an upper bound then S contains a maximal element.

Ex: $S = \{A \subseteq \mathbb{N} \mid A \text{ finite}\}$ no max'l element. (no finite set M with $M \subseteq A \Rightarrow M = A$)

$A \le B \Leftrightarrow A \subseteq B$

Sep 6

Recall: $(S, \le)$ poset.

"$a \le b$" for $(a, b) \in R \subseteq S \times S$.

(i) $a \le a \ \forall a \in S$

(ii) $a \le b, b \le a \Rightarrow a = b$

(iii) $a \le b, b \le c \Rightarrow a \le c$

Def: $m \in (S, \le)$ is a maximal element if $m \le x \Rightarrow x = m$.

Ex: $S = \mathcal{P}(\mathbb{N})$, $A \le B$ if $A \subseteq B$ then $\mathbb{N}$ is (!) maximal element.
(unique)

Ex: $S = \{A \mid A \subsetneq \mathbb{N}\}$ $\{2, 3, 4, \ldots\}$ is a maximal element.

In fact, for $m \in \mathbb{N}$, $\mathbb{N} \setminus \{m\}$ is maximal. $\exists \infty$-ly many max. element

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Theorem (ZORN'S LEMMA)

If S is a nonempty poset s.t. every chain in S has an upper bound, then S has a maximal element.

$\Leftrightarrow$ axiom of choice
$\Leftrightarrow$ well-ordering thm [every set can be well-ordered → every non-empty subset has a least [strict total order]]

Def: ${}_RM \neq 0$. $N \leq_R M$ is a maximal submodule if $\underline{N \neq M}$ and $\underline{N \leq_R X \leq M}$ implies $\underline{X = N \text{ or } X = M}$.

[N is maximal in poset of proper submodules]

Prop: Assume ${}_RM$ is finitely generated and $A \subsetneq M$ submodule. Then $\exists$ maximal submodule of M containing A.

↑ for submod to be f.g.

R Noetherian ?!

Proof: Let $M = Rm_1 + \_ + Rm_t$ (f.g.). $P = \{X \mid A \subseteq X \subsetneq M,\ X \text{ submodule}\}$

$X \leq Y$ if $X \subseteq Y$

$P \neq \emptyset$ since $A \in P$. $A \subseteq A \subsetneq M$, A submod ✓

Let $\mathcal{C} = \{X_i : i \in I\}^{\subseteq P}$ be a chain in P.
I some index set $\qquad$ [$C \subseteq P$, C linearly ordered]

Let $Y = \bigcup_{i \in I} X_i$.

Have to prove $\mathcal{C}$ has an upper bound. Claim: Y is ~~the~~ a upper bound for $\mathcal{C}$

Claim: $Y \in P$.

Check Y is a submodule: $Y \neq \emptyset$ because $\mathcal{C} \neq \emptyset$ [otherwise: every element in P is an upper bound for $\mathcal{C}$]

If $y \in Y$, $y \in X_i$ some i. $\Rightarrow ry \in X_i \subseteq Y$ for all $r \in R$.

If $x, y \in Y \Rightarrow x \in X_i$, $y \in X_j$. wlog $X_i \subseteq X_j$ ($\mathcal{C}$ linearly ordered!) $\overset{x \in X_j}{\Rightarrow} x + y \in X_j \subseteq Y$.

$\underline{\underline{Y \neq M}}$ if $Y = M \Rightarrow m_1 \in X_{i_1}, m_2 \in X_{i_2}, \_, m_t \in X_{i_t}$

$\mathcal{C}$ is a chain, so one X_{i_j} contains all other X_{i_l}, $l = 1, \_, t$. because there are only finitely many

$\Rightarrow m_1, \_, m_t \in X_{i_j} \Rightarrow M = Rm_1 + \_ + Rm_t \subseteq X_{i_j}$ $\lightning$ $[X_{i_j} \subsetneq M]$
X_{i_j} submodule

$\Rightarrow Y \neq M$

$\underline{Y \geq A}$ because $X_i \geq A$ for all $i \in I$ $\qquad \Rightarrow Y \in P$

Y is an upper bound for $\mathcal{C}$. $\quad X_i \subseteq Y$ for all $i \in I$ ✓

$\overset{\text{ZORN}}{\Rightarrow}$ P has a $\underline{\text{maximal element}}$ which is a maximal submodule containing A.

↘ $\tilde{M} \in P \rightarrow A \subseteq \tilde{M} \subsetneq M$
submod

if $X \in P$
$\tilde{M} \subseteq X \subseteq M \rightarrow \tilde{M} = X$ or $M = M$

Sep 6

? **Exercise:** R integral domain, not a field.
non zero, commutative ring with no non zero zero divisors

Let $K = Q(R)$ be quotient field of R.

Show ${}_R K$ has no maximal submodule.

Example: A, B 2 sets $\Rightarrow$ Either $\exists\ f: A \to B$ 1-1 or $\exists g: B \to A$ 1-1.

? **[Exer:** If both $\exists\ h: A \to B$ bijective.]

Sketch: Let $P = \{ (X, h_X) \mid X \subseteq A,\ h_X : \overset{X}{\cancel{A}} \to B \text{ 1-1} \}$
injective

$(X, h_X) \leq (Y, h_Y)$ if $X \subseteq Y$ and $h_Y|_X = h_X$.

If $\mathcal{C} = \{ (X_i, h_i) \mid i \in I \}$ be a chain in P.

Let $A_0 = \bigcup_{i \in I} X_i$, $h_0 : A_0 \to B$
$h_0(x) = h_i(x)$ if $x \in X_i$ makes sense because if $x \in X_i \cap X_j$ then $h_i(x) = h_j(x)$

(A_0, h_0) is an upper bound. $\overset{\text{ZORN}}{\Rightarrow} \exists$ maximal element $(A_1, h_1) \in P$.
$(A_0, h_0) \in P$ ✓
$(X_i, h_i) \leq (A_0, h_0)\ \forall (X_i, h_i) \in \mathcal{C}$ $[X_i \subseteq A_0 = \bigcup_{i \in I} X_i$ ✓ $h_0|_{X_i} = h_i$ ✓$]$

If $A_1 = A$ done $\Rightarrow h_1 : A_1 = A \to B$ 1-1

If $A_1 \subsetneq A$, then h_1 must be onto.

[If $a' \in A \setminus A_1$, $b' \in B \setminus h_1(A_1)$ (assume h_1 not onto $\Rightarrow h_1(A_1) \neq B$)

let $h' : A_1 \cup \{a'\} \to B$

$$x \mapsto \begin{cases} h_1(x) & \text{if } x \in A_1 \\ b' & \text{if } x = \cancel{A}\, a' \end{cases}$$

Now $(A_1, h_1) \lneqq (A_1 \cup \{a'\}, h')$ $\lightning$ ← (A_1, h_1) is a max'l el of P]
$A_1 \subset A_1 \cup \{a'\}$, $h'|_{A_1} = h_1$

Now let $g = \text{"}\underset{h_1^{-1}}{f^{-1}}\text{"} : B \to A$ is 1-1. because h_1 is a fct (left-inverse) ▫

h_1 is onto $\Rightarrow h_1^{-1}$ def $\forall b \in B$ (can be def'd by taking a pre-image of b)

Note: If $I \subsetneq R$ is a proper left ideal then $\exists I \subseteq M$, $M \subseteq R$ maximal left ideal.
$\forall M$?

$[R = R \cdot 1_R$ is finitely generated.$]$

Free modules

Recall: Given $m \in {}_R M$ $\exists$ R-hom $\varphi : {}_R R \to {}_R M$,
$r \mapsto r \cdot m$.

Really: φ is the unique R-hom $\varphi : R \to M$ s.t. $\varphi(1) = m$.

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$(\varphi(r) = \varphi(r\cdot 1) = r\,\varphi(1) = r\,m]$

Def: Given ${}_R F$ and $X \subseteq F$ subset. We say ${}_R F$ is a free module with basis X if given any function $f: X \to M$, M an R-module $\exists\,(!)$ R-hom $\varphi: F \to M$ s.t. $\varphi|_X = f$.

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[ld, dashed, "\exists(!)\,\varphi"] \\
M &
\end{tikzcd}$$

Ex: ${}_R R$ is free with basis $\{1\} \subseteq R$.

given $f: \{1\} \to M$
def. $\varphi: R \to M$
$m \in M$, $r \mapsto r\cdot\cancel{m}f(1)$

Ex: $R = k$ field if V has k-basis X, V is a free k-mod w/ basis X.

given $f: X \to M$ one can extend f to $\varphi: V \to M$ by $\varphi(\sum r_i x_i) = \sum r_i\,\varphi(x_i)$ [X k-basis

Ex: $F = R^{(n)} = R \oplus R \oplus \_ \oplus R$ is free w/ basis $\{e_1, e_2, \_, e_n\}$
$= \{(r_1, \_, r_n) \mid r_i \in R\}$

where $e_i = (0, \_, 0, 1, 0, \_, 0)$ same as for Ex before
$\uparrow$ ith position

Theorem: Given any set X $\exists$ a free R-mod w/ basis X.

"Proof": $F \overset{\text{def}}{=} \{\sum\limits_{x\in X} r_x x \mid r_x = 0$ almost everywhere$\}$
[formal sums] $\uparrow$ (a.e.)
(except for finitely many)

$$\sum_x r_x x = \sum_x s_x x \overset{\text{def}}{\iff} r_x = s_x \quad \forall x \in X$$

$$\left(\sum_x a_x x + \sum_x b_x x\right) \overset{\text{def}}{=} \sum_{x\in X} (a_x + b_x)x$$

$$r\left(\sum_x a_x x\right) \overset{\text{def}}{=} \sum_x (r a_x) x$$

Given $f: X \to M$ def. $\varphi: F \to M$
$$\sum_x r_x x \longmapsto \sum_x r_x f(x)$$
[finite sum!]

Clearly $\varphi(1\cdot x) = 1\cdot f(x) = f(x)$.

Identify x with $1_R \cdot x$, done! $\square$

Note: $F = \{\alpha: X \to R \mid \alpha = 0 \text{ a.e.}\}$, $\alpha(x) = r_x$.

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Theorem: Every ~~a~~ R-mod is a homomorphic image of a free module.

$[\exists_R \ {}_R F$ free $\pi : F \to_R M$ onto.$]$

Proof: Let $S = \{m_i \mid i \in I\} \subseteq M$ be a generating set. (Take $S = M \setminus 0$).

Take new set $X = \{x_i \mid i \in I\}$ and let $f : X \to M$
$x_i \mapsto m_i$.

Let F be the free mod on X.

$f(X) = S \subseteq M$ generates.

$\exists !$ R-hom $\pi : F \to M$ s.t. $\pi|_X = f$.

π is onto since $\pi(F) \supseteq f(X) = S$, S generates ${}_R M$.

(Now, $M \cong \frac{F}{\ker \pi}$, by 1^{st} isom. thm.) □

Ex: $R = \mathbb{Z}$, $\mathbb{Z}_2$ is not free.

If $x \in \mathbb{Z}_2$, $2x = 0$.

$f : X \to \mathbb{Z}$
$x \mapsto 1$
$\varphi : \mathbb{Z}_2 \to \mathbb{Z}$
$\varphi|_{\{x\}} = ($
$\Rightarrow \varphi(\underbrace{2x}_{=0}) = 2\varphi(x) = 2$ ↯

Artinian rings

Def: (1) ${}_R M$ is Artinian if given a descending chain of submods

$M_1 \supseteq M_2 \supseteq \dots$ $\exists$ m s.t. $M_m = M_{m+1} = M_{m+2} = \dots$

Ex: (2) R is Left Artinian if ${}_R R$ is Artinian.

Ex: $\mathbb{Z}$ is not left Art. $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 2^2\mathbb{Z} \supsetneq \dots$

Ex: D division ring (D skew field) non-comm. $R = M_n(D)$ matrices is left Art.

Every left ideal of R is a D-subspace of R.

$(\alpha [a_{ij}] = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} [a_{ij}])$

R is fin. dim. / D $\dim_D R = n^2 < \infty$.

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Prop: Let $A, B, C \subseteq_R M$ where $A \subseteq B$, $A+C = B+C$, $A \cap C = B \cap C$.

Then $A = B$.

Proof: It suffices to show $B \subseteq A$. Let $b \in B \subset B+C = A+C$.

$\Rightarrow b = a+c$, some $a \in A$, $c \in C$ $\Rightarrow c = \underset{\in B}{b} - \underset{\in A \subseteq B}{a} \in B \cap C = A \cap C \subseteq A$.

Now, $b = a+c \in A$. □

Theorem: If $N \subseteq_R M$ submod, then M is Artinian iff N and M/N are Artinian. (same for Noetherian!)

Pf: If M is Art. $\Rightarrow$ $N \subseteq M$ is Art. and M/N is Art. by correspondence thm.

Conversely assume N, M/N are Art.

Let $K_1 \supseteq K_2 \supseteq \dots$ desc. chain in ${}_RM$.

[For $i >> 0$
(sufficiently large)]

$K_1 \cap N \supseteq K_2 \cap N \supseteq \dots$

N is Art. $\Rightarrow$ For $i \geq 0$ $K_i \cap N = K_{i+1} \cap N$.

$\frac{K_1+N}{N} \supseteq \frac{K_2+N}{N} \supseteq \dots$

M/N Art. $\Rightarrow$ For $i >> 0$ $(K_i+N)/N = \frac{K_{i+1}+N}{N}$

$\Rightarrow$ -"- -"- $K_i + N = K_{i+1} + N$
by corresponding thm.

(i = max{of i's before})
Let $A = K_{i+1}$, $B = K_i$, $C = N$ in prop.

Get $K_{i+1} = K_i$ for $i >> 0$. □

Corollary: (1) If $M_1, \dots, M_t$ are Art. Then $M_1 \oplus \dots \oplus M_t$ is Art.

(2) If R is left Art. and ${}_RM$ is fin. gen. Then ${}_RM$ is Art.

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Pf: (1) Induction on t.

$t=1$ ✓

$t>1$: $\dfrac{M_1 \oplus \_ \oplus M_t}{M_1 \oplus - \oplus M_{t-1} \oplus 0} \cong M_t$

We know $M_1 \oplus \_ \oplus M_{t-1}$ Art by ind.

M_t Art. Done. (Thm M/N Art, N Art $\Rightarrow M$ Art)

(2) R left Art. $R^{(t)} = R \oplus \_ \oplus R$ is Art.

If $M = Rm_1 + \_ + Rm_t$ is fin. gen. $\exists \pi : R^{(t)} \to M$ onto.

$(r_1, \_, r_t) \mapsto r_1 m_1 + \_ + r_t m_t$

$\Rightarrow M = \dfrac{R^{(t)}}{\operatorname{Ker}\pi}$ is Art. by Thm. [$\operatorname{Ker}\pi \subset R^{(t)}$ submod] $\square$

Exercise: (HW 2) $\exists$ ring R left Art. not right Artinian. $\begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$

Jacobson radical

Def: ① If $X \subseteq {}_R M$ the annihilator of X (in R) is

$\operatorname{Ann}_R(X) = \{ r \in R \mid rx = 0 \ \forall x \in X \}$.

② If $X \subseteq R$ left annihilator is $\text{l-ann}_R(X) = \{ r \in R \mid rx = 0 \ \forall x \in X \}$

Exercise: (1) $\operatorname{Ann}_R(x)$, $x \in {}_R M$

$\text{l-ann}_R(x)$, $x \subseteq R$ are left ideals of R.

(2) If $X \subseteq {}_R M$ submod or $X \subseteq R$ left ideal then annih. is an ideal.

Def: (1) If $p \triangleleft R$, p is a prime ideal if given $A, B \triangleleft R$ s.t.

$AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$.

[Note: $AB = \{ \sum_{i=1}^{n} (a_i b_i) \mid n \geq 1, a_i \in A, b_i \in B \}$]

(2) $p \triangleleft R$ is a (left) primitive if $\exists$ simple left R-module ${}_R S$ w/

$p = \operatorname{ann}_R(S)$.

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Ex: k field. $R = M_2(k)$ has only 2 ideals 0 and R.

$\Rightarrow$ 0 is prime but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0_R$.

$\Rightarrow$ prime doesn't imply $a \cdot b \in \underline{\underline{p}} \Rightarrow a \in p / b \in p$

Prop: If $P \triangleleft R$ is left primitive then P is prime.

Proof: Suppose $A, B \triangleleft R$, $AB \subseteq P$. $P = \operatorname{ann}_R(S)$

Suppose $B \not\subseteq P$. $\Rightarrow 0 \neq BS = \{ \sum_{i=1}^{n} b_i s_i \mid b_i \in B, s_i \in S, n \geq 1 \}$

where $p = \operatorname{ann}_R(S)$, ${}_R S$ simple.

$0 \neq BS \subseteq S$

S simple $\Rightarrow BS = S$

Now, $0 = (AB)S$ since $AB \subseteq P$ & $P = \operatorname{ann}_R(S)$

$= A(BS)$

$= AS$ $\Rightarrow A \subseteq \operatorname{ann}_R(S) = P$. □

Def: R ring. The Jacobson radical of R is

$$J(R) = \bigcap_{\substack{M \subseteq R \\ \text{max'l} \\ \text{left ideal}}} M.$$

Ex: $R = \mathbb{Z}$. $p\mathbb{Z}$ maximal left ideal if p prime.

$\bigcap_p p\mathbb{Z} = 0$

$[\ n \in \bigcap_p p\mathbb{Z} \Rightarrow p \mid n \ \forall$ primes $p \Rightarrow n = 0.\ ]$ $J(\mathbb{Z}) = 0$.

Exercise: If $n = p_1^{k_1} \cdots p_t^{k_t}$ $p_1, \ldots, p_t$ distinct primes, $k_i \geq 0$

~~and~~ find $J(\mathbb{Z}_n) = J(\mathbb{Z}/n\mathbb{Z})$

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Recall: $J(R) = \bigcap_{\substack{M \subseteq R \\ \text{max'l L.} \\ \text{ideal}}} M$ Jacobson radical.

Prop: $x \in J(R) \Leftrightarrow 1 + rx$ has a left inverse $\forall r \in R$

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" $\frac{1}{1+rx} = \sum_{n=0}^{\infty} (-1)^n (rx)^n$ "

Proof: Let $x \in J(R)$, $r \in R$. ~~Let~~ If $1+rx$ has no left inverse then $1 \notin R(1+rx) \subsetneq R$. ${}_R R = R \cdot 1_R$ is finitely generated $\Rightarrow \exists {}_R M \subsetneq R$ maximal left ideal s.t. $R(1+rx) \subseteq M$. Now, $1 = (1+rx) + (-r)x \in M$ since $x \in J(R) \subseteq M$ $\Rightarrow M = R$. Contradiction! $\rightarrow$ $1+rx$ has a left inverse.

If $x \notin J(R)$ we need $1+rx$ has no left inverse for some $r \in R$.

$x \notin J(R) \Rightarrow x \notin M \subsetneq R$, M maximal left ideal

Now, $M \subsetneq \underbrace{M + Rx = R}_{\text{since } M \text{ max'l}}$ $\Rightarrow 1 = m + sx$, some $s \in R$.

$\Rightarrow m = 1 - sx = 1 + rx$ where $r = -s$. Since $m \in M$ a proper left ideal

$\Rightarrow m = 1+rx$ has no left inverse. □

Ex: $R = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \text{ odd}\} \subseteq \mathbb{Q}$.

All Ideals of R are $R \supseteq 2R \supseteq 2^2 R \supseteq \ldots$

$J(R) = 2R$ (!) maximal ideal.

Ex: $R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ k field. $\Rightarrow J(R) = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$.

Theorem: Let $I \triangleleft R$, $I \subseteq J(R)$. Then $J(R/I) = J(R)/I$.

Proof: all maximal left ideals contain I.

maximal l. ideals of R/I are M/I, $M \subseteq R$ max'l L. ideal

$\Rightarrow J(R/I) = \bigcap_{I} (M/I) = \frac{\bigcap_M M}{I} = J(R)/I$. □

$\rightarrow$ use in HW

Theorem: R ring.

(1) $J(R) = \bigcap_{\substack{P \text{ prim.} \\ \text{L. ideal}}} P$, in particular $J(R) \triangleleft R$.

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(2) $J(R)$ is (!) largest ideal $I \lhd R$ s.t. $1+x \in \underset{\substack{\uparrow \\ \text{units of } R}}{\mathcal{U}}(R)$ $\forall x \in I$.

(3) $J(R) = \bigcap\limits_{\substack{Q \lhd R \\ \text{right} \\ \text{primitive}}} Q$

(4) $J(R) = \bigcap\limits_{\substack{M \\ \text{max'l} \\ \text{right ideal}}} M$

<u>Proof:</u> (1) Let $P \lhd R$ be left primitive. $\Rightarrow$ $P = \operatorname{ann}_R(S)$, ${}_R S$ simple.

$$= \bigcap_{x \in S \setminus \{0\}} \operatorname{ann}_R(x)$$

But $\operatorname{ann}_R(x) \subseteq R$ is a maximal ideal $[\, S \cong R/\operatorname{ann}_R(x) \,]$

S simple $\Rightarrow$ $S \cong R/I$, $I \trianglelefteq R$ max'l l. ideal

$\Rightarrow$ $J(R) \subseteq P$

$\Rightarrow$ $J(R) \subseteq \bigcap\limits_{\substack{P \\ P \text{ l. primitive}}} P$

Conversely, let $x \in \bigcap\limits_{\substack{P \\ P \text{ l. primitive}}} P$. If $M \subseteq R$ is a max'l left ideal we need $x \in M$. Then $x \in J(R)$.

R/M is a simple R-mod. $\Rightarrow$ $x \in \operatorname{ann}_R(R/M) \subseteq \operatorname{ann}_R(1+M) = M$.

(2) Part (1) shows $J(R) \lhd R$.

If $I \lhd R$ s.t. $1+x \in \mathcal{U}(R)$ $\forall x \in I$ $\Rightarrow$ $1+rx$ has a left inverse $\forall r \in R, x \in I$

$\Rightarrow$ $x \in J(R)$ by prop.

Finally, we need to show $J(R)$ has this property.

If $x \in J(R)$ $\Rightarrow$ $1+x$ has a left inverse, $1+y$ say.

$\Rightarrow$ $1 = (1+y)(1+x)$

$x + y + yx = 0$

$y = -(1+y)x \in J(R)$ $\Rightarrow$ $1+y$ has left inverse, necessarily $1+x$.

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$\Rightarrow 1+x \in U(R)$ w/ $(1+x)^{-1} = 1+y$. □

[**Note:** $uv = 1 = wu \Rightarrow v = w$

$\uparrow$

$w = \overset{w(uv)}{\cancel{(uv)w}} = \cancel{u(vw)}(wu)v = v.$]

Def: ① $a \in R$ is nilpotent if $a^n = 0$, some $n \geq 1$.

② $I \triangleleft R$ is a nil ideal if every $x \in I$ is nilpotent.

③ $N \triangleleft R$ is nilpotent if $N^n = 0$, some $n \geq 1$.

Recall: $A, B, C \triangleleft R$. $AB = \{ \sum_{\substack{\cancel{\text{rean}} \\ \text{finite}}} a_i b_i \mid a_i \in A, b_i \in B \} \triangleleft R$

$(AB)C = A(BC)$.

If $N \triangleleft R$ has $N^n = 0$ then $x_1 x_2 \cdots x_n = 0 \quad \forall x_1, \ldots, x_n \in N$.

In particular $x^n = 0 \quad \forall x \in N$.

Ex:

$$R = \begin{bmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix} \quad k \text{ field}$$

$$N = \begin{bmatrix} 0 & k & k \\ 0 & 0 & k \\ 0 & 0 & 0 \end{bmatrix} \triangleleft R \qquad N^2 = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^3 = 0.$$

(**Prop:** If $I \triangleleft R$ is nil then $I \subseteq J(R)$.

Proof: It suffices to know I is a left (or right) ideal.

If I is a left ideal ~~then~~ ~~Fx~~ and $x \in I$, $r \in R$ then

$(rx)^n = 0$, some n. Now, $[1 - rx + (rx)^2 - (rx)^3 \ldots](1+rx) = 1$.

$\Rightarrow 1+rx$ has a left inverse $\Rightarrow x \in J(R)$ by prop. $\Rightarrow I \subseteq J(R)$. □)

Theorem: If R is left (or right) Artinian then $J(R)$ is nilpotent.

Proof: Let $J = J(R)$.

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$J \supseteq J^2 \supseteq J^3 \supseteq \dots \quad \Rightarrow J^n = J^{n+1} = J^{n+2} = \dots$ some n.

Suppose $J^n \neq 0$.

Let $S = \{ I \subseteq R \mid 0 \neq I$ left ideal w/ $J^n I \neq 0 \}$

Since $0 \neq J^n = J^{2n} = (J^n)(J^n) \Rightarrow J^n \in S$, so S is not empty.

T.B.C.

Prop: $J(R)$ contains every nil one-sided ideal. Sep 13

Proof: Assume $I \subseteq R$ nil left. If $r \in R, x \in I$ then $rx \in I$. It suffices to show $1+y$ is left invertible $\forall y \in I$.

$y^n = 0$, some $n > 0$. $\Rightarrow (1 - y + y^2 - \dots + (-1)^{n-1} y^{n-1})(1+y) = 1$

$$= (1+y)(1 - y + y^2 + \dots + (-1)^{n-1} y^{n-1})$$

$\Rightarrow 1+y \in U(R) \; \forall y \in I \Rightarrow I \subseteq J(R)$. □

Note: R ring. ${}_R M$ is Artinian iff every nonempty set of submodules has a minimal element.

Why? (⇐): If $M_1 \supseteq M_2 \supseteq \dots$ are submods then $\{ M_i \mid i \geq 1 \}$ has a minimal element M_n, say. $\Rightarrow M_n = M_{n+1} = \dots$

(⇒): Pick M_1 in set. if minimal, done. Otherwise pick $M_2 \subsetneq M_1$. Etc. We get $M_1 \supsetneq M_2 \supsetneq M_3 \dots$ This process must stop w/ M_n minimal in set. □

Theorem: If R is left (or right) Artinian then $J(R)$ is nilpotent.

Proof: Let $J = J(R)$ $\quad J \supseteq J^2 \supseteq J^3 \supseteq \dots$ If R is left Artinian, then $J^n = J^{n+1} = \dots$, some n. Suppose $J^n \neq 0$.

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Let $S = \{ I \subseteq R \mid I \text{ left ideal}, J^n I \neq 0 \}$. $J^n \in S \Rightarrow S \neq \emptyset$

This has minimal $I_0 \subseteq R$. $\Rightarrow J^n x \neq 0$, some $x \in I_0$.

Now, $J^n(J^n x) = J^{2n} x = J^n x \neq 0$

Since $J^n x \subseteq I_0$ and $J^n x \in S$, we get $J^n x = I_0$.

$\Rightarrow jx = x$ for some $j \in J^n \subseteq J$.

$\Rightarrow (1-j)x = 0 \Rightarrow (1-j)^{-1}(1-j)x = (1-j)^{-1} 0 = 0$ since $1-j \in U(R)$

$\Rightarrow x = 1 \cdot x = 0 \Rightarrow I_0 = J^n x = J^n 0 = 0$ C!

$\Rightarrow J^n = 0$.

$\Rightarrow J(R)$ is nilpotent. □

<u>Corollary:</u> If R is left (or right) Artinian every nil 1-sided ideal is nilpotent.

<u>why?</u> If $I \subseteq R$ is a nil left or right ideal, then $I \subseteq J(R)$

$\Rightarrow I^n \subseteq J(R)^n = 0$ some n. □

<u>Theorem</u> (<u>Nakayama's lemma</u>)

Let ${}_R M$ be finitely generated w/ $JM = M$, where $J = J(R)$, then $M = 0$.

<u>Proof:</u> $M = 0$ done.

If not, pick $m_1, \ldots, m_t$ a generating set w/ t minimal.

Now, $m_t \in M = JM = J(Rm_1 + \cdots + Rm_t) = Jm_1 + \cdots + Jm_t$

$\Rightarrow m_t = j_1 m_1 + \cdots + j_t m_t$, $j_i \in J(R)$

$\Rightarrow \underbrace{(1-j_t)}_{\text{unit since } j_t \in J(R)} m_t = j_1 m_1 + \cdots + j_{t-1} m_{t-1}$

$\Rightarrow m_t = (1-j_t)^{-1} j_1 m_1 + \cdots + (1-j_t)^{-1} j_{t-1} m_{t-1} \in Rm_1 + \cdots + Rm_{t-1}$

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Contradiction! $\Rightarrow M=0.$ $\square$

Case $t=1$ → $u \cdot m_1 = 0$ compare w/ Thm before

<u>Example:</u> k field. $k[x_1, x_2, x_3, \_\_]$ poly's in infinitely many variables.

Let $D = \langle x_1^2, x_2^3, x_3^4, \dots \rangle$, $R = k[x_1, x_2, \_]/D$.

Let $I = \langle x_1, x_2, x_3, \_\rangle/D \lhd R$. → no constants $(1 \notin \langle x_1, x_2, \_\rangle/D)$

$(x_i + D)^i \neq 0$ in R $\Rightarrow I$ is not nilpotent.

If $r \cancel{f} \in I$, then $r = f + D$ where $f \in k[x_1, \_, x_n]$ (only fin. many)

is a poly w/ 0 constant term.

Now, $f^{(1+1)+(2+1)+\dots+(n+1)} = f^{\frac{n(n+3)}{2}} \in D \Rightarrow r = f + D$ has $r^{\overset{\in\mathbb{Z}}{\frac{n(n+3)}{2}}} = 0$

$\Rightarrow I \lhd R$ is nil, but <u>not</u> nilpotent. $\Rightarrow R$ <u>not</u> Artinian

<u>Ex:</u> $(x_1 + x_2)^{(1+1)+(2+1)} = (x_1 + x_2)^5 \in D.$

<u>Def:</u> ${}_R M$ is called completely reducible if given $A \leq M$ submod.

$\exists\, {}_R B \leq M$ such that $A \oplus B = M$.

<u>Ex:</u> Every vector space is a completely red. module over k.

<u>Ex:</u> $2\mathbb{Z} \subset \mathbb{Z}$. If $2\mathbb{Z} \oplus B = \mathbb{Z}$ we need $\underset{B \cong}{\mathbb{Z}}/2\mathbb{Z} \cong \mathbb{Z}_2$

But $\mathbb{Z}$ has no elements of finite order! [but ~~B~~ $\mathbb{Z}_2$ has]
$(a + a + \dots + a + a \neq 0 \ \forall a \neq 0)$

$\Rightarrow B$ does not exist.

<u>Def:</u> ① A short exact sequence of R-modules is

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where
f is 1-1,
g is onto,
$\mathrm{Im}(f) = f(A) = \mathrm{Ker}\, g.$

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② $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ is exact (where A_1, A_2, A_3 are R-mods, f_1, f_2 R-homs) if $\operatorname{Im} f_1 = \operatorname{Ker} f_2$.

Note: In ① f 1-1 says $0 \xrightarrow{f} A \xrightarrow{f} B$ is exact.

g onto says $B \xrightarrow{g} C \to 0$ is exact.

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Recall: A short exact sequence (SES)

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

f 1-1

g onto

$f(A) = \operatorname{Ker}(g)$

$\bar{g}: B|_{f(A)} \cong C$

$B = \dot{\bigcup} [c' + f(A)]$

↑

disjoint union

$\Rightarrow |B| = |A||C| = |A \times C| = \max(|A|, |C|)$, if $|A|$ or $|C|$ is infinite.

Conclusion: We can fix A, C and view SES $s: 0 \to A \to B \to C \to 0$ as a set.

s is equivalent to SES $s': 0 \to A \to B' \to C \to 0$

if $\exists$ commutative diagramm

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal, "I_A"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, equal, "I_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C \arrow[r] & 0
\end{tikzcd}$$

It follows (check?) h is an isomorphism. → equivalence relation

The set of equivalence classes are dentoted by $e(C, A)$.

It is a group!

Prop: Let $0 \to A \underset{\dashleftarrow\, r}{\overset{f}{\to}} B \underset{\dashleftarrow\, s}{\overset{g}{\to}} C \to 0$ be a SES. The following are equivalent:

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(1) $\exists\, \underset{\text{R-hom}}{r : B \to A}$ s.t. ~~[illegible]~~ $r \circ f = I_A$.

(2) $\exists\, \underset{\text{R-hom}}{s : C \to B}$ s.t. $g \circ s = I_C$

(3) $B = f(A) \oplus C'$ (Then $g|_{C'} : C' \to C$ is an isom.), some submod C'.
So $C' = s(C)$ if s exists as in (2)

<u>Proof:</u> <u>(1) ⇒ (3):</u> Let $C' = \operatorname{Ker} r$. If $b \in f(A) \cap C'$ ⇒ $b = f(a)$, some a

and $r(b) = rf(a) = 0$ ~~$\Rightarrow fr(b) = f(0) = 0$~~

$\Rightarrow a = 0 \Rightarrow b = f(0) = 0$

If $b \in B$ $\quad b = \underbrace{fr(b)}_{\in f(A)} + \underbrace{[b - fr(b)]}_{\in C'}$

$r(b - fr(b)) = r(b) - \underbrace{rf}_{= I_A} r(b) = r(b) - r(b) = 0$

$\Rightarrow B = f(A) \oplus C'$ $\quad \Leftarrow$

<u>(3) ⇒ (2):</u> By (3), $B = f(A) \oplus C'$ where $g|_{C'} : C' \to C$ is an isom.

If $c \in C$, let $s(c) \in C'$ be (!) $c' \in C'$ w/ $g(c') = c$.
(g onto)

Check s is an R-hom. Clearly $g \circ s = I_C$.

<u>(3) ⇒ (1):</u> Def $r : f(A) \oplus C' = B \to A$

$f(a) + c' \longmapsto a$ $\quad$ well-def'd since f inj.

Check R-hom. Clearly $r \circ f = I_A$.

<u>(2) ⇒ (3):</u> Let $C' = s(C)$. If $b \in f(A) \cap C'$ ⇒ $b = f(a)$, $a \in A$
$= s(c)$, $c \in C$

$\Rightarrow g(b) = \underset{\substack{\| \;\leftarrow \operatorname{Im} f = \operatorname{Ker} g \\ 0}}{g(f(a))} = g \circ s(c) = c$

$\Rightarrow c = 0 \Rightarrow b = s(c) = 0$.

If $b \in B$ then $\quad b = \underbrace{[b - sg(b)]}_{\in \operatorname{Ker} g = f(A)} + \underbrace{sg(b)}_{\in C'}$

$\Rightarrow B = f(A) \oplus C'$.

□

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Def: We say SES is split if one (hence all) condition hold.

"Really": $0 \to A \xrightarrow{f} A \oplus C \xrightarrow{g} C \to 0$

$a \mapsto (a,0)$

$(a,c) \mapsto c$

Ex: $R = k$ field. $A = C = k$, $B = k^{(2)}$

$0 \to A \to B \to C \to 0$

$\alpha \mapsto (\alpha, 0)$

$(\alpha, \beta) \mapsto \beta$

SES of v.s.

$s : C \to B$ is a splitting map

$\beta \mapsto (0, \beta)$

But: $s_1 : C \to B$ is also a splitting map.

$\beta \mapsto (\beta, \beta)$

Def: $_RP$ module is called projective if given a diagram

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dashed, "h'"'] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

of R-homs w/ row exact (g onto) $\exists\, h' : P \to B$ s.t. $g \circ h' = h$.

Prop: Let $0 \to A \to B \to P \to 0$ be a SES w/ P projective.

Then the SES splits.

Proof:

$$\begin{tikzcd} & P \arrow[d, equal] \arrow[dl, dashed, "s"'] & \\ B \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

P proj. $\Rightarrow \exists\, s : P \to B$ s.t. $g \circ s = I_P$ $\Rightarrow$ SES splits!

$\square$

Def: ① $_RA \subseteq {}_RB$ R-modules. A is a direct summand of B if $B = A \oplus C$, some submod $C \leq B$. [write $A \mid B$].

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② ${}_R M$ is completely reducible if every submod is a direct summand.

③ socle of ${}_R M$ is $\operatorname{soc}(M) = \sum_{\substack{S \subseteq M \\ \text{simple}}} S$

④ M is semisimple if $M = \operatorname{soc}(M)$

Ex: v.s. are semisimple (gen by 1-dim subspaces)

<u>Theorem:</u> For ${}_R M$ tfae (the follow. are equiv.)

(1) ${}_R M$ is completely reducible

(2) M is a direct sum of simple submodules.

(3) M is semisimple.

<u>$(2) \Rightarrow (3)$</u> clear!

<u>$(\overset{3}{\cancel{2}}) \Rightarrow (1)$</u> Let $A \subseteq M$ submod. Let $P = \{X \subseteq M \mid X \text{ submod}, X \cap A = 0\}$

P poset by inclusion.

ZORN $\Rightarrow$ $\exists$ max'l element B.

We claim $M = A \oplus B$. Clearly $A \cap B = 0$.

$M = \overset{\sum_{i\in I}}{\cancel{\bigoplus}}_{i \in I} S_i$, S_i simple. If $A + B \neq M \Rightarrow S_i \not\subseteq A + B$, some i.

$\Rightarrow (A \oplus B) \cap S_i = 0 \Rightarrow (A \oplus B) \oplus S_i \subseteq M$.

Now, $B \subsetneq B \oplus S_i \in P$. Contradicts maximality of B.

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<u>$(1) \Rightarrow (2)$</u> Let $\{S_i : i \in I\}$ be the set of simple submodules.

Let $\mathcal{P} = \{J \subseteq I \mid \sum_{j \in J} S_j$ is a direct sum$\}$.

ZORN $\Rightarrow$ $\mathcal{P}$ has a maximal element (check!)

Call maximal element T.

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Suppose $W = \sum_{t \in T} S_t \neq M$

$W \oplus B = M$, some $B^{\neq 0}$ by (1).

Choose $0 \neq x \in B$. Now $W \oplus Rx \subseteq M$ submod.

$\Rightarrow M = W \oplus Rx \oplus C$ some submod C, by (1).

$0 \neq Rx$ is fin. generated $\Rightarrow Rx$ has a max'l submod L.

$W \oplus L \oplus C \subseteq W \oplus Rx \oplus C = M$.

$$\frac{M}{W \oplus L \oplus C} = \frac{W \oplus Rx \oplus C}{W \oplus L \oplus C} \cong 0 \oplus \frac{Rx}{L} \oplus 0 \cong \frac{Rx}{L}$$

$\Rightarrow \frac{M}{W \oplus L \oplus C}$ is simple.

Now, $W \oplus L \oplus C \oplus S_0 = M$, some S_0, by (1).

$S_0 \cong \frac{M}{W \oplus L \oplus C}$ is simple

Now $W_T \oplus S_0 = \left[\bigoplus_{t \in T} S_t\right] \oplus S_0 \subseteq P$, since S_0 is simple submod this contradicts maximality of T.

$\Rightarrow W = M$. □

Note: Every R-module is completely red. iff R is left Art.

and $J(R) = 0$.

$\exists$ lots of equivalent conditions!

Corollary: Assume ${}_R M$ w/ $N \subseteq M$. If M is completely reducible then so are N and M/N.

Proof: $M = \sum_{i \in I} S_i$ is a sum of simple submodules.

Now, $\frac{M}{N} = \sum_{i \in I} \frac{S_i + N}{N} = \cancel{\sum_{i \in I} \frac{N}{S_i \cap N}}$

Now, $\frac{S_i + N}{N} \cong \frac{S_i}{S_i \cap N}$ (2nd isom. thm.) $\cong \begin{cases} S_i & \text{if } S_i \cap N = 0 \\ 0 & \text{if } S_i \cap N = S_i \end{cases}$

S_i simple, N sub

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$\Rightarrow \frac{S_i + N}{N}$ is simple (or 0)

$\Rightarrow \frac{M}{N}$ is Semisimple / compl. red.

<u>Next</u> $M = N \oplus X$ some submod X

$N \cong \frac{M}{X} = \frac{N \oplus X}{0 \oplus X} = \frac{N}{0} \oplus 0$

By first part $\frac{M}{X}$ is Comp. red. $\Rightarrow$ N is S.S. (Semisimple). ■
(⟺ comp. red.)

<u>Recall:</u> ${}_R P$ is projective if given any diagram of R-modules (w/ exact row) (and R-homs)

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'"'] \arrow[d, "h"] \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

G

then $\exists\, h' : P \to B$ s.t. $g \circ h' = h$.

h' is a <u>lifting</u> of h.

<u>Theorem:</u> ${}_R P$ is projective iff $P \mid F$ for some free module F.

<u>Proof:</u> Assume ${}_R P$ is projective. We can find a SES

$$\begin{tikzcd} 0 \arrow[r, leftrightarrow] & K \arrow[r, hook, "i"] & F \arrow[r, "g"] & P \arrow[l, dashed, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

(every R-mod is a <u>hom.</u> <u>image</u> of a free mod)

We have surj. $g : F \to P$, F free, $K = \operatorname{Ker} g$, i inclusion map.

$$\begin{tikzcd} & P \arrow[dl, dashed, "s"'] \arrow[d, equal] \\ F \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

P proj $\Rightarrow \exists\, s : P \to F$ s.t. $g \circ s = I_P$ $\Rightarrow$ SES splits

$\Rightarrow F = K \oplus s(P)$ But $s : P \to s(P)$ is an isom.

$F = \underbrace{i(K)}_{=K} \oplus C'_{\cong P}$ $C' = s(P)$ bc s exists

$\Rightarrow$ P is a direct Summand of a free mod. $[K \oplus P \cong F]$.

Conversely, assume $P \oplus Q = F$ is free (w/ basis X), some ${}_R Q$.

Consider

$$\begin{tikzcd} & P \arrow[d, "h"] \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Let $i : P \to F$ be inclusion.

$\pi : F = P \oplus Q \to P$ is projection.

$\pi \circ i = I$

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$$\begin{tikzcd}
& P \oplus Q = F \arrow[d, shift right, "\pi"'] \arrow[ddl, dashed, "\varphi"'] & \\
& P \arrow[u, shift right, "i"'] \arrow[d, "h"] \arrow[dl, dashed, "h'"'] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

~~Consider~~

For each $x \in X^{\subset F}$ we can choose $\varphi(x) \in B$ s.t.

$(h \circ \pi)(x) = g(\varphi(x))$ [possible since g is onto]

Now, φ extends to a (!) $\varphi : F \to B$
(since F free)

$(g \circ \varphi)(x) = (h \circ \pi)(x)\ \forall x \in X$

$\Rightarrow g \circ \varphi = h \circ \pi$

Let $h=' = \varphi \circ i$. Now $g \circ h' = g \circ (\varphi \circ i) = (g \circ \varphi) \circ i = (h \circ \pi) \circ i$

$= h \circ (\pi \circ i) = h \circ I_P = h.$

$\Rightarrow h'$ lifts h. $\Rightarrow P$ projective □

Note: P is projective to every SES $0 \to A \to B \to P \to 0$ being split. (Exercise)

Def: ${}_R E$ is injective if given any diagram w/ exact row

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "h'"] \\
& E &
\end{tikzcd}$$

$\exists\ h' : B \to E$ s.t. $h' \circ f = h$.

Think of $A \subseteq B$, f inclusion map. $\Rightarrow h'$ is an extension of h to B.

Works for v. spaces

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Ex: $R = M_2(k)$ Then $\mathrm{Col}_2(k) = (M, \text{say})$ is a projective left R-module

$M \oplus M \cong R$.

Prop (1) A direct summand of ~~A~~ a proj. module is projective

(2) A direct sum of proj. mods is proj.

Pf: (1) Suppose ${}_RA \mid {}_RP$, P projective.

Now, $P = A \oplus B$, some $B \subseteq P$. Also $P \oplus Q$ is free, some Q.
[Thm Sep 18]

Now $P \oplus Q = (A \oplus B) \oplus Q = A \oplus (B \oplus Q)$ $\Rightarrow A \mid P \oplus Q$ is a direct summ. of a free module $\overset{\text{Thm}}{\Rightarrow}$ A projective.

(2) If $\{P_i \mid i \in I\}$, P_i proj. $\Rightarrow \forall i \ \exists i$ s.t. $P_i \oplus Q_i$ is free.

Let $Q = \bigoplus_i Q_i$. $(\bigoplus_i P_i) \oplus Q = (\bigoplus_i P_i) \oplus (\bigoplus_i Q_i) \cong \bigoplus_i [P_i \oplus Q_i]$ is a direct sum of frees and hence free. □

Injective modules

Recall: ${}_RE$ inj. if given any diag.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "h'"] \\ & E & \end{tikzcd}$$

w/ exact row $\exists h'$ s.t. $h' \circ f = h$.

f inj. $\rightsquigarrow A \subseteq B$

(Think of $A \subseteq B$ and $h: A \to E$ and you want extension to $h': B \to E$.)

Prop: (1) If E is injective and $E = E_1 \oplus E_2$, where $E_1, E_2 \subseteq E$.

Then E_1 is injective.

(2) If M_1 and M_2 are injective, then $M_1 \oplus M_2$ is injective. does not work for infinitely many

Pf: (Exercise)! (1) Mimic proof for $P \mid F$, ~~free module~~ Free is proj.

(2) easy comp.

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Thm: Given ${}_RE$. ${}_RE$ is injective iff given $I \subseteq R$ left ideal and a diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "h"] & R \arrow[dl, dashed, "h'"] \\ & E & \end{tikzcd}$$

, $\exists\ h': R \to E$ s.t. $h'|_I = h$.

(Baer's criterion)

Proof: If ${}_RE$ is inj. we can complete diags by defn of inj.

Now, assume condition is satisfied.

Consider

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\iota"] & B \arrow[dl, dashed, "\iota'"] \\ & E & \end{tikzcd}$$

where the row is exact.

WLOG assume $A \subseteq B$ and f is the inclusion map.

$A \cong f(A) \subseteq B$

Let $\mathcal{P} = \{(C,t) \mid A \subseteq C \subseteq B,\ t: C \to E$ R-hom w/ $t|_A = \iota\}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d, "\iota"] & C \arrow[r, hook] \arrow[dl, dashed, "t"] & B \\ & E & & \end{tikzcd}$$

Say $(C_1, t_1) \leq (C_2, t_2)$ if $C_1 \subseteq C_2$ and $t_2|_{C_1} = t_1$.

If $\mathcal{C} = \{(C_i, t_i) \mid i \in I\}$ is a chain.

Note: $(\bigcup C_i, t) \in \mathcal{P}$, where $t(x) = t_i(x)$, if $x \in C_i$, is an upper bound for $\mathcal{C}$.

ZORN $\Rightarrow \exists$ max'l elt $(B_0, t_0) \in \mathcal{P}$.

Suppose $B_0 \subsetneq B$ Choose $x \in B \setminus B_0$. We will extend t_0 to $B_0 + Rx$.

Let $I = \{r \in R \mid rx \in B_0\} = \operatorname{ann}(x + B_0)$, $x + B_0 \in B/B_0$

I is a left ideal. as an annihilator

Consider

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "h"] & R \arrow[dl, dashed, "h'"] \\ & E & \end{tikzcd}$$

where $h(r) = t_0(\underset{\in B_0}{rx})\ \forall r \in I$

h is an R-hom. We now have $h': R \to E$ s.t. $h'|_I = h$

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Def. $L_1 : B_0 + Rx \to E$

$b_0 + rx \mapsto t_0(b_0) + h'(r)$

Suppose $b_0 + rx = b_0' + r'x \Rightarrow b_0 - b_0' = (r'-r)x$

$\Rightarrow r'-r \in I$ ~~$\Rightarrow h'(r'-r) = h(r'-r) = t_0((r'-r)x)$~~

~~$\Rightarrow h'(r') - h'(r) = t_0(r'x) - t_0(rx)$~~

$t_0(b_0) - t_0(b_0')$ ~~t_0~~ $(b_0 - b_0') = t_0((r'-r)x) = h'((r'-r)) = h'(r') - h'(r)$

$\Rightarrow t_0(b_0) + h'(r) = t_0(b_0') + h'(r')$

$\Rightarrow L_1$ is an ~~R-hom~~ well-def. Clearly an R-hom.

Now, $(B_0, t_0) \lneq (B_0 + Rx, L_1)$ C! $\Rightarrow B_0 = B$. □

Def: (1) $c \in R$ is left regular if $\text{l-ann}_R(c) = 0$ $(ac = 0 \Rightarrow a = 0)$

(2) ${}_R M$ is divisible if given any $x \in M$ and $c \in R$ left reg. $\exists y \in M$ s.t. $cy = x$.

Prop: If ${}_R E$ is inj. then ${}_R E$ is divisible. need R domain? (no non zero divisors)

why? Rc free ~~R-mod~~ R-mod, basis $\{c\}$.

If $x \in E$ $\exists$

$$\begin{tikzcd} 0 \arrow[r] & Rc \arrow[r, hook] \arrow[d, "h"] & R \arrow[dl, dashed, "h'"] \\ & E & \end{tikzcd}$$

where $h(rc) = rx$

extend to h'. $ch'(1) = h'(c) = h(c) = h(1 \cdot c) = 1x = x$

Let $y = h'(1)$.

Thm: If R is a PID and ${}_R M$ is divisible then ${}_R M$ is injective.

(I.O.U. I owe you)

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Ex: ${}_{\mathbb{Z}}\mathbb{Q}$ is injective since $\mathbb{Z}$ PID, $\mathbb{Q}$ is divisible.

~~Thm~~:

Def: ${}_RA \leq {}_RB$ R-modules. A is essential in B if $A \cap C \neq 0$ $\forall\, 0 \neq C \leq B$.

Ex: $\mathbb{Z}$ essential in ${}_{\mathbb{Z}}\mathbb{Q}$.

If $0 \neq C \leq_{\mathbb{Z}} \mathbb{Q}$ choose $0 \neq \frac{p}{q} \in C$. Then $p = q\left(\frac{p}{q}\right) \in C \cap \mathbb{Z}$. $\Rightarrow C \cap \mathbb{Z} \neq 0$.

Ex: A ess. A
0 ess 0.

Notation: A ess B

or B is an essential extension of A.

Thm: If ${}_RM$ then $\exists$ inj. module E w/ $M \leq_R E$. Furthermore, we can choose E w/ ${}_RM \operatorname{ess}_R E$.

Proof as for fields → Galois Theory

Dictionary:

Modules	Fields
ess. ext.	alg. ext.
inj.	alg. closed
$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "h"'] & R \arrow[dl, dashed] \\ & M \arrow[d, hook] & \\ & E & \end{tikzcd}$	polynomial
$h'(1)$	root

Proof of Thm: I.O.U. [I owe you]

Recall: If H_i, $i \in I$ a family of modules. Then

① $\bigoplus_{i \in I} H_i = \{ (h_i)_{i \in I} \mid h_i \in H_i,\ h_i = 0 \text{ a.e.} \}$

$= \{ h : I \to \bigcup_{i \in I} H_i \mid h(i) = h_i \in H_i\ \forall i,\ h = 0 \text{ a.e.} \}$

② $\prod_{i \in I} H_i = \{ (h_i)_{i \in I} \mid h_i \in H_i \}$

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I finite "same".

Prop: Let A, B be R-modules and $\{A_i \mid i \in I\}$, $\{B_j \mid j \in J\}$ be families of R-mods.

Then ① $\mathrm{Hom}_R(\oplus A_i, B) = \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$

② $\mathrm{Hom}_R(A, \prod_{j\in J} B_j) = \prod_{j \in J} \mathrm{Hom}(A, B_j)$.

$$\mathrm{Hom}_R(\underset{\substack{\uparrow \\ \text{screwed} \\ \text{up variable}}}{\cdot}\ ,\ \underset{\substack{\nwarrow \\ \text{nice}}}{\cdot})$$

Proof: ① Let $\alpha_t : A_t \hookrightarrow \bigoplus_i A_i$

$$a_t \mapsto (a_i)_{i\in I}, \quad a_i = \begin{cases} 0, & \text{if } i \neq t \\ a_t & \text{if } i = t. \end{cases}$$

$$p_t : \bigoplus_i A_i \longrightarrow A_t$$

$$(a_i)_{i\in I} \mapsto a_t.$$

$$p_t \circ \alpha_t = I_{A_t} \quad \text{and} \quad \sum_{t\in I} \alpha_t \circ p_t = I_{\bigoplus_{i\in I} A_i}$$

If $f \in \mathrm{Hom}_R(\oplus A_i, B)$. Let $f_i = f \circ \alpha_i : A_i \to B$.

Let $\theta : \mathrm{Hom}_R(\bigoplus_i A_i, B) \to \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$.

$$f \mapsto (f_i)_{i\in I}$$

and $\chi : \prod_i \mathrm{Hom}_R(A_i, B) \to \mathrm{Hom}_R(\bigoplus_i A_i, B)$.

$$\chi((g_i)_{i\in I})((a_i)_{i\in I}) = \underbrace{\sum_{i\in I} g_i(a_i)}_{\text{finite sum}}$$

Check $\theta \circ \chi = I$, $\chi \circ \theta = I$.

② Similar. Exercise □

Ex: $M = M_1 \oplus M_2$ R-modules

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Think of (m_1, m_2) as $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$.

Let $f_{ij} : M_j \to M_i$, $i,j = 1,2$.

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} f_{11}(m_1) + f_{12}(m_2) \\ f_{21}(m_1) + f_{22}(m_2) \end{bmatrix} \in \begin{matrix} M_1 \\ \oplus \\ M_2 \end{matrix}$$

$$\text{End}(M_1 \oplus M_2) = \text{Hom}(M_1 \oplus M_2, M_1 \oplus M_2) \cong \bigoplus_{i,j=1}^{2} \text{Hom}(M_j, M_i)$$

Situation: $M = \bigoplus_{i=1}^{n} M_i$ $\quad (m_1, \dots, m_n) = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \in M$.

$E = [\text{Hom}_R(M_j, M_i)]_{i,j}$ set matrices w/ (i,j)-entry contain elt of $\text{Hom}_R(M_j, M_i)$.

E is a ring.

Thm: In above situation, $\varphi : \text{End}_R(\bigoplus_{i=1}^{n} M_i) \to E$

$f \mapsto [f_{ij}]$

where $f_{ij} = p_i \circ f \circ \alpha_j : M_j \to M_i$, φ is an isom. of rings.

Proof: φ is clearly additive.

$$\varphi(fg) = [p_i fg \alpha_j]_{i,j} = [p_i f I g \alpha_j]_{i,j} \overset{*}{=} [\sum_l p_i f \alpha_l p_l g \alpha_j]_{i,j}$$

$$= \varphi(f)\varphi(g)$$

$\varphi(I) = I \Rightarrow \varphi$ is a ring hom.

Let $\psi : E \to \text{End}_R(\bigoplus_{i=1}^{n} M_i)$

$$[f_{ij}]_{i,j} \to \sum_{i,j} \alpha_i f_{ij} p_j$$

$\underline{\psi \circ \varphi = I_{\text{End}_R(\oplus M_i)}}$:

$$\psi \circ \varphi(f) = \psi([p_i \circ f \circ \alpha_j]_{i,j}) = \sum_{i,j} \alpha_i (p_i f \alpha_j) p_j = \sum_{i,j} (\alpha_i p_i) f (\alpha_j p_j)$$

$$= IfI = f.$$

$\underline{\varphi \circ \psi = I_E}$ similar. $\Rightarrow \varphi$ and ψ are inverse. □

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$$\mathrm{End}_R(\underbrace{S\oplus S\oplus \_ \oplus S}_{\overleftrightarrow{\ n\ }}) \cong M_n(\mathrm{End}_R(S)).$$

S simple $\Rightarrow$ $\mathrm{End}_R(S)$ div. ring

$$\Big(\mathrm{End}_R(\underbrace{S\oplus \_ \oplus S}_{n}) = E = [\underbrace{\mathrm{Hom}_R(S,S)}_{=\mathrm{End}_R(S)}]_{i,j}\Big)$$

<u>Sep 25</u>

<u>Prop:</u> $\mathrm{End}_R({}_RR) \cong R^{op}$.

<u>Proof.</u> If $f \in \underset{=:E}{\mathrm{End}_R({}_RR)}$ then $f(r) = rf(1) = rx$, $x \overset{def}{=} f(1)$.

$$\varphi : E \to R$$
$$f \mapsto f(1)$$

If $z \in R$ then $g: R \to R$, $r \mapsto rz$ is in E. $\Rightarrow$ φ is onto

$[g(sr) = srz = s(rz) = sg(r).]$

Clearly, φ is ~~onto~~ 1-1.

If $\varphi(f) = x$, $\varphi(g) = y$. $\quad \varphi(f\circ g) \overset{=(fg)(1)}{=} f(g(1)) = f(y) = yf(1) = y\cdot x = \varphi(g)\varphi(f)$

$= \varphi(f) * \varphi(g)$ where $*$ is opp. mult.

$\varphi(f)+\varphi(g) \overset{\frown}{=} \varphi(f+g)$ clear. $\square$

<u>Ex:</u> $F = R^{(n)}$ free.

Friday: $\mathrm{End}(F) = M_n(\mathrm{End}_R(R)) \cong M_n(R^{op}) \cong M_n(R)^{op}$ (see HW 4). $\square$

<u>Prop:</u> (Schur's Lemma) Let ${}_RS$ be a simple R-mod. Then $\mathrm{End}_R(S) = D$ is a division ring.

<u>Proof:</u> need to show $D\setminus 0 = U(D)$. $\Leftarrow$ If $f \in D\setminus 0$ then f is an isom.

$f \neq 0 \Rightarrow \mathrm{Ker}\, f \neq S \Rightarrow \mathrm{Ker}\, f = 0$ (S simple) $\Rightarrow f$ is 1-1.

$f \neq 0 \Rightarrow f(S) \neq 0 \Rightarrow f(S) = S$ (S simple) $\Rightarrow$ f is onto.

$\Rightarrow f$ is an isom. $\square$

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<u>Ex:</u> ${}_R S$ simple $\quad S^{(n)} = \underbrace{S \oplus \dots \oplus S}_{n}$

$\mathrm{End}_R(S^{(n)}) \cong M_n(D)$, $D = \mathrm{End}_R(S) = \mathrm{Hom}_R(S, S)$

Theorem (Artin-Wedderburn)

Ring R. T.F.A.E

(1) R is left Art. and $J(R) = 0$ (semiprimitive)

(2) ${}_R R$ is semisimple

(3) Every R-module is semisimple

(4) -"- -"- -"- completely reducible

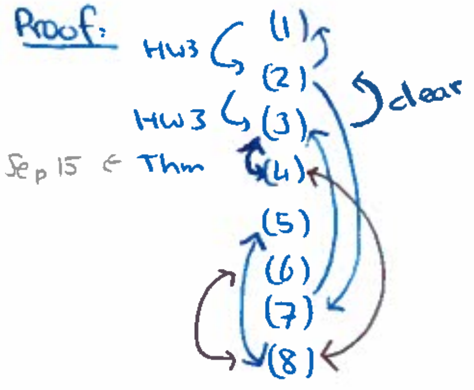
(5) Every R-module is projective

(6) -"- -"- -"- injective

(7) $R \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_t}(D_t)$ where $D_1, \dots, D_t$ division rings.

(8) Every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules splits.

<u>Proof:</u>



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<u>(2) ⇒ (1):</u> (2) says $R = \bigoplus_{i \in I} L_i$ where each L_i is a minimal left ideal.

$1 \in R$, $1 = \sum_{i \in I} l_i$, $l_i = 0$ a.e. Let $I_0 = \{ i \in I \mid l_i \neq 0 \} \subseteq I$ finite.

$\Rightarrow R = R \cdot 1 = R \sum_{i \in I_0} l_i \subseteq \sum_{i \in I_0} \overset{O_x}{\underset{\subseteq L_i}{R l_i}} = \sum_{i \in I_0} L_i$ $\Rightarrow I = I_0$ is finite.

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Each ${}_R L_i$ is Art. (bc simple) $\Rightarrow R = L_1 \oplus \_ \oplus L_n$, $I_0 = \{1, -, n\}$ is left Art.

Let $M_i = L_1 \oplus \_ \oplus L_{i-1} \oplus L_{i+1} \oplus \_ \oplus L_n \subseteq R$ M_i left ideal.

$R/_{M_i} \cong L_i$ simple $\Rightarrow$ ${}_R M_i \subseteq R$ max left ideal.

$\Rightarrow J(R) \subseteq \bigcap_i M_i = 0.$ $\Rightarrow J(R) = 0.$

<u>(2)⇒(7):</u> as above ${}_R R$ is a direct sum of fin. many simple left R-modules.

$${}_R R \cong S_1^{(n_1)} \oplus \_ \oplus S_t^{(n_t)}$$

where $S_1, \_, S_t$ are simple R-modules $S_i \not\cong S_j$, if $i \neq j$.

$\mathrm{Hom}_R(S_j, S_i) = 0$ if $i \neq j$ (Ker, Im submods S_i, S_j simple, not isom.)

Now, $R^{op} \cong \mathrm{End}_R(R) \cong \mathrm{End}_R(S_1^{(n_1)} \oplus \_ \oplus S_t^{(n_t)})$

$$\underset{(\text{Friday})}{\cong} \begin{bmatrix} M_{n_1}(F_1) & 0 & \cdots & 0 \\ 0 & M_{n_2}(F_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_{n_t}(F_t) \end{bmatrix} \cong M_{n_1}(F_1) \oplus \_ \oplus M_{n_t}(F_t)$$

where $F_1, \_, F_t$ div. rings

where $F_i = \mathrm{End}_R(S_i)$ div. ring

Now, $R \cong (R^{op})^{op} \underset{\substack{\uparrow \\ \text{HW4}}}{\cong} M_{n_1}(D_1) \oplus \_ \oplus M_{n_t}(D_t)$ where $D_i \cong \mathrm{End}_R(S_i)^{op}$.

<u>(7)⇒(3):</u> Note $0 \oplus \_ \oplus 0 \oplus L_{ij} \oplus 0 \oplus \_ \oplus 0$ where $L_{i,j} \in M_{n_i}(D_i)$ is set

$\begin{bmatrix} 0 & \begin{matrix} D_i \\ \vdots \\ D_i \end{matrix} & 0 \end{bmatrix}$ ^j-th row

L_{ij} is a min'l left ideal and $R = \bigoplus_{\substack{i=1 \\ 1 \leq j \leq n_i}}^{t} L_{ij}$ is semisimple.

<u>(5)⇒(8):</u> $0 \to A \to B \to \overset{\overset{C}{\parallel}}{C} \to 0$. By (5) ${}_R C$ proj. $\Rightarrow \exists\ s : C \to B :\ g \circ s = I_C$.

$\Rightarrow$ SES splits.

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(8) ⇒ (5): Given ${}_RM$ $\exists\, \pi : F \to M$ onto w/ F free.

$0 \longrightarrow K \hookrightarrow F \xrightarrow{\pi} M \longrightarrow 0$ w/ $K = \operatorname{Ker} \pi$ is a SES.

splits by (8) $\Rightarrow F \cong \underset{i(K)}{K} \oplus M$ ← M bc SES splits $\Rightarrow M$ direct summand of a free $\Rightarrow M$ is proj.

We have (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇔ (7) and (5) ⇔ (8).

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(8) ⇒ (4): Let ${}_RA \subseteq {}_RB$.

Take SES $0 \to A \overset{i}{\hookrightarrow} B \xrightarrow{\pi} C \to 0$ $\quad C = B/A$.

<u>Splits</u> $\Rightarrow$ $B = i(A) \oplus C' = A \oplus C'$ some $C' \cong C$ $\Rightarrow {}_RA \mid_R B$.

(8) ⇒ (6): For ${}_RA$ choose inj. module E w/ $A \subseteq E$.

$0 \to A \overset{i}{\hookrightarrow} E \xrightarrow{\pi} C \to 0$, $C = E/A$

<u>splits</u> $\Rightarrow$ $E = A \oplus C'$, some C'. A is a direct summand of an inj. $\Rightarrow A$ is inj.

great exam problem

(6) ⇒ (8):

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[dl, dashed, "r"] & C \arrow[r] & 0 \\
 & A & & &
\end{tikzcd}$$

SES.

A inj. $\Rightarrow \exists\, r : B \to A$ s.t. $r \circ f = I_A$ $\Rightarrow$ SES splits.

(4) ⇒ (8): $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ SES

$f(A) \mid B$ by (4).

$\Rightarrow$ $B = f(A) \oplus C'$, some C'.

$f(A) = \operatorname{Ker} g$

$\Rightarrow B/\operatorname{Ker} g = B/f(A) \cong g(B)$ ~~B~~

$\cong C'$

It follows $g|_{C'} : C' \to C$ is an isom.

Now, $f : A \underset{\bar{f}}{\rightleftarrows} f(A)$ is an isom. Inverse map $\bar{f}$.

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Let $s = \bar{s}\circ\pi$, where $\pi : B = f(A)\oplus C' \to f(A)$ is proj.

$s\circ f = I_A$. SES splits. $\square$

Category

Def: A category $\mathcal{C}$ consists of

(1) a class of objects $Ob(\mathcal{C})$

(2) If $A, B \in Ob(\mathcal{C})$ a set $Hom_{\mathcal{C}}(A,B)$ or $\mathcal{C}(A,B)$

(3) $Hom_{\mathcal{C}}(A,B)\times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$

$$(f,g) \mapsto g\circ f.$$

satisfying

(1) $Hom(A',B')\cap Hom(A,B) = \emptyset$ unless $A=A'$, $B=B'$.

(2) Composition is associative

$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ $\quad h\circ(g\circ f) = (h\circ g)\circ f \quad \forall f,g,h, A,B,C,D.$

(3) For each $A\in Ob(\mathcal{C})$ $\exists$ $I_A \in Hom_{\mathcal{C}}(A,A)$ s.t.

$I_A\circ f = f$ and $g\circ I_A = g$ whenever composition is defined.

Ex: ① Set category of sets $Hom_{Set}(A,B) = \{\text{function } f: A\to B\}$

② Top category of topological spaces

$Hom(X,Y) = \{f: X\to Y \text{ continuous}\}$

③ Ab Cat. of abelian groups $Hom_{Ab}(A,B) = \{f: A\to B \mid f \text{ group hom.}\}$

④ Grp cat. of groups, group homs

⑤ R-mod $= {}_R\mathfrak{M}$ cat. of left R-modules, R-mod homs (R fixed)

⑥ k-Vect v.s./k k-linear transformations

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⑦ Ring cat. of rings.

Write $A \xrightarrow{f} B$ or $f: A \to B$ to say $A, B \in Ob(\mathcal{C})$ and $f \in Hom_{\mathcal{C}}(A,B)$.

Ex: Let G be a directed graph.

$Ob(\mathcal{C})$ = set of vertices

$Hom_{\mathcal{C}}(p,q)$ = set of paths from p to q.

Comp. is concatination

Ex:

$e_1 = I_1$ is "empty" path from 1 to 1.

$Hom_{\mathcal{C}}(1,3) = \{ba, ca\}$, $Hom_{\mathcal{C}}(4,4) = \{I_4, e, e^2, e^3, e^4, \_\}$

$Hom_{\mathcal{C}}(3,1) = \emptyset$

Def: Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{D}$ assigns $F(A) \in Ob(\mathcal{D})$ to each $A \in Ob(\mathcal{C})$ and $F(f): F(A) \to F(B)$ to each $f: A \to B$ in $\mathcal{C}$ s.t.

$$F(g \circ f) = F(g) \circ F(f) \quad \forall \quad A \xrightarrow{f} B \xrightarrow{g} C \text{ in } \mathcal{C},$$

and $F(I_A) = I_{F(A)}$.

Ex: $F: \mathcal{T}op \to \mathcal{S}et$

$X \mapsto X$

$f \mapsto f.$

(forgetful functor)

Ex: $F: R\text{-mod} \to Ab$ A fixed

$B \mapsto Hom_R(A,B)$

$F(g) = g_*$ $F = Hom_R(A, \cdot)$.

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Recall: $\mathcal{C}, \mathcal{D}$ cat.

$F: \mathcal{C} \to \mathcal{D}$ assigns $F(A) \in \mathcal{D}$ to each $A \in \mathcal{C}$.

$F(f) \in$ to each morphism f in $\mathcal{C}$.

↖ morphism in $\mathcal{D}$

respects comp.,

$F(I_A) = I_{F(A)}$ $\forall$ $A \in \mathcal{C}$.

Ex: $\mathcal{C} = R$-mod, $\mathcal{D} = Ab$

Fix $A \in R$-mod.

$F(B) = \mathrm{Hom}_R(A, B)$

$F(f) = f_*$ where for $f: B_1 \to B_2$ in $\mathcal{C}$

$$f_*: \mathrm{Hom}(A, B_1) \to \mathrm{Hom}(A, B_2)$$

$$g \mapsto f \circ g$$

Clear $F(I_B) = I_{\mathrm{Hom}_R(A,B)}$ $\forall B$.

$F = \mathrm{Hom}_R(A, -)$.

Note: Suppose we try $\mathcal{C} = R$-mod, $\mathcal{D} = Ab$

Fix B, define $G: \mathcal{C} \to \mathcal{D}$

$$A \longmapsto \mathrm{Hom}_R(A, B)$$

If $f: A_1 \to A_2$ in $\mathcal{C}$ $G(f) = ??!?$

$$\mathrm{Hom}(A_1, B) \leftarrow \mathrm{Hom}(A_2, B)$$

$$f^* = G(f) = g \circ f \longleftarrow g$$

$$g \longmapsto ?$$

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & A_2 \arrow[r, "g"] & B \end{tikzcd}$$

G is not exactly a covariant functor.

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Def. $G: \mathcal{C} \to \mathcal{D}$ is a contravariant functor if

(1) G assigns $G(A) \in \mathcal{D}$ to each $A \in \mathcal{C}$

(2) G assigns $G(f):$ ~~to each~~ $G(Y) \to G(X)$ for all $f: X \to Y$ in $\mathcal{C}$

s.t. $G(f \circ h) = G(h) \circ G(f)$ where $\overset{f\circ h}{\cancel{h \circ f}}$ is def.

$$\begin{tikzcd}
X \arrow[r, "h"] & Y \arrow[r, "f"] & Z \\
G(X) & G(Y) \arrow[l, "G(h)"'] & G(Z) \arrow[l, "G(f)"']
\end{tikzcd}$$

$\Downarrow G$

(3) $G(I_X) = I_{G(X)} \quad \forall X \in \mathcal{C}$.

Ex: $\mathrm{Hom}_R(-, B): R\text{-mod} \to Ab$ is a contravariant functor.

Note: Cat $\mathcal{C}$, $\mathcal{C}^{op}$ has same objects, $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$

$$\underset{\text{in } \mathcal{C}^{op}}{\underset{\uparrow}{f \circ g}} = \underset{\text{in } \mathcal{C}}{\underset{\uparrow}{g \circ f}}$$

$\mathcal{C}^{op}$ is a cat. $I_x = I_x$

係 $G: \mathcal{C} \to \mathcal{D}$ is a contravariant

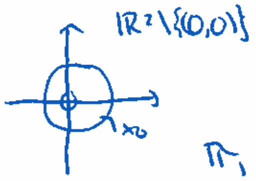
$\Leftrightarrow$ $G: \mathcal{C} \to \mathcal{D}^{op}$ is a covariant functor.

Ex: $\mathcal{C}$ cat. of pointed topological spaces

Object (X, x_0), X top. space, $x_0 \in X$

$\pi_1((X, x_0)) =$ Homotopy classes $\{f: I \to X \mid f(0) = f(1) = x_0\}$

$I = [0, 1]$ unit interval



$\pi_1(X, x_0) = \mathbb{Z}$.

$f: (X, x_0) \to (Y, y_0)$

Sep 29

Def: $\mathcal{C}$ cat. ① $i \in \mathcal{C}$ is an initial object if $\mathrm{Hom}_{\mathcal{C}}(i, A)$ is a singleton $\forall A \in \mathcal{C}$.

only one element

② $t \in \mathcal{C}$ is a terminal object if $\mathrm{Hom}_{\mathcal{C}}(A, t)$ is a singleton $\forall A \in \mathcal{C}$.

③ $0 \in \mathcal{C}$ is a zero object if it's both a terminal and an initial object.

Ex: ① Graph

1 is initial
3 is terminal

no zero object

②

No initial or terminal objects

③ Group 1 is a zero object.

④ Ab 0 is a zero object

Note: If i is initial $\mathrm{Hom}_{\mathcal{C}}(i, i) = I_i$

t is terminal $\mathrm{Hom}_{\mathcal{C}}(t, t) = I_t$.

Def: $f: A \to B$ a morphism in $\mathcal{C}$ is an isomorphism if $\exists g: B \to A$ s.t.

$f \circ g = I_B$, $g \circ f = I_A$.

two zero obj. are isomorphic

zero obj

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

$f \circ g \overset{!}{=} \mathrm{Id}$ since $\mathrm{Hom}(A, A) = \{\mathrm{Id}\}$

Proposition: Let R, S, T be rings.

(1) Given ${}_R A_S, {}_R B$, then $\mathrm{Hom}_R({}_R A_S, {}_R B)$ is a left S-module

bimodule

via $(sf)(a) = f(as)$ $\forall a \in A, s \in S$, $f \in \mathrm{Hom}_R(A, B)$

(2) ${}_R A$, ${}_R B_T$ $\mathrm{Hom}_R({}_R A, {}_R B_T)$ is a right T-module.

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(3) $_RA_S,\ _RB_T$ $\quad \mathrm{Hom}_R(A,B)$ is a S-T-bimodule.

Pf: ① $s_1(s_2 f) \overset{?}{=} (s_1 s_2)f$

$(s_1(s_2f))(a) = s_2 f(as_1) = f((as_1)s_2) = f(a(s_1s_2)) = (s_1s_2)f(a) \ \forall a\in A.$

$\Rightarrow s_1(s_2 f) = (s_1 s_2)f.$

$(sf)(a_1+a_2) = sf(a_1) + sf(a_2).$ (Check!)

$(sf)(ra) = f((ra)s) = f(r(as))$ (A bimod)

$\overset{f\ R\text{-hom}}{=} rf(as) = r(sf)(a) \quad \Rightarrow sf \in \mathrm{Hom}_R(A,B)$

② $f\in \mathrm{Hom}_R(A,B)$. let $(ft)(a) = f(a)t$ Check!

③ $\mathrm{Hom}_R(_RA_S,\ _RB_T)$ is a left S-, right T-module

Check $(sf)(t) = s(ft)$.

$[(sf)(t)](a) = (sf)(a)\cdot t = f(as)\cdot t$

$s(ft)(a) = (ft)(as) = f(as)\cdot t$ $\quad /\!/ \quad \forall a\in A.$ □

Ex: $\mathrm{Hom}_R(_RA_S, -) : R\text{-mod} \to S\text{-mod}$ is a functor.

What about $\mathrm{Hom}_R(-,\ _RB_T)$?

Def: $\mathcal{C}$ cat. $\mathcal{F} = \{X_i \mid i\in I\}$ family in $\mathcal{C}$, I set.

A product of $\mathcal{F}$ is an object P together w/ morphs $\{p_i : P\to X_i \mid i\in I\}$ s.t. given any family of morphisms $\{f_i : A\to X_i \mid i\in I\}$ $\exists(!)$ morphism $\theta: A\to P$ s.t. $p_i\circ\theta = f_i \ \forall i\in I.$

Ex: $\mathcal{C} = R\text{-mod}$ $\{M_i \mid i\in I\}$. $P = \prod_{i\in I} M_i$, $P_i : P\to M_i$ proj. onto i-th component

Given $\{f_i : A_i \to M_i \mid i\in I\}$ $\quad \theta(a) = (f_i(a))_{i\in I}$

$\left[I = \{1,2,3\} \mid \theta(a) = (f_1(a), f_2(a), f_3(a)) \right]$ Sep. 29

Check that's a product

Check: what happens if you reverse arrows?

<u>Recall</u> $_RR$ semisimple is equiv. to 8 conditions. Oct 2

One was $R \cong M_{n_1}(D_1) \times \_ \times M_{n_t}(D_t)$ symmetric condition!

It follows $_RR$ s.s. $\Leftrightarrow$ R_R s.s.

All conditions can be replaced by "right".

<u>Ex:</u> $_RR$ s.s. iff every S.E.S. $0 \to A \to B \to C \to 0$ of right R-modules splits.

<u>Prop:</u> Assume $I \trianglelefteq R$, $_RM$ and $IM = 0$. Then M is a left R/I-module via $\bar{r}m = rm$ $\forall$ $\bar{r} = r + I \in R/I$, $m \in M$.

<u>Proof:</u> (Recall $IM = \{ \sum_{i=1}^{n} x_i m_i \mid n \geq 1, x_i \in I, m_i \in M \}$ is always an R-module.)

Let $r, r' \in R$ w/ $\bar{r} = \bar{r'}$ and $m \in M$. $\Rightarrow r - r' \in I$.

Now, $\bar{r}m - \bar{r'}m = rm - r'm = (r - r')m = 0$, $r - r' \in I$ $\Rightarrow \bar{r}m = \bar{r'}m$.

$\Rightarrow$ $R/I \times M \to M$,
$(\bar{r}, m) \mapsto \bar{r}m$ is well-defined.

Module properties follow immediately. □

<u>Note:</u> ① If $\varphi : R \to S$ ring hom., $I \trianglelefteq R$, $I \subseteq \operatorname{Ker} \varphi \trianglelefteq R$ then φ induces

$\bar{\varphi} : R/I \to S$
$r + I \mapsto \varphi(R)$, a ring hom.

② In Prop. $_RM \Rightarrow \exists\, \varphi : R \to \operatorname{End}_{\mathbb{Z}}(M)$

$IM = 0 \Rightarrow I \subseteq \operatorname{Ker} \varphi = \operatorname{ann}_R(M)$. We get $\bar{\varphi} : R/I \to \operatorname{End}_{\mathbb{Z}}(M)$.

<u>Theorem:</u> If $_RR$ s.s. then R has fin. many simple left R-modules up to isomorphism. (see HW 5 solution).

Oct 2

Thm: If R is left Artinian then R is left Noetherian. [Hopkin's thm]

Proof: Let $N = J(R)$. We know that $N^n = 0$, some n. We have

$0 = N^n \subseteq N^{n-1} \subseteq \dots \subseteq N \subseteq R$. The quotients are

$N^i / N^{i+1} =: M_i$. Notice $N(M_i) = 0 \Rightarrow M_i$ is an R/N-mod.

R/N left Art., semiprimitive. $\left(J(R/N) = \frac{J(R)}{N} = \frac{N}{N} = 0\right)$.
(quot.)

$\Rightarrow {}_R M_i = {}_{R/N} M_i$ is s.s.

(Art since quot of Art N^i)

R Art. $\Rightarrow {}_R N^i$ art. $\Rightarrow {}_R\left(N^i/N^{i+1}\right)$ is s.s.
(submod)

← English

$\Rightarrow M_i = N^i/N^{i+1} = \bigoplus_{i\in I} S_i$, S_i simple R-mod.

$\bigoplus_{i\in I} S_i$ Noeth (Art) $\Leftrightarrow$ $\forall i$ S_i Noeth (Art)

${}_R M_i$ is Art. $\Rightarrow I$ fin. $\Rightarrow M_i$ is Noetherian.
S_i Noeth

$M_{m-1} = \frac{N^{n-1}}{N^n} = N^{n-1}$ is Noeth.

$$0 \to N^{n-1} \hookrightarrow N^{n-2} \to N^{n-2}/N^{n-1} \to 0$$

notice N^{n-1} Noeth., $\frac{N^{n-2}}{N^{n-1}} = M_{n-1}$ is Noeth $\Rightarrow N^{n-2}$ is Noeth.

Similarly, $0 \to N^{n-2} \to N^{n-3} \to N^{n-3}/N^{n-2} \to 0$

$$N^{n-3}/N^{n-2} = M_{n-3}$$

Noeth

$\Rightarrow N^{n-3}$ is Noeth. etc.

Eventually $N^0 = R$ is Noeth. ▯

Exercise: Assume ${}_R R$ is left Artinian and R_R is right Noetherian.

$\Rightarrow R$ is right Art.

[Hint: adapt proof: $\left(\frac{N^i}{N^{i+1}}\right)_R$ is s.s. and Noeth. $\Rightarrow \frac{N^i}{N^{i+1}}$ is Art.]

Example: $\mathcal{C}_k = k\text{-Vect}_{fin.}$ Cat. of fin.-dim. v.s./k, a field

Define $F: {}_k\mathcal{C} \to {}_k\mathcal{C}$

$V \mapsto V^{**}$ double dual.

Oct 2

If $T: V \to W$ morphism. $F(T) = T^{**}: V^{**} \to W^{**}$

Recall $V^* = \mathrm{Hom}_k(V, k)$

If $T: V \to W$ lin. transformation

$T^*: W^* \to V^*$

$T^*(f) = f \circ T \in V^*$

$V \xrightarrow{T} W \xrightarrow{f \in W^*} k$

We have an isom. $\tau: V \to V^{**}$
$v \mapsto \hat{v}$, $\hat{v}(\ell) = \ell(v)$.

We have two functors $I, F: \mathcal{C} \to \mathcal{C}$

We have an isom. $\tau_V: I(V) = V \longrightarrow F(V) = V^{**}$
$v \longmapsto \hat{v}$.

Note Given $T: V \to W$ in $\mathcal{C}$

$$\begin{tikzcd} \tau_V: I(V) \arrow[r, "\tau_V"] \arrow[d, "I(T)"] \arrow[dr, phantom, "\circlearrowleft"] & F(V) \arrow[d, "F(T)"] \\ I(W) \arrow[r, "\tau_W"'] & F(W) \end{tikzcd}$$

diag. commutes

$\{\tau_V \mid V \in \mathcal{C}\}$ nat. transf. from I to F.
$\tau_V: V \to V^{**}$

~~Note: There is no natural~~

Def. $\mathcal{C}, \mathcal{D}$ 2 cat. $F, G: \mathcal{C} \to \mathcal{D}$ 2 (covariant) functors.

A natural transformation $\tau: F \to G$ consists of morphisms in $\mathcal{D}$ $\{\tau_V: F(V) \to G(V) \mid V \in \mathcal{C}\}$ s.t. for any $f: V \to W$ in $\mathcal{C}$ the diag.

$$\begin{tikzcd} F(V) \arrow[r, "\tau_V"] \arrow[d, "F(f)"] & G(V) \arrow[d, "G(f)"] \\ F(W) \arrow[r, "\tau_w"'] & G(W) \end{tikzcd}$$

commutes.

Ex: $F = \mathrm{Hom}_R(A_1, -): R\text{-mod} \to Ab$

$G = \mathrm{Hom}_R(A_2, -): R\text{-mod} \to Ab$

Oct 2

Fix $h: A_1 \to A_2$ R-hom.

For each $B \in R\text{-mod}$, we get

$$\tau_B : \overset{= G(B)}{\mathrm{Hom}(A_2, B)} \to \overset{= F(B)}{\mathrm{Hom}_R(A_1, B)}$$

$$f \mapsto f \circ h$$

$$\begin{tikzcd}
\mathrm{Hom}(A_2,B) = G(B) \arrow[r, "\tau_B"] \arrow[d, "G(f)"] \arrow[dr, phantom, "\circlearrowright"] & F(B) \arrow[d, "F(f)"] \\
\mathrm{Hom}(A_2,D) = G(D) \arrow[r, "\tau_D"'] & F(D)
\end{tikzcd}$$

diag. commutes (check!)

for any $f: B \to D$ in R-mod.

$\tau = \{\tau_B \mid B \in R\text{-mod}\}$ is a nat. transf. from G to F.

Oct 4

Def. $\underset{\text{covariant}}{F: \mathcal{C} \to \mathcal{D}}$ is an isomorphism of categories

if $\exists\, \underset{\text{Cov.}}{G: \mathcal{D} \to \mathcal{C}}$ s.t. $G \circ F = Id_{\mathcal{C}}$ identity functor on $\mathcal{C}$.
and $F \circ G = I_{\mathcal{D}}$

or both contravariant

Recall $\tau: F \to G$ is a natural transformation from F to G if

$F, G: \mathcal{C} \to \mathcal{D}$ ~~[illegible]~~ functors where $\tau = \{\tau_A : F(A) \to G(A)\}$ morphisms in $\mathcal{D}$

s.t. given any $f: A \to B$ in $\mathcal{C}$

$$\begin{tikzcd}
F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"', shift right] \arrow[dr, phantom, "\circlearrowleft"] & G(A) \arrow[d, "G(f)", shift left] \\
F(B) \arrow[r, "\tau_B"'] \arrow[u, dotted, shift right] & G(B) \arrow[u, dotted, shift left]
\end{tikzcd}$$

Commutes.

(covariant version!)

(---> Contravar. version)

τ is an equivalence if τ_A is an isom. $\forall A \in \mathcal{C}$.

We write $F \simeq G$.

Oct 4

Def: If $F : \mathcal{C} \to \mathcal{D}$
$G : \mathcal{D} \to \mathcal{C}$

s.t. $G \circ F \simeq I_{\mathcal{C}}$ and $F \circ G \simeq I_{\mathcal{D}}$, we call F, G equivalences

and $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories.

Example: $\mathcal{C} = k\text{-Vect.}_{\text{fin.}}$, $\mathcal{D} = \{0\} \cup \{k^{(n)} \mid n \geq 1\}$.

$F : \mathcal{C} \to \mathcal{D}$
$V \longmapsto k^{(n)}$ $\quad n = \dim_k(V)$
$0 \longmapsto 0$

For each $V \in \mathcal{C}$ choose an isom. $\tau_V : V \to k^{(n)}$ w/ $\tau_{k^{(n)}} = I_{k^{(n)}} : k^{(n)} \to k^{(n)}$

If $f : V \to W$ lin. transf.

$$\begin{tikzcd} V \arrow[r, "\tau_V"] \arrow[d, "f"] & k^{(n)} \arrow[d, dashed] \\ W \arrow[r, "\tau_W"'] & k^{(n)} \end{tikzcd}$$

(*) $\quad F(f) \overset{\text{def}}{=} \tau_W f \tau_V^{-1}$

Check taking $F(f) = \tau_W f \tau_V^{-1}$ makes F a functor.

$G : \mathcal{D} \to \mathcal{C}$

$k^{(n)} \mapsto k^{(n)}$

$G(g) = g$ $\quad$ (inclusion functor)

$F \circ G = I_{\mathcal{D}}$

(*) becomes

$$\begin{tikzcd} I_{\mathcal{C}}(V) \arrow[r, "\tau_V"] \arrow[d, "f = I(f)"] & k^{(n)} = G \circ F(V) \arrow[d, "(G \circ F)(f) = F(f)"] \\ I_{\mathcal{C}}(W) \arrow[r, "\tau_W"] & k^{n} = G \circ F(W) \end{tikzcd}$$

$\Rightarrow G \circ F \simeq I_{\mathcal{C}}$

Note: F is <u>not</u> an isom. It sends every v.s. of dim 7 to $k^{(7)}$.

there are no natural isom. of cat. ;-)

Oct 4

Prop: ① Let $0 \xrightarrow{\cancel{f}} A \xrightarrow{f} B \xrightarrow{g} C$ be exact in R-mod, and $X \in R$-mod.

Then $0 \to \operatorname{Hom}_R(X, A) \xrightarrow{f_*} \operatorname{Hom}_R(X, B) \xrightarrow{g_*} \operatorname{Hom}_R(X, C)$ is exact in Ab.

(g has not to be onto)

② Let ~~$0 \to$~~ $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact in R-mod and $Z \in R$-mod. Then

$0 \to \operatorname{Hom}_R(C, Z) \xrightarrow{g^*} \operatorname{Hom}_R(B, Z) \xrightarrow{f^*} \operatorname{Hom}_R(A, Z)$ is exact in Ab.

Proof: ① Let $h \in \operatorname{Ker} f_*$ $\Rightarrow$ $0 = f_*(h) = f \circ h$

$\Rightarrow$ $h = 0$ since f is 1-1. $\to$ $\operatorname{Ker} f_* = 0$.

Claim: $\operatorname{Im} f_* = \operatorname{Ker} g_*$

$g_* f_* = (gf)_* = 0_* = 0$ $\to$ $\operatorname{Im} f_* \subseteq \operatorname{Ker} g_*$.

Let $l \in \operatorname{Ker} g_*$.

$$\begin{tikzcd}
 & & X \arrow[dll, "t"'] \arrow[dl, dashed, "l'"'] \arrow[d, "l"] \\
A \arrow[r, "\bar{f}", "\cong"'] & f(A) \arrow[r, hook, "i"'] & B
\end{tikzcd}$$

(since f inj)
$f: A \to f(A)$ onto

$0 = g_*(l) = g \circ l$ $\Rightarrow$ $\operatorname{Im} l \subseteq \operatorname{Ker} g = \operatorname{Im} f$

$\exists\, l': X \to f(A)$ s.t. $i \circ l' = l$.

$\bar{f}$ is an isom. so $t = \bar{f}^{-1} l'$ exists to make diag. commute.

Now, $l = \underbrace{i \circ \bar{f}}_{= f} \circ t = f \circ t = f_*(t)$ $\to$ $l \in \operatorname{Im} f_*$. $\Rightarrow$ $\operatorname{Ker} g_* \subseteq \operatorname{Im} f_*$.

② Let $h \in \operatorname{Ker}(g^*)$ $\Rightarrow$ $0 = g^*(h) = h \circ g$ $\Rightarrow$ $h = 0$ since g onto.

$f^* g^* = (gf)^* = 0^* = 0$ $\Rightarrow$ $\operatorname{Im} g^* \subset \operatorname{Ker} f^*$.

$\operatorname{Im} f = \operatorname{Ker} g$

Let $l \in \operatorname{Ker} f^*$. $\to$ $0 = f^*(l) = l \circ f$.

$$\begin{tikzcd}
B \arrow[r, "l"] \arrow[d, "\pi"] & Z \\
B/\operatorname{Im} f \arrow[r, "\sim", "\bar{g}"'] \arrow[ur, dashed, "l'"] & g(B) = C \arrow[u, "s"']
\end{tikzcd}$$

$\operatorname{Im} f = \operatorname{Ker} g$, g onto

$\Rightarrow L|_{\operatorname{Im} f = \operatorname{Ker} g} = 0$

$\Rightarrow L'$ exists to make diag. commute.

$\bar{g}$ is an isom.

$\Rightarrow \exists s: g(B) \to Z$ making diag. commute.

$L' = s \circ \bar{g}$

$L = L' \circ \pi = s \circ \underbrace{\bar{g} \circ \pi}_{= g} = s \circ g = g^*(s) \quad \Rightarrow L \in \operatorname{Im}(g^*)$

$\Rightarrow \operatorname{Ker} f^* \subset \operatorname{Im} g^*$.

□

Example $R = \mathbb{Z}$ $\qquad 0 \to \mathbb{Z} \xrightarrow{\cdot 2} 2\mathbb{Z} \to \mathbb{Z}_2 \to 0$

apply $\operatorname{Hom}(\mathbb{Z}_2, \cdot)$ to get

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, 2\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ & 0 & 0 & \mathbb{Z}_2 & \end{tikzcd}$$

$\uparrow$
not exact

Exercise: apply $\operatorname{Hom}(-, \mathbb{Z}_2)$ and see new sequ. not exact.

Hint: $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2$
$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$

$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$ can't be exact.

$\operatorname{Hom}_R(-, Y)$
$\operatorname{Hom}_R(X, -)$ are left exact.

Theorem: $\operatorname{Hom}_R(P, -)$ preserves SES iff ${}_R P$ is proj.

Theorem: $\operatorname{Hom}_R(-, E)$ ″ ″ iff ${}_R E$ is inj.

Oct 6

Theorem: ${}_R P$ is projective iff $0 \to \text{Hom}_R(P,A) \xrightarrow{f_*} \text{Hom}_R(P,B) \xrightarrow{g_*} \text{Hom}_R(P,C) \to 0$ is exact for every SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

Proof: $(\Leftrightarrow)$ We only need to consider whether g_* is onto.

Let $h \in \text{Hom}_R(P, C)$.

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dashed, "h'"'] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Does $\exists\, h' \in \text{Hom}_c(P,B)$ s.t. $g_*(h') = g \circ h' = h$?

Always yess iff P proj. by def. $\square$

Theorem: ${}_R E$ inj. iff $0 \to \text{Hom}(C,E) \xrightarrow{g^*} \text{Hom}_R(B,E) \xrightarrow{f^*} \text{Hom}(A,E) \to 0$ is exact for every SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

Proof: Exercise!.

Tensor products

Def: Given A_R, ${}_R B$ and $(D,+)$ an abelian group a function $\beta: A \times B \to D$ is R-balanced

if (1) β is bi-additive: $\beta(a_1 + a_2, b) = \beta(a_1, b) + \beta(a_2, b)$
$\beta(a, b_1 + b_2) = \beta(a, b_1) + \beta(a, b_2)$

(2) $\beta(ar, b) = \beta(a, rb)$. $\forall a \in A, b \in B, r \in R$.

"Sort of" assoc. from $A \times R \times B \to D$.

Ex: $k = R$ field, V, W v.s. $D = (k, +)$.

Bilinear map $\beta: V \times W \to k$ is k-balanced.

Ex: $R = M_m(k)$, $A = M_{n \times m}(k)$, $B = M_{m \times l}(k)$.

Oct 6

$A = A_R, \quad B = {}_R B$

$A \times B \longrightarrow M_{n \times n}$

$(a,b) \mapsto ab$ is R-balanced Matrix mult additive, associative ↳ $\beta(ar,b) = \beta(a,rb)$

<u>Def:</u> Given A_R, ${}_R B$ a tensor product of A and B is a pair (T, τ) where T is an abelian group, $\tau : A \times B \to T$ is R-balanced and given any R-balanced function $\beta : A \times B \to D$ $\exists\,!$ group hom. $\theta : T \to D$ s.t. the diagram

$$\begin{tikzcd}
A \times B \arrow[r, "\tau"] \arrow[dr, "\beta"'] \arrow[dr, phantom, "\circlearrowleft", pos=0.3, shift left=3] & T \arrow[d, dashed, "\theta(!)"] \\
& D
\end{tikzcd}$$

Commutes.

<u>Theorem:</u> Let (T, τ), (T', τ') be two tensor products of A_R, ${}_R B$. Then $\exists$ isom $\theta : T \to T'$ s.t. $\theta \circ \tau = \tau'$.

<u>Proof:</u>

$$\begin{tikzcd}
& T \arrow[d, dashed, "\theta"] \\
A \times B \arrow[ur, "\tau"] \arrow[r, "\tau'"] \arrow[dr, dashed, "\tau"'] & T' \arrow[d, dashed, "\theta'"] \\
& T
\end{tikzcd}$$

$\exists (!)$ group hom. $\theta : T \to T'$ since (T, τ) tensor product of A, B (def'n)

$\exists (!)$ group hom. $\theta' : T' \to T$.

$$\begin{tikzcd}
& T \arrow[d, dashed, "\theta' \circ \theta"] \\
A \times B \arrow[ur, "\tau"] \arrow[dr, "\tau"'] \arrow[r, phantom, "\circlearrowleft"] & T
\end{tikzcd}$$

$[\theta' \circ \theta \circ \tau = \theta' \circ \tau' = \tau]$

We also have

$$\begin{tikzcd}
& T \arrow[d, equal, "Id"] \\
A \times B \arrow[ur, "\tau"] \arrow[dr, "\tau"'] & T
\end{tikzcd}$$

By def'n of tensor product the group hom making the diag. commute is (!) $\Rightarrow \theta' \circ \theta = I_T$. Similarly, $\theta \circ \theta' = I_{T'}$. $\square$

Oct 6

Theorem: Given A_R, ${}_R B$ a tensor product $\overset{(T,\tau)}{\text{exists}}$

Proof: Let ~~F=~~ F = Free abelian group on set $A \times B$

$\equiv$ Free $\mathbb{Z}$-module w/ basis $A \times B$

R is subgroup gen. by

~~$R=$~~ $\{ (a_1+a_2, b) - (a_1, b) - (a_2, b) \mid a_1, a_2 \in A, b \in B \}$

$\cup \{ (a, b_1 + b_2) - (a, b_1) - (a, b_2) \mid a \in A, b_1, b_2 \in B \}$

$\cup \{ (ar, b) - (a, rb) \mid a \in A, b \in B, r \in R \}$

$T = \frac{F}{R}$, $\tau : (a,b) \mapsto (a,b) + R \in T$.

Notation: $(a,b) + R = a \otimes b = a \otimes_R b$

By construction $\tau : A \times B \to T$ is R-balanced.

Suppose $\beta : A \times B \to D$ is R-balanced.

We have a unique group hom. $\bar{\theta} : F \to D$ s.t. $\bar{\theta}(a,b) = \beta(a,b)$.
(def'd by generators of F)

Since β is R-balanced we get $\bar{\theta}(R) = 0$.

$\to \bar{\theta}$ induces $\theta : T = \frac{F}{R} \to D$

and

$$\begin{tikzcd}
(a,b) \arrow[r, mapsto] \arrow[dr] & (a,b)+R \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\
\beta(a,b) = \bar{\theta}(a,b) = \theta(a,b) &
\end{tikzcd}$$

$\Rightarrow$ θ is (!) group hom. $T \to D$ s.t.

$$\begin{tikzcd}
A \times B \arrow[r, "\tau"] \arrow[dr, "\beta"'] & T \arrow[d, dashed, "\theta"] \arrow[dl, phantom, "\circlearrowleft"] \\
& D
\end{tikzcd}$$

Commutes.

$\square$

Caution: Elements of $T = A \otimes_R B$ are of the form
↖ Notation

$\sum_{i=1}^{n} a_i \otimes b_i$, $a_i \in A$, $b_i \in B$

not only $a_i \otimes b_i$!

Oct 6

$A\otimes B = < a\otimes b \mid a\in A, b\in B >$ as abelian groups.

$a_1\otimes b + a_2\otimes b = (a_1+a_2)\otimes b$

$ar\otimes b = a\otimes rb$

$a\otimes (b_1+b_2) = a\otimes b_1 + a\otimes b_2.$

Ex: $R=\mathbb{Z}$, $A=\mathbb{Z}_n$, $B=\mathbb{Z}_m$, $n,m\in\mathbb{N}$, $\gcd(n,m)=1.$

$\Rightarrow \exists\ s,t\in\mathbb{Z}$ s.t. $sn+tm=1$

If $a\otimes b\in A\otimes B$ then $a\otimes b = a1\otimes b = a(sn+tm)\otimes b$

$= asn\otimes b + atm\otimes b = (an)s\otimes b + at\otimes mb.$

$= 0\otimes b + a\otimes 0 = 0 \quad \Rightarrow \mathbb{Z}_n\otimes\mathbb{Z}_m = 0.$

Ex: $R = k\times k$ $\quad k$ field

$P_1 = k\times 0$

proj. modules

$P_2 = 0\times k$

$P_1\otimes P_2 = ?$

$(\alpha,0)\otimes(0,\beta) = (\alpha,0)(1,0)\otimes(0,\beta) = (\alpha,0)\otimes(1,0)(0,\beta) = (\alpha,0)\otimes(0,0) = 0$

$\Rightarrow P_1\otimes P_2 = 0.$

$[\, a\otimes 0 = a\otimes 0+0 = a\otimes 0 + a\otimes 0 \Rightarrow a\otimes 0 = 0 \,]$

Ex: $A_R\otimes R\cong A$

$R\otimes_R B\cong B$

| elements don't have unique form!

Oct 9

$A_R, {}_RB \qquad A\otimes_R B = \dfrac{\mathcal{F}}{\mathcal{R}} \qquad \mathcal{F}$ free ab. group w/ basis $A\times B$

$\mathcal{R} = \{ (a_1+a_2,b)-(a_1,b)-(a_2,b) \mid \ldots \} \cup \{ (a,b_1+b_2)-(a,b_1)-(a,b_2) \mid \ldots \}$

$\cup \{ (ar,b)-(a,rb) \mid \ldots \}$

Oct 9

$$\begin{tikzcd} A\times B \arrow[r, "\tau"] \arrow[dr, "\beta"] & A\otimes_R B \arrow[d, dashed, "\exists(!)\text{ group hom}"] \\ & D \end{tikzcd}$$

$\nearrow$ R-balanced   abel. g h.

Elements: $a\otimes b = (a,b)+R \in \frac{F}{R} = A\otimes B$

$\{a\otimes b \mid a\in A, b\in B\}$ generates $(A\otimes B,+)$

typical elements $\sum_{i=1}^{n} a_i \otimes b_i$

Example: $A = R_{R,R}\ B.$ Then $R\otimes_R B \cong B$. $\leftarrow$ as ab. groups

$f: R\times B \to B$
$(r,b) \mapsto rb.$

f is R-balanced since B R-mod.

We get unique map $\theta: R\otimes B \to B$
$r\otimes b \mapsto rb$

If $r\in R$, $b\in B$ then $r\otimes b = 1r\otimes b = 1\otimes rb$

$\Rightarrow R\otimes B = \langle 1\otimes b \mid b\in B\rangle = \{1\otimes b \mid b\in B\}$

Similarly, $A_R\otimes R \cong A$

Define: $\alpha: B \to R\otimes B$
$b \mapsto 1\otimes b$

$\alpha(b_1+b_2) = 1\otimes(b_1+b_2) = 1\otimes b_1 + 1\otimes b_2 = \alpha(b_1)+\alpha(b_2)$

Check $\alpha\circ\theta = I_{R\otimes B}$

$\theta\circ\alpha = I_R$

Prop: ① Given A_1, A_2 right R-modules
B_1, B_2 left R-modules

$\left.\begin{array}{l} f: A_1\to A_2 \\ g: B_1\to B_2 \end{array}\right]$ R-homs we get $f\otimes g: A_1\otimes B_1 \to A_2\otimes B_2$ (notation)
$a\otimes b \mapsto f(a)\otimes g(b)$

a group hom.

Oct 9

② If $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ in Mod-R

$B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3$ in R-Mod

then $(f_2 \otimes g_2)(f_1 \otimes g_1) = f_2 f_1 \otimes g_2 g_1$

③ If $f, h : A_1 \to A_2$ in Mod-R

$g, l : B_1 \to B_2$ in R-Mod

Then $(f+h) \otimes g = f \otimes g + h \otimes g$

and $f \otimes (g+l) = f \otimes g + f \otimes l$

④ $I_A \otimes I_B = I_{A \otimes B} : A \otimes B \to A \otimes B$.

Pf: ① F_1 free abel. group on $A_1 \times B_1$

F_2 -"- -"- -"- $A_2 \times B_2$

∃(!) group hom. $f \times g : F_1 \to F_2$
$(a_1, b_1) \mapsto (f(a_1), g(b_1))$

Let R_i be the appropriate subgp of F_i

$\frac{F_i}{R_i} = A_i \otimes B_i$

We check $(f \times g)(R_1) \le R_2$.

$$(f \times g)[(a_1 + a_1', b_1) - (a_1, b_1) - (a_1', b_1)] = (f(a_1 + a_1'), g(b_1)) - (f(a_1), g(b_1))$$

$$- (f(a_1'), g(b_1)) = (f(a_1) + f(a_1'), g(b_1)) - (f(a_1), g(b_1)) - (f(a_1'), g(b_1)) \in R_2$$

Similarly for other two types of generators of R_1

$\Rightarrow f \times g$ induces $f \otimes g : A_1 \otimes B_1 = \frac{F_1}{R_1} \to \frac{F_2}{R_2} = A_2 \otimes B_2$

② $(f_2 \otimes g_2)[(f_1 \otimes g_1)(a_1 \otimes b_1)] = (f_2 \otimes g_2)[f_1(a_1) \otimes g_1(b_1)] = f_2 f_1(a_1) \otimes g_2 g_1(b_1)$

$= (f_2 f_1) \otimes (g_2, g_1)(a_1 \otimes b_1)$ $\forall a_1 \in A_1, b_1 \in B_1$

Since $\{a_1 \otimes b_1 \mid a_1 \in A_1, b_1 \in B_1\}$ generates $(A_1 \otimes B_1, +)$, we get ②.

Oct 9

③, ④ similar (check!) □

Prop: Let M be an R-mod. and $\{M_i : i \in I\}$ a family of R-modules. Assume $\exists$ R-homs

$$\{\alpha_i : M_i \to M \mid i \in I\},\ \{P_i : M \to M_i \mid i \in I\} \text{ s.t.}$$

(1) $P_i \circ \alpha_i = I_{M_i}$, $P_j \circ \alpha_i = 0$ if $i \neq j$

(2) If $m \in M$, $P_i(m) = 0$ a.e.

(3) $m = \sum_i \alpha_i P_i(m)$ $\forall m \in M$ ↖ finite sum

Then $M = \bigoplus_i \alpha_i(M_i)$ and $\alpha_i : M_i \to \alpha(M_i)$ is an isom. $\forall i \in I$.

"Proof": By (3), $M = \sum_i \alpha_i(M_i)$. If $0 = \sum_{i\in I} y_i$, $y_i = 0$ a.e. $\overset{=\alpha_i(x_i)}{y_i} \in \alpha_i(M_i)$

then $0 = P_j(0) = \sum_{i \in I} P_j(y_i) = \overset{x_j}{y_j}$ $\forall j \in I$ $\Rightarrow 0$ has unique form $0 = \sum_{i\in I} 0$

$\Rightarrow M = \bigoplus_{i \in I} \alpha_i(M_i)$

$\alpha_i : M \to \alpha_i(M_i)$ is onto by def. $P_i\alpha_i = I_{M_i} \Rightarrow \operatorname{Ker} \alpha_i = 0 \Rightarrow \alpha_i : M_i \to \alpha_i(M_i)$

is an isom. □

Theorem: Let $A = \bigoplus_{i \in I} A_i$ be a direct sum of right R-modules, $B \in R$-Mod.

Then $A \otimes B = (\bigoplus_{i\in I} A_i) \otimes B \cong \bigoplus_{i \in I} [A_i \otimes B]$.

Proof: $\alpha_i : A_i \to A$ inclusion
$P_i : A \to A_i$ is proj.

Let $\bar{\alpha}_i = \alpha_i \otimes I_B : A_i \otimes B \to A \otimes B$

$\bar{P}_i = P_i \otimes I_B : A \otimes B \to A_i \otimes B$

Note $\bar{P}_i \circ \bar{\alpha}_i = (P_i \otimes I_B)(\alpha_i \otimes I_B) = (P_i\alpha_i) \otimes I_B = I_{A_i} \otimes I_B = I_{A_i \otimes B}$

$\Rightarrow$ $\{\bar{\alpha}_i \mid i \in I\}$, $\{\bar{P}_i \mid i \in I\}$ satisfies conditions of prev. Prop. □

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Note: $A_i \otimes B$ is identified with $\alpha_i(A_i \otimes B)$ or with

$\langle a_i \otimes b \mid a_i \in A_i,\ b \in B \rangle \subseteq A \otimes B$

Note: ① $A \otimes (\bigoplus_{j \in J} B_j) = \bigoplus_{j\in J} [A \otimes B_j]$

② $(\bigoplus_i A_i) \otimes (\bigoplus_j B_j) = \bigoplus_{i,j} [A_i \otimes B_j]$

Ex: $R^{(n)} \otimes B \cong B^{(n)}$

$R^{(n)} \otimes B = \bigoplus_{i=1}^{n} (R \otimes B) \cong \bigoplus_{i=1}^{n} B = B^{(n)}$.

$A \in$ Mod-R

Note: $F = A \otimes \_ :$ R-Mod $\to$ Ab
$B \mapsto A \otimes B$

with $(A \otimes \_)(g) = I_A \otimes g : A \otimes B_1 \to A \otimes B_2$ where $g : B_1 \to B_2$ is an R-hom. gives a functor!

$F(g_1 + g_2) = F(g_1) + F(g_2)$.

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Exercise: If ~~$0\to$~~ $\underset{(*)}{A \xrightarrow{f} B \xrightarrow{g} C \to 0}$ in R-Mod satisfies

$0 \to \mathrm{Hom}(C,D) \xrightarrow{g^*} \mathrm{Hom}(B,D) \xrightarrow{f^*} \mathrm{Hom}(A,D)$ is exact in Ab $\forall D$,

then $(*)$ is exact.

If $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ $(**)$ satisfies
$0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C)$
exact $\forall X \in$ R-Mod $\Rightarrow (**)$ is exact.

Def: A $\mathcal{C}$ is preadditive if $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an abelian group $\forall A, B \in \mathcal{C}$ and ~~if~~

(1) If $A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$ then $(g+h)\circ f = g\circ f + h \circ f$

(2) If $A \underset{l}{\overset{f}{\rightrightarrows}} B \xrightarrow{g} C$ then $g \circ (f + l) = g\circ f + g \circ l$

(3) $\mathcal{C}$ has a 0-object.

Ex: R-Mod.

Oct 11

Ex: Ab.

Def: If $F : \mathcal{C} \to \mathcal{D}$ $\mathcal{C}, \mathcal{D}$ preadd. cat.

is a functor s.t. $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ is a hom. of

abelian groups, we say F is a (cov.) add. functor.

[If $F : \mathcal{C} \to \mathcal{D}$ is contravar. we require $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(\underbrace{F(B)}_{=FB}, \underbrace{F(A)}_{FA})$
(notation)

to be a group hom.]

Ex: $\mathcal{C} = R\text{-Mod}$. $\mathcal{D} = Ab$.

Fix $X, Y \in R\text{-Mod}$.

$F : \mathcal{C} \to \mathcal{D}$

$B \mapsto Hom_R(X, B)$ covar. add. functor

$G : \mathcal{C} \to \mathcal{D}$

$A \mapsto GA = Hom_R(A, Y)$

Contravar. add. functor.

$B_1 \overset{g_1}{\underset{g_2}{\rightrightarrows}} B_2$ $F(g_1 + g_2) = (g_1 + g_2)_* = (g_1)_* + g_{2*}$

Recall: $F(g_1) = g_{1*}$ $[h \longmapsto g_1 \circ h]$

Prop: Let R, S, T be rings.

① Given ${}_S A_R, {}_R B$ then $A \otimes_R B$ is a left S-module via

$s(a \otimes b) = (sa) \otimes b$.

② Given $A_R, {}_R B_T$ then $A \otimes_R B$ is a right T-mod. via

$(a \otimes b)t = a \otimes bt$.

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③ Given ${}_SA_R, {}_RB_T$ then $A\otimes_R B$ is a S-T-bimod.

via $s(a\otimes b)t = (sa)\otimes(bt)$.

<u>Proof:</u> (1) $\mathcal{F}$ free ab. group on $A\times B$.

$\bar{s} : A\times B \to A\times B$
$(a,b) \mapsto (sa, b)$

We get group hom. $\bar{s} : \mathcal{F} \to \mathcal{F}$.

If $s_1, s_2 \in S$ check $\bar{s_1}\circ\bar{s_2} = \overline{s_1 s_2}$ and $\bar{s_1} + \bar{s_2} = \overline{s_1+s_2}$, $\bar{1} = \mathrm{Id}_{\mathcal{F}}$

Recall $A\otimes_R B = \dfrac{\mathcal{F}}{\mathcal{R}}$

If we show $\bar{s}(\mathcal{R}) \subseteq \mathcal{R}$ then we get $s : A\otimes B \to A\otimes B$
$a\otimes b \mapsto (sa)\otimes b$

and $S \to \mathrm{End}_{\mathbb{Z}}(A\otimes_R B)$
$s \mapsto s$
is a ring hom. $\Rightarrow A\otimes_R B$ is a S-mod.

$$\bar{s}((ar,b)-(a,rb)) = (s(ar),b) - (sa,rb) = (\underbrace{(sa)}_{t}r, b) - (sa, rb) \quad ({}_SA_R)$$
$$= (tr,b)-(t,rb) \in R$$

Similarly, $\bar{s}$ sends other generators of R into R.

$\Rightarrow {}_S(A\otimes B)$ as described.

(2) Similar

(3) It suffices to show that actions on pure tensors commute.

$[s(a\otimes b)]t = ((sa)\otimes b)t = (sa)\otimes(bt) = s(a\otimes bt) = s((a\otimes b)t)$.

$[(sw)t = s(wt) \ \forall w\in A\otimes B]$ ▫

<u>Prop:</u> Fix ~~${}_RA_S$~~ ${}_SA_R$ and ${}_RB_T$. Then

(1) F: R-Mod $\to$ S-Mod
$X \mapsto$ ~~$\mathrm{Hom}_R(A,X)$~~ $A\otimes_R X$ is a (cov.) add. functor.

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(2) $G: \text{Mod-}R \to \text{Mod-}T$

$X \mapsto X \otimes B$

is a contravar. add. functor.

<u>Proof:</u> (1) We know that $A \otimes X = F(X)$ is a left S-mod.

Let $g: X_1 \to X_2$ be an R-hom.

$$I_A \otimes g = Fg : A\otimes X_1 \to A \otimes X_2$$

$$(Fg)(s(a\otimes b)) = (Fg)(sa\otimes b) = sa\otimes g(b) \quad \forall b \in X$$

$$= s(a\otimes g(b)) = s\,F(g)(a\otimes b) \Rightarrow F(g) = 1_A\otimes g$$

is an S-hom. □

<u>Note:</u> If R is comm. $A_R = {}_RA_R$.

<u>Example:</u> If V is a v.s. $/k$ w/ basis $\{v_i \mid i \in I\}$

and W - - - '' - - - $\{w_j \mid j \in J\}$

then $V\otimes W$ is a v.s $/k$ with basis $\{v_i \otimes w_j \mid i \in I, j \in J\}$

why (?) $V = \bigoplus_i kv_i$, $W = \bigoplus_j kw_j$

$$V\otimes W = \bigoplus_{i,j} (kv_i \otimes kw_j) = \bigoplus_{i,j} k(v_i \otimes w_j)$$

$kv_i \cong k$

$kw_j \cong k$

$kv_i \otimes kw_j \cong k\otimes k \cong k$

<u>Example:</u> Let V, W be a fin. dim. v.s. $/k$

Then $\overset{W}{V} \otimes_k \overset{V^*}{W^*} \to \text{Hom}(V, W)$ →Prop later

$$v\otimes f \mapsto \widehat{(v\otimes f)}(u) = f(u)v$$

is an isom. of v.s.

[Exercise!] first proof k-balance, biaddlitive map $V\times W \to$ ___

2nd surj, 3rd dims are the same ⇒ isom

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Theorem: Let $X \in R\text{-Mod}$. $F: R\text{-Mod} \to Ab$.
$A \mapsto A \otimes X$.

If $A \to B \to C \to 0$ is exact in Mod-R.

Then $A \otimes X \to B \otimes X \to C \otimes X \to 0$ is exact in Ab

[Try.]

Oct 13

Prop: k field, V, W finite-dimensional. Then $\varphi: W \otimes V^* \to \mathrm{Hom}_k(V,W)$

$w \otimes f \mapsto (\widehat{w \otimes f})[: v \mapsto f(v)w]$

is an isom. of v.s.

Proof:
$$W \times V^* \xrightarrow{\bar{\varphi}} \mathrm{Hom}_k(V,W)$$
$$(w,f) \mapsto (\widehat{w,f})[:v \mapsto f(v)w]$$

$(\widehat{w,f}) \in \mathrm{Hom}_k(V,W)$: $(\widehat{w,f})(\alpha v) = f(\alpha v)w = \alpha f(v)w = \alpha[(\widehat{w,f})(v)]$

$\bar{\varphi}$ is k-balanced: $(\widehat{\alpha w,f})(v) = f(v)(\alpha w) = (\alpha f)(v)w = (\widehat{w,\alpha f})(v)$. $\forall v \in V$

$= \alpha f(v) \cdot w$
$\uparrow \uparrow$
scalars in k

Check $\bar{\varphi}$ is additive in both variables.

We get $\varphi: W \otimes V^* \to \mathrm{Hom}_k(V,W)$ group hom.

φ is k-linear map Check.

φ 1-1: Let $t \in \mathrm{Ker}\,\varphi$. We can write $t = \sum_{i=1}^{n} w_i \otimes f_i$

W.L.O.G $w_1, w_2, \ldots, w_n$ is lin. indep.

If $w_3 = \alpha_1 w_1 + \alpha_2 w_2$ $\quad w_1 \otimes f_1 + w_2 \otimes f_2 + w_3 \otimes f_3 = w_1 \otimes f_1 + w_2 \otimes f_2$

$+ (\alpha_1 w_1 + \alpha_2 w_2) \otimes f_3 = w_1 \otimes (f_1 + \alpha_1 f_3) + w_2 \otimes (f_2 + \alpha_2 f_3)$

In fact if n is minimal it follows $w_1, \ldots, w_m$ is lin. indepent.

[also $f_1, \ldots, f_n$ are lin. indep.!]

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Now, $\varphi(f)(v) = 0 \ \forall v \in V.$ $\Rightarrow \sum_{i=1}^{n} f_i(v) w_i = 0 \ \forall v \in V$

$\Rightarrow f_i(v) = 0 \ \forall i, v \in V$ since $w_1, \dots, w_n$ lin. indep.

$\Rightarrow \operatorname{Ker} \varphi = 0 \Rightarrow \varphi$ ~~is~~ 1-1.

Finally, $\dim(W \otimes V^*) = (\dim W)(\dim V^*) = (\dim W)\dim(V)$

$= \dim(\operatorname{Hom}_K(V, W))$

$\Rightarrow \varphi$ is onto by dim. theory. □

$[V \otimes V^* = M_n(V)]$

<u>Recall</u> $F = {}_S A_R \otimes \_ : R\text{-Mod} \to S\text{-Mod}$ is an add. functor.

Check given $B_1 \overset{g_1}{\underset{g_2}{\rightrightarrows}} B_2$ $F(g_1 + g_2) = I_A \otimes (g_1 + g_2) = I_A \otimes g_1 + I_A \otimes g_2$

$= F(g_1) + F(g_2).$

<u>Theorem</u>: Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact in R-Mod and $X \in$ Mod-R.

Then $X \otimes A \xrightarrow{I_X \otimes f} X \otimes B \xrightarrow{I_X \otimes g} X \otimes C \to 0$, is exact.

<u>Proof:</u> Let $x \otimes c \in X \otimes C$. $\exists\, b \in B.\ g(b) = c$

$\Rightarrow (I \otimes g)(x \otimes b) = x \otimes g(b) = x \otimes c$

$\Rightarrow \operatorname{Im}(I \otimes g) \supseteq \langle x \otimes c \mid x \in X, c \in C \rangle = X \otimes C.$ $\Rightarrow I \otimes g$ is onto.

<u>$\operatorname{Im}(I \otimes f) \subseteq \operatorname{Ker}(I \otimes g)$:</u>

$(I \otimes g)(I \otimes f) = I \otimes (g \circ f) = I \otimes 0 = 0.$ $\Rightarrow \operatorname{Im}(I \otimes f) \subseteq \operatorname{Ker}(I \otimes g).$

$\Big[\ \underset{\text{exact}}{L \xrightarrow{f} M \xrightarrow{g} N \to 0} \Rightarrow \frac{M}{f(M)} \underset{\text{well def.}}{\xrightarrow{\bar{g}}} F$ since $f(M) \subseteq \operatorname{Ker}(g)$ and $\cong$ $\Big]$

Let $E = \operatorname{Im}(I \otimes f) \subseteq \operatorname{Ker}(I \otimes g)$.

Define $\alpha: \frac{X \otimes B}{E} \to X \otimes C$

$(x \otimes b) + E \mapsto x \otimes g(b) = (I \otimes g)(x \otimes b)$ well-def'd group hom.

Define $\beta : X \times C \longrightarrow \frac{X \otimes B}{E}$

$(x,c) \longmapsto (x\otimes b)+E$ where $g(b)=c$.

Suppose $g(b') = g(b) = c \Rightarrow b'-b \in \operatorname{Ker} g = \operatorname{Im} f$

$\Rightarrow b'-b = f(a)$, some $a \in A$

$x\otimes b' - x\otimes b = x\otimes(b'-b) = x\otimes f(a) \in \operatorname{Im}(I\otimes f) = E$

$\Rightarrow \beta$ is well-defined.

$\beta(xr,c) = xr\otimes b = x\otimes rb$ ~~$= x\otimes r g(c) = x\otimes g(rc$~~$)$

& $\beta(x,rc) = x\otimes rb$ since $g(rb) = rg(b) = rc$

$\Rightarrow \beta(xr,c) = \beta(x,rc)$.

Sm. β is bi-additive (add. in each var.)

We get $\beta : X\otimes C \to \frac{X\otimes B}{E}$

$(\beta\circ\alpha)(x\otimes b+E) = \beta(x\otimes g(b)) = (x\otimes b)+E$

$\Rightarrow \beta\circ\alpha = I$

Similarly, $\alpha\circ\beta = I_{X\otimes C}$ $\quad (\alpha\circ\beta)(x\otimes c) = \alpha((x\otimes b)+E) = x\otimes g(b) = x\otimes c$
$g(b)=c$

$\Rightarrow \alpha$ is an isom. $\Rightarrow \operatorname{Im}(I\otimes f) = E \underset{\text{isom}}{=} \operatorname{Ker}(I\otimes g)$ Δ

Exercise: ① If $F : \mathcal{C} \to D$ is an equivalence of cat., then

$F : \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_D(FA,FB)$ is a bijection.

② If $F: \mathcal{C}\to D$ is an add. functor of preadd. cat. then $F(0_{\mathcal{C}}) = 0_D$.

$I_{0_{\mathcal{C}}} = 0_{0_{\mathcal{C}}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & D \arrow[l, shift left, "G"] \end{tikzcd}$$ isom. of cat.

$\operatorname{Hom}_D(FA,B) \overset{\text{Bijection}}{\underset{\cong}{\longleftrightarrow}} \operatorname{Hom}_{\mathcal{C}}(GFA, GB) = \operatorname{Hom}_{\mathcal{C}}(A, GB)$

F, G adjoint functors

Oct 16

$F : \mathcal{C} \to \mathcal{D}$
$G : \mathcal{D} \to \mathcal{C}$ inverse isom.

$\mathrm{Hom}_{\mathcal{D}}(FA, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, GB)$

Recall: A_R, ${}_R B_S$, C_S (bi-)modules over Ring R,S

$- \otimes_R B_S : \text{Mod-}R \to \text{Mod-}S$. functor

$\mathrm{Hom}_S({}_R B_S, -) : \text{Mod-}S \to \text{Mod-}R$ functor

$[(fr)(b) = f(rb)]$

Thm: (Adjoint isom. thm) In above situation,

$$\varphi = \varphi_{A,B,C} : \mathrm{Hom}_S(\underbrace{A_R \otimes_R B_S}_{\in \text{Mod-}S}, C_S) \longrightarrow \mathrm{Hom}_R(A_R, \underbrace{\mathrm{Hom}_S({}_R B_S, C_S)}_{\in \text{Mod-}R})$$

$$f \mapsto f^*[: a \mapsto f^*_a] \quad \text{where } f^*_a(b) = f(a \otimes b)$$

↖ "one variable (a) fixed"

is an isomorphism which is natural in A, B and C.

Proof: (1) $f^*_a \in \mathrm{Hom}_S(B,C)$: $\quad f^*_a(bs) = f(a \otimes bs) = f((a \otimes b)s)$

$= f(a \otimes b)s$ (f S-linear) $= f^*_a(b) \cdot s$

$f^*_a(b_1 + b_2) = f^*_a(b_1) + f^*_a(b_2)$ (check)

(2) $\varphi(f) = f^* : A_R \to \mathrm{Hom}_S({}_R B_S, C_S)$ is an R-hom.:

$f^*(a_1 + a_2)(b) = f((a_1 + a_2) \otimes b) = f(a_1 \otimes b + a_2 \otimes b) = f(a_1 \otimes b) + f(a_2 \otimes b)$

$= f^*(a_1)(b) + f^*(a_2)(b) = (f^*(a_1) + f^*(a_2))(b) \quad \forall b \in B$

$\Rightarrow f^*$ additive

$f^*(ar)(b) = f^*_{ar}(b) = f(ar \otimes b) = f(a \otimes rb) = f^*_a(rb)$

$= (f^*_a r)(b) \quad \forall b \in B \Rightarrow f^*(ar) = f^*(a) r \qquad$ (recall $f^*(a) = f^*_a$)

(3) To show that φ is an isomorphism we construct inverse ψ.

Oct 16

$\psi: \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)) \to \mathrm{Hom}_S(A\otimes B, C)$

Let $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_{\underset{R}{S}}(B_S, C_S))$

Def. $\hat{g}: A\times B \to C$
$(a,b) \mapsto g(a)(b)$ $\hat{g}$ is clearly bi-additive.

$\hat{g}(ar, b) = g(ar)(b) = (g(a)r)(b) = g(a)(rb) = \hat{g}(a, rb)$

Thus $\hat{g}$ induces (!) $\psi(g): A\otimes B \to C$ group hom. s.t.

$$\begin{tikzcd} A\times B \arrow[r, "\tau"] \arrow[dr, "\hat{g}"'] & A\otimes B \arrow[d, dashed, "\exists!\,\psi(g)"] \\ & C \end{tikzcd}$$

$\psi(g)(a\otimes bs) = \hat{g}(a, bs) = g(a)(bs) = g(a)(b)s = \hat{g}(a,b)s$

$= [\psi(g)(a\otimes b)]s$

$\Rightarrow \psi(g) \in \mathrm{Hom}_S(A\otimes_R B, C)$
↑
S linear

(4) $\psi\circ\varphi = I_{\mathrm{Hom}_S(A\otimes_R B, C)}$

$(\psi\circ\varphi)(f) = \psi(f^*): a\otimes b \to f^*_a(b) = f(a\otimes b)$ $\forall a\in A, b\in B$

$\to \psi\circ\varphi(f) = f.$

(5) $\varphi\circ\psi(g) = g.$ Exercise!

$\varphi\circ\psi(g) = \varphi(\psi(g)) = (\psi(g))^*: a \to (\psi(g))^*_a$.
$(\psi(g))^*_a(b) = \psi(g)(a\otimes b)$
$= g(a)(b)$
$\Rightarrow (\psi(g))^* = g \Rightarrow \varphi(\psi(g)) = g$

Conclude $\varphi_{A,B,C}$ (and hence ψ) is an ism.

(6) Naturality in A: Let $\alpha: A_1 \to A_2$. Then $\alpha^*: \mathrm{Hom}(A_2, \mathrm{Hom}_S(B,C))$
$\to \mathrm{Hom}_R(A_1, \mathrm{Hom}_S(B,C))$

similarly $(\alpha\otimes I_B)^*: \mathrm{Hom}_S(A_2\otimes B, C) \to \mathrm{Hom}_S(A_1\otimes B, C)$

Claim diagram

$$\begin{tikzcd} \mathrm{Hom}_S(A_1\otimes B, C) \arrow[r, "\varphi_{A_1,B,C}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_R(A_1, \mathrm{Hom}_S(B,C)) \\ \mathrm{Hom}(A_2\otimes B, C) \arrow[u, "(\alpha\otimes I)^*"] \arrow[r, "\varphi_{A_2,B,C}"] & \mathrm{Hom}_R(A_2, \mathrm{Hom}_S(B,C)) \arrow[u, "\alpha^*"'] \end{tikzcd}$$

commutes. (check!)

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(7) (8) naturality using $\beta: B_1 \to B_2$
$\gamma: C_1 \to C_2$ check!

□

Thm: (2nd proof!)

Let $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_3} A_3 \to 0$ be exact in Mod-R, and $B \in R$-mod.

Then $A_1 \otimes B \to A_2 \otimes B \to A_3 \otimes B \to 0$ is exact in $\mathcal{A}b$.

Proof: Let $S = \mathbb{Z}$, B is an R-S-bimodule.

For any $C \in \mathbb{Z}$-mod (ab.) we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(A_3 \otimes B, C) \arrow[r, "(\alpha_2 \otimes I_B)^*"] \arrow[d, "\varphi"', "\wr"] & \mathrm{Hom}(A_2 \otimes B, C) \arrow[r, "(\alpha_1 \otimes I)^*"] \arrow[d, "\wr\,\varphi"] & \mathrm{Hom}(A_1 \otimes B, C) \arrow[d, dash, "\wr\,\varphi"] \\
0 \arrow[r] & \mathrm{Hom}_R(A_3, \mathrm{Hom}(B,C)) \arrow[r, "\alpha_3^*"] & \mathrm{Hom}_R(A_2, \mathrm{Hom}(B,C)) \arrow[r, "\alpha_1^*"] & \mathrm{Hom}_R(A_1, \mathrm{Hom}(B,C))
\end{tikzcd}$$

bottom row is exact since $A_1 \to A_2 \to A_3 \to 0$ is exact, $\mathrm{Hom}_S(B, C)$ is a left R-mod. (Prop on Oct 4)

$\Rightarrow$ top row is exact $\forall C \in S$-mod

$\Rightarrow A_1 \otimes B \to A_2 \otimes B \to A_3 \otimes C \to 0$ is exact in S-mod. (Exer, Oct 11)

Remark: Last day A_R, ${}_R B_S$, C_S <u>Oct 18</u>

$\exists$ nat. ~~hom.~~ nat. tarsf $\mathrm{Hom}_S(A \otimes_R B_S, C_S) \xrightarrow{\overset{\cong}{\varphi}} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$

similarly given ${}_S A_R$, ${}_R B$, ${}_S C$

$\exists$ nat. isom. $\mathrm{Hom}_S({}_S A_R \otimes_R B, {}_S C) \to \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$

$[\, {}_{R^{op}} A_{S^{op}}, B_{R^{op}}, C_{S^{op}}$ $\quad (A \otimes B)_{S^{op}} \cong [B \otimes A]_{S^{op}}$ apply prev. result. $]$

$\overset{\uparrow}{\text{left } S\text{-mod}}$
$\Rightarrow$ right S^{op}-mod

Kronecker product

Let $A = [a_{ij}] \in M_{m \times n}(k) = V$ (say)
$B = [b_{st}] \in M_{p \times q}(k) = W$ (say)

Oct 18

Def: The Kronecker product of A and B is

$$A \odot B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \\ a_{m1}B & \cdots & \cdots & a_{mn}B \end{bmatrix} \in M_{pm\times qn}(k) = T \text{ (say)}.$$

Prop:

$$\begin{aligned} \mathcal{K}: V\times W &\to T \\ (A,B) &\mapsto A\odot B \end{aligned}$$

is a tensor product.

Proof: Recall $V\otimes W$ is a k-vs. of dimension $(\dim V)(\dim W) = (mn)(pq)$
$= (mp)(nq)$.

$\mathcal{K}$ is clearly k-balanced.

$\Rightarrow \exists(!)$ group hom.

$$\begin{tikzcd} V\times W \arrow[r] \arrow[dr] & V\otimes W \arrow[d, dashed, "\theta(!)"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T \end{tikzcd}$$

θ is additive and k-linear.

Since $\operatorname{span}\{E_{ij}\odot E_{st} \mid {1\le i\le m \atop 1\le j\le n} \; {1\le s\le p \atop 1\le t\le q}\} = T$

$\Rightarrow \theta$ is onto $\Rightarrow \theta$ is an isom. since $V\otimes W$ and T have same dimension $\Rightarrow (T, \mathcal{K})$ is a tensor product. □

Ex: $m=n=p=q=2$

$E_{32} = E_{2,1}\odot E_{12}$
$\in M_{4\times 4}(k)$

$$\begin{bmatrix} 0\begin{bmatrix}0&1\\0&0\end{bmatrix} & 0\begin{bmatrix}0&1\\0&0\end{bmatrix} \\ 1\begin{bmatrix}0&1\\0&0\end{bmatrix} & 0\begin{bmatrix}0&1\\0&0\end{bmatrix} \end{bmatrix}$$

Group algebras

~) likewise; group ring

Def: G group, k field. The group algebra of G over k is the vector space over k with basis G. (= free vs on G over k)

$$k[G] = \{\sum_{g\in G}\alpha_g g \mid \alpha_g\in k,\ \alpha_g = 0 \text{ a.e.}\}$$

$G = 1_k G \le U(k[G])$ units in $k[G]$

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$1_{k[G]} = 1_k 1_G = 1$

$k = k\,1_G \subseteq k[G]$. $\quad k \hookrightarrow k[G]$

Now if we extend product $(\alpha g)(\beta h) = (\alpha\beta)(gh)$ $\forall \alpha,\beta \in k,\ g,h \in G$

using distributive law $k[G]$ is a ring.

<u>Ex:</u> $G = \langle g \rangle = C_n$ cyclic group of order n.

$$\varphi : k[x] \to k[G]$$
$$f(x) \mapsto f(g)$$

$x^n - 1 \in \operatorname{Ker} \varphi$.

φ is onto. $\Rightarrow \dim\left(\frac{k[x]}{\operatorname{Ker}(\varphi)}\right) = \dim_k k[G] = |G| = n$.

But $\dim_k\left(\frac{k[x]}{(x^n-1)}\right) = n \quad \Rightarrow \operatorname{Ker}\varphi = (x^n-1)$.

$$\Rightarrow \frac{k[x]}{(x^n-1)} \cong k[G].$$

<u>Theorem</u> (Maschke's Thm $\frac{?}{?}$ 18~~88~~98)

Assume G is a finite group. Then $k[G]$ is semisimple iff $\operatorname{char} k \nmid |G|$.

(I.O.U)

<u>Prop:</u> If D is a division ring with $\mathbb{C} \subseteq Z(D)$ center of D.

and $\dim_{\mathbb{C}}(D) < \infty \Rightarrow D = \mathbb{C}$.

<u>Why?</u> $\alpha \in D$ $\mathbb{C}(\alpha)$ is a field, fin. dim $/\mathbb{C}$ $\Rightarrow \mathbb{C}(\alpha)/\mathbb{C}$ algebraic.

$\mathbb{C}$ alg. closed $\Rightarrow \mathbb{C}(\alpha) = \mathbb{C}$. $\overset{\Rightarrow D \subseteq \mathbb{C},\ \mathbb{C} \subseteq D \text{ - hypoth}}{}$ and $D = \mathbb{C}$.

<u>Ex:</u> $G = S_3$ $\quad \mathbb{C}[G]$ is s.s. by Maschke.

$\mathbb{C}[G] \cong M_{n_1}(D_1) \times \_ \times M_{n_t}(D_t)$, D_i div. rings. $\mathbb{C} \subseteq Z(D_i)$

$\dim_{\mathbb{C}} \mathbb{C}[G] < \infty \overset{\Rightarrow \dim_{\mathbb{C}} D_i < \infty}{\Rightarrow} D_i = \mathbb{C}$.

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$\mathbb{C}[G]$ not comm. $\Rightarrow t_i \geq 2$, some i.

But $6 = \dim_{\mathbb{C}} \mathbb{C}[G] = n_1^2 + n_2^2 + \_\_ + n_t^2$

$\Rightarrow t = 3,\ n_1 = n_2 = 1,\ n_3 = 2$

$\mathbb{C}[G] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

$AB = 0 \not\Rightarrow A = 0$ or $B = 0$

$R = \mathbb{Z}_{30}$

$A = \frac{15\mathbb{Z}}{30\mathbb{Z}}$, $B = \frac{10\mathbb{Z}}{30\mathbb{Z}}$ $\triangleleft$ $\frac{\mathbb{Z}}{30\mathbb{Z}} = \mathbb{Z}_{30}$

$AB = 0$ but $A \neq 0,\ B \neq 0$

! $\underline{J = J(R)}$ ${}_R S$ simple $\Rightarrow JS = 0$ (looking at ann)

$JA + X = M$

$\overset{\psi}{\sum_{i=1}^{n} j_i a_i + x}$

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Injective hulls

alg. closures:

$$\begin{tikzcd} \bar{k} \arrow[r, dashed, "\alpha", "\cong"'] \arrow[d] & L \arrow[d] \\ k \arrow[r, phantom, "="] & k \end{tikzcd}$$

$\alpha|_k = \mathrm{Id}$

alg. ext.	Inj.
alg. ext.	ess ext.

Prop: Every abelian group $(A, +)$ embeds in a divisible group $(D, +)$.

Proof: (See homework!)

Note: D is an inj. $\mathbb{Z}$-module. ($\mathbb{Z}$ P.I.D.)

Let M be any module.

$(M, +)$ abelian group. We get exact sequence in $\mathcal{A}b$

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$$0 \to M \xrightarrow{i} D$$

$$\Rightarrow \quad 0 \to \mathrm{Hom}_{\mathbb{Z}}(R, H) \xrightarrow{i_*} \mathrm{Hom}_{\mathbb{Z}}(R, D) \quad \text{is exact.}$$

<u>Note:</u> $R = {}_{\mathbb{Z}}R_R$ bimodule.

$\Rightarrow$ $E \overset{def}{=} \mathrm{Hom}_{\mathbb{Z}}(R, D) \to$ is a left R-module.

<u>Theorem:</u> Every R-module embeds in an inj. R-module.
(Baer's Thm)

<u>Proof:</u> Use notation above.

<u>Strategy:</u> ① M embeds in $E = \mathrm{Hom}_{\mathbb{Z}}(R, D)$ as an R-module.

② ${}_R E$ is inj.

<u>①</u>: Define
$$\varphi: M \to E$$
$$m \mapsto \hat{m} \qquad \text{where } \hat{m}(r) = rm \quad (\text{recall } M \leq D)$$

$$\hat{m}(r_1 - r_2) = (r_1 - r_2)m = r_1 m - r_2 m = \hat{m}(r_1) - \hat{m}(r_2) \Rightarrow \hat{m} \in E.$$

Similarly $\varphi(m_1 - m_2) = \widehat{m_1 - m_2} = \hat{m}_1 - \hat{m}_2$

$\Rightarrow \varphi: M \to E$ is a $\mathbb{Z}$-hom.

$$\varphi(rm)(s) = \widehat{rm}(s) = srm = (sr)m = \hat{m}(sr) = \varphi(m)(sr) = [r\varphi(m)](s) \quad \forall s \in R$$

$\Rightarrow \varphi(rm) = r\varphi(m)$ and φ is an R-hom.

If $\varphi(m) = 0 \Rightarrow \hat{m}(r) = rm = 0 \quad \forall r \in R \Rightarrow 1m = m = 0 \Rightarrow \varphi$ is 1-1, so

M embeds in E.

<u>②</u>: We need to show that $\overbrace{\mathrm{Hom}_R(-, E)}^{\text{contravar. add. functor}} : R\text{-Mod} \to Ab$ preserves exact sequences.

Let $0 \to A \to B \to C \to 0$ be SES in R-mod.

We get diagram

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$$\begin{tikzcd}
0 \arrow[r] & R\otimes A \arrow[r] \arrow[d, phantom, "\wr"] & R\otimes B \arrow[r] \arrow[d, phantom, "\wr"] & R\otimes C \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

which commutes. Bottom row exact $\Rightarrow$ Top row is exact.
(also as $\mathbb{Z}$-mod.)

~~We get~~ Apply $Hom_{\mathbb{Z}}(-, D)$ to ~~bottom~~ top row. to get

$$\begin{tikzcd}
0 \arrow[r] & Hom_{\mathbb{Z}}(R\otimes C, D) \arrow[r] \arrow[d, "\varphi_{R,C,D}"] & Hom_{\mathbb{Z}}(R\otimes B, D) \arrow[r] \arrow[d, "\varphi"] & Hom_{\mathbb{Z}}(R\otimes A, D) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & Hom_R(C, Hom_{\mathbb{Z}}(R,D)) \arrow[r] & Hom_R(B, Hom_{\mathbb{Z}}(R,D)) \arrow[r] & Hom_R(A, Hom_{\mathbb{Z}}(R,D))
\end{tikzcd}$$

Use adjoint isoms to get 2^{nd} row.

1^{st} row exact since ${}_{\mathbb{Z}}D$ is injective.

$\Rightarrow$ 2^{nd} row exact since diagram commutes, vertical maps are isoms. $\Rightarrow$ $Hom_R(-, E)$ is exact. $\Rightarrow$ ${}_RE$ is inj. $\square$

Lemma: ${}_RE$ is inj. iff E has no nontrivial essential extension.

Pf: Assume ${}_RE$ inj. and ${}_RE \subseteq X$. The SES

$0 \to E \to X \to X/E \to 0$ splits. $\Rightarrow X = E \oplus Y$, some Y.

E ess X $\iff$ $Y = 0$. $\Rightarrow$ E has no proper ess. extension.

Conversely assume ${}_RE$ ess $Z \Rightarrow E = Z$.

Choose inj. ${}_RQ$ with $E \subseteq Q$.

~~Let $\mathcal{P} = \{{}_RS \subseteq Q \mid E \text{ ess } S\}$ order by inclusion.~~

~~If $\mathcal{C} = \{S_i \mid i \in I\}$ is a chain in $\mathcal{P}$ then $\bigcup_{i\in I} S_i \in \mathcal{P}$ (check!)~~

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~~is an upper bound for $\mathcal{C}$ (check!).~~

$\Rightarrow$ By ~~Zorn~~ Lemma (see below) $\exists_R W \leq Q$ s.t. $E \oplus W$ ess_R Q.

$$E \cong \frac{E+W}{W} \subseteq \frac{Q}{W}$$

Claim $\frac{E+W}{W}$ ess $\frac{Q}{W}$.

Let $0 \neq \frac{X}{W} \subseteq \frac{Q}{W}$ $\Rightarrow$ $W \subsetneq X$ $\Rightarrow$ $E \cap X \neq 0$ by maximality of W.

$\Rightarrow$ $0 \neq \frac{(X \cap E)+W}{W} \subseteq \frac{E+W}{W} \cap \frac{X}{W}$. $\Rightarrow$ $\frac{E+W}{W}$ ess $\frac{Q}{W}$

Now $E \cong \frac{E+W}{W}$ so $\frac{E+W}{W}$ has no (nontriv.) ess ext.

$\Rightarrow$ $\frac{E+W}{W} = \frac{Q}{W}$ $\Rightarrow$ $E+W = Q$ $\Rightarrow$ $E \oplus W = Q$. $\Rightarrow$ ${}_R E \mid_R Q$

$\Rightarrow$ ${}_R E$ is inj. (check!). $\square$

<u>Lemma:</u> Let $A \leq B$ be R-modules. Then $\exists$ ${}_R C \leq B$ s.t.

$A \oplus C$ ess B.

<u>Pf:</u> $\mathcal{P} = \{ {}_R X \leq B \mid X \cap A = 0 \}$ $0 \in \mathcal{P} \Rightarrow \mathcal{P} \neq \emptyset$.

If $\mathcal{C} = \{ X_i \mid i \in I \}$ is a chain in $\mathcal{P}$, $\bigcup_{i \in I} X_i$ is an upper bound.

By Zorn $\exists$ a max'l elt $C \in \mathcal{P}$. $A \oplus C \leq B$.

Suppose $A \oplus C$ ~~ess~~ B. $\Rightarrow$ $(A+C) \cap X = 0$ some $0 \neq {}_R X \leq B$.

Now $A \oplus C \oplus X$ is a direct sum. $\Rightarrow C \subsetneq C \oplus X$ and $C+X \in \mathcal{P}$ $C!$

$\Rightarrow$ $A \oplus C$ ess B.

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<u>Caution</u>

If $A \leq B$ in Mod-R, $M \in R$-Mod

then $A \otimes M$ and $B \otimes M$ are abelian groups

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$a \otimes m \in B \otimes M$ can be 0

while $a \otimes m \in A \otimes M$ is not zero!

Ex: $R = \mathbb{Z}$, $A = \mathbb{Z}$, $B = \mathbb{Q}$, $M = \mathbb{Z}_2$

$0 \neq 1 \otimes 1 \in A \otimes M \cong \mathbb{Z}_2$

$0 = 1 \otimes 1 = \frac{1}{2} \otimes 2 \cdot 1 \in B \otimes M.$

Thm: If ${}_R M$ is an R-mod then $\exists$ an injective module E_0 s.t. M ess E_0 and E_0 is inj. Furthermore, E_0 is (!) up to an isom. which is identity on M.

Proof. Let Q be any inj. module containing M, by Baer's Thm.

Let $\mathcal{P} = \{ E \mid M \text{ ess } E, {}_R E \subseteq Q \}$. $M \in \mathcal{P} \Rightarrow \mathcal{P} \neq \emptyset$.

If $\{ E_i \mid i \in I \}$ is a chain in $\mathcal{P}$ then $\bigcup_{i \in I} E_i$ is an R-submod.

If $0 \neq X \subset \bigcup_{i \in I} E_i \Rightarrow \exists\, 0 \neq x \in X \Rightarrow x \in E_i$, some i. Now, $0 \neq Rx \subseteq E_i \subseteq \bigcup_{j \in I} E_j$

$\Rightarrow 0 \neq M \cap Rx \subseteq M \cap [\bigcup_{j \in J} E_j] \Rightarrow \bigcup_{i \in I} E_i$ is an upper bound for chain in $\mathcal{P}$.

By Zorn $\exists$ a maximal element E_0 in $\mathcal{P}$.

We claim E_0 is inj. It suffices to show that E_0 has no ess. extensions.

Suppose E_0 ess L, some L.

We have a diag.

$$\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r, hook] \arrow[d, "i"] & L \arrow[dl, dashed] \\
 & Q &
\end{tikzcd}$$

Q inj $\Rightarrow \exists h : L \to Q$ s.t. $h(x) = x$ $\forall x \in E_0$.

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$\mathrm{Ker}(h) \wedge E_0 = 0$. But E_0 ess L $\Rightarrow \mathrm{Ker}(h) = 0$. $\Rightarrow h$ is 1-1.

But now E_0 ess $h(L)$ $\Rightarrow E_0 = h(L) \Rightarrow\Leftarrow$ by maximality of E_0 in P.

We get E_0 has no nontrivial ess ext. and E_0 is inj. [Note $E_0 \mid Q$]

<u>Uniqueness:</u> Assume M ess E_0
M ess E_1

where E_1, E_0 are inj. We need $\varphi : E_0 \to E_1$ isom s.t. $\varphi(x) = x$ $\forall x \in M$.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook] \arrow[d, hook] & E_0 \arrow[dl, dashed, "\varphi"] \\ & E_1 & \end{tikzcd}$$

$\exists\, \varphi : E_0 \to E_1$ s.t. $\varphi(m) = m$ $\forall m \in M$

$\mathrm{Ker}(\varphi) \wedge M = 0$. But M ess E_0. $\Rightarrow \mathrm{Ker}\,\varphi = 0$.

$\Rightarrow \varphi$ 1-1. Note $\varphi(E_0) \cong E_0$ is inj.

$\Rightarrow E_1 = \varphi(E_0) \oplus X$, some X. But $M \subseteq \varphi(E_0)$ $\Rightarrow$ $\varphi(E_0)$ ess E_1

M ess E_1

$\Rightarrow X = 0$ and φ is onto. ▯

<u>Fact:</u> ${}_R E$ inj., ${}_R E \subseteq {}_R A$. $\Rightarrow {}_R E \mid {}_R A$.

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, hook] \arrow[d, equal] & A \arrow[dl, dashed, "r"] \\ & E & \end{tikzcd}$$

$\exists r : A \to E$
s.t. $r(x) = x$ $\forall x \in X$

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, shift left] & A \arrow[l, dashed, shift left, "r"] \arrow[r] & A/E \arrow[r] & 0 \end{tikzcd}$$

SES splits.

[<u>Prop:</u> ${}_R E$ inj. $\Leftarrow$ Every SES $0 \to E \to A \to B \to 0$ of R-mods splits.]

<u>Think about:</u> [Fun problems]

① We write $E(M)$ for E_0 in last theorem and call it the injective envelope or injective hull of M.

② Show $E({}_{\mathbb{Z}}\mathbb{Z}) = \mathbb{Q}$.

③ Full 5-lemma:

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Assume
$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

is a Commutative diag in R-mod with
(0) exact rows
(1) α_5 inj.

(2) $\alpha_{\cancel{5}1}$ surj.

(3) α_2, α_4 isomorphisms

Then α_3 is an isomorphism.

Supp. $\alpha_3(a_3)=0$.
$$\begin{tikzcd}
{} \arrow[d] & a_2 \arrow[r] \arrow[d] & a_3 \arrow[r] \arrow[d] & f_3(a_3) \arrow[d, "\alpha_4"] \\
? \arrow[r] & \alpha_2(a_2) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$
isom

$\Rightarrow f_3(a_3)=0$ since α_4 isom.

$\Rightarrow a_3 = f_2(a_2)$.

Jordan-Hölder-Thm

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Def: ① A series for $_RM$ is a fin. sequence of submods

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n = 0$$

The factors are $\frac{M_i}{M_{i+1}}$, $i = 0, \dots, n-1$.

② A series is a composition series if all factors are simple modules.

Ex: $\mathbb{Z}_4 \supseteq 2\mathbb{Z}/4\mathbb{Z} \supseteq 0$

Factors: $\frac{\mathbb{Z}/4\mathbb{Z}}{2\mathbb{Z}/4\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2$, $\frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}_2$

Factors $\mathbb{Z}_2, \mathbb{Z}_2$

Ex: If $0 \to A \to B \to C \to 0$ is a SES w/ A, C simple then

$B \supseteq A \supseteq 0$ is a comp. series.

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Def: A Series $M = A_0 \geq A_1 \geq \dots \geq A_t = 0$ is a refinement of $M = M_0 \geq M_1 \geq \dots \geq M_r = 0$ if each $M_i = A_j$ for some j.

Ex: $$\mathbb{Z}_{12} \geq \frac{3\mathbb{Z}}{12\mathbb{Z}} \geq \frac{6\mathbb{Z}}{12\mathbb{Z}} \geq 0$$

factors: $\mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ comp. series

$$\mathbb{Z}_{12} \geq \frac{2\mathbb{Z}}{12\mathbb{Z}} \geq \frac{6\mathbb{Z}}{12\mathbb{Z}} \geq 0$$

factors: $\mathbb{Z}_2$ $\mathbb{Z}_3$ $\mathbb{Z}_2$

Def: 2 series for M: $M = M_0 \geq M_1 \geq \dots \geq M_r = 0$

$M = N_0 \geq N_1 \geq \dots \geq N_s = 0$

are equivalent if the modules $\left\{ \frac{M_i}{M_{i+1}} \mid i \right\}$ and $\left\{ \frac{N_j}{N_{j+1}} \mid j \right\}$ are in bijective correspondence, where corresponding modules are isomorphic.

Ex: In previous example the comp. series are equivalent.

Prop: ${}_RM$ has a composition series iff M is both Art. and Noeth.

Pf: Assume M is both Noeth and Artinian. Choose M_1, maximal submod. of M. (M Noeth $\Rightarrow M_1$ exists compare. M Art. → min'l submod in set of subm. exists)

If $M_1 = 0$, done. Choose M_2 a max'l submod of M_1. etc.

We get $M \geq M_1 \geq M_2 \geq \dots$ Since M Art. this must stop but it only stops when $M_r = 0$.

Now, $M = M_0 \geq M_1 \geq \dots \geq M_r = 0$ and $\frac{M_i}{M_{i+1}}$ simple since M_{i+1} is maximal in M_i. We have a comp. series.

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Conversely, let $M = M_0 \supseteq M_1 \supseteq \_ \supseteq M_r = 0$ be a comp. series.

Use induction on r. $r = 1 \Rightarrow M = M_0/0$ simple $\Rightarrow$ Art & Noeth

M_1 has comp. series $M_1 \supseteq \dots \supseteq M_r = 0$ By ind., M_1 is Art & Noeth.

Also M/M_1 simple, so also Art. and Noeth.

$\Rightarrow M$ is Art and Noeth. $\square$

<u>Theorem</u> (Schrier refinement thm)

Let $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_s = 0$ (1) and $M = N_0 \supseteq N_1 \supseteq \dots \supseteq N_r = 0$ (2)

be 2 series for M. Then (1) and (2) have equivalant refinements.

<u>Proof:</u> Use (2) to refine (1). Replace $M_i \supseteq M_{i+1}$ by

$$M_i = (M_i \cap N_0) + M_{i+1} \supseteq (M_i \cap N_1) + M_{i+1} \supseteq \dots \supseteq (M_i \cap N_r) + M_{i+1} = M_{i+1}.$$

If $A_{i,j} = (M_i \cap N_j) + M_{i+1}$ then $A_{i,0} = M_i = A_{i-1,r}$

$\| \qquad \|$

$(M_i \cap M) + M_{i+1}$

$$M = A_{00} \supseteq A_{0,1} \supseteq \_ \supseteq A_{0,r} = A_{1,0} \supseteq A_{1,1} \supseteq \dots$$

Similarly, we can refine (2) using (1).

Result follows from Zassenhaus Lemma

$$\frac{M_{i+1} + (M_i \cap N_j)}{M_{i+1} + (M_i \cap N_{j+1})} \cong \frac{N_{j+1} + (M_i \cap N_j)}{N_{j+1} + (M_{i+1} \cap N_j)} \qquad \square$$

<u>Thm</u> (Zassenhaus Lemma)

Assume $\begin{matrix} A \subseteq A' \\ B \subseteq B' \end{matrix}$ are submods of ${}_R M$. Then $\dfrac{A + (A' \cap B')}{A + (A' \cap B)} \cong \dfrac{B + (A' \cap B')}{B + (A \cap B')}$

<u>Pf:</u> Define $\varphi : A + (A' \cap B') \longrightarrow \dfrac{A' \cap B'}{(A \cap B') + (A' \cap B)} = \dfrac{A' \cap B'}{E}.$

$a + x \mapsto x + E \quad \forall a \in A,\ x \in A' \cap B'$

<u>φ is well defined:</u> If $a + x = a_1 + x_1$, $a, a_1 \in A$, $x, x_1 \in A' \cap B'$.

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$\Rightarrow x - x' = a - a_1 \in A \cap (A' \cap B') = A \cap B' \subseteq E.$

$\Rightarrow \varphi$ well def'd.

Clearly onto.

If $a + x \in \operatorname{Ker} \varphi \quad \Rightarrow a + x \in A + E = A + [\underbrace{\cancel{(A' \cap B')}}_{\subset A} + (A' \cap B)]$

$\Rightarrow a + x \in A + (A' \cap B) \quad \Rightarrow A + (A' \cap B) = \operatorname{Ker} \varphi.$

1st isom. $\Rightarrow$ $$\frac{A + (A' \cap B')}{A + (A' \cap B)} \cong \frac{A' \cap B'}{(A' \cap B) + A \cap B'}.$$

Similarly, $$\frac{B + (A' \cap B')}{B + (A \cap B')} \cong \frac{A' \cap B'}{E}. \qquad \square$$

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Recall: 2 series $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_s = 0$
$M = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_t = 0$

have equiv. refinements

$A_{i,j} = (M_i \cap N_j) + M_{i+1}$
$B_{j,i} = (M_i \cap N_j) + N_{j+1}$

We saw $\dfrac{A_{ij}}{A_{i,j+1}} \cong \dfrac{B_{j,i}}{B_{j,i+1}}$ by Z-lemma. We got a common refinement

Thm: (Jordan-Hölder)

If ${}_RM$ has a composition series, the length of ~~th~~ this series is (!) and the factor modules are (!) counting multiplicity.

Proof: Let $M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_s = 0$
$M = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_t = 0$ be 2 comp. series.

We know these have equiv. refinements. Since M_i/M_{i+1}, N_j/N_{j+1} are
↳ factors in bij. correspondence

all simple, we can only refine by repeating modules (adding ${}_R0$ factor modules) $\Rightarrow s = t$ and factors are same counting multiplicity. $\square$

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Exer: If $_RR$ is Art. then $_RR$ has a comp. series and every simple R-mod occurs as a factor. → HW 9#$\boxed{2}$

Def: If $_RM$ has comp. series $M=M_0 \supset \ldots \supset M_r = 0$, then r is the composition length and $\{ M_i/M_{i+1} \mid i \geq 0 \}$ are comp. factors.

Ex: $n = p_1^{k_1} \cdots p_r^{k_r} \in \mathbb{N}$, $p_1, \ldots, p_r$ dist. primes, $k_i \geq 1$.

comp. series for $\mathbb{Z}_n$ is $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}} \supseteq \frac{q_1\mathbb{Z}}{n\mathbb{Z}} \supseteq \frac{q_1q_2\mathbb{Z}}{n\mathbb{Z}} \supseteq \ldots \supseteq \frac{q_1q_2\cdots q_t\mathbb{Z}}{n\mathbb{Z}} = 0$

where $t = k_1 + k_2 + \ldots + k_r$ and $q_1, \ldots, q_t$ all primes with p_i occuring exactly k_i times,

$$\frac{\frac{q_1\cdots q_i\mathbb{Z}}{n\mathbb{Z}}}{\frac{q_1\cdots q_{i+1}\mathbb{Z}}{n\mathbb{Z}}} \cong \mathbb{Z}_{q_{i+1}}$$

We recover (!) of prime factorization in $\mathbb{Z}$.

Jacobson density thm

Situation: $_RS$ simple module. $D = \mathrm{End}_R(S)$ is a div. ring.

$\Rightarrow$ S is a left D-v.s. via $f \cdot x = f(x)$ $\forall f \in D, x \in S$.

"**Most**" lin. alg. results hold for a v.s. over a division ring.

Prop: Every v.s. over a div. ring D has a basis.

Proof: Same! (ZORN!)

Thm: Let $_RS$ be simple and $D = \mathrm{End}_R(S)$. If $x_1, \ldots, x_n \in S$ are lin. independent over D and $y_1, \ldots, y_n \in S$ then $\exists$ $r \in R$ s.t. $rx_1 = y_1, \ldots, rx_n = y_n$. (Jacobson density thm (JDT))

Note: $R \to \mathrm{End}_D(S)$
$r \mapsto f$ with $f(x) = rx$ is a ring hom.

Lemma: Let $_RS$ be simple, $x_1, \ldots, x_m \in S$, $z \in S$. Let $L = \mathrm{ann}_R(x_1, \ldots, x_m) \leq R$. If $Lz = 0$, then $z \in Dx_1 + \ldots + Dx_m$, ~~then~~ where $D = \mathrm{End}_R(S)$, a div. ring.

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Proof: Use induction on m.

$m=1$: Define $f: S \to S$, $rx_1 \mapsto rz$. If $rx_1 = r'x_1 \Rightarrow (r-r')x_1 = 0 \Rightarrow r-r' \in L = \mathrm{ann}_R(x_1)$

$\Rightarrow (r-r')z = 0 \Rightarrow rz = r'z \Rightarrow f$ is well-def'd.

f is an R-hom. and $f(x_1) = f(1\cdot x_1) = 1z = z \Rightarrow z \in Dx_1$.

$f\cdot x_1 = f(x_1)$, $f \in D = \mathrm{End}_R(S)$

Assume true for $m-1$.

Let $I = \mathrm{ann}_R(x_1, \dots, x_{m-1})$

If $Ix_m = 0 \Rightarrow x_m \in Dx_1 + \dots + Dx_{m-1} \subseteq Dx_1 + \dots + Dx_m$.

If $Ix_m \neq 0 \Rightarrow Ix_m = S$. Def. $f: S \to S$, $rx_m \mapsto rz$ $\forall r \in I$.

If $rx_m = r'x_m \Rightarrow (r-r')x_m = 0 \Rightarrow r-r' \in I \cap \mathrm{ann}_R(x_m) = L$.

$\Rightarrow (r-r')z = 0 \Rightarrow rz = r'z$. $\Rightarrow f$ is well-def'd R-hom.

Consider $z - f(x_m)$.

If $r \in I$, then $r(z - f(x_m)) = rz - f(rx_m) = rz - rz = 0$.

$\Rightarrow I(z - f(x_m)) = 0$.

By induction, $z - f(x_m) \in Dx_1 + \dots + Dx_{m-1} \Rightarrow z - f(x_m) = f_1x_1 + \dots + f_{m-1}x_{m-1}$

$\Rightarrow z = f_1x_1 + \dots + f_{m-1}x_{m-1} + \underbrace{f(x_m)}_{= fx_m} \in Dx_1 + \dots + Dx_m$ $\square$

Proof of J.D.T.: $n=1$ Exercise. $x_1 \in S$ lin. indep. $\Rightarrow x_1 \neq 0$, $y_1 \in S$. ~~Claim: $\exists r \in R$, $rx_1 = y_1$.~~ ~~If $y_1 = 0$ take $r=0$. If $y_1 \neq 0$~~, $Rx_1 = S \Rightarrow y_1 \in Rx_1$

By ind. $\exists$ $t \in R$ s.t. $tx_1 = y_1, \dots, tx_{n-1} = y_{n-1}$.

Let $L = \mathrm{ann}_R(x_1, \dots, x_{n-1})$. Since $x_1, \dots, x_n$ lin. ~~dep.~~ indep. / D $Lx_n \neq 0$. by lemm

$\Rightarrow Lx_n = S$

$\exists$ $u \in L$ s.t. $ux_n = y_n - tx_n$. Let $r = t+u$.

$\Rightarrow rx_i = tx_i + \underbrace{ux_i}_{=0} = tx_i = y_i$ if $1 \leq i \leq n-1$, since $u \in L = \mathrm{ann}_R(x_1, \dots, x_{n-1})$.

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$rx_n = (t+u)x_n = tx_n + ux_n = tx_n + y_n - tx_n = y_n.$ □

Nov 1

Thm: (Maschke's thm) (1898)

Let G be a finite group. Then $k[G]$ is s.s. iff char $k \nmid |G|$.

Proof: Assume that char $k \mid |G|$. Let $\hat{G} = \sum_{g\in G} g = \sum_{g\in G} 1_k \cdot g \in k[G]$.

If $x \in G$ $\quad x\hat{G} = \sum_{g\in G} xg = \hat{G}$

$$\Rightarrow \left(\sum_{x\in G} \alpha_x x\right)\hat{G} = \left(\sum_{x\in G} \alpha_x\right)\hat{G} \in k\hat{G}$$
$$= \hat{G}\left(\sum_{x\in G} \alpha_x x\right)$$

$\Rightarrow k\hat{G}$ is a 1-dim. ideal of $k[G]$.

$$\hat{G}\hat{G} = \left(\sum_{x\in G} x\right)\hat{G} = \sum_{x\in G}\hat{G} = \cancel{\sum_{x\in G}}\, |G|\hat{G} \overset{\text{char } k \mid |G|}{=} 0 \quad \text{since char } k \mid |G|$$

$\Rightarrow |G| = 0$ in k

$\Rightarrow \quad 0 \neq k\hat{G} \subseteq J(k[G])$ since $(k\hat{G})^2 = 0 \overset{\text{Artin-Wed.}}{\Rightarrow} k[G]$ not semisimple

[s.s. ⟺ left Art & J(R)=0]

Conversely (Pf 1)

Assume char $k \nmid |G| \Rightarrow |G|^{-1} \in k$. Suppose $J(k[G]) \neq 0$.

$\Rightarrow \exists\, 0 \neq t \in J(k[G])$, $t = \sum_{g\in G} \alpha_g g$.

Replacing t by tx^{-1}, some $x \in G$, if necessary, we can

assume $\alpha_1 \neq 0$. $\quad [t = 0\cdot 1 + \alpha_x x + \ldots \Rightarrow tx^{-1} = \alpha_x \cdot 1 + \ldots]$

Def. $L: k[G] \to \mathrm{End}_k(k[G])$.

$$w \mapsto L_w \quad \text{where} \quad L_w(v) = wv.$$

$$L(t) = L\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \alpha_g L_g$$

Compute trace: (trace of basis repr. as matrix)

Nov 1

$$tr(L(t)) = \sum_{g \in G} \alpha_g \, tr(L_g).$$

If $g \neq 1$ then L_g permutes the basis G with no fixed points.

$\Rightarrow tr(L_g) = 0$ if $g \neq 1$.

$\Rightarrow tr(L(t)) = \alpha_{g_1} tr(L_1) = \alpha_{g_1} |G| \neq 0$.

But $k[G]$ is left Art. $\Rightarrow J(k[G])$ nilpotent $\Rightarrow L_t$ nilpotent.
since $t \in J(k[G])$)

$\Rightarrow tr(L_t) = 0$ C!

tr = product of eigenvalues . eigenval. of nilpot. are 0

Proof 2: Assume $char(k) \nmid |G|$. Show every $k[G]$-mod. is completely reducible.

great exam problem

Assume $W \subseteq V$ $k[G]$-modules. We can write $V = W \oplus U$ as k-v.s.
$k \subset k[G] \to$ k-v.s.

1.

Let $\pi: V \to W$ be the projection rel. to this composition.

Define $\varphi: V \to W$
$$v \mapsto \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gv).$$

$\varphi: V \to W$ is a k-linear map.

Let $x \in G$.
$$\varphi(xv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gxv)$$
$$= x \left[\frac{1}{|G|} \sum_{g \in G} (gx)^{-1} \pi(gxv) \right]$$
$$= x \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi(hv) = x \varphi(v).$$

$\Rightarrow \varphi: V \to W$ is a $k[G]$-hom.

If $w \in W \Rightarrow gw \in W \; \forall g \in G$ $\Rightarrow \varphi(w) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(\underbrace{gw}_{\in W})$

$= \frac{1}{|G|} \sum_{g \in G} g^{-1} g w = w.$ $\Rightarrow \varphi|_W = I|_W$ $\xRightarrow[\text{Midterm Part 1,1}]{}$ $W | V$ as $k[G]$-module

Nov 1

Note: If $\alpha : R \to S$ is a ring hom, then ${}_S M$ becomes ${}_R M$ via $r \cdot m = \alpha(r) \cdot m$. $\forall m \in M, r \in R$.

Ex: $\beta : G \to H$ group hom.

then $\beta : k[G] \to k[H]$
$g \mapsto \beta(g)$ is a ring hom.

We have $\delta : G \to G \times G$
$g \mapsto (g,g)$

$$k[G \times G] \cong k[G] \otimes k[G]$$

If ${}_{k[G]}V$, ${}_{k[G]}W$ then $V \otimes W$ is a $k[G] \otimes k[G]$-module via $(g_1 \otimes g_2)(v \otimes w) = g_1 v \otimes g_2 w$

Now, $\Rightarrow$ $V \otimes W$ is a $k[G]$-module via δ.

$$g(v \otimes w) = (g \otimes g)(v \otimes w) = gv \otimes gw.$$

Recall: $k[G]$ group alg.

Nov 3

$\tau : k[G] \to k[G]^{op}$

$\sum_{g \in G} \alpha_g g \to \sum_{g \in G} \alpha_g g^{-1}$ is an isom.

$$\tau(gh) = (gh)^{-1} = h^{-1} g^{-1} = \tau(g) \underset{\text{op mult.}}{\underset{\uparrow}{*}} \tau(h)$$

① If ${}_{k[G]}V$ consider $V^* = \mathrm{Hom}_k[{}_{k[G]}V_k, {}_k k_k]$

V^* is a right $k[G]$-module.

In fact, V^* is a left $k[G]$-mod via $(gf)(v) = f(g^{-1}v)$

② If V, W $k[G]$-modules then V^*, W are $k[G]$-modules, then $W \otimes V^*$ is a $k[G]$-module

Nov 3

$$\Big[\ g : W \times V^* \to W \otimes V^* \qquad (w, v) \mapsto g(w) \otimes g(v) \ \text{k-bal.}$$
this gives action on $W \otimes V\ \Big]$

③ If $\rho \in \mathrm{Hom}_k(V, W)$ define $g \cdot \rho$ by $(g \cdot \rho)(v) = g\,\rho(g^{-1} v)$, $g \in G$

check $g \cdot \rho \in \mathrm{Hom}_k(V, W)$ and $h \cdot (g \cdot \rho) = (hg) \cdot \rho$.

$\Rightarrow \mathrm{Hom}_k(V, W)$ is a $k[G]$-module.

④ V, W $\underbrace{\text{fin. dim.}}_{\text{over } k}$, $k[G]$-modules.

We saw
$$W \otimes V^* \to \mathrm{Hom}_k(V, W)$$
$$w \otimes f \mapsto \widehat{w \otimes f}\,[: v \mapsto f(v) w]$$
is an isom. of v.s. $/k$

<u>Exercise:</u> Check this is an isom. of $k[G]$-modules.

[need only check $g\widehat{(w \otimes f)} = g(\widehat{w \otimes f}).$]

<u>Def:</u> If G is a group, a representation of G over k is a group homomorphism $\Phi : G \to GL_k(V)$, V k v.s.

If $\underset{n}{\dim_k V} < \infty$, we have $\Phi : G \to GL_n(k)$.

<u>Note:</u> V becomes a $k[G]$-module via
$$\Big(\sum_{g \in G} \alpha_g g\Big) v = \sum \alpha_g\, \Phi(g)(v)$$

But if we start with $k[G]$ V we get rep. $\Phi : G \to GL_k(V)$
$$g \mapsto \Phi(g)\,[: v \mapsto gv]$$

If $\Phi : G \to GL_n(k)$, $\Psi : G \to GL_m(k)$ are 2 representations of G, then the Kronecker product of Φ and Ψ is

Nov 3

$(\Phi \odot \Psi)(g) = \Phi(g) \odot \Psi(g) \in GL_{mn}(k)$.

This is precisely same as taking tensor product of 2 modules. ∎

Assume M is a s.s. R-module $\Rightarrow JM = 0$ $\Rightarrow M$ is an $\frac{R}{J}$-module,

$J = J(R)$

If we assume R is left art, then $R/J(R)$ is a s.s. ring

W.L.O.G. $R = M_{n_1}(D_1) \times \ldots \times M_{n_t}(D_t)$ ($= R/J(R)$ really (assume $J(R) = 0$))

We get t central idempotents $e_1 = (I_{n_1}, 0, \ldots, 0)$

$\vdots$

$e_t = (0, \ldots, 0, I_{n_t})$

$M = M_1 \oplus M_2 \oplus \ldots \oplus M_t$, where $M_i = e_i M$. $[1_R = e_1 + \ldots + e_t]$

Exercise: $M_i = \sum_{\substack{S_i \subseteq M \\ S_i \cong (!) \text{ simple mod}/M_{n_i}(D_i)}} S_i$ homog. component (homogeneous)

Ex: $k[G]$, $|G| < \infty$, char $k \nmid |G|$. $(0 \neq |G| \in k)$

Let $e = \frac{1}{|G|} \hat{G} = \frac{1}{|G|} \sum_{g \in G} g$ $\Rightarrow e^2 = \frac{1}{|G|^2} \hat{G}\hat{G} \underset{\substack{\uparrow \\ \text{last time}}}{=} \frac{1}{|G|^2} |G| \hat{G} = e$.

Consider eV, where V is a $k[G]$-module.

Prop: $eV = \{ v \in V \mid gv = v \;\; \forall g \in G \} \overset{\text{def}}{=} V^G$.

Proof: If $v \in V^G \Rightarrow ev = \frac{1}{|G|} \sum_g \underbrace{gv}_{=v} = \frac{1}{|G|} |G| v = v$ $\Rightarrow v = ev \in eV$

$\Rightarrow V^G \subseteq eV$.

If $v \in V$, $x \in G$. $x(ev) = \frac{1}{|G|} \sum_{g \in G} (xg)v = \frac{1}{|G|} \sum_{h \in G} hv = ev$.

$\Rightarrow ev \in V^G \;\; \forall ev \in eV \Rightarrow eV \subseteq V^G$.

Exercise: V, W $k[G]$-modules, fin. dim.

$W \otimes V^* \xrightarrow{\cong} \mathrm{Hom}_k(V, W)$ Then

Nov 3

$\mathrm{Hom}_k(V,W)^G = \mathrm{Hom}_{k[G]}(V,W)$. (see pf. 2 of Maschke's thm)

Krull - Schmidt Theorem

Lemma: If ${}_RM$ is Art. and Noeth. then $M = M_1 \oplus \_ \oplus M_t$ where each M_i is indecomposible.

Def: ${}_RA$ is decomposible if $A = A_1 \oplus A_2$, A_1, A_2 nonzero.

Otherwise indecomposible.

Proof: Choose B max'l in set of proper submodules that are direct summands of M. [0 is a direct summand & $M \neq 0$ Noeth.]

Now, $M = A_1 \oplus B_1$ some $A_1 \subseteq M$. Suppose A_1 is not indecomposable.

Then $A_1 = X \oplus Y$. Now $M = A_1 \oplus B_1 = X \oplus \underbrace{Y \oplus B_1}$

But $B_1 \subsetneq Y \oplus B_1$ and $Y \oplus B_1$ is a summand of M. C!

$\Rightarrow A_1$ is indecomp.

Now, B_1 is Art/Noeth. $\Rightarrow B_1 = A_2 \oplus B_2$, A_2 indecomp by same arg.

We get $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ Process must stop. This happens only when $B_r = 0 \Rightarrow M = A_1 \oplus A_2 \oplus \_ \oplus A_r$. ▯

Note: If M is only Noeth. then $A_1 \subsetneq A_2 \subsetneq \dots$, so process only stops when $M = A_1 \oplus A_2 \oplus \_ \oplus A_r$.

Nov 6

Def: If ${}_RM$ has comp. series $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_r = 0$, the simple modules $\frac{M_i}{M_{i+1}}$, $0 \le i \le r$, are the composition factors of M (counting multiplicity).

Nov 6

Ex: $\underset{M_2}{\underset{\shortparallel}{\mathbb{Z}_4}} \supseteq \underset{M_1}{\underset{\shortparallel}{\frac{2\mathbb{Z}}{4\mathbb{Z}}}} \supseteq \underset{M_0}{\underset{\shortparallel}{0}}$ has $\mathbb{Z}_2$ as a comp. factor twice.

Exercise: If $_RR$ is Art. then $_RR$ is Noetherian (recall Hopkins's thm.).

Thus $_RR$ has a comp. series. Show every simple R-mod occurs as a composition factor. → HW 9, #2

$H, K \leq G$. $HK \leq G \iff HK = KH$.

Notation problem

$A, B \subseteq M$ R-mods $M = A \oplus B$. $B \cong M/A$ is more like $\underbrace{\text{"}M-A\text{"}}_{\text{classical notation! Jacobson}}$

$M-A$ is also $\{m-a \mid m \in M, a \in A\} = M$.

3rd isom. thm $A \subseteq B \subseteq M$ R-mod. $(M-A)-(B-A) \cong M-B$

HW 8, Ex 1: $\prod_{m \in M} \alpha_m$ does not work

Lemma: $_RM$ Noeth. $\Rightarrow M = M_1 \oplus M_2 \oplus \dots \oplus M_r$ where M_i indecomposable.

Note: If $M = 0$, $r = 0$.

Def: R is local if $R/J(R)$ is a division ring.

Ex: $R = \{\frac{a}{b} \mid b \text{ odd}\} \subseteq \mathbb{Q}$. $m = 2R$ is the unique max'l ideal of R.

$\frac{R}{m} \cong \mathbb{Z}_2$ div. ring (field)

Prop: Ring R. T.F.A.E.

(1) R local

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(2) $R \setminus U(R) \subseteq J(R)$

(3) $R \setminus U(R)$ is closed under +.

<u>Pf:</u> <u>(1) ⇒ (2):</u>

Let $a \in R \setminus U(R)$. Suppose $a \notin J(R)$. $\Rightarrow \bar{a} \in U\left(\frac{R}{J(R)}\right)$ ← division ring (all nonzero el. are units)

$\Rightarrow \exists b \in R$ s.t. $\bar{a}\bar{b} = \bar{b}\bar{a} = \bar{1} = 1_{R/J(R)}$ $\Rightarrow ab = 1+x$, $ba = 1+y$, $x, y \in J(R)$

But $1+x$, $1+y \in U(R)$. $\Rightarrow ab(1+x)^{-1} = 1$, $(1+y)^{-1}ba = 1$.

<u>(2)⇒(3):</u> $J(R)$ contains no units. $\Rightarrow J(R) \subseteq R \setminus U(R)$.

(2) implies $R \setminus U(R) = J(R)$. $\Rightarrow R \setminus U(R) \triangleleft R$ and hence is closed under +.

<u>(3)⇒(1):</u> ~~Let $a \in R \setminus U(R)$. $\Rightarrow 1+ax, 1+ya \in U(R)$ $\forall x, y \in R$.~~

~~[If $\left.\begin{matrix}1+ax \\ 1+ya\end{matrix}\right] \notin U(R)$ $\Rightarrow$ $1 = (1+ax) + (-ax)$ ~~$\notin U(R)$~~~~

~~$1 = (1+ya) + (-ya)$ ~~$\notin U(R)$~~~~

~~If both $1+ax$, $1+ya$ are not units then a is a unit.~~

~~$\Rightarrow 1 \notin U(R)$ by (3) C!~~

~~$\Rightarrow a \notin J(R)$.~~

~~Let $a \in R \setminus U(R)$ $\Rightarrow 1+ax$, $1+ya \in U(R)$ $\forall x, y \in R$.~~ [(3)⇒(2) ...]

~~$\Rightarrow ax \in U(R)$ and $ya \in U(R)$~~

~~[If $ax \notin U(R)$ $\Rightarrow$ $1 = \underbrace{1+ax}_{\in U(R)} + \underbrace{(-ax)}_{\notin U(R)} \in U(R)$~~

Let $a \in R \setminus J(R)$. $\Rightarrow 1+ax$, $1+ya \notin U(R)$, some x, y.

If $ax \notin U(R)$, then $1 = 1+ax + (-ax) \notin U(R)$ by (3). C! $\Rightarrow ax \in U(R)$.

$(ax)y = 1$ $\Rightarrow a(xy) = 1$

Similarly, $ya \in U(R)$. $\Rightarrow ax$ has a right inverse. $\Rightarrow a$ has right inverse. ya has a left inv. $\Rightarrow$ a left inverse. $\Rightarrow a \in U(R)$

COMPLETION OF PROOF OF MON 11/6 PROPOSITION!

<u>PROPOSITION</u>: IF R IS A RING, TFAE.

ⓐ R IS LOCAL
ⓑ $R \setminus \mathcal{U}(R) \subseteq J(R)$
ⓒ $R \setminus \mathcal{U}(R)$ IS CLOSED UNDER $+$.

Proof of ⓒ $\Rightarrow$ (a)

Let $a \in R \setminus J(R)$, SO THAT $a + J(R) \in \frac{R}{J(R)}$ IS NONZERO. WE NEED TO SHOW $\bar{a} = a + J(R)$ IS A UNIT IN $\frac{R}{J(R)}$.

$a \notin J(R)$ SO THAT $1 + ax$, $1 + ya$ ARE NOT UNITS IN R, FOR SOME $x, y \in R$.

IF ax WERE <u>NOT</u> A UNIT IN R, THEN $1 = (1 + ax) + (-ax)$ IS NOT A UNIT BY ⓒ. <!
THUS $ax \in \mathcal{U}(R)$. $\Rightarrow$ a HAS A RIGHT INVERSE IN R.

SIMILARLY $ya \in \mathcal{U}(R)$ SO THAT a HAS A LEFT INVERSE IN R.
$\Rightarrow a \in \mathcal{U}(R)$
$\overset{\otimes}{\Rightarrow} \bar{a} \in \mathcal{U}\left(\frac{R}{J(R)}\right)$. ∎

$\otimes$ a unit in R $\Leftrightarrow$ $a + J(R) \in \mathcal{U}\left(\frac{R}{J(R)}\right)$.
Check!

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$\Rightarrow \bar{a} \in R/_{J(R)}$ is a unit. $\Rightarrow R/_{J(R)}$ div. ring □

Lemma: $_RM$. $\varphi: M \to M$ R-hom. If either

(1) M is Noeth and φ is onto

(2) M is Art. and φ is 1-1

then φ is an isomorphism.

Ex: $R = k$ field. $_kM$ fin. dim. $\varphi: M \to M$

φ is 1-1 $\Leftrightarrow$ φ is onto $\Leftrightarrow$ φ is an isom.

Proof: (1) $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ker}(\varphi^2) \subseteq \ldots$

If $\mathrm{Ker}\,\varphi \neq 0$ $\exists\, 0 \neq x \in \mathrm{Ker}\,\varphi$ $\Rightarrow \exists\, y \in M$ s.t. $\varphi(y) = x$ since φ is onto.

Now, $\varphi^2(y) = 0$ but $\varphi(y) \neq 0$.

$y \in \ker \varphi^2$, $y \in \mathrm{Ker}\,\varphi$

$\Rightarrow \mathrm{Ker}\,\varphi \subsetneq \mathrm{Ker}\,\varphi^2$

Sim. $\mathrm{Ker}\,\varphi^n \subsetneq \mathrm{Ker}\,\varphi^{n+1}$. This contradicts $_RM$ is Noeth.

(2) Exercise. $\varphi(M) \supseteq \varphi^2(M) \supseteq \ldots$

Nov 8

Def. let k be a commutative ring.

A k-algebra R is (1) ring
(2) k-module s.t.

$\alpha(ab) = (\alpha a)b = a(\alpha b)$ $\forall a, b \in R, \alpha \in k$

[Note: $1_k a = a$ $\forall a \in R$]

Ex: $k = \mathbb{Z}$. Every ring is a k-alg. $ma = \begin{cases} a + \ldots + a & (m \text{ times}) \text{ if } m \geq 0 \\ -a + \ldots + (-a) & (-) \text{ if } m < 0 \end{cases}$

or $(R, +)$ is an abelian group, so R is a $\mathbb{Z}$-module.

Ex: $k[G]$ k field
free k-mod w/ basis G.

Ex: Recall $Z(R) = \{ a \in R \mid ab = ba \ \forall b \in R \}$

Nov 8

Note $1 \in Z(R)$ and $Z(R)$ is a commutative subring of R.

Take $k = Z(R)$ $\alpha \cdot a = \alpha a \ \forall \alpha \in k, a \in R$.

Then R is a k-algebra.

Why bother? often we put cond. on R k and ${}_kR$.

Ex: k Art. ${}_kR$ is fin. gen. $\Rightarrow R$ is Art.

Finally If R is a k-alg. we get
$$\varphi: k \to R \qquad \text{ring hom.}$$
$$\alpha \mapsto \alpha \cdot 1_R$$

$\alpha a = \alpha(1a) = (\alpha 1)a = \varphi(\alpha)a$, we can replace k by $\varphi(k)$ and view R as a $\varphi(k)$-alg.

Ex: $R = \mathbb{Z}_6$ is a $\mathbb{Z}$-alg but $\varphi: \mathbb{Z} \to Z(\mathbb{Z}_6)$ is not 1-1.

Ex: $k[G]$ k-alg. k field
$$\varphi. \ k \to k[G]$$
$$\alpha \mapsto \alpha \cdot 1_G$$

Ex: $R = \mathrm{End}_k(V)$ where k is a field, V has basis $\{v_1, v_2, -\}$, countably infinite basis. $T, S \in R$. $T(v_i) = v_{i+1} \ \forall i \geq 1$

$$S(v_i) = \begin{cases} v_{i-1} & \text{if } i > 1 \\ 0 & \text{if } i = 1 \end{cases}$$

$(ST)(v_i) = S(v_{i+1}) = v_i \ \forall i \geq 1. \Rightarrow ST = 1_R$.

$TS \neq 1_R$ since $TS(v_1) = T(0) = 0$.

Note: If $a, b \in R$ with $ab = 1$ then $e = ba$ is an idempotent.

$e^2 = (ba)(ba) = b(ab)a = b1a = ba = e$.

Ex: F is free group on $X = \{x_i \mid i \in I\}$.

$w(k[F]) = \{\sum_t \alpha_t g_t \mid \sum_t \alpha_t = 0\} \trianglelefteq k[F]$.

$w(k[F])$ is a free $k[F]$-module with basis

Nov 8

$\{x_i - 1 \mid i \in I\}$.

<u>Ex:</u> $R = M_\infty(k)_{fin}$, the set of countably infinite matrices where each row and column have finitely many nonzero entries.

$\begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix} \in R$ R is a ring with identity

AB Each column is a lin. comb. of col's of B with coeffs coming from a row of A. Each row of A has only fin. many nonzero entries. $\Rightarrow$ Each column of AB has only fin. many entries.

Similarly, each row has fin. many nonzero entries.

I claim $R \oplus R \cong R$, as left R-modules.

$\varphi: R \oplus R \to R$

$(A, B) \mapsto [A_1, B_1, A_2, B_2, \_]$ where $A_1, A_2, \_$ are col's of A in order
$B_1, B_2, \_$ -"- B -"-

φ is bijective and an R-module-hom.

If $r \in R$ $rA = [rA_1, rA_2, \ldots]$ $\Rightarrow \varphi(r(A,B)) = r\varphi(A,B)$.

Now, $R \cong R \oplus R \cong R \oplus (R \oplus R) = R^3$, $R \cong R^n$ $\forall n \geq 1$

Is $\underset{\uparrow}{\bigoplus_{i=1}^{\infty} R} \cong \underset{\nwarrow}{R}$?

not fin. gen. fin. gen. as R-mod (by 1)

<u>Recall:</u> R is local if $R/J(R)$ is a div. ring.

<u>Prop:</u> R local $\Leftrightarrow$ $R \setminus U(R) \subseteq J(R)$ $\Leftrightarrow$ $R \setminus U(R)$ is closed under $+$

<u>Prop:</u> Let R be ring where every element is a unit or nilpotent.

Then R is local.

<u>Proof:</u> Let $a \in R$, not a unit, $a \neq 0$. Need $a \in J(R)$.

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We know $a^n = 0$, choose $n \geq 2$ minimal.

If ba is not nilpotent, for $b \in R$ $\Rightarrow$ ba is a unit.

$\Rightarrow 0 = ba^n = (ba)a^{n-1} \Rightarrow a^{n-1} = 0$ since ba is a unit. C!

$\Rightarrow ba$ is nilpotent $\Rightarrow Ra \leq R$ is a nil left ideal. $\Rightarrow Ra \leq J(R)$

$\Rightarrow a \in J(R)$. $\square$

<u>Recall:</u>

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<u>Lemma:</u> $\varphi: M \to N$ R-hom.

If either
1) φ onto, M Noeth
2) φ 1-1, M Art.

$\Rightarrow \varphi$ is an isom.

<u>Theorem</u> (Fitting's lemma):

Assume M is Artinian and Noeth and $\varphi \in \operatorname{End}_R(M)$. Then $M = P \oplus Q$ s.t.

(a) $\varphi(P) \leq P$, $\varphi(Q) \leq Q$

(b) $\varphi|_P : P \to P$ is an isom.

(c) $\varphi|_Q : Q \to Q$ is nilpotent.

<u>Proof:</u> $\exists m > 0$ s.t. $\operatorname{Im}(\varphi^t) = \operatorname{Im}(\varphi^m)$ $\forall t \geq m$

$\operatorname{Im}\varphi \geq \operatorname{Im}\varphi^2 \geq \ldots$ stops since Art

$\operatorname{Ker}\varphi \leq \operatorname{Ker}\varphi^2 \leq \ldots$ stops since Noeth $\operatorname{Ker}(\varphi^t) = \operatorname{Ker}(\varphi^m)$ $\forall t \geq m$.

Let $\cancel{p}\, P = \varphi^m(M)$
$Q = \operatorname{Ker}(\varphi^m)$

$\varphi(P) = \varphi(\varphi^m(M)) = \varphi^{m+1}(\cancel{P}M) = P$

$\varphi(Q) = \varphi(\operatorname{Ker}(\varphi^m)) = \varphi(\operatorname{Ker}(\varphi^{m+1})) \subseteq \operatorname{Ker}\varphi^m = Q$

$\left[x \in \operatorname{Ker}(\varphi^{m+1}) \Rightarrow \varphi^{m+1}(x) = 0 = \varphi^m(\varphi(x)) \Rightarrow \varphi(x) \in \operatorname{Ker}\varphi^m\right]$

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(b) $\varphi|_P : P \to P$ onto. P Noeth $\Rightarrow \varphi_P : P \to P$ is an isom. by lemma.

(c) $\varphi^m(Q) = 0$ by def'n.

It remains to show $M = P \oplus Q$.

$\varphi|_{P\cap Q}$ is nilpotent and 1-1. $\Rightarrow P \cap Q = 0$.

$A \subseteq f^{-1}(f(A))$

[**Aside:** $f \in \mathrm{Hom}_R(A,B)$ $\quad f^{-1}(f(A)) = A + \mathrm{Ker}(f)$] Put $f = \varphi^m$, $A = \varphi^m(M)$

$y \in f^{-1}(f(A)) \Rightarrow f(y) \in f(A) \Rightarrow f(y) = f(x)$, $x \in A \Rightarrow y - x \in \mathrm{Ker}(f) \Rightarrow y \in A + \mathrm{Ker}(f)$ □

$$M = (\varphi^m)^{-1}\underbrace{\varphi^m(M)}_{=\varphi^m(\varphi^m(M))} = \varphi^m(M) + \mathrm{Ker}\,\varphi^m = P + Q.$$

Lemma: If ${}_R M = M_1 \oplus M_2$, $M_1, M_2 \neq 0$. Then $\mathrm{End}_R(M)$ is not local.

Pf. $P_i : M \to M_i$, $\alpha_i : M_i \to M$ usual projection and inj.

$I_M = \alpha_1 P_1 + \alpha_2 P_2 = \pi_1 + \pi_2$ where $\pi_i = \alpha_i P_i$

$\mathrm{Ker}\,\pi_1 = M_2 \neq 0$ $\qquad \pi_1, \pi_2 \notin U(\mathrm{End}_R(M))$

$\mathrm{Ker}\,\pi_2 = M_1 \neq 0$ $\qquad$ but $\pi_1 + \pi_2 = I_M \in U(\mathrm{End}_R(M))$

$\Rightarrow \mathrm{End}_R(M)$ not local. □

Corollary: (To Fitting)

Assume ${}_R M$ is ~~indecomposable~~ Art. and Noeth. Then

${}_R M$ is indecomp. $\Leftrightarrow$ $\mathrm{End}_R(M)$ is local.

Pf: $\mathrm{End}_R(M)$ local $\Rightarrow$ ${}_R M$ is indecomp. always [Lemma]

Conversely, assume ${}_R M$ indecomp. If $\varphi \in \mathrm{End}_R(M)$, & $M = P \oplus Q$ as in Fitting. $\Rightarrow$ Either $P = M$ and $\varphi \in U(\mathrm{End}_R(M))$

or $Q = M$ and φ is nilpotent.

$\overset{\text{Prop}}{\Rightarrow}$ $\mathrm{End}_R(M)$ is local. □

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Ex: $R = \mathbb{Z}$, $M = \mathbb{Q}$, ${}_RM$ is not simple

$0 \neq \mathbb{Z} \subseteq_{\mathbb{Z}} \mathbb{Q}$.

If $A, B \subseteq {}_RM$ are nonzero, then $\exists\, 0 \neq \frac{a}{b} \in A$, $0 \neq \frac{c}{d} \in B$,

$a, b, c, d \in \mathbb{Z}$. $0 \neq (bc)\left(\frac{a}{b}\right) = (ad)\frac{c}{d} \in A \cap B$. $\Rightarrow A \cap B \neq 0$.

$\Rightarrow M$ indecomposable

Def: ${}_RM$ is uniform if

(1) $M \neq 0$

(2) If $0 \neq A \subseteq M$ is any submod, then A ess M.

Ex: Show ${}_{\mathbb{Z}}\mathbb{Q}$ is uniform.

M s.s., indecomp $\Rightarrow$ M simple

Ex: If ${}_RM = S_1 \oplus S_2 \oplus \dots \oplus S_r = W_1 \oplus W_2 \oplus \dots \oplus W_s$ where S_i and W_j are simple submodules, then $r = s$ and (after relabelling)

$S_1 \cong W_1, \dots, S_r \cong W_r$.

Exercise: why? Hint: Consider: $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_r = 0$

$M_1 = S_2 \oplus \dots \oplus S_r$, $M_2 = S_3 \oplus \dots \oplus S_r$ comp. series + use Jordan-Hölder

Thm (Krull-Schmidt (Remak-Azumaya))

~~Assume ${}_RM$ is Art and Noeth.~~ Given ${}_RM$. If $M = M_1 \oplus \dots \oplus M_r \cong N_1 \oplus \dots \oplus N_s$

where $\mathrm{End}_R(M_i)$ and $\mathrm{End}_R(N_j)$ are local for all i, j.

Then $r = s$ and after relabelling $M_1 \cong N_1, \dots, M_r \cong N_r$.

Caution: (L. Levy) Fix $n \in \mathbb{Z}^+$ $\exists\, {}_RM$ s.t. M can be written as a direct sum of

$2, 3, 4, \dots, n$ indecomp. modules.

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Lemma: Assume $M = M_1 \oplus M_2 = N_1 \oplus N_2$ and $\varphi \in \mathrm{End}_R(M)$ s.t.

$p_1' \varphi \alpha_1 : M_1 \to N_1$ is an isom, then $M_2 \cong N_2$. $\qquad p_j' : M \to N_j$ proj.

why?

$$\mathrm{End}_R(M) \underset{\text{Sep 22}}{\cong} \begin{bmatrix} \mathrm{Hom}(M_1, N_1) & \mathrm{Hom}(M_2, N_1) \\ \mathrm{Hom}(M_1, N_2) & \mathrm{Hom}(M_2, N_2) \end{bmatrix}$$

$m \in M = M_1 \oplus M_2$
$m = m_1 + m_2 \cong \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$. $\qquad \varphi \longrightarrow \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ where $\varphi_{11} = p_1' \varphi \alpha_1$

$\uparrow$
inv. element

$$\searrow \begin{bmatrix} \varphi_{11} & 0 \\ 0 & * \end{bmatrix} = \psi$$

$\to$ isom $M \to M$
$\to$ $* : M_2 \to N_2$ isom.

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$\varphi : M = M_1 \oplus M_2 \to N_1 \oplus N_2 = N$ an isom.

$$\left.\begin{array}{l} \alpha_i : M_i \to M_{\ell} \\ p_i : M \to M_i \\ \alpha_j' : N_j \to N \\ p_j' : N \to N_j \end{array}\right\} (*)$$

Canonical incl.'s/ proj.'s

$m = m_1 + m_2$
$\varphi(m_1 + m_2) = \varphi\alpha_1(m) + \varphi\alpha_2(m)$
etc, etc.

Recall: $M = M_1 \oplus M_2$ internal direct sum. We can think of $m = m_1 + m_2$ as $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$. We saw

$$\mathrm{End}_R(M) \cong \begin{bmatrix} \mathrm{Hom}(M_1, M_1) & \mathrm{Hom}(M_2, M_1) \\ \mathrm{Hom}(M_1, M_2) & \mathrm{Hom}(M_2, M_2) \end{bmatrix}$$

$$\varphi \mapsto \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \hat{\varphi}$$

$\varphi(m_1 + m_2)$ is $\hat{\varphi}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$

Prop: Let $\varphi : M_1 \oplus M_2 \to N_1 \oplus N_2$ be an isom, where $\varphi_{11} = p_1' \varphi \alpha_1 : M_1 \to N_1$ is an isom. where notation $(*)$ is used. Then $M_2 \cong N_2$.

Aside: a^{-1} exists $\qquad \begin{bmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & * \\ 0 & * \end{bmatrix}$ repeat w/ col's

to get $\begin{bmatrix} a & 0 \\ 0 & * \end{bmatrix}$

Pf: We view $\mathrm{Hom}_R(M, N)$ as

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$$\begin{bmatrix} \mathrm{Hom}(M_1, N_1) & \mathrm{Hom}(M_2, N_1) \\ \mathrm{Hom}(M_1, N_2) & \mathrm{Hom}(M_2, N_2) \end{bmatrix}$$

$$\varphi \leftrightarrow \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \quad \text{where } \varphi_{11} = p_1' \varphi \alpha_1 \text{ is an isom.}$$

We have $\rho = \begin{bmatrix} I_{N_1} & 0 \\ -\varphi_{21}\varphi_{11}^{-1} & I_{N_2} \end{bmatrix}$ autom. of $N_1 \oplus N_2$

inv: $\begin{bmatrix} I_{N_1} & 0 \\ \varphi_{21}\varphi_{11}^{-1} & I_{N_2} \end{bmatrix}$

$\psi = \begin{bmatrix} I_{M_1} & -\varphi_{11}^{-1}\varphi_{12} \\ 0 & I_{M_2} \end{bmatrix}$ autom. of $M_1 \oplus M_2$

$\begin{bmatrix} I_{M_1} & -\varphi_{11}^{-} \\ 0 & I \end{bmatrix}$

Now $\rho \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \psi = \begin{bmatrix} I_{N_1} & 0 \\ -\varphi_{21}\varphi_{11}^{-1} & I_{N_2} \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \psi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ 0 & -\varphi_{21}\varphi_{11}^{-1}\varphi_{12} + \varphi_{22} \end{bmatrix}$

$= \begin{bmatrix} \varphi_{11} & 0 \\ 0 & \omega \end{bmatrix} : M_1 \oplus M_2 \to N_1 \oplus N_2$ is an isom. since ρ, ψ, φ are.

$\Rightarrow \omega : M_2 \to N_2$ is an isom.

$\square$

<u>Thm</u> (K-S-R-A): Assume ${}_R M$ is an R-mod, $M = M_1 \oplus \dots \oplus M_r$ $= N_1 \oplus \dots \oplus N_s$ where M_i, N_j are submodules with $\mathrm{End}_R(M_i)$, $\mathrm{End}_R(N_j)$ local for all i, j. Then $r = s$ and after relabelling $M_1 \cong N_1, \dots, M_r \cong N_r$.

<u>Proof:</u> $p_i, p_j', \alpha_i, \alpha_j'$ are as usual.

$$I_M = \sum_{j=1}^{s} \alpha_j' p_j'$$

$$I_{M_1} = p_1 \alpha_1 = p_1 I_M \alpha_1 = \sum_{j=1}^{s} p_1 \alpha_j' p_j' \alpha_1$$

$\mathrm{End}_R(M_1)$ is local $\Rightarrow$ Nonunits closed under $+$.

$\Rightarrow p_1 \alpha_j' p_j' \alpha_1 : M_1 \to M_1$ is an isom. for some j.

Relabel, so $\underbrace{p_1 \alpha_1' p_1' \alpha_1}_{=\gamma}$ is an isom.

Let $\gamma = p_1 \alpha_1' p_1' \alpha_1 : M_1 \to M_1$ isom, Nov. 13

$\psi = \gamma^{-1} P_1 \alpha_1' : N_1 \to M_1$

and $\chi = P_1' \alpha_1 : M_1 \to N_1$.

γ is an isom $\Rightarrow \chi$ is 1-1.

$$\begin{tikzcd} (*) \quad 0 \arrow[r] & M_1 \arrow[r, "\chi"] & N_1 \arrow[l, dashed, bend left, "\psi"] \arrow[r] & N_1/M_1 \arrow[r] & 0 \end{tikzcd}$$

$\psi \circ \chi = \gamma^{-1} P_1 \alpha_1' P_1' \alpha_1 = \gamma^{-1}\gamma = I_{M_1}$

$\Rightarrow$ (*) splits $\Rightarrow N_1 = \chi(M_1) \oplus \mathrm{Ker}(\psi)$.

$\mathrm{End}_R(N_1)$ is local $\Rightarrow N_1$ indecomp.

$\chi(M_1) \neq 0$ since χ 1-1 $\Rightarrow \mathrm{Ker}\,\psi = 0 \Rightarrow \chi = P_1'\alpha_1$ is an isom.

Apply prev. prop. with $\varphi = I_M$, $M_2 \oplus \_ \oplus M_r$ for M_2
$N_2 \oplus \_ \oplus N_s$ for N_2

<u>Note</u> $P_1' \varphi \alpha_1 = P_1' \alpha_1 = \chi : M_1 \to N_1$ is an isom.

$\Rightarrow M_2 \oplus \_ \oplus M_r \cong N_2 \oplus \_ \oplus N_s$. WLOG this is an equality.

By induction on r, we get the result.

[trivial to start ind. at $r=0$, $M=0$
or $r=1$] □

<u>Thm</u> (K-S) If M is Art and Noeth. then $M = M_1 \oplus \_ \oplus M_r$ where each M_i is indecomp. In addition, r is (!) and ~~Same~~ so are isom. classes of $M_1, \_, M_r$ counting multiplicity. <u>Pf:</u> KS RA + Cor (To Fitting)

<u>Ex:</u> $R = \mathbb{Z}$ fin. irred. modules are $\mathbb{Z}_{p^n}$, p prime, $n \geq 1$.

Recall fund. thm of finite abelian groups:

$G \cong \mathbb{Z}_{p_1^{n_1}} \times \_ \times \mathbb{Z}_{p_t^{n_t}}$. $p_1, \_ p_t$ prime, $n_i \geq 1$ and decomp. is unique

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[Exer: infinite ab. gp Art. ?]

Exercise: If $_{\mathbb{Z}}A$ is infinite, could A be Art? [No]

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Exercise:

$$R = \begin{bmatrix} k & k & k & k \\ 0 & k & k & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{bmatrix} \qquad k \text{ field}$$

(1) Find all simple R-mods → HW9 #1

(2) ... comp. series for $\mathrm{Col}_4(k)$.

Exercise: If R left Art. show every simple module is a comp. factor of ${}_R R$. → HW9 #2

Localization

R commutative for a while.

Def: $S \subseteq R$ is multiplicatively closed if

(1) $1 \in S$

(2) $s, t \in S \Rightarrow s \cdot t \in S$

(3) $0 \notin S$

(1) not needed. Replace S by $S \cup \{1\}$.

Ex: ① R domain. $S = R \setminus 0$.

② $P \triangleleft R$ prime. $S = R \setminus p$. [① is ② with $p = 0 \triangleleft R$]

③ $R = \mathbb{Z}_{15}$, $S = \{\bar{5}, \bar{10}, \bar{1}\}$

④ R any commut. ring, $a \in R$ not nilpotent $\{a^n \mid n \geq 0\}$.

Def: R, $S \subseteq R$ mult. closed. A localization of R at S ~~is a~~ ~~ring~~ denoted $S^{-1}R$ (or R_S or RS^{-1}) is a ring and a ring hom $\varphi: R \to S^{-1}R$ s.t. (1) given any ring $f: R \to T$ s.t. $f(S) \subseteq U(T)$, $\exists!$ ring hom.

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$g : S^{-1}R \to T$ s.t. $g \circ \varphi = f$.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "!", "g"'] \\ & T \end{tikzcd}$$

(2) $\varphi(S) \subseteq U(S^{-1}R)$.

Ex: $R = \mathbb{Z}_{15}$, $S = \{\overline{1}, \overline{5}, \overline{10}\}$.

$\overline{3}\,\overline{5} = 0 \Rightarrow \varphi(\overline{3})\varphi(\overline{5}) = \varphi(\overline{15}) = \varphi(\overline{0}) = 0 \Rightarrow \varphi(\overline{3}) = 0$

since $\varphi(\overline{5}) \in U(S^{-1}R)$.

Consider $\varphi : \mathbb{Z}_{15} \to \mathbb{Z}_{\cancel{5}3} \cong \dfrac{\mathbb{Z}/_{15\mathbb{Z}}}{3\mathbb{Z}/_{15\mathbb{Z}}}$

$a + 15\mathbb{Z} \to a + 3\mathbb{Z}$

$\varphi(\overline{5}) = 2 + 3\mathbb{Z} \in U(\mathbb{Z}_3) \Rightarrow \varphi(\overline{5}) \in U(\mathbb{Z}_3)$.

Check $\mathbb{Z}_3 = S^{-1}R$ $R = \mathbb{Z}_{15}$
$S = \{\overline{1}, \overline{5}, \overline{10}\}$

If we take $\varphi' : R \to S^{-1}R$ $\exists!$ hom. $g : S^{-1}R \to \mathbb{Z}_3$

s.t.

$$\begin{tikzcd} R \arrow[r, "\varphi'"] \arrow[dr, "\varphi"'] & S^{-1}R \arrow[d, shift left, "g"] \\ & \mathbb{Z}_3 = T \arrow[u, dashed, shift left] \end{tikzcd}$$

Horrible notation: If $S = R \setminus p$, $p \triangleleft' R$ ← prime ideal

$S^{-1}R = R_S = R_p$
↖ ntn (dreadful!)

Thm: Given $S \subseteq R$ mult. closed. $S^{-1}R$ exists and is ! up to isom.

Proof: Let $W = \{(r,s) \mid r \in R, s \in S\}$. Def. relation $\sim$ on W by $(r,s) \sim (r',s')$ if $(rs' - r's)t = 0$, some $t \in S$.

Mindless drudge $\Rightarrow \sim$ is an equiv. relation.

Details from proof of the existence of the localization $S^{-1}R$.

Recall $W = R\times S$ and $(r,s)\sim(r',s')$ if $(rs' - r's)t = 0$, for some $t \in S$.

$\Leftarrow (rs-rs)1=0$

<u>This is an equivalence relation:</u> It is clear that $(r,s)\sim(r,s)$ and $(r,s)\sim(r',s')$ implies $(r',s')\sim(r,s)$, since 1 $\in S$ and R is commutative. Thus, transitivity is the real challenge.

$\to t(rs'-r's)=0 \Rightarrow \Leftarrow t(r's-rs')=0$

Assume $(r,s)\sim(r',s')$ and $(r',s')\sim(r'',s'')$. Then $(rs'-r's)t \underset{(1)}{=} 0$ and $(r's''-r''s')u \underset{(2)}{=} 0$, for some $t,u \in S$. We need to show $(r,s)\sim(r'',s'')$.

We have $rs''s'ut = (rs't)s''u \overset{(1)}{=} (r'st)s''u = (r's''u)st \overset{(2)}{=} (r''s'u)st$. Each equality follows from either reordering the factors, or replacing the term in parentheses using two equations resulting from the relations in the previous paragraph. Rewriting the first and last terms we get the equality $rs''(s'ut)=r''s(s'ut)$ or equivalently $(rs''-r''s)(s'ut)=0$. This gives $(r,s)\sim(r'',s'')$, since $s'ut \in S$.

Note that this was just a long way to get common denominators without every dividing or cancelling. We now have the set of equivalence classes $S^{-1}R = W/\sim$.

To show that addition and multiplication are well defined, we only need to change one representative of an equivalence class at a time. Writing r/s for $[(r,s)]$, we need only show that if $(r,s)\sim(r',s')$ and (r_1,s_1) is in W, then $(rs_1+r_1s)/ss_1 = (r's_1+r_1s')/s's_1$, to check that ⊗ addition is well defined. This is easily done as above. Similarly - and more easily - we can check that multiplication is well defined.

Finally, showing ~~that~~ that $S^{-1}R$ is a commutative ring is routine using the properties of R. Note that $1_{S^{-1}R} = 1/1 = s/s$ and $0_{S^{-1}R} = 0/1 = 0/s$, for any $s \in S$.

⊗ $\exists t: (rs'-r's)t=0 \Leftrightarrow rs't=r'st$

$\frac{(rs_1+r_1s)}{ss_1} = \frac{r's_1+r_1s'}{s's_1}$ ~~$\Leftrightarrow \exists u: ((rs_1+r_1s)s's_1 - (r's_1+r_1s')ss_1)u=0$~~

~~$\Leftrightarrow (rs_1s's_1 + r_1ss's_1 - r's_1ss_1 - r_1s'ss_1)u=0$~~

~~$\Leftarrow rs'ts_1^2$~~

$(rs_1+r_1s)s's_1t = rs_1s's_1t + r_1ss's_1t \quad = (rs't)s_1^2 + r_1ss's_1t$

$= r'sts^2 + r_1ss's_1t$

$= (ss_1)(r's_1t + r_1s't)$

$= (ss_1t)(r's_1+r_1s')$

Take $u=t$.

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$[(r,s)] \overset{\text{ntn}}{=} \frac{r}{s}$. (really $\underset{\varphi(r)\varphi(s^{-1})}{\cancel{rs^{-1}}}$)

Define: $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$, $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$, well-def'd!

Def. $\varphi : R \to S^{-1}R := \frac{W}{\sim}$

$r \longmapsto \frac{r}{1} = \frac{rs}{s}$, $s \in S$. is a ring hom.

$\varphi(S) = \frac{s}{1}$, $\frac{s}{1} \cdot \frac{1}{s} = \frac{1}{1} = 1_{S^{-1}R}$ $\forall s \in S \Rightarrow \varphi(S) \subseteq U(S^{-1}R)$

If $f : R \to T$ s.t. $f(S) \subseteq U(T)$, def. $g : S^{-1}R \to T$,

$g(\frac{r}{s}) := f(r) f(s)^{-1}$.

Check:
$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T \end{tikzcd}$$

unique: $r \in R \Rightarrow g(\frac{r}{1}) \overset{!}{=} f(r)$
since $f(r) = g(\varphi(r))$
$= g(\frac{r}{1})$

$f(s) \overset{!}{=} g(\varphi(s)) = g(\frac{s}{1})$

Finally, if $\varphi' : R \to L$ is another localization

$$\begin{tikzcd} & L \arrow[d, dashed, "f'"] \arrow[dd, bend left=60, "!"] \\ R \arrow[ur, "\varphi'"] \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & S^{-1}R \arrow[d, dashed, "f"] \\ & L \end{tikzcd}$$

$\leftarrow$ $f \circ f'$ and I_L both work

$\Rightarrow f \circ f' = I_L$. Sim. $f' \circ f = I_{S^{-1}R}$.

□

Note: R, T rings. $\varphi : R \to T$ ring-hom. Then any ${}_T M$ becomes an R-module via $r \cdot m := \varphi(r) \cdot m$, $\forall r \in R, m \in M$.

Thus any $S^{-1}R$-module is an R-mod.

In particular, $S^{-1}R$ is a left R-mod.

If $I \triangleleft R$ then $I(S^{-1}R) \subseteq S^{-1}R$ is an R-submod.

Lemma: $I(S^{-1}R) \triangleleft S^{-1}R$.

Exercise: $\operatorname{Ker}(\varphi : R \to S^{-1}R) = \{ r \in R \mid rs = 0 \text{ some } s \in S \}$.

Show $1 \notin \operatorname{Ker} \varphi$.

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Recall: $S \subseteq R$ mult. closed:

Rings comm. until announced

(1) $1 \in S$

(2) $s, t \in S \Rightarrow st \in S$

(3) $0 \in S$

$W = R \times S$

Def. $\sim$ by $(r, s) \sim (r_1, s_1)$ if $(s_1 r - r_1 s)t = 0$, some $t \in S$

equivalence relth.

$W/_\sim = S^{-1}R$. $[(r,s)]$ is written $\frac{r}{s}$.

Def. $+$ via $\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$.

$\cdot$ via $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$.

ring hom.
$$R \to S^{-1}R$$
$$r \mapsto \frac{r}{1}$$
localization map.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "\exists!\, g"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T \end{tikzcd}$$

$f(S) \subseteq U(T)$

Ex: R domain, $S = R \setminus 0$, $S^{-1}R = Q(R)$ field of fractions.

Ex: $R = \mathbb{Z}$, $P = p\mathbb{Z}$, $p \in \mathbb{N}$ prime. $S = R \setminus \mathfrak{p}$

$S^{-1}R$ is denoted R_p.

$R_p = \{ \frac{a}{b} \mid a \in R, \mathbb{Z}\, p \nmid b \} \subseteq \mathbb{Q}$.

$p = 2$: $R_p = \{ \frac{a}{b} \mid b \text{ odd} \}$ (!) max'l ideal $2R_p$.

Ex: $S = \langle s \rangle = \{1, s, s^2, \_\}$ S not nilpotent.
$\subseteq R$.

$\frac{R[t]}{\langle st - 1 \rangle} = R_s$.

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Change of rings

Let $\alpha: R \to R_0$ be a ring hom. We viewed ${}_{R_0}M$ as an R-module via $rm = \alpha(r)m$ $\forall m \in M, r \in R$.

We get functor $F: R_0\text{-Mod} \to R\text{-Mod}$
$M \mapsto FM = M_0$, $M_0 = M)$ with this action of R

$F: R_0\text{-Mod} \to R\text{-Mod}$ is an $\underbrace{\text{exact}}$ additive functor.
↳ Oct 11 $\quad F: \text{Hom}(A,B) \to \text{Hom}(FA,FB)$
hom of ab. gps.
R preadd.

preserves SES's

$f: M \to N$ in R_0-mod. $\quad Ff = f: M \to N$ in R-Mod.

Note: $\varphi: R \to S^{-1}R$ $\quad F: S^{-1}R\text{-Mod} \to R\text{-Mod}$

Start w/ $S \subseteq R$ mult. closed. It does not change things if we either

(1) Replace S by $U(R)S = \{ ~~us~~ \; us \mid u \in U(R), s \in S\}$

(2) replace S by its saturation S',
$S' = \{w \in R \mid wt \in S, \text{ some } t \in R\}$.

[Check! suffices to show $\left.\begin{matrix}\varphi(U(R)S) \\ \varphi(S')\end{matrix}\right\} \subseteq U(S^{-1}R)$].

Exercise: A localization of $S^{-1}R$ is a localization of R.

Prop: $\text{Ker}\,\varphi = \{r \in R \mid rs = 0 \text{ some } s \in S\}$ where $\varphi: R \to S^{-1}R$ is loc. morphism. & $1 \notin \text{Ker}\,\varphi$

Pf: why?: $0_{S^{-1}R} = \frac{0}{s}$ any $s \in S$. Suppose $r \in \text{Ker}\,\varphi$

$\Rightarrow \frac{r}{1} = \frac{0}{s} \Rightarrow (rs - 01)t = 0$, some $t \in S$.

$\Rightarrow r(st) = 0$ where $st \in S$.

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Conversely, if $sr=0$, some $s\in S$. $\frac{r}{1}=\frac{0}{s}$ since

$(rs-0\cdot 1)1=0$. $\Rightarrow$ $r\in \operatorname{Ker}\varphi$. □

Since $0\notin S$, $1s\neq 0$ for any $s\in S$. $\Rightarrow 1\notin \operatorname{Ker}\varphi$.

So $S^{-1}R$ is a ring.

Def. Given $S\subseteq R$ mult. closed, M an R-module.

A localization of M at S is a pair $(S^{-1}M, h)$ where $S^{-1}M$ is an $S^{-1}R$-module, $h: M\to S^{-1}M$ is an R-mod hom. s.t. given any R-hom. $f: M\to N$, where N an $S^{-1}R$-mod., $\exists!$ $g: S^{-1}M\to N$ an $S^{-1}R$-hom. s.t.

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "f"', "(R\text{-hom})"'] & S^{-1}M \arrow[d, dashed, "\exists! g"] \arrow[dl, phantom, "\circlearrowleft"] \\ & N \ (\in S^{-1}R\text{-mod}) \end{tikzcd}$$

Commutes.

How to get $S^{-1}M$?

Bad way: $X=M\times S$. Def. $\sim$ by $(m,s)\sim(m',s')$ if $t(s'm-sm')=0$, some $t\in S$.

$\sim$ is an equiv. relation.

$[(m,s)]$ is denoted $s^{-1}m$.

Everything works out nicely. $h: M\to S^{-1}M$
$m\mapsto 1^{-1}m$.

Lemma: $\operatorname{Ker} h=\{m\in M \mid sm=0 \text{ some } s\in S\}$.

Good way: $S^{-1}M=S^{-1}R\otimes_R M$
$h: M\longrightarrow S^{-1}R\overset{R}{\otimes} M$
$m\longmapsto 1\otimes m$.

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Lemma: (1) If $M \in S^{-1}R\text{-Mod}$, then $M \cong S^{-1}M = S^{-1}R \otimes_R M$ via $m \longmapsto 1 \otimes m$.

(2) $M \in R\text{-Mod}$ then $S^{-1}R \otimes_R M$ is a loc. of M at S.

Def: $\overset{R\text{ not comm.}}{(1)\ {}_R F}$ is flat if $\_ \otimes F : \text{Mod-}R \to Ab$ is exact.

(2) Similarly for F_R.

Fact: (HW?) $S^{-1}R$ is a flat R-Mod.

Recall: R_R is flat. $\Rightarrow$ any free R-Mod is flat. ⊛

(Recall $\otimes$ respects direct sums.)

$\Rightarrow$ any proj. module is flat.

Exercise: (1) A direct summand of a flat module is flat.

(2) $_{\mathbb{Z}}\mathbb{Q}$ is flat but <u>not</u> projective.
$\uparrow$ localization
not free : basis must have 1 elt &

⊛ F free R-Mod w/ $\overset{\text{left free}}{\text{basis}}$ $\{x_i \mid i \in I\}$

$$\begin{tikzcd} \bigoplus_{i\in I} R \arrow[r, "\sim"] & F \\ (r_i)_{i\in I} \arrow[r, mapsto] & \sum_i r_i x_i \end{tikzcd}$$

[Google.
Prop $0 \to A \to B \to C \to 0$ A, B, C left R-mods.

B Noeth (Art) $\Leftrightarrow$ A & C Noeth (Art)

Prop: R left Noeth (Art) $\Rightarrow$ Every f.g. left R-mod is Noeth (Art)

Pf: (Art). R^n left Art $\forall n$, M f.g. left R-mod $\Rightarrow \exists n \in \mathbb{N}$, $\varphi : R^n \to M$,
s.t. φ surj. $\overset{R\text{-mod}}{\text{hom}}$ $\Rightarrow M \cong R/\ker\varphi$ $\Rightarrow$ left Art as quotient

rem: subrings of Art(Noeth) rings are not nec Art (Noeth)

ex: $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$ left Noeth, not right Noeth

$I_n = \{\begin{pmatrix} 0 & \frac{a}{2^n} \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}\}$ right ideal, $I_n = \{\begin{pmatrix} 0 & \frac{1}{2^n} \\ 0 & 0 \end{pmatrix} \ldots\}$,
$I_n \subsetneq C, \subsetneq$

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Inverse limits and completions

[(X,d) metric space. Completion of X is $\hat{X} \supseteq X$ s.t.

(1) X is dense in $\hat{X}$.
(2) Every Cauchy-seq in $\hat{X}$ converges. (complete)

$Y = \overline{\{ (a_n)_{n\geq 1} \mid \text{Cauchy seq} \}}$ $\qquad$ $\{a_n\} \sim \{b_m\}$ if $|a_n - b_n| \to 0$ as $n \to \infty$

$\hat{X} = Y/\sim$

Identifies $x \in X$ w/ seq. $a_1 = x,\ a_2 = x, \ldots$

We end up with $X \subseteq \hat{X}$. Check $\hat{X}$ is complete]

Recall: Poset $(I, \leq)$ is a category $\quad \mathrm{Hom}_I(i,j) = \begin{cases} \{\varkappa^i_j\} & \text{if } i \leq j \\ \emptyset & \text{if } i \not\leq j \end{cases}$

$(\varkappa^i_i = I_i)$

Def: An inverse system (over I) of R-mods is a contravariant functor

$F: I \to \text{R-mod}$
$i \mapsto M_i$
$\varkappa^i_j \mapsto \psi^j_i : M_j \to M_i$

Note: i, j changed places (catravariant)

Ex: I discrete ($i \leq j$ iff $i = j$). Inverse system is a fam. of R-mods $\{M_i \mid i \in I\}$

Ex: $I = \{1, 2, 3\}$ $\qquad 1 \leq 3,\ 1 \leq 2 \qquad$ (2, 3 unrelated!)

$$\begin{tikzcd}
 & 2 & {} \arrow[r, "F"] & {} & & A \arrow[d, "g"] \\
3 & 1 \arrow[l, "\varkappa^1_3"'] \arrow[u, "\varkappa^1_2"'] & & & B \arrow[r, "f (= \psi^3_1)"'] & C
\end{tikzcd}$$

Note: If $i \leq j \leq l$ $\qquad \psi^j_i \psi^l_j = \psi^l_i$.

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Ex: $\mathbb{N}$ usual order

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, "K^1_2"] & \underset{2}{\bullet} \arrow[r, "K^2_3"] & \underset{3}{\bullet} \arrow[r, "K^3_4"] & \underset{4}{\bullet} \arrow[r] & \cdots
\end{tikzcd}$$

$$\downarrow F$$

$$\begin{tikzcd}
M_1 & M_2 \arrow[l, "\psi^2_1"'] & M_3 \arrow[l, "\psi^3_2"'] & M_4 \arrow[l, "\psi^4_3"'] & \cdots
\end{tikzcd}$$

Def: An inverse limit $(\varprojlim_i M_i, \alpha_i)_{i\in I}$ of an inverse system over I is

(1) R-mod $\varprojlim_i M_i$ and R-homs $\alpha_i : \varprojlim_i M_i \to M_i$

s.t.

$$\begin{tikzcd}
 & M_j \arrow[d, "\psi^j_i"] \\
\varprojlim_i M_i \arrow[ur, "\alpha_j"] \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, "\alpha_i"'] & M_i \\
 &
\end{tikzcd}$$

commutes $\forall i \leq j$.

(2) Given X in R-Mod and $f_i : X \to M_i$ $\forall i \in I$ s.t.

$$\begin{tikzcd}
 & M_j \arrow[d, "\psi^j_i"] \\
X \arrow[ur, "f_j"] \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, "f_i"'] & M_i
\end{tikzcd}$$

commutes $\forall i \leq j$,

then $\exists(!)$ $\theta : X \to \varprojlim_i M_i$ s.t.

$$\begin{tikzcd}
X \arrow[rr, dashed, "\theta"] \arrow[d, "f_j"] \arrow[dd, bend right, "f_i"'] & \circlearrowleft \leftarrow \text{new} & \varprojlim_i M_i \arrow[dll, "\alpha_j"'] \arrow[ddll, "\alpha_i"] \\
\circlearrowleft M_j \arrow[d, "\psi^j_i"] & \circlearrowleft & \\
M_i & &
\end{tikzcd}$$

commutes $\forall i \in I$

Ex: I discrete ($i \leq j$ iff $i = j$), Inv. syst.: fam. of R-mods $\{M_i \mid i \in I\}$

$\varprojlim_i M_i = \prod_{i\in I} M_i$, $\alpha_j : \prod_{i \in I} M_i \to M_j$ projection.

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$\Theta: X \to \prod_{i\in I} M_i$ universal prop. of direct product

$x \longmapsto (f_i(x))_{i\in I}$.

Theorem: Given inverse system (M_i, ψ_i^j) over $(I, \leq)$.

Then $\varprojlim_i M_i$ exists and is (!).

Proof: Let $L = \{ (m_i)_{i\in I} \in \prod_{i\in I} M_i \mid \psi_i^j(m_j) = m_i \ \forall i \leq j \}$

Let $\alpha_j = p_j|_L$ where $p_j: \prod_i M_i \to M_j$ is usual proj.

L is clearly a submod.

By construction

$$\begin{tikzcd}
 & M_j \arrow[d, "\psi_i^j"] \\
L \arrow[ur, "\alpha_j"] \arrow[r, "\alpha_i"'] & M_i
\end{tikzcd}$$

commutes.

Suppose $\{f_i: X \to M_i \mid i \in I\}$ R-homs s.t.

$$\begin{tikzcd}
 & M_j \arrow[d, "\psi_i^j"] \\
X \arrow[ur] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & M_i
\end{tikzcd} \qquad (*)$$

commutes $\forall i \leq j$.

univ. prop of $\prod_i M_i$ $\Rightarrow \exists(!)\ \bar{\Theta}: X \to \prod_{i\in I} M_i$ s.t.

$$\begin{tikzcd}
X \arrow[r, "\bar{\Theta}"] \arrow[dr, "f_j"'] \arrow[r, phantom, "\circlearrowleft", shift right=4] & \prod_{i\in I} M_i \arrow[d, "p_j"] \\
 & M_j
\end{tikzcd}$$

commutes $\forall j \in J$.

Compatibility conditions (*) ensures $\bar{\Theta}(X) \subseteq L$.

Hence we get $\Theta: X \to L$ $(\subseteq \prod_{i\in I} M_i)$ s.t.

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "f_j"'] \arrow[r, phantom, "\circlearrowleft", shift right=4] & L \arrow[d, "\alpha_j"] \\
 & M_j
\end{tikzcd}$$

commutes $\forall j \in J$.

If (L', α_i') is another inv. limit, $\exists(!)\ \rho: L \to L'$ w/ comm. Δ's

(!) $\chi: L' \to L$ -"- -"- -"-

s.t. $\chi \circ \rho: L \to L$ satisfies

$$\begin{tikzcd}
L \arrow[r, "\overset{I_L}{\cancel{\chi\circ\rho}}"] \arrow[dr, "\alpha_j"'] & L \arrow[d, "\alpha_j"] \\
 & M_j
\end{tikzcd}$$

$\forall j \in J$. But I_L also makes diagram commute.

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By def $I_L = \chi \circ \rho$. Sim. $\rho \circ \chi = I_{L'}$.

$$\begin{tikzcd}
L \arrow[rr, "\rho"] \arrow[dr, "\alpha_j"'] & \arrow[d, phantom, "\circlearrowleft"] & L' \arrow[dl, "\alpha'_j"] \\
& M_j &
\end{tikzcd}$$

$\forall j$.

□

Note: If $i<j$ in I, $m_i = \psi^j_i (m_j)$ $\forall$ $(m_i)_{i\in I} \in L$.

Thus we do not need M_i !

Example: $I = \{1,2,3\}$, $1 \le 2$, $1 \le 3$

$$\begin{tikzcd}
& A \ (=M_2) \arrow[d] \\
B \arrow[r, "f"'] \arrow[d, phantom, "\shortparallel"] & C \ (=M_1) \\
M_3 &
\end{tikzcd}$$

$$L = A \oplus B \oplus \cancel{C} = \{ (a,b,c) \mid f(b)=c, g(a)=c \}$$
$$= \{ (a,b) \in A \oplus B \mid g(a)=f(b) \}.$$

$$\begin{tikzcd}
X \arrow[dr, dashed, "\theta"] \arrow[drr, "\delta"] \arrow[ddr, "\gamma"'] & & \\
& L \arrow[r, "\alpha_2"] \arrow[d, "\alpha_3"'] & A \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

Given X, $\gamma: X \to B$, $\delta: X \to A$ s.t. $g \circ \delta = f \circ \gamma$

then $\exists(!)$ $\theta: X \to L$ to make diag. commute.

L is called the pullback of f and g.

cov. functor comp. ⇝ direct limit (= direct sum in discrete case)
(otherwise quot. module)

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I poset $\quad i \leq j$

$$\begin{tikzcd} i \arrow[r, "k^i_j"'] & j \end{tikzcd}$$

$\Downarrow$ F contravariant

R-Mod

$F(i) = M_i$

$$\begin{tikzcd} M_i & M_j \arrow[l, "F(k^i_j) = \psi^j_i"] \end{tikzcd}$$

$$\varprojlim_i M_i = \{ (m_i)_i \mid \psi^j_i(m_j) = m_i \ \forall \ i \leq j \}$$

$\alpha_j : \varprojlim_i M_i \to M_j \qquad \alpha_j = P_j|_L$ = restriction of $P_j : \prod_{i \in I} M_i \to M_j$

$$\begin{tikzcd} X \arrow[r, dashed, "\theta(!)"] \arrow[dr, "f_i"'] & \varprojlim_i M_i \arrow[d, "\alpha_j"] \\ & M_j \end{tikzcd}$$

[If I discrete, $\varprojlim_i M_i = \prod_{i \in I} M_i$, $\alpha_j = P_j$]

Ex: $J \triangleleft R$ (or left ideal) $\quad M_i = J^i M \subseteq M$ submod, $\quad I = \mathbb{N}$

$$\begin{tikzcd} \frac{M}{JM} & \frac{M}{J^2M} \arrow[l, "\psi^2_1"'] & \frac{M}{J^3M} \arrow[l, "\psi^3_2"'] \end{tikzcd}$$

Notice $JM \supseteq J^2M \supseteq J^3M \dots$

$\Rightarrow$ If $i \leq j$ let $\psi^j_i : \frac{M}{J^jM} \to \frac{M}{J^iM}$ R-hom

well-def'd since $J^jM \subseteq J^iM$

$x + J^jM \to x + J^iM$

We have $M_i = \frac{M}{J^iM}$ and $\psi^j_i : M_j \to M_i \quad \forall i \leq j$

$\varprojlim_i M_i$ is [J-adic completion of M.]

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<u>Example:</u> R comm. ring, $J \triangleleft R$ s.t. $\bigcap_{i=0}^{\infty} J^i = 0$.

Take $M = R$ in last example.

$$\frac{R}{J^i} \xleftarrow{\psi_i^j} \frac{R}{J^j} \quad \forall i \le j \qquad \text{Note: } \psi_i^j \text{ is a ring hom.}$$

$\varprojlim_i \frac{R}{J^i}$ is the J-adic completion of R.

This is a ring!

<u>2 special examples</u>

① $R = k[x]$ k comm. ring, $J = \langle x \rangle$ $\bigcap_{i=0}^{\infty} J^i = 0$.

$J^i \neq 0$ for all i. Every elt of $\frac{R}{J^n}$ has (!) form $a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + J$

[$k[x]$ has basis $1, x, x^2, \dots$ as free k-module]

[$J^i = kx^i + kx^{i+1} + \dots$]

An element of $\varprojlim \frac{R}{J^i}$ looks like

$$(a_0 + J,\ a_0 + a_1 x + J^2,\ a_0 + a_1 x + a_2 x^2 + J^3, \dots)$$

We identify this with <u>formal</u> power series $\sum_{i=0}^{\infty} a_i x^i$

This is a ring isomorphism. [Check!]

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{i=0}^{\infty} c_i x^i \quad \text{where} \quad c_i = \sum_{l=0}^{i} a_l b_{i-l}$$

② $R = \mathbb{Z}$, $J = \mathbb{Z}p$ p prime.

Every elt of $\frac{R}{J^n}$ has (!) form

$$a_0 + a_1 p + a_2 p^2 + \dots + a_{n-1} p^{n-1} \quad \text{where} \quad 0 \le a_i < p.$$

(write coset rep. (less than p^n) in base p).

$$\varprojlim_i \frac{R}{J^i} = \{(a_0 + J,\ a_0 + a_1 p + J^2,\ a_0 + a_1 p^2 + a_2 p^2 + J^3, \dots)\}$$

This gives a formal sum $\sum_{i=0}^{\infty} a_i p^i$, $0 \le a_i < p$.

This corresponds to a Cauchy-sequence

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$a_0, a_0+a_1p, a_0+a_1p+a_2p^2, \dots$ in $\mathbb{Z}$

w/ a special metric.

Notation: $\varprojlim_i R/J^i$ is denoted $\mathbb{Z}_p$.

Bad!

$\mathbb{Z}_p$ can be $\varprojlim_i \frac{R}{J^i}$

or $\mathbb{Z}/p\mathbb{Z}$

or $S^{-1}\mathbb{Z}$, $S = \mathbb{Z} \setminus p\mathbb{Z}$.

Ex: $\mathbb{Z}_p$ is an integral domain. no nzd's

$\mathbb{Q}_p = Q(\mathbb{Z}_p)$ is field of p-adic numbers

Note: R comm, $J \triangleleft R$, $\bigcap_{i=1}^{\infty} J^i = 0$

If $r=0$, $|r|=0$.

If $r \neq 0$, $|r| = 2^{-n}$ if $r \in J^n$ but $r \notin J^{n+1}$.

Now define d a metric on R by $d(r,s) = |r-s|$.

d is a metric.

$|rs| \le |r|\,|s|$. and $|r+s| \le \max\{|r|, |s|\}$.

In $\varprojlim_i \frac{R}{J^i} \subseteq \prod_{i \in I} \frac{R}{J^i}$, $(x_i + J^i)_{i \ge 1}$ gives $\{x_i\}_{i \ge 1}$, a Cauchy sequence.

Conversely, any Cauchy seq. in R can be give an element of $\varprojlim_i R/J^i$.

$\varprojlim_i R/J^i$ is the completion of (R, d).

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Finally: If $I \lhd R$, $\bigcap_{i=1}^{\infty} J^i = 0$, then $R \hookrightarrow \varprojlim_i \frac{R}{J^i}$

$r \longmapsto (r+J, r+J^2, r+J^3, \_\_)$

Dez 01

Recall: $S \subseteq R$ mult. closed

$S^{-1}R = \frac{R\times S}{\sim} = \{ \frac{r}{s} \mid r \in R, s \in S \}$

$\varphi: R \to S^{-1}R$ ring hom

$r \longmapsto \frac{r}{1} = \frac{rs}{s}$

$\varphi(S) \subseteq U(S^{-1}(R))$ If $f: R \to T$ ring hom
$f(S) \subseteq U(T)$ $\exists!\, g$:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "g"] \arrow[dl, phantom, "\circlearrowright"] \\ & T \end{tikzcd}$$

$S^{-1}R$ is an R-mod via $r\cdot q = \varphi(r)\cdot q$ $\forall r \in R, q \in S^{-1}R$.

Theorem: 1) If $I \lhd R$ then $I(S^{-1}R) \lhd S^{-1}R$, with

$I(S^{-1}R) = S^{-1}R$ iff $I \cap S \neq \emptyset$.

2) If $J \lhd S^{-1}R$, $J = I(S^{-1}R)$, some $I \lhd R$.

Proof: If $x \in I$, $\frac{r}{s} \in S^{-1}R$ then $x(\frac{r}{s}) = \frac{xr}{s}$. Thus

$I(S^{-1}R) = \{ \frac{x}{s} \mid x \in I, s \in S \}$ Write $S^{-1}I$ for this set.

Check $S^{-1}I \lhd S^{-1}R$. ~~$I(S^{-1}R) = S^{-1}R \Leftrightarrow \underset{= \frac{s}{s}}{1} \in I(S^{-1}R) \Leftrightarrow s \in I \cap S \Leftrightarrow I \cap S \neq \emptyset$~~

Suppose $J \lhd S^{-1}R$. Let $I = \{ r \in R \mid \frac{r}{s} \in J$, some $s \in S \}$.

It is "easy" to see $I \lhd R$.

If $x,y \in I \Rightarrow \frac{x}{t}, \frac{y}{s} \in J$, some $s,t \in S$ $\Rightarrow \frac{x}{1} = \frac{x}{t}\frac{t}{1}$, $\frac{y}{1} = \frac{y}{s}\frac{s}{1} \in J$

$\Rightarrow \frac{x}{1} + \frac{y}{1} = \frac{x+y}{1} \in J \Rightarrow x+y \in I$

Write $\underbrace{J \cap R}_{\text{horrible}}$ for I. Clearly, $J = I(S^{-1}R)$.

Suppose $I(S^{-1}R) = S^{-1}R$ $\Rightarrow \frac{x}{s} = \frac{1}{1}$, some $x \in I$.

$\Rightarrow (x-s)t = 0$ some $t \in S$. $\Rightarrow xt = st \in S \cap I$.

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Conversely if $u \in S \cap I \Rightarrow \frac{u}{1} \in I(S^{-1}R) \lhd S^{-1}R$. $\frac{u}{1} \in U(S^{-1}R)$

$\Rightarrow I(S^{-1}R) = S^{-1}R$.

□

Def. ① If R is a ring, $\operatorname{Spec} R = \{p \lhd R \mid p \text{ prime}\}$ is the spectrum of R.

② If R commutative, $S \subseteq R$ mult. closed, then

$$\operatorname{Spec}_S(R) = \{p \lhd R \mid p \text{ prime}, \ p \cap S = \emptyset\} \subseteq \operatorname{Spec} R.$$

$p \subsetneq R$ prime if $AB \subseteq P$, $A, B \lhd P$ then either $A \subseteq P$ or $B \subseteq P$.

Theorem: $\psi : \operatorname{Spec}_S(R) \to \operatorname{Spec}(S^{-1}R)$

$$p \longmapsto p(S^{-1}R) \overset{\text{Ntn}}{=} S^{-1}p$$

is an inclusion preserving bijection.

Proof. Let $\mathfrak{P} \lhd S^{-1}R$ be a prime ideal.

Let $p = \{r \in R \mid \frac{r}{s} \in \mathfrak{P} \text{ some } s \in S\}$

Let $a, b \in R$ with $ab \in p$. $\Rightarrow \frac{a}{1}\,\frac{b}{s} = \frac{ab}{s} \in \mathfrak{P}$, some $s \in S$.

$\mathfrak{P}$ prime $\Rightarrow \frac{a}{1}$ or $\frac{b}{s} \in \mathfrak{P} \Rightarrow a \in p$ or $b \in p$

$\Rightarrow \mathfrak{P} = p(S^{-1}R)$ where $p \in \operatorname{Spec}(R)$.

We know $p \cap S = \emptyset$ since $p(S^{-1}R) \neq S^{-1}R$. $\Rightarrow p \in \operatorname{Spec}_S(R)$.

Conversely, suppose $Q \in \operatorname{Spec}_S(R)$.

$Q(S^{-1}R) = \{\frac{a}{s} \mid a \in Q, s \in S\}$. Suppose $\frac{x}{t}\,\frac{y}{u} \in Q(S^{-1}R)$

$\Rightarrow \frac{xy}{tu} = \frac{a}{s}$, some $a \in Q$, $s \in S$. $\Rightarrow (xys - atu)v = 0$, some $v \in S$.

$\rightarrow atuv = xysv \Rightarrow (xy)sv \in Q$ since $a \in Q$. $Q \lhd R$ prime

$\Rightarrow xy \in Q$ since $sv \in S$ and $S \cap Q = \emptyset$.

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$\Rightarrow x \in Q$ or $y \in Q$ since $Q \triangleleft' R$ $\leftarrow$ prime.

$\Rightarrow \frac{x}{t}$ or $\frac{y}{u} \in Q(S^{-1}R)$.

It remains to show if $p_1 \neq p_2$, $p_1, p_2 \in \mathrm{Spec}_S(R)$ then $p_1(S^{-1}R) \neq p_2(S^{-1}R)$ See homework. $\square$

Galois Theory

Def: Group G acts on a ring R if we have a group hom $\varphi: G \to \mathrm{aut}(R)$ = set of ring automorphisms of R

$\varphi(g)(r)$ is written $g \cdot r$.

We have

$$(gh)\cdot r = g(h\cdot r)$$
$$g\cdot(r+s) = g\cdot r + g\cdot s$$
$$g\cdot(rs) = (g\cdot r)(g\cdot s)$$
$$g\cdot 1_R = 1_R$$
$$1_G \cdot r = r \qquad \forall g,h \in G,\ r,s \in R.$$

Example: k field, $G \leq \mathrm{aut}(k)$. $\varphi: G \to \mathrm{aut}(k)$ inclusion map.

Def: If G acts on R, the skew group ring of G over R is the free module $RG = \bigoplus_{g \in G} Rg$ over R with G as a basis.

mult: $(rg)(sh) = r(g\cdot s)(gh)$

extend using dist. law. (extend linearly)

Dec 4

$\alpha: G \to \mathrm{aut}(R)$ group hom
$\alpha(g)(r) =: g\cdot r \quad \forall g \in G, r \in R$

$g\cdot(rs) = (g\cdot r)(g\cdot s)$
$g(r+s) = g\cdot r + g\cdot s$
$g\cdot 1_R = 1_R$
$g\cdot h(r) = (gh)\cdot r \qquad \forall g,h \in G, r \in R$

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Theorem: Let $G \leq \operatorname{aut}(K)$, K a field, be finite. Then the skew group ring KG is a simple ring with $Z(KG) = k1_G = k$ where $k := K^G = \{\alpha \in \overset{K}{G} \mid g(\alpha) = \alpha \ \forall g \in G\}$

Furthermore, $C_{KG}(K) = K$, where $K = K1_G \subseteq KG$ and

$$C_{KG}(K) = \{t \in KG \mid t\alpha = \alpha t \ \forall \alpha \in K\}$$

$$[x \subseteq R: \ C_R(x) = \{t \in R \mid tx = xt \ \forall x \in \overset{X}{K}\}, \quad Z(R) = C_R(R)].$$

Recall: In RG $\quad (rg)(sh) = r(g \cdot s)gh \ \forall \ r,s \in R, \ g,h \in G.$

RG free R-mod with basis G.

To check RG is associative it suffices to show,

$$[(rg)(sh)](t \cdot l) = (rg)[(sh)(t l)] \quad \forall r,s,t \in R, \ g,h,l \in G.$$

LHS: $\quad [(rg)(sh)](t l) = [r(g \cdot s)gh][t \cdot l] = r(g \cdot s)[(gh) \cdot t]gh \cdot l$

$$= r(g \cdot s)(gh \cdot t)(ghl)$$

RHS: $\quad (rg)[(sh)(tl)] = (rg)[s(h \cdot t)hl] = r(g \cdot (s(h \cdot t)))ghl$

$$= r(g \cdot s)(gh) \cdot t \ (ghl)$$

$g \cdot (rs) = (g \cdot r)(g \cdot s)$

RG is assoc. $\quad$ <u>and</u> $\quad 1_{RG} = 1_R 1_G.$ $\qquad (rg)s = r(g \cdot s)g$

$R \to RG$ ring embedding
$r \mapsto r1$
$G \to RG \qquad G \hookrightarrow U(RG).$ $\qquad$ Note: $g = 1_R g \in U(RG)$
$g \mapsto 1_R \cdot g$

If $r \in R$ $\quad r = r1_G$

$(1_R g)(rg^{-1}) = 1 \cdot (g \cdot r)gg^{-1} \longrightarrow$ $\qquad (g r)g^{-1} = (g \cdot r)g \, g^{-1} = g(r)1_G$

$$\overset{= g \cdot r}{}$$

Action of g on R has become Conj~~unction~~gation.

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<u>Proof:</u> Let $I \trianglelefteq KG$, $I \neq 0$. If $t = \sum_{g \in G} \alpha_g g \in KG$, the <u>support</u> of t is $\{ g \mid \alpha_g \neq 0 \} =: \mathrm{Supp}(t)$.

Pick $0 \neq t \in I$ s.t. $|\mathrm{Supp}(t)|$ is minimal

If $g \in \mathrm{Supp}\ t$ then $tg^{-1} \in I$. $\rightsquigarrow$ W.L.O.G. $1 \in \mathrm{Supp}\, t$.

$t = \alpha_1 1 + \alpha_2 g_2 + \dots + \alpha_s g_s$ where $\mathrm{Supp}(t) = \{1, g_2, \dots, g_s\}$

$g_2 \neq 1_G \Rightarrow \quad \Rightarrow \exists \beta \in K$ s.t. $g_2 \cdot \beta \neq \beta$.

Consider $I \ni \beta \cdot t - t\beta = \underbrace{\cancel{(\beta\alpha_1 - \alpha_1\beta) +}}_{=0}$

$= \beta\,(\alpha_1 \cdot 1 + \alpha_2 g_2 + \dots + \alpha_s g_s) - (\alpha_1 1 + \alpha_2 g_2 + \dots + \alpha_s g_s)\beta$

$= \underbrace{(\overset{\in K}{\beta\alpha_1} - \overset{\in K}{\alpha_1\beta})}_{=0} 1 + [\beta_2 \alpha_2 - \alpha_2 (g_2 \cdot \beta)] g_2 + \dots + (\dots) g_s$

$\beta t - t\beta \in I$ but $|\mathrm{Supp}(\beta t - t\beta)| < |\mathrm{Supp}\ t|$

$\Rightarrow \beta t - t\beta = 0$ by choice of $t \in I$. $\Rightarrow g_2 = \dots = g_s = 0$

$\Rightarrow \mathrm{Supp}(t) = \{1\}$.

$t = \alpha_1 \cdot 1 \in U(KG)$. $\Rightarrow I = kG$.

Clearly, $K \subseteq C_{KG}(K)$.

If $t = \sum_{g \in G} \alpha_g g \in C_{KG}(K)$, $0 = \beta t - t\beta = \sum_{g \in G} [\beta \alpha_g - \alpha_g (g \cdot \beta)] g$

$\forall \beta \in K$.

If $\alpha_g \neq 0 \Rightarrow \beta = g \cdot \beta \ \ \forall \beta \in K \Rightarrow g = 1 \Rightarrow C_{KG}(K) \subseteq K 1_G = K$.

Clearly, $Z(KG) = C_{KG}(KG) \subseteq C_{KG}(K)$. $\Rightarrow Z(KG) \subseteq K$.

K commutes with K.

$g\alpha = (1_K g)(\alpha 1_G) = 1_K (g \cdot \alpha) g 1_G$
$= (g \cdot \alpha) g$

$\rightarrow Z(KG) = \{ \alpha \in K \mid \overset{\alpha = g\alpha g^{-1} = g \cdot \alpha}{\overbrace{\alpha g = g\alpha}} \ \forall g \in G \} = \{ \alpha \in K \mid \alpha g = (g \cdot \alpha) g \ \forall g \in G \}$
$= \{ \alpha \in K \mid g \cdot \alpha = \alpha \ \forall g \in G \} = k$ $\square$

Dec 4

$|K:k| = |G|$ where $|K:k| = \dim_k K$ as a v.s.

<u>Def:</u> A is a central simple k-algebra if

(1) A is a simple ring

(2) $\dim_k A < \infty$.

<u>Thm:</u> Let A be a central k-alg. If $B \subseteq A$ is a simple subalgebra, then $\mathbb{C}_A(B)$ is again a simple subalgebra and $\mathbb{C}_A(\mathbb{C}_A(B)) = B$. (Double centralizer thm).
(D.C.T.)

<u>Example:</u> (Fund. thm of Galois theory).

$G \leq \mathrm{aut}(K)$ fin, $k = K^G$.

Simple subalg. B with $k \subseteq B \subseteq K$ are the <u>intermediate fields</u>

Similarly, simple subalgebra $K \subseteq B \subseteq KG$ are of form KH, $H \leq G$.

$\mathbb{C}_{KG}(K) = K$ from thm

$\mathbb{C}_{KG}(KH) = K^H = \{\alpha \in K \mid h\alpha = \alpha \ \forall \ h \in H\}$

If $k \subseteq L \subseteq K$ int. field, $\mathbb{C}_{KG}(L) = KH$, $H = \{h \in G \mid h\cdot\alpha = \alpha \ \forall \alpha \in L\}$.

We get Galois correspondence by D.C.T.

<u>Dec 8</u>

~~MAT~~ $\boxed{\text{Exam}}$ 12^{45}, 100 CARN, 2h

<u>Recall:</u> ① KG $(\alpha g)(\beta h) = \alpha(g\cdot\beta)gh$

$K \leq KG$.
$\alpha \mapsto \alpha 1_G$

$G \leq U(KG)$
$g \mapsto 1_K g$

Action is in KG $g(\alpha)g^{-1} = g\cdot\alpha$.

Dec 5

② $\mathbb{C}_{KG}(K)=K$. If $k \overset{def}{=} K^G = \{\alpha \in K \mid g(\alpha)=\alpha\ \forall g \in G\}$.

Simple subalgebras L, $k \le L \le K$ are intermediate fields.

(Recall: $\underset{\dim_k K}{\underset{\uparrow}{|K:k|}} = |G|$)

Def: (1) A, a k-alg. is called a central k-alg. if $Z(A)=k=k1_A$.

(2) A central simple k-alg if
(a) A is central
(b) A is simple
(c) $\underset{\dim_k A}{\underset{\uparrow}{|A:k|}}<\infty$

k a field.

Note: If A, B are k-algebras, then $A \otimes_k B$ is a k-alg. via

$(a\otimes b)(a'\otimes b') = aa'\otimes bb'$.

why? Mult. in A: $\mu_A: A\otimes A \to A$, $a\otimes b \mapsto ab$

--- B: $\mu_B: B\otimes B \to B$

Associativity in A:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "\mu_A\otimes I"] \arrow[d, "1\otimes\mu_A"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "\mu_A"] \\
A\otimes A \arrow[r, "\mu_A"'] & A
\end{tikzcd}$$

assoc!

$$\begin{tikzcd}
k\otimes A \arrow[r, "\mu\otimes I"] \arrow[d, "\wr"] & A\otimes A \arrow[dl, "\mu_A"] \arrow[l, phantom, "\circlearrowleft", shift right=4] \\
A &
\end{tikzcd}$$

$\rightsquigarrow \exists$ left identity

$$\begin{tikzcd}
A\otimes k \arrow[r, "I\otimes\mu"] \arrow[d, "\wr"] & A\otimes A \arrow[dl, "\mu_A"] \\
A &
\end{tikzcd}$$

$\rightsquigarrow$ right identity

$$\mu_{A\otimes B} \overset{?}{=} \quad \begin{tikzcd}
A\otimes B\otimes A\otimes B \arrow[r, "I_A\otimes T\otimes I_B"] \arrow[rr, bend right, dashed, "\mu_{A\otimes B}"'] & A\otimes A\otimes B\otimes B \arrow[r, "\mu_A\otimes\mu_B"] & A\otimes B
\end{tikzcd}$$

$\swarrow$ twist

$T: B\otimes A \to A\otimes B$
$b\otimes a \mapsto a\otimes b$

$a\otimes b\otimes a'\otimes b' \to a\otimes a'\otimes b\otimes b' \to aa'\otimes bb'$

$\mu_{A\otimes B}$ is as expected. Rest is easy!

Ex: A k-alg. ① $A\otimes M_n(k) \cong M_n(A)$
$\sum_{i,j} a_{ij}\otimes E_{ij} \mapsto [a_{ij}]$

Dec 8

② $A \otimes k[x] \cong A[X]$

$$\sum_{i=0}^{n} a_i \otimes X^i \to \sum_{i=0}^{n} a_i x^i$$

If $\varphi : A \to B$
$\varphi' : A' \to B'$ alg. maps then $\varphi \otimes \varphi' : A \otimes A' \to B \otimes B'$
is an algebra map.

<u>Ex:</u> $C = k[x]$. $\Delta : C \to C \otimes C$
$x \mapsto x \otimes 1 + 1 \otimes x$ extends to alg map

If $\varphi . A \to B$

$$\begin{tikzcd} \varphi \otimes I_{M_n(k)} \cdot A \otimes M_n(k) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & B \otimes M_n(k) \arrow[d, phantom, "\cong" rotate=90] \\ M_n(A) & M_n(B) \end{tikzcd}$$

<u>Thm:</u> Let A be a central simple k-alg. and B a simple k-alg.

Then $A \otimes_k B$ is a simple k-alg.

<u>Ex:</u> $\mathbb{C}$ simple k-alg. $k = \mathbb{R}$. <u>not</u> central

$\mathbb{C} \otimes_k \mathbb{C}$ has basis $a = 1 \otimes 1$, $b = i \otimes 1$,
$c = 1 \otimes i$, $d = i \otimes i$.

$\dim_{\mathbb{R}} \mathbb{C} \otimes \mathbb{C} = 4$
If simple, $\cong M_n(D_i)$
C! ?

$b^2 = c^2$, $(b-c)(b+c) = 0$, but $b \neq \pm c$.

What is algebra structure of $\mathbb{C} \otimes_k \mathbb{C}$? ⇝ not simple

<u>Pf: (of Thm):</u> Suppose $0 \neq I \triangleleft A \otimes B$. Let $0 \neq t = \sum_{i=1}^{n} a_i \otimes b_i \in I$, where n is minimal. It follows $a_1, \ldots, a_n$ are lin independent <u>and</u> $b_1, \ldots, b_n$ —"— —"— —"—

$a_1 \neq 0$, so $\exists$ $(x_l, y_l)^{\in A \times A}$ $l = 1, \ldots, s$ s.t. $\sum_{l=1}^{s} x_l a_1 y_l = 1_A$. ⇐ $(a_1) = A$ since A simple

Now, $\sum_{l=1}^{\infty} (x_l \otimes 1_B) t (y_l \otimes 1_B) = 1 \otimes b_1 + a_2' \otimes b_2 + \ldots + a_n' \otimes b_n \in I$.

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This is <u>not</u> 0 since $b_1, \dots, b_n$ lin. indep/k.

$\Rightarrow$ W.L.O.G. $a_1 = 1$.

If $c \in A$ $\quad (c\otimes 1)t - t(c \otimes 1) = 0 \otimes b_1 + (c a_2 - a_2 c) \otimes b_2$

$+ \dots + (c a_n - a_n c) \otimes b_n \in I$.

By minimality of n, $\quad c a_i - a_i c = 0 \;\; \forall i \geq 2$.

$\Rightarrow a_i \in Z(A) \overset{A \text{ central}}{=} k \;\; \forall i \geq 2$.

$\Rightarrow t = \sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{n} 1 \otimes a_i b_i = 1_A \otimes \left(\sum_{i=1}^{n} a_i b_i \right) \in I. \Rightarrow n = 1$

Now, $t = 1 \otimes \underset{\underset{1}{\shortparallel}}{a_1} b_1 \in I$, $b_1 \neq 0$. $\quad$ Now, $1_A \otimes 1_B \in \cancel{I}\, 1_A \otimes B$

$= \underset{\substack{B \text{ simple} \\ Bb_1B = B}}{(1 \otimes B)}\, \underbrace{(1 \otimes b_1)}_{\in I} (1 \otimes B) \in I.$

$\Rightarrow I = A \otimes B$. $\quad A \otimes B$ simple. $\square$

<u>Ex:</u> A central simple $\Rightarrow$ So is A^{op}

A is a left mod over $A \otimes A^{op}$ via

$(a \otimes b) x = a x b \quad \forall a \in A, b \in A^{op}, x \in A$.

$A \otimes A^{op}$ is simple of dim. $(|A:k|^2)$.

$A \otimes A^{op} \longrightarrow \mathrm{End}_k(A) \cong M_n(k), \quad n = |A:k|$.

We get $A \otimes A^{op} \cong \mathrm{End}_k(A) \cong M_n(k)$. $\square$

<u>Dec 8</u>

<u>Recall:</u> A central simple k-alg.
B simple k-alg

Then $A \otimes_k B$ is a simple k-alg.

<u>Prop:</u> If A, B are central simple $/k$, then $A \otimes_k B$ is central simple.

<u>Pf:</u> Exercise.

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Def: If A, B central simple k-alg.

Say $A \sim B$ if $A \otimes M_n(k) \cong B \otimes M_l(k)$, some $n, l \geq 1$.

Recall: $J(A) = 0$, A simple $\Rightarrow A \cong M_t(D)$ some div. ring D.

Sim. $B \cong M_s(E)$ some div. ring E.

Def: A product on isom. classes of central simple k-alg.

$$[A] \cdot [B] = [A \otimes_k B]$$

Thm: $\underline{\text{Central simple } k\text{-alg.}}_{\sim} = B(k)$ is a group (Brauer Group)

Pf: "$\cdot$" is well defined a~~nd~~s above.

$$1_{B(k)} = [k] = \{M_n(k) \mid n \geq 1\}$$

[well def: $A \sim A'$, $B \sim B'$

$A \otimes M_l(k) \cong A' \otimes M_s(k)$

$B \otimes M_t(k) \cong B' \otimes M_u(k)$

$$(A \otimes \cancel{B}\, M_l(k)) \otimes (B \otimes M_t(k))$$

[$A \otimes B \cong B \otimes A$
$a \otimes b \mapsto b \otimes a$]

$$\cong A \otimes B \otimes (M_l(k) \otimes M_t(k))$$

$$\cong (A \otimes B) \otimes M_{lt}(k)$$

$$\cong (A' \otimes M_s(k)) \otimes (B' \otimes M_u(k)) \cong (A' \otimes B') \otimes M_{su}(k)$$

$$\Rightarrow A \otimes B \overset{\sim}{\cancel{\cong}} A' \otimes B' \quad]$$

$[A]^{-1} = [A^{op}]$: Recall $A \otimes A^{op} \cong \text{End}_k(A) \cong M_n(k)$

$$[A] \cdot [A^{op}] = 1_{B(k)}. \qquad \square$$

Ex: $k = \mathbb{C}$. $B(\mathbb{C}) = 1$.

Ex: $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ 4-dim $/\mathbb{R}$.

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$i^2 = j^2 = k^2 = -1$

$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$

If $q = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$

Let $\bar{q} = a - bi - cj - dk$

$q\bar{q} = a^2 + b^2 + c^2 + d^2$

If $q \neq 0$ then $q\left(\frac{1}{q\bar{q}}\bar{q}\right) = 1.$

$\mathbb{H}$ is a division ring.

$\mathbb{R} \subseteq \mathbb{C} = \mathbb{R} + \mathbb{R}i \subseteq \mathbb{H}.$

$[\mathbb{C} : \mathbb{R}] = 2 = [\mathbb{H} : \mathbb{C}].$ — $\mathbb{H}$ as a left (or right) vs over $\mathbb{C}$ [not the same!]

$B(\mathbb{R}) = \mathbb{Z}_2 = \langle [\mathbb{H}] \rangle.$ $\qquad \mathbb{H} \cong \mathbb{H}^{op}$

Suppose $k = Z(D)$, D a div. ring, and $\dim_k D < \infty$.

If $k \subsetneq D$ and $\alpha \in D \setminus k$, then $k(\alpha) = k[\alpha]$ is a field.

We can choose $k \leq L \leq D$ where L is a max'l subfield.

Note $D \otimes_k L$ simple. central simple $\otimes$ simple $\quad$ D simple as div. ring

${}_{D\otimes L}D$ via $(d \otimes l)(x) = dxl \quad \forall d \in D, l \in L, x \in D$

$\mathrm{End}_{D\otimes L}(D) = ?$ $\qquad \mathrm{End}_D({}_D D) = \{\hat{c} : x \to xc\}$

$\subseteq \mathrm{End}_D({}_D D)$

$\mathrm{End}_{D\otimes L}(D) = \{\hat{c} \mid c \in C_D(L) \underset{\uparrow}{=} L\}$

max'l of L

JDT: (Since D is fin. dim / L)

Says $D \otimes_k L \cong \mathrm{End}(D_L) \cong M_n(L)$, $n = [D:L]$

$\Rightarrow [D:k][L:k] = n^2 \dim[L:k] \quad \Rightarrow [D:k] = n^2$

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and we can show $[L:k]=n.$
$$\overset{[L:k]=n.}{\underset{[D:L]}{\|}}$$

$D = Lx_1 \oplus \dots \oplus Lx_n$ as L-v.s.
$D \cong L \oplus \dots \oplus L$ as k v.s. $\Rightarrow [D:k] = [L:k][D:L]$

Similarly on right. $[{}_kD:k] = [D_k:k]$
$\Rightarrow [{}_LD \cdot L] = [D_L : L]$

[If $\dim_{Z(D)} D = \infty$ $\Rightarrow$ can happen that $[{}_kD:k] = \infty$, $[D_k:k] < \infty$]

$\operatorname{ann}_R(M) = \{ r \in R \mid rm = 0 \ \forall m \in M \}$ simple $\Rightarrow M$ faithful ?

R noncommutative:

$c \in R$ is regular if $\begin{cases} rc = 0 \Rightarrow r = 0 \\ cr = 0 \Rightarrow r = 0 \end{cases}$

$C \subseteq R$ mult. closed set of regular elements. $1 \in C$ $[0 \notin C]$

Quotient ring RC^{-1} is a ring and a ring hom

$\varphi : R \to RC^{-1}$

such that ① $RC^{-1} = \{ \varphi(r)\, \varphi(c)^{-1} \mid r \in R, c \in C \}$

Note: $\varphi(C) \subseteq U(RC^{-1})$

② $\ker \varphi = 0$

Def: $C \subseteq R$ is a right ore set if given $r \in R, c \in C$

$\exists r' \in R, c' \in C$ s.t. $rc' = cr'$.

Thm: If R is right Noeth and $C \subseteq R$ is a mult closed set of reg. elements RC^{-1} exists.

In particular, you can take C to be set of all reg. elements

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Cor: If R is right Noeth and a domain, then R has a right division ring of quotients.

[c reg. $cR \cong R$, (R Noeth)]

[c reg in $R \Rightarrow cR$ ess R.]