"final exam" is actually another lecture

cover ch 1 (but not section 8) some ch 2, some ch 3

hand in correct 10 problems (see syllabus) for an A
↳ multiple tries OK

Some conventions

- A *ring* is a commutative ring with identity (unless otherwise stated)
- All ring homomorphisms are assumed to take 1 to 1
- For a ring R, the whole ring is considered to be an ideal but it is not considered to be a prime ideal (even though it is)
- $\{0\}$ is allowed to be a prime ideal if it is prime
- From page 55 "To define algebraic geometry, we could say it is the study of the solutions of systems of polynomial equations..."

Chapter 1 Varieties

This is the old fashioned pre-1960's way of doing algebraic geometry.

"Let k be a fixed algebraically closed field"

For the really basic stuff you do not need that.

k any field or maybe integral domain or even ring.

Later on you might add other assumptions of char 0, char $p \neq 0$, infinite, or $k = \mathbb{C}$

Denote k^n the set of ordered n-tuples of elements of k by $\mathbb{A}^n_k$ or $\mathbb{A}^n(k)$ or just $\mathbb{A}^n$ and call it affine n-space over k.

To save space we often denote the polynomial ring $k[x_1, \dots, x_n]$ by A.

Note that if $P \in \mathbb{A}^n$, $f \in A$, then $f(P) \in k$

Elements of A are functions from $\mathbb{A}^n$ to k.

Definition:

For $f \in A$ we define the zeros of f, denoted $Z(f)$ by $Z(f) = \{P \in \mathbb{A}^n \mid f(P) = 0\}$

For $T \subseteq A$ we define the zeros of T denoted $Z(T)$ by $Z(T) = \bigcap_{f \in T} Z(f)$

$= \{P \in \mathbb{A}^n \mid f(P) = 0 \ \forall f \in T\}$

A subset Y of $\mathbb{A}^n$ is called an algebraic set iff $\exists$ a subset $T \subseteq A$ s.t. $Y = Z(T)$

Homework Exercise (not in book, counts for chapter 1 section)

Let $\mathbb{R}$ = the real numbers

(a) Prove that in $\mathbb{A}^2_{\mathbb{R}}$ the graph of $y = \sin(x)$ is not an algebraic set. Hint: The graph intersects the x-axis in infinitely many points yet a nonconstant polynomial in one variable has at most finitely many roots

(b) Prove that in $\mathbb{A}^2_{\mathbb{R}}$ the graph of $y = e^x$ is not an algebraic set. Hint: e^x grows faster than any power of x. That is for any nonnegative integer n,

$$\lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty$$

Proposition:

(a) Let $T \subseteq A$ and let J be the ideal of A generated by T, then $Z(T) = Z(J)$. Thus every algebraic set in $\mathbb{A}^n$ is of the form $Z(J)$ for some ideal J of A.

(b) Every algebraic set in $\mathbb{A}^n$ is of the form $Z(T)$ for some finite set $T \subseteq A$

Proof: part (a) first sentence

WTS $Z(T) \subseteq Z(J)$

Let $P \in Z(T)$

This says $\forall f \in T$, $f(P) = 0$

we wish to show $\forall g \in J$, $g(P) = 0$

Let $g \in J$. Then $g = \sum_{i=1}^m a_i f_i$

$a_i \in A$, $f_i \in T$

$g(P) = \sum_{i=1}^m a_i(P) f_i(P)$

$= \sum_{i=1}^m a_i(P) \cdot 0 = 0$

WTS $Z(J) \subseteq Z(T)$

$T \subseteq J$

If everything in J vanishes at P, so does everything in T.

second sentence of part (a) follows immediately from the first.

part (b).

This follows from part (a) and the fact that $A = k[x_1, \dots, x_n]$ is a Noetherian ring so that every ideal is finitely generated.

Hilbert Basis Theorem says that R Noetherian implies $R[x_1, \dots, x_n]$ Noetherian

Proposition 1.1 (modified)

(a) $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$

Thus the union of any finite number of algebraic sets is algebraic.

(b) $Z\left(\bigcup_{i \in I} T_i\right) = \bigcap_{i \in I} Z(T_i)$

Thus the intersection of any "number" of algebraic sets is an algebraic set

(c) $\emptyset = Z(1)$ Thus the empty set is an algebraic set

(d) $\mathbb{A}^n = Z(\{0\})$. Thus $\mathbb{A}^n$ is an algebraic set.

(e) The algebraic sets in $\mathbb{A}^n$ can be taken to be the closed sets of a topology on $\mathbb{A}^n$. It is called the Zarisky topology.

proof:

(a) WTS $Z(T_1) \cup Z(T_2) \subseteq Z(T_1 T_2)$

$P \in Z(T_1) \cup Z(T_2)$

$\Rightarrow P \in Z(T_1)$ or $P \in Z(T_2)$

say $P \in Z(T_1)$. Any element of $T_1 T_2$ is of the form fg

$f \in T_1$, $g \in T_2$

$(fg)(P) = f(P) g(P) = 0 \cdot g(P) = 0$

so $P \in Z(T_1 T_2)$

WTS $Z(T_1 T_2) \subseteq Z(T_1) \cup Z(T_2)$

pick $p \in Z(T_1 T_2)$

we will show if $p \notin Z(T_1)$ then $p \in Z(T_2)$

$p \notin Z(T_1) \Rightarrow \exists f \in T_1$ s.t. $f(p) \neq 0$

pick any $g \in T_2$.

Then $fg \in T_1 T_2$

so $0 = fg(p)$

$= f(p) g(p)$

since $f(p) \neq 0$, so long as k an integral domain

$g(p) = 0$ g was arbitrary $\Rightarrow p \in Z(T_2)$

$\Rightarrow p \in Z(T_1) \cup Z(T_2)$

(b) $p \in Z\left(\bigcup_{i \in I} T_i\right) \Leftrightarrow$ every polynomial in every T_i vanishes at p

$\Leftrightarrow p \in \bigcap_{i \in I} Z(T_i)$

(c) obvious. 1 does not vanish anywhere

(d) obvious. 0 vanishes everywhere

(e) obvious if you know def of a topology

Example 1.1.1 (modified)

The algebraic sets of $\mathbb{A}^1$ are exactly $\emptyset$, $\mathbb{A}^1$ and any finite set.

show these are algebraic

$\emptyset$, $\mathbb{A}^1$ (prop 1.1)

$Y = \{a_1, a_2, \dots, a_n\}$

$Y = Z(\{(x-a_1)(x-a_2)\cdots(x-a_n)\})$

$\{a_1\} = Z(x - a_1)$

We have shown that in $\mathbb{A}^1$, $\emptyset$, $\mathbb{A}^1$ and all finite subsets are algebraic.

Let's show they are the only ones. Assume k is a field.

Any algebraic set is of the form $\bigcap_{i \in I} Z(f_i)$ $f_i \in k[x]$

What can $Z(f_i)$ be?

$f_i = 0$ $\mathbb{A}^1$

$f_i = c \neq 0$ $\emptyset$

f_i non-constant poly, it has at most finitely many roots so $Z(f_i)$ is finite.

Any finite set in any $\mathbb{A}^n$ is algebraic.

Single point $p = (a_1, a_2, \ldots, a_n)$

$P = Z(\{x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n\})$

$\{p\}$ algebraic

prop 1.1 (a) says finite unions of algebraic are algebraic.

k finite, the Zariski topology on $\mathbb{A}^n$ is the discrete topology

When k is infinite, the Zariski topology on $\mathbb{A}^1_k$ is not T_2, any two nonempty open sets intersect

Definition: A nonempty subset Y of a topological space X is irreducible iff it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each of which is closed in Y (in the induced topology). The empty set is not considered to be irreducible

Example 1.1.2

When k is infinite, $\mathbb{A}^1$ is irreducible. Its only proper closed subsets are finite yet it is infinite.

Example: In $\mathbb{A}^2$, $Z(xy) = Z(x) \cup Z(y)$

Definition: An affine algebraic variety (or simply affine variety) is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology).

An open subset of an affine variety is a quasi-affine variety

Definition

Let $Y \subseteq \mathbb{A}^n$ be any subset. We define the ideal of Y, denoted $I(Y)$, by $I(Y) = \{f \in A \mid f(p) = 0 \ \forall p \in Y\}$

Two functions

$$\{\text{subsets of } \mathbb{A}^n\} \underset{Z}{\overset{I}{\rightleftarrows}} \{\text{subsets of } k[x_1, \ldots, x_n]\}$$

(Z: common zeros)

(not inverses though)

Prop

Let $Y \subseteq \mathbb{A}^n$ be any subset. $I(Y)$ is an ideal.

Proof:

$0 \in I(Y)$ because 0 vanishes everywhere.

Let $f, g \in I(Y)$, $h \in A$

For any $p \in Y$, $(f+g)(p) = f(p) + g(p)$
$= 0 + 0$
$= 0$

$f + g \in I(Y)$

$(hf)(p) = h(p) f(p) = h(p) \cdot 0 = 0$

$hf \in I(Y)$ □

prop 1.2

(a) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$

(c) For any two subsets Y_1, Y_2 of $\mathbb{A}^2$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(d) For any ideal $J \subseteq A$, $I(Z(J)) = \sqrt{J}$, the radical of J. $\sqrt{J} = \{f \in A \mid f^r \in J \text{ for some } r > 0\}$

(e) For any subset $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the closure of Y in the Zariski topology.

Proof:

(a) $p \in Z(T_2) \Rightarrow f(p) = 0 \ \forall f \in T_2$
$\Rightarrow f(p) = 0 \ \forall f \in T_1 \Rightarrow p \in Z(T_1)$

(b) $f \in I(Y_2) \Rightarrow f(p) = 0 \ \forall p \in Y_2$
$\Rightarrow f(p) = 0 \ \forall p \in Y_1 \Rightarrow f \in I(Y_1)$

(c) $f \in I(Y_1 \cup Y_2) \Leftrightarrow f(p) = 0 \ \forall p \in Y_1 \cup Y_2$
$\Leftrightarrow f(p) = 0 \ \forall p \in Y_1$ and $f(p) = 0 \ \forall p \in Y_2$
$\Leftrightarrow f \in I(Y_1)$ and $f \in I(Y_2)$
$\Leftrightarrow f \in I(Y_1) \cap I(Y_2)$

(d) wait

(e) (i) $Z(I(Y))$ is closed because it is Z of something.

(ii) WTS $Z(I(Y)) \supseteq Y$

$p \in Y \Rightarrow f(p) = 0 \ \forall f \in I(Y)$
$\Rightarrow p \in Z(I(Y))$

(iii) WTS Let W be any closed set, say $W = Z(J)$, Then $I(W) \supseteq J$.
$f \in J \Rightarrow f(p) = 0 \ \forall p \in Z(J) = W$
$\Rightarrow f \in I(W)$.

Thus $Z(I(W)) \subseteq Z(J) = W$ by part (a)

from (ii) with $Y = W$, $Z(I(W)) \supseteq W$

$\Rightarrow W = Z(I(W))$

(iv) Let W be any closed set containing Y. $W \supseteq Y \Rightarrow$ (b) $I(W) \subseteq I(Y)$
$\Rightarrow$ (a) $Z(I(W)) \supseteq Z(I(Y))$

(iii) says $W = Z(I(W))$

so $W \supseteq Z(I(Y))$

(v) Putting all this together, $Z(I(Y))$ is a closed set (i) that contains Y (ii) and is contained in any closed set containing Y (iv)

done.

Thm 1.3A (Hilbert's Nullstellensatz)

Let k be an algebraically closed field, let J be an ideal in $A = k[x_1, \ldots, x_n]$ and let $f \in A$ be a polynomial which vanishes at all points of $Z(J)$. Then $f^r \in J$ for some integer $r > 0$.

This needs k algebraically closed.

Examples:

(1) $k = \mathbb{R}$ $J = (x^2 + x + 1) \subseteq \mathbb{R}[x]$

$Z(J) = \emptyset$.

1 vanishes at all points of $\emptyset = Z(J)$ but no power of 1 is in $(x^2 + x + 1)$

(2) $k = \mathbb{R}$ $J = (x^2 + y^2) \subseteq \mathbb{R}[x, y]$

$Z(J) = \{(0,0)\}$

x vanishes at $(0,0)$ but no power of x is in $(x^2 + y^2)$

Proof of (d) $I(Z(J)) = \sqrt{J}$

$f \in \sqrt{J} \Leftrightarrow f^r \in J$ some $r > 0$ (def of $\sqrt{\ }$) $\Leftrightarrow$ for this r $f^r(p) = 0$ for all $p \in Z(J)$ ($\Rightarrow$ def Z $\Leftarrow$ Null)

$\Leftrightarrow f(p) = 0$ for all $p \in Z(J)$ (fields have no zero divisors)

$\Leftrightarrow f \in I(Z(J))$ (def I) □

A radical ideal is an ideal J s.t. $J = \sqrt{J}$.

$$\{\text{subsets of } \mathbb{A}^n\} \underset{Z}{\overset{I}{\rightleftarrows}} \{\text{subsets of } k[x_1, \ldots, x_n]\}$$

$$\cup \qquad\qquad\qquad \cup$$

$$\{\text{closed subsets of } \mathbb{A}^n\} \underset{Z}{\overset{I}{\rightleftarrows}} \{\text{radical ideals of } k[x_1, \ldots, x_n]\}$$

inverse bijections

Cor 1.4 There is a one-to-one inclusion reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A, given by

$Y \to I(Y)$ $J \to Z(J)$

for Y closed, J radical

Furthermore, Y is irreducible iff J is prime.

$$\begin{array}{ccc} \{\text{subsets of } \mathbb{A}^n\} & \underset{Z}{\overset{I}{\rightleftarrows}} & \{\text{subsets of } A = k[x_1,\dots,x_n]\} \\ \cup & & \cup \\ \{\text{algebraic subsets}\} & \underset{Z}{\overset{I}{\rightleftarrows}} & \{\text{radical ideals}\} \end{array}$$

J radical ideal $\Rightarrow Z(J)$ is an algebraic subset because $Z(\text{anything})$ is an algebraic subset

If Y algebraic subset, we saw $I(\text{anything})$ is an ideal. It is radical $f^r \in I(Y) \Rightarrow f^r$ vanishes on all of $Y \Rightarrow f$ vanishes on all of Y (field, no zerodivisors) $\Rightarrow f \in I(Y)$ so $I(Y)$ is radical.

Did not use Y algebraic.

So restrictions make sense.

(d) If J is a radical ideal $I(Z(J)) = \sqrt{J} = J$

(e) says if Y algebraic $Z(I(Y)) = \overline{Y} = Y$

Parts (a) and (b) show inclusion reversing.

Last thing to show:

If Y is an algebraic set, then Y is irreducible $\Leftrightarrow I(Y)$ is prime

The proof is easiest in the form

Y is reducible $\Leftrightarrow I(Y)$ is not prime

$\Rightarrow$

Say $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are both proper closed subsets of Y.

That implies Y_1, Y_2 both closed in $\mathbb{A}^n$

Find some $P \in Y_1$, $P \notin Y_2$

$Q \notin Y_1$, $Q \in Y_2$

Y_2 being closed says $Z(I(Y_2)) = \overline{Y_2} = Y_2$

so there must be $f \in I(Y_2)$, $f(P) \neq 0$

Similarly, $g \in I(Y_1)$, $g(Q) \neq 0$

$P, Q \in Y_1 \cup Y_2 = Y$ $f, g \notin I(Y)$

but fg vanishes on all of Y_1 and Y_2

so $fg \in I(Y)$

Shows $I(Y)$ not prime.

$\Leftarrow$

$I(Y)$ not prime. Find $f, g \notin I(Y)$ with $fg \in I(Y)$

$f \notin I(Y) \Rightarrow Z(f) \cap Y$ must be a proper closed subset of Y.

Similarly, $Z(g) \cap Y$ must be a proper closed subset of Y.

However $fg \in I(Y) \Rightarrow Y \subseteq Z(fg)$

$Y = Y \cap Z(fg) = Y \cap (Z(f) \cup Z(g))$

$= (Y \cap Z(f)) \cup (Y \cap Z(g))$

Shows Y is reducible. □

<u>Example 1.4.1</u>:

$\mathbb{A}^n$ is irreducible, since it corresponds to the zero ideal in $A = k[x_1, \dots, x_n]$ which is prime.

k algebraically closed. You can get that from the bijective correspondence.

$$Z(\{0\}) = \mathbb{A}^n \xleftarrow{Z} \{0\}$$

<u>proposition</u>

Let k be an infinite field and $f \in k[x_1, \dots, x_n]$, $f \neq 0$

Then there exists a point P in $\mathbb{A}^n$ such that $f(P) \neq 0$.

<u>proof</u>:

Proceed by induction on n.

$n=1$ A polynomial in one variable has finitely many roots and the field is infinite.

Assume true for n. Prove for $n+1$.

$f \in k[x_1, \dots, x_{n+1}]$, $f \neq 0$

we can write f as $a_m x_{n+1}^m + a_{m-1} x_{n+1}^{m-1} + \dots + a_1 x_{n+1} + a_0$

$a_i \in k[x_1, \dots, x_n]$

$k[x_1, \dots, x_{n+1}] = k[x_1, \dots, x_n][x_{n+1}]$

Some $a_i \neq 0$

by induction find $Q = (b_1, \dots, b_n)$ $b_i \in k$

$a_i(Q) \neq 0$

$f(b_1, \dots, b_n, x_{n+1})$ is a nonzero polynomial in one variable.

Can find a b_{n+1} with

$f(b_1, \dots, b_n, b_{n+1}) \neq 0$ □

Obviously false for finite fields.

<u>Examples 1.4.2 and 1.4.3</u>

If $f \in A$ is an irreducible polynomial then (f) is a prime ideal. (in UFD, irred $\Rightarrow$ prime)

$\Rightarrow Z(f)$ is irreducible

$n=2$ $Z(f)$ is called a curve

$n=3$ $Z(f)$ is called a surface

in general, $Z(f)$ is called a hypersurface

<u>Example 1.4.4</u>

A maximal ideal m of $A = k[x_1, \dots, x_n]$ corresponds to a minimal irreducible closed set.

That's a point.

Say $P = (a_1, a_2, \dots, a_n)$

$\{P\} = Z(\langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle)$

$\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is maximal

because $k[x_1, \dots, x_n] / \langle x_1 - a_1, \dots, x_n - a_n \rangle \cong k$

All maximal ideals in $k[x_1, \dots, x_n]$ are of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$

Used algebraically closed because we used the bijection.

$\mathbb{R}[x] / \langle x^2 + 1 \rangle \cong \mathbb{C}$ max'l not of that form

<u>Def</u>

If $Y \subseteq \mathbb{A}^n$ is an algebraic set, we define the <u>affine coordinate ring</u> $A(Y)$ of Y to be $A / I(Y)$.

You can think of elements of $A(Y)$ as functions from Y to k.

$Y \subseteq \mathbb{A}^n$ $f \in A = k[x_1, \dots, x_n]$

f certainly gives a function $f: \mathbb{A}^n \to k$. By restriction you get a function $f: Y \to k$

$f, g \in k[x_1, \dots, x_n]$ induce the same function from $Y \to k \Leftrightarrow$

$f - g \in I(Y)$

Modding out by $I(Y)$ does exactly the right equivalence.

$A(Y)$ is the ring of all functions $f: Y \to k$ that are restrictions of polynomials.

<u>Remark 1.4.6</u>

$A(Y)$ is an integral domain $\Leftrightarrow$ $I(Y)$ is prime $\Leftrightarrow Y$ is irreducible.

Similar to an exercise

$A(Y)$ is finitely generated as a k-algebra because it is a quotient of $k[x_1, \dots, x_n]$

Now let B be a domain that is a f.g. k-algebra.

$$\varphi: k[x_1, \dots, x_n] \xrightarrow{\text{surjective homomorphism}} B$$

kernel prime, call it I

$Y = Z(I)$ $\quad I(Y) = I(Z(I)) = \sqrt{I} = I$

$A(Y) = B$

<u>Def</u>

A topological space X is <u>Noetherian</u> iff it satisfies the descending chain condition for closed subsets:

for any sequence $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \dots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \dots$

<u>Example 1.4.7</u>

$\mathbb{A}^n$ with Zariski top is Noetherian.

$Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \dots$

Take $I(Y_1) \subseteq I(Y_2) \subseteq I(Y_3) \subseteq \dots$

$k[x_1, \dots, x_n]$ is Noetherian so satisfies acc

Eventually $I(Y_r) = I(Y_{r+1}) = \dots$

take Z

get $Y_r = Y_{r+1} = \dots$

<u>Question</u>: Is $\mathbb{R}^n$ with its usual metric topology Noetherian?

A: no

Let $\{a_n\}$ be a sequence of distinct positive numbers $\to 0$

Y_i = closed sphere centered at origin of radius a_i

not Noetherian

<u>Prop</u>:

In a Noetherian topological space, every nonempty set of closed subsets has a minimal element.

<u>proof</u>:

S nonempty set of closed subsets.

Pick $Y_1 \in S$ If minimal, done.

If not $Y_2 \in \subsetneq Y_1$. If Y_2 minimal, done.

If not pick $Y_3 \subsetneq Y_2 \subsetneq Y_1$

etc.

Noetherian $\Rightarrow$ eventually stops.

<u>Prop 1.5</u>

In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \dots \cup Y_r$ of irreducible closed subsets of Y.

If we require $Y_i \not\supseteq Y_j$ for $i \neq j$ (which we can), then the Y_i are uniquely determined. They are called the irreducible components of Y.

Proof: First we show the existence of such an expression.

Let S = the set of all closed subsets of X (other than $\emptyset$) that cannot be expressed as a finite union of irreducible closed sets.

We want to show $S = \emptyset$

S must have a minimal element, unless $S = \emptyset$

Let $Y \in S$ minimal. Then Y is not irreducible or it would be a finite union of irreducibles (itself)

$Y = A \cup B$, A, B proper closed subsets of Y.

By minimality of Y, $A, B \notin S$

$A = A_1 \cup \dots \cup A_s$ irred

$B = B_1 \cup \dots B_t$ irred

$Y = A_1 \cup \dots \cup A_s \cup B_1 \cup \dots \cup B_t$

contradicts $Y \in S$

$Y = Y_1 \cup \dots \cup Y_s$ Y_i irred

if $Y_i \supseteq Y_j$ $i \neq j$

you could drop Y_j from the expression. Do that.

Assume $Y_i \not\supseteq Y_j$ $i \neq j$.

Now show uniqueness.

Assume $Y = Y_1 \cup \dots \cup Y_s$

$= Y_1' \cup \dots \cup Y_t'$

$i \neq j$ $Y_i \not\supseteq Y_j$ $Y_i' \not\supseteq Y_j'$

$Y_1 = Y \cap Y_1 = (Y_1' \cup \dots \cup Y_t') \cap Y_1$

$= (Y_1' \cap Y_1) \cup \dots \cup (Y_t' \cap Y_1)$

But Y_1 is irred.

Could not all be proper closed subsets.

$Y_1 = Y_i' \cap Y_1$ some i.

$Y_1 \subseteq Y_i'$

Similarly $Y_i' \subseteq Y_j$ for some j.

$Y_1 \subseteq Y_i' \subseteq Y_j$

so $j = 1$ and $Y_1 = Y_i'$

Renumber so $Y_1 = Y_1'$

Claim: $\overline{(Y - Y_1)} = Y_2 \cup \dots \cup Y_s$ *

and $\overline{(Y - Y_1)} = Y_2' \cup \dots \cup Y_t'$

$Y_2 \cup \dots \cup Y_s$ is a closed set containing $Y - Y_1$

Now let W be any closed set containing $Y - Y_1$

Fix any $i \geq 2$ $W \supseteq Y_i - Y_1$

$Y_i = (Y_i \cap W) \cup (Y_i \cap Y_1)$

Y_i irred.

so either $Y_i = Y_i \cap W$ or $Y_i = Y_i \cap Y_1$

$Y_i = Y_i \cap Y_1 \Rightarrow Y_i \subseteq Y_1$ contradiction

so $Y_i = Y_i \cap W \Rightarrow Y_i \subseteq W$

$\Rightarrow Y_2 \cup \dots \cup Y_s \subseteq W$

Finishes proof of *

$Y_2 \cup \dots \cup Y_s = Y_2' \cup \dots \cup Y_t'$

repeat

how can that end?

If $s \leq t$

$\emptyset = Y_{s+1}' \cup \dots \cup Y_t'$

Shows $s = t$ and $Y_i = Y_i'$ all i □

<u>Corollary 1.6</u>

Every algebraic set in $\mathbb{A}^n$ can be expressed uniquely as a union of varieties, no one containing another

<u>Definition</u>: If X is a topological space, we define the <u>dimension</u> of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space

<u>Example 1.6.1</u>

$\dim(\mathbb{A}^1) = 1$

Closed sets:

$\emptyset$, $\mathbb{A}^1$, finite sets

irred:

$\mathbb{A}^1$, point

point $\subsetneq \mathbb{A}^1$

$\Rightarrow$ dimension is 1

<u>Remark</u>: Trying to prove the dimension of something directly from the definition is usually very difficult. You can often get a lower bound by finding a chain. Unless you have a very good description of all the irreducible closed sets it is very hard to prove you have the longest possible chain.

<u>Exercise 1.10(e)</u>

Give an example of a Noetherian topological space of infinite dimension.

A maximal chain does not have to a maximum chain.

For algebraic sets, algebra helps.

<u>Definition</u>

In a ring A, the <u>height</u> of a prime ideal p is the supremum of all integers n such that there exists a chain $p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_n = p$ of distinct prime ideals.

We define the <u>dimension</u> (or Krull dimension) of A to be the supremum of the heights of all prime ideals.

<u>prop 1.7</u>

If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y)$

<u>proof</u>:

If Y is an algebraic set in $\mathbb{A}^n$, then the closed irreducible subsets of Y correspond to prime ideals of $A = k[x_1, \dots, x_n]$ containing $I(Y)$. These in turn correspond to prime ideals of $A(Y) = \frac{k[x_1, \dots, x_n]}{I(Y)}$

Hence $\dim Y$ is the length of the longest chain of prime ideals in $A(Y)$, which is $\dim A(Y)$. □

<u>Theorem 1.8A</u>

Let k be a field and let B be an integral domain that is finitely generated as a k-algebra. Then:

(a) the dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k

(b) For any prime ideal p in B, we have height p + $\dim B/p = \dim B$

<u>prop 1.9</u>

The dimension of $\mathbb{A}^n$ is n

<u>Proof</u>

According to (1.7) this is equivalent to the claim that the dimension of the polynomial ring $k[x_1, \dots, x_n]$ is n, which follows from part (a) of the theorem.

$k(x_1, \dots, x_n)$ has transcendence degree n over k.

<u>Remark</u>:

when you have $Y \subsetneq X$, Y closed, and you know $\dim X = m < \infty$ you can sometimes get $\dim Y$ directly from def. without too much difficulty.

Get lower bounds on $\dim Y$ by finding $Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_t \subseteq Y$

Get upper bounds on $\dim Y$ by finding chains $Y \subseteq W_1 \subsetneq W_2 \subsetneq \dots \subsetneq W_s = X$

If you can get upper = lower, you are done.

<u>prop 1.10</u>

If Y is a quasi-affine variety then $\dim Y = \dim \overline{Y}$

quasi-affine variety = open subset of an affine variety.

This is something special about algebraic sets.

<u>Exercise 1.10(c)</u>

Give an example of a topological space X and a dense open set U with $\dim U < \dim X$

<u>Two examples skipped</u>:

<u>Example 1.1.3</u>

Any nonempty open subset of an irreducible space is irreducible and dense.

In Zariski topology, open sets are very big.

<u>Example 1.1.4</u>

If Y is an irreducible subset of X, then its closure $\overline{Y}$ in X is also irreducible.

<u>Cor</u>

If Y is irred and U_1, U_2 are nonempty open subsets of Y, then $U_1 \cap U_2 \neq \emptyset$

<u>proof</u>:

Suppose $U_1 \cap U_2 = \emptyset$

then $Y - U_1$ is a closed subset of Y.

Not all of Y. It contains U_2

contradict U_2 dense. □

<u>proof of 1.1.3</u>:

Y is irred. U open in Y, $U \neq \emptyset$

Work in induced top on Y.

Show U is dense. Assume U is not dense $\overline{U} \subsetneq Y$

$Y - U \subsetneq Y$

$Y = \overline{U} \cup (Y - U)$

Contradicts Y irred.

Show U irred.

If not, there exist closed subsets Y_1, Y_2 of Y s.t.

$U = (U \cap Y_1) \cup (U \cap Y_2)$

and each of $U \cap Y_i$ is a proper subset of U.

$U \nsubseteq Y_i$

Example 1.1.3: Any nonempty open subset of an irreducible space is irreducible and dense.

proof: Dense ✓

For irreducible:

if not ∃ closed subsets Y_1 and Y_2 of Y s.t. $U=(U\cap Y_1)\cup(U\cap Y_2)$ and each of $U\cap Y_i$ is a proper subset of U meaning $U\not\subseteq Y_i$. This means each Y_i is a proper subset of Y.

$U=(U\cap Y_1)\cup(U\cap Y_2)$

$\Rightarrow U\subseteq Y_1\cup Y_2$

but $\bar{U}=Y$

$\Rightarrow Y\subseteq Y_1\cup Y_2$

$\Rightarrow Y=Y_1\cup Y_2$

contradicting irreducibility of Y.

Example 1.1.4

If Y is an irreducible subset of X, then its closure $\bar{Y}$ in X is also irreducible.

Proof: Assume $\bar{Y}$ is not irreducible.

∃ closed subsets Y_1, Y_2 of X with $\bar{Y}\not\subseteq Y_1$, $\bar{Y}\not\subseteq Y_2$,

$\bar{Y}=(\bar{Y}\cap Y_1)\cup(\bar{Y}\cap Y_2)$

Since $Y\subseteq\bar{Y}$ we certainly have

$Y=(Y\cap Y_1)\cup(Y\cap Y_2)$

Because Y irred,

$Y=Y\cap Y_1$ or $Y=Y\cap Y_2$

Say $Y=Y\cap Y_1$

This says $Y\subseteq Y_1$

which implies, since Y_1 closed,

$\bar{Y}\subseteq Y_1$

which is a contradiction □

Thm 1.11A (Krull's Hauptidealsatz)

let A be a Noetherian ring, and let $f\in A$ be an element which is neither a zero divisor nor a unit. Then every minimal prime ideal p containing f has height 1.

prop 1.12A

A Noetherian integral domain A is a unique factorization domain iff every prime ideal of height 1 is principal

prop 1.13

A variety Y in $\mathbb{A}^n$ has dimension $n-1$ iff it is the zero set of a single irreducible polynomial in $A=k[x_1,\dots,x_n]$

proof

If f is an irreducible polynomial we have already seen that $Z(f)$ is a variety. Its ideal is the prime ideal $p=(f)$. Since $p=(f)$ is certainly the minimal prime containing f. Thm 1.11A says p has height 1. Put this into theorem 1.18A part b.

$\text{height } p + \dim \frac{k[x_1,\dots,x_n]}{p} = \dim k[x_1,\dots,x_n]$ (height p = 1, right side = n)

conclude $\dim \frac{k[x_1,\dots,x_n]}{p} = n-1$

But $\frac{k[x_1,\dots,x_n]}{p}=A(Y)$

prop 1.7 says that $\dim Y=\dim A(Y)=n-1$

Now suppose Y has dim $n-1$ and say $I(Y)=p$. Thm 1.8A (b) ⇒

$\text{height } p + \dim \frac{k[x_1,\dots,x_n]}{p} = \dim k[x_1,\dots,x_n]$ (the quotient has dim $n-1$, right side = n)

conclude height $p=1$

Now the polynomial ring is a UFD, so by 1.12A, p is principal necessarily generated by a single irreducible poly f.

Clearly $Y=Z(f)$.

★ Two exercises that are important and easy.

1.8 let Y be an affine variety of dimension r in $\mathbb{A}^n$. let H be a hypersurface in $\mathbb{A}^n$, and assume $Y\not\subseteq H$. Then every irreducible component of $Y\cap H$ has dimension $r-1$.

(use some algebra thms prev showed)

1.9 let $J\subseteq A=k[x_1,\dots,x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(J)$ has dimension $\geq n-r$.

(induction, 1 gen almost like Prop 1.13 use prop 1.13 on factors, for induction step use exercise 1.8)

2 Projective Varieties

First some facts about graded rings.

Def: let R be a ring. A *grading* of R is an expression of the additive group $(R,+)$ as an internal direct sum $R=\bigoplus_{i=0}^{\infty}R_i$ with the property that if $a\in R_e$, $b\in R_d$ then $ab\in R_{e+d}$ written $R_e\cdot R_d\subseteq R_{e+d}$

A graded ring is a ring together with a given grading on it. An element of R_d is said to be homogeneous of degree d.

Example:

$R=k[x_1,\dots,x_n]$

$R_i=\{f\in R \mid \text{all monomials of } f \text{ have degree } i\}\cup\{0\}$

0 is homogeneous of every degree.

$\mathbb{C}[x,y]$

$x^2+4xy-17y^2$ homogeneous degree 2

x^3-xy not homogeneous

Remark: You can replace $\bigoplus_{i=0}^{\infty}R_i$ with $\bigoplus_{i\in M}R_i$ where M is any monoid. (since need addition)

let R be a graded ring.

If each $f\in R$ has unique expression $f=f_0+f_1+\dots+f_d$, f_i homogeneous of degree i.

That's just def of ⊕.

prop/def:

let R be a graded ring and $I\subseteq R$ an ideal. The following conditions are equivalent:

(a) I can be generated by homogeneous elements.

(b) $I=\bigoplus_{i=0}^{\infty}(I\cap R_i)$ as a group

(c) Given any $f\in R$, write $f=f_0+f_1+\dots+f_d$, f_i homg of degree i. Then $f\in I \Leftrightarrow$ all $f_i\in I$

An ideal satisfying any one, hence all of these conditions is called a homogeneous or graded ideal.

Proof:

(b) ⇔ (c) is just the def of direct sum

In (c) note that all $f_i\in I$ certainly implies $f\in I$ in any case

The thing to check is $f\in I\Rightarrow$ all $f_i\in I$

(c) ⇒ (a)

write every $f\in I$ as $f=f_0+f_1+\dots+f_d$, f_i homog deg i

Certainly I is generated by all the f_i that appear as f varies over all $f\in I$.

(a) ⇒ (c)

Say I is generated by $\{f_\alpha\}_{\alpha\in A}$ where f_α is homogeneous of degree d_α.

Pick $F\in I$, then

$F=\sum_{i=1}^{n}a_if_{\alpha_i}$

write $F=F_0+F_1+\dots+F_d$

F_j homog deg j

What is F_j in terms of the expression $F=\sum_{i=1}^{n}a_if_{\alpha_i}$

$F_j=\sum_{i=1}^{n}\big((\text{degree } j-\underbrace{\deg(f_{\alpha_i})}_{d_{\alpha_i}})\text{ piece of } a_i\big)f_{\alpha_i}$

$\in I$ □

prop. let R be a graded ring.

(a) If I and J are homogeneous ideals then so are $I+J$, IJ, $I\cap J$, and $\sqrt{J}$

(b) let I be a homogeneous ideal. Then I is a prime ⇔ for any two homogeneous elements $f,g\in R$, if $fg\in I$ then either $f\in I$ or $g\in I$.

(c) If $f,g\in R$, $f\neq0$, $g\neq0$ and R is an integral domain then fg is homogeneous iff both f and g are homogeneous.

proof

(b) "⇒" obvious

"⇐"

We have that for any two homogeneous $f,g\in R$,

$fg\in I\Rightarrow$ either $f\in I$ or $g\in I$

we need for any two $f,g\in R$ homog or not $fg\in I\Rightarrow f\in I$ or $g\in I$.

$f,g\in R$

$f=f_0+f_1+\dots+f_d$ f_i homog deg i

$g=g_0+g_1+\dots+g_e$ g_i homog deg i

$fg=(f_dg_e)+(f_{d-1}g_e+f_dg_{e-1})$
$+(f_{d-2}g_e+f_{d-1}g_{e-1}+f_dg_{e-2})$
$+\dots$

$f = f_0 + f_1 + \dots + f_d$

$g = g_0 + g_1 + \dots + g_e$

$$fg = (f_d g_e) + (f_{d-1} g_e + f_d g_{e-1}) + (f_{d-2} g_e + f_{d-1} g_{e-1} + f_d g_{e-2}) + \dots$$

If all $f_i \in I$ then $f \in I$ and we are done.

If all $g_i \in I$, then $g \in I$ and we are done.

Thus we assume some $f_i \notin I$ and some $g_j \notin I$.

Let s be the largest number s.t. $f_s \notin I$ and let t be the largest number s.t. $g_t \notin I$.

Look at the degree $s+t$ piece of fg.

Because $fg \in I$ and I is homogeneous it is in I. It will have the form $f_s g_t$ + terms of the form $f_i g_j$ with either $i > s$ or $j > t$.

All these extra terms are in I the whole sum is in the ideal thus $f_s g_t \in I$

$\Rightarrow$ either $f_s \in I$ or $g_t \in I$

Contradiction □

For I a homogeneous ideal of R we often denote $I \cap R_i$ by I_i.

It then follows that

$$R/I \cong \frac{\oplus R_i}{\oplus I_i} \cong \bigoplus_{i=0}^{\infty} \frac{R_i}{I_i}$$

Thus R/I is naturally a graded ring with

$$(R/I)_i = R_i/I_i$$

Now suppose R is a Noetherian graded ring and $I \subset R$ is a homogeneous ideal. We know I can be generated by homogeneous elements because it is homogeneous. Because R is Noetherian, I can be generated by finitely many elements.

Can it be generated by finitely many homogeneous elements?

Yes. If you set up the proof of acc $\Rightarrow$ every ideal f.g. you can prove acc $\Rightarrow$ every set of generators for an ideal has a finite subset that generates.

Now for projective space.

You may have heard "parallel lines meet at infinity". We will eventually see what this means in projective space.

The basic idea is to "compactify" $\mathbb{A}^n$ by adding "points at infinity". This is similar to the way $\mathbb{R}^1$ is compactified to S^1 or $\mathbb{R}^2$ to S^2.

This is not apparent at all from the initial definition of projective space, but will eventually become clear.

Consider $\mathbb{A}^{n+1}$ with coordinates $X_0, X_1, \dots, X_n$ (note that the funny numbering).

On $\mathbb{A}^{n+1} \setminus \{(0,\dots,0)\}$ we define the following equivalence relation.

$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \iff$ there exists $\lambda \in k^*$ s.t.

$$(a_0, \dots, a_n) = (\lambda b_0, \dots, \lambda b_n)$$

The equivalence class of $(a_0, \dots, a_n)$ is the one dimensional subspace of $\mathbb{A}^{n+1}$ spanned by $(a_0, \dots, a_n)$ minus $(0, \dots, 0)$

= the line through the origin and $(a_0, \dots, a_n)$ minus the origin.

Def

Projective n-space over k denoted $\mathbb{P}^n_k$ or $\mathbb{P}^n(k)$ or $\mathbb{P}^n$

$$= \frac{\mathbb{A}^{n+1}_k \setminus \{0,\dots,0\}}{\sim}$$

A point $P \in \mathbb{P}^n$ is an equivalence class of some point $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus \{0,\dots,0\}$

It is denoted by $[a_0, \dots, a_n]$

The a_i are called the homogeneous coordinates of the point P.

They are only well defined up to a nonzero constant multiple

$\mathbb{P}^n$ = lines through the origin in k^{n+1}

= one dimensional subspaces of k^{n+1}

Denote the polynomial ring $k[X_0, \dots, X_n]$ by S.

For $f \in S$ and $P \in \mathbb{P}^n$, $f(P)$ is not well defined, because the coordinates of P are not well defined.

But suppose $f \in S$ is homogeneous.

Then $f(\lambda a_0, \dots, \lambda a_n) = \lambda^{\deg f} f(a_0, \dots, a_n)$

$f(P)$ still not well defined but since $\lambda \in k^*$ whether $f(P) = 0$ whether $f(P) = 0$ is well defined.

Def:

For homogeneous $f \in S$ we define the zeros of f denoted $Z(f)$ by $Z(f) = \{P \in \mathbb{P}^n \mid f(P) = 0\}$

For a set of homogeneous elements $T \subseteq S$ we define the zeros of T denoted $Z(T)$ by

$$Z(T) = \bigcap_{f \in T} Z(f) = \{P \in \mathbb{P}^n \mid f(P) = 0 \ \forall f \in T\}$$

A subset $Y \subseteq \mathbb{P}^n$ is called algebraic iff $Y = Z(T)$ for some subset $T \subseteq S$ of homogeneous elements. If $I \subseteq S$ is a homogeneous ideal, we define

$$Z(I) = \{P \in \mathbb{P}^n \mid f(P) = 0 \ \forall \text{ homogeneous } f \in I\} = \bigcap_{\substack{f \in I \\ f \text{ homogeneous}}} Z(f)$$

Prop

Let $T \subseteq S$ be a set of homog. elements and let $I \subseteq S$ be the homog. ideal generated by T.

Then $Z(T) = Z(I)$

Thus every algebraic set in $\mathbb{P}^n$ is of the form $Z(I)$ for a homogeneous ideal and $Z(\{f_1, \dots, f_r\})$ for finitely many homog f_i.

The algebraic subsets of $\mathbb{P}^n$ satisfy the properties needed to be the closed subsets of a topology on $\mathbb{P}^n$. We call it the Zariski topology.

Def

A projective algebraic variety (or simply projective variety) is an irreducible algebraic set in $\mathbb{P}^n$ with the induced topology. An open subset of a projective variety is a quasi-projective variety. The dimension of a variety or quasi-variety is its dimension as a topological space.

If Y is any subset of $\mathbb{P}^n$, we define the homogeneous ideal of Y in S denoted by $I(Y)$ to be the ideal generated by $\{f \in S \mid f \text{ is homogeneous and } f(P) = 0 \ \forall P \in Y\}$

If Y is an algebraic set we define the homogeneous coordinate ring of Y to be $S(Y) = S/I(Y)$

Note that $I(Y)$ is a homogeneous ideal so $S(Y)$ is a graded ring.

Exercises 2.1 – 2.7 are to show that many of the things we proved about affine algebraic sets have their projective analogues.

There is one place where there is a funny difference. S is clearly a homogeneous ideal of S and $Z(S) = \emptyset$. $S = k[X_0, \dots, X_n]$

$(X_0, \dots, X_n)$ is also clearly a homogeneous ideal of S.

It is often called S_+ because it contains all homogeneous elements of positive degree.

$Z(S_+) = \emptyset$. That's the only thing that goes wrong.

Exercise 2.4 (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $J \mapsto Z(J)$.

Note that since S_+ does not occur in the correspondence, it is sometimes called the irrelevant maximal ideal.

2.1 – 2.7 mostly easy, 2.6 hard

Now we work on seeing how $\mathbb{P}^n$ is $\mathbb{A}^n$ with points added at ∞.

In $\mathbb{P}^n$ consider closed sets of the form $Z(f)$ with f a nonconstant polynomial. These are called hypersurfaces. When f is linear, it is called a hyperplane. Of particular interest is when $f = X_i$ for some i. Set $H_i = Z(X_i)$ and $U_i = \mathbb{P}^n \setminus H_i$ $\quad i = 0, \dots, n$

U_i is open. The U_i are an open cover of $\mathbb{P}^n$.

$$\left(\bigcup_{i=0}^n U_i\right)^c = \bigcap_{i=0}^n U_i^c = \bigcap_{i=0}^n H_i$$

$$= \{[a_0, \dots, a_n] \mid a_i = 0 \ \forall i\} = \emptyset$$

$H_i = Z(x_i) = \{[a_0,\dots,a_n] \mid a_i = 0\}$

$U_i = \mathbb{P}^n \setminus H_i = \{[a_0,\dots,a_n] \mid a_i \neq 0\}$

U_i $i=0,\dots,n$ form an open cover of $\mathbb{P}^n$

$P = [a_0,\dots,a_n] \in U_i \iff P = \left[\frac{a_0}{a_i}, \frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, 1, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}\right]$ (1 in the ith spot)

Each point in U_i has a unique set of homogeneous coordinates s.t. the i^{th} coordinate is 1.

Define a map $\varphi_i : \mathbb{A}^n \to U_i$ by

$\varphi_i(a_1,\dots,a_n) = [a_1, a_2, \dots, 1, \dots, a_n]$ (1 in the ith spot; n+1 now)

clearly a bijection.

<u>Prop 2.2</u>

The map φ is a homeomorphism of U_i with its induced topology to $\mathbb{A}^n$ with its Zarisky topology.

The proof is based on homogenization and dehomogenization of polynomials.

To make notation easier assume $i=0$

$S = k[x_0,\dots,x_n]$

$A = k[x_1,\dots,x_n]$

S^h = the homogeneous elements of S

dehomogenization

There is a ring homomorphism $\alpha : S \to A$ defined by evaluation at $x_0 = 1$

$\alpha(f(x_0,\dots,x_n)) = f(1, x_1, \dots x_n)$

Since α is a ring homomorphisms

$\alpha(f+g) = \alpha(f) + \alpha(g)$

$\alpha(fg) = \alpha(f)\alpha(g)$

Called dehomogenization because even if f is homogeneous, $\alpha(f)$ might not be.

ex $\alpha(x_0^2 + x_0 x_1) = 1 + x_1$

homogenization

$\beta : A \to S^h$

$\beta(f(x_1,\dots,x_n)) = x_0^{\deg f} f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)$

$\beta(f)$ is homogeneous of degree $\deg f$

Look at a monomial of f.

$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ $\quad i_1 + i_2 + \dots + i_n \le \deg f$

$x_0^{\deg f} \left(\frac{x_1}{x_0}\right)^{i_1} \left(\frac{x_2}{x_0}\right)^{i_2} \cdots \left(\frac{x_n}{x_0}\right)^{i_n}$

$x_0^{\deg f - (i_1 + i_2 + \dots + i_n)} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$

has degree $\deg f$

$\beta(fg) = \beta(f)\beta(g)$

$\beta(fg) = x_0^{\deg f + \deg g} f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) g\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)$

$= x_0^{\deg f} f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) x_0^{\deg g} g\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)$

$= \beta(f)\beta(g)$

$\beta(f+g)$ might not equal $\beta(f) + \beta(g)$

if $\deg f \neq \deg g$, $\beta(f) + \beta(g)$ will not even be homogeneous

If $\deg f = \deg g = \deg f+g$ then

$\beta(f+g) = \beta(f) + \beta(g)$

$\alpha(\beta(f)) = \alpha\left(x_0^{\deg f} f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)\right) = 1^{\deg f} f\left(\frac{x_1}{1},\dots,\frac{x_n}{1}\right) = f$

$\beta(\alpha(f)) = \beta(\alpha(x_0^{\ell} f))$

Assume you have a homogeneous F that is not divisible by x_0.

$F = \sum a_{i_0 \dots i_n} x_0^{i_0} \cdots x_n^{i_n}$ $\quad i_0 + \dots + i_n = d$

$\alpha(F) = \sum a_{i_0 \dots i_n} x_1^{i_1} \cdots x_n^{i_n}$

and in at least one case $i_1 + \dots + i_n = d$

$\beta(\alpha(F)) = x_0^d \sum a_{i_0 \dots i_n} \left(\frac{x_1}{x_0}\right)^{i_1} \cdots \left(\frac{x_n}{x_0}\right)^{i_n}$

$= F$

Consider a point $(a_1,\dots,a_n) \in \mathbb{A}^n$

$\varphi_0(a_1,\dots,a_n) = [1, a_1, \dots, a_n] \in U_0 \subseteq \mathbb{P}^n$

$f \in k[x_1,\dots,x_n]$

$F \in k[x_0,\dots,x_n]$ $\quad F$ homogeneous

$f(a_1,\dots,a_n) = 1^{\deg f} f\left(\frac{a_1}{1},\dots,\frac{a_n}{1}\right) = \beta(f)(1, a_1, \dots a_n)$

$F(1, a_1, \dots a_n) = \alpha(F)(a_1,\dots,a_n)$

f vanishes on $P \in \mathbb{A}^n \iff \beta(f)$ vanishes on $\varphi_0(P) \in U_0$

F vanishes on $\varphi_0(P) \in U_0 \iff \alpha(F)$ vanishes on $P \in \mathbb{A}^n$

It is now easy to see that

$\varphi_0 : \mathbb{A}^n \to U_0 \subseteq \mathbb{P}^n$

is a homeomorphism.

Suppose $Y \subseteq \mathbb{A}^n$ is closed.

$Y = Z(T)$ $\quad T \subseteq k[x_1,\dots,x_n]$

Set $\beta(T) = \{\beta(f) \mid f \in T\}$

$\varphi_0(Y) = Z(\beta(T)) \cap U_0$

$W \subseteq U_0$ closed in U_0

$W = U_0 \cap \overline{W}$ $\quad \overline{W}$ closed in $\mathbb{P}^n$

$\overline{W} = Z(T)$ $\quad T \subseteq S^h$

$\alpha(T) = \{\alpha(F) \mid F \in T\}$

$\varphi_0^{-1}(W) = Z(\alpha(T))$

φ_0 bijective, φ_0 and φ_0^{-1} both take closed sets to closed sets. φ_0 is a homeomorphism

<u>Corollary 2.3</u>

If Y is a projective (respectively quasi-projective) variety then Y is covered by the open sets

$Y \cap U_i$ $i=0,\dots,n$ which are homeomorphic to affine (respectively quasi-affine) varieties via the map φ_i.

$\mathbb{A}^n \cong U_i \subseteq \mathbb{P}^n$ $\quad U_i = \mathbb{P}^n \setminus H_i$ $\quad H_i = Z(H_i) = \{[a_0, a_1, \dots, \underset{i^{th}}{0}, \dots a_n] \mid \text{not all } a_i = 0\} \cong \mathbb{P}^{n-1}$

$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$

Parallel lines meet at infinity

$\mathbb{A}^2$ coordinates x, y

$Ax + By + C = 0$ $\quad$ at least one of $A, B \neq 0$

$Ax + By + D = 0$ $\quad C \neq D$

Homogenize

$z\left(A\left(\frac{x}{z}\right) + B\left(\frac{y}{z}\right) + C = 0\right)$

$Ax + By + Cz = 0$

$-(Ax + By + Dz = 0)$

$(C-D)z = 0$

$z = 0$

$Ax + By = 0$

$x = \lambda B$ $\quad y = -\lambda A$

$[\lambda B, -\lambda A, 0]$ $\quad \lambda \neq 0$

one point in $\mathbb{P}^2$

$[B, -A, 0]$

it is in H_2

$x_0 \quad x_1 \quad x_2$

$x \quad\; y \quad\; z$

Slope of $Ax + By + C = 0$ is the ratio $-\frac{A}{B}$

Looking at a curve in different patches and seeing how they fit together.

Hyperbola $xy - 1 = 0$

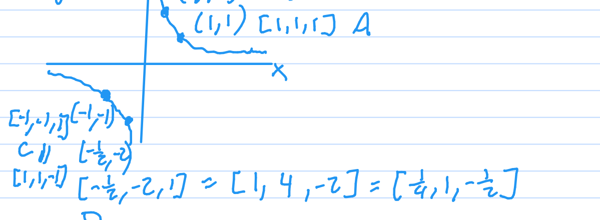
Homogenize that $xy - z^2 = 0$

$z = 1$ $\quad xy - 1 = 0$

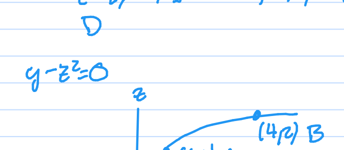
$y = 1$ $\quad x - z^2 = 0$

$x = 1$ $\quad y - z^2 = 0$

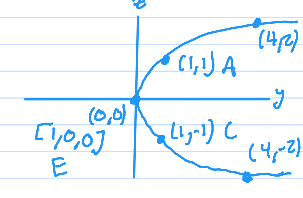
$xy - 1 = 0$



$y - z^2 = 0$



$x - z^2 = 0$



$x^2 - y^2 = 1$ v.s. $x^2 + y^2 = 1$

$y \mapsto iy$ become the same in $\mathbb{C}$

Exercise 2.12

The d-Uple Embedding (also called the Veronese embedding)

For given $n, d > 0$ let $M_0, M_1, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$ where $N = \binom{n+d}{n} - 1$

we define a mapping $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ (also denoted ν_d) by sending the point $P = [a_0, \dots, a_n]$ to $\rho_d(P) = [M_0(P), M_1(P), \dots, M_N(P)]$ obtained by substituting the a_i in the monomials M_i.

This is called the d-Uple or Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$

$\rho_d(P) = [M_0(P), M_1(P), \ldots, M_N(P)]$

First note that the map is well-defined.

$\rho_d([\lambda a_0, \lambda a_1, \ldots, \lambda a_n]) = \lambda^d \rho_d([a_0, a_1, \ldots, a_n])$

because all of the M_i are homogeneous of degree d.

what is wrong with:
$f : \mathbb{P} \to \mathbb{P}^2$

$f([x_0, x_1, x_2]) = [x_0 x_1, x_0 x_2, x_0^2]$

$0, 0, 1 \to 0, 0, 0$ illegal point

The exercise asks you to check various things.
ρ_d is one-to-one, $\rho_d(\mathbb{P}^n)$ is a closed subset of $\mathbb{P}^N$, if we give $\rho_d(\mathbb{P}^n)$ its induced topology as a subset of $\mathbb{P}^N$ then
$\rho_d : \mathbb{P}^n \to \rho_d(\mathbb{P}^n)$ is a homeomorphism.
It seems reasonable to call it an embedding.

Example:

$n=1 \quad d=2$

$[a_0, a_1] \mapsto [a_0^2, a_0 a_1, a_1^2]$ (x, y, z)

Image is contained in $Z(xz - y^2)$

In fact $=$.

Exercise 2.14

The Segre Embedding

Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair
$[a_0, \ldots, a_r] \times [b_0, \ldots, b_s] \mapsto [\ldots, a_i b_j, \ldots]$
all $a_i b_j$

$N = rs + r + s = (r+1)(s+1) - 1$

This is well defined

$[\lambda a_0, \ldots, \lambda a_n] \times [\mu b_0, \ldots, \mu b_s]$
$= \lambda\mu[\ldots, a_i b_j, \ldots]$

Example:

$\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$

$[a_0, a_1] \times [b_0, b_1] \mapsto [a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1]$ (w, x, y, z)

image lies on $wz - xy = 0$

in fact $=$

Book asks you to show ψ is one-to-one and image is closed.
The book does not ask you to show that ψ is a homeomorphism onto its image when you give the image its induced topology. Why? We have not defined what the Zarisky topology on $\mathbb{P}^r \times \mathbb{P}^s$ is.

Exercise 1.4

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zarisky topology on $\mathbb{A}^2$ is not the product topology of the Zarisky topologies on the two $\mathbb{A}^1$'s.

Hint: In $\mathbb{A}^2$ with coordinates x, y
$Z(x-y)$ is closed in $\mathbb{A}^2$ with Zarisky topology, but not product topology.

on product topology / not closed on product topology

Three ways to define the Zarisky topology on $\mathbb{P}^r \times \mathbb{P}^s$.

(1) $\mathbb{P}^r$ is covered by U_i's with each U_i homeomorphic to $\mathbb{A}^r$.
$\mathbb{P}^s$ is covered by U_j's with each U_j homeomorphic to $\mathbb{A}^s$.
$\mathbb{P}^r \times \mathbb{P}^s$ will be covered by $U_i \times U_j$'s.
Give $U_i \times U_j$ the Zarisky topology of $\mathbb{A}^{r+s}$
$U_i \times U_j \cap U_k \times U_\ell$
The topology induced from $U_i \times U_j$ is the same as the topology induced from $U_k \times U_\ell$.
So you can get a topology on $\mathbb{P}^r \times \mathbb{P}^s$
$X \subseteq \mathbb{P}^r \times \mathbb{P}^s$ is closed iff $X \cap U_i \times U_j$ closed all i, j.

(2) On $\mathbb{P}^r$, take homogeneous coordinates $x_0, \ldots, x_r$ and on $\mathbb{P}^s$ take homogeneous coordinates $y_0, \ldots, y_s$. A polynomial $f \in k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ is said to be bihomogeneous of bidegree d, e iff every monomial of f has degree d in the x_i's and degree e in the y_j's.
In this case, $f(\lambda a_0, \ldots, \lambda a_r, \mu b_0, \ldots, \mu b_r)$
$= \lambda^d \mu^e f(a_0, \ldots, a_r, b_0, \ldots, b_r)$
Whether f vanishes on $(P, Q) \in \mathbb{P}^r \times \mathbb{P}^r$
Define $Y \subseteq \mathbb{P}^r \times \mathbb{P}^s$ is closed iff
$Y = Z(T)$ for some subset $T \subseteq k[x_0, \ldots y_s]$ of bihomogeneous polynomials.

(3) Given $\mathbb{P}^r \times \mathbb{P}^s$ the unique topology it must have for the segre map
$\psi : \mathbb{P}^r \times \mathbb{P}^s \to \psi(\mathbb{P}^r \times \mathbb{P}^s) \subseteq \mathbb{P}^N$ to be a homeomorphism when $\psi(\mathbb{P}^r \times \mathbb{P}^s)$ is given the induced topology as a subset of $\mathbb{P}^N$.

All three give the same topology.

3 Morphisms

Note: Any affine variety is also a quasi-affine variety. Same thing holds for projective and quasi-projective

Definition

Let Y be a quasi-affine variety in $\mathbb{A}^n$. A function $f : Y \to k$ is regular at a point $P \in Y$ iff there is an open neighborhood U with $P \in U \subseteq Y$, and polynomials $g, h \in A = k[x_1, \ldots, x_n]$ such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. We say f is regular on Y iff it is regular at every point of Y.
"A function is regular iff it is locally a rational function."

Lemma 3.1

A regular function is continuous, when k is identified with $\mathbb{A}^1_k$ with its Zarisky topology.

Note: It is important to remember this is not iff. Continuous does not imply regular.
$f : \mathbb{A}^1 \to \mathbb{A}^1$ every bijection is continuous.
not every bijection is locally a rational function.

proof:

It is enough to show that f^{-1} of closed is closed. $f^{-1}(\emptyset) = \emptyset$. $f^{-1}(\mathbb{A}^1_k) = Y$.
Need f^{-1}(finite) closed.
Since finite unions of closed are closed, we just need to show f^{-1}(point) closed.
This can be checked locally:
a subset Z of a topological space Y is closed if there exists an open cover of Y s.t. $Z \cap U$ is closed in U for every U in cover.
let U be an open set on which f can be represented as $f = \frac{g}{h}$
g, h polynomials, $h \neq 0$ on U.
Such U form an open cover.
$f^{-1}(a) \cap U = \{P \in U \mid \frac{g(P)}{h(P)} = a\}$
$\frac{g(P)}{h(P)} = a$ iff $\underbrace{(g - ah)}_{\text{polynomial}}(P) = 0$
that is closed. □

Functions on projective spaces are more tricky.
$P \in \mathbb{P}^n \quad f \in k[x_0, \ldots, x_n]$
$f(P)$ not well defined even if f is homogeneous.
$f, g \in k[x_0, \ldots, x_n]$ both homogeneous of the same degree d. Also assume that
$g(P) \neq 0$. $P = [a_0, \ldots, a_n]$

$$\frac{f(\lambda a_0, \ldots \lambda a_n)}{g(\lambda a_0, \ldots \lambda a_n)} = \frac{\lambda^d f(a_0, \ldots, a_n)}{\lambda^d g(a_0, \ldots, a_n)} = \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)}$$

$\frac{f}{g}(P)$ is well defined.

Definition:

Let Y be a quasi-projective variety in $\mathbb{P}^n$. A function $f : Y \to k$ is regular at a point $P \in Y$ iff there is an open neighborhood U with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S = k[x_0, \ldots, x_n]$ of the same degree, such that h is nowhere zero on U and $f = \frac{g}{h}$ on U.
We say f is regular iff it is regular at every point.

proposition:

Let $Y \subseteq \mathbb{P}^n$ be a quasi-projective variety.
$f : Y \to k$ a function, $P \in Y$ and assume $P \in U_i = \mathbb{P}^n \setminus Z(x_i)$. Then when we think of U_i as $\mathbb{A}^n$, $Y \cap U_i$ is a quasi-affine variety and by restriction we get a function $f|_{Y \cap U_i} : Y \cap U_i \to k$
Then f is regular at P under the projective definition iff $f|_{Y \cap U_i}$ is regular at P under the affine definition.

(out sick)
Notes from Silvia

Proof: To make notation easier assume $i = 0$.
Assume f is regular at P under the projective definition. Thus we have an open set U with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S = k[x_0, \dots, x_n]$ of the same degree, s.t. h is nowhere 0 on U and $f = g/h$ on U.
On $U \cap U_0$ which is open in $Y \cap U_0$ $\frac{g(a_0, \dots, a_n)}{h(a_0, \dots, a_n)} = \frac{g(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})}{h(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})}$

$g(1, x_1, \dots, x_n)\, h(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ w/ $h \neq 0$ on $Y \cap U_0$, so $f|_{Y \cap U_0}$ is regular at P under the affine definition.

Assume $f|_{Y \cap U_0}$ is regular at P under the affine definition. There is an open neighborhood U with $P \in U \subseteq Y \cap U_0$ and polynomials $g, h \in k[x_1, \dots, x_n]$ s.t. h is nowhere 0 on U and $f = g/h$ on U.

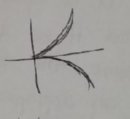
Note that U is open in Y not just $Y \cap U_0$.
Let G and H be the homogenizations of g and h. If they do not have the same degree multiply the one of lower degree by an appropriate power of x_0 to make them both have the same degree. Now call them $\bar{G}, \bar{H}$. For any point $P = [1, a_1, \dots, a_n] \in U$ $f(P) = \frac{g(P)}{h(P)} = \frac{\bar{G}(1, a_1, \dots, a_n)}{\bar{H}(1, a_1, \dots, a_n)} = \frac{\bar{G}(\lambda, \lambda a_1, \dots, \lambda a_n)}{\bar{H}(\lambda, \lambda a_1, \dots, \lambda a_n)}$
so f regular at P under proj. def. □

Fact: X a variety, f, g regular functions on X.
Suppose $f = g$ on some nonempty $U \subseteq X$. Then $f = g$ on all of X.
Proof: $Z(f-g)$ is closed in X and since in an irreducible space nonempty opens are dense it is also dense
$Z(f-g) = X$. □

Category of varieties:
Definition: Let k be a fixed algebraically closed field. A variety over k is an affine, quasi-affine, projective or quasi-projective variety. If X and Y are two varieties, a morphism $\varphi: X \to Y$ is a continuous map such that for every open $V \subseteq Y$ and for every regular function $f: V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.

A morphism $\varphi: X \to Y$ is an isomorphism iff there exists a morphism $\psi: Y \to X$ s.t. $\varphi \circ \psi =$ identity on Y and $\psi \circ \varphi =$ identity on X.
isomorphism $\Rightarrow$ bijective, bicontinuous hence homeomorphism
however "$\Leftarrow$" does not hold
$\exists$ bijective, bicont. that are morphisms in one direction but not in the other.

Ex. 3.2: $X = \mathbb{A}^1_t$, $Y = Z(y^2 - x^3) \subseteq \mathbb{A}^2_{x,y}$



$\varphi: X \to Y$, $\varphi(t) = (t^2, t^3)$
φ is a morphism, one-to-one, onto, φ^{-1} exists
check it is continuous, but φ^{-1} is not a morphism.
There exists a regular fct. $f: X \to k$ s.t. $f \circ \varphi^{-1} : Y \to k$ is not regular. f is locally a rational function but $f \circ \varphi^{-1}$ is not. Problem at the cusp.

Alternative definition of morphism:
Proposition: Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties and $f: X \to Y$ a function. Then f is a morphism iff the following condition is satisfied.
Given any $P \in X$ there exists a neighborhood U of P, $P \in U \subseteq X$ such that $f(U)$ is

contained in one of the affine opens $U_i \subseteq \mathbb{P}^m$. To make notation easier call it U_0. Now think of $f: U \to \mathbb{A}^m$. We require further that there exists m regular functions $g_i: U \to k$ s.t. for all $Q \in U$: $f(Q) = (g_1(Q), \dots, g_m(Q))$
Further since regular fct. are locally rational by shrinking U you may assume each g_i is a rational function.

A morphism is a fct. locally given by tuples of rational functions.

We have shown polynomials are continuous in Zariski topology. Easily follows rational functions are continuous where defined.

A composition of rational functions is rational

Ex.: Consider the Veronese $\nu: \mathbb{P}^1 \to \mathbb{P}^2$,
$\nu([x_0, x_1]) = [x_0^2, x_0 x_1, x_1^2]$
Is a morphism. On $U_0 \subseteq \mathbb{P}^1$ $\nu(U_0) \subseteq U_0'$
$\nu(1, \frac{x_1}{x_0}) = [1, \frac{x_1}{x_0}, (\frac{x_1}{x_0})^2]$ $\nu(y) = [y, y^2]$

$U_1 \subseteq \mathbb{P}^1$, $\nu(U_1) \subseteq U_2'$
$\nu(\frac{x_0}{x_1}, 1) = [(\frac{x_0}{x_1})^2, \frac{x_0}{x_1}, 1]$
$\nu(y) = [y^2, y]$

Proposition: Let $X \subseteq \mathbb{P}^n$ & $Y \subseteq \mathbb{P}^m$ be quasi projective varieties and $f: X \to Y$ a function. Then f is a morphism iff the following condition is satisfied:
Given any $P \in X$ there is an open neighb.hood U of P, $P \in U \subseteq X$ and m polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ all homogeneous of the same deg. s.t. $\forall Q \in U$: $f(Q) = [F_0(Q), \dots, F_m(Q)]$

Example: We can find the inverse of the Veronese embedding $\nu: \mathbb{P}^1 \to \mathbb{P}^2$, $\nu([x_0, x_1]) = [\underset{y_0}{x_0^2}, \underset{y_1}{x_0 x_1}, \underset{y_2}{x_1^2}]$
on $U_0 \subseteq \mathbb{P}^2$ $\nu^{-1}[y_0, y_1, y_2] = [y_0, y_1]$
$U_0 \cap \nu(\mathbb{P}^1)$ On $U_2 \cap \nu(\mathbb{P}^1) \subseteq \mathbb{P}^2$ $\nu^{-1}([y_0, y_1, y_2]) = [y_1, y_2]$
U_0 & U_2 cover $\nu(\mathbb{P}^1)$

- $\underset{\in U_0}{[x_0, x_1]} \longmapsto [x_0^2, x_0 x_1, x_1^2] \longmapsto [x_0^2, x_1 x_0] = [x_0, x_1]$ (divide by x_0, $\in \mathbb{P}^1$)
- $\underset{\in U_1}{[x_0, x_1]} \longmapsto [x_0^2, x_0 x_1, x_1^2] \longmapsto [x_0 x_1, x_1^2] = [x_0, x_1]$
- $\underset{\in U_0}{[y_0, y_1, y_2]} \longmapsto [y_0, y_1] \longmapsto [y_0^2, y_0 y_1, y_1^2] = [y_0^2, y_0 y_1, y_0 y_2] = [y_0, y_1, y_2]$
  only working on $\nu(\mathbb{P}^1)$ $y_0 y_2 = y_1^2$
- $\underset{\in U_2}{[y_0, y_1, y_2]} \longmapsto [y_1, y_2] \longmapsto [y_1^2, y_1 y_2, y_2^2] = [y_0 y_2, y_1 y_2, y_2^2] = [y_0, y_1, y_2]$

$\nu(\mathbb{P}^1) \cap U_0 \cap U_2$ on $\mathbb{P}^2$
$[y_0, y_1, y_2] \longmapsto [y_0, y_1] = [y_0 y_2, y_1 y_2]$
$\longmapsto [y_1, y_2] = [y_1^2, y_1 y_2]$

$y_0 y_2 = y_1^2$

Definition: Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the local ring of P on Y, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the ring of germs of regular fct. on Y near P.
Equivalence classes of pairs $\langle U, f \rangle$ where U is an open neighborhood of P in Y and f is a regular function on U. $\langle U, f \rangle \sim \langle V, g \rangle \Leftrightarrow f = g$ on $U \cap V$

$\langle U, f \rangle + \langle V, g \rangle = \langle U \cap V, f+g \rangle$
$\langle U, f \rangle \cdot \langle V, g \rangle = \langle U \cap V, fg \rangle$
Y is a variety hence irreducible.

Notice that for $\langle U, f\rangle \in \mathcal{O}_{P,Y}$ $f(P)$ is well-defined since all U contain P.

The evaluation map $ev: \mathcal{O}_{P,Y} \to k$

$ev(\langle U,f\rangle) = f(P)$ is easily seen to be a ring homomorphism.

Surjective $\langle Y, a\rangle \in \mathcal{O}_{P,Y}$ a is the constant function a for $a \in k$. The kernel is a maximal ideal $m_{P,Y} = \{\langle U, f\rangle \in \mathcal{O}_{P,Y} ; f(P) = 0$

$$\mathcal{O}_{P,Y} / m_{P,Y} \cong k$$

Prop

$m_{P,Y}$ is the only maximal ideal in $\mathcal{O}_{P,Y}$.

Proof:

Say $\langle U, f\rangle \notin m_{P,Y}$

This says $f(P) \neq 0$. On a perhaps smaller open $p \in U_1 \subseteq U$ we may write $f = \frac{g}{h}$ with $g, h \in k[x_1, \dots, x_n]$ $h \neq 0$ on U_1 $g(P) \neq 0$. Let $U_2 = U_1 \setminus Z(g)$

$p \in U_2$. $\frac{h}{g}$ is regular on U_2

so $\langle U_2, \frac{h}{g}\rangle \in \mathcal{O}_{P,Y}$.

$\langle U_2, \frac{h}{g}\rangle \cdot \langle U, f\rangle = \langle U_2, \frac{h}{g}\rangle \cdot \langle U_2, \frac{g}{h}\rangle$
$= \langle U_2, 1\rangle$

$\langle U, f\rangle$ is invertible in $\mathcal{O}_{P,Y}$

$\langle U, f\rangle$ is not contained in any proper ideal. So $m_{P,Y}$ is the unique maximal ideal.

Recall: For any commutative Noetherian ring R we call R a local ring iff it has only one maximal ideal m. we call R/m the residue field of R.

Def:

If Y is a variety, we define the function field of $K(Y)$ of Y as follows: an element of $K(Y)$ is an equivalence class of pairs $\langle U, f\rangle$, where U is a nonempty open subset of Y, f is a regular function on U, and we define

$\langle U, f\rangle \sim \langle V, g\rangle$ iff $f = g$ on $U \cap V$.

$\langle U, f\rangle + \langle V, g\rangle = \langle U \cap V, f+g\rangle$

$\langle U, f\rangle \langle V, g\rangle = \langle U \cap V, fg\rangle$

The elements of $K(Y)$ are called rational functions on Y.

Need Y irreducible so that any two opens intersect. otherwise $+, \times$ do not even make sense.

$K(Y)$ is a field.

$\langle U, f\rangle$ $f \neq 0$

$\langle V, \frac{g}{h}\rangle$ $V' = V \setminus Z(g)$

$\langle V', \frac{g}{h}\rangle$ $\langle V', \frac{g}{h}\rangle^{-1} = \langle V', \frac{h}{g}\rangle$

We have natural ring homomorphisms:

$k \hookrightarrow \mathcal{O}(Y) \hookrightarrow \mathcal{O}_{P,Y} \hookrightarrow K(Y)$

$a \mapsto$ constant function a

$f \mapsto \langle U, f\rangle \mapsto \langle U, f\rangle$

easy to check ring homomorphism and injective.

$\mathcal{O}(Y)$, $\mathcal{O}_{P,Y}$, $K(Y)$ are all k-algebras

Often think of them as subrings

Let $f: X \to Y$ be a morphism of varieties. Since by definition, under a morphism, regular functions pull back to regular functions, it is easy to see we get the following ring homomorphisms.

$f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$

If $P \in X$, $f(P) \in Y$

$f^*: \mathcal{O}_{f(P),Y} \to \mathcal{O}_{P,X}$

$f^*\langle U, g\rangle = \langle f^{-1}(U), f^*(g)\rangle$

k-algebra homomorphisms.

When f is an isomorphism these are both isomorphisms.

$(f^{-1})^* = (f^*)^{-1}$

$\mathcal{O}(Y)$ and $\mathcal{O}_{P,Y}$ are isomorphism invariants.

Why did we not talk about $f^*: K(Y) \to K(X)$?

Because it is not always defined.

If $f(X) \subseteq$ proper closed subset of Y, there will be some open $U \subseteq Y$ s.t. $f^{-1}(U) = \emptyset$

Very quick review of localization in commutative rings.

Def:

Let R be a ring and $S \subseteq R$.

We say S is multiplicatively closed iff $1 \in S$ and if $a, b \in S$ then $ab \in S$.

Given a ring R and a multiplicatively closed set $S \subseteq R$, we define the localization of R with respect to S, denoted $S^{-1}R$ as follows:

As a set $S^{-1}R$ = equivalence classes of elements $\frac{r}{s}$, $r \in R$, $s \in S$ under the equivalence relation

$\frac{r}{s} \sim \frac{a}{t} \Leftrightarrow \exists w \in S$ with

$w(rt - sa) = 0$

We make $S^{-1}R$ into a ring by:

$\frac{r}{s} + \frac{a}{t} = \frac{rt + as}{st}$

$\frac{r}{s} \cdot \frac{a}{t} = \frac{ra}{st}$

The most common examples of S are:

R an integral domain $S = R \setminus \{0\}$

$S^{-1}R = \mathrm{Frac}(R)$

P is a prime ideal and $S = R \setminus P$

$S^{-1}R$ is denoted R_P called the localization of R at P.

$f \in R$, f not nilpotent

$S = \{1, f, f^2, f^3, \dots\}$

$S^{-1}R$ is denoted R_f called the localization of R at f.

Warning: If f is prime so that (f) is prime. $R_f \neq R_{(f)}$

Theorem 3.2

Let $Y \subseteq \mathbb{A}^n$ be an affine variety with affine coordinated ring $A(Y)$.

Then:

(a) $\mathcal{O}(Y) \cong A(Y)$

(b) for each point $P \in Y$, let $m_P \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \mapsto m_P$ gives a 1-1 correspondence between the points of Y and the maximal ideals of $A(Y)$;

(c) for each P, $\mathcal{O}_{P,Y} \cong A(Y)_{m_P}$ and $\dim \mathcal{O}_{P,Y} = \dim Y$;

(d) $K(Y)$ is isomorphic to the quotient field of $A(Y)$ and hence $K(Y)$ is a finitely generated field extension of k, of transcendence degree $= \dim Y$.

Proof:

(b) Points of $\mathbb{A}^n$ are in bijective correspondence with maximal ideals of $k[x_1, \dots, x_n]$ (*)

$P \in Y \Leftrightarrow I(P) \supseteq I(Y)$ ($I(P)$ maximal)

Points of $Y \leftrightarrow$ maximal ideals of $k[x_1, \dots, x_n] \supseteq I(Y)$

$\leftrightarrow$ maximal ideals of $A(Y) = \frac{k[x_1, \dots, x_n]}{I(Y)}$

(*) we saw those maximal ideals

$P = (a_1, \dots, a_n)$

$I(P) = (x_1 - a_1, \dots, x_n - a_n)$

= Poly vanishing at P

Part of (a) easy

we get an injective homomorphism

$\alpha: A(Y) \to \mathcal{O}(Y)$

Start with the homomorphism

$f: k[x_1, \dots, x_n] \to \mathcal{O}(Y)$

regular functions need to be locally rational and poly is globally rational

$\ker = I(Y)$

injective

$\alpha: \frac{k[x_1, \dots, x_n]}{I(Y)} \to \mathcal{O}(Y)$, $\frac{k[x_1, \dots, x_n]}{I(Y)} = A(Y)$

surjective later.

(c) For each P there is a natural map $A(Y)_{m_P} \to \mathcal{O}_{P,Y}$

$\frac{a}{b} \in A(Y)_{m_P}$ $a, b \in A(Y)$ $b(P) \neq 0$

replace a, b by polynomials that represent them

$Y \setminus Z(b)$ is an open neighborhood of P in Y

$\frac{a}{b} \mapsto \langle Y \setminus Z(b), \frac{a}{b}\rangle \in \mathcal{O}_{P,Y}$

injective because α is injective

surjective $\langle U, f\rangle \in \mathcal{O}_{P,Y}$

rep by $\langle V, \frac{g}{h}\rangle$ g, h poly $h \neq 0$ on V

$\frac{g}{h} \in A(Y)$

That gives $\mathcal{O}_{P,Y} \cong A(Y)_{m_P}$

Thm 3.2
Let $Y \subseteq \mathbb{A}^n$ be an affine variety with coordinate ring $A(Y)$. Then

a) $\mathcal{O}(Y) \cong A(Y)$

b) for each point $P \in Y$ let $m_P \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \mapsto m_P$ gives a 1-1 correspondence between the points of Y and the maximal ideals $A(Y)$.

c) for each P, $\mathcal{O}_P \cong A(Y)_{m_P}$ and $\dim \mathcal{O}_P = \dim Y$

d) $K(Y)$ is isomorphic to the quotient field of $A(Y)$ and hence $K(Y)$ is a finitely generated extension field of k of transcendence degree $= \dim Y$

Had done:

$\alpha : A(Y) \to \mathcal{O}(Y)$ injective

(b) ✓

(c) $\mathcal{O}_{P,Y} \cong A(Y)_{m_P}$

$\dim \mathcal{O}_{P,Y} = \dim A(Y)_{m_P}$

$= \text{height of } m_P \text{ in } A(Y)_{m_P}$

$= \text{height } m_P \text{ in } A(Y)$

$A(Y)/m_P = k$

1.8A

$\text{height } m_P + \dim A(Y)/m_P = \dim A(Y)$

($A(Y)/m_P \cong k$, $\dim 0$)

1.7 $\dim Y = \dim A(Y)$

done with (c)

Work on (d)

$\text{Frac } A(Y) = \text{Frac } A(Y)_{m_P}$ any P

(inverted all nonzero; first invert some nonzero, invert the rest)

$\mathcal{O}_P \cong A(Y)_{m_P}$

$\text{Frac } A(Y) \cong \text{Frac } \mathcal{O}_P$ any P

(*) $\text{Frac } \mathcal{O}_P = K(Y)$

$A(Y)_{m_P} \cong \mathcal{O}_{P,Y} \subseteq K(Y)$

Since $K(Y)$ is a field

$\text{Frac } \mathcal{O}_P \subseteq K(Y)$

but $\text{Frac } \mathcal{O}_P \cong \text{Frac } A(Y)_{m_P} = \text{Frac } A(Y)$

so all the $\text{Frac } \mathcal{O}_P$ are the same.

Also $\bigcup_P \mathcal{O}_P = K(Y)$

$\langle U, f \rangle \in K(Y)$

$\langle U, f \rangle \in \mathcal{O}_P$ for any $p \in U$

All $\text{Frac } \mathcal{O}_{P,Y}$ are the same and their union is all $K(Y)$

so each must equal $K(Y)$

$A(Y)$ f.g. k-algebra

$\frac{k[x_1, \ldots, x_n]}{I(Y)}$

so $\text{Frac } A(Y)$ f.g.

$\dim Y = \dim A(Y)$ (1.7)

1.8A $\dim A(Y) = $ transcendence degree of $\text{Frac } A(Y)$ over k

$\text{Frac } A(Y) = K(Y)$

done with (d)

(a) $\mathcal{O}(Y) \subseteq \bigcap_{P \in Y} \mathcal{O}_{P,Y}$

$\langle Y, f \rangle \in$ every $\mathcal{O}_{P,Y}$

$A(Y) \subseteq \mathcal{O}(Y) \subseteq \bigcap_m A(Y)_m$

m maximal ideal

$\mathcal{O}(P,Y) = A(Y)_{m_P}$

Prop

Let B be an integral domain with fraction field K. Then $B \subseteq K$ as $b \mapsto \frac{b}{1}$ and for any maximal ideal m, $B_m \subseteq K$ as $\frac{a}{b} \mapsto \frac{a}{b}$

Then $B = \bigcap_m B_m$ where m runs over all maximal ideals.

Proof:

From Advanced Modern Algebra by Joseph J Rotman Prop 11.30

$B \subseteq \bigcap_m B_m$ clear

For the opposite inclusion suppose $b \in \bigcap_m B_m$ let $I = \{x \in B : xb \in B\}$

claim I is an ideal in B.

nonempty $0 \in I$

closed under + $xb \in B$, $yb \in B$

$(x+y)b = xb + yb \in B$

swallows up multiplication

$xb \in B$ $y \in B$

$(xy)b = y(xb) \in B$

If $I = B$ then $1 \in I$ so $b = 1b \in B$.

If $I \neq B$, it is a proper ideal

so it is contained in some maximal ideal m $I \subseteq m$

$b = \frac{a}{c}$ $a \in B$, $c \notin m$

$cb = c\left(\frac{a}{c}\right) = \frac{a}{1} \in B$

so $c \in I \subseteq m$ a contradiction

so $I = B$. done. □

Prop 3.3

Let $U_i \subseteq \mathbb{P}^n$ be the open set defined by the equation $x_i \neq 0$

Then the mapping $\varphi_i : U_i \to \mathbb{A}^n$ of (2.2) is an isomorphism of varieties.

Proof:

we already saw φ_i is a homeomorphism need to show it and its inverse pull back regular functions to regular functions.

That is just the proposition we did showing affine and projective definition of regular are equivalent. □

Localization in graded rings

Let S be a graded ring and $T \subseteq S$ a multiplicatively closed subset. Further assume that T consists only of homogeneous elements.

We make $T^{-1}S$ into a graded ring (sort of) as follows:

$R \cong \bigoplus_{i=0}^{\infty} R_i$

$T^{-1}S = \bigoplus_{i=-\infty}^{\infty} (T^{-1}S)_i$

Prop:

Let $f, g \in S$ be homogeneous elements and $a, b \in T$. Assume that $\frac{f}{a} = \frac{g}{b}$ as elements in $T^{-1}S$

Then $\deg f - \deg a = \deg g - \deg b$

or $\frac{f}{a} = \frac{g}{b} = \frac{0}{1}$

proof

There exists $t \in T$ s.t. $t(fb - ga) = 0$

$tfb - tga = 0$

each of tfb, tga are homogeneous

either $tfb = tga = 0$ or $\deg(tfb) = \deg(tga)$

$tfb = 0$

$\Rightarrow \frac{f}{a} = \frac{0}{b} = 0$

$tga = 0 \Rightarrow \frac{g}{b} = \frac{0}{a} = 0$

$\deg(tfb) = \deg(tga)$

~~$\deg t$~~ $+ \deg f + \deg b =$ ~~$\deg t$~~ $+ \deg g + \deg a$

□

Def:

For $f \in S$ homogeneous and nonzero $a \in T$ define degree $\frac{f}{a} \in T^{-1}S$ $= \deg f - \deg a$

Let $(T^{-1}S)_i =$ the set of all elements of $T^{-1}S$ of degree i union $\{0\}$.

Prop

As an additive group $T^{-1}S = \bigoplus_{i=-\infty}^{+\infty} (T^{-1}S)_i$ and $(T^{-1}S)_i (T^{-1}S)_j \subseteq (T^{-1}S)_{i+j}$

Thus this makes $T^{-1}S$ into a graded ring.

proof $(T^{-1}S)_i (T^{-1}S)_j \subseteq (T^{-1}S)_{i+j}$

obvious

$\frac{f}{a} \cdot \frac{g}{b} = \frac{fg}{ab}$

$(T^{-1}S)_i$ closed under +

$\frac{f}{a} + \frac{g}{b} = \frac{fb + ga}{ab}$

$\deg f - \deg a = \deg g - \deg b$

$\Rightarrow \deg f + \deg b = \deg g + \deg a$

shows that the numerator is homogeneous of degree

$\deg f + \deg b = \deg g + \deg a$

$\deg \frac{fb + ga}{ab} = \deg f +$ ~~$\deg b$~~ $- \deg a -$ ~~$\deg b$~~

direct sum

$\frac{F}{a} \in T^{-1}S$

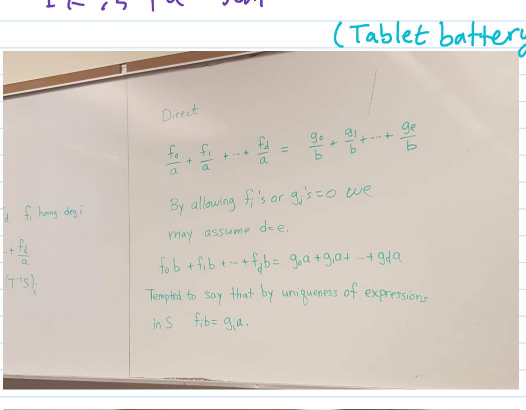
$F = f_0 + f_1 + \cdots + f_d$ f_i homog deg i

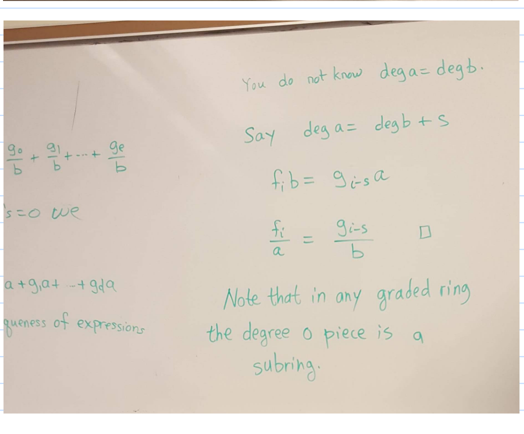
$\frac{F}{a} = \frac{f_0}{a} + \frac{f_1}{a} + \cdots + \frac{f_d}{a}$

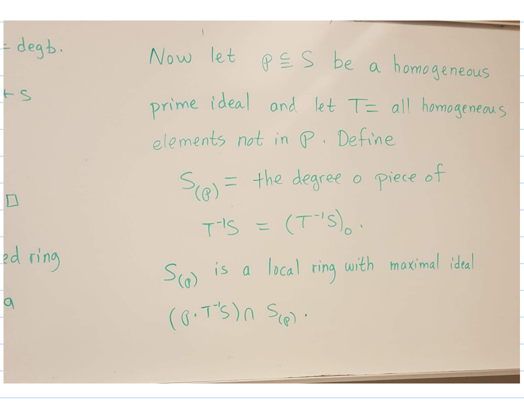
each is in some $(T^{-1}S)_i$

It is the sum

(Tablet battery died)







$S_{(\mathfrak{p})}$ is a local ring with maximal ideal $(\mathfrak{p}\cdot T^{-1}S) \cap S_{(\mathfrak{p})}$.

$(\mathfrak{p}\cdot T^{-1}S)\cap S_{(\mathfrak{p})}$ is just things of the form $\frac{f}{a}$ $\deg f = \deg a$ $f \in \mathfrak{p}$ $a \notin \mathfrak{p}$, f, a homogeneous.

Something in $S_{(\mathfrak{p})}$ not in there has $f \notin \mathfrak{p}$ so it is invertible.

If S is a domain then for $\mathfrak{p} = (0)$ we obtain a field $S_{((0))}$. Similarly if $f \in S$ is homogeneous we denote by $S_{(f)}$ the subring of elements of degree 0 in the localized ring S_f

Inverting $\{1, f, f^2, \dots\}$

Bad notation: if f is a prime element so that (f) is prime ideal $S_{(f)}$ also means $S_\mathfrak{p}$ $\mathfrak{p}$ prime

<u>Theorem 3.4</u> (Read proof in book)

Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$ then:

(a) $\mathcal{O}(Y) = k$ ← Complex analysis: only global holomorphic fxns are constants

(b) for any point $p \in Y$ let $m_p \subseteq S(Y)$ be the ideal generated by the set of homogeneous $f \in S(Y)$ s.t. $f(P) = 0$ Then $\mathcal{O}_{P,Y} = S(Y)_{(m_p)}$

(c) $K(Y) \cong S(Y)_{((0))}$

The proof is largely based on the following which he proves and we have listed as a corollary

<u>Corollary</u>

Suppose $P \in Y$, $P \in U_i = \{x_i \neq 0\}$. Set $Y_i = Y \cap U_i$. Then Y_i is an affine variety and $\mathcal{O}_{P,Y_i} = \mathcal{O}_{P,Y}$ and $K(Y_i) = K(Y)$

More generally let Y be any variety and $U \subseteq Y$ a nonempty open, $P \in U$.

$\mathcal{O}_{P,U} = \mathcal{O}_{P,Y}$

$K(U) = K(Y)$

<u>Proof:</u>

elements of $\mathcal{O}_{P,Y}$ pairs $\langle V, f\rangle$

V open neighborhood of P in Y

f regular function on V

$\mathcal{O}_{P,U}$ $\langle W, g\rangle$

W open neighborhood of p in U

g regular on W.

W open in $U \Rightarrow W$ open in Y

V open in $Y \Rightarrow V \cap U$ is open in U

$\varphi: \mathcal{O}_{P,Y} \to \mathcal{O}_{P,U}$ by

$\langle V, f\rangle \mapsto \langle V\cap U, f|_{V\cap U}\rangle$

$\psi: \mathcal{O}_{P,U} \to \mathcal{O}_{P,Y}$

$\langle W, g\rangle \mapsto \langle W, g\rangle$ ∎

<u>Prop 3.5</u>

Let X be any variety let Y be an affine variety. Then there is a natural bijective map of sets

$\alpha: \mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(A(Y), \mathcal{O}(X))$

where the left Hom means morphism of varieties and the right Hom means homomorphisms of k-algebras

<u>★ related to hw problem</u>

Suppose X projective

Y be any affine space $\mathbb{A}^n$

looking for morphisms

$\varphi: X \to \mathbb{A}^n$

look for k-algebra homomorphisms

$\psi: k[x_1, \dots, x_n] \to k$ (Thm 3.4)

$x_i \to 0$ (only one of these)

<u>Proof</u>: First a very elementary result.

Let X and Y be any two sets and R any ring.

Let $F(X)$ and $F(Y)$ be the sets of all functions $f: X \to R$

$g: Y \to R$

we have seen under pointwise addition and multiplication $F(X)$ and $F(Y)$ become rings. In fact they are R-algebras with $R \hookrightarrow F(X)$ $R \hookrightarrow F(Y)$ as constant functions. Assume $X, Y \neq \emptyset$

Now let $\varphi: X \to Y$ be map of sets.

Then pull back φ^* induces an R-algebra homomorphism

$\varphi^*: F(Y) \to F(X)$

$\varphi^*(g)(a) = g(\varphi(a))$

$a \in X$ $g \in F(Y)$

Now when X and Y are varieties, by definition of morphism if φ is a morphism

$\varphi^*(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)$

$\mathcal{O}(Y)$ subring of $F(Y)$

$\mathcal{O}(X)$ subring of $F(X)$

$R = k$.

so $\varphi^* \in \mathrm{HOM}(\mathcal{O}(Y), \mathcal{O}(X))$

but in this prop Y is affine so we may replace $\mathcal{O}(Y)$ by $A(Y)$. That gives α.

Must find inverse of α.

We have a homomorphism

$h: A(Y) \to \mathcal{O}(X)$ of k-algebras

Suppose that Y is a closed subset of $\mathbb{A}^n$. So $A(Y) \cong k[x_1, \dots, x_n]/I(Y)$

Let $\overline{x_i}$ be the image of x_i in $A(Y)$

set $\xi_i = h(\overline{x_i}) \in \mathcal{O}(X)$

these are global regular functions on X so we can use them to define a mapping

$\psi: X \to \mathbb{A}^n$ by

$\psi(P) = (\xi_1(P), \dots, \xi_n(P))$ for $P \in X$

Given the alternate defs of morphism we have covered this is obviously a morphism.

Need to show $\psi(X) \subseteq Y$

Y is $Z(I(Y))$

Need to show that for any $P \in X$, $f \in I(Y)$, $f(\psi(P)) = 0$

$f(\psi(P)) = f(\xi_1(P), \dots, \xi_n(P))$

f is a polynomial. h is a homomorphism of k-algebras.

$f(\xi_1(P), \dots, \xi_n(P)) = f(h(\overline{x_1}), \dots, h(\overline{x_n}))(P)$

$= h(f(\overline{x_1}, \dots, \overline{x_n}))(P)$

Since $f \in I(Y)$ $f(\overline{x_1}, \dots, \overline{x_n})(P) = 0$

$= h(0)(P) = 0$

Check this gives inverse to α.

<u>Corollary 3.7</u>

If X and Y are two affine varieties, then X and Y are isomorphic iff $A(X)$ and $A(Y)$ are isomorphic as k-algebras.

<u>Corollary 3.8</u>

The functor $X \mapsto A(X)$ induces an arrow reversing equivalence of categories between the category of affine varieties over k and the category of finitely generated integral domains over k.

You need to check that the α from prop 3.5 respects compositions

$X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z$

Does $\alpha(\varphi \circ \psi) = \alpha(\psi) \circ \alpha(\varphi)$

$A(Z) \xrightarrow{f} A(Y) \xrightarrow{g} A(X)$

Does $\alpha^{-1}(g \circ f) = \alpha^{-1}(f) \circ \alpha^{-1}(g)$?

Both fairly easy

$X \cong Y$

$\varphi: X \to Y$

$\psi: Y \to X$

$\varphi \circ \psi = id_Y$ $\psi \circ \varphi = id_X$

$\alpha(\varphi \circ \psi) = \alpha(id_Y)$ $= id_{A(Y)}$

$\alpha(\psi) \circ \alpha(\varphi)$

Some category theory:

references:

- Categories for the working mathematician
- Wikipedia
- Mark Kleiner

<u>Definition:</u>

A functor $S: A \to C$ is an equivalence of categories iff there exists a functor $T: C \to A$ s.t. $S \circ T$ is naturally isomorphic to the identity on C and $T \circ S$ is naturally isomorphic to the identity on A.

If you replace naturally isomorphic with equal you get isomorphism of categories.

<u>Theorem:</u>

A functor $f: A \to C$ is an equivalence of categories iff it is full, faithful, and dense.

<u>full</u>: for every $a, a' \in A$ and every $g: Sa \to Sa'$, there exists $f: a \to a'$ s.t. $Sf = g$

surjective on morphisms

<u>faithful</u>: For every $a, a' \in A$ and every $f_1, f_2: a \to a'$

$Sf_1 = Sf_2 \Rightarrow f_1 = f_2$

injective on morphisms

dense: $S : A \to C$
For any $c \in C$ there is an $a \in A$
s.t. Sa is isomorphic to c.
On objects surjective up to isomorphism
$X \to A(X)$ affine varieties
affine Variety $\to$ finitely generated integral domains over k

bijection
$\alpha : \mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(A(Y), \mathcal{O}(X))$
if X affine $\to \cong A(X)$
full & faithful come from that being a bijection
dense (Remark 1.4.6) showed that any finitely generated integral domain over k was isomorphic to $A(X)$ some X.

Automorphisms of $\mathbb{P}^n$
let M be an invertible $(n+1)\times(n+1)$ matrix with entries in k. M induces a map

$$m : \mathbb{P}^n \to \mathbb{P}^n$$
$$[a_0, \dots, a_n] \to M\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$
$$[\lambda a_0, \dots, \lambda a_n] \to \lambda M \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$

Since M is invertible the only $[a_0, \dots, a_n]$ mapping to $[0, \dots, 0]$ is $[0, \dots, 0]$
Its a morphism because if you multiply out $M\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$ you see each entry is a homogeneous linear poly in the a_i
The inverse morphism is mult by M^{-1}.
That is an automorphism.
If $D = \begin{bmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{bmatrix}$ $d \neq 0$ then
M and DM induce the same map
$$DM\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = dM\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$
$GL_k(n+1)$ group of all $(n+1)\times(n+1)$ invertible matrices
operation is matrix mult.
general linear group
The set of scalar matrices is a normal subgroup.
Define $PGL_k(n+1) = \dfrac{GL_k(n+1)}{\text{scalar matrices}}$
projective general linear group

It can be shown that
$$\mathrm{Aut}(\mathbb{P}^n_k) \cong PGL_k(n+1)$$
Linear Varieties (subspaces) in $\mathbb{P}^n$
$$\mathbb{P}^n = \frac{\mathbb{A}^{n+1} - \{(0,\dots,0)\}}{\sim}$$
$(a_0, \dots, a_n) \sim \lambda(a_0, \dots, a_n)$, $\lambda \in k^*$ — lines through origin in $\mathbb{A}^{n+1}$

Go back to thinking of $\mathbb{A}^{n+1}$ as k^{n+1} a vector space.
If $V \subseteq k^{n+1}$ is any subspace of dimension $n+1$, let $X = \{P \in \mathbb{P}^n \mid \text{the line to which } P \text{ corresponds lies in } V\}$

$X \cong \mathbb{P}^m$ $\quad V \cong k^{m+1}$
X is called a linear subvariety of $\mathbb{P}^n$ sometimes it is called a linear subspace
This is confusing because $\mathbb{P}^n$ is not a vector space
$V \subseteq k^{n+1}$ is the zero locus of $n+1-(m+1)$ homogeneous linear equations
X will also be the zero locus of these equations.
In fact they generate its ideal
By change of variables you can assume that equations are
$$x_0 = x_1 = \dots = x_{(n+1)-(m+1)-1} = 0$$
Then not too hard to see.
X is called an m-plane
$X \cap U_i \subseteq \mathbb{A}^n$
it really is an m-plane

Some important exercises in Hartshorne

3.1(e) If an affine variety is isomorphic to a projective variety, then it consists of one point
(done in notes a while ago)

3.14 Projection from a point
Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$, and let $P \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$
Define a mapping
$$\varphi : \mathbb{P}^{n+1} \setminus \{P\} \to \mathbb{P}^n$$
by $\varphi(Q)$ = the intersection of the unique line containing P and Q with $\mathbb{P}^n$
(a) show that φ is a morphism
(Among other things you need to show it is defined for all $Q \in \mathbb{P}^{n+1} \setminus \{P\}$. The line always meets the plane in a unique point)
Might want to show by change of variables you can assume:
$$\mathbb{P}^n = \{x_0 = 0\}$$
$$P = [1, 0, \dots, 0]$$
$$\varphi([a_0, a_1, \dots, a_n]) = [a_1, \dots, a_n]$$

3.15 Products of affine varieties
let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.
(a) Show $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. (You should also show it is closed)

3.16 Products of Quasi-Projective Varieties
use the Segre embedding (Ex 2.14) to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it the structure of a projective variety
Now for any two quasi projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$
(a) show $X \times Y$ is a quasi-projective variety

3.21 Group Varieties
A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$ s.t. the points of Y with the operation given by μ is a group s.t. the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \to Y$
Example: $\mathbb{A}^n = k^n$ with vector addition.
$$\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$$
$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_1 + y_1, \dots, x_n + y_n)$$
morphism because given by polynomials
$$(x_1, \dots, x_n) \to (-x_1, \dots, -x_n)$$
Topological Group consists of a topological space Y together with a continuous map $\mu : Y \times Y \to Y$ s.t. the points of Y w/ operation given by μ is a group, and such that the inverse map $y \to y^{-1}$ is also a continuous map $Y \to Y$
Since morphisms are continuous maps in Zarisky topology, is a group variety automatically a topological group in Zariski topology? No

4 Rational Maps

Lemma 4.1
Let X and Y be varieties, let φ and ψ be two morphisms from X to Y and suppose that there is a nonempty open subset $U \subseteq X$ s.t. $\varphi|_U = \psi|_U$
Then $\varphi = \psi$
(Remark 3.1.1 If you have two regular functions $f, g : X \to k$ and an nonempty open set such that $f = g$ on U then $f = g$ on X)

Definition
let X and Y be varieties.
A rational map $\varphi : X \to Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where U is a nonempty open subset of X and φ_U is a morphism of U to Y.
$\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle \Leftrightarrow \varphi_U = \varphi_V$ when restricted to $U \cap V$. The rational map φ is dominant iff for some (and hence every) pair $\langle U, \varphi_U \rangle$ the image of φ_U is dense in Y.

Let's convince ourselves that the and hence every is true.

We have $\langle U, \varphi_U \rangle$ s.t. $\varphi_U(U)$ is dense in Y. This means that for any nonempty open $V \subset Y$

$\varphi_U(U) \cap V \neq \emptyset$

Now suppose we have another pair $\langle W, \varphi_W \rangle$. We need to show that for any open $V \subseteq Y$

$\varphi_W(W) \cap V \neq \emptyset$

$\varphi_U(U) \cap V \neq \emptyset$ and since morphisms are continuous

$\varphi_U^{-1}(V)$ is nonempty open in U hence open in X

This means $\varphi_U^{-1}(V) \cap W \neq \emptyset$ since X is irreducible. Pick

$P \in \varphi_U^{-1}(V) \cap W$

note $P \in U \cap W$

By def φ_U and φ_W agree on $U \cap W$

$\varphi_W(P) = \varphi_U(P) \in V$

Show $\varphi_W(W) \cap V \neq \emptyset$

A rational map might not be defined at some points of X. There may be some points of X not in any U s.t. $\langle U, \varphi_U \rangle$ is in the rational map.

<u>ex</u> projection from a point

$\mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$

$\langle \mathbb{P}^n \setminus \{P\}, \varphi \rangle$

Sometimes rational maps are denoted by dotted arrows

"Let $\varphi: - - - \to Y$ be a rational map"

This means we cannot always compose rational maps

$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ φ, ψ rational maps

It could be that

$$\varphi(X) = \bigcup_{\langle U, \varphi \rangle} \varphi_U(U)$$

is contained in the set of points where ψ is not defined

This will not happen if φ is dominant.

In fact we can check that if φ and ψ are dominant then so is $\psi \circ \varphi$

So you can make a category where the objects are varieties and the morphisms are dominant rational maps

<u>Definition:</u>

A birational map $\varphi: X \to Y$ is a rational map that admits an inverse, namely a rational map $\psi: Y \to X$ such that $\psi \circ \varphi = id_X$ and $\varphi \circ \psi = id_X$ as rational maps (equal to identity where defined)

If there is a birational map from X to Y we say that X and Y are birationally equivalent or birational or birationally isomorphic.

birational maps are also sometimes called birational isomorphisms.

What is a birational morphism? That is a morphism that has a rational inverse. It is a birational isomorphism.

<u>Examples</u>

(1) $\mathbb{A}^n$ and $\mathbb{P}^n$ are birationally isomorphic. (They are not isomorphic)

$\varphi: \mathbb{A}^n \to \mathbb{P}^n$

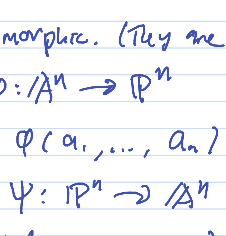
$\varphi(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$

$\psi: \mathbb{P}^n \to \mathbb{A}^n$ defined on U_0

$\psi(1, a_1, \ldots, a_n) = (a_1, \ldots, a_n)$

birational morphism

(2) $X = Z(Y^2 - X^2(X+1)) \subseteq \mathbb{A}^2$



nodal cubic

X is birationally isomorphic to $\mathbb{A}^1$.

$\varphi: \mathbb{A}^1 \to X$

$\varphi(t) = (t^2-1, t(t^2-1))$

$[t(t^2-1)]^2 - (t^2-1)^2(t^2-1+1)$

$= t^2(t^2-1)^2 - (t^2-1)^2(t^2) = 0$

$\varphi(\mathbb{A}^1) \subseteq X$

$\psi: X \to \mathbb{A}^1$

$\psi(x, y) \to \frac{y}{x}$ only defined when $x \neq 0$

$\psi \circ \varphi(t) = \psi(t^2-1, t(t^2-1))$

$= \frac{t(t^2-1)}{t^2-1} = 1$

away from $t = \pm 1$

$\varphi \circ \psi = \varphi(\frac{y}{x}) = (\frac{y^2}{x^2} - 1, \frac{y}{x}(\frac{y^2}{x^2} - 1))$

but on X

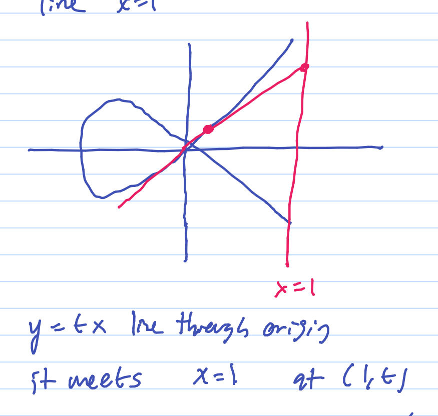
$\frac{y^2}{x^2} = x+1$ as long as $x \neq 0$

$= (x+1-1, \frac{y}{x}(x+1-1)) = (x, y)$

away from $x=0$

Geometric interpretation of map

Projecting from the origin to the line $x=1$



$y = tx$ line through origin

it meets $x=1$ at $(1, t)$

It meets $y^2 - x^2(x+1)$ at

$t^2x^2 - x^2(x+1) = 0$

$x^2(t^2 - (x+1)) = 0$

$x=0$ twice

$x = t^2 - 1$

$y = tx = t(t^2-1)$

Remark: Clearly a morphism is a rational map. Thus X isomorphic to Y implies X birationally isomorphic to Y. Those examples show reverse does not hold.

$\mathcal{O}(\mathbb{A}^n) = k[x_1, \ldots, x_n]$

$\mathcal{O}(\mathbb{P}^n) = k$

$\mathbb{A}^n$ not isomorphic to $\mathbb{P}^n$

$A(\mathbb{A}^1) = k[x]$

$A(X) = \frac{k[x,y]}{y^2 - x^2(x+1)}$

$k[x]$ is a PID

In $A(X)$ look at the ideal generated by $\bar{x}, \bar{y}$

Anything in $(y^2 - x^2(x+1))$ has all its monomials of degree at least 2.

$(\bar{x}, \bar{y}) = (h(\bar{x}, \bar{y}))$

h would have to have no constant term because $h \in (\bar{x}, \bar{y})$

$\bar{x}$ and $\bar{y}$ would have to be constant multiples of degree 1 term

Birational is a weaker equivalence relation than isomorphism. Ideally we would like to be able to classify varieties up to isomorphism

This is very$^{\infty}$ hard.

Break it into steps.

First classify up to birational isomorphism then for each birational equivalence class try to classify up to isomorphism.

Confusing terminology:

When we say X is an affine variety, we mean X is a closed irreducible set of some affine space, $\mathbb{A}^n$.

When we say a variety X is affine, we mean that X is isomorphic to an affine variety.

Example:

$\mathbb{A}^1 \setminus \{0\}$ not an affine variety but it is affine

$\mathbb{A}^1_t \setminus \{0\} \cong X = Z(xy-1) \subseteq \mathbb{A}^2_{x,y}$

$\varphi: \mathbb{A}^1_t \setminus \{0\} \to X$

$\varphi(t) = (t, \frac{1}{t})$

$\psi: X \to \mathbb{A}^1 \setminus \{0\}$

$\psi(x, y) = x$

$t(\frac{1}{t}) - 1 = 0$ $t \neq 0$

$\psi(\varphi(t)) = \psi(t, \frac{1}{t}) = t$

$\varphi(\psi(x,y)) = \varphi(x) = (x, \frac{1}{x})$ *

but on $xy - 1 = 0$ $y = \frac{1}{x}$

so * $= (x, y)$

Lemma 4.2

Let Y be a hypersurface in $\mathbb{A}^n$ given by the equation $f(x_1, \ldots, x_n) = 0$

f irred

Then $\mathbb{A}^n \setminus Y$ is isomorphic to the hypersurface H in $\mathbb{A}^{n+1}$ given by $x_{n+1} f - 1 = 0$

In particular $\mathbb{A}^n \setminus Y$ is affine and its coordinate ring is $k[x_1, \ldots, x_n]_f$.

In previous example $f = x$ so $x_{n+1} f - 1$ becomes $xy = 1$

<u>Proof:</u>

$\varphi: \mathbb{A}^n \setminus Y \to H$

$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)})$

$\frac{1}{f(x_1, \ldots, x_n)} \cdot f(x_1, \ldots, x_n) - 1 = 0$

$\psi: H \to \mathbb{A}^n \setminus Y$

$\psi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$

Check $\varphi \circ \psi = id$ $\psi \circ \varphi = id$

$A(H) \cong k[x_1, \ldots, x_n]_f$

$\frac{k[x_1, \ldots, x_{n+1}]}{(x_{n+1} f - 1)}$

$x_{n+1} f - 1 = 0$

$x_{n+1} f = 1$

$f = \frac{1}{x_{n+1}}$

$k[x_1, \ldots, x_n, x_{n+1}] \to k[x_1, \ldots, x_n]_f$

$x_i \mapsto x_i$ $1 \leq i \leq n$

$x_{n+1} \mapsto \frac{1}{f}$

Obviously a surjective ring homomorphism

Show kernel is $(x_{n+1} f - 1)$

kernel contains ideal easy

$k[x_1, \dots, x_n, x_{n+1}] \longrightarrow k[x_1, \dots, x_n]_f$

$x_i \mapsto x_i \quad i=1,\dots,n$
$x_{n+1} \mapsto \frac{1}{f}$

Obviously surjective

We wish to show the kernel is $(x_{n+1}f-1)$

Write an arbitrary element of $k[x_1,\dots,x_n,x_{n+1}]$ as a polynomial in x_{n+1} with coefficients in $k[x_1,\dots,x_n]$

$$\sum_{i=0}^{\ell} g_i x_{n+1}^i \longmapsto \sum_{i=0}^{\ell} \frac{g_i}{f^i} = \sum_{i=0}^{\ell} \frac{f^{\ell-i} g_i}{f^\ell} = \frac{0}{1}$$

$$\Longleftrightarrow \sum_{i=0}^{\ell} f^{\ell-i} g_i = 0$$

$$\sum_{i=0}^{\ell} g_i x_{n+1}^i = (x_{n+1}f-1)\sum_{i=0}^{\ell-1} h_i x_{n+1}^i$$

right sum only goes to $\ell-1$ not ℓ so the degrees in x_{n+1} match up.

On the right the coefficient of x_{n+1}^j

$-h_j + h_{j-1}f$

need $g_j = -h_j + h_{j-1}f$

$j=0 \quad -h_0 = g_0$

$j=1 \quad -h_1 + h_0 f = g_1$

$h_1 = -g_1 + h_0 f$

$= -g_1 - g_0 f$

$j=2 \quad -h_2 + h_1 f = g_2$

$h_2 = -g_2 + h_1 f$

$= -g_2 - g_1 f - g_0 f^2$

in general

$$h_i = -\sum_{j=0}^{i} g_j f^{i-j}$$

works only if $h_\ell = 0$

$$\sum_{i=0}^{\ell} f^{\ell-i} g_i = 0$$

exactly the same condition to be in the kernel. □

Theorem 3.4

Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$.

Then

(a) $\mathcal{O}(Y) = k$

Introduction to commutative algebra by Atiyah and Macdonald chapter 5

B a ring, A a subring, $1 \in A$

$x \in B$ is said to be integral over A iff it satisfies a monic polynomial with coefficients in A.

$x^n + a_1 x^{n-1} + \dots + a_n = 0 \qquad a_i \in A$

Proposition 5.1

The following are equivalent:

(i) $x \in B$ is integral over A

(ii) $A[x]$ is a finitely generated A-module

(iii) $A[x]$ is contained in a subring C of B such that C is a finitely generated A-module

Proposition 6.5

Let A be a Noetherian (resp. Artinian) ring, M a finitely generated A-module. Then M is Noetherian (resp. Artinian).

A submodule of a finitely generated module over a Noetherian ring is finitely generated.

Now try to prove 3.4(a).

Let $f \in \mathcal{O}(Y)$ be a global regular function. Then for each i, f is regular on $Y_i = Y \cap U_i$ which is an affine variety.

$f \in A(Y_i)$

Another part of theorem 3.4 said

$A(Y_i) \cong S(Y)_{(x_i)}$

we can write $f = \frac{g_i}{x_i^{N_i}}$ where $g_i \in S(Y)$ is homogeneous of degree N_i.

$S(Y)_{(x_i)}$ was the degree 0 piece of $S(Y)_{x_i}$

Thinking of $\mathcal{O}(Y)$, $K(Y)$ and $S(Y)$ all as subrings of the quotient field L of S

$\mathcal{O}(Y) \hookrightarrow K(Y)$

$K(Y) \cong S(Y)_{((0))}$ = the degree 0 piece of $S(Y)_{(0)} \cong L$

This means that $x_i^{N_i} f \in S(Y)_{N_i}$ for each i.

$f = \frac{g_i}{x_i^{N_i}} \Rightarrow x_i^{N_i} f = g_i$

g_i is homogeneous of degree N_i

Now choose $N \geq \sum_{i=0}^{n} N_i$

$S(Y)_N$ is spanned as a k-vector space by monomials of degree N in $\bar{x}_0, \dots, \bar{x}_n$ and in any such monomial at least one x_i occurs to a power $\geq N_i$ (*)

Thus we have $S(Y)_N f \subseteq S(Y)_N$

f has degree 0

(*) i is that power

(monomial)$\cdot f \in S(Y)_{N_i} \cdot$ something

poly of right degree

iterating, we have $S(Y)_N \cdot f^q \subseteq S(Y)_N$ for all $q \geq 0$

$S(Y)_N \cdot f \subseteq S(Y)_N$ multiply by f

$S(Y)_N f^2 \subseteq S(Y)_N f \subseteq S(Y)_N$

In particular, $x_0^N f^q \in S(Y) = \bigcup_{N=0}^{\infty} S(Y)_N$ for all $q > 0$

This shows that the subring $S(Y)[f]$ of L is contained in $x_0^{-N} S(Y)$ which is a finitely generated $S(Y)$-module gen $x_0^{-N} \cdot 1$

Since $S(Y)$ is a Noetherian ring $S(Y)[f]$ is a finitely generated $S(Y)$-module

and therefore f is integral over $S(Y)$

This means that there are elements $a_1, \dots, a_m \in S(Y)$ s.t.

$f^m + a_1 f^{m-1} + \dots + a_m = 0$

Since f has degree 0 we can replace the a_i by their degree 0 homog pieces and we still have a valid equation.

But $S(Y)_0 = k$ so may assume $a_i \in k$ and f is algebraic over k.

But k is algebraically closed so $f \in k$ □

Proposition 4.3

On any variety Y, there is a base for the topology consisting of open affine subsets.

Any variety = affine, quasi-affine, projective, quasi-projective

By an open affine subset is meant an open set isomorphic to an affine variety = closed irreducible subset of an affine space.

We had just shown that for irred $f \in k[x_1,\dots,x_n]$

$\mathbb{A}^n \setminus Z(f) \cong Z(x_{n+1}f-1) \subseteq \mathbb{A}^{n+1}$

Proof:

We must show for any point $P \in Y$ and any open subset U of Y containing P, there exists an open set V with $P \in V \subseteq U$ and V is isomorphic to an affine variety.

First since U is also a variety, we may assume $U = Y$. Secondly since any variety is covered by quasi-affine varieties (2.3) we may assume Y is quasi-affine in $\mathbb{A}^n$.

{$\mathbb{P}^n$ is covered by $U_i \cong \mathbb{A}^n$ $\quad Y \cap U_i$ is quasi-affine}

choose $P \in U_i$.

Let $Z = \bar{Y} \setminus Y$ which is closed in $\mathbb{A}^n$.

{Y is open in $\bar{Y}$}

Then since Z is closed and $P \notin Z$

{$P \in Y$} we can find a polynomial $f \in k[x_1,\dots,x_n]$ s.t. $f(P) \neq 0$ but $f \in I(Z)$

$Z(I(Z)) = Z$

$H = Z(f)$

$Z \subseteq H \quad P \notin H$

so $P \in Y \setminus (Y \cap H)$

which is an open subset of Y

$Y \setminus (Y \cap H)$ is a closed subset of $\mathbb{A}^n \setminus H \cong$ closed in $\mathbb{A}^{n+1}$

so $Y \setminus (Y \cap H) \cong$ closed in $\mathbb{A}^{n+1}$

Answer to question from last class.

In Lemma 4.2 it is not needed that f be irreducible.

Let $\varphi: X \to Y$ be a dominant rational map. We define a k-algebra homomorphism $\varphi^*: K(Y) \to K(X)$ as follows:

say $\langle U, f_U \rangle \in K(Y)$

Since φ is dominant, take any $\langle V, \varphi_V \rangle \in \varphi$ $\quad \varphi_V(V)$ is dense in Y

so that $\varphi_V(V) \cap U \neq \emptyset$

That means $\varphi_V^{-1}(U)$ is nonempty open in X.

$\langle \varphi_V^{-1}(U), f_U|_{U \cap \varphi_V(V)} \circ \varphi_V|_{\varphi_V^{-1}(U)} \rangle \in K(X)$

That is $\varphi^* \langle U, f_U \rangle$

check lots of things.

Theorem 4.4

For any two varieties X and Y the above construction gives a bijection between

(i) the set of dominant rational maps from X to Y and

(ii) the set of k-algebra homomorphisms from $K(Y)$ to $K(X)$

Furthermore, this correspondence gives an arrow reversing equivalence of categories of the category of varieties and dominant rational maps with the category of finitely generated field extensions of k.

Remember for fields, ring homomorphisms are 0 or injective.

Since they are k-algebra homomorphisms, they are identity on k so they are not 0. They are all embeddings.

Proof:

Construct how to go from (ii) to (i)

$\theta: K(Y) \to K(X)$

To get a rational map $X \to Y$ we only need to do it on open sets.

Varieties are covered by affine open sets.

Assume Y affine.

Let $y_1, \dots, y_n$ be generators for $A(Y)$ as a k-algebra.

Then $\theta(y_1), \dots, \theta(y_n)$ are rational functions on X. Each is a pair $\langle$open, regular$\rangle$ intersect the opens and you get a nonempty open on which they are all regular.

gives injective homomorphism of k-algebras from

$A(Y) \to \mathcal{O}(U)$

injective because θ was.

By (3.5) this gives morphism

$\varphi: U \to Y$

this is a rational map $\varphi: X \to Y$

Check that it is dominant.

Suppose there was a nonempty open $V \subseteq Y$ with $\varphi(U) \cap V = \emptyset$

Then $\varphi(U) \subseteq Z = Y \setminus U$ proper closed in Y. Pick $f \in I(Z) \subseteq A(Y)$, $f \neq 0$

$\varphi^*(f) = 0$ contradicts injective

$A(Y) \to \mathcal{O}(U)$

Do lots more checking.

Corollary 4.5

For any two varieties X, Y the following conditions are equivalent:

(i) X and Y are birationally equivalent

(ii) there are open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V.

(iii) $K(X) \cong K(Y)$ as k-algebras

proof:

(i) $\Rightarrow$ (ii)

We have rational maps $\varphi: X \to Y$, $\psi: Y \to X$ that are inverses as rational maps

$\varphi \circ \psi = id_Y \qquad \psi \circ \varphi = id_X$ as rational maps

choose reps $\langle U, \varphi_U \rangle$ $\langle V, \psi_V \rangle$

maybe shrink U, V

(ii) $\Rightarrow$ (iii)

$U \subseteq X \quad V \subseteq Y$

$U \cong V$

$$\begin{array}{ccc} K(U) & \cong & K(V) \\ \wr\wr & & \wr\wr \\ K(X) & & K(Y) \end{array}$$

(iii) $\Rightarrow$ (i) equivalence of categories

Prop 4.9 Any variety X of dimension r is birational to a hypersurface Y in $\mathbb{P}^{r+1}$ ($\mathbb{A}^{r+1}$)

Blowing Up

A specific birational morphism that comes up a lot.

First we construct the blowing-up of $\mathbb{A}^n$ at the point $O = (0, \dots, 0)$

Consider the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$ which is a quasi-projective variety {open subset of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ by segre was a closed set in some $\mathbb{P}^N$}

If $x_1, \dots, x_n$ are coordinates on $\mathbb{A}^n$ and $y_1, \dots, y_n$ are homogeneous coordinates on $\mathbb{P}^{n-1}$ (note nonstandard numbering), closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ are defined by polynomials in x_i, y_j homogeneous in the y_j.

We now define the blowing-up of $\mathbb{A}^n$ at the point O to be the closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the equations $\{x_i y_j = x_j y_i \mid i, j = 1, \dots, n\}$

$$X \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1}$$
$\varphi: X \to \mathbb{A}^n$ (the diagram commutes; the vertical arrow $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ is projection on first factor)

Study X and φ

(1) if $P \in \mathbb{A}^n$, $P \neq O$ then $\varphi^{-1}(P)$ consists of a single point. In fact φ gives an isomorphism of $X \setminus \varphi^{-1}(O)$ with $\mathbb{A}^n \setminus \{O\}$

Let $P = (a_1, \dots, a_n)$ say $a_i \neq 0$

If $P \times (y_1, \dots, y_n) \in \varphi^{-1}(P)$ meaning $\in X$ then for each j $y_j = \frac{a_j}{a_i} y_i = a_j \left(\frac{y_i}{a_i}\right)$

$a_i y_j = a_j y_i$

Since all y_j are multiples of $\frac{y_i}{a_i}$ just one point

In fact setting $y_i = a_i$ we can take $(y_1, \dots, y_n) = (a_1, \dots, a_n)$

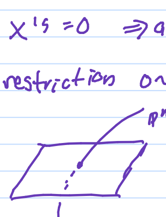
Inverse map

$\psi((a_1, \dots, a_n)) = (a_1, \dots, a_n) \times (a_1, \dots, a_n)$

(2) $\varphi^{-1}(O) = \mathbb{P}^{n-1}$

all x's $= 0 \Rightarrow$ all equations are $0 = 0$.

No restriction on y's.



No.

(3) The points of $\varphi^{-1}(O)$ are in 1-1 correspondence with the set of lines through O in $\mathbb{A}^n$.

Indeed a line through O in $\mathbb{A}^n$ can be given parametrically by

$x_i = a_i t \quad i = 1, \dots, n \quad a_i \in K$ not all 0, $t \in \mathbb{A}^1$

Set $L' = \varphi^{-1}(L \setminus O)$ in $X - \varphi^{-1}(O)$

$x_i = a_i t \quad y_i = a_i t \quad t = \mathbb{A}^1 \setminus \{0\}$

y's homogeneous $\quad x_i = a_i t \quad y_i = a_i$

Now they make sense even when $t = 0$

Now you can let $t = 0$, also

That gives parametric equations for $\bar{L'}$

$\bar{L'}$ meets $\varphi^{-1}(O)$ at the point $(0, \dots, 0) \times (a_1, \dots, a_n)$

(4) X is irreducible

$X = \overline{\varphi^{-1}(\mathbb{A}^n \setminus O)}$

$\mathbb{A}^n \setminus O$ irred

closure of irred is irred.

(5) X is covered by open sets each isomorphic to $\mathbb{A}^n$.

$X \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1}$, $\varphi: X \to \mathbb{A}^n$ (projection)

$X = Z(\{x_i y_j = x_j y_i \mid i,j = 1,\dots,n\})$

(5) X is covered by open sets each isomorphic to $\mathbb{A}^n$. $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is covered by $\mathbb{A}^n \times U_i \cong \mathbb{A}^n \times \mathbb{A}^{n-1} \cong \mathbb{A}^{2n-1}$

Set $V_i = X \cap \mathbb{A}^n \times U_i$

On V_i, $Y_i = 1$

$X_i Y_j = X_j Y_i$ becomes $X_i Y_j = X_j$, $j = 1,\dots,n$

The points of X are of the form

$(X_i Y_1, X_i Y_2, \dots, X_i, \dots X_i Y_n) \times (Y_1, Y_2, \dots, 1, \dots Y_n)$

Note that this satisfies all the other equations

$X_s Y_t = X_t Y_s \qquad X_s = X_i Y_s$

$\qquad\qquad\qquad\quad X_t = X_i Y_t$

$X_i Y_s Y_t = X_i Y_t Y_s$

$\mathbb{A}^n \to V_i$

$(Y_1, Y_2, \dots, X_i, \dots Y_n) \mapsto$

Note that on V_i $\varphi^{-1}(0)$ is given by $X_i = 0$

This again shows $\varphi^{-1}(0)$ fits nicely into X.

Definition. If Y is a closed subvariety of $\mathbb{A}^n$ passing through O, we define the blowing-up of Y at the point O to be $\tilde{Y} = \overline{\varphi^{-1}(Y \setminus \{O\})}$ where $\varphi: X \to \mathbb{A}^n$ is the blowing up of $\mathbb{A}^n$ at O.

We denote by $\varphi: \tilde{Y} \to Y$ the restriction of φ to $\tilde{Y}$. To blow up any other point P on $\mathbb{A}^n$ make a linear change of variables sending P to O.

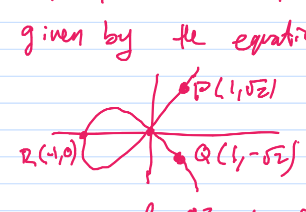
Blow-up points of $\mathbb{P}^n$ by going to an affine patch.

$\tilde{Y}$ is also sometimes called the proper transform of Y.

<u>Example 4.9.1</u>

Let Y be the plane cubic curve given by the equation $y^2 = x^2(x+1)$

nodal cubic



Blow up of $\mathbb{A}^2$ at $(0,0)$ is covered by two $\mathbb{A}^2$'s.

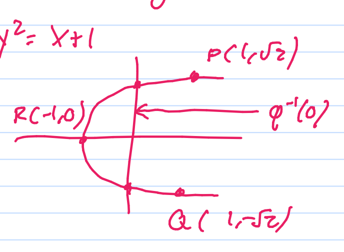
On one, ① $x = X$ $y = XY$ $\varphi^{-1}(0)$ $X = 0$

other ② $x = xy$ $y = Y$ $\varphi^{-1}(0)$ $y = 0$

① $X^2Y^2 = X^2(X+1)$

divide out by X^2

$Y^2 = X + 1$



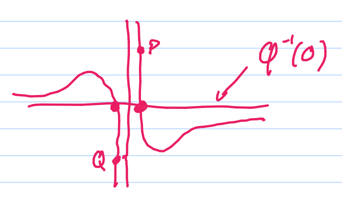
② $Y^2 = x^2y^2(XY+1)$

$1 = x^2(XY+1)$

$\frac{1}{x^2} = XY + 1$

$\frac{1}{x^2} - 1 = XY$

$Y = \frac{1-x^2}{x^3}$



A Theorem proven by the classical Italian algebraic geometers late 1800's early 1900's

Assume $k = \mathbb{C}$ and let X and Y be nonsingular projective surfaces (surface means dimension 2, nonsingular means they are complex manifolds)

Let $\varphi: X \to Y$ be a birational isomorphism.

Then there exists a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \overset{\psi}{\underset{\psi^{-1}}{\longleftrightarrow}} & \tilde{Y} \\ f \downarrow & & \downarrow g \\ X & \overset{\varphi}{\underset{\varphi^{-1}}{\dashleftarrow\dashrightarrow}} & Y \end{array}$$

where ψ is an isomorphism of nonsingular complex projective surfaces and f and g are both finite sequences of blow-ups.

Every birational isomorphism of nonsingular projective surfaces can be factored into a finite sequence of blow ups followed by a finite sequence of blow downs.

Probably ~20 years ago, Mori got a fields medal for sort of generalizing this to three-folds.

Example:

Consider $Q = Z(XY - ZW) \subseteq \mathbb{P}^3$

Q is the image of the segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$

$(s,t) \times (u,v) \mapsto (su, sv, tu, tv)$ [w x y z]

How do we quickly know Q is birationally isomorphic to $\mathbb{P}^2$?

$U_i \subseteq \mathbb{P}^2 \qquad U_i \cong \mathbb{A}^2$

$U_i \times U_j \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \qquad U_i \times U_j \cong \mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$

They have isomorphic sets

Two families of lines on Q.

$P \times \mathbb{P}^1$ vary P

$\mathbb{P}^1 \times R$ vary R

fix s,t vary u,v

$tw - sy = 0$
$tx - sz = 0$ } intersection of 2 planes = line

$tw - sy = 0$
$tsu - stu$

$tx - sz = 0$
$tsv - stv$

fix u,v vary s,t

$vw - ux = 0$

$vy - uz = 0$

Consider $P = [1,0,0,0] \in Q$

Project from P onto $w = 0$

$\varphi([w,x,y,z]) = [x,y,z]$

Restricting we get a morphism

$\varphi: Q \setminus P \to \mathbb{P}^2 = \{w = 0\}$

That is a birational isomorphism

$\varphi^{-1}[x,y,z] = [xy, xz, yz, z^2]$

φ is defined except when $x = y = z = 0$

φ^{-1} is defined except when

$z = 0$ and either $x = 0$ or $y = 0$

$[0,*,0] \qquad [*,0,0]$

two points

$[0,1,0] \qquad [1,0,0]$

$[x,y,z] \to [xy, xz, yz, z^2]$

$\to [xz, yz, z^2] = [x,y,z]$ if $z \neq 0$

$[w,x,y,z] \to [x,y,z]$

$\to [xy, xz, yz, z^2]$

but on Q $xy = wz$ $= [wz, xz, yz, z^2]$

$= [w,x,y,z]$ if $z \neq 0$

Since φ is projection, the two lines on Q through P each map to a point.

These two lines are $[1,0] \times [u,v] \to [u,v,0,0]$

$[s,t] \times [1,0] \to [s,0,t,0]$

where do they map to in $\mathbb{P}^2$

$[v,0,0] = [1,0,0]$

$[0,t,0] = [0,1,0]$

These are the two points at where φ^{-1} is undefined.

Look at where the two families of lines on Q land in $\mathbb{P}^2$

$(s,t) \times (u,v) \to (su, sv, tu, tv)$

$\downarrow \varphi$

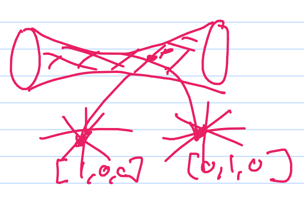
(sv, tu, tv) [x y z]

Fix s,t let u,v vary

$tx - sz = 0$

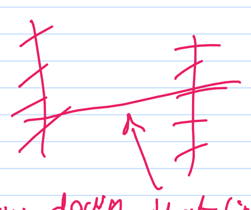
all these lines go through $[0,1,0]$

Fix u,v $\qquad [1,0,0]$



need to blow up $[1,0,0]$ and $[0,1,0]$ so the lines separate out.

The line through $[1,0,0]$ and $[0,1,0]$ is in both families but the families on Q are disjoint.



Blow down that line

Each family of lines is now missing a line.

It gets replaced by the other $\varphi^{-1}(0)$.

<u>Some interesting exercises to look at.</u>

<u>4.5</u> Show that the quadric surface $Q: xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$.

- Already showed that they are birational
- suggestions for showing that they are not isomorphic. Show they are not even homeomorphic. Exercise 3.7(a) shows that $\mathbb{P}^2$ has a topological property that you can show Q does not have.

(out of town)
Notes from
Silvia

4.6 Plane Cremona Transformation

03/21/2019

Another example of a birational isomorphism to analyse.

4.7 Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ s.t. the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of U to V which sends P to Q.

Since Zariski open sets are very big this shows that local rings are not very local.

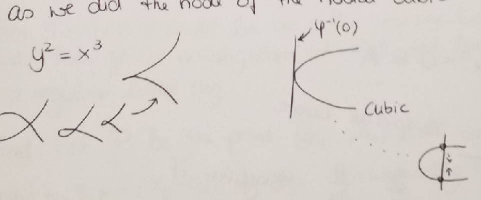
The proof of a weaker statement is easy

$$\mathcal{O}_{P,X} \cong \mathcal{O}_{Q,Y} \Rightarrow \text{Frac}\,\mathcal{O}_{P,X} \cong \text{Frac}\,\mathcal{O}_{Q,Y}$$

$\Rightarrow K(X) \cong K(Y) \Rightarrow$ X and Y are birationally equiv.

$\Rightarrow$ they have isomorphic open sets.

4.10 Blow up the cusp of the cuspidal cubic $y^2 = x^3 \subseteq \mathbb{A}^2$ as we did the node of the nodal cubic

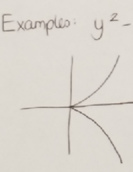


Chapter 5:
Nonsingular Varieties

Definition: Let $Y \subseteq \mathbb{A}^n$ be an affine variety and let $f_1, \dots, f_t \in A = k[x_1, \dots, x_n]$ be a set of generators for the ideal of Y. Y is nonsingular at a point $P \in Y$ if and only if the rank of the matrix $\| \frac{\partial f_i}{\partial x_j}(P) \|$ is $n-r$ where r is the dimension of Y. Y is nonsingular if it is nonsingular at every point.

Remarks: (1) Formal derivatives of polynomials
(2) Corresponds to manifold in analysis.
(3) From the definition it is not clear it is independent of the choice of generators for the ideal. It is and in fact it is an isomorphism invariant. We shall see this soon.

Examples: $y^2 - x^3 = 0 \subseteq \mathbb{A}^2$

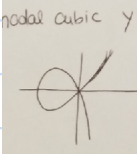


Cuspidal cubic

$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$
$-3x^2$ $2y$

Singular if $\begin{Bmatrix} -3x^2 = 0 \\ 2y = 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix}$ and

nodal cubic $y^2 - x^2(x+1) = 0 \Leftrightarrow y^2 - x^3 - x^2 = 0$



$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$
$-3x^2 - 2x$ $2y$

$\begin{Bmatrix} -3x^2 - 2x = 0 \\ 2y = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x = 0,\ x = -2/3 \\ y = 0 \end{Bmatrix}$ sing. pt.

$\rightsquigarrow (0,0)$ ← only, $(-\frac{2}{3}, 0) \notin$ curve

Definition: Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. A is a regular local ring if and only if $\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2} = \dim A$.

dim A is Krull dimension

$\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2}$ vector space dimension

$\frac{\mathfrak{m}}{\mathfrak{m}^2}$ is a module over $A/\mathfrak{m} = k$

$(f + \mathfrak{m})(g + \mathfrak{m}^2) = fg + \mathfrak{m}^2$ ($g \in \mathfrak{m}$)

Independent of choices say $f_1 \in \mathfrak{m}, g_1 \in \mathfrak{m}^2$

$(f + f_1) + \mathfrak{m} = f + \mathfrak{m}$

$(g + g_1) + \mathfrak{m}^2 = g + \mathfrak{m}^2$

$(f + f_1)(g + g_1) = fg + \underbrace{fg_1 + f_1 g + f_1 g_1}_{\in \mathfrak{m}^2}$

Thm. 5.1: Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Let $P \in Y$ be a point. Then Y is nonsingular at P iff the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.

Proof: Let P be the point $(a_1, \dots, a_n) \in \mathbb{A}^n$ and let $\mathfrak{a}_P = (x_1 - a_1, \dots, x_n - a_n)$ be the corresponding maximal ideal in $A = k[x_1, \dots, x_n]$. We define a map

$$\theta : A \longrightarrow k^n \text{ by}$$
$$\theta(f) = \langle \frac{\partial f}{\partial x_1}(P), \dots, \frac{\partial f}{\partial x_n}(P) \rangle, \quad f \in A$$

θ is linear over k.

$\theta(x_i - a_i) = \langle 0, \dots, 0, 1, 0, \dots, 0 \rangle$ (1 in the i^{th} spot)

$\theta(x_i - a_i), i = 1, \dots, n$, is the standard basis for k^n.

$\theta(\mathfrak{a}_P^2) = 0$

product rule

$d(fg) = f\,dg + g\,df$

θ' induces an isomorphism $\theta' : \frac{\mathfrak{a}_P}{\mathfrak{a}_P^2} \longrightarrow k^n$

Now let $\mathfrak{b}$ be the ideal of Y in A and let $f_1, \dots, f_t$ be a set of generators for $\mathfrak{b}$.

$\mathfrak{b} \subseteq \mathfrak{a}_P$. Assume $P \in Y$. Then the rank of the Jacobian matrix $J = \| \frac{\partial f_i}{\partial x_j}(P) \|$ is just the dimension of $\theta(\mathfrak{b})$ as a subspace of k^n.

{The rows of the Jacobian matrix are $\theta(f_1), \theta(f_2), \dots, \theta(f_t)$.
Claim: these span $\theta(\mathfrak{b})$

$$\theta\left(\sum_{i=1}^{t} g_i f_i\right) = \langle \sum_{i=1}^{t} \frac{\partial f_i}{\partial x_1}(P) g_i(P) + f_i(P) \frac{\partial g_i}{\partial x_1}(P), \dots$$

$f_i(P) = 0$: $= \sum_{i=1}^{t} g_i(P)\,\theta(f_i)$

Using the isomorphism θ' this is the same as the dimension of the subspace $\frac{\mathfrak{b} + \mathfrak{a}_P^2}{\mathfrak{a}_P^2}$ of $\frac{\mathfrak{a}_P}{\mathfrak{a}_P^2}$

{The image of $\mathfrak{b}$ under θ is $\frac{\mathfrak{b}}{\mathfrak{b} \cap \mathfrak{a}_P^2} \cong \frac{\mathfrak{b} + \mathfrak{a}_P^2}{\mathfrak{a}_P^2}$ via the second isomorphism thm.

On the other hand, the local ring $\mathcal{O}_P$ of P in Y is obtained from A by dividing by $\mathfrak{b}$ and localizing at the maximal ideal $\mathfrak{a}_P$. Thus if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$, we have

$$\mathfrak{m}/\mathfrak{m}^2 \cong \frac{\mathfrak{a}_P}{\mathfrak{b} + \mathfrak{a}_P^2}$$

{First forget about the localizing

$\frac{\mathfrak{m}}{\mathfrak{m}^2}$ where $\mathfrak{m}$ is the max. ideal of P in A(Y)

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(\mathfrak{a}_P/\mathfrak{b})}{(\mathfrak{a}_P/\mathfrak{b})^2} = \frac{(\mathfrak{a}_P/\mathfrak{b})}{\frac{\mathfrak{a}_P^2}{\mathfrak{b} \cap \mathfrak{a}_P^2}} = \frac{(\mathfrak{a}_P/\mathfrak{b})}{\frac{\mathfrak{a}_P^2 + \mathfrak{b}}{\mathfrak{b}}} = \frac{\mathfrak{a}_P}{\mathfrak{a}_P^2 + \mathfrak{b}}$$

Now the localizing doesn't change things.

$\frac{\mathfrak{m}}{\mathfrak{m}^2}$ vs. $\mathfrak{m}_{\mathfrak{a}_P}/(\mathfrak{m}_{\mathfrak{a}_P})^2$

$\cong$

$\frac{\mathfrak{m}}{\mathfrak{m}^2}$ $\mathfrak{m}_{\mathfrak{a}_P}/(\mathfrak{m}_{\mathfrak{a}_P})^2$

$f + \mathfrak{m}^2$ $\frac{f}{g} + (\mathfrak{m}_{\mathfrak{a}_P})^2, g \notin \mathfrak{a}_P$

$f \in \mathfrak{m}$

isomorphism:

$$g(P)^{-1} f + \mathfrak{m}^2 \longrightarrow \frac{f}{g} + (\mathfrak{m}_{\mathfrak{a}_P})^2$$

$$\frac{g(P)^{-1} f}{1} - \frac{f}{g} = \frac{g(P)^{-1} fg - f}{g} = \frac{f(g(P)^{-1} g - 1)}{g} \in (\mathfrak{m}_{\mathfrak{a}_P})^2$$

$f \in \mathfrak{m}, g(P)^{-1} g - 1 \in \mathfrak{m}$

Counting dimensions of vector spaces we have

$$\dim \frac{\mathfrak{m}}{\mathfrak{m}^2} + \text{rank } J = n$$

{$\dim \frac{\mathfrak{a}_P}{\mathfrak{a}_P^2} = n$

$\dim \frac{\mathfrak{b} + \mathfrak{a}_P^2}{\mathfrak{a}_P^2} = \text{rank } J$

$\dim \frac{\mathfrak{m}}{\mathfrak{m}^2} = \dim \frac{\mathfrak{a}_P}{\mathfrak{b} + \mathfrak{a}_P^2} = \dim \left(\frac{\mathfrak{a}_P}{(\mathfrak{a}_P)^2} \Big/ \left(\frac{\mathfrak{b} + \mathfrak{a}_P^2}{\mathfrak{a}_P^2} \right) \right) = n - \text{rank } J$

Now let $\dim Y = r$. Then $\mathcal{O}_P$ is a local ring of dim r (3.2), so $\mathcal{O}_P$ is regular if and only if $\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2} = r$. But this is equivalent to rank $J = n - r$

That was definition of nonsingular point. □

Definition: Let Y be any variety. Y is nonsingular at a point $P \in Y$ if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring. Y is nonsingular if and only if it is nonsingular at every point.

Y is singular if and only if it is not nonsingular.

Proposition 5.2A

If A is a Noetherian local ring with maximal ideal m and residue field k, then $\dim_k \frac{m}{m^2} \geq \dim A$

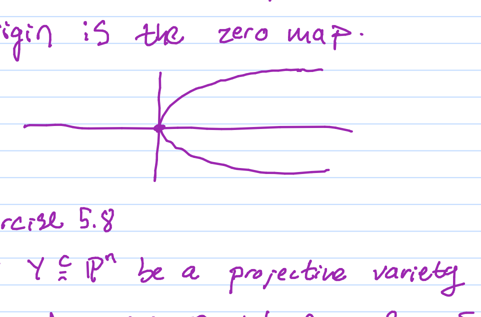
Exercise 5.10

For a point P on a variety X, let m be the maximal ideal of the local ring $\mathcal{O}_{P,X}$. We define the zarisky tangent space $T_P(X)$ at P to be the dual k-vector space of m/m^2

(a) For any point $P \in X$, $\dim_P(X) \geq \dim X$ with equality iff P is nonsingular.

(b) For any morphism $\varphi: X \to Y$, there is a natural induced k-linear map $T_P(\varphi): T_P(X) \to T_{\varphi(P)}Y$

(c) If φ is the vertical projection of the parabola $x=y^2$ onto the x-axis. Show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.



Exercise 5.8

Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let $f_1, \dots, f_t \in S = k[X_0, \dots, X_n]$ be homogeneous polynomials which generate the ideal of Y. Let $P \in Y$ be a point with homogeneous coordinates $P = (a_0, \dots, a_n)$. Show that p is nonsingular on Y iff the rank of the matrix $\left\| \frac{\partial f_i}{\partial x_j}(a_0, \dots, a_n) \right\|$ is $n-r$.

Hint: You will need Euler's Lemma, which says if f is a homogeneous polynomial of degree d, then

$$\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = df$$

$$f = \sum_{i_0+\dots+i_n=d} a_{i_0 \dots i_n} x_0^{i_0} \cdots x_n^{i_n}$$

$$\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^n x_i \sum_{i_0+\dots+i_n=d} i_i a_{i_0 \dots i_n} \cdot x_0^{i_0} \cdots x_i^{i_i - 1} \cdots x_n$$

$$= \sum_{i_0+\dots+i_n=d} \underbrace{\left(\sum_{i=0}^n i_i\right)}_{d} a_{i_0 \dots i_n} x_0^{i_0} \cdots x_n^{i_n}$$

Theorem 5.3

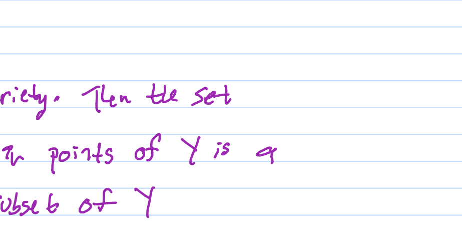
Let Y be a variety. Then the set $\operatorname{Sing} Y$ of singular points of Y is a proper closed subset of Y

Completion

We mentioned before that local rings are not very local

$y^2 - x^2(x-1) = 0$ nodal cubic

$xy = 0$



Let A be a ring and $m_1 \supset m_2 \supset \dots$ a decreasing sequence of ideals in A. We will define $\hat{A}$ the completion of A with respect to $m_1 \supset m_2 \supset \dots$

Notice that we have a sequence of homomorphisms $A/m_1 \xleftarrow{\varphi_1} A/m_2 \xleftarrow{\varphi_2} A/m_3 \leftarrow \dots$

$\hat{A}$ is defined to be the inverse limit $\varprojlim A/m_n$.

The inverse limit is defined as a subring of the infinite direct product of rings $\prod_{i=1}^\infty A/m_i$

$\hat{A} = \{(a_1, a_2, a_3, \dots) \in \prod_{i=1}^\infty A_{m_i}$ s.t. $\forall i \ \varphi_{i-1}(a_i) = a_{i-1}\}$

When A is a local ring with maximal ideal m, by $\hat{A}$ we mean the completion with respect to $m \supset m^2 \supset m^3 \supset \dots$

Example: Take $A = k[t]$ $m_i = (t^i)$ then $\hat{A} = k[[t]]$ ring of formal power series.

Map $k[[t]] \to \hat{A}$

an element of $k[t]/(t^n)$ can be written as $a_0 + a_1 t + \dots + a_{n-1} t^{n-1}$

$\sum_{i=0}^\infty a_i t^i \in k[[t]]$ maps to $(g_1, g_2, g_3, \dots)$

$g_i \in k[t]/(t^i)$

$g_i = \sum_{j=0}^{i-1} a_j t^j$

More generally if $A = k[x_1, \dots, x_n]$ $m_i = (x_1, \dots, x_n)^i$

$\hat{A} \cong k[[X_1, \dots, X_n]]$

Also works you first localize at $(X_1, \dots, X_n)$

Definition.

Let Y be a variety and $P \in Y$. The complete local ring of Y at P is $\hat{\mathcal{O}}_{P,Y}$. Given two points on two varieties $P \in Y$, $Q \in X$ we say P is analytically isomorphic to Q iff $\hat{\mathcal{O}}_{P,Y}$ and $\hat{\mathcal{O}}_{P,X}$ are isomorphic as k-algebras.

We can extend the definition of local ring of a variety at a point to local ring of a closed set at a point.

$P \in X \subseteq \mathbb{A}^n$, X closed

$\mathcal{O}_{P,X} = \left(k[X_1, \dots, X_n]/I(X)\right)_{m/I(X)}$

where m is the maximal ideal of P in $k[X_1, \dots, X_n]$

Computation of local rings

Since affine opens are a base for the topology assume P is a point on an affine closed set $X \subseteq \mathbb{A}^n$, say $I(X) = (f_1, \dots, f_t)$ and assume $P = (0, \dots, 0)$

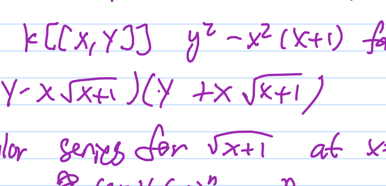
$\hat{\mathcal{O}}_{P,X} = k[[x_1, \dots, x_n]]/(f_1, \dots, f_t)$

Example 5.6.3

$\mathbb{A}^2$ $P = (0,0)$

$X = Z(Y^2 - X^2(X+1))$

$Y = Z(XY)$



$\mathcal{O}_{P,X}$ is still an integral domain but $\mathcal{O}_{P,Y}$ is not

$\hat{\mathcal{O}}_{P,X} \cong \hat{\mathcal{O}}_{P,Y}$ as k-algebras

$$\frac{k[[X,Y]]}{(Y^2 - X^2(X+1))} \qquad \frac{k[[X,Y]]}{(XY)}$$

In $k[[X,Y]]$ $Y^2 - X^2(X+1)$ factors

$(Y - X\sqrt{X+1})(Y + X\sqrt{X+1})$

Taylor series for $\sqrt{x+1}$ at $x=0$ is

$$1 - \sum_{n=1}^\infty \frac{(2n)!(-1)^n}{(2^n n!)^2(2n-1)} x^n \qquad \text{char } k \neq 2$$

$X \mapsto Y - X\sqrt{X+1}$

$Y \mapsto Y + X\sqrt{X+1}$

6 Nonsingular Curves

Resolution of Singularities

Definition.

Let X and Y be varieties and $f: X \to Y$ a morphism. f is called a projective morphism iff there exists a commutative diagram

$$\begin{array}{ccc} X & \xhookrightarrow{g} & Y \times \mathbb{P}^n \\ & {\searrow}^{f} & \downarrow p \\ & & Y \end{array}$$

where $g(X)$ is closed in $Y \times \mathbb{P}^n$, g is an isomorphism onto $g(X)$ and P is projective onto the first factor.

Definition

Let X be a variety $X_0 < X$ the open subset of nonsingular points. A resolution of singularities of X is a nonsingular variety $\tilde{X}$ together with a surjective projective morphism $f: \tilde{X} \to X$ s.t. f induces an isomorphism between $f^{-1}(X_0)$ and X_0.

Examples:

(1) Blowing up is a projective morphism.

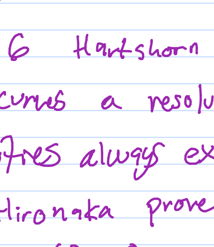
$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{A}^n \times \mathbb{P}^{n-1} \\ & \searrow & \downarrow \\ & & \mathbb{A}^n \end{array}$$

$Z(Y^2 - X^2(X+1)) < \mathbb{A}^2$

$\tilde{\mathbb{A}}^2$

$\downarrow$

$\mathbb{A}^2$



Most books require f to be proper not projective. projective $\Rightarrow$ proper and we can't define proper until we do schemes.

In section 6 Hartshorn proves that for curves a resolution of singularities always exists

In 1964 Hironaka proves that in characteristic 0 resolutions of singularities always exist and got a Field's Medal for it.

unsolved for char p

Hironaka did it by blowing up and normalizing.

Outline of an alternate way for curves.

Normalization of Varieties

$A \subseteq B$ both integral domains $b \in B$ is said to be integral over A iff b satisfies a monic polynomial with coefficients in A

$P(X) = X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0$

$a_i \in A$.

Set $\bar{A} = \{b \in B \mid b$ is integral over $A\}$

$\bar{A}$ is an integral domain

It is called the integral closure of A in B.

Normalization of varieties

Let X be an affine variety with affine coordinate ring $A(X) = k[x_1, \dots, x_n]/I(X)$

Let B be the integral closure of $A(X)$ in its fraction field $\mathrm{Frac}\, A(X)$. It can be proven that B is still a finitely generated k-algebra. So it must correspond to some affine variety $\tilde{X}$.

The containment $A(X) \subseteq B$ must correspond to a morphism $\tilde{X} \to X$. $\tilde{X}$ is called the normalization of X.

For general X, cover X with affine sets, do normalization to each. Then glue back together to get $\tilde{X}$.

Properties of $f: \tilde{X} \to X$

(1) f is projective (proper)

(2) f induces an isomorphism between $X \setminus \mathrm{sing}\, X$ and $f^{-1}(X \setminus \mathrm{sing}\, X)$

(3) The singularities of $\tilde{X}$ have codimension at least 2

Thus for curves you have a desingularization

For higher dims often $\tilde{X}$ is still singular

7. Intersections in Projective space

Recall the following result from linear algebra.

Let V be a vector space of dimension n and let U, W be subspaces of dimension r, s.

Then $\dim(U \cap W) \geq r+s-n$

There are similar results for intersections of varieties in affine and projective space.

Proposition 7.1

Affine Dimension Theorem

Let Y, Z be varieties of dimensions r, s that are closed subsets in $\mathbb{A}^n$.

Then every irreducible component W of $Y \cap Z$ has dimension $\geq r+s-n$

This includes the possibility that $Y \cap Z$ is empty.

proof:

We proceed in several steps.

First suppose that Z is a hypersurface defined by an equation $f=0$. This is Exercise I.1.8.

$r=n-1$

$n-1+s-n = s-1$

Now for the general case.

We consider the product $Y \times Z \subseteq \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ which is a variety of dimension $r+s$

Exercise I.3.15.

Let Δ be the diagonal $\{P \times P \mid P \in \mathbb{A}^n\} \subseteq \mathbb{A}^{2n}$

Then $\mathbb{A}^n$ is isomorphic to Δ by the map $P \to P \times P$

Δ is closed in $\mathbb{A}^{2n}$

take coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$

x_i coord on first $\mathbb{A}^n$

y_i coord on second $\mathbb{A}^n$

$\Delta = Z(x_1-y_1, x_2-y_2, \dots, x_n-y_n)$

$\varphi: \mathbb{A}^n \to \mathbb{A}^{2n}$

$\varphi(x_1, \dots, x_n) = (x_1, \dots, x_n, x_1, \dots, x_n)$

morphism because given by polynomials

$\varphi^{-1}(x_1, \dots, x_n, x_1, \dots, x_n) = (x_1, \dots, x_n)$

and under this isomorphism $Y \cap Z$ corresponds to $(Y \times Z) \cap \Delta$

$(a_1, \dots, a_n) \in Y \cap Z \iff$

$(a_1, \dots, a_n, a_1, \dots, a_n) \in (Y \times Z) \cap \Delta$

Since Δ has dimension n ($\cong \mathbb{A}^n$)

and since $r+s-n = \underset{\underset{Y \times Z}{\uparrow}}{(r+s)} + \underset{\underset{\Delta}{\uparrow}}{n} - \underset{\mathbb{A}^n}{2n}$

We reduce to proving the result for the two varieties $Y \times Z$ and Δ in $\mathbb{A}^{2n}$.

Recall $\Delta = Z(x_1-y_1, \dots, x_n-y_n)$.

Apply the First case n-times.

dim goes down by at most 1 for each. $\square$

Theorem 7.2 (Projective Dimension Theorem)

Let Y, Z be varieties of dimensions r, s that are closed subsets in $\mathbb{P}^n$. Then every irreducible component of $Y \cap Z$ has dimension $\geq r+s-n$. Furthermore, if $r+s-n \geq 0$, then $Y \cap Z$ is nonempty.

Proof:

The first statement follows from the previous result, since $\mathbb{P}^n$ is covered by affine n-spaces.

For the second result, let $C(Y)$ and $C(Z)$ be the cones over Y, Z in $\mathbb{A}^{n+1}$,

Exercise I.2.10.

I.2.10 The cone over a projective variety.

Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta: \mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \dots, a_n)$ to the point with homogeneous coordinates $[a_0, \dots, a_n]$ we define the cone over Y to be

$C(Y) = \theta^{-1}(Y) \cup \{(0, \dots, 0)\}$

(a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$ whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \dots, x_n]$

(b) $C(Y)$ is irreducible iff Y is.

(c) $\dim C(Y) = \dim Y + 1$

Then $C(Y), C(Z)$ have dimensions $r+1, s+1$ respectively.

Furthermore $C(Y) \cap C(Z)$ is nonempty because it contains the origin $P = (0, \dots, 0)$. By the affine dimension theorem,

$C(Y) \cap C(Z)$ has dimension $\geq (r+1)+(s+1)-(n+1)$
$= r+s-n+1$
> 0

$C(Y) \cap C(Z)$ must have points other than $(0, \dots, 0)$

These correspond to points of $Y \cap Z$

$(r+1)+(s+1)-(n+1)-1 = r+s-n$

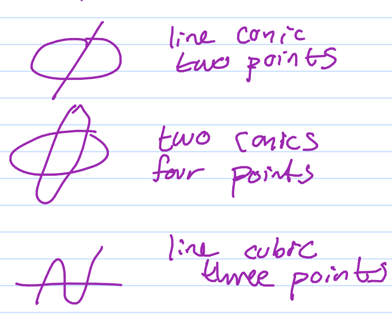
Some inspiration for the rest of the section.

Recall the following result. Let $f \in k[x]$ be a nonconstant polynomial of degree d. Then f has d roots.

Of course to make that true you need k algebraically closed and roots are counted with multiplicity.

Suppose you want a similar statement about polynomials in two variables.

To get a finite number of points you want to intersect two polynomial curves.



f, g # points of intersection $= (\deg f)(\deg g)$

need some assumptions

f, g no common factors

k algebraically closed

$y = x^2+1 \quad y=0$

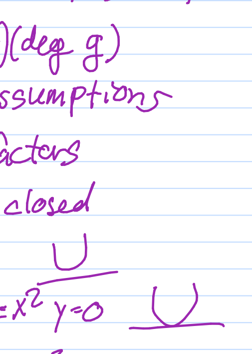
multiplicity $y = x^2 \quad y=0$

work in $\mathbb{P}^2$ not $\mathbb{A}^2$

parallel lines

curve asymptote $xy=1$

$y=x^2 \quad x=0$



Definition

A numerical polynomial is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(n) \in \mathbb{Z}$ for all $n \gg 0$ $n \in \mathbb{Z}$

Proposition 7.3

(a) If $P \in \mathbb{Q}[z]$ is a numerical polynomial then there are integers $c_0, c_1, \dots, c_r$ s.t.

$P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \dots + c_r$

where $\binom{z}{r} = \frac{1}{r!} z(z-1) \cdots (z-r+1)$ is the binomial coefficient function.

In particular $P(n) \in \mathbb{Z} \quad \forall n \in \mathbb{Z}$

(b) If $f: \mathbb{Z} \to \mathbb{Z}$ is any function and if there exists a numerical polynomial $Q(z)$ s.t. the difference function $\Delta f = f(n+1) - f(n)$ is equal to $Q(n)$ for all $n \gg 0$ then there exists a numerical polynomial $P(z)$ s.t. $f(n) = P(n)$ for all $n \gg 0$.

Prop 7.3

(a) If $P \in \mathbb{Q}[z]$ is a numerical polynomial then there are integers $c_0, c_1, \ldots, c_r$ s.t.

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where $\binom{z}{r} = \frac{1}{r!} z(z-1)\cdots(z-r-1)$

is the binomial coefficient. In particular, $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$

(b) If $f: \mathbb{Z} \to \mathbb{Z}$ is any function, and if there exists a numerical polynomial $Q(z)$ s.t. the difference function $\Delta f(n) = f(n+1) - f(n)$ is equal to $Q(z)$ $\forall n \gg 0$ then there exists a numerical polynomial $P(z)$ s.t. $f(n) = P(n)$ $\forall n \gg 0$.

proof

(a) By induction on the degree of P, the case of degree 0 being obvious.

{ degree 0 means constant so $P(z) = c_r$, $c_r \in \mathbb{Q}$

But $P(n) \in \mathbb{Z}$ for $n \gg 0$ but $P(n) = c_r$ all n so $c_r \in \mathbb{Z}$ }

Since $\binom{z}{r} = \frac{z^r}{r!}$ + lower terms we can express any polynomial $P \in \mathbb{Q}[z]$ of degree r in the above form with unique $c_0, \ldots, c_r \in \mathbb{Q}$

{Start from z^r work down}

For any polynomial define the difference function ΔP by

$\Delta P(z) = P(z+1) - P(z)$

Since $\Delta\binom{z}{r} = \binom{z}{r-1}$

This is the Pascal triangle identity $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

$\Delta P(z) = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \cdots + c_{r-1}$

$\Delta P(n) \in \mathbb{Z}$ $\forall n \gg 0$ since $P(n) \in \mathbb{Z}$ $\forall n \gg 0$

$\Delta P(z)$ is a numerical polynomial.

By induction $c_0, c_1, \ldots, c_{r-1} \in \mathbb{Z}$

claim $\binom{z}{r} \in \mathbb{Z}$ $\forall z \in \mathbb{Z}$

For $z \geq r$ that is because it is a binomial coefficient

For $0 \leq z < r$ it is 0

For $z < 0$ $-z + r - 1 \geq r$

so that $\binom{-z+r-1}{r} \in \mathbb{Z}$ by prev case

$\binom{-z+r-1}{r} = \frac{1}{r!}(-z+r-1)(-z+r-2)\cdots(-z+r-1-r)(-z+r-1-(r+1))$

$(-1)^r\binom{z}{r} \in \mathbb{Z}$

$P(z) = \underbrace{c_0\binom{z}{r} + c_1\binom{z}{r+1} + \cdots + c_{r-1}\binom{z}{1}}_{\text{always in } \mathbb{Z}} + c_r$ for $z \in \mathbb{Z}$

By assumption $P(n) \in \mathbb{Z}$ for $n \gg 0$ gives $c_r \in \mathbb{Z}$

Then $P(n)$ always $\in \mathbb{Z}$ for $n \in \mathbb{Z}$

(b) Using (a) we write

$Q = c_0\binom{z}{r} + \cdots + c_r$

$c_0, \ldots, c_r \in \mathbb{Z}$

let $P(z) = c_0\binom{z}{r+1} + \cdots + c_r\binom{z}{1}$

$\Delta P = Q$

$\Delta(f - P) = 0$ $\forall n \gg 0$

$\Delta f - \Delta P$ ($= Q - Q$)

$(f-P)(n) = c_{r+1}$ for all $n \gg 0$

$f = P + c_{r+1}$ for all $n \gg 0$ ∎

Def.

Let S be a ring. A grading of S is an expression of the additive group $(S,+)$ as an internal direct sum

$S = \bigoplus_{i=0}^{\infty} S_i$ with the property that if $a \in S_e$ $b \in S_d$ then $ab \in S_{e+d}$

written $S_e \cdot S_d \subseteq S_{e+d}$

A graded ring is a ring together with a given grading on it.

An ideal $I \subseteq S$ is called graded or homogeneous if $I = \bigoplus_{i=0}^{\infty}(I \cap S_i)$ as a group.

Now let S be a graded ring and M an S-module. We say that M is a graded S-module iff M as a group can be expressed as an internal direct sum $M = \bigoplus_{i=-\infty}^{+\infty} M_i$ s.t. if $f \in S_e$ and $m \in M_d$ then $fm \in M_{e+d}$ written

$S_e \cdot M_d \subseteq M_{e+d}$

Examples:

If S is a graded ring it is a graded module over itself.

Any homogeneous ideal of S is a graded module over S. Also any quotient of S by a homogeneous ideal is a graded module over S.

Let S be a graded ring and M a graded S-module. We define the twisted module $M(l)$ be setting $M(l)_i = M_{l+i}$

Say $a \in S_e$ $b \in M(l)_d$

Then $b \in M_{l+d}$ so $ab \in M_{e+l+d}$ $= M(l)_{e+d}$. So this is still a grading.

Def

Let S be a graded ring and M and N graded modules over S. An S-module homomorphism $\varphi: M \to N$ is said to be graded or homogeneous of degree d iff $\forall e$ $\varphi(M_e) \subseteq N_{d+e}$. Two graded modules are considered isomorphic iff they are isomorphic via an isomorphism that is graded of degree 0.

The most interesting graded homomorphisms are those graded of degree 0. In fact usually when one says the homomorphism is graded without saying of degree d, one means graded of degree 0. This is where twisting comes in.

Any homomorphism that is graded of degree d can be made into one graded of degree 0.

$\varphi: M \to N$ graded of degree d

$\varphi: M(-d) \to N$ or $\varphi: M \to N(d)$ are graded of degree 0.

$\varphi(M(-d)_e) = \varphi(M_{-d+e})$

$\subseteq M_{d-d+e} = N_e$

$\varphi(M_e) \subseteq N_{d+e} = N(d)_e$

Def:

Let A be a ring and M an A-module

The length of M over A denoted $l_A(M)$ is equal to the length of the largest increasing sequence of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_d = M$

length = number of steps

Example: If A is a field this is just vector space dimension.

Proposition/Definition

If M is a graded S-module, we define the annihilator of M, $\text{ann}\, M = \{ s \in S \mid sM = 0 \}$ This is a homogeneous ideal in S.

proof:

ideal

$(s+t)m = sm + tm$

$sm = 0$ $tm = 0$ $\Rightarrow (s+t)m = 0$

$(as)m = a(sm) = 0$

homogeneous

$s = s_0 + s_1 + \cdots + s_d \in \text{Ann}\, M$

we need to show $s_i \in M$.

$s \in \text{Ann}\, M \Leftrightarrow sm = 0$ $\forall m \in M$

$\Leftrightarrow sm_i = 0$ $\forall m_i \in M$ s.t. m_i homogeneous

{ "$\Rightarrow$" obvious "$\Leftarrow$" any $m \in M$ is $\sum m_i$ }

So we want to show $s_j m_i = 0$ $0 \leq j \leq d$ and m_i homogeneous of degree i.

$sm_i = 0 \Rightarrow s_0 m_i + s_1 m_i + \cdots + s_d m_i = 0$

but each term in sum is homog of different degree $\Rightarrow s_j m_i = 0$ all j ∎

Prop 7.4

Let M be a finitely generated graded module over a Noetherian graded ring S.

Then there exists a filtration

$0 = M^0 \subsetneq M^1 \subsetneq M^2 \subsetneq \cdots \subsetneq M^r = M$

by graded submodules such that for each i, $M^i/M^{i-1} \cong (S/p_i)(l_i)$

where p_i is a homogeneous prime ideal, and $l_i \in \mathbb{Z}$

The filtration is not unique but for any such filtration we have:

(a) if p is a homogeneous prime ideal of S then $p \supseteq \text{Ann}\, M$ $\Leftrightarrow$ $p \supseteq p_i$ for some i.

In particular the minimal elements of the set $\{p_1, \ldots, p_r\}$ are just the minimal primes of M. i.e. the primes which are minimal containing $\text{Ann}\, M$.

(b) for each minimal prime of M the number of times which p occurs in the set $\{p_1, \ldots, p_r\}$ is equal to the length of M_p over the local ring S_p (and hence is independent of the filtration)

Skip proof.

Def:

If p is a minimal prime of a graded S-module M, we define the multiplicity of M at p denoted $\mu_p(M)$ to be the length of M_p over S_p

Now say $S = k[x_0, \ldots, x_n]$

M a graded S-module.

Notice this makes each M_i a vector space over k.

Define the Hilbert function of M denoted φ_M as

$\varphi_M(l) = \dim_k M_l$

$\varphi_M: \mathbb{Z} \to \mathbb{Z}^{\geq 0}$

Theorem 7.5 (Hilbert-Serre)
Let M be a finitely generated graded $S=k[x_0, \dots, x_n]$-module. Then there is a unique polynomial $P(z) \in \mathbb{Q}[x]$ s.t. $\varphi_M(l) = P_M(l)$ for all $l \gg 0$. Furthermore, $\deg(P_M(z)) = \dim Z(\operatorname{Ann} M)$, where Z denotes the zero set in $\mathbb{P}^n$ of a homogeneous ideal.

proof: If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence then $\varphi_M = \varphi_{M'} + \varphi_{M''}$ and
$Z(\operatorname{Ann} M) = Z(\operatorname{Ann} M') \cup Z(\operatorname{Ann} M'')$
He is assuming in the short exact sequence, maps are graded of degree 0.
So for any fixed l we get a short exact sequence of vector spaces
$0 \to (M')_l \to M_l \to (M'')_l \to 0$
so that $\varphi_M = \varphi_{M'} + \varphi_{M''}$ is just rank + nullity = dim domain
For the statement about annihilators notice that
$\operatorname{Ann} M' \supset \operatorname{Ann} M$ and
$\operatorname{Ann} M'' \supset \operatorname{Ann} M$. Thus
$Z(\operatorname{Ann} M) \supset Z(\operatorname{Ann} M')$
and $Z(\operatorname{Ann} M) \supset Z(\operatorname{Ann} M'')$
$\Rightarrow Z(\operatorname{Ann} M) \supset Z(\operatorname{Ann} M') \cup Z(\operatorname{Ann} M'')$
For the other inclusion we first show $\operatorname{Ann} M \supset \operatorname{Ann} M' \cdot \operatorname{Ann} M''$
Pick any $m \in M$, $s \in \operatorname{Ann} M'$, $t \in \operatorname{Ann} M''$
Consider the coset $m + M' \in M''$
$t(m+M') = 0 + M'$
$\Rightarrow tm + M' = 0 + M'$
$\Rightarrow tm \in M' \Rightarrow stm = 0$
$Z(\operatorname{Ann} M) \subset Z(\operatorname{Ann} M' \cdot \operatorname{Ann} M'')$
$= Z(\operatorname{Ann} M') \cup Z(\operatorname{Ann} M'')$

So if the theorem is true for M' and M'', it is true for M
We have $P_{M'}(z), P_{M''}(z) \in \mathbb{Q}[x]$
$P_{M'}(l) = \varphi_{M'}(l)$ $P_{M''}(l) = \varphi_{M''}(l)$ $l \gg 0$
$\varphi_M(l) = \varphi_{M'}(l) + \varphi_{M''}(l)$
$= P_{M'}(l) + P_{M''}(l)$ all $l \gg 0$
Now $\varphi_{M'}(l)$ $\varphi_{M''}(l)$ are never negative
This means the leading coefficients of $P_{M'}(z)$ $P_{M''}(z)$ are positive
$\deg P_M(z) = \deg(P_{M''}(z) + P_{M'}(z))$
$= \max\{\deg P_{M''}(z), \deg P_{M'}(z)\}$
$\dim(Z(\operatorname{Ann} M)) = \dim(Z(\operatorname{Ann} M'') \cup Z(\operatorname{Ann} M'))$
$= \max\{\dim Z(\operatorname{Ann} M'), \dim Z(\operatorname{Ann} M'')\}$
The dimension of any algebraic set is the dimension of its largest irreducible component.
The longest increasing sequence of irreducible subsets lies in one component.
By 7.4 M has a filtration with quotients of the form $(S/p)(l)$ where p is a homogeneous prime ideal and $l \in \mathbb{Z}$. So we reduce to the case $M \cong (S/p)(l)$
$0 = M^0 \subseteq M^1 \subseteq \dots \subseteq M^r = M$
If we knew it for $(S/p_i)(l_i)$ we would know it for
$M^1 \cong M^1/M^0 \cong (S/p_1)(l_1)$
Knowing it for M^i gives it for M^{i+1} because
$0 \to M^i \to M^{i+1} \to M^{i+1}/M^i \to 0$
(assume true for this: M^i; $M^{i+1}/M^i \cong (S/p_{i+1})(l_{i+1})$)
The shift l corresponds to a change of variables $z \mapsto z+l$ so it is sufficient to consider the case $M = S/p$
$S/p(l)$ ← only changes the grading by a shift. In the poly, it is just that change of variables. Does not affect annihilator or degree
If $p = (x_0, \dots, x_n)$ then $\varphi_M(l) = 0$ for $l \gg 0$
so $P_M = 0$ is the corresponding polynomial, $\deg P_M = \dim Z(p)$
where we make the convention that the zero polynomial has degree -1 and empty set has dimension -1.
If $p \neq (x_0, \dots, x_n)$ choose $x_i \notin p$ and consider the exact sequence
$0 \to M \xrightarrow{\times x_i} M \to M' \to 0$
$\times x_i$ that map is mult by x_i
$M' = M/x_iM$
Multiplication by x_i is injective because $M \cong S/p$ $x_i \notin p$
so for a coset $f + p \neq 0 + p$ so that $f \notin p$ we have $x_if \notin p$ because p is prime, so $x_if + p \neq 0 + p$
Then $\varphi_{M'}(l) = \varphi_M(l) - \varphi_M(l-1)$
$= (\Delta\varphi_M)(l-1)$
To make all maps in the sequence graded of degree 0, you actually want
$0 \to M(-1) \xrightarrow{\times x_i} M \to M' \to 0$
$\varphi_M(l) = \varphi_{M(-1)}(l) + \varphi_{M'}(l)$, where $\varphi_{M(-1)}(l) = \varphi_M(l-1)$
On the other hand $Z(\operatorname{Ann} M') = Z(p) \cap H$ where H is the hyperplane $x_i = 0$
$M \cong S/p$ so $M/x_iM \cong S/(p, x_i)$
and $Z(p) \not\subseteq H$ by choice of x_i
so by 7.2 $\dim Z(\operatorname{Ann} M') = \dim Z(p) - 1$
Now using induction on $\dim Z(\operatorname{Ann} M)$ we may assume that $\varphi_{M'}$ is a polynomial function corresponding to a polynomial $P_{M'}$ of degree $= \dim Z(\operatorname{Ann} M')$
Now by 7.3 it follows that φ_M is a polynomial function corresponding to a polynomial of degree $\dim Z(p)$ ⊛
⊛ means for $l \gg 0$, using 7.3b Δf eventually poly $\Rightarrow f$ eventually poly
The statement of 7.3b does not mention degree but proof does.
The uniqueness of P_M is clear. $P_M(l) = \varphi_M(l)$ $l \gg 0$ two polynomials that agree infinitely often are equal. □

Definition
The polynomial P_M of the theorem is the Hilbert polynomial of M.

Definition
If $Y \subseteq \mathbb{P}^n$ is an algebraic set of dimension r, we define the Hilbert polynomial of Y to be the Hilbert polynomial P_Y of its homogeneous coordinate ring $S(Y)$ (By the theorem, it is a polynomial of degree r)
We define the degree of Y to be $r!$ times the leading coefficient of P_Y.

Side remark:
Given an algebraic set $Y \subseteq \mathbb{P}^n$ there are two obvious $S = k[x_0, \dots, x_n]$ modules to associate to it.
$I(Y)$ and $S(Y) = \frac{k[x_0, \dots, x_n]}{I(Y)}$
Why choose $S(Y)$ to get the Hilbert function? Lots of results are easier to state.
Both choices are equivalent.
$0 \to I(Y) \to S \to S(Y) \to 0$
(at S:) Hilbert function here is easily combinatorially computed.

Proposition 7.6
(a) If $Y \subseteq \mathbb{P}^n$, $Y \neq \emptyset$ then the degree of Y is a positive integer.
(b) Let $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 have the same dimension r and where $\dim(Y_1 \cap Y_2) < r$ then $\deg Y = \deg Y_1 + \deg Y_2$
(c) $\deg \mathbb{P}^n = 1$
(d) If $H \subseteq \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then $\deg H = d$. (In other words, this definition is consistent with the degree of a hypersurface as defined earlier (I.4.2))

Prop 7.6 (a) If $Y \subseteq \mathbb{P}^n$, $Y \neq \emptyset$ then the degree of Y is a positive integer.
(b) Let $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 have the same dimension r and where $\dim(Y_1 \cap Y_2) < r$. Then $\deg Y = \deg Y_1 + \deg Y_2$
(c) $\deg \mathbb{P}^n = 1$
(d) If $H \subsetneq \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then $\deg H = d$.
(In other words, this definition of degree is consistent with the degree of a hypersurface defined earlier in (1.4.2))

Proof (a)
$Y \neq \emptyset \Rightarrow P_Y$ is a polynomial of degree $r = \dim Y \geq 0$
7.3 It looks like $c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots$
$c_i \in \mathbb{Z}$
$\deg Y = c_0 \in \mathbb{Z}$
positive because for $l \gg 0$ $P_Y(l) = \varphi_{S/I}(l)$
$= \dim_k (S/I)_l \geq 0$
Since $Y \neq \emptyset$, $I(Y) \neq (x_0, \dots, x_n)$
In fact $\dim(S/I)_l > 0$
(b)
Let I_1, I_2, be the ideals of Y_1 and Y_2.
Then $I = I_1 \cap I_2$ is the ideal of Y.
we have an exact sequence
$0 \to S/I \to S/I_1 \oplus S/I_2 \to S/I_1+I_2 \to 0$
check:
Map $S/I_1 \oplus S/I_2 \to S/I_1+I_2$
by $(f+I_1, g+I_2) \mapsto (f+g) + (I_1+I_2)$
well defined: Replace f by $f+f_1$, $f_1 \in I_1$, g by $g+g_1$ $g_1 \in I_2$. Then $f+g$ is replaced by $(f+g) + (f_1+g_1)$ with $f_1+g_1 \in I_1+I_2$
onto: $f + (I_1+I_2)$ gets mapped onto by $(f+I_1, 0+I_2)$
homomorphism: obvious
What is the kernel?
$(f+I_1, g+I_2) \to (f+g) + (I_1+I_2) = 0$
$\Leftrightarrow f+g \in I_1+I_2 \Leftrightarrow f+g = f_1+g_1$, $f_1 \in I_1$, $g_1 \in I_2$ $\Leftrightarrow$
$f - f_1 = -g + g_1$ but f is only defined up to elements of I_1 and g is only defined up to elements of I_2.
Thus we may assume $f = -g$
Thus the homomorphism
$S \to S/I_1 \oplus S/I_2$ defined by
$f \mapsto (f+I_1, -f+I_2)$ surjects onto the kernel. What's the kernel of that map
$(f+I_1, -f+I_2) = 0 \Leftrightarrow f \in I_1$ and $-f \in I_2$
$\Leftrightarrow f \in I_1 \cap I_2$ Thus $S/I_1 \cap I_2$ maps isomorphically onto kernel
All maps have degree 0.
$Z(I_1+I_2) = Y_1 \cap Y_2$ which has dim $< r$

$$\underset{\substack{\uparrow \\ P}}{\varphi_{S/I_1 \oplus S/I_2}} = \underset{\substack{\uparrow \\ P \\ \text{start degree } r}}{\varphi_{S/I}} + \underset{\substack{\uparrow \\ P}}{\varphi_{I_1+I_2}} \leftarrow \text{starts degree } r-1$$

$$\| $$

$$\underset{\substack{\uparrow \\ P}}{\varphi_{S/I_1}} + \underset{\substack{\uparrow \\ P}}{\varphi_{S/I_2}}$$

start in degree r
Look at leading coefficients, get result.
(c) $S(\mathbb{P}^n) = k[x_0, \dots, x_n]$
$P_{\mathbb{P}^n}(z) = \binom{z+n}{n}$
counting monomials
$n!$ $\dfrac{(z+n)(z+n-1)\dots(z+n-n+1)}{n!}$
$= z^n + \dots$
(d) Let f be a generator of degree d
exact sequence
$0 \to \underset{\substack{\uparrow \\ \text{makes map of degree } 0}}{S(-d)} \xrightarrow{\times f} S \to S/(f) \to 0$

$P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$
$\frac{1}{n!}(z+n)(z+n-1)(z+n-2)\dots(z+n-n+1)$
$-\frac{1}{n!}(z-d+n)(z-d+n-1)(z-d+n-2)\dots(z-d+n-n-1)$
The z^n terms cancel
Look at z^{n-1} terms
How do you get a z^{n-1}
take z in all factors except (when you take the constant.
$\frac{1}{n!}z^{n-1}(n+(n-1)+(n-2)+\dots+2+1)$
$-\frac{1}{n!}z^{n-1}((-d+n)+(-d+n-1)+(-d+n-2)\dots(-d+1))$
$= \frac{1}{n!}z^{n-1}\underbrace{(d+d+\dots+d)}_{n \text{ times}}$
$= \frac{nd}{n!}z^{n-1} = \frac{d}{(n-1)!}z^{n-1}$
H has dim $n-1$ degree $(n-1)!\frac{d}{(n-1)!} = d$

Intersection Multiplicities
Let $Y \subsetneq \mathbb{P}^n$ be a projective variety of dimension r. Let H be a hypersurface not containing Y. Then by 7.2 $Y \cap H = Z_1 \cup \dots \cup Z_s$ where Z_j are varieties of dimension $r-1$
Let P_j be the homogeneous prime ideal of Z_j
We define the intersection multiplicity of Y and H along Z_j to be
$i(Y, H; Z_j) = \mu_{P_j}(S/I_Y+I_H)$
where I_Y and I_H are the homogeneous ideals of Y and H.
The module $M = S/I_Y+I_H$ has annihilator I_Y+I_H and
$Z(I_Y+I_H) = Y \cap H = Z_1 \cup \dots \cup Z_s$
so P_j is a minimal prime of M
(minimal prime corresponds to maximal irreducible subvariety = irreducible component
$\mu_P(M) = l_{S_P}(M_P)$
minimal prime of M makes that finite)
Theorem 7.7 Let Y be a variety of dimension ≥ 1 in $\mathbb{P}^n$, and let H be a hypersurface not containing Y. Let $Z_1, \dots, Z_s$ be the irreducible components of $Y \cap H$. Then
$\sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j = (\deg Y)(\deg H)$
"when you intersect a variety with a hypersurface the degree of the intersection is the product of the degrees"
Proof
Say $I(H) = (f)$ and $\deg f = d = \deg H$
exact sequence
$0 \to S/I_Y(-d) \xrightarrow{\times f} S/I_Y \to M \to 0$
$M = S/I_Y+I_H$
$\times f$ injective because f does not vanish on Y, which is irreducible so I_Y prime. $f(g+I_Y) = 0$
$fg + I = 0 \Leftrightarrow fg \in I_Y$
but $f \notin I_Y$ and I_Y prime
$\Rightarrow g \in I_Y$
$P_M(z) = P_Y(z) - P_Y(z-d)$
Our result comes from comparing the leading coefficients of both sides of this equation.
Let Y have dimension r degree e
then $P_Y(z) = \frac{e}{r!}z^r + \dots$
On the right
$\left(\frac{e}{r!}\right)z^r + \dots - \left[\left(\frac{e}{r!}\right)(z-d)^r + \dots\right]$
z^r's cancel
need to look at z^{r-1}
$\frac{e}{r!}z^r + \alpha z^{r-1} - \left[\frac{e}{r!}(z-d)^r + \alpha(z-d)^{r-1} + \dots\right]$

$\alpha z^{r-1} - \alpha z^{r-1} + \frac{der}{r!}z^{r-1}$
$= \frac{de}{(r-1)!}z^{r-1} + \dots$

P_M side
by 7.4 we have a filtration
$0 = M^0 \subsetneq M^1 \subsetneq \dots \subsetneq M^q = M$
whose quotients are M^i/M^{i-1}
are of the form
$(S/q_i)(l_i)$. Hence $P_M = \sum_{i=1}^{q} P_i$ where P_i is the Hilbert polynomial of $(S/q_i)(l_i)$
$0 \to M^1 \to M^2 \to M^2/M^1 \to 0$
$P_{M^2} = P_{M^1} + P_{M^2/M^1}$
$M^1 \cong (S/p_i)(l_i)$ $M^2/M^1 \cong (S/p_j)(l_j)$
If $Z(q_i)$ is a projective variety of dim r_i and deg f_i then
$P_i(z) = \frac{f_i}{r_i!}z^{r_i} + \dots$
shift l_i does not change leading coefficient
we only care about the P_i where $r_i = (r-1)$ Those are the minimal primes that correspond to the irreducible components Y_i's
Each of these occurs $\mu_{P_j}(M)$ times.
$\| $
$l_{S_{P_j}}(M_{P_j})$
when you localize M at P_j all other primes disappear.
The number of times that min appears is the length
coefficient z^{r-1}
$\sum_{j=1}^{s} \frac{i(Y, H, Z_j)\deg Z_j}{(r-1)!}$

$Y \cap H = Z_1 \cup \cdots \cup Z_s$

$\sum_{j=1}^{s} i(Y, H; Z_j) \deg Z_j = (\deg Y)(\deg H)$

Corollary 7.8 (Bezout's theorem)
Let Y, Z be distinct curves in $\mathbb{P}^2$ having degrees d, e. Let $Y \cap Z = \{P_1, \ldots, P_s\}$.
Then $\sum_{j=1}^{s} i(Y, H, P_j) = de$

Proof:
A curve is a hypersurface so in theorem we can let either curve be the hypersurface and the other curve the variety. "We have only to observe that a point has Hilbert polynomial 1, hence degree 1"
By linear change of variables we may assume $P = [1, 0, \ldots, 0]$
Then $I(P) = (x_1, \ldots, x_n)$

$$S(P) = \frac{K[x_0, \ldots, x_n]}{I(P)} = \frac{K[x_0, \ldots, x_n]}{(x_1, \ldots, x_n)} \cong K[x_0]$$

Hilbert poly $= 1$

Remark: Intersection multiplicities are always positive integers
Y variety, H hypersurface $\not\supseteq Y$
$Y \cap H = Z_1 \cup \cdots \cup Z_s$ irred comp.
P_j is the prime ideal corresponding to Z_j

$$i(Y, H; Z_j) = \mu_{P_j}(S/I_Y + I_H) = \ell_{S_{P_j}}((S/I_Y + I_H)_{P_j})$$

The annihilator of $S/I_Y + I_H$ is $I_Y + I_H$
$Z(I_Y + I_H) = Z_1 \cup \cdots \cup Z_s$
Since P_j corresponds to Z_j, P_j is a minimal prime containing $I_Y + I_H$
minimal primes $\Leftrightarrow$ maximal irreducible subsets $\Leftrightarrow$ irreducible components
Thus in the filtration

$$0 = M^0 \subsetneq M^1 \subsetneq \cdots \subsetneq M^r = \frac{S}{I_Y + I_H}$$

at least one of the $M^i/M^{i-1} \cong (S/P_j)(\ell_i)$
length is equal to number of times

Corollary to Thm 7.7
Let Y be a variety in $\mathbb{P}^n$ and let H be a hypersurface not containing Y.
Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then
$\sum_{j=1}^{s} \deg Z_j \leq (\deg Y)(\deg H)$

Corollary to Bézout's Theorem
Let Y, Z be distinct curves in $\mathbb{P}^2$, having degrees d, e.
− $Y \cap Z = \{P_1, \ldots, P_s\}$
Then $s \leq de$

One way to use this.
Suppose you have two curves Y, Z of degrees d, e but you do not know whether they are irreducible
If you can find more than de points of intersection then you know they are reducible with a common component.

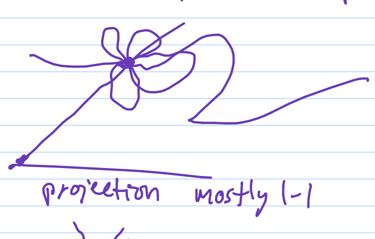
Remark 7.8.1
Our definition of intersection multiplicity in terms of the homogeneous coordinate ring is different from the local definition given earlier (Ex 5.4)
However it is easy to show that they coincide in the case of intersections of plane curves.

Exercise I.5.4
Intersection Multiplicity
If $Y, Z \subseteq \mathbb{A}^2$ are two distinct curves given by equations $f=0$, $g=0$ and if $P \in Y \cap Z$ we define the intersection multiplicity $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_{P,\mathbb{A}^2}$ module $\frac{\mathcal{O}_{P,\mathbb{A}^2}}{(f,g)}$

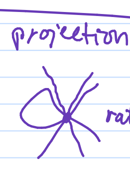
In fact can even go to complete local ring

Exercise 7.1
a) Find the degree of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ (Ex 2.12)
[Answer: d^n]
Hint: Use the information in Ex 2.12 to compute the Hilbert function and Hilbert polynomial of the image of the Veronese embedding
(b) Find the degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$ (Ex 2.14)
[Answer: $\binom{r+s}{r}$]
Similar hint.

Exercise 7.5
(a) Show that an irreducible curve Y of degree $d > 1$ in $\mathbb{P}^2$ cannot have a point of multiplicity $\geq d$. (Ex. 5.3)
curve $X \subseteq \mathbb{A}^2$, $P \in X$.
Assume $P = (0, \ldots, 0)$
ideal of X generated by f. multiplicity of X at P = lowest degree of a monomial in f
If X, Y both contain P intersection multiplicity of X and Y at $P \geq$ product of the multiplicities of X and Y at P.
Use Bézout. If you had points of multiplicity d or greater you could find a line that meets the curve in more than d points counting multiplicity. Find line thru pt of multiplicity d and another pt on the curve.
(b) If Y is an irreducible curve of degree > 1 having a point of multiplicity $d-1$ then Y is a rational curve.
A variety of dim n is rational iff it is birationally isomorphic to $\mathbb{P}^n$
Project from the point of multiplicity $d-1$ to a line not through that point.



projection mostly 1-1



rational, nodal cubic using this idea previously

Chapter II Schemes
Why schemes?
1. A scheme is an abstract variety. A variety that is not embedded in $\mathbb{A}^n$ or $\mathbb{P}^n$, though it is also more general
Having a notion of an abstract vector space, rather than just looking at subspaces of K^n was useful.
2. The section on intersections showed that just looking at sets was not enough.
There we looked at sets with multiplicities.
Schemes are sets with even more added information.
3. The bijection {algebraic subsets of $\mathbb{A}^n$} $\underset{V}{\overset{I}{\rightleftarrows}}$ {radical ideals in $K[x_1, \ldots, x_n]$} is not very satisfying.
You would like to consider all ideals.
Want a bijection
geometric objects — all ideals in $K[x_1, \ldots, x_n]$
schemes do that.
Again looking at sets alone is not enough

1 Sheaves
Definition:
Let X be a topological space. A presheaf $\mathcal{F}$ of Abelian groups on X consists of the data
(a) for any open subset $U \subseteq X$, an Abelian group $\mathcal{F}(U)$, and
(b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of Abelian groups $\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$
subject to the conditions
(a) $\mathcal{F}(\emptyset) = 0$ where $\emptyset$ is the empty set
(b) ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$ and
(c) if $W \subseteq V \subseteq U$ are three open sets then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Remark: A sheaf or presheaf is supposed to relate local and global data. The group $\mathcal{F}(U)$ should contain some information about U. The homomorphism $\rho_{U,V}: \mathcal{F}(U) \to \mathcal{F}(V)$ relates information about U to information about V.
Since $V \subseteq U$ you can think of information about U as global (especially if $U = X$) and information about V as local.

Examples

Let G be a fixed Abelian group and X a topological space. For any open $U \subseteq X$ set $\mathcal{F}(U) =$ all functions $f: U \to G$ with the convention that $\mathcal{F}(\emptyset) = 0$

For $V \subseteq U$ opens define $\rho_{UV}(f) = f|_V$ restriction of functions

$\mathcal{F}(U)$ is a group under pointwise addition $(f+g)(x) = f(x) + g(x)$

restriction of functions is a group homomorphism

$W \subseteq V \subseteq U$

First restricting to V then W is same as restricting directly to W

There are several special cases of this

(i) $G = \mathbb{R}$ and set $\mathcal{F}(U) =$ all cts functions $f: U \to \mathbb{R}$

o.k. because restriction of cts is cts, and cts + cts = cts

(ii) If X is a differentiable manifold

$\mathcal{F}(U) =$ all differentiable functions $f: U \to \mathbb{R}$

(iii) X complex manifold

$\mathcal{F}(U) =$ all holomorphic functions

(iv) suppose X is a variety over k

$\mathcal{F}(U) =$ all regular functions

restriction of regular is regular

& regular + regular is regular

This presheaf is denoted

$\mathcal{O}$ or $\mathcal{O}_X$ and is called the structure sheaf on X or the sheaf of regular functions on X

Remarks: (i) – (iv) $\mathcal{F}(U)$ is actually a ring and the restriction maps are ring homomorphisms

We say these are presheaves of rings.

You can have presheaves of lots of things, groups, rings, fields, modules, v.s. even non algebraic things

Perhaps the best way to define this is with category theory

Given a topological space X we can make a category $\mathcal{T}op(X)$ as follows.

Objects are open subsets of X

The morphisms are inclusions

For open sets U, V if $V \subseteq U$ $\mathrm{Hom}(V, U)$ has one element the inclusion $V \subseteq U$, if $V \not\subseteq U$ $\mathrm{Hom}(V, U) = \emptyset$

For any category $\mathcal{C}$ we say $\mathcal{C}$ has a terminal object T iff $\exists$ an object T in $\mathcal{C}$ s.t. for any object S in $\mathcal{C}$ $\mathrm{Hom}(S, T)$ consists of a single element

In other words, every object has a unique morphism to T

For example at the category of Abelian groups the zero group is the terminal object

Now let X be a topological space and $\mathcal{C}$ a category with a terminal object T. A presheaf of $\mathcal{C}$'s on X is an arrow reversing functor $F: \mathcal{T}op(X) \to \mathcal{C}$ s.t. $F(\emptyset) = T$

Some terminology/notation

$\mathcal{F}$ a presheaf on X

$\mathcal{F}(U)$ is called the sections of $\mathcal{F}$ over U. Sometimes denoted $\Gamma(U, \mathcal{F})$

The maps ρ_{UV} are called restriction homomorphisms.

sometimes Write $s|_V$ instead of $\rho_{UV}(s)$ if $s \in \mathcal{F}(U)$

The maps ρ_{UV} relate local & global info.

A sheaf is a presheaf that satisfies additional conditions that tie local & global together more tightly

Definition

A presheaf $\mathcal{F}$ on a topological space X is a sheaf iff it satisfies the following supplementary conditions:

(3) if U is an open set, if $\{V_i\}$ is an open covering of U and if $s \in \mathcal{F}(U)$ is an element s.t. $s|_{V_i} = 0$ for all i, then $s = 0$.

(4) If U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each i, j $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ then there is an element $s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$ for each i.

Informally (3) A section that is locally everywhere 0 is 0

(4) sections on little opens that look like they should patch together to give a section on the big open do.

Condition (3) implies the s in (4) is unique.

s, t both satisfy $s|_{V_i} = s_i$ all i, $t|_{V_i} = s_i$ all i

$(s-t)|_{V_i} = 0$ all i

so by (3) $s - t = 0$ or $s = t$

Examples

basic $\mathcal{F}(U) =$ all functions $f: U \to G$

(3) functions that are locally everywhere 0 are 0.

(4) functions that look like they should patch together do

(i) – (iv) (3) is just set theory

(4) You use that you can check whether a function is cts, diff, holomorphic, reg locally

Nonexamples

presheaves that are not sheaves

X topological space, G fixed group

(a) $\mathcal{F}$

for $U \neq \emptyset$, $\mathcal{F}(U) = G$ $\mathcal{F}(\emptyset) = 0$

For any $V \subseteq U$ $V \neq \emptyset$

$\rho_{UV}: G \to G$ is identity

$\rho_{U\emptyset} = 0$

This is called the constant presheaf

(b) $\mathcal{G}$ For $U \neq \emptyset$, $\mathcal{G}(U) = G$, $\mathcal{G}(\emptyset) = \emptyset$

all restriction maps except ρ_{UU} are the 0 map

$\rho_{UU} = id$

This is so stupid it does not have a name

If X has any open set U with an open covering of U by $\{U_i\}$ with no $U_i = U$ and G has at least 2 elements then $\mathcal{G}$ violates both (3) & (4)

(3) Pick some $s \neq 0$ $s \in \mathcal{G}(U) = G$

$\rho_{UU_i}(s) = 0$ all i $s \neq 0$ but restricts to 0 everywhere

(4) $g \in G$, $g \neq 0$ let $s_i \in \mathcal{G}(U_i)$

$s_i = g$ $s_i|_{U_i \cap U_j} = 0$ $s_j|_{U_i \cap U_j} = 0$

should be $s \in \mathcal{G}(U)$ with $s|_{U_i} = s_i$

but $s|_{U_i} = 0$ and $s_i \neq 0$

$\mathcal{F}$ will satisfy (3)

only thing that can ever restrict to 0 is 0.

if X has two open sets U_1, U_2 with $U_1 \cap U_2 = \emptyset$ $U_1, U_2 \neq \emptyset$ and G has at least 2 elements then $\mathcal{F}$ violates (4)

$\{U_1, U_2\}$ is an open cover of $U_1 \cup U_2$

Pick $g_1, g_2 \in G$ $g_1 \neq g_2$ $s_1 \in \mathcal{F}(U_1)$ $s_1 = g_1$ $s_2 \in \mathcal{F}(U_2)$ $s_2 = g_2$ $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2} = 0$

Because the restriction from $\mathcal{F}(U_1 \cup U_2) \to \mathcal{F}(U_1)$ or $\mathcal{F}(U_2) = id$ no element of $\mathcal{F}(U_1 \cup U_2)$ would restrict to s_1 on U_1 and s_2 on U_2

There is a modified version of the constant presheaf that is a sheaf

Example 1.0.3

Let X be a topological space. We define the constant sheaf $\mathcal{A}$ on X determined by A (an Abelian group) as follows. Give A the discrete topology, for any open set $U \subseteq X$ let $\mathcal{A}(U) =$ all cts maps $f: U \to A$. The restriction maps are just restrictions of functions. This is a sheaf

If U is connected $\mathcal{A}(U) = A$

Definition

If $\mathcal{F}$ is a presheaf on X and if P is a point of X we define the stalk $\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all opens U containing P via the restriction maps ρ.

If $\mathcal{F}$ is actually a sheaf we also define stalk that way.

Thus an element of $\mathcal{F}_P$ is represented by a pair $\langle U, s\rangle$ U open containing P, $s \in \mathcal{F}(U)$

$\langle U, s\rangle \sim \langle V, t\rangle$ iff there is an open $W \subseteq V \cap U$, $P \in W$ with $s|_W = t|_W$

$\mathcal{F}_P$ germs of sections at P

X variety $p \in X$ $(\mathcal{O}_X)_p \approx \mathcal{O}_{X,p}$

Def: If $\mathcal{F}, \mathcal{G}$ are presheaves on X, a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ consists of a morphism of Abelian groups $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U, whenever $V \subseteq U$ is an inclusion the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array} \quad (*)$$

is commutative.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, we use the same definition for a morphism of sheaves.

An isomorphism is a morphism with a two-sided inverse.

This just amounts to saying all the $\varphi(U)$'s are isomorphisms. Suppose $(*)$ commutes and $\varphi(U)$ and $\varphi(V)$ are isomorphisms. Does this commute?

$$\begin{array}{ccc} \mathcal{F}(U) & \xleftarrow{\varphi(U)^{-1}} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xleftarrow{\varphi(V)^{-1}} & \mathcal{G}(V) \end{array}$$

$g \in \mathcal{G}(U)$

$\rho_{UV}\varphi(U)^{-1}(g) = \varphi(V)^{-1}\rho'_{UV}(g)$

Note that a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $\varphi_P: \mathcal{F}_P \to \mathcal{G}_P$ on the stalks for any $P \in X$.

$\varphi_P(\langle U, s\rangle) = \langle U, \varphi_U(s)\rangle$

$s \in \mathcal{F}(U)$

The following proposition (which would be false for presheaves) illustrates the local nature of sheaves.

Prop 1.1 Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_P: \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for every $P \in X$.

Two examples

(1) Show Hartshorne's statement that this proposition is false for presheaves is true

(2) The following similar sounding proposition is false for sheaves (false proposition)

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Then $\mathcal{F} \cong \mathcal{G}$ iff $\mathcal{F}_P \cong \mathcal{G}_P$ for all P.

(1) $X = \mathbb{P}^1$

Let $\mathcal{F}$ be the constant presheaf associated to a group G with at least two elements.

Clearly $\mathcal{F}_P \cong G$ for all P.

Now let $\mathcal{G}$ be the same as $\mathcal{F}$ except take $\mathcal{G}(X) = 0$, all other $\mathcal{G}(U) = \mathcal{F}(U)$.

$\varphi_{\mathbb{P}^1} = 0$, all other φ's the same.

The stalks are still G.

$\varphi: \mathcal{F} \to \mathcal{G}$

$\varphi(U) = \text{id}$ except when $U = \mathbb{P}^1$

$\varphi_{\mathbb{P}^1}(0) \to G$

induces the isomorphism on stalks.

$\mathcal{F} \ncong \mathcal{G}$ $\mathcal{F}(\mathbb{P}^1) = G$ $\mathcal{G}(\mathbb{P}^1) = 0$

(2) $X = \mathbb{P}^1$ $\mathcal{O}$ structure sheaf

fix one point say $P = [0,1] \in X$

Define the sheaf $\mathfrak{a}(P)$

$\mathfrak{a}(P)(U) = \{ f \in \mathcal{O} : f(P) = 0 \}$

use restriction maps from $\mathcal{O}$

$\mathcal{O}_P = \mathcal{O}_{P,X}$ local rings

$\mathfrak{a}(P)_Q$ $P \neq Q$

$= \mathcal{O}_{Q,X}$

$\mathfrak{a}(P)_P = \mathfrak{m}_{P,X} \subset \mathcal{O}_{P,X}$

$\mathcal{O}_{P,\mathbb{P}^1} \cong k[x]_{(x)}$, $\supseteq \mathfrak{m}_{P,X}(x)_{(x)}$

$\quad \|$ t coordinate $k[t]_{(t)}$ $\quad \|$ (t)

As a module (group)

$(t) \subset k[t]_{(t)} \cong k[t]_{(t)}$

$\mathcal{O}(\mathbb{P}^1) = k$

$\mathfrak{a}(P)(\mathbb{P}^1) = 0$

Def

Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves we define the presheaf kernel of φ, presheaf cokernel of φ and presheaf image of φ to be the presheaves given by

$U \mapsto \ker(\varphi(U))$

use $\mathcal{F}$ restriction maps

need to show $V \subseteq U$

$\rho_{UV}(\ker \varphi(U)) \subseteq \ker(\varphi(V))$

$U \mapsto \operatorname{coker}(\varphi(U))$

use $\mathcal{G}$ restriction maps

$U \mapsto \operatorname{Im}(\varphi(U))$

use $\mathcal{G}$ restriction maps

If $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the presheaf kernel of φ is a sheaf.

But presheaf cokernel and presheaf image might not be sheaves. This leads the notion of sheaf associated to a presheaf

Example where presheaf image is not a sheaf

$X = \mathbb{P}^1$

$Y \subsetneq \mathbb{P}^1$ $Y = \{[0,1], [1,0]\}$

(actually any two points will do)

define the sheaf $\mathcal{F}$ by

$\mathcal{F}(U)$ = regular functions on $U \cap Y$

$\mathcal{F}$ sheaf on $\mathbb{P}^1$

restriction maps are just restriction of functions.

$\mathcal{F}(U) = 0$ if $U \cap Y = \emptyset$

$\mathcal{F}(U) = k$ if $U \cap Y$ is one point

$\mathcal{F}(U) = k \oplus k$ if $U \cap Y$ is 2 points

$\varphi: \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F}$

by restriction of functions

$\mathcal{O}_{\mathbb{P}}(U) \to \mathcal{F}(U)$

Call $\mathcal{G}$ the presheaf image of φ.

$\mathcal{G}(\mathbb{P}^1) = k$

$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{F}(\mathbb{P}^1)$ regular functions on Y

$\quad \|$ $\quad\quad\quad k \oplus k$

$\quad k$

$a \longmapsto (a, a)$

consider the following open cover of $\mathbb{P}^1$

$\mathbb{P}^1 \setminus [0,1]$, $\mathbb{P}^1 \setminus [1,0]$

$\mathcal{G}(\mathbb{P}^1 \setminus [0,1]) = k$

$\mathcal{G}(\mathbb{P}^1 \setminus [1,0]) = k$

$\mathcal{G}((\mathbb{P}^1 \setminus [0,1]) \cap (\mathbb{P}^1 \setminus [1,0])) = 0$

Pick 2 unequal elements of k, t & s

$t \in \mathcal{G}(\mathbb{P}^1 \setminus [0,1])$

$s \in \mathcal{G}(\mathbb{P}^1 \setminus [1,0])$

t, s both restrict to 0 on intersection.

If $\mathcal{G}$ was a sheaf we could find $y \in \mathcal{G}(\mathbb{P}^1)$ that restricts to t on $\mathbb{P}^1 \setminus [0,1]$ and s on $\mathbb{P}^1 \setminus [1,0]$

$\mathcal{G}(\mathbb{P}^1) = k$ via diagonal

Anything in $\mathcal{G}(\mathbb{P}^1)$ must restrict to the same element of k in $\mathcal{G}(\mathbb{P}^1 \setminus [1,0])$ and $\mathcal{G}(\mathbb{P}^1 \setminus [0,1])$

violates property 4 of sheaf.

Proposition - Definition 1.2

Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta: \mathcal{F} \to \mathcal{F}^+$ with the property that for any sheaf $\mathcal{G}$ and morphism $\varphi: \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi: \mathcal{F}^+ \to \mathcal{G}$ s.t. $\varphi = \psi \circ \theta$. Furthermore the pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism

$\mathcal{F}^+$ is called the sheaf associated to the presheaf $\mathcal{F}$. (It is also called the sheafification of $\mathcal{F}$.)

Property 3 wants to reduce the number of sections

Property 4 wants to increase the number of sections.

Both happen.

proof

we construct the sheaf $\mathcal{F}^+$ as follows.

For any open set U, let $\mathcal{F}^+(U)$ be the set of functions s from U to the union $\bigcup_{P \in U} \mathcal{F}_P$ of the stalks of $\mathcal{F}$ over points of U s.t.

(1) for each $P \in U$

$s(P) \in \mathcal{F}_P$ and

(2) for each $P \in U$ there is a neighborhood V of P contained in U and an element $t \in \mathcal{F}(U)$ s.t. for all $Q \in V$ the germ t_Q of t at Q is equal to $s(Q)$

{for any $Q \in V$, $\langle V, t\rangle \in \mathcal{F}_Q$}

{Given $s \in \mathcal{F}(U)$ $\langle U, s\rangle \in \mathcal{F}_P$ all $P \in U$}

Now one can verify immediately that $\mathcal{F}^+$ with the natural restriction maps (restriction of functions) is a sheaf.

Definition

A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ s.t. for every open set $U \subseteq X$ $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$

$$V \subseteq U \quad \mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$$
$$\mathcal{F}'(U) \subseteq \mathcal{F}(U), \quad \mathcal{F}'(V) \subseteq \mathcal{F}(V)$$

$$\rho_{UV}(\mathcal{F}'(U)) \subseteq \mathcal{F}'(V)$$

It follows that for any point P, the stalk $\mathcal{F}'_P$ is a subgroup of $\mathcal{F}_P$.

If $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the kernel of φ, denoted $\ker\varphi$ to be the presheaf kernel of φ (which is a sheaf). Thus $\ker\varphi$ is a subsheaf of $\mathcal{F}$.

We say that a morphism of sheaves $\varphi: \mathcal{F} \to \mathcal{G}$ is injective iff $\ker\varphi = 0$.

Thus φ is injective iff the induced map $\varphi(U): \mathcal{F}(U) \to \mathcal{G}$ is injective for every open set of X.

If $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the image of φ denoted $\operatorname{im}\varphi$, to be the sheaf associated to the presheaf image of φ. By the universal property of the sheaf associated to a presheaf, there is a natural map $\operatorname{im}\varphi \to \mathcal{G}$. In fact this map is injective (see Exercise 1.4) and thus $\operatorname{im}\varphi$ can be identified with a subsheaf of $\mathcal{G}$. We say a morphism of sheaves $\varphi: \mathcal{F} \to \mathcal{G}$ is surjective iff $\operatorname{im}\varphi = \mathcal{G}$.

We say a sequence

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \to \cdots$$

of sheaves and morphisms is exact iff at each stage, $\ker\varphi^{i} = \operatorname{im}\varphi^{i-1}$

Thus $0 \to \mathcal{F} \to \mathcal{G}$ is exact iff φ is injective $\mathcal{F} \to \mathcal{G} \to 0$ is exact iff φ is surjective.

Now let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. We define the quotient sheaf $\mathcal{F}/\mathcal{F}'$ to be the sheaf associated to the presheaf $U \to \mathcal{F}(U)/\mathcal{F}'(U)$.

Restriction maps come from $\mathcal{F}$.

It follows that for any point P, the stalk $(\mathcal{F}/\mathcal{F}')_P$ is the quotient $\mathcal{F}_P/\mathcal{F}'_P$.

If $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the cokernel of φ, denoted $\operatorname{coker}\varphi$ to be the sheaf associated to the presheaf cokernel of φ.

Caution 1.2.1

We saw that a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ of sheaves is injective iff the map of sections $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each U. The corresponding statement for surjective morphisms is not true. If $\varphi: \mathcal{F} \to \mathcal{G}$ is surjective, the maps of $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective.

Example:

$X = \mathbb{P}^1 \quad Y \subseteq X \quad Y = \{[0,1], [1,0]\}$

$\mathcal{O}_{\mathbb{P}^1}$ = sheaf of regular functions on $\mathbb{P}^1$.

$\mathcal{F}$ on $\mathbb{P}^1$ $\mathcal{F}(U)$ = regular functions on $U \cap Y$

$\mathcal{F}(U) = 0$ if $U \cap Y = \emptyset$

$\mathcal{F}(U) = k$ if $U \cap Y$ is one point

$\mathcal{F}(U) = k \oplus k$ if $U \cap Y$ is two points

restriction maps are just restriction of functions.

$\varphi: \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F}$

by restriction of functions

Call $\mathcal{G}$ the presheaf image of φ.

$\mathcal{G}(\mathbb{P}^1) = k$ because $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$

$\varphi(\mathbb{P}^1): \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{F}(\mathbb{P}^1)$ not surjective
$\mathcal{F}(\mathbb{P}^1) = k \oplus k$

$k \to k$

For any other U $\varphi(U)$ will be surjective.

As soon as you delete one point from $\mathbb{P}^1$ you get an $\mathbb{A}^1$.

$\mathcal{O}_{\mathbb{P}^1}(U)$ is at least all polynomials $U \neq \mathbb{P}^1$

You can always find a polynomial taking prescribed values at any finite set of points.

The stalks of $\mathcal{G}$ at either [0,1] or [1,0] are k. Because for small enough open sets containing only one of the points $\mathcal{O}_{\mathbb{P}^1}(U) = k$.

Sections of $\mathcal{G}^+(\mathbb{P}^1)$ only need to locally come from $\mathcal{G}$. By taking the two open sets $\mathbb{P}^1 \setminus [1,0]$, $\mathbb{P}^1 \setminus [0,1]$ you get that $\mathcal{G}^+(\mathbb{P}^1) = k \oplus k$

Thus $\mathcal{G}^+ = \mathcal{F}$

φ is surjective even though $\varphi(\mathbb{P}^1)$ not.

Back to Caution 1.2.1

However, we can say that φ is surjective iff the maps $\varphi_P: \mathcal{F}_P \to \mathcal{G}_P$ are surjective for each P. More generally, a sequence of sheaves is exact iff it is exact on stalks (Ex 1.2)

Definition

Let $f: X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.

For $V \subseteq U$ open in Y then $f^{-1}(V) \subseteq f^{-1}(U)$ so you can use the restriction maps from $\mathcal{F}$. Not too hard to check this gives a sheaf on Y.

For any sheaf $\mathcal{G}$ on Y, we define the inverse image sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf

$$U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V)$$

$U' \subseteq U$ any $V \supseteq f(U)$ contains $f(U')$

$f^{-1}\mathcal{G}$ is different than $f^*\mathcal{G}$.

$f^*\mathcal{G}$ will be defined later for certain sheaves on certain spaces.

f_* functor from sheaves on X to sheaves on Y.

f^{-1} functor from sheaves on Y to sheaves on X.

Definition

If Z is a subset of X regarded as a topological space with the induced topology. Let $i: Z \to X$ be the inclusion map. For a sheaf $\mathcal{F}$ on X, $i^{-1}\mathcal{F}$ is called the restriction of $\mathcal{F}$ to Z, $\mathcal{F}|_Z$

$p \in Z \quad (\mathcal{F}|_Z)_p = \mathcal{F}_p$

Exercise 1.1 Show that the sheaf associated to the constant presheaf is the constant sheaf.

Sheafify the constant presheaf

Exercise 1.8 (very important) For any open set $U \subseteq X$ show that the functor $\Gamma(U, \cdot)$ from sheaves on X to Abelian groups is a left exact functor.

recall: $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$

i.e. if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is an exact sequence of sheaves then $0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$ is an exact sequence of groups.

note: the presheaf kernel is a sheaf

The functor $\Gamma(U, \cdot)$ need not be exact (see exercise 1.21)

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact $\not\Rightarrow$

$0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'') \to 0$ exact

(circled: $\to 0$) might not work here specifically

presheaf image not a sheaf example and surjectivity counterexample useful for 1.21

This is the start of cohomology.

2. Schemes

Definition

Let A be any commutative ring with identity. We define Spec A to be the set of all prime ideals of A. For any ideal I of A we define V(I) to be the set of all prime ideals that contain I.

Remember: A is not a prime ideal, but it is an ideal.

Remark: We could have defined for any subset $S \subseteq A$, V(S) = the set of all prime ideals containing S. It is clear that if $\langle S \rangle$ is the ideal generated by S then $V(S) = V(\langle S \rangle)$. That is why they only define V for ideals.

Lemma 2.1

(a) If I and J are two ideals of A then $V(IJ) = V(I) \cup V(J)$

(b) If $\{I_i\}$ is any set of ideals of A, then $V(\sum I_i) = \bigcap V(I_i)$

(c) If I and J are two ideals of A, $V(I) \subseteq V(J)$ iff $\sqrt{I} \supseteq \sqrt{J}$

$V(A) = \emptyset$

$V(0) = \operatorname{Spec} A$

+(a) +(b) ⇒ V(I)'s can be taken as closed sets of a topology on Spec A called the Zarisky topology

Lemma 2.1 (a) If I and J are two ideals of A, then $V(IJ) = V(I) \cup V(J)$

(b) If $\{I_i\}$ is a set of ideals of A then $V(\sum I_i) = \cap V(I_i)$

(c) $V(I) \subseteq V(J)$ iff $\sqrt{I} \supseteq \sqrt{J}$

proof (a) $p \supseteq IJ \Rightarrow p \supseteq I$ or $p \supseteq J$

$p \not\supseteq I$ but $p \not\supseteq J$ pick $a \in I$, $a \notin p$, $b \in J$, $b \notin p$

$ab \notin p$ for any ... $ab \in IJ$

$p \not\supseteq IJ$

(b) p contains $\sum I_i$ iff p contains each I_i because $\sum I_i$ is the smallest ideal containing all the I_i

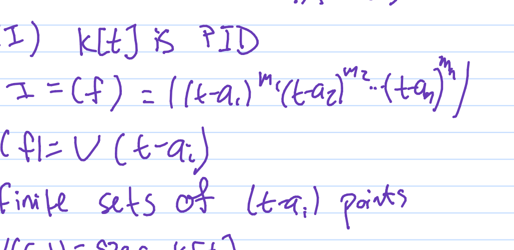
(c) $\sqrt{I} = \cap$ all primes containing I.

Ex 1) k field Spec k = one point

$k[x]$ k algebraically closed field

primes $(x-a)$ $a \in k$ (0)

looks like



$V(I)$ $k[t]$ is PID

$I = (f) = ((t-a_1)^{m_1} \cdots (t-a_n)^{m_n})$

$V(I) = \cup V(t-a_i)$

finite sets of $(t-a_i)$ points

$V((0)) = \text{Spec } k[t]$

$\{(0)\}$ its closure is all of Spec $k[t]$

(0) is called the generic point

(2) Spec $\mathbb{Z}$

primes (p) p prime

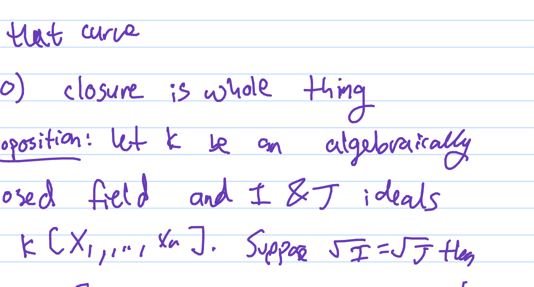
(0)

(4) Spec $k[x,y]$ k alg closed

max ideals $(x-a, y-b) \to \mathbb{A}^2$

non max primes are curves

(0)



for each irreducible curve you have an extra pt whose closure is that curve

(0) closure is whole thing

proposition: Let k be an algebraically closed field and $I \subseteq J$ ideals in $k[X_1, \ldots, X_n]$. Suppose $\sqrt{I} = \sqrt{J}$ then Spec $k[X_1,\ldots,X_n]/I$, Spec $k[X_1,\ldots,X_n]/J$ and Spec $k[X_1,\ldots,X_n]/\sqrt{I}$ are all homeomorphic.

pf

primes in $k[X_1,\ldots,X_n]/I$ correspond to primes $I \subseteq p$ in $k[X_1,\ldots,X_n]$ correspond to primes in $k[X_1,\ldots,X_n]/\sqrt{I}$

Obvious bijection between Spec $k[X_1,\ldots,X_n]/I$ and Spec $k[X_1,\ldots,X_n]/\sqrt{I}$

Let K be an ideal of $k[X_1,\ldots,X_n]/I$ corresponding to an ideal $K' \supseteq I$ of $k[X_1,\ldots,X_n]$

$V(K)$ = primes containing K those correspond to primes containing K' which are the same as primes containing $\sqrt{I}\,K'$

so $\sqrt{K'}$ (corresponds) to an ideal L in $k[X_1,\ldots,X_n]/\sqrt{I}$

Thus in the bijection between Spec $k[X_1,\ldots,X_n]/I$ and Spec $k[X_1,\ldots,X_n]/\sqrt{I}$

$V(K)$ goes to $V(L)$.

closed in Spec $k[X_1,\ldots,X_n]/I$ goes to closed in Spec $k[X_1,\ldots,X_n]/\sqrt{I}$

reverse similar and easier.

Spec does not distinguish between ideals with the same radical

You need to put a sheaf on it.

Def

Next we will define a sheaf of rings $\mathcal{O}$ on Spec A.

For each prime ideal $p \subseteq A$ let A_p be the localization of A at p. For an open set $U \subseteq \text{Spec} A$ we define $\mathcal{O}(U)$ to be the set of all functions

$s: U \to \coprod_{p \in U} A_p$ {disjoint union}

s.t. $s(p) \in A_p$ for each p and s.t. s is locally a quotient of elements of A: to be precise we require that for each $p \in U$ there is a neighborhood V of p contained in U and elements $a, f \in A$ s.t. for each $q \in V$

$f \notin q$ and $s(q) = \frac{a}{f}$ in A_q

easy to see it is a ring.

restriction maps are just restriction of functions

presheaf obvious

(3) easy because restriction of functions.

(4) because of local nature of def.

Let $A = A(X)$ the coordinate ring of an affine variety

Spec $A \supseteq$ max Spec A

only look at maximal ideals

max Spec A with induced top is homeomorphic to X,

If m is maximal, $(A_m)_m = k$

$s: U \to \coprod_{m \in U} A_m \to k$

That is an old fashioned regular function

Def

Let A be a ring. The spectrum of A is the pair consisting of the topological space Spec A together with the sheaf of rings $\mathcal{O}$ defined above.

For $f \in A$ denote by $D(f) = \text{Spec} A \setminus V(f)$ = primes that do not contain f, open

These $D(f)$ are a base for the topology Spec A.

$V(I)$ closed $p \notin V(I)$

$p \not\supseteq I$ so there is $f \in I$ $f \notin p$

$p \in D(f)$ $D(f) \cap V(I) = \emptyset$

prop 2.2

Let A be a ring and $(\text{Spec} A, \mathcal{O})$ its spectrum.

(a) For any $p \in \text{Spec} A$, the stalk $\mathcal{O}_p$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring A_p.

(b) For any element $f \in A$ the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f.

(c) In particular

$\Gamma(\text{Spec} A, \mathcal{O}) \cong A$

$\cong \mathcal{O}(\text{Spec} A)$

Remarks

(a) should remind you of the fact that in the old fashioned variety world you could compute local rings by localization

(b) Another way to picture $\mathcal{O}(U)$ at least for the $D(f)$'s.

(c) This shows that the pair $(\text{Spec} A, \mathcal{O})$ can now distinguish between $k[X_1,\ldots,X_n]/I$ and $k[X_1,\ldots,X_n]/\sqrt{I}$ when $I \neq \sqrt{I}$

proof

a) Define a homomorphism

$\varphi: \mathcal{O}_p \to A_p$

$\langle U, s \rangle$ $s \in \mathcal{O}(U)$ U neighborhood of p

$\mapsto s(p) \in A_p$

φ is surjective.

Any element of A_p has the form $\frac{a}{f}$ with $a, f \in A$, $f \notin p$

$D(f)$ will be an open neighborhood of p.

$\langle D(f), \frac{a}{f} \rangle$

injective

U neighborhood of p

s.t. $\in \mathcal{O}(U)$ with $s(p) = t(p) \in A_p$

By shrinking U if needed we may assume $s = \frac{a}{f}$ $t = \frac{b}{g}$ on U

where $a, b, f, g \in A$ $f, g \notin p$

$\frac{a}{f} = \frac{b}{g}$ in A_p

there exists $h \notin p$

$h(ga - fb) = 0$ in A

therefore $\frac{a}{f} = \frac{b}{g}$ in every local ring A_q s.t. $f, g, h \notin q$

The set of such q is $D(f) \cap D(g) \cap D(h)$ which contains p.

This is a smaller open set on which $s = t$

so they are equal in $\mathcal{O}_p$.

(b) (c)

(c) is (b) with $f = 1$

$D(1) = \text{Spec} A$

$\psi: A_f \to \mathcal{O}(D(f))$

$\frac{a}{f^n} \to s$ which assigns $s(p) = \frac{a}{f^n} \in A_p$

check that is a homomorphism

injective

$\psi\left(\frac{a}{f^n}\right) = \psi\left(\frac{b}{f^m}\right)$

$\Rightarrow$ For every $p \in D(f)$ $\frac{a}{f^n}$ and $\frac{b}{f^m}$ have same image in A_p

$h \notin p$ $h(f^m a - f^n b) = 0$ in A

Let I be the annihilator of $f^m a - f^n b$

Then $h \in I$ and $h \notin p$ so $I \not\subseteq p$

This holds for every $p \in D(f)$

Office Hours Thu / Mon 2-3

Definition: A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X.

A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a cts map $f: X \to Y$ and $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on Y.

The ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for each point $p \in X$ the stalk $\mathcal{O}_{X,p}$ is a local ring.

A morphism of locally ringed spaces is a morphism of ringed spaces s.t. for each point $p \in X$ the induced map of local rings $f_p^\#: \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{(X,p)}$ is a local homomorphism of local rings.

An element of $\mathcal{O}_{Y,f(p)}$ is an equivalence class of pairs $\langle U, s \rangle$ where U is an open neighborhood of $f(p)$ in Y and $s \in \mathcal{O}_Y(U)$.

We have $f^\#(U): \mathcal{O}_Y(U) \to (f_* \mathcal{O}_X)(U)$
$$= \mathcal{O}_X(f^{-1}(U))$$

Define $f_p^*(\langle U, s \rangle) = \langle f^{-1}(U), f^\#(U)(s) \rangle \in \mathcal{O}_{X,p}$

If A and B are local rings with max'l ideals m_A and m_B respectively, a homomorphism $\varphi: A \to B$ is called a local homomorphism iff $\varphi^{-1}(m_B) = m_A$

An isomorphism of locally ringed spaces is a morphism with a two sided inverse. Thus a morphism $(f, f^\#)$ is an isomorphism iff f is a homeomorphism of the underlying topological spaces and $f^\#$ is an isomorphism of sheaves

Prop 2.3 (a) If A is a ring then $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space

(b) If $\varphi: A \to B$ is a homomorphism of rings then φ induces a natural morphism of locally ringed spaces $(f, f^\#): (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

(c) If A and B are rings, then any morphism of locally ringed spaces from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ is induced by a homomorphism of rings $\varphi: A \to B$ as in (b)

Proof:

(a) ✓

(b) Given a homomorphism $\varphi: A \to B$ we define a map $f: \operatorname{Spec} B \to \operatorname{Spec} A$ by $f(p) = \varphi^{-1}(p)$ for any $p \in \operatorname{Spec} B$. If I is an ideal of A, then it is immediate that $f^{-1}(V(I)) = V(\varphi(I))$ so f is continuous

$q \in f^{-1}(V(I)) \Rightarrow f(q) \in V(I)$
$\Rightarrow \varphi^{-1}(q) \in V(I)$
$\Rightarrow \varphi^{-1}(q) \supset I$
$\Rightarrow \varphi(\varphi^{-1}(q)) \supset \varphi(I)$
(but $\varphi(\varphi^{-1}(q)) = q$
$\Rightarrow q \supset \varphi(I) \Rightarrow q \in V(\varphi(I))$

$q \in V(\varphi(I)) \Rightarrow q \supset \varphi(I) \Rightarrow$
$\varphi^{-1}(q) \supset \varphi^{-1}(\varphi(I))$ but $\varphi^{-1}(\varphi(I)) \supset I$
$\Rightarrow \varphi^{-1}(q) \supset I \Rightarrow f(q) \in V(I)$
$\Rightarrow q \in f^{-1}(V(I))$

For each $p \in \operatorname{Spec} B$ we can localize φ to obtain a local homomorphism of local rings $\varphi_p: A_{\varphi^{-1}(p)} \to B_p$

Something in $A_{\varphi^{-1}(p)}$ is of the form $\frac{a}{b}$ where $b \notin \varphi^{-1}(p)$, $a \in A$. Map this to $\frac{\varphi(a)}{\varphi(b)}$, since $b \notin \varphi^{-1}(p)$, $\varphi(b) \notin p$

Now for any open set $V \subseteq \operatorname{Spec} A$ we obtain a homomorphism of rings $f^\#: \mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$ by the definition of $\mathcal{O}$, composing with the maps f and φ_p

$f^\#: \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$
$f^\#(V): \mathcal{O}_{\operatorname{Spec} A}(V) \to f_* \mathcal{O}_{\operatorname{Spec} B}(V)$
$$= \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$$

$s \in \mathcal{O}_{\operatorname{Spec} A}(V)$
$s: V \to \coprod_{p \in V} A_p$

$t \in \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$
$t: f^{-1}(V) \to \coprod_{q \in f^{-1}(V)} B_q$

Say $t = f^\#(s)$ what would t be.
$f^{-1}(V) = \{q : f(q) \in V\}$
$= \{q : \varphi^{-1}(q) \in V\}$

set $t(q) = \varphi_q(s(\varphi^{-1}(q)))$
$q \in f^{-1}(V) \Rightarrow \varphi^{-1}(q) \in V$
$\Rightarrow s(\varphi^{-1}(q)) \in A_{\varphi^{-1}(q)}$
$\Rightarrow \varphi_q(s(\varphi^{-1}(q))) \in B_q$

This gives the morphism of sheaves $f^\#: \mathcal{O}_{\operatorname{Spec} A} \to f_*(\mathcal{O}_{\operatorname{Spec} B})$

The induced maps on the stalks are just the local homomorphisms φ_p, so $(f, f^\#)$ is a morphism of locally ringed spaces

(c) Conversely suppose given a morphism of locally ringed spaces $(f, f^\#)$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$. Taking global sections $f^\#$ induces a homomorphism of rings $\varphi: \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$

$f^\#(\operatorname{Spec} A)$ $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to f_* \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} A)$ $f_* \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} A)$
$\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$
$\mathcal{O}_{\operatorname{Spec} B}(f^{-1} \operatorname{Spec} A)$
$= \operatorname{Spec} B$

2.2 c. $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$
$\Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) = B$

given $(f, f^\#)$ we get $\varphi: A \to B$

You need to check when you do the construction in (b) to φ you get $(f, f^\#)$ back, and other way

When we have a cts map $f: X \to Y$ and a sheaf $\mathcal{F}$ on Y we defined $f^{-1}(\mathcal{F})$ as the sheaf associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{F}(V)$

When f is the inclusion of an open set Y into Y this becomes much simpler.

$X = W$ $i: W \hookrightarrow Y$

any U open in W is open in Y

$i(U) = U$ open in Y

$\lim_{V \supseteq i(U)} \mathcal{F}(V) = \mathcal{F}(U)$ $U = i(U)$

and it is already a sheaf

$i^{-1}\mathcal{F}$ often written $\mathcal{F}|_{X=W}$

Definition.

an <u>affine scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring

A <u>scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U s.t. the topological space U together with the restricted sheaf $\mathcal{O}_X|_U$ is an affine scheme.

We call X the underlying topological space of the scheme $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$ its structure sheaf. By abuse of notation we often write simply X for the scheme $(X, \mathcal{O}_X)$

If we wish to refer to the underlying topological space without its scheme structure, we write $sp(X)$ read "space of X"

A morphism of schemes is a morphism of locally ringed spaces.

An isomorphism is an isomorphism as locally ringed spaces.

Remark: Remember that the same topological space can have many different structure sheaves on it and thus many different schemes

Ex (1) L, K two non-isomorphic fields

$\operatorname{Spec} L \qquad \operatorname{Spec} K$

$\mathcal{O}(\operatorname{Spec} L) = L \qquad \mathcal{O}(\operatorname{Spec} K) = K$

(2) $\operatorname{Spec} \frac{K[x_1, \dots, x_n]}{I_1 \text{ or } I_2}$

I_1, I_2 have same radical but $I_1 \neq I_2$

Alternate Less Categorical Definition of Scheme

(a) A scheme is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X such that there exists an open cover $\{U_i\}$ of X with the following property. There exist rings A_i and homeomorphisms $f_i : \operatorname{Spec} A_i \to U_i$ s.t. $f_{i*}(\mathcal{O}_{\operatorname{Spec} A_i}) = \mathcal{O}_X|_{U_i}$.

(b) A morphism of schemes from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves that satisfies

There exists an open covering $\{U_i\}$ of Y and for each i an open covering $\{U_{ij}\}$ of $f^{-1}(U_i)$ and well as rings $\{A_i\}$ $\{B_{ij}\}$ with homeomorphisms $f_i : \operatorname{Spec} A_i \to U_i$

$f_{ij} : \operatorname{Spec} B_{ij} \to U_{ij}$ with $f_{i*} \mathcal{O}_{\operatorname{Spec} A_i} \cong \mathcal{O}_Y|_{U_i}$

$f_{ij*} : \mathcal{O}_{\operatorname{Spec} B_{ij}} \cong \mathcal{O}_X|_{U_{ij}}$

Furthermore for each ij we have a composition

$$\begin{tikzcd}
\operatorname{Spec} B_{ij} \arrow[r, "f_{ij}"] \arrow[rrr, bend right] & U_{ij} \arrow[r, "f|_{U_{ij}}"] & U_i \arrow[r, "f_i^{-1}"] & \operatorname{Spec} A_i
\end{tikzcd}$$

Call this composition g_{ij}

It is continuous.

We construct a morphism of sheaves

$g_{ij}^\# : \mathcal{O}_{\operatorname{Spec} A_i} \to g_{ij*} \mathcal{O}_{\operatorname{Spec} B_{ij}}$

as follows

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A_i} \arrow[rr, "g_{ij}^\#"] \arrow[d, phantom, "\cong"] & & g_{ij*} \mathcal{O}_{\operatorname{Spec} B_{ij}} \arrow[d, phantom, "\cong"] \\
\mathcal{O}_Y|_{U_i} \arrow[dr, "f^\#|_{U_{ij}}"'] & & f_{i*}^{-1} f|_{U_{ij}*} \mathcal{O}_X|_{U_{ij}} \\
& f|_{U_{ij}*} \mathcal{O}_X|_{U_{ij}} \arrow[ur, "f_{i*}^{-1}"'] &
\end{tikzcd}$$

We have made a pair

$(g_{ij}, g_{ij}^\#)\ (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$

$\to (\operatorname{Spec} B_{ij}, \mathcal{O}_{\operatorname{Spec} B_{ij}})$

We require that comes from a ring homomorphism $\varphi_{ij} : A_i \to B_{ij}$

Ex2.1 let A be a ring. let $X = \operatorname{Spec} A$ let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$

Exercise 2.2

Let $(X, \mathcal{O}_X)$ be a sheme and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. Remember that the $D(f)$'s are a base for the topology.

Chapter III Cohomology

1 Derived Functors

2 Cohomology of Sheaves

Exercise II.1.8

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on X then setting $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ we have that

$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$

is exact.

However $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ need not be exact.

$\Gamma(X, \cdot)$ is left exact but not right exact.

Denote $\Gamma(X, \cdot)$ by $H^0(X, \cdot)$

$H^0(X, \mathcal{F}) = \mathcal{F}(X)$

zeroth cohomology group of $\mathcal{F}$

Cohomology deals with the question: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, how do you continue the exact sequence

$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to ?$

Sections 1 and 2 say the following:

$H^0(X, \cdot)$ is a functor from the category of sheaves of Abelian groups on X to the category of Abelian groups. It is left exact. Various category theory and homological algebra facts say that when you have a left exact functor between two categories that satisfy certain conditions you get left derived functors that give a long exact sequence.

Thus if you have a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of sheaves on X you get a long exact sequence of Abelian groups.

$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'')$

$\to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}'')$

$\to H^2(X, \mathcal{F}') \to \dots$

$H^i(X, \mathcal{F})$ is the i^{th} cohomology group of $\mathcal{F}$ on X.

Very Important Fact: $H^i(X, \mathcal{F})$ depends only on i, X, and $\mathcal{F}$, not on which short exact sequence $\mathcal{F}$ appeared in.

4 Čech Cohomology

A method for computing cohomology groups.

Let X be a topological space and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Fix, once and for all, a well ordering of the index set I.

For any finite set of indices $i_0, \dots, i_p \in I$ we denote the intersection $U_{i_0} \cap \dots \cap U_{i_p}$ by $U_{i_0, \dots, i_p}$

Now let $\mathcal{F}$ be a sheaf of Abelian groups on X. We can define a complex of Abelian groups as follows. For each $p \geq 0$ let

$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0, \dots, i_p})$

Thus an element $\alpha \in C^p(\mathfrak{U}, \mathcal{F})$ is determined by giving an element $\alpha_{i_0, \dots, i_p} \in \mathcal{F}(U_{i_0, \dots, i_p})$ for each $p+1$-tuple $i_0 < \dots < i_p$ of elements of I.

We define the coboundary map

$d : C^p \to C^{p+1}$ by setting

$$(d\alpha)_{i_0, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}}|_{U_{i_0, \dots, i_{p+1}}}$$

Here the notation $\hat{i}_k$ means omit i_k.

Check that $d^2 = 0$.

Definition: Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of Abelian groups $\mathcal{F}$ on X, we define the p^{th} Čech cohomology group of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$ to be

$\check{H}^p(\mathfrak{U}, \mathcal{F}) = h^p(C^\cdot(\mathfrak{U}, \mathcal{F})) = \frac{\ker d^i}{\operatorname{im} d^{i-1}}$

Theorem 4.5: Let X be a Noetherian separated scheme, let $\mathfrak{U}$ be an open affine covering of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, the natural maps of (4.4) give isomorphisms.

$\check{H}^p(\mathfrak{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$

Example: Let's compute the cohomology groups of $\mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^1$. Take the affine open U_0, U_1.

$C^0 = \mathcal{O}_{\mathbb{P}^1}(U_0) \times \mathcal{O}_{\mathbb{P}^1}(U_1) = k[\frac{X_1}{X_0}] \times k[\frac{X_0}{X_1}]$

X_0, X_1 homogeneous coordinates

$C^1 = \mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1)$

$= k[\frac{X_1}{X_0}, \frac{X_0}{X_1}]$

$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} 0$

$d^0(a_0 + a_1(\frac{X_1}{X_0}) + a_2(\frac{X_1}{X_0})^2 + \dots,$
$b_0 + b_1(\frac{X_0}{X_1}) + b_2(\frac{X_0}{X_1})^2 + \dots)$

$= a_0 - b_0 + a_1(\frac{X_1}{X_0}) + a_2(\frac{X_1}{X_0})^2 + \dots - b_1(\frac{X_0}{X_1}) - b_2(\frac{X_0}{X_1})^2 - \dots$

$\ker d^0 \quad (a, a) \quad a \in k \quad \cong k$

$\operatorname{im} d^0 = C^1 = k[\frac{X_1}{X_0}, \frac{X_0}{X_1}]$

$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \frac{\ker d^0}{\operatorname{im} d^{-1}} = \frac{k}{0} = k$

$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \frac{\ker d^1}{\operatorname{im} d^0} = \frac{k[\frac{X_0}{X_1}, \frac{X_1}{X_0}]}{k[\frac{X_0}{X_1}, \frac{X_1}{X_0}]} = 0$

$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $i > 1$