MAT 732 - Homological Algebra
Spring 2018

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Lecture: MW 8:00–9:20, Carnegie 120.
Office hours: TBA; also by appointment and any time my door is open.

Text: No specific text; your notes from lecture will be the main source. They can be supplemented by the following books (on reserve in the Math. Library):

- Rotman: *An Introduction to Homological Algebra.*
- Weibel: *An Introduction to Homological Algebra.*
- Gelfand-Manin: *Methods of Homological Algebra.*

Course Description: MAT 732 is an introduction to homological algebra. Homological algebra is an important tool for solving various problems in other areas of algebra, as well as geometry and topology. Prerequisites: MAT 631, 632 and 731.

The plan is to cover at least the following topics (not necessarily in this order):

- ✓ Quick review of homological concepts from MAT 731. Projective modules, injective modules. Tensor products, flatness.
- ✓ Categories and functors. Definitions, examples, additive and abelian categories, adjoint functors. Limits, pushouts and pullbacks.
- ✓ Chain complexes, homology and homotopy. Projective and injective resolutions. Derived functors.
- ✓ Double complexes. Mapping cones. Tor and Ext.
- ✓ Dimensions. Projective, injective, and global dimension.
- ✓ Triangulated categories.
- ✓ Derived categories.

Note that triangulated and derived categories are not covered in Rotman's book, but they are discussed in the books of Gelfand-Manin and Weibel.

Grading and homework: Your course grade will be based on the homework. You are encouraged to work together to solve the homework problems, but each person must submit his/her own writeup.

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`http://disabilityservices.syr.edu`

located in Room 309 of 804 University Avenue, or call (315) 443-4498 for an appointment to discuss your needs and the process for requesting accommodations. ODS is responsible for coordinating disability-related accommodations and will issue students with documented disabilities Accommodation Authorization Letters, as appropriate. Since accommodations may require early planning and generally are not provided retroactively, please contact ODS as soon as possible. You are also welcome to contact your instructor privately to discuss your academic needs although I cannot arrange for disability-related accommodations. Making arrangements with ODS takes time. Do not wait until just before the first test.

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`http://supolicies.syr.edu/emp_ben/religious.observance.htm`

Under the policy, students are provided an opportunity to make up any examination, study, or work requirement that may be missed due to a religious observance provided they notify their instructors before the end of the second week of classes. There is an online notification process available on MySlice.

Summary

complex: seq. of homoms in A (cat.)

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r, "d_{n-1}"] & \cdots \end{tikzcd}$$

differentials

with $d_n d_{n+1} = 0$ ∀n

($\Leftrightarrow \operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n$ in R-Mod)

Complex is exact / acyclic if ∀n $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$

Subcomplex of complex $(C_\cdot, d^C)$ is a complex $(C_\cdot', d^{C'})$ s.t. $C_n' \subseteq C_n$

and ∀n have comm. diagr. with vertical maps incl.

$$\begin{tikzcd} C'_{n+1} \arrow[r, "d^{C'}_{n+1}"] \arrow[d, hook] & C'_n \arrow[d, hook] \\ C_{n+1} \arrow[r, "d^{C}_{n+1}"'] & C_n \end{tikzcd}$$

and ∀n $d^{C'}_{n+1} = d^{C}_{n+1}\big|_{C'_{n+1}}$.

morphism | chain map $(A_\cdot, d^A) \to (B_\cdot, d^B)$: family of morph. $\{f_n\}$,

$f_n: A_n \to B_n$ s.t. have

$$\begin{tikzcd} \cdots \arrow[r] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\circlearrowright"] & A_n \arrow[r, "d^A_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & B_{n+1} \arrow[r, "d^B_{n+1}"'] & B_n \arrow[r, "d^B_n"'] & B_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

⇝ cat. of chain compl. $\mathrm{Com}(A)$ ($\mathrm{Comp}(A)$) $(gf)_n = g_n f_n$, $1_{C_\cdot} = \{1_{C_n}\}$

Thm: A abelian ⇒ $\mathrm{Com}(A)$ abelian

e.g. direct sum of $(A_\cdot, d^A), (B_\cdot, d^B)$:

$$\begin{tikzcd}[column sep=large] \cdots \arrow[r] & A_{n+1} \oplus B_{n+1} \arrow[r, "\begin{bmatrix} d^A_{n+1} & 0 \\ 0 & d^B_{n+1} \end{bmatrix}"] & A_n \oplus B_n \arrow[r] & \cdots \end{tikzcd}$$

Ker f

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \ker f_{n+1} \arrow[r, dashed, "\exists!"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \ker f_n \arrow[d] \\ A_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & A_n \arrow[d, "f_n"] \\ B_{n+1} \arrow[r] & B_n \end{tikzcd}$$

univ. prop. kernel

isom: ∀n f_n isom.

chain maps

exact seq. of compl.: $0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$ s.t. have

follw. diag. with exact rows:

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ 0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^A_{n+1}"] \arrow[dr, phantom, "\circlearrowright"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^B_{n+1}"] \arrow[dr, phantom, "\circlearrowright"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\ 0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "d^A_n"] \arrow[dr, phantom, "\circlearrowright"] & B_n \arrow[r, "g_n"] \arrow[d, "d^B_n"] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r] \arrow[d, "d^C_n"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

(each $0 \to A_n \to B_n \to C_n \to 0$ exact ∀n
and ↻)

(columns not nec. exact since
$(A_\cdot, d^A)$ complex)

$\boxed{\text{Homology}}$ $(C_\bullet, d^\bullet)$ complex in $\mathcal{A}$ $\dots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots$

$\leadsto$ nth homology object of $(C_\bullet, d)$ is $H_n(C) = \frac{\operatorname{Ker}(d_n)}{\operatorname{Im}(d_{n+1})}$ $\left(\overset{\text{Rotman}}{=} \frac{Z_n(C)}{B_n(C)} \right)$

$\operatorname{Ker} d_n =$ n-cycles, $\operatorname{Im} d_{n+1} =$ n-boundaries

Have $0 \to \operatorname{Im} d_{n+1} \to \operatorname{Ker} d_n \to H_n(C) \to 0$
$0 \to \operatorname{Ker} d_n \to C_n \to \operatorname{Im} d_n \to 0$

$(C_\bullet, d)$ exact $\Leftrightarrow$ $H_n(C) = 0$ $\forall n$

ex: $0 \to 0 \xrightarrow{d_2} A \xrightarrow{d_1} B \xrightarrow{d_0} 0 \to 0$ $\to H_1(C) = \operatorname{Ker} d_1$, $H_0(C) = \operatorname{coker} d_1$, $H_n(C) = 0$ $\forall n \neq 0,1$

Prop: $f = \{f_n\}_n : A_\bullet \to B_\bullet$ chain map $\Rightarrow$ $\forall n \in \mathbb{Z}$ f induces homom's

$H_n(f) = \bar{f}_n : H_n(A) \to H_n(B)$ $\mathcal{A} = \operatorname{Mod} R$: $\bar{f}_n(x_n + \operatorname{Im} d^A_{n+1}) = f_n(x_n) + \operatorname{Im} d^B_{n+1}$

$\leadsto$ If $\mathcal{A}$ ab. cat., then $\underline{H_n}$ $\operatorname{Comp}(\mathcal{A}) \to \mathcal{A}$, $M_\bullet \mapsto H_n(M_\bullet)$
$f \mapsto H_n(f) = \bar{f}_n$

is an <u>additive functor</u>

sketch of pf: Have induced morph. $\operatorname{Ker} d_n^A \to \operatorname{Ker} d_n^B$ by univ. prop. of kernel and $\operatorname{Im} d^A_{n+1} \to \operatorname{Im} d^A_{n+1}$ (by univ. prop of coker & $\underline{\downarrow \,|\, \uparrow}$)
exer

get

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Im} d^A_{n+1} \arrow[r] \arrow[d] & \operatorname{Ker} d^A_n \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d, dashed, "\exists ! \bar{f}_n"] & 0 \\ 0 \arrow[r] & \operatorname{Im} d^B_{n+1} \arrow[r] & \operatorname{Ker} d^B_n \arrow[r] & H_n(B) \arrow[r] & 0 \end{tikzcd}$$

univ. prop. of cokernel

Thm (<u>Long exact seq. in homology</u>): $\mathcal{A}$ ab. cat., $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$

SES in $\operatorname{Comp}(\mathcal{A})$ $\Rightarrow$ $\exists$ induced exact seq. in homology

$$\dots \to H_{n+1}(C) \xrightarrow{\bar{\partial}_{n+1}} H_n(A) \xrightarrow{\bar{f}_n} H_n(B) \xrightarrow{\bar{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\bar{f}_{n-1}} H_{n-1}(B) \to \dots$$

$\partial_n : H_n(C) \to H_{n-1}(A)$ connecting homomorphism

$$\begin{tikzcd} & H_n(A) \arrow[dr] & \\ H_n(C) \arrow[ur] & & H_n(B) \arrow[ll] \end{tikzcd}$$

sketch of pf: apply <u>Snake Lemma</u> to

$$\begin{tikzcd} 0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d"] & C_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_n \arrow[r, "f_n"] & B_n \arrow[r] & C_n \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow 0 \to \operatorname{Ker} d^A_{n+1} \to \operatorname{Ker} d^B_{n+1} \to \operatorname{Ker} d^C_{n+1} \xrightarrow{\partial_n} A_n/\operatorname{Im} d^A_{n+1} \to B_n/\operatorname{Im} d^B_{n+1} \to C_n/\operatorname{Im} d^C_{n+1} \to 0$ $\forall n$

univ. prop. of coker $(0 \to \operatorname{Im} d^A_{n+1} \to A_n \to A_n/\operatorname{Im} d^A_{n+1} \to 0)$

$$\begin{tikzcd} \leadsto & A_n/\operatorname{Im} d^A_{n+1} \arrow[r] \arrow[d] & B/\operatorname{Im} d^B_n \arrow[r] \arrow[d] & C_n/\dots \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Ker} d^A_{n-1} \arrow[r] & \operatorname{Ker} d^B_{n-1} \arrow[r] & \operatorname{Ker} d^C_{n-1} & \end{tikzcd}$$

$\operatorname{Ker} \tilde{d}^A_n \overset{?}{=} \dots$ $\xrightarrow{\text{Snake}}$ $\leadsto$ $\operatorname{Ker} \tilde{d}_n \to \operatorname{Ker} \tilde{d}_n B \to \operatorname{Ker} \tilde{d}^C_n$
$\qquad /\!/ H_n(A) \qquad \cong H_n(B) \qquad \cong H_n(C)$
$\to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \dots$
$\quad \| \operatorname{coker} \tilde{d}^A_n$

□

Cor: $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ SES in $\operatorname{Com}(\mathcal{A})$, $\mathcal{A}$ ab., $\Rightarrow$ $A_\bullet, C_\bullet$ exact $\Rightarrow$ $B_\bullet$ exact

$(0 \to H_n(B_\bullet) \to 0$ exact $\Rightarrow H_n(B_\bullet) = 0 \Rightarrow B_\bullet$ exact$)$

Thm (<u>Naturality of long ex. seq. in hom.</u>): Given comm. diag.

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "f"] \arrow[d, "u"] & B_\bullet \arrow[r, "g"] \arrow[d, "v"] & C_\bullet \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r, "f'"'] & B'_\bullet \arrow[r, "g'"'] & C'_\bullet \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ get comm. diag.

$$\begin{tikzcd}
\dots \arrow[r] & H_n(A) \arrow[r, "H_n(f)"] \arrow[d, "H_n(u)"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(B) \arrow[r] \arrow[d, "H_n(v)"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(A) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r] & H_{n-1}(A') \arrow[r] & \dots
\end{tikzcd}$$

(use Long exact seq for exactness of rows, H_n functor $\Rightarrow$ first two squares commute, to show 3rd square commutes (elementwise → Rotman p. 336),

Homotopy

$f, g : A_\bullet \to B_\bullet$ chain maps are homotopic ($f \sim g$) if $\forall n \ \exists \ s_n : A_n \to B_{n+1}$ s.t.

$d_{n+1}^B s_n + s_{n-1} d_n^A = f_n - g_n$.
$[\ ds + sd = f - g]$

$$\begin{tikzcd}
\dots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}^A"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & A_n \arrow[r, "d_{n-1}^A"] \arrow[dl, "s_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & A_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}^B"'] & B_n \arrow[r, "d_n^B"'] & B_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

- $A_\bullet, B_\bullet \in \mathrm{Com}(\mathcal{A}) \Rightarrow \sim$ equiv. rel'n in $\mathrm{Hom}_{\mathcal{A}}(A_\bullet, B_\bullet)$
- $A_\bullet \xrightarrow[h]{} B_\bullet \overset{f}{\underset{g}{\rightrightarrows}} C_\bullet \xrightarrow{k} D_\bullet$, $f \sim g \Rightarrow fh \sim gh$, $kf \sim kg$
- $f \sim f'$, $g \sim g'$; $f, f', g, g' : A_\bullet \to B_\bullet \Rightarrow f + g \sim f' + g'$

$\rightsquigarrow K(\mathcal{A})$ "quot." of $\mathrm{Com}(\mathcal{A})$; $\mathrm{Ob}_j K(\mathcal{A}) = \mathrm{Ob}\,\mathrm{Com}\,\mathcal{A}$, morph: homotopy classes ($\sim$) of morph. in $\mathrm{Com}(\mathcal{A})$ ($\rightarrow K(\mathcal{A})$ add. cat.)

Prop: $f, g : A_\bullet \to B_\bullet$, $f \sim g \Rightarrow H_n(f) = H_n(g)$ show $(f_n - g_n) \mathrm{Ker}\, d_n^A \in \mathrm{Im}\, d_{n+1}^B$

$f : A_\bullet \to B_\bullet$ nullhomotopic if $f \simeq 0$ ← zero chain map

$(C_\bullet, d)$ has a contracting homotopy if 1_A is nullhomotopic $[1 = sd + ds]$

Prop: $A_\bullet$ has contracting homotopy $\Rightarrow A_\bullet$ exact $1_A \sim 0$, 1_A induces $1_{H_n(A)} \Rightarrow 1_{H_n(A)} = 0 \Rightarrow H_n(A) = 0$

Ex of complexes

e.g. → resolutions

- $M \in \mathrm{Ob}(\mathcal{A})$, $\mathcal{A}$ ab. cat. $C_\bullet : \dots \to 0 \to \underset{\overset{\downarrow}{n}}{M} \to 0 \to 0 \to \dots$ concentrated in degree n
- complex concentrated in degree 0 $\rightsquigarrow$ the stalk complex of M

($\rightsquigarrow \mathcal{A} \overset{\text{"incl"}}{\hookrightarrow} \mathrm{Comp}(\mathcal{A})$, $M \mapsto$ stalk complex, $\begin{matrix} M \\ \downarrow f \\ N \end{matrix} \mapsto$

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & M \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "0"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & N \arrow[r] & 0 & \dots
\end{tikzcd}$$

- free resolution : long exact seq. $P : \dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$, P_n free

Construction for ${}_R M$: $\exists$ free ${}_R F_0$ and $F_0 \xrightarrow{f_0} A \to 0 \rightarrow 0 \to K_0 \to F_0 \xrightarrow{f_0} A \to 0$, $\exists$ free $F_1 \xrightarrow{f_1} K_0 \to 0$

$$\begin{tikzcd}
\rightsquigarrow & & & F_1 \arrow[rr, "d_1"] \arrow[dr, "f_1"'] & & F_0 \arrow[r, "f_0"] & A \arrow[r] & 0 \\
& 0 \arrow[r] & K_1 \arrow[ur] & & K_0 \arrow[ur, "\text{incl}"'] & & &
\end{tikzcd}$$

$\mathrm{Im}\, d_1 = \mathrm{Im}\, f_1 = K_0 = \mathrm{Ker}\, f_0$ ✓

$d_1 = \text{incl} \circ f_1$
$= \text{comp.}$

continue

- (free ⇒ proj. ⇒ flat) ⇝ projective resolution, flat resolution
  (exact seq. $\dots \to P_1 \to P_0 \to A \to 0$, P_n all proj. / flat)

(P_M) $\quad \dots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} P_0 \to M \to 0$ proj. res'n

⇝ deleted resolution $\quad \dots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} P_0 \to 0$ complex

(not exact if $M \neq 0$ ($\operatorname{Im} d_1 = \operatorname{Ker} d_0 \neq \operatorname{Ker}(P_0 \to 0) = P_0$))

($M \cong \operatorname{coker} d_1 = P_0 / \operatorname{Im} d_1 \cong P_0 / \operatorname{Ker} d_0 \cong M$)

<u>Prop:</u> Every ${}_R M$ has a free resolution. <u>Prop:</u> $\mathcal{A}$ ab. cat. w/ enough proj. ⇒ every $A \in \operatorname{Ob}(\mathcal{A})$ has a proj. res'n.

- injective resolution of $A \in \operatorname{Ob}(\mathcal{A})$, $\mathcal{A}$ ab. cat.: $\quad E: \; 0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \dots$

Construction:
(if $\mathcal{A}$ has enough injectives)

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "j"] & E^0 \arrow[r, "d^0"] \arrow[dr, "\pi_0"'] & E^1 \arrow[r, "d^1"] \arrow[dr, "\pi_1"] & E^2 \arrow[r, "d^2"] & \dots \\
 & & 0 \arrow[r] & \operatorname{coker} j \arrow[u, "j'"] \arrow[r] & 0 & \operatorname{coker} d^0 \arrow[r] & 0
\end{tikzcd}$$

$\operatorname{Im} j = \operatorname{Ker} \pi_0 \doteq \operatorname{Ker}(j' \circ \pi_0)$
$= \operatorname{Ker}(d^0)$ ✓

⇝ deleted inj. res'n: $\quad E^A$, $\quad 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \dots$ $\quad [\leadsto A \cong \operatorname{Ker} d^0]$

[$P \in \operatorname{Ob} \mathcal{C}$, $\mathcal{C}$ cat is projective if whenever we have

$$\begin{tikzcd}
B \arrow[r, "g"] & C \arrow[r] & 0 \\
 & P \arrow[ul, dashed, "\exists h"] \arrow[u, "f"'] &
\end{tikzcd}$$

in $\mathcal{C}$ with g epi, $\exists h \in \operatorname{Hom}_{\mathcal{C}}(P, B)$ s.t. $gh = f$

$E \in \operatorname{Ob} \mathcal{C}$ injective :

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "g"'] & B \arrow[ul, dashed, "h"']
\end{tikzcd}$$

$f = hg$

cat $\mathcal{C}$ has enough projectives if $\forall M \in \operatorname{Ob} \mathcal{C}$ $\exists$ proj. $P \in \operatorname{Ob}(\mathcal{C})$ & epim. $P \to M$.
enough injectives if $\forall M \in \operatorname{Ob} \mathcal{C}$ $\exists$ inj. $E \in \operatorname{Ob}(\mathcal{C})$ and monom $M \to E$.

<u>Prop</u> (<u>Comparison Thm</u>): $\mathcal{A}$ ab. cat. (w/ enough proj.) with diagram

$$\begin{tikzcd}
\dots \arrow[r] & P_n \arrow[r, "d_n"] & \dots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, "p"] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
 & & \dots \arrow[r, "\delta^2"] & Q_1 \arrow[r, "\delta^1"] & Q_0 \arrow[r, "q"] & B \arrow[r] & 0
\end{tikzcd}$$

rows complexes, P_i proj. $\forall i$
bottom complex exact

⇒ ∃ induced chain map $\{f_i\}_{i \geq 0}$. If ∃ second chain map $\{g_i\}_{i \geq 0}$ ⇒ $\{f_i\}, \{g_i\}$ homotopic

pf by induction (for 1st statement): q epi, $f p \in \operatorname{Hom}(P_0, B)$, P_0 proj ⇒ $\exists f_0$, ; $\delta_i f_i \, d_{i+1} = 0$ ⇝ $\exists f_{i+1}: P_{i+1} \to Q_{i+1}$, $f_i d_{i+1}$; $Q_{i+1} \to \operatorname{Im} \delta^{i+1} \to 0$

<u>dually</u> [Rotman]

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & E^2 \arrow[r] & \dots \\
0 \arrow[r] & A' \arrow[r] \arrow[u, "g"] & X^0 \arrow[r] \arrow[u, dashed, "g_0"] & X^1 \arrow[r] \arrow[u, dashed, "g_1"] & X^2 \arrow[r] \arrow[u, dashed, "g_2"] & \dots
\end{tikzcd}$$

complexes, E^n inj., bottom row exact
⇒ $\exists \{g_i\}$

Pushouts and pullbacks

$\mathcal{C}$ cat.
$$\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$
A ~~pushout~~ pullback : triple (D,α,β) s.t. $g\alpha = f\beta$
$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

with universal property: $\forall (D',\alpha',\beta')$ with $g\alpha' = f\beta'$
$\exists!\, h$ with $\alpha h = \alpha'$, $\beta h = \beta'$
$$\begin{tikzcd} D' \arrow[dr, dashed, "\exists!h"] \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] & & \\ & D \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

Prop. If pullback exists, then unique up to isom.

Prop: Pullbacks exists in Mod R. $\rightarrow D = \{(b,c) \mid f(b) = g(c)\} \leq B \oplus C$, $\alpha(b,c) = c$, $\beta(b,c) = b$, $h = (\beta', \alpha')$

$\mathcal{C}$ cat.
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$
pushout: (D,α,β) s.t. $\alpha g = \beta f$
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \arrow[ddr, bend left, "\beta'"] & \\ C \arrow[r, "\alpha"'] \arrow[drr, bend right, "\alpha'"'] & D \arrow[dr, dashed, "\exists!h"] & \\ & & D' \end{tikzcd}$$

with universal prop.: $\forall (D',\alpha',\beta')$ with $\alpha' g = \beta' f$
$\exists!\, h$ s.t. $h\beta = \beta'$, $h\alpha = \alpha'$

Prop. If pushout exists, than unique up to isom.

Prop: Pushouts exist in Mod R. $\rightarrow D = B \oplus C / \underbrace{\{(f(a), -g(a)) \mid a \in A\}}_{=:N}$

$h((b,c)+N) = \alpha'(b) + \beta'(c)$?

$\alpha(c) = (0,c)+N$, $\beta(b) = (b,0)+N \rightarrow \alpha g = \beta f$

Prop:
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C \arrow[r] & 0 \end{tikzcd}$$
SES $\rightarrow$ get pushout
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$
exact

[pushout $\rightsquigarrow f'(a') = (0,a')+N \Rightarrow (0,a') \in N \Rightarrow a' = u(a), f(a) = 0 \Rightarrow a = 0 \Rightarrow a' = 0 \rightsquigarrow f'$ 1-1]

Cor.: If have pushout
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\alpha"'] & D \end{tikzcd}$$
f mono $\Rightarrow$ α mono
g mono $\Rightarrow$ β mono

(make lemma) applied to diag.

rem.:
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r] \arrow[d, "v"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r, "g"'] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$
Comm. diag. with exact rows $\Rightarrow$
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$
is a pushout

Dually:
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f'"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d, "t"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$
exact

exact

$\rightarrow$ get pullback
$$\begin{tikzcd} B' \arrow[r, "g'"] \arrow[d, "t"'] & C' \arrow[d, "h"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$
g onto $\Rightarrow g'$ onto
h onto $\Rightarrow t$ onto

Lemma: If
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$
pushout, then $\exists$ SES $0 \rightarrow A \xrightarrow{\begin{bmatrix} i \\ f \end{bmatrix}} B \oplus C \xrightarrow{[g, -j]} D \rightarrow 0$

Schanuel's Lemma: $M \in \operatorname{Mod} R$,
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & P \arrow[r, "p"] & M \arrow[r] & 0 \\ 0 \arrow[r] & L \arrow[r, "j"'] & Q \arrow[r, "q"'] & M \arrow[r] & 0 \end{tikzcd}$$
exact rows, P, Q proj.

$\Rightarrow P \oplus L \approx Q \oplus K$

get
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "g"] & P \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] & Q \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$
f since P proj., q onto
g from exer.
$\rightarrow$ pushout (1^{st} square) Lemma $\Rightarrow$ SES $0 \rightarrow K \rightarrow L \oplus P \rightarrow Q \rightarrow 0$, Q proj. $\Rightarrow$ split $\Uparrow$

$\boxed{\text{Derived functors}}$

$A \xrightarrow{f} B$ in ab. cat w/ enough proj., P_A, P_B proj. res'n of A, B ⇝ look at deleted res'ns $\overline{P_A}, \overline{P_B}$

comparison thm $\Rightarrow$ $\exists$ induced chain map $\{f_i\}_{i\geq 0}$ ⇝ a map over f / induced by f
$(f\varepsilon^A = \varepsilon^B f_0)$ $(: \overline{P_A} \to \overline{P_B})$ (unique to homotopy) (also denoted $\hat{f}$)

$\boxed{\text{Left derived functors}}$

$T: \mathcal{A} \to \mathcal{C}$ additive covariant functor, $\mathcal{A}, \mathcal{C}$ ab. cats, $\mathcal{A}$ enough projectives

$\forall n \in \mathbb{Z}$ construct/def. the nth left derived functor of T $L_nT: \mathcal{A} \to \mathcal{C}$:

On objects: $A \in \mathcal{A}$ choose proj. res'n of A $\quad P_A: \quad \dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$

⇝ look at deleted res'n $\overline{P_A}: \quad \dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$

⇝ apply T, get complex in $\mathcal{C}$ $\quad T(\overline{P_A}): \quad \dots \to T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} T(P_0) \to 0$

$$\to \text{def.} \quad \boxed{(L_nT)(A) = H_n(T(\overline{P_A}))} = \frac{\operatorname{Ker} T(d_n)}{\operatorname{Im} T(d_{n+1})} \quad \forall n \in \mathbb{Z} \qquad L_nT(A) = 0 \ \forall n \text{ negati}$$

Prop: $(L_nT)(A)$ does not depend on the choice of P_A

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] \arrow[d, equal] & 0 \\
\dots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \\
\dots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \\
\dots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A \arrow[r] &
\end{tikzcd}$$

Q_A $\quad \dots \to Q_2 \to Q_1 \to Q_0 \to A \to 0$ another proj. res'n ⇝ get

⇝ $\exists$ $\{f_i\}_{i\geq 0}: \overline{P_A} \to \overline{Q_A}$, $\{g_i\}_{i\geq 0}: \overline{Q_A} \to \overline{P_A}$

⇝ get $\underset{\widehat{gf}}{\{g_if_i\}_{i\geq 0}}$, $\underset{\hat{1}_{\overline{P_A}}}{\{1_{P_i}\}_{i\geq 0}}$ $: \overline{P_A} \to \overline{P_A}$ comp. thm $\Rightarrow$ they are homotopic, $\widehat{gf} \sim \hat{1}_{\overline{P_A}} \overset{\text{(lemma/below)}}{\Rightarrow} T(\widehat{gf}) \sim T(\hat{1}_{\overline{P_A}})$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \rotatebox{90}{=}$ $T(\hat{g})T(\hat{f})$

$\left.\begin{matrix}\text{Prop } f \sim g \\ \Rightarrow H_n(f) = H_n(g)\end{matrix}\right\} \Rightarrow \overline{T(g_n)}\ \overline{T(f_n)} = 1_{H_n(\overline{P_A})}$, $\overline{T(f_n)}\ \overline{T(g_n)} = 1_{H_n(T(\overline{Q_A}))}$ $\Rightarrow H_n(T(\overline{P_A})) \cong H_n(T(\overline{Q_A}))$

use Lemma/HW: $f = \{f_i\}$, $g = \{g_i\}: C \to C'$ chain maps, $C, C' \in \operatorname{Comp}(\mathcal{A})$, $\mathcal{A}$ add. cat.,

$F: \mathcal{A} \to \mathcal{B}$ add. functor, $\mathcal{B}$ add., $f \sim g \Rightarrow F(f) \sim F(g)$ where $F(f), F(g): F(C) \to F(C')$

On morphisms: $A, B \in \mathcal{A}$, $f \in \operatorname{Hom}_{\mathcal{A}}(A, B)$, choose P_A, P_B proj. res'ns and a chain map $\{f_i\}_{i\geq 0}$ over f, get

$$\begin{tikzcd}
P_A & \dots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
P_B & \dots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

⇝

$$\begin{tikzcd}
\overline{P_A} & \dots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
\overline{P_B} & \dots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & 0
\end{tikzcd}$$

⇝ get

$$\begin{tikzcd}
T(\overline{P_A}) & \dots \arrow[r] & T(P_2) \arrow[r] \arrow[d, "T(f_2)"] & T(P_1) \arrow[r] \arrow[d, "T(f_1)"] & T(P_0) \arrow[r] \arrow[d, "T(f_0)"] & 0 \\
T(\overline{P_B}) & \dots \arrow[r] & T(Q_2) \arrow[r] & T(Q_1) \arrow[r] & T(Q_0) \arrow[r] & 0
\end{tikzcd}$$

$\{Tf_i\}_{i\geq 0}$ chain map (diag commutes since T functor)
in $\mathcal{C}$: $\quad T(\overline{P_A}) \xrightarrow{\{Tf_i\}_i} T(\overline{P_B})$

⇝ get induced maps in hom.:

$\forall n$ get $\overline{T(f_n)}: \underset{(L_nT)(A)}{\underset{\parallel}{H_n(T(\overline{P_A}))}} \xrightarrow{\overline{\{T(f_i)\}_i}} \underset{(L_nT)(B)}{\underset{\parallel}{H_n(T(\overline{P_B}))}}$

⇝ def. $\forall n$ $\quad \boxed{\begin{aligned}(L_nT)(f) &= \overline{T(f_n)} \\ &= H_n(T(\{f_i\}_i))\end{aligned}}$

Check: • def'n of $(L_nT)(C)$ doesn't depend on choice of P_A, P_B and not on the choice of the liftings (use comparison thm)

Important examples: The Tor functors

1) A_R, R ring, $T = A \otimes_R -$: left R-mod's $\to Ab$ "A tensor blank"
add., cov.

$T(B) = A \otimes_R B$, $B \xrightarrow{f} C \Rightarrow T(f) = 1_A \otimes f : A \otimes_R B \to A \otimes_R C$

$$\forall n \quad \boxed{\mathrm{Tor}_n^R(A,B) := L_n(A \otimes_R \overline{P_B}) = \frac{\mathrm{Ker}(1_A \otimes d_n)}{\mathrm{Im}(1_A \otimes d_{n+1})}} \quad = H_n(A \otimes_R \overline{P_B}) = H_n(T(\overline{P_B}))$$

construction: ${}_RB \rightsquigarrow P_B \rightsquigarrow \overline{P_B} \to T(\overline{P_B}) = A \otimes_R \overline{P_B}$:

$$\ldots \to A \otimes_R P_2 \xrightarrow{1_A \otimes d_2} A \otimes_R P_1 \xrightarrow{1_A \otimes d_1} A \otimes_R P_0 \xrightarrow{1_A \otimes 0} \overset{=0}{0}$$

$\mathrm{Ker}(1_A \otimes 0) = A \otimes_R P_0$

Have $\mathrm{Tor}_0^R(A,B) = A \otimes_R B$

$= \frac{A \otimes P_0}{\mathrm{Im}(1_A \otimes d_1)} = \frac{A \otimes P_0}{\mathrm{Ker}(1_A \otimes d_0)} \cong A \otimes B$
$1_A \otimes d_0$ onto

$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0$ exact $\Rightarrow$ $A \otimes_R P_1 \xrightarrow{1_A \otimes d_1} A \otimes_R P_0 \xrightarrow{1_A \otimes d_0} A \otimes B \to 0$
exact

$\Rightarrow \mathrm{Im}(1_A \otimes d_1) = \mathrm{Ker}(1_A \otimes d_0)$

2) ${}_RB$, R ring, $T = - \otimes_R B$: right R-mod's $\to Ab$, $T(A) = A \otimes_R B$, $T(f) = f \otimes 1_B$
add., covariant

$$\forall n \quad \boxed{\widehat{\mathrm{Tor}}_n^R(A,B) = (L_nT)(A) = \frac{\mathrm{Ker}(d_n \otimes 1_B)}{\mathrm{Im}(d_{n+1} \otimes 1_B)} = L_n(\overline{P_A} \otimes_R B)}$$

$\widehat{\mathrm{Tor}}_0^R(A,B) = A \otimes_R B$

Thm: R ring, A_R, ${}_RB$. $\forall n \geq 0$ we have $\mathrm{Tor}_n^R(A,B) \cong \widehat{\mathrm{Tor}}_n^R(A,B)$.

The horseshoe lemma:

Prop: $\mathcal{A}$ ab. cat., enough proj, $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ SES in $\mathcal{A}$, P_A, P_C proj. res'ns

$\Rightarrow \exists$ proj. res'n P_B of B and chain maps s.t. we get a SES in $\mathrm{Comp}(\mathcal{A})$:

$0 \to P_A \to P_B \to P_C \to 0.$

$$\begin{tikzcd}
0 \arrow[r] & P_0^A \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "\varepsilon"] & P_0^A \oplus P_0^C \arrow[r, "{\left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]}"] \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] & P_0^C \arrow[r] \arrow[d, "\mu"] \arrow[dl, dashed, "\exists s_0"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

P_0^C proj $\Rightarrow \exists\, s_0 : P_0^C \to B$, $g s_0 = \mu$
$\pi(x,y) := f\varepsilon(x) + s_0(y)$ $\rightsquigarrow$ $\circlearrowleft$
$\rightsquigarrow \pi$ onto by snake lemma, ind. step similar

Quest. related to the horseshoe lemma:

• Assume have P_A, P_B. Can we construct P_C? Yes.

$0 \to \Omega_0 \to \Omega_? \to A \to 0$ (snake lemma)
$P_0^B = P_0^B$

$\hookrightarrow \left\{ \begin{array}{l} 0 \to A \to B \to C \to 0 \\ \rightsquigarrow 0 \to \Omega B \to \Omega C \to A \to 0 \end{array} \right.$

$\ldots \to P_2 B \to P_1 B \to \Omega_B' \to 0$
proj res'n [illegible]

• Assume have $P_B, P_C \Rightarrow$ can construct P_A.

The dimension shift therem:

$\mathcal{A}$ ab. cat., enough proj., $T: \mathcal{A} \to \mathcal{C}$ cov. add. functor, $\mathcal{C}$ ab. cat.

$A \in Ob(\mathcal{A})$, P: $\dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$ proj. resolution,

$K_i := \operatorname{Im} d_{i+1} = \operatorname{Ker} d_i \quad \forall i \geq 0 \quad \Rightarrow \forall n \geq 1$:

$$(L_{n+1}T)(A) \cong (L_nT)(K_0) \cong (L_{n-1}T)(K_1) \cong \dots \cong (L_1T)(K_{n-1})$$

$\dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} K_0 \to 0$ proj. res'n of K_0 $\Rightarrow (L_nT)(K_0) = \dfrac{\operatorname{Ker}(Td_{n+1})}{\operatorname{Im} T(d_{n+1})} = L_{n+1}(A)$

<u>Cor:</u> • $T = A \otimes_R -$, ${}_RB \Rightarrow \operatorname{Tor}^R_{n+1}(A,B) \cong \operatorname{Tor}^R_n(A, K_0) \cong \operatorname{Tor}^R_{n-1}(A, K_1) \cong \dots \cong \operatorname{Tor}^R_1(A, K_{n-1})$

• $T = - \otimes_R B \Rightarrow \widehat{\operatorname{Tor}}^R_{n+1}(A,B) \cong \widehat{\operatorname{Tor}}^R_n(K_0, B) \cong \dots \cong \widehat{\operatorname{Tor}}^R_1(K_{n-1}, B)$.

<u>Thm:</u> $\forall n \geq 0 \quad \operatorname{Tor}^R_n(A,B) \cong \widehat{\operatorname{Tor}}^R_n(A,B)$.

The long exact seq. of left derived functors

$T: \mathcal{A} \to \mathcal{C}$ cov., add., $\mathcal{A}, \mathcal{C}$ ab., $\mathcal{A}$ enough proj. $\quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ SES

$\Rightarrow \exists$ long exact seq. in $\mathcal{C}$: $\quad \dots \to (L_nT)(A) \to (L_nT)(B) \to (L_nT)(C)$

$\to (L_{n-1}T)(A) \to \dots \to (L_0T)(A) \to (L_0T)(B) \to (L_0T)(C) \to 0$.

P_A, P_C proj. resns of A, C, horseshoe lemma $\rightsquigarrow P_B \to$ exact seq in $\mathcal{C}$:
$0 \to T(\overline{P_A}) \to T(\overline{P_B}) \to T(\overline{P_C}) \to 0$. apply long exact seq. in homology
$(L_nT)(X) = 0$ for $n < 0$ $\to$ ends at 0 on the right

<u>Rem:</u> long exact seq $\Rightarrow$ dimension shift $\quad 0 \to K_0 \to P_0 \to A \to 0$

$$\rightsquigarrow \underset{\substack{\shortparallel \\ 0 \ \forall n \geq 1}}{(L_nT)(P_0)} \to (L_nT)(A) \to (L_{n-1}T)(K_0) \to \underset{\substack{\shortparallel \\ 0 \ \forall n \geq 2}}{(L_{n-1}T)(P_0)} \to \dots$$

$\Rightarrow \forall n \geq 2: (L_nT)(A) \cong (L_{n-1}T)K_0$

[$0 \to 0 \to P_0 \to P_0 \to 0$ is a proj. resn]

<u>Cor.:</u> • $0 \to L \to M \to N \to 0$ exact, A_R $\Rightarrow$ Get long exact sequence

$\dots \to \operatorname{Tor}^R_1(A, M) \to \operatorname{Tor}$

MAT 732 1/17/18

Short review of 731

Noetherian and Artinian rings and modules

R ring (always with 1). $_RM$ or M_R
left right

Def: An R-module $_RM$ is Noetherian if it satisfies the
left Noetherian
following equivalent conditions:

(1) ACC - the ascending chain condition:

$\forall$ chain of submodules of M: $M_1 \subseteq M_2 \subseteq \dots \subseteq M$

$\exists k$ with $M_k = M_{k+1} = \dots$

(2) Every nonempty family of submodules of M has a maximal element.

(3) Every submodule of M is fin. gen.

Def: $_RM$ is an artinian module if it satisfies the following
(left artinian)
equiv. cond:

(1) M satisfies DCC - the desc. chain cond:

$\forall$ chain $M \supseteq M_1 \supseteq M_2 \supseteq \dots$ $\exists k$ with $M_k = M_{k+1} = \dots$

(2) Every nonempty fam. of submod's of M has a minimal element.

Similar def'n ~~fo~~ right Noetherian and right artinian modules.

Def: A ring R is left (right) Noetherian if $_RR$ is Noetherian
—''— —''— —''— —''— artinian —''— —''— artinian
(resp. as a right module)
(—''—).

Thm: R artinian ring $\underset{\not\Leftarrow}{\Rightarrow}$ R is Noetherian

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Remark: $_RM$ Noetherian $\underset{\not\Leftarrow}{\not\Rightarrow}$ $_RM$ artinian

Def: A diagram of R-modules and R-homomorphisms

$$(*) \quad \cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots$$

is a complex if $\forall i$ we have $f_{i+1} f_i = 0$

(equivalently $\operatorname{Im} f_i \subseteq \operatorname{Ker} f_{i+1}$ $\forall i$).

The complex $(*)$ is exact (or acyclic) if

$\forall i$ $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$.

also called "(long) exact sequence"

Examples:

① $B \xrightarrow{g} C \to 0$ is exact $\Leftrightarrow$ g onto.

② $0 \to A \xrightarrow{f} B$ exact $\Leftrightarrow$ f 1-1.

③ Let $B = A \oplus C$. Then the sequence

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & A \oplus C & \xrightarrow{[0\ 1]} & C & \to & 0 \\ & & a & \mapsto & (a,0) & & & & \\ & & & & (a,c) & \longmapsto & c & & \end{array}$$

is exact.

Prop: Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules.

Then B is Noetherian (artinian) $\Leftrightarrow$ A, C are both Noetherian

(artinian respectively).

Corollary: $\bigoplus_{i=1}^{n} M_i$ is Noetherian (artinian) $\Leftrightarrow$ each M_i is Noeth

(artinian).

look at ③ & prop

Prop: Let R be a Noetherian (artinian) ring and let M be fin. gen.

Then M is Noetherian (artinian respectively).

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[Reason: $\exists n>0$ with $\underbrace{R\oplus\dots\oplus R}_{n} \to M \to 0$ Apply prev. prop.]

$\downarrow$ Noeth (art.)

Split exact sequences

Def: A homom. $A \xrightarrow{f} B$ is a splittable mono if $\exists k: B\to A$ s.t. $kf = 1_A$.

"$\Rightarrow$" $D=\mathrm{Ker}\,k$, $b\in f(A)\cap D \Rightarrow k(b) \underset{\exists a}{=} k(f(a)) = a = 0$, $b\in B \Rightarrow b = \overset{\in f(A)}{fk(b)} + \underbrace{(b-fk(b))}_{\in \mathrm{Ker}(k) = \Gamma}$

"$\Leftarrow$" $k: f(A)\oplus D \to A$, $f(a)+d \mapsto a$ f inj $\Rightarrow$ welldef'd

(f splittable mono $\Leftrightarrow$ $B = f(A)\oplus D$ for some $D \le B$.)

A homom. $B \xrightarrow{g} C$ is a splittable ~~mono~~ epi if $\exists j: C\to B$ with $gj = 1_C$. (g splittable epi $\Leftrightarrow$ $B = \mathrm{Ker}(g) \oplus C'$) $C' \le B$

"$\Leftarrow$": $c\in C \Rightarrow$ let $j(c)\in C'$ be ! $c'\in C'$ w/ $g(c')=c$ [$C' \cong B/\mathrm{Ker}\,g = \mathrm{Im}(g)$]

"$\Rightarrow$": $C' = j(C)$. $b\in \mathrm{Ker}(g)\cap C' \Rightarrow g(b) = 0 = g(j(c)) = c$,

$b\in B \Rightarrow b = \underbrace{[b-jg(b)]}_{\in \mathrm{Ker}(g)} + \underbrace{jg(b)}_{\in j(C)}$ $\exists c$

Prop: Let $0\to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact.

$\Rightarrow \mathrm{Ker}\, g = f(A) \Rightarrow f$ splittable $\Leftrightarrow B = f(A)\oplus D$, $D\le B$ $\Leftrightarrow$ g splittable

Then f is a splittable mono ~~iff~~ $\Leftrightarrow$ g is a splittable epi.

If this happens, the sequence "splits" and is also called a split exact sequence.

Constructions

all left or all right mod's

Let M be an R-module. Let $X \xrightarrow{h} Y$ be a homom. of R-modules.

Then $\mathrm{Hom}_R(M,X)$ and $\mathrm{Hom}_R(M,Y)$ are both abelian groups.

We have a homom of abelian gps

$\mathrm{Hom}(M,h)$: $\mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(M,Y)$ given by

$$\begin{tikzcd}
& X \arrow[r, "h"] & Y \\
M \arrow[ur, "f"] & &
\end{tikzcd}$$

$$\boxed{\mathrm{Hom}(M,h)(f) = hf.}$$

Similarly we can define a homom of ab gps

$\mathrm{Hom}(h,M)$: $\mathrm{Hom}_R(Y,M) \to \mathrm{Hom}_R(X,M)$, $\boxed{\mathrm{Hom}(h,M)(g) = gh}$

$$\begin{tikzcd}
X \arrow[r, "h"] & Y \arrow[dr, "g"] & \\
& & M
\end{tikzcd}$$

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Thm: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ (*) be a SES. Let M be an R-module. ① Then the sequence

$$0 \to \operatorname{Hom}_R(M,A) \xrightarrow{\operatorname{Hom}(M,f)} \operatorname{Hom}_R(M,B) \xrightarrow{\operatorname{Hom}(M,g)} \operatorname{Hom}_R(M,C)$$ is exact.

② The sequence $0 \to \operatorname{Hom}_R(C,M) \xrightarrow{\operatorname{Hom}(g,M)} \operatorname{Hom}_R(B,M) \xrightarrow{\operatorname{Hom}(f,M)} \operatorname{Hom}_R(A,M)$

is exact.

③ If (*) splits, then the induced sequences

$$0 \to \operatorname{Hom}(M,A) \to \operatorname{Hom}(M,B) \to \operatorname{Hom}(M,C) \to 0$$ and

$$0 \to \operatorname{Hom}(C,M) \to \operatorname{Hom}(B,M) \to \operatorname{Hom}(A,M) \to 0$$ are both exact.

Projectives and injective modules

Def: (and thm) Let R be a ring. Let P be an R-module.

P is projective if it satisfies the following equivalent properties:

(1) $\forall$ onto homom $B \xrightarrow{g} C \to 0$ and $h: P \to C$, $\exists$ $k: P \to B$

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u, "h"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dashed, "\exists k"'] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

with $h = gk$ (equiv. $\operatorname{Hom}(P,g): \operatorname{Hom}_R(P,B) \to \operatorname{Hom}_R(P,C)$ is onto)

(2) P is isomorphic to a direct summand of a free module.

(3) $\forall$ homom onto $B \xrightarrow{g} P \to 0$ split (ie. it is a splittable epi)

$$\begin{tikzcd} & P \arrow[dl, dashed, "s"'] \arrow[d, equal] & \\ B \arrow[r, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

$g \circ s = 1_P$

Examples

① Every free module is projective.

② If R is a PID, then P is proj. $\Leftrightarrow$ P is free.

If R is local, then P is proj. $\Leftrightarrow$ P is free (Kaplansky's thm)

If $R = k[x_1, \dots, x_n]$, k field. Then every f.g. proj. module is free.

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(Quillen - Suslin)

③ Let $R = \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$, k field

↓ lower triangular matrices with entries in k.

as a v.s. over k, $\dim_k R = 3$.

By a dimension argument, R is artinian (and Noetherian).

[ideals are k-subspaces → of dim 0,1,2,3 ⇝ only finite chains]

Every finitely gen. free R-module is of the form R^n has dimension $3n$.

But may write
$$R = \underset{\overset{\|}{P_1}}{\begin{bmatrix} k & 0 \\ k & 0 \end{bmatrix}} \oplus \underset{\overset{\|}{P_2}}{\begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}}$$

left submodules of R

$\Rightarrow R = P_1 \oplus P_2$ P_1, P_2 are projective, not free.
(as direct summ. of a free mod (R))

More generally

Say R is artinian not local. $1 = \underbrace{e_1 + \dots + e_n}$

primitive orthogonal idempotents
($e_i^2 = e_i$ $\forall i$, $e_i e_j = 0$ $\forall i \neq j$)

Then $R = Re_1 \oplus Re_2 \oplus \dots \oplus Re_n$

proj. submod of R; not free.

[In the previous example, $1_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{e_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} + \underset{e_2}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$

$e_1 e_2 = e_2 e_1 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$. $Re_1 = P_1$, $Re_2 = P_2$]

④ If in 2 we look at $R = D[X, Y]$ then there are f. gen. projectives not free.
↳ division ring (non commut.)

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Facts

$\begin{bmatrix} k & 0 \\ k & 0 \end{bmatrix} \subset \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$ not proj. (see previous page) quot. $\begin{bmatrix} k & 0 \\ k & k \end{bmatrix} / \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ not proj.

① Submodules and quotients of projective modules need not be projective (exercise: give some example)

But if $0 \to A \to B \to C \to 0$ exact and if A, C proj. $\Rightarrow$ B is projective.

② • Any direct sum of projective modules is proj. — use (2) in def'n

• A direct summ. of a proj. is proj.

• Direct product of proj. modules need not be projective.

example : $\mathbb{Z} \times \mathbb{Z} \times \dots \times ~~\mathbb{Z}~~ \dots = \prod \mathbb{Z}$ module over $\mathbb{Z}$.

countably many factors

not projective.

To prove it, show that ~~it~~ is not a free $\mathbb{Z}$-module.

Def: A module E is injective if it satisfies the equiv. cond.:

(1) Every monomorphism $0 \to E \xrightarrow{f} M$ splits (ie, splittable mono)

(2) For each monomorphism $0 \to A \xrightarrow{f} B$ the homom.

$\mathrm{Hom}_R(B, E) \xrightarrow{\mathrm{Hom}(f, E)} \mathrm{Hom}(A, E)$ is onto, that is the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "\exists h"] \\ & E & \end{tikzcd}$$

can be completed to $h: B \to E$ s.t. $hf = g$.

It looks like "projective" and "injective" are dual notions.

Remarks:

① Submod's and quotients of injective modules need not be injective.

② If $0 \to A \to B \to C \to 0$ exact and A, C are injective, then B is injective.

↑ the easiest pf would be using Baer's criterion (will talk about it Monday).

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Recall:

Quillen-Suslin: Every f.g. proj. module over $k[x_1, \ldots, x_n]$ is free. } in Rotman

Bass: Same result as above but for proj. modules that are not f.g.

Injective modules

A module ${}_R E$ is injective if ∀ exact sequence of R-modules

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ the induced secence of abelian gps

$0 \to \mathrm{Hom}_R(C, E) \xrightarrow[\mathrm{Hom}(g,E)]{} \mathrm{Hom}_R(B, E) \xrightarrow[\mathrm{Hom}(f,E)]{} \mathrm{Hom}_R(A, E) \to 0$ is exact

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[dl, dashed, "\exists h"] \\
& E \arrow[ur, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "k"'] & R \arrow[dl, dashed, "\exists h"] \\
& E \arrow[ur, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

Baer's criterion

A left module E is injective ⟺ ∀ left ideal I, every homom.

$I \to E$ can be extended to R.

Using Baer's criterion we can prove easily:

Thm: Let $\{E_i\}$ be a family of injective modules. Then $\prod_i E_i$ is injective. In particular, a finite direct sum of injective modules is injective.

Also we can prove:

Prop: Any direct summand of an injective module is injective.

— " — proj. — " — — " — proj.

Thm: Let R be a ring. Then R is Noetherian ⟺ every direct sum of inj. modules is injective. **Pf:** in Rotman's "... Homolog. Algebra"

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Kaplansky: R PID
f gen. proj. = f. gen free } in Rotman

infinitely gen. case S. Lang "Algebra"

Notation: f.g. ~ f. gen. = finitely generated

$R(\ , \ ) = \mathrm{Hom}_R(\ , \ )$

Thm: Let R be a ring. Every R-module can be embedded as a submodule in an injective module. □

Def: (1) A homom. of modules $A \xrightarrow{f} B$ is left minimal if whenever we have a commutative triangle

$$\begin{tikzcd} & B \arrow[dd, "h"] \\ A \arrow[ur, "f"] \arrow[dr, "f"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & B \end{tikzcd}$$

then h is an isomorphism.

(2) A homom $g: B \to C$ is right minimal if whenever we have a commut. diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "h"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

then h is an isomorphism.

Thm: Let R be a ring and let M be an R-module. Then, there exists an injective module E containing M such that the inclusion $i: M \to E$ is left minimal.

Def: The injective module E defined in the prev. thm is called the injective envelope of M, or the injective hull.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[dr, "i"'] & E \arrow[d, phantom, "\circlearrowleft"] \arrow[dl, "h"] \\ & E & \end{tikzcd}$$

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Characterization of injective envelope

Call $E(M)$ = inj. envelope of M.

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & M \arrow[r, hook, "f"] \arrow[d] & E(M) \arrow[dl, dashed, "\exists h"] \\ & E & \end{tikzcd}$$

Let E be any inj. module. Let f be any embedding $\Rightarrow$ h 1-1.

So the injective envelope is "the smallest" injective module containing M.

A similar concept is that of projective cover.

Def: Let M be an R-module. Then a homom $P \xrightarrow{g} M \to 0$ with P projective is a projective cover, if g is right minimal.

Proj. covers are unique up to isomorphism (if they exist).

Intuitively, the proj. covers are "smallest" projective modules mapping onto the module. They don't always exist.

? **Exercise:** Over the ring $\mathbb{Z}$, the module $\mathbb{Z}/2\mathbb{Z}$ has no proj. cover.

(Anderson-Fuller useful reading)

Assume $\mathbb{Z}/2\mathbb{Z}$ has a proj. cover ?
$\Rightarrow \exists g_{\mathbb{Z}} P \to \mathbb{Z}/2\mathbb{Z}$ onto, P proj.
Look at

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, dashed, "h"'] \arrow[d, "\pi \text{ onto}"] & \\ P \arrow[r, "g"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

${}_{\mathbb{Z}}\mathbb{Z}$ free $\Rightarrow$ $\mathbb{Z}$ proj. $\Rightarrow \exists h\, \mathbb{Z} \to P$
with $gh = \pi$
$\Rightarrow g$ onto

Exercises

✓ ① [1] Assume $\exists$ commut. diagram of modules

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "u"] \arrow[d, dashed, "\exists ! f"'] & B_1 \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & C_1 \arrow[d, "h"] \\ 0 \arrow[r] & A_2 \arrow[r, "w"'] & B_2 \arrow[r, "t"'] & C_2 \end{tikzcd}$$

with exact rows. Then $\exists ! f : A_1 \to A_2$ commuting the 1st square.

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Moreover, if g, h are isomorphisms, then f is an isom.

② Assume ∃ comm. diag. with exact rows

✓ $\boxed{2}$

$$\begin{tikzcd}
A_1 \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B_1 \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & C_1 \arrow[r] \arrow[d, dashed, "\exists! h"] & 0 \\
A_2 \arrow[r, "w"'] & B_2 \arrow[r, "t"'] & C_2 \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \exists!\ h : C_1 \to C_2$ commuting the correspond. square.

If f, g are isomorphisms, then h is an isomorphism.

③ 2.3.1 in Rotman

✓ $\boxed{4}$

Assume

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

commutes and the vertical maps are isomorphisms.

Prove that the top row is exact iff the bottom row is exact.

one direction not enough (diag w/ inverses may not commute)

Review of tensor products

Let R be a ring, let M_R and ${}_R N$. The tensor product $M \otimes_R N$ is an abelian group constructed as follows:

First let $\mathcal{F}$ = free abelian gp on the basis $M \times N$.

then let $\mathcal{S}$ = subgp generated by all the elements of the form

$(x_1 + x_2, y) - (x_1, y) - (x_2, y)$ x's in M

$(x, y_1 + y_2) - (x, y_1) - (x, y_2)$ y's in N

$(xr, y) - (x, ry)$ $r \in R$

Then $M \otimes_R N = \mathcal{F}/\mathcal{S}$. We denote by $x \otimes y$, the equiv. class of (x, y).

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The elements of $M\otimes_R N$ are sums $\sum\limits_i^{\text{finite}} x_i\otimes y_i$ $\quad x_i\in M$, $y_i\in N$.

We also have a "canonical map": $\quad M\times N \xrightarrow{f} M\otimes_R N$

$$(m,n)\mapsto m\otimes n.$$

f is "<u>biadditive</u>", that is

$$f(m_1+m_2,n)=f(m_1,n)+f(m_2,n)$$
$$f(m,n_1+n_2)=f(m,n_1)+f(m,n_2)$$
$$f(mr,n)=f(m,rn)$$

<u>Remark:</u> The tensor product may also be defined using its <u>universal property</u>:

"The tensor product of M,N is an abelian gp $M\otimes N$ together with a biadditive map $f: M\times N\to M\otimes N$ such that $\forall$ biadditive map $M\times N\to G$ where G is an abelian gp

$$\begin{tikzcd} M\times N \arrow[r, "g"] \arrow[d, "f"', "\text{canonical}"] & G \\ M\otimes_R N \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

$\exists!$ h homom. of abelian gps with $hf=g$."

<u>Fact:</u> Universal property $\Rightarrow$ the tensor product is unique up to isom.

<u>Remark:</u> Let R,S be two rings. Assume ${}_SM_R$, ${}_RN$.

$\downarrow$

"M is an S-R-bimodule"
$(sx)r=s(xr)$
$\forall s\in S, \forall r\in R\ \forall x\in M$

Then $M\otimes_R N$ is a left S-module.

(On generators $s(x\otimes y)\overset{\text{def}}{=}sx\otimes y$.)

Similarly we may have M_R, ${}_RN_S$. Then $M\otimes_R N$ is a right S-module via $(x\otimes y)s=x\otimes ys$.

In particular, if R is commutative, then $M\otimes_R N$ is an R-module via $r(x\otimes y)=rx\otimes y=x\otimes ry$.

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Constructions involving tensor products

Let $A_R \xrightarrow{f} A_R'$ and ${}_RB \xrightarrow{g} {}_RB'$ be homoms.

Then $\exists!$ homom of abelian gps $A \otimes_R B \to A' \otimes_R B'$ taking each $a \otimes b$ into $f(a) \otimes g(b)$.

This homom is denoted $f \otimes g$.

Remark:

Assume we have $A_R \xrightarrow{f} A'_R \xrightarrow{f'} A''_R$

${}_RB \xrightarrow{g} {}_RB' \xrightarrow{g'} {}_RB''$

$$\Rightarrow (f' \otimes g')(f \otimes g) = f'f \otimes g'g : A \otimes_R B \to A'' \otimes_R B''$$

Thm: ("Right exactness of tensor product")

(1) Let $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ be an exact sequence of left R-modules.

Let M be a right R-module.

Then we have an induced exact sequence of abelian gps:

$$M \otimes_R A_1 \xrightarrow{1_M \otimes f} M \otimes_R A_2 \xrightarrow{1_M \otimes g} M \otimes_R A_3 \to 0$$

(in particular, $1_M \otimes$ (onto) $=$ onto)

(2) Let $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ be an exact sequence of right R-modules and let M be a left module. The induced sequence

$$A_1 \otimes_R M \xrightarrow{f \otimes 1_M} A_2 \otimes_R M \xrightarrow{g \otimes 1_M} A_3 \otimes_R M \to 0$$ is exact.

(3) Let $0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ be a split exact sequence of left R-modules. Then, $\forall M_R$, the induced sequence

$$0 \to M \otimes_R A_1 \xrightarrow{1_M \otimes f} M \otimes_R A_2 \xrightarrow{1_M \otimes g} M \otimes_R A_3 \to 0$$ is split exact

(good exercise to do). ✓ $\boxed{6}$

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Question:

What about tensoring with a monomorphism?

Example: Look at $0 \to \mathbb{Z} \xrightarrow{\text{incl}} \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ over $\mathbb{Z}$.

$M = \mathbb{Z}/_{2\mathbb{Z}}$. Tensor with M.

$$\underbrace{\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}}_{\cong \mathbb{Z}/2\mathbb{Z}} \xrightarrow{1_{\mathbb{Z}/2\mathbb{Z}} \otimes \text{incl}} \underbrace{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Q}}_{=0}$$

since for each generator

$x \otimes y \neq 0$ $x = 1$

so such a gen. is $1 \otimes y$

$y = \frac{a}{b} = \frac{2a}{2b}$ $\Rightarrow 1 \otimes y = 1 \otimes \frac{2a}{2b} = 2 \otimes \frac{a}{2b} = 0 \otimes \frac{a}{2b}$

$= 0$

So, $\mathbb{Z}/_{2\mathbb{Z}} \otimes \mathbb{Q} = 0$

$\Rightarrow 1_{\mathbb{Z}/2\mathbb{Z}} \otimes \text{incl} = 0$

So, tensoring with a mono need not be a mono.

Def: An R-module M_R is flat if $\forall$ monomorphism

$0 \to {}_R A \xrightarrow{g} {}_R B$ the induced map $0 \to M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B$ is a mono.

Remark: M_R <u>flat</u> $\Leftrightarrow$ $\forall$ short exact sequence

$0 \to {}_R A \xrightarrow{f} {}_R B \xrightarrow{g} {}_R C \to 0$ the induced sequence

$0 \to M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B \xrightarrow{1_M \otimes g} M \otimes_R C \to 0$ is exact.

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HW exer: #3 Prove that $\prod_{i \geq 1} \mathbb{Z}$ is not projective over $\mathbb{Z}$.

Since over $\mathbb{Z}$, proj. = free, ETS $\prod_{i \geq 1} \mathbb{Z}$ is not free.

Dummit & Foote 3d edition Ex 24 p. 328 given hints

Some more results about tensor products

Prop: Let M be a left R-module. Then $\exists$ natural isomorphism of left R-modules $R \otimes_R M \xrightarrow{\ell_M} M$ taking every $r \otimes m \mapsto rm$.

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⌈ "Natural" means that $\forall$ homom. $M \xrightarrow{f} N$ we have the following comm. diagram:

$$\begin{tikzcd}
R \otimes_R M \arrow[r, "\varphi_M", "\sim"'] \arrow[d, "1_R \otimes f"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "f"] \\
R \otimes_R N \arrow[r, "\sim", "\varphi_N"'] & N
\end{tikzcd}$$

⌋

Note that we also have the similar statement for right modules M_R, i.e. $\exists$ nat. isoms of R-modules $\varphi_M : M \otimes_R R \to M$.

<u>Remark:</u> A typical elt of $R \otimes_R M$ is of the form

$$\sum r_i \otimes m_i = \sum 1 \otimes r_i m_i = 1 \otimes \underbrace{\sum r_i m_i}_{\text{in } M} = 1 \otimes x \quad \text{for some } x \in M$$

The inverse of φ_M is $\psi_M : {}_R M \to {}_R R \otimes_R M$, $\psi_M(x) = 1 \otimes x$.

Check, φ_M and ψ_M are inverse to each other & that ψ_M is a homo. ✓

Defined "flat" module.

<u>Example:</u> R_R is flat. Pf: Let ${}_R M \xrightarrow{f} {}_R N$ be a monomorphism.

Look at the diagram

$$\begin{tikzcd}
 & R \otimes_R M \arrow[r, "1_R \otimes f"] \arrow[d, "\varphi_M", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & R \otimes_R N \arrow[d, "\varphi_N", "\wr"'] \\
0 \arrow[r] & M \arrow[r, "f"] & N
\end{tikzcd}$$

$f\varphi_M$ is 1-1 (φ_M isom, f 1-1)

$\parallel$

$\varphi_N(1_R \otimes f)$ mono too $\overset{\varphi_N \text{ isom.}}{\Rightarrow}$ $1_R \otimes f$ is a mono.

<u>Consequence:</u> Every free R-module is flat.

This will follow from the following considerations:

<u>Thm:</u> Let M_R and $\{A_i\}_{i \in I}$ be a family of left R-modules. Then $\exists$ isom of abelian gps $M \otimes_R (\bigoplus_i A_i) \xrightarrow[\sim]{\psi_{M,A_i}} \bigoplus_i (M \otimes_R A_i)$

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that is, the tensor product commutes with arbitrary direct sums.

Moreover, this isomorphism is functorial in M and in the A_i-s as follows:

$\forall$ homoms $M \xrightarrow{f} N$ we have a comm. diagram

$$\begin{tikzcd}
M \otimes_R (\bigoplus_i A_i) \arrow[r, "\varphi_{M,A_i}", "\sim"'] \arrow[d, "f \otimes 1_{\oplus A_i}"] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_i (M \otimes_R A_i) \arrow[d, "\oplus(f\otimes 1_{A_i}) \text{ -- diagonal map}"] \\
N \otimes_R (\bigoplus_i A_i) \arrow[r, "\varphi_{N,A_i}"'] & \bigoplus_i (N \otimes_R A_i)
\end{tikzcd}$$

Also, assuming that $\forall i$ we have homoms $A_i \xrightarrow{f_i} B_i$ we have induced commutative diagrams

$$\begin{tikzcd}
M \otimes_R (\bigoplus_i A_i) \arrow[r, "\varphi_{M,A_i}", "\sim"'] \arrow[d, "1_M \otimes (\oplus f_i)"] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_i (M \otimes A_i) \arrow[d, "\oplus(1_M \otimes f_i)"] \\
M \otimes_R (\bigoplus_i B_i) \arrow[r, "\varphi_{M,B_i}", "\sim"'] & \bigoplus_i (M \otimes B_i)
\end{tikzcd}$$

$\oplus A_i \xrightarrow{\oplus f_i} \oplus B_i$

$M \otimes_R A_i \xrightarrow{1_M \otimes f_i} M \otimes_R B_i$

Thm: Let $\{M_i\}$ be a family of right R-modules. Then $\bigoplus_i M_i$ is flat $\Leftrightarrow$ each M_i is flat.

(another way of putting it: "a module is flat $\Leftrightarrow$ every direct summand is flat")

Pf: Let ${}_R A \xrightarrow{f} {}_R B$ be a monomorphism. Look at the comm. diagram of abelian gps.

$$\begin{tikzcd}
\bigoplus_i (M_i \otimes_R A) \arrow[r, "\oplus(1_{M_i} \otimes f)", "\text{\textemdash\ ``diagonal map''}"'] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_i (M_i \otimes_R B) \arrow[d, "\wr"] \\
(\bigoplus_i M_i) \otimes_R A \arrow[r, "1_{\oplus M_i} \otimes f"] & (\oplus M_i) \otimes_R B
\end{tikzcd}$$

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$\bigoplus_i M_i$ flat $\Leftrightarrow$ $\forall$ $f: A \to B$ mono the map $1_{\bigoplus_i M_i} \otimes f$ is mono.

vertical $\Leftrightarrow$ maps are isoms $\quad \forall f: A \to B$ $\bigoplus_i (1_{M_i} \otimes f)$ is mono

this map is $\Leftrightarrow$ the diagonal map $\quad \forall f: A \to B$ each of $1_{M_i} \otimes f$ is mono

$\Leftrightarrow$ $\quad$ each M_i is flat. $\square$

Thm: (1) Every free module is flat

(2) Every projective module is flat.

Pf: (1) F free $\Rightarrow$ $F \simeq \bigoplus_{i \in I} R$, R_R flat. So we may apply the previous thm. & use that R_R is flat

(2) P projective $\Rightarrow$ P iso to a direct summand of a free module
$\Rightarrow$ P is flat by other previous thm.

flat
direct summ of flat is flat

Examples

① $\mathbb{Q}$ is flat over $\mathbb{Z}$ but not projective.

This is a special case of the following:

Let R be a commutative domain and let S be a multiplicative subset of R (that is, if $a, b \in S \Rightarrow ab \in S$, $0 \notin S$ and $1 \in S$).

Let $S^{-1}R$ or R_S denote the ring of fractions. It is a flat R module.

$S^{-1}R$ flat: $0 \to M \xrightarrow{f} N$ exact $\to$ $0 \to S^{-1}R \otimes M \to S^{-1}R \otimes N$

In fact we have a natural isomorphism of R_S modules.

$0 \to S^{-1}M \to S^{-1}N$

$\frac{m}{s} \mapsto \frac{f(m)}{s}$

1-1 since f 1-1

$\to$ bottom exact

$\Rightarrow$ top exact

$$\begin{tikzcd}
M \arrow[d, "f"] & R_S \otimes_R M \arrow[r, "\varphi_M", "\sim"'] \arrow[d, "1_{R_S} \otimes f"] & M_S \arrow[d, "f_S"] \\
N & R_S \otimes_R N \arrow[r, "\varphi_N", "\sim"'] & N_S
\end{tikzcd}$$

$\rightarrow$ module of fractions

Or show directly $1_{S^{-1}R} \otimes f$ 1-1 (Quinn)

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[We also have a isom of R_S-modules $\mathrm{Hom}_R(M,N)_S \xrightarrow{\sim} \mathrm{Hom}_{R_S}(M_S, N_S)$]

② $\prod_{i\in I}\mathbb{Z}$ is flat over $\mathbb{Z}$ (justification coming later)

free $\underset{\not\Leftarrow}{\Rightarrow}$ projective $\underset{\not\Leftarrow}{\Rightarrow}$ flat

<u>Thm:</u> Let R be a ring and let $I \trianglelefteq R$ be a 2-sided ideal. Then $\forall$ left R-modules M, we have a natural isomorphism. of R/I-modules,

$$R/I \otimes_R M \xrightarrow{\varphi_M} M/IM$$

taking $(r+I)\otimes x \longmapsto rx + IM.$

[natural means that $\forall\ f : {}_RM \to {}_RN$ we have a comm. diagram

$$\begin{tikzcd}
R/I \otimes_R M \arrow[r, "\varphi_M", "\sim"'] \arrow[d, "1_{R/I}\otimes f"] & M/IM \arrow[d, "\bar{f}"] \\
R/I \otimes_R N \arrow[r, "\varphi_N", "\sim"'] & N/IN
\end{tikzcd}$$

$\bar{f} = ?$

$M \xrightarrow{f} N$

$f(IM) \subseteq IN$

$\Downarrow$

f induces $\overset{\text{unique}}{\text{a}}$ homom

$\bar{f} : M/IM \to N/IN$

s.t. the diagram

Exer 2
1/22/18 ←

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
IM \arrow[r, "f|_{IM}"] \arrow[d] & IN \arrow[d] \\
M \arrow[r, "f"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[d] \\
M/IM \arrow[r, dashed, "\exists!"] \arrow[d] & N/IN \arrow[d] \\
0 & 0
\end{tikzcd}$$

┘

1/24/18

Very useful result:

The Snake Lemma

Assume we have the following commutative diagram of R-modules:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C'
\end{tikzcd}$$

with exact rows. Then, there exists an exact sequence of R-modules

$$\mathrm{Ker}\, f \to \mathrm{Ker}\, g \to \mathrm{Ker}\, h \xrightarrow{\textcircled{\delta}} \mathrm{Coker}\, f \to \mathrm{Coker}\, g \to \mathrm{Coker}\, h$$

→ "connecting homomorphism"

show existence w/ univ. prop. of kernel

univ. prop of cokernel

Prove its existence by diagram chasing

Moreover, if u is 1-1 then $\mathrm{Ker}\, f \to \mathrm{Ker}\, g$ is also 1-1 and, if v' is onto, then $\mathrm{Coker}\, g \to \mathrm{Coker}\, h$ is also onto.

In particular, if we have a comm. diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

⇒ ∃ ~~ses~~ exact sequence $0 \to \mathrm{Ker}\, f \to \mathrm{Ker}\, g \to \mathrm{Ker}\, h \to \mathrm{Coker}\, f$

$\to \mathrm{Coker}\, g \to \mathrm{Coker}\, h \to 0.$

"idea of pf"

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{Ker}\, f \arrow[r, dashed] \arrow[d] & \mathrm{Ker}\, g \arrow[r, dashed] \arrow[d] & \mathrm{Ker}\, h \arrow[d] & x \arrow[d] \\
 & A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[d, "h"] & b \arrow[r] \arrow[d, "g"] \arrow[r, phantom] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[d] & a' \arrow[r] \arrow[d] & b' \\
 & \mathrm{Coker}\, f \arrow[r, dashed] \arrow[d] & \mathrm{Coker}\, g \arrow[r, dashed] \arrow[d] & \mathrm{Coker}\, h \arrow[d] & \textcircled{*} & \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

← show ex. w/ univ. prop. kernel
then show exactness
$\mathrm{Ker}\, f \to \mathrm{Ker}\, g \to \mathrm{Ker}\, h$

← show does not depend on choice of b

← def map $\mathrm{Ker}\, h \xrightarrow{\delta} \mathrm{Coker}\, f$
show: does not depend on choice

then show exactness

✓ 7-12

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Thm: Let R be a ring and let $I \triangleleft R$. Let

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of left R-modules.

f inclusion map

Then the sequence

$$0 \to R/I \otimes_R A \xrightarrow{1_{R/I}\otimes f} R/I \otimes_R B \xrightarrow{1_{R/I}\otimes g} R/I \otimes_R C \to 0 \quad \text{is exact}$$

$\Leftrightarrow \; IB \cap A = IA.$

Remark:

① The sequence $R/I \otimes_R A \xrightarrow{1_{R/I}\otimes f} R/I \otimes_R B \xrightarrow{1_{R/I}\otimes g} R/I \otimes_R C \to 0$

is always exact.

So the problem is with $1_{R/I}\otimes f$ being mono.

② $IB = \{ \overset{\text{finite}}{\sum} \alpha_i b_i \mid \alpha_i \in I, b_i \in B \}$

Similarly $IA = \{ \sum \alpha_i a_i \mid \alpha_i \in I, a_i \in A \}$

③ Always have the inclusion $IA \subseteq IB \cap A$.

$A \subseteq B, \; IA \subseteq A$ ✓

$\Downarrow$

$IA \subseteq IB$

④ Have an exact seq. of left R-modules

$$0 \to IB \cap A \xrightarrow{\text{incl}} IB \xrightarrow{g|_{IB}} IC \to 0$$

exercise ✓ $\boxed{13}$

⑤ Look at the diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & IB \cap A \arrow[r] \arrow[d, "\text{incl}"] \arrow[dr, phantom, "\circlearrowleft"] & IB \arrow[r] \arrow[d, "\text{incl}"] \arrow[dr, phantom, "\circlearrowleft"] & IC \arrow[r] \arrow[d, "\text{incl}"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/IB\cap A \arrow[r] \arrow[d] & B/IB \arrow[r] \arrow[d] & C/IC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

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The Snake lemma ⇒ ∃ short exact sequence which is the bottom one

$$\begin{tikzcd} 0 \arrow[r] & A/_{IB\cap A} \arrow[r] \arrow[d, squiggly] & B/_{IB} \arrow[r] \arrow[d] & C/_{IC} \arrow[r] \arrow[d, no head] & 0 \\ 0 \arrow[r] & R/_I \otimes A \arrow[r] & R/_I \otimes B \arrow[r] & R/_I \otimes C \arrow[r] & 0 \end{tikzcd}$$

⑥ use this and previous results to finish the proof → see 1/29/18

Thm: Let $I \triangleleft R$. Let $0 \to A \xhookrightarrow{f} B \xrightarrow{g} C \to 0$ be exact with f= inclusion. Then the induced sequence

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$$0 \to R/_I \otimes_R A \xrightarrow{1_{R/I}\otimes f} R/_I \otimes_R B \xrightarrow{1_{R/I}\otimes g} R/_I \otimes_R C \to 0$$ is exact ⟺ $IA = A \cap IB$.

Pf: We always have a comm. exact diagram:

$$\begin{tikzcd} R/_I \otimes A \arrow[r, "1_{R/I}\otimes f"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & R/_I \otimes B \arrow[r, "1_{R/I}\otimes g"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & R/_I \otimes C \arrow[r] \arrow[d, "\wr"] & 0 \\ R\ A/_{IA} \arrow[r, "\bar{f}"'] & B/_{IB} \arrow[r, "\bar{g}"'] & C/IC \arrow[r] & 0 \end{tikzcd}$$

Use the fact that we have a natural isomorphism

$R/_I \otimes_R M \to M/_{IM}$

$(r+I)\otimes x \mapsto rx + IM$

∀ mod's M.

Since the vertical maps are isomorphisms, it follows that

$1_{R/I} \otimes f$ is 1-1 ⟺ $\bar{f}$ is 1-1.

So what we have to prove is that the sequence

$$0 \to A/_{IA} \xrightarrow{\bar{f}} B/_{IB} \xrightarrow{\bar{g}} C/_{IC} \to 0$$ is exact ⟺ $IA = A \cap IB$. (*)

Look at the comm. exact diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & IB\cap A \arrow[r, "f|_{IB\cap A}"] \arrow[d, "f'"] & IB \arrow[r, "g|_{IB}"] \arrow[d, "g'"] & IC \arrow[r] \arrow[d, "h'"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

(*)

top row exact by remark ④
→ turn page

Apply the snake lemma to it. Get

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A/_{IB\cap A} \arrow[r] \arrow[d, equal] & B/IB \arrow[r] \arrow[d, equal] & C/_{IC} \arrow[r] \arrow[d, equal] & 0 \\ \ker h' & A/\operatorname{Im}(f') & \operatorname{coker} g' & \operatorname{coker} h' & \end{tikzcd}$$ is exact.

$= \operatorname{coker} f'$

Prove (*): "⇐" 1/29/18

Assuming $IB \wedge A = IA \Rightarrow$ The induced sequence

$$0 \to A/IA \to B/IB \to C/IC \to 0$$ is exact.

"⇒" Assume now that the induced seq. $0 \to A/IA \to B/IB \to C/IC \to 0$ is exact.

Look at the diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\pi_A"] & B \arrow[r, "g"] \arrow[d, "\pi_B"] & C \arrow[r] \arrow[d, "\pi_C"] & 0 \\
0 \arrow[r] & A/IA \arrow[r] \arrow[d] & B/IB \arrow[r] \arrow[d] & C/IC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

← vertical maps are the canonical projections.

It is an exact comm. diagram. Apply the snake lemma to it.

Have an exact sequence: $0 \to \overset{\ker \pi_A}{\overset{\shortparallel}{IA}} \xrightarrow{f|_{IA}} \overset{\ker \pi_B}{\overset{\shortparallel}{IB}} \xrightarrow{g|_{IB}} \overset{\ker \pi_C}{\overset{\shortparallel}{IC}} \to \underset{\operatorname{coker}\pi_A}{\underset{\shortparallel}{0}}$

(by snake lemma) $\uparrow$ since coker $\pi_A = 0$, π onto

Get a comm. diagram:

$$\begin{tikzcd}
0 \arrow[r] & IB \wedge A \arrow[r] \arrow[d, "i"] & IB \arrow[r] \arrow[d, equal] & IC \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & IA \arrow[r] & IB \arrow[r] & IC \arrow[r] & 0
\end{tikzcd}$$

← always exact rem. ④

"=" are isos ⇒ by exer ① from 1/22/18 i is an isom.

⇒ left most inclusion is an isomorphism.

$\Rightarrow IB \wedge A = IA$.

□

<u>Adjoint isomorphism theorem</u> (Rotman p. 92-93)

<u>First version:</u> Given rings R, S and A_R, ${}_R B_S$, C_S, there exists a natural isomorphism (in each variable)

$$\tau_{A,B,C} : \operatorname{Hom}_S(A \otimes_R B, C) \xrightarrow{\sim} \operatorname{Hom}_R(A, \operatorname{Hom}_S({}_R B_S, C_S))$$

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Where, if $A \otimes_R B \xrightarrow{f} C$ is an S-homom, then

$$\tau(f)(a)(b) = f(a \otimes b) \quad \forall a \in A, b \in B.$$

[Natural in A means that if $\exists g: A \to A'$ then $\exists$ induced

$A \otimes B \xrightarrow{g \otimes 1_B} A' \otimes B$ and a comm. diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(A \otimes B, C) \arrow[r, "\tau_{A,B,C}"] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)) \\
\mathrm{Hom}_S(A' \otimes_R B, C) \arrow[u, "\mathrm{Hom}_S(g \otimes 1_B, C)"'] \arrow[r, "\tau_{A',B,C}"] & \mathrm{Hom}_R(A', \mathrm{Hom}_S(B,C)) \arrow[u, "\mathrm{Hom}_R(g, \mathrm{Hom}_S(B,C))"']
\end{tikzcd}$$

Similar comm. diagrams come from naturality in B and C.

Pf:

Claim: τ is a homom of abelian groups.

Pf of claim:

- $\tau(f_1+f_2)(a)(b) = (f_1+f_2)(a \otimes b) = f_1(a \otimes b) + f_2(a \otimes b)$

$$= \tau(f_1)(a)(b) + \tau(f_2)(a)(b) = (\tau(f_1)+\tau(f_2))(a)(b)$$

$\forall a \in A \ \forall b \in B \Rightarrow \tau(f_1+f_2) = \tau(f_1)+\tau(f_2)$.

Need an inverse for this τ:

Let $\sigma = \sigma_{A,B,C} : \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)) \to \mathrm{Hom}_S(A \otimes_R B, C)$ be defined as follows: Let $g: A \to \mathrm{Hom}_S(B,C)$ be a homom of R-modules, let $\boxed{\sigma(g)(a \otimes b) = g(a)(b).}$

$\rightsquigarrow$ that is $\sigma(g)(\Sigma a_i \otimes b_i) = \Sigma g(a_i)(b_i)$

Is σ well defined? We have

$$\begin{tikzcd}
A \times B \arrow[r, "h"] \arrow[d, "\text{canonical map}"'] & C \\
A \otimes_R B &
\end{tikzcd}$$

given by $h(a,b) = g(a)(b)$

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If we show that h is biadditive then the universal property of tensor products will tell us that $\exists!$ $A \otimes_R B \to C$, $a \otimes b \mapsto g(a)(b)$

So σ will be well-defined:

g homom of R-mods

Is σ biadditive? Verify $h(ar, b) = g(ar)(b) \overset{\circledast}{=} (g(a)r)(b)$
$r \in R$

$\overset{\circledast}{=} g(a)(rb) = h(a, rb)$

$\circledast$ $\mathrm{Hom}_S({}_R B_S, C_S)$ is a right R-module, via

$(gr)(b) = g(rb)$ $\qquad g: {}_R B_S \to C_S$

Easy to check that h is additive in each variable.

Exercise: $\sigma_{A,B,C}$ and $\tau_{A,B,C}$ are inverse to each other. $\square$

$(\sigma_{A,B,C}(\tau_{A,B,C}(f)))(a \otimes b) = (\tau_{A,B,C}(f))(a)(b) = f(a \otimes b)$

$\Rightarrow (\sigma \circ \tau)(f) = f \Rightarrow \sigma \circ \tau = 1$...

$(\tau(\sigma(g)))(a)(b)$

$= \sigma(g)(a \otimes b) = g(a)(b)$

$\Rightarrow \tau \circ \sigma(g) = g$

$\Rightarrow \tau \circ \sigma = 1$...

Basic things about categories and functors

To define a category $\mathcal{C}$ we need 3 pieces of data:

(1) A class (or collection) of objects denoted $\mathrm{Ob}\,\mathcal{C}$:

$A, B, C, \ldots, X, Y, \ldots$ [write $A \in \mathrm{Ob}\,\mathcal{C}$ or abuse notation and write $A \in \mathcal{C}$]

(2) $\forall A, B \in \mathcal{C}$, there exists a set of morphism "from A to B" denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
$\downarrow$
Note: This could be empty!

satisfying:

[I] $\forall$ triple $A, B, C \in \mathrm{Ob}\,\mathcal{C}$ we have a mapping

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(A, C)$$

$$(f, g) \longmapsto f \circ g \quad (= fg)$$

called the composition.

This composition satisfies the "associativity" rule,

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that is $\forall f \in \mathrm{Hom}_{\mathcal{C}}(C,D)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(A,B)$

we have $(fg)h = f(gh)$.

(II) $\forall A \in \mathrm{Ob}\,\mathcal{C}$, $\exists$ an element $\underline{\underline{1_A}} \in \mathrm{Hom}_{\mathcal{C}}(A,A)$

satisfying $f \cdot 1_A = f$ $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $1_A \cdot g = g$ $\forall g \in \mathrm{Hom}_{\mathcal{C}}$

Examples:

(1) $\mathcal{C} = \mathrm{Mod}\, R$ modules over a ring R.

$\mathrm{Ob}\,\mathcal{C} =$ all the R-modules

morphisms $=$ all the R-~~h~~module homoms

(2) $\mathcal{C} = \mathrm{mod}\, R$ f. generated R-modules.

(3) $\mathcal{C} = \mathrm{Gr}$ the category of groups

Objects: all the groups

Morphisms: the group homoms.

(4) $\mathcal{C} = \mathrm{Ab}$ $\mathrm{Ob}\, A =$ all the abelian groups

$= \mathrm{Mod}\, \mathbb{Z}$.

Def: Let $\mathcal{C}$ be a category. A subcategory $\mathcal{D}$ of $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}\,\mathcal{D} \subseteq \mathrm{Ob}\,\mathcal{C}$.

For all $A, B \in \mathcal{D}$ we have that $\mathrm{Hom}_{\mathcal{D}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$

$\mathcal{D}$ is called a full subcategory if $\forall A, B \in \mathrm{Ob}\,\mathcal{D}$ we have

$\mathrm{Hom}_{\mathcal{D}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$

Examples:

① $\mathrm{mod}\, R \overset{\text{full}}{\subset} \mathrm{Mod}\, R$ subcat.

② $\mathrm{Ab} \overset{\text{full}}{\subset} \mathrm{Gr}$.

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Other examples

(5) **Sets** : the category of sets   Objects = sets
Morphism = maps between sets

(6) **Top**   Objects = all topological spaces
Morphisms = all continuous maps

Axiom: If $\mathcal{C}$ is a category, then the sets $\mathrm{Hom}_{\mathcal{C}}(A,B)$, $\mathrm{Hom}_{\mathcal{C}}(A',B')$ are equal $\Leftrightarrow$ $A=A'$ and $B=B'$!

(7) Let $X \neq \emptyset$ be a partially ordered set

(That is, $\exists$ partial order $<$ on X satisfying
$x<x \quad \forall x \in X$
$x<y$ and $y<x \Leftrightarrow x=y$
$x<y, y<z \Rightarrow x<z$)

View X as a category with $\mathrm{Ob}\,X$ = the elements of X.

if $x, y \in X$

$$\mathrm{Hom}(x,y) = \begin{cases} \emptyset & \text{if } x \not< y \\ \{e_{xy}\} & \text{if } x<y \end{cases}$$

~~they are not comparable~~

↓

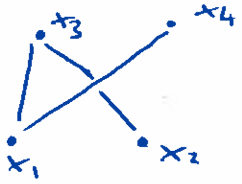
Set consisting of <u>one</u> element called e_{xy}

Example

Notation 

means $x<y$

Consider $X = \{x_1, x_2, x_3, x_4\}$



$x_1 < x_4$
$x_1 < x_3$
$x_2 < x_3$

$\mathrm{Hom}(x_1,x_3) = \{e_{13}\}$ $\mathrm{Hom}(x_2,x_4) = \emptyset$. ...

Definitions Let $\mathcal{C}$ be a category. Let $X, Y \in \mathrm{Ob}\,\mathcal{C}$ and let $\underline{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)}$

Then

(1) f is an isomorphism if $\exists\, \underline{g} \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that

$\underline{gf = 1_X}$ and $\underline{fg = 1_Y}$.

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(2) f is a monomorphism (or monic) if $\forall Z \in \mathrm{Ob}\,\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$, if $fg = fh$ then $g = h$.

$f: B \to C$ monom $\Leftrightarrow$ $f_*: \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$ injection $\forall A$

add. cat.: $\mathrm{Hom}(A,B)$ ab. gp $\rightsquigarrow$ f mono $\Leftrightarrow$ ($fg = 0$ $\Rightarrow g = 0$)

in Sets & $_R$Mod: f mono $\Leftrightarrow$ f 1-1

(3) f is an epimorphism if $\forall Z \in \mathrm{Ob}\,\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, if $gf = hf \Rightarrow g = h$.

f epi $\Leftrightarrow$ f^* injection

(4) f is a split monomorphism if $\exists g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ with $gf = 1_X$.

Similary may define split epimorphism. f split epi: $\exists j$: $fj = 1_Y$

Fact:

f isomorphism $\Rightarrow$ f split mono $\Rightarrow$ f mono

f isomorphism $\Rightarrow$ f split epi $\Rightarrow$ f epi

$\left\{ \begin{array}{l} \exists g: gf = 1_X \\ \text{Let } fh = fh' \\ \Rightarrow \underset{h}{\overset{\parallel}{gfh}} = \underset{h'}{\overset{\parallel}{gfh'}} \Rightarrow f \text{ mono} \end{array} \right.$

$\exists j$: $fj = 1_Y$ Let $gf = hf$

$\Rightarrow \underset{g}{\overset{\parallel}{gfj}} = \underset{h}{\overset{\parallel}{hfj}}$ $\Rightarrow g = h \Rightarrow f$ epi

Remarks

① If $\mathcal{C}$ = Sets, Modules, groups, then all monos are injective,

$f: A \to B$, $f(a) = 0$ $\Rightarrow$ $f \circ i_{(a) \hookrightarrow A} = f \circ 0$ $\Rightarrow i = 0$ $\Rightarrow a = 0$

and all epis are surjective.

$g: A \to B$ $\pi: B \to B/g(A)$ proj. $\Rightarrow$ $0 = \pi \circ g = 0 \circ g: A \to B/g(A)$

$\Rightarrow \pi = 0 \Rightarrow B/g(A) = 0 \Leftrightarrow g(A) = B \Rightarrow g$ onto

② Let $\mathcal{C}$ = Rings with 1. Morphisms are all ring homoms.

Exercise: The inclusion $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category of rings. Clearly not onto.

If $\varphi: \mathbb{Q} \to R$ is a ring hom.

$\forall \frac{p}{q}, q \neq 0$: $\varphi(\frac{p}{q}) = \varphi(p)\varphi(q)^{-1}$

$\rightsquigarrow \varphi$ determined by $\varphi|_{\mathbb{Z}}$

If $\psi: \mathbb{Q} \to R$ any R-hom and

$\psi \circ i = \varphi \circ i \Rightarrow \varphi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} \Rightarrow \varphi = \psi \Rightarrow i$ epic

[$\varphi(1) = 1$, $\varphi(n) = n\varphi(1)$ $\rightsquigarrow$ can be only the hom. $\varphi: \mathbb{Q} \to R$]

Special objects

- an object $A \in \mathcal{C}$ is an initial object if $\forall B \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has precisely one element.

A initial $\to$ $\mathrm{Hom}_{\mathcal{C}}(A, A) = \{1_A\}$

- an object $A \in \mathcal{C}$ is terminal if $\forall A \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(B, A)$ has precisely one element.

A terminal $\to$ $\mathrm{Hom}_{\mathcal{C}}(A, A) = \{1_A\}$

- a zero element of $\mathcal{C}$ is an element that is both initial and terminal.

two zero obj. are isom.

$$\begin{tikzcd} & \text{zero obj} \arrow[dl] \arrow[dr] & \\ A \arrow[rr, shift left, "f"] & & B \arrow[ll, shift left, "g"] \end{tikzcd}$$

$\rightsquigarrow f \circ g = 1_B$, $g \circ f = 1_A$ since $\mathrm{Hom}(B, B$

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Examples

① in Mod R, mod R 0 is a zero element.

$\underline{0 \text{ terminal:}}$ $\mathrm{Hom}(A,0)=\{0\}$ since if $f: A\to 0 \rightsquigarrow f(a)=0$ $\forall a\in A$ $\Rightarrow f\equiv 0$

$\underline{0 \text{ initial:}}$ $\mathrm{Hom}(0,A)=\{0\}$ since if $f: 0\to A$, f nonzero $\to f(0)\neq 0$ $\to f$ no homom. $(f(0)=0)$

② in the partially ordered set example, there are no initial (terminal) objects.

for example, $\mathrm{Hom}(x_1,x_3)$ & $\mathrm{Hom}(x_1,x_4)$ have only one elem
but $\mathrm{Hom}(x_1,x_2)=\emptyset$

Exercise: If they exist the initial (terminal) objects are unique up to isomorphism.

$\underline{\text{initial:}}$ Supp. A, B are initial $\Rightarrow \mathrm{Hom}_{\mathcal{C}}(A,B)=\{f\}$, $\mathrm{Hom}_{\mathcal{C}}(B,A)=\{g\}$

we have $g\circ f \in \mathrm{Hom}_{\mathcal{C}}(A,A)\underset{\substack{\uparrow\\ A \text{ initial}}}{=}\{1_A\}$, $f\circ g\in \mathrm{Hom}_{\mathcal{C}}(B,B)\underset{\substack{\uparrow\\ B \text{ initial}}}{=}\{1_B\}$ $\Rightarrow f\circ g=1_B, g\circ f=1_A$

$\Rightarrow A\cong B$.

$\underline{\text{terminal:}}$ Same argument

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We also have the notion of direct products and direct sums in categories.

Def: Let $\mathcal{C}$ be a category. Let I be a set and let $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$.

1) An object X together with a family of morphisms $\{\pi_i\}_{i\in I}$ where $\pi_i\in \mathrm{Hom}_{\mathcal{C}}(X,A_i)$ is a product (or direct product) if it satisfies the following universal property:

For each object $Y\in\mathcal{C}$ and family of homoms $\varphi_i\in \mathrm{Hom}_{\mathcal{C}}(Y,A_i)$ $\exists!$ $h\in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ s.t. each diagram

$$\begin{tikzcd}
 & Y \arrow[dl, dashed, "\exists h"'] \arrow[d, "\varphi_i"] \\
X \arrow[r, "\pi_i"'] \arrow[ur, phantom, "\circlearrowleft"] & A_i
\end{tikzcd}$$

commutes.

Can prove: if product exists in $\mathcal{C}$, they are unique up to isomorphism. ✓ $\rightsquigarrow \boxed{B}$

Notation: $\prod_{i\in I} A_i$.

[**Note:** They exist in Mod R, but do not $\overset{\text{necessarily}}{\text{exist}}$ in mod R.]

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or, if $\mathcal{C}$ = all free abelian groups

then products <u>need not exist there</u>

(2) An object C in a $\mathcal{C}$, together with a family of morphisms $\{k_i\}_{i\in I}$, $k_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, C)$ is a coproduct (or direct sum) if $\forall$ objects $Y \in \mathcal{C}$ and morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, Y)$ $\exists!$ $h \in \mathrm{Hom}_{\mathcal{C}}(C, Y)$ commuting the diagrams:

$$\begin{tikzcd}
C \arrow[dr, dashed, "\exists! h"] & \\
A_i \arrow[u, "k_i"] \arrow[r, "\varphi_i"'] \arrow[r, phantom, bend left, "\curvearrowright"] & Y
\end{tikzcd}$$

<u>Notation:</u> $C = \coprod_{i\in I} A_i$ (or $C = \bigoplus_{i\in I} A_i$)

If they exist, coproducts are unique up to isom. ✓ $\rightsquigarrow \boxed{14}$

<u>Examples:</u>

- Coproducts exist in the category of free R-modules.
  (full subcategory of Mod R)
- Coproducts need not exist in mod R.

<u>Facts:</u> $\mathcal{C}$ category. Assume products and coproducts exist. Then

① $\coprod_{i=1}^{n} A_i \cong \prod_{i=1}^{n} A_i$

② $\mathrm{Hom}_{\mathcal{C}}(\coprod_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A_i, B)$

③ $\mathrm{Hom}_{\mathcal{C}}(B, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(B, A_i)$.

<u>Exer:</u> due Monday. (HW2)

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Examples: ① $\mathcal{C}$ = Sets.   Products = cartesian products
coproducts = disjoint union.

② $\mathcal{C}$ = k-algebras where k is a field   v.s. over k, also ring

product = tensor product of algebras.

"Special types of categories"

A category $\mathcal{C}$ is additive if

(1) it has a zero object 0, ~~and~~ finite products and finite coproducts
= zero el. (= initial + terminal)

(2) For all objects $A, B \in \mathcal{C}$, the sets $\text{Hom}_{\mathcal{C}}(A, B)$ are additive abelian groups and the addition in them is distributive w.r.t. the composition, that is
↳ $0 \in \text{Hom}_{\mathcal{C}}(A, B)$ exists

$\forall A, B, C, D \in \text{Ob}\,\mathcal{C}$

if

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \arrow[r, "\ell"] & D \end{tikzcd}$$

then $(h+g)\circ f = hf + gf$ and $\ell(g+h) = \ell g + \ell h$.

if $\mathcal{C}$ satisfies just (2), it is called "preadditive"

functor $F : \mathcal{C} \to \mathcal{D}$ between add. cat. is additive if $F(f+g) = F(f) + F(g)$.
⤳ $\text{Hom}_{\mathcal{C}}(A,B) \to \text{Hom}_{\mathcal{D}}(FA, FB)$
$f \mapsto Tf$ is homom. of ab. gps
→ $F(0) = 0$

Examples

mod R and Mod R are additive. ⤳ zero obj = zero mod
$\text{Hom}_R(A,B)$ ab. gp w/ composition

The category of free modules is also additive.

groups not additive!! (p. 303 Rotman)

Def: A category $\mathcal{C}$ is abelian if it is additive and also satisfies:

(AB1) Every morphism in $\mathcal{C}$ has a kernel and a cokernel.

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(AB2) Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

(Rotman, p. 307: instead of (AB2) Every monom. is a kernel & every epim is a cokernel)

where,

Recall: ① If $B \xrightarrow{f} C$ is a morphism in additive $\mathcal{C}$, a kernel of f is a morphism $A \xrightarrow{k} B$ s.t. $fk=0$ and $\forall$ morphisms $g \in Hom_{\mathcal{C}}(X,B)$ with $fg=0$, $\exists!\, h \in Hom_{\mathcal{C}}(X,A)$ with $g=kh$.

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists! h"'] \arrow[d, "g"] & \\ A \arrow[r, "k"] & B \arrow[r, "f"] & C \end{tikzcd}$$

$fg=0$

$fk=0$

Facts: ① If $A \xrightarrow{k} B$ is a kernel of $f \in Hom_{\mathcal{C}}(B,C)$, then k is a monomorphism. → HW1

Prop. 5.89 Rotman, p. 306.
u: A → B morph. in add cat A
(a) If ker u exists, then u mono ⟺ ker u = 0
(b) If coker exists, then u epi ⟺ coker u = 0 → HW1

② The kernel is unique up to isomorphism.

② What is the cokernel $f \in Hom_{\mathcal{C}}(A,B)$?

It is $\pi \in Hom(B,C)$

$$A \xrightarrow{f} B \xrightarrow{\pi} C$$

s.t. $\forall\; B \xrightarrow{\pi'} C'$ with $\pi' f = 0$, $\exists!\, h \in Hom(C,C')$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\pi"] \arrow[d, "\pi'"'] & C \arrow[dl, dashed, "\exists! h"] \\ & C' & \end{tikzcd}$$

with $h\pi = \pi'$.

π: B → B/Im f fulfills univ. prop. of cokernel & is epi
↓
$\boxed{15}$

If it exists, it is unique up to isom, (and π is epi).

use univ. prop → C → C' → C has to be identity

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AB2 If "$0 \to A \underset{\text{mono}}{\xrightarrow{k}} B \underset{\text{coker of } k}{\xrightarrow{\pi}} C \to 0$" if we had modules.

this happens automatically for modules.

Notion of exactness in abelian categories

Assume we have $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category $\mathcal{A}$.

This is a complex if $gf = 0$.

May define the image as follows:

if $f \in \mathrm{Hom}(A, B)$ then $\mathrm{im}\, f \underset{\downarrow}{=} \ker$ of the cokernel of f

easy to define up to isom.

Read from Rotman about "subobjects in an abelian category"

then "exactness -"- -"- " → $\boxed{19}$, $\boxed{20}$

Def: $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\ker g = \mathrm{im} f$.

Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

A covariant functor $F: \mathcal{C} \to \mathcal{D}$ associates to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ satisfying the following axioms: (1) $F(g \cdot f) = F(g) \cdot F(f)$ where $A \xrightarrow{f} B \xrightarrow{g} C$

(2) $F(1_A) = 1_{F(A)}$ $\forall A \in \mathcal{C}$

A contravariant functor assigns to each A in $\mathcal{C}$, $F(A) \in \mathcal{D}$, and $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$

Here, instead of $F(g \cdot f) = F(g) \cdot F(f)$, we get $F(g \circ f) = F(f) \circ F(g)$.

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Examples

① identity functor $1 : \mathcal{C} \to \mathcal{C}$

② $\text{Mod } R \xrightarrow{F} \text{Sets}$ F = forgetful functor

$\overset{\text{module}}{M} \mapsto$ "set" M

module morphism $\mapsto$ set map

③ Let $\mathcal{C} =$ (left) $\text{Mod } R$ $\mathcal{D} = Ab$

if ${}_R M$ is a module, $F = \text{Hom}_R(M, -)$ is a covariant functor

$F(A) = \text{Hom}_R(M, A)$

if $A \xrightarrow{f} B$ $F(f) = \text{Hom}(M, f) : \text{Hom}_R(M, A) \to \text{Hom}_R(M$

$M \xrightarrow{\alpha} A \mapsto f\alpha$

Similarly $\overset{\text{have}}{\vee}$ the functor $G = - \otimes_R M$: right R-mod's → abelian gps

$G(A) = A \otimes_R M$

if $A \xrightarrow{f} B$, $G(f) = f \otimes 1_M$

This functor is covariant.

The functor $F = \text{Hom}_R(-, M) : \text{Mod } R \to Ab$ is contravariant

Natural transformations

Let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$. A natural transformation $\eta : F \to G$

is a collection of morphisms $\{\eta_A\}_{A \in \mathcal{C}}$ where

$\eta_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$ so that $\forall A, B \in \mathcal{C}$ and $f \in \text{Hom}_{\mathcal{C}}(A, B$

the following diagrams commute:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\circlearrowright"] & F(B) \arrow[d, "\eta_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

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η is called a natural isomorphism if each η_A is an isomorphism in $\mathcal{D}$.

Examples:

① Let R be a ring. Let $I \trianglelefteq R$.

Let $F: \text{Mod } R \to \text{Mod } R/I$ be $F(M) = M/IM$

$G: \text{Mod } R \to \text{Mod } R/I$ be $G(M) = R/I \otimes_R M$

Then from MAT 731 we get that $\exists$ natural isomorphism $F \to G$ and one from $G \to F$.

$f: M \to N$

$$\begin{tikzcd} M/IM \arrow[r, "F(f)=\bar{f}"] & N/IN \\ R/I \otimes_R M \arrow[u, "\varphi_M", "\wr"'] \arrow[r, "G(f)=R/I\otimes f"'] & R/I \otimes_R N \arrow[u, "\varphi_N"', "\wr"] \end{tikzcd}$$

$f(IM) \subseteq IN$ $\to$ induces uniq. $\bar{f}$ (see 1/24)

$\varphi_M: (r+I)\otimes x \mapsto rx + IM$

φ_M nat. isom. $G \to F$

Exercise: what are they?

$\psi_M: F \to G$

$m + IM \mapsto 1 + I \otimes m$

well def'd:

$m - m' \in IM$

$\Rightarrow 1+I \otimes m - 1+I \otimes m'$

$= 1+I \otimes (m-m')$

$= 1+I \otimes \sum a_i m_i$, $a_i \in I$

$= \sum \underline{a_i + I} \otimes m_i$

$= I \otimes m$

$= 0 \in R/I \otimes M$

Some "special types" of functors

① Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. F is called faithful if for all $A, B \in \mathcal{C}$, the mapping $f \to F(f)$ from $H_{\mathcal{C}}(A, B)$ to $\text{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective.

② The functor F is full if for all $A, B \in \mathcal{C}$ the map taking f in $\text{Hom}_{\mathcal{C}}(A, B)$ to $F(f) \in \text{Hom}_{\mathcal{D}}(F(A), F(B))$ is surjective.

"fully faithful functor" = full + faithful

③ A functor $F: \mathcal{C} \to \mathcal{D}$ is an isomorphism if $\exists$ functor $G: \mathcal{D} \to \mathcal{C}$ with $FG = 1_{\mathcal{D}}$, $GF = 1_{\mathcal{C}}$.

A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence if $\exists G: \mathcal{D} \to \mathcal{C}$ such that FG is naturally isomorphic to $1_{\mathcal{D}}$ and GF is naturally isomorphic to $1_{\mathcal{C}}$.

$\exists \{\eta_A\}$ s.t.

$$\begin{tikzcd} GF(A) \arrow[r, "GF(f)"] \arrow[d, "\eta_A"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & GF(B) \arrow[d, "\eta_B", "\wr"'] \\ 1_{\mathcal{C}}(A)=A \arrow[r, "f"] & 1_{\mathcal{C}}(B)=B \end{tikzcd}$$

Rotman p. 315. $\mathcal{A}$ ab. cat. $F: \mathcal{A} \to \mathbf{Ab}$ is exact if $A' \to A \to A''$ exact in $\mathcal{A}$ implies $FA' \to FA \to FA''$ exact in $\mathbf{Ab}$

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④ A functor $F: \mathcal{C} \to \mathcal{D}$ is dense if $\forall X \in \mathcal{D}$, $\exists A \in \mathcal{C}$ with $F(A) = X$.

⑤ A category is a k-category if k is a ring and every object in $\mathcal{C}$ is also a k-module and every morphism is also a morphism of k-modules.

Then have the following:

Let $F: \underset{\searrow\ \swarrow}{\mathcal{C} \to \mathcal{D}}$ functor

k-categories

Then F is an equivalence $\Leftrightarrow$ F is full, faithful and dense.

Fact: isomorphism of categories $\Rightarrow$ equivalence $\overset{?}{\Rightarrow}$ full + faithful

$FG = 1_{\mathcal{D}}$, $GF = 1_{\mathcal{C}}$ $\Rightarrow$ $FG \simeq 1_{\mathcal{D}}$, $GF \simeq 1_{\mathcal{C}}$ via nat. isom = identity

faithful: $F(f) = F(g) \Rightarrow G(F(f)) = G(F(g))$

Have

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$$\begin{tikzcd} GF(A) \arrow[r, "\overset{GF(g)}{\overset{\shortparallel}{GF(f)}}"] \arrow[d, "\eta_A"'] \arrow[rd, phantom, "\circlearrowleft"] & GF(B) \arrow[d, "\eta_B"] \\ A = 1_{\mathcal{C}}(A) \arrow[r, "1_{\mathcal{C}}(f) = f = g"'] & 1_{\mathcal{C}}(B) = B \end{tikzcd}$$

$\Rightarrow f = g \; (= \eta_B \circ \underset{GF(g)}{GF(f)} \circ \eta_A^{-1})$

full: Let $h \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$
$\Rightarrow G(h) \in \mathrm{Hom}_{\mathcal{C}}(G(F(A)), G(F(B)))$ ⇝

$$\begin{tikzcd} G(F(A)) \arrow[r, "(G(h))"] \arrow[d, "\eta_A"'] & GF(B) \arrow[d, "\eta_B"] \\ (A) \arrow[r, dashed, "k"] & (B) \end{tikzcd}$$

$\exists k: A \to B$, $k := \eta_B \circ G(h) \circ \eta_A^{-1}$
$\Rightarrow F(k): F(A) \to F(B)$
$F(k) = h$? $F(G(h)) = F(k)$?
$= h$

Remark on exercise:

$M = \underset{\substack{\downarrow \\ \text{inf. count. many}}}{\prod} \mathbb{Z}$ Show M is not free / $\mathbb{Z}$

Hint: Assume M free. Let $N = \bigoplus \mathbb{Z} < M$.

$\downarrow$

all families (x_i) where only finitely many elements are $\neq 0$

M free $\Rightarrow$ N free (but we already know this)

$\downarrow$

submod of free (over $\mathbb{Z}$)

Pick a basis B_1 of N.

B_1 is a linear indep. subset of M. B_1 can be extended to a basis $\overset{B}{\checkmark}$of M.

so $B = B_1 \overset{\circ}{\cup} B_2$. $\Rightarrow M = \langle B \rangle = \langle B_1 \rangle \oplus \langle B_2 \rangle$

$\downarrow$ disjoint union

$=: N_1 \oplus \overline{M}$ $\to \overline{M}$ is free with basis B_2.

(Also $\overline{M} \cong M/N$).

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Adjoints

Let $\mathcal{C}, \mathcal{D}$ be two categories and let F, G be covariant functors:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

We say that (F, G) is an adjoint pair if $\forall A \in \mathrm{Ob}\,\mathcal{C}$, $\forall B \in \mathrm{Ob}\,\mathcal{D}$ we have a bijection

$$\tau = \tau_{AB} : \mathrm{Hom}_{\mathcal{D}}(F(A), B) \to \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$

that is natural in A and B. That is for all $A_1, A_2 \in \mathcal{C}$ and $B_1, B_2 \in \mathcal{D}$ and $f \in \mathrm{Hom}(A_1, A_2)$, $g \in \mathrm{Hom}_{\mathcal{D}}(B_1, B_2)$ we have the following commutative diagrams:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(A_1), B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(A_1, G(B)) \\
\mathrm{Hom}_{\mathcal{D}}(F(A_2), B) \arrow[r] \arrow[u, "\mathrm{Hom}(F(f), B)"] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathcal{C}}(A_2, G(B)) \arrow[u, "\mathrm{Hom}_{\mathcal{C}}(f, G(B))"']
\end{tikzcd}$$

$A_1 \xrightarrow{f} A_2$ $\rightsquigarrow \mathrm{Hom}(F(f), B)(g) = g \circ F(f)$

$F(A_1) \xrightarrow{F(f)} F(A_2) \dashrightarrow B$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(A), B_1) \arrow[r] \arrow[d, "\mathrm{Hom}(F(A), g)"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{C}}(A, G(B_1)) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(A, G(g))"] \\
\mathrm{Hom}_{\mathcal{D}}(F(A), B_2) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(A, G(B_2))
\end{tikzcd}$$

$g : B_1 \to B_2$
$G(g) : G(B_1) \to G(B_2)$

Def: If (F, G) is an adjoint pair, then $F \overset{\text{def}}{=}$ left adjoint (of G) and G is a right adjoint (of F).

Example: Recall the "adjoint isom thm".

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Version A: R, S rings , A_R, ${}_R B_S$, C_S.

$\overset{\exists \text{ natural}}{\underset{\text{iso}}{\Longrightarrow}}$ $\quad \mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$.

Let $F = - \otimes_R B$: $\underset{\mathcal{E}}{\underset{\parallel}{\text{right } R\text{-mod's}}} \to \underset{\mathcal{D}}{\underset{\parallel}{\text{right } S\text{-mod's}}}$

$G = \mathrm{Hom}_S(B, -)$: right S-mods $\to$ right R-mod's

$\mathrm{Hom}_{\mathcal{D}}(F(A), C) \cong \mathrm{Hom}_{\mathcal{E}}(A, G(C))$

So $(- \otimes_R B_S, \mathrm{Hom}_S({}_R B, -))$ is an adjoint pair.

Version B: R, S rings ${}_R A$, ${}_S B_R$, ${}_S C$.

$\Rightarrow \exists$ natural isms $\quad \mathrm{Hom}_S(B \otimes A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$

Q: What is the adjoint pair here?

$F = B \otimes -$: left R-mods $\to$ left S-mods

$G = \mathrm{Hom}_S(B, -)$: left S-mod's $\to$ left R-mods

Chain complexes and homology

Note: everything is true in an arbitrary abelian category.

To make life easier assume that we are dealing with

Mod R where R is a ring.

(seq. of objs and morph's in an ab. cat)

Def: A complex (or chain complex) of modules $(C_\bullet, d_\bullet)$ is a sequence of homoms:

$C_\bullet$ short for $(C_\bullet, d_\bullet)$ or even C

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

are called the "differentials" of $C_\bullet$.

with $d_n d_{n+1} = 0 \quad \forall n.$ $\to$ $\mathrm{Im}\, d_{n+1} \subset \mathrm{Ker}\, d_n$

This complex is exact (or acyclic) if $\forall n \in \mathbb{Z}$ we have $\mathrm{Ker}\, d_n = \mathrm{Im}\, d_{n+1}$

A subcomplex of a complex $(C_\bullet, d^C)$ is a complex

$(C'_\bullet, d^{C'})$ where for each n, $C'_n \subseteq C_n$ (submod in Mod R) and where $\forall n$ we have

Comm. diagrams where the vertical maps are inclusions

(in ab. cat. each is monic)

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$$\begin{tikzcd}
C'_{n+1} \arrow[r, "d^{C'}_{n+1}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & C'_n \arrow[d, hook] \\
C_{n+1} \arrow[r, "d^{C}_{n+1}"] & C_n
\end{tikzcd}$$

and $\forall n$ $\quad d^{C'}_{n+1} = d^{C}_{n+1}|_{C'_{n+1}}$, (in Mod_R)

More generally, a morphism between the chain complexes $(A_\cdot, d^A) \to (B_\cdot, d^B)$ (or a chain map) is a family of morphisms $\{f_n\}_{n\in\mathbb{Z}}$ where $f_n : A_n \to B_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] \arrow[u, phantom, "\circlearrowleft"] & B_{n+1} \arrow[r, "d^B_{n+1}"] \arrow[ur, phantom, "\circlearrowleft"] & B_n \arrow[r, "d^B_n"] \arrow[ur, phantom, "\circlearrowleft"] & B_{n-1} \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \cdots
\end{tikzcd}$$

May form the category of chain complexes denoted $\underline{\text{Com}(\mathcal{A})}$ (where $\mathcal{A}$ is an abelian category)
or Comp($\mathcal{A}$)

where the objects are complexes and the morphisms are the chain maps.

composite gf of two chain maps $f_\cdot, g_\cdot$ is a chain map, where $(gf)_n = g_n f_n$. $(1_{C_\cdot})_n = 1_{C_n}$

Rotman, p.349

Thm: $\mathcal{A}$ abelian $\Rightarrow$ Com($\mathcal{A}$) is also abelian.

→ additive, kernels, cokernels exist
R (AB3) → 1/31/18

[For instance, let $A_\cdot, B_\cdot$ be two complexes $(A_\cdot, d^A), (B_\cdot, d^B)$.

Their direct sum is

$$\cdots \longrightarrow A_{n+1} \oplus B_{n+1} \xrightarrow{\begin{bmatrix} d^A_{n+1} & 0 \\ 0 & d^B_{n+1} \end{bmatrix}} A_n \oplus B_n \xrightarrow{\begin{bmatrix} d^A_n & 0 \\ 0 & d^B_n \end{bmatrix}} A_{n-1} \oplus B_{n-1} \to \cdots$$

(composition is usual multipl. of matrices $\begin{bmatrix} d^A_n & 0 \\ 0 & d^B_n \end{bmatrix} \begin{bmatrix} d^A_{n+1} & 0 \\ 0 & d^B_{n+1} \end{bmatrix}$

$$= \begin{bmatrix} d^A_n d^A_{n+1} & 0 \\ 0 & d^B_n d^B_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad]$$

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example: If $f: A_\bullet \to B_\bullet$ the kernel of f is the complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots & \operatorname{Ker} f_{n+1} \arrow[r, dashed, "\exists!"] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker} f_n \arrow[r, dashed, "\exists!"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker} f_{n-1} \arrow[d] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d^B_{n+1}"] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

univ. prop. of kernel

$0 = f_n \circ i : \operatorname{Ker} f_n \to B_n$, $i: \operatorname{Ker} f_n \to A_n$ kernel

$f_n \circ \underbrace{d^A_{n+1} \circ j}_{g} = d^B_{n+1} \circ \underbrace{f_{n+1} \circ j}_{=0} = 0$

$g: \operatorname{Ker} f_{n+1} \to A_n$

Univ. prop. of kernel $\Rightarrow \exists!$ $h: \operatorname{Ker} f_{n+1} \to \operatorname{Ker} f_n$ s.t. $i \circ h = g = d^A_{n+1} \circ j$

(for modules the maps are the restrictions to the kernels)

↳ kernel is $\operatorname{Ker} f = \{a \in A : f(a) = 0\} \hookrightarrow A$,

if $g: L \to A$ s.t. $fg = 0$ $\Rightarrow \operatorname{Im} g \subset \operatorname{Ker} f$ → can def. $h: L \to \operatorname{Ker} f$, $L \mapsto g(L)$ ⇝ $h = g|_L$ (w/ restricted image)

"exactness": use "exactness" of $\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$ (like in pf of snake lemma)

(need only $\operatorname{Im} \cdots \subset \operatorname{Ker} \cdots$

Remark: The Snake Lemma holds also in an arbitrary abelian category.

Homology: Let $(C_\bullet, d_\bullet)$ be a complex in $\mathcal{A}$.

$C_\bullet$ exact $\Leftrightarrow$ $H_n(C) = 0$ $\forall n$

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

The nth homology object of $(C_\bullet, d)$ is $\boxed{H_n(C) = \dfrac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}}$

$\operatorname{Ker} d_n$ = "n-cycles" $= Z_n(C)$

$\operatorname{Im} d_{n+1}$ = "n-boundaries" $= B_n(C)$

[In the abelian cat. $\mathcal{A}$, think of $\operatorname{Im} d_{n+1}$ as a subobject of $\operatorname{Ker} d_n$,
⇝ exists map $\operatorname{Im} d_{n+1} \to \operatorname{Ker} d_n$
then $\dfrac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$ = Coker of $\operatorname{Im} d_{n+1} \to \operatorname{Ker} d_n$.
⊗

Note: "homology groups" will turn out to be important invariants.

two fund. ~~exact~~ seq. arise from a complex:

$0 \to \operatorname{Im} d_{n+1} \to \operatorname{Ker} d_n \to H_n(C) \to 0$

and $0 \to \operatorname{Ker} d_n \to C_n \to \operatorname{Im} d_{n} \to 0$

$d_n = 0$ $\forall n$
$\Rightarrow H_n(C) = C_n$ $\forall n \in \mathbb{Z}$
$\operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ $= \operatorname{Ker} d_n = C_n$

Proposition: Let $f = \{f_n\}_n : A_\bullet \to B_\bullet$ be a chain map.

Then, for each $n \in \mathbb{Z}$ f induces homomorphisms

$$\bar{f}_n : H_n(A) \to H_n(B).$$

Fut. book f_n / f_s

Pf: Have

⊗ $\operatorname{Im} d_{n+1} = \operatorname{Ker}(\operatorname{coker} d_{n+1})$

quot. "B/C" is an equiv. class $[(f, C)]$
$f: B \to C$ epi

regard cokernels as quotients

⇝ $H_n(C)$ lies in $\operatorname{Ob}(\mathcal{A})$ if quotients are viewed as objects.

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$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d^B_{n+1}"] & B_n \arrow[r, "d^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\otimes$ We have induced morphisms $\operatorname{Ker} d^A_n \to \operatorname{Ker} d^B_n$ for all n.

Similarly one can show (exercise!) that we have induced morphisms $\operatorname{Im} d^A_{n+1} \to \operatorname{Im} d^B_{n+1}$ $\rightarrow\boxed{14}$ $\forall n$ and comm. diagrams $\searrow \boxed{15}$ with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} d^A_{n+1} \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & \operatorname{Ker} d^A_n \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowright"] & H_n(A) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
0 \arrow[r] & \operatorname{Im} d^B_{n+1} \arrow[r] & \operatorname{Ker} d^B_n \arrow[r] & H_n(B) \arrow[r] & 0
\end{tikzcd}$$

(univ. prop. of the cokerne

We denote by $\bar{f}_n$ the induced morphism $H_n(A) \to H_n(B)$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} d_n^A \arrow[r, "i"] \arrow[d, dashed, "\exists!"] \arrow[rd, phantom, "\circlearrowleft"] & A_n \arrow[r, "d_n^A"] \arrow[d, "f_n"] & A_{n-1} \arrow[d, "f_{n-1}"] \\
0 \arrow[r] & \operatorname{Ker} d_n^B \arrow[r, "j"] & B_n \arrow[r, "d_n^B"] & B_{n-1}
\end{tikzcd}$$

$\otimes$

univ. prop. of kernel

$f: A \to B$

$j: \operatorname{Ker} f \hookrightarrow A$ $fj = 0$

$\forall h: L \to A$ s.t. $fh = 0$

$\exists!$ $s: L \to \operatorname{Ker} f$ s.t. $js = h$

$d_n B \circ \underbrace{f_n \circ i}_{h} \overset{S}{=} f_{n-1} \circ \underbrace{d_n^A \circ i}_{=0}$

$= 0 \Rightarrow h = f_n \circ i : \operatorname{Ker} d_n^A \to B_n$ s.t. $d_n^B \circ h = 0$

$\overset{\text{univ. prop of } \underline{\operatorname{Ker} d_n^B}}{\Rightarrow} \exists! s: \operatorname{Ker} d_n^A \to \operatorname{Ker} d_n^B$ s.t. $j \circ s = h = f_n \circ i$

~~In this way~~,

⌈ Specifically, for ~~A~~ $\mathcal{A}$ = Mod R, $\underline{\bar{f}_n(x_n + \operatorname{Im} d^A_{n+1})} = f_n(x_n) \underline{+ \operatorname{Im} d^B_{n+1}}$

$x_n \in \operatorname{Ker} d^A_n \subseteq A_n$ $\searrow \boxed{15}$ ⌋

□

<u>Observation:</u> We also have the notion of "<u>exact sequences of complexes</u>" (since <u>Com(A)</u> is an <u>abelian</u> <u>category</u>).

To be precise, an exact sequence of chain complexes is

$$0 \to A. \xrightarrow{f} B. \xrightarrow{g} C. \to 0$$

chain maps

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such that we have the following comm. diagram with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^A_{n+1}"] \arrow[dr, phantom, "\circlearrowleft"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^B_{n+1}"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "d^A_n"] \arrow[dr, phantom, "\circlearrowleft"] & B_n \arrow[r, "g_n"] \arrow[d, "d^B_n"] \arrow[dr, phantom, "\circlearrowleft"] & C_n \arrow[r] \arrow[d, "d^C_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$\operatorname{Im} f_n = \operatorname{Ker} g_n \ \forall n \in \mathbb{Z}$

Theorem: (the long exact sequence in homology):

Let $\mathcal{A}$ be an abelian category and let

$$0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$$

be a short exact sequence of chain complexes in $\mathcal{A}$.

Then, there exists an induced exact sequence in homology

$$\cdots H_n(A) \xrightarrow{\overline{f_n}} H_n(B) \xrightarrow{\overline{g_n}} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \to$$
$$\xrightarrow{\overline{f_{n-1}}} H_{n-1}(B) \xrightarrow{\overline{g_{n-1}}} H_{n-1}(C) \xrightarrow{\partial_{n-1}} H_{n-2}(A) \to \cdots$$

(∂_n is called the "connecting homomorphism").

Notation: We think of this long exact sequence in homology as an "exact triangle"

will come up when we talk about "triangulated categories".

really maps to $H_{n-1}(A)$

$$\begin{tikzcd}
 & H_n(A) \arrow[dr] & \\
H_n(C) \arrow[ur] & & H_n(B) \arrow[ll]
\end{tikzcd}$$

Pf of thm elementwise ~~for~~ modules ⟶ Rotman

Note: $\forall n$ have functors (additive) $\text{Com}\,\mathcal{A} \to \mathcal{A}$, $M_\cdot \mapsto H_n(M_\cdot)$, $f \mapsto H_n(f) = \overline{f_n}$

chain map

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The long exact sequence in homology

Let $\mathcal{A}$ be an ab. cat. and let $0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$ be a SES of chain complexes in $\mathcal{A}$. Then $\exists$ long exact sequence:

$$\cdots \to H_n(A_\cdot) \to H_n(B_\cdot) \to H_n(C_\cdot) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$$

Proof: (Rotman has a proof if $\mathcal{A} = \text{Mod } R$ "elementwise")

Will use the snake lemma. Use first the commutative exact diagrams in $\mathcal{A}$.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^A_{n+1}"] \arrow[dr, phantom, "\circlearrowleft"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^B_{n+1}"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f_n"] & B_n \arrow[r, "g_n"] & C_n \arrow[r] & 0
\end{tikzcd}$$

Snake lemma: Get an exact sequence

$$\text{Ⓧ} \quad 0 \to \text{Ker } d^A_{n+1} \to \text{Ker } d^B_{n+1} \to \text{Ker } d^C_{n+1} \to \underbrace{A_n/\text{Im } d^A_{n+1}}_{= \text{Coker } d^A_{n+1}} \to \underbrace{B_n/\text{Im } d^B_{n+1}}_{= \text{Coker } d^B_{n+1}} \to \underbrace{C_n/\text{Im } d^C_{n+1}}_{= \text{Coker } d^C_{n+1}} \to 0 \quad \forall n$$

We also have $\forall n$:

$$\begin{tikzcd}
0 \arrow[r] & \text{Im } d^A_{n+1} \arrow[r, "\text{incl.}"] \arrow[dr, phantom, "=0!"] & A_n \arrow[r, "\pi"] \arrow[d, "d_n^A"] & A_n/\text{Im } d^A_{n+1} \arrow[r] \arrow[dl, dashed, "?"] \arrow[dl, dashed, swap, "\exists!\, \tilde{d}_n^A"] & 0 \\
& & \text{Ker } d^A_{n-1} & &
\end{tikzcd}$$

Since $d_n^A d_{n+1}^A = 0$

$\exists!\ \tilde{d}_n^A$ (univ. prop. of cokernel)
see 1/31/18

$\pi \circ \text{incl} = 0$
$d_n^A \circ \text{incl} = 0$
$\Rightarrow \exists!\ s: A_n/\text{Im } d^A_{n+1} \to \text{Ker } d^A_{n-1}$
s.t. $s \circ \pi = d_n^A$

Ⓧ $\Rightarrow$

$$\begin{tikzcd}
& A_n \arrow[r, "f_n"] \arrow[d, "\pi_A"] \arrow[dr, phantom, "(\text{snake}) \circlearrowright"] \arrow[dd, bend right, "d_n^A"'] & B_n \arrow[r, "g_n"] \arrow[d, "\pi_{B_n}"] \arrow[dr, phantom, "(\text{snake}) \circlearrowright"] \arrow[dd, bend right, "d_n^B"'] & C_n \arrow[d] & \\
& A_n/\text{Im } d^A_{n+1} \arrow[r, "s"] \arrow[d, "\tilde{d}_n^A"] \arrow[dr, phantom, "\circlearrowleft"] & B_n/\text{Im } d^B_{n+1} \arrow[r] \arrow[d, "\tilde{d}_n^B"] \arrow[dr, phantom, "\circlearrowleft"] & C_n/\text{Im } d^C_{n+1} \arrow[r] \arrow[d, "\tilde{d}_n^C"] & 0 \\
0 \arrow[r] & \text{Ker } d^A_{n-1} \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "(\text{snake}) \circlearrowright"] & \text{Ker } d^B_{n-1} \arrow[r] \arrow[d, "i_{n-1,B}"] \arrow[dr, phantom, "(\text{snake}) \circlearrowright"] & \text{Ker } d^C_{n-1} \arrow[d] & \\
& A_{n-1} \arrow[r, "f_{n-1}"] & B_{n-1} \arrow[r, "g_{n-1}"] & C_{n-1} &
\end{tikzcd}$$

$\boxed{\text{Ⓧ Ⓧ}}$

$\boxed{\tilde{d}_n^B \circ s = t \circ \tilde{d}_n^A:}$

$\tilde{d}_n^A \circ \pi = d_n^A$

$d_n^B \circ f_n = f_{n-1} \circ d_n^A$
$(0 \to A_\cdot \to B_\cdot \to C_\cdot \to 0 \text{ SES})$

$\Rightarrow \tilde{d}_n^B \circ s \circ \pi_{A_n}$
$= \underbrace{\tilde{d}_n^B \circ \pi_{B_n}}_{= d_n^B} \circ f = d_n^B \circ f$

$i_{n-1,B} \circ \tilde{d}_n^B \circ s \circ \pi_{A_n} f_A$
$= d_n^B \circ f_n = f_{n-1} \circ d_n^A$
$= i_{n-1,B} \circ t \circ d_n^A$
$= i_{n-1,B} \circ t \circ \tilde{d}_n^A \circ \pi_{A_n}$

$\overset{i \text{ mono}}{\Longrightarrow} \tilde{d}_n^B \circ s \circ \pi_{A_n}$
$= t \circ \tilde{d}_n^A \circ \pi_{A_n}$
$\overset{\pi \text{ epi}}{\Longrightarrow} \tilde{d}_n^B \circ s = t \circ \tilde{d}_n^A$

$\boxed{16} \leftarrow$ Compute now $\text{Coker } \tilde{d}_n^A$. But $\text{Im } \tilde{d}_n^A = \text{Im } d_n^A$ since $A_n \to A_n/\text{Im } d^A_{n+1}$ is epi. $\Rightarrow \text{Coker } \tilde{d}_n^A = \dfrac{\text{Ker } d^A_{n-1}}{\text{Im } d_n^A} = H_{n-1}(A)$.

Applying the snake lemma to $\boxed{\text{Ⓧ Ⓧ}}$ yields $\forall n$

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$$\operatorname{Ker}\tilde{d}_n^A \to \operatorname{Ker}\tilde{d}_n^B \to \operatorname{Ker}\tilde{d}_n^C \to \overset{\operatorname{Coker}\tilde{d}_n^{A}}{\overset{\shortparallel}{H_{n-1}(A)}} \to \overset{\operatorname{coker}\tilde{d}_n^{B}}{\overset{\shortparallel}{H_{n-1}(B)}} \to \overset{\operatorname{coker}\tilde{d}_n^{C}}{\overset{\shortparallel}{H_{n-1}(C)}} \to \ldots$$

Need to compute $\operatorname{Ker}\tilde{d}_n^A$, $\operatorname{Ker}\tilde{d}_n^B$, $\operatorname{Ker}\tilde{d}_n^C$.

Diagram chasing shows that

$$\operatorname{Ker}\tilde{d}_n^A = H_n(A_\cdot)$$
$$\operatorname{Ker}\tilde{d}_n B = H_n(B_\cdot)$$
$$\operatorname{Ker}\tilde{d}_n C = H_n(C_\cdot)$$

and the proof follows. $\to \boxed{18}$ for modules $\to \boxed{17}$ "try" in general? □

Corollary: Let $0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$ be a short exact sequence in $\mathrm{Com}(\mathcal{A})$ where $\mathcal{A}$ is abelian.

Assume $A_\cdot$, $C_\cdot$ are exact. Then $B_\cdot$ is exact.

Pf: To show $B_\cdot$ is exact is equivalent to showing that $H_n(B_\cdot) = 0$ $\forall n$. $\Leftrightarrow \operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$

Have $\forall n$

$$\underset{\underset{0}{\shortparallel}}{H_n(A_\cdot)} \to H_n(B_\cdot) \to \underset{\underset{0}{\shortparallel}}{H_n(C_\cdot)} \quad \text{exact}$$

So from the exactness of $0 \to H_n(B_\cdot) \to 0$ we get $B_\cdot$ exact.

(exact iff $H_n(B_\cdot) = 0$ $(\forall n)$ $\Leftrightarrow B_\cdot$ exact)

Thm: (Naturality of the long exact sequence in homology):

Assume we have the following commutative exact diagram in $\mathrm{Com}(\mathcal{A})$ of $\mathcal{A}$ abelian:

$$\begin{tikzcd}
0 \arrow[r] & A_\cdot \arrow[r, "f"] \arrow[d, "u"] & B_\cdot \arrow[r, "g"] \arrow[d, "v"] & C_\cdot \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & A_\cdot' \arrow[r, "f'"] & B_\cdot' \arrow[r, "g'"] & C_\cdot' \arrow[r] & 0
\end{tikzcd}$$

Then we have a comm. diagram in $\mathcal{A}$:

$$\begin{tikzcd}
\ldots & H_n(A) \arrow[r, "H_n(f)"] \arrow[d, "H_n(u)"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(B) \arrow[r, "H_n(g)"] \arrow[d, "H_n(v)"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(C) \arrow[r, "\partial_n"] \arrow[d, "H_n(w)"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(u)"] & \ldots \\
\ldots & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial_n'"] & H_{n-1}(A') \arrow[r] & \ldots
\end{tikzcd}$$

↑ ↗ commute since H_n functor ↑ Rotman 335f.

no pf here □

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Homotopy

Let $f, g : A_\bullet \to B_\bullet$ be two chain maps. We say that "f is homotopic to g" and write $f \sim g$, if $\forall n$, $\exists s_n : A_n \to B_{n+1}$ (degree $+1$) such that

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}^A"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & A_n \arrow[r, "d_n^A"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "s_n"'] & A_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}^B"'] & B_n \arrow[r, "d_n^B"'] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

triangles do not commute

$$d_{n+1}^B s_n + s_{n-1} d_n^A = f_n - g_n$$ (abelian/additive cat. ⇝ sum makes sense)

⌈ we usually write (abusing the language)

$f - g = ds + sd$ for short ⌋

Exercise: ① Given $A_\bullet, B_\bullet \in \mathrm{Com}(\mathcal{A})$, $\sim$ is an equivalence relation in $\mathrm{Hom}_{\mathcal{A}}(A_\bullet, B_\bullet)$. → $\boxed{17}$

② Assume we have $A_\bullet \xrightarrow{h} B_\bullet \overset{f}{\underset{g}{\rightrightarrows}} C_\bullet \xrightarrow{k} D_\bullet$ and $f \sim g$.

Then $fh \sim gh$ and $kf \sim kg$. → $\boxed{18}$

③ ($\mathcal{A}$ abelian cat!) Assume $A_\bullet \overset{f, f', g, g'}{\rightrightarrows} B_\bullet$ chain maps such that $f \sim f'$ and $g \sim g'$.

Then $f + g \sim f' + g'$. → $\boxed{19}$

Let $K(\mathcal{A})$ be the "quotient" of $\mathrm{Com}(\mathcal{A})$ defined as follows:

$\mathrm{Obj}\, K(\mathcal{A}) = \mathrm{Ob}\, \mathrm{Com}(\mathcal{A})$

The morphisms in $K(\mathcal{A})$ are the homotopy classes (equivalence classes under $\sim$) of morphisms in $\mathrm{Com}(\mathcal{A})$.

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Fact: $K(\mathcal{A})$ is also an additive category.

Prop: Let $f, g : A_\bullet \to B_\bullet$ and assume that $f \sim g$. Then f, g induce the same morphisms in homology. That is, $\forall n$

$$\underset{\overline{f_n}}{\underset{\parallel}{H_n(f)}} = \underset{\overline{g_n}}{\underset{\parallel}{H_n(g)}}.$$

Pf: It is enough to show that $H_n(f-g) = 0 \ \forall n$. (why?)] ?

$H_n(f-g) = H_n(f) - H_n(g)$?

Writing $\overline{f_n} = H_n(f)$, $\overline{g_n} = H_n(g)$, it is enough to show

$$(f_n - g_n)(\operatorname{Ker} d_n^A) \subseteq \operatorname{Im} d_{n+1}^B$$

$$\begin{tikzcd}
{} \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r, "d_n^A"] \arrow[dl, "s_n"'] \arrow[d, "f_n-g_n"] & A_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] & \cdots \\
{} \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}^B"'] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$f_n - g_n\big|_{\operatorname{Ker} d_n^A} = \underset{0}{\underset{\parallel}{s_{n-1} d_n^A}} + d_{n+1}^B s_n\big|_{\operatorname{Ker} d_n^A}$$

when applied to $\operatorname{Ker} d_n^A$

$$\Rightarrow f_n - g_n\big|_{\operatorname{Ker}} = d_{n+1}^B s_n \in \operatorname{Im} d_{n+1}^B$$

$$\Rightarrow \overline{f_n - g_n} = 0 \Rightarrow \overline{f_n} = \overline{g_n}.$$

for modules:

$\overline{f_n}(x_n + \operatorname{Im} d_{n+1}^A) = f_n(x_n) + \operatorname{Im} d_{n+1}^B$

$x_n \in \operatorname{Ker} d_n^A$

If $(f_n - g_n)(\operatorname{Ker} d_n^A) \subseteq \operatorname{Im} d_{n+1}^B$

$$\Rightarrow \underset{H_n(f-g)}{\underset{\parallel}{\overline{f_n - g_n}}} = 0$$

general case?

□

Def: A map of complexes is nullhomotopic if it is homotopic to the zero morphism.

A ~~chain map~~ Complex ~~f: A.~~ A. has ~~→ A. is~~ a contracting homotopy if $1_{A_\bullet}$ is nullhomotopic. $[\overset{1-0}{1} = sd + ds]$

$$\begin{tikzcd}
\cdots & A_{n+1} \arrow[r, "d_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[dl] \arrow[d, "1"] & A_{n-1} \arrow[r] \arrow[dl] & \cdots \\
\cdots & A_{n+1} \arrow[r, "d_{n+1}"'] & A_n \arrow[r, "d_n"'] & A_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Prop: Assume $A_\bullet$ has a contracting homotopy. Then $A_\bullet$ is exact.

Pf: Know $1_{A_\bullet}$ induces $1_{H_n(A)}$. At the same time $1_A \sim 0$

(for modules clear; in general?)

and they induce the same map in homology, so $1_{H_n(A)} = 0$.
(by Prop. before)

This can happen only if $H_n(A) = 0$. ▯

element wise:

$$\begin{tikzcd} & \overset{\in \operatorname{Ker} d_n}{x} \arrow[dl] \arrow[r] \arrow[d, "1"] & 0 \arrow[dl] \\ s(x) \arrow[r, "d_{n+1}"] & x & \end{tikzcd}$$

$x = d_{n+1}s(x) + 0$
$= d_{n+1}(s(x))$

$x \in \operatorname{Im} d_{n+1} \Rightarrow H_n(A) = 0$

Examples of complexes we'll play with

① $\mathcal{A}$ abelian category. Let $M \in \operatorname{Ob} \mathcal{A}$. Fix n. Let $C_\bullet$ be the complex that has M in degree n and 0 elsewhere.

So all the morphisms are zero.

$$\cdots \to 0 \to \underbrace{M}_{n} \to 0 \to 0 \to \cdots$$

We say that $C_\bullet$ is concentrated in degree n.

If we make $n = 0$, so we get a complex concentrated in degree 0, then we call this complex the stalk complex of M.

$$\underset{\text{abelian}}{\mathcal{A}} \xrightarrow{\text{"incl"}} \underset{\text{also abelian}}{\operatorname{Comp}(\mathcal{A})}$$

$M \longmapsto$ stalk complex of M

$$\begin{tikzcd} M \arrow[d, "f"] & & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & M \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "0"] \cdots \\ N \arrow[rr, phantom, "\longmapsto"] & & \cdots \arrow[r] & 0 \arrow[r] & N \arrow[r] & 0 \cdots \end{tikzcd}$$

$\downarrow$ degree 0

② R ring M R-module. Know $\exists$ free module F s.t. we

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have $F_0 \xrightarrow{f_0} M \to 0$. But free $\Rightarrow$ proj. $\Rightarrow$ flat.

Look at $0 \to \overset{= \text{ker} f}{K_0} \xrightarrow{\text{incl}} F_0 \xrightarrow{f_0} M \to 0$

$\exists$ free $F_1 \xrightarrow{f_1} K_0 \to 0$. Let $K_1 = \text{Ker}\, f_1$. Continue:

Get: $\forall\, M \in \text{Mod}\, R$, $\exists$ long exact sequence

$$\dots \to F_2 \to F_1 \to F_0 \to M \to 0$$

where each F_i is free.

$$\begin{tikzcd}
& & F_1 \arrow[rr, "d_1"] \arrow[dr, "f_1"'] & \arrow[d, phantom, "\circlearrowleft"] & F_0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & K_1 \arrow[ur] & & K_0 \arrow[ur, hook, "\text{incl}"'] & & &
\end{tikzcd}$$

$d_1 = \text{incl} \circ f_1$

$$\begin{tikzcd}
& K_1 \arrow[d, hook, "\text{incl}_1"] & & & \\
& F_1 \arrow[d, "f_1"] & & & \\
0 \arrow[r] & K_0 \arrow[r, hook, "\text{incl}_0"] \arrow[d] & F_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
& 0 & & &
\end{tikzcd}$$

exact

exact

get induced exact seq.

$$\begin{tikzcd}
& K_2 \arrow[d, hook] & & & & \\
& F_2 \arrow[d] & & & & \\
0 \arrow[r] & K_1 \arrow[r, "\text{incl}_1"] \arrow[d] & F_1 \arrow[r, "\overset{\text{incl}_0 \circ f_1}{\overset{\parallel}{\text{comp}}}"] & F_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
& 0 & & & &
\end{tikzcd}$$

at F_1 exact: $\text{incl}_0 \circ \overbrace{f_1 \circ \text{incl}_1}^{=0} = 0$

$\text{Im}\, \text{incl}_1 = K_1 = \text{Ker}\, f_1 = \text{Ker}(\text{incl}_0 \circ f_1)$ ✓

at F_0 exact: $f_0 \circ \text{comp} = \underbrace{f_0 \circ \text{incl}_0}_{=0} \circ f_1 = 0$

$\text{Im}\, \text{incl}_0 \circ f_1 = \text{Im}\, f_1 \underset{f_1 \text{ onto}}{=} K_0 = \text{Ker}\, f_0$

Continue

Such a long exact sequence is called a free resolution of M.

This way we can also construct "projective resolutions" and "flat resolutions".

free $\Rightarrow$ proj. $\Rightarrow$ flat

Cor. 6.3, p.376 Rotm.: If $\mathcal{A}$ is an ab. cat. w/ enough proj. $\Rightarrow$ every $A \in \text{Ob}(\mathcal{A})$ has a proj. res'n

These complexes are exact.

A proj. resolution of $A \in \text{Ob}(\mathcal{A})$, $\mathcal{A}$ ab. cat. is an exact seq. $P_\bullet: \ \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\overset{d_0}{\overset{\parallel}{\varepsilon}}} A \to 0$ in which each P_n is projective

We will look at projective resolutions:

$$(P_M) \qquad \dots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \to P_0 \xrightarrow{d_0} M \to 0$$

The deleted resolution is $\dots \to P_n \to P_{n-1} \to \dots \to P_0 \to 0$

only proj. mod's left

deleting A loses no information: $A \cong \text{coker}\, d_1 \overset{\text{def}}{=} P_0/\text{Im}\, d_1 = P_0/\text{Ker}\, d_0 \overset{\leftarrow P_0 \text{ onto}}{\cong} A$

inverse op.: restoring A to $P_\bullet$, is called augmenting

in degree 0 the homology need not be zero.

So this is not acyclic.

no longer exact if $A \neq 0$ since $\text{Im}\, d_1 = \text{Ker}\, d_0 \neq \text{Ker}(P_0 \to 0) = P_0$

$(\text{Ker}\, d_0 = P_0 \Rightarrow d_0 : P_0 \to 0 \subset A \Rightarrow A = 0)$

2/12/18

Def: Let $\mathcal{C}$ be a category. An object $P \in \mathrm{Ob}\,\mathcal{C}$ is called projective if whenever we have a diagram in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, phantom, "\circlearrowright"] & C \\ & P \arrow[ul, dashed, "\exists h"] \arrow[u, "f"'] \end{tikzcd}$$

~~P~~

with g being an epimorphism, then $\exists h \in \mathrm{Hom}_{\mathcal{C}}(P,B)$ s.t. $gh=f$.

Similarly we may define injective objects. We say that the category $\mathcal{C}$ has enough projectives if $\forall M \in \mathrm{Ob}\,\mathcal{C}$, $\exists$ projective object $P \in \mathrm{Ob}\,\mathcal{C}$ and an epimorphism $P \xrightarrow{f} M$.

Prop: (Comparison Thm)

needed in pf?

Let A be an abelian category (with enough projectives) and consider the following diagram where the horizontal lines are complexes.

$$\begin{tikzcd} \cdots & P_n \arrow[r, "d^n"] & \cdots \arrow[r, "d^2"] & P_1 \arrow[r, "d^1"] & P_0 \arrow[r, "p"] & A \arrow[r] \arrow[d, "f"] & 0 \\ & & \cdots \arrow[r, "\delta^2"] & Q_1 \arrow[r, "\delta^1"] & Q_0 \arrow[r, "q"] & B \arrow[r] & 0 \end{tikzcd}$$

Assume that each P_i is a projective object, and that bottom Complex is acyclic. Then, there exists an induced chain map
(exact)
$\{f_i\}_{i\geq 0}$ between these complexes.

Assume $\exists$ a second chain map between these complexes $\{g_i\}_{i\geq 0}$.

Then, these chain maps are homotopic.

Pf: Have $fp \in \mathrm{Hom}(P_0, B)$. q epi $\Rightarrow$ $\exists$ $f_0 \in \mathrm{Hom}(P_0, Q_0)$ s.t.
P_0 proj

~~gf_0~~ $qf_0 = fp$.

2/12/18

$$\begin{tikzcd}
P_1 \arrow[r, "d^1"] & P_0 \arrow[r, "p"] \arrow[d, dashed, "f_0"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r] & Q_0 \arrow[r, "q"'] & B \arrow[r] & 0
\end{tikzcd}$$

We construct the f_i inductively. Assume we have constructed $f_0, \ldots, f_i$.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "d^{i+1}"] \arrow[d, dashed, "\exists f_{i+1}"'] & P_i \arrow[r, "d^i"] \arrow[d, "f_i"] \arrow[dr, phantom, "\circlearrowleft"] & P_{i-1} \arrow[d, "f_{i-1}"] \\
Q_{i+1} \arrow[r, "\delta^{i+1}"'] & Q_i \arrow[r, "\delta^i"'] & Q_{i-1}
\end{tikzcd}$$

$$\left(\begin{tikzcd} Q_i \arrow[r, "\delta_i"] & \operatorname{Im} \delta^i \arrow[r] & 0 \end{tikzcd} \right)$$

Look at $\delta_i f_i d^{i+1} = 0$

$= f_{i-1} \underbrace{d^i d^{i+1}}_{=0}$ $\uparrow$ Commutativity + top sequence is ~~exact~~ a complex

$\Rightarrow \operatorname{Im} f_i d^{i+1} \subseteq \operatorname{Ker} \delta^i_{\xi} = \operatorname{Im} \delta^{i+1}$.

$$\begin{tikzcd}
 & P_{i+1} \arrow[d, "f_i d^{i+1}"] \arrow[dl, dashed, "\exists f_{i+1}"'] & \\
Q_{i+1} \arrow[r, "\delta^{i+1}"] & \operatorname{Im} \delta^{i+1} \arrow[r] & 0
\end{tikzcd}$$

$(P_{i+1}$ proj $\Rightarrow \exists f_{i+1})$

Continue and the existence of the chain map $\{f_i\}_{i \geq 0}$ is proved.

Assume now that we have:

$$\begin{tikzcd}
\cdots P_{i+1} \arrow[r, "d^{i+1}"] & P_i \arrow[r, "d^i"] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] & P_1 \arrow[r, "d^1"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & P_0 \arrow[r, "p"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \arrow[dl, "s_0"'] & A \arrow[r] \arrow[d, "f"] \arrow[dl, "\overset{\text{def}}{s_{-1}=0}"'] & 0 \\
\cdots Q_{i+1} \arrow[r, "\delta^{i+1}"] & Q_i \arrow[r, "\delta^i"] & \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r, "\delta^1"] & Q_0 \arrow[r, "q"] & B \arrow[r] & 0
\end{tikzcd}$$

and we have $\forall i$

$$f_{i-1} d^i = \delta^i f_i$$
$$g_{i-1} d^i = \delta^i g_i.$$

and that $f p = q f_0 = q g$

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Look for a homotopy:

$q(f_0 - g_0) = 0$

P_0 proj

univ. prop. of kernel

$$\begin{tikzcd}
 & Q_1 \arrow[d, "\delta' \text{ epi}"'] & P_0 \arrow[l, dashed, "\exists \text{ lifting } s_0"'] \arrow[dl, dashed, "2"] \arrow[d, "f_0 - g_0"] & & \\
0 \arrow[r] & \operatorname{Ker} q \arrow[r, "\text{incl.}"'] \arrow[dr] & Q_0 \arrow[r, "q"] & B \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

$\exists s_0$ with $\underbrace{\delta_1 s_0}_{= \text{incl} \circ \delta_1 s_0} = f_0 - g_0$

So get $f_0 - g_0 = \delta' s_0 + \overset{=0}{s_{-1}} p$.

Induction step:

$$\begin{tikzcd}
P_{i+1} \arrow[r] & P_i \arrow[r, "d^i"] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & P_{i-1} \arrow[r] \arrow[dl, "s_{i-1}"'] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & {} \arrow[dl, "s_{i-2}"] & {} \arrow[dl] \cdots \\
Q_{i+1} \arrow[r] & Q_i \arrow[r, "\delta^i"'] & Q_{i-1} & {} &
\end{tikzcd}$$

Need to find s_i. Have $f_{i-1} - g_{i-1} = s_{i-2} d^{i-1} + \delta^i s_{i-1}$.

Note $f_i - g_i - s_{i-1} d^i \in \operatorname{Hom}(P_i, Q_i)$.

$$\delta^i (f_i - g_i - s_{i-1} d^i) = f_{i-1} d^i - g_{i-1} d^i - \delta^i s_{i-1} d^i$$

$$= (f_{i-1} - g_{i-1} - \delta^i s_{i-1}) d^i$$

$$= s_{i-2} \underbrace{d^{i-1} d^i}_{=0} = 0$$

$\Rightarrow \operatorname{Im}(f_i - g_i - s_{i-1} d^i) \subset \operatorname{Ker} \delta^i = \operatorname{Im} \delta^{i+1}$.

P_i proj $\Rightarrow$

$$\begin{tikzcd}
 & P_i \arrow[dl, dashed, "\exists s_i"'] \arrow[d, "f_i - g_i - s_{i-1} d^i"] & \\
Q_{i+1} \arrow[r, "\delta^{i+1}"'] & \operatorname{Im} \delta^{i+1} \arrow[r] & 0
\end{tikzcd}$$

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$\exists s_i : \quad \delta^{i+1} s_i = f_i - g_i - s_{i-1} d^i.$

or $\quad f_i - g_i = \delta^{i+1} s_i + s_{i-1} d^i.$

In short " $f - g = \delta s + s d$ ". $\square$

Intermezzo

Pushouts and pullbacks

Def: Let $\mathcal{C}$ be a category. Let $f \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ be two morphisms.

$$\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

A pullback is a triple (D, α, β) where $D \in \mathrm{Ob}\,\mathcal{C}$, and $\alpha \in \mathrm{Hom}(D, C)$ and $\beta \in \mathrm{Hom}(D, B)$ and $g\alpha = f\beta$

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

satisfying the following universal property:

$\forall$ triple (D', α', β') where $\alpha' \in \mathrm{Hom}(D', C)$, $\beta' \in \mathrm{Hom}(D', B)$

with $g\alpha' = f\beta'$

$$\begin{tikzcd} D' \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] \arrow[dr, dashed, "\exists! h"] & & \\ & D \arrow[r, "\alpha"] \arrow[d, "\beta"] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

$\exists!\, h \in \mathrm{Hom}(D', D)$ with $\alpha h = \alpha'$, $\beta h = \beta'$.

Prop: If the pullback exists, then it is unique up to isomorphism

(Exercise : turn in)

Prop: Pullbacks exist in the category of modules.

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Pf: Start

$$\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

Let $D^{\subseteq B\oplus C}$ be the submodule (check) $D=\{(b,c) \mid f(b)=g(c)\}$.

Define α, β as follows: $\alpha(b,c)=c$
$\beta(b c) = b$

By def'n the square

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

Commutes.

Check the universal property:

$$\begin{tikzcd} X \arrow[rrd, bend left, "u"] \arrow[rdd, bend right, "v"'] & & \\ & D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

Assume $gu=fv$.

Let $h: X\to D$ be $h(x)=(v(x),u(x))$. This will work.

$\to \alpha h(x)=\alpha(v(x),u(x))=u(x) \Rightarrow \alpha h=u$; $\beta h=v$ ✓

Uniqueness of h is easy. assume $\alpha h'=u$, $\beta h'=v$
$\Rightarrow u(x)=\alpha h'(x)=\alpha h(x)$, $\beta h'(x)=\beta h(x)$
def'n of $\alpha,\beta \Rightarrow h(x)\equiv h'(x)$

$\square$

Pushout

Consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

in $\mathcal{A}$ abelian cat.

A pushout is a triple (D,α,β) such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[rd, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\alpha"'] & D \end{tikzcd}$$

the square commutes and

we have the following universal property:

$\forall$ triple (D', α', β') with $\alpha' \in \mathrm{Hom}(C, D')$, $\beta' \in \mathrm{Hom}(B,D')$

s.t. $\alpha' g = \beta' f$

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$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \arrow[ddr, bend left, "\beta'"] & \\
C \arrow[r, "\alpha"'] \arrow[drr, bend right, "\alpha'"'] & D \arrow[dr, dashed, "\exists! h"] & \\
& & D'
\end{tikzcd}$$

$\Rightarrow \exists!$ h s.t. $h\beta = \beta'$, $h\alpha = \alpha'$.

Prop: If the pushout exists, then it is unique up to isom.

Prop: Pushouts exist in Mod R.

Pf: Start with

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\
C \arrow[r, "\alpha"'] & D
\end{tikzcd}$$

Let $D = B \oplus C / \overbrace{\{(f(a), -g(a)) \mid a \in A\}}^{=: N}$

$\alpha: C \to D$, $\alpha(c) = (0, c) + N$
$\beta: B \to D$, $\beta(b) = (b, 0) + N$

exercise: Complete the pf. (What is α, β? Show it satisfies the univ. prop.) □

$\alpha g(a) = (0, g(a)) + N$
$\beta(f(a)) = (f(a), 0) + N$ $\Big\}$ $=$ in D

$h(b, c) = \alpha'(b) + \beta'(c)$?

Very useful application

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a SES.

$$\begin{tikzcd}
A \arrow[d, "u"'] \\
A'
\end{tikzcd}$$

Then, there exists a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C \arrow[r] & 0
\end{tikzcd}$$

Start with
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \\
A'
\end{tikzcd}$$

get pushout
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u'"'] & B \arrow[d, "v"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

where

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

is a pushout.

f' 1-1:
⊗ $f'(a') = (0, a') + N = 0$
$\Rightarrow (0, a') \in N \Rightarrow a' = u(a)$, $f(a) = 0$
$\Rightarrow a = 0 \Rightarrow a' = 0$

→ have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u'"'] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r] & 0 \\
0 \arrow[r, "⊗"'] & A' \arrow[r, "f'"'] & B' & &
\end{tikzcd}$$

?

Consequence: If have a pushout

$$\begin{tikzcd}
A \arrow[r, "\text{mono}"] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

$\Rightarrow$ $C \to D$ is mono.

Similarly, if $A \to C$ (down) is mono $\Rightarrow$ $B \to D$ (down) is mono (apply snake lemma to the diagram above.)

$$\begin{tikzcd}
A \arrow[d] & B \arrow[d] \\
C & D
\end{tikzcd}$$

$\rightsquigarrow$ $0 \to \ker u \to \ker v \to \ker \mathrm{id} \to \operatorname{coker} u \to \dots$ $\ker u = 0 \Rightarrow \ker v = 0$

0 (coker → coker)

2/12/18

Remark:

The "converse" is also true, in the following sense:

Let
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r] \arrow[d, "v"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r, "g"'] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

be a comm. diag. with exact rows. Then
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$
is a pushout.

Dually:

Assume
$$\begin{tikzcd} & & & C' \arrow[d, "h"] & \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$
exact

⟹ Can complete to
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f'"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d, "t"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows and
$$\begin{tikzcd} B' \arrow[r, "g'"] \arrow[d, "t"'] & C' \arrow[d, "h"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$
is a pullback.

g onto ⟹ g' onto
h onto ⟹ t onto

(use snake lemma)

Schanuel's Lemma

Let $M \in$ Mod R and let $0 \to K \xrightarrow{i} P \xrightarrow{p} M \to 0$

and $0 \to L \xrightarrow{j} Q \xrightarrow{q} M \to 0$

be exact with P, Q projective modules.

"K = Ker p
L = Ker q"

Then we have an isomorphism $P \oplus L \simeq Q \oplus K$.

Pf: Draw a picture
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, dashed, "\exists! g"'] & P \arrow[r, "p"] \arrow[d, dashed, "\exists! f"'] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "j"'] & Q \arrow[r, "q"'] & M \arrow[r] & 0 \end{tikzcd}$$

f exists since P proj.
g exists from exer ① (1/22/18)

Get a "pushout diagram" so the square
$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "g"'] & P \arrow[d, "g"] \\ L \arrow[r, "j"'] & Q \end{tikzcd}$$
is a pushout.

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Now, we the following

Lemma: If
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$
is a pushout, then ∃ a SES

$$0 \longrightarrow A \xrightarrow[\$]{\begin{bmatrix} i \\ f \end{bmatrix}} B \oplus C \xrightarrow{[g,-j]} D \longrightarrow 0$$

exercise: why? $[g,-j]\begin{bmatrix} i \\ f \end{bmatrix}(a) = [g,-j](i(a), f(a)) = g(i(a)) - j(f(a)) \overset{gi=jf}{=} 0 \Rightarrow \mathrm{Im} \ldots \subset \mathrm{Ker}.$

If $(b,c) \in \ker([g,-j]) \Rightarrow g(b) - j(c) = 0 \in B \oplus C/_N \Rightarrow \exists a.\ g(b) = i(a),\ j(c) = f(a)$ ✓

Apply this to our situation, get exact seq.

$$0 \to K \to L \oplus P \to Q \to 0$$

Q proj. ⇒ ~~SES~~ seq. splits ⇒ $L \oplus P \cong K \oplus Q$. □

Derived functors

2/14/18

Def.: Let $A \xrightarrow{f} B$ be a homom in abelian category that has enough projectives. Then, let P_A, P_B be two projective resolutions of A, B:

$$\begin{tikzcd} P_A & \cdots \arrow[r] & P_2^A \arrow[r] \arrow[d, dashed, "\exists f_2"] \arrow[dr, phantom, "\circlearrowleft"] & P_1^A \arrow[r] \arrow[d, dashed, "\exists f_1"] \arrow[dr, phantom, "\circlearrowleft"] & P_0^A \arrow[r, "\varepsilon^A"] \arrow[d, dashed, "\exists f_0"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, "f"] & 0 \\ & \cdots \arrow[r] & P_2^B \arrow[r] & P_1^B \arrow[r] & P_0^B \arrow[r, "\varepsilon^B"'] & B \arrow[r] & 0 \end{tikzcd}$$

by comparison thm (and unique up to homotopy)

There exists an induced chain map between the complexes: $\{f_i\}_{i \geq 0}$. We only look at the deleted resolutions $\overline{P}_A, \overline{P}_B$:

$$\begin{tikzcd} \cdots \arrow[r] & P_1^A \arrow[r] \arrow[d, "f_1"] & P_0^A \arrow[r] \arrow[d, "f_0"] & 0 \\ \cdots \arrow[r] & P_1^B \arrow[r] & P_0^B \arrow[r] & 0 \end{tikzcd}$$

We call this chain map a map "over f" or "induced by f". (a chain map is over f if $f\varepsilon^A = \varepsilon^B f_0$)

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Left derived functors (will follow Rotman)

Let $T: \mathcal{A} \to \mathcal{C}$ be an additive covariant functor between two abelian categories and assume $\mathcal{A}$ has enough projectives.

For each $n \in \mathbb{Z}$ will define the nth left derived functor of T

$$L_nT : \mathcal{A} \to \mathcal{C}$$

as follows:

On objects: Let $A \in \mathcal{A}$. There exists a projective resolution of A:

$\mathbb{P}\ (P_A)$ $$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

where $\{P_i\}$ family of proj. objects. Look at the deleted resolution

$\mathbb{P}_A\ (\overline{P}_A)$ $$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

deg 0 deg -1

Apply T to this. Get $T(\overline{P}_A)$ $$\cdots \to T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} \overset{\downarrow}{T(P_0)} \to 0 \to 0 \to \cdots$$

- a complex in $\mathcal{C}$. (not nec. exact!)

Then, define $\boxed{(L_nT)(A) = H_n(T(\overline{P}_A))}$ $\forall n \in \mathbb{Z}$

Obs.: $\boxed{(L_nT)(A) = 0}$ $\forall n$ negative

Need to show well defined. Show that $(L_nT)(A)$ does not depend on the choice of the proj. resolution.

Will use this lemma:

Let $\underset{\{f_i\}}{\overset{}{f}}, \underset{\{g_i\}}{g} : C \to C'$ be two chain maps between two chain complexes in an additive category $\mathcal{A}$. Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor, with $\mathcal{B}$ being additive.

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Then, if $f \sim g \Rightarrow F(f) \sim F(g)$ where $F(f), F(g): F(C) \to F(C')$.

(exer. 6.6 p. 339) $\rightsquigarrow$ HW 3

<u>Prop:</u> L_nT does not depend on the choice of the proj. res'n of A.

<u>Pf:</u> Let $Q_A: \cdots \to Q_2 \to Q_1 \to Q_0 \to A$ be another projective resolution. Have the following comm. diagram with exact rows:

$$\begin{tikzcd}
P_A & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
Q_A & \cdots \arrow[r] & Q_2 \arrow[r] \arrow[d, "g_2"] & Q_1 \arrow[r] \arrow[d, "g_1"] & Q_0 \arrow[r] \arrow[d, "g_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
P_A & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_1"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
Q_A & \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

(by comparison thm)

There exists a chain map $\{f_i\}_{i\geq 0}: \overline{P_A} \to \overline{Q_A}$ and $\exists$ chain map $\{g_i\}_{i\geq 0}: \overline{Q_A} \to \overline{P_A}$.

Look at

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] \arrow[d, "g_2"] & Q_1 \arrow[r] \arrow[d, "g_1"] & Q_0 \arrow[r] \arrow[d, "g_0"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0
\end{tikzcd}$$

"merge" these as

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "g_2f_2"] & P_1 \arrow[r] \arrow[d, "g_1f_1"] & P_0 \arrow[r] \arrow[d, "g_0f_0"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0
\end{tikzcd}$$

So we get two chain maps $\overline{P_A} \to \overline{P_A}$, namely $\{g_if_i\}_{i\geq 0}$ and $\{1_{P_i}\}_{i\geq 0}$ (since

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "1_{P_1}"] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r, "d^A"] \arrow[d, "1_{P_0}"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$
)

We know that these chain maps are homotopic by the comparison thm

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If we write $\widehat{gf} = (g_i f_i)_{i \in \mathbb{Z}}$ and $\hat{1}_{\bar{P}_A} = (1_{P_i})_{i \in \mathbb{Z}}$, then

$$\widehat{gf} \sim \hat{1}_{\bar{P}_A} \overset{\text{lemma}}{\Longrightarrow} \underset{\substack{\shortparallel \\ T(\hat{g})T(\hat{f})}}{T(\widehat{gf})} \simeq T(\hat{1}_{\bar{P}_A}) = 1_{T(\bar{P}_A)}$$

$$\Rightarrow T(\hat{g})\ T(\hat{f}) \simeq 1_{T(\bar{P}_A)}$$

Similarly, $T(\hat{f})\ T(\hat{g}) \sim 1_{T(\bar{Q}_A)}$. by the lemma.

But, homotopic chain maps induce the same maps in homology.
(Prop 2/7/18)

So, for each $n \in \mathbb{Z}$ we get

$$\overline{T(g_n)}\ \overline{T(f_n)} = 1_{H_n(P(\bar{P}_A))}, \ \& \ \overline{T(f_n)}\ \overline{T(g_n)} = 1_{H_n(T(\bar{Q}_A))}$$

So we get isomorphisms for all n.

$$H_n(T(\bar{P}_A)) \cong H_n(T(\bar{Q}_A)).$$

So $(L_n T)(A)$ does not depend on the choice of the resolution.

□

Define <u>$L_n T$ on morphisms</u> now: Let $A, B \in \mathcal{A}$ and let $f \in \mathrm{Hom}_{\mathcal{A}}(A,B)$. Construct projective resolutions of A, B and a chain map over f: (which exists by comparison thm)

$$\begin{tikzcd}
P_A & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
P_B & \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Get

$$\begin{tikzcd}
\bar{P}_A & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
\bar{P}_B & \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
T(\bar{P}_A) & \cdots \arrow[r] & T(P_2) \arrow[r] \arrow[d, "T(f_2)"] & T(P_1) \arrow[r] \arrow[d, "T(f_1)"] & T(P_0) \arrow[r] \arrow[d, "T(f_0)"] & 0 \\
T(\bar{P}_B) & \cdots \arrow[r] & T(Q_2) \arrow[r] & T(Q_1) \arrow[r] & T(Q_0) \arrow[r] & 0
\end{tikzcd}$$

$\leadsto$ chain map
(diag commutes
(since T
functor))

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in $\mathcal{C}$: $T(\bar{P}_A) \xrightarrow{T(f_i)_i} T(\bar{P}_B)$

get induced maps in homologies:

$\forall n$ get $H_n(T(\bar{P}_A)) \xrightarrow{\{\overline{T(f_i)}\}_i} H_n(T(\bar{P}_B))$

$$\begin{tikzcd}
H_n(T(\bar{P}_A)) \arrow[r, "\{\overline{T(f_i)}\}_i"] \arrow[d, phantom, "\shortparallel"] & H_n(T(\bar{P}_B)) \arrow[d, phantom, "\shortparallel"] \\
(L_n T)(A) & (L_n T)(B)
\end{tikzcd}$$

so get homoms $\forall n$ $\overline{T(f_n)} : (L_nT)(A) \to (L_nT)(B)$

↳ call it $(L_nT)(f)$.

Remains to check:

- the def'n of $(L_nT)(f)$ does not depend on the choice of the resolution for A, B.
- the def of $(L_nT)(f)$ does not depend on the choice of the liftings

→ use the comparison thm exer

Very important examples: the Tor functors

Example 1: Let R be a ring and let A_R.

Let $T = A \otimes_R \_$: left R-modules $\to Ab$

"A tensor blank"

be given by $T(B) = A \otimes_R B$ and if $B \xrightarrow{f} C$ then

$T(f) = 1_A \otimes f : A \otimes_R B \to A \otimes_R C$. covariant additive.

Construct $(L_nT)(B)$:

$\bar{P}_B$ $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0$ proj. res'n

$A \otimes_R \bar{P}_B$ $\cdots \to A \otimes_R P_2 \xrightarrow{1_A \otimes d_2} A \otimes_R P_1 \xrightarrow{1_A \otimes d_1} A \otimes_R P_0 \xrightarrow{1_A \otimes 0 = 0} \cancel{A \otimes_R}\ 0$

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Take homologies $\forall n$

$$\boxed{L_n(A\otimes_R \overline{P}_B) = \frac{\operatorname{Ker}(1_A\otimes d_n)}{\operatorname{Im}(1_A\otimes d_{n+1})} \overset{def}{=} \operatorname{Tor}_n^R(A,B)}$$

$= H_n(A\otimes_R \overline{P}_0)$ read "Tor-n, A, B"

Easy obs:

$$\operatorname{Tor}_0^R(A,B) = ? \quad = \frac{A\otimes P_0}{\operatorname{Im}(1_A\otimes d_1)} \quad = \operatorname{Ker}(1_A\otimes \overset{=0}{0})$$

Recall that $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0$ exact

$$\Rightarrow A\otimes_R P_1 \xrightarrow{1_A\otimes d_1} A\otimes_R P_0 \xrightarrow{1_A\otimes d_0} A\otimes B \to 0 \text{ exact}$$

$$\Rightarrow \operatorname{Im}(1_A\otimes d_1) = \operatorname{Ker}(1_A\otimes d_0)$$

$1_A\otimes d_0$ onto (1^{st} isom thm)

$$\Rightarrow \frac{A\otimes P_0}{\operatorname{Im}(1_A\otimes d_1)} = \frac{A\otimes P_0}{\operatorname{Ker}(1_A\otimes d_0)} \cong A\otimes_R B$$

So $\boxed{\operatorname{Tor}_0^R(A,B) = A\otimes_R B}$

Example 2: $\widehat{\operatorname{Tor}}$ Let R be a ring and let ${}_RB$.

Let $T = -\otimes_R B$: right R-modules $\to$ Ab

$T(A) \overset{def}{=} A\otimes_R B$, and if $A \xrightarrow{f} C$ then $T(f) = f\otimes 1_B$:

$A\otimes_R B \to C\otimes_R B$.

T is additive and covariant.

Define $\boxed{\widehat{\operatorname{Tor}}_n^R(A,B) = (L_nT)(A) \text{ for all } n}$

$P_A \qquad \cdots \to P_1 \xrightarrow{d_1} P_0 \to A \to 0$

$\overline{P}_A\otimes_R B \qquad \cdots \to P_1\otimes_R B \xrightarrow{d_1\otimes 1_B} P_0\otimes_R B \xrightarrow{0\otimes 1_B = 0} 0$ Then take

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homology.

$$\boxed{\hat{\operatorname{Tor}}_n^R(A,B) = \frac{\operatorname{Ker}(d_n \otimes 1_B)}{\operatorname{Im}(d_{n+1} \otimes 1_B)}}$$

We get again $\hat{\operatorname{Tor}}_0^R(A,B) = A \otimes_R B$.

Next time will show that $\forall n$ we have $\boxed{\operatorname{Tor}_n^R(A,B) \cong \hat{\operatorname{Tor}}_n^R(A,B}$

$$\begin{tikzcd}
 & P_1^A \arrow[d] & & P_1^C \arrow[d] & \\
 & P_0^A \arrow[d] & & P_0^C \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

The horseshoe lemma

Prop: Prop. 6.24, p. 344 Let $\mathcal{A}$ be an abelian category with enough projectives.

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{A}$. Assume P_A and P_C are projective resolutions of A and ~~B~~ C, respectively.

Then, $\exists$ a projective resolution P_B of B, and chain maps such that we get a short exact sequence in $\operatorname{Com}(\mathcal{A})$:

$$0 \to P_A \to P_B \to P_C \to 0.$$

Pf:

$$\begin{tikzcd}
0 \arrow[r] & P_0^A \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\circlearrowleft"] & P_0^A \oplus P_0^C \arrow[r, "{[0\ 1]}"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowleft"] & P_0^C \arrow[r] \arrow[d, "\mu"] \arrow[dl, dashed, "\exists s_0"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

$(P_A) = (P_i^A)$

$(P_C) = (P_i^C)$

P_0^C proj. $\Rightarrow \exists\, s_0 : P_0^C \to B$. $\boxed{g s_0 = \mu}$

Define π: $\pi(x,y) \overset{\text{def}}{=} f\varepsilon(x) + s_0(y)$

Claim: 1st and 2nd square commute

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & (x,0) \arrow[d, mapsto, "\text{since } s_0(0)=0"] \\
\varepsilon(x) \arrow[r, mapsto] & f\varepsilon(x)
\end{tikzcd}$$

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2nd square:

$$\begin{tikzcd}
(x,y) \arrow[r, "{[0,\ 1]}"] \arrow[d, "\pi"'] & y \arrow[d, "\mu"] \\
f\varepsilon(x)+s_0(y) \arrow[dr, "g"'] & \mu(y) \arrow[d, equal] \\
gf=0 & gs_0(y)
\end{tikzcd}$$

$\leftarrow gs_0=\mu$ ✓

Claim: π is onto by the snake lemma. or from five lemma since ε, μ onto, $0 \to 0$ 1-

$$\begin{tikzcd}
0 \arrow[r] & P_1^A \arrow[rr] \arrow[dr] & 0 \arrow[d] & P_1^A \oplus P_1^C \arrow[rr] \arrow[dr, dashed] & 0 \arrow[d] & P_1^C \arrow[r] \arrow[dr] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \operatorname{Ker}\varepsilon \arrow[rr] \arrow[d] \arrow[dr] & & \operatorname{Ker}\pi \arrow[rr] \arrow[d] & & \operatorname{Ker}\mu \arrow[r] \arrow[d] \arrow[dr] & 0 & \\
 & 0 \arrow[r] & P_0^A \arrow[rr, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "\varepsilon"] & 0 & P_0^A \oplus P_0^C \arrow[rr, "{[0\ 1]}"] \arrow[d, "\pi"] & & P_0^C \arrow[rr] \arrow[d, "\mu"] & 0 & 0 \\
 & 0 \arrow[r] & A \arrow[rr, "f"'] \arrow[d] & & B \arrow[rr, "g"'] \arrow[d] & & C \arrow[rr] \arrow[d] & & 0 \\
 & & 0 & & 0 & & 0 & &
\end{tikzcd}$$

exists by snake Lemma, second map onto since coker $\varepsilon = 0$

$\leftarrow$ exact since bottom two rows are exact by 3×3 Lemma (Ex 2.32 HW 3)

Continue by induction.

exercise: verify the inductive step. $\to \boxed{21}$

Write down P_B with the differentials as wells
$\to 2\times2$

(the terms are: $P_n^B \simeq P_n^A \oplus P_n^C$.)

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The dimension shift theorem

Let $\mathcal{A}$ be an abelian category with enough projectives and let $T: \mathcal{A} \to \mathcal{C}$ be a covariant additive functor where $\mathcal{C}$ is an ab. cat. Let $A \in \mathrm{Ob}\,$~~$\mathcal{C}$~~$\mathcal{A}$ and let $P_\cdot \quad \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} A \to 0$ be a proj. resolution, and denote by $K_i = \operatorname{Im} d_{i+1} = \operatorname{Ker} d_i \quad \forall i \geq 0$.

$\Rightarrow \forall n \geq 1$ we have $(L_{n+1}T)(A) \cong (L_nT)(K_0) \cong (L_{n-1}T)(K_1) \cong \dots \cong (L_1T)K_{n-1}$

Pf: Just observe that $P_{K_0} \quad \dots \to \overset{\substack{\text{in degree 1}\\\downarrow}}{P_2} \overset{d_2}{\to} \overset{\substack{\text{in degree 0}\\\downarrow}}{P_1} \overset{d_1}{\to} K_0 \to 0$ part of the res'n of A, shifted to the right

is a proj. res'n of K_0. We get $(L_nT)(K_0) = \dfrac{\operatorname{Ker}(T d_{n+1})}{\operatorname{Im} T(d_{n+2})} = L_{n+1}(A)$. $\square$

$= H_n(TP_{K_0})$

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Corollary: (Dimension shift in Tor)

- For $T = A \otimes_R -$ and ${}_R B$ with projective resolution

$$\begin{tikzcd}
\mathbf{P}_B: \cdots \arrow[r] & Q_2 \arrow[rr] \arrow[dr] & & Q_1 \arrow[rr] \arrow[dr] & & Q_0 \arrow[r] & B \arrow[r] & 0 \\
& & L_1 \arrow[ur] \arrow[dr] & & L_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$\forall n \geq 1$ $\quad \mathrm{Tor}^R_{n+1}(A, B) \cong \mathrm{Tor}^R_n(A, L_0) \cong \mathrm{Tor}^R_{n-1}(A, L_1) \cong \ldots \cong \mathrm{Tor}^R_1(A, L_{n-1})$

- For $T = - \otimes_R B$ and $\mathbf{P}_A$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & A \arrow[r] & 0 \\
& & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

is a proj. res'n

we get $\quad \widehat{\mathrm{Tor}}^R_{n+1}(A, B) \cong \widehat{\mathrm{Tor}}^R_n(K_0, B) \cong \ldots \cong \widehat{\mathrm{Tor}}^R_1(K_{n-1}, B)$.

Let R be a ring and A_R, ${}_R B$.

Thm: $\forall n \geq 0$ we have $\mathrm{Tor}^R_n(A, B) \cong \widehat{\mathrm{Tor}}^R_n(A, B)$.

HW: read the pf on p. 354-356 of Rotman. (A. Zacks)

$T = \mathrm{Tor}$ nat. isom. to $L_0 T$
$\Rightarrow \mathrm{Tor}^R_0(A, B) \cong \widehat{\mathrm{Tor}}^R_0(A, B)$
pf by ind. & dimension shift

The long exact sequence of left derived functors

Let $T: \mathcal{A} \to \mathcal{C}$ be covariant and additive where $\mathcal{A}, \mathcal{C}$ are abelian and $\mathcal{A}$ has enough projectives. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence. Then, $\exists$ a long exact sequence in $\mathcal{C}$:

$$\cdots (L_n T)(A) \to (L_n T)(B) \to (L_n T)(C) \to (L_{n-1} T)(A) \to \cdots$$

$$\to \cdots \to (L_0 T)(A) \to (L_0 T)(B) \to (L_0 T)(C) \to 0.$$

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Pf: Let P_A, P_c be projective resolutions of $\mathcal{A}$ and of C.

Use the horseshoe lemma to fill in a proj. res'n of B.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
0 \arrow[r] & P_2^A \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "d_2^A"] & P_2^A \oplus P_2^C \arrow[r, "{[0\ 1]}"] \arrow[d] & P_2^C \arrow[r] \arrow[d, "d_2^C"] & 0 & \\
0 \arrow[r] & P_1^A \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "d_1^A"] & P_1^A \oplus P_1^C \arrow[r, "{[0\ 1]}"] \arrow[d] & P_1^C \arrow[r] \arrow[d] & 0 & \text{split exact} \\
0 \arrow[r] & P_0^A \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "d_0^A"] & P_0^A \oplus P_0^C \arrow[r, "{[0,1]}"] \arrow[d, "d_0^B"] & P_0^C \arrow[r] \arrow[d, "d_0^C"] & 0 & \text{split exact} \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & 0 & & \text{(commutes)}
\end{tikzcd}$$

Get an exact seq. of complexes in $\mathcal{C}$

$$0 \to T\left(\begin{array}{c}\text{deleted}\\ \text{proj. res'n}\\ \text{of } A\end{array}\right) \to T\left(\begin{array}{c}\text{deleted}\\ \text{proj. res'n}\\ \text{of } B\end{array}\right) \to T\left(\begin{array}{c}\text{deleted}\\ \text{proj. res'n}\\ \text{of } C\end{array}\right) \to 0$$

not split exact ✓ (by lifting maps might not be chain maps)

Since if we have a split exact seq. in $\mathcal{A}$ and we apply T to it

Apply the long exact seq. of homology to this sequence and the theorem follows from here. The sequence ends at 0 on the right since $(L_nT)(X) = 0$ if n is negative. □

Remark: The dimension shift follows immediately now (as an alternative pf) as follows:

Start with $A \in \mathcal{A}$. Look at $0 \to K_0 \to P_0 \to A \to 0$.

Apply the long exact seq. in homology:

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$$\to (L_nT)(K_0) \to \boxed{(L_nT)(P_0)} \to (L_nT)(A) \to (L_{n-1}T)(K_0)$$
$$\to \boxed{(L_{n-1}T)(P_0)} \to (L_{n-1}T)(A) \to \dots$$

But P_0 is projective so a proj. resolution is $0 \to \dots \to 0 \to 0 \to P_0 \xrightarrow{=} P_0 \to 0$

But $(L_nT)(P_0)$ does not depend on the proj. res'n so $(L_nT)(P_0) = 0$ $\forall n \geq 1$.

Get $0 \to (L_nT)(A) \xrightarrow{} (L_{n-1}T)(K_0) \to 0$ $\forall n \geq 2$.

so this is an isom.

□

<u>Corollary:</u> R ring, ~~A_R, $_RB$ Let~~ ~~$0 \to K_0 \to \underset{\text{proj.}}{P_0} \to A \to 0$ and let~~

~~$0 \to L \to Q_0 \to B \to 0$~~

(1) Let $0 \to L \to M \to N \to 0$ be exact. Then, if A_R we have a long exact sequence

$$\dots \to \mathrm{Tor}_1^R(A,M) \to \mathrm{Tor}_1^R(A,N) \to A \otimes_R L \to A \otimes_R M \to A \otimes_R N \to 0.$$

(2) Let $0 \to X_R \to Y_R \to Z_R \to 0$ and $_RB$. Then we have a long exact sequence

$$\dots \to \mathrm{Tor}_1^R(X,B) \to \mathrm{Tor}_1^R(Y,B) \to \mathrm{Tor}_1^R(Z,B) \to X \otimes_R B \to Y \otimes_R B \to Z \otimes_R B \to 0$$

<u>Notation used frequently:</u>

Suppose $M \in \mathrm{Mod}\, R$ and P_M: $\dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$

is a proj. resolution.

We denote by $\Omega^i = \ker d_{i-1} = \mathrm{Im}\, d_i$ $\quad i \geq 1$.

called "the ith <u>syzygy</u>" of M

$\ker d_0 =$ 1st syzygy of M.

(Sometimes we use this terminology if P_M is a "special" projective resolution, namely a <u>minimal projective</u> resolution.)

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⌈ Example of non minimal res'n:

$$\underbrace{\begin{matrix}P_{i+1}\\ \oplus\\ Q\end{matrix} \xrightarrow{d_{i+1}} \begin{matrix}P_i\\ \oplus\\ Q\end{matrix}} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

Let $Q \neq 0$ be proj.

the new differential is
$$\begin{bmatrix} d_{i+1} & 0 \\ 0 & 1_Q \end{bmatrix}$$

$$0 \to 0 \to 0 \to Q \xrightarrow{1_Q} Q \to 0 \cdots$$

exact seq.

this is still a resolution. ↓ proj. res'n of M. ⌋

Question related to the horseshoe lemma

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a SES in an abelian cat. with enough projectives

Know: • if have P_A, P_C ⇒ may construct P_B.

• Assume have P_A, P_B. Can we construct a proj. res'n of C?

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \Omega'_B \arrow[r] \arrow[d] & \Omega'_C \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & * & P_0^B \arrow[r, equal] \arrow[d] & P_0^B \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

← apply the Snake lemma

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[d] & \\
0 \arrow[r] & \Omega'_B \arrow[r] \arrow[d] & P_0^B \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega'_C \arrow[r] \arrow[d] & P_0^B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & A & 0 & 0 &
\end{tikzcd}$$

$$0 \to 0 \to A \to A \to 0 \to 0$$

Get a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Omega B \arrow[r] \arrow[d] & \Omega C \arrow[r] & A \arrow[r] \arrow[d] & 0 \\
 & \cdots \to P_2^B \to P_1^B \to \Omega'_B \to 0 & & \text{have } P_A \text{ by assumption} &
\end{tikzcd}$$

is a proj. res'n.

Horseshoe lemma ⇒ get a resolution of ΩC.

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to \Omega C \to 0$$

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"splice" with $0 \to \Omega C \to P_0^B \to C \to 0$ and get a proj. res'n of C:
(verkleben)

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to P_0^B \to C \to 0.$$

Question: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. Assume we have P_B, P_C
P_B ↓ proj. res'n of B
P_C → proj. res'n of C

Can we construct explicitly a projective resolution of A? exercise (due Monday)

exercise! (HW)

Corollary to the long exact sequence of derived functors

Let $T: \mathcal{A} \to \mathcal{C}$ be additive covariant where $\mathcal{A}, \mathcal{C}$ are abelian and $\mathcal{A}$ has enough projectives. Then $L_0T: \mathcal{A} \to \mathcal{C}$ is right exact.

Pf: Start with $0 \to A \to B \to C \to 0$ exact, get

$$\cdots \to (L_0T)(A) \to (L_0T)(B) \to (L_0T)(C) \to 0.$$

Axioms for left derived functors

Thm: (Rotman p. 358) Let $\{T_n\}_{n\geq 0}$ and $\{T_n'\}_{n\geq 0}$ be sequences of covariant additive functors $\mathcal{A} \to \mathcal{C}$ where $\mathcal{A}, \mathcal{C}$ are abelian and $\mathcal{A}$ has enough projectives. Assume the following:

(1) $\forall$ SES in $\mathcal{A}$: $0 \to A \to B \to C \to 0$ there are long exact sequences with natural connecting homomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & T_{n+1}(C) \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & T_n(A) \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & T_n(B) \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & T_n(C) \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & T_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & T'_{n+1}(C') \arrow[r] & T'_n(A') \arrow[r] & T'_n(B') \arrow[r] & T'_n(C') \arrow[r] & T'_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

whenever we have a comm. diagram in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

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and the same thing with T_n' instead of T_n.

(2) T_0 is naturally isomorphic to T_0', that is, if $f: A \to B$ in $\mathcal{A}$ we have a comm diagram:

$$\begin{tikzcd}
T_0(A) \arrow[r, "T_0(f)"] \arrow[d, "\wr", "\eta_A"'] & T_0(B) \arrow[d, "\wr"', "\eta_B"] \\
T_0'(A) \arrow[r, "T_0'(f)"'] & T_0'(B)
\end{tikzcd}$$

(3) $T_n(P) = T_n'(P) = 0 \quad \forall n \geq 1$ and projective objects P.

Then $\forall n \geq 0$ T_n is naturally isomorphic to T_n'.

<u>Corollary:</u> R commutative. $\Rightarrow \widehat{\mathrm{Tor}}_n(A,B) \cong \mathrm{Tor}_n(A,B) \quad \forall n \geq 0$.

! <u>HW:</u> read the pf in Rotman or, in Waibel (Thm 2.4.7) on p. 47

Right derived functors on Wednesday

the functor Ext.

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Right derived $\overset{\text{covariant}}{\text{functors}}$

$\underset{\text{additive}}{T}^{-\boxed{\text{covariant}}} : \underset{\text{abelian}}{\mathcal{A} \to \mathcal{C}}$ where $\mathcal{A}$ has enough* ~~projectives~~ injectives

($\forall A \in \mathcal{A}$ $\exists$ inj. object $E \in \mathcal{A}$ and a mono $A \to E$)

We construct $\{R_n T\}$ nth right derived functors of T.

Let $B \in \mathrm{Ob}\,\mathcal{A}$.

Start with $0 \to B \xrightarrow{j} E^0 \xrightarrow{\pi_0} \mathrm{coker}\, j \to 0$

then $\exists E'$ inj., $j': \mathrm{coker}\, j \to E'$

Then $\exists$ an injective resolution of B:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "j\cancel{d^0}"] & E^0 \arrow[rr, "d^0"] \arrow[dr, dashed, "\pi_0"'] & \arrow[d, phantom, "\circlearrowleft"] & E^1 \arrow[r, "d^1"] \arrow[dr, "\pi_1"] & E^2 \arrow[r, "d^2"] & \cdots \\
& & & \mathrm{coker}\, \cancel{d^0}\, j \arrow[ur, "j'"] \arrow[dr] & & \mathrm{coker}\, d^0 \arrow[r] & 0 \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

exact!

$d^0 = j'\pi$

$\mathrm{Im}\, j = \mathrm{Ker}\, \pi_0 = \mathrm{Ker}(j' \circ \pi_0) = \mathrm{Ker}(d^0)$

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deleted complex

$\mathbb{E}_B$: $0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \dots$

deleting B loses no information, since $B \cong \ker d_0$. $\left[B \cong \operatorname{Im}(j) = \ker d_0\right]$

Apply T: $0 \to TE^0 \xrightarrow{Td^0} TE^1 \xrightarrow{Td^1} TE^2 \to \dots$ complex

<u>Def:</u> $\boxed{(R^nT)(B) = \dfrac{\ker Td^n}{\operatorname{Im} d^{n-1}}}$ $\left(= H_n(T\mathbb{E}_B)\right)$

<u>Prop:</u> $(R^nT)(B)$ does not depend on the choice of the injective resolution.

<u>Pf:</u> Show that any two injective res.^{s} of B $\mathbb{E}_1$ and $\mathbb{E}_2$ are homotopic. Then show that the complexes $T\mathbb{E}_1$ and $T\mathbb{E}_2$ are homotopic. Then the homologies will be the same.

Define R^nT on homoms: Let $B \xrightarrow{f} C$ be in $\mathcal{A}$.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & E^0_B \arrow[r, "d^0_B"] \arrow[d, "f^0"] \arrow[dr, phantom, "\circlearrowleft"] & E^1_B \arrow[r] \arrow[d, "f^1"] \arrow[dr, phantom, "\circlearrowleft"] & E^2_B \arrow[r] \arrow[d, "f^2"] & \dots \\
0 \arrow[r] & C \arrow[r] & E^0_C \arrow[r] & E^1_C \arrow[r] & E^2_C \arrow[r] & \dots
\end{tikzcd}$$

Get a chain map

$$\begin{tikzcd}
0 \arrow[r] & TE^0_B \arrow[r] \arrow[d, "Tf^0"] \arrow[dr, phantom, "\circlearrowleft"] & TE^1_B \arrow[r] \arrow[d, "Tf^1"] \arrow[dr, phantom, "\circlearrowleft"] & TE^2_B \arrow[r] \arrow[d, "Tf^2"] & \dots \\
0 \arrow[r] & TE^0_C \arrow[r] & TE^1_C \arrow[r] & TE^2_C \arrow[r] & \dots
\end{tikzcd}$$

Then we have $\forall n$, induced homoms: $(R^nT)(B) \to (R^nT)(C)$
(in homology $\overline{Tf^i}$)

$$\begin{tikzcd}
(R^nT)(B) \arrow[r] & (R^nT)(C) \\
& (R^nT)(f) \arrow[u, squiggly]
\end{tikzcd}$$

To check: This definition does not depend on the choice of the injective resolutions and it does not depend on the liftings

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for f.

Thm: $\forall n \geq 0$ the functors $R^nT: \mathcal{A} \to \mathcal{C}$ are additive.

Example: **Ext**

$\mathcal{A} = \text{Mod}\, R$
$\mathcal{C} = \mathcal{A}b$

Fix $A \in \text{Mod}\, R$.

$T = \text{Hom}_R(A, -) : \text{Mod}\, R \to \mathcal{A}b$

Let $B \in \text{Mod}\, R$.

$$\boxed{\text{Ext}^n_R(A, B) \overset{\text{def}}{=} (R^n \text{Hom}_R(A, -))(B)}$$

"ext - n - A - B"

Pick an injective resolution of B:

$$0 \to B \xrightarrow{j} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

[mod R need not have enough injectives but it has enough projectives]

for Art. rings mod R has enough inj.
for Noeth. rings need not be true

Take $\text{Hom}(A, -)$ into this:

$$0 \to \text{Hom}(A, I^0) \xrightarrow{\text{Hom}(A, d^0)} \text{Hom}(A, I^1) \xrightarrow{\text{Hom}(A, d^1)} \cdots$$

Then

$$\text{Ext}^n(A, B) = \frac{\text{Ker}\, \text{Hom}(A, d^n)}{\text{Im}\, \text{Hom}(A, d^{n-1})} \qquad n \geq 1$$

Important fact: $\text{Ext}^0(A, B) = \text{Hom}_R(A, B)$

look at the exact sequence

$$0 \to \text{Hom}(A, B) \xrightarrow{\text{Hom}(A, j)} \text{Hom}(A, I^0) \xrightarrow{\text{Hom}(A, d^0)} \text{Hom}(A, I^1)$$

exact

(use the fact that $\text{Hom}_R(A, -)$ is left exact)

$\to \text{Ker}\, \text{Hom}(A, d^0) = \text{Im}\, \text{Hom}(A, j) \underset{\text{mono}}{=} \text{Hom}(A, B)$

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Remarks:

① We also have the "injective" version of the horseshoe lemma:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & I^0_A \arrow[d] & & I^0_C \arrow[d] & \\
 & I^1_A \arrow[d] & & I^1_C \arrow[d] & \\
 & \vdots \;\; \mathbb{E}_A & & \vdots & \\
 & & & \mathbb{E}_C &
\end{tikzcd}$$

short exact seq in $\mathcal{A}$ abelian with enough injectives

injective resolutions

$\Rightarrow$ may "fill in" an injective resolution $\mathbb{E}_B$ of B so that we get an exact seq. of complexes of injective objects:

$$0 \to \mathbb{E}_A \to \mathbb{E}_B \to \mathbb{E}_C \to 0$$

deleted injective resolutions

② Apply the "long exact seq. in homology" and get the corresponding long exact seq. for right derived functors:

fill in ... $(\ldots) \longrightarrow (R_n T(A) \to (R_n T)(B) \to (R_n T)(C) \to (R_{n+1} T)(A) \to \ldots$

In the case of Ext-functor:

$M \in \mathrm{Mod}\, R \qquad T = \mathrm{Hom}_R(M, -)$

Let $0 \to A \to B \to C \to 0$ short exact in $\mathrm{Mod}\, R$.

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to$$

$$\to \mathrm{Ext}^1_R(M,A) \to \mathrm{Ext}^1(M,B) \to \mathrm{Ext}^1(M,C) \to \ldots$$

$$\vdots$$

$$\to \mathrm{Ext}^n_R(M,A) \to \mathrm{Ext}^n_R(M,B) \to \mathrm{Ext}^n_R(M,C) \to \ldots$$

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③ We also have a "dimension shift" result:

$$0 \to A \to I \to B \to 0$$

I: injective

write $0 \to A \to I \to \Omega^{-1}A \to 0$

$0 \to I \xrightarrow{=} I \to 0 \to 0 \to \cdots$

injective resolution of I $\quad I^0 = I$

So $\mathrm{Ext}^n_R(A,I) = 0 \quad \forall n \geq 1$

$\mathrm{Ext}^0_R(A,I) = \mathrm{Hom}_R(A,I)$

Since $\boxed{\mathrm{Ext}^n_R(M,I) = 0 \ \forall M \text{ and } \forall n \geq 1 \text{ because } I \text{ is injective}}$

we get from the long exact seq. in Ext

$$\boxed{\mathrm{Ext}^{n}(M,B) \cong \mathrm{Ext}^{n+1}(M,A) \qquad \forall n \geq 1}$$

④ We also have "axioms" for right derived functors, see Rotman p. 367.

<u>Right derived functors using contravariant functors</u>

$\mathcal{A} \xrightarrow{T} \mathcal{C}$ $\qquad \mathcal{A}, \mathcal{C}$ abelian

T: additive, contravar.

$\mathcal{A}$: with enough projectives

Let $A \in \mathcal{A}$. Pick a proj. resolution of A:

$$\mathbb{P} \qquad \cdots \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A \to 0$$

$$\mathbb{P}_A \qquad \cdots \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \to 0$$

deleted res'n

Apply T:

$$0 \to TP^0 \xrightarrow{Td^1} TP^1 \xrightarrow{Td^2} TP^2 \to \cdots$$

complex

Then take homology: $\quad H^n(T\mathbb{P}_A) \overset{\text{def}}{=} (R^nT)(A).$

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Again we must go through the entire procedure:

- Show $(R^n T)(A)$ does not depend on the choice of the projective resolution of A
- show $(R^n T)(f)$, $f: B \to C$, does not depend on the choice of resolutions for B and C, and also does not depend on the liftings
- Have $\boxed{(R^n T)(\text{proj. object}) = 0 \quad \forall n \geq 1}$

<u>Example:</u> $\overline{\mathrm{Ext}}$

$\mathcal{A} = \mathrm{Mod}\, R$ Fix $B \in \mathrm{Mod}\, R$.
$\mathcal{C} = \mathcal{A}b$

$T = \mathrm{Hom}_R(-, B)$ additive contravariant

Then $\boxed{\overline{\mathrm{Ext}}^n_R (A, B) \overset{\text{def}}{=} (R^n \mathrm{Hom}_R(-, B))(A)}$

$$\cdots \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A \to 0$$

deleted res.: $\mathbb{P}_A \quad \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \to 0$

$\mathrm{Hom}(\mathbb{P}_A, B) \qquad 0 \to \mathrm{Hom}_R(P^0, B) \xrightarrow{\mathrm{Hom}(d^1, B)} \mathrm{Hom}_R(P^1, B) \xrightarrow{\mathrm{Hom}(d^2, B)} \mathrm{Hom}_R(P^2, B) \to \cdots$

$$\Rightarrow \boxed{\overline{\mathrm{Ext}}^n_R (A, B) \overset{\text{def}}{=} \frac{\mathrm{Ker}\ \mathrm{Hom}(d^{n+1}, B)}{\mathrm{Im}\ \mathrm{Hom}(d^n, B)}}$$

<u>Thm:</u> R ring, $A, B \in \mathrm{Mod}\, R$.

Then, $\forall n \geq 0$ $\mathrm{Ext}^n_R(A, B) \cong \overline{\mathrm{Ext}}^n_R (A, B)$. read pf

From now on, we will use Ext^n.

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Always remember:

$\underline{\mathrm{Ext}^n_R(P, M) = 0}$ ∀ proj. module P ∀M ∀n ≥ 1.

$\underline{\mathrm{Ext}^n_R(M, I) = 0}$ ∀ inj. module I ∀M ∀n ≥ 1

Extension groups

Def: Let $M, N \in \mathrm{Mod}\, R$.

Let $\xi: \; 0 \to N \to E_1 \to M \to 0$,

let $\mu: \; 0 \to N \to E_2 \to M \to 0$

be two short exact sequences. We say that $\underline{\xi \sim \mu}$ (" ξ is equivalen to μ ") if ∃ a commutative diagram:

$$\begin{tikzcd}
\xi & 0 \arrow[r] & N \arrow[r] \arrow[d, equal] \arrow[rd, phantom, "\circlearrowright"] & E_1 \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, equal] & 0 \\
\mu & 0 \arrow[r] & N \arrow[r] & E_2 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

equality, not $\underline{\text{isom}}$!

Def: A $\underline{\text{short exact seq.}}$ $\underline{0 \to N \to E \to M \to 0}$ is also called an "$\underline{\text{extension}}$ of $\underline{M \text{ by } N}$".

Prop: ~ is an equivalence relation on the set of extensions of M by N.

Pf: reflexive and transitive are clear.

"symmetric": assume $\xi \sim \mu$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f_1"] \arrow[d, equal] & E_1 \arrow[r, "g_1"] \arrow[d, shift left, "h"] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "f_2"] & E_2 \arrow[r, "g_2"] \arrow[u, shift left, "k"] & M \arrow[r] & 0
\end{tikzcd}$$

(h onto: $e_2 \in E_2$. g_1 onto ⇒ ∃ $e_1 \in E_1$: $g_1(e_1) = g_2(e_2)$
$= g_2(h(e_1))$ ⇒ $e_2 - h(e_1) \in \ker g_2 = \operatorname{Im} f_2$
⇒ ∃ $n \in N$: $f_2(n) = h(f_1(n)) = e_2 - h(e_1)$
⇒ $e_2 = h(f_1(n) - e_1)$)

⇒ h is an isomorphism. Let k = inverse isom of h.

Five Lemma

$hf_1 = f_2$, $g_1 = g_2 h$. Look at the composition:

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$k h f_1 = k f_2 \Rightarrow f_1 = k f_2$

$g_1 k = g_2 h k \Rightarrow g_1 k = g_2$

$$\begin{tikzcd}
\mu & 0 \arrow[r] & N \arrow[r, "f_2"] \arrow[d, equal] \arrow[rd, phantom, "\circlearrowleft"] & E_2 \arrow[r, "g_2"] \arrow[d, "k"] \arrow[rd, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, equal] & 0 \\
\xi & 0 \arrow[r] & N \arrow[r, "f_1"'] & E_1 \arrow[r, "g_1"'] & M \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \mu \sim \xi.$

□

For the time being let $\mathcal{E}$ = the set of equivalence relations of extensions of M by N.

Make $\mathcal{E}$ into an abelian group.

The Baer Sum

~~Let~~ "add" $0 \to N \xrightarrow{f_1} E_1 \xrightarrow{g_1} M \to 0$

to $0 \to N \xrightarrow[f_2]{} E_2 \xrightarrow[g_2]{} M \to 0$

First: take the direct sum of the 2 extensions:

$$\xi \oplus \mu \qquad 0 \to N \oplus N \xrightarrow[\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}]{} E_1 \oplus E_2 \xrightarrow[\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}]{} M \oplus M \to 0$$

pullback

$$\begin{tikzcd}
 & 0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & M \arrow[d, "\Delta_M = \begin{bmatrix} 1 \\ 1 \end{bmatrix}"] & \\
\xi \oplus \mu & 0 \arrow[r] & N \oplus N \arrow[r] & E_1 \oplus E_2 \arrow[r] & M \oplus M \arrow[r] & 0
\end{tikzcd}$$

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$\Delta_M(x) = (x, x)$
"the diagonal map"

Take pullback diagram

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$$\begin{tikzcd}
0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "\nabla_N = {[1\ 1]}"'] & V \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

pushout

Define $[\xi]+[\mu] = [0 \to N \to E \to M \to 0]$

Last time: 2/26/18

$\mathcal{E} = \mathcal{E}(M,N)$ = the set of $\overset{\text{equiv. relations on}}{\vee}$ "extensions of M by N"

$= \{ [0 \to N \to E \to M \to 0] \text{ in } \mathcal{M}od\ R \}$

$\xi \sim \eta$ $\qquad \xi \quad 0 \to N \to E_1 \to M \to 0$

$\qquad\qquad \eta \quad 0 \to N \to E_2 \to M \to 0$

$\iff \exists$ commut. diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E_2 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Introduce an addition on $\mathcal{E}(M,N)$.

if ξ, η as above (representation in $\mathcal{E}(M,N)$)

first $\xi \oplus \eta$ $\qquad 0 \to N \oplus N \xrightarrow{\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}} E_1 \oplus E_2 \xrightarrow{\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}} M \oplus M \to 0$

Then take pullback

$$\begin{tikzcd}
0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\Delta = {[1;1]}"] & 0 \\
0 \arrow[r] & N \oplus N \arrow[r] & E_1 \oplus E_2 \arrow[r] & M \oplus M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "{[1\ 1]} = \nabla"'] & X \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] & V \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

pushout

Then we define $[\xi]+[\eta] = [0 \to N \to V \to M \to 0]$.

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To check: This + is well defined, that is if $\xi \sim \xi'$,

$$\eta \sim \eta' \Rightarrow [\xi]+[\eta] = [\xi']+[\eta']$$

Read for details: Rotman or <u>MacLane</u> ("Homology"?)

<u>Obs:</u> We may define the addition of extension also as follows:

Start again with $\xi \oplus \eta$

$$\begin{tikzcd} 0 \arrow[r] & N \oplus N \arrow[r, "{\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}}"] \arrow[d, "{[1\ 1] = \nabla}"'] & E_1 \oplus E_2 \arrow[r, "{\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}}"] \arrow[d] & M \oplus M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & X \arrow[r] & M \oplus M \arrow[r] & 0 \\ 0 \arrow[r] & N \arrow[r] \arrow[u, equal] & V \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "{\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}"'] & 0 \end{tikzcd}$$

Then take pullback

$\rightsquigarrow$ take equiv. class

<u>Prop:</u> get again $[\xi]+[\eta]$.

<u>Fact:</u> addition on $\mathcal{E}(M,N)$ is ~~associated~~ associative and commutative.

There is an <u>identity element</u>, namely the equivalence class of $0 \to N \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} N \oplus M \xrightarrow{[0\ 1]} M \to 0$

that is, the class of ξ: $0 \to N \to E \to M \to 0$

is "zero" $\Leftrightarrow$ ξ is a split exact sequence.

$\xi \equiv \mathrm{id} \in \mathcal{E}$, $\exists$ comm. d

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & N \oplus M \arrow[r] \arrow[d, "h"] & M \arrow[d, equal] \\ 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \end{tikzcd}$$

$\Rightarrow h$ isom $\Rightarrow N \oplus M \cong$

$\Rightarrow \xi$ splits

What is $-[\xi]$? Let

$$\begin{tikzcd} \xi: 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "-1"] & E \arrow[r, "g"] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\ -\xi: 0 \arrow[r] & N \arrow[r, "-f"'] & E \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd}$$

Then $[-\xi] \overset{\text{def}}{=} -[\xi]$.

So $\mathcal{E}(M,N)$ is an abelian group under the Baer Sum.

<u>Aim:</u> There exists an isomorphism of abelian groups

$$\mathrm{Ext}^1_R(M,N) \leftrightarrows \mathcal{E}(M,N).$$

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Remarks about $\mathrm{Ext}^n_R(M,N)$

Take a projective resolution of M:

$$\cdots \to P^{n+1} \to P^n \xrightarrow{d^n} P^{n-1} \cdots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

Then

$$0 \to \mathrm{Hom}(P^0,N) \to \mathrm{Hom}(P^1,N) \to \cdots \to \mathrm{Hom}(P^{n-1},N) \xrightarrow{\mathrm{Hom}(d^n,N)} \mathrm{Hom}(P^n,N) \xrightarrow{\mathrm{Hom}(d^{n+1},N)} \mathrm{Hom}(P^{n+1},N) \to \cdots$$

$$\mathrm{Ext}^n_R(M,N) = \frac{\mathrm{Ker}\,\mathrm{Hom}(d^{n+1},N)}{\mathrm{Im}\,\mathrm{Hom}(d^n,N)}$$

Will "rewrite" this.

$$\begin{tikzcd} P^{n+1} \arrow[r, "d^{n+1}"] & P^n \arrow[r] \arrow[d] & \Omega^n M \arrow[r] \arrow[dl, dashed, "\exists ! s"] & 0 \\ & N & & \end{tikzcd}$$

$$\Omega^n M = \mathrm{Im}\, d^n = \mathrm{Ker}\, d^{n-1}$$
$$\parallel$$
$$\mathrm{Coker}\, d^{n+1}$$

if ↻ then ∃! s

if ∃s then so $d^n \circ d^{n+1} = 0$

$P^n \to N \Rightarrow$ ↻ & $\begin{matrix} P^n \\ \downarrow \\ N \end{matrix}$ exi

So, $\mathrm{Ker}\,\mathrm{Hom}(d^{n+1},N) \cong \mathrm{Hom}(\Omega^n M, N)$

Want $\mathrm{Im}\,\mathrm{Hom}(d^n, M)$

$$\begin{tikzcd} P^n \arrow[r, "d^n"] \arrow[dr] & P^{n-1} \arrow[r] \arrow[drr, "g"] & \Omega^{n-1} M \arrow[r] & 0 \\ 0 \arrow[r] & \Omega^n M \arrow[u, "\varepsilon_n"'] \arrow[r] & 0 & N \end{tikzcd}$$

$$\mathrm{Im}\,\mathrm{Hom}(d^n, M) = \mathrm{Im}\,\mathrm{Hom}(\varepsilon_n, N).$$

So we can rewrite $\forall n$

$$\boxed{\mathrm{Ext}^n_R(M,N) = \frac{\mathrm{Hom}(\Omega^n_* M, N)}{\mathrm{Im}\,\mathrm{Hom}(\varepsilon_n, N)}}$$

where $\varepsilon_n : \Omega^n M \to P^{n-1}$ is the inclusion

In particular, we get

$$\boxed{\mathrm{Ext}^1_R(M,N) = \frac{\mathrm{Hom}_R(\Omega M, N)}{\mathrm{Im}\,\mathrm{Hom}(\varepsilon, N)}} \quad ⊛$$

Where $0 \to \Omega M \xrightarrow{\varepsilon} P^0 \to M \to 0$ $(\Omega M := \Omega^1$

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Thm: $\exists$ an isomorphism of abelian groups

$$\mathrm{Ext}^1_R(M,N) \leftrightarrows \mathcal{E}(M,N)$$

"**Sketch of Pf**"

Let $[f] \in \mathrm{Ext}^1_R(M,N)$ so $f: \Omega M \to N$.

Take pushout

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "\varepsilon"] \arrow[d, "f"] & P^0 \arrow[r, "d^0"] \arrow[d, "g"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd} \quad \xi_f$$

We define $\Psi: \mathrm{Ext}^1_R(M,N) \to \mathcal{E}(M,N)$

$$\Psi([f]) = [\xi_f]$$

Helpful **Note 1:** Assume $[f]=[0]$ in $\mathrm{Ext}^1(M,N)$.

Then ξ_f is split exact, so $[\xi_f]=[0]$.

[What does it mean that $[f]=[0]$ in Ext^1?

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "\varepsilon"] \arrow[d, "f"] & P^0 \arrow[r] \arrow[dl, dashed, "\exists! s"] & M \arrow[r] & 0 \\
 & N & & &
\end{tikzcd}$$

It means that $f \in \mathrm{Im}\,\mathrm{Hom}(\varepsilon, N)$, or that f factors through P^0.

So we have:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "\varepsilon"] \arrow[d, "f"'] & P^0 \arrow[r, "d^0"] \arrow[d, "g"] \arrow[dl, "\exists"', "s"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha"'] & E \arrow[r, "\beta"'] & M \arrow[r] \arrow[l, bend right, "t"'] & 0
\end{tikzcd} \qquad s\varepsilon = f$$

[Useful fact about pushouts:

Given

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varepsilon"] \arrow[d, "f"] & B \arrow[r, "d"] \arrow[d, "g"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] & V \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

pushout seq.

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then the pushout sequence splits $\Leftrightarrow$ $\exists\, s: B \to D$ s.t. $s\varepsilon = f$. $\rfloor$

In our case: $s\varepsilon = f$

$$\underline{\alpha s \varepsilon} = \alpha f = \underline{g\varepsilon}$$

$$(g - \alpha s)\varepsilon = 0$$

$$\begin{tikzcd} 0 \arrow[r] & \Omega M \arrow[r, "\varepsilon"] & P^0 \arrow[r, "d^0"] \arrow[d, "g-\alpha s"] & M \arrow[r] \arrow[dl, dashed, "\exists! t"] & 0 \\ & & \forall E & & \end{tikzcd}$$

$\exists!\, t: M \to E$ s.t. $t d^0 = g - \alpha s$

Compute βt: $\beta t d^0 = \beta g - \beta \alpha s$
$= \beta g = d^0$

$\Rightarrow$ $\boxed{\beta t = 1}$ since d^0 is onto. $\rfloor$

<u>Note 2:</u> We also have to show that Ψ does not depend on the choice of the projective module mapping onto M.

This follows from the isom. of abelian groups

$$\frac{\mathrm{Hom}((\Omega M)', N)}{\mathrm{Im}\ \mathrm{Hom}(\varepsilon', N)} \cong \frac{\mathrm{Hom}(\Omega M, N)}{\mathrm{Im}\ \mathrm{Hom}(\varepsilon, N)}$$

where $0 \to \Omega M \xrightarrow{\varepsilon} P^0 \xrightarrow{d^0} M \to 0$

and $0 \to (\Omega M)' \xrightarrow{\varepsilon'} P^{0\prime} \xrightarrow{d^{0\prime}} M \to 0$

⋮

Ψ is well-defined.

Want $\Phi: \mathcal{E}(M, N) \to \mathrm{Ext}^1_R(M, N)$.

Let $\xi \in [\xi]$ $\xi: 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$

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Draw

$$\begin{tikzcd}
 & 0 \arrow[r] & \Omega M \arrow[r, "\varepsilon"] \arrow[d, "\exists ! u_\xi"'] & P^0 \arrow[r, "d^0"] \arrow[d, dashed, "\exists"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, equal] & 0 \\
\xi & 0 \arrow[r] & N \arrow[r, "f"] & E \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

$\leftarrow P^0$ proj.

Define $\Phi([\xi]) = [u_\xi]$.

To show Φ is well defined, that Φ and Ψ are inverses to each other and that they are additive.

Homological dimensions

Def: A module $M \in \operatorname{Mod} R$ has projective dimension at most n, and we write $\operatorname{pd} M \leq n$, if there exists a projective resolution of M:

$$0 \to P^n \to \cdots \to P^1 \to P^0 \to M \to 0$$

If no such resolution exist, then we say that M has infinite projective dimension and we write $\operatorname{pd} M = \infty$.

We say that $\operatorname{pd} M = n$, if $\operatorname{pd} M \leq n$ but $\operatorname{pd} M \nleq n-1$.

Examples & remarks

① Every projective module has pd equal to 0.

② Let R be a PID. So projective modules = free modules.

Also, every submodule of a free module is free.

Let $M \in \operatorname{Mod} R$ not free, so $\operatorname{pd} M \neq 0$.

$$0 \to \underset{\text{free}}{\Omega M} \to \underset{\text{free}}{F} \to M \to 0$$

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So $\operatorname{pd} M = 1$. So, over a <u>PID</u> every module has <u>$\operatorname{pd} = 0, 1$.</u>

③ Look at $R = \frac{k[x]}{\langle x\rangle^2} = k \oplus k\cdot\bar{x}$, k field

$\dim_{\cancel{k}} R = 2$

Unique max'l ideal is $\frac{\langle x\rangle}{\langle x\rangle^2} = \langle \bar{x}\rangle$.

Over R, every projective module is free, since R is local.

$M := R/_{\text{max ideal}} \simeq k$ so it is 1-dimensional.

Assume that $\operatorname{pd} M < \infty$. Then, $\exists$ an exact sequence

$$0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$$

free
finitely generated

<u>Exer:</u> Why can ~~the~~ we choose them to be finitely generated?

<u>Hint:</u> use R Noetherian + generalization of Schanuell's lemma.
exer 3.15?!

<u>Turn in</u> Fr/ Monday

Each P_i is fin. gen. + free, so it is a direct sum of ~~fin. gen.~~ finitely many copies of R. $\Rightarrow \dim P_i = \text{even} \ \# \ \forall i \geq 1$.

<u>exer:</u> Assume we have a long exact sequence of f. dim. vectorspaces.

$\cancel{\Rightarrow}$ $0 \to V_1 \to V_{i-1} \to \dots \to V_0 \to V \to 0$

exer 3.16?!

$\Rightarrow \dim V = \sum_{i=0}^{n} (-1)^i \dim V_i$

Euler characteristic of V

In our case $V = M = k$ 1-dim. and RHS is even. $\Rightarrow \operatorname{pd} M = \infty$.

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__Fact:__ Over $R = k[x]/\langle x\rangle^n$ $\forall n \geq 2$ __every__ non projective module has __infinite projective__ dimension.

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__Monday:__ proj. dim. of a module

$\mathrm{pd}\, M \leq n \Leftrightarrow \exists$ a proj. resolution of M: $0 \to \underbrace{P_n \to P_{n-1} \to \dots \to P_0}_{\text{projective}} \to M \to 0$

$\mathrm{pd}\, M = n \Leftrightarrow$ if $\mathrm{pd}\, M \leq n$ and $\mathrm{pd}\, M \not\leq n-1$.

__Thm:__ TFAE for a module M:

(1) $\mathrm{pd}\, M \leq n$

(2) $\mathrm{Ext}^i_R(M,N) = 0$ $\forall i > n$, $\forall N \in \mathrm{Mod}\, R$

(3) $\mathrm{Ext}^{n+1}_R(M.\ N) = 0$ $\forall N \in \mathrm{Mod}\, R$

(4) Every projective resolution of M has a projective nth syzygy.

__Pf:__ __(1) ⇒ (2):__ Let $0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \to P_0 \to M$ $\mathbb{P}$, be a proj. res'n of M. Take $\mathrm{Hom}_R(\text{deleted res'n},\ N)$

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \dots \to \mathrm{Hom}_R(P_n. N) \to 0 \to 0$$

So for $i \geq n+1$ the homology gps are 0 $\Rightarrow \mathrm{Ext}^i_R(M.N) = 0$ $\forall i \geq n+1$.

__(2) ⇒ (3):__ trivial

__(3) ⇒ (4):__ Look at $\dots \to P_t \xrightarrow{d_t} P_{t-1} \to \dots \to M \to 0$, a proj. resolution,

and let $\Omega^n M = \mathrm{Ker}\ d_{n-1}$.

So have $0 \to \Omega^{n+1} M \xrightarrow{\varepsilon} P_{n-1} \to P_{n-2} \to \dots \to M \to 0$, where ε = inclusion.

From Monday $\underset{0\,=}{\mathrm{Ext}^{n+1}_R}(M\ N) = \dfrac{\mathrm{Hom}_R(\Omega^{n+1}M,\ N)}{\mathrm{Im}\ \mathrm{Hom}(\varepsilon, N)}$

So every homom f

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{n+1} M \arrow[r, "\varepsilon"] \arrow[d, "f"] & P_{n-1} \arrow[dl, dashed] \\ & N & \end{tikzcd}$$

factors through ε. True $\forall N$

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In particular, the identity map

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{n+1}M \arrow[r] \arrow[d, "1_{\Omega^{n+1}M}"] & P_n \arrow[dl] \\
 & \Omega^{n+1}M &
\end{tikzcd}$$

So $\Omega^{n+1}M$ is a direct summand of P_n, so it is projective.

Look at

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{n+1}M \arrow[r] & P_n \arrow[r] \arrow[d, equal] & \dots \arrow[r] & M \arrow[r] & 0 \\
 & & \Omega^{n+1}M \oplus Q & & &
\end{tikzcd}$$

$$\Rightarrow \begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d] & P_{n-1} \arrow[r] & \dots \arrow[r] & M \arrow[r] & 0 \\
 & \text{projective} & & & &
\end{tikzcd}$$

(4) ⇒ (1) trivial. $\square$

Def: Let R be a ring. The left projective global dimension of R is $\text{Lpgldim } R = \sup\{\text{pd } M \mid {}_R M \in \text{Mod } R\}$.

Examples:

① $\text{Lpgldim } R = 0 \iff \text{pd } M = 0 \ \forall \, {}_R M \in \text{Mod } R$

$\iff M$ projective $\forall \, {}_R M \in \text{Mod } R$

$\iff R$ is semisimple.

$\iff M$ projective $\forall \, M_R \in \text{Mod } R$

$\iff \dots \iff \text{rpgldim } R = 0$
(right proj. gldim R)

Note: semisimple ~~module~~ rings are abelian and noetherian.

② Analyse $\text{Lpgldim } R = 1$. This means that $\forall \, {}_R M$ that is not projective, $\text{pd } M = 1$. In particular,

$\text{Lpgldim } R = 1 \iff$ every submod of a projective module is proj.

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Def: A ring is called left hereditary if every left ideal is projective.

Thm (Kaplansky): Let R be left hereditary. Then every submodule of a free module is isomorphic to a direct sum of left ideals of R.

In particular, every submod of a projective module is projective.

So, R left hereditary $\Leftrightarrow$ lpgldim $R \leq 1$.
R not s.s. $\qquad = 1$

Remark: There are rings that are left hereditary but not right hereditary. (in this case R is not Noetherian)

Remark: If R is commutative, then R is hereditary $\Leftrightarrow$ gl dim $R = 1$
$\Leftrightarrow$ R is a Dedekind ring. (Rotman)

Remark: If R is a finite dimensional k-algebra, where k is a field $(*)$, then hereditary algebras are well-understood.

[$(*) \Leftrightarrow$ k is contained in the center of R that is $\alpha r = r\alpha$ $\forall \alpha \in k$, $r \in R$
R is a ring + vector space over k]

$\rightarrow$ they are "path algebras" of finite directed graphs.

Injective dimension:

We say that the injective dimension of a module M is at most n and write $\mathrm{id}\, M \leq n$ if $\exists$ an injective resolution of M:

$$0 \to M \to \underbrace{I_0 \to I_1 \to \cdots \to I_n}_{\text{injective}} \to 0$$

We say $\mathrm{id}\, M = n$ if $\mathrm{id}\, M \leq n$ but $\mathrm{id}\, M \not\leq n-1$. If $\forall n$ $\mathrm{id}\, M \not\leq n$, we say that $\mathrm{id}\, M = \infty$.

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Thm: TFAE for ${}_R M$:

(1) id $M \leq n$

(2) $\mathrm{Ext}^k_R(A, M) = 0$ $\forall\, {}_R A$ and $\forall k > n$.

(3) $\mathrm{Ext}^{n+1}_R(A, M) = 0$ $\forall\, {}_R A$.

(4) In every injective resolution of M, the $n+1$ st cosyzygy is injective.

Def: $\mathrm{L\,igl\,dim}\, R = \sup\{\mathrm{id}\, M \mid {}_R\underset{\underset{\mathrm{Mod}\, R}{\wedge}}{M}\}$

Examples: ① $\mathrm{L\,igldim}\; R = 0 \iff \mathrm{id}\, M = 0\ \forall\, {}_R M \iff {}_R M$ is injective $\forall\, {}_R M \iff R$ left semisimple $\iff R$ right semisimple

$= \mathrm{Lgl\ p\ dim}\, R = \mathrm{rgl\ pd}\, R = \mathrm{rgl\ id}\, R$

Want to prove the following:

Thm: $\mathrm{Lgl\ idim}\, R \leq 1 \iff R$ is left hered $\underset{\text{already know}}{\iff}$ $\mathrm{Lgl\ p\,dim}\, R \leq 1$.

We want to show that the following statements are equivalent:

(1) every submodule of a projective module is projective.

(2) every quotient of an injective module is injective.

Note: the pf for this gldim = 0 is easy since R is s.s. then and the result follows.

Lemma: (A characterization of proj. (injective) modules) (MacLane)

${}_R P$ is projective $\iff \forall$ diagrams

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed, "\exists"'] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] & \\ E \arrow[r] \arrow[d, squiggly] & E_1 \arrow[r] & 0 \\ \text{injective} & & \end{tikzcd}$$

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The "dual statement" for injective modules is also true.

Pf: "⇒": trivial.

"⇐": Start with

$$\begin{tikzcd} & & & P \arrow[d, "f"] \arrow[dll, dashed, "\exists s"'] & \\ 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "k"] \arrow[d, hook, "u"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r] \arrow[d, dashed, "l"] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "u \circ i"] & E \arrow[r, "g"] & D \arrow[r] & 0 \end{tikzcd}$$

E = injective envelope of B $\qquad D = \operatorname{coker} u \circ i$

$\exists s: P \to E$ with $gs = lf$.

Then $\operatorname{Im} s \subseteq \operatorname{Im} u$. (exercise) Then since u is an inclusion

$\Rightarrow \operatorname{Im} u = B$, so $\operatorname{Im} s \subseteq B$, so f can be lifted to B. □

Now, assume R is left hered. To show that quotient of injective is injective. Let E be injective; consider $E \to E' \to 0$

We will use the "dual" of lemma, that is, if

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & Q \arrow[u, "f"] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & P \arrow[ul, dashed, "F"'] \end{tikzcd}$$

P proj. $\quad \Leftarrow E$ injective

Have the following picture:

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "i"] \arrow[d, "f"] \arrow[dl, dashed, "\exists s"'] & P \arrow[dll, "\exists t", bend left] \\ E \arrow[r, "p"'] & E' \arrow[r] & 0 \end{tikzcd}$$

inj

with $t \circ i = s$ since E injective

Q proj since it is a submod of a proj. mod P.

E is inj. ⇝ $\exists t \Rightarrow pt : P \to E'$

$pti = ps = f$ ✓

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So in the case of left (right) hereditary (non s.s.)

we have $\quad L\,g\,pd\,R = lglidim\,R = 1$

$\quad rgl\,pd\,R = rglidim\,R = 1$

__Remark:__ There is no semi-decent characterization of rings of

$lpgldim\,R = 2$ or higher.

This is a hopeless problem.

__Lemma:__ Let R be a ring and ${}_RM \in Mod\,R$. Then

M is injective $\Leftrightarrow$ $Ext^1_R(R/_I, M) = 0$ $\forall$ left ideals I.

__Pf:__ Let I be a left ideal. Look at

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "f_I"] & R \arrow[r] \arrow[d] \arrow[dl, dashed] & R/_I \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & R/_I \arrow[r] & 0
\end{tikzcd}$$

Baer's criterion: M is injective $\Leftrightarrow$ f factors through R

$\Leftrightarrow$ the pushout sequence $0 \to M \to E \to R/_I \to 0$ splits.

Now, $Ext^1(R/_I, M) = 0 \Leftrightarrow$ every extension $0 \to M \to X \to R/_I \to 0$

splits. $\Rightarrow M$ injective by above.

"$\Rightarrow$" is also $\square$

__Remark:__ The cyclic R-modules are the modules of the form $R/_I$.

__Thm (Auslander):__ Let R be a ring.

$\Rightarrow$ $lgl\,pdim\,R = \sup\{pd\,R/_I \mid I \text{ left ideal}\}$

So $lgl\,pdim\,R$ can be computed only using cyclic modules.

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Pf: "≥" is clear. To show "≤":

If $\sup\{\text{pd } R/I\} = \infty$, we are done.

So assume $\sup_{I \text{ left ideal}}\{\text{pd } R/I\} = n < \infty$

So, for every left ideal I we have $\text{Ext}^{n+1}_R(R/I, M) = 0$ ∀ modules M.

$\Updownarrow$

since $\text{pd } R/I \leq n$

using "dimension shift" $\searrow$ $= \text{Ext}^1_R(R/I, \Omega^{-n}M) = 0$

So $\Omega^{-n}M$ is injective ∀ modules M.

$\Rightarrow \text{id } M \leq n$. To be continued. (We want the following:)

Thm (Auslander) R any ring. Then $\text{Lglpd } R = \text{lglid } R$.

R fin. dim k-alg. or simply a 2-sided artinian ring.

We may compute $\text{id}({}_R R)$ and $\text{id}(R_R)$.

Known: if both $\text{id}_R R$, $\text{id } R_R$ are finite, then $\text{id}_R R = \text{id } R_R$.

Conjecture: if either one of $\text{id}\,{}_R R$, $\text{id } R_R$ is finite, the other one is, too.

Thm (Auslander) 03/05/18

Let R be a ring. Then $\text{lpgldim } R = \text{l idgl dim } R$.

Pf: Recall $\text{pd } A \leq n \Leftrightarrow \text{Ext}^k_R(A, B) = 0 \ \forall k > n$

[In particular, $\text{pd } A = n$ if $\exists B$ with $\text{Ext}^n(A, B) \neq 0$ but $\text{Ext}^k_R(A, B) = 0 \ \forall k > n, \forall B$]

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[Similarly, $\mathrm{id}\, B \leq n \Leftrightarrow \mathrm{Ext}^k_R(A,B)=0 \ \forall A, \forall k>n$

and $\mathrm{id}\, B = n$ if $\exists A$ with $\mathrm{Ext}^n_R(A,B) \neq 0$ but

$\mathrm{Ext}^k_R(A,B) = 0 \ \forall k>n, \ \forall A$.]

Since $\mathrm{lpgldim}\, R = \sup\{\mathrm{pd}\, A \mid A \in \mathrm{Mod}\, R\}$

and $\mathrm{Ligldim}\, R = \sup\{\mathrm{id}\, B \mid B \in \mathrm{Mod}\, R\}$ $\Rightarrow$ get equality. $\square$

Last time: Lemma: ${}_R M$ is injective $\Leftrightarrow \mathrm{Ext}^1_R(R/I, M)=0$ $\forall$ left ideals I.

Thm: R ring. Then $\mathrm{Lpgldim}\, R = \sup\{\mathrm{pd}\ R/I \mid I \text{ left ideal of } R\}$ In particular, the left global dim can be computed by looking at finitely generated modules.

Notation: $\mathrm{Lgldim}\, R \overset{\mathrm{def}}{=} \mathrm{Lpgldim}\, R = \mathrm{Ligldim}\, R$

$\downarrow$

"left global dimension"

Soon: show that if R is left and right Noetherian, then $\mathrm{Lgldim}\, R = \mathrm{rgldim}\, R$. We'll call this $\mathrm{gldim}\, R$ ("global dim. of R").

Pf of the thm:

Clear "$\geq$". To show "$\leq$". If RHS $= \infty$, then clear.

So assume $\sup\{\mathrm{pd}\ R/I \mid I \text{ left ideal}\} = n < \infty$.

$\Rightarrow \forall$ left ideals I $\mathrm{Ext}^{n+1}_R(R/I, M)=0 \ \forall M$.

Dimension shift (use notation: $0 \to M \to \text{injective} \to \Omega^{-1}M \to 0$

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$0 \to \Omega^{-1}M \to$ injective $\to \Omega^{-1}M \to 0$)

Then $\mathrm{Ext}_R^{n+1}(R/I, M) = \mathrm{Ext}_R^{n}(R/I, \Omega^{-1}M) = \mathrm{Ext}_R^{1}(R/I, \Omega^{-n}M)$

$\Rightarrow \forall$ left ideals I, get $\mathrm{Ext}_R^1(R/I, \Omega^{-n}M) = 0 \Rightarrow \Omega^{-n}M$ injective

$\forall M$ $\Rightarrow$ id $M \leq n$ for all M. Then Lgldim $R \leq n$.

$\Rightarrow$ Lpgldim $R \leq n$.

HW Rotman p. 466

ex. 8.3 (they want: "if pd $M = n < \infty$, show that $\mathrm{Ext}_R^n(M, F) \neq 0$ for some module F")

Hint: induction

Do also 8.4, 8.5, 8.6, 8.7. → End of Spring Break

Thm: exer 8.9 Let $\{A_i\}_{i \in I}$ be a collection of R-modules. Let $A = \bigoplus_{i \in I} A_i$. Then pd $A = \sup\{$ pd $A_i \mid i \in I\}$.

Remark: We can make Mod R or mod R to be $\{A_i\}_{i \in I}$.

The thm tells us if Lgldim $R = \infty$, then there exists an R-module of infinite projective dimension.

Remark: Let R be an artinian ring. Write

fpd $R = \sup\{$ pd $A \mid$ pd $A < \infty$, $A \in$ mod $R\}$

↳ "finitistic projective dimension of R"

Conjecture: fpd $R < \infty$.

Open

Pf: First establish some notation:

For each $i \in I$, let $A_i^0 = A_i$ and define inductively projective modules

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P_i^j and modules A_i^j:

$$0 \to A_i^{j+1} \to P_i^j \to A_i^j \to 0 \quad \forall i$$

[So $0 \to A_i^1 \to P_i^0 \to \underset{\substack{\| \\ A_i}}{A_i^0} \to 0$ —]

Let $A^j = \bigoplus_i A_i^j$, $P^j = \bigoplus_i P_i^j$ projective $\forall j$

$\Rightarrow \forall j$, exact sequence $0 \to A^{j+1} \to P^j \to A^j \to 0$

In particular, by "splicing", we get:

$\forall j$: $0 \to A^{j+1} \to P^j \to P^{j-1} \to \dots \to P^0 \to \underset{\substack{\| \\ A}}{A^0} \to 0$

Fix $n \geq 0$. If $\operatorname{pd} A_i \leq n$ $\forall i$ $\underset{\substack{\updownarrow \\ A_i = A_i^0}}{\Rightarrow}$ $\operatorname{pd} A_i^0 \leq n$ $\forall i$ $\Rightarrow$ each A_i^n is projective. $\Rightarrow A^n = \bigoplus_i A_i^n$ is projective. $\Rightarrow \operatorname{pd} A \leq n \Rightarrow \operatorname{pd} A \leq \sup\{\operatorname{pd} A_i\}$. So proved "$\leq$".

Conversely, if $\operatorname{pd} A \leq n$, since A_i^n is a summand of A^n for all i, A_i^n is projective $\forall i \Rightarrow \operatorname{pd} A_i \leq n$ $\forall i$

$\Rightarrow \sup\{\operatorname{pd} A_i\} \leq n \Rightarrow$ get "$\geq$".

Remark: (exer from HW) If $0 \to A \to B \to C \to 0$, then

$\operatorname{pd} B \leq \max\{\operatorname{pd} A, \operatorname{pd} C\}$. long exact seq / horseshoe lemma

Thm: Assume R is left Artinian.

$\operatorname{lgldim} R = \max\{\operatorname{pd} S \mid S \text{ simple left module}\}$

Remark: Assume R is left Art.

(1) Every simple R-module is cyclic.

(2) Up to isomorphism, there are finitely many simple

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R-modules $S_1, \to S_n$.

(3) The thm says that we need only to compute pd $S_1, \to$ pd S_n.

<u>Pf of thm:</u> Clearly we have "$\geq$". To prove "$\leq$":

WLOG we may assume that our modules are f.g.

But ${}_R M$ f.g. has finite <u>composition length</u>.

Prove "$\leq$" by induction on the length of M.

- if $\ell(M)=1$, then M is simple $\Rightarrow$ pd $M \leq$ RHS.
- if $\ell(M)=n > 1$, then there exists a proper simple submod of M. Let it be S. Look at.

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, squiggly] & M \arrow[r] \arrow[d, squiggly] & M/S \arrow[r] \arrow[d, squiggly] & 0 \\ & \ell(S)=1 & \ell(M)=n & \ell(M/S)=n-1 & \end{tikzcd}$$

Induction: pd $S \leq$ RHS
pd $M/S \leq$ RHS

exercise: pd $M \leq$ RHS.

So $\forall M$ f.gen. we get pd $M \leq$ RHS. $\Rightarrow \underset{\text{lgldim } R}{\underset{\shortparallel}{\sup\{\text{pd } M \mid M \text{ f.g.}\}}} \leq \text{RHS}$. $\square$

<u>Hilbert's Syzygy thm</u>

<u>Original version:</u>

gl dim $\mathbb{C}[x_0, \ldots, x_n] = n$.

<u>More general version:</u>

<u>Thm:</u> Let R be a ring, (not necessarily commutative).
Then lgldim $R[x_1, \ldots, x_n]$ = lgldim $R + n$.

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[We assume $x_i x_j = x_j x_i$ $\forall i, j$, so have "commuting variables"

and $r x_i = x_i r$ $\forall i$ $\forall r \in R$.]

[**Note:** if the variables do not commute, life can be more complicated. For example, let $R = \mathbb{C}\langle x_1, x_2, \ldots, x_n \rangle$ polynomial ring in n non commuting variables, or the free algebra in n variables. Then $\text{lgldim } R = \text{rgldim } R = 1$.]

in P.M. Cohn's book.

To prove Hilbert's syzygy thm, it is enough to prove

$$\text{lgldim } R[x] = \text{lgldim } R + 1.$$

and then apply induction, since $R[x_1, \ldots, x_n] = R[x_1, \ldots, x_{n-1}][x_n]$.

Remarks: ① $R[x]$ is a free R-module both left and right with basis $\{1, x, x^2, \ldots\}$

② Let ${}_R M$. Define $M[x] \overset{\text{def}}{=} R[x] \otimes_R M$, so $M[x]$ is a left $R[x]$-module.

Note: $R[x] = \bigoplus_{i \geq 0} x^i R$

$$\Rightarrow M[x] = \left(\bigoplus_{i \geq 0} x^i R\right) \otimes_R M = \bigoplus_{i \geq 0} \underbrace{x^i \otimes_R M}_{\overset{\text{def}}{=:} M_i}$$

As R-modules, $M_i \cong M$ $\forall i$.

③ $M[x]$ is both an $R[x]$-module and also an R-module.

As an R-module, $M[x] \cong \bigoplus M_i$ each $M_i \cong M$

so $\text{pd}_R M[x] = \sup\{\text{pd}_R M_i\} = \text{pd}_R M$ $\Rightarrow$ $\boxed{\text{pd}_R M[x] = \text{pd}_R M}$

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④ **Lemma:** $\mathrm{pd}_R M = \mathrm{pd}_{R[x]} M[x]$.

Sublemma: Let $R \to S$ be a homom of rings. S can be viewed as an R-module. ~~Assume~~ ~~S_R is projective~~. Let ${}_RM$ be projective. Then the "induced" module $S \otimes_R M$ is projective over S.

Pf of sublemma: ${}_RM$ proj. $\Rightarrow$ $\exists\, {}_RN$ s.t. $M \oplus N = R^{(I)}$ free module.

$$\begin{array}{l} S \otimes_R (M \oplus N) \cong S \otimes_R R^{(I)} \cong S^{(I)} \text{ free } S\text{-mod} \\ \wr\!\wr \\ S \otimes_R M \oplus S \otimes_R N \end{array}$$

$\Rightarrow S \otimes_R M$ is projective over S. □

Pf of lemma:

Let $n \geq 0$, and assume $\mathrm{pd}_R M \overset{=}{\cancel{\leq}} n$. $\Rightarrow$ $\exists$ proj. res'n over R:

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \to M \to 0$$

~~Then~~ Tensor over R with $R[x]$. ~~Ke~~ Keeping in mind that $R[x]$ ~~is~~ is free over R, so it is flat. So $R[x] \otimes_R -$ preserves exactness:

$$0 \to \underbrace{R[x] \otimes_R P_n \to \dots \to R[x] \otimes_R P_0}_{\substack{\text{Using the sublemma with } S = R[x] \\ \text{these are all proj. } R[x]\text{-modules.}}} \to \underbrace{R[x] \otimes_R M}_{\overset{\text{def}}{=} M[x]} \to 0 \quad \text{is exact}$$

$\Rightarrow \mathrm{pd}_{R[x]} M[x] \leq n.$ $\Rightarrow$ $\mathrm{pd}_{R[x]} M[x] \leq \mathrm{pd}_R M$.

Assume $\mathrm{pd}_{R[x]} M[x] \overset{=}{\cancel{\leq}} n$.

Let $$0 \to Q_n \to Q_{n-1} \to \dots \to Q_0 \to M[x] \to 0 \qquad (*)$$

be a proj. res'n of $M[x]$ ~~over~~ $R[x]$. Using $R \to R[x]$ we view this $(*)$ as ~~a~~ an exact seq. of R-modules.

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As R-modules $M[x] = \oplus M_i$ and each $M_i \cong M$.

Each Q_i is a direct summand of a free $R[x]$-module.

But $R[x] \cong \bigoplus_{i\geq 0} R$. So each Q_i is projective when viewed as an R-module.
$\downarrow$
as an R-mod

$\Rightarrow$ (*) is a proj. res'n ~~of~~ over R of $\bigoplus_{i\geq 0} M_i$. $\Rightarrow pd_R(\oplus M_i) \leq n$.
$\quad\quad\quad\quad \| $
$pd_R M$ since it equals $\sup\{pd M_i\}$

$\Rightarrow pd_R M \leq n$.

So, $pd_R M = pd_{R[x]} M[x]$. □

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Hilbert's Syzygy Thm

Let R be a ring. Then $\mathrm{lgldim}\, R[x_1, \dots, x_n] = \mathrm{lgldim}\, R + n$.

Have seen: It is enough to prove $\mathrm{lgldim}\, R[x] = \mathrm{lgldim}\, R + 1$.

Last time: Given ${}_R M \rightsquigarrow M[x] \overset{def}{=} R[x] \otimes_R M$
$\downarrow$
$R[x]$-module

Fact: The elements of $M[x]$ have the form $\sum_{i=0}^{k} x^i \otimes m_i$, $m_i \in M$.

Since we can view $M[x] \cong \oplus M_i$ each $M_i \cong M$ ($R[x]$ is free $/R$ with basis $1, x, x^2, \dots$) $\Rightarrow \sum x^i \otimes m_i = 0 \iff$ each $m_i = 0$.

Proved: Let $M \in \mathrm{Mod}\, R$. Then $pd_R M = pd_{R[x]} M[x]$.

Cor: If $\mathrm{lgldim}\, R = \infty \Rightarrow \mathrm{lgldim}\, R[x] = \infty$

Pf: Assume $\mathrm{lgldim}\, R = \infty$. $\underset{\text{last time}}{\Longrightarrow}$ $\exists$ an R-module M with $pd_R M = \infty$.

$\Rightarrow pd_{R[x]} M[x] = \infty$ since ⊛ we get a contradiction to the previous if this

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was not the case.

Unrelated remarks:

① ⊛ To find a module of infinite proj. dim, we simply look at

$A = \bigoplus_{i \in I} A_i$ where $\{A_i\}_{i \in I}$ is the family of all R-modules.

Note that A is infinitely generated. But we also know that the global dim. can be attained on the set of cyclic modules. So we can find a cyclic module of infinite proj. dim.

② Assume $\text{lgl dim } R = \infty$.

$\Rightarrow \exists$ module M with $\text{pd } M = \infty$, and $\exists$ module N with $\text{id } N = \infty$.

Let $L = M \oplus N$. Then $\text{pd } L = \text{id } L = \infty$.

Question: is there an Indecomposable R-module X s.t. $\text{pd } X = \text{id } X = \infty$?

[If R is artinian, the answer is yes. In general, it is not known.]

Lemma: Let M be an R[x]-module. $\Rightarrow \exists$ short exact seq. of R[x]-modules

$$0 \to M[x] \to M[x] \to M \to 0$$

Pf: Look at $\varepsilon: M[x] \to M$ given by $\varepsilon\left(\sum_{i=0}^{k} x^i \otimes m_i\right) = \sum_{i=0}^{k} x^i m_i$

It's easy to see that ε is homom of R[x]-modules that is onto.

Let $K = \text{Ker } \varepsilon$, so have $0 \to K \to M[x] \xrightarrow{\varepsilon} M \to 0$.

So we have to show that $\exists$ isom $M[x] \cong K$.

Let $\beta: M[x] \to K$ be defined as follows:

$$\beta\left(\sum_{i=0}^{k} x^i \otimes m_i\right) = \sum_{i=0}^{k} x^i (1 \otimes x - x \otimes 1) m_i.$$

Remarks:

① $(1 \otimes x - x \otimes 1) m_i = 1 \otimes x m_i - x \otimes m_i \in M[x]$, so RHS makes sense.

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② $\varepsilon((1\otimes x - x\otimes 1)m_i) = \varepsilon(1\otimes xm_i - x\otimes m_i) = xm_i - xm_i = 0$

$\Rightarrow \varepsilon(\operatorname{Im}\beta) = 0$, so $\operatorname{Im}\beta \subseteq K$.

③ β is a homom. of R-modules.

④ We now <u>rewrite β</u>:

$$\beta\left(\sum_{i=0}^{k} x^i \otimes m_i\right) = (1\otimes x - x\otimes 1)m_0 + x(1\otimes x - x\otimes 1)m_1$$

$$+ \dots + x^k(1\otimes x - x\otimes 1)m_k$$

$$= 1\otimes xm_0 + x\otimes xm_1 - x\otimes m_0 + x^2\otimes m_1 + \dots$$

$$= 1\otimes \underbrace{xm_0}_{\text{in } M} + x\otimes \underbrace{(xm_1 - m_0)}_{\text{in } M} + x^2\otimes(\underbrace{xm_2 - m_1}_{\text{in } M}) + \dots + x^k\otimes\underbrace{(xm_k - m_{k-1})}_{\text{in } M} - x^{k+1}\otimes \underbrace{m_k}_{\text{in } M}$$

<u>Cl. 1:</u> β is 1-1.

<u>Pf:</u> Assume $\beta\left(\sum_{i=0}^{k} x^i \otimes m_i\right) = 0 \underset{\text{in } M[x]}{\overset{\text{we are}}{\Rightarrow}}$

$$\begin{aligned} xm_0 &= 0\\ xm_1 - m_0 &= 0\\ xm_2 - m_1 &= 0\\ &\vdots\\ xm_k - m_{k-1} &= 0\\ m_k &= 0\end{aligned}$$

$m_k = 0 \Rightarrow m_{k-1} = 0 \Rightarrow \dots \Rightarrow m_1 = 0 \Rightarrow m_0 = 0 \Rightarrow \sum x^i\otimes m_i = 0$ so $\operatorname{Ker}\beta = 0$

$\Rightarrow \beta$ is 1-1.

<u>Cl. 2:</u> β is onto.

<u>Pf:</u> Let $\sum_{i=0}^{k} x^i \otimes \overline{m_i} \in K \subseteq M[x]$ so $\overline{m_i} \in M$, so $\varepsilon\left(\sum_{i=0}^{k} x^i \otimes \overline{m_i}\right) = 0$

$\Rightarrow \sum_{i=0}^{k} x^i\overline{m_i} = 0 \Rightarrow \overline{m_0} + x\overline{m_1} + \dots + x^k\overline{m_k} = 0$ in M.

Look at the equations:

$$\begin{aligned} xm_0 &= \overline{m_0}\\ xm_1 - m_0 &= \overline{m_1}\\ xm_2 - m_1 &= \overline{m_2}\\ &\vdots\\ -m_{k-1} &= \overline{m_k}\end{aligned}$$

$$\begin{aligned} \overline{m_0} &= xm_0\\ x\overline{m_1} &= x^2m_1 - xm_0\\ x^2\overline{m_2} &= x^3m_2 - xm_1\\ &\vdots\\ x^k\overline{m_k} &= -x^k m_{k-1}\\ \hline 0 &= 0\end{aligned}$$

add

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Solve backwards for $m_{k-1}, m_{k-2}, —, m_1, m_0 \in M$

__Fact:__ $\sum_{i=0}^{k-1} x^i \otimes m_i \in M[x]$ and $\beta\left(\sum^{k-1} x^i \otimes m_i\right) = \sum_{i=0}^{k} x^i \otimes \overline{m_i}$.

so β is onto.

How to use this lemma. Have $0 \to M[x] \to M[x] \to M \to 0$ over $R[x]$

Want to compare $\operatorname{pd}_{R[x]} M$ and $\operatorname{pd}_{R[x]} M[x]$.

__Related remarks:__ Assume we have a short exact seq. of S-modules

$0 \to A \to B \to C \to 0$

One of the axioms says that $\operatorname{pd} A = \operatorname{pd} B \Rightarrow \operatorname{pd} C \leq \operatorname{pd} A + 1$

$\downarrow$

use the long exact seq.

This can also be done without the long exact seq.

Look at the following picture.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & K_1^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & K_1^C \arrow[r] \arrow[d] & \overset{\operatorname{coker}(0\to A)}{\overset{\shortparallel}{A}} \arrow[r] & 0 & \text{by the Snake lemma} \\
 & 0 \arrow[r] \arrow[d] & P_0^B \arrow[r, equal] \arrow[d] & P_0^B \arrow[r] \arrow[d] & 0 & & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & & \\
 & & 0 & 0 & & &
\end{tikzcd}$$

Using the horseshoe lemma

$\operatorname{pd} K_1^C \leq \max\{\operatorname{pd} K_1^B, \operatorname{pd} A\}$

But $\operatorname{pd} B = \operatorname{pd} A$, $\Rightarrow \operatorname{pd} K_1^B \leq \operatorname{pd} B \mp 1 = \operatorname{pd} A \mp 1$

$\Rightarrow \underset{\operatorname{pd} C - 1}{\overset{\shortparallel}{\operatorname{pd} K_1^C}} \leq \max\{\operatorname{pd} A - 1, \operatorname{pd} A\} = \operatorname{pd} A$

so $\operatorname{pd} C \leq \operatorname{pd} A + 1$.

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$pd(A \oplus B) = \max\{pd\, A,\ pd\, B\}$

$$\underset{pd\ 1000}{0 \to A} \to \underset{pd\ 1000}{A \oplus C} \to \underset{pd\ 5}{C} \to 0$$

Prop: $\operatorname{lgldim} R[x] \le \operatorname{lgldim} R + 1$

Pf: WLOG we may assume $\operatorname{lgldim} R = n < \infty$.

Let M be an $R[x]$-module. Have a ses of $R[x]$-modules

$$0 \to M[x] \to M[x] \to M \to 0$$

$$\Rightarrow pd_{R[x]} M \underset{\substack{\downarrow \\ \text{by previous} \\ \text{remark}}}{\le} pd_{R[x]} M[x] + 1 \underset{\substack{\downarrow \\ \text{from last} \\ \text{time}}}{=} pd_R M + 1 \le \operatorname{lgldim} R + 1.$$

$$\Rightarrow \sup\{pd_{R[x]} M\} \le \operatorname{lgldim} R + 1 \Rightarrow \operatorname{lgldim} R[x] \le \operatorname{lgldim} R + 1. \quad \square$$

Want to prove $\ge$ also. Then we'll be done.

Will prove a more general result. First, we need some definitions.

Remember want to prove: $\operatorname{lgldim} R[x] \ge \operatorname{lgldim} R + 1$.

Def: S ring.

① An element $x \in S$ is central if $xr = rx$ $\forall r \in S$

② A central element x in S is a unit if $\exists u \in S$ with $xu = ux = 1$.

③ We will want central elements $x \in S$ that are not zero divisors ($xy = 0 \to y = 0$) and are not units.

Example: $S = R[X]$. X is central, not a zero divisor and not a unit.

With this notation, let $R = S/_{xS}$. Note that if $S = R[x] \Rightarrow S/_{xS} = S/_{\langle x \rangle} = \frac{R[x]}{\langle x \rangle} \cong R$.

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We will need to prove the following thm due to Rees.

Thm: Let $x \in S$ be central, non unit, not a zero divisor. Let $R = S/xS$.

Let B be an S-module such that the maps $\mu_x : B \to B$ given by

$\mu_x(b) = xb$ is a mono. Then, $\forall$ R-module A and every n, we have

$$\mathrm{Ext}^n_R(A, B/xB) \cong \mathrm{Ext}^{n+1}_S(A, B).$$

Remarks:

① $\mu_x(b) = xb$ μ_x is an S-homom since x is a central element.

② We have a ring surjection $S \xrightarrow{\pi} S/xS \cong R$. If B is an S-mod then B/xB is an R module. A is an R-module so it is also an S-module by restriction of scalars.

③ Here is how we will use Rees' thm:

use $S = R[X]$ and $x = X$. Let ${}_RM$ and assume $\mathrm{pd}_R M = n$.

$\Rightarrow \exists$ free R-module F s.t. $\mathrm{Ext}^n_R(M, F) \neq 0$ (one of the exercises)

But $F = \oplus R = \oplus S/xS = \dfrac{\oplus S}{x(\oplus S)}$

Putting $\hat{F} = \oplus S$ (free S-module), get $F = \hat{F}/x\hat{F}$.

In Rees' Theorem, take $B = \hat{F}$, $M = A$. $\rightsquigarrow B/xB = F/x\hat{F} = F$

So $\underset{\substack{\not\parallel \\ 0}}{\mathrm{Ext}^n_R(M, F)} \cong \mathrm{Ext}^{n+1}_{R[x]}(M, \hat{F})$

$\Rightarrow \mathrm{pd}_{R[x]} M \geq n+1$. $\Rightarrow \mathrm{lgldim}\ R[X] \geq n+1$.

$\Rightarrow \mathrm{lgldim}\ R[x] \geq \mathrm{lgldim}\ R + 1$ which is what we wanted □

Normally there is no relation between gldim of a ring S and over a quotient ring R.

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Example of a ring of infinite global dimension

R local artinian not semisimple $\rightsquigarrow \ell(R)>1$

$\Downarrow$

(so R is also Noetherian)

F finitely generated free $\Rightarrow F = R^n$

so length F = multiple of $\ell(R)$

Look at $k = R/\underline{m}$ simple/$_R$ so $\ell(k)=1$.

Assume ∃ a finite proj. res'n of k.

R local ⇒ ∃ finite free res'n of k over R. $0 \to F_n \to \dots \to F_1 \to F_0 \to k \to 0$

$$\begin{array}{l} \ell(k) = \underbrace{\ell(F_0) - \ell(F_1) + \ell(F_2) - \dots}_{\text{multiple of } \ell(R)} \\ \| \\ 1 \end{array}$$

But 1 is not divisible by $\ell(R)$. $\Rightarrow$ gldim $R=\infty$.

Go back to last time:

Wanted to prove: $\boxed{\text{lgldim } R[x] = 1+\text{lgldim } R}$

(Then lgldim $R[x_1, \dots, x_n] = n +$ gldim R.)

"Hilbert's syzygy thm"

Proved:

- lgldim $R[x] \leq 1+$ gldim R
- if lgldim $R=\infty \Rightarrow$ lgldim $R[x]=\infty$.

Remains to show: if lgldim $R < \infty \Rightarrow$ lgldim $R[x] \geq 1 +$ gldim R.

This follows from the following:

Thm (Rees): Let S be a ring and let $x \in S$ central, not a zero divisor,

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not a unit.

Let $R = S/xS$. Let B be an S-module s.t. $\mu_x : B \to B$ is one-to-one where $\mu_x(b) = xb$. Then, $\forall$ R-modules A and $n \geq 0$, we have:

$$\mathrm{Ext}^n_R(A, B/xB) \cong \mathrm{Ext}^{n+1}_S(A, B)$$

Pf of Rees' thm:

Step 1: $\mathrm{Hom}_S(A, B) = 0$ (view A as an S-module by restriction of scalars $S \to R$)

Pf of step 1: Let $f : A \to B$. Let $a \in A$. $\underset{\underset{B}{\cap}}{x f(a)} = f(\underset{\text{=0 since } A \text{ is } R\text{-mod}, \ =S/xS}{xa}) = f(0) = 0$

$x f(a) = 0 \Rightarrow f(a) = 0$ since mult. by x on B is 1-1. $\Rightarrow f = 0$.

Step 2: Look at the following exact seq. of S-modules:

$$0 \to B \xrightarrow{\mu_x} B \to B/xB \to 0$$ since $\mu_x(B) = xB$. $M := \mu_x$

Viewing A as an S-module, we apply $\mathrm{Hom}_S(A, \_)$ to this sequence.

We get a long exact seq:

$$0 \to \mathrm{Hom}_S(A, B) \xrightarrow{\overline{M}} \mathrm{Hom}_S(A, B) \to \mathrm{Hom}_S(A, B/xB)$$
$$\to \mathrm{Ext}^1_S(A, B) \xrightarrow{\overline{M}} \mathrm{Ext}^1_S(A, B) \to \mathrm{Ext}^1_S(A, B/xB) \to \mathrm{Ext}^2_S(A, B) \to \dots$$

Notation: Denote by $\overline{\mu}$ all the induced maps $\mathrm{Ext}^n_S(A, B) \to \mathrm{Ext}^n_S(A, B)$ $n \geq 0$.

Claim: Each induced $\overline{\mu}$ is also multiplication by x.

[In fact if $x \in Z(S)$ and $\mu =$ mult. by x $\Rightarrow$ ~~∀m~~ $\nearrow$ either $A \to A$ $\searrow$ or $B \to B$

$\Rightarrow \forall n$ $\overline{\mu} : \mathrm{Ext}^n_S(A, B) \to \mathrm{Ext}^n_S(A, B)$ is also mult. by x.]

Pf of claim: Look at

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mu = \cdot x"'] & I_0 \arrow[r, "\delta_0"] \arrow[d, "\cdot x"] \arrow[l, phantom, "\circlearrowleft"] & I_1 \arrow[r, "\delta_1"] \arrow[d, "\cdot x"] \arrow[l, phantom, "\circlearrowleft"] & \dots \\
0 \arrow[r] & B \arrow[r] & I_0 \arrow[r, "\delta_0"'] & I_1 \arrow[r, "\delta_1"'] & \dots
\end{tikzcd}$$

injective resolution of B

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Observe that multiplication by x yields a chain map. Apply $\mathrm{Hom}_S(A,-)$ to the deleted resolution.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(A, I_0) \arrow[r, "\delta_0^*"] \arrow[d, "\cdot x"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_S(A, I_1) \arrow[r, "\delta_1^*"] \arrow[d, "\cdot x"] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_S(A, I_0) \arrow[r, "\delta_0^*"] & \mathrm{Hom}(A, I_1) \arrow[r, "\delta_1^*"] & \cdots
\end{tikzcd}$$

Reason it commutes:

$$\begin{tikzcd}
f: A \to I_0 \arrow[r, mapsto] \arrow[d, "\cdot x"] & \delta_0^*(f) = \delta_0 \circ f \arrow[d, "\cdot x"] \\
xf \arrow[r] & \delta_0 x f
\end{tikzcd}$$

$x\delta_0 f(a) = x\delta_0(f(a))$
$\delta_0(xf)(a) = \delta_0(xf(a))$
$= x\delta_0(f(a))$

If $g \in \mathrm{Ker}\,\delta_n^* \Rightarrow \delta_n g = 0 \Rightarrow \underset{\delta_n x g}{\underset{\parallel}{x\delta_n g}} = 0 \Rightarrow xg \in \mathrm{Ker}\,\delta_n^*$

Similarly if $g \in \mathrm{Im}\,\delta_{n-1}^* \Rightarrow xg \in \mathrm{Im}\,\delta_{n-1}^*$.

So mult. by x is the induced map by μ_x $\mathrm{Ext}^n_S(A,B) \xrightarrow{\bar{\mu}} \mathrm{Ext}^n_S(A,B)$

Cl: For each $n \geq 0$ the induced map $\bar{\mu}: \mathrm{Ext}^n_S(A,B) \to \mathrm{Ext}^n_S(A,B)$ is the zero map.

Pf: Have the picture

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(A, I_0) \arrow[r] \arrow[d, "\cdot x"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_S(A, I_1) \arrow[r] \arrow[d, "\cdot x"] & \cdots & \mathrm{Hom}_S(A, I_n) \arrow[d, "\cdot x"] \\
0 \arrow[r] & \mathrm{Hom}_S(A, I_0) \arrow[r] & \mathrm{Hom}_S(A, I_1) \arrow[r] & \cdots & \mathrm{Hom}_S(A, I_n)
\end{tikzcd}$$

Let $g: A \to I_n$. Then $\underset{a \in A}{(xg)(a)} = xg(a) = g(xa) = g(0) = 0 \Rightarrow xg = 0$

So the chain map has all its components equal to zero!

$\Rightarrow$ the induced $\bar{\mu}: \mathrm{Ext}^n_S(A,B) \to \mathrm{Ext}^n_S(A,B)$ is zero $\forall n \geq 0$. $\Rightarrow$ Cl.

Go back to the long exact seq. in Ext.

$\mathrm{Im} \ldots = \mathrm{Ker}\ 0 = \mathrm{Ext}^1_S(A,B) \Rightarrow$ onto

$$0 \to \mathrm{Hom}_S(A, B/xB) \xrightarrow[\cong]{\text{onto}} \mathrm{Ext}^1_S(A,B) \xrightarrow{0} \mathrm{Ext}^1_S(A,B) \to \mathrm{Ext}^1_S(A, B/xB)$$
$$\xrightarrow[\cong]{\text{onto}} \mathrm{Ext}^2_S(A,B) \xrightarrow{0} \mathrm{Ext}^2_S(A,B) \to \mathrm{Ext}^2(A, B/xB) \to \ldots$$

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Get isomorphisms $\forall n \geq 0$ $\boxed{\mathrm{Ext}^n_S(A, B/xB) \cong \mathrm{Ext}^{n+1}_S(A, B)}$.

In particular also get $\boxed{\begin{aligned}\mathrm{Ext}^1_S(A, B) &\cong \mathrm{Hom}_S(A, B/xB) \\ &= \mathrm{Hom}_R(A, B/xB).\end{aligned}}$

Step 3: We will use the "axiomatic" description of the derived functors here.

Recall: Assume $\{G^n\}_{n\geq 0}$ is a sequence of additive contravariant functors $G^n : \mathrm{Mod}\ R \to \mathrm{Ab}$ satisfying

- every SES of R-modules gives rise to a long exact seq. having naturality, that is,

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & V_1 \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & Z_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_2 \arrow[r] & V_2 \arrow[r] & Z_2 \arrow[r] & 0
\end{tikzcd}$$ exact diag commut.

$$\Rightarrow \begin{tikzcd}
0 \arrow[r] & G^0(Z_1) \arrow[r] \arrow[rd, phantom, "\circlearrowleft"] & G^0(V_1) \arrow[r] \arrow[rd, phantom, "\circlearrowleft"] & G^0(M_1) \arrow[r] & G^1(Z_1) \arrow[r] & G^1(V_1) \arrow[r] & G^1(M_1) \arrow[r] & G^2(Z_1) \arrow[r] & \cdots \\
0 \arrow[r] & G^0(Z_2) \arrow[r] \arrow[u] & G^0(V_2) \arrow[r] \arrow[u] & G^0(M_2) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

- $\exists$ R-module $\overline{B}$ with $G^0 \cong \mathrm{Hom}_R(-, \overline{B})$
- $G^n(F) = 0$ $\forall$ free R-module F and $\forall n \geq 1$ ⊛

⊛ $G^n(F) = 0$ $\forall$ free mod F $\forall n \geq 1$ $\Leftrightarrow$ $G^n(P) = 0$ $\forall$ proj. mod's and all $n \geq 1$

The thm tells us that in this case, we have
(axiomatic ...)
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$$G^n = \mathrm{Ext}^n_R(-, \overline{B}) \quad \forall n \geq 0.$$

Apply this to our case: Choose $\overline{B} = B/xB$.

Choose $G^n = \mathrm{Ext}^{n+1}_S(-, B)$ applied to Mod R.

To check that the axioms are satisfied.

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Clear that each G^n is additive + contravariant.

$G^0(A) = \mathrm{Ext}^1_S(A, B) \cong \mathrm{Hom}_R(A, B/xB) = \mathrm{Hom}_R(A, \bar{B})$

$A \in \mathrm{Mod}\ R$ by above remark.

Consider the exact seq. $0 \to U \to V \to Z \to 0$ in Mod R.

Do we get $0 \to G^0(Z) \to G^0(V) \to G^0(U) \to \ldots$? naturality: exer

Know we have a seq. $\underset{\text{Is this 1-1? ⊛}}{0 \to} \underbrace{\mathrm{Ext}^1_S(Z, B) \to \mathrm{Ext}^1_S(V, B) \to \mathrm{Ext}^1_S(U, B)}$

$\underline{\to \mathrm{Ext}^2_S(Z, B) \to \ldots}$

Know it is exact since it is part of the long exact seq. in $\mathrm{Ext}_S(-, R)$.

⊛ Yes, because we have a comm. diagram

$$\begin{tikzcd} & \mathrm{Ext}^1_S(Z,B) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Ext}^1_S(V,B) \arrow[d, phantom, "\cong" rotate=90] \\ 0 \arrow[r] & \mathrm{Hom}_S(Z, B/xB) \arrow[r] & \mathrm{Hom}_S(V, B/xB) \end{tikzcd}$$

since $\mathrm{Hom}(-, B/xB)$ left exact

$\Rightarrow$ top row map 1-1

Show now that $G^n(F) = 0$ $\forall$ free R-module F and $\forall n \geq 1$.

<u>Cheat a little:</u> Assume F is <u>finitely generated</u> free.

Suffices to show that $G^n(R) = 0$ $\forall n \geq 1$.

That is, we need to show $\boxed{\mathrm{Ext}^{n+1}_S(R, B) = 0 \quad \forall n \geq 1}$

Look at $0 \to S \xrightarrow{x} S \to S/xS \to 0$ or $0 \to S \xrightarrow{x} S \to R \to 0$

Apply $\mathrm{Hom}_S(-, B)$.

$\ldots \mathrm{Ext}^1_S(R, B) \to \mathrm{Ext}^1_S(S, B) \to \overset{0}{\cancel{\mathrm{Ext}^1_S}}(S, B) \to \mathrm{Ext}^2_S(R, B) \to \overset{0}{\cancel{\mathrm{Ext}^2_S}}(R, B)$

$\to \ldots \to \overset{0}{\cancel{\mathrm{Ext}^n_S}}(S, B) \to \mathrm{Ext}^{n+1}_S(R, B) \to \overset{0}{\cancel{\mathrm{Ext}^{n+1}_S}}(S, B) \to \ldots$

$\Rightarrow \forall n \geq 2$ $0 \to \mathrm{Ext}^n_S(R, B) \to 0$ exact $\Rightarrow$ Get what we wanted

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By the thm $G^n = \text{Ext}^n_R(-, B/xB)$ or

$$\text{Ext}^{n+1}_S(A,B) \cong \text{Ext}^n_R(A, B/xB) \quad \forall A \in \text{Mod } R \quad \forall n \geq 0. \quad \square$$

Consequence: Under ~~the~~ our assumptions on x, we can show now that if

$\text{lgldim } R = n < \infty \Rightarrow \text{gldim } S \geq n+1.$

Pf: Let $A \in \text{Mod } R$ of maximal proj-dim. So $\text{Ext}^n_R(A,R) \neq 0$ and

$\text{Ext}^{n+1}_R(A,M) = 0 \quad \forall M \in \text{Mod } R.$

But $R = S/xS$. $\Rightarrow 0 \neq \text{Ext}^n_R(A, S/xS) \underset{\substack{\text{Rees lemma} \\ \text{with } B=S}}{\cong} \text{Ext}^{n+1}_S(A,S)$

$\Rightarrow \text{pd}_S A \geq n+1.$

$\Rightarrow \text{lgldim } S \geq \text{lgldim } R + 1.$

Then, for our situation, let $S = R[X]$. X plays the role of x.

Have $R = S/\langle X \rangle$.

Next time: Auslander's thm:

R Noeth: $\text{lgldim } R = \text{rgldim } R.$

also about flat resolutions, Tor dimension.

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See Tim's notes

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Then look at

$$\underbrace{\cdots \longrightarrow A \otimes_R P_n \longrightarrow \cdots \longrightarrow A \otimes_R P_0} \longrightarrow 0 \qquad \left(A \otimes_R \bar{P}_B \right)$$

Since A is flat ↗ all of this is exact.

$\Rightarrow H_n(A \otimes \bar{P}_B) = 0$ for $n \geq 1$

$\Rightarrow \operatorname{Tor}_n(A, B) = 0$ for $n \geq 1$.

(ii) $\Rightarrow$ (iii) ✓

(iii) $\Rightarrow$ (i) Let $0 \to L \to M \to N \to 0$ be exact.

Apply $A \otimes_R \square$. Get LES in Tor,

$$\begin{tikzcd} \operatorname{Tor}^1(A,N) \arrow[r] \arrow[d, equal] & A \otimes L \arrow[r] & A \otimes M \arrow[r] & A \otimes N \arrow[r] & 0 \\ 0 & & & & \end{tikzcd}$$

~~Since A~~

by assumption

So, $A \otimes \square$ is an exact functor R-Mod $\to$ Ab

$\Rightarrow$ A is flat. $\square$

Note that we also have the left version: TFAE ${}_R B$

(i) B is flat

(ii) $\operatorname{Tor}_n^R(A, B) = 0 \ \forall n \geq 1, \forall A_R$

(iii) $\operatorname{Tor}_1^R(A, B) = 0 \ \forall A_R$.

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Aim: To prove if R is Noetherian on both sides, then lgldim R = rgldim R.

For this we need "Tor dimension", also called weak dimension.

Quick review of flat modules.

Def: A_R is flat if $\forall$ mono $0 \to {}_RL \xrightarrow{i} {}_RM$,

the induced map $A \otimes_R L \xrightarrow{1_A \otimes i} A \otimes_R M$ is a mono.

Or equivalently, the functor $A \otimes_R \square$: R-Mod $\longrightarrow$ Ab is exact.

Similar definition for left flat modules.

Recall also, every projective module is flat.

Lemma. TFAE for A_R:

(i) A is flat

(ii) $\operatorname{Tor}_n^R(A, B) = 0 \quad \forall n \geq 1, \forall {}_RB$

(iii) $\operatorname{Tor}_1^R(A,B) = 0 \quad \forall {}_RB$

Proof. (i) ⇒ (ii) Consider a proj res of B, P_B

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

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Corollary. Let $0 \to A \to B \to C \to 0$ be exact of R-mods. Assume C is flat. Then, A is flat if and only if B is flat.

pf: (exercise due Monday) $\square$

Some background material from 731.

(1) $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

(2) If M_R, then its character module, also called its Pontrjagin dual, is $M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a left R-module via

$$(rf)(x) = f(xr) \quad \forall\, r \in R,\ f: M \to \mathbb{Q}/\mathbb{Z} \ \ \mathbb{Z}\text{-homomorphism and } \forall\, x \in M.$$

(3) $M_R = 0 \iff M^* = 0$.

Proof. $[\Rightarrow]$ ✓

$[\Leftarrow]$ Assume $M \neq 0$. So, $\exists\, x \in M$, $x \neq 0$. Look at M as an abelian group. If x has finite order n ($nx = 0$), then let $f: \langle x \rangle \to \mathbb{Q}/\mathbb{Z}$ be given by $x \longmapsto \frac{1}{n} + \mathbb{Z}$.

If x has infinite order, let $f: \langle x \rangle \to \mathbb{Q}/\mathbb{Z}$ by $x \longmapsto \frac{1}{2} + \mathbb{Z}$ (any $\frac{1}{n} + \mathbb{Z}$ would work).

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So, in either case, since $\mathbb{Q}/\mathbb{Z}$ is injective.

$$\begin{tikzcd}
0 \arrow[r] & \langle x \rangle \arrow[r] \arrow[d, "f \neq 0"'] & M \arrow[dl, dashed, "\exists f \text{ (use same name)}"] \\
& \mathbb{Q}/\mathbb{Z} &
\end{tikzcd}$$

$\Rightarrow f \in M^*$ and is nonzero $\Rightarrow M^* \neq 0$.

Note that we proved more that

$\forall\ 0 \neq x \in M,\ \exists\ f \in M^*$ w/ $f(x) \neq 0$

This implies that (w/ some work) $\mathbb{Q}/\mathbb{Z}$ is an <u>injective cogenerator</u> of Ab.

(4) The sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-mods is exact if and only if the induced sequence

$$0 \to {}_R C^* \xrightarrow{\beta^*} {}_R B^* \xrightarrow{\alpha^*} {}_R A^* \to 0 \quad \text{is exact.}$$

<u>Proof.</u> [$\Rightarrow$] trivial. $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-mod.

~~[⇐]~~ So, $\operatorname{Hom}(\text{—}, \mathbb{Q}/\mathbb{Z})$ is exact.

[$\Leftarrow$] Look at $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$. <u>Claim 1.</u> $\operatorname{Ker}\alpha^* = \operatorname{Im}\beta^*$

$\Rightarrow \operatorname{Im}\alpha \subseteq \operatorname{Ker}\beta$.

(The fact that β^* is mono and α^* is epi is not used.)

Let $a \in A$ and assume $\alpha(a) \notin \operatorname{Ker}\beta$.

$\Rightarrow \underbrace{\beta\alpha(a)}_{\in C} \neq 0$. We just showed that $\exists$ a ~~function~~ $\mathbb{Z}$-hom

$f : C \to \mathbb{Q}/\mathbb{Z}$ s.t. $f(\beta\alpha(a)) \neq 0$.

So, $\exists f \in C^*$ w/ $\alpha^*\beta^*(f) \neq 0$ but this contradicts that $\operatorname{Im}\beta^* = \operatorname{Ker}\alpha^*$, really only used that $\operatorname{Im}\beta^* \subseteq \operatorname{Ker}\alpha^*$.

<u>Claim 2</u>. $\operatorname{Ker}\alpha^* \subseteq \operatorname{Im}\beta^* \Rightarrow \operatorname{Ker}\beta \subseteq \operatorname{Im}\alpha$.

(Again, we don't use that β^* is mono and α^* is epi)

Assume $\exists\, b \in \operatorname{Ker}\beta$ but $b \notin \operatorname{Im}\alpha$. So, $b + \operatorname{Im}\alpha \neq 0$ in $B/\operatorname{Im}\alpha$. From previous result #3 $\exists\ \overset{g}{\cancel{f}} \neq 0$ ~~∈ Hom~~,

$g\,\cancel{f} : B/\operatorname{Im}\alpha \to \mathbb{Q}/\mathbb{Z}$ s.t. $\overset{g}{\cancel{f}}(b + \operatorname{Im}\alpha) \neq 0$.

Look at $A \xrightarrow{\alpha} B \xrightarrow{\pi} B/\operatorname{Im}\alpha \xrightarrow{g} \mathbb{Q}/\mathbb{Z}$ (all $\mathbb{Z}$-homs)

Let $\bar{g} = g\pi \in B^*$. Since π is onto, $\bar{g} \neq 0$. But

$\underset{\overset{\parallel}{0}}{\bar{g}\alpha} = \alpha^*(\bar{g})$ Since $\pi\alpha = 0 \Rightarrow \bar{g} \in \operatorname{Ker}\alpha^* \subseteq \operatorname{Im}\beta^*$

$\Rightarrow \bar{g} = \beta^*(h)$ for some $h \in C^*$

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$\Rightarrow \bar{g} = h\beta \Rightarrow \bar{g}(b) = h\beta(b) = 0$ since $b \in \operatorname{Ker}\beta$

At the same time $\bar{g}(b) = g(b + \operatorname{Im}\alpha) \neq 0$ from before.

Contradiction. So, we showed that

$$C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \text{ exact} \Rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \text{ exact.}$$

Renaming our modules

$$0 \to C^* \xrightarrow{\beta^*} B^* \quad \text{exact}$$

$$\Downarrow$$

$$B \xrightarrow{\beta} C \to 0 \quad \text{exact}$$

And renaming again ~~A~~ $B^* \xrightarrow{\alpha^*} A^* \to 0$ exact

$$\Downarrow$$

$$0 \to A \xrightarrow{\alpha} B \quad \text{exact}.$$

$\square$

Proposition. M_R is flat if and only if ${}_RM^*$ is injective.

Proof. M_R we think of as ${}_{\mathbb{Z}}M_R$.

// F

[$\Rightarrow$] Need to show, $\operatorname{Hom}_R(\square, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$ is exact.

Pick an Arbitrary module ${}_RA$. Then,

$$\operatorname{Hom}_R(A, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(M \otimes_R A, \mathbb{Q}/\mathbb{Z})$$

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Recall that $G = M \underset{R}{\otimes} \square : {}_R\text{Mod} \to \text{Ab}$ is exact since M is flat.

$H = \text{Hom}_{\mathbb{Z}}(\square, \mathbb{Q}/\mathbb{Z}) : \text{Ab} \to \text{Ab}$ is exact

since $\mathbb{Q}/\mathbb{Z}$ is injective.

$\Rightarrow$ $F = HG$ is exact since composition of exact is exact.

$[\Leftarrow]$ Assume ${}_RM^*$ is injective. Suppose

$0 \to {}_RA \xrightarrow{i} {}_RB$ is exact ~~$A/0 \to B^*$~~

~~$\Rightarrow$ $B^* \xrightarrow{i^*} A^* \longrightarrow 0$ is an epi by previous Remarks.~~

~~Then, we have~~

~~$\text{Hom}_{\mathbb{Z}}(B,$~~

Apply $\text{Hom}_R(\square, M^*)$ which is exact since M^* injective.

$$\begin{tikzcd}
\text{Hom}_R(B, M^*) \arrow[r] \arrow[d, phantom, "\|"] \arrow[dr, phantom, "\circlearrowleft"] & \text{Hom}_R(A, M^*) \arrow[r] \arrow[d, phantom, "\|"] & 0 \\
\text{Hom}_R(B, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \arrow[r] \arrow[d, phantom, sloped, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \text{Hom}_R(A, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\
\text{Hom}_{\mathbb{Z}}(M \underset{R}{\otimes} B, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \text{Hom}_{\mathbb{Z}}(M \underset{R}{\otimes} A, \mathbb{Q}/\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Top row exact $\Rightarrow$ bottom row is exact.

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$\Longrightarrow$ As $\mathbb{Z}$-modules we get that

$$(M \otimes_R B)^* \longrightarrow (M \otimes_R A)^* \longrightarrow 0$$

#4

Apply one of the previous results w/ "$R = \mathbb{Z}$" get

$$0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B$$ is exact in Ab

$\Longrightarrow M \otimes_R \square$ exact $\Rightarrow$ M is flat.

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Last time:

① Pontrjagin dual (character module)

if M_R, then $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$

② Proved: $M=0 \iff M^*=0$.

and $0 \to A_R \xrightarrow{\alpha} B_R \xrightarrow{\beta} C_R \to 0$ is exact

$$\Updownarrow$$

$0 \to {}_R C^* \to {}_R B^* \to {}_R A^* \to 0$ is exact

③ Also proved M_R is flat $\iff {}_R M^*$ is injective.

④ TFAE for A_R: (a) A_R is flat

(b) $\mathrm{Tor}_1(A, B) = 0 \ \forall_R B$

(c) $\mathrm{Tor}_n(A, B) = 0 \ \forall_R B \ \forall n \geq 1$

(version for left modules is also true.)

Lemma: Let R and S be two rings. Assume we have $({}_R A, {}_R B_S, C_S)$.

Assume ${}_R A$ is finitely generated free (true also if we just assume ${}_R A$ f.g projective)

$\Rightarrow$ $\exists$ natural isom (in each variable)

$$\mathrm{Hom}_S(B, C) \otimes_R A \simeq \mathrm{Hom}_S(\mathrm{Hom}_R(A, B), C)$$

Pf: Let σ: LHS $\to$ RHS be given by

$$\sigma(f \otimes a)(g) = f(g(a))$$

$f: B \to C$

$a \in A$

$g: A \to B$

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Facts: (1) σ is a homom. of abelian gps.

(2) if ${}_R A = B$, then σ is an isomorphism using $X \otimes_R R \cong X$, $\mathrm{Hom}_R(R, B) \cong B$.

(3) if $A \simeq R^n \Rightarrow \sigma$ is again an isom. using the result just above

(4) if A is fin. gen. proj. $\Rightarrow A \oplus A' = R^n$ for some n

σ restricted to R^n is an isom., then we ~~com~~ may deduce that $\sigma|_A$ is an isom.

Def: An R-module M is finitely presented or finitely related if

$\exists$ an exact sequence: $F_1 \to F_0 \to M \to 0$

where F_0, F_1 are finitely generated free modules.

finitely presented $\Longrightarrow$ finitely generated

$\not\Longleftarrow$ not always true

exer: find example
chose $F_0 \to M \to 0$ as desired
$\to$ Ker F_0 not nec. f.g.

Thm: Every finitely presented flat module is projective.

Pf: Let M be f. presented and flat.

$$\begin{tikzcd} F_1 \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

f.gen free

Let $B \to C \to 0$ be onto.

To show that the induced map $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is onto.

From last time, this is equivalent to showing that:

$$0 \to \mathrm{Hom}_R(M, C)^* \to \mathrm{Hom}_R(M, B)^*$$

is exact.

That is, to show

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$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M,C), \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr ?"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M,B), \mathbb{Q}/\mathbb{Z}) \arrow[d, phantom, "\wr ?"] & \text{is exact.} \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \otimes_R M \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R M & \text{is exact}
\end{tikzcd}$$

? = is this true

If true, we need to show that the bottom sequence is exact and that will suffice.

If $B \to C \to 0$ then, since $\mathbb{Q}/\mathbb{Z}$ is injective over $\mathbb{Z}$, we get

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \text{ is exact.}$$

M is flat $\Rightarrow$ bottom seq. is exact.

To show that the vertical maps are isomorphisms.

Apply $\mathrm{Hom}_R(-,C)$ to the presentation of M:

$$0 \to \mathrm{Hom}_R(M,C) \to \mathrm{Hom}_R(F_0,C) \to \mathrm{Hom}_R(F_1,C)$$

Since $\mathrm{Hom}_R(-,C)$ is left exact.

$\xrightarrow[\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})]{\text{apply}}$ get exactness by lemma from last time

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(F_1,C), \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(F_0,C), \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M,C), \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \otimes F_1 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \otimes F_0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

isos from previous lemma since F_0, F_1 are f.g. free.

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The bottom seq. is exact since it is $F_1 \to F_0 \to M \to 0$ tensored over R with $\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$.

So, our picture is a commutative exact diagram, and the 2 vertical maps are isoms.

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From an earlier exercise, $\exists$ induced isom $\mathrm{Hom}_R(M,C)^* \to$ ~~Hom~~ $C^* \otimes_R M$.

□

Thm: Let R be a Noetherian ring, and let M be a fin. gen. R-module. Then M is projective $\Leftrightarrow$ M is flat.

Pf: Over a Noetherian ring, every f.g. module is finitely presented. □

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, dash] & F_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
\text{ker} & \text{free f.g.} & &
\end{tikzcd}$$

f.g. as submod

Flat resolutions, flat dimension

Def: A flat resolution of a module M is an exact sequence:

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where each F_i is flat.

Remark: We have the following "dimension shift" type result:

Lemma: Let $0 \to Y_R \to F_R \to A_R \to 0$ be exact with F flat. Then, for every $k \geq 1$, and $\forall_R B$ we have

$$\mathrm{Tor}_k(Y,B) \simeq \mathrm{Tor}_{k+1}(A,B)$$

[Have seen this being true in the case that F is projective.]

Pf: Tensor our seq. with B: Get the long exact seq. in Tor:

$$\cdots \to \overset{0}{\cancel{\mathrm{Tor}(F,B)}} \to \mathrm{Tor}(A,B) \to Y \otimes_R B \to F \otimes_R B \to A \otimes_R B \to 0$$

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From the long exact sequence:

$$\cdots \to \overset{0}{\cancel{\mathrm{Tor}_{k+1}}}(F,B) \to \underset{\text{dim}}{\underset{\wr}{\mathrm{Tor}_{k+1}(A,B)}} \to \mathrm{Tor}_k(K,B) \to \overset{0}{\cancel{\mathrm{Tor}_k}}(F,B) \to \cdots$$

$\square$

Thm: $\mathrm{Tor}_n(A,B)$ can be computed using just flat resolutions in either variable.

Pf: Let $\mathbb{F}: \quad \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to A \to 0$ be a flat resolution of A, and let $\mathbb{F}_A: \quad \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$ be the deleted flat resolution. To show $H_n(\mathbb{F}_A \otimes_R B) = \mathrm{Tor}_n(A,B)$, $\forall n \geq 0$.

Remark: If we denote by $A' = \ker d_0 = \mathrm{Im}\, d_1$, then $\cdots \to F_2 \to F_1 \to A' \to 0$ is a flat resolution of A' and then, it is easy to see that

$$H_n(\mathbb{F}_{A'} \otimes_R B) = H_{n+1}(\mathbb{F}_A \otimes B)$$

So, if we show $H_0(\mathbb{F}_A \otimes_R B) = A \otimes_R B \quad \left(\underset{\text{know}}{=} \mathrm{Tor}_0(A,B)\right)$

and $H_1(\mathbb{F}_A \otimes_R B) = \mathrm{Tor}_1(A,B)$,

then we are done since we can use dimension shift using $H_n(\mathbb{F}_A \otimes B)$ and Tor using flat modules as earlier today.

Have $F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \to 0$ which yields

$$\begin{tikzcd}
F_1 \otimes B \arrow[r, "d_1 \otimes B"] & F_0 \otimes B \arrow[r] \arrow[dr] & A \otimes B \arrow[r] & 0 \\
 & & 0 &
\end{tikzcd}$$

$H_0(\mathbb{F}_A \otimes B) = \mathrm{Coker}(d_1 \otimes B) = A \otimes B$ ✓

We also have $H_1(\mathbb{F}_A \otimes_R B) = \dfrac{\ker(d_1 \otimes B)}{\mathrm{Im}(d_2 \otimes B)}$

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Look first at

$$\begin{tikzcd}
F_1 \arrow[rr, "d_1"] \arrow[dr, "d_1'"'] & & F_0 \\
& Y \arrow[ur, "i"'] \arrow[dr] & \\
& & 0
\end{tikzcd}$$

$Y = \mathrm{Im}\, d_1 = \mathrm{Ker}\ d_0$

$$0 \to Y \xrightarrow{i} F_0 \xrightarrow{d_0} A \to 0$$

Tensor this sequence with B.

$$\overset{0}{\cancel{\mathrm{Tor}_1(F_0,B)}} \to \mathrm{Tor}_1(A,B) \to Y \otimes B \xrightarrow{i\otimes B} F_0 \otimes B \to A \otimes B \to 0$$

since F_0 is flat

so get $0 \to \mathrm{Tor}_1(A,B) \to Y \otimes B \xrightarrow{i\otimes B} F_0 \otimes B \to A \otimes B \to 0$ is exact

so $\mathrm{Tor}_1(A,B) \cong \mathrm{Ker}(i \otimes B)$

Want to show that $H_1(\mathbb{F}_A \otimes B)$ is also $\mathrm{Ker}(i \otimes B)$.

Have

$$\begin{tikzcd}
F_2 \arrow[d, "d_2"] & & \\
F_1 \arrow[rr, "d_1"] \arrow[dr, "d_1'"'] & & F_0 \\
& Y \arrow[ur, "i"'] \arrow[dr] & \\
& & 0
\end{tikzcd}$$

tensor with B

$\mathrm{coker}\ d_2 = F_1/\mathrm{Im}\, d_2 = F_1/\mathrm{Ker}\ d_1 \cong \mathrm{Im}\, d_1 = Y$

$$\begin{tikzcd}
& & F_2 \otimes B \arrow[d, "d_2 \otimes B"] & & \\
0 \arrow[r] & \mathrm{Ker}(d_1 \otimes B) \arrow[r] & F_1 \otimes B \arrow[rr, "d_1 \otimes B"] \arrow[dr, "\partial = d_1' \otimes B"'] & & F_0 \otimes B \\
& & & Y \otimes B \arrow[ur, "i \otimes B"'] \arrow[dr] & \\
& & & & 0
\end{tikzcd}$$

$\mathrm{Ker}(i \otimes B) = \partial(\mathrm{Ker}(d_1 \otimes B))$

↓

inside $Y \otimes B = \dfrac{F_1 \otimes B}{\mathrm{Im}(d_2 \otimes B)}$

$$\mathrm{Ker}(i \otimes B) = \frac{\mathrm{Ker}(d_1 \otimes B)}{\mathrm{Im}(d_2 \otimes B)} = H_1(\mathbb{F}_A \otimes B)$$

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Conclusion: we may compute Tor using flat resolutions.

This does not depend on the choice of the flat resolution.

Flat dimension

Def: Let A_R. Say that the flat dimension of A is less or equal than n, and write $\operatorname{fd} A \leq n$

if $\exists$ a flat resolution of A: $0 \to F_n \to F_{n-1} \to \dots \to F_0 \to A \to 0$.

If no such resolution exists for any n, we say $\operatorname{fd} A = \infty$.

We say $\operatorname{fd} A = n < \infty$ if $\operatorname{fd} A \leq n$ but $\operatorname{fd} A \nleq n-1$.

In particular $\operatorname{fd} A = 0 \Leftrightarrow A$ is flat.

Remark: Every projective resolution is a flat resolution, so $\operatorname{fd} A \leq \operatorname{pd} A$.

Obs.: Schanuel's Lemma need not hold when playing with flat resolutions. For instance, let $R = \mathbb{Z}$, $A = \mathbb{Q}/\mathbb{Z}$.

Have

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[dr] & F_0 \arrow[r] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
 & & \text{free} & &
\end{tikzcd}$$

But $\underbrace{F_0 \oplus \mathbb{Z}}_{\text{free}} \ncong \underbrace{\mathbb{Q} \oplus F_1}_{\text{not free}}$

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Thm: The following are equiv. for A_R:

(1) $\underset{\hookrightarrow \text{ flat dim.}}{\operatorname{fd}} A \leq n$

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(2) $\operatorname{Tor}_k(A,B)=0 \quad \forall k\geq n+1 \quad \forall_R B$

(3) $\operatorname{Tor}_{n+1}(A,B)=0 \quad \forall_R B$.

(4) Every flat resolution of A has a flat nth kernel

Pf: Using Monday's results and the dimension shift using flat modules in the middle the result follows.

Have the corresponding statement for left R-modules.

Remarks We saw that Tor can be computed using flat resolutions and it does not depend on the choice of the flat resolution.

Def: R ring. right weak (Tor) dimension of $R \overset{\text{def}}{=} \sup\{\operatorname{fd} A_R \mid A_R \in \operatorname{Mod} R\}$

Similarly can define the leaft weak dim.

Prop: Let R be a ring. Then the right and left weak dimensions coincide.

Pf: Pick n. right weak dim $R\leq n \iff \operatorname{Tor}_{n+1}(A,B)=0 \quad \forall A_R, {}_RB$

$\iff$ left weak dim $R\leq n$.

Def: Let weak dim $R=\operatorname{wdim} R$ be the common value. This is also called the Tor dimension of R.

Prop: Let R be any ring. $\Rightarrow \operatorname{wdim} R\leq \min\{\operatorname{lgldim} R, \operatorname{rgldim} R\}$.

Pf: Result is clear if $\operatorname{lgldim} R=\infty$, or $\operatorname{rgldim} R=\infty$.

Assume $\operatorname{lgldim} R=n<\infty$. So $\forall_R B$, $\operatorname{pd} B\leq n$. But we saw that $\operatorname{pd} B\leq n \Rightarrow \operatorname{fd} B\leq n$ since a proj. res'n is also a flat res'n.

$\Rightarrow \operatorname{wdim} R\leq \operatorname{lgldim} R$. Similarly $\operatorname{wdim} R\leq \operatorname{rgldim} R$.

Examples & Definitions

A ring is called von Neumann regular if $\forall a\in R\ \exists b\in R$ s.t. $a=aba$.

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Characterization: R is von Neumann regular $\Leftrightarrow$ every R-module is flat

$\Leftrightarrow$ weak dim $R=0$.

Examples: ① It can be shown that if F is a field, and V is a vector sp. ofer F, then $R=\text{End}_F(V)$ is von Neumann-regular. If V is infinite dimensional this is not semisimple, so gldim $R\neq 0$ but wdim $R=0$, so the inequality in the proposition can be strict.

② Boolean rings (i.e. ring R where $r^2=r$ $\forall r\in R$) are also von Neumann regular.

Thm: Let R be a right Noetherian ring.

(1) Let A_R be finitely generated. Then pd A_R = fdim A_R.

(2) w dim R = rgldim R.

(3) If R is also left Noetherian then lgldim R = rgldim R.

Pf: (2) and (3) follow immediately from (1) and the corresponding (1)-statement for left Noetherian rings.

(1): We know fd $A_R\leq$ pd A_R. To show "$\geq$".

But we know that over a Noeth. ring, every f.g. flat module is proj.

So (1) follows also quickly. □

Def: If R is left and right Noetherian ring, we say

gldim R = common value of lgldim R, rgldim R

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Next part: "Triangulated categories, derived categories"

Operations with complexes

$\mathcal{A}$ = your favorite abelian category.

$\mathrm{Com}(\mathcal{A})$ = category of complexes in $\mathcal{A}$. It is abelian.

we can specialize (later) to only certain types of complexes.

Translation (Shift)

If $A: \quad \cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \to \cdots$ is a complex and $k \in \mathbb{Z}$, then $A[k]$ is the complex $(A[k])_i = A_{k+i} \; \forall i$ and the differential of $A[k]$ is $(-1)^k d$.

For instance:

$$\begin{array}{lccccccccc}
\text{degree} & & 2 & & 1 & & 0 & & -1 & \\
A: & \cdots & A_2 & \xrightarrow{d_1} & A_1 & \xrightarrow{d_1} & A_0 & \xrightarrow{d_0} & A_{-1} & \xrightarrow{d_{-1}} \cdots \\
A[1]: & \xrightarrow{-d_3} & A_2 & \xrightarrow{-d_2} & A_1 & \xrightarrow{-d_1} & A_0 & \xrightarrow{-d_0} \cdots & &
\end{array}$$

(shift to right one unit)

$$A[-2]: \quad \cdots \; A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \xrightarrow{d_{-2}} \cdots$$

(shift to left 2 units)

Obs.: The translation is an auto equivalence of $\mathrm{Com}(\mathcal{A})$.

Obs.: $\boxed{H_n(A[k]) = H_{n+k}(A) \quad \forall n, k}$

The mapping cone

Let $A, B \in \mathrm{Com}(\mathcal{A})$ and let $f: A \to B$ be a chain map.

The cone of f, denoted $C(f)$ is the complex

$$\boxed{C(f)_i = (A[-1] \oplus B)_i = A_{i-1} \oplus B_i}$$

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and the differential is given by the following:

$$\begin{tikzcd} A_i \arrow[r, "-d_i^A"] \arrow[dr, "f"] & A_{i-1} \\ \oplus & \oplus \\ B_{i+1} \arrow[r, "d_{i+1}"] & B_i \end{tikzcd}$$

$\swarrow d_i^{A[-1]}$

degree $i+1$ ⠀⠀⠀⠀ degree i

$$d_{i+1}^{C(f)} = \begin{bmatrix} -d_i^A & 0 \\ f & d_{i+1}^B \end{bmatrix} = \begin{bmatrix} -d^A & 0 \\ f & d^B \end{bmatrix}$$

↙ in some places, people put $-f$ there

So $d_{i+1}^{C(f)}(a,b) = (-d_i^A(a), d_{i+1}^B(b) + f_i(a))$ ⠀ $f_i: A_i \to B_i$

Lemma: $(C(f), d^{C(f)}_{\cdot})$ is a complex.

Pf. $$d_i^{C(f)} d_{i+1}^{C(f)} = \begin{bmatrix} -d_{i-1}^A & 0 \\ f_{i-1} & d_i^B \end{bmatrix} \begin{bmatrix} -d_i^A & 0 \\ f_i & d_{i+1}^B \end{bmatrix} = \begin{bmatrix} \overset{0}{\cancel{d_{i-1}^A d_i^A}} & 0 \\ \underset{0}{\cancel{-f_{i-1} d_i^A + d_i^B f_i}} & \overset{0}{\cancel{d_i^B d_{i+1}^B}} \end{bmatrix}$$

A, B Complexes

$$\begin{tikzcd} A_i \arrow[r, "d_i^A"] \arrow[d, "f_i"] \arrow[dr, phantom, "\circlearrowleft"] & A_{i-1} \arrow[d, "f_{i-1}"] \\ B_i \arrow[r, "d_i^B"'] & B_{i-1} \end{tikzcd}$$

since f is a chain map

$\Rightarrow d_i^B f_i = f_{i-1} d_i^A$ ⠀ $\square$

Obs: Note that, if $A =$
- → all complexes
- → all the "bounded" complexes
- → all the "bounded from left"
- → -"- -"- right

then the cone of a chain map is again in the same category.

Prop: Let $f: A \to B$ be a chain map. Then $\exists$ a SES of complexes:

$$0 \to B \xrightarrow{j} C(f) \xrightarrow{\delta} A[-1] \to 0$$

Pf: We define $j: B \to C(f)$ as follows:

$$C(f) = A[-1] \oplus B$$

$\forall i$ ⠀ $j_i: B_i \to A_{i-1} \oplus B_i$ is $b \mapsto (0, b)$, so $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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We define δ as follows:

$$\begin{tikzcd} C(f)_i \arrow[r] \arrow[d, phantom, "\shortparallel"] & A[-1]_i \arrow[d, phantom, "\shortparallel"] \\ A_{i-1} \oplus B_i \arrow[r] & A_{i-1} \\ (a,b) \arrow[r, mapsto] & a \end{tikzcd}$$

so $\delta = [\,1\ 0\,]$.

Exercise: j, δ are chain maps and this is a SES of complexes.

Obs.: Note that in every degree, $0 \to B_i \to C(f)_i \to A[-1]_i \to 0$ splits. But the sequence of complexes does not split if $f \neq 0$.

Mapping cylinder

Let $A, B \in Com(\mathcal{A})$ and let $f: A \to B$ be a chain map.

The (mapping) cylinder of f is the following complex:

$$Cyl(f) = A \oplus A[-1] \oplus B$$

$$Cyl(f)_i = A_i \oplus \underset{A_{i-1}}{\underset{\shortparallel}{A[-1]_i}} \oplus B_i$$

Need the differentials:

degree i degree $i-1$

$$\begin{tikzcd} A_i \arrow[r, "d_i^A"] & A_{i-1} \\ \oplus & \oplus \\ A_{i-1} \arrow[uur, "1"] \arrow[r, "-d_{i-1}^A"] \arrow[ddr, "f_{i-1}"] & A_{i-2} \\ \oplus & \oplus \\ B_i \arrow[r, "d_i^B"'] & B_{i-1} \end{tikzcd}$$

so

$$D^{Cyl(f)} = \begin{bmatrix} d^A & \begin{matrix} 1 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} -d^A & 0 \\ f & d^B \end{matrix}} \end{bmatrix}$$

← diff. from C(f)

Lemma: D is a differential, that is, $D^2 = 0$.

Pf.

$$\begin{bmatrix} d^A & 1 & 0 \\ 0 & -d^A & 0 \\ 0 & f & d^B \end{bmatrix} \begin{bmatrix} d^A & 1 & 0 \\ 0 & -d^A & 0 \\ 0 & f & d^B \end{bmatrix} = \begin{bmatrix} \overset{0}{\cancel{d^A d^A}} & \overset{=0}{d^A - d^A} & 0 \\ 0 & +\overset{0}{\cancel{d^A d^A}} & 0 \\ 0 & \underset{f \text{ chain map}}{\boxed{-fd^A + d^B f}^{=0}} & \underset{B \text{ complex}}{\overset{0}{\cancel{d^B d^B}}} \end{bmatrix}$$

A complex

□

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Lemma: $\forall$ chain map $A \xrightarrow{f} B$, $\exists$ a commutative diagram with exact rows in $\mathrm{Com}(\mathcal{A})$: [Gelfand-Manin]

$$\begin{tikzcd}
 & 0 \arrow[r] & B \arrow[r, "j"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & C(f) \arrow[r, "\delta"] \arrow[d, equal] & A \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\bar{f}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\
0 \arrow[r] & A \arrow[r, "f"] & B & & &
\end{tikzcd}$$

Moreover, α and β are quasiimorphisms and we have $\beta\alpha = 1_B$ and $\alpha\beta$ is homotopic to $1_{Cyl(f)}$.

Pf: Let $\bar{f}$ be $\bar{f} = \begin{bmatrix} 1_A \\ 0 \\ 0 \end{bmatrix}$. $A \to \begin{matrix} A \\ \oplus \\ A[-1] \\ \oplus \\ B \end{matrix}$

Let $\pi: \begin{matrix} Cyl(f) \\ \| \\ A \\ \oplus \\ A[-1] \\ \oplus \\ B \end{matrix} \longrightarrow \begin{matrix} C(f) \\ \| \\ A[-1] \\ \oplus \\ B \end{matrix}$, $\pi = \begin{bmatrix} 0 & 1_{A[-1]} & 0 \\ 0 & 0 & 1_B \end{bmatrix}$

$\beta = [-f \quad 0 \quad 1_B]$

$$\begin{tikzcd}
A \arrow[drrrr, "-f"] & & & & \\
\oplus & & & & \\
A[-1] & & & & B \\
\oplus & & & & \\
B \arrow[rrrruu, "1_B"'] & & & &
\end{tikzcd}$$

$\alpha = \begin{bmatrix} 0 \\ 0 \\ 1_B \end{bmatrix}$ Immediately get $\beta\bar{f} = f$.

$\pi\alpha = \begin{bmatrix} 0 \\ 1_B \end{bmatrix} = j.$

Exer: Check middle seq. is exact.

$\beta \cancel{A} \alpha = 1_B$. ✓

Def: A chain map $f: A \to B$ is a quasiisomorphism if every induced

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map $f_n^*: H_n(A) \to H_n(B)$ is an isomorphism.

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do ex 1 part 1

[in ex 3 do only ∄ quasi-isom ↓ , do rest later (after Wedn)]

Mapping Cone $A \xrightarrow{f} B$ in $\mathrm{Com}(\mathcal{A})$ (or $\mathrm{Com}^+(\mathcal{A})$, $\mathrm{Com}^-(\mathcal{A})$) (*)

Then the cone of f is $C(f) = A[-1] \oplus B$ (so in degree i, $C(f)_i = A_{i-1} \oplus B_i$)

and $d^{C(f)} = \begin{bmatrix} d^{A[-1]} & 0 \\ \oplus & d^B \end{bmatrix}$

Seen: ∃ a seq S.e.s. of complexes: $0 \to B \xrightarrow[\begin{bmatrix}0\\1\end{bmatrix}]{} C(f) \xrightarrow[\begin{bmatrix}1 & 0\end{bmatrix}]{} A[-1] \to 0$

(*) $\mathrm{Com}^+(\mathcal{A})$ = all complexes bounded from ~~below~~ left

($A \in \mathrm{Com}^+(\mathcal{A}) \Leftrightarrow \exists i_0$ s.t. $A_i = 0 \quad \forall i > i_0$)

$\mathrm{Com}^-(\mathcal{A})$ = all complexes bounded from right

$\mathrm{Com}^b(\mathcal{A})$ = all bounded complexes : $0 \to 0 \to * \to * \to \dots \to * \to 0 \to 0 \to$.

Remark: When using the horseshoe lemma, we got a differential for the middle resolution that is a triangular matrix and the diagonal entries are differentials of the ends.

Q: is this a mapping cone of sth?

Or If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

Know P_A, P_B. Can we construct P_C as a mapping cone?

→ Come talk to him about this

Mapping cylinder $A \xrightarrow{f} B$
chain map

$\mathrm{Cyl}(f) = A \oplus A[-1] \oplus B$ with differential

$$D = \begin{bmatrix} d^A & 1 & 0 \\ 0 & d^{A[-1]} & 0 \\ 0 & f & d^B \end{bmatrix}$$

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Lemma: $\exists$ a commutative diagram of complexes with exact rows.

The vertical maps α, β are quasi-isomorphisms with $\beta\alpha = 1_B$, $\alpha\beta \sim 1_{Cyl(f)}$

$$\begin{tikzcd}
 & 0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "\alpha"] & C(f) \arrow[r, "\delta"] \arrow[d, equal] & A[-1] \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\bar{f}"] \arrow[d, equal] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\
0 \arrow[r] & A \arrow[r, "f"] & B & & &
\end{tikzcd}$$

where $\pi = \begin{bmatrix} 0 & 1_{A[-1]} & 0 \\ 0 & 0 & 1_B \end{bmatrix}$, $\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\beta = [-f \ 0 \ 1]$, $\bar{f} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

easy to check $\beta\alpha = 1$.

Check $\alpha\beta \sim 1_{Cyl(f)}$. Need $s_{i-1} : Cyl(f)_{i-1} \to Cyl(f)_i$

$$\begin{tikzcd}
A_{i-1} \arrow[rdd, "1_{A_{i-1}}"] & A_i \\
\oplus & \oplus \\
A_{i-2} & A_{i-1} \\
\oplus & \oplus \\
B_{i-1} & B_i
\end{tikzcd}$$

Let $s = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Claim: $sD - Ds = 1 - \alpha\beta$ **Pf:** Compute $\alpha\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f & 0 & 1 \end{bmatrix}$

Then check that $sD + Ds = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f & 0 & 0 \end{bmatrix}$ ✓

Def: A triangle in $Com(\mathcal{A})$ is a diagram of complexes
$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[-1]$

A triangle is distinguished if it is isomorphic as a triangle to a triangle of the form $A \to Cyl(f) \to C(f) \to A[-1]$ for some $f: A \to B$.

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Def: A chain map $f: A \to B$ is a quasi-isomorphism if it induces isomorphisms in homology, that is $\forall n \in \mathbb{Z}$ the induced maps

$H_n(f): H_n(A) \to H_n(B)$ are isomorphisms.

Example: R ring. Look at $\mathrm{Com}(R)$. Let $M \in \mathrm{Mod}\, R$. Let $\mathbb{P}_M$ be ~~the~~ a (deleted) projective resolution of M and let I_M be a (deleted) injective resolution of M.

$$\begin{tikzcd}
 & & & & \text{degree } 0 \arrow[d] & & & \\
\mathbb{P}_M: \arrow[d, "\text{chain map}"] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "0"] & P_1 \arrow[r] \arrow[d, "0"] & P_0 \arrow[r] \arrow[d, "\varepsilon_0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
M: \arrow[d, "\text{chain map}", "\text{stalk complex}"'] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] \arrow[d, "\delta_0"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
I_M: & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I_0 \arrow[r, "\delta_{-1}"] & I_1 \arrow[r, "\delta_{-2}"] & I_2 \arrow[r] & \cdots
\end{tikzcd}$$

The chain maps are quasi-isomorphisms.

Def Let $\mathcal{A}$ be an abelian category.

The derived category of $\mathcal{A}$, denoted $D(\mathcal{A})$ is a category equipped with a functor $\varphi: \mathrm{Com}(\mathcal{A}) \to D(\mathcal{A})$ such that:

(1) φ takes every quasi-isom. into an isom. in $D(\mathcal{A})$.

(2) φ is "universal" with this property, that is:

$$\begin{tikzcd}
\mathrm{Com}(\mathcal{A}) \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & D(\mathcal{A}) \arrow[d, dashed, "\exists! F", "?"'] \\
 & \mathcal{B}
\end{tikzcd}$$

$\psi(\text{quasi-isom}) = \text{isom in } \mathcal{B}$

$\Rightarrow \exists!\ D(\mathcal{A}) \xrightarrow{F} \mathcal{B}$ s.t. $F\varphi = \psi$.

Comments:

(1) Derived categories need not always exist (Freyd 1964).

(2) If $D(\mathcal{A})$ exists then it is unique up to isom.

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(3) If $\mathcal{A} = \text{Mod } R$, then $D(\mathcal{A})$ exists. Then, every module can be identified in $D(\mathcal{A})$ with a proj. res'n of M, and also with an inj. res'n.

(4) $D(\mathcal{A})$ is always a triangulated category.

Def: A suspended category $\mathcal{T}$ is an additive category together with an auto-equivalence Σ or $[-] : \mathcal{T} \to \mathcal{T}$, called suspension, or shift. Notation: $(\mathcal{T}, [\ ])$

A triangle in $(\mathcal{T}, [-])$ is a diagram $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$

Draw it

$$\begin{tikzcd}
 & C \arrow[dl, "w"'] & \\
A \arrow[rr, "u"'] & & B \arrow[ul, "v"']
\end{tikzcd}$$

u is called the "base" of the triangle

Def: A morphism of triangles $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} A[1]$ is a triple (f, g, h) of morphisms in ~~E~~ $\mathcal{T}$ such that the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, "w"] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1]
\end{tikzcd}$$

The morphism (f, g, h) is an isomorphism of triangles if each of f, g, h is an isomorphism.

Def (Triangulated categories) → formally introduced by Puppe? and Verdier?

A triangulated category is a triple $(\mathcal{T}, [\ ], \Delta)$ where $(\mathcal{T}, [\ ])$ is a suspended category, and Δ is a $\overset{\text{collection}}{\text{class}}$ of triangles in $\mathcal{T}$ called distinguished or exact satisfying the following axioms (TR1, TR2, TR3, TR4).

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TR1: ① Δ is closed under isom. of triangles.

② $\forall\ A \xrightarrow{u} B$ in $\mathcal{T}$ is the <u>base</u> of some distinguished triangle, that is,

$\exists$ triangle in Δ: $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$

③ $\forall\ A \in \mathcal{T}$, the triangles $A \xrightarrow{1_A} A \to 0 \to A[1]$ are distinguished.

TR2: A triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is distinguished

$\Leftrightarrow$ the triangle $B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1]$ is distinguished.

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} & A[1] \arrow[dl] & \\ B \arrow[rr, "v"'] & \circlearrowright & C \arrow[ul] \end{tikzcd}$$

in Δ in Δ

TR3: Any comm. diagram with the rows being distinguished triangles:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, dashed, "\exists h"] \arrow[dr, phantom, "\circlearrowleft"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

can be completed to a homom of triangles.

TR4: (Verdier's octahedral axiom).

Def: Let $\mathcal{T}, \mathcal{T}'$ be triangulated categories with shifts $[\ ], [\ ]'$.

An additive functor $F : \mathcal{T} \to \mathcal{T}'$ is <u>exact</u> if $\exists$ natural isom

α: $F \circ [\ ] \to [\ ]' \circ F$ such that, a triangle $F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \xrightarrow{\alpha \circ F(w)} F(A)[1]'$

is distinguished in $\mathcal{T}'$, if the triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is distinguished in $\mathcal{T}$.

Know $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$

$$\begin{tikzcd} F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r] & F(A[1]) \arrow[r, squiggly, "(F\circ[1])(A)"] \arrow[d, "\alpha_A"] & {} \\ & & & F(A)[1]' & \end{tikzcd}$$

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$F: \mathcal{T} \to \mathcal{T}'$ is an equivalence of triangulated categories if it is an <u>exact</u> equivalence. We say that $\mathcal{T}, \mathcal{T}'$ are "triangle equivalent" or, "equivalent as triangulated categories".

<u>Def:</u> Let $\mathcal{T}$ be a triangulated category. An additive covariant functor $H: \mathcal{T} \to \mathcal{A}$ where $\mathcal{A}$ is an abelian category is a homological functor, if $\forall$ distinguished (exact) triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ $\exists$ a long exact sequence in $\mathcal{A}$:

$$\dots \to H(A) \to H(B) \to H(C) \to H(A[1]) \to H(B[1]) \to H(C[1]) \to \dots$$

Similary, one may define a contravariant cohomological functor

$H: \mathcal{T} \to Ab$ $\quad A \to B \to C \to A[1] \longrightarrow \quad \dots H(A[i+1]) \to H(C[i]) \to H(B[i]) \to \dots$

<u>Lemma:</u> Let $\mathcal{T}$ be a triangulated.

① Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be a distinguished triangle. Then $vu = 0$ and $wv = 0$.

② Let $M \in \mathcal{T}$. Then $Hom_{\mathcal{T}}(M, -): \mathcal{T} \to Ab$ is a homological functor and $Hom_{\mathcal{T}}(-, M'): \mathcal{T} \to Ab$ is a cohomological functor.

③ Let

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1] \end{tikzcd}$$

$\to$ in Δ:

$\to$ in Δ

(f, g, h) morphism. Then f, g isom. $\Rightarrow$ h isom.

<u>Axiom TR4 for triangulated categories</u> "Octahedral axiom" <u>4/4/18</u>

Given $A \xrightarrow{u} B \xrightarrow{v} C$ in $\mathcal{T}$ and 3 distinguished triangles:

$$A \xrightarrow{u} B \xrightarrow{i} C' \xrightarrow{i'} A[1]$$

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$B \xrightarrow{v} C \xrightarrow{j} A' \xrightarrow{j'} B[1]$

$A \xrightarrow{vu} C \xrightarrow{k} B' \xrightarrow{k'} A[1]$

$\exists\, f, g$ where $f: C' \to B'$, $g: B' \to A'$

s.t. the following diagram commutes:

$$\begin{tikzcd}
B'[-1] \arrow[r, "{k'[-1]}"] \arrow[d, "{g[-1]}"'] & A \arrow[r, equal] \arrow[d, "u"] & A \arrow[d, "vu"] & & \\
A'[-1] \arrow[r, "{j'[-1]}"] & B \arrow[r, "v"] \arrow[d, "i"'] & C \arrow[r, "j"] \arrow[d, "k"] & A' \arrow[r, "j'"] \arrow[d, equal] & B[1] \arrow[d, "{i[1]}"] \\
 & C' \arrow[r, "f"] \arrow[d, "i'"'] & B' \arrow[r, "g"] \arrow[d, "k'"] & A' \arrow[r, "{i[1]\circ j'}"] & C'[1] \\
 & A[1] \arrow[r, equal] & A[1] & &
\end{tikzcd}$$

and the third row is also a distinguished triangle

One way to remember it is the this. Think of $A \subset B \subset C$ where A, B, C are modules.

$$\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d, hook] & A \arrow[d] & \\
 & B \arrow[r, hook] \arrow[d] & C \arrow[r] \arrow[d] & C/B = A' \arrow[d, equal] \\
\exists & C' = B/A \arrow[r, hook, "f"] & C/A = B' \arrow[r, "g"] & C/B
\end{tikzcd}$$

Pf of the last stated lemma: Δ = all the distinguished triangles

① If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ in Δ then we need to show $vu = 0 = wv$

Look at

$$\begin{tikzcd}
B \arrow[r, "v"] \arrow[d, "v"'] & C \arrow[r, "w"] \arrow[d, equal, "1_C"', "\wr"] & A[1] \arrow[r, "{-u[1]}"] \arrow[d, dashed, "\exists h"] & B[1] \arrow[d, "{v[1]}"] \\
C \arrow[r, equal, "1_C"] & C \arrow[r] & 0 \arrow[r] & C[1]
\end{tikzcd}$$

TR3

comm. diag. with rows in Δ (by TR1 ③, TR2)

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By TR3, $\exists$ h s.t. last square commutes.

$\Rightarrow -v[1]u[1] = 0 \Rightarrow (vu)[1] = 0 \Rightarrow vu = 0$

(Using the fact that $\mathcal{T}$ triangulated and [ ] is an autoequiv of $\mathcal{T}$).

Using another axiom

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1] \in \Delta$$

So by what we just proved we get $wv = 0$.

② We have to show that if $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ in Δ and $M \in \mathcal{T}$, then $\operatorname{Hom}_{\mathcal{T}}(M,-)$ is a homological functor.

Claim: 1: $\operatorname{Hom}_{\mathcal{T}}(M,A) \xrightarrow{\operatorname{Hom}(M,u)} \operatorname{Hom}_{\mathcal{T}}(M,B) \xrightarrow{\operatorname{Hom}(M,v)} \operatorname{Hom}_{\mathcal{T}}(M,C)$

is exact in Ab.

Pf. Clear that $\operatorname{Hom}(M,v) \circ \operatorname{Hom}(M,u) = \operatorname{Hom}(M, vu) = \operatorname{Hom}(M,0) = 0$

so $\operatorname{Im} \operatorname{Hom}(M,u) \subseteq \operatorname{Ker}(M,v)$.

Let $g \in \operatorname{Ker} \operatorname{Hom}(M,v)$ $\quad M \xrightarrow{g} B \xrightarrow{v} C$

so $vg = 0$
$\quad \|$
$\operatorname{Hom}(M,v)(g)$

Look at the following diagram:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] & M[1] \arrow[r, "-1_{M[1]}"] \arrow[d, dashed, "\exists h!"'] \arrow[dr, phantom, "\circlearrowright"] & M[1] \arrow[d, "g[1]"] \\
B \arrow[r, "v"] & C \arrow[r, "w"] & A[1] \arrow[r, "-u[1]"] & B[1]
\end{tikzcd}$$

(get this by rotating $M \xrightarrow{1_M} M \to 0 \to M[1]$)

rotation of triangle
So rows are in Δ

By TR3, $\exists h: M[1] \to A[1]$ with $g[1]\, 1_{M[1]} = u[1]\, h$

Apply $[-1]$: $g = u\, h[-1]$ where $h[-1]: M \to A$

This means that $g = \operatorname{Hom}(M,u)(h[-1])$

So $\operatorname{Ker}(M,v) \subseteq \operatorname{Im}(M,u)$. So claim is proved.

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Claim 2: Rotating our triangle i times we get

$$A[i] \xrightarrow{\pm u[i]} B[i] \xrightarrow{\pm v[i]} C[i] \xrightarrow{\pm w[i]} A[i+1] \text{ in } \Delta$$

Apply $\mathrm{Hom}(M, -)$ to this, and by claim 1 we obtain that

$\mathrm{Hom}(M, A[i]) \to \mathrm{Hom}(M, B[i]) \to \mathrm{Hom}(M, C[i])$ is exact $\forall i$.

Putting cl. 1 and cl. 2 + rotations we get a long exact seq. as needed.

③ We have to show that if we have a morphism (f, g, h) of triangles in Δ and two of the maps are isomorphisms, then so is the third.

Using rotation, it is enough to prove that:

$$(*) \quad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"', "\wr"] & B \arrow[r, "v"] \arrow[d, "g", "\wr"'] & C \arrow[r, "w"] \arrow[d, "h"] & A[1] \arrow[d, "f[1]", "\wr"'] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1] \end{tikzcd}$$

and ~~f~~ f, g are isoms then h isom.

Apply $\mathrm{Hom}_{\mathcal{T}}(\cancel{A}\, C', -)$ to this picture. Get

$$\begin{tikzcd} \mathrm{Hom}(C', A) \arrow[r] \arrow[d, "\mathrm{Hom}(C', f)", "\wr"'] & \mathrm{Hom}(C', B) \arrow[r] \arrow[d, "\mathrm{Hom}(C', g)", "\wr"'] & \mathrm{Hom}(C', C) \arrow[r] \arrow[d, "\mathrm{Hom}(C', h)"] & \mathrm{Hom}(C', A[1]) \arrow[r] \arrow[d, "\mathrm{Hom}(C', f[1])", "\wr"'] & \mathrm{Hom}(C', B[1 \arrow[d, "\mathrm{Hom}(C', g[1"', "\wr"] \\ \mathrm{Hom}(C', A') \arrow[r] & \mathrm{Hom}(C', B') \arrow[r] & \mathrm{Hom}(C', C') \arrow[r] & \mathrm{Hom}(C', A'[1]) \arrow[r] & \mathrm{Hom}(C', B'[1, \end{tikzcd}$$

The rows are exact by the previous part and the diagram is commutative.

By the Five Lemma $\Rightarrow$ $\mathrm{Hom}(C', h)$ is an isom.

So it is onto, so given $1_{C'}$, there is $\varphi \in \mathrm{Hom}(C', C)$ s.t.

$\mathrm{Hom}(C', h)(\varphi) = 1_{C'}$ or $\boxed{h\varphi = 1_{C'}}$

Applying part ② ($\mathrm{Hom}_{\mathcal{T}}(-, M)$ cohomological functor), we

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get by applying $\mathrm{Hom}(-, C)$ to $(*)$, and using again the Five Lemma, that $\mathrm{Hom}(h, C)$ an isomorphism and $\exists\, \psi : C \to C'$ s.t. $\boxed{\psi h = 1_C}$ $\Rightarrow$ h isom.

Remark: So far, in each proof we did, we only ~~rotated~~ in one direction.

Moreover, the second (2) axiom TR2 was stated as follows:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1] \text{ is in } \Delta \Leftrightarrow B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1] \text{ is in } \Delta$$

It turns out that TR2 can be restated only as "⇒" and "⇐" follows.

Prop: Assume TR2 is only "⇒" and let $B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1]$ be in Δ $\Rightarrow$ $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is also in Δ

Pf: When we play with triangulated categories, to show that a triangle is distinguished, we normally show that it is isomorphic to a distinguished triangle and ~~let~~ apply the axiom. TR1 ①

Apply rotation to the given distinguished triangle. Get:

$$C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1] \xrightarrow{-v[1]} C[1] \text{ in } \Delta$$

rotate again $A[1] \xrightarrow{-u[1]} B[1] \xrightarrow{-v[1]} C[1] \xrightarrow{-w[1]} A[2]$ in Δ

Look at $A \xrightarrow{u} B$. By axiom (TR1 ②), $\exists$ triangle in Δ

$$A \xrightarrow{u} B \xrightarrow{v'} C' \xrightarrow{w'} A[1]$$

rotate this enough times (3 times) to get the following diagram:

$$\begin{tikzcd}
A[1] \arrow[r, "-u[1]"] \arrow[d, equal] & B[1] \arrow[r, "-v[1]"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & C[1] \arrow[r, "-w[1]"] \arrow[d, dashed, "\exists h"] \arrow[dr, phantom, "TR3\ \circlearrowleft"] & A[2] \arrow[d, equal] & \text{in } \Delta \\
A[1] \arrow[r, "-u[1]"] & B[1] \arrow[r, "-v'[1]"] & C'[1] \arrow[r, "-w'[1]"] & A[2] & \text{in } \Delta
\end{tikzcd}$$

h isom.

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Get rid of the -. Apply $[-1]$

We get

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] & B \arrow[r, "v"] \arrow[d, equal] & C \arrow[r, "w"] \arrow[d, "h[-1]"] & A[1] \arrow[d, equal] \\ A \arrow[r, "u"] & B \arrow[r, "v'"] & C' \arrow[r, "w'"] & A[1] \end{tikzcd}$$

$\rightsquigarrow$ know is distinguished

and we get an isom. of triangles, since $h[-1]$ is also an isom.

$\Rightarrow$ $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is also in Δ. $\square$

Lemma: $\mathcal{T}$ triangulated. Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be in Δ.

TFAE:

(1) $w=0$

(2) u is a section that is $\exists t: B \to A$ with $tu = 1_A$.

$\hookrightarrow$ ("splittable mono")

(3) v is a ~~retract~~, that is $\exists j: C \to B$ with $vj = 1_C$.

$\hookrightarrow$ ("splittable epi")

Pf. (1) $\Rightarrow$ (2):

Have

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "1_A"'] & B \arrow[r] \arrow[d, dashed, "t"] \arrow[l, phantom, "\circlearrowleft"] & C \arrow[r, "0"] \arrow[d, "0"] & A[1] \arrow[d, "1_{A[1]}"] \arrow[l, phantom, "\circlearrowleft"] \\ A \arrow[r, "1_A"'] & A \arrow[r] & 0 \arrow[r] & A[1] \end{tikzcd}$$

in Δ (top row), in Δ (bottom row)

So $\exists t$ with $tu = 1_A$.

exercise: (1) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1): Start with

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "1_A"'] & B \arrow[r, "v"] \arrow[d, "t"] \arrow[l, phantom, "\circlearrowleft"] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & A[1] \arrow[d, equal] \\ A \arrow[r, "1_A"'] & A \arrow[r] & 0 \arrow[r] & A[1] \end{tikzcd}$$

in Δ (top row), in Δ (bottom row)

$\exists h$ (which we know is 0) s.t. $0 = w$.

exercise: prove (3) $\Rightarrow$ (1).

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Recall:

Lemma: $\mathcal{T}$ triangulated. Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be distinguished

TFAE: (1) $w=0$
(2) u is a splittable mono
(3) v is a splittable epi. □

[exact seq: $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$
u split mono $\Leftrightarrow$ v split epi
$\Leftrightarrow$ $B \cong A \oplus C$]

Note that in this case we can also get $B \cong A \oplus C$.

To prove this, we need:

Lemma: A direct sum of distinguished triangles is a dist. triangle.

Pf: Let $\emptyset \geq 0$ $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$, and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} A'[1]$ be in Δ.

Look at $A \oplus A' \xrightarrow{\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix}} B \oplus B' \xrightarrow{\begin{bmatrix} v & 0 \\ 0 & v' \end{bmatrix}} C \oplus C' \xrightarrow{\begin{bmatrix} w & 0 \\ 0 & w' \end{bmatrix}} (A \oplus A')[1]$.

We will show that it is isom. to a disting. triangle.

Know by TR1 that $A \oplus A' \xrightarrow{\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix}} B \oplus B'$ is the basis of some distinguished triangle:

$$\begin{tikzcd}
A \oplus A' \arrow[r, "\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix}"] \arrow[d, "{[1\ 0]}"] \arrow[dd, bend right, "{[0\ 1]}"'] & B \oplus B' \arrow[r] \arrow[d, "{[1\ 0]}"] \arrow[dd, "{[0,1]}"', bend right] & D \arrow[r] \arrow[d, dashed, "\exists h"] \arrow[dd, dashed, bend right, "\exists g"'] & (A \oplus A')[1] \arrow[d, "{[1\ 0]}"] \arrow[dd, bend right, "{[0\ 1]}"'] & \text{in } \Delta \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & A[1] & \text{in } \Delta \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r] & A'[1] & \text{in } \Delta
\end{tikzcd}$$

Use this to get:

$$\begin{tikzcd}
A \oplus A' \arrow[r, "\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix}"] \arrow[d, equal] & B \oplus B' \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d, "\begin{bmatrix} h \\ g \end{bmatrix}"] & (A \oplus A')[1] \arrow[d, equal] & \text{in } \Delta \\
A \oplus A' \arrow[r, "\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix}"] & B \oplus B' \arrow[r, "\begin{bmatrix} v & 0 \\ 0 & v' \end{bmatrix}"] & C \oplus C' \arrow[r, "\begin{bmatrix} w & 0 \\ 0 & w' \end{bmatrix}"] & {[A \oplus A'][1]}
\end{tikzcd}$$

Easy to check that we get an isom. of triangles, so the direct sum is distinguished. □

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Remark:

Assume we are in the situation

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1] \qquad \text{where } w = 0.$$

Look at the distinguished triangle

$$0 \to C \xrightarrow{1} C \to 0 \qquad (0[1] = 0)$$

[the rotation of $C \xrightarrow{1} C \to 0 \to C[1]$]

$\to C \to 0 \to C[1] \to C[1]$

$\to 0 \to C[1] \to C[1] \to 0[1]$, $0[1] = 0$

$\xrightarrow{[-1]} 0 \to C \to C \to 0$

and $A \xrightarrow{1_A} A \to 0 \to A[1]$

Take the direct sum:

$$\begin{tikzcd}
A \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[d, equal] & A \oplus C \arrow[r, "{[0,1]}"] \arrow[d, dashed, "\exists h"] & C \arrow[r, "w"] \arrow[d, equal] & A[1] \arrow[d, equal] & \text{distinguished} \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & A[1] & \text{Since } w = 0
\end{tikzcd}$$

Know $\exists h$ commuting everything.

So $(1_A, h, 1_C)$ is a morph. of dist. triangles.

From last time we also get that h is an isom. $\Rightarrow B \cong A \oplus C$. $\square$

Prop: Let $\mathcal{T}$ be triangulated. Then $A \xrightarrow{u} B \to 0 \to A[1]$ is distinguished $\Leftrightarrow$ u is an isom.

Pf. "$\Leftarrow$":

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "u"] & B \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & A[1] \arrow[d, "{u[1]}"] \\
B \arrow[r, "1_B"] & B \arrow[r] & 0 \arrow[r] & B[1] & \text{in } \Delta
\end{tikzcd}$$

This is an isom of triangles, so top triangle is also distinguished

"$\Rightarrow$": Assume $A \xrightarrow{u} B \to 0 \to A[1]$ in Δ. $\underset{\text{lemma}}{\Rightarrow}$ u split mono.

Also $B \to 0 \to A[1] \xrightarrow{-u[1]} A[1]$ is also in Δ by rotation.

rotate again $0 \to A[1] \xrightarrow{-u[1]} A[1] \to 0$ in Δ by rotation

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$\Rightarrow -u[1]$ split epi $\Rightarrow u[1]$ split epi $\Rightarrow u$ split epi

<u>exercise:</u> u is an isom. □

<u>First example</u> of triang. cat.

<u>The homotopy category</u>

$\mathcal{A}$ abelian category. Look at $Com^*(\mathcal{A})$ = the category of complexes, where $*$ means one of $\{$all complexes ($Com(\mathcal{A})$), complexes bounded from above ($Com^-(\mathcal{A})$), ... bounded from below ($Com^+(\mathcal{A})$), bounded compl ($Com^b(\mathcal{A})$)$\}$

<u>Complex:</u> $\dots A_n \xrightarrow{d_n^A} A_{n+1} \xrightarrow{d_{n+1}^A} A_{n+2} \to \dots$

Need to adjust the notation for the cone, cylinder.

<u>The cone:</u>

$A \xrightarrow{f} B$ morphism of complexes

then its cone is $C(f) = A[1] \oplus B$ that is $(C(f))_i = A_{i+1} \oplus B_i$

$$d^{C(f)} = \begin{bmatrix} d^{A[1]} & 0 \\ f & d^B \end{bmatrix} = \begin{bmatrix} d^A & 0 \\ f & d^B \end{bmatrix}$$

Have an exact seq. of complexes:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}", "j"'] & C(f) \arrow[r, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}", "S"'] & A[1] \arrow[r] & 0 \end{tikzcd}$$

For each $Com^*(A)$ we consider the homotopy category $K^*(A)$

where $Ob\, K^*(\mathcal{A}) = Ob\, Com^*(A)$

$$Hom_{K^*(A)}(A,B) = \frac{Hom_{Com^*(A)}(A,B)}{\sim}$$

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where $\sim$ denotes homotopy equivalence.

We show that $K^*(\mathcal{A})$ is triangulated.

<u>Observation</u>

~~the~~ The shift functor $[\ ]: Com^*(\mathcal{A}) \xrightarrow{\sim} Com^*(\mathcal{A})$ induces an equivalence $[\ ]: K^*(\mathcal{A}) \xrightarrow{\sim} K^*(\mathcal{A})$.

<u>Recall:</u> we have the following diagram in $Com^*(\mathcal{A})$, where $Cyl(f)$ denotes the cylinder of a chain map $f: A \to B$

$$\boxed{Cyl(f) = A \oplus A[1] \oplus B}$$

with differential
$$D = \begin{bmatrix} d^A & 1 & 0 \\ 0 & d^{A[1]} & 0 \\ 0 & f & d^B \end{bmatrix} = \begin{bmatrix} d^A & 1 & 0 \\ 0 & -d^A & 0 \\ 0 & f_i & d^B \end{bmatrix}$$

$$\begin{tikzcd}
 & 0 \arrow[r] & B \arrow[r, "j"] \arrow[d, "\alpha"] & C(f) \arrow[r, "\delta"] \arrow[d, equal] & A[1] \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\bar{f}"] \arrow[d, equal, "A"] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] \arrow[d, equal] & 0 & \\
 & A \arrow[r, "f"] & B \arrow[r, "j"] & C(f) \arrow[r, "\delta"] & A[1] &
\end{tikzcd}$$

where $\beta\alpha = 1_B$ and $\alpha\beta \sim 1_{Cyl(f)}$

So, in $K^*(\mathcal{A})$ α and β are isomorphisms inverse to each other.

<u>Lemma:</u> $j\beta \sim \pi$.

<u>Pf:</u> Know $\pi\alpha = j \Rightarrow \pi\alpha\beta = j\beta$ But $\alpha\beta \sim 1_{Cyl(f)}$

$\Rightarrow \pi\alpha\beta \sim \pi \Rightarrow j\beta \sim \pi$. $\square$

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So, the "new square" commutes in the homotopy category.

Def: A diagram $A \to B \to C \to A[1]$ is a distinguished triangle in $K^{\bullet}(A)$ if it is isomorphic in $K^{\bullet}(A)$ to a diagram of the form: $A \xrightarrow{u} B \xrightarrow{j} C(u) \xrightarrow{\delta} A[1]$

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, squiggly] & C(u) \arrow[r, "\delta"] & A[1] \\ {[0\ 1_B]} & {[1_A\ 0]} \arrow[from=1-2, to=2-2, phantom] & \end{tikzcd}$$

Note: Using our homotopy equivalences, this is equiv. to saying that the distinguished Δ are of the form:

$$A \xrightarrow{\bar{f}} Cyl(f) \xrightarrow{\pi} C(f) \xrightarrow{\delta} A[1]$$

Thm: $K^*(A)$ is a triangulated category with the shift $[\ ]$.

Pf: **TR1:** ① Trivial to show that a triangle isomorphic to a distinguished triangle is again distinguished. Comp. of isom is isom

② Clear, that if have $u: A \to B$, then $\exists$ a diagram

$$A \xrightarrow{u} B \xrightarrow{\begin{bmatrix} 0 \\ 1_B \end{bmatrix}} C(u) \xrightarrow{[1_A\ 0]} A[1]$$

So $\exists$ a distinguished triangle with base u

③ We also have to show that $\forall\ A \in K^*(A)$ $A \xrightarrow{1_A} A \to 0\ A[1]$ is distinguished.

Look at $0 \to A[-1]$. Then know $0 \to A[-1] \xrightarrow{1_{A[-1]}} A[-1] \to 0$ is distinguished because $\underset{\substack{\uparrow \\ \text{cone}}}{C}(0 \to A[-1]) = A[-1]$.

Assuming the rotation axiom TR2 we get that

$A \xrightarrow{1_A} A \to 0 \to A[1]$ is distinguished.

So we need to prove "one half" of TR2. (Know from last time

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that the other half of TR2 follows from the axioms)

means: TR2: $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is in Δ

$\Downarrow \quad \Uparrow$

$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1]$ in Δ

"$\Uparrow$" follows from the axioms where the older TR2 is replaced by the newer TR2.

<u>Preparation for proving TR2:</u>

Start with $A \xrightarrow{f} B$. Have $0 \to B \xrightarrow{j} C(f) \xrightarrow{s} A[1] \to 0$.

Look at $C(j) = B[1] \oplus C(f)$.

$C(j) = B[1] \oplus A[1] \oplus B$.

$$d^{C(j)} = \begin{bmatrix} -d^B & 0 \\ j & d^{C(f)} \end{bmatrix} = \begin{bmatrix} -d^B & 0 & 0 \\ 0 & -d^A & 0 \\ 1 & f & d^B \end{bmatrix}$$

<u>Prop:</u> Consider $f: A \to B$ and $0 \to B \xrightarrow{j} C(f) \xrightarrow{s} A[1] \to 0$ in $\mathrm{Com}^*(\mathcal{A})$

Then, $\exists$ $\varphi: A[1] \to C(j)$ chain map s.t.

(a) φ is an isom. when viewed in $K^*(\mathcal{A})$

(b) The diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, equal] & C(f) \arrow[r, "\pi"] \arrow[d, equal] & A[1] \arrow[r, "-f{[1]}"] \arrow[d, "\varphi"'] & B[1] \arrow[d, equal] \\ B \arrow[r, "j"] & C(f) \arrow[r, "\begin{bmatrix} 0 \\ 1_{C(f)} \end{bmatrix}"'] & C(j) \arrow[r, "\begin{bmatrix} 1_{B[1]} & 0 \end{bmatrix}"'] & B[1] \end{tikzcd}$$

"canonical maps"

Commuting in $K^*(\mathcal{A})$.

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$\mathcal{A}$ ab. cat.

Let $\mathrm{Com}(\mathcal{A}^*)$ where $\mathrm{Com}(\mathcal{A}^*)$ → all compl. over $\mathcal{A}$

→ all complexes bounded from above → $\mathrm{Com}^{+}(\mathcal{A})$

below → $\mathrm{Com}^{-}(\mathcal{A})$

→ all <u>bounded</u> compl. → $\mathrm{Com}^{b}(\mathcal{A})$.

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Consider the corresponding homotopy category $K^*(\mathcal{A})$.

We say that the diagram $A \xrightarrow{f} B \xrightarrow{\cancel{g}} C \xrightarrow{h} A[1]$ is a <u>distinguished triangle</u> in $K^*(\mathcal{A})$, if it is isomorphic (as a triangle) in $K^*(\mathcal{A})$ to a diagram $A \xrightarrow{f} B \underset{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\xrightarrow{j}} C(f) \underset{[1\ 0]}{\longrightarrow} A[1]$

where $C(f)$ = cone of f $\boxed{C(f) = A[1] \oplus B}$

<u>Thm:</u> $K^*(\mathcal{A})$ is a triangulated category with shift functor $[\ ]$.

<u>Pf:</u> We still need to prove one half of the rotation axiom.

Start with $A \xrightarrow{f} B \mapsto 0 \to B \xrightarrow{j} C(f) \xrightarrow{\delta} A[1] \to 0$
$\overset{\shortparallel}{[1\ 0]}$

Want $C(j) = B[1] \oplus C(f) = B[1] \oplus A[1] \oplus B$

$$d^{(j)} = \begin{bmatrix} -d^B & 0 & 0 \\ 0 & -d^A & 0 \\ 1 & f & d^B \end{bmatrix}$$

<u>Prop:</u> With the above notation, $\exists\ \varphi: A[1] \to C(j)$ in $Com^*(\mathcal{A})$ s.t.

(a) φ is an isom. in $K^*(\mathcal{A})$

(b) The diagram

$$\begin{tikzcd}
B \arrow[r, "j"] \arrow[d, equal] & C(f) \arrow[r, "\delta"] \arrow[d, equal] & A[1] \arrow[r, "-f[1]"] \arrow[d, "\varphi"] & B[1] \arrow[d, equal] \\
B \arrow[r, "j"] & C(f) \arrow[r, "\begin{bmatrix} 0 & 0 \\ 1_A & 0 \\ 0 & 1_B \end{bmatrix}"'] & C(j) \arrow[r, "{[1\ 0\ 0]}"'] & B[1]
\end{tikzcd}$$

is distinguished triangle in $K^*(\mathcal{A})$ (by def)

commutes in $K^*(\mathcal{A})$.

If we prove the proposition, the rotation axiom will follow since the top triangle is isomorphic to a distinguished Δ, so it is distinguished (top Δ is rotation of $A \xrightarrow{f} B \xrightarrow{j} C(f) \xrightarrow{\delta} A[1]$ which is distinguished)

(so TR1, TR2 will be proved then)

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Define $\varphi : A[1] \to C(j)$

$$\varphi = \begin{bmatrix} -f[1] \\ 1_{A[1]} \\ 0 \end{bmatrix}$$

Cl: φ is a chain map. (exercise)

Define $\psi : C(j) \to A[1]$ $\psi = [0 \ 1_A \ 0]$

Cl: ψ is a chain map.

$$\psi\varphi = [0 \ 1 \ 0]\begin{bmatrix} -f \\ 1 \\ 0 \end{bmatrix} = 1_{A[1]}$$

$$\varphi\psi = \begin{bmatrix} -f \\ 1 \\ 0 \end{bmatrix}[0 \ 1 \ 0] = \begin{bmatrix} 0 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\varphi\psi : C(j) \to C(j)$ Want to show $\varphi\psi \sim 1_{C(j)}$

$$\begin{tikzcd}
 & B_{i+1}\oplus A_{i+1}\oplus B_i & B_{i+2}\oplus A_{i+2}\oplus B_{i+1} \\[-2em]
 & C(j)_i \arrow[r] \arrow[dl, "s"'] \arrow[d, "1_{C(j)}-\varphi\psi"] & C(j)_{i+1} \arrow[dl, "s"] \\
C(j)_{i-1} \arrow[r, "d^{C(j)}"] & C(j)_i &
\end{tikzcd}$$

Want s s.t. $sd+ds = 1_{C(j)} - \varphi\psi$

$$= \begin{bmatrix} 1 & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $s = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $sd+ds = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -d^B & 0 & 0 \\ 0 & -d^A & 0 \\ 1 & f & d^B \end{bmatrix}$

$$+ \begin{bmatrix} -d^B & 0 & 0 \\ 0 & -d^A & 0 \\ 1 & f & d^B \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ (a).

(b) To check commutativity of the diagram.

Only 2 squares need checking:

$$\begin{tikzcd}
C(f) \arrow[r, "{[1 \ 0]}"] \arrow[d, equal] & A[1] \arrow[r, "-f[1]"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowright"] & B[1] \arrow[d, equal] \\
C(f) \arrow[r, "\begin{bmatrix} 0 & 0 \\ 1_A & 0 \\ 0 & 1_B \end{bmatrix}"'] & C(j) \arrow[r, "{[1 \ 0]}"'] & B[1]
\end{tikzcd}$$

$\uparrow$ commutes in $Com^*(\mathcal{A})$

where $\varphi = \begin{bmatrix} -f \\ 1_A \\ 0 \end{bmatrix}$

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Show $\varphi[1\ 0] \sim \begin{bmatrix} 0 & 0 \\ 1_A & 0 \\ 0 & 1_B \end{bmatrix} : C(f) \to C(j)$

Have picture

$$\begin{tikzcd}
C(f)_{k-1} \arrow[r] & C(f)_k \arrow[r, "d^{C(f)}"] \arrow[dl, "s"'] \arrow[d, shift right, "{\varphi[1\,0]}"'] \arrow[d, shift left, "{\begin{bmatrix} 0 & 0 \\ -1_A & 0 \\ 0 & -1_B \end{bmatrix}}"] & C(f)_{k+1} \arrow[dl, "s"] \\
C(j)_{k-1} \arrow[r, "d^{C(j)}"'] & C(j)_k &
\end{tikzcd}$$

Let $s = \begin{bmatrix} 0 & 1_B \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $ds + sd = \varphi[1\ 0] - \begin{bmatrix} 0 & 0 \\ 1_A & 0 \\ 0 & 1_B \end{bmatrix}$.

<u>exercise.</u>

<u>TR3:</u> Assume we have

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[d, phantom, shift left=6, "\circlearrowleft"] & B \arrow[r, "j"] \arrow[d, "g"] & C(u) \arrow[r, "\delta"] \arrow[d, red, "\exists h"] \arrow[d, phantom, shift right=4, red, "?"] \arrow[d, phantom, shift left=6, red, "?"] & A[1] \arrow[r] \arrow[d, "{f[1]}"] & \text{dist. in } K^*(A) \\
A' \arrow[r, "u'"'] & B' \arrow[r, "j'"'] & C(u') \arrow[r, "\delta'"'] & A'[1] \arrow[r] & \text{dist. in } K^*(A)
\end{tikzcd}$$

commutes in $K^*(A)$

~~dist. in $K^*(A)$~~

To show $\exists h : C(u) \to C(u')$ commuting everything in $K^*(A)$

We know: $gu \sim u'f : A \to B'$.

$$\begin{tikzcd}
& A_k \arrow[r, "d^A"] \arrow[dl, "s"'] \arrow[d, "gu-u'f"] & A_{k+1} \arrow[dl, "s"] \\
B'_{k-1} \arrow[r, "d^{B'}"'] & B'_k &
\end{tikzcd}$$

So $\exists s$ with $sd^A + d^{B'}s = gu - u'f$

Let h: $C(u) \to C(u')$ $\qquad h = \begin{bmatrix} f & 0 \\ s & g \end{bmatrix}$

$$\begin{matrix} C(u) & \to & C(u') \\ \shortparallel & & \shortparallel \\ A[1] \oplus B & & A[1] \oplus B' \end{matrix}$$

<u>exercise:</u> h is a chain map in $Com^*(A)$.

<u>Claim:</u> the 2 squares commute in $Com^*(A)$.

$j'g = \begin{bmatrix} 0 \\ 1 \end{bmatrix} g = \begin{bmatrix} 0 \\ g \end{bmatrix}$, $hj = \begin{bmatrix} f & 0 \\ s & g \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$ $\Rightarrow j'g = hj$.

$\delta' h = [1\ 0] \begin{bmatrix} f & 0 \\ s & g \end{bmatrix} = [f\ 0]$, $f[1]\delta = f[1\ 0] = [f\ 0] \Rightarrow \delta' h = f[1]\delta$.

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So (f,g,h) is a homom of distinguished triangles in $K^*(\mathcal{A})$.

TR4: Start with $A \xrightarrow{u} B \xrightarrow{v} C$.

Draw:

$$\begin{tikzcd}
C(vu) \arrow[r, "k'{[-1]}"] \arrow[d, "g{[-1]}"'] & A \arrow[r, equal] \arrow[d, "u"] \arrow[rd, phantom, "\hookrightarrow"] & A \arrow[d, "vu"] & & \\
C(v)[-1] \arrow[r, "j'{[-1]}"] & B \arrow[r, "v"] \arrow[d, "i"] \arrow[rd, phantom, "\circlearrowright"] & C \arrow[r, "j"] \arrow[d, "k"] \arrow[rd, phantom, "\circlearrowright"] & C(v) \arrow[r, "j'"] \arrow[d, equal] \arrow[rd, phantom, "\hookrightarrow"] & B[1] \arrow[d, "i{[1]}"] \\
& C(u) \arrow[r, "\exists f"] \arrow[d, "i'"] \arrow[rd, phantom, "\circlearrowright"] & C(vu) \arrow[r, "\exists g"] \arrow[d, "k'"] & C(v) \arrow[r, "i{[1]}j'"] & C(u)[1] \\
& A[1] \arrow[r, equal] & A[1] & &
\end{tikzcd}$$

To show $\longrightarrow$ $\exists f, g$ commuting the corresp. 4 squares in $K^*(\mathcal{A})$

$\searrow$ the third row is a distinguished triangle in $K^*(\mathcal{A})$.

Def. of f: $f: C(u) \longrightarrow C(vu)$ $\quad f = \begin{bmatrix} 1_{A[1]} & 0 \\ 0 & v \end{bmatrix}$

$A[1]\oplus B \qquad A[1]\oplus C$

Cl: f chain map. exercise.

Def. of g: $g: C(vu) \rightarrow C(v)$

$A[1]\oplus C \qquad B[1]\oplus C$ $\quad g = \begin{pmatrix} u[1] & 0 \\ 0 & 1 \end{pmatrix}$

exercise: g is a chain map.

Cl: all the new squares commute in $\overset{\text{Com}}{K^*(\mathcal{A})}$

$$gk = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1_C \end{bmatrix} = \begin{bmatrix} 0 \\ 1_C \end{bmatrix} = j$$

$$fi = \begin{bmatrix} 1_A & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 \\ 1_B \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}, \quad kv = \begin{bmatrix} 0 \\ 1_C \end{bmatrix} v = \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

$$k'f = [1 \ \ 0]\begin{bmatrix} 1_A & 0 \\ 0 & v \end{bmatrix} = [1 \ \ 0],$$

Exer: Check remaining one.

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Remains to show that the 3rd row is distinguished in $K^*(\mathcal{A})$.

In particular, we need to show that $C(v) \simeq C(f)$ in $K^*(\mathcal{A})$.

Let $\varphi : \underset{\overset{\|}{B[1] \oplus C}}{C(v)} \to \underset{\overset{\|}{A[2] \oplus B[1] \oplus A[1] \oplus C}}{C(f)}$

and $\psi : C(f) \to C(v)$ be

$$\varphi = \begin{bmatrix} 0 & 0 \\ 1_B & 0 \\ 0 & 0 \\ 0 & 1_C \end{bmatrix} \qquad \psi = \begin{bmatrix} 0 & 1_B & u & 0 \\ 0 & 0 & 0 & 1_C \end{bmatrix}$$

get $\psi\varphi = \begin{bmatrix} 1_B & 0 \\ 0 & 1_C \end{bmatrix} = 1_{C(v)}$

$$\varphi\psi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_B & u & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_C \end{bmatrix} : C(f) \to C(f)$$

To show $\varphi\psi \sim 1_{C(f)}$.

Also have:

$$\begin{tikzcd}
C(u) \arrow[r, "f"] \arrow[d, equal] & C(vu) \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\ast"] & C(v) \arrow[r, "{i[1] \circ j'}"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\hookrightarrow"] & C(u)[1] \arrow[d, equal] \\
C(u) \arrow[r, "f"] & C(vu) \arrow[r, swap, "{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1_A & 0 \\ 0 & 1_C \end{bmatrix}}"] & C(f) \arrow[r, swap, "{\begin{bmatrix} 1_A & 0 & 0 & 0 \\ 0 & 1_B & 0 & 0 \end{bmatrix}}"] & C(u)[1]
\end{tikzcd}$$

commutes in $Com^*(\mathcal{A})$

$\ast$ Commutes in $K^*(\mathcal{A})$.

To show this, $\varphi g = \begin{bmatrix} 0 & 0 \\ u & 0 \\ 0 & 0 \\ 0 & 1_C \end{bmatrix}$. To show $\varphi g \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1_A & 0 \\ 0 & 1_C \end{bmatrix} : C(vu) \to C(f)$

$$\begin{tikzcd}
 & C(vu)_n \arrow[r, "d^{C(vu)}"] \arrow[dl, swap, "s"] \arrow[d, shift left] \arrow[d, shift right] & C(vu)_{n+1} \arrow[dl, "s"] \\
C(f)_{n-1} \arrow[r, "d^{C(f)}"] & C(f)_n &
\end{tikzcd}$$

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$$d^{C(f)} = \begin{bmatrix} d^{A[2]} & 0 & 0 & 0 \\ -u & d^{B[1]} & 0 & 0 \\ 1_A & 0 & -d^A & 0 \\ 0 & v & vu & d^C \end{bmatrix}$$

$d^{A[2]} = "d^A"$

$d^{B[1]} = -d^B$

Let $s = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Cl.: This will work.

So, if we show $\varphi\psi \sim 1_{C(f)}$, then the bottom triangle is isomorphic to the top one, so the top triangle is also distinguished.

$$\begin{tikzcd} & C(f)_k \arrow[r, "d^{C(f)}"] \arrow[dl, "s"'] \arrow[d, shift right, "\varphi\psi"'] \arrow[d, shift left, "1_{C(f)}"] & C(f)_{k+1} \arrow[dl, "s"] \\ C(f) \arrow[r, "d^{C(f)}"'] & C(f)_k & \end{tikzcd}$$

Take $s = \begin{bmatrix} 0 & 0 & 1_A & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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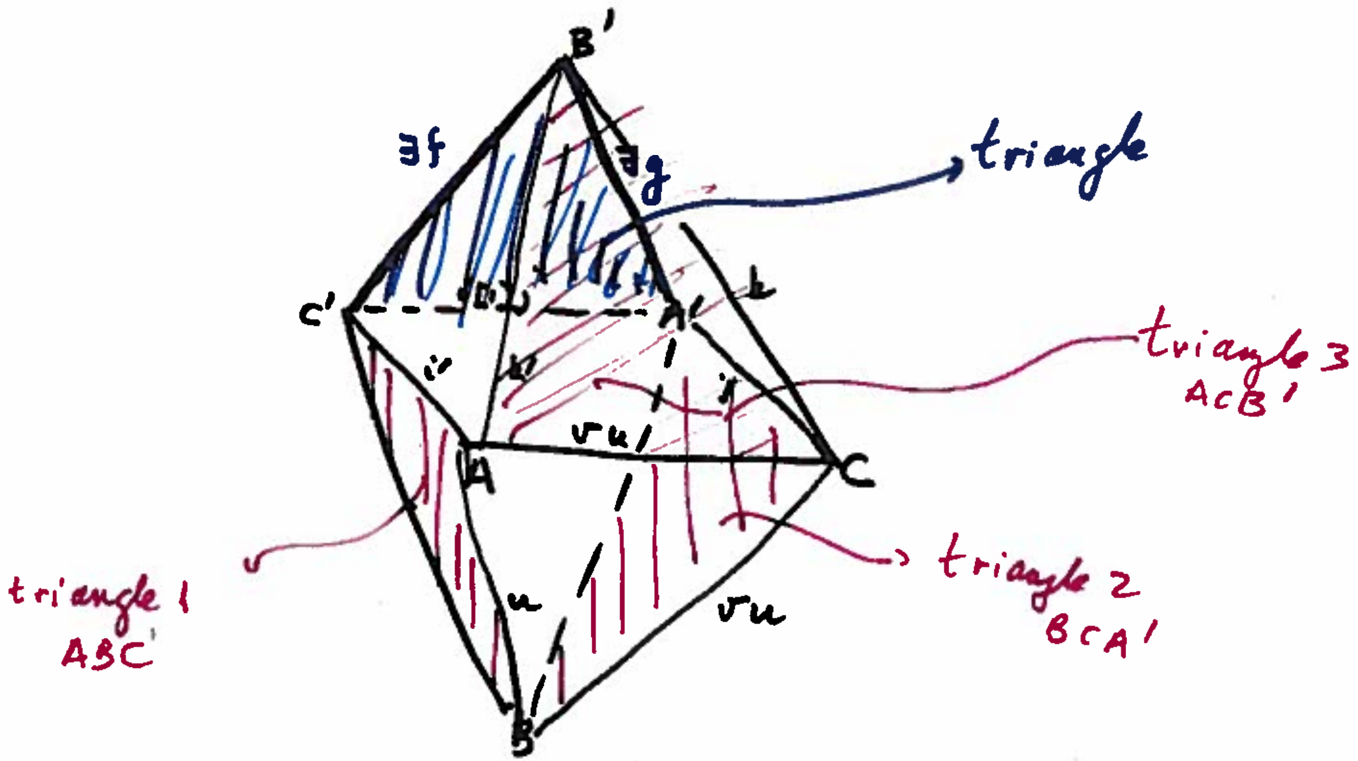

Have the 3 triangles in shaded red color.

Remark: existence of f => ∃ morphism $(1_A, v, f)$ from triangle 1 to triangle 3

existence of g => ∃ morphism $(u, 1_C, g)$ from triangle 3 to triangle 2.

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Odds and ends

Prop: Let $\mathcal{T}$ be a triangulated category. Let $u: A \to B$ be a monomorphism (or an epimorphism). Then u is a splittable mono (or splittable epi, resp.)

Pf: Deal with the mono case (the epi case is similar). Know $\exists$ distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$

Rotation $\Rightarrow$ $C[-1] \xrightarrow{-w[-1]} A \xrightarrow{u} B \xrightarrow{v} C$ is a distinguished triangle.

$\Rightarrow u \circ (-w[-1]) = 0 \Rightarrow u \circ w[-1] = 0 \underset{u \text{ mono}}{\Longrightarrow} w[-1] = 0 \Rightarrow w = 0.$

This means that u is a split mono. □

Corollary: If an abelian cat. is triangulated $\Rightarrow$ $\mathcal{T}$ is semisimple, that is every short exact seq in $\mathcal{T}$ splits.

Pf: Let $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ be a short exact seq. in $\mathcal{T}$. u mono $\Rightarrow u$ splits. So the seq. splits. □

Recall that if $\mathcal{T}$ triangulated, then a covariant functor $F:$ ~~$\mathcal{T} \to$~~ $\mathcal{T} \to \mathcal{A}$ abelian is homological if $\forall$ distinguished triangles

$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ we have an induced long exact sequence

$$\cdots \to F(A) \to F(B) \to F(C) \to F(A[1]) \to F(B[1]) \to \cdots$$

Obs: The rotation axiom tells us that $F: \mathcal{T} \to \mathcal{A}$ is homological $\Leftrightarrow$ $\forall$ dist. triangles $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ the seq. $F(A) \to F(B) \to F(C)$ is exact.

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Let $\mathcal{A}$ be abelian. Look at $H_0: Com(\mathcal{A}) \to \mathcal{A}$ be the homology in degree 0 (but we may use H_i $\forall i \in \mathbb{Z}$).

Know that if $0 \to A \to B \to C \to 0$ exact in $Com(\mathcal{A}) \Rightarrow \exists$ long exact seq. in homology:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{-1}(C) \arrow[r] \arrow[d, phantom, "\parallel"] & H_0(A) \arrow[r] & H_0(B) \arrow[r] & H_0(C) \arrow[r] & H_1(A) \arrow[r] \arrow[d, phantom, "\parallel"] & \cdots \\
& H_0(C[-1]) & & & & H_0(A[1]) &
\end{tikzcd}$$

Note: may rewrite this seq. using only H_0.

Prop: Let $K(\mathcal{A})$ be the homotopy category. Then H_0 induces a homological functor $H_0: K(\mathcal{A}) \to \mathcal{A}$.

Pf: A distinguished triangle in $K(\mathcal{A})$ looks like $A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{\delta} A[1]$

Coming from the exact seq. $0 \to B \to C(f) \to A[1] \to 0$

Apply H_0 to this seq. yields

$$\begin{tikzcd}
\cdots \arrow[r] & H_{-1}(A[1]) \arrow[r] \arrow[d, phantom, "\parallel"] & H_0(B) \arrow[r] & H_0(C(f)) \arrow[r] & \cdots \\
& H_0(A) & & &
\end{tikzcd}$$

So get exact seq.: $H_0(A) \to H_0(B) \to H_0(C)$

$\Rightarrow H_0$ is a homological functor on $K(\mathcal{A})$.

HW:

① Let $A \xrightarrow{f} B$ in $Com(\mathcal{A})$. Then f is a quasi-isom $\Leftrightarrow C(f)$ is exact. !

Multiplicative systems (or localizing systems)

Let $\mathcal{C}$ be a category. Let S be a collection of morphisms in $\mathcal{C}$.

Def: S is a multiplicative system if it satisfies the following:

S1. $\forall X \in \mathcal{C}$ $1_X \in S$.

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S2: $\forall f, g$ and $g \circ f$ exist, then $g \circ f \in S$.

S3: Every diagram

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $g \in S$ can be completed to a commutative square with $h \in S$

[Read: $g^{-1}f = kh^{-1}$]

Also, every diagram

$$\begin{tikzcd} W & Z \arrow[l, "k"'] \\ X \arrow[u, "h"] & Y \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

with $g \in S$ can be completed to a commutative diagram with h in S.

[Read $fg^{-1} = h^{-1}k$.]

S4: If $f, g \in \mathrm{Hom}(X, Y)$ then TFAE:

(a) $\exists\, t \in S$, $t: Y \to Y'$ s.t. $tf = tg$

(b) $\exists\, s \in S$, $s: X' \to X$ s.t. $fs = gs$.

$$\begin{tikzcd} X' \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "t"] & Y \end{tikzcd}$$

Assumption: $\mathcal{C}$ category and S a multiplicative system in $\mathcal{C}$.

The localization of $\mathcal{C}$ at S is a category $\mathcal{C}[S^{-1}]$ or $\mathcal{C}_S$ defined as follows: $\boxed{\mathrm{Ob}\,\mathcal{C}_S = \mathrm{Ob}\,\mathcal{C}}$

For the morphisms we first need a def'n:

A roof in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

where $s \in S$.

Think of it as fs^{-1}.

Morphisms in $\mathcal{C}_S$: $\mathrm{Hom}_{\mathcal{C}_S}(A, B)$ = roofs

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd} \Big/ \sim$$

where $\sim$ is the equiv. relation below.

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd} \sim \begin{tikzcd} & C' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

$s, t \in S$ $\overset{\text{def}}{\iff}$ $\exists$ commut. diag. with $u \in S$ and $C'' \in \mathcal{C}$

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & C'' \arrow[l, "u"'] \arrow[u] \arrow[d] & B \\ & C' \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

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Exercise: $\sim$ is an equiv. relation. tedious ! (HW3.1)

Remark: Composition of morphisms in $\mathcal{C}_S$:

Will use the notation (C, s, f) for the roof
$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

Consider the roofs
$$\begin{tikzcd} & & \exists Z \arrow[dl, "u"'] \arrow[dr, "k"] \arrow[d, phantom, "\circlearrowright"] & & \\ & X \arrow[dl, "s"'] \arrow[dr, "f"] & {} & Y \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A & & B & & C \end{tikzcd}$$
$\longrightarrow$ by $\underline{S3}$, and $u \in S$.
$su \in S$ by $S2$.

Compose $(Y, t, g) \circ (X, s, f)$, $s, t \in S$

So the composition will be the equiv class of
$$\begin{tikzcd} & Z \arrow[dl, "su"'] \arrow[dr, "gk"] & \\ A & & B \end{tikzcd}.$$

Remark: Think of it also as $g t^{-1} f s^{-1}$ rewritten as the fraction

$gk(su)^{-1}$ or $gku^{-1}s^{-1}$.

Q: Is this true?
$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "s"] & \\ A & & A \end{tikzcd} \sim \begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "1_A"] & \\ A & & A \end{tikzcd}$$
$s \in S$

exercise

answer: yes

Thm: $\mathcal{C}_S$ is a category. ∎

Define now a **functor** $Q: \mathcal{C} \to \mathcal{C}_S$ as follows

$Q(A) = A$ for every $A \in \mathrm{Ob}\,\mathcal{C}$.

To define Q on morphisms. let
$$Q(A \xrightarrow{f} B) = \begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd} \Big/ \sim$$

(i.e. the equiv. class of the above ro

Prop: Q is a functor.

Sketch of Pf: Assume have $A \xrightarrow{f} B \xrightarrow{g} C$. To show

$Q(gf) = Q(g)\,Q(f)$

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$$\begin{tikzcd}
& & A \arrow[dl, equal] \arrow[dr, "f"] & & \\
& A \arrow[dl, equal] \arrow[dr, "f"] & \circlearrowleft & B \arrow[dl, equal] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd} = Q(g)\, Q(f)$$

get

$$\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, "gf"] & \\
A & & C
\end{tikzcd}$$

which is $Q(gf)$

Lemma: Let $s \in S$. Then $Q(s)$ is invertible in $\mathcal{C}_S$.

Pf: Let $s: A \to B$ in S.

$$Q(s) \quad \begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, "s"] & \\
A & & B
\end{tikzcd}$$

Cl.: $Q(s)^{-1} = \begin{tikzcd} B & A \arrow[l, "s"'] \arrow[r, equal] & A \end{tikzcd}$

~~What is~~ Check: $Q(s)^{-1} Q(s) = ?$

$$\begin{tikzcd}
& & A \arrow[dl, equal] \arrow[dr, equal] & & \\
& A \arrow[dl, equal] \arrow[dr, "s"] & \circlearrowleft & A \arrow[dl, "s"'] \arrow[dr, equal] & \\
A & & B & & A
\end{tikzcd}$$

get

$$\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, equal] & \\
A & & A
\end{tikzcd} = 1_{Q(A)}$$

Check $Q(s)\, Q(s)^{-1}$

$$\begin{tikzcd}
& & A \arrow[dl, equal] \arrow[dr, equal] & & \\
& A \arrow[dl, "s"'] \arrow[dr, equal] & & A \arrow[dl, equal] \arrow[dr, "s"] & \\
B & & A & & B
\end{tikzcd}$$

get

$$\begin{tikzcd}
& A \arrow[dl, "s"'] \arrow[dr, "s"] & \\
B & & B
\end{tikzcd} \underset{\text{yes}}{\overset{\sim}{?}} \begin{tikzcd}
& B \arrow[dl, equal] \arrow[dr, equal] & \\
B & & B
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dl, "s"'] \arrow[dr, "s"] & \\
B \arrow[dr, equal] & A \arrow[u, "1_A"'] \arrow[l, "s"'] \arrow[d, "s"] \arrow[ul, phantom, "\circlearrowleft"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[r, phantom, "?"] & B \arrow[dl, equal] \\
& B &
\end{tikzcd}$$

Thm: (Universality of the localization)

Let $\mathcal{C}$ be a cat. with localizing system S, and let $\mathcal{C}_S$ be its localization at S.

Let $Q: \mathcal{C} \to \mathcal{C}_S$ be as above. Let $\mathcal{C}'$ be another cat., and let

$F: \mathcal{C} \to \mathcal{C}'$ be a functor s.t. $F(s)$ is invertible in $\mathcal{C}'$ for every $s \in S$.

Then $\exists!$ $G: \mathcal{C}_S \to \mathcal{C}'$ s.t. $GQ = F$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{C}_S \arrow[d, dashed, "\exists!\, G"] \arrow[dl, phantom, "?"] \\
& \mathcal{C}'
\end{tikzcd}$$

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Pf: existence ~~Uniqueness~~: Since $\mathrm{Ob}\ \mathscr{C}_S = \mathrm{Ob}\ \mathscr{C}$, we define $Q(A) \overset{\text{def}}{=} F(A)$.

$\forall A \in \mathrm{Ob}\ \mathscr{C}_S = \mathrm{Ob}\ \mathscr{C}$, so $GQ = F$ on objects.

Need to show they agree on morphisms.

Let
$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd} : A \to B \text{ in } \mathscr{C}_S.$$

Claim: $(C, s, f) \circ Q(s) = Q(f)$. $(*)$

Pf of claim:

$$\begin{tikzcd} & & C \arrow[dl, equal] \arrow[dr, equal] & & \\ & C \arrow[dl, equal] \arrow[dr, "s"] & & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ C & & A & & B \\ & Q(s) & & (C,s,f) & \end{tikzcd}$$

Composition is
$$\begin{tikzcd} & C \arrow[dl, equal] \arrow[dr, "f"] & \\ C & & B \end{tikzcd} = Q(f)$$

(don't yet know what G is, but if it existed)

If we apply G to $(*)$, get

$G(C,s,f) \circ F(s) = F(f)$
$GQ = F$

But $F(s)$ is inv. in $\mathscr{C}'$ $\Rightarrow$ $\boxed{G(C,s,f) = F(f) \circ F(s)^{-1}}$

we would get this formula

So we simply define $G(C,s,f) \overset{\text{def}}{=} F(f) \circ F(s)^{-1}$.

With these definitions $G : \mathscr{C}_S \to \mathscr{C}$ is unique with the prop. that $GQ = F$. $\square$

4/18/2018

See Tim's notes

Localizing triangulated categories

4/23/2018

Set up: $\mathcal{T}$ triangulated. S localizing (multiplicative) system in $\mathcal{T}$.

Construct $\mathcal{T}_S = \mathcal{T}[S^{-1}]$.

Assume also ① $s \in S \Leftrightarrow s[1] \in S$

② Assume we have
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] & \text{in } \Delta \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] & \text{in } \Delta \end{tikzcd}$$

Then $f, g \in S \Rightarrow h \in S$

4/28/18

Last time: $\mathcal{C}$ is a category, S is a localizing ~~category~~ system.

Constructed $\mathcal{C}_S$ and a functor $Q : \mathcal{C} \to \mathcal{C}_S$ s.t. $Q(s)$ is an iso in $\mathcal{C}_S$ $\forall s \in S$. Also $\forall$ $\mathcal{C}'$ and F w/ $F(s)$ is iso in $\mathcal{C}'$

$$\begin{tikzcd}
\mathcal{C}\cancel{X} \arrow[r, "Q"] \arrow[dr, "G", "F"'] & \mathcal{C}_S \arrow[d, "\exists ! G"] \\
 & \mathcal{C}'
\end{tikzcd}$$

Specialize our discussion to the case where $\mathcal{C}$ is the homotopy category, $\mathcal{C} = K(A)$, and S = quasi isomorphisms.

<u>Thm</u>: The set S of quasi isomorphisms forms a localizing system in $K(A)$.

<u>Proof</u>. <u>S1</u>. Let A be a complex. We know 1_A is a quasi isomorphism. ✓

<u>S2</u>. Assume $f : A \to B$, $g : B \to C$ and both f, g are q.i.s. (quasi iso).

$\Rightarrow gf$ is also quasi iso (why?)

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S3. Assume have

$$\begin{tikzcd}
 & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

where f is a qis.

Need to complete to a comm square in $K(\mathcal{A})$,

$$\begin{tikzcd}
D \arrow[r, dashed, "u"] \arrow[d, dashed, "K"'] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

w/ u a qis.

Construct the following diagram: Use that $K(A)$ is triangulated.

$$\begin{tikzcd}
C(jg)[-1] \arrow[r, "h"] \arrow[d, dashed, "K"'] \arrow[dr, phantom, "\circlearrowleft \text{ in } K(A)"] & C \arrow[r, "jg"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & C(f) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & C(jg) \arrow[d, dashed] & \text{dist } \Delta \\
A \arrow[r, "f"'] & B \arrow[r, "j"'] & C(f) \arrow[r, "\delta"'] & A[1] & \text{dist } \Delta
\end{tikzcd}$$

$\nearrow$ in $K(A)$ $\searrow$

So, let $D = C(jg)[-1]$. Remains to show, h is a qis.

f is a quasi iso $\Rightarrow C(f)$ is exact. Look at top dist Δ

$$C(jg)[-1] \xrightarrow{h} C \xrightarrow{jg} C(f) \longrightarrow C(jg)$$

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Applying $H_0(-)$ to this triangle yields a long exact sequence where $H_0(C(f)_i) = 0$ $\forall i$, so this implies $\forall i$ $H_i(h) : C(jg)[-1]_i \longrightarrow H_i(C)$ is an isomorphism.

$\Rightarrow h$ is a qis.

<u>2nd Part of S3</u>. Start w/

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

w/ $f \in S \Leftrightarrow f$ is qis.

Then,

$$\begin{tikzcd}
C(f)[-1] \arrow[r, "-\delta[-1]"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\underset{\text{in } K(A)}{\circlearrowright}"] & B \arrow[r, "j"] \arrow[d, dashed, "\exists K"] & C(f) \arrow[r, "\delta"] \arrow[d, equal] & A[1] \\
C(f)[-1] \arrow[r] & C \arrow[r, "u"'] & C(-g\delta[-1]) \arrow[r] & C(f) &
\end{tikzcd}$$

Have to u is a qis again.

f is qis $\Rightarrow C(f)$ is exact $\Rightarrow u$ is qis by same method as above.

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<u>S4</u>. Since $K(\mathcal{A})$ is additive, we can replace S4 by the following:

If $f : A \to B$, the TFAE:

$$\exists \underset{\nearrow}{t} : B \longrightarrow B' \text{ s.t. } tf = 0 \text{ in } K(\mathcal{A})$$

a qis

$$\Updownarrow$$

$$\exists s : A' \longrightarrow A \text{ s.t. } fs = 0 \text{ in } K(\mathcal{A})$$

qis

<u>Proof of one direction</u>

Since $tf = 0$ in $K(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, dashed, "\exists h"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d, "0"] & A[1] \arrow[d, dotted] & \text{dist } \Delta \\
C(t)[-1] \arrow[r, "-\delta[-1]"'] & B \arrow[r, "t"'] & B' \arrow[r, "j"] & C(t) \arrow[r, "\delta"] & \text{dist } \Delta
\end{tikzcd}$$

Thus, $\boxed{f = -\delta[-1]h}$ in $K(\mathcal{A})$. Look at the following dist Δ

$$C(h)[-1] \xrightarrow{s} A \xrightarrow{h} C(t)[-1] \longrightarrow C(h)$$

t is a qis $\Rightarrow$ $C(t)$ is exact $\Rightarrow$ s is a qis

by same method as before

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Need to show $fs = 0$

But $hs = 0$ in $K(\mathcal{A})$ b/c consecutive maps in a dist Δ compose to 0. Then, $hs = 0$

$$\Rightarrow fs = -\delta[-1]hs = 0 .$$

Good exercise to prove bottom implies top. $\square$

~~Let τ be triangulated, τ_S its localization at S.~~

~~Def: A triangle $A_1 \to A_2 \to A_3 \to A_1[1]$ in τ_S is distinguished~~

We specialize to $\mathcal{C} = K^*(\mathcal{A})$ and S is the set of quasi isomorphisms. Define $D^*(\mathcal{A}) := \mathcal{C}_S$.

$D^*(\mathcal{A})$ is called the Derived Category. Its objects are the same as $K^*(\mathcal{A})$ so they are complexes. We define the shift in $D^*(\mathcal{A})$ to be the same as in $K^*(\mathcal{A})$ and $Com^*(\mathcal{A})$. It is again an equivalence from $D^*(\mathcal{A})$ to itself. $K^*(A)$ is triangulated, we would like $D^*(\mathcal{A})$ to be triangulated as well.

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Def: A triangle $A_1 \to A_2 \to A_3 \to A_1[1]$ in $D^*(\mathcal{A})$ is distinguished if it is isomorphic in $D^*(\mathcal{A})$ to the image under the functor $Q : K^*(\mathcal{A}) \to D^*(\mathcal{A})$ of a distinguished triangle in $K^*(\mathcal{A})$

$$A \xrightarrow{f} B \xrightarrow{j} C(f) \xrightarrow{\delta} A[1] \in K^*(\mathcal{A})$$

Apply Q :

$$A \xrightarrow{Q(f)} B \xrightarrow{Q(j)} C(f) \xrightarrow{Q(\delta)} A[1] \in D^*(\mathcal{A})$$

The difficulty is that the morphisms in the localization are not honest morphisms, they are roofs.

With this definition,

$$Q \circ [\ ]_{\text{in } K^*(\mathcal{A})} = [\ \ ]_{\text{in } D^*(\mathcal{A})} \circ Q$$

So, if we show $D^*(\mathcal{A})$ is triangulated, Q will be an "exact functor of triangulated categories"

takes Δ's $\to$ Δ's.

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Thm. With the above def of distinguished triangles, $D^*(\mathcal{A})$ is a triangulated category and the natural functor

$Q: K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ is an exact functor of triangulated categories.

Remark. The homology functors $H_i : K^*(\mathcal{A}) \to \mathcal{A}$ take quasi isomorphisms to isomorphisms of $\mathcal{A}$.

$\forall i$

$$\begin{tikzcd}
K^*(\mathcal{A}) \arrow[dr, "Q"] \arrow[dd, "H_i"'] \arrow[ddr, phantom, "\circlearrowleft"] & \\
 & D^*(\mathcal{A}) = K^*(\mathcal{A})_S \arrow[dl, dashed, "\exists!\, G_i"] \\
\mathcal{A} &
\end{tikzcd}$$

So, the H_i factor uniquely through the derived category.

Remark. We want to show $D^*(\mathcal{A})$ is additive also.

So, we want to show that we can add roofs.

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So,

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd} \quad + \quad \begin{tikzcd} & C' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ A & & B \end{tikzcd}$$

$\varphi : A \to B$ in $D^*(A)$ $\qquad$ $\varphi' : A \to B$ in $D^*(A)$

[want $fs^{-1} + f'(s')^{-1}$

these are like your denominators } get a "common denominator"]

I.e. if we had

$$\begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad + \quad \begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "g'"] & \\ A & & B \end{tikzcd} \quad = \quad \begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "g+g'"] & \\ A & & B \end{tikzcd}$$

$$\varphi \quad + \quad \varphi' \quad := \quad \varphi + \varphi'$$

Have

$$\begin{tikzcd} C & & C' \\ & A \arrow[ul, "s"] \arrow[ur, "s'"'] & \end{tikzcd} \Longrightarrow \exists D \text{ s.t. } \begin{tikzcd} & D \arrow[dl, "u"'] \arrow[dr, "v"] & \\ C & \circlearrowright & C' \\ & A \arrow[ul, "s"] \arrow[ur, "s'"'] & \end{tikzcd}$$

and u is a qis

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So far, $su = s'v$ w/ s, s', u are qis.

Claim, this implies v is qis (exercise).
↑
think about previous remark.

Let $t = su = s'v$. Have

$$\begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "f'v"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "fu"] & \\ A & & B \end{tikzcd}.$$

Claim: $\varphi \sim \begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "fu"] & \\ A & & B \end{tikzcd}$ and $\varphi' \sim \begin{tikzcd} & D \arrow[dl, "t"'] \arrow[dr, "f'v"] & \\ A & & B \end{tikzcd}$.

If we prove this, we are done. Show it for φ:

$$\begin{tikzcd} & D \arrow[dl, "su = t"'] \arrow[d, equal] \arrow[dr, "fu"] & \\ A & D \arrow[l, "t"'] \arrow[d, "u"] & B \\ & C \arrow[ul, "s"] \arrow[ur, "f"'] & \end{tikzcd}$$

and $t = su$ is a qis

So we are done.

<u>Proposition</u>. $D^*(A)$ is an additive category.

Still to show, things like $+$ above is well defined ...

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Localizations of triangulated categories.

Are they also triangulated?

$\mathcal{T}$ is triangulated, S is a localizing system.

$\mathcal{T}_S = \mathcal{T}[S^{-1}]$. Assume also the following:

(i) $\forall\ s \in S$, $s[1]$ and $s[-1] \in S$

(ii) Assume we have a comm diagram of comm triangles in $\mathcal{T}$, i.e. a morphism of triangles f, g, h

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

Then, if $f, g \in S$, then $h \in S$ also.

Remark. In the case where $\mathcal{T} = K^*(\mathcal{A})$ and S is the quasi isomorphisms, then these two conditions are always satisfied

[to show (b) apply H_0 to the diagram and use 5-lemma to the associated LES diagram.]

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Thm: Assume (1), (2) are satisfied. Define $[\ ]_S$ in $\mathcal{T}_S$ to be $[\ ]$ on objects. A distinguished triangle in $\mathcal{T}_S$ is defined to be the image under Q of a distinguished triangle in $\mathcal{T}$. Then $\mathcal{T}_S$ is triangulated too.

Pf: Need to define $[\ ]_S$ on morphisms.

Let $f: A \to B$ in $\mathcal{T}_S$

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

f: with $s \in S$.

Define $f[1]_S = [f]_S$ to be the roof

$$\begin{tikzcd} & C[1] \arrow[dl, "s{[1]}"'] \arrow[dr, "g{[1]}"] & \\ A[1] & & B[1] \end{tikzcd}$$

exer: $[\ ]_S$ preserves equivalence classes of roofs.

TR1: Let $A \xrightarrow{f} B$ in $\mathcal{T}_S$. To show f is the base of a dist. triangle in $\mathcal{T}_S$.

$$f = \begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

$s \in S$. g is the base of a dist. triangle in $\mathcal{T}$.

Have the following:

$$\begin{tikzcd} C \arrow[r, "Q(g)"] \arrow[d, "Q(s)"'] & B \arrow[r, "Q(v)"] \arrow[d, equal, "1_B"] & D \arrow[r, "Q(u)"] \arrow[d, equal, "1_D"] \arrow[dr, phantom, "?"] & C[1] \arrow[d, "Q(s)[1]"] \\ A \arrow[r, "f"] & B \arrow[r] & D \arrow[r] & C[1] \end{tikzcd}$$

dist. in $\mathcal{T}$

Commute in $\mathcal{T}_S$

To show left most square commutes in $\mathcal{T}_S$.

$$\begin{tikzcd}[column sep=huge, row sep=huge] C \arrow[r, "\begin{tikzcd}[sep=tiny] & C \arrow[dl, equal] \arrow[dr, "g"] & \\ C & & B \end{tikzcd}"] \arrow[d, "\begin{tikzcd}[sep=tiny] & C \arrow[dl, equal] \arrow[dr, "g"] & \\ C & & A \end{tikzcd}"'] & B \arrow[d, equal, "\begin{tikzcd}[sep=tiny] & B \arrow[dl, equal] \arrow[dr, equal] & \\ B & & B \end{tikzcd}"] \\ A \arrow[r, "\begin{tikzcd}[sep=tiny] & C \arrow[dl, "s"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}"'] & B \end{tikzcd}$$

exer: show this commutes in $\mathcal{T}_S$.

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Then $(Q(s), 1_B, 1_B)$ is an isom of triangles, so the bottom is a dist. triangle in $\mathcal{T}_S$.

- triangles "isomorphic to dist. triangles" are dist. triangles by def
- $A \overset{1_A}{\to} A \to 0 \to A[1]$, $\quad 1_A \to 1_A = Q(1_A)$ is simply $Q(\underbrace{A \overset{1_A}{\to} A \to 0 \to A[1]}_{\text{dist. in } \mathcal{T}})$

TR2: The rotation axiom:

Look at $A \xrightarrow{Q(u)} B \xrightarrow{Q(v)} C \xrightarrow{Q(w)} A[1]_S$ $\Leftarrow$

(dist in $\mathcal{T}_S$ and the image under Q of $A \overset{u}{\to} B \overset{v}{\to} C \overset{w}{\to} A[1]$)

which is dist in $\mathcal{T}$.

Then we claim that $B \xrightarrow{Q(v)} C \xrightarrow{Q(w)} A[1]_S \xrightarrow{-Q(u)[1]_S} B[1]_S$ is dist. in $\mathcal{T}_S$

(exer)

TR3, TR4: Gelfand–Manin p. 252–...

$\blacksquare$

Corollary: If $\mathcal{A}$ is abelian, then $D^*(\mathcal{A})$ is triangulated.

Remarks, examples and exercises

① $f \overset{\text{def}}{=}$
$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "0"] & \\ A & & B \end{tikzcd}$$
: $A \to B$ in $D^*(\mathcal{A})$ is the zero morphism.

"
$$\begin{tikzcd} A & \cdot \arrow[l, "s^{-1}"'] \arrow[r, "0"] & \cdot \end{tikzcd}$$
"

exer Know $D^*(\mathcal{A})$ is additive, so $\mathrm{Hom}_{D^*(\mathcal{A})}(A, B)$ is an ab. gp.

Prove that $f+h = h \ \forall \ h: A \to B$ in $D^*(\mathcal{A})$.

$\underset{\text{roof}}{\wr\ s} \qquad \wr$

② Let $f: A \to B$ in $\mathrm{Com}^*(\mathcal{A})$. Then $Q(f) = 0 \Leftrightarrow$

$\exists \ M \underset{\text{quasi-isom.}}{\overset{u}{\to}} A$ with $\underset{\hookrightarrow \text{homotopic}}{fu \sim 0}$

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Pf:

"⇒": Assume (by ①)

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] & & & C \arrow[dl, "s"'] \arrow[dr, "0"] & \\ A & & B \arrow[r, phantom, "\sim"] & A & & B \end{tikzcd}$$

⇒

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] & \\ A \arrow[r, phantom, "\circlearrowleft"] & H \arrow[l, "u"'] \arrow[u, "u"] \arrow[d, dashed, "\exists H"] & B \\ & C \arrow[ul, "s"] \arrow[ur, "0"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

$fu = 0$ in $K^\circ(\mathcal{A})$ that is $fu \sim 0$.

③ Let $f: A \to B$ in $\mathrm{Com}^*(\mathcal{A})$ be s.t. $Q(f)=0$ $\Rightarrow \exists u: M \to A$ qis s.t. $fu \sim 0$. $M \xrightarrow{u} A \xrightarrow{f} B$. ~~Apply Q~~ $fu \sim 0 \Rightarrow \forall i$ $H_i(f)\underset{\substack{\downarrow \\ \text{iso } \forall i}}{H_i(u)}=0$.

$\Rightarrow H_i(f)=0$ $\forall i$

So:

$$\boxed{\begin{array}{c} A \xrightarrow{f} B \text{ in } \mathrm{Com}^*(\mathcal{A}) \text{ with } Q(f)=0 \\ \Downarrow \\ H_i(f): H_i(A) \to H_i(B) \text{ is zero } \forall i \end{array}}$$

Note: The converse is not true in general:

Example: $\mathcal{A} = \text{mod } \mathbb{Z}$.

$$\begin{tikzcd} & & & & H_0 & H_1 & & \\ A: \arrow[d, "f"'] & \cdots & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 3"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[r] \arrow[d, "\beta"] & 0 \arrow[r] & \cdots \\ B: & \cdots & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

in $\mathrm{Com}^b(\text{mod } \mathbb{Z})$

Where $\alpha: 1 \longmapsto 1+3\mathbb{Z}$

$\beta: 1 \longmapsto 2+3\mathbb{Z}$

$$\begin{tikzcd} H_0 & H_1 \\ 0 \arrow[d, "0"'] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "0"] \\ 3\mathbb{Z} & 0 \end{tikzcd}$$

The induced maps in homology are all zero.

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Claim: $Q(f) \neq 0$.

exercise: prove it using ③ above.

④ **exercise:** Let $A \in \mathrm{Com}^*(\mathcal{A})$. Then $A = 0$ in $D^*(\mathcal{A})$

$\Updownarrow$

A is acyclic.

⑤ **Example:** Let K be a field. Describe ② $D(\mathrm{mod}\, K)$

Let $\mathcal{A} = \mathrm{mod}\, K$.

Facts we know:

(1) $\mathrm{mod}\, K$ is semisimple as a category, that is, every exact sequence splits

(2) $\mathrm{Com}_0(\mathcal{A})$ = all the complexes in $\mathcal{A}$ where the differentials are all zero. It is an abelian category.

$$\mathrm{Com}_0(\mathcal{A}) \hookrightarrow \mathrm{Com}(\mathcal{A}) \xrightarrow{H_*} \mathrm{Com}_0(\mathcal{A})$$

the homology functor

where $H_*(A) = \ldots \; H_i(A) \xrightarrow{0} H_{i+1}(A) \to \ldots$

(we also write it as $H_*(A) = \prod_i H_i(A)$)

(3) If $s: A \to B$ qis $\Rightarrow$ $H_i(s)$ is an isom $\forall i$ $\Rightarrow H_*(s): H_*(A) \to H_*$

is an isom in $\mathrm{Com}_0(\mathcal{A})$.

Have a diagram

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[d, "H_*"'] \arrow[rr, "Q"] & & D(\mathcal{A}) \arrow[dl, "\exists ! G"] \\
\mathrm{Com}_0(\mathcal{A}) & \circlearrowright &
\end{tikzcd}$$

$H_*(\mathrm{qis})$ = invertible.

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__Claim:__ G is an equivalence of categories.

This will show that $\mathcal{D}(\mathcal{A}) \simeq \mathrm{Com}_0(\mathcal{A})$

Let $A \in \mathrm{Com}(\mathcal{A})$. Let $Z_i = \mathrm{Ker}\, d_i^A$
$B_i = \mathrm{Im}\, d_{i-1}^A$

$$A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \to \cdots$$

$$\underset{\displaystyle B_i}{\underset{\|}{\mathrm{Im}\, d_{i-1}}} \subseteq Z_i = \mathrm{Ker}\, d_i$$

Have exact sequence $\forall i$ $\quad 0 \to B_i \to Z_i \to H_i \to 0.$

$$0 \to Z_i \to A_i \to B_{i+1} \to 0.$$

$\mathcal{A}$ semisimple so these seq. split.

$$A_i = Z_i \oplus B_{i+1} = B_i \oplus B_{i+1} \oplus H_i(\mathcal{A})$$

Rewrite the differential d_i.

$$\begin{tikzcd} d_i : B_i \oplus B_{i+1} \oplus H_i(A) \arrow[d, "{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}"] \\ B_{i+1} \oplus B_{i+2} \oplus H_{i+1} \end{tikzcd}$$

Let $f_{\mathcal{A}} : \mathcal{A} \to \prod_i H_i(\mathcal{A}) = H_*(\mathcal{A})$

$$(f_{\mathcal{A}})_i = [0 \quad 0 \quad 1]$$

Let $g_{\mathcal{A}} : \quad \prod H_i(\mathcal{A}) \to \mathcal{A} \qquad g_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let F : $\mathrm{Com}_0(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ be the composition

$$\mathrm{Com}_0(\mathcal{A}) \hookrightarrow \mathrm{Com}(\mathcal{A}) \to K(\mathcal{A}) \xrightarrow{Q} \mathcal{D}(\mathcal{A}).$$

Then $FG(A) = H_*(A) = \prod H_i(A) \qquad (\cong A)$

$GF \simeq 1_{\mathrm{Com}_0(\mathcal{A})}$

Use the following $\qquad g_A f_A = 1_{H_*(A)}$ in $\mathrm{Com}(\mathcal{A})$.

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Claim: $Q(g_A f_A)$ is an isom. in $D(A)$

(Show $g_A f_A$ is a quasi-isom)

Then $D(\text{mod } K) \cong \Pi(\text{mod } K)$.

Corollary: $A = \text{mod } K$. $D^b(A) \cong \oplus \text{ mod } K$.

Example of triangle categories

(The stable cat. and BGG)

Let R be a ring. Let mod R be the f. gen. modules.

For $A, B \in \text{mod } R$, let $P(A,B)$ = all the homoms from A to B factoring through a projective module $\subseteq \text{Hom}_R(A,B)$

Look at $\underline{\text{mod}}\ R$: the category whose objects are the objects of mod R and where $\underline{\text{Hom}}(A,B) \overset{\text{def}}{=} \dfrac{\text{Hom}_R(A,B)}{P(A,B)}$

$\underline{\text{mod}}\ R$ = the stable cat. of R.

In a similar way, one can define $I(A,B) = \ldots$ factoring through injective

and $\overline{\text{mod}}\ R$ where $\overline{\text{Hom}}_R(A,B) = \dfrac{\text{Hom}_R(A,B)}{I(A,B)}$

Def: A ring R is self injective if it satisfies.

M is projective $\Leftrightarrow$ M is injective.

Examples: ① every semisimple ring is selfinjective.

② Gorenstein commutative rings of dimension 0 are selfinjective.

③ $R = k[X]/\langle x \rangle^n$ $n \geq 1$ is selfinjective.

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④ G finite gp K field $\Rightarrow$ KG is selfinjective.

Fact: If R is selfinjective $\Rightarrow$ $\underline{\text{mod}}\ R = \overline{\text{mod}}\ R$.

Claim: Let R be selfinjective. Then $\underline{\text{mod}}\ R$ is triangulated.
⇝ "st mod R"

Shift functor:

Def: If $A \in \text{mod } R$. Know $\exists\ \underset{\text{proj.}}{P} \to A \to 0$

Look at the kernel. Call it ΩA.

A can also map into an injective module I.

$$0 \to A \to I \to \underset{\substack{\downarrow \\ \text{the cokernel}}}{\Omega^{-1}A} \to 0$$

Thm: (Heller) Let R be selfinjective. Then Ω and Ω^{-1}: $\underline{\text{mod}}\ R \to \underline{\text{mod}}\ R$ are equivalences to each other.

Important: in $\underline{\text{mod}}\ R$, the projective and injective modules are zero objects. ⇝ Ω, Ω^{-1} well-def'd

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega A \arrow[r, "\Omega f"] \arrow[d] & \Omega_B \arrow[d] \\
P_A \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & P_B \arrow[d] \\
A \arrow[r, "f"'] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

Ωf uniquely determined in $\underline{\text{mod}}\ R$ by f.

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Selfinjective rings

$$\text{R selfinj.} \\ \Downarrow \\ \text{"\{proj. modules\} = \{inj. modules\}"}$$

Thm: Ω, Ω^{-1}: $\underline{\text{mod}}\ R \to \underline{\text{mod}}\ R$ are equivalences inverse to each other

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Thm: $\underline{\mathrm{mod}}\, R$ is triangulated with shift functor Ω^{-1}.

Comments: Let $A \xrightarrow{f} B$ homom of R-mods.

Get the following comm. diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I(A) \arrow[r] \arrow[d] & \Omega^{-1}A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "\text{pushout}"'] & C \arrow[r] & \Omega^{-1}A \arrow[r] & 0
\end{tikzcd}$$

$\exists$ injective $I(A)$ s.t. $A \hookrightarrow I(A)$

A distinguished triangle in $\underline{\mathrm{mod}}\, R$ is a diagram: $A \to B \to C \to \Omega^{-1}A$

Coming from a commutative diagram as above.

Thm: (BGG equivalence)

Let $\mathbb{P}^n(\mathbb{C})$ be the projective n-space over $\mathbb{C}$. Let $\mathrm{coh}($~~$\mathbb{R}$~~$\mathbb{P}^n)$ be the coherent sheaves.

Then $D^b(\mathrm{coh}\, \mathbb{P}^n)$

$$\wr\wr$$

$\underline{\mathrm{mod}}^{\mathbb{Z}} R$ → R where R is the Grassmann algebra in n+1 variable

⇝ the "graded stable category"

This is also equivalent to $D^b(\mathrm{mod}\, \mathbb{C}[x_0, \dots, x_n])$.

noncommuting var.

$$R = \frac{\mathbb{C}\langle x_0, x_1, \dots, x_n\rangle}{\langle x_i x_j + x_j x_i,\ x_i^2\rangle_{i \neq j}}$$

[dim 2^{n+1}]

Thm: Assume the abelian cat. $\mathcal{A}$ has enough injectives (for instance $\mathcal{A} = \mathrm{mod}\, R$ for a ring R.)

Let $I \underset{\text{full}}{\subseteq} \mathcal{A}$ be the subcategory consisting of injective objects.

Let $K^+(I)$ be the homotopy category of injective objects.

$\Rightarrow K^+(I)$ is equivalent to $D^+(\mathcal{A})$ via the functor Q.

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There is a dual to this:

Thm: Assume $\mathcal{A}$ has enough projectives (for instance $\mathcal{A} = \mathrm{mod}\, R$).

Then Q induces an equivalence $K^-(\mathcal{P})$ and $D^-(\mathcal{A})$ where $\mathcal{P} \subseteq \mathcal{A}$ is the full subcat. consisting of proj objects.

Some "special" truncations

Def: Let $A \in \mathcal{A}$ be a complex. Let $k \in \mathbb{Z}$

$$\tau^{\leq}(A) : \quad \cdots \to A_{k-2} \xrightarrow{d_{k-2}} A_{k-1} \to \operatorname{Ker} d_k \to 0 \to 0 \cdots$$

$$\tau^{\geq k}(A) : \quad \cdots 0 \to 0 \to \operatorname{coker} d_{k-1} \to A_{k+1} \xrightarrow{d_{k+1}} A_{k+2} \to \cdots$$

Remarks:

① Let A be a complex with $H_i(A) = 0 \ \forall i < k$

Then

$$\begin{tikzcd}
A \arrow[d, "\mathrm{qis}"] & \arrow[r] & A_{k-2} \arrow[r, "d_{k-2}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A_{k-1} \arrow[r, "d_{k-1}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A_k \arrow[r, "d_k"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A_{k+1} \arrow[r] \arrow[d, equal] & \cdots \\
\tau^{\geq k}(A) & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \operatorname{Coker} d_{k-1} \arrow[r] & A_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

② Let A be a complex with $H_i(A) = 0 \ \forall i > k$

Then

$$\begin{tikzcd}
\tau^{\leq k}(A) \arrow[d, "\mathrm{qis}"] & \cdots \arrow[r] & A_{k-2} \arrow[r, "d_{k-2}"] \arrow[d, equal] & A_{k-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker} d_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
A & \cdots \arrow[r] & A_{k-2} \arrow[r, "d_{k-2}"] & A_{k-1} \arrow[r] & A_k \arrow[r, "d_k"] & A_{k+1} \arrow[r] & 0
\end{tikzcd}$$

Prop: (1) $D^b(\mathcal{A})$ is equivalent to the full subcat of $\mathcal{D}(\mathcal{A})$ consisting of complexes A for which $H_i(A) = 0$ for $|i| \gg 0$

(2) $D^+(\mathcal{A})$ is equivalent to the full subcat. of $D(\mathcal{A})$ consisting of complexes A for which $H_i(A) = 0$ for $i \ll 0$.

(3) $D^-(\mathcal{A})$ is equivalent to the full subcat of $D(\mathcal{A})$ consisting of complexes A for which $H_i(A) = 0$ for $i \gg 0$.

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(4) $\mathcal{A}$ is equivalent to the full subcat. of $D(\mathcal{A})$ consisting of all complexes A where $H_i(A) = 0$ $\forall i \neq 0$.

<u>Obs:</u> $D^*(\mathcal{A})$ comes from $\mathrm{Com}^*(\mathcal{A})$ via $K^*(\mathcal{A})$

Dist. Δ are of the form $A \xrightarrow{f} B \to C(f) \to A[1]$

But we also know that if $0 \to A \to B \to C \to 0$ is exact in $\mathrm{Com}^*(\mathcal{A})$

$\Rightarrow$ get a long exact seq. in homology.

Is it true that we can get distinguished Δ $A \to B \to C \to A[1]$?

<u>Prop:</u> Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact in $\mathrm{Com}^*(\mathcal{A})$.

Let $\alpha : C(f) \to C$ be given by $\alpha = [0 \ \ g]$. Then α is a qis and $\alpha j = g$

where j is the "canonical embedding" in

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "j = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"] & C(f) \arrow[r, "s = {[1 \ \ 0]}"] \arrow[d, equal] & A[1] \arrow[r] & 0 \\
 & & A[1] \oplus B & &
\end{tikzcd}$$

<u>Pf:</u> Recall $d^{C(f)} = \begin{bmatrix} -d^A & 0 \\ f & d^B \end{bmatrix}$

$$\begin{tikzcd}
C(f) \arrow[r, "d^{C(f)}"] \arrow[d, "\alpha"'] & C(f) \arrow[d, "\alpha"] \\
C \arrow[r, "d^C"] & C \\
B \arrow[u, "g"] \arrow[r, "d^B"'] & B \arrow[u]
\end{tikzcd}$$

easy to show

$\alpha d^{C(f)} = d^C \alpha$ so α is a homom of complexes

and easy to see $\alpha j = g$.

Note that we have exact sequence in $\mathrm{Com}^*(\mathcal{A})$:

$$\begin{tikzcd}
0 \arrow[r] & C(1_A) \arrow[r, "\left[\begin{smallmatrix} 1_A & 0 \\ 0 & f \end{smallmatrix}\right]"] & C(f) \arrow[r, "\alpha"] & C \arrow[r] & 0 \\
 & A[1] \oplus A & A[1] \oplus B & &
\end{tikzcd}$$

$C(1_A)$ is exact since 1_A is a quasiisom $\Rightarrow$ α is a quasiisom.

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Consequence: with the above notation, if $0 \to A \xrightarrow{f} B \to C \to 0$ is exact in $\mathrm{Com}^+(\mathcal{A})$, the triangle $A \xrightarrow{Q(f)} B \xrightarrow{Q(g)} C \xrightarrow{``\delta\alpha^{-1}"} A[1]$ is distinguished in $D^+(\mathcal{A})$ and is isomorphic to $A \xrightarrow{Q(f)} B \xrightarrow{Q(\iota)} C(f) \xrightarrow{Q(\delta)} A[1]$

Pf: Work in $K^+(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "\iota"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & C(f) \arrow[r, "\delta"] \arrow[d, "\alpha"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1]
\end{tikzcd}$$

Commutes in $D^+(\mathcal{A})$

$$\begin{tikzcd}
& C(f) \arrow[dl, "\alpha"', "\text{qis}"] \arrow[dr] & \\
C & & A[1]
\end{tikzcd}$$

$\downarrow$

this is a homom $C \to A[1]$ in $D^+(\mathcal{A})$ "$\delta\alpha^{-1}$"

But in $D^+(\mathcal{A})$ $(1_A, 1_A, Q(\alpha))$ is an isom of triangles. So get our result.

Def: An abelian cat. $\mathcal{A}$ has enough projectives if $\forall A \in \mathrm{Ob}\,\mathcal{A}$ $\exists$ proj. object $P \in \mathrm{Ob}\,\mathcal{A}$ and an epi $P \to A \to 0$.

("projective object P" means that $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Ab}$ is exact.)

Similar def. for "enough injectives".

Obj:

1. Assume $\mathcal{A}$ has "enough injectives". Then every object $A \in \mathcal{A}$ has an injective resolution : exact complex $0 \to A \to \underbrace{I_0 \to I_1 \to I_2 \to \dots}_{\text{injective objects}}$

$$\begin{tikzcd}[
{[}\ 0 \arrow[r] & A \arrow[r, "i"] & I_0 \arrow[dr] & & & & & \\
& & 0 \arrow[r] & \mathrm{coker}\, i \arrow[r, hook] \arrow[dr] & I_1 \arrow[drr] & & & \\
& & & & 0 & 0 \arrow[r] & \mathrm{coker}\, i_0 \arrow[r, hook] \arrow[dr] & I_2 \\
& & & & & & & 0
\end{tikzcd}$$

$\rightarrow 0 \to A \to I_0 \to I_1 \to I_2 \to \dots]$

2. If $\mathcal{A}$ has enough projectives $\Rightarrow$ every $A \in \mathcal{A}$ has a projective resolution

Sublemma: Every chain map from an acyclic complex to a complex of bounded from below

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of injective modules in $\mathrm{Com}^+(\mathcal{A})$ is null homotopic.

Pf. Start WLOG

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, "d_0^A"] \arrow[d, "f_0"] & A_1 \arrow[r, "d_1^A"] \arrow[d, "f_1"] \arrow[dl, dashed, "\exists s_1"'] & A_2 \arrow[r] \arrow[d, "f_2"] & \cdots \\
0 \arrow[r] & I_0 \arrow[r, "d_0^I"'] & I_1 \arrow[r, "d_1^I"'] & I_2 \arrow[r] & \cdots
\end{tikzcd}$$

$\exists s_1$ with $s_1 d_0^A = f_0$

Take $f_1 - d_0^I s_1 : A_1 \to I$ Note that $(f_1 - d_0^I s_1) d_0 = 0$

$\Rightarrow \exists s_2 : A_2 \to I_1$ s.t. $s_2 d_1^A = f_1 - d_0^I s_1$.

$\Rightarrow f_1 = d_0^I s_1 + s_2 d_1^A$

Continue, build $s_3, s_4, \ldots$ Get $f = d^I s + s d^A$ so $f \sim 0$. □

Lemma: Let $s: I \to A$ be a quasi-isom where I is a complex of injective modules, $I \in K^+(I)$, and $A \in K^+(A)$

$\Rightarrow \exists t: A \to I$ chain map with $ts \sim 1_I$.

Pf. Have a dist triangle in $K^+(A)$. $I \xrightarrow{s} A \to C(s) \to I[1]$.

s qis. $\Rightarrow C(s)$ is acyclic. By the sublemma, the map $C(s) \to I[1]$ is zero in $K^+(A)$.

$\overset{\text{stuff before}}{\Longrightarrow}$ $A \simeq I \oplus C(s) \Rightarrow \exists t : I \to A$ to be left inverse of s in $K^+(A)$.

$\Rightarrow ts = 1$ in $K^+(A)$

Thm: The functor $Q: K^+(I) \to D^+(A)$ is fully faithful.

If A has enough injectives then Q is dense, that is, Q is an equivalence.

Pf: Full: To show $Q: \mathrm{Hom}_{K^+(A)}(I, J) \to \mathrm{Hom}_{D^+(A)}(I, J)$ is onto.

That is, to show that given a roof qis $\begin{tikzcd}[column sep=small] & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ I & & J \end{tikzcd}$ $\exists g: I \to J$ in $K^+(J)$

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s.t. $Q(g)$ = our roof. That is

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, "g"] & \\ I & & J \end{tikzcd} \sim \begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ I & & J \end{tikzcd}$$

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ I \arrow[dr, equal] & & J \\ & I \arrow[ur, "g"'] & \end{tikzcd}$$

$$\boxed{\mathcal{D}(\mathrm{mod}\,K) \simeq \prod_{-\infty}^{\infty} \mathrm{mod}\,K[i]}$$

Let K be a field and let $\mathcal{A} = \mathrm{mod}\,K$. A is a <u>semisimple</u> category.

Let $\mathrm{Com}_0(\mathcal{A})$ = complexes over $\mathcal{A}$ with <u>zero differentials</u>.

Immediate fact: $\mathrm{Com}_0(\mathcal{A})$ is isomorphic to $\prod_{-\infty}^{\infty} \mathrm{mod}\,K[i]$ where $A[i]$ means that we have objects of $\mathcal{A}$ in the ith entry.

To prove: $\mathrm{Com}_0(\mathcal{A}) \simeq \mathcal{D}(\mathcal{A})$. Have the following comm diagram

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[d, "H_*"'] \arrow[dr, "Q"] & \\
\mathrm{Com}_0(\mathcal{A}) & \mathcal{D}(\mathcal{A}) \arrow[l, dashed, "\exists!\,G"'] \arrow[ul, phantom, "\circlearrowright"]
\end{tikzcd}$$

$\cdots H_i(A) \xrightarrow{0} H_{i+1}(A) \to$

where $H_*(A) = \prod_i H_i(A)$ (with 0 differentials).

Since $s : A \to B$ qis $\Rightarrow$ each induced map $H_i(A) \to H_i(B)$ is an iso, we get

$H_*(\text{qis})$ = isomorphisms in $\mathrm{Com}_0(\mathcal{A})$

$\Rightarrow \exists! G : \mathcal{D}(\mathcal{A}) \to \mathrm{Com}_0(\mathcal{A})$; $\boxed{GQ = H_*}$

Let F be the composition

$$\mathrm{Com}_0(\mathcal{A}) \xhookrightarrow[\text{ind}]{} \mathrm{Com}(\mathcal{A}) \xrightarrow[\text{can}]{} K(\mathcal{A}) \xrightarrow[Q]{} \mathcal{D}(\mathcal{A})$$

So $F : \mathrm{Com}_0(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$.

$$\boxed{\underline{\text{Claim}}:\quad GF = 1_{\mathrm{Com}_0(\mathcal{A})} \qquad \text{(This is trivial)}}$$

Use now the fact that $\mathcal{A}$ is semisimple to show $FG \simeq 1_{\mathcal{D}(\mathcal{A})}$.

If A is a complex, $FG(A) : \ \cdots \to H_i(A) \xrightarrow{0} H_{i+1}(A) \to \cdots$

So we must show that $FG(A)$ and A are quasi'isom.

For $d_i : A_i \longrightarrow A_{i+1}$, let $Z_i = \mathrm{Ker}\, d_i$

$B_{i+1} = \mathrm{Im}\, d_i$

$\Longrightarrow$ Have (split) exact sequences:

$$0 \to B_i \to Z_i \to H_i(A_i) \to 0 \qquad \forall i$$

and

$$0 \to Z_i \to A_i \to B_{i+1} \to 0$$

$$\Longrightarrow \boxed{\begin{tikzcd} A_i = B_i \oplus H_i \oplus B_{i+1} \arrow[d, "d_i"] \\ A_{i+1} = B_{i+1} \oplus H_{i+1} \oplus B_{i+2} \end{tikzcd} \qquad d_i = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Let $f_A : A \longrightarrow FG(A)$ be defined as follows:

$$\boxed{(f_A)_i : A_i = B_i \oplus H_i(A) \oplus B_{i+1} \longrightarrow H_i(A) \qquad f_A = [0 \; 1 \; 0]}$$

chain map!

and let $g_A : FG(A) \longrightarrow A$ be the map

$$H_i(A) \longrightarrow A_i = B_i \oplus H_i(A) \oplus B_{i+1}$$

$$g_A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

chain map!

Then $f_A g_A = 1_{FG(A)}$ and $g_A f_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then observe that $g_A f_A \sim 1_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where the homotopy Δ is given by $\boxed{\Delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$

$$\begin{tikzcd}
 & A_i \arrow[r, "d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}"] \arrow[dl, "\Delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}"'] \arrow[d, shift right, "1_A"'] \arrow[d, shift left, "g_A f_A"] & A_{i+1} \arrow[dl, "\Delta"] \\
A_{i-1} \arrow[r, "d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}"'] & A_i &
\end{tikzcd}$$

and $\Delta d + d\Delta = 1_A - g_A f_A$

$$\Downarrow$$

$g_A f_A \sim 1_A$

$\Rightarrow$ the complexes are quasi isomorphic.

Lemma Let $s : A \longrightarrow B$ be a quasi-isomorphism in $K^+(\mathcal{A})$ where $\mathcal{A}$ is an abelian category.

$\Longrightarrow$ $\forall$ complex $I \in K^+(\mathcal{J})$, the induced map

$$\mathrm{Hom}_{K(\mathcal{A})}(B,I) \xrightarrow[\mathrm{Hom}(s,I)]{} \mathrm{Hom}_{K(\mathcal{A})}(A,I)$$

is bijective.

so I is a complex of injectives bounded from below.

Proof

$\exists$ distinguished triangle in $K^+(\mathcal{A})$:

$$\boxed{A \xrightarrow{s} B \longrightarrow C(s) \longrightarrow A[1].}$$

$C(s)$ exact since s is qis!

Apply $\mathrm{Hom}_{K^+(\mathcal{A})}(-,I)$ which is a cohomological functor (!)

$\Rightarrow$ get exact sequence

$$\underbrace{\mathrm{Hom}_{K^+(\mathcal{A})}(C(s),I)}_{=0} \longrightarrow \mathrm{Hom}_{K^+(\mathcal{A})}(B,I) \longrightarrow \mathrm{Hom}_{K^+(\mathcal{A})}(A,I) \longrightarrow \underbrace{\mathrm{Hom}_{K^+(\mathcal{A})}(C(s)[-1],I}_{=0}$$

since ~~every~~ maps from exact complexes to complexes of injectives are nullhomotopic.

$\Longrightarrow$ $\mathrm{Hom}_{K^+(\mathcal{A})}(B,I) \xrightarrow{\sim} \mathrm{Hom}_{K^+(\mathcal{A})}(A,I)$.

Lemma Let $A, I \in K^+(\mathcal{A})$ with $I \in K^+(\mathcal{J})$. Then $\boxed{\mathrm{Hom}_{K^+(\mathcal{A})}(A,I) = \mathrm{Hom}_{D^+(\mathcal{A})}(A,I}$

Pf Q induces a ~~surjection~~ map $\mathrm{Hom}_{K^+(\mathcal{A})}(A,I) \longrightarrow \mathrm{Hom}_{D^+(\mathcal{A})}(A,I)$.

To show that $\forall$ roof $$\begin{tikzcd} & B \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & I \end{tikzcd}$$ in $D^+(\mathcal{A})$, there is a unique map $A \xrightarrow{g} I$ of complexes so that $g s \sim f$. This follows from above. ∎

THM Assume $\mathcal{A}$ contains enough injectives. Then, the localization functor $Q: K^+(\mathcal{J}) \longrightarrow \mathcal{D}^+(\mathcal{A})$ is an equivalence.

Proof

"Full": assume
$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ I & & J \end{tikzcd}$$
is a morphism in the derived category between $I, J \in K^+(\mathcal{J})$, so it is in $\mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(I,J)$. But from the previous lemma we have seen that we have
$$\mathrm{Hom}_{K^+(\mathcal{A})}(I,J) = \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(I,J).$$
(after relabelling)

"Faithful": let $f: I \to J$ in $K^+(\mathcal{J})$. Assume $Q(f)=0$

$$\Longrightarrow \begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, "f"] & \\ I & & J \end{tikzcd} \sim \begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, "0"] & \\ I & & J \end{tikzcd}$$

$\Longrightarrow$ have comm diagram
$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, "f"] & \\ I & M \arrow[l, "t"'] \arrow[u, "t"'] \arrow[d, "t"] & J \\ & I \arrow[ul, equal] \arrow[ur, "0"'] & \end{tikzcd}$$

t qis

$\Longrightarrow ft = 0$ in $K^+(\mathcal{A})$. Since t is a qis $\Longrightarrow f \sim 0$ in $K^+(\mathcal{A})$.
why?
(exercise)

We prove now that Q is "dense", that is $\forall\, A \in \mathcal{D}^+(\mathcal{A})$, $\exists\, I \in K^+(\mathcal{J})$ s.t. $Q(I) = A$ in $\mathcal{D}^+(\mathcal{A})$.

-C-

THM Assume $\mathcal{A}$ has enough injectives.
$\Rightarrow Q : K^+(\mathcal{J}) \longrightarrow \mathcal{D}^+(\mathcal{A})$ is an equivalence.

Proof

It remains to prove Q is "dense". That is, we must show that $\forall A \in K^+(\mathcal{A})$, $\exists I \in K^+(\mathcal{J})$ and a quasi-isomorphism

$$\boxed{A \xrightarrow[q.is]{t} I}$$

WLOG may assume $A: \cdots 0 \to \cdots \to 0 \to A_0 \xrightarrow{d_0^A} A_1 \xrightarrow{d_1^A} \cdots$

Construct $I: \to 0 \cdots \to 0 \to I_0 \to I_1 \to \cdots$ inductively.

Start with $t_0 : A_0 \longrightarrow I_0$ mono $I_0 \in \mathcal{J}$.

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, "d_0^A"] \arrow[d, "t_0"'] & A_1 \arrow[d, dashed, "\alpha"] & \\
0 \arrow[r] & I_0 \arrow[r, dashed, "\beta"'] & B_1 \arrow[dr, "\gamma"'] & \\
& & & I_1
\end{tikzcd}$$

"pushout"

γ mono, $I_1 \in \mathcal{J}$.

Define $d_0^I = \gamma\beta$
$t_1 = \gamma\alpha$

Have
$$\begin{tikzcd}
\cdots 0 \arrow[r] & A_0 \arrow[r, "d_0^A"] \arrow[d, "t_0"'] & A_1 \arrow[r, "d_1^A"] \arrow[d, "t_1"] & A_2 \\
\cdots 0 \arrow[r] & I_0 \arrow[r, "d_0^I"'] & I &
\end{tikzcd}$$

Inductive step: Assume we have constructed.

$0 \to I_0 \to \dots \to I_k \quad \in K^+(\mathcal{J})$. Look at

$$\begin{tikzcd}
 & \cdots A_k \arrow[r, "d^A_k"] \arrow[d, "t_k"'] \arrow[dr, "pt_k"] \arrow[dr, phantom, "\circlearrowright", shift right=4] & A_{k+1} \arrow[dr, "\alpha"] \arrow[d, phantom, "\circlearrowright"] & & \\
\cdots I_{k-1} \arrow[r, "d^Z_{k-1}"] & I_k \arrow[r, "p"] & \operatorname{Coker} d^Z_{k-1} \arrow[r, "\beta"'] & B_{k+1} \arrow[r, "\text{pushout}"] \arrow[dr, "\gamma"] & \text{} \\
 & & & & I_{k+1}
\end{tikzcd}$$

Define $d^Z_k = \gamma \beta p$

$t_{k+1} = \gamma \alpha$.

In this way we get a comm diagram

$$\begin{tikzcd}
A \arrow[d, "t"] & \cdots & 0 \arrow[r] & A_0 \arrow[r, "d^A_0"] \arrow[d, "t_0"'] \arrow[dr, phantom, "\circlearrowright"] & A_1 \arrow[r] \arrow[d, "t_1"] & \cdots & \in K^+(A) \\
I & \cdots & 0 \arrow[r] & I_0 \arrow[r, "d^Z_0"'] & I_1 \arrow[r] & \cdots & \in K^+(\mathcal{J})
\end{tikzcd}$$

The bottom sequence is a complex since the d_i^Z factors through cokernels of previous differentials. (check!).

We have to show t is a quasi-isomorphism.

First, observe $\boxed{H_0(A) = \operatorname{Ker} d^A_0.}$ We also have $\boxed{\underset{\boxed{\underset{\displaystyle H_0(Z)}{\shortparallel}}}{\operatorname{Ker} d^Z_0} = \operatorname{Ker} \beta}$

From the "pushout exercise", it follows that

$$\boxed{t_0(\operatorname{Ker} d_0^A) = \operatorname{Ker} d_0^Z}$$

$\Longrightarrow$ t_0 induces an isomorphism $\operatorname{Ker} d_0^A \approx \operatorname{Ker} d_0^Z$
or t_0 induces $\boxed{H_0(A) \xrightarrow{\sim} H_0(I).}$

<u>Claim</u>: $\boxed{\forall k,\ H_k(t) \text{ is an epi.}}$ Look at the diagram where we use the fact that

$$\operatorname{Ker} d_k^Z = \operatorname{Ker} \beta p \quad (\text{as } \gamma \text{ is a mono})$$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} d_k^A \arrow[r] & A_k \arrow[r, "d_k^A"] \arrow[d, "pt_k"'] & A_{k+1} \arrow[d, "\kappa"] \\
0 \arrow[r] & \operatorname{Ker} \beta \arrow[r] & p(I_k) \arrow[r, "\beta"'] & B_{k+1}
\end{tikzcd}$$

pushout

$p(I_k) = \operatorname{Coker} d_{k-1}^Z$

Note that $H_k(I) = \dfrac{\operatorname{Ker} d_k^Z}{\operatorname{Im} d_{k-1}^Z} \subseteq \operatorname{coker} d_{k-1}^Z = p(I_k)$

let $z \in H_k(I) \Longrightarrow z = p(x) \quad x \in I_k$

$p(x) = x + \operatorname{Im} d_{k-1}^Z$ where $x \in \operatorname{Ker} d_k^Z$

$\Longrightarrow \beta(z) = 0$ since $\beta p(x) = d_k^Z(x) = 0$. So $\beta(z) \in \operatorname{Ker} \beta$.
By the "$i=0$ case" discussion <u>and</u> the pushout exercise

$\Longrightarrow z = pt_k(a),\ a \in \operatorname{Ker} d_k^A.$ $\Longrightarrow z = H_k(t)(a + \operatorname{Im} d_{k-1}^A)$

$\Longrightarrow \boxed{H_k(t) \text{ epi } \forall k.}$

Claim 2 : $\boxed{H_{k+1}(t) \text{ is a mono } \forall k}$. This, together with the base case and claim 1 should imply t is a qis.

Let $z \in H_{k+1}(A)$ s.t $H_{k+1}(t)(z) = 0$, also have $z = x + \mathrm{Im}\, d_k^A$, $x \in \mathrm{Ker}\, d_{k+1}^A$

But $t_{k+1} : A_{k+1} \longrightarrow Z_{k+1}$ is given by $t_{k+1} = \gamma\alpha$ (and γ is a mono), and so

$$t_{k+1}(x) = \gamma\alpha(x) \in \mathrm{Im}\, d_k^Z$$

Since $d_k^Z = \gamma\beta p$

$$\Downarrow$$

$$\gamma\alpha(x) = \gamma\beta p(a) \quad \text{some } a \in A_k$$

$\Longrightarrow$ γ mono $\quad \alpha(x) = \beta p(a) \Longrightarrow [\alpha, -\beta](x, p(a)) = 0$

where we view $[\alpha, -\beta] : A_{k+1} \oplus \mathrm{Coker}\, d_{k-1}^Z \longrightarrow B_{k+1}$

But

$$\begin{tikzcd} A_k \arrow[r, "d_k^A"] \arrow[d, "pt_k"] & A_{k+1} \arrow[d, "\alpha"] \\ \mathrm{Coker}\, d_{k-1}^Z \arrow[r, "\beta"'] & B_{k+1} \end{tikzcd}$$

is a pushout diagram so the sequence

$$0 \to A_k \xrightarrow{\begin{bmatrix} d_k^A \\ -pt_k \end{bmatrix}} \begin{matrix} A_{k+1} \\ \oplus \\ \mathrm{Coker}\, d_{k-1}^Z \end{matrix} \xrightarrow{[\alpha \; -\beta]} B_{k+1} \to 0$$

is exact.

Since $(x, p(a)) \in \mathrm{Ker}\, [\alpha, -\beta]$

$\Longrightarrow \exists\, b \in A_k$ with $d_k^A(b) = x \Longrightarrow z = 0$ in $H_{k+1}(A)$

$\Longrightarrow$ $H_{k+1}(t)$ is mono. ∎

Remark : One may also prove the following "dual" statement :

THM . Assume $\mathcal{A}$ has enough projectives. Then Q induces an equivalence $K^-(\mathcal{P}) \longrightarrow \mathcal{D}^-(\mathcal{A})$ where $\mathcal{P}$ denotes the full subcategory of $\mathcal{A}$ consisting of projective objects ∎

Remark. We may want to apply the previous thms to categories of modules over a ring. Here is what we may get if we denote our ring by R.

- $\mathcal{D}^+(\text{Mod}R) \approx K^+(\text{Injective modules})$
- $\mathcal{D}^-(\text{Mod}R) \approx K^-(\text{Projective modules})$
- $\mathcal{D}^-(\text{mod}R) \simeq K^-(\text{fin.gen.proj.modules})$