Algebraic Topology

01/12/2015

Question: Given spaces X, Y, are they homeomorphic? That is, do there exist

$f: X \to Y$

$f^{-1}: Y \to X$

such that f, f^{-1} are continuous.

Def: An n-manifold is a topological space, 2^{nd} countable, Hausdorff, and locally homeo. to $\mathbb{R}^n$.

Classify Manifolds

Classify the connected, compact manifolds

0-manifolds: Point. All $\cong$

1-manifold: S^1

2-manifolds: $S_{(p)}, T^{(g)}_{(p)}, D^{(n)}_{(p)}$

3-manifolds: Poincaré Conjecture

4-manifolds: Impossible

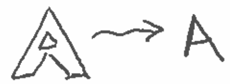
5, 6, ...-manifolds: Completed in the 60's & 70's.

Def: A knot K is the image of an embedding from $S \to \mathbb{R}^3$

Problem: Classify knots up to isotopy.

Our goal is to answer topological question by developing algebraic tools.

Homotopy:

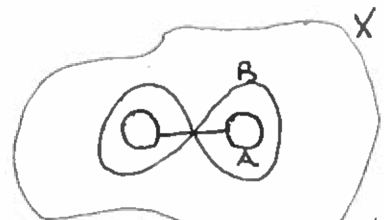


Def: Deformation retraction of a space X onto a subspace A is a continuous family of functions $f_t : X \to X$ such that

(i) $f_0 = 1_X$

(ii) $f_1(X) = A$

(iii) $f_t|_A = 1_A$



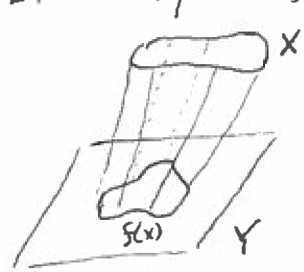
$X \xrightarrow{dr} A$, $X \xrightarrow{dr} B$ But $A \not\xrightarrow{dr} B$, $B \not\xrightarrow{dr} A$

Deformation gives a relation but not an equivalence relation. This is seen above.

Def: $f: X \to Y$. The mapping cylinder is

$$M_f \overset{def}{=} \left(X \times [0,1] \sqcup Y \right) / x \times 1 \sim f(x)$$

for all x.



Ex: $M_f \xrightarrow{dr} Y$

$g_t : M_f \to M_f$

$g_t(x \times s) = x \times s(1-t)$

$g_t(y) = y$

Def: Homotopy is a map $F: X \times [0,1] \to Y$ and a continuous family of maps

$f_t = F(-, t)$

* Homotopy is an equivalence relation.

Def: Let $A \subset X$. A retraction of X onto A is a map $X \xrightarrow{r} A$ such that $r(X) = A$ and $r|_A = 1$. That is, r is onto and $r^2 = r$.

Prop: A deformation retraction is stronger ~~than~~ homotopy ~~th~~ from id_X to retraction of X onto A, but only slightly stronger.

Ex: $\left. \begin{array}{l} X = B \\ A = \bullet \end{array} \right\}$ As before

$r: X \to A$ the constant map. $r(a) = a$

This is a retraction but not a deformation retraction.

Observe $\pi_1(A) = 1$

but $\pi_1(B) \cong S^1 \vee S^1$
$\mathbb{Z} \times \mathbb{Z}$

Relative Homotopy

Let X, Y be spaces with $A \subset X, Y$.

A homotopy $F: X \times I \to Y$ is a homotopy rel A if $F(-,t)|_A = \mathrm{id}_A$ for all t.

We can rephrase this to get an if and only if in the previous proposition.

Def: Let $f: X \to Y$ be a map. We say f is a homotopy equivalence if there is a $g: Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X$$
$$f \circ g \simeq \mathrm{id}_Y$$

There is a class of homotopy; that is, there is a category of these objects.

Then X is homotopic equivalent to Y.

The fact that this is an equivalence relation is obvious.

* You cannot say spaces are homotopic. Maps are homotopic.

Ex: $X \overset{drs}{\Rightarrow} A$ then $X \simeq A$. However, the converse is false.

∞ vs. o—o

These are $\simeq$ but are not deformation retractions.

* $\simeq$ - \simeq

Def: A space X is contractible if X is homotopic to a point. That is, if there is a deformation retraction to the constant map.

Ex: $\mathbb{R}^n$, X (the letter)

Def: A map is nulhomotopic if it is homotopic to a constant map.

CW complex

1. Start with a discrete set X^0, whose points are 0-cells.

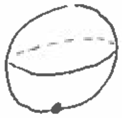
2. Inductively form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha: S^{n-1} \to X^{n-1}$. So X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under ident. $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Hence, $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ each e^n_α is an open n-disk.

3. Stop and set $X = X^n$ for $n < \infty$ or $X = \cup X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

The smallest such n is the dimension of $X = X^n$ - the maximum dimension of cells of X.

01/14/2015

CW complex sphere:

$e^0 \cup e^2$ $\qquad$ $e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^2 \cup e^2$

We can think of S^n as $\mathbb{R}^n \cup \infty$

point @ infinity

$\mathbb{RP}(0)$ is a point. So CW structure is e^0 alone. We build up the CW structure for $\mathbb{RP}(n)$.

$\mathbb{RP}(n) = \mathbb{RP}(n-1) \cup e^n \to D^n$

$\varphi_n : S^{n-1} \to \mathbb{RP}(n-1)$

the quotient map. Use antipodes.

$\mathbb{RP}(0)$ $\bullet$

$\mathbb{RP}(1)$  $\rightsquigarrow$  $\approx S^1 \approx \mathbb{RP}(1)$

$\mathbb{RP}(2)$  e^2

Exercise: Why $S^1 \approx \mathbb{RP}(1)$

$e^0 = \bullet$

Both open & closed, naturally.

Ex:

$\mathbb{RP}(n) :=$ Real projective plane

$:=$ line through 0 in $\mathbb{R}^{n+1}$

$:= \{\vec{x} \in \mathbb{R}^{n+1} \setminus 0 \mid \vec{x} \sim \lambda \vec{x} \text{ for } \lambda \neq 0\}$

$\Downarrow \wr$

$S^n / \vec{x} \sim -\vec{x}$ 

OR

We use this for first $\quad D^n / \vec{x} \sim -\vec{x} \;\; \forall x \in \partial D^n$

So $\mathbb{RP}(n)$ has a CW structure with one n-cell in each dimension:

$e^0 \cup e^1 \cup \dots \cup e^n$

where the attaching maps are quotient maps by antipodes.

$\mathbb{RP}(j) \subset \mathbb{RP}(n)$ if $j \leq n$

In fact, $\mathbb{RP}(j)$ is $(\mathbb{RP}(n))^j$

You can prove a similar statement for S^n cells.

Note $\mathbb{RP}(j)$ is a closed subspace of $(\mathbb{RP}(n))^j$

Def: Let X be a CW complex and $A \subset X$. We say A is a subcomplex of X if A is closed in X and A is a CW complex. We say (X, A) is a CW pair.

Ex: X is a CW complex
Then (X, X^n) is a CW pair.

(S^n, S^j) ; $j \le n$

$(\mathbb{RP}(n), \mathbb{RP}(j))$; $j \le n$

Def: Given two spaces, X, Y with chosen points $x \in X$ and $y \in Y$, the wedge sum (or wedge product)

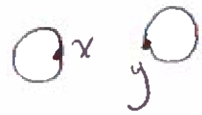
$$X \vee Y$$

is $X \sqcup Y / x \sim y$

$S^2 \vee T^2$

Proposition: If X is a CW complex & Y is a CW complex with $x \in X^0$, $y \in Y^0$, then $X \vee Y$ is a CW complex. Furthermore, X, Y are subcomplexes of $X \vee Y$.

Ex:



$X \vee Y$

$e^0 \cup e^1 \cup e^1$

$S^0 \to e^0$ $\quad$ $S^0 \to e^0$

Homotopy Equivalence Property

Say X is a space and $A \subset X$. Suppose $f_0 : X \to Y$ is a map and $f_t : A \to Y$ is a homotopy with $f_0|_A = f_t|_A\big|_{t=0}$

Can we extend f_t to X?

Def: The pair (of spaces) (X, A) has the homotopy ext. prop. if f_t can be extended to X for all Y, f_0, f_t.

That is, (X, A) has the homotopy ext. property if and only if for all $X \times 0 \cup A \times I \xrightarrow{g} Y$ $\exists \tilde{g} \ni$ the following diagram commutes

$$\begin{tikzcd} X \times 0 \cup A \times I \arrow[r, "g"] \arrow[d, hook] & Y \\ X \times I \arrow[ur, dashed, "\exists \tilde{g}"'] & \end{tikzcd}$$

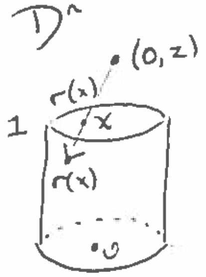
Prop: If (X,A) is a CW pair, then $X\times 0 \cup A\times I$ is a deformation retraction of $X\times I$. Hence, (X,A) has the homotopy ext. property.

Note if there is a deformation retraction then there is a retraction

$$r: X\times I \to X\times 0 \cup A\times I$$

Pf (sketch-cells): WTS $D^n\times I$ retracts to $D^n\times 0 \cup \partial D^n\times I$

$z>1$



$r: D^n\times I \to D^n\times 0 \cup \partial D^n\times I$

Observe that r is continuous. Check that r is a retract. Homotopy is straight line homotopy.

Criterion for Homotopy Equivalence

Collapsing: If (X,A) is a CW pair and A is contractible, then the quotient map

$$q: X \to X/A$$

is a homotopy equivalence.

Pf: There is a contraction $g_t: A\to A$ with $g_0 = 1_A$ and g_1 a constant map. Extend g_0 to map $g_0: X\to X$ by $g_0 = 1_X$. Use homotopy ext. property on (X,A).

So g_t extends to all of X.

$F: X\times I \to X$.

$F(-,0) = 1_X$

$F(-,1)|_A = \text{constant}$.

WTS quotient map is homo. equiv. F takes A to A: As F extends g_t.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "q\circ F(-,t)"] & \\ X/A \arrow[r, dashed] & X/A \end{tikzcd}$$

Univ. property quotient map, $\exists!\ F_t \ni$ the diagram commutes. So we have family maps.

$$f_t\circ q = q\circ F(-,t)$$

Take $t=1$ Commutes

$$\begin{tikzcd} X \arrow[r, "F(-,1)"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, dashed, "f"] \arrow[r, "f_1"'] & X/A \end{tikzcd}$$

$g_1(A) = \text{point}$. Preimage A under map is point. $\exists!\ f \ni f\circ q = F(-,1)$.

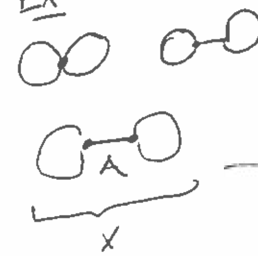
~~[illegible]~~ Choose $x \ni q(x) = \bar{x}$

$(q f)(\bar{x}) = (q f q)(x) = q F(x,1) = f_1(q(x)) = f_1(\bar{x})$.

f is homotopy inverse of q.

~~[illegible]~~ Check $fq \simeq 1_X,\ qf \simeq 1_{X/A}$ simple.

Ex:



A CW complex. A contractible.

q

$X/A \simeq S^1\vee S^1$

We had this before using deformation retraction.

Ex:

Think of $e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^2$

$A\subset X$; A, B contractible.

$B\subset X$

$X/B \simeq$

$\simeq S^2\vee S^1$

$X/A \simeq$

$\simeq \to$ Homotopy alone

$S^2\vee S^1$

Ex:

$X/A \simeq$

Attaching Criterion

Given X and $A \subset Y$, $f : A \to X$

Define,

$$X \sqcup_f Y = X \sqcup Y /_{f(a) \sim a}$$

for all $a \in A$.

Attaching Criterion: If (Y, A), a CW pair and $f, g : A \to X$ are homotopic gluing maps, then

$$X \sqcup_f Y \underset{\substack{\uparrow \\ \text{homotopy}}}{\simeq} X \sqcup_g Y$$

Pf (idea):

~~Take~~ Let $F : Y \times I \to X$ be homotopy between f, g.

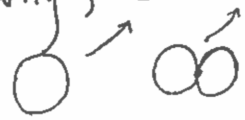
$$\begin{tikzcd}[column sep=small]
& M = X \sqcup_F Y \times I & \\
X \sqcup_f Y \arrow[ur, hook] & & X \sqcup_g Y \arrow[ul, hook]
\end{tikzcd}$$

Collapse I to 0 or 1 to get deformation retracts onto $X \sqcup_f Y$ and $X \sqcup_g Y$. $\blacksquare$

Fundamental Group

01/21/2015

Why is $S^1 \not\cong S^1 \vee S^1$



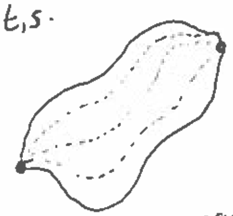
Space $X \longrightarrow \underbrace{A(X)}_{\text{algebraic object}}$

(cont.) map $f: X \to Y \longmapsto$ map $\varphi: A(X) \to A(Y)$

Def: A path in X is a map $f: I \to X$ with endpoints $f(0)$ & $f(1)$.

Recall here: $I = [0,1]$
map $\to$ cont.

Def: A path homotopy is a homotopy $h_t: I \to X$ for $t \in [0,1]$ such that $h_t(0) = h_s(0)$ for all t, s and $h_t(1) = h_s(1)$ for all t, s.



If h_0, h_1 are path homotopic, we write $h_0 \simeq h_1$

Prop: Path homotopy is an equivalence relation.

Pf (sketch):

$f \simeq f$: $h_t(x) = f(x)$

$f \simeq g \to g \simeq f$:

$h_t(x)$
$h_0 = f$, $h_1 = g$

$H_t = h_{1-t}$ will work.

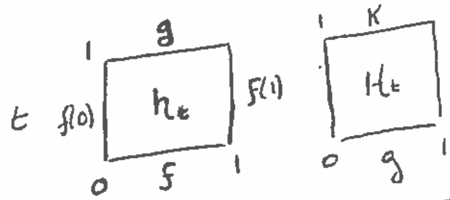
$f \simeq g,\ g \simeq k \to f \simeq k$:

h_t p.h. from $f \to g$
H_t p.h. from $g \to k$

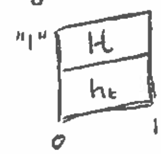
Let $\bar{H}_t$ be given by

$$H_t(x) = \begin{cases} h_{2t}(x), & t \in [0, 1/2] \\ H_{2t-1}(x), & t \in [1/2, 1] \end{cases}$$

$\blacksquare$



Stack homotopy



Example: $\mathbb{R}^n$, any two paths are homotopic. Let f, g be paths. Then

$h_t(x) = (1-t)f(x) + t g(x)$

the straight line homotopy.

Def: Suppose f, g are paths in X with $f(1) = g(0)$

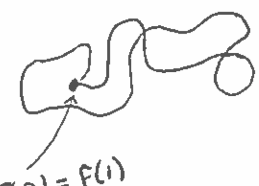
The composition is the path $f \cdot g$ is given by

$$f \cdot g \overset{\text{def}}{=} \begin{cases} f(2x), & x \in [0, 1/2] \\ g(2x-1), & x \in [1/2, 1] \end{cases}$$

* Path homotopies can be composed in a way that gives homotopies between compositions of paths.

Exercise: $f(1) = g(0)$
$f \simeq f'$ & $g \simeq g'$
then $f \cdot g \simeq f' \cdot g'$

Def: A loop is a path f with $f(0) = f(1)$



Two loops that are path homotopic are automatically "loop homotopic"

Def: Given a space X with basepoint x_0, the fundamental group $\pi_1(X, x_0)$ is

$$\pi_1(X, x_0) = \left\{ [f] \ \middle|\ \begin{array}{l} f \text{ is a loop at} \\ x_0 \in X \end{array} \right\}$$

The group operation is $\cdot$, composition of paths. Note $[f]$ is the path homotopy class of loop f.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, two heads] & X \\ S \cong I/\sim \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

loop

$\sim : 0=1$

$\tilde{f} : S^1 \to X$

$\tilde{f}(0) = \tilde{f}(1) = x_0 = f(0) = f(1)$.

Thm: $\pi_1(X, x_0)$ is a group.

Closed: Composition of two loops at x_0 is a loop at x_0.

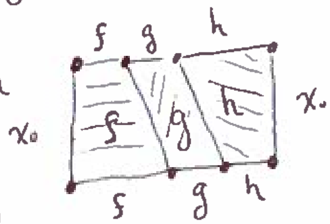
Associative: $([f]\cdot[g])\cdot[h] = [f]\cdot([g]\cdot[h])$

We define $[f]\cdot[g] \overset{\text{def}}{=} [f\cdot g]$

Choose rep. f, g, h for classes. WTS,

$(f\cdot g)\cdot h \simeq f\cdot(g\cdot h)$

$(f\cdot g)\cdot h$



$f\cdot(g\cdot h)$

Just reparametrize.

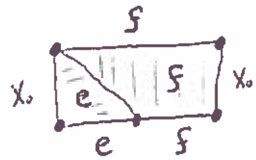
Identity: Constant map to point.

$e(x) = x_0$

$[e]\cdot[f] = [e\cdot f]$

Proof same for

$[f]\cdot[e] = [f\cdot e]$



$$h_t(x) = \begin{cases} e\left(\frac{2}{1-t}x\right), & x\in\left[0, \frac{1-t}{2}\right] \\ f\left(\frac{2}{1+t}\left(x-\frac{1-t}{2}\right)\right), & x\in\left[\frac{1-t}{2}, 1\right] \end{cases}$$

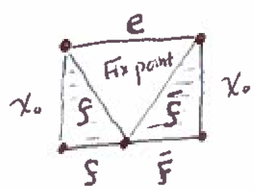
So that $e\cdot f \simeq f$. Then

$[e]\cdot[f] = [f]$

Inverses: The reverse of a path f is path $\bar{f}(x) = f(1-x)$

$[\bar{f}] = [f]^{-1} \leftarrow$ will show

That is, $f\cdot g\bar{f} \simeq e$



"Indecisive walker." The homotopy is

$h_t = \{$

Need left/right inverse, cont. but all trivial now. ☒

Ex: $\pi_1(\mathbb{R}^n, x_0) \cong \{[e]\} \cong \{1\} = 1$ as straight line homotopy shows.

Ex: $\pi_1(S^1, x_0) = \mathbb{Z}$

$n>1$: $\pi_1(S^n, x_0) = 1$

Applications of π_1

- Fundamental Theorem of Algebra
- Brower Fixed Point Theorem
- Borsuk-Ulam Theorem

Cor: $S^2 \not\leftrightarrow \mathbb{R}^2$

Cor: "Ham & Cheese" Theorem

$\ast\ \pi_1(X\times Y, (x_0, y_0)) \cong \pi_1(X, x_0)\times\pi_1(Y, y_0)$

Ex: $\pi_1(T, x_0) = \pi_1(S\times S, x_0) \cong \pi_1(S, x_0')\times\pi_1(S, x_0'') \cong \mathbb{Z}\times\mathbb{Z}$

Role of basepoint

One makes a choice for $x_0\in X$ to compute $\pi_1(X, x_0)$. If $x_0 \neq x_0'$,

$\pi_1(X, x_0) \overset{?}{\cong} \pi_1(X, x_0')$

Yes, if there is a path from x_0 to x_0', say h.

$\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_0')$

where

$\beta_h([f]) \overset{\text{def}}{=} [h\cdot f\cdot\bar{h}]$

Then β is an isomorphism.

Homomorphism:

$\beta_h([f]\cdot[g]) =$

$\beta_h([f\cdot g]) =$

$[h\cdot f\cdot g\cdot \bar{h}]$

But

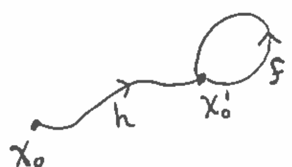
$\beta_h([f])\cdot\beta_h([g]) =$

$[h\cdot f\cdot\bar{h}]\cdot[h\cdot g\cdot\bar{h}] =$

$[h\cdot f\cdot\bar{h}\cdot h\cdot g\cdot \bar{h}] =$

$[h\cdot f\cdot g\cdot\bar{h}] =$

$\beta_h([f\cdot g])$



Easy to check that $\beta_{\bar{h}}$ is inverse of β_h.

Induced Homomorphisms

$f: X\to Y$ a map.

$x_0 \mapsto f(x_0)$

Define map induced by f to be

$$f_*: \pi_1(X,x_0)\longrightarrow \pi_1(Y,y_0)$$

given by

$$f_*([\gamma]) = [f(\gamma)]$$

Proposition: f_* is a homomorphism

"Pf":

$$\begin{aligned} f_*([\gamma]\cdot[\varphi]) &= f_*([\gamma\cdot\varphi]) \\ &= [f(\gamma\cdot\varphi)] \\ &= [f(\gamma)\cdot f(\varphi)] \\ &= [f(\gamma)]\cdot[f(\varphi)] \\ &= f_*([\gamma])\cdot f_*([\varphi]) \end{aligned}$$

01|26|2015

Punctured torus def. retracts to wedge of 2-circles

CW structure for torus:

$e^0 \cup e^1 \cup e^1 \cup e^2$

Poke hole here

Claim d.r. $\odot$ onto S^1 extends to d.r. of punctured torus onto its 1-skeleton.

∞

$t\dfrac{\vec{x}}{|\vec{x}|} + (1-t)\vec{x}$

Van Kampen Theorem

Free Product of Groups:

Let $\{G_\alpha\}_{\alpha\in A}$ be a collection of groups.

The free product G of G_α's is $G = \underset{\alpha\in A}{*} G_\alpha$ satisfying the univ. Property

$\exists$ homomorphisms

$\varphi_\alpha: G_\alpha \to G$

such that for any group H and homomorphisms $\psi_\alpha: G_\alpha\to H$ $\exists!$ homo. $\varphi: G\to H$ such that $\varphi\varphi_\alpha = \psi_\alpha$

$$\begin{tikzcd} G_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "\psi_\alpha"'] & G \arrow[dl, dashed, "\varphi"] \\ H & \end{tikzcd}$$

Thm (Exercise):

Each φ_α is a monomorphism

If the free product exists, it is unique up to isomorphism. In fact, it is uniquely unique, ie unique isomorphism.

Existence: A word ω in $\{G_\alpha\}_{\alpha\in A}$ is an expression

$\omega = g_1 \cdots g_n$

$g_i \in G_{\alpha_j}$ for some j

$\omega = 1$ empty word.

Then we concatenate in the obvious way.

$G_A = \{\text{all words in } G_\alpha\text{'s} /\sim\}$

concatenation forward/reverse

Thm: G_A is a group under concatenation of words.

~ Each word has unique red. rep.

Thm: $G_A \cong \ast_{\alpha \in A} G_A$

Ex: $G = \mathbb{Z}_2 * \mathbb{Z}_2$

a $\langle a | a^2 \rangle$ b $\langle b | b^2 \rangle$

What is $\mathbb{Z}_2 * \mathbb{Z}_2$

$$\left\{ \begin{matrix} a, b, \ aba, abab, ababa \dots \\ \text{or } ba, bab, baba \dots \end{matrix} \right\}$$

Free Group: Given set A, the free group on A is the group

$F(A) = \ast_{\alpha \in A} \mathbb{Z}$

Prop: Given any group H and function $g: A \to H$ $\exists!$ homo. $\varphi: F(A) \to H$ such that $\varphi f = g$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & H \\ F(A) \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Cor: Free groups are torsion free.

Ex: $\mathbb{Z}_2 * \mathbb{Z}_2$ is not free

Suppose $\mathbb{Z}_2 * \mathbb{Z}_2 \cong F(A)$ for some A.

$$\mathbb{Z}_2 * \mathbb{Z}_2 \underset{\phi}{\overset{\varphi}{\rightleftarrows}} F(A)$$

$\varphi\phi = 1_{F(A)}$

$\phi\varphi = 1_{\mathbb{Z}_2 * \mathbb{Z}_2}$

Let $x = \varphi(a)$

$\phi(x) = a$

$\phi(x^2) = a^2 = 1 \in \mathbb{Z}_2 * \mathbb{Z}_2$

ϕ iso. so $x^2 = 1$ in $F(A)$

So $x^2 = 1$ in some $\mathbb{Z}$ given by some $\alpha \in A$. $\therefore x = 1$

$\mathbb{Z} = \langle t \rangle$

True for all a. But clearly can't be true for all a as $F(A)$ has more than 1 element. or $\mathbb{Z}_2 * \mathbb{Z}_2$ has more than 1 element $\Rightarrow\Leftarrow$.

Rank of free group is

$\text{rank } F(A) \overset{\text{def}}{=} \text{Card } A$

$= \text{rank } F(A_{ab})$

abelianization

G

$\downarrow$

G/G'

largest abelian quotient

Cor: Two free groups are isomorphic if and only if

$\text{Card } A = \text{Card } B$

Def: Let X be a set. $R \subset F(X)$. A group presentation $\langle X | R \rangle$

where

$$\langle X | R \rangle \overset{\text{def}}{=} F(X) \Big/ \begin{matrix}\text{smallest normal subgroup} \\ \text{gen. by } R\end{matrix}$$

Notation here

$$= F/\bar{R}$$

* Every group has a presentation.

X is generating set.

R is relating set.

Ex: $\langle x \mid x^5 \rangle \cong \mathbb{Z}_5$

$$\langle x,y \mid \underbrace{xyx^{-1}y^{-1}}_{xy=yx} \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$$

$$\langle x,y \mid x^2, y^3 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

$$\langle x,y \mid \underbrace{x^2yx^{-1}y^{-1}}_{x^2y=yx}, \underbrace{xyx^{-1}y^{-2}}_{xy=y^2x} \rangle \cong G$$

$$G/G' = 1$$

That is, G is "perfect"

$$G \cong \langle x,y \mid x, y, \underbrace{xyx^{-1}y^{-1}}_{xy=yx} \rangle$$

$$= 1$$

Seifert van Kampen Theorem

Let X be a space with open, nonempty path connected subspaces $U, V, U \cap V$ such that $U \cup V = X$

$$\begin{tikzcd} & U \arrow[dr] & \\ U \cap V \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & X \\ & V \arrow[ur] & \end{tikzcd}$$

induced homo.

pushouts

$$\begin{tikzcd} & \pi_1(U,x_0) \arrow[dr, "j_1"] \arrow[drr, "\Phi_1"] & & \\ \pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] \arrow[rr, phantom, "\circlearrowleft"] & & \pi_1(X,x_0) \arrow[r, dashed] & H \\ & \pi_1(V,x_0) \arrow[ur, "j_2"'] \arrow[urr, "\Phi_2"'] & & \end{tikzcd}$$

Suppose Φ_1, Φ_2 are homomorphisms to a group $H \ni \Phi_1 i_1 = \Phi_2 i_2$, then

$\exists!$ φ homo. $\varphi : \pi_1(X,x_0) \to H$

such that $\varphi j_1 = \Phi_1$

$\varphi j_2 = \Phi_2$

That is,

$$\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0) / \sim$$

$$\sim : \langle i_1(x)\, i_2(x)^{-1} \mid x \in \pi_1(U \cap V, x_0) \rangle$$

01/28/2015

$X = \mathcal{U} \cup V$

$\mathcal{U}, V, \mathcal{U} \cap V$

open, nonempty, path connected

$x \in \mathcal{U} \cap V$ the basepoint

$$\begin{tikzcd} & \pi_1(\mathcal{U}) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\ \pi_1(\mathcal{U}\cap V) \arrow[ur] \arrow[dr] & \pi_1(X) \arrow[r, dashed, "\exists! \phi"] & G \\ & \pi_1(V) \arrow[u, "j_2"] \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

pushout

i.e. $\pi_1(X) \approx \pi_1(\mathcal{U}) * \pi_1(V) \big/ \overline{\langle i_1(\gamma)\, i_2(\gamma)^{-1} \mid \gamma \in \pi_1(\mathcal{U}\cap V)\rangle}$

~~Cor: If $\pi_1(\mathcal{U}\cap V)$, $\pi_1(\mathcal{U}\cap V)$ are trivial~~

Cor: If $\pi_1(\mathcal{U}\cap V)$ is trivial then

$\pi_1(X) \approx \pi_1(\mathcal{U}) * \pi_1(V)$

Cor: If i_1, i_2 are trivial, then $\pi_1(X) \approx \pi_1(\mathcal{U}) * \pi_1(V)$

Cor: If i_1, i_2 injective then so too are j^1, j^2 and $\pi_1(X) \approx$ amalgamated free product.

$\pi_1(X) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U}\cap V)} \pi_1(V)$

Remark: $A \subset X$

$i : A \hookrightarrow X$ inj.

But $\pi_1(A) \xrightarrow{i_*} \pi_1(X)$ does not need to be injective.

Ex: $S^1 \subset D^2$

But $\pi_1(S) = \mathbb{Z}$

$\pi_1(D^2) = 0$

Ex: $X = S^1 \vee S^1$

$\pi_1(X, x_0)$?

$\mathcal{U}$ = neigh. left O

V = neigh. right O

Def. retract to left circle.

$\pi_1(\mathcal{U}, x_0) = \langle a \rangle$

$\pi_1(V, x_0) = \langle b \rangle$

Sim for V

$\mathcal{U} \cap V = \{x_0\}$

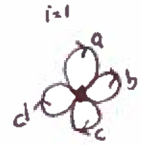
$\pi_1(\mathcal{U}\cap V) = 0$

$\pi_1(X) = \underbrace{\mathbb{Z}}_{a} * \underbrace{\mathbb{Z}}_{b} = \langle a, b \rangle$

Ex:

$X = \bigvee_{i=1}^{n} S$

eg. $n = 4$



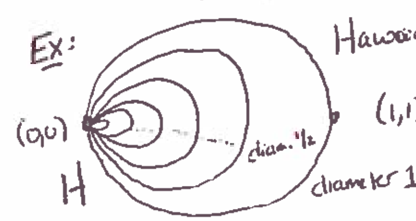
$\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

$= \langle a, b, c, d \rangle$

Ex: $X = \bigvee_{\alpha \in A} S \approx \mathop{*}_{\alpha \in A} \mathbb{Z}$

countable

Ex:



Hawaiian earring

Not a CW complex

$\pi_1(H) \to \prod_{i \in \mathbb{Z}} \mathbb{Z}$

Ex: If $n \geq 2$ $\pi_1(S^n) = 0$

$S^n = D^n \cup_{S} D^n$

$\supset S^{n-1}$

for $n \geq 2$ S^{n-1} path connected

D^n $\underbrace{\text{simply connected}}$

path ~~conn~~ connected & $\pi_1 = 0$

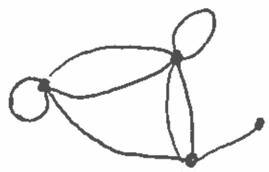
* Find example where connected but $\pi_1 \neq 0$.

Graphs

Any (connected) graph has a CW structure

0-cells $\rightleftharpoons$ vertices

1-cells $\rightleftharpoons$ edges



Trees

~ Every graph has a maximal tree.

Pick T to be a maximal tree.

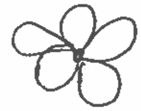
T is a contractible subcomplex of G.

$G \to G/T$ is a homotopy equivalence.

Induce iso. of fundamental group.

$\pi_1(G) \approx \pi_1(G/T)$

$G \supset T$ 

G/T 

$\pi_1(G) \approx \pi_1(G/T) =$ free group of rank # of edges in T.

~ # of edges not in some maximal tree of G is an invariant of G.

Ex: $X = \mathbb{R}^3 -$ unknot

u

$\mathbb{R}^3 - u \simeq$ 

$\mathbb{B}^3 - u$

So $\pi_1(\mathbb{R}^3 | u) \cong \mathbb{Z}$

Thm: $\pi_1(\bigvee_i^n X_i) = \mathop{*}_{i=1}^{n} \pi_1(X_i)$

$\overbrace{\text{Seifert van Kampen}}^{\text{SVK}}$ Thm for CW complexes

X is a 2-dim. CW complex

Let $U = X^2$ (just a graph)

S with only one 2-cell

$V = D^2$

u

φ

gluing map

$\varphi: \partial D^2 \to X' = U$

By SVK,

$\pi_1(X) \cong \pi_1(U) * \pi_1(V) / \langle - \rangle$

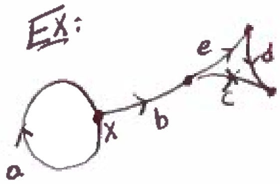
$= \pi_1(U) / \langle \text{"}\varphi(\partial D^2)\text{"} \rangle$

Thm: Let X be a 2-dim CW complex. $U = X'$, $X = U \cup_\alpha D^2_\alpha$ via attaching maps φ_α. Choose basepoint in maximal tree, $\forall \alpha$, choose path p_α from x to $\varphi_\alpha(1_\alpha)$. Then

$\pi_1(X, x) \cong \pi_1(U) / \sim$

$\sim : \langle [p_\alpha \varphi_\alpha(S^1) \bar{p}_\alpha] \ \forall \alpha \rangle$

Ex:



$\varphi: D^2 \to \mathcal{U}$

$c\bar{d}\bar{e}c\bar{d}\bar{e}$

twice around Δ

$X = \mathcal{U} \cup_\varphi D^2$

$\pi_1(X,x_0) = \pi_1(\mathcal{U},x_0) \Big/ \overline{\langle bc\bar{d}\bar{e}c\bar{d}\bar{e}\bar{b} \rangle}$

$= \langle \underbrace{a}_{\text{one loop}}, \underbrace{bc\bar{d}\bar{e}\bar{b}}_{\text{other loop}} \mid bc\bar{d}\bar{e}c\bar{d}\bar{e}\bar{b} \rangle$

$\approx \langle a \rangle * \langle \gamma \mid 2\gamma \rangle$

$\gamma = bc\bar{d}\bar{e}\bar{b}$

Ex:

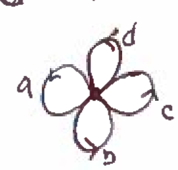
glue D^2 by $aba^{-1}b$ } Klein bottle

$\pi_1(K,x) \approx \langle a,b \mid aba^{-1}b \rangle$

Not abelian, it surjects to D_3.

Ex:

$G = \langle a,b,c,d \mid a^2b^3, cd \rangle$

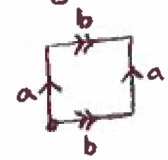


Glue disks by a^2b^3 & by cd.

Surfaces (2-manifolds)

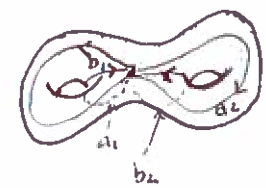
M_g = orientable surface of genus $g \geq 0$ =

Ex: $g=1$: Torus



Claim

$M_2 =$



4 cells : one 2-cell

$\pi_1(M_2) = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle$

Further claim

$\pi_1(M_g) \cong \langle a_1, \dots, a_{2g} \mid [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] \rangle$

Thm: $M_g \cong M_h$ iff $g = h$

Pf: Abelianize $\pi_1(M_g)$ to get $\mathbb{Z}^{2g}$ so if $h \neq g$ then different π_1 $\Rightarrow\Leftarrow$.

Nonorientable Surface

N_g : $\underbrace{\text{genus -}g}_{\text{non orientable surface}}$

glue disks by $a_1^2 a_2^2 \cdots a_g^2$

$\pi_1(N_g) \approx \langle a_1, \dots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 \rangle$

Thm: $N_g \cong N_h$ iff $g=h$

Pf: Abelianize $\pi_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$

If $g \neq h$ then not isomorphic. $\Rightarrow\Leftarrow$ ▨

Cor: The list $M_0, M_1, \dots$
$N_1, N_2, \dots$
contains no "duplicates"

Thm: Every closed 2-manifold is on that list.

Exercise: Which on list is K?

02/02/2015

Covering Space

Ex: $p: \mathbb{R} \to S^1$

$p(\theta) \overset{\text{def}}{=} e^{i\theta} = (\cos\theta, \sin\theta)$

$\downarrow p \to$ locally a homeomorphism

Def: A covering space of X is a map $p: \tilde{X} \to X$ such that there is an open cover of X, $\{U_\alpha\}$, such that

i) $p^{-1}(U_\alpha) = \bigsqcup_\beta V_\beta$ where V_β open in $\tilde{X}$

ii) $p|_{V_\beta} : V_\beta \to U_\alpha$ is a homeomorphism.

p surjective? Need? Want?

$\tilde{X}$: covering space (upstairs space)

X: base space

U_α: Evenly covered neigh.

$\bigsqcup V_\beta$: Stack of pancakes over U_α

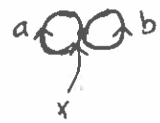
$x \in X$, $p^{-1}(x)$: Fiber over x.

Ex: $n \in \mathbb{Z} \setminus \{0\}$

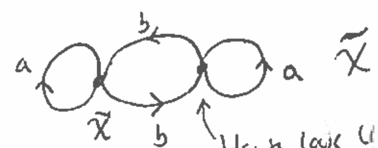
$p_n : S \to S$ given by $z \mapsto z^n$

p_n map, surjective. p_n a covering space.

Ex: X =



Covering 1:



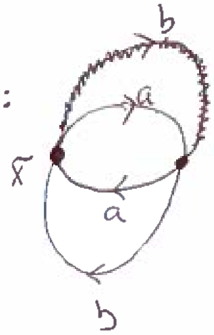
Loop e a to b to a point point go to X. loops at b go to b

$\pi_1(\tilde{X}, \tilde{x}) = \langle a, bab^{-1}, b^2 \rangle$

$\pi_1(X, x) = \langle a, b \rangle$

$\pi_1(X) \trianglelefteq \pi_1(\tilde{X}, \tilde{x})$?

Ex:



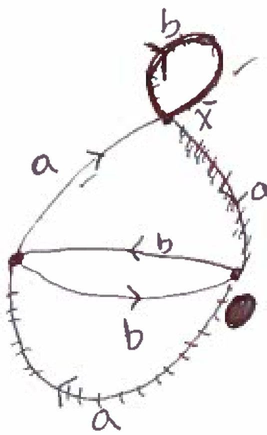
Covering space of X

$\pi_1(\tilde{X}, \tilde{x}) = \langle ba^{-1}, ba, bb \rangle$

$\pi_1(\tilde{X}) \trianglelefteq \pi_1(X) = \langle a, b \rangle$

Process: Use maximal trees and go around loops at $\tilde{x}$ using b & all other paths not used.

Ex:

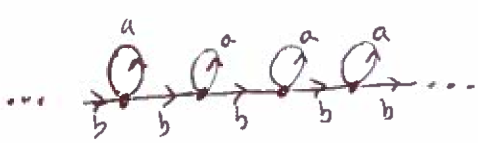


$\tilde{X}$ covering space of X.

$\pi_1(\tilde{X}, \tilde{x}) = \langle b, aaa, aba^{-1}a^{-1}, aaba^{-1} \rangle$

$\pi_1(\tilde{X}, \tilde{x}) \ntrianglelefteq \pi(X)$

Ex. Can have inf covering space



Lifting Properties

Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$

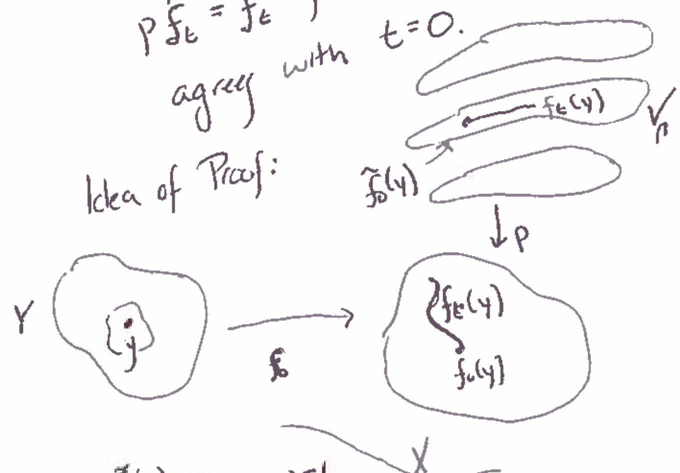
A lift, $\tilde{f}$, of f is a map f is a map $\tilde{f}: Y \to \tilde{X}$ such that $p\tilde{f} = f$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Homotopy Lifting Property

Let $p: \tilde{X} \to X$ be a covering space. $f_t: Y \to X$ a homotopy and a lift $\tilde{f}_0: Y \to \tilde{X}$ of f_0. Then $\exists!$ $\tilde{f}_t: Y \to \tilde{X}$ be such that $p\tilde{f}_t = f_t$ for all t and agrees with $t=0$.

Idea of Proof:



Define $\tilde{f}_t(y) = (p|_{V_\beta})^{-1} f_t(y)$ for all y, t. Use Pasting Lemma. Given about & p homeo, immediate

$p\tilde{f}_t = f_t$

Cor: Path Lifting

A path in X is just a homotopy of a point.

So given a path γ "c" X and a lift $\tilde{\gamma}(0)$ of $\gamma(0)$, $\exists!$ path $\tilde{\gamma}$ in $\tilde{X}$ starting at $\tilde{\gamma}(0)$ & lifting γ.

Cor: Path Homotopy Lifting

Let γ, γ' be homotopic paths in X. Suppose $\exists$ lifts with $\tilde{\gamma}(0) = \tilde{\gamma}'(0)$. Then $\tilde{\gamma} \simeq \tilde{\gamma}'$ in $\tilde{X}$. (path hom $\xrightarrow{\text{lift}}$ path homo.)

Hence, $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$.

* Two above suggests relationship between $\pi_1(X)$ & $\pi_1(\tilde{X})$.

Thm: Let p be a covering map: $\tilde{X} \xrightarrow{p} X$. with $p(\tilde{x}) = x$. Then

$p_*: \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$

is a monomorphism.

$p_*(\pi_1(\tilde{x}, \tilde{x}))$ is ~~the~~ a subgroup of $\pi_1(X, x)$. of homotopy classes of loops in $\tilde{X}$ at $\tilde{x}$.

Pf:

Injective: $p_*([\tilde{\gamma}]) = 1$ in $\pi_1(X, x)$

in $\pi_1(X, \tilde{x})$. $\tilde{\gamma}$ with $\tilde{x}$ b.p.

We know

$p_*([\tilde{\gamma}]) = [p\tilde{\gamma}]$
$= [\gamma]$
$= 1$

Let $\gamma = p\tilde{\gamma}$ to be loop in X at x b.p.

That is, $\gamma \simeq e_x$ constant path (loop)

path homotopy

We have $\tilde{\gamma}$ a lift of γ.

We have $e_{\tilde{x}}$ a lift of e_x.

By path homotopy lifting, $\exists!$ path homotopy in $\tilde{X}$ giving $\tilde{\gamma} \simeq e_{\tilde{x}}$. But then $[\tilde{\gamma}] = 0$.

Let $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$ then $\exists [\tilde{\gamma}]$ at $p_*([\tilde{\gamma}]) = [\gamma]$

that is, $[p\tilde{\gamma}] = [\gamma]$.

that is, $p\tilde{\gamma} \simeq \gamma$ in X. By path lifting, $\tilde{\gamma}$ is a loop at $\tilde{x}$. $\exists$ $\boxed{\text{path}}$ $\tilde{\tilde{\gamma}}$ in $\tilde{X}$ starting at $\tilde{x}$ and $\tilde{\tilde{\gamma}}$ is a lift that lifts γ. of loop $p\tilde{\gamma}$ at $\tilde{x}$. So Path Homotopy Lifting gives $\tilde{\gamma} \simeq \tilde{\tilde{\gamma}}$ at $\tilde{x}$ lifting γ. So $\tilde{\tilde{\gamma}}$ is a loop at $\tilde{x}$ lifting γ. ∎

1. Paths aren't loops
2. Basepoint matters
3. Use of same 2 cor.

Takeaway points

02/04/2015

Prop: Let $\tilde{X} \xrightarrow{p} X$ be a covering space, the cardinal $\# p^{-1}(x)$ is locally constant. Furthermore, if X is connected then $\# p^{-1}(x)$ is globally constant.

Def: If X is connected, $\# p^{-1}(x) = n$ is the number of sheets of the covering, we say $\tilde{X}$ is an n-fold cover of X.

Note: X need be connected for this to be well-defined.

Prop: Let $p: \tilde{X} \to X$ be a covering space with X & $\tilde{X}$ path connected $\Big\}$ $\tilde{x} \in \tilde{X}$ & $p(\tilde{x}) = x$ then $\# p^{-1}(x)$ is given by $\underbrace{[\pi_1(X,x) : p_* \pi_1(\tilde{X}, \tilde{x})]}_{\text{akin to Galois Theory}}$.

Bigger cover is (in # sheets) the smaller the subgroup of $\pi_1(X,x)$

Recall 2 & 3 fold cover of

∞ of index 2 & index 3 subgroups of $\underbrace{\text{rank}}$ rank 3 & 4 ?

Pf:

$$\phi: H \backslash G \to p^{-1}(x)$$

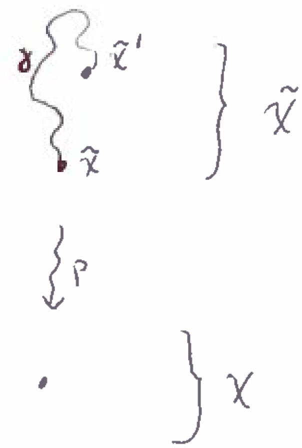
where $H = p_* \pi_1(\tilde{X}, \tilde{x})$ and $G = \pi_1(X, x)$.

$\phi(H[g]) \overset{def}{=} \tilde{g}(1)$, where $\tilde{g}$ is the unique lift of g to $\tilde{X}$ starting at $\tilde{x}$.

Why is this well defined?

Epi: Need fiber of x mapped onto by ϕ. Choose $\tilde{x}' \in p^{-1}(x)$. Let γ be a path in $\tilde{X}$ from $\tilde{x}$ to $\tilde{x}'$. Then $p(\gamma)$ is a loop in X at x. Then

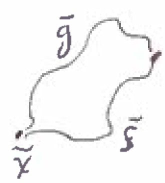
$$\tilde{x}' = \underbrace{\phi(H[p(\gamma)])}_{\text{uses def \& uniqueness of lifts.}}$$



Mono: Suppose

$$\phi(H[g]) = \phi(H[f])$$

Then the lifts $\tilde{g}$ and $\tilde{f}$ (starting at $\tilde{x}$) have $\tilde{g}(1) = \tilde{f}(1)$.



Then $\tilde{g} \cdot \bar{\tilde{f}}$ is a loop at $\tilde{x}$.

$$p_*([\tilde{g} \cdot \bar{\tilde{f}}]) \in H$$
$$\| $$
$$[g][f]^{-1}$$
$$H[g] = H[f]. \quad \blacksquare$$

Def: A space X is locally path connected if $\forall x \in X$ and for all neigh U of x, there is a neigh V of x with $V \subset U$ and V is path connected.

<u>Lifting Criterion</u>

Suppose

$$\begin{tikzcd} & (\tilde X, \tilde x) \arrow[d, "p"] \\ (Y, y) \arrow[r, "f"] & (X, x) \end{tikzcd}$$

with p a covering map.

Assume Y path connected & locally path connected

Then $\exists$ a lift $\tilde f$ of f if and only if

$$f_*(\pi_1(Y,y)) < p_*(\pi_1(\tilde X, \tilde x))$$

Pf:

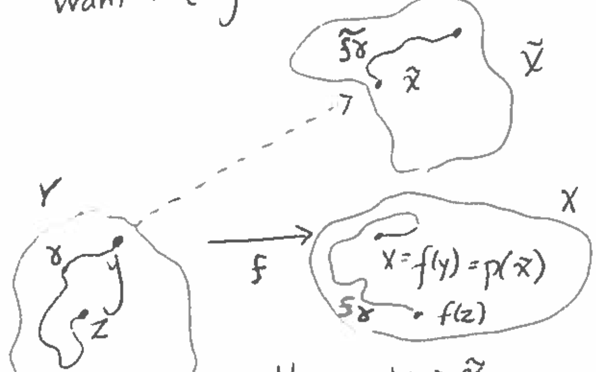
$\Rightarrow$ If $\tilde f$ exists, then $p_* \tilde f_* = f_*$ by functoriality so that

$$\text{im} f_* < \text{im}\, p_*$$

$\Leftarrow$: Suppose

$$\text{im} f_* < \text{im}\, p_*$$

Want to construct $\tilde f : Y \to \tilde X$



Know $y \mapsto \tilde x$

Y is path connected so $\exists$ path γ from y to z. Push the path forward to $f\gamma$. Then

$$\tilde f(z) = \widetilde{f\gamma}(1), \text{ i.e. end of } \widetilde{f\gamma}$$

Automatic that the diagram commutes. Is it well defined? What if chose different γ?

Let φ be another path y to z. $f\varphi$ lifts to $\widetilde{f\varphi}$ starting at $\tilde x$.

Is $\widetilde{f\varphi}(1) = \widetilde{f\gamma}(1)$?

Know that $\gamma\bar\varphi$ is a loop in Y at y. But $f\gamma \cdot \overline{f\varphi}$ is a loop in X at x. But

$$f\gamma \cdot \overline{f\varphi} = f(\gamma \cdot \bar\varphi)$$

Consider $f_*[\gamma \cdot \bar\varphi] \in \pi_1(X,x)$

We assumed $\text{im} f_* < \text{im}\, p_*$. So

$\exists\ [\tilde\eta] \in \pi_1(\tilde X, \tilde x) \ni f_*[\gamma \cdot \bar\varphi] = p_*[\tilde\eta]$

Since p_* is mono, $f\gamma \cdot \overline{f\varphi}$ lifts to a loop in $\tilde X$ at $\tilde x$. This loop is $\simeq$ to $\tilde\eta$. Therefore, $\widetilde{f\gamma} \cdot \overline{\widetilde{f\varphi}}$ is a loop so same endpoint and have well defined.

Only remains to show $\tilde f$ cont. Skip proof of this but it uses fact Y is locally path connected and fact $\tilde X$ is locally ~~path~~ homeomorphic to X. ∎

Good exercise

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ S^n \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}?"] & S^1 \end{tikzcd}$$

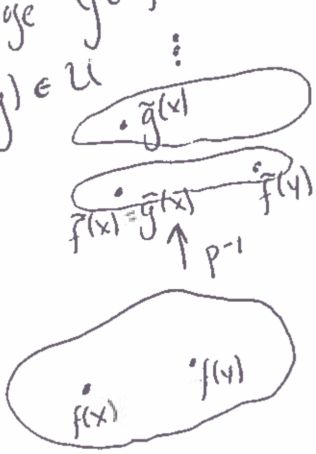
Uniqueness of Lifts

Let $p: \tilde{X} \to X$ be a covering and $f: Y \to X$ be a map. Suppose also $\tilde{f}$ and $\tilde{g}$ are two lifts of f that agree at a point. If Y is also connected, then $\tilde{f} = \tilde{g}$.

Generalizes uniqueness of path & homotopy lifting

Pf: Let $\bar{X} = \{x \in Y \mid \tilde{f}(x) = \tilde{g}(x)\}$
We know $\bar{X} \neq \phi$. WTS that $\bar{X}$ is clopen.

$\bar{X}$ is open: Take $x \in \bar{X}$.
Let $U \subset X$ be open neigh of $f(x)$ with U evenly covered.
We want to show $f^{-1}(U) \subset \bar{X}$.
Suppose $y \in f^{-1}(U) \setminus \bar{X}$.
$f(y) \in U$



$\tilde{f}$ and $\tilde{g}$ are cont.

If $\tilde{g}(x)$ not in same pancake, then violating continuity. "☒"

WTS closed, same but on complement.

Classifying Covering Spaces

Idea: Given space X

$$\left\{ \begin{matrix} \text{covering spaces} \\ \tilde{X} \text{ of } X \end{matrix} \right\} \xrightarrow{\text{epi}} \left\{ \begin{matrix} \text{Subgroups} \\ \text{of} \\ \pi_1(X, x_0) \end{matrix} \right\}$$

$$p: \tilde{X} \to X \quad \mapsto \quad p_* \pi_1(\tilde{x}, \tilde{x}_0)$$

Then take equiv relations of both to make bijection.

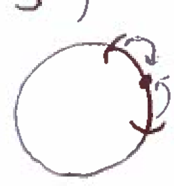
$$id: X \to X \longleftrightarrow \pi_1(X, x)$$

$$? \longleftrightarrow 1$$

"Def": A space is "nice" if it is path connected, locally path connected, and semilocally simply connected.

$\forall x \in X$, $\exists$ open neigh U of x where inclusion induced map on π_1 is trivial, ie
$\operatorname{im}\left[\pi_1(U, x) \to \pi_1(X, x)\right] = 1$.

Ex: S^1 is nice



All connected CW complexes nice.

All connected manifolds are nice.

Thm: Let X be nice. Then $\exists$ covering space $\tilde{X}$ of X that is simply connected. In particular, $\pi_1(\tilde{X}, \tilde{x}) = 1$.

Fix $x \in X$

Pf: $\tilde{X} = \{[\gamma] \mid \gamma \text{ path in } X \text{ at } x\}$

set

$p : \tilde{X} \to X$

$p([\gamma]) = \gamma(1)$

Need topology on $\tilde{X}$. This is clearly surjective. Build top. on $\tilde{X}$ using that of X (after all, need be locally homeomorphic). Consider $\mathcal{U}$ the collection of open sets in X $\mathcal{U}$ with $\text{im}(\pi_1(\mathcal{U}, x) \to \pi_1(X, x)) = 1$

Here, $x \in \mathcal{U}$. where $\mathcal{U}$ is path connected.

Claim: $\mathcal{U}$ is a basis for topology on X. This is because X is nice.

Let $\mathcal{U} \in \mathcal{U}$ and γ be path in X from x to $\mathcal{U}$. Define

$$\mathcal{U}_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path inside } \mathcal{U} \text{ and } \gamma \cdot \eta \text{ defined}\}$$

Claim: $\{\mathcal{U}_{[\gamma]}\}$ basis for some topology on $\tilde{X}$.

Many details to check here!

This is called the universal cover.

What is a path in $\tilde{X}$?

It is a one-family parameter of points.

If γ be a path in X starting at x.

Let γ_t denote $\gamma|_{[0,t]}$.

For all t, $[\gamma_t] \in \tilde{X}$ and

$[0,1] \to \tilde{X}$

$t \mapsto [\gamma_t]$

is a path from $[\gamma_0] \to [\gamma]$

What is a loop in $\tilde{X}$ and why is it nullhomotopic?

02/09/2015

Geometric why
cover big $\to \pi_1$ smaller
Look @ π_1 p_* prop

Covering Spaces Cont.

X is "nice"

Want to prove "Galois-ish" correspondence between

$$\left\{\begin{matrix}\text{covering spaces} \\ \text{of } X\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{subgroups of} \\ \pi_1(x)/\sim\end{matrix}\right\}$$

$$\tilde{X} \longmapsto \text{im } \pi_1(\tilde{X})$$

$$p^{-1}(x) = [\pi_1 X ; p_* \pi_1 \tilde{X}]$$

Prop: Let X be a "nice" space. Let $x \in X$ and $H \leq \pi_1(X, x)$. Then $\exists$ cover $p: \tilde{X} \to X$ and $\tilde{x} \in \tilde{X} \ni H = p_*(\pi_1(\tilde{X}, \tilde{x}))$

Pf: Let X_u = universal cover of X, ie simply connected cover $\tilde{X}$.

$$X_u = \{[\gamma] \mid \gamma \text{ path in } X \text{ with } \gamma(0) = x\}$$

Define an equivalence relation:

$$\underbrace{[\gamma] \sim [\psi] \text{ if } [\gamma \cdot \bar{\psi}] \in H < \pi_1(X)}_{}$$

That is if γ starts where $\bar{\psi}$ ends and loop at x_0.

$\sim$ is an equiv. relation.

1. $[\gamma] \sim [\gamma] \Leftrightarrow [\gamma \cdot \bar{\gamma}] \in H$:

   $\gamma \cdot \bar{\gamma} = 1$ (that is, e_x)

   $1 \in H$ by assumption. ✓

   $\subset H < \pi_1$

2. $[\gamma] \sim [\psi] \to [\psi] \sim [\gamma]$:

   $[\gamma \cdot \bar{\psi}] \in H$. H closed under inverses.

   $[\gamma \cdot \bar{\psi}]^{-1} = [\psi \cdot \bar{\gamma}]$

   But this is $[\psi] \sim [\gamma]$

3. You check (follows from group closure).

$\exists$ quotient map $q: X_u \to \underbrace{X_u/\sim}_{\text{def } \tilde{X}}$

In fact, q a covering map. } We have found our space $(\tilde{X}, q)$. Just want to show a covering space.

Let U_γ be a basis open set.

$U_\gamma = \{[\gamma \cdot \eta] \mid \gamma \text{ starts } x \text{ ends in } U,\ \eta \text{ path in } U\}$

Take U_ψ.

$$[\gamma] \sim [\psi] \Leftrightarrow [\gamma \cdot \eta] \sim [\psi \cdot \eta]$$
for all η in U.

If any points of $U_{[\gamma]}$ are identified via q to point in $U_{[\psi]}$

Then all of $U_{[\gamma]}$ is identified with all of $U_{[\psi]}$ under q.

Then every covered neighb. are the $\underbrace{q(U_{[\gamma]})}_{\text{Basis for top.}}$.

Have

$$\begin{tikzcd} & X_u \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \tilde{X} \arrow[rr, dashed, "\tilde{p}"'] & & X \end{tikzcd}$$

Need covering map $\tilde{p}: \tilde{X} \to X$.

HW: In diagram, $\exists\ \tilde{p}: \tilde{X} \to X$.

Verify $\tilde{p}_* \pi_1(\tilde{X}, q(\varphi)) = H$

1. $H \subset \tilde{p}_*(\pi_1 \tilde{X}, q(\tilde{x}))$

Take loop γ with $[\gamma] \in H$

$\gamma \cdot e_x = \gamma$

$[\gamma \cdot \bar{e}_x] \in H$

$[\gamma] \sim [e_x]$

Let $\gamma_t = \gamma|_{[0,t]}$. Then $[\gamma_t]$ is a path in X_u. Know $[\gamma_0] = [e_x]$ & $[\gamma_1] = [\gamma]$

So $q([\gamma_t])$ is a loop in $\tilde{X}$

As $\tilde{p}q = p$ and $p([\gamma_t]) = [\gamma]$ then

$\tilde{p}_*([q(\gamma_t)]) = [\gamma] \in H$

Not proving other direction (same trick).

Take loop in X. In image of $\tilde{p}_*$
Pull back then show in H
(go around clockwise).

Ex: $X = S^1$
$x = 1$

$\pi_1(X) \cong \mathbb{Z}$

$2\mathbb{Z} < \mathbb{Z}$

Should exist covering space of S with $p_*\pi_1 = 2\mathbb{Z}$

equal not isomorphic

$$p : \tilde{X} \to X$$
$$\tilde{X} = S$$

$$z \mapsto z^2.$$

Uniqueness

Df: Let $p_1 : X_1 \to X$
$p_2 : X_2 \to X$
be covering space of X.
An isomorphism (called a deck translation or covering transformation) is a homeomorphism $f : X_1 \to X_2$ commuting the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & X_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

ie homeo respecting covering map.

Lem: Let x_1 & x_2 be lifts of x to X_1, X_2. Then $\exists$
$x_1 \mapsto x_2$ iso. $f : X_1 \to X_2$ iff
$$p_{1*}(\pi_1(x_1)) = p_{2*}(\pi_1(x_2))$$

"Pf":
$\Rightarrow$: $p_{2*} f_* = p_{1*}$ & f_* iso.
as $\operatorname{im} p_{1*} = \operatorname{im} p_{2*}$

$\Leftarrow$:

$$\begin{tikzcd} & X_2 \arrow[d, "p_2"] \arrow[dl, dotted] \\ X_1 \arrow[r, "p_1"'] \arrow[ur, dotted] & X \end{tikzcd}$$

$\operatorname{im} p_{1*} \subset \operatorname{im} p_{2*}$
So $\exists!$ lift $f_{12} : X_1 \to X_2$ with
$p_2 f_{12} = p_1$
Sim. $\exists!$ lift $f_{21} : X_2 \to X_1$ with
$p_1 f_{21} = p_2$

Use
$$\begin{tikzcd} & X_1 \arrow[d, "p_1"] \\ X_1 \arrow[r, "p_1"'] \arrow[ur, dashed, "id"] & X \end{tikzcd}$$

$f_{21} f_{12} : X_1 \to X_1$ so
$f_{21} f_{12} = id_{X_1}$
& sim.
$f_{12} f_{21} = id_{X_2}$
By uniqueness of lifts.

Need check $f_{12}(x_1) = x_2$. ∎

Ex:
$$\begin{tikzcd} S \arrow[rr, dashed, "f"] \arrow[dr, "p_1"'] & & S \arrow[dl, "p_2"] \\ & S & \end{tikzcd}$$

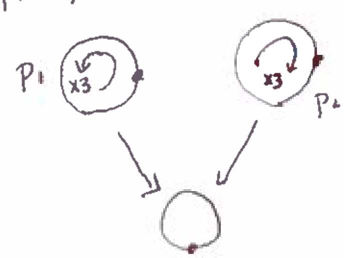
$3\mathbb{Z} = -3\mathbb{Z}$

$p_1(z) = z^3$

$p_2(z) = z^{-3}$

Find hom. $f: S \to S \ni$

$p_1 = p_2 f$



flip, look in mirror $\to$ $f: S \to S$ given by

$$\underbrace{z \mapsto z^{-1} = \bar{z}}_{\text{Cont. but not analytic.}}$$

* "Covering Trans: Symmetry taking basepoint to basepoint."

Thm: If X is "nice" and $x \in X$.

Then

1.) $\exists$ bij. s.t. $\pi_1(X,x)$ & basepoint preserv. iso. classes of covering spaces of X.

2.) $\exists$ bijection between conj. class of subgroups of $\pi_1(X)$ and (not nec. basepoint prepr.) iso. of classes of covering spaces of X.

(42) Ex:

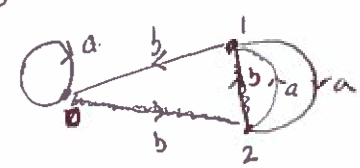
$$\begin{tikzcd} E_1 \arrow[rr, "f"', "e_0 \mapsto e_1"] \arrow[dr, "p_1 \text{ Cover}"'] & & E_1 \arrow[dl, "p_2 \text{ Cover}"] \\ & B & \end{tikzcd}$$

Chang. basepoints is (act) conj. on fundamental groups.

Cor: Only covering spaces of S^1 are

$S^1 \xrightarrow{z^n} S^1$ & $\mathbb{R} \xrightarrow{e^{i\theta}} S^1$

Ex: $X = S^1 \vee S^1 =$ 

Let $X_1 =$ 

$\exists p: X_1 \to X$

$\{0,1,2\} = p^{-1}(x)$

$\pi_1(X) = \langle a, b \rangle$

Check: $p_* \pi_1(X_1, 0) =$
$\langle a, bbb, bab^{-2}, b^2ab^{-1} \rangle$

Hw: Write down

$p_* \pi_1(X,1) = \langle ? \rangle$

$a \in p_* \pi(X_1, 0)$

Not in $p_* \pi_1(X_1, 1)$ or $p_* \pi_2(X_1, 2)$

(otherwise would lift to loop)

$\nexists$ iso. X_1 to itself taking 0 to 1.

02/11/2015

But if on b/o. act basepoint pres. so A & B conj.

HW: Find $x \in \langle a,b\rangle$

$$xAx^{-1} = B$$

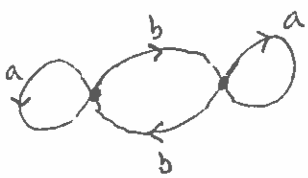
Ex: $F(2) = \langle a,b\rangle$

Find all index 2 subgroups.

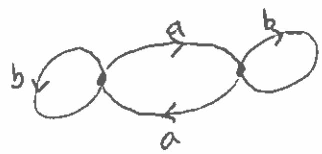
(Alg. problem)

$\Updownarrow$

Find all 2 sheeted (connected) cover of ∞

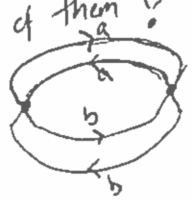


$\longrightarrow \langle a, b^2, bab^{-1}\rangle$



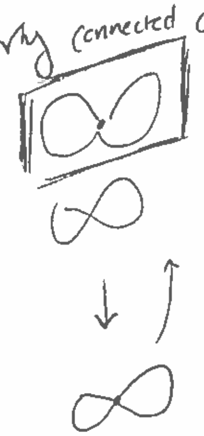
$\longrightarrow \langle b, a^2, aba^{-1}\rangle$

All of them!



and variants $\longrightarrow \langle a^2, ab, ab^{-1}\rangle$

Why connected covers?



Only recovers part with basepoint.

Yet more covering spaces....

Prop: $n = \cancel{1}, 2, 3, \dots, \infty$

$$F(n) \triangleleft F(2)$$

"Pf": (~~subgroup alone, see normal later~~)

$n=2$: Trivial

$X =$ 

$n \geq 3$

Start with $(n-1)$-gon. Add a loop at each corner. Call this X_n

Ex: $n=7$

$$p_*\pi_1(X_n, \tilde{x}) = \underbrace{\langle a, bab^{-1}, b^2ab^{-2}, \dots, b^{n-2}ab^{n-2}, b^{n-1}\rangle = F(n)}$$

π_1 connected graph free.

Normal as contains all conjugates of generators.

π_1 injects under covering space.

For ∞ take ∞-gon and same idea. ∎

Prop: Every subgroup of free group free.

Idea: $F(A)$ build CW complex. Get graph. Find covering space of subgroup (also a graph).

Covering Spaces & Representation Theory

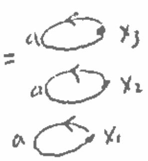
Covering spaces & permutations

How does $\pi_1(X,x)$ act on $p^{-1}(x)$ if $p:\tilde{X}\to X$ is a covering.

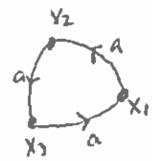
\# We need consider disconnected coverings as well for this.

Ex: $X = S^1$ $x = 1$

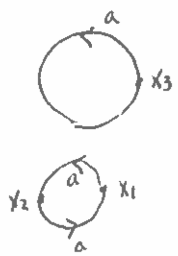
$X_1 =$



$Y_3 =$



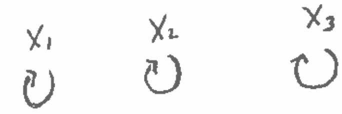
$X_2 =$



In each example, each lift x_j of x, $\exists!$ lift of a starting at x_j. Given such a lift x_j of x and given $a \in \pi_1(X,x)$ let $\rho(x_j)$ be end of lift $\tilde{a}$ starting at x_j. Clearly, this must be in $p^{-1}(x)$.

$\underline{X_1}$

ρ:

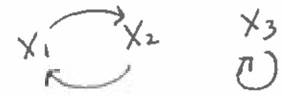


So identity permutation

(1)(2)(3)

$\underline{Y_2}$

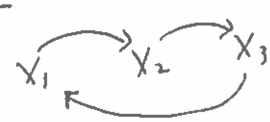
ρ:



(1 2)(3)

$\underline{X_3}$

ρ:



(1 2 3)

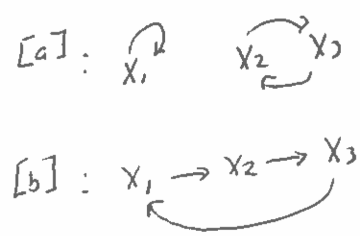
Only permutations on 3 things up to label change.

Idea is that ρ determines $\tilde{X}$.

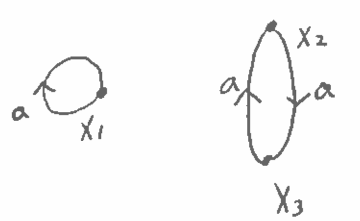
Ex: $X =$ as before

$p^{-1}(x) = \{x_1, x_2, x_3\}$

ρ:

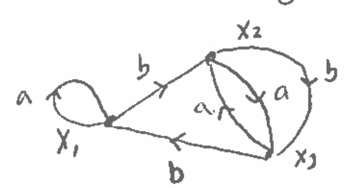


$\tilde{X}$



} From a

Now adding b



Ex:

$X = T^2 =$ 

$p^{-1}(x) = \mathbb{Z} \times \mathbb{Z}$

just a set, could be any set.

$\pi_1(X,x) \cong \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}}$

$\downarrow \rho$

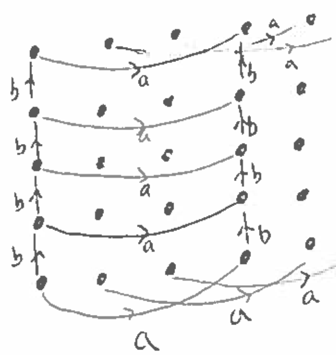
$\underbrace{S}_{\text{permutation group on}}(\mathbb{Z} \times \mathbb{Z} = p^{-1}(x))$

Define $\rho(a) : (n,m) \mapsto (n+3, m)$

$\rho(b) : (n,m) \mapsto (n, m+1)$

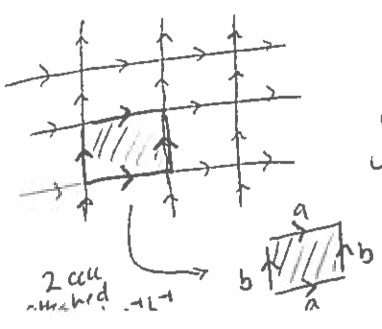
Think about translation to see it is a homomorphism. But what covering space is this?

$\mathbb{Z} \times \mathbb{Z}$



Don't draw all m.

So really only have 3 copies of



So all fills.

$\mathbb{R}^2$

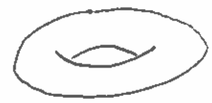
So we have

$\mathbb{R}^2$

$\mathbb{R}^2$

$\mathbb{R}^2$

$\downarrow p$



$p(x,y) = (e^{2\pi i x}, e^{2\pi i y})$

Thm: (Action of $\pi_1(X)$ on fiber $p^{-1}(x)$ Thm)

Let X be "nice" and $p : \tilde{X} \to X$ a covering. $F = p^{-1}(x)$. Then

i) $\exists$ homo. $\rho : \pi_1(X,x) \to \mathrm{Perm}(F)$

ii) ρ determines $p : \tilde{X} \to X$ up to covering space isomorphism.

So more uniqueness of covering than anything

Pf:

1. $\forall \tilde{x} \in F$

$\forall [\gamma] \in \pi_1(X,x)$

$\exists !$ lift $\tilde{\gamma}$ of γ starting at $\tilde{x}$.

$L_{[\gamma]} : F \to F$

$\tilde{x} \mapsto$ end of lift $\tilde{\gamma}$ starting at $\tilde{x}$.

Claims: $L_{[\gamma]}$ perm. (bij. $F \to F$)

Why?: $(L_{[\gamma]})^{-1} = L_{[\bar{\gamma}]} = L_{[\gamma]^{-1}}$ } NTS

Def: $\rho[\gamma] = L_{[\gamma]}$

Claims: $\rho([\gamma]^{-1}) = \rho([\gamma])^{-1}$. So ρ respects inverses.

$L_{[\gamma]\cdot[\eta]} = \underbrace{L_{[\eta]} L_{[\gamma]}}_{\text{b/c functions compose right to left.}}$

Deck Transformations

Given covering space $p: \tilde{X} \to X$, a deck transformation (covering translation) is a self isomorphism $f: \tilde{X} \to \tilde{X}$ such that $pf = p$

* These form a group: $G(\tilde{X})$ (Automorphism). Sometimes called $\underbrace{\text{Deck}}_{\text{deck group}}(\tilde{X})$ or $\mathcal{C}(\tilde{X})$. We use $\mathrm{Aut}(p)$

map, not prime

Remark: $\mathrm{Aut}(p)$ acts on fiber $p^{-1}(x)$ by virtue of $pf = p$.

Ex: $p_n: S \to S$
$z \mapsto z^n$

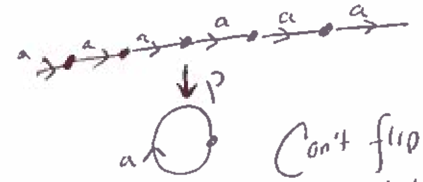
eg $n=3$

$$\underbrace{\mathrm{Aut}(p_n) \cong \mathbb{Z}_n}_{\langle r \rangle \leq D_n}$$

given by rotations $(2\pi/n)$
flips do not respect arrows

Both quotients of $\mathbb{Z}$!

Ex: $p: \mathbb{R} \to S^1$
$\theta \mapsto e^{i\theta}$



$$\underbrace{\mathrm{Aut}(p) \cong \mathbb{Z}}_{\text{Translation by } 2\pi}$$

Can't flip but can shift by any amount

Prop: Given an $f \in \mathrm{Aut}(p)$, f is completely determined by value at a single point.

Compare with giving single perm (perm) in S_n

Pf:
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, dashed, "f"] \arrow[r, "p"'] & X \end{tikzcd}$$
Lifting Lem.

$f \in \mathrm{Aut}(p)$ is a lift of p.
That is, $pf = p$. Lifts are equal if the agree at one point. ∎

Prop: Let $\tilde{x}$ & $\tilde{x}'$ be lifts of x.
If $p_*(\pi_1(\tilde{X}, \tilde{x})) = p_*(\pi_1(\tilde{X}, \tilde{x}'))$ then $\exists!$ $f \in \mathrm{Aut}(p)$ $\ni$ $f(\tilde{x}) = \tilde{x}'$.

Pf: Lifting lem & uniqueness of lifts.

Cor: The only automorphism of $p: \tilde{X} \to X$ that fixes a point is $\mathbb{1}_{\tilde{X}}$.

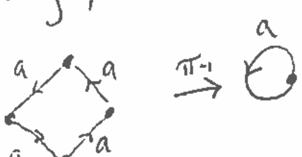
02/16/2015

Rem: $Aut(p)$ acts on fiber $p^{-1}(x)$.

Transitivity

Def: Given a covering space $\tilde{X} \xrightarrow{p} X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$. (for all x)
If for all such pairs $\exists$ $f \in Aut(p) \ni f(\tilde{x}) = \tilde{x}'$
(ie $Aut(p)$ acts trans. on fiber)
we call p a regular/normal covering space.

Ex: $\mathbb{R} \xrightarrow{p} S$
$S \xrightarrow{p} S$
Covers of S are normal. (?)✓

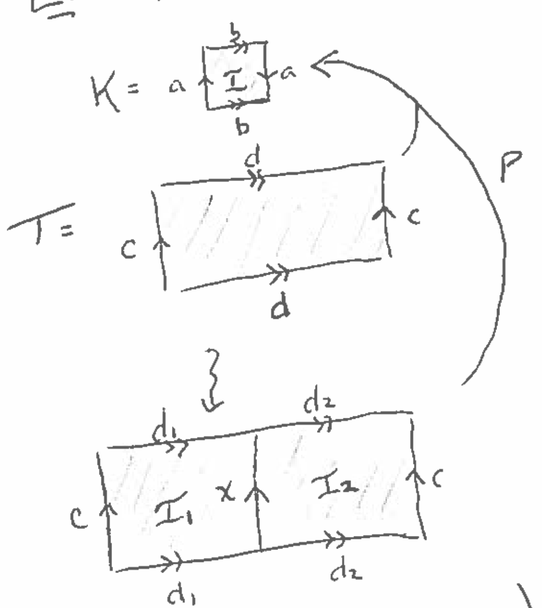


$Aut(p_4) = \mathbb{Z}/4\mathbb{Z}$
acts trans. on $p^{-1}(x)$.

P_4 is a normal covering

(Ex) Thm: X & $\tilde{X}$ are path connected, then $Aut(p)$ acts trans. on all fibers if it acts trans. on one fiber.

Ex: $T^2 \xrightarrow{p} K$ (2-1 cover)



p: $\vec{c}$ onto $\hat{a}$ (in directed sense)
$\vec{x}$ onto $-\vec{a}$

Left: $\vec{d_1}, \vec{d_2}$ onto $\vec{b}$
$I_1 \to I$

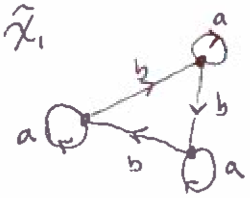
Right: $\vec{x}$ onto $\vec{a}$
$\vec{d_1}, \vec{d_2}$ onto $\vec{b}$
$\vec{c}$ to $-\vec{a}$
$I_2 \to I$

Is this map normal?

Ex:

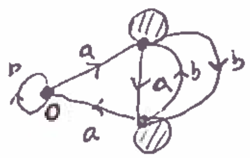
$X =$ 

$\tilde{X}_1$



$Aut(p_1) = \mathbb{Z}/3\mathbb{Z}$
rotation by $2\pi/3$
Act transitely on $p_1^{-1}(x)$

LaTeX
\wedge $\wedge$
\vee $\vee$

$\tilde{X}_2 =$ 

If $f \in Aut(p_2)$, then
$f(0) = 0$
since only one loop b in graph.
By prev. cor., fixes point
so $Aut(p_2) = \mathbb{1}$
So does not act transitely on a fiber. So this cannot be normal.

* Notice the symmetry here (but need symmetry respecting loops).

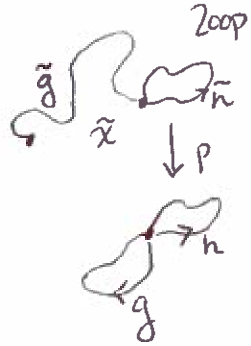
Prop: X "nice"
$p: \tilde{X} \to X$ a covering map
$\tilde{x} \in p^{-1}(x)$
$H = p_*(\pi_1(\tilde{X}, \tilde{x}))$
$G = \pi_1(X, x)$
(so $H \leq G$)

i) $H \trianglelefteq G$ iff $\tilde{X}$ is a normal cover.
ii) $Aut(p) \cong N(H)/H$
normalizer of H
ie largest subgroup of G in which H is normal.

Cor:
i) $Aut(p) \cong G/H \Leftrightarrow H \trianglelefteq G$
ii) $Aut(\text{univ. cover}) \cong G$
$Aut(X)$

* Think about this deep connection between paths & automorphisms.

"Pf": Normal cover $\to H \trianglelefteq G$
i) Take $[h] \in H$
$[g] \in G$
WTS $[g][h][g]^{-1} \in H$
$[h] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$ lifts to loop $\tilde{h}$ at $\tilde{x}$.

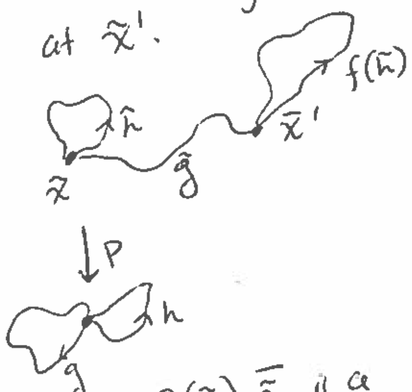


$\tilde{g}$ path at $\tilde{x}'$

$\tilde{h}$ loop

By assumption, $\exists f \in Aut(p)$
$\ni f(\tilde{x}) = \tilde{x}'$

Furthermore,
$f(\tilde{h})$ is a loop at $\tilde{x}'$.



So $\tilde{g} f(\tilde{h}) \bar{\tilde{g}}$ is a loop at $\tilde{x}$. Then

$p_*([\tilde{g}][f(\tilde{h})][\tilde{g}]^{-1}) =$
$[g][h][g^{-1}]$ } using fact $pf = p$ for $[f(\tilde{h})]$

So $H \trianglelefteq G$

Other direction same idea and use lifting correspondence.

ii) Construct homo.

$\varphi : N(H) \to Aut(p)$

$[g] \mapsto f \in Aut(p)$

where f takes $\tilde{x}$ to $\tilde{x}'$ end of $\tilde{g}$ lift of g at $\tilde{x}$.

Why does such an f exist? (Hm.)

Why does this work in $N(H)$ and not G? (Hm.)

Check: Surjective with Kernel H. Follows then by 1st Iso. Thm.

Ex:

$S = M_{2k+1}$, $\Sigma = M_2$ | $S = M_k$; $k \geq 2$, $\Sigma = M_2$

$k = 6$:  Cuts. Glue cuts to get back to Σ.

$\downarrow p$

$\Sigma =$  Cut here

$M_k \xrightarrow{p} M_2$

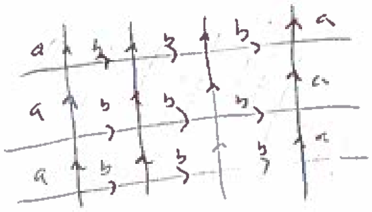
$\forall k \geq 2$.

$Aut(p_k) = \mathbb{Z}/\underbrace{k-1}_{\text{rotations}}$

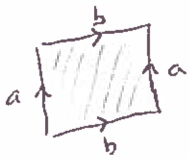
Could find
$p_* \pi_1(S) < \pi_1(\Sigma)$
$\uparrow$
Since trans., we know $\trianglelefteq$.

$[\pi_1(\Sigma) : p_* \pi_1(S)] = k-1$

Ex: $\mathbb{R}^2 \to T^2$



$\big\downarrow p$



$\mathrm{Aut} \cong \pi_1(T^2)/\mathbb{1} \cong \mathbb{Z} \oplus \mathbb{Z}$

$\tau : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (x+1, y)$

$\sigma : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (x, y+1)$

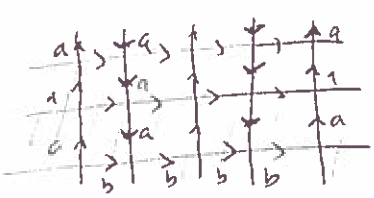
$\mathbb{Z} \oplus \mathbb{Z}$

$= \langle \sigma \rangle \oplus \langle \tau \rangle$

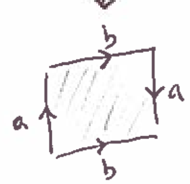
$*$
$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dl, "p_T"'] \arrow[dr, "p_K"] & \\ T^2 \arrow[rr, "2:1"] & & K \end{tikzcd}$$

$[\mathrm{Aut}(p_K) : \mathrm{Aut}(p_T)] = 2$

Ex:



$\big\downarrow p$



$\mathrm{Aut}(p) = \underbrace{\pi_1(K)}_{\text{not abelian}}$

$\tau : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (x, y+1)$

$\sigma : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (x+1, -y)$

τ, σ do not commute but are generators (as trans. on fiber)

02/18/2015

Quotients of Abelian Groups

use this to move towards Homology

$$\underbrace{X}_{\text{space}} \rightsquigarrow \underbrace{H_n(X)}_{\substack{n \geq 0 \\ \text{abelian groups}}}$$

Thm: Any finitely generated free abelian group is $\cong \mathbb{Z}^n$ for some $n < \infty$. n is called the rank.

Any subgroup of $\mathbb{Z}^n$ is $\cong \mathbb{Z}^m$ for some $m \leq n$.

If you have a linear map
$$f: \mathbb{Z}^m \to \mathbb{Z}^n$$
Choosing basis for each, can represent f by a matrix. $f(\vec{x}) = M\vec{x}$ where M is a $n \times m$ matrix over $\mathbb{Z}$.

Def: Let G be a fin. gen. free abelian group $\mathbb{Z}^n$. An element $g \in G$ is called primitive if $g = nx$ then $n = \pm 1$

$n \in \mathbb{Z}$

$x, g \in G$

$1, 1^{-1}$

Ex: In $\mathbb{Z}^2$, the element $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is not primitive
$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
However, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is primitive. Another example is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Here, only gcd matters.

Thm: Let $x \in \mathbb{Z}^n$ be primitive, then $\{x\}$ can be extended to a basis for $\mathbb{Z}^n$.

Ex: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ primitive and extends to basis, e.g.
$$\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

"Pf": Essentially Euclidean Alg. on $x = (a_1, \cdots, a_n)$
$$\rightsquigarrow (1, 0, 0, \cdots, 0)$$

Thm: If $y = mx$ and x is primitive in $\mathbb{Z}^n$, then
$$\mathbb{Z}^n / \langle x \rangle \cong \mathbb{Z}^{n-1}$$

Pf: Follows directly from prev. thm.

and $\mathbb{Z}^n / \langle y \rangle \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}/m$

Pf: x "=" $(1, 0, \cdots, 0)$

basis $\mathbb{Z}^n = \{e_1, \cdots, e_n\}$

Killing of mx does not kill e_1. Leaves $e_1, 2e_1, \cdots, (m-1)e_1$ alive.

Ex: $\mathbb{Z}^2/\langle\binom{2}{3}\rangle \cong \mathbb{Z} = \langle\binom{1}{2}\rangle$

$$\mathbb{Z}^2/\langle\binom{2}{4}\rangle \cong \underset{\binom{2}{3}}{\underset{\uparrow}{\mathbb{Z}}} \oplus \underset{\binom{1}{2}}{\underset{\uparrow}{\mathbb{Z}/2\mathbb{Z}}}$$

Ex: 2 primitive elements need not extend to a basis

$\binom{1}{3}, \binom{2}{1}$

these primitive in $\mathbb{Z}^2$.

If extends to basis, must be

$\{\binom{1}{3}, \binom{2}{1}\}$

but this is not a basis as

$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5 \neq \pm 1$

Verify $\mathbb{Z}^2/\langle\binom{1}{3},\binom{2}{1}\rangle \cong \mathbb{Z}/5\mathbb{Z}$

Example: $G = \mathbb{Z}^3/\langle \underbrace{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}}_{\text{lin. indep \& primitive}} \rangle$

Does this have torsion?

$G \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$G = \operatorname{coker}\begin{pmatrix}1 & 1\\ 1 & -1\\ -1 & 1\end{pmatrix}$

$= \mathbb{Z}^3/\text{col space}\begin{pmatrix}1 & 1\\ 1 & -1\\ -1 & 1\end{pmatrix}$

Recall: $\operatorname{coker} = \overset{Y}{\operatorname{im}}/\operatorname{im} f$

$f: X \to Y$

Know $f: \mathbb{Z}^2 \to \mathbb{Z}^3$ given by matrix $\begin{pmatrix}1 & 1\\ 1 & -1\\ -1 & 1\end{pmatrix}$

Know "Y". Need to find $\operatorname{im} f$. Then $G = \mathbb{Z}^3/\operatorname{im} f$

Operations need be in $\mathbb{Z}$!

Reduce $\begin{pmatrix}1 & 1\\ 1 & -1\\ -1 & 1\end{pmatrix}$

add colums

$\begin{pmatrix}2 & 1\\ 0 & -1\\ 0 & 1\end{pmatrix}$

add rows

$\begin{pmatrix}2 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$

$\operatorname{im} f = \text{col. space} = \left\langle \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

So

$G = \operatorname{coker} f = \mathbb{Z}^3/\operatorname{im} f$

$= \langle e_1, e_2, e_3\rangle / \langle 2e_1, e_2\rangle$

$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

There is a change of basis in domain corresponding to column operation(s).

There is a change of basis in codomain corresponding to row operation

What is the change of basis.

Column operations

Ex: $f: \mathbb{Z}^3 \to \mathbb{Z}$

$\{x, y, z\}$ basis $\quad \{1\}$

$f(x) = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} x$

ie $x \mapsto 1$
$y \mapsto -1$
$z \mapsto 2$

Swap columns

$\begin{pmatrix} -1 & 1 & 2 \end{pmatrix}$. New basis?

$\underline{y} \mapsto -1$
$\underline{x} \mapsto 1$
$\underline{z} \mapsto 2$

swap column $\rightleftharpoons$ swap elements in basis.

(Column 3) $\cdot -1$

$(1, -1, -2)$

$\underline{x} \longmapsto 1$
$\underline{y} \longmapsto -1$
$\underline{-z} \longmapsto -2$

Mult column by $c \rightleftharpoons$ mult. basis el by -1.

$2(\text{colm } 2) + (\text{col } 3)$

$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$

$\underline{x} \mapsto 1$
$\underline{y} \mapsto -1$
$\underline{2y+z} \mapsto 0$

add mult. colu $\rightleftharpoons$ add mult. basis.

Row Operations

1. Swap 2 rows $\rightleftharpoons$ swap element in basis
2. Mult basis el $\rightleftharpoons$ mult. element in basis
3. Lin sum in row $\rightleftharpoons$ replacing y with ...

$f: \mathbb{Z} \to \mathbb{Z}^2$

$\{1\} \quad \{x, y\}$

$v \mapsto \begin{pmatrix} 3 \\ -1 \end{pmatrix} v$

$1 \mapsto 0x' - 1y' = 3x - 1y$

$x' = ax + by \qquad c = -3$
$y' = cx + dy \qquad d = 1$

$\begin{vmatrix} a & b \\ -3 & 1 \end{vmatrix} = a + 3b$

Take $a = 1, b = 0$

$x' = x$
$y' = -3x + y$

Ex: $f: \mathbb{Z}^2 \to \mathbb{Z}^3$

Basis: $\{x, y\}$ $\{a, b, c\}$

f:

$x \mapsto a+b-c$

$y \mapsto a-b+c$

$$\begin{array}{c|cc} & x & y \\ \hline a & 1 & 1 \\ b & 1 & -1 \\ c & -1 & 1 \end{array}$$

$\downarrow$

$$\begin{array}{c|cc} & x+y & y \\ \hline a & 2 & 1 \\ b & 0 & -1 \\ c & 0 & 1 \end{array}$$

$\downarrow$

$$\begin{array}{c|cc} & x+y & y \\ \hline a & 2 & 1 \\ b-c & 0 & -1 \\ c & 0 & 0 \end{array}$$

$\downarrow$

$$\begin{array}{c|cc} & x+y & y \\ \hline a & 2 & 0 \\ b-c-a & 0 & -1 \\ c & 0 & 0 \end{array}$$

$\downarrow$

$$\begin{array}{c|cc} & x+y & y \\ \hline a & 2 & 0 \\ a+c-b & 0 & 1 \\ c & 0 & 0 \end{array}$$

$\operatorname{im} f = \text{col space} = \langle 2a, a+c-b \rangle$

$(= \langle a+b-c, a-b+c \rangle)$

$\operatorname{Ker} f = 0$

$\Leftarrow$ rank nullity Thm.

or $\{x+y, y\}$ basis $x+y \not\mapsto 0$, $y \not\mapsto 0$ so no lin. comb. goes to 0.

*Think Smith Normal Form!

$\mathbb{Z}^2 / \operatorname{Ker} f = \mathbb{Z}^2/0$

$= \langle x+y, y \rangle / 0$

$\cong \langle x+y, y \rangle$

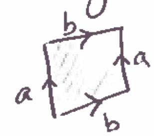
While

$\operatorname{Coker} f = \mathbb{Z}^3 / \operatorname{im} f = \langle a, a+c-b, c \rangle \big/ \langle 2a, a+c-b \rangle$

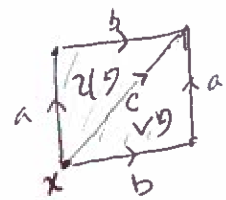
$\cong \underset{\text{gen. by } a}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{\text{gen by } c}{\mathbb{Z}}$

Ex: (Looking way ahead)

T^2



Adding a one cell with orientation.



$A = \mathbb{Z}^2 = \langle u, v \rangle$ ~~All~~

$B = \mathbb{Z}^3 = \langle a, b, c \rangle$

$C = \mathbb{Z} = \langle x \rangle$

$f: A \to B$ $\quad f(u) = c-b-a$

$g: B \to C$ $\quad f(v) = b+a-c$

$g(a) = x-x = 0$

$g(b) = 0$

$g(c) = 0$

So g is zero map.

$\operatorname{im} f \le B$. $g(\operatorname{im} f) = 0$ so

$\operatorname{im} f \le \operatorname{Ker} g$

$$f: \begin{array}{c|cc} & u & v \\ \hline a & -1 & 1 \\ b & -1 & 1 \\ c & 1 & -1 \end{array} \rightsquigarrow \begin{array}{c|cc} & u+v & v \\ \hline a & 0 & 1 \\ b & 0 & 1 \\ c & 0 & -1 \end{array}$$

$$\begin{array}{c|cc} & u+v & v \\ \hline a-c & 0 & 1 \\ b & 0 & 1 \\ c & 0 & 0 \end{array} \rightarrow \begin{array}{c|cc} & u+v & v \\ \hline a+b-c & 0 & 1 \\ b & 0 & 0 \\ c & 0 & 0 \end{array}$$

So $\operatorname{im} f \cong \mathbb{Z} = \langle a+b-c \rangle$

$\operatorname{Ker} f / \operatorname{im} f = \langle a, b, c \rangle / \langle a+b-c \rangle$

$= \langle a, b, a+b-c \rangle / \langle a+b-c \rangle$

$= \underbrace{\mathbb{Z}}_{a} \oplus \underbrace{\mathbb{Z}}_{b}$

We computed $H_1(T^2) = \operatorname{Ker} g / \operatorname{im} f$

$= \underbrace{\mathbb{Z}}_{a} \oplus \underbrace{\mathbb{Z}}_{b}$

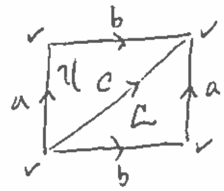
Inside $\operatorname{Ker} g / \operatorname{im} f = H_1(T^2)$

$c \equiv a+b$

See above $a+b-c \approx 0$.

02/23/2015

T^2



$$\underset{\langle U, L \rangle}{\mathbb{Z}^2} \xrightarrow{f} \underset{\langle a, b, c \rangle}{\mathbb{Z}^3} \xrightarrow{g} \underset{\langle v \rangle}{\mathbb{Z}}$$

$f = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad \operatorname{im} f \subset \operatorname{Ker} g$

$g = 0 \qquad c = a+b$

Homology

$H_1(T^2) = \operatorname{Ker} g / \operatorname{im} f$

$= \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}}$

Def: Suppose C_n is a collection of free abelian groups and $\partial_n : C_n \to C_{n-1}$ are homomorphisms. Then $\{C_n, \partial_n\}_{n \geq 0}$ is a chain complex over $\mathbb{Z}$ if $\underbrace{\partial_{n-1} \partial_n}_{\partial^2} = 0 \ \forall n \geq 0$.

$\operatorname{im} \partial_n \subset \operatorname{Ker} \partial_{n-1}$

So can take quotient

$H_n(C_*) = \operatorname{Ker} \partial_n / \operatorname{im} \partial_{n+1} = Z_n / B_n$

$a \in \underset{\overset{\shortparallel}{Z_n}}{\operatorname{Ker} \partial_n}$ is an n-cycle

$b \in \underset{\overset{\shortparallel}{B_n}}{\operatorname{im} \partial_{n+1}}$ is an n-boundary

$x \in H_n(C_*)$ is an n-dim. homology class.

∂ is boundary map or boundary homomorphism.

Functional Goal

$$\text{Top} \rightsquigarrow \underbrace{\text{Chain Complex}}_{\text{Homology}}$$

Homology $=$ **Abelian Groups** H_n ; $n \geq 0$

Simplices

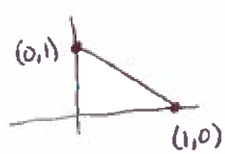
Def: The standard n-simplex

$$\Delta^n = \left\{ (t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum t_i = 1,\ t_i \geq 0 \right\}$$

In $\mathbb{R}^1$, we have $\Delta^0 = \underset{1}{\bullet}$

In $\mathbb{R}^2$, we have

$$\Delta^1 = \left\{ (1,0), (0,1), \underbrace{\cdots}_{\substack{\text{all points} \\ \text{"in-between"}}} \right\}$$



Def: Let $v_0, \cdots, v_n$ be some points in $\mathbb{R}^m$ such that $v_1 - v_0, v_2 - v_0, \cdots, v_n - v_0$ are linearly independent. The ordered n-simplex with these $v_0, \cdots, v_n$ as vertices is

$$[v_0, \cdots, v_n] = \underbrace{\left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; \sum t_i = 1,\ t_i \geq 0 \right\}}_{\text{Convex Hull}}$$

The ith face is $[v_0, \cdots, \hat{v}_i, \cdots, v_n] = [v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n]$ ordered $(n-1)$-simplex

$= [v_0, v_1, v_2]$

$[v_0, v_1] = v_0 \bullet\!\!\longrightarrow\!\bullet\, v_1$

Ex:

$\{ \bullet v_0$ ordered 0-simplex

$\{ v_0 \bullet\!\!\longrightarrow\!\bullet\, v_1$ ordered 1-simplex $[v_0, v_1]$

$[v_0, v_1, v_2]$

3 faces: $[v_0, v_1]$, $[v_0, v_2]$, $[v_1, v_2]$

Exercise: Draw ordered 4-simplex and all ordered faces.

Def: Given an ordered n-simplex $S = [v_0, \cdots, v_n]$, the algebraic boundary

$$\partial_n S = \sum_{i=0}^{n} (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

Prop: $\partial_{n-1}\partial_n = 0$

Pf:

$$\partial_{n-1}\partial_n S = \partial_{n-1} \sum_0^n (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$
$$= \sum_0^n (-1)^i \partial_{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n]$$
$$= \sum_{i=0}^n (-1)^i \left(\sum_{j<i} (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] \right)$$
$$+ \sum_{i=0}^n (-1)^i \left(\sum_{j>i} (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n] \right)$$

$j-1^{th}$ spot

Swap j & i and everything cancels

$= 0$ ∎

We want to "fill space" in such a way so that the above proof works.

Def: Let X be a space. A Δ-complex (Δ-structure) on X is a collection of simplices (maps of)
$$\mathcal{A} = \{ \sigma_\alpha : \Delta^{n_\alpha} \to X \}_{\alpha \in \Lambda}$$
such that

i) "Injectivity": $\forall \sigma_\alpha \in \mathcal{A}$ $\sigma_\alpha|_{\mathring{\Delta}^{n_\alpha}}$ is injective

ii) "Coverage": $\forall x \in X$, $\exists! \sigma_\alpha \in \mathcal{A} \ni x \in \operatorname{im} \sigma_\alpha|_{\mathring{\Delta}^{n_\alpha}}$

iii) (For $\partial^2 = 0$) $\forall \sigma_\alpha \in \mathcal{A}$, each "face" $\sigma_\alpha|_{\text{Face } \Delta^{n_\alpha}} \in \mathcal{A}$

iv) $\forall A \subset X$, A open $\iff \sigma_\alpha^{-1}(A)$ is open. $\forall \sigma_\alpha \in \mathcal{A}$.

Remark

i) $X \cong \coprod_{\sigma_\alpha \in \mathcal{A}} \Delta^{n_\alpha} / \sim$

$\sim$ given by σ_α

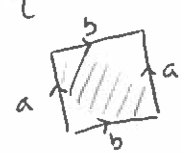
ii) Any Δ-complex is a CW complex (but not other way around).

Trick: $\Delta^n \cong D^n$

$\mathring{\Delta}^n \cong \mathring{D}^n$

$\partial \Delta^n \cong \partial D^n$

Converse Counterexample

T^2  This is not a Δ. ∎

Question: Given X, how do we find a Δ-structure?

Ex: $X = \bullet$

$\mathcal{A} = \{ \Delta^0 \to X \}$

Ex: $X = S^2 = \partial D^2 = \partial \Delta^2$

$\mathring{\Delta}^2 \cong S^1$ $\quad \mathcal{A}' = \left\{ \mathcal{A} \text{ from } \Delta^n \,\middle|\, \Delta^n \xrightarrow{id} \Delta^n \right\}$

For S^{n-1}

Ex: $X = D^n \cong \Delta^n$

$\mathcal{A} = \{ \Delta^n \xrightarrow{id} D^n$, all faces, faces of faces, $\dots \}$

Ex:

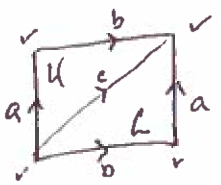
$S^1 =$  1 in $\mathbb{C}$

$A = \{ \Delta^0 \xrightarrow{f} S^1, \ \Delta^1 \xrightarrow{g} S^1 \}$, $\Delta^1 = [0,1]$

$f(\cdot) = 1$

$g(\theta) = e^{2\pi i \theta}$

Ex: T^2



0-simplices: $\Delta^0 \to v$

1-simplices: $\Delta^1 \mapsto a$

$\Delta^1 \mapsto b$

$\Delta^1 \mapsto c$

2-simplices: $\Delta^2 \xrightarrow{f} U$

$\Delta^2 \xrightarrow{e} L$

$f \begin{cases} [v_0, v_1] \mapsto a \\ [v_1, v_2] \mapsto b \\ [v_0, v_2] \mapsto c \end{cases}$

$g \begin{cases} [v_0, v_1] \mapsto b \\ [v_0, v_2] \mapsto c \\ [v_1, v_2] \mapsto a \end{cases}$

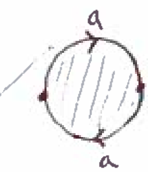
Still need to check topology matches.

02/25/2015

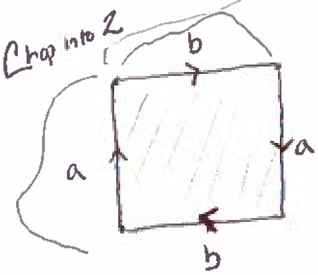
Sym-pligh-al

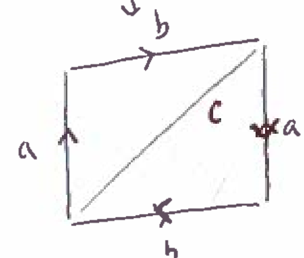
↳ Simplicial Homology

Top ⇝ Δ-complex ⇝ Chain Complex ⇝ Homology

Ex: $\mathbb{RP}(2) =$ 

Chop into 2

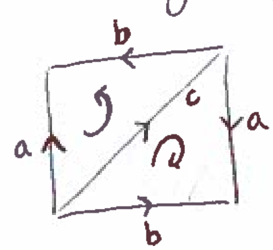




Can't orient c "up" as then forms a loop cab. That cannot be 2-simplex.

Down on c works on lower Δ but not upper ⊕. So doing it this way just won't work.

Need something else.



This one works

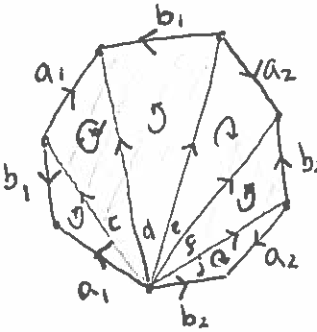
2 0-simplices

3 1-simplices

2 2-simplices

Thm: Any surface M_g or N_g have Δ-complex structures.

Ex: N_2



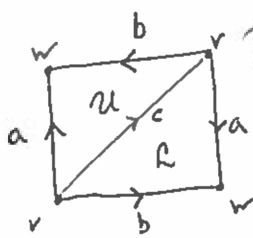
Def: Given a Δ-complex X with $A = \{\sigma_\alpha : \Delta^{n_\alpha} \to X\}_{\alpha \in \Lambda}$ the simplicial chain group (of dimension n) for X is

$$\Delta_n(X) = \text{free abelian group generated by } \sigma_\alpha \in A \text{ where } n_\alpha = n.$$

$$= \left\{ \sum k_\alpha \sigma_\alpha \;\middle|\; \sigma_\alpha \in A,\ n_\alpha = n,\ \text{finite support on } k_\alpha\text{'s.} \right\}$$

integers

$$\approx \bigoplus_{\#n \text{ simplices}} \mathbb{Z}$$



$\mathbb{R}P(2)$

Labeling gives names to the maps.

$$\Delta_0(\mathbb{R}P_2) = \underset{v}{\mathbb{Z}} \oplus \underset{w}{\mathbb{Z}}$$

$$\Delta_1(\mathbb{R}P_2) = \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}} \oplus \underset{c}{\mathbb{Z}}$$

$$\Delta_2(\mathbb{R}P_2) = \underset{u}{\mathbb{Z}} \oplus \underset{l}{\mathbb{Z}}$$

Def: The simplicial boundary map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined by $\sigma : \Delta^n \to X$, $\Delta^n = [v_0, \dots, v_n]$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Prop: $\partial_n \partial_{n+1} = 0$

Pf: Already done, mutatis mutandis.

Def: $\{\Delta_n(X), \partial_n\}_{n \geq 0}$ is the simplicial chain complex of X (associated to A).

The n^{th} simplicial homology group of X is $H_n^\Delta(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

Ex: T^2

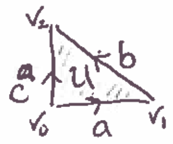
$H_0^\Delta(T^2) \cong \mathbb{Z}$

$H_1^\Delta(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_2^\Delta(T^2) \cong \mathbb{Z}$

$H_3^\Delta(T^2) \cong 0$

Ex:

$X = \Delta^2 =$ 

$\Delta_0(X) = \mathbb{Z}^3$
gen. by $[v_0][v_1][v_2]$

$\Delta_1(X) = \mathbb{Z}^3$
gen. by $\underset{a}{[v_0, v_1]}, \underset{b}{[v_1, v_2]}, \underset{c}{[v_0, v_2]}$

$\Delta_2(X) = \mathbb{Z}$
generated by $\underset{u}{[v_0, v_1, v_2]}$

Hence, see why we use labels for the maps ♡!

Chain Complex

$$\begin{array}{lccccccccc} \text{dim:} & 3 & & 2 & & 1 & & 0 & & -1 \\ & 0 & \to & \mathbb{Z} & \to & \mathbb{Z}^3 & \to & \mathbb{Z}^3 & \to & 0 \end{array}$$

$$\partial_2 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} u \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{array}$$

Comes from $[v_0, v_1]$, $[v_1, v_2]$, $[v_0, v_2]$ not loops

$$\partial_1 = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \end{matrix} & \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

Only 3 choices for each entry.

$$\partial_1 a = \partial_1 [v_0, v_1]$$
$$= (-1)^0 v_1 + (-1)^1 v_0$$
$$= v_1 - v_0$$

$\operatorname{Ker} \partial_0 \cong \mathbb{Z}^3 = \langle v_0, v_1, v_2 \rangle$ as ∂_0 0-map.

$$\partial_1 : \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \end{matrix} & \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

Then

$-C_1 + C_3 \to C_3$

$-C_2 + C_3 \to C_3$

$R_3 + R_2 \to R_2$

$-R_2 + R_1 \to R_1$

$$\begin{array}{c|c} & \begin{matrix} a & b & c-a-b \end{matrix} \\ \begin{matrix} v_0 \\ v_1 - v_0 \\ v_2 - v_1 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Switch to get diagonal.

$\operatorname{im} \partial_1 = \langle v_1 - v_0, v_2 - v_1 \rangle$

$\operatorname{Ker} \partial_1 = \langle c - a - b \rangle$

$$\partial_2 : \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} u \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{array} \rightsquigarrow \begin{array}{c} \\ a \\ b-a \\ a+c \end{array} \begin{array}{c} u \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

or

$$\begin{array}{c} \\ a+b-c \\ b \\ c \end{array} \begin{array}{c} u \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$\operatorname{im} \partial_2 = \langle a + b - c \rangle$

$\operatorname{Ker} \partial_2 = 0$

D^2

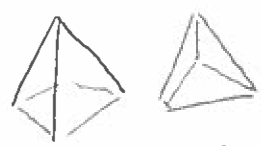
$$H_0(\Delta^2) = \operatorname{Ker} \partial_0 / \operatorname{im} \partial_1 = \langle v_0, v_1, v_2 \rangle / \langle v_1 - v_0, v_2 - v_1 \rangle$$
$$= \langle v_0, v_1 - v_0, v_2 - v_1 \rangle / \langle v_1 - v_0, v_2 - v_1 \rangle$$
$$= \underset{\langle v_0 \rangle}{\mathbb{Z}}$$

$$H_1(\Delta^2) = \ker\partial_1 / \operatorname{im}\partial_2 = \langle c-a-b\rangle \big/ \langle a+b-c\rangle$$
$$= 0$$

$$H_2(\Delta^2) = 0$$

or ≥ 2

Ex: $S^2 = \partial\Delta^3$



Simplicial Chain Complex

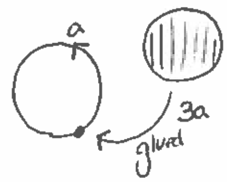
$$0 \to \mathbb{Z}^4 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^4 \to 0$$

or viewing as 2 Δ's glued along boundary

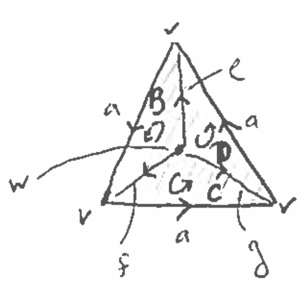
$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to 0$$

Ex: $H_n(S^1)$?

Ex: D_3 = 3-fold Dunce Cap



or



Chain Complex

$$0 \to \underset{B,C,D}{\overset{(2)}{\mathbb{Z}^3}} \xrightarrow{\partial_2} \underset{a,e,f,g}{\overset{(1)}{\mathbb{Z}^4}} \xrightarrow{\partial_1} \underset{v,w}{\overset{(0)}{\mathbb{Z}^2}} \to 0$$

$$\partial_1: \begin{array}{c|cccc} & a & e & f & g \\ \hline v & 0 & 1 & 1 & 1 \\ w & 0 & -1 & -1 & -1 \end{array}$$

$$\begin{array}{c} v \\ w \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c|cccc} & a & e & f-e & g-e \\ \hline v-w & 0 & 1 & 0 & 0 \\ w & 0 & 0 & 0 & 0 \end{array}$$

$$\ker\partial_1 = \langle a, f-e, g-e\rangle$$

$$\operatorname{im}\partial_1 = \langle v-w\rangle$$

Check $H_1(D_3) \cong \mathbb{Z}/3\mathbb{Z}$

03/02/2015

Singular Homology

$H_n^{\Delta}(X)$ good for computation.

Singular Homology $H_n(X)$ good for proving theorems.

Def: A singular n-simplex in a space X is a map $\sigma: \Delta^n \to X$. The group of singular n-chains

$C_n(X)$ = free abelian group gen. by sing. n-chains in X.

$$= \left\{ \sum k_\alpha \sigma_\alpha \;\middle|\; \begin{array}{l} \sigma_\alpha \text{ a sing. } n\text{-simplex in } X \\ k_\alpha = 0 \text{ for all but fin. } \alpha\text{'s} \end{array} \right\}$$

$$\cong \bigoplus_{\#\text{ sing. } n\text{-simplices}} \mathbb{Z}$$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_i$$

$\partial^2 = 0$ (Check)

$\partial_n : C_n \to C_{n-1}$ ✓

So $H_n(X) = \operatorname{Ker} \partial_n / \operatorname{im} \partial_{n+1}$

Thm: If $X \cong Y$, then $H_n(X) \cong H_n(Y)$; $n \geq 0$

* Good for showing gr spaces $\ncong$

Pf: Let $f: X \xrightarrow{\cong} Y$

Then f gives 1-1 correspondence between sing n-simplices in X & sing. n-simplices in Y.

$$\begin{tikzcd} & \Delta^n \arrow[dl, "\sigma"'] \arrow[dr, dashed, "\sigma'"] & \\ X \arrow[rr, shift left, "f"] & & Y \arrow[ll, dashed, shift left, "f^{-1}"] \end{tikzcd}$$

Finish yourself. ■

Thm: Let X be a space with path components $\{X_\alpha\}$ then

$H_n(X) \cong \bigoplus H_n(X_\alpha)$; $n \geq 0$

Pf: $\sigma: \Delta^n \to X$

path connected

So $\operatorname{im} \sigma \subset X_\alpha$ (path connected)

Also $\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} \subset X_\alpha$

$\partial_n \sigma$ is in X_α.

Let $\partial_{n,\alpha} = \partial_n |_{\text{subgroup of } C_n(X) \text{ gen by } n\text{-simplex in } X_\alpha} =: C_n(X_\alpha)$

So $C_n(X) = \bigoplus C_n(X_\alpha)$

and $\partial_n = \bigoplus \partial_{n,\alpha}$

$\operatorname{Ker} \partial_n = \bigoplus \operatorname{Ker} \partial_{n,\alpha}$

$\operatorname{im} \partial_{n+1} = \bigoplus \operatorname{im} \partial_{n+1,\alpha}$

So $H_n(X) = \dfrac{\operatorname{Ker} \partial_n}{\operatorname{im} \partial_{n+1}} = \dfrac{=}{=} = \bigoplus \dfrac{\operatorname{Ker} \partial_{n,\alpha}}{\operatorname{im} \partial_{n+1,\alpha}} = \bigoplus H_n(X_\alpha)$ ■

* Homology "sees" all the path components at the same time whereas the fundamental group does not.

Ex: $X = T^2 \sqcup S^1$

$H_2(X) \cong H_2(T^2) \oplus H_2(S^1)$

But what are these???

Thm: If X is path connected (p.c.) space, then $H_0(X) \cong \mathbb{Z}$

Pf:

$$\cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$$H_0(X) = \operatorname{Ker} \partial_0 / \operatorname{im} \partial_1 = C_0(X) / \operatorname{im} \partial_1$$

$$\begin{array}{ccc} & C_0(X) & \\ \swarrow \pi & & \searrow \epsilon \\ H_0(X) & & \mathbb{Z} \end{array}$$

$\epsilon : C_0(X) \to \mathbb{Z}$ "augmentation map"

$\epsilon\left(\sum k_\alpha \sigma_\alpha\right) = \sum k_\alpha$

Claim: $\operatorname{Ker} \pi = \operatorname{Ker} \epsilon$
$\qquad \operatorname{Ker}\pi = \operatorname{im} \partial_1$

$\operatorname{im} \partial_1 \subset \operatorname{Ker} \epsilon$:

$x = \partial_1 \sum k_\alpha \sigma_\alpha$

$\sigma_\alpha : \Delta^1 \to X$, $\Delta^1 = [v_0, v_1]$

$$x = \sum (-1)^0 k_\alpha \sigma_\alpha|_{[v_1]} + (-1)^1 k_\alpha \sigma_\alpha|_{[v_0]}$$

$$\epsilon(x) = \sum k_\alpha - k_\alpha = 0$$

$x \in \operatorname{Ker} \epsilon$

(works even not path connected)

$\operatorname{Ker} \epsilon \subset \operatorname{im} \partial_1$:

$x = \sum k_\alpha \sigma_\alpha \in C_0(X)$

and $\epsilon(x) = 0$

"induction"

Suppose $k_\alpha > 0$ then must be some $k_\beta < 0$. Take path

$\Gamma_{\alpha\beta} : k_\alpha \to k_\beta$

$\partial \Gamma_{\alpha\beta} = x_\beta - x_\alpha$
$\qquad\qquad \uparrow$ point

and continue to find $\sum$ paths with

$\partial \sum \Gamma_{ij} = x$. ∎

Cor: $H_0(X) = \bigoplus_{\#\text{ path comp of } X} \mathbb{Z}$

Thm: $X = \cdot$. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Pf: Take $\Delta^n \to X$ be n-simplex in X. There is only one, so

$C_n(X) = \mathbb{Z}$

$$\cdots \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_i$$

$$= \sum_{i=0}^{n} (-1)^i \text{ gen. of } C_{n-1}$$

$$= \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \geq 2 \end{cases}$$

$H_n(X)$; $n \geq 1$

$\ker \partial_{2n} = 0$ $\qquad \ker \partial_{2n+1} = \mathbb{Z}$

$\operatorname{im} \partial_{2n+1} = 0$ $\qquad \operatorname{im} \partial_{2n+2} = \mathbb{Z}$

So $H_n(X) \cong 0$ ∎

Def: The augmented chain complex for X is

$$\cdots \to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

hom. degree -1

The red. h. group of X are

$\tilde{H}_n(X)$

Thm:

$\tilde{H}_n(X) \cong H_n(X)$; $n \geq 1$

$H_0(X) \cong \mathbb{Z} \oplus \tilde{H}_0(X)$

Pf: Automatic for $n \geq 1$

$\pi : C_0(X)/\operatorname{im} \partial_1 \to C_0(X)/\ker \varepsilon$

$\operatorname{im} \partial_1 \subset \ker \varepsilon$

$\ker \pi = \ker \varepsilon / \operatorname{im} \partial_1$

Exact seq:

$$\begin{array}{ccccccccc} 0 & \to & \ker \pi & \hookrightarrow & C_0(X)/\operatorname{im}\partial_1 & \xrightarrow{\pi} & C_0(X)/\ker\varepsilon & \to & 0 \\ & & \updownarrow & & \updownarrow & & & & \\ 0 & \to & \tilde{H}_0(X) & \hookrightarrow & H_0(X) & \to & \mathbb{Z} & \to & 0 \end{array}$$

Want to show splits

Take something mapping to $1 \in \mathbb{Z}$ and use generating. obv.

$H_0(X) \cong \mathbb{Z} \oplus \tilde{H}_0(X)$ ∎

Question: How does π_1 relate to H_1?

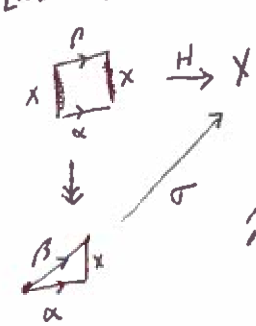
Loops are singular 1-simplex in X.

$\partial \text{loop} = \text{end} - \text{beg} = 0$

So loops $\in \ker \partial_1$

If $\alpha \simeq \beta$ loops. WTS $\alpha - \beta \in \operatorname{im} \partial_2$

so $[\alpha] = [\beta] \in \ker \partial_1 / \operatorname{im} \partial_2 = H_1(X)$



$\exists \sigma : \Delta^2 \to X$

$\partial_2 \sigma = \alpha + x - \beta$

↳ constant 1-simplex @ x.

$\exists\ \sigma_x : \Delta^2 \to X$

Constant map to x

$\partial_2 \sigma_x = x - x + x = x$

So $\partial_2(\sigma - \sigma_x) = \alpha - \beta$

So $\alpha - \beta \in \operatorname{im} \partial_2$

$X \neq \emptyset$ & p.c.

Thm: $\exists$ (Hurewicz) homomorphism

$h : \pi_1(X) \twoheadrightarrow H_1(X)$

and $\ker h = [\pi_1, \pi_1]$, ie

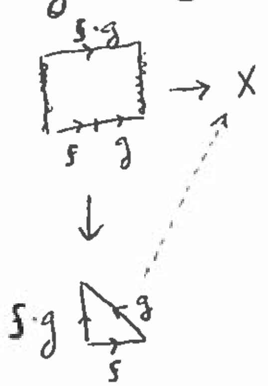
H_1 is the abelianization of π_1.

"Pf": 1. Disguised but verified

$e_x \mapsto [0]$

2. $h([f][g]) = f+g$?

$f \cdot g \simeq f \cdot g$



$\partial_2 \sigma = f + g - f\cdot g$

in H_1

$[f+g] = [f \cdot g]$

Homotopy Invariance

A map $f : X \to Y$ induces a homomorphism

$f_\# : C_n(X) \to C_n(Y)$

for all $n \geq 0$.

For $\sigma : \Delta^n \to X$

$f_\#(\sigma) = f\sigma \in C_n(Y)$

$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$

Prop: $f_\# : C_n(X) \to C_n(Y)$ is a chain map, ie

$\partial f_\# = f_\# \partial$

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1}(X) \arrow[d, "f_\#"] \\ C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \end{tikzcd}$$

Pf: Let $\sigma : \Delta^n \to X$ be a generator of $C_n(X)$.

$$\begin{aligned} \partial f_\# \sigma &= \partial(f\sigma) \\ &= \sum_i (-1)^i (f\sigma)|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \\ &= \sum_i (-1)^i f(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}) \\ &= \sum_i (-1)^i f_\#(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}) \\ &= f_\# \partial \sigma \qquad \blacksquare \end{aligned}$$

$= f_\#(\Sigma \text{—})$

Next, a chain map $\varphi: C_1 \to C_2$

(C_1, C_2 chain cmplx)

Induces a homomorphism

$\varphi_*: H_n(C_1) \to H_n(C_2)$

for all $n \geq 0$

Lem: $f_\#(Z_n(X)) \subset Z_n(Y)$ (actually, could be any chain map)

$f_\#(B_n(X)) \subset B_n(Y)$

for all $n \geq 0$

"Pf": Take $z \in Z_n(X)$, ie

$\partial z = 0$

$f_\#(z) \in C_n(Y)$

$\partial f_\#(z) = f_\# \partial z = f_\# 0 = 0$

So $f_\#(z) \in Z_n(Y)$.

Other follows sim. ∎

Prop: $f_\#$ the chain map induces $f_\#: H_n(X) \to H_n(Y)$ for all $n \geq 0$.

Pf:

$$\begin{array}{ccc} f_\#: Z_n(X) & \xrightarrow{f_\#} & Z_n(Y) \\ \downarrow \pi & & \downarrow \pi \\ Z_n(X)/B_n(X) & \xrightarrow{f_*} & Z_n(Y)/B_n(Y) \end{array}$$

$\operatorname{im} \partial_{n+1} \subset \ker \partial_n$

$B_n \qquad Z_n$

$f_\#(B_n) \subset B_n$ By lem.

By property of quotients, $\exists!\ f_*: Z_n(X)/B_n(X) \to Z_n(Y)/B_n(Y)$

$f_*([\sigma]) = [f(\sigma)]$ ∎

$f: X \to Y$ cont.

$\rightsquigarrow$

$f_\#: C_n(X) \to C_n(Y)$ chain map

$\rightsquigarrow$

$f_*: H_n(X) \to H_n(Y)$ homo.

$f: X \to Y$ cont.

$g: Y \to Z$

Do we have $g_\# \circ f_\# = (g \circ f)_\#$?

$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y \xrightarrow{g} Z$

$(g \circ f)_\#(\sigma) = (gf)\sigma$

$= g(f\sigma)$

$= g_\#(f\sigma)$

$= g_\# \circ f_\# \sigma$

Thm: $X \xrightarrow{f} Y \xrightarrow{g} Z$

then $(g \circ f)_* = g_* \circ f_*$

as maps $H_n(X) \to H_n(Z)$

Furthermore, $id: X \to X$

then $id_* = id: H_n(X) \to H_n(X)$ $\forall n \geq 0$

Furthermoremore, $f: X \xrightarrow{\simeq} Y$, then

$f_*: H_n(X) \xrightarrow{\simeq} H_n(Y)$. $\forall n \geq 0$.

So sing. hom. is a topological invariant.

Homotopy Invariance

Def: Let (C_*, ∂) and (C'_*, ∂') be chain complexes and $f_\#, g_\#$ chain maps from $C_* \to C'_*$. A chain homotopy ($g_\#$ to $f_\#$) is a homo.

$$H: C_* \to C'_*$$

such that

i) $H(C_n) \subset C'_{n+1}$

ii) $\partial' H + H\partial = f_\# - g_\#$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & C_n \arrow[r, "\partial"] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] \arrow[dl, "H"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] \arrow[dl, "H"'] & \cdots \arrow[dl] \\
C'_{n+1} \arrow[r, "\partial'"'] & C'_n \arrow[r, "\partial'"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lem: If $f_\#$ is chain homotopic to $g_\#$, then $f_* = g_*$ as homomorphisms.

Pf: $[\sigma] \in H_n(X)$

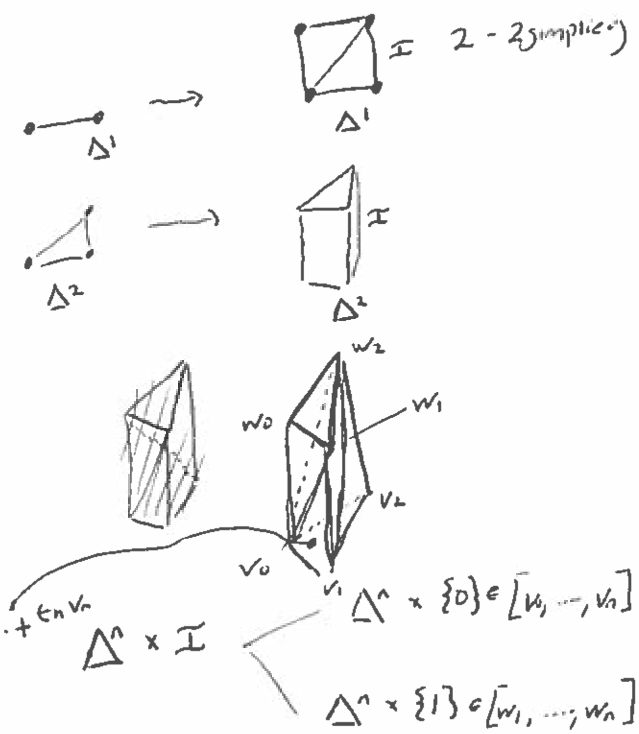
$$\begin{aligned}(f_* - g_*)[\sigma] &= [(f_\# - g_\#)\sigma] \quad \text{sharp} \\ &= [\partial' H\sigma + H\partial\sigma] \\ &= [\partial' H\sigma + H0] \\ &= [\underbrace{\partial' H\sigma}_{\text{Boundary so 0 in homology}}] \\ &= [0]\end{aligned}$$

Lem: Homotopic maps $f \simeq g: X \to Y$ have chain homotopic maps

$$f_\# \simeq_{c.H.} g_\#$$

Pf: Given a homotopy $F: X \times I \to Y$ from g to f. We use F to map n-simplex in X to Σ $(n+1)$-simplices in Y

IDEA:



$\varphi_i: \Delta^n \to I$

$\varphi_i(t_0, \dots, t_n) = t_{i+1} + \dots + t_n$

Exercise: φ_i homeo. onto its ~~image~~ graph.

$[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ is the graph

Linear functions. So only need check "corners"

Ex: $\varphi_1(v_2) = 1$

$v_2 \mapsto (v_2, 1) \text{"="} w_2$

φ_i takes

$v_0, \dots, v_i \mapsto v_0, \dots, v_i$

$v_{i+1} \dots v_n \mapsto w_{i+1}, \dots, w_n$

$\overset{\text{"}}{0}\ \cancel{\varphi_n} \le \varphi_{n-1} \le \dots \le \varphi_1 \le \varphi_0 \le \varphi_{-1} = 1$

"Triangle" move up prism, one corner at a time.

Graphs of φ_i seq. of rising Δ^n's.

Region between φ_i & φ_{i+1} is a $(n+1)$-simplex (add a corner and area in between).

So decomposed $\Delta^n \times I$ as a union of Δ^{n+1}'s glued along Δ^n's.

Define the prism operator $P: C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times id)\Big|_{[v_0 \dots v_i, w_i, \dots w_n]}$$

$$\Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$$

$\Delta^n \xrightarrow{\sigma} X$

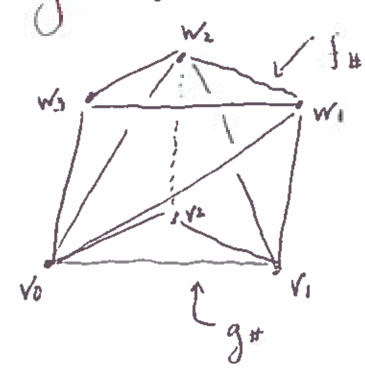
Claim: $\partial P + P\partial = f_\# - g_\#$

$$\underbrace{\partial P}_{\text{boundary of prism}} = \underbrace{f_\#}_{\text{top of prism}} - \underbrace{g_\#}_{\text{bottom of prism}} - \underbrace{P\partial}_{\text{sides}}$$

03/18/2015

Homotopy $F: X \times I \to Y$ from g to f

gave rise to prism operator



$P: C_n(X) \to C_{n+1}(Y)$

satisfying $P\partial + \partial P = f_\# - g_\#$

Thm: If $f \simeq g : X \to Y$ then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

Cor: If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n, where $f_*[\sigma] = [f(\sigma)]$

Ex: $X = \mathbb{R}^k$ (hom. eq. to point)

then $H_n(\mathbb{R}^k) = 0$ for $n > 0$

$= \mathbb{Z}$ for $n = 0$

Exact Sequence

Def: Suppose we have a sequence of groups and homo.

$$\cdots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots$$

is an exact sequence if $\forall n$

$\operatorname{im} f_{n+1} = \operatorname{Ker} f_n$

For hom, only need $\operatorname{im} f_{n+1} \subset \operatorname{Ker} f_n$

So exactness implies trivial homology groups and vice versa.

Prop: Let A, B, C be groups and f, g homo. Then

i) $0 \to A \xrightarrow{f} B$ is exact iff f is a monomorphism

ii) $B \xrightarrow{g} C \to 0$ is exact iff g is an epimorphism

iii) $0 \to A \xrightarrow{f} B \to 0$ is exact iff f is an isomorphism

Short exact sequence (ses) $\to$ iv) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff f mono & g epi & $\operatorname{im} f = \operatorname{ker} g$

Pf: Trivial exercise

Ex: $\mathbb{Z} \overset{g}{\twoheadrightarrow} \mathbb{Z}/5\mathbb{Z}$

So can write

$\mathbb{Z} \overset{g}{\twoheadrightarrow} \mathbb{Z}/5\mathbb{Z} \to 0$

Can make longer

$0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to 0$

That is,

$0 \to 5\mathbb{Z} \hookrightarrow \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to 0$

In general, $M \overset{f}{\twoheadrightarrow} N$ then get ses

$0 \to \operatorname{Ker} f \hookrightarrow M \to N \to 0$

Even better, G & $N \trianglelefteq G$

$0 \to N \hookrightarrow G \to G/N \to 0$

Ex: Suppose

$$0 \to \mathbb{Z} \to B \to \mathbb{Z} \to 0$$

is exact. What is B? B must be $\cong \mathbb{Z} \oplus \mathbb{Z}$

Relative Homology

Suppose $A \subset X$

$i : A \hookrightarrow X$ the inclusion map

i is continuous. So i induces a homo. on homologies.

$$\begin{tikzcd} H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[d, "s"] \\ & \boxed{?} \arrow[d, "g"] \\ H_{n-1}(A) \arrow[r] & H_{n-1}(X) \end{tikzcd}$$

Want to find group ? and maps s, g so that this is exact.

$i_\# : C_n(A) \hookrightarrow C_n(X)$

$\sigma_\alpha : \Delta^n \to A$

Suppose $x = \sum_\alpha n_\alpha \sigma_\alpha \in C_n(A)$ with $i_\#(x) = 0$

$i_\#(x) = \sum_\alpha n_\alpha (i \circ \sigma_\alpha) = 0$

But then $n_\alpha = 0 \ \forall \alpha$ as $i \circ \sigma_\alpha \in C_n(X)$ (free abelian)

So $i_\#$ must be injective. However,

$i_* : H_n(A) \to H_n(X)$ need NOT be an injection.

Def: The group of n-chains of X relative to A, the relative chain group, is

$$C_n(X, A) \overset{\text{def}}{=} C_n(X) / C_n(A)$$

The ∂ map on $C_n(X)$ induces $\bar{\partial}$ on $C_n(X)$

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X) \arrow[d] \\ C_n(X)/C_n(A) \arrow[r, "\exists!\, \bar{\partial}"'] & C_{n-1}(X)/C_{n-1}(A) \end{tikzcd}$$

Prop: $\bar{\partial}^2 = 0$

Pf: Follows easily from

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d] \arrow[rr, bend left] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d] & C_{n-2}(X) \arrow[d] \\ C_n(X)/C_n(A) \arrow[r, "\bar{\partial}"] \arrow[rr, bend right] & C_{n-1}(X)/C_{n-1}(A) \arrow[r, "\bar{\partial}"] & C_{n-2}(X)/C_{n-2}(A) \end{tikzcd}$$

$\blacksquare$

So $C_n(X, A)$ and $\bar{\partial}$ form the relative chain complex and

$$H_n(X, A) \overset{\text{def}}{=} \operatorname{Ker} \bar{\partial} / \operatorname{im} \bar{\partial}_{n+1}$$

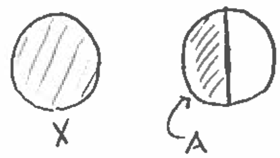
is the n^{th} homology of X relative to A

A relative n-cycle x has $\partial x \in C_{n-1}(A)$
$c \in C_n(X, A)$

A relative n-boundary $c \in C_{n-1}(X, A)$ is $c = \partial b + a$, where $b \in C_n(X)$ and $a \in C_{n-1}(A)$.

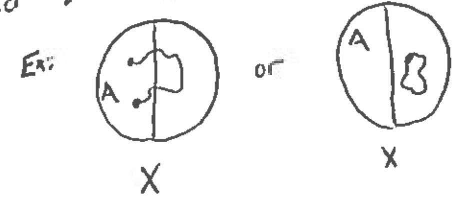
Ex: $X = D^2$

$A = \{(x,y) \in X \mid x \leq 0\}$

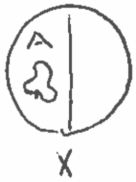


Relative 1-cycle if Σ 1-simplices and $\partial \subset A$.

Ex:



or



Thm: Suppose $A \subset X$.
Then $\exists$ long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

So this fills in the $\boxed{?}$ and the maps from before:

i inclusion

$j: (X, \phi) \to (X,A)$ inclusion
(ϕ empty set)

∂ is connecting map from snake lemma.

Snake Lemma:

Suppose M', M, M'' abelian groups
N, N', N''

f, g, f', g' homo. with exact rows and commuting diagram

$$\begin{array}{ccccccc} & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \to 0 \\ & & \downarrow d' & & \downarrow d & & \downarrow d'' & \\ 0 & \to & N' & \xrightarrow[f']{} & N & \xrightarrow[g']{} & N'' & \end{array}$$

Then there is a short exact sequence

$$\ker d' \xrightarrow{\bar{f}} \ker d \xrightarrow{\bar{g}} \ker d'' \xrightarrow{\delta} \operatorname{coker} d' \xrightarrow{\bar{f'}} \operatorname{coker} d \xrightarrow{\bar{g'}} \operatorname{coker} d''$$

where $\bar{\ }$ = induced map and $\partial = (f')^{-1} d g^{-1}$

"Pf:" We only bother to define ∂

$\partial : \ker d'' \to \operatorname{coker} d'$
$\hookrightarrow N'/\operatorname{im} d'$

Let $a \in \ker d''$. So $d''(a) = 0$

g surj., so choose $m \in M \ni g(m) = a$

$0 = d''(g(m))$. But diagram commutes so

$0 = d''(g(m)) = g'(d(m))$

ie $d(m) \in \ker g'$. Row exact $\ker g' = \operatorname{im} f$

Unique as f' inj. → Choose $c \in N' \ni f'(c) = d(m)$.

Define $\partial(a) = [c]$. So need check choice independent of m. $\blacksquare$

Pf: (Thm) This works $\forall n$. Choose one n.

$$0 \to C_n(A) \overset{i_\#}{\hookrightarrow} C_n(X) \xrightarrow{j_\#} C_n(X,A) \to 0$$

The above sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, hook, "i_\#"] \arrow[d, "\partial_A"] & C_n(X) \arrow[r, "j_\#"] \arrow[d, "\partial_X"] & C_n(X,A) \arrow[r] \arrow[d, "\bar\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

exact

Check diagram commutes. Apply Snake lemma. Get

exact sequence

$$\underbrace{\ker \partial_A \xrightarrow{i_\#} \ker \partial_X \xrightarrow{j_\#} \ker \bar\partial}_{\text{exact}}$$

$\ker \partial_A = Z_n(A)$, $\ker \partial_X = Z_n(X)$, $\ker \bar\partial = Z_n(X,A)$

$$\underbrace{\to \operatorname{coker} \partial_A \xrightarrow{i_\#} \operatorname{coker} \partial_X \xrightarrow{j_\#} \operatorname{coker} \bar\partial}_{\text{exact}}$$

$\operatorname{coker} \partial_A = C_{n-1}(A)/B_{n-1}$, $\operatorname{coker} \partial_X = C_{n-1}(X)/B_{n-1}$, $\operatorname{coker} \bar\partial = C_{n-1}(X,A)/B_{n-1}$

So...

show $j_\#$ surj.

$$\begin{tikzcd}
 & C_{n-1}(A)/B_{n-1} \arrow[r, "i_\#"] \arrow[d, "\partial_A"] & C_{n-1}(X)/B_{n-1} \arrow[r, "j_\#"] \arrow[d, "\partial_X"] & C_{n-1}(X,A)/B_{n-1} \arrow[r] \arrow[d, "\bar\partial"] & 0 \text{ exact} \\
0 \arrow[r] & Z_{n-2}(A) \arrow[r, "i_\#"] & Z_{n-2}(X) \arrow[r, "j_\#"] & Z_{n-2}(X,A) &
\end{tikzcd}$$

exact

show $i_\#$ inj.

Apply Snake lemma again!

$$\ker \partial_A \xrightarrow{i_\#} \ker \partial_X \xrightarrow{j_\#} \ker \bar\partial$$

$\ker \partial_A = Z_{n-1}(A)/B_{n-1}(A) = H_{n-1}(A)$

$\ker \partial_X = Z_{n-1}(X)/B_{n-1}(X) = H_{n-1}(X)$

$\ker \bar\partial = Z_{n-1}(X,A)/B_{n-1}(X,A) = H_{n-1}(X,A)$?

$$\to \operatorname{coker} \partial_A \xrightarrow{i_*} \operatorname{coker} \partial_X \xrightarrow{j_*} \operatorname{coker} \bar\partial$$

$\operatorname{coker} \partial_A = Z_{n-2}(A)/B_{n-2}(A) = H_{n-2}(A)$

$\operatorname{coker} \partial_X = Z_{n-2}(X)/B_{n-2}(X)$

$\operatorname{coker} \bar\partial = Z_{n-2}(X,A)/B_{n-2}(X,A) = H_{n-2}(X,A)$

03/23/2015

Relative Homology

$0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0$

for $A \subset X$. Get a long exact seq. on homology

$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*}$
$H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$
$H_0(A) \to H_0(X) \to H_0(X,A) \to 0$

The l.e.s. for relative homology is natural:
If $f: (X,A) \to (Y,B)$ is a map of pairs,
then

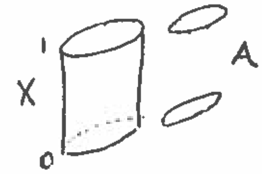
$$\begin{array}{ccccc} H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) \\ f_* \downarrow & \circlearrowleft & \downarrow f_* & \circlearrowleft & \downarrow f_* \\ H_n(B) & \xrightarrow[i_*]{} & H_n(Y) & \xrightarrow[j_*]{} & H_n(Y,B) \end{array}$$

What is δ doing? Came from Snake Lemma.

$$\begin{array}{ccccccccc} 0 \to & C_n(A) & \to & C_n(X) & \to & C_n(X,A) & \to 0 \\ & \downarrow \partial_A & & \downarrow \partial_X & & \downarrow \bar{\partial} & \\ 0 \to & C_{n-1}(A) & \to & C_{n-1}(X) & \to & C_{n-1}(X,A) & \to 0 \end{array}$$

$\partial: H_n(X,A) \to H_{n-1}(A)$ is essentially just ∂_X.

Ex: $X = D^2 \times I$
$A = D^2 \times \partial I$



$\partial: Z_1(X,A) \to \text{coker}$ —

$\sigma \in Z_1(X,A)$

Take $\sigma: \Delta \to X$ given by
$\sigma(t) = (0,t)$

Is it a relative 1-cycle?
$\partial_X \sigma = \sigma|_{0\text{-face}} - \sigma|_{1\text{-face}}$
$= v - u \in C_0(A)$

So $\sigma \in Z_1(X,A)$

$\partial: H_1(X,A) \to H_0(A)$
$\partial[\sigma] = [\partial_X \sigma]$
$= [v-u]$
$= [v] - [u] \in H_0(A)$

Ex: $X = D^2$; $A = \partial D^2 = S^1$
$\sigma: \Delta^2 \to D^2$ a homeo.

$\partial_X \sigma \in C_1(A)$, $\sigma \in Z_2(X,A)$

$\partial[\sigma] = [\partial_X \sigma] = [\text{triangle } A] \in H_1(A)$

Thm: Let $B \subset A \subset X$.

We call (X,A,B) a triple.

Then $\exists$ a (natural) l.e.s. for relative homology.

$$\cdots \to H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$$

Cor: If X is path connected and $A \neq \emptyset$ then $H_0(X,A) = 0$

Pf: Let $B = \overset{a}{pt} \in A$

Then (X,A,B) triple

$$\cdots \to \underset{\underset{\tilde{H}_0(A)}{\cong}}{H_0(A,B)} \xrightarrow{i_*} \underset{\underset{\tilde{H}_0(X) = 0}{\cong}}{H_0(X,B)} \xrightarrow{j_*} H_0(X,A) \to 0$$

$\tilde{H}_0(X) = 0 \leftarrow X$ path connected

Sequence exact. So j^* onto.

So $H_0(X,A) = 0$. ∎

Cor: $\exists$ l.e.s. for reduced relative homology of pair (X,A). $(A \neq \emptyset)$.

Pf: $B = a \in A$. LES for (X,A,B)

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X,A) \to \tilde{H}_{n-1}(A) \to \cdots$$ ∎

$\cong H_n(X,A)$ if $n > 0$.

Def: Let (X,A) be a pair. We say (X,A) is a good pair if A is closed in X and is a def. retract of some open set.

Ex: X is a CW complex. A subcomplex. Then (X,A) is a good pair.

Ex: Let A be a smoothly embedded S^1 in $\mathbb{R}^3$, i.e. a knot.

$(\mathbb{R}^3, \text{Knot})$ is a good pair.

Thm: Let (X,A) be a good pair. $A \hookrightarrow X$. $q: X \to X/A$ quotient map.

Then $\exists$ l.e.s.

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \cdots$$

Thm: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & i = n \\ 0 & \text{otherwise} \end{cases}$

Pf: $X = D^n$; $A = \partial D^n \cong S^{n-1}$

A subcomplex of X.

$(X,A) \to$ good pair

$q: X \to X/A \cong S^n$

Exact seq.

$$\tilde{H}_i(S^{n-1}) \xrightarrow{i_*} \underset{0}{\tilde{H}_i(D^n)} \xrightarrow{q_*} \tilde{H}_i(S^n) \xrightarrow{\partial} \cdots$$

So ∂ iso. So $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$ $\forall i, n$. Cont. down ~~to n=0~~.

Get to $\tilde{H}_{i-n}(S^0)$

$S^0 = \{-1, 1\}$

$$\tilde{H}_{i-n}(S^0) \cong \begin{cases} \mathbb{Z}, & i=n \\ 0, & \text{otherwise} \end{cases}$$ ▨

Cor: $S^n \simeq S^m$ iff $m=n$.

True since $\tilde{H}_*$ invariant of homotopy type. $\tilde{H}_*(S^m)$ "sees" m.

Cor: (Invariance of Dimension)

$\mathbb{R}^m \cong \mathbb{R}^n$ iff $m=n$.

Pf: Suppose $\exists\, f: \mathbb{R}^m \to \mathbb{R}^n$ homeo. Then $\exists$ homeo.

$f: \mathbb{R}^m \setminus 0 \xrightarrow{\sim} \mathbb{R}^n \setminus f(0)$

d.r. $\to$ S^{m-1} $\qquad$ $\mathbb{R} \to$ d.r. S^{n-1}

f induces hom. equiv. S^{m-1} to S^{n-1}

so $m-1 = n-1 \to m=n$. ▨

No Retraction Thm:

~~[illegible]~~ $n \geq 1$

There is no retraction D^n to S^{n-1}

Pf: Suppose $\exists$ retraction $r: D^n \to S^{n-1}$

$i: S^{n-1} \hookrightarrow D^n$

$ri \simeq 1_S$

$r_* i_* = 1$ on hom. so

So r_* surjective.

So $\underbrace{\tilde{H}_{n-1}(D^n)}_{\cong 0} \twoheadrightarrow \underbrace{\tilde{H}_{n-1}(S^{n-1})}_{\cong}$

$0 \twoheadrightarrow \mathbb{Z}$

But $0 \not\twoheadrightarrow \mathbb{Z}$ so no such r can exist. ▨

Brouwer ~~Brouwer~~ Fixed Point Thm:

Any map $f: D^n \to D^n$ has a fixed point.

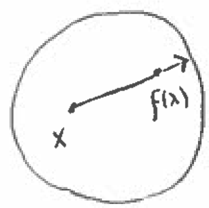
Pf: Suppose has no fixed point.

Define $g: D^n \to S^{n-1}$ by

$$g(x) = \frac{x - f(x)}{|x - f(x)|} \in S^{n-1}$$

$x \neq f(x)$

$\to \|x\| = 1$



If $x \in S^{n-1}$, then

$g(x) = x$

$g|_{S^{n-1}} = 1_{S^{n-1}}$

So g is a retract onto S^{n-1}.

But cannot exist by no retraction theorem.

03/25/2015

Final Fri May 1
10:15 - 12:15
Rm 115

Excision

$H_n(X,A)$

"$H_n(X,A)$ is homology of X 'ignoring' A"

$Z \subset A \subset X$

Can you 'ignore' Z?

Thm: (Excision Thm)
Let $Z \subset A \subset X$ with $\bar{Z} \subset \mathring{A}$. Then inclusion
$i: (X-Z, A-Z) \to (X,A)$
induces an isomorphism
$i_*: H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X,A)$

Let $\mathcal{U} = \{U_j\}$, where $U_j \subset X$ (not nec. open)
and $X = \bigcup \mathring{U}_j$.

Let $C_n^{\mathcal{U}}(X) = \{ \sum n_\alpha \sigma_\alpha \in C_n(X) \mid \forall \alpha, \operatorname{im} \sigma_\alpha \subset U_j$ for some $U_j \in \mathcal{U} \}$

∂ from X restricts to $C_n^{\mathcal{U}}(X)$.
So $C_n^{\mathcal{U}}(X) \xrightarrow{\partial \mathcal{U}} C_{n-1}^{\mathcal{U}}(X) \cdots$
is a chain complex.
Let $H_n^{\mathcal{U}}(X)$ be its homology.

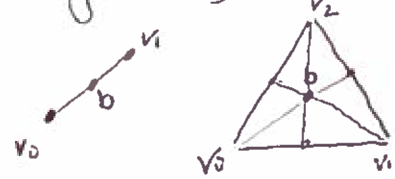
There is an inclusion
$i: C_n^{\mathcal{U}}(X) \to C_n(X)$

Lem: $\exists$ a chain map
$\rho: C_n(X) \to C_n^{\mathcal{U}}(X) \ni$
$\rho i = 1_{C_n^{\mathcal{U}}(X)}$
$i\rho$ is chain homotopic to $1_{C_n(X)}$.

So inclusion induces isomorphism on homology on $H_n^{\mathcal{U}}(X) \to H_n(X)$.

Pf: Hard and long. Hard to construct the chain map.

Barycenter: Given $[v_0, \dots, v_n]$
barycenter is $\frac{1}{n+1}(v_0 + \cdots + v_n)$



$\rho: C_n(X) \to C_n^{\mathcal{U}}(X)$
by $\sigma : \sum_i \sigma_\alpha$

Pf of Excision:

$B = X \setminus Z$
$\mathring{B} = X \setminus \bar{Z}$
$\bar{B} = X \setminus \mathring{Z}$

Excision $\overset{?}{=}$ Let $A, B \subset X$ with $\mathring{A} \cup \mathring{B} = X$. Then

$i : (B, B \cap A) \to (X, A)$

induces an isomorphism on homology groups, i_*.

Let $\mathcal{U} = \{A, B\}$

Denote $C_n(A+B) = C_n(A) + C_n(B)$
$= C_n^{\mathcal{U}}(X)$

By lem., $\exists$ chain map

$\rho : C_n(X) \to C_n(A+B)$

Observe

$\rho(C_n(A)) \in C_n(A)$

and

$i(C_n(A)) \subset C_n(A)$

There exist induced maps

$\bar{\rho} : C_n(X)/C_n(A) \to C_n(A+B)/C_n(A)$

$\ni \bar{\rho}\bar{i} = \mathbb{1}$

Furthermore, chain hom. D from lem

$\partial D + D\partial = i\rho - id$

Induces chain homotopy $\bar{D}$

$\ni \partial\bar{D} + \bar{D}\partial = \bar{i}\bar{\rho} - id$

ie $\bar{i}\bar{\rho} \simeq \mathbb{1}$

So $\bar{i}$ induces iso. on homology level

$H_*(C_*(X)/C_*(A)) \Rightarrow H_*(C_*(A+B)/C_*(A))$

$H_*(C_*(X)/C_*(A)) = H_*(X, A)$

What is $H_*(C_*(A+B)/C_*(A))$?

$C_n(A+B)/C_n(A) \overset{\text{def}}{=}$

$\left\{ \sum \lambda_\alpha \sigma_\alpha \in C_n(X) \mid (\text{im}\, \sigma_\alpha \in A \text{ or } \text{im}\, \sigma_\alpha \in B) / C_n(A) \right\}$

$\cong \left\{ \sum \lambda_\alpha \sigma_\alpha \in C_n(X) \mid (\text{im}\, \sigma_\alpha \in A \text{ or } (\text{im}\, \sigma \in B \wedge \text{im}\, \sigma_\alpha \notin A) / C_n(A) \right\}$

$\cong \left\{ \sum \lambda_\alpha \sigma_\alpha \in C_n(X) \mid \text{im}\, \sigma_\alpha \in B \wedge \text{im}\, \sigma_\alpha \notin A \right.$

$\cong C_n(B)/C_n(A \cap B)$

$\overset{\text{def}}{=} H_*(B, B \cap A)$ ☒

Recall what we know so far:

$H_n(\overset{\text{pt.}}{*}) = 0$

$H_n(X, *) \cong \tilde{H}_n(X)$ (HW)

$H_n(S^n)$ L.E.S. for rel. hom.

Thm: If (X, A) is a good pair, then $\exists$ les

$q : X \to X/A$ quotient map

$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$

Pf: Have les (for rel. hom.)

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$$

WTS $H_n(X,A) \cong \tilde{H}_n(X/A)$

Let V open set in X that d.r. onto A

$$A \subset V \subset X$$

Need $\bar{A} \subset \mathring{V} = V$

$\bar{A} = A \subset \mathring{V} = V$

So have it.

$$\begin{tikzcd}
(X-A, V-A) \arrow[r, "i"', "②"] \arrow[d, "q ③"] & (X,V) \arrow[d, "q"] & (X,A) \arrow[l, "i"', "①"] \arrow[d, "q ⑤"] \\
(X/A - A/A, V/A - A/A) \arrow[r, "i ④"] & (X/A, V/A) & (X/A, A/A) \arrow[l, "i"]
\end{tikzcd}$$

Get same diagram on homology level (ie, i_* & q_*'s between $H_n(—)$'s.)

② is $\cong$ by excision.

④ is $\cong$ by excision

① ⑤ } $A \simeq V$, these are iso.

③ $q: X-A \to X/A - A/A$ cont., surj., inj. Need to check open map. (quotient maps are open)

So homeomorphism, so induces iso on hom. level.

In X/A, A/A is a point, so

$$H_n(X/A, A/A) \cong H_n(X/A, *) \cong \tilde{H}_n(X/A) \quad \blacksquare$$

Cor: Let X be a CW complex with subcomplexes A and B with $X = A \cup B$. Then inclusion induces an iso. $H_n(B, A \cap B) \xrightarrow{\sim} H_n(X,A)$ $\forall n$

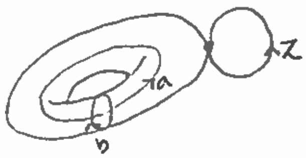
Pf: Excision.

Cor: Let X_α be a collection of spaces with points $x_\alpha \in X_\alpha$ $\ni$ (X_α, x_α) is a good pair. Let $X = \bigvee_\alpha X_\alpha$ along x_α. Then the inclusion maps $i_\alpha: X_\alpha \to \bigvee_\alpha X_\alpha$ induce an iso. $\bigoplus i_{\alpha *}$:

$$\bigoplus \tilde{H}_n(X_\alpha) \xrightarrow{\sim} \tilde{H}_n(\bigvee_\alpha X_\alpha)$$

Pf: $(\sqcup X_\alpha, \sqcup \{x_\alpha\})$ is a good pair. $\sqcup X_\alpha / \sqcup x_\alpha \underset{(def)}{\cong} \bigvee_\alpha X_\alpha$

Ex: $X = T^2 \vee S^1$



$H_1(X) \cong \mathbb{Z}^3$ gen by a, b, z

$H_2(X) \cong \mathbb{Z}$ gen by $H_2(T^2)$

$H_n(X) = 0$ for $n \geq 3$

$H_0(X) = ?$ has to be $\mathbb{Z}$ as space path connected

Cor. doesn't apply b/c on reduced hom.

03/30/2015

Generators of $H_n(D^n, \partial D^n)$ and $H_n(S^n)$.

Know $H_n(X, A) \cong H_n(X/A)$ if (X, A) is a good pair.

Know both of above are $\mathbb{Z}$ if $n > 0$.

$H_n(D^n, \partial D^n) = H_n(\Delta^n, \partial\Delta^n)$

Let $1 : \Delta^n \to \Delta^n \cong D^n$.

$\partial 1 = \sum_i (-1)^i$ faces

face of simplex $\Delta^n \subset \partial\Delta^n$

So $1 \in Z_n(D^n, \partial D^n)$

Claim: $[1]$ is a generator for $H_n(D^n, \partial D^n) \cong \mathbb{Z}$

Pf: By induction.

$n = 0$: $\Delta^0 = \cdot$

$\partial\Delta = \phi$

$H_0(\Delta^0) \cong \mathbb{Z}$ "generated by a point."

$= H_0(\Delta^0, \partial\Delta^0)$.

Choose a $(n-1)$-dim. face of Δ^n.

$f : \Delta^{n-1} \hookrightarrow \partial\Delta^n$

Let $\Lambda = \bigcup_{\substack{\text{faces}\\ \text{not}\\ f}} (n-1)\text{-dim faces}$

Λ is path connected.

$\Lambda \simeq \partial D^n \setminus \text{point} \simeq S^{n-1} \simeq \text{point}$

(1) $\Lambda \simeq$ point

(2) $\Lambda \cong D^{n-1}$

(3) $(\Delta^n, \Lambda) \simeq (\Lambda, \Lambda)$

(4) (Δ^n, Λ), (Λ, Λ) are good pairs.

Triple: $(\Delta^n, \partial\Delta^n, \Lambda)$

Get l.e.s.

$$\cdots \to H_n(\partial\Delta^n, \Lambda) \to \overbrace{H_n(\Delta^n, \Lambda)}^{\text{trivial by (3)}} \to H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\delta} H_{n-1}(\partial\Delta^n, \Lambda) \to \underbrace{H_{n-1}(\Delta^n, \Lambda)}_{\text{trivial by (3) as } (\Lambda, \Lambda)} \to \cdots$$

So δ iso.

Notice f induces a homeo.

$\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$

$\partial[1] \overset{\text{Snake lem}}{=} [\partial 1]$ in $H_{n-1}(\partial\Delta^n, \Lambda)$

$= [``f"]$

$= 1\,[id_{n-1}]$

$\partial\, y \cong \partial(1_N) =$

Gen. for S^n =

$S^n = \Delta^n_1 \cup \Delta^n_2$

$[\Delta^n_1 = \Delta^n_2]$

generating

It gen. $H_n(S^n)$

$H_n(S^n) \cong H_n(S^n, \Delta^n_2)$

$$\uparrow \wr$$

$H_n(\Delta^n, \partial\Delta^n_1)$

$[\Delta^n_1 - \Delta^n_2] \mapsto [\Delta^n_1] = [1]$

So $[\Delta^n_1 - \Delta^n_2]$ generating $H_n(S^n)$ $\boxed{}$

$n=1$

$= S^1$ generator for $H_1(S^1)$.

Ex: $S^1 \vee S^2$

$H_0(S^1 \vee S^2) = \mathbb{Z}$ generated by $\star$

$H_1(S^1 \vee S^2) = \mathbb{Z}$ generated by S^1 from $S^1 \vee S^2$.

$H_2(S^1 \vee S^2) = \mathbb{Z}$ generated by S^2 in $S^1 \vee S^2$.

$H_n(S^1 \vee S^2) = 0$ for $n \geq 3$.

Thm: Better Invariance of Dim.
Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open set. If $U \cong V$ then $n = m$.

Pf: Let $x \in U$.

$$\underset{\overset{\shortparallel}{B}}{\mathbb{R}^m \setminus U} \subset \underset{\overset{\shortparallel}{A}}{\mathbb{R}^m \setminus \{x\}} \subset \underset{\overset{\shortparallel}{X}}{\mathbb{R}^m}$$

Excision:

$\tilde{H}_k(X, A) \cong H_k(X \setminus Z, X \setminus A)$

$= H_k(U, U \setminus \{x\})$

$\tilde{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$

$\tilde{H}_k(\mathbb{R}^n) \to \tilde{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \xrightarrow{\partial\, \sim} \tilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \to \tilde{H}_{k-1}(\mathbb{R}^m)$

R dr S^{m-1}

So $\tilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \cong \tilde{H}_{k-1}(S^{m-1})$

ranging over k, "sees" m.

So over K,
$H_*(U, U \setminus \{x\})$ "sees" m.

$\hookrightarrow$ Local homologies of space at x.

So if $U \cong V$,
$H_*(V, V \setminus \{x\})$ "sees" n.

So $m = n$. ■

Degree

If $h : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism. the degree of h is $\deg(h) \overset{\text{def}}{=} h(1)$.

Notice degree determines the homomorphism.

Def: If $f : S^n \to S^n$ then identify $H_n(S^n) = \mathbb{Z}$

Declare $[\Delta_1^n - \Delta_2^n]$ to be positive generator. Then map induced on homology

$$f_* : \underset{\mathbb{Z}}{\underset{\parallel}{H_n(S^n)}} \to \underset{\mathbb{Z}}{\underset{\parallel}{H_n(S^n)}}$$

the degree of f is $\deg f \overset{\text{def}}{=} \deg f_*$

Properties: $f, g : S^n \to S^n$

i) Homotopy Invariance: If $f \simeq g$, then $\deg f = \deg g$

Pf: Induce same map on homology as $f \simeq g$.

ii) $\deg(f \circ g) = \deg f \deg g$

"Pf:" $(fg)_* = f_* g_*$

iii) If f constant or not onto then $\deg f = 0$.

"Pf:" Stereographic proj. then collapse to pt

iv) $\deg 1 = 1$

Pf: Take 1 to 1

v) Degree of reflection is -1.

"Pf": Only need consider equatorial ref.

vi) Antipodal map has degree $(-1)^{n+1}$

$S^n \ni (x_1, \cdots, x_{n+1}) \to$ Pf: $\deg f \circ g = \deg f \deg g$

vii) If f has no fixed point then $f \simeq$ antipodal map so has deg $(-1)^{n+1}$.

Things you can do with this:

Thm: S^n has a nonvanishing tangent vector field $\rightleftarrows$ n is odd.

Pf: n odd $x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}$ with $|x| = 1$. Def $v(x) = (-x_2, x_1, \cdots, -x_{n+1}, x_n)$

Check $x \cdot v_x = 0$ so $x \perp v_x$ so v_x tang. vector field never 0. Suppose $\exists$ nonvan. tang. vector field v. $H(x,t) = (\cos t)x + (\sin t)v(x)$.

Assuming $|v(x)| = 1$ $\forall x$. (otherwise ... can b/c nonzero)

Then $|H(x,t)| = \sqrt{\cos^2 + \sin^2} = 1$

So $H: S^n \times I \to S^n$

$H(x,0) = x$

$H(x,\pi) = -x$

So $1 \simeq$ antipodal map so same degree so $(-1)^{n+1}$ so n must be odd.

04/01/2015

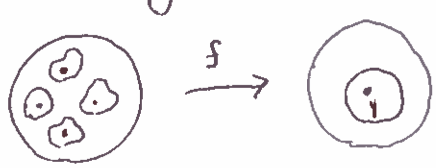
Final Exam: Fri. May 1
10:15-12:15

Local vs. Global Degree

Suppose $f: S^n \to S^n$; $n > 0$ and let $y \in S^n$ with

$f^{-1}(y) = x_1 \cup \cdots \cup x_m$

Pick disjoint neigh U_i of x_i each mapping into neigh V of y.



So $f(U_i - x_i) \subset V \setminus y$. We have

$$\begin{tikzcd}
H_n(U_i, U_i|x_i) \arrow[r, "f_*"] \arrow[d, leftrightarrow, "\text{excision}"'] & H_n(V, V|y) \arrow[d, leftrightarrow, "\text{excision}"] \\
H_n(S^n, S^n|x) & H_n(S^n, S^n|f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n|y) \\
H_n(S^n) \arrow[u, leftrightarrow, "\text{les}"] \arrow[ur, "j_*"] \arrow[rr, "f_*"] & & H_n(S^n) \arrow[u, leftrightarrow, "\text{les}"]
\end{tikzcd}$$

k_i ; $H_n(S^n, \emptyset)$; $H_n(S^n, \emptyset)$

Mod by shaded "stuff" (shrink to point) get

Redrawn

$$\begin{tikzcd}
& H_n(U_i, U_i|x_i) \arrow[dl, "\sim"'] \arrow[r, "f_*"] \arrow[d, "k_i"] & H_n(V, V|y) \arrow[d, "\wr"] \\
H_n(S^n, S^n|x_i) & H_n(S^n, S^n|f^{-1}(y)) \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(S^n, S^n|y) \\
& H_n(S^n) \arrow[ul, "\sim"] \arrow[u, "j_*"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\wr\ m"']
\end{tikzcd}$$

$H_n(S^n) = \mathbb{Z}$, $H_n(S^n) = \mathbb{Z}$

$*$ Note: Hatcher says whole diagram commutes but does not. The individual sections commute.

Def: The local degree of f at x_i is the ~~loc.~~ $\deg f|_{x_i} \overset{def}{=}$

$\deg(f_*: H_n(U_i, U_i|x_i) \to H_n(V, V|y))$

Fact: $H_n(S^n, S^n|f^{-1}(y)) \cong \mathbb{Z}^m$

By argument.

k_i is inclusion into i^{th} factor of $\mathbb{Z}^m$.

$j_*(1) = (1, 1, \cdots, 1)$

p_i projection onto i^{th} factor of $\mathbb{Z}^m$

Prop: $\deg f = \sum_i \deg f|_{x_i}$

Pf: $f_*(1)$, $1 \in H_n(S^n)$

$m(f_*(1))$

$f_*(1) = f_*(j_*(1))$

$= f_*((1, 1, \cdots, 1))$

$= f_*(\sum_i k_i(1))$

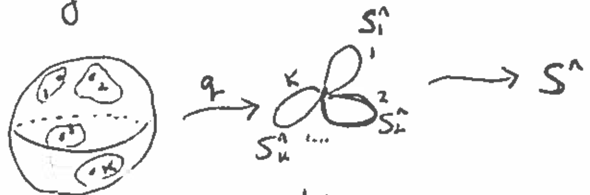
$= \sum_i f_*(1)$ in $H_n(U_i, U_i|x_i)$

$= \sum \deg f|_{x_i}$ ∎

Ex: Given $K \in \mathbb{Z}$, $n > 0$

Construct map $f_K : S^n \to S^n$ with $\deg f_K = K$

Idea:



Choose $x_1, \cdots, x_K$. Quotient by neigh. (disjoint) of x_i's.

If $K > 0$, map each copy of sphere by id. onto S^n. If $K < 0$, map each S_i^n onto S^n by reflection.

If $K = 0$, use constant map.

Each local degree is 1 (or -1)

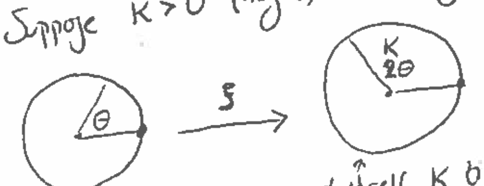
$\sum_1^K 1 = K$ or $\sum_1^K -1 = -K$

Global degree is sum of local degrees so done.

Ex: $f : S^1 \to S^1$
$z \mapsto z^K$

map of degree K.

Suppose $K > 0$ (neg. case ref. through x-axis)



Wraps circle around itself K times.

q

id.

So $\deg(z \mapsto z^K) = K$ as expected.

Cellular Homology

X is a CW complex.

Lem: For a CW complex X we have

i) $H_K(X^n, X^{n-1}) = \begin{cases} 0, & K \neq n \\ \mathbb{Z}^m, & K = n \end{cases}$

$m = \#$ of n-cells

ii) $H_K(X^n) = \begin{cases} 0 & \text{if } K > n \end{cases}$

iii) The inclusion induces an iso.
$i_* : H_K(X^n) \to H_K(X)$
for all $n > K$.

Pf:

i) (X^n, X^{n-1}) is a good pair so
$H_K(X^n, X^{n-1}) \cong \underbrace{\tilde{H}_K(X^n / X^{n-1})}_{\cong \bigvee_{\# \text{cells}} S^n} \cong \mathbb{Z}^{\# n \text{ cells}}$ for $K = n$

ii) Look at "L.e.s." of pair (X^n, X^{n-1})

$H_{K+1}(X^n, X^{n-1}) \xrightarrow{\partial} H_K(X^{n-1}) \to H_K(X) \to H_K(X^n, X^{n-1})$

$H_{K+1}(X^n, X^{n-1}) = 0$ as $K > n$ by (i); $H_K(X^n, X^{n-1}) = 0$ as $K > n$ by (i)

So $\cong$

Then continue downwards by induction.

Notice skeleta degree goes down, NOT homology

Then $H_K(X^n) \cong H_K(X^{n-1}) \cong \cdots \cong \underbrace{H_K(X^0)}_{\text{point}} = 0$

iii) Same idea as above but going up.

Rem: To calculate $H_k(X)$, you only need X^{k+1}.

Def: The n^{th} cellular chain group of a CW complex X is $C_n^{CW}(X) \overset{def}{=} H_n(X^n, X^{n-1})$
We know this is a free abelian group of rank #n-cells

Is there a $d_n : C_n^{cw}(X) \to C_{n-1}^{cw}(X)$ with $d_n^2 = 0$?
Use les for pairs (X^e, X^{e-1})

$$\begin{tikzcd}
 & H_n(X^{n+1}, X^n) = 0 \ \text{(lemma)} & & \\
 & H_n(X^{n+1}) \ (\cong H_n(X)) \arrow[u] & & \\
0 = H_n(X^{n-1}) \arrow[r] & H_n(X^n) \arrow[u, two heads] \arrow[dr, hook, "j_n"] & & \\
 & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial"] & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & C^{cw}_{n+1} & C^{cw}_n & H_{n-1}(X^{n-1}) \arrow[u, hook, "j_{n-1}"] \\
 & & & H_{n-1}(X^{n-2}) = 0 \arrow[u]
\end{tikzcd}$$

H, ||, \ exact

Define:

$d_n : C_n^{cw}(X) \to C_{n-1}^{cw}(X)$

by $d_n = j_{n-1}\partial$

Prop: $d_n d_{n+1} = 0$

$$d_n d_{n+1} = j_{n-1} \underbrace{\partial j_n}_{=0 \text{ by exactness}} \partial$$
$$= j_{n-1} 0 \partial$$
$$= 0 \quad \blacksquare$$

Def: The n^{th} cellular homology group of CW complex X is
$H_n^{cw}(X) = \ker d_n / \operatorname{im} d_{n+1}$

Thm: For all n,
$$\underbrace{H_n^{cw}(X) \cong H_n(X)}_{\text{prove this}} \cong H_n^{\Delta}(X)$$

Pf: By the diagram
$H_n(X) \cong H_n(X^n)/\operatorname{im}\partial$

j_n takes $\operatorname{im}\partial$ injectively into $\ker \partial$ ← diff

Since j_{n-1} is inj.
$\ker\partial = \ker d_n$

So j_n induces map
$$\bar{j}_n : \underset{\substack{\| \\ H_n(X)}}{H_n(X^n)/\operatorname{im}\partial} \to \underset{\substack{\|\text{def} \\ H_n^{cw}(X)}}{\ker d_n / \operatorname{im} d_{n+1}}$$

Check iso. (1-1, you modded out ker ↟)

So just argue onto.

Computing d_n

$$\begin{array}{ccc}
C_n^{cw}(X) & \xrightarrow{d_n} & C_{n-1}^{cw}(X) \\
\| & & \| \\
H_n(X^n, X^{n-1}) & " & \text{---} \ " \\
\| & & \| \\
\tilde{H}_n(\bigvee_{\#} S^n) & " & \text{---} \ " \\
\| & & \| \\
\bigoplus_{\#} \mathbb{Z} & " & \bigoplus_{\#} \mathbb{Z} \ " \\
\uparrow & & \uparrow \\
\alpha & & \beta
\end{array}$$

What is $d_n(e_\alpha^n)$?

Write $d_n(e_\alpha^n) = \sum_\beta \underbrace{d_{\alpha\beta}}_{\in \mathbb{Z}} e_\beta^{n-1}$

So the question is $d_{\alpha\beta}$. What is it?

$e^2 \cong \mathring{D}^2$

glued onto X^1 by attaching map

$\varphi_\alpha : S^1_\alpha \to X^1$

$$\begin{array}{c}
\searrow q \\
X^1 / X^0 \cong \bigvee_\beta S^1 \\
\downarrow q_n \\
X^1 / (X^1 \setminus e^1_\beta) \\
\downarrow \cong \\
S^1_\beta
\end{array}$$

$q_n \varphi_\alpha : S^1 \to S^1$

Thm: $d_{\alpha\beta} = \deg(q_n \varphi_\alpha)$

04/06/2015

X is a CW complex with n-cells $\{e_\alpha^n\}$ and $(n-1)$-cells $\{e_\beta^{n-1}\}$

$d_n : C_n^{cw}(X) \to C_{n-1}^{cw}(X)$

$d(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$

where $d_{\alpha\beta} \in \mathbb{Z}$

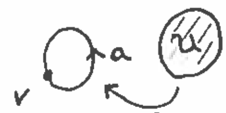
Prop: $d_{\alpha\beta} = \deg(q_\beta \circ \varphi_\alpha)$

where $\varphi_\alpha : \underbrace{\partial D_\alpha^n}_{\cong S_\alpha^{n-1}} \to X^{n-1}$

is the attaching map and

$q_\beta : X^{n-1} \to X^{n-1} / (X^{n-1} \setminus e_\beta^{n-1}) \cong S_\beta^{n-1}$

Ex: $\mathbb{RP}(2) = e^0 \cup e^1 \cup e^2$



wrap around a twice

$a \mapsto a^2$

Circle to itself.

CW chain complex

$0 \to C_2^{CW}(\mathbb{RP}(2)) \xrightarrow{d_2} C_1^{cw}(-) \xrightarrow{d_1} C_0^{cw}(-) \to 0$

No cell deg. 3 or higher

$C_0^{cw}(\mathbb{RP}(2)) = \mathbb{Z} = \langle v \rangle$

$C_1^{cw}(\mathbb{RP}(2)) = \mathbb{Z} = \langle a \rangle$

$C_2^{cw}(\mathbb{RP}(2)) = \mathbb{Z} = \langle \alpha \rangle$

What is map d_2? Map is $\times 2$ as has degree 2.

$d_2 : \alpha \mapsto 2a$ $\qquad$ $\operatorname{im} d_1 = 0$, $\operatorname{Ker} d_1 = \langle a \rangle$

$d_1 : 0$ $\qquad$ $\operatorname{Ker} d_2 = 0$, $\operatorname{im} d_2 = 2\mathbb{Z} = \langle 2a \rangle$

$H_0 = \langle v \rangle / 0 = \langle v \rangle \cong \mathbb{Z}$ $\qquad$ $H_2 = 0/0 = 0$

$H_1 = \langle a \rangle / \langle 2a \rangle = \mathbb{Z}/2\mathbb{Z}$ $\qquad$ $H_n = 0$ for $n \geq 3$.

$\underline{Ex}$: $X =$ 

Union 2 2-cells

U $\varphi_1 : \partial D^2 \to X'$

$S^1 \mapsto a^5 b^{-3}$

T $\varphi_2 : \partial D^2 \to X'$

$S^1 \mapsto b^3 (ab)^{-2}$

CW Chain complex

$$\cdots\; \underset{(3)}{0} \to \underset{\langle U, T\rangle}{\overset{(2)}{\mathbb{Z}^2}} \xrightarrow{d_2} \underset{\langle a,b\rangle}{\overset{(1)}{\mathbb{Z}^2}} \to \underset{\langle v\rangle}{\overset{(0)}{\mathbb{Z}}} \to 0$$

$d_2 : U \mapsto 5a - 3b$

(Kill b and see what happens a^5
Kill a see what happens b^{-3})

$T \mapsto -2a + b$

$$\begin{array}{cc} U & T \end{array}$$
$$\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}\begin{array}{c} a \\ b \end{array}$$

$3C_2 + C_1$

$$\begin{array}{cc} U+3T & T \end{array}$$
$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}\begin{array}{c} a \\ b \end{array}$$

* Colums work way you want. Rows not so much.

Over $\mathbb{Z}$, this is onto as has det 1

So d_2 is $\cong$ (iso.)

So $\operatorname{im} d_2 = \mathbb{Z}^2$

$\ker d_2 = 0$

So $H_2(X) = 0$

$H_n(X) = 0$ for $n \geq 3$

$H_1 = \langle a,b\rangle / \langle a,b\rangle = 0$

$H_0 = \langle v\rangle / 0 = \mathbb{Z}$

A space X is called acyclic if $\tilde{H}_n(X) = 0$ $\forall n$.

Homology can't see anything about these spaces.

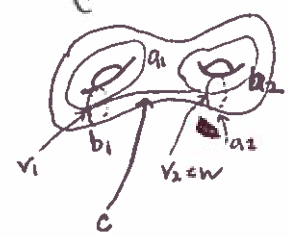
From above,

$\pi_1(X, v)$ is icosahedral group.

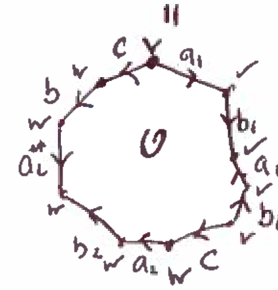
$\hookrightarrow \cong A_5 \ltimes$ Reflection ?

It's abelianization is trivial.

$\underline{Ex}$: $M_2 = T^2 \# T^2$



$=$



Cut open get 2 side for each letter

Decagon

$$\underset{(3)}{0} \to \underset{\langle U\rangle}{\overset{(2)}{\mathbb{Z}}} \xrightarrow{d_2} \underset{\langle a_1,a_2,b_1,b_2,c\rangle}{\overset{(1)}{\mathbb{Z}^5}} \xrightarrow{d_1} \underset{\langle v,w\rangle}{\overset{(0)}{\mathbb{Z}^2}} \xrightarrow{d_0} 0$$

$d_2 = 0$

$d_1 :$ $a_i \mapsto 0$
$b_i \mapsto 0$
$c \mapsto w - v$

$H_0 = \langle v,w\rangle / \langle w-v\rangle = \langle v, w-v\rangle / \langle w-v\rangle = \mathbb{Z}$ $= \langle v\rangle$ or $\langle w\rangle$

$H_1 = \langle a_1,a_2,b_1,b_2\rangle / 0 = \mathbb{Z}^4$

$H_2 = \langle U\rangle / 0 = \mathbb{Z}$

Sometimes written $\langle [M_2]\rangle$

Remark: $\mathbb{C}P(k)$, $k<n$, sits inside $\mathbb{C}P(n)$ as $2k$-skeleton

Thm: If $M_g = \#_{i=1}^{g} T^2$

Then
$$H_i(M_g) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}^{2g}, & i=1 \\ \mathbb{Z}, & i=2 \\ 0, & \text{otherwise} \end{cases}$$

Ex: $\mathbb{C}P(n)$ = "Lines in $\mathbb{C}^{n+1}$ through origin"

$\cong \{\mathbb{C}^{n+1} \setminus \{0\}\} / \{x \sim \lambda x$ for $\lambda \in \mathbb{C}\setminus 0\}$

$\cong S^{2n+1} / x \sim \lambda x$ for $x \in S^{2n+1}$ & $|\lambda| = 1$

$\star \cong D^{2n} / v \sim \lambda v$ for $v \in S^{2n-1}$ & $|\lambda|=1$

$\mathbb{C}P(0) = \bullet$ (point) $= \bullet$

$\mathbb{C}P(1)$ = Riemann sphere, aka $S^2 = \bullet \cup$

$\vdots$

$\mathbb{C}P(n) = e^0 \cup e^2 \cup \cdots \cup e^{2n}$

where gluing map given in $\star$

CW chain complex

$$0 \to \overset{(2n)}{\mathbb{Z}} \to \overset{(2n-1)}{0} \to \overset{(2n-2)}{\mathbb{Z}} \to \cdots \to \overset{(2)}{\mathbb{Z}} \to \overset{(1)}{0} \to \overset{(0)}{\mathbb{Z}} \to 0$$

So
$$H_k(\mathbb{C}P(n)) = \begin{cases} \mathbb{Z}, & k \text{ even } \& \ k \le 2n \leftarrow \text{gen. by } \mathbb{C}P(k/2) \\ 0, & \text{otherwise} \end{cases}$$

Ex: $\mathbb{R}P(n) = \{\mathbb{R}^{n+1} \setminus 0\} / x \sim \lambda x$ for $\lambda \in \mathbb{R}^\times$

$\approx S^n / x \sim -x$

$\approx D^n / x \sim -x$ for $x \in \partial D^n$

Let $q_n : S^n \to \mathbb{R}P(n)$ be quotient map

$\mathbb{R}P(n) = \mathbb{R}P(n-1) \cup e^n$

$= e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$

How are they glued on? By matching $\to$ using antipodes.

Gluing:

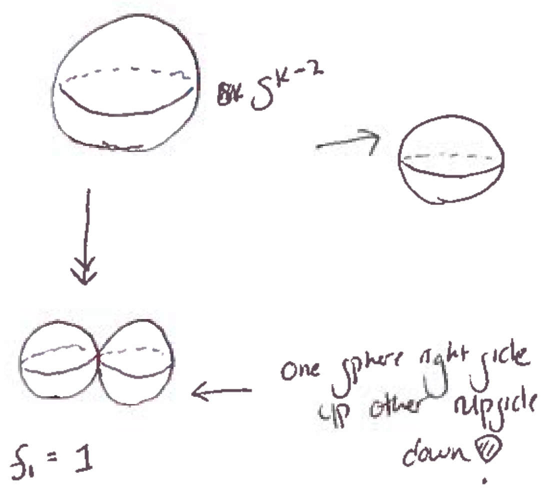
$q_k = \varphi_k : \underset{\cong S^{k-1}}{\partial e^k} \to \mathbb{R}P(k-1)$

$$\begin{array}{ccccc} \partial e^k = S^{k-1} & \xrightarrow{\varphi_k} & \mathbb{R}P(k-1) & \xrightarrow{\text{quotient}} & \mathbb{R}P(k-1)/\mathbb{R}P(k-2) \\ \downarrow & & & & \cong \\ S^{k-1}/\text{equatorial } S^{k-2} & & & & S^{k-1} \\ \cong & & & \overset{\text{using quotient}}{\dashrightarrow} & \exists \ s_1 \vee s_2 \\ S^{k-1} \vee S^{k-1} & & & & \end{array}$$

Gluing map is quotient map. Choose equator going to zero $\to$ way.

So Degree of $q_n \varphi_k$ is degree going around bottom of diagram.

$\therefore = \underbrace{\deg f_1 + \deg f_2}_{\text{Global sum of local degree}}$



$f_1 = 1$

$f_2 =$ antipodal

$\deg f_1 + \deg f_2 = \deg 1 + \deg \text{ant}$

$= 1 + (-1)^k$

$= \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$

So $d_k : C_k^{cw}(\mathbb{RP}(n)) \to C_{k-1}^{cw}(\mathbb{RP}(n))$

is either $\times 2$ or 0.

$\uparrow$ $k \leq n$ and k even

$\uparrow$ otherwise

n even:

$$\cdots \underset{(n+1)}{0} \to \underset{(n)}{\mathbb{Z}} \xrightarrow{2} \underset{(n-1)}{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \to \cdots \to \underset{(2)}{\mathbb{Z}} \xrightarrow{2} \underset{(1)}{\mathbb{Z}} \xrightarrow{0} \underset{(0)}{\mathbb{Z}} \to 0$$

$H_k = 0$ for $k \geq n+1$

$H_k = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ for k odd, $< n$

$H_0 = \mathbb{Z}/0 = \mathbb{Z}$

$H_k = 0$ otherwise

For odd n

$$H_k(\mathbb{RP}(n)) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2\mathbb{Z} & k = \text{odd}, < n \\ 0 & \text{otherwise} \end{cases}$$

04/08/2015

X is a CW complex. (fin many cells)

Def: The Euler characteristic of the CW space X is

$$\chi(X) = \sum_i (-1)^i \# \text{ of } i\text{-cells in } X$$

Chi $\to$

Ex: $S^1 =$ 

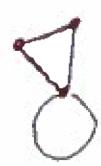
$\chi(S^1) = 1 - 1 = 0$

or 

$\chi(S^1) = 2 - 2 = 0$

Well-defined?

Ex: (Euler)



$v - e + f = 2$

$\chi(X) = \chi(S^2) =$

$v - e + f = 2$

Ex: $S^1 \vee S^1 =$ 

$\chi(S^1 \vee S^1) = 1 - 2 = -1$

Ex: $T^2 = S^1 \times S^1$

$\chi(T^2) = 1 - 2 + 1 = 0$

Ex: $M_g =$ 

$\chi(M_g) = 1 - 2g + 1 = 2 - 2g$

* Notice numbers adding and subtracting is rank of homology groups! (Kind of)

Ex: If $\tilde{X} \to X$ is a k-fold cover, then

$\chi(\tilde{X}) = k\,\chi(\tilde{X})$

Ex:  $\chi = -6$

$\downarrow 3:1$

 $\chi = -2$

"Pf": Give X a CW complex, that lift cells to $\tilde{X}$ and get a CW complex for $\tilde{X}$ ▨

Thm: For a finite CW complex $\tilde{X}$,

$$\chi(\tilde{X}) = \sum_i (-1)^i \operatorname{rank}_{\mathbb{Z}} H_i(X)$$

(So indep. of choice of CW complex)

Lemma: If A, B, C are f.g. abelian groups with s.e.s.

$$0 \to A \to B \to C \to 0$$

then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$

(rank nullity for abelian groups)

Pf (Thm):

$\operatorname{rank} C_i^{CW}(X) = \#$ i-cells in X

$Z_i = \ker d_i$

$B_{i-1} = \operatorname{im} d_i$

$d_i : C_i^{CW} \to C_{i-1}^{CW}$

We have the exact sequence

$$0 \to Z_i \to C_i^{CW} \xrightarrow{d_i} B_{i-1} \to 0$$

Also have exact sequence

$$0 \to B_i \to Z_i \twoheadrightarrow H_i(X) \to 0$$

So $\chi(\tilde{X}) \overset{\text{def}}{=} \sum_i (-1)^i \operatorname{rank} C_i^{CW}$

$= \sum_i (-1)^i (\operatorname{rank} B_{i-1} + \operatorname{rank} Z_i)$

$= \sum_i (-1)^i (\operatorname{rank} B_{i-1} + \operatorname{rank} H_i(\tilde{X}) + \operatorname{rank} B_i)$

Telescopes so

$= \sum_i (-1)^i \operatorname{rank} H_i(\tilde{X})$ ▨

Remark In S^2

$\chi = v - e + f = 2$

Using homology only get $1 - 0 + 1 = 2$

Def: $\operatorname{rank}_{\mathbb{Z}} H_i(X)$ is the i^{th} Betti Number: $B_i(X)$.

Now X is not necessarily CW

Mayer-Vietoris Sequence

(Seifert van Kampen for homology)

Thm: Let $X = \mathring{A} \cup \mathring{B}$ $\quad A \cap B \neq \phi$

Then the following is exact:

$$\cdots \to H_n(A \cap B) \xrightarrow{i_*} H_n(A) \oplus H_n(B) \xrightarrow{j_*} H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

$i_*([x]) = i_{A*}[x] \oplus -i_{B*}[x]$

$j_*([a] \oplus [b]) = j_{A*}[x] + j_B[b]$

∂ from Snake Lemma.

$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A \cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$

Pf: Let $\mathcal{U} = \{\mathring{A}, \mathring{B}\}$ open cover of X. Recall

Excision $\left\{ \begin{array}{l} C_n^{\mathcal{U}}(X) = \{\sum n_\alpha \sigma_\alpha : \sigma_\alpha : \Delta^n \to X \text{ with } \operatorname{im} \sigma \text{ is in } \mathring{A} \text{ or } \mathring{B}\} \text{ in } C_n(X) \\ \\ H_n^{\mathcal{U}}(X) \cong H_n(X) \end{array} \right.$

$$0 \to C_n(A \cap B) \xrightarrow{i_\#} C_n(A) \oplus C_n(B) \xrightarrow{j_{A\#} + j_{B\#}} C_n^{\mathcal{U}}(X) \to 0$$

is exact. Are $i_\#, j_\#$ chain maps? ie $i_\# \partial = \partial i_\#$?

Yes! (Why?)

Apply Snake Lemma. Done except for

$\partial : H_n(X) \to H_{n-1}(A \cap B)$

$\partial[y] = [\partial a]$ in $H_{n-1}(A \cap B)$. ▨

Thm: $\exists$ Mayer Vietoris with reduced hom. (same as prev. just with $\tilde{=}$)

Cor: "Abelianized SVK"

Assume A, B, X & $A \cap B$ path conn.

Then $H_1(X) \cong H_1(A) \oplus H_1 B / \operatorname{im} H_1(A \cap B)$

Pf: Use MV

Exact

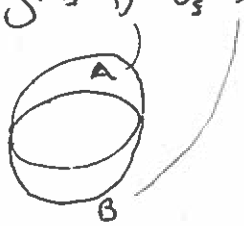
$$\tilde H_1(A\cap B) \to \tilde H_1(A)\oplus \tilde H_1(B) \to \tilde H_1(X) \to \underbrace{\tilde H_0(A\cap B)}_{0}$$

Because $A\cap B$ path connected ■

MV is good for computations.

Ex:

$S^n = D^n \cup_S D^n$



$A\cap B = S^{n-1}$

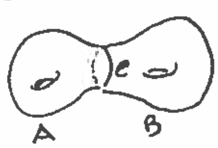
MV says have exact sequence

$$\tilde H_k(A\cap B) \to \underbrace{\tilde H_k(A)\oplus \tilde H_k(B)}_{D^n \quad D^n} \to \tilde H_k(X) \to \tilde H_{k-1}(A\cap B) \to \tilde H_{k-1}(A)\oplus \tilde H_{k-1}(B)$$

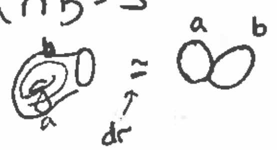
$$0 \to \tilde H_k(S^n) \xrightarrow{\partial} \tilde H_{k-1}(S^{n-1}) \to 0$$

So $\tilde H_k(S^n) \cong \tilde H_{k-1}(S^{n-1})$

Do induction down to $\tilde H_k(S^0)$.

Ex: $M_2 =$ 

$A\cap B = S^1$

$A =$ 

$H_1(A) \cong \mathbb{Z}^2 = \langle a,b\rangle$

$H_1(B) \cong \mathbb{Z}^2 = \langle c,d\rangle$

MV gives exact sequence

line 1: $H_2(\underset{A\cap B}{S^1}) \to \tilde H_2 A \oplus \tilde H_2 B \to \tilde H_2 M_2 \to \tilde H_1 A\cap B \to$

line 2: $0 \to 0 \to \tilde H_2 M_2 \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*}$, $\mathbb{Z} = \langle e\rangle$

line 1: $\to \tilde H_1 A \oplus \tilde H_1 B \xrightarrow{j_*} \tilde H_1(M_2) \to \tilde H_0(S^1) = 0$

line 2: $\to \mathbb{Z}^2 \oplus \mathbb{Z}^2 \twoheadrightarrow \tilde H_1(M_2)$, $\langle a,b,c,d\rangle$

j_* surj | $\tilde H_1(M_2) =$ ker | im = ker

∂ is inj.

$\operatorname{im} i_* = \langle i_*(e)\rangle = \langle 0 \oplus -0\rangle = 0$

Need ?

So $i_* = 0$

$\ker i_* = \langle e\rangle = \operatorname{im}\partial$

So ∂ iso so

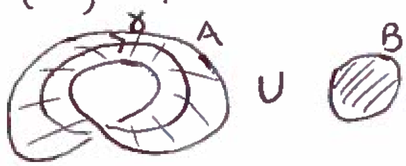
$\tilde H_2(M_2) \cong \mathbb{Z}$ and $\tilde H_1(M_2) \cong \mathbb{Z}^4$ } and have generators!

$e = \partial a$ where

$y = a + b$, $a \in A$, $b \in B$

Ex: $\mathbb{RP}(2) =$ 

$=$ Möbius strip $\cup$ disk



$A \cap B = \partial$ disk $= \partial$ Möbius $= S^1$

$H_1(A\cap B) = \langle \gamma \rangle$

$i_A[\gamma] = [2a] \in H_1(A) \cong \mathbb{Z} = \langle a \rangle$

$i_B[\gamma] = 0 \in \underbrace{H_1(B)}_{\text{Bk disk}} = 0$

$H_1 \mathbb{RP}^2 = H_1(A) \oplus H_1(B) / \operatorname{im} H_1(A\cap B)$

$= \mathbb{Z}/2\mathbb{Z}$

04/13/2015

Effect of attaching n-cells

Thm: Let X be a space.
$Y = X \cup_\varphi D^n$: $\varphi: \partial D^n \to X$
Then the following is exact:

$$\cdots \to \tilde H_k(S^{n-1}) \xrightarrow{\varphi_*} \tilde H_k(X) \xrightarrow{i_*} \tilde H_k(Y) \xrightarrow{\partial} \tilde H_{k-1}(S^{n-1}) \to \cdots$$

Pf: Use MV

$A = X$

$B = D^n$

$A \cap B = ?$

$A \cap B = \varphi(S^{n-1}) \cong S^{n-1}$

$A \cup B = Y$

Rank can only change when adding n-cell
H_n, H_{n-1}

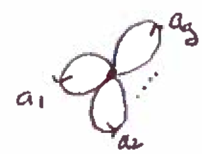
Then we have a "s.e.s."

$$\cdots \to \tilde H_k(A\cap B) \to \tilde H_k(A) \oplus \tilde H_k(B) \to \tilde H_k(A\cup B) \xrightarrow{\partial} \tilde H_{k-1}(A \cap B)$$

So...

$$\cdots \to \tilde H_k(S^{n-1}) \xrightarrow{\varphi_*} \tilde H_k(X) \oplus \underbrace{\tilde H_k(D^n)}_{0} \to \tilde H_k(Y) \xrightarrow{\partial} H_{k-1}(S^{n-1})$$

$[x] \mapsto [x] \oplus [-x]$ ∎

Ex: $X = \bigvee_{i=1}^{g} S^1$ 

Add a D^2 along $a_1^2 a_2^2 \cdots a_g^2$

To get $Y =$  $2g$-gon

$= \#_{i=1}^{g} \mathbb{RP}(2) = N_g$

H_0 & H_3 change by prev. Thm.

$$H_2(Y) \to \underbrace{H_1(\partial D^2)}_{=\mathbb{Z}} \xrightarrow{\varphi_*} H_1(X) \xrightarrow{j_*} H_1(Y) \to 0$$

$$\cdots 0 \to H_2(X) \xrightarrow{j_*} H_2(Y) \xrightarrow{\partial} \underbrace{H_1(\partial D^2)}_{=\mathbb{Z}}$$

$H_1(Y) = \mathbb{Z}^{2g}/\varphi_*(1)$

$\varphi_*(1) = 2a_1 + 2a_2 + \cdots + 2a_g$

$= \langle a_1, \dots, a_g \rangle / \langle 2(a_1, \dots, a_g)\rangle$

$= \underbrace{\mathbb{Z}^{g-1}}_{\langle a_1,\dots,a_{g-1}\rangle} \oplus \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\sum_i a_i}$

$H_2(Y) = ?$

φ_* is injective

$\ker \varphi_* = 0 \underset{\text{exact}}{=} \operatorname{im} \partial$

So $\partial = 0$

$H_2(Y) = \ker \partial / \operatorname{im} j^*$

But $\ker \partial = H_2(Y) = \operatorname{im} j^*$

So j^* is an iso.

$\therefore H_2(Y) = 0$ (as $H_2(X) = 0$)

EX: $Y = \mathbb{R}P(5)$ (Going backwards)

$X = \mathbb{R}P(5) \setminus \mathring{D}^5$

$Y = X \cup_{id} D^5$ → one cell in each dim attaching maps ct. 0 & deg 2

i	$H_i(\mathbb{R}P^5)$
0	$\mathbb{Z}$ → path connected
1	$\mathbb{Z}/2\mathbb{Z}$
2	0
3	$\mathbb{Z}/2\mathbb{Z}$
4	0
5	$\mathbb{Z}$
6	0

Only H_4 & H_5 can be different.

$H_0, H_1, H_2, H_6, H_7, H_8$ ..... of X match.

What about H_4 & H_5?

$\tilde{H}_5(X) \to \tilde{H}_5(Y) \xrightarrow{\partial} \tilde{H}_4(S^4) \xrightarrow{\varphi_*} \tilde{H}_4(X) \twoheadrightarrow \cancel{\tilde{H}_4(Y)} \to 0$ ($\tilde{H}_4(Y) = 0$)

$0 \to \cancel{\tilde{H}_5(S^4)} \to \tilde{H}_5(X) \to \tilde{H}_5(Y) \xrightarrow{\partial} \tilde{H}_4(S^4) \xrightarrow{j^*} \tilde{H}_4(X) \to 0$ ($\tilde{H}_5(S^4) = 0$)

So $0 \to \tilde{H}_5(X) \to \underset{\mathbb{Z}}{\tilde{H}_5(Y)} \xrightarrow{\partial} \underset{\mathbb{Z}}{\tilde{H}_4(S^4)}$

$0 \to \tilde{H}_5(X) \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{j^*} \tilde{H}_4(X) \to 0$

Sphere S^4 gen. for $\tilde{H}_4(S^4)$

5-cell gen. for $\tilde{H}_5(Y)$

$\partial[5\text{-cell}] = \partial[a+b] = [\partial b]$ (or $[\partial a]$) $= [S^4]$

a in 5-cell

b in D^5

So ∂ is an isomorphism

$\operatorname{im} j = 0$ & $\ker j^* = \mathbb{Z}$ by exactness

$\tilde{H}_4(X) = \mathbb{Z}/\mathbb{Z} = 0$

$\tilde{H}_5(X) = 0$

Homology with coefficients

Let G be an abelian group

X a space.

Recall $C_n(X) = \left\{ \sum_i n_\alpha \sigma_\alpha \;\middle|\; \begin{array}{l} \sigma_\alpha : \Delta^n \to X \\ n_\alpha \in \mathbb{Z} \\ \text{finite support} \end{array} \right\}$

Def: The group of n-chains with coefficients in G is

$C_n(X; G) \overset{\text{def}}{=} \left\{ \sum_i n_\alpha \sigma_\alpha \;\middle|\; \begin{array}{l} \sigma_\alpha : \Delta^n \to X \\ n_\alpha \in G \\ \text{finite support} \end{array} \right\}$

Note: Need these to be abelian groups ??

$$\partial\sigma = \sum_{i,j} (-1)^j n_i \sigma_i\big|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}$$

$x \in C_n(X;G)$

$$x = \sum_{i=1}^{k} g_i \sigma_i$$

$$\partial x = \sum_{i=1}^{k} g_i \left(\sum_j (-1)^j \, j^{\text{th}} \text{ face of } \sigma_i\right)$$

$\partial^2 = 0$ with same proof.

This gives us homology with G-coefficients

$$H_n(X;G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

* G abelian so $\mathbb{Z}$-module so integers (like ± 1) act in the "normal" way.

Remark: "All" the theorems we have proved "work" in $H_n(X;G)$.

i) MV
ii) Excision
iii) Les relative hm.

Ex Thm: If X space, then $H_0(X;G) \cong G^{\#\text{path components}}$

"Pf": B/c $C_1 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

$\sum n_\alpha \sigma_\alpha \mapsto \sum n_\alpha$

$C_1(X;G) \to C_0(X;G) \xrightarrow{\varepsilon} G \to 0$

$\sum g_\alpha \sigma_\alpha \mapsto \sum g_\alpha$

Ex Thm: $H_k(\text{point}; G) \cong \begin{cases} G, & k=0 \\ 0 & \text{otherwise} \end{cases}$

"Pf": Same as before

Ex Thm: $\tilde{H}_k(S^n, G) = \begin{cases} G, & k=n \\ 0, & k \neq n \end{cases}$

Lem: If $f: S^n \to S^n$ has degree m, then

$$f_*: H_n(S^n;G) \to H_n(S^n;G)$$
$$g \mapsto mg$$

Ex: $\mathbb{R}P(n) = X$

$G = \mathbb{Z}/2\mathbb{Z}$

Calculate $H_n(X;G)$

CW complex has 1 cell in each dim up to n.

$$\begin{array}{ccccccccccccccc} (n+1) & & (n) & & (n-1) & & (n-2) & & & & 2 & & 1 & & 0 \\ 0 \to & & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \cdots & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} 0 \end{array}$$

So $H_k(X;G) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k \leq n \\ 0, & k > n \end{cases}$

Smart choice of G. Easy calculation and still distinguishes between the $\mathbb{R}P(n)$'s.

Ex: $\mathbb{R}P(n) = X$ (n odd for simplicity)

$\mathbb{Q} = G$

$$\overset{(n)}{\mathbb{Q}} \xrightarrow{0} \overset{(n-1)}{\mathbb{Q}} \xrightarrow{2} \cdots \xrightarrow{0} \overset{(2)}{\mathbb{Q}} \xrightarrow{2} \overset{(1)}{\mathbb{Q}} \xrightarrow{0} \overset{(0)}{\mathbb{Q}} \xrightarrow{0} 0$$

x2 map iso. (over $\mathbb{Q}$)

$\therefore \ker \times 2 = 0$

$\operatorname{im} \times 2 = \mathbb{Q}$

$$H_k(X; G) = \begin{cases} \mathbb{Q}, & k = 0, n \\ 0, & k \neq 0, n \end{cases}$$

rank $\tilde{H}_k(\mathbb{R}P(n\text{ even})) = 0 \ \forall k$.

Ex: $X = S^n$

$Y = S^n \cup_\varphi D^{n+1}$

$\varphi: \underset{\overset{\shortparallel}{S^n}}{\partial D^{n+1}} \to S^n$

where φ has degree m.

Calculate $H_k(Y; \mathbb{Z}/m\mathbb{Z})$

Effect of attaching $(n+1)$-cell (MV)

So we have an exact sequence

$\cdots \to \tilde{H}_k(S^n; G) \to \tilde{H}_k(X; G) \to \tilde{H}_k(Y; G) \xrightarrow{?} \tilde{H}_{k-1}(S^n; G) \to \cdots$

$$0 \to \underset{\overset{\shortparallel}{0}}{\tilde{H}_{n+1}(X; G)} \to \tilde{H}_{n+1}(Y; G) \to \underset{\overset{\shortparallel}{G}}{\tilde{H}_n(S^n; G)} \xrightarrow[\overset{\shortparallel}{0}]{\varphi_*\ \ m} \underset{\overset{\shortparallel}{G}}{\tilde{H}_n(X; G)} \xrightarrow{j_*} \tilde{H}_n(Y, G) \to 0$$

Know ∂ inj., $\operatorname{im} \partial = \ker m = \tilde{H}_n(S^n; G) = G$

So $\tilde{H}_{n+1}(Y; \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$

$\tilde{H}_n(Y; \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$

$\ker j_* = 0$
but j_* onto
So iso

So $\mathbb{Z}/m\mathbb{Z}$ probably not the group to use.

04/15/2015

Final Exam:
May 1st Friday
10:15 - 12:15

Eilenberg - Steenrod Axioms

Def: A category is a collection of objects and morphisms between them
$\mathcal{C} = (\text{obj.}, \text{Mor})$

For $X, Y \in \text{Obj } C$

$\text{Mor}(X,Y)$ = collection of morphisms from X to Y.
(could be empty).

For $X, Y, Z \in \text{Obj } C$
there is an associative composition.

$\text{Mor}(X,Y) \times \text{Mor}(Y,Z) \to \text{Mor}(X,Z)$

There is (by assumption)
$1_X \in \text{Mor}(X,X) \ni$
$f 1_X = f \quad \forall Y \in \text{Obj } C$
& $f \in \text{Mor}(X,Y)$ and
$1_X g = g \quad \forall Z \in \text{Obj } C$
& $\forall g \in \text{Mor}(Z,X)$.

Ex:

1. (Sets, functions)
2. (Groups, homo.)
3. (Rings, ring homo.)
4. (Rmod, mod. homo)
5. (Top. spaces, cont. functions)
6. (CW complex, cont.)
7. (vector spaces over $\mathbb{K}$, lin trans.)
8. (Chain complexes over $\mathbb{Z}$, chain maps)

Def: Given two categories C, D a (covariant) functor is an assignment

$\forall X \in \text{Obj } C \xrightarrow[F]{} F(X) \in \text{Obj } D$

and

$\forall X, Y \in \text{Obj } C$ & $\forall f \in \text{Mor}(X,Y)$
$F(f) \in \text{Mor}_D(F(X), F(Y))$

But we need

i) $F(1_X) = 1_{F(X)} \quad \forall X \in \text{Obj } C$
ii) $F(g \circ f) = F(g) F(f) \quad \forall X, Y, Z \in \text{Obj } C$ and $\forall f \in \text{Mor}(X,Y)$, $g \in \text{Mor}(Y,Z)$

Ex:

1. F: Top with basepoint $\xrightarrow{\pi}$ Groups
2. Top $\xrightarrow{C_n}$ Chain complexes $\mathbb{Z}$
3. Top $\xrightarrow{H_n}$ Ab. groups

Axioms for Homology

Consider the category of CW pairs and morphisms maps of pairs.

Let h be a functor to the category of ($\mathbb{Z}$-graded) abelian groups

$h_n(X,A)$ abelian group $\}$ Fixed $n \in \mathbb{Z}$

$f_* = h_n(f)$ morphism.

We call h a homology functor if it satisfies

i) $1_{(X,A)*} = 1_{h_n(X,A)}$

ii) $f_* g_* = (f \circ g)_*$

iii) $\exists$ homo. $\partial: h_n(X,A) \to h_n(A)$ that is natural in the sense that if $f: (X,A) \to (Y,B)$ then

$$\begin{tikzcd} h_{n+1}(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"] & h_{n+1}(Y,B) \arrow[d, "\partial"] \\ h_n(A) \arrow[r, "f_*"] & h_n(B) \end{tikzcd}$$

iv) For inclusions $i: (A,\emptyset) \to (X,\emptyset)$ and $j: (X,\emptyset) \to (X,A)$, $\exists$ a les

$$\cdots \to \underbrace{h_n(A)}_{w/\emptyset} \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X,A) \xrightarrow{\partial} h_{n-1}(A) \to \cdots$$

v) $f \simeq g$ then $f_* = g_*$

vi) $\bar{U} \subset \mathring{A} \subset X$ then inclusions induce iso. $\forall n$ $h_n(X \setminus U, A \setminus U) \cong h_n(X,A)$

vii) $h_n(\text{point}) = 0 \quad \forall n \neq 0$

Rem: $h_0(\text{point})$ is called coefficient group of this homology theory.

Thm: $\forall$ abelian groups G

"Existence" $\{$ $\exists$ homology theory h_n having G as its coefficient group.

Pf. $h_n(X,A) = H_n(X,A;G)$ ∎

Thm: Let G be the coeff. group of h_n. Then $\tilde{h}_i(S^n) = G$ if $i = n$ and 0 otherwise.

Pf: There is les

$(X,A) = (D^n, \partial D^n)$ $\quad D^n/\partial D^n \cong S^n$ $\quad \partial D^n \cong S^{n-1}$

$$\begin{tikzcd} \underbrace{\tilde{h}_k(D^n)}_{0} \arrow[r] & \tilde{h}_k(D^n, \partial D^n) \arrow[r, "\partial"] & \tilde{h}_{k-1}(\partial D^n) \arrow[d] \\ & & \underbrace{\tilde{h}_{k-1}(D^n)}_{0} \end{tikzcd}$$

induct downwards ∎

Thm: On the category of CW pairs, any homology theory is uniquely determined by its coefficient group.

"Pf":

1. Construct a theory h^{cw} for homo. theory h.
2. Prove it is h.
3. Glue on cells and calculate effect on homology

Glue n-cell on

$\varphi: S^{n-1} \to X^{n-1}$

What is

$\varphi_*: h^{cw}_{n-1}(S^{n-1}) \to h^{cw}_{n-1}(X)^{n-1}$

So induct downwards to G to figure out these maps.

$$G \to h^{cw}_{n-1}(X^{n-1})$$

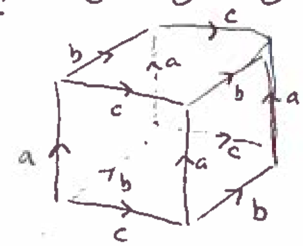
h sing. hom. given by degrees. Is there a notion of degrees here?

This is the difficult part.

Use idea from homotopy - suspensions

Turns n-sphere to n+1-sphere so can work upwards through dimensions.

Ex: $S^1 \times S^1 \times S^1$



There is a cell in the middle

1 0-cell

3 2-cells

3 1-cells

CW chain complex

$$0 \to \overset{(3)}{\mathbb{Z}} \xrightarrow{0} \overset{(2)}{\mathbb{Z}^3} \xrightarrow{0} \overset{(1)}{\mathbb{Z}^3} \xrightarrow{0} \overset{(0)}{\mathbb{Z}} \xrightarrow{0} 0$$

All faces have trivial boundary

$$H_i(T^3) = \begin{cases} \mathbb{Z} & i=0,3 \\ \mathbb{Z}^3, & i=1,2 \\ 0, & \text{otherwise} \end{cases}$$

$$T^k = \underbrace{S^1 \times \cdots \times S^1}_{k}$$

and T^0 = point

K	rank H_i				
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Pascal?

Do $H_i(T^k)$ & $H_{i+1}(T^k)$ combine to give $H_{i+1}(T^{k+1})$?

04/20/2015

Assume submanifolds have product neigh

Prop: If $D \subset S^n$ with $D \cong D^k$ for some $k \geq 0$, then $\tilde{H}_*(S^n \setminus D) = 0$.
Furthermore, if $S \subset S^n$ with $S \cong S^k$ for some $k \geq 0$ then
$$\tilde{H}_i(S^n - S) = \begin{cases} \mathbb{Z}, & i = n-k-1 \\ 0, & \text{otherwise} \end{cases}$$

Cor: Jordan Curve Thm: Any simple closed curve $\to S^1$ in S^2 separates S^2 into two components.

Pf: $\tilde{H}_0(S^2 \setminus S^1) = \mathbb{Z}$ if $i = 2-1-1 = 0$

Cor: Higher Jordan Sep. Thm: Any smooth $S^n \subset S^{n+1}$ separates S^{n+1} into two components.

Cor: $S^n \setminus D^k$ is connected.

Cor: $K: S^1 \hookrightarrow S^3$ is a Knot. Then $H_i(S^3 \setminus \operatorname{im} K) =$
$$\begin{cases} \mathbb{Z}, & i = 0,1 \\ 0, & \text{otherwise} \end{cases}$$

Pf: Use prop ~~of~~ MV or excision

Pf (Prop): We use induction on k.

If $k=0$
$$S^n \setminus D^0 = S^n \setminus \{\ ,1\} = \mathbb{R}^n$$
And red. hom of $\mathbb{R}^n$ is 0.

Assume true up to $k-1$. $D \cong D^k$

$D \subset S^n$. Denote

Square region of dijk $\to$ $h: I^k \to D$

a homeomorphism. Let
$$A = S^n \setminus h(I^{k-1} \times [0, 1/2])$$
$$B = S^n \setminus h(I^{k-1} \times [1/2, 1])$$

$$A \cap B = S^n \setminus D$$
$$A \cup B = S^n \setminus h(\underbrace{I^{k-1} \times \{1/2\}}_{\approx D^{k-1}})$$

MV says have exact sequence
$$\cdots \to \tilde{H}_{i+1}(A \cup B) \xrightarrow{\partial} \tilde{H}_i(A \cap B) \xrightarrow{i_*} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \xrightarrow{j_*} \tilde{H}_i(A \cup B) \to$$

"Know": $\tilde{H}_{i+1}(A \cup B) = 0$ by inductive hypothesis so i_* is an isomorphism. Suppose $\tilde{H}_i(A \cap B) \neq 0$. So $\exists\ [x] \neq 0$.

$i_*([x]) = [x] \oplus -[x]$, so as is so:

Know $[x]$ is nonzero in A or B. WLOG, $[x] \neq 0$ in $\tilde{H}_i(A)$. Get a sequence

$I_j \supset I_{j+1} \supset \cdots$

claim $I_j = 1/2^j \to 0$

$I_0 = [0,1]$

$I_1 = [0, 1/2]$

Repeat MV to get

$[x] \neq 0$ in

$\tilde{H}_i(S^n \setminus h(I^{k-1} \times I_j))$

$A = S^n \setminus h(I^{k-1} \times I_1)$

$\cap$

$S^n \setminus h(I^{k-1} \times I_j)$

$\cap$

$S^n \setminus h(I^{k-1} \times pt)$

to

$= \cap I_j$

So we have a commutative diagram

$$\begin{tikzcd} \tilde{H}_i(A) \arrow[r, "i_*"] \arrow[dr, "i_*"'] & \tilde{H}_i(S^n \setminus h(I^{k-1} \times t_0)) \\ & \tilde{H}_i(S^n \setminus h(I^{k-1} \times I_j)) \arrow[u, "i_*"'] \end{tikzcd}$$

$$\begin{tikzcd} {[x]} \arrow[rr] \arrow[dr] & & 0 \text{ by induction hypothesis} \\ & \pm[x] \arrow[ur] & \end{tikzcd}$$

So $[x] = 0$. So $x = \partial B$ for some $B \in C_{i+1}(\underbrace{S^n \setminus h(I^{k-1} \times t_0)}_{\text{Call this } J})$

So $B = \sum_{\text{finite}} n_\alpha \sigma_\alpha$, where $\sigma_\alpha : \Delta^{i+1} \to J$
compact

So $\bigcup_{\text{finite}} \operatorname{im}(\sigma_\alpha)$ is compact. (finite union compact compact).

$\{S^n \setminus h(I^{k-1} \times I_j)\}_{j=1}^{\infty}$ is an "increasing cover" of, by compactness $\exists j$

$\ni \operatorname{Im} \sigma_\alpha \subset S^n \setminus h(I^{k-1} \times I_j)$

Therefore, $B \in C^{i+1}(S^n \setminus h(I^{k-1} \times I_j))$

So $[x] = 0$ in $\tilde{H}_i(S^n \setminus h(I^{k-1} \times I_j))$

a contradiction. ☒

For the next part, induction?

$S^n \setminus S^0 = S^n \setminus \{-1, 1\} =$

$\mathbb{R}^n \setminus \{1\} \underset{\text{def}}{\simeq} S^{n-1}$

Only nontriv. hom if $n-1=0$, $n-1$

Assume true for $\dots$ $k-1$.

Suppose $S \cong S^k$ and $S \subset S^n$

$S = D_1 \cup D_2$

$D_1 \cong D^k$, $D_2 = D^k$

Let $A = S^n \setminus D_1$

$B = S^n \setminus D_2$

Then $A \cap B = S^n - (D_1 \cup D_2) = S^n \setminus S$

$A \cup B = S^n \setminus S^{k-1}$

$A \cap B \cong S^{k-1}$

Again, use MV. SES

$$\cancel{\tilde{H}_{i+1}(A)} \oplus \tilde{H}_{i+1}(B) \to \tilde{H}_{i+1}(A \cup B) \xrightarrow{?} \tilde{H}_i(A \cap B) \hookrightarrow \cancel{\tilde{H}_i(A)} \oplus \tilde{H}_i(B)$$

0   By prev. result

$H_i(A \cap B) \cong H_{i+1}(A \cup B)$

By hypothesis, nontrivial when

$i+1 = n-(k-1)-1$

$i+1 = n-k+1-1$

$i = n-k-1$

Invariance of Domain

Prop: If $\mathcal{U}, V \subset S^n$ with $\mathcal{U} \cong V$ and $\mathcal{U}$ is open, then V is open.

Not true usually!

$I = [0,1]$

then $(1/2, 1]$ open

and $(0, 1/2]$ not open

but homeomorphic.

Cor: True in $\mathbb{R}^n$ as well.

Rem: $S^n \setminus \text{point} = \mathbb{R}^n$ open in S^n.

Pf (Prop): Assume hypothesis.

Let $y \in V$. Will build a V-neigh Y of y and show Y is open in S^n. (shows V open)

Let $h: \mathcal{U} \xrightarrow{\sim} Y$ a homeo.

and $x \in \mathcal{U}$ the point $\ni h(x) = y$.

$\mathcal{U}$ is open in S^n, so $\exists$ neigh of x that is homeo. to $\mathbb{R}^n$ and $\mathbb{R}^n \subset \mathcal{U}$. Inside $\mathbb{R}^n$, put a closed disc $\underset{\cong\, D^n}{A}$ cont. x.

$\partial A \cong \partial D^n \approx S^{n-1}$

$\mathring{A} = A \setminus \partial A$

Let $B = h(A) \approx D^n$

$\partial B = h(\partial A)$

Has product neigh in V.

$\underbrace{\mathring{B} = B \setminus \partial B}_{\text{interior in } V}$

WTS $h(\mathring{A})$ is open in S^n.

Consider $S^n \setminus \partial B$ open in S^n

$S^n \setminus \partial B$ has 2 path components by prev. prop.

$\underbrace{B \setminus \partial B}_{\text{Yes, nonempty}} \sqcup \underbrace{S^n \setminus B}_{S^n \setminus D^n \text{ connected by prev. prop.}} = S^n \setminus \partial B$

So $B \setminus \partial B$ a path component and open. So $B \setminus \partial B$ in $S^n \setminus \partial B$ is open in S^n. Done.

So $\mathring{B}$ is open in S^n. ∎

$Y = \mathring{B}$

Cor: If M is a compact n-manifold and N is a connected n-manifold then any embedding $f: M \hookrightarrow N$ is a surjection.

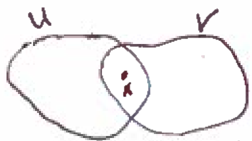
Pf: M is a compact subspace of Hausdorff space N so M is closed in N.

Now only need show M is open (as N is connected).

Let $x \in M$ so

$\underbrace{f(x)}_{\in N} = \underbrace{x}_{\in M}$

$\exists$ open set U in M and V in N $\ni$ $x \in U \cong \mathbb{R}^n$ and $x \in V \cong \mathbb{R}^n$



Pick U inside $\underbrace{f^{-1}(V)}_{\text{open as } f \text{ cont.}}$

$U \cong \underbrace{V}_{\text{open}} = \mathbb{R}^n$

So U open in V open in N by invariance of domain. ∎

04/22/2015

Cor: If M is a compact manifold and $f: M \hookrightarrow N$ is an embedding then f is surjective.

Def: A homeomorphism is a cont. bijection that is open.

Prop: If $f: X \to Y$ is cont. and bijective and X, Y compact Hausdorff then f is open. Hence, f is a homeomorphism.

Cor: $S^n \not\hookrightarrow \mathbb{R}^n$

Cor: $\mathbb{R}^n$ contains no compact n-manifolds.

Cor: $\mathbb{R}^m$ does not embed in $\mathbb{R}^n$ if $m > n$.

*Fields act on vector spaces by scalar multiplication.

Def: An algebra A over $\mathbb{R}$ is a v.s. over $\mathbb{R}$ that has a multiplication (not nec. associative/comm.)

$A \times A \xrightarrow{\times} A$

This mult. has to be compatible with scalar mult. by $\mathbb{R}$.

If in addition, $\forall a \in A$ & $b \neq 0 \in A$ have equations $a = bx$ and $a = yb$ unique solutions $x \in A$ & $y \in A$, we call A a division algebra.

Ex: $A = \mathbb{R}^3$
$\times$ = cross (vector) product

Not associative or commutative but is compatible with scalar-multiplication.

Is $(\mathbb{R}^3, \times)$ a division algebra?

No, $a = 0$, $b = 2i$; $a = b \times x$ has inf. many solutions. $x = ri$ solution

Ex: $\mathbb{C}$ is a division algebra over $\mathbb{R}$ and $\mathbb{C}$.

Ex: $\mathbb{H}$ div. algebra over $\mathbb{R}$. Not Commutative but associative.

Thm (Hopf): The only finite dimensional div. algs. over $\mathbb{R}$ are $\mathbb{R}$ & $\mathbb{C}$.

Pf: Let A be such an algebra. $A \underset{v.s.}{\cong} \mathbb{R}^n$

$\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\times} \mathbb{R}^n$

$(x+y) \times z \longmapsto xz + yz$

Bilinear function. These are continuous.

Consider $f: S^{n-1} \to S^{n-1}$ by $f(x) = \dfrac{x^2}{\|x^2\|}$

Cont. map sphere to itself.

Why $\|x^2\| \neq 0$?

Need check $x \cdot x \neq 0$

$\underbrace{x}_{\text{unit vec. so nonzero}} \cdot x = 0$?

So this has unique solution

A solution is $x = 0$

$\cancel{\Rightarrow}$.

$$f(-x) = \frac{(-x)^2}{\|(-x)^2\|} = \frac{x^2}{\|x^2\|} = f(x)$$

Modding by antipodes, $\exists$ cont.

$\bar{f}: \mathbb{R}P(n-1) \to S^{n-1}$

Claim $\bar{f}$ is injective.

Suppose $f(x) = f(y)$ ie

$$\frac{x^2}{|x^2|} = \frac{y^2}{|y^2|}$$

So $x^2 = \underbrace{\dfrac{|x^2|}{|y^2|}}_{>0} y^2$

Let $\alpha = \sqrt{\dfrac{|x^2|}{|y^2|}} \in \mathbb{R}$

Commutivity ?! Consider
$$x^2 - \alpha^2 y^2 = 0$$
$$(x - \alpha y)(x + \alpha y) = 0$$

Clear one of these is 0 (only one by unique solvability).

$x = \pm \alpha y$

x has norm 1 so $\alpha = \pm 1$

So $\bar{f}$ injective.

$\bar{f}: \underbrace{\mathbb{R}P(n-1)}_{\text{compact}} \longrightarrow \underbrace{S^{n-1}}_{\text{connected if } n-1>0}$

$(n-1)$-manifolds

Either $n = 1$ or $\mathbb{R}P(n-1) \cong S^{n-1}$

$A \underset{v.s.}{\cong} \mathbb{R}$

$\mathbb{R}P(1) \cong S^1$

$\underset{\text{nontriv. hom.}}{\mathbb{R}P(k)}$ vs $\underset{\text{triv. hom.}}{S^k}$

So not homeo

$n - 1 = 1$

$A \underset{v.s.}{\cong} \mathbb{R}^2$

Is $(A, \times) \cong (\mathbb{C}, \cdot)$?

Is there an exotic div. algebra structure? on $\mathbb{R}^2$

No: Let $\{1, v\}$ basis for $\mathbb{R}^2$ so span $\approx \mathbb{R} \subset \mathbb{R}^2$

$|v^2| \neq 0$

Scale v to have $|v^2| = 1$.

$\vdots$

$v \cdot v = a > 0$

$v^2 - a = 0$

Modules, Rings, & Tor

Modules $\rightsquigarrow$ Rings

V.S. $\rightsquigarrow$ Fields

Def: Let M be an R-mod. An element $m \in M$ is R-torsion if $\exists\ 0 \neq r \in R \ni rm = 0$. M is a torsion R-mod if all elements $\in M$ are R-torsion.

Ex: $\mathbb{Z}/5\mathbb{Z}$ is a torsion $\mathbb{Z}$-module but is not a torsion $\mathbb{Z}/5\mathbb{Z}$-torsion module.

If only trivial torsion, then it is torsion free.

Def: A free module is one that has a lin. indep. generating set.

Thm: If R is a PID then free module $\Leftrightarrow$ torsion free.

Tensor Product Let M & N be modules over a ring R. The tensor product is $M \otimes_R N$ is the abelian group generated by $\{m \otimes n\}$ for $m \in M$ and $n \in N$ with relations

$(m + m') \otimes n = m \otimes n + m' \otimes n$

Same on n side

and

$r(m \otimes n) = rm \otimes n = m \otimes rn$

This gives the tensor product an R-module structure.

Ex: $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Q}$ are $\mathbb{Z}$-modules

$\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$

Note $0 \otimes 0 = 0 \otimes m = n \otimes 0$

$i \otimes r/s = i \otimes 3\frac{r}{3s} = 3i \otimes r/3s$
$= 0 \otimes r/3s$
$= 0$

Ex: $\mathbb{Z} \otimes_{\mathbb{Z}} G$

G abelian group

$\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$

$\otimes$ properties of $\mathbb{Z}$-modules (Abelian groups)

$\otimes = \otimes_{\mathbb{Z}}$

i) $A \otimes B \cong B \otimes A$

ii) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

iii) $\left(\bigoplus_i A_i\right) \otimes B \cong \bigoplus_i (A_i \otimes B)$

iv) $\mathbb{Z} \otimes A \cong A$

v) $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$

vi) $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$

vii) If $f: A \to C$, $g: B \to D$ are homomorphisms

$f \otimes g : A \otimes B \to C \otimes D$

by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$

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Class Cancelled