Algebraic Topology

Office Hours: TTh 3:30-4:30

Hatcher + Bredon, 'Top. & Geo.'

08/30/2016

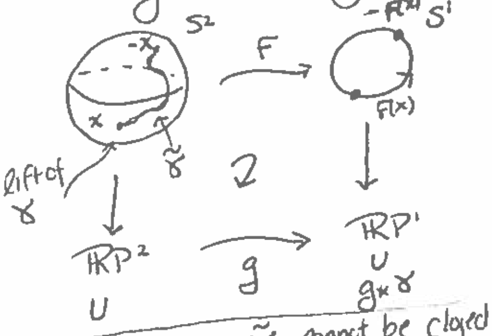
Applications

Borsuk-Ulam Theorem: $\nexists$ cont. $F: S^2 \to S^1$ such that $F(-x) = -F(x)$ $\forall x \in S^2$.

Pf: Suppose $\exists$ $F: S^2 \to S^1$. Then F descends to a map $g: \mathbb{R}P^2 \to \mathbb{R}P^1$. g is continuous. So g induces map $g_*: \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$

$$\begin{tikzcd} \pi_1(\mathbb{R}P^2) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_1(\mathbb{R}P^1) \arrow[d, phantom, "\shortparallel"] \\ \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

So $g_* = 0$. But $g_* \neq 0$:



loop $\gamma \neq 0$. $\tilde{\gamma}$ cannot be closed as $\gamma \neq 0$ (all closed loops on sphere ?) $F_* \tilde{\gamma}$. Clear $g_*\gamma$ is a loop. $g_*\gamma$ has a lift which is not closed so $g_*\gamma \neq 0$. Then $g_* \neq 0$.

$\Rightarrow\Leftarrow$. ☒

Fact (Thm): There is no such cont map $F: S^{n+1} \to S^n$ for $n > 0$ taking antipodes to antipodes. Proof could use Homology, eg Hatcher pp 174. There is a much easier proof using cohomology, makes use of ring structure.

Now for another application:

Def: An H-space is a top. space X with a cont map $\mu: X \times X \to X$ and a dist. element $e \in X$ such that

i) $x \mapsto \mu(x,e)$

ii) $x \mapsto \mu(e,x)$

are homotopic to 1_X. See Hatcher pp. 281. This gives a 'mult.', where e serves as an 'identity' up to homotopy.

Ex: All topological groups are H-spaces, e.g. all Lie groups.

Thm: For n even and $n > 0$, S^n cannot be an H-space.

The proof will use cohomology.

A stronger thm. is due to Adams: S^n is an H-space only for $n = 0, 1, 3, 7$.

Mentioned in Hatcher on pp. 428

S^0: $\quad \underset{0}{\bullet} \quad \underset{1}{\bullet}$ "$\cong$" $\mathbb{Z}/2\mathbb{Z}$ $\uparrow$ Disc. top. grp

S^1: 

$= SO(2, \mathbb{R})$

$e^{i\theta} = U(1)$

S^3: $Spin(3) = SU(2)$

$=$ unit sphere in $\mathbb{H}^\times$

S^7: Unit sphere in $\mathbb{O}^\times$

In fact, S^7 cannot be endowed with a Lie group structure.

Now review graded rings and multilinear algebra.

F field (though can be relaxed)

Def: A F-algebra is a F vector space with a bilinear mult.

$A \times A \to A$

Rem: Can relax with F comm. ring and A an F-module. We will assume A is unital and associative.

Note: Unital associative $\mathbb{Z}$-algebra is a ring.

Def: A F-algebra is graded if

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

such that $A_i A_j \subseteq A_{i+j}$

A_i called homogeneous component of degree i.

Notation: If $a \in A_i$, we write $\deg a = i$ or $|a| = i$. We say 'a' is homogeneous of degree i.

Remark:

1) $0 \in A_i$ does not have a well defined degree

2) We will often assume

$$A = \bigoplus_{n \geq 0} A_n$$

ie, $A_n = 0$ for $n < 0$.

3) If $\exists\, 1 \in A$ then $1 \in A_0$

Ex: $A = \mathbb{C}[x]$ via $\deg x^n = n$

$$\mathbb{C}[x] = \bigoplus_{n \geq 0} \mathbb{C}x^n$$

Def: If A is graded, then we say is graded commutative if...

$$ab = (-1)^{ij} ba$$

$\forall a \in A_i$ and $b \in B_j$. This means even degreed elements commute with everything. All odd elements anticommute. So if a is odd then $a^2 = a \cdot a = -a \cdot a = -a^2$ then $2a^2 = 0$.

We will later see cohomology rings of top. spaces are commutative graded.

Tensor Product

Let V, W be v.s. over F

Def: The tensor product of V, W is an F-vector space $V \otimes W$ with an F-lin. map

$$i: V \times W \to V \otimes W$$

such that for all bilinear maps $B: V \times W \to \mathcal{U}$, $\exists!$ lin. map $\tilde{B}: V \otimes W \to \mathcal{U}$ such that

$$B = \tilde{B} i$$

$$\begin{tikzcd}
 & V \otimes W \arrow[d, dashed, "\exists \tilde{B}"] \\
V \times W \arrow[ur, "i"] \arrow[r, phantom, "\circlearrowleft"] \arrow[r, "B"'] & \mathcal{U}
\end{tikzcd}$$

Notation: $v \in V$ and $w \in W$ then $v \otimes w$ for $i(v, w)$.

Thm: If $V \otimes W$ exists, it is unique up to canonical iso. $\hookrightarrow$ compatible with bilin. maps

Thm: $V \otimes W$ exists.

⌜

$$\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & V \otimes W \arrow[d, "\Phi"] \\
V \times W \arrow[r, "B'"'] & (V \otimes W)'
\end{tikzcd}$$

Φ iso

⌟

Notice if $\dim_F V = n$ & $\dim_F W = m$ then $\dim_F V \otimes W = nm$

Recall:

1) $F \otimes V \cong V \otimes F \cong V$

2) $V \otimes W \cong W \otimes V$

3) $(V \otimes W) \otimes \mathcal{U} \cong V \otimes (W \otimes \mathcal{U})$

Let V be an F-vector space

$$V^{\otimes 0} \overset{\text{def}}{=} F$$

$$V^{\otimes n} \overset{\text{def}}{=} \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \overset{\text{def}}{=} V^{\otimes n-1} \otimes V$$

Def: $V^{\otimes n}$ is called the n^{th} tensor power.

Rem: If $\beta = \{e_i\}_{i \in I}$ is a basis for V then $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{i_j \in I}$ is a basis for $V^{\otimes n}$.

Def: The tensor algebra

$$T(V) \overset{\text{def}}{=} F \oplus V \oplus V^{\otimes 2} \oplus \cdots = \bigoplus_{n \geq 0} V^{\otimes n}$$

with mult. given by

$$(v_1 \otimes \cdots \otimes v_n) \times (w_1 \otimes \cdots \otimes w_m) \overset{\text{def}}{=} v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

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V vector space over field F

$V^{\otimes 0} = F$

$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$

Def: The tensor algebra $T(V)$ is $T(V) = F \oplus V \oplus V^{\otimes 2} \oplus \cdots$

$$= \bigoplus_{n \geq 0} V^{\otimes n}$$

with mult.

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) \overset{def}{=}$$

$$v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$$

Note if $\lambda \in F$ is a scalar and $v \in V$ then $\lambda \otimes v \overset{def}{=} \lambda v$. Consistent with fact $F \otimes V \cong V$.

The tensor algebra is graded with homogeneous component $T(V)_i = V^{\otimes i}$

1) $T(V)$ is not commutative
2) $T(V)$ is associative
3) $T(V)$ is unital : $1 \in F \} T(V)_0$
4) If $\{e_1, \cdots, e_n\}$ basis for V then

$$T(V) \cong \underset{\sim}{F\langle x_1, \cdots, x_n \rangle}$$

noncommutative poly. algebra in $x_1, \cdots, x_n$

ie free associative unitive F-algebra generated by $x_1, \cdots, x_n$.

$$e_{i_1} \otimes \cdots \otimes e_{i_n} \mapsto x_{i_1} \cdots x_{i_n}$$

Symmetric Algebra: $\mathrm{Sym}(V)$

Let $I^{\bullet}(V)$ denote two-sided ideal in $T(V)$ generated by $v \otimes w - w \otimes v$ for $v, w \in V$

$$\mathrm{Sym}(V) \overset{def}{=} T(V) / I^{\bullet}(V)$$

Notation: If $v_1, \cdots, v_n \in V$ then we write $v_1 \cdots v_n$ for equiv. class of $v_1 \otimes \cdots \otimes v_n$ in this quotient.

Note: In Symmetric Algebra

$$v \cdot w - w \cdot v = 0$$

so $v \cdot w = w \cdot v$ so $\mathrm{Sym}(V)$ is commutative.

1) $\mathrm{Sym}(V)$ is commutative
2) $\mathrm{Sym}(V)$ is unital and associative
3) $\mathrm{Sym}(V)$ is graded (~~commutative~~)

$$\mathrm{Sym}^n(V) = \mathrm{Span}\{v_1 \cdots v_n \mid v_i \in V\} \subseteq \mathrm{Sym}\, V$$

If $\{e_1, \cdots, e_n\}$ is a basis for V then

$$\mathrm{Sym}\, V \cong F[x_1, \cdots, x_n]$$

$$e_{i_1} \cdots e_{i_n} \mapsto x_{i_1} \cdots x_{i_n}$$

Exterior Algebra $\Lambda(V)$ ← wedge

See Hatcher pp 217

Let $I^\wedge(V)$ be two-sided ideal in $T(V)$ generated by $v\otimes v$ for $v\in V$.

$$\Lambda(V) \overset{\text{def}}{=} T(V)/I^\wedge(V)$$

Notation: If $v_1,\dots,v_n\in V$ we write $v_1\wedge\cdots\wedge v_n$ for equiv. class $v_1\otimes\cdots\otimes v_n$ in the quotient.

Note: $v\wedge v = 0$ $\forall\, v\in V$.

and

$$v\wedge w = -(w\wedge v)$$

Pf: $(v+w)\wedge(v+w)=0$

$$\overset{\|}{\cancelto{0}{v\wedge v} + v\wedge w + w\wedge v + \cancelto{0}{w\wedge w}}\ \square$$

1) $\Lambda(V)$ is unital and associative

2) $\Lambda(V)$ is graded commutative

$$\underset{\sim}{\Lambda^n} = \text{span}\{v_1\wedge\cdots\wedge v_n \mid v_i\in V\}\subseteq\Lambda(V)$$

n^{th} ext. power

$$a\wedge b = (-1)^{ij}\, b\wedge a \qquad a\in\Lambda^i(V),\ b\in\Lambda^j(V)$$

3) If $\{e_1,\dots,e_n\}$ basis for V then

$$\Lambda(V)\cong \frac{F\langle x_1,\dots,x_n\rangle}{\langle x_ix_j=-x_jx_i,\ x_i^2=0\rangle}$$

$$e_{i_1}\wedge\cdots\wedge e_{i_n}\mapsto x_{i_1}\cdots x_{i_n}$$

In Hatcher, this is denoted $\Lambda_p[x_1,\dots,x_n]$. See page 217.

Rem: If $\{e_1,\dots,e_n\}$ is a basis for V then

$$\{e_{i_1}\wedge\cdots\wedge e_{i_k} \mid 1\le i_1 < \cdots < i_k \le n\}$$

is a basis for $\Lambda^k(V)$.

Pf: LTR $\square$

Cor: If V is fin. dim. then $\dim\Lambda^k(V) = \binom{n}{k}$

In particular, $\Lambda^k(V)=0$ for $k>n$.

Cor: If V fin. dim. then

$$\begin{aligned}\dim\Lambda(V) &= \dim \bigoplus_i \Lambda^i(V)\\ &= \sum_i \binom{n}{i}\\ &= (1+1)^n\\ &= 2^n\end{aligned}$$

In $\Lambda(V)$, $v\wedge v=0$ for $v\in V$. But generally, $a\in\Lambda(V)$ then $a\wedge a$ can be nonzero.

Ex: $V=\mathbb{R}^4 = \text{Span}_{\mathbb{R}}\{e_1,e_2,e_3,e_4\}$

$$a = e_1\wedge e_2 + e_3\wedge e_4 \in \Lambda^2(V)$$

$$a\wedge a = 2\,\underbrace{e_1\wedge e_2\wedge e_3\wedge e_4}_{}$$

basis vector for $\Lambda^4(V)$ so is nonzero.

Rem: All these constructions of $T(V)$, $\text{Sym}\,V$, & $\Lambda(V)$ are functorial, i.e. $g: V\to W$ is F-lin., $\exists$ induced maps

$$T(g): T(V)\to T(W)$$

$$\text{Sym}(g): \text{Sym}\,V\to\text{Sym}\,W$$

$$\Lambda(g): \Lambda(V)\to\Lambda(W)$$

all alg. homo.

IF $V=W$ and dim $V<\infty$ then

$$\Lambda^n(g) : \Lambda^n(V) \to \Lambda^n(V)$$

is given by mult. by det g.

Kind of matrix free version of det.

Categories (Hatcher pp 162-165)

Def: A category $\mathcal{C}$ consists of obj. ← class of and morph., denoted obj C or C and Morph or sometimes Hom along with a composition

$$\circ : Mor(b,c) \times Mor(a,b) \to Mor(a,c)$$
$$(f,g) \mapsto f\circ g$$

such that $(f\circ g)\circ h = f\circ(g\circ h)$ and $\forall A \in Obj\ \mathcal{C}$, $\exists$ identity 1_A

$f\circ 1_A = 1_A \circ f \quad \forall f \in Hom_C(A,B)$

Def: A category is called small if Ob C is a set.

Ex:

Set: Obj: sets
Morph: functions

Ab: Obj: abelian grp
Morph: homo.

Gp: Obj: groups
Morph: group homo

R-mod: Obj: left R-mod
Morph: R-homo.

Top: Obj: Top spaces
Morp: cont. maps

All of these are not small.

Ex: X a top. space

Def: the fundamental groupoid $\pi_1(X)$ is category with

Obj: points of X

Morph: paths in X up to path hmty.

This category is small.
Composition is given by path concatenation.

$$Hom_{\pi_1(X)}(x_0, x_1) = \pi_1(X, x_0)$$

Def: Opposite category of category $\mathcal{C}$, denoted, $\mathcal{C}^{op}$ given by obj $\mathcal{C}^{op}$ = obj $\mathcal{C}$ and $Hom_{\mathcal{C}^{op}}(A,B) = Hom_{\mathcal{C}}(B,A)$ with order comp reversed.

Functors

Def: Categories $\mathcal{C}$ and D. A (covariant) Functor $F: \mathcal{C} \to D$ assigns obj $\mathcal{C}$ to obj D and morph in $\mathcal{C}$ to morph in D.

with $F(A) \in Obj\ D$, $f \in Hom_C(A,B)$ and $F(f) \in Hom_D(F(A),F(B))$

with $F(1_A) = 1_{F(A)}$ and

$$F(f\circ g) = F(f)\circ F(g)$$

Ex: N^{th} singular homology

$H_n : Top \to Ab$

Def: A contravariant functor is just a functor which reverses the order of composition.

Ex: N^{th} singular cohomology is a contravariant functor

$H^n : Top \to Ab$

Natural Transformations

$F, G : C \to D$ are functors

Def: A natural transformation $\tau : F \to G$ is a collection of morph. $\tau_A : F(A) \to G(A)$ for $A \in Obj\, C$ with

$$\tau_B F(f) = G(f) \tau_A$$

for $f \in Hom_C(A, B)$

Ex: $F : \overset{\to \text{vector space}}{V} \mapsto T(V)$
$G : V \mapsto \Lambda(V)$
} Functors from cat. of V.S to cat. of algebras

Then quotient map $T(V) \to \Lambda(V)$ induces a natural transformation $\tau : F \to G$

$$\tau_V : T(V) \underset{\text{quotient map}}{\to} \Lambda(V)$$

Direct Limits

(Hatcher pp 243-244)

I be a directed poset (directed: $\forall i, j \in I$, $\exists k \geq i, j$)

Suppose given abelian group G_i for each $i \in I$ and a group homo. $F_{ij} : G_i \to G_j$ for $i \leq j$. such that

1) $F_{ii} = 1_{G_i}$

2) F_{ij} compatible with comp.

$$F_{jk} F_{ij} = F_{ik}$$

Def: $\{G_i, G_j\}_{i,j}$ is called a direct system over the index set I.

Def: The direct limit of $\{G_i, F_{ij}\}$ is

$$G = \varinjlim_{i \in I} G_i = \coprod G_i / \sim$$

where $\sim$ equiv. relation given by $a \sim b$ if $\exists k \geq i, j$ $F_{ik}(a) = F_{jk}(b)$ $a \in G_i$, $b \in G_j$, $k \geq i, j$

So the elements "eventually" become equal under the maps.

Notation: If $a \in G_i$ then $[a]$ is the class of a in $G = \varinjlim G_i$. If G_i ab gps then $G = \varinjlim G_i$ is an ab grp.

The group structure:

$a \in G_i$
$b \in G_j$

Choose $k \geq i, j$

$F_{ik}(a) \in G_k$
$F_{jk}(b) \in G_k$

Define $[a] + [b] \overset{\text{def}}{=}$

$[F_{ik}(a) + F_{jk}(b)]$

Check well defined defines a group operation. ▨

Claim ∃ iso. between $\varinjlim G_i \cong \dfrac{\oplus G_i}{\langle a - F_{ij}(a) \mid a \in G_i, j \geq i \rangle}$

Can easily see this is a group.

"Pf": In sum every element is a finite sum of elements of $a_i \in G_i$ so in quotient is equivalent to an element in single G_k. So can replace direct sum by disjoint union

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There are natural maps $\phi_i : G_i \to \varinjlim G_i$ defined by $\phi_i(a) = [a]$. Note

$$\phi_i = \phi_j F_{ij} \quad \text{For } i \leq j.$$

The direct limit has the universal property:

Let H be an abelian group and $\psi_i : G_i \to H$ maps such that $\psi_i = \psi_j F_{ij}$ For $i \leq j$.

Then $\exists!$ map $u : G \to H \ni \psi_i = u \circ \phi_i$

So we have....

$$\begin{tikzcd}
\varinjlim G_i \arrow[rr, dashed, "u"] & & H \\
& G_i \arrow[ul, "\phi_i"'] \arrow[ur, "\psi_i"] \arrow[d, "F_{ij}"] & \\
& G_j \arrow[uul, bend left, "\phi_j"] \arrow[uur, bend right, "\psi_j"'] &
\end{tikzcd}$$

Define $u([a]) \overset{\text{def}}{=} \psi_i(a)$

$[a] = \phi_i(a)$

Remark: The direct limit can also be defined for direct systems of sets, groups, rings, and chain complexes.

Using Univ. Property, $\varinjlim$ can be defined axiomatically and works in any category but such a direct limit might not exist.

Categorical Perspective

I directed poset
$\{G_i, F_{ij}\}$

Think of I as a category with obj $I = I$ and one non-id. morphism $i \to j$ $\forall i \leq j$

A direct system $\{G_i, F_{ij}\}$ in $\mathcal{C}$ can be viewed as a functor

$$F : I \to \mathcal{C}$$
$$i \mapsto G_i$$
$$(i \to j) \mapsto F_{ij}$$

The universal property can be generalized to the case where I is arbitrary category. This leads to the colimit and other generalized objects like coproducts & pushouts.

Remark: One can define categorical limits (generalized inverse limits, products, and pullbacks).

Ex: (Direct limits)

1) S_i subset of $\mathcal{U}$

$F_i : S_i \to \mathcal{U}$ inclusion

$F_{ij} : S_i \to S_j$ inclusion

$\varinjlim S_i = \bigcup S_i \subseteq \mathcal{U}$

2) X top. space. $x \in X$

$\mathcal{U}$ open neigh of x

$C(\mathcal{U}) = \{$cont. functions $F: \mathcal{U} \to \mathbb{R}\}$

open neigh of x form directed poset by $\mathcal{U} \geq V \iff \mathcal{U} \subseteq V$

$\varinjlim C(\mathcal{U}) = \{$germs of functions at $x\}$

$= C_x$ = stalk of C at x.

3) X top space. Compact subspaces of X form a directed poset:

$K \geq K' \iff K \supseteq K'$

Later: $\varinjlim_{K \text{ compact}} H^n(X, X \setminus K) \cong \underbrace{H^n_c(X)}_{\text{cohomology with compact support}}$

Suppose $\{C_i, F_{ij}\}$ a direct system of chain complexes of abelian groups (with the F_{ij} chain maps). We get an induced direct system $\{H_*(C_i), (F_{ij})_*\}$.

Thm: $\varinjlim H_*(C_i) \cong H_*(\varinjlim C_i)$

eg. direct limits commute with homology.

PF (sketch): Up to isomorphism, H_* can be defined in terms of exact sequences and $\varinjlim$ preserves exact sequences and then must preserve homology. ⊞

In more detail, $(C_\cdot, \partial)$ a chain complex.

$Z_n = \ker \partial_n$: $B_n = \operatorname{im} \partial_{n+1}$: $H_n = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$.

$i_n : Z_n \hookrightarrow C_n$ inclusion.

$\partial'_{n+1} : C_{n+1} \to B_n$ restriction of ∂_{n+1}.

$j_n : B_n \hookrightarrow Z_n$ inclusion.

(1) (Z_n, i_n) is determined by $\partial_n : C_n \to C_{n-1}$ and by....

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} C_{n-1}$$

(2) (B_n, ∂'_{n+1}) is determined by $i_{n+1} : Z_{n+1} \to C_{n+1}$ and by

$$Z_{n+1} \xrightarrow{i_{n+1}} C_{n+1} \xrightarrow{\partial'_{n+1}} B_n \to 0$$

(3) $j_n : B_n \to Z_n$ determined by

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial'_{n+1}"'] & C_n \\ B_n \arrow[r, "j_n"'] & Z_n \arrow[u, "i_n"'] \end{tikzcd}$$

- (4) H_n is determined by $j_n: B_n \to Z_n$ and $B_n \xrightarrow{j_n} Z_n \to H_n \to 0$

Singular Homology

$v_0, \cdots, v_n \in \mathbb{R}^m$ (vertices)
(ordered)

Def: $[v_0, \cdots, v_n]$ - n simplex formed by these points, ie the convex hull of $v_0, \cdots, v_n$.

Notation: $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ is n-simplex with v_i removed, ie a specific n-1 simplex. Called the i^{th} face of $[v_0, \cdots, v_n]$.

Def: $\Delta^n = [e_0, \cdots, e_n]$; e_i standard basis vectors of $\mathbb{R}^{n+1}$. Explicitly,

$$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}$$

$t_0, t_1, \cdots, t_n$ are barycentric coordinates.

Ex: Δ^0

Δ^1

Δ^2

Note: If $[v_0, \cdots, v_n]$ is an n-simplex then $\exists$ a distinguished affine map

$$\Delta^n \longrightarrow [v_0, \cdots, v_n]$$

$$(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i$$

Def: The i^{th} face map $F_i^n: \Delta^{n-1} \to \Delta^n$ is the distinguished affine map

$$\Delta^{n-1} \to [e_0, \cdots, \hat{e}_i, \cdots, e_n]$$

Singular Simplices: X top. space

Def: A singular n-simplex in X is a continuous map

$$\sigma: \Delta^n \to X$$

$S_n(X) = \{$sing n-simplices in X$\}$

Ex: $S_0(X) = \{$maps $\underset{\sim}{\text{points}} \to X\}$
Δ^0 - single point
$= \{$points in X$\}$

$S_1(X) = \{$cont. map $[0,1] \to X\}$
$= \{$paths in X$\}$

Notation: If $\sigma \in S_n(X)$ then

$$\sigma|_{[e_0, \cdots, \hat{e}_i, \cdots, e_n]} = \sigma \circ F_i^n$$

Singular Chain Complex

$C_n(X)$ = Free abelian group (ℤ-mod.) by $S_n(X)$

$= \{$ Fin. formal sums $\sum_i n_i \sigma_i \mid_{\sigma_i \in S_n(X)}^{n_i \in \mathbb{Z}}$

Note: $n \geq 0$. For $n < 0$ $C_n(X) = 0$.

singular n-chains

∂_n the linear map

$\partial_n : C_n(X) \to C_{n-1}(X)$

given by....

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma F_i^n$$

Fact: $\partial_n \partial_{n+1} = 0$

(ie $\partial^2 = 0$) so forms a chain complex.

Def: $\underset{\text{nth sing. homology}}{H_n(X)} = H_n(C_*(X), \partial_*)$

$= \ker \partial_n / \text{im}\, \partial_{n+1}$

$= \underset{\text{cycle}}{Z_n} / \underset{\text{Boundary}}{B_n}$

Ex: $C_*(\{pt\})$ looks like....

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \underbrace{\mathbb{Z}}_{0^{th}\text{ spot}} \to 0 \to 0 \to \cdots$$

So $H_0(pt) = \mathbb{Z}$

$H_n(pt) = 0$ for $n > 0$.

Thm: $H_0(X) = \text{Span}_{\mathbb{Z}}\{\text{path components of } X\}$

$$= \bigoplus_{\#\text{path comp.}} \mathbb{Z}$$

Pf: $H_0(X) = \dfrac{\ker \partial_0}{\text{im}\, \partial_1} = \dfrac{C_0(X)}{\text{im}\, \partial_1} = \dfrac{\text{span points in } X}{\text{boundary of } \underbrace{\text{pair points}}_{\text{paths}}}$ ⊠

Induced Maps:

Every cont. map $f: X \to Y$

induces $f_\# : C_n(X) \to C_n(Y)$

$\sigma \mapsto f\sigma$

Check:

$f_\#(\partial\sigma) = \partial(f_\# \sigma)$

↳ Both sides $=$

$$\sum_{i=0}^{n} (-1)^i f \circ \sigma \circ F_i^n$$

There is an induced map

$f_* : H_n(X) \to H_n(Y)$

Remark: The assignment $f \mapsto f_*$ is compatible with composition, ie $(\cancel{fo}\, f \circ g)_* = f_* g_*$ and $1_* = 1_{H_n(X)}$

ie H_n is a functor Top to Ab

So if f is a homeo. then f_* is an iso.

09/08/2016

$S_n(X) = \{\text{sing. n-simplices in } X\}$

$C_*(X) = \{\text{sing chain complexes}\}$

H_n is a functor Top → Ab

In fact, H_n descends to a functor Top/~ → Ab, where Top/~ is the category with:

1) Obj = Top spaces

2) Morph: Cont. maps up to homotopy

Cor: If X, Y homotopic equiv. then $H_n(X) \cong H_n(Y)$ for all n.

Homology with coefficients

G ab. group | R-mod

$C_n(X;G) = \{$ Formal finite sum $\sum m_i \sigma_i \mid m_i \in G, \sigma_i \in S_n(X) \}$

$= C_n(X) \otimes_{\mathbb{Z}} G$

The differential is given by $\partial_n^G = \partial_n \otimes 1_G$

Def: $H_n(X;G) = n^{th}$ homology of $C_n(X;G)$. $H_n(X;G) \overset{def}{=} H_n(C_*(X;G), \partial_n^G\}$

Rem: If G is an R-mod then
$C_n(X;G) = C_n(X) \otimes_{\mathbb{Z}} G \cong C_n(X;R) \otimes_R G$

Univ. Coefficient Theorem (UCT) for Homology

(Hatcher pp. 264) If G is a module over a PID R, then

$$H_n(X;G) \cong H_n(X;R) \otimes_R G \oplus \mathrm{Tor}_1^R(H_{n-1}(X;R), G)$$

In particular, if $R = \mathbb{Z}$, we have

$$H_n(X;G) \cong H_n(X) \otimes_{\mathbb{Z}} G \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G)$$

The isomorphisms are not canonical. Furthermore, the isomorphisms are not natural.

Rem: There is a refinement of UCT involving a short exact sequence.

Rem: UCT holds for $H_n(C_* \otimes_R G)$, where C_* can be any complex of free modules over a PID R.

$\mathrm{Tor} = \mathrm{Tor}_1^R$

Def: $\mathrm{Tor}(H,G) = H_1(\tilde{H}_* \otimes_R G)$
$\cong H_1(H \otimes_R \tilde{G})$

where $\tilde{H}_* \to H \to 0$ and $\tilde{G}_* \to G \to 0$ are free resolutions of H, G.

Prop: (Prop. of Tor, $R = \mathbb{Z}$)

1) $\mathrm{Tor}(H,G) \cong \mathrm{Tor}(G,H)$
2) $\mathrm{Tor}(\oplus H_i, G) \cong \oplus \mathrm{Tor}(H_i, G)$
3) $\mathrm{Tor}(H,G)$ if H proj (eg free) or if H torsion free (eg in $\mathbb{Z}$)
4) $\mathrm{Tor}(H, \mathbb{Z}_n) = \mathrm{Ker}(H \xrightarrow{n} H)$

"n torsion part of H" → mult. by n.

If p is prime then
$\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$
$\mathrm{Tor}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$
$k > 0$

Suppose we have a short exact sequence of ab. groups (or R-mod.)

$$0 \to G' \to G \to G'' \to 0$$

tensor with $C_*(x)$ exact as $C_*(x)$ free vn.

then there is an induced short exact seq:

$$0 \to C_*(X;G') \to C_*(X;G) \to C_*(X;G'') \to 0$$

So of course we then obtain a long exact sequence in homology.

We can then give a (partial) proof of the UCT.

Let G be a module over a PID R

Let $0 \to F_1 \to F_0 \to G \to 0$ be a free resolution. So we get a long sequence in homology in these modules.

$$\cdots \to \underbrace{H_n(X;F_1) \to H_n(X;F_0)}_{\shortparallel} \to H_n(X;G) \to \underbrace{H_{n-1}(X;F_1) \to H_{n-1}(X;F_0)} \to \cdots$$

coker of this map is $\mathrm{Tor}_0^R(H_n(X;R),G)$

$= H_n(X;R) \otimes_R G$

ker of this map is $\mathrm{Tor}_1^R(H_{n-1}(X;R),G)$

So we get a short exact sequence....

$$0 \to H_n(X;R) \otimes_R G \to H_n(X;G) \to \mathrm{Tor}_1^R(H_{n-1}(X;R),G) \to 0$$

This sequence splits and the UCT follows.

<u>Relative Homology</u>

X top. space

$A \subseteq X$ subspace

Can view $S_n(A)$ as a subset of $S_n(X)$. So $C_*(A) = C_*(X)$ is a subcomplex.

$C_*(X,A) = C_*(X)/C_*(A)$; quotient complex

Def: $H_n(X,A) \overset{def}{=} H_n(C_*(X,A))$

'n^{th} relative homology of (X,A)'

Rem: There is a s.e.s.

$$0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0$$

So that there is an induced long exact sequence in homology.

Rem: If $(X,A) \xrightarrow{F} (Y,B)$ cont. map of pairs $(F(A) \subseteq B)$ there is an induced map of homology:

$$F_*: H_n(X,A) \to H_n(Y,B)$$

Thm: If F, g are homotopic
↑
through maps of pairs
then $F_* = g_*$

Pf: Hatcher pp 118.

Rem: For each n, $C_*(A)$ is a direct summand of $C_n(X) =$

$$\underbrace{\mathrm{Span}\, S_n(A)}_{C_n(A)} \oplus \mathrm{Span}(S_n(X) \setminus S_n(A))$$

$\rightsquigarrow$

Rem: But differential does not respect this direct summand.

$$C_n(X,A;G) = \frac{C_n(X;G)}{C_n(A;G)} \cong C_n(X,A) \otimes G$$

Then $H_n(X,A;G)$ is the homology of this sequence (complex).

<u>Good pairs</u>:

Def: The pair (X,A) is good if $\emptyset \neq A \subseteq X$, A closed, and there exists $A \subseteq \mathfrak{U} \subseteq X$ open that def. retracts to A.

Thm: If (X,A) is good then
$$H_n(X,A) \cong \tilde{H}_n(X/A)$$

Here, $\underset{\text{reduced homology}}{\underbrace{\tilde{H}_n}}(Y) \overset{\text{def}}{=} H_n(Y, \text{point})$ $(= H_n(Y)$ for $n>0)$

induced by the quotient map
$$(X,A) \to (X/A, \underset{pt}{\underbrace{A/A}})$$

Pf: Hatcher pp 124.

Ex: (Homology of spheres)

$n>0$

$X = D^n$, closed unit ball in $\mathbb{R}^n$

$A = \partial D^n = S^{n-1}$

$(D^n, \partial D^n)$ is good so
$$H_i(D^n, \partial D^n) \cong \tilde{H}_i(D^n/\partial D^n)$$
$$= \tilde{H}_i(S^n)$$

Look at the long exact sequence of $(D^n, \partial D^n) \xrightarrow{} (i>1)$

$$\cdots \to \underset{\substack{\| \\ 0}}{H_i(D^n)} \to H_i(D^n, \partial D^n) \to H_{i-1}(\partial D^n) \to \underset{\substack{\| \\ 0}}{H_{i-1}(D^n)} \to \cdots$$

iso

$$\tilde{H}_i(S^n) \cong \underbrace{H_{i-1}}(S^{n-1})$$

since $i>0$ same

So $\tilde{H}_i(S^1) \cong \tilde{H}_{i-1}(S^{n-1})$ For $i>1$, $n>0$

For $i=1$, easy to see result still holds.

$n \backslash i$	$\tilde{H}_0$	$\tilde{H}_1$	$\tilde{H}_2$	$\tilde{H}_3$
S^0	$\mathbb{Z}$	0	0	0
S^1	0	$\mathbb{Z}$	0	0
S^2	0	0	$\mathbb{Z}$	0
S^3	0	0	0	$\mathbb{Z}$

$H_0(S^0) = \mathbb{Z}^2$, reduced rank 1 smaller so $\mathbb{Z}$

2 points

By equation, $\tilde{H}_i(S^n)$ is constant along the diagonals in table.

Thm: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & i=n \\ 0, & i \neq n \end{cases}$

Cor: $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} \mathbb{Z}, & i=n \\ 0, & i \neq n \end{cases}$

Pf (1): Look at l.e.s. of pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ and use $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ ⊠

Pf (2): $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_i(\mathbb{R}^n, S^{n-1})$

↑ Use 5 lemma to show id: $(\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ Induces an iso.

But $H_i(\mathbb{R}^n, S^{n-1}) \cong H_i(D^n, \underset{\substack{\| \\ \partial D^n}}{S^{n-1}})$

$$\cong \tilde{H}_i(\underset{S^n}{\underbrace{D^n/\partial D^n}})$$

then it follows by preceeding thm.

Rem: $(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$ is $\underline{\underline{\text{not}}}$ a good pair.

Local Homology

X top. space.

$x_0 \in X$

Def: $H_n(X|x_0) \overset{def}{=} H_n(X, X\setminus\{x_0\})$ is the 'local homology of X at x_0'.

Using excision, one can see: If $U \subseteq X$ is an open neigh of x_0, then $H_n(X|x_0) = H_n(U|x_0)$

So $H_n(X|x_0)$ only depends on topology of X "near" x_0.

Simplicial Homology

X = a O-complex.
= top space X with collection C of sing simplices such that

1) $\sigma \in C$ is injective on $\Delta^n\setminus\partial\Delta$ and as a set X is disjoint union of all $\sigma(\Delta^n\setminus\partial\Delta^n)$ for $\sigma \in C$.
$\uparrow$ n-simplex

2) $\sigma \in C \to \sigma_n \circ F_i^n$ is in C.

3) $\sigma^{-1}(U)$ open in Δ^n for all $\sigma \in C$ then U is open in X. $(U \subseteq X)$

Let (X, C) be a Δ-complex

$S_n^\Delta(X) \overset{def}{=} S_n(X) \cap C$

$C_n^\Delta(X) = \mathrm{Span}_{\mathbb{Z}}\, S_n^\Delta(X) \subseteq C_n(X)$

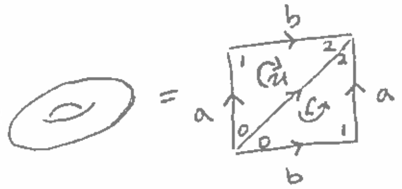
By (2), $C_*^\Delta(X)$ is a subcomplex.

Def: $H_n^\Delta(X) \overset{def}{=} H_n(C_*^\Delta(X))$ is 'Simplicial homology'.

Thm: $C_*^\Delta(X) \hookrightarrow C_*(X)$ induces an isomorphism in homology.

Pf: Hatcher pp. 178

$T^2 =$



$S_2^\Delta = \{U, L\}$

$S_1^\Delta = \{a, b, c\}$

$S_0^\Delta = \{0\}$

$\partial U = a + b - c = \partial L$

$$\to \mathbb{Z}^2 \to \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$$

$$\begin{array}{cc|c} U & L & \\ 1 & 1 & a \\ 1 & 1 & b \\ -1 & -1 & c \end{array}$$

09/13/2016

Just a brief word on Cellular homology:

Let X be a CW complex.

X^n the n-skeleton of $X = \bigcup$ cells dim $\leq n$

X^0 0-skeleton = discrete space

X^n obtained from X^{n-1} by attaching disks D_α^n via:

$\Phi_\alpha : \partial D_\alpha^n \to X^{n-1}$

Topology on X is such that $\mathcal{U} \subseteq X$ is open IFF $\mathcal{U} \cap X^n \subseteq X^n$ is open.

Check: $X = \varinjlim X^n$ in Top.

Moreover, by prop A.1 from Hatcher pp 520, every compact subspace $A \subseteq X$ is contained in an X^n. So

$$C_*(X) = \varinjlim C_*(X^n) = \bigcup C_*(X^n)$$

Cellular Homology

$$C_n^{cw}(X) = H_n(X^n, X^{n-1})$$
$$= H_n(\underbrace{X^n/X^{n-1}}_{\text{wedge } n\text{-spheres}})$$
$$= \operatorname{Span}_{\mathbb{Z}}\{n\text{-cells}\}$$

$$\partial_n e_\alpha = \sum_\beta \deg F_{\alpha\beta}\, e_\beta$$

where e_α is an n-cell

e_β is an $(n-1)$-cell

$F_{\alpha\beta}$ is the map

$$\begin{tikzcd} \partial D^n_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, equal] & X^{n-1} \arrow[r] & X^{n-1}/X^{n-1}-e_\beta \arrow[d, equal] \\ S^{n-1} \arrow[rr, "F_{\alpha\beta}"'] & & S^{n-1} \end{tikzcd}$$

Ex: (Homology of $\mathbb{RP}^\infty$)

$\mathbb{R}^\infty = \{(x_1; x_2; \dots) \mid x_i \in \mathbb{R}$, only fin. many nonzero$\}$

Topologize $\mathbb{R}^\infty$ by dec. $\mathcal{U} \subseteq \mathbb{R}^\infty$ to be open if $\mathcal{U} \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all n.

$$\mathbb{R}^n = \{x \in \mathbb{R}^\infty \mid x_i = 0 \text{ for } i > n\}$$

For $x \in \mathbb{R}^\infty$, define $\|x\| = \sqrt{\sum x_i^2}$. $S^\infty = \{x \in \mathbb{R}^\infty \mid \|x\| = 1\}$.

Equip S^∞ with subspace topology.

$$\mathbb{RP}^\infty = S^\infty / x \sim (-x)$$

Fact: $\mathbb{RP}^\infty$ has cell decomposition of the form

$$e_0 \cup e_1 \cup e_2 \cup e_3 \cup \cdots$$

e_i an i-cell and $\deg(F_{e_i}, e_{i-1})$

$$= \begin{cases} 2, & i \text{ even} > 0 \\ 0, & \text{otherwise} \end{cases}$$

$C_*^{cw}(\mathbb{RP}^\infty)$ looks as follows

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

Homology of $\mathbb{RP}^\infty$ with coeffic. in G

$G \backslash i$	H_0	H_1	H_2	H_3	H_4	H_5
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Q}$	$\mathbb{Q}$	0	0	0	0	0

map $\rightarrow$ 0 in $\mathbb{Z}/2\mathbb{Z}$

$\nearrow$ map iso.
$C^{cw}(\mathbb{RP}^\infty; \mathbb{Q}) \cong C(pt, \mathbb{Q})$

Subdivision: Suppose X is a top space

$\mathcal{U} = \{$subsets $A_i \subseteq X$ whose int. cover $X\}$

Def: $C_n^{\mathcal{U}}(X) = \operatorname{Span}\left\{\begin{matrix}\text{sing. } n\text{-simplices cont.} \\ \text{in one of the } A_i\end{matrix}\right\}$

in

$C_n(X)$

Thm: The inclusion $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a homotopy equivalence. In fact, $C_*^{\mathcal{U}}(X)$ is a deformation retract of $C_*(X)$.

Pf: Hatcher pp 119-124

Application 1: $\mathcal{U} = \{A, B\}$ whose interiors cover X; $\text{Int } A \cup \text{int } B = X$. Then $\exists$ a ses:

$$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*^{\{A,B\}}(X) \to 0$$

This gives a long exact sequence in homology - the Mayer-Vietoris sequence.

Rem: The Mayer-Vietoris seq. also exists for the case where A, B are cellular subcomplexes of a CW complex X such that $A \cup B = X$.

Thm (Version 1): If $\text{Int } A \cup \text{Int } B = X$, then $H_n(X, A) \cong H_n(B, A \cap B)$ for all n.

Pf:
$$C_*(X,A) = C_*(X)/C_*(A) \overset{\text{Thm}}{\cong} \frac{C_*^{\{A,B\}}(X)}{C_*(A)} = \frac{C_*(B)}{C_*(A \cap B)} = C_*(B, A \cap B)$$

⊠

Thm (Version 2): If $\bar{Z} \subseteq \text{Int } A \subseteq X$ then $H_n(X, A) \cong H_n(X \setminus Z, A \setminus Z)$

Pf: Use version 1 for $B = X \setminus Z$ ⊠

Cor: If $\mathcal{U} \subseteq X$ is an open neigh, then $H_n(X, X\setminus\{x_0\}) \cong H_n(\mathcal{U}, \mathcal{U}\setminus\{x_0\})$

Pf: Use version 1 for $A = X \setminus \{x_0\}$ and $B = \mathcal{U}$ ⊠

Thm: If (X, A) is good then $H_n(X, A) \cong \tilde{H}_n(X/A)$.

* Cohomology *

Let X be a top. space. The n^{th} cochain group is

$$C^n(X) = \text{Hom dual } C_n(X) = \text{Hom}(C_n(X), \mathbb{Z}) \cong \text{Maps}(S_n(X), \mathbb{Z})$$

because $C_n(X) = \text{Span } S_n(X)$. So we have functors:

Adjoints $\begin{cases} S \in \text{Set} \mapsto \text{Span } S \in \text{Ab} \\ Z \in \text{Ab} \mapsto Z \in \text{Set} \end{cases}$

Forgetful Functor

Elements of $C^n(X)$ are called singular n-cochains in X. So singular n-cochain is a map $\phi : S_n(X) \to \mathbb{Z}$

The n^{th} coboundary map is the map

$$\delta^n : C^n(X) \to C^{n+1}(X)$$
$$\phi \mapsto \phi \partial_{n+1}$$

So δ^n is the dual of ∂_{n+1}.

Explicitly, if $\phi \in C^n(X)$ and $\sigma \in S_{n+1}(X)$ then

$$(\delta^n \phi)(\sigma) = \phi(\partial_{n+1} \sigma)$$
$$= \sum_{i=0}^{n+1} (-1)^i \phi(\sigma \circ F_i^{n+1})$$

Def: The n^{th} cohomology of $(C^n(X), \delta^n)$ is $H^n = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}} = \frac{Z^n}{B^n}$ ← n-cocycles ← n-coboundaries

Ex: C^*(point) looks as follows

$$\cdots \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

So $H^n(pt) = 0$ for $n > 0$. So this is the same as homology.

Thm: $H^0(X)$ = set of all maps from path components of X to $\mathbb{Z}$: $\prod \mathbb{Z}$, one copy per path component

*Note: In homology, get direct sum so H^0, H_0 same for fin. # path components

PF: $H^0(X) = \frac{\ker \delta^0}{\operatorname{im} \delta^{-1}} = \frac{\ker \delta^0}{0} = \ker \delta^0$

$$= \{ \phi \in C^0(X) \mid \underbrace{\delta^0 \phi}_{\phi \circ \delta_1} = 0 \}$$

$$= \{ \phi : S_0(X) \to \mathbb{Z} \mid \phi(\partial_1 \sigma) = 0 \ \forall \sigma \in S_1(X) \}$$

$$* \quad S_0(X) = \{\text{points}\}$$
$$\sigma \in S_1(X) = \{\text{paths}\}$$

$$= \left\{ \phi : \{\text{points}\} \to \mathbb{Z} \;\middle|\; \begin{array}{l} \phi(x_1) - \phi(x_0) = 0 \\ \exists \text{ path } x_0 \to x_1 \end{array} \right\}$$

So ϕ takes same values on points connected by paths so constant on each path component. Result then follows. ▨

Induced Homomorphisms

Every cont. map $F: X \to Y$ induces a map

$$F^\# : C^n(Y) \to C^n(X)$$
$$\psi \mapsto \psi F_n$$

So $F^\#$ is the dual of $F_\#$.

Explicitly, if $\phi \in C^n(Y)$ and $\sigma \in S_n(X)$ then

$$(F^\# \phi)(\sigma) = \phi(F \circ \sigma)$$

09/15/2016

Since $F_\#$ is a chain map, $F^\#$ is a chain map and we get a long exact sequence in homology and a induced map in cohomology

$$f^* : H^n(Y) \to H^n(X)$$
$$[\psi] \mapsto [\underbrace{\psi \circ F_n}]$$

pull back of $[\psi]$ along f

Properties:

i) $(f \circ g)^{\#} = g^{*} \circ f^{*}$

ii) $(1_X)^{*} = 1_{H^n(X)}$

Cor: H^n is a contravariant functor from Top to Ab.

$$\begin{tikzcd} \mathrm{Top} \arrow[r] & \mathrm{Ch}(\mathrm{Ab}) \arrow[r] & \mathrm{CCh}(A) \arrow[r] & \mathrm{Ab} \\ X \arrow[r, mapsto] & C_*(X) \arrow[r, mapsto] & \mathrm{Hom}(C_*, \mathbb{Z}) \arrow[r, mapsto] & H^n(X) \end{tikzcd}$$

Thm: If f, g are homotopic then $f^{\#}, g^{\#}$ are homotopic.

Pf: $f \simeq g \rightsquigarrow f_{\#} \simeq g_{\#}$, dualizing preserves homotopies because it is an additive functor. $\boxtimes$

Cor: H^n descends to a contravariant functor $\mathrm{Top}/\sim \;\to \mathrm{Ab}$

Cor: If X, Y are homotopy equiv. then $H^n(X), H^n(Y)$ are isomorphic.

Cohomology with coefficients

If G is an abelian group or R-mod

$$C^n(X; G) = \mathrm{Hom}(C_n(X), G)$$
$$= \mathrm{Maps}(S_n(X), G)$$

δ^n_G is defined as before ϕ to $\phi \circ \partial_{n+1}$

Def: $H^n(X; G) = n^{\text{th}}$ cohomology of $(C^*(X; G), \delta^*_G)$ - cohomology with coefficients in G.

Rem: If G is an R-module (R comm.)

$$C^n(X; G) = \mathrm{Hom}(C_n(X), G)$$
$$= \mathrm{Hom}_R(C_n(X; R), G)$$

as both are given by maps from $S_n(X)$ to G.

Relative Cohomology

X top. space.
$A \subseteq X$ subspace

$$C^n(X, A) = \mathrm{Hom}(C_n(X, A), \mathbb{Z})$$

$$\overset{\shortparallel}{C_n(X)/C_n(A)}$$

$$= \{\phi \in C^n(X) \mid \phi|_{C_n(A)} = 0\}$$

$$= \mathrm{Ker}\left(C^n(X) \xrightarrow[i^*]{\text{restrict}} C^n(A)\right)$$

where $i : A \hookrightarrow X$ inclusion.

So $C^*(X, A) \subseteq C^*(X)$ is a subcomplex

Def: $H^n(X, A) = H^n(C^*(X, A))$ the n^{th} relative cohomology of (X, A).

Rem: By def of relative chain complex, there is a s.e.s.:

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

Moreover, $C_n(X) = \underbrace{\mathrm{span}\ S_n(A)}_{= C_n(A)} \oplus \underbrace{\mathrm{Span}\ S_n(X) \setminus S_n(A)}_{\cong C_n(X, A)}$

So For each n, the short exact sequence

$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$

splits. Hence, the dual sequence is still exact. So we get ses of cochain complexes which induces a les in cohomology.

$$\cdots \to H^n(X,A) \to H^n(X) \to H^n(A) \xrightarrow{\delta} H^{n+1}(X,A) \to \cdots$$

$$\delta: H^n(A) \to H^{n+1}(X,A)$$

$$[\phi] \mapsto [\tilde{\phi}\, \partial_{n+1}]$$

$$\tilde{\phi} = \begin{cases} \phi & \text{on } S_n(A) \\ 0 & \text{on } S_n(X) \setminus S_n(A) \end{cases}$$

Rem: If $f: (X,A) \to (Y,B)$ is a cont. map of pairs, then $\exists$ induced map

$f^*: H^n(Y,B) \to H^n(X,A)$

Thm: If f, g homotopic (through maps of pairs) then $f^* = g^*$

Pf: If $f \simeq g \to f_\# \simeq g_\#$

$: C_*(X,A) \to C_*(Y,B)$

and dualizing preserves homotopies. □

Reduced Cohomology

Hatcher pp. 119.

X top space (nonempty)

Let $\epsilon \in C^0(X)$ be 0-cochain given by $\epsilon(\sigma) = 1$ $\forall S_0(X)$

Let $\delta^{-1}: \mathbb{Z} \to C^0(X)$ be the map $1 \mapsto \epsilon$.

Def: $\tilde{H}^n(X)$ is n^{th} cohomology of

$$\cdots \xleftarrow{\delta^1} C^1(X) \xleftarrow{\delta^0} C^0(X) \xleftarrow{\delta^{-1}} \mathbb{Z} \leftarrow 0$$

Rem: For $n > 0$, $\tilde{H}^n(X) = H^n(X)$.

Rem: The inclusion $C^*(X, pt) \hookrightarrow C^*(X)$ induces an isomorphism

$H^n(X, pt) \cong \tilde{H}^n(X)$

Thm: If (X,A) is good then $H^n(X,A) \cong \tilde{H}^n(X/A)$.

Pf: Same as for homology using excision for cohomology. □

Rem: Define $\tilde{H}^n(X; G)$ one has to replace δ^{-1} by

$$\delta^{-1}: \mathrm{Hom}(\mathbb{Z}, G) \to C^0(X; G)$$

$$\phi \mapsto \phi \circ \epsilon$$

map $S_0(X) \to \mathbb{Z}$

$\sigma \mapsto 1$

Simplicial Cohomology

X Δ-complex

$S_n^\Delta(X) = \{\text{dist. } n\text{-simplices}\}$

$C_n^\Delta(X) = n^{th}$ simplicial chain group $=$ span $S_n^\Delta(X)$

$C^n_\Delta(X) = \mathrm{Hom}(C^\Delta_n(X), \mathbb{Z})$

δ^n_Δ = dual of ∂^Δ_{n+1}

Thm: $H^n_\Delta(X) = H^n(C^n_\Delta(X), \delta^*_\Delta)$
$= \dfrac{\mathrm{Ker}\ \delta^n_\Delta}{\mathrm{im}\ \delta^{n-1}_\Delta}$

Thm: $H^n_\Delta(X) \cong H^n(X)$

PF: Follows from UCT for Cohomology and from $H^\Delta_n(X) \cong H_n(X)$. ▨

Result From Lin. Alg:

V, W vector spaces (Free R mod., R comm.) which are n, m dim., respectively.

V basis: $\{e_1, \cdots, e_n\}$

W basis: $\{F_1, \cdots, F_m\}$

We have dual basis

$\{e_1^*, \cdots, e_n^*\}$, $\{F_1^*, \cdots, F_m^*\}$ for V^*, W^*

$L: V \to W$ linear

$L^*: W^* \to V^*$ linear

$\mu \mapsto \mu L$

Thm: matrix of L^* = (matrix of L)^T with respect to the chosen basis.

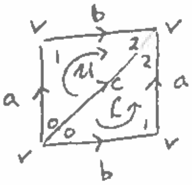
$$\begin{pmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ddots & \vdots \\ L_{m1} & \cdots & L_{mn} \end{pmatrix} \leftarrow$$

i^{th} row contains coordinates of $L^*(F_i^*)$ wrt to $\{e_j^*\}$

j^{th} column contains coordinates of $L(e_j)$ wrt $\{F_i\}$

Ex: X =

=



Recall $C^\Delta_*(X)$ looks as follows

$$0 \to \mathrm{Span}\{u, L\} \xrightarrow{\partial^\Delta_2} \mathrm{Span}\{a,b,c\} \xrightarrow{0} \mathrm{Span}\{v\} \to 0$$

Saw $\partial^\Delta_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$

$C^*_\Delta(X)$ is the dual complex

$$\mathrm{Span}\{u^*, L^*\} \xleftarrow{\delta^1_\Delta} \mathrm{Span}\{a^*, b^*, c^*\} \xleftarrow{0} \mathrm{Span}\{v^*\} \leftarrow 0$$

eg, $u^*(u) = 1$, $u^*(L) = 0$
$L^*(u) = 0$, $L^*(\ell) = 1$

δ^1_Δ (by thm) is $\begin{array}{cc} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} & \begin{matrix} u^* \\ \ell^* \end{matrix} \\ \begin{matrix} a^* & b^* & c^* \end{matrix} & \end{array}$

$$\sim \begin{array}{c|c} \begin{matrix} a^*+c^* & b^*+c^* & -c^* \end{matrix} & \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} u^* \\ L^* \end{matrix} \end{array}$$

$$\sim \begin{array}{c|c} \begin{matrix} a^*+c^* & b^*+c^* & -c^* \end{matrix} & \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} u^*+\ell^* \\ L^* \end{matrix} \end{array}$$

$\mathrm{Ker}\ \delta^1_\Delta = \mathrm{Span}\{a^*+c^*, b^*+c^*\}$

$\mathrm{im}\ \delta^1_\Delta = \mathrm{Span}\{u^*+\ell^*\}$

$H^2(X) = \dfrac{\mathrm{Ker}\ \delta^2_\Delta}{\mathrm{im}\ \delta_\Delta} = \dfrac{\mathrm{Span}\{u^*+\ell^*, \ell^*\}}{\mathrm{Span}\{u^*+\ell^*\}}$

$\cong \mathrm{Span}\{\ell^*\}$

$\cong \mathbb{Z}$

$H^1(X) = \dfrac{\mathrm{Ker}\ \delta^1_\Delta}{\mathrm{Im}\ \delta^0_\Delta} = \mathrm{Span}\{a^*+c^*, b^*+c^*\}$

$\cong \mathbb{Z}^2$

$H^0(X) = \dfrac{\mathrm{Ker}\ \delta^0_\Delta}{\underset{\overset{\shortparallel}{0}}{\mathrm{im}\ \delta^{-1}_\Delta}} = \mathrm{Span}\{v^*\} \cong \mathbb{Z}$

In particular,

$$H^*(S^1 \times S^1) \cong H^\bullet_\times(S^1 \times S^1)$$

Could compute δ_Δ^1 directly by using definition of the dual map

eg:

$$\delta_\Delta^1(a^*)(u) = a^*(\partial_2 u)$$
$$= a^*(\underbrace{a+b-c}_{\partial u})$$
$$= 1$$

$$\delta_\Delta^1(a^*)(e) = a^*(\partial_2 e)$$
$$= a^*(\underbrace{a+b-c}_{\partial e})$$
$$= 1$$

So $\delta_\Delta^1(a^*) = u^* + e^*$

You can compute the others in the same way.

Cellular Cohomology

X a CW complex

X^n the n-skeleton

Def: $C^n_{cw}(X) = H^n(X^n, X^{n-1})$

δ^n_{cw} is the connecting homo in les in triple (X^{n+1}, X^n, X^{n-1}). Equivalently,

$\delta^n_{cw} = c_n \circ j_n$ where

$$\begin{tikzcd} H^n(X^n, X^{n-1}) \arrow[rr, "\delta^n_{cw}"] \arrow[dr, "j_n"] & \circlearrowleft & H^{n+1}(X^{n+1}, X^n) \\ & H^n(X^n) \arrow[ur, "c_n"] & \end{tikzcd}$$

From les from pair (X^n, X^{n-1})

les for pair (X^{n+1}, X^n) ie connecting homo.

Note: $\delta^n_{cw}\,\delta^{n-1}_{cw} = j_n \underbrace{c_n j_{n-1}}_{=0} c_{n-1}$

$= 0$

Def: $H^n_{cw}(X) = H^n(C^*_{cw}(X), \delta^*_{cw})$

Thm: $H^n_{cw}(X) \cong H^n(X)$

Thm: $C^*_{cw}(X) \cong$ dual cellular chain complex

Cor: If X has only fin. many n-cells e_α for each n, then

$$C^n_{cw}(X) = \text{span}\{(e_\alpha)^*\} \quad e_\alpha \text{ } n\text{-cells}$$

and ...

$$\delta^n_\Delta(e_\alpha)^* = \sum \deg(F_{\beta\alpha})(e_\beta)^*$$

where e_α is an n-cell and e_β is an $(n+1)$-cell.

$F_{\beta\alpha}$ is defined as in the case of cellular homology

$$\begin{tikzcd} \partial D^{n+1}_\beta \arrow[r, "\phi_\beta"] \arrow[d, equal] & X^n \arrow[r, two heads] \arrow[d, phantom, "\circlearrowleft"] & \frac{X^n}{X^n \setminus e_\alpha} \arrow[d, equal] \\ S^n \arrow[rr, "F_{\beta\alpha}"'] & & S^n \end{tikzcd}$$

Need for HW

$$H^n(X) \cong \text{Hom}(H_n(X), \mathbb{Z}) \oplus \underbrace{\text{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), \mathbb{Z})}$$

torsion subgroup of $H_{n-1}(X)$ for $H_{n-1}(X)$ F.g.

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Universal Coefficient Thm for Cohomology

X top. space
G ab. group or R-mod

Recall $C^*(X;G) = \text{Hom}(C_*(X), G)$
$= \text{Hom}_R(C_*(X;R), G)$
if G is an R-module

Thm (UCT for Cohomology):
If G is a module over a PID R,
then

$$H^n(X;G) \cong \text{Hom}_R(H_n(X;R), G) \oplus \text{Ext}^1_R(H_{n-1}(X;R), G)$$

In particular, if $R = \mathbb{Z}$

$$H^n(X;G) \cong \text{Hom}(H_n(X), G) \oplus \text{Ext}(H_{n-1}(X), G)$$

If $R = G = \mathbb{Z}$, then

$$H^n(X) \cong \text{Hom}(H_n(X), \mathbb{Z}) \oplus \text{Ext}(H_{n-1}(X), \mathbb{Z})$$

This isomorphism is not canonical or natural

More, there is an exact sequence:

$$0 \to \text{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \text{Hom}(H_n(X), G) \to 0$$

This sequence splits but not naturally.

$$h(\underset{n\text{-cocycle}}{[\phi]})(\overset{n\text{-cycle}}{[c]}) = \phi(c) \in G$$

Properties of Ext for $R = \mathbb{Z}$

1) $\text{Ext}(H,G) \cong \text{Ext}(G,H)$ generally
2) $\text{Ext}(\oplus H_i, G) \cong \prod \text{Ext}(H_i, G)$
3) $\text{Ext}(H, \prod G_i) = \prod \text{Ext}(H, G_i)$
4) $\text{Ext}(H,G) = 0$ if H is free or G inj.
5) $\text{Ext}(\mathbb{Z}_n, G) = G/nG$

Consequences....

- If $H_n(X), H_{n-1}(X)$ are fin gen then $H^n(X) \cong \underbrace{F_n(X)}_{\text{free part of } H_n(X)} \oplus \underbrace{T_{n-1}(X)}_{\text{torsion part of } H_{n-1}(X)}$

- So $H^n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Q})$

- $H_n(X)$ arb. and F $^{\cdot\,\mathbb{Q}}$ field
  $H^n(X;\mathbb{Q}) \cong \text{Hom}(H_n(X); \mathbb{Q})$

Generally, if F field then

$$H^n(X;F) = \text{Hom}_F(H_n(X;F), F)$$

Def (Ext):

$$\text{Ext}(H,G) = H^1(\text{Hom}(\tilde{H}_n, G))$$
$$\cong H^1(\text{Hom}(H, \tilde{G}))$$

where $\tilde{H}_* \to H \to 0$ is a free resolution of H and $0 \to G \to \tilde{G}$ is an injective resolution of G.

Injective Object

Def: I is injective if for all monomorphism $i: A \hookrightarrow B$ and morphism $f: A \to I$, $\exists$ a morphism $\tilde{f}: B \to I$ such that $F = \tilde{f} i$

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\ I & \end{tikzcd}$$

Equivalently, the functor $\mathrm{Hom}_{\mathcal{C}}(-, I)$ sends monomorphisms to surjections.

Ex: $\mathbb{Z}$ is not injective in Ab

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "i"] \arrow[d, "id"'] & \mathbb{Z} \arrow[dl, dashed, "\times"' , "\leftarrow \text{no} \frownie"] \\ \mathbb{Z} & \end{tikzcd}$$

Let R be a commutative ring.

Fact: If I is injective R-mod then I is divisible.

Pf: Exercise ☒

Fact: If R is a PID, then I injective $\rightleftarrows$ I divisible } Weibel pp 39

Recall: Divisible if $\forall r \in R$, $r \neq 0$ r not zero divisor, $\forall m \in I$ $\exists \tilde{m} \in I$ with $m = r\tilde{m}$

Ex: $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are ~~not~~ injective in Ab.

Fact: R-mod has "enough injectives", ie every R-mod has an injective resolution. See Weibel pp 40-42

Ex: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution for $\mathbb{Z}$ in Ab.

More generally, if G is a module over a PID R, then we can embed G into an injective module I

$$0 \to G \to \underbrace{I}_{\text{div.}} \to \underbrace{\overset{\curvearrowright}{G/I}}_{\text{div.} \to \text{inj.}} \to 0$$

is an injective resolution of G.

Change of Coefficients

Suppose we have ses:

$$0 \to G' \to G \to G'' \to 0$$

Apply $\mathrm{Hom}(\overset{\text{Free}}{C_*(X)}, -)$:

$$0 \to C^*(X; G') \to C^*(X; G) \to C^*(X; G'') \to 0$$

So there is a les in cohomology the connecting homo. is called the Bockstein homo, Hatcher pp 303.

"Pf:" (UCT for Cohomology)

Let G be a module over PID R,

Consider an injective resolution

$$0 \to G \to I_0 \to I_1 \to 0$$

We get a les in cohomology

$$\underbrace{H^{n-1}(X;I_0) \to H^{n-1}(X;I_1)}_{\text{coker of map is}} \to H^n(X,G) \to \underbrace{H^n(X;I_0) \to H^n(X;I_1)}_{\text{ker of this map is}}$$

coker of map is $\mathrm{Ext}^1_R(H_{n-1}(X),G)$

ker of this map is $\mathrm{Ext}^0_R(H_0(X;R),G)$

$$\| $$

$\mathrm{Hom}_R(H_n(X;R);G)$

"$\square$"

Pf (UCT for Cohomology):

Observe $B_n = B_n(X;R)$
$Z_n = Z_n(X;R)$

are submodules of free module $C_n = C_n(X;R)$.

Since R is a PID, this implies that B_n and Z_n are also free. Then

$$0 \to B_n \hookrightarrow Z_n \twoheadrightarrow H_n \to 0$$

is a free resolution of $H_n = H_n(X;R)$. Consider ses:

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$$

View as chain complex with trivial differential.

Since B_{n-1} is free, this sequence splits $\forall n$. So applying functor $\mathrm{Hom}_R(-,G)$ gives ses

$$0 \to \mathrm{Hom}_R(B_{n-1},G) \to \mathrm{Hom}_R(C_n,G) \to \mathrm{Hom}_R(Z_n,G) \to 0$$

We get a les in cohomology.

$$\overbrace{H_R(Z_{n-1};G) \to \mathrm{Hom}(B_{n-1},G)}^{\text{coker of this map is } \mathrm{Ext}^1_R(H_{n-1}(X;R),G)} \hookrightarrow H^n(X;G) \to \underbrace{\mathrm{Hom}(Z_n,G) \to \mathrm{Hom}_R(B_n,G)}_{\text{ker of this map is}}$$

ker of this map is $\mathrm{Ext}^0_R(H_n(X;R),G)$

$$\|$$

$\mathrm{Hom}_R(H_n(X;R),G)$

which gives:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_R(H_{n-1}(X;R),G) \arrow[r] & H^n(X;G) \arrow[d, "h"'] & \\
& & \mathrm{Hom}_R(H_n(X;R),G) \arrow[r] \arrow[u, bend right, "s"'] & 0
\end{tikzcd}$$

Why does this split? We can construct a section s of h as follows:

Choose left inverse $r: C_n \to Z_n$ of $i: Z_n \hookrightarrow C_n$

Given $\phi \in \mathrm{Hom}_R(H_n, G)$ regard ϕ as a map

$$\phi: Z_n \to G$$

such that $\phi|_{B_n} = 0$.

The composition

$$\phi r: C_n \to G$$

is a cocycle because $\phi|_{B_n} = 0$

Define $s(\phi) = [\phi r] \in H^n(X;G)$

Check: s is a section of h

The UCT ses splits. $\square$

Remark: The ses in the UCT just constructed is natural but the splitting is not!

Remark: The UCT also holds for relative cohomology.

Remark: If (X,A,B) is a triple of spaces, then the following diagram commutes:

les connecting map for (X,A,B)

$$\begin{tikzcd}
H^n(A,B) \arrow[r, "\delta"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & H^{n+1}(X,A) \arrow[d, "h"] \\
\mathrm{Hom}(H_n(A,B),\mathbb{Z}) \arrow[r, "\delta"'] & \mathrm{Hom}(H_{n+1}(X,A),\mathbb{Z})
\end{tikzcd}$$

So $C^*_{cw}(X) = \text{dual } C^{cw}_*(X)$; $(X,A,B) = (\underbrace{X^{n+1}, X^n, X^{n-1}}_{\text{skeleton}})$

Remark: Kronecker Pairing: If G is an R-mod. $\langle\ ,\ \rangle : H^n(X;G) \times H_n(X;R) \to G$

$[\phi] \times [\psi] \mapsto \langle[\phi],[\psi]\rangle$

where $\langle[\phi],[\psi]\rangle \overset{\text{def}}{=} \phi(\psi) = \underset{\text{equiv. classes}}{h(\phi)(\psi)}$

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UCT also holds for relative cohomology.

Excision for Cohomology

Thm: If $\mathrm{int}\,A \cup \mathrm{int}\,B = X$, then

$$H^n(X,A) \cong H^n(B, A\cap B)$$

the isomorphism is induced by $(B, A\cap B) \hookrightarrow (X,A)$

Pf: Excision for homology together with UCT and five lemma. ▨

Thm: If $\bar{Z} \subseteq \mathrm{int}\,A \subseteq X$ then

$$H^n(X,A) \cong H^n(X\setminus Z, A\setminus Z)$$

Pf: Set $B = X\setminus Z$. ▨

Thm: If (X,A) is good then

$$H^n(X,A) \cong \tilde{H}^n(X\setminus A).$$

Pf: Let $A \neq \emptyset$ be closed and N = nbd A be open nbd of A which deformation retracts to A.

$$H^n(X,A) \cong H^n(X,N)$$

$$\underset{\text{excision}}{\cong} H^n(X\setminus A, N\setminus A)$$

$$\underset{\text{equality}}{=} H^n(X/A \setminus A/A,\ N/A \setminus A/A)$$

$$\underset{\text{excision}}{\cong} H^n(X/A, N/A)$$

$$\cong H^n(X\setminus A, pt)$$

$$\cong \tilde{H}^n(X\setminus A)$$ ▨

Subdivision

X is a top. space

$\mathcal{U} = \{$subsets $A_i \subseteq X$ whose int. cover $X\}$

Def: $C^n_{\mathcal{U}}(X) = \mathrm{Hom}(C^{\mathcal{U}}_n(X), \mathbb{Z})$

Recall $C^{\mathcal{U}}_n(X) = \text{span}\left\{\begin{array}{l}\text{sing. } n\text{-simplices} \\ \text{that are in one} \\ \text{of the } A_i\end{array}\right\}$

Recall: $C^{\mathcal{U}}_*(X)$ is homotopy equivalent to $C_*(X)$. So $C^*_{\mathcal{U}}(X)$ is homotopy equivalent to $C^*(X)$.

Mayer-Vietoris for Cohomology

$X = \text{int}\, A \cup \text{int}\, B$

$$0 \to C_*(A\cap B) \to C_*(A) \oplus C_*(B) \to C_*^{\{A,B\}}(X) \to 0$$

This short exact sequence splits for each n as $C_n^{\{A,B\}}(X)$ is free. So we have....

$$0 \to C^*_{\{A,B\}}(X) \to C^*(A) \oplus C^*(B) \to C^*(A \cap B) \to 0$$

← Also split exact. Above was, Hom preserves sums - functor

So we get a long exact sequence

$$\dots \to H^n(X) \to H^n(A) \oplus H^n(B) \to H^n(A\cap B) \to H^{n+1}(X) \to \cdots$$

Why is Cohomology interesting?

$H^{(*)}(X)$ has a natural (with respect to cont. maps) graded ring structure - the cup product.

Here $H^{(*)}(X)$ means $H^0(X) \oplus H^1(X) \oplus \cdots$

So H^* is a contravariant functor:

$$H^* : \text{Top}/\!\sim \;\longrightarrow\; \text{graded commutative rings}$$

↑ homotopic maps identified

If M is a closed orientable n-manifold, then $H^i(M) \cong H_{n-i}(M)$. This is called the Poincaré duality.

As a consequence of this and UCT

$$F_i(M) \cong F_{n-i}(M)$$
$$T_{i-1}(M) \cong T_{n-i}(M)$$

where $F_i(M)$ free part of $H_i(M)$ and $T_i(M)$ torsion part of $H_i(M)$.

Roughly follows from $H^i(M) \overset{UCT}{\cong} F_i(M) \oplus T_{i-1}(M)$.

Eg:

$H_i(S^3)$:

$$\underset{i=3}{\mathbb{Z}} \qquad 0 \;\updownarrow\; 0 \qquad \underset{i=0}{\mathbb{Z}}$$

$H_i(\mathbb{RP}^3)$:

$$\mathbb{Z} \qquad 0 \qquad \mathbb{Z}_2 \qquad \mathbb{Z}$$

$F_i(\mathbb{RP}^3)$ $\quad \mathbb{Z} \qquad 0 \;\updownarrow\; 0 \qquad \mathbb{Z}$

$T_i(\mathbb{RP}^3)$ $\quad 0 \qquad 0 \qquad \underset{\updownarrow}{\mathbb{Z}_2} \qquad 0$

Furthermore, Poincaré duality gives

For M as above, one can use P.D. to define an "intersection product"

$$H_i(M) \times H_j(M) \to H_{i+j-n}(M)$$

e.g. if $n = 2i$, one gets a "product"

$$H_i(M) \times H_i(M) \to H_0(M) = \underline{\mathbb{Z}}$$

If M connected and nonempty

If M is a smooth manifold, the intersection product is related to the intersection of (smooth, closed) submanifolds.

eg. $M = S^1 \times S^1$

$[c] \cdot [c'] =$ alg. number of intersection points between c and c'.

$- \cdot [c'] \in \mathrm{Hom}(H_i(M), \mathbb{Z})$

It turns out $- \cdot [c^{-1}] = h(PD^{-1}([c']))$

More on this later.

de Rham Cohomology

If M is a (fin. dim.) smooth manifolds.

then $H^i(M; \mathbb{R}) \cong \underset{\sim}{H^i_{dR}(M)}$

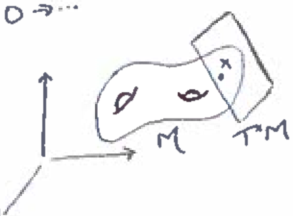
i^{th} de Rham cohomology

defined in terms of differential calculus.

We give a cochain complex $\Omega^*(M)$

$$\Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \cdots \to \Omega^n(M) \to 0 \to \cdots$$

$\Omega^i(M) = \{$(smooth) differential i-forms on $M\}$



0-forms = smooth functions $f: M \to \mathbb{R}$

1-forms = smooth functions $x \in M \mapsto \omega_x \in \underset{\sim}{(T_xM)}^*$

tangent space of M at x.

⋮

i^{th}-forms = smooth functions $x \in M \mapsto \omega_x \in (\Lambda^i T_x M)^*$

If ω is an i-form on M and $N \subseteq M$ is a submanifold then one can integrate ω over N.

Stokes' Theorem

$$\int_{\partial N} \omega = \int_N d\omega$$

ω i-form

N $(i+1)$-manifold

Can define a map

smooth i-chains

$$\Psi: \Omega^i(M) \longrightarrow \mathrm{Hom}(C_i^S(M), \mathbb{R})$$

$\overset{\in}{\omega}$ i-Form $\longmapsto \Psi(\omega)$

via $\Psi(\omega)(\sigma)$ ← σ smooth i-simplex

$\sigma: \underset{\sigma^*\omega}{\Delta^i} \to \underset{\omega}{M}$

$\| \text{def}$

$$\int_{\Delta^i} \sigma^* \omega \in \mathbb{R}$$

Stokes' Thm gives this is a chain map.

Turns out Ψ induces an isomorphism

$H^i_{dR}(M) \to H^i(M, \mathbb{R})$

Relationship with Vector Fields:

g_x Riemannian metric (inner product for tangent vectors) depending smoothly on point x.

Smooth Vector Field V $\longleftrightarrow$ $\omega_x(-) = g_x(v_x, -)$ $\overset{\in}{}$ $(T_xM)^*$

depends on choice of metric.

curl $v = 0 \longleftrightarrow d^1\omega = 0$ ("ω-closed")

$v = \nabla f \longleftrightarrow \omega = d^0 f$ (ω "exact")

Rem: curl, ∇ depend on choice of g. But d^1, d^0 do not depend on choice of metric.

$$H^1_{dR}(M) = \frac{\ker d^1}{\operatorname{im} d^0} = \frac{\{\text{closed 1-Form}\}}{\{\text{exact 1-Form}\}}$$

$$= \frac{\{\text{curl-Free vector Field}\}}{\{\text{gradient Field conservative}\}}$$

$$\cong H^1(M;\mathbb{R})$$

$\overline{\sqcap}$?

Cup Product

Δ^n = standard n-simplex = $[e_0,\dots,e_n] \subseteq \mathbb{R}^{n+1}$

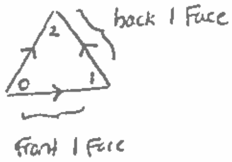
Let $n = i+j$

$[e_0,\dots,e_i]$ is called the Front i Face of Δ^{i+j}

$[e_i,\dots,e_{i+j}]$ is called the back j Face of Δ^{i+j}

Ex:

$\Delta^2 =$



Let

$f_i : \Delta^i \hookrightarrow \Delta^{i+j}$

$b_j : \Delta^j \hookrightarrow \Delta^{i+j}$

be the map sending Δ^i, Δ^j to the Front i Face and the Back j-Face.

Let X be a topological Face and $\sigma \in S_{i+j}(X)$

$\sigma : \Delta^{i+j} \to X$

Def:

$\sigma|_{[e_0,\dots,e_i]} = \sigma \circ f_i$

$\sigma|_{[e_i,\dots,e_{i+j}]} = \sigma \circ b_j$

Bredon notation:

$\sigma\rfloor_i \overset{def}{=} \sigma|_{[e_0,\dots,e_i]}$

${}_j\lfloor\sigma \overset{def}{=} \sigma|_{[e_i,\dots,e_{i+j}]}$

Let $\phi \in C^i(X)$, $\psi \in C^j(X)$

$\sigma \in S_{i+j}(X)$

$$(\phi \smile \psi)(\sigma) \overset{def}{=} \underbrace{\underset{\in\mathbb{Z}}{\phi(\sigma|_{[e_0,\dots,e_i]})}\underset{\in\mathbb{Z}}{\psi(\sigma|_{[e_i,\dots,e_{i+j}]})}}_{\text{mult in }\mathbb{Z}}$$

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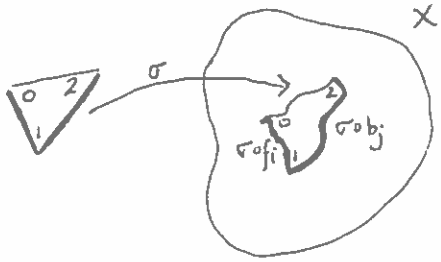
$$C^i(X) \times C^j(X) \to C^{i+j}(X)$$

$$(\phi,\psi) \longmapsto \phi \smile \psi$$

Rem: $\smile$ is bilinear as product in $\mathbb{Z}$ is bilinear.



Rem: The same works for coefficients in a commutative ring (i.e. $\smile$ bilin.) R.

Rem: If X is a Δ-complex, the same definition can be used to define $\smile$ for simplicial cochains. In part,

$$C^*(X) \to C_0^*(X)$$

is compatible with $\smile$. So simplicial structures are sufficient.

Thm: $\smile$ is unital and associative.

Pf: $\smile$ unital: Define $\epsilon \in C^0(X)$ by $\epsilon(\sigma) = 1$ $\forall \sigma \in S_0(X)$. Let $\phi \in C^i(X)$ & $\sigma \in S_i(X)$.

$$(\phi \smile \epsilon)(\sigma) = \phi(\underbrace{\sigma|_{[e_0,\dots,e_i]}}_{\sigma})\, \epsilon(\underbrace{\sigma_{[e_i]}}_{\text{0-simplex}})$$

$$= \phi(\sigma) \cdot 1$$

$$= \phi(\sigma)$$

So $\phi \smile \epsilon$. Other order similiar.

$\smile$ associative: $\phi \in C^i(X), \psi \in C^j(X), \omega \in C^h(X)$
$\sigma \in S_{i+j+k}(X)$.

$$((\phi \smile \psi) \smile \omega)(\sigma) = \phi(\sigma|_{e_0 \dots e_i})\, \psi(\sigma|_{e_i \dots e_{i+j}})\, \omega(\sigma|\_\_)$$

Changing parenthesis results in same formula. $\boxtimes$

Rem. The same holds true for any coefficients in any comm (unital, $\overset{\wedge}{\text{assoc.}}$) ring R.
$\underline{\hphantom{xx}}$ need

Thm: $\smile$ is natural w.r.t. cont. maps
That is, $F: X \to Y$ is cont. $\phi \in C^i(Y)$
$\psi \in C^j(Y)$ then

$$f^\#(\phi \smile \psi) = f^\#\phi \smile f^\#\psi$$

Pf: Let $\sigma \in S_{i+j}(X)$.

$$f^\#(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(F \circ \sigma)$$

$$= \phi(\underset{\substack{\uparrow \\ \text{that, } \circ f_i}}{F\sigma|\_\_})\ \psi(\underset{\substack{\uparrow \\ \text{that is, } \circ b_j}}{F\sigma|\_\_})$$

$$= (F^\#\phi)(\sigma \circ f_i)(F^\#\psi)(\sigma \circ b_j)$$

$$= ((F^\#\phi) \smile (f^\#\psi))(\sigma)$$

$\boxtimes$

Note: If $\epsilon \in C^0(X)$ is def. by $\epsilon(\sigma) = 1$ $\forall \sigma \in S_0(X)$
$\epsilon' \in C^0(Y)$ is def. by $\epsilon'(\sigma') = 1$ $\forall \sigma' \in S_0(Y)$ then

$$f^\#\epsilon' = \epsilon$$

because both $F^\#\epsilon'$ and ϵ are constant & $=$.

Special Case: If $X = \emptyset$, then $C^0(X) = \{0\}$ and $\epsilon = 0$.
Formula still holds as $f^\# = 0$.

Summary: $C^*(X) = \bigoplus C^i(X)$ is a graded ring and $f^\#$ is a graded ring homomorphism.

Connection between $\smile$ & δ

Thm: If $\phi \in C^i(X)$ & $\psi \in C^j(X)$ then $\delta(\phi \smile \psi) =$

$$(\delta\phi) \smile \psi + (-1)^i \phi \smile \delta\psi$$

Pf: Let $\sigma \in S_{i+j+1}(X)$.

$$\delta(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial \sigma)$$
$$= \sum_{k=0}^{i+j+1} \underbrace{(-1)^k (\phi \smile \psi)(\sigma|_{e_0 \cdots \hat{e}_k \cdots e_{i+j+1}})}_{*}$$

IF $\underline{k \le i}$:

$$* = (-1)^k \phi(\sigma|_{e_0 \cdots \hat{e}_k \cdots e_{i+1}}) \psi(\sigma|_{e_{i+1} \cdots e_{i+j+1}})$$

IF $\underline{k > i}$:

$$* = \underset{\substack{\| \\ (-1)^i (-1)^{k-i}}}{(-1)^k} \phi(\sigma|_{e_0 \cdots e_{i+1}}) \psi(\sigma|_{e_{i+1} \cdots \hat{e}_k \cdots e_{i+j+1}})$$

Easy to check

$$((\delta\phi) \smile \psi)(\sigma) = \sum \{* \text{ For } k \le i\} + a$$

$$(-1)^i (\phi \smile \delta\psi)(\sigma) = \sum \{* \text{ For } k > i\} + b$$

$$a = (-1)^{i+1} \phi(\sigma|_{e_0 \cdots \hat{e}_{i+1}}) \psi(\sigma|_{e_{i+1} \cdots e_{i+j+1}})$$

$$b = (-1)^i \phi(\sigma|_{e_0 \cdots e_i}) \psi(\sigma|_{e_i \cdots e_{i+j+1}})$$

Note: $a = -b$ so adding terms above gives the original sum at top of the page. ☒

Cor: IF $\delta\phi = 0$ & $\delta\psi = 0$ then $\delta(\phi \smile \psi) = 0$

Cor: IF $\delta\phi = 0$ & $\psi = \delta\omega$ then $\phi \smile \psi = \delta((-1)^i \phi \smile \omega)$

Cor: IF $\phi = \delta\omega$ and $\delta\psi = 0$ then $\phi \smile \psi = \delta(\omega \smile \psi)$

Cor: $Z^i \smile Z^j \subseteq Z^{i+j}$
$Z^i \smile B^j \subseteq B^{i+j}$
$B^i \smile Z^j \subseteq B^{i+j}$

Cor: Cup product induces a well defined map

$$\smile : H^i(X) \times H^j(X) \to H^{i+j}(X)$$
$$([\phi], [\psi]) \mapsto [\phi \smile \psi]$$

Def: $H^*(X) = \bigoplus_{i \ge 0} H^i(X)$ is a graded ring called the singular cohomology ring of X.

Rem: $[\epsilon]$ is the id. where $\epsilon(\sigma) = 1 \ \forall S_0(X)$.

IF $f: X \to Y$ is cont., then

$$f^*: H^*(Y) \to H^*(X)$$

is a ring homo.

IF R comm ring $H^*(X, R) = \bigoplus_{i \ge 0} H^i(X, R)$ is a graded unital associative R-alg., as before.

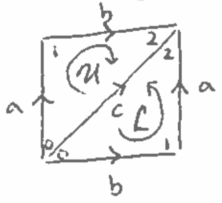
Later, $H^*(X)$ is graded commutative

$$[\phi] \smile [\psi] = (-1)^{ij} [\psi] \smile [\phi]$$

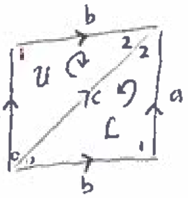
But not true on chain level.

Ex: $T =$

$=$



$a^*, b^* \in C_0^1(T)$



What is $a^* \smile b^*$?

$(a^* \smile b^*)(U) = a^*(U|_{e_0 e_1})\, b^*(U_{e_1, e_2})$

$= a^*(a)\; b^*(b)$

$= 1 \cdot 1 = 1$

$(a^* \smile b^*)(L) = a^*(L_{e_0 e_1})\, b^*(L_{e_1 e_2})$

$= a^*(b)\; b^*(a)$

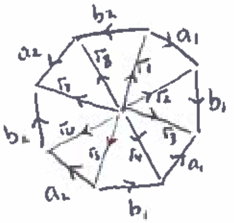
$= 0 \cdot 0 = 0$

$a^* \cup b^* = U^*$

a^*, b^* are not cocycles so do not rep. for cohomology classes

HW: Compute $\smile$ on $H^*_\Delta(T)$

Ex: $M =$ 



Can check:

$H^1_\Delta(M) \cong \mathbb{Z}^4 = \text{Span}\{[\alpha_1], [\beta_1], [\alpha_2], [\beta_2]\}$

$\alpha_1 = a_1^* + \sigma_2^* + \sigma_3^*$

$\alpha_2 = a_2^* + \sigma_6^* + \sigma_7^*$

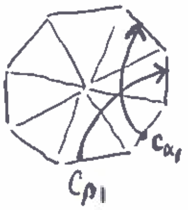
$\beta_1 = b_1^* + \sigma_3^* + \sigma_4^*$

$\beta_1 = b_2^* + \sigma_2^* + \sigma_8^*$

We can 'visualize' $\alpha_i\ \beta_i$ by drawing curves $C_{\alpha_i}, C_{\beta_i} \subseteq M$ such that for each edge e,

$\alpha_i(e) = e \cdot C_{\alpha_i}$

$\beta_i(e) = \underbrace{e \cdot C_\beta}_{\text{alg. int \#}}$



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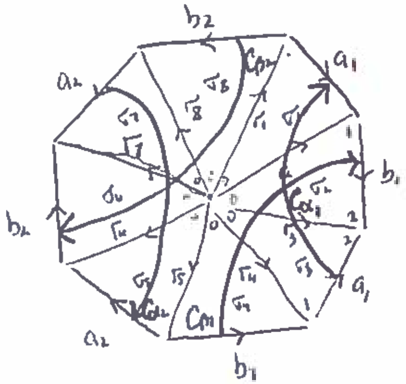


$H^1_\Delta(M) \underset{\text{Span}}{\cong} \{[\alpha_1], [\beta_1], [\alpha_2], [\beta_2]\}$

$\alpha_1 = a_1^* + \sigma_2^* + \sigma_3^*$

$\vdots$

As on the left.

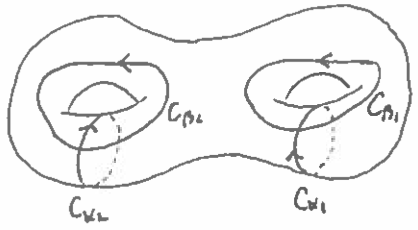
Draw closed curves $\underbrace{C_{\alpha_i}, C_{\beta_i}}_{\text{Rep. Poincaré duals of } \alpha_i, \beta_i} \subseteq M$ such that for every edge e,

$\alpha_i(e) = e \cdot C_{\alpha_i}$

$\beta_i(e) = e \cdot C_{\beta_i}$

↖ algebraic integer #

On the surface M



$$(\alpha_1 \smile \beta_1)(\sigma_2) = \alpha_1(\sigma_2|_{e_0,e_1})\, \beta_1(\sigma_2|_{e_1,e_2})$$

↑ really $[e_0,e_1]$

$$= \alpha_1(\underset{\sigma_2 \cdot C_{\alpha_1}}{\sigma_2})\, \beta_1(\underset{b_1 C_{\beta_1}}{b_1})$$

$$= 1 \cdot 1$$

$$= 1$$

Sim., check that

$(\alpha_1 \smile \beta_1)(\sigma_i) = 0$ for $i \neq 2$. So $\alpha_1 \smile \beta_1 = \sigma_2^*$

$$(\beta_1 \smile \alpha_1)(\sigma_3) = \beta_1(\sigma_3|_{e_0,e_1})\, \alpha_1(\sigma_3|_{e_1,e_2})$$

$$= \beta_1(\sigma_4)\, \alpha_1(a_1)$$

$$= (\sigma_4\, C_{\beta_1})(a_1 \cdot C_{\alpha_1})$$

$$= 1 \cdot 1$$

$$= 1$$

Sim., Check that

$(\beta_1 \smile \alpha_1)(\sigma_i) = 0$ for $i \neq 3$

So $\beta_1 \smile \alpha_1 = \sigma_3^*$

In $H^2_\Delta(M)$,

$$[\sigma_1^*] = [\sigma_2^*]$$

$\longleftarrow \delta\sigma_2^* = \sigma_2^*\delta = \sigma_2^* - \sigma_1^*$

$$[\sigma_5^*] = [\sigma_6^*] = -[\sigma_7^*] = -[\sigma_8^*]$$

Notation:

$$[M^*] = [\sigma_3^*]$$

Later $[M^*]$ will be called the Fundamental class of M.

$$H^2_\Delta(M) = \text{Span}\{[M]^*\} \cong \mathbb{Z}$$

$$[\alpha_1] \smile [\beta_1] = [\sigma_2^*] = \boxed{-[M]^*}$$

$$[\beta_1] \smile [\alpha_1] = [\sigma_3^*] = [M]^*$$

$$[\beta_i] \smile [\alpha_j] = \begin{cases} [M]^*, & i=j \\ 0, & i \neq j \end{cases}$$

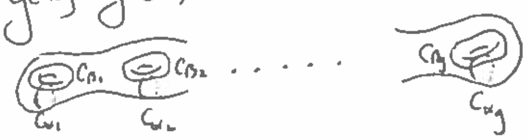
$$= -[\alpha_j] \smile [\beta_i]$$

$$[\alpha_i] \smile [\alpha_j] = 0 = [\beta_i] \smile [\beta_j] = 0$$

$$[\beta_i] \smile [\alpha_j] = (C_{\beta_i} \cdot C_{\alpha_j})[M]^*$$

Cup product is dual to the intersection product.

IF M is a closed, oriented, connected genus g surface



$H^1_\Delta(M)$ Freely span $[\alpha_i], [\beta_i]$; $i = 1, \dots, g$

↑ Poincaré Duals of C_{α_i}, C_{β_i}

is given by the formula:

$$[\beta_i] \smile [\alpha_j] = \begin{cases} [M]^*, & i=j \\ 0, & i\neq j \end{cases}$$

$$= -[\alpha_j]\smile[\beta_i]$$

Again, $[\beta_i]\smile[\alpha_j] = (C_{\beta_i}\, C_{\alpha_j})[M]^*$

Ex: $H^*(S^n)$, $n>0$ $\quad$ $\epsilon(\sigma)=1$ for $\sigma\in S_0(S^n)$

$H^0(S^n)\cong \mathbb{Z} \cong \text{Span}\{[\epsilon]\}$

$H^n(S^n)\cong \mathbb{Z} \cong \text{Span}\{[S^n]^*\}$ "dual of Fund. class of S^n"

$H^i(S^n)=0$ for $i\neq 0,n$

$[\epsilon]\smile[\epsilon]=[\epsilon]$ because ϵ id. for $\smile$

$\underbrace{[S^n]^*\smile[S^n]^*}_{\text{so } =0} \in H^{2n}(S^n)=0$

$[\epsilon]\smile[S^n]^* = [S^n]^* = [S^n]^*\smile[\epsilon]$

Conclusion: There is an iso:

$$H^*(S^n) \longrightarrow \mathbb{Z}[\alpha]/(\alpha^2)$$

$\deg\alpha = n$

$[\epsilon]\longmapsto 1$

$[S^n]^*\longmapsto \alpha$

Rem: Poincare Dual of $[S^n]^* = [x]\in H_n(S^n)$

For any point $x\in S^n$

Check: $[x]\cdot[x]=0$

Reason if $x'\neq x''$ are generic points in S^n then $x'\,x''=0$

Thm: If $[\phi]\in H^i(X)$ and $[\psi]\in H^j(X)$ then

$$[\phi]\smile[\psi] = (-1)^{ij}[\psi]\smile[\phi]$$

PF: Let $r_n: \underset{\Delta^n}{[e_0,\dots,e_n]} \to \underset{\Delta^n}{[e_0,\dots,e_n]}$

be the affine map given by

$e_i\mapsto e_{n-i}$

Further let

$\rho: C_n(X)\to C_n(X)$

be the map given by

$\sigma\longmapsto \epsilon_n\,\sigma\, r_n$

where $\epsilon_n = (-1)^{n\frac{(n+1)}{2}}$

Fact: ρ is a chain map homotopic to the identity

PF: Hatcher pp 216-217.

Idea: r_n is composition of $n\frac{(n+1)}{2}$ transpositions of form $e_i\leftrightarrow e_{i+1}$, each of which reverses orientation of simplex Δ^n.

Let $\phi\in C^i(X)$, $\psi\in C^j(X)$

$\delta\phi=0$, $\delta\psi=0$ (rep cohom classes)

and let $\sigma\in S_{i+j}(X)$

$(\rho^*\phi\smile\rho^*\psi)(\sigma)=$

$(\rho^*\phi)(\sigma\circ f_i)\,(\rho^*\psi)(\sigma\circ b_j)=$

$\epsilon_i\,\epsilon_j\,\phi(\sigma f_i r_i)\,\psi(\sigma b_j r_j)=$

$\epsilon_i\,\epsilon_j\,\phi(\sigma|_{e_i,\dots,e_0})\,\underbrace{\psi(\sigma|_{e_{i+j}\cdots e_i})}_{}$

$$\boxed{\rho(\sigma) = \epsilon_n \sigma r_n}$$

$$\rho^*(\psi \smile \phi)(\sigma)$$

$$= \epsilon_{i+j}(\psi \smile \phi)(\sigma \circ r_{i+j})$$

$$= \epsilon_{i+j} \, \psi(\sigma \underbrace{r_{i+j} F_j}_{= b_j r_j}) \, \phi(\sigma \underbrace{r_{i+j} b_i}_{= F_i r_i})$$

r_{i+j} exchanges the front face w/ corresponding back face

$$= \epsilon_{i+j} \, \psi(\sigma b_j r_j) \, \phi(\sigma F_i r_i)$$

$$= \epsilon_{i+j} \, \underbrace{\psi(\sigma_{e_{i+j}, \dots, e_i}) \, \phi(\sigma|_{e_i \dots e_0})}$$

Which contains same term as org. computation

We then get $\epsilon_i \epsilon_j (\rho^* \phi \smile \rho^* \psi) = \epsilon_{i+j} \rho^*(\psi \smile \phi)$

By Fact $\rho = 1$, $\rho^* = 1$ so on homology level 'forgotten' so

$$[\phi] \smile [\psi] = \frac{\epsilon_{i+j}}{\epsilon_i \epsilon_j} [\psi] \smile [\phi]$$

$$(-1)^{\frac{(i+j)(i+j+1)}{2} - \frac{i(i+1)}{2} - \frac{j(j+1)}{2}}$$

which is $(-1)^{ij}$ ▨

Rem: Thm remains true for cohomology in any comm. ring R.

There is an alternative proof which uses "acyclic models" (later).

Rem: In de Rham Cohomology the cup product corresponds to the wedge product.

Relative Cup Product

X top. space

$A, B \subseteq X$ open

Recall,

$$C^i(X,A) = \{ \phi \in C^i(X) \mid \phi|_{S_i(A)} = 0 \}$$

$$C^j(X,B) = \{ \phi \in C^j(X) \mid \phi|_{S_j(B)} = 0 \}$$

Let $\phi \in C^i(X,A)$, $\psi \in C^j(X,B)$

then $\phi \smile \psi$ vanish on $S_{i+j}(A)$ and on $S_{i+j}(B)$

So $\smile$ restricts to a map

$$C^i(X,A) \times C^j(X,B) \to C^{i+j}(X, A+B)$$

where $C^{i+j}(X, A+B) = \{ \phi \in C^{i+j}(X) \mid \underbrace{\phi|_{S_{i+j}(A)} = 0, \ \phi|_{S_{i+j}(B)} = 0}_{} \}$

ϕ vanish on both A & B $i+j$ simplices

Using subdivision, one can show :

$$H^{i+j}(C^*(X, A+B)) \cong H^{i+j}(X, A \underset{\substack{\uparrow \\ \text{product}}}{\cup} B)$$

So $\smile$ induces a map

$$H^i(X,A) \times H^j(X,B) \to H^{i+j}(X, A \cup B)$$

Called the relative cup product. For $A = B \subseteq X$, one gets a map

$$H^i(X,A) \times H^j(X,A) \to H^{i+j}(X,A)$$

So...

$$H^*(X,A) = H^0(X,A) \oplus H^1(X,A) \oplus \cdots$$

is a graded associative, graded comm. $\mathbb{Z}$-alg (not unital).

10/04/2016

Tensor Products of Chain Complexes

Let C, C' be chain complexes.

Def: $C \otimes C'$ is defined as

$$(C \otimes C')_n = \bigoplus_{i+j=n} C_i \otimes C_j'$$

with differential

$$\partial_\otimes (c \otimes c') = \partial c \otimes c' + (-1)^i c \otimes \partial c'$$

for $c \in C_i$, $c' \in C_j'$

Check:

1) $\partial_\otimes^2 = 0$

2) $Z_i \otimes Z_j' \subseteq Z_{i+j}(C \otimes C')$

3) $Z_i \otimes B_j' \subseteq B_{i+j}(C \otimes C')$

4) $B_i \otimes Z_j' \subseteq B_{i+j}(C \otimes C')$

In particular, $\otimes$ induces a map

$$H_i(C) \otimes H_j(C') \to H_{i+j}(C \otimes C')$$
$$[z] \otimes [z'] \mapsto [z \otimes z']$$

Rem: The same is true for cochain complexes.

Note also $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^i \phi \smile \psi$ implies that $\smile$ defines a chain map

$$\smile : \underbrace{C^*(X) \otimes C^*(X)}_{\text{maps } \delta \otimes \delta} \to \underbrace{C^*(X)}_{\text{map } \delta}$$

On cohomology level

$$H^i(X) \otimes H^j(X) \to H^{i+j}(C^*(X) \otimes C^*(X))$$
$$[\phi] \otimes [\psi] \mapsto [\phi \otimes \psi]$$

and map

$$H^{i+j}(C^*(X) \otimes C^*(X)) \xrightarrow[\smile]{} H^{i+j}(X)$$
$$[\phi \otimes \psi] \longmapsto [\phi \smile \psi]$$

Long term goal: Understand cohomology rings of product spaces.

We will need to introduce more products:

Cohomology Cross Product

X, Y be top. spaces

We have:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Define bilinear map

$$\times : C^i(X) \times C^j(X) \to C^{i+j}(X \times Y)$$

as follows: let $\phi \in C^i(X)$, $\psi \in C^j(Y)$

$\sigma \in S_{i+j}(X \times Y)$

$$(\phi, \psi) \mapsto \phi \times \psi$$

$$(\phi \times \psi)(\sigma) = \phi(\pi_X \circ \sigma \circ f_i)\, \psi(\pi_Y \circ \sigma \circ b_j)$$

Note: We can write this as

$$(\phi \times \psi)(\sigma) = \underbrace{(p_X^\# \phi)(\sigma f_i)\, (p_Y^\# \psi)}_{\text{projections}}(\sigma \circ b_j)$$

$$= (\pi_X^\# \phi \smile \pi_Y^\# \psi)\sigma$$

So $\phi \times \psi = \pi_X^\# \phi \smile \pi_Y^\# \psi$

Then $\times$ is a chain map

$$C^*(X) \otimes C^*(Y) \to C^*(X \times Y)$$

(Follows as $\pi_X^\#, \pi_Y^\#$ are chain maps)

Then $\exists$ induced map

$$\times : H^i(X) \otimes H^j(Y) \longrightarrow H^{i+j}(X \times Y)$$

$$[\phi] \otimes [\psi] \longmapsto [\phi \times \psi]$$

$$\shortparallel$$

$$[\phi] \times [\psi]$$

Called the cohomology cross product.

Explicitly: $[\phi] \times [\psi] = \pi_X^*[\phi] \smile \pi_Y^*[\psi]$

Properties:

$1_X, 1_Y$ identity elements in $H^*(X), H^*(Y)$

then $[\phi] \times 1_Y = \pi_X^*[\phi]$

$1_X \times [\psi] = \pi_Y^*[\psi]$

Thm: Let $\phi \in C^i(X)$, $\phi' \in C^{i'}(X)$

$\psi \in C^j(X)$, $\psi' \in C^{j'}(Y)$

Then

$$\left[(\phi \times \psi) \overset{\text{in } X \times Y}{\overset{\downarrow}{\smile}} (\phi' \times \psi')\right] =$$

$$(-1)^{ji'}\left[(\phi \underset{\text{in } X}{\underset{\uparrow}{\smile}} \phi') \times (\psi \underset{\text{in } Y}{\underset{\uparrow}{\smile}} \psi')\right]$$

Pf: $[(\phi \times \psi) \smile (\phi' \times \psi')]$

$$= [\pi_X^\# \phi \smile \underbrace{\pi_Y^\# \psi \smile \pi_X^\# \phi'}_{\text{graded comm.}} \smile \pi_Y^\# \psi']$$

$\downarrow$ naturality

$$= (-1)^{ji'} [\pi_X^\# \phi \smile \pi_X^\# \phi' \smile \pi_Y^\# \psi \smile \pi_Y^\# \psi']$$

$$= (-1)^{ji'} [\pi_X^\# (\phi \smile \phi') \smile \pi_Y^\# (\psi \smile \psi')]$$

$$= (-1)^{ji'} [(\phi \smile \phi') \overset{\times}{\smile} (\psi \smile \psi')] \quad \blacksquare$$

We have the cross product in terms of the cup product. But the reverse is also possible:

$$D : X \to X \times X$$

$$x \mapsto x \times x$$

Thm: Let $\phi \in C^i(X)$, $\psi \in C^j(X)$, then

$$\phi \smile \psi = D^\#(\phi \times \psi)$$

Diagrammatically:

$$\begin{tikzcd} C^i(X) \otimes C^j(X) \arrow[r, "\times"] \arrow[rr, bend right, "\smile"'] & C^{i+j}(X \times X) \arrow[r, "D^\#"] & C^{i+j}(X) \end{tikzcd}$$

Pf: Let $p_1, p_2 : X \times X \to X$ be projection onto the factors.

$$D^\#(\phi \times \psi) = D^\#(p_1^\# \phi \smile p_2^\# \psi)$$

naturality $= (D^\# p_1^\#) \phi \smile (D^\# p_2^\#) \psi$

$$= \underbrace{(p_1 D)}_{1_X}{}^\# \phi \smile \underbrace{(p_2 D)}_{1_Y}{}^\# \psi$$

$$= \phi \smile \psi \quad \blacksquare$$

Alexander - Whitney Map:

X, Y top. spaces

Recall: $(C_*(X) \otimes C_*(Y))_n = \bigoplus_{\substack{i+j=n \\ i,j \geq 0}} C_i(X) \otimes C_j(Y)$

Define a linear map

$$\Theta : C_n(X \times Y) \to (C_*(X) \otimes C_*(Y))_n$$

via $\sigma \longmapsto \sum_{\substack{i+j=n \\ i,j \geq 0}} (p_X \sigma f_i) \otimes (p_Y \sigma b_j)$

This map Θ is called the Alexander-Whitney map.

Notations: If $x \in X$, write σ_x for 0-simplex sending Δ^0 to X

If $f: X \to X'$, $g: Y \to Y'$ are cont. then $\langle f, g\rangle$ for the product map

$$\langle f,g\rangle : X \times Y' \to X' \times Y'$$
$$(x,y) \mapsto (fx, gy)$$

Thm:

1) Θ is a chain map

2) Θ is natural in X & Y, ie

$$(f_\# \otimes g_*)\underset{\substack{\uparrow\\ \text{for } X,Y}}{\Theta} = \underset{\substack{\uparrow\\ \text{for } X',Y'}}{\Theta}\langle f, g\rangle_\#$$

3) Θ is the canonical map

$$\sigma_{(x,y)} \longmapsto \sigma_x \otimes \sigma_Y$$

in degree 0.

4) Up to homotopy, Θ is the unique map satisfying 1 & 2 & 3.

Pf (sketch):

(1) Similar to proof for $\smile$ chain map.

(2) Let $f: X \to X'$, $g: Y \to Y'$ be cont. $\sigma \in S_*(X \times Y)$. Then

$$(f_\# \otimes g_\#)(\Theta(\sigma)) = \sum_{i+j=n} (f p_X \sigma f_i) \otimes (g p_Y \sigma b_j)$$

$$= \sum_{i+j=n} (p_{X'} \circ \underbrace{\langle f, g\rangle \circ \sigma}_{\langle f,g\rangle_\# \sigma} \circ f_i) \otimes (p_{Y'} \underbrace{\langle f, g\rangle}_{\langle f,g\rangle_\# \sigma} \sigma b_j)$$

$$= \Theta(\langle f, g\rangle_\#(\sigma))$$

3) Let $(x,y) \in X \times Y$. Then

$$\Theta(\sigma_{(x,y)}) = \underbrace{\sum_{\substack{i+j=0\\ i,j\geq 0}}}_{\substack{1 \text{ term}\\ i=j=0}} (\underbrace{p_X \sigma_{x,y}}_{\substack{\uparrow\\ \sigma_x}} \overset{f_0}{\overset{\nearrow}{f_i}}) \otimes (\underbrace{p_Y \sigma_{x,y}}_{\substack{\uparrow\\ \sigma_Y}} \overset{b_0}{\overset{\nearrow}{b_j}})$$

$$= \sigma_x \otimes \sigma_Y$$

4) Perhaps later: uses acyclic modules ☒

Connection with $\times$:

Let $\Phi \in C^i(X)$, $\Psi \in C^j(X)$. Then $\Phi \otimes \Psi \in C^i(X) \otimes C^j(Y)$, we can view $\Phi \otimes \Psi$ as a linear form on $C_i(X) \otimes C_j(Y)$

$$\underline{\cap}$$

$$(C_*(X) \otimes C_*(Y))_{i+j}$$

We can extend $\Phi \otimes \Psi$ to all of $(C_*(X) \otimes C_*(Y))_{i+j}$ by

$$(\Phi \otimes \Psi)\Big|_{C_{i'}(X) \otimes C_{j'}(Y)} = 0$$

for $(i', j') \neq (i,j)$

Thm: $\overset{\in C^{i+j}(X \times Y)}{\Phi \times \Psi} = (\Phi \otimes \Psi)\underset{\substack{\nearrow\\ C^{i+j}(X\times Y) \to (C_*(X) \otimes C_*(Y))_{i+j}}}{\Theta}$

$= \Theta^*(\Phi \otimes \Psi)$

That is, cohomology cross product is dual of Alexander-Whitney map.

Pf: $\sigma \in S_{i+j}(X \times Y)$

$$(\phi \otimes \psi)(\Theta(\sigma)) = (\phi \otimes \psi)\Big(\sum_{i'+j'=i+j} (p_X \sigma f_{i'}) \otimes (p_Y \sigma b_{j'})\Big)$$

$$= (\phi \otimes \psi)((p_X \sigma f_i) \otimes (p_Y \circ \sigma \circ b_j))$$

$$= \phi(p_X \sigma f_i)\psi(p_Y \sigma b_j)$$

$$= (\phi \times \psi)(\sigma). \quad \blacksquare$$

So far we have....

Cohomology Cross Product:

$\times : C^i(X) \otimes C^j(Y) \to C^{i+j}(X \times Y)$

Alexander - Whitney Map:

$\Theta : C_{i+j}(X \times Y) \to (C_*(X) \otimes C_*(Y))_{i+j}$

And $\times$ & Θ are somehow dual to each other.

Next we shall see a homology cross product:

$$p : C_i(X) \otimes C_j(Y) \to C_{i+j}(X \times Y)$$

$$(\underset{\underset{S_i(X)}{\cap}}{\sigma}, \underset{\underset{S_j(Y)}{\cap}}{\tau}) \mapsto p(\sigma, \tau)$$

Problem is product of simplices are not simplices.

10/06/2016

Homology Cross Product:

X, Y top spaces

X × Y top. spaces

Goal: Define a map

$$C_i(X) \otimes C_j(Y) \xrightarrow{P} C_{i+j}(X \times Y)$$

Would like to define p as "product" of simplices σ and τ.

Problem: Products of simplices are not simplices, ie ∃ no identification $\Delta^i \times \Delta^j = \Delta^{i+j}$

Possible Solutions:

Only for CW complexes → ~ Use Cellular homology, c.f. Hatcher 3.B

Not discussed → ~ Use cubical sing. homology

~ Use acyclic models.

Thm: There exists $p : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ ∋

1) P chain map

2) p natural in both entries

$$\langle F, g \rangle_\# \, p = p \circ (F_\# \otimes g_\#)$$

in degree 0, p is the canonical map

$$\sigma_x \otimes \sigma_y \mapsto \sigma_{(x,y)} \quad \forall \, \begin{matrix} x \in X \\ y \in Y \end{matrix}$$

Moreover, the map p is unique up to homotopy.

Note: (1) means $p(\sigma, \tau) = p(\partial\sigma, \tau) + (-1)^i p(\sigma, \partial\tau)$ *

For $(\sigma, \tau) \in S_i(X) \times S_j(Y)$

PF: We will define $p : C_i(X) \times C_j(Y) \to C_{i+j}(X \times Y)$ simultaneously for all pairs (X,Y) by induction on $i+j$

IF $i+j = 0 \to i,j = 0$ as nonnegative. Hence, p is completely determined by (3).

IF $i+j = 1$, then $(i,j) = (1,0)$ or $(i,j) = (0,1)$ So choose $(\sigma,\tau) \in S_i(X) \times S_j(Y)$, then either σ or τ 0-simplex. Hence, we can define $p(\sigma,\tau) = \underbrace{\sigma \times \tau}_{\text{top. product.}}$ - the product with a 0-simplex is again a simplex. One can check this def of p is compatible with $*$.

Now suppose $i+j > 1$. Suppose p is already defined for all pairs (i',j') with $i'+j' < i+j$. Define p for (i,j), we proceed in 2 steps:

Step 1: Define $p(\imath_i, \imath_j)$, where $\imath_i : \Delta^i \to \Delta^i$, $\imath_j : \Delta^j \to \Delta^j$ are the identity maps viewed as simplices in $\Delta^i \times \Delta^j$, respectively

Step 2: Define p for arbitrary $\sigma : \Delta^i \to X$ $\tau : \Delta^j \to Y$.

- -

Step 1: By $*$, $p(\imath_i, \imath_j)$ must satisfy

$$\partial p(\imath_i, \imath_j) = \underbrace{p(\partial \imath_i, \imath_j) + (-1)^i p(\imath_i, \partial \imath_j)}_{\text{already defined by induction}}$$

Let z = right hand side (again, defined by induction).

A computation shows z is a cycle. (uses $*$), ie $\partial z = 0$

$$z \in Z_{i+j-1}(\Delta^i \times \Delta^j)$$

Since $\Delta^i \times \Delta^j$ is contractible and since $i+j > 1$, this implies that

$$z \in B_{i+j-1}(\Delta^i \times \Delta^j)$$

So $z = \partial b$ for $b \in C_{i+j}(\Delta^i \times \Delta^j)$. Then Define $p(\imath_i, \imath_j) = b$. Then $*$ is satisfied by construction of $p(\imath_i, \imath_j)$.

Step 2: Let $\sigma : \Delta^i \to X$, $\tau : \Delta^j \to Y$ be arbitrary. Then $\sigma = \sigma \circ \imath_i$ & $\tau = \tau \circ \imath_j$ & $\sigma = \sigma_\#(\imath_i)$ & $\tau = \tau_\#(\imath_j)$. By (2), we must define $p(\sigma,\tau)$ by....

$$p(\sigma,\tau) = \langle \sigma,\tau \rangle_\# (\underbrace{p(\imath_i, \imath_j)}_{\text{Defined in Step 1.}})$$

$$\parallel$$

$$p(\sigma_\#(\imath_i), \tau_\#(\imath_j))$$

Easy to see that $p(\sigma,\tau)$ defined as above satisfies (2).

It remains to show that $p(\sigma,\tau)$ satisfies $*$

$$\partial p(\sigma,\tau) = \partial(\langle\sigma,\tau\rangle_\#(p(z_i,z_j)))$$

$$= \langle\sigma,\tau\rangle_\#(\partial p(z_i,z_j))$$; $\langle\sigma,\tau\rangle_\#$ is a chain map

$$= \langle\sigma,\tau\rangle_\#(p(\partial z_i,z_j) + (-1)^i p(z_i,\partial z_j))$$; construction

lin. of $\langle\sigma,\tau\rangle_\#$ & nat. of p ↪

$$= p(\underbrace{\sigma_\#(\partial z_i)}_{\partial \underbrace{\sigma_\# z_i}_{\sigma}}, \tau_\#(z_j)) + (-1)^i p(\sigma_\#(z_i), \underbrace{\tau_\#(\partial z_j)}_{\partial \underbrace{\tau_\# z_j}_{\tau}})$$

$$= p(\partial\sigma,\tau) + (-1)^i p(\sigma,\partial\tau)$$

← Unique to homotopy?

The proof of uniqueness statement is similar. ▨

Def: $P: C_i(X)\otimes C_j(Y) \to C_{i+j}(X\times Y)$ is called the homology cross product. This is called (denoted) by $\times$.

Rem: In the above proof, we used complexes $C_r(\Delta^i\times\Delta^j)$ are acyclic in the sense that

$$H_n(\Delta^i\times\Delta^j)=0 \quad \text{for } n\neq 0.$$

More precisely, we only used ← why we needed $(i,j) = (1,0),(0,1)$ case

$$H_{i+j-1}(\Delta^i\times\Delta^j)=0 \quad \text{for } i+j-1\neq 0$$

Rem: In step 2, we used every tensor product $\sigma\otimes\tau \in C_i(X)\otimes C_j(Y)$ can be written as

$$\sigma\otimes\tau = \sigma_\#(z_i)\otimes\tau_\#(z_j)$$

In this sense, pairs z_i, z_j are what we call "models". It turns out these are a form of a more general method: the method of acyclic models.

This is related to the Fundamental Lemma of Homological Algebra.

Acyclic Models:

Notations:

- $Ch_{\geq 0}(Ab)$ category with objects chain complexes in Ab which zero in neg. degree, morph are chain maps. Assume over Ab gps here.
- If C category & $F: C \to Ch_{\geq 0}$ is a functor then F_n will denote functor $F_n: C \to Ab$ given by $F_n(X)$ = n^{th} chain group of $F(X)$ $\forall X\in$ obj C.

Def: A category with models is a category C together with 'set' of objects $\mathcal{M} \subseteq$ obj C. Elements of $\mathcal{M}$ are called "models."

Def: A functor $F: C \to Ch_{\geq 0}$ is called acyclic for $\mathcal{M}$ if

$$H_n(F(M))=0 \quad \text{for } n>0$$

and $M\in\mathcal{M}$.

Def: A functor $F: C\to Ch_{\geq 0}$ is called free for $\mathcal{M}$ if $\forall n\geq 0$ there is

1) Indexed family $\{M_\alpha\}_{\alpha\in I_n}$ of objects $M_\alpha\in\mathcal{M}$
2) For each $\alpha\in I_n$, an element $z_\alpha\in F_n(M_\alpha)$

$\ni$ for every $X\in$ obj C, $F_n(X)$ freely gen as ab grp by set $\left\{F_n(f)(z_\alpha) \;\middle|\; \begin{matrix} f\in Hom(M_\alpha,X) \\ \alpha\in I_n\end{matrix}\right\}$

Ex: $\mathcal{C}$ = Top

F = Functor $\underset{\uparrow \text{ top space}}{X} \mapsto \underset{\uparrow \text{ sing. chain cxc}}{C_*(X)}$

Let $\mathcal{M} = \{\Delta^0, \Delta^1, \Delta^2, \dots\}$
$\subseteq$ Ob Top

Let $I_n = \{n\}$
← only one index ∇!

$M_n = \Delta^n$

$z_n = \Delta^n \xrightarrow{id} \Delta^n$ viewed as element of $S_n(\underset{\Delta^n}{X}) \subseteq C_n(\underset{\Delta^n}{X}) = F_n(\underset{\Delta \in M^n}{X})$

Then this is free for $\mathcal{M}$ because
$C_n(X) = F_n(X)$ is freely gen by $\{\sigma_\#(z_n) \mid \sigma \in Hom_{Top}(\underset{M_n}{\Delta^n}, X)\}$

Note F is also acyclic for $\mathcal{M}$ ~~because~~ because

$$H_n(\Delta^i) = 0 \text{ for } n > 0 \text{ \& } \forall i$$
$$\|$$
$$H_n(F(\Delta^i))$$
$$\|$$
$$C_n(\Delta^i)$$

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$\mathcal{C}$ category

$\mathcal{M} \subseteq$ obj $\mathcal{C}$ set of models

$F: \mathcal{C} \to \underbrace{Ch_{\geq 0}}_{\text{positive chain cxc.}}$

Def: $F: \mathcal{C} \to Ch_{\geq 0}$ is acyclic for $\mathcal{M}$ if $H_n(F(M)) = 0$ for $n \neq 0$ and $\forall M \in \mathcal{M}$

Def: $F: \mathcal{C} \to Ch_{\geq 0}$ is free for $\mathcal{M}$ if $\forall n \geq 0$, we have

- $\{M_\alpha\}_{\alpha \in I_n}$ Family of models $M_\alpha \in \mathcal{M}$
- for each $\alpha \in I_n$, an element $z_\alpha \in F_n(M_\alpha)$

such that $\forall X \in$ Obj $\mathcal{C}$, $F_n(X)$ is a free abelian group freely generated by

$\{F_n(f)(z_\alpha) \mid f \in Hom_{\mathcal{C}}(M_\alpha, X), \alpha \in I_n\}$

Ex: $\mathcal{C} = Top^2 \overset{def}{=}$ category of pairs of pairs of top spaces and pairs of cont maps

$(f,g) \circ (f',g') = (ff', gg')$

Let $F: Top^2 \to Ch_{\geq 0}$ be the functor

$(X,Y) \mapsto C_*(X) \otimes C_*(Y)$

$(f,g) \mapsto f_\# \otimes g_\#$

Let $\mathcal{M} = \{(\Delta^i, \Delta^j) \mid i,j \geq 0\}$

$I_n = \{(i,j) \mid i+j = n; i,j \geq 0\}$

$M_{(i,j)} = (\Delta^i, \Delta^j)$

$z_{(i,j)} = z_i \otimes z_j$ as in ex.

$\left.\begin{array}{l} z_i : \Delta^i \xrightarrow{id} \Delta^i \\ z_j : \Delta^j \xrightarrow{id} \Delta^j \end{array}\right\}$ viewed as simplices in Δ^i & Δ^j.

Then $C_*(X) \otimes C_*(Y)$ is freely generated by

$\{\underbrace{(\sigma_\# \otimes \tau_\#)(\underbrace{z_i \otimes z_j}_{z_{(i,j)}})}_{\sigma \otimes \tau} \mid (\sigma, \tau) \in Hom_{Top^2}((\underbrace{\Delta^i, \Delta^j}_{\substack{i+j=n \\ i,j \geq 0}}), (X,Y))\}$

is free for $\mathcal{M}'$. F acyclic for $\mathcal{M}'$ as $H_n(C_*(\Delta^i) \otimes C_*(\Delta^j)) = 0$ for $n \neq 0$, $\forall i,j$

Ex: $\mathcal{C} = Top^2$

Let $G : Top^2 \to Ch_{\geq 0}$ be the functor

$(X,Y) \mapsto C_*(X \times Y)$

$(f,g) \mapsto \langle f,g \rangle_\#$

$f \times g \overset{def}{=} \langle f,g \rangle$ the product map

Note: G is acyclic for $\mathcal{M}'$ because $H_n(\Delta^i \times \Delta^j) = 0$ for $n \neq 0, \forall i,j$

$\mathcal{M}' = \{ (\Delta^i, \Delta^j) \mid i,j \geq 0 \}$

$\mathcal{I}_n = \{n\}$, $M_n = (\Delta^n, \Delta^n) \in \mathcal{M}'$

z_n = diagonal embedding $D_n : \Delta^n \to \Delta^n \times \Delta^n$, $x \mapsto (x,x)$

viewed as a singular n-simplex in $\Delta^n \times \Delta^n$.

Then $C_n(X \times Y)$ is freely generated by

$\{ \langle \sigma_1, \sigma_2 \rangle_\# (z_n) \mid (\sigma_1, \sigma_2) \in Hom_{Top^2}((\Delta^n,\Delta^n),(X,Y)) \}$

where $\sigma_1 = p_X \circ \sigma$, $\sigma_2 = p \circ \sigma$ as $\sigma \in S_n(X \times Y)$ can write σ as $\sigma = \langle \sigma_1, \sigma_2 \rangle_\# (D_n)$ as above.

Therefore, G is free for $\mathcal{M}'$.

Thm (Acyclic Model Thm) Let $(\mathcal{C}, \mathcal{M})$ be a category of models. Let $F : \mathcal{C} \to Ch_{\geq 0}$ be free for $\mathcal{M}$ and let $G : \mathcal{C} \to Ch_{\geq 0}$ be acyclic for $\mathcal{M}$. Let $\Phi : H_0 F \to H_0 G$ be a natural transformation. Then

1) $\exists$ a natural trans $\tau : F \to G \ni$ τ induces Φ on H_0

2) τ unique up to natural homotopy

Pf: Will come later. ▨

Applications

We can prove the existence of the homology cross product:

$C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$

Let $\mathcal{C} = Top^2$

$\mathcal{M} = \{ (\Delta^i, \Delta^j) \mid i,j \geq 0 \}$

$F : (X,Y) \mapsto C_*(X) \otimes C_*(Y)$

$G : (X,Y) \mapsto C_*(X \times Y)$

We saw F, G are free and acyclic for $\mathcal{M}$. One can check the canonical map $\sigma_x \otimes \sigma_y \mapsto \sigma_{(x \times y)}$ induces a natural transformation $\Phi : H_0 F \to H_0 G$. By thm, Φ extends to natural transformation between F & G, say τ. Then define $\times \overset{def}{=} \tau$. ▨

Thm (Eilenberg-Zilber) The Alexander-Whitney map

$\Theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$

and the homology cross product

$\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$

are homotopy inverses of each other.

Pf: $\Theta \circ \times$ is a natural transformation $F \to F$ induced by identity on H_0 $\to$ $\Theta \circ \times \simeq 1_F$ by thm. Likewise, $\times \circ \Theta \simeq 1_G$. ▨

Cor: $C_*(X \times Y)$ and $C_*(X) \otimes C_*(Y)$ are homotopy equivalent

We also get a new proof that $H_*(X)$ is homotopy ~~equivalent~~ invariant

Lem: $X \subseteq \mathbb{R}$ convex then $H_n(X) = 0$ for $n \neq 0$

Pf: $X \neq \emptyset$ for then trivial. Choose $x_0 \in X$. Define maps

$$h : C_*(X) \to C_{*+1}(X)$$

$$p : C_*(X) \to C_*(X)$$

via

$$\underset{\substack{\uparrow \\ n\text{-simplex}}}{h(\sigma)}(t_0, \dots, t_{n+1}) = \underbrace{(1-t_0)x_0 + t_0\sigma\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right)}_{\in X}$$

$$p(\sigma) = \begin{cases} \sigma_{x_0}, & \sigma \text{ 0-simplex} \\ 0, & \text{otherwise} \end{cases}$$

The reader can check:

1) p is a chain map
2) $\operatorname{im} p = C_*(\{x_0\}) \subseteq C_*(X)$
3) $p|_{C_*(x_0)} = 1$
4) $1 - p = \partial h + h\partial$

So $C_*(X)$ def retracts to $C_*(\{x_0\})$, which has 0 homology groups. ▨

Let $f, g : X \to Y$ be cont. maps and let

$$H : X \times I \to Y$$

be a homotopy between f & g. Then $f = H \circ i_0$, $g = H \circ i_1$; i_j inclusions

To show $f_\# = g_\#$, it is enough to show $(i_0)_\# \simeq (i_1)_\#$

Let $C = \text{Top}$

F functor $X \mapsto C_*(X)$

G functor $X \mapsto C_*(X \times I)$

$\mathcal{M} = \{\Delta^0, \Delta^1, \dots\}$

Have seen for $\mathcal{M}$, F free & acyclic for $\mathcal{M}$. G is acyclic for $\mathcal{M}$ by the lemma. Moreover, $(i_0)_\#$ & $(i_1)_\#$ can be viewed as natural transformations $F \to G$

On H_0, $(i_0)_\#$, $(i_1)_\#$ both induce the canonical isomorphism

$$H_0(X) \cong \mathbb{Z}\langle (\text{path comp. of } X) \rangle$$

$$\xrightarrow{\text{interval path connected}} \cong \mathbb{Z}\langle \text{path comp of } X \times I \rangle$$

$$\cong H_0(X \times I)$$

Then $(i_0)_\# \simeq (i_1)_\#$

Thm (Acyclic Models Thm)

Pf: Since F free for $\mathcal{M}$, $\exists$ $\{M_\alpha\}_{\alpha \in I_n}$ family of models $M_\alpha \in \mathcal{M}$ for $\alpha \in I_n$, $z_\alpha \in F_n(M_\alpha)$ $\ni$ $X \in \operatorname{Ob} C$ $F_n(X)$ freely gen by

$$\{F_n(f)(z_\alpha) \mid f \in \operatorname{Hom}_C(M_\alpha, X), \alpha \in I_n\}$$

Goal: Show that Φ extends to a $\tau : F \to G$

Explicitly, we have to show $\exists$ natural trans $\tau_n : F_n \to G_n$ $\ni$ the following commutes

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d, "\tau_2"] & F_1 \arrow[r] \arrow[d, "\tau_1"] & F_0 \arrow[r] \arrow[d, "\tau_0"] & H_0 F \arrow[r] \arrow[d, "\Phi"] & 0 \\
\cdots \arrow[r] & G_2 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & H_0 G \arrow[r] & 0
\end{tikzcd}$$

We proceed by induction on n:

$n=-1$: Define $\tau_{-1} = \phi$

$n \geq -1$: Proceed in two steps

1) Define $\tau_n(z_\alpha) \in G_n(M_\alpha)$ for each $\alpha \in I_n$

2) Define $\tau_n : F_n(X) \to G_n(X)$ for all obj X.

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1) Consider the diagram

$$\begin{tikzcd}
z_\alpha \in F_n(M_\alpha) \arrow[r, "\partial"] \arrow[d, dashed, "\tau_n"] \arrow[drr, bend right] & F_{n-1}(M_\alpha) \arrow[r, "\partial"] \arrow[d, "\tau_{n-1}"] \arrow[r, shift right] & F_{n-2}(M_\alpha) \arrow[d, "\tau_{n-2}"] \\
G_n(M_\alpha) \arrow[r, "\partial"'] & G_{n-1}(M_\alpha) \arrow[r, "\partial"'] & G_{n-2}(M_\alpha) \\
b \arrow[r, mapsto] & z \arrow[r, mapsto] & 0
\end{tikzcd}$$

Define $z = \tau_{n-1}(\partial z_\alpha)$ Then

$\partial z = \partial\tau_{n-1}\partial z = \tau_{n-2}\partial^2 z_\alpha = 0$

Since G acyclic for $\mathcal{M}$, bottom row exact, $\exists$ $b \in G_n(M_\alpha) \ni \partial b = z$

Define $\tau_n(z_\alpha) = b$. Then $\partial\tau_n(z_\alpha) = \partial b = z = \tau_{n-1}\partial z_\alpha$.

2) Let $X \in \text{Ob}\, C$, then $F_n(X)$ is freely generated by $F_n(f)(z_\alpha)$ for $f \in \text{Hom}(M_\alpha, X)$, $\alpha \in I_n$.

Define $\tau_n(F_n(f)(z_\alpha)) \overset{\text{def}}{=} G_n(f)(\underbrace{\tau_n(z_\alpha)}_{\text{via step 1.}})$

Extend τ_n to $F_n(X)$ by linearity.

We have to verify the desired qualities.

Naturality is simple to see.

Remains to show $\partial_G \tau_n = \tau_{n-1}\partial_F$

$\underline{\partial\tau_n}(F_n(f)(z_\alpha)) = \partial G_n(f)(\tau_n(z_\alpha))$

G(F) Chain map $\longrightarrow$ $= G_{n-1}(f)\,\partial(\tau_n(z_\alpha)$

(1) $\longrightarrow$ $= G_{n-1}(f)(\tau_{n-1}(\partial z_\alpha))$

Naturality of τ_{n-1} $\longrightarrow$ $= \tau_{n-1}(F_{n-1}(f)(\tau_{n-1}(\partial z_\alpha))$

F(F) chain map $= \underline{\tau_{n-1}(}\partial(F_n(f)(z_\alpha))$

Then $\partial\tau_n = \tau_{n-1}\partial$

Proof of (2) is similar. ▨

Recall, $F : C \to Ch_{\geq 0}$ is free for $\mathcal{M}$.

It turns out the thm holds under the weaker assumption that for each n F_n is (naturally) is a direct summand of a $\underset{\sim}{\text{free}}$ functor $\tilde{F}_n$.

for $\mathcal{M}$.

$\exists$ natural trans

$i : F_n \to \tilde{F}_n$

$p : \tilde{F}_n \to F_n$

such that

$pi = 1_{F_n}$

so $\tilde{F}_n = \text{im}\, i \oplus \text{Ker}\, p$

Pf: Modify original proof as follows

In the inductive step, replace

$$F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} F_{n-2}$$

with

$$\tilde{F}_n \xrightarrow[\partial p]{} F_{n-1} \xrightarrow[\partial]{} F_{n-2}$$

Since $\tilde{F}_n$ is free, the org. argument yields a $\tilde{\tau}_n : \tilde{F}_n \to G$ to get $\tau_n : F_n \to G_n$, compare τ_n with i, ie $\tau_n = \tilde{\tau}_n \circ i$. ▨

Connection with the Fund. Lemma of Homological Algebra:

R ring

P complex of R-modules (projective)

Q complex of R-modules such that $H_n(Q) = 0$ for $n \neq 0$

& $P_n, Q_n = 0$ for $n < 0$.

Lem: $f : H_0(P) \to H_0(Q)$ be a map of R-modules. Then $\exists$ a chain map $\tau : P \to Q$ which induces f on H_0. This τ is unique ~~&~~ up to homotopy.

Pf: We use acyclic model thm.

Define C to be category with single object M and $\mathrm{End}_C\, \underbrace{M}_{\text{mult by } R} = R$

Define $F : C \to Ch_{\geq 0}$, $G : C \to Ch_{\geq 0}$

$$M \mapsto P \qquad M \mapsto Q$$
$$r \in R \mapsto r|_P \qquad r \mapsto r|_Q$$

Let $\mathcal{M} = \{M\}$. Then since P_n is projective, F_n is a direct summand of a free functor $\tilde{F}_n$. $H_n(Q) = 0$ for $n \neq 0$, G acyclic for $\mathcal{M}$.

Moreover, given map $f : H_0(P) \to H_0(Q)$ can be viewed as trans $\Phi : H_0 F \to H_0 G$

The lemma follows from acyclic models thm by noting that f R-module map iff Φ is natural. Similar for τ & h ◻

The reverse implication holds in a sense

C category, $\mathcal{M} \in \mathrm{obj}\, C$ set of models

$F : C \to Ch_{\geq 0}$ free for $\mathcal{M}$

$G : C \to Ch_{\geq 0}$ acyclic for $\mathcal{M}$

$\Phi : H_0 F \to H_0 G$ natural. Define

$$R = \bigoplus_{M, N \in \mathcal{M}} \mathrm{Span}_{\mathbb{Z}}(\mathrm{Hom}_C(M, N))$$

$$P = \bigoplus_{M \in \mathcal{M}} F(M)$$

$$Q = \bigoplus_{M \in \mathcal{M}} G(M)$$

[Use models as a set.]

Then R is an associative "idempotented" $\mathbb{Z}$-algebra.

The product is composition of morphisms if composable & 0 otherwise.

Lack ring because of lack of id.

P, Q are complexes of R-modules, action of $r \in \mathrm{Hom}_C(M, N) \subseteq R$ on P_n & Q_n is given by $F_n(r)$ & $G_n(r)$. Reader can check since F free for $\mathcal{M}$

$$P \cong \bigoplus_{\alpha \in I_n} \underbrace{\bigoplus_{M \in \mathcal{M}} \mathrm{Span}_{\mathbb{Z}}(\mathrm{Hom}_C(M_\alpha, M))}_{\substack{\text{direct summand of } R \\ \text{viewed as a left-module over itself}}}$$

So P is a projective R-module.

Moreover, $H_n Q = 0$ for all n as G acyclic for $\mathcal{M}$.

Therefore, Fund. Lem. of Hom. Alg. implies that transformation $\phi : H_0F \to H_0G$ induces a chain map $t : P \to Q$

One can check that t induces a natural transformation

$$\tau_D : F|_D \to G|_D$$

where D is full subcategory of C with obj $D = \mathcal{M}$. To extend τ_D to a $\tau : F \to G$, one can argue as in (2) of proof of acyclic models thm.

Künneth Thm

C, C' complexes of free modules over a PID R. $\rightsquigarrow$ Only need one free, say C

<u>Thm</u> (Alg. version) : $\exists$ natural ~~transformation~~ short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(C') \to H_n(C \otimes C') \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i C, H_j C') \to 0$$

$$[z] \otimes [z'] \mapsto [z \otimes z']$$

<u>Pf</u>: Regard cycles & boundaries as subcomplexes with trivial differentials. These subcomplexes fit into s.e.s.

$$0 \to Z_* \hookrightarrow C_* \xrightarrow{\partial} B_{*-1} \to 0$$

This short exact sequence splits as B_{*-1} free. Tensoring with C' yields a ses

$$0 \to Z_* \otimes C'_* \to C_* \otimes C' \to B_{*-1} \otimes C'_* \to 0$$

The induced les in homology looks as follows:

$$\cdots \xrightarrow{c_n} \bigoplus_{i+j=n} Z_i \otimes H_j(C') \to H_n(C_* \otimes C'_*) \to \bigoplus_{i+j=n-1} B_i \otimes H_j(C') \xrightarrow{c_{n-1}} \cdots$$

Using $0 \to B_n \hookrightarrow Z_n \to H_n(C) \to 0$ is a free resolution of $H_n(C)$, one can see that

$$\mathrm{coker}\, c_n = \bigoplus_{i+j=n} \underbrace{\mathrm{Tor}_0^R(H_i(C), H_j(C'))}_{H_i(C) \otimes H_j(C')}$$

$$\mathrm{Ker}\, c_{n-1} = \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C), H_j(C'))$$

▨

<u>Note</u>: UCT for homology arises as a special case of Künneth Thm.

<u>10/18/2016</u>

X, Y top spaces

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \arrow[r] & H_n(C_*(X) \otimes C_*(Y)) \arrow[r] \arrow[d, shift right, "\times"'] & \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \\
& & H_n(X \times Y) \arrow[u, shift right, "\theta"'] &
\end{tikzcd}$$

Recall on chain level, $\times, \theta$ are homotopy inverses

Cor : $\exists$ natural ses (Hatcher pp 275 Thm 3B 6)

$$0 \to \bigoplus H_i(X) \otimes H_j(Y) \xrightarrow{\times} H_n(X \times Y) \to \bigoplus \mathrm{Tor}(H_i(X), H_j(Y)) \to$$

Cor: If $H_i(X)$, $H_j(Y)$ flat then $\overset{\to \text{inv. } \theta}{\times}$ is an isomorphism. In particular, when either are free.

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Cor: (Künneth Thm for Cohomology)

If X, Y are spaces such that $H_i(X)$, $H_j(Y)$ are fin. gen. and free, then

$$H^n(X\times Y) \cong \bigoplus_{i+j=n} H^i(X)\otimes H^j(Y)$$

$\times$ ← cohom. cross product, eg dual Alex. Whit. map $\theta^{\#}$

Pf: By Cor, $H_n(X\times Y) \cong \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y)$ $*$

So $H_n(X\times Y)$ is f.g. and free. Then

$$H^n(X\times Y) \cong \mathrm{Hom}(H_n(X\times Y,\mathbb{Z})) \text{ by UCT}$$

$$\overset{\theta^{\#}}{\cong} \mathrm{Hom}\Big(\bigoplus_{i+j=n} H_i(X)\otimes H_j(Y),\mathbb{Z}\Big) \text{ by } *$$

$$\cong \bigoplus_{i+j=n} \mathrm{Hom}(H_i(X)\otimes H_j(Y),\mathbb{Z})$$

$$\cong \bigoplus_{i+j=n} \mathrm{Hom}(H_i(X),\mathbb{Z})\otimes \mathrm{Hom}(H_j(Y),\mathbb{Z}) \quad \text{free, f.g.}$$

$$\cong \bigoplus_{i+j=n} H^i(X)\otimes H^j(Y) \quad \text{UCT}$$

To see $\times$ is the iso, just trace through the maps. ☒

So this describes $H^n(X\times Y)$ as an abelian group, but what about as a ring?

Thm: If $H_i(X)$, $H_j(Y)$ free & f.g. then

$$H^*(X\times Y) \cong H^*(X) \,\hat{\otimes}\, H^*(Y)$$

cohom. cross product → $\times$

$\hat{\otimes}$ ← graded comm. tensor product

Pf: By Cor, know $\times$ is iso of graded abelian groups.

We know if $[\Phi], [\Phi'] \in H^*(X)$ and $[\Psi], [\Psi'] \in H^*(Y)$ then

$$([\Phi]\times[\Psi]) \smile ([\Phi']\times[\Psi'])$$

$$= (-1)^{ji'} ([\Phi]\smile[\Phi']) \times ([\Psi]\smile[\Psi'])$$

sign in def of $\hat{\otimes}$ ↗ $(-1)^{ji'}$; in $H^*(X)$ ↗ first $\smile$; in $H^*(Y)$ ↑ second $\smile$

where $j = \deg[\Psi]$ & $i' = \deg[\Phi']$

So $\times$ is an iso. of graded rings. ▨

Application

Compute $H^*(T^n)$:

$$H^*(T^n) = \underbrace{H^*(S^1)\,\hat{\otimes}\, H^*(S^1)\,\hat{\otimes}\cdots\hat{\otimes}\, H^*(S^1)}_{n \text{ factors}}$$

Claim $\hat{\bigotimes} H^*(S^1) \cong \Lambda \mathbb{Z}^n$

$$\cong \frac{\mathbb{Z}\langle \alpha_1,\dots,\alpha_n\rangle}{\left\langle \begin{matrix}\alpha_i\alpha_j = -\alpha_j\alpha_i \\ \alpha_i^2 = 0\end{matrix}\right\rangle}$$

$\mathbb{Z}^n$ in deg 1, $\deg\alpha_i = 1$

Pf (sketch): $H^*(S^1) = \mathbb{Z}[\alpha]/(\alpha^2)$

α gen. of $H^1(S^1)$.

$$= \underbrace{\mathbb{Z}\cdot 1}_{H^0} \oplus \underbrace{\mathbb{Z}\alpha}_{H^1}$$

$$= \Lambda^0(\mathbb{Z}) \oplus \Lambda^1\mathbb{Z}$$

$$= \Lambda\mathbb{Z}$$

For $n>1$, simply use induction.

$$\hat{\bigotimes} H^*(S^1) = \underbrace{H^*(S^1)\,\hat{\otimes}\cdots\hat{\otimes}\, H^*(S^1)}_{n-1}\,\hat{\otimes}\, H^*(S^1)$$

$$= \Lambda\mathbb{Z}^{n-1} \otimes \Lambda\mathbb{Z}$$

HW1 ⟶ $= \Lambda(\mathbb{Z}^{n-1}\oplus\mathbb{Z})$

$$= \Lambda\mathbb{Z}^n$$

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Similarly, $H^*(\underbrace{S^2 \times \cdots \times S^2}_{n}) \cong \dfrac{\mathbb{Z}[x_1, \dots, x_n]}{\langle x_i^2 = 0 \rangle}$

$\deg x_i = 2$ $\qquad \cong \dfrac{\mathrm{Sym}\, \mathbb{Z}^n}{e_i^2 = 0}$

i^{th} basis vector

Cap Product

Hatcher pp. 239

Let $m \geq i$. Define a linear map,

denoted $\frown : C_m(X) \otimes C^i(X) \to C_{m-i}(X)$

$$c \otimes \phi \to c \frown \phi$$

defined as follows:

$$\sigma \frown \phi \overset{def}{=} \phi(\sigma|_{[e_0, \dots, e_i]})\, \sigma|_{[e_i, \dots, e_m]}$$

for every $\sigma \in S_m(X)$. Note if $m = i$

then $\sigma \frown \phi = \phi(\sigma)\, \sigma|_{[e_m]}$

$$\overset{\shortparallel}{\langle \phi, \sigma \rangle} \left.\begin{matrix}\text{Kronecker pairing}\\ \text{evaluation pairing}\end{matrix}\right\}$$

Thm: $\frown$ is natural: if $f: X \to Y$ cont

then for all $\sigma \in S_m(X)$, $\phi \in C^i(Y)$

$$(f_\# \sigma) \frown \phi = f_\#(\sigma \frown f^\# \phi)$$

Pf: Both sides compute to

$$\phi((f_\# \sigma)|_{[e_0, \dots, e_i]})\ (f_\# \sigma)|_{[e_i, \dots, e_m]} \quad ☒$$

$$\begin{tikzcd}
& c \otimes f^\# \phi \arrow[rrrdd, bend left] & & & \\
C_m(X) \arrow[d, "f_\#"'] & \otimes & C^i(X) \arrow[r, "\frown"] & C_{m-i}(X) \arrow[d, "F_\#"] & \\
C_m(Y) & \otimes & C^i(Y) \arrow[u, "f^\#"'] \arrow[r, "\frown"] & C_{m-i}(Y) & = \\
& f_\# c \otimes \phi \arrow[rrru, bend right] & & &
\end{tikzcd}$$

Thm:

$$\partial(\sigma \frown \phi) = (-1)^i [\partial\sigma \frown \phi - \sigma \frown \delta\phi]$$

for $\sigma \in S_m(X)$, $\phi \in C^i(X)$

Pf: Computation

$$\partial(\sigma) \frown \phi = \sum_{k=0}^{m} (-1)^k \sigma|_{\underbrace{[e_0, \dots, e_i]}_{e_k \text{ missing}}} \frown \phi$$

$$= \sum_{k=0}^{i} (-1)^k \phi(\sigma|_{e_0, \hat{e}_k, e_{i+1}})\, \sigma|_{[e_{i+1}, \dots, e_m]}$$

$$+ \sum_{k=i+1}^{m} \underbrace{(-1)^k}_{(-1)^r (-1)^{k-i}} \phi(\sigma|_{e_0, \dots, e_i})\, \sigma|_{[e_i, \dots, \hat{e}_k, e_m]}$$

$$= \cdots = (\sigma \frown \delta\phi) + (-1)^i \partial(\sigma \frown \phi)$$

- extra $\overset{\text{two}}{\wedge}$ terms which cancel

Move things over

$$(\partial\sigma) \frown \phi - (\sigma \frown \delta\phi) = (-1)^i \partial(\sigma \frown \phi) \quad ☒$$

Cor:
$Z_m \frown Z^i \subseteq Z_{m-i}$
$Z_m \frown B^i \subseteq B_{m-i}$
$B_m \frown Z^i \subseteq B_{m-i}$

Cor: $\frown$ induces a well defined map

$$\frown : H_m(X) \otimes H^i(X) \to H_{m-i}(X)$$
$$[c] \otimes [\phi] \mapsto [c \frown \phi]$$
$$\overset{def}{=} [c] \frown [\phi]$$

Note: If $m = i$ and X path connected,

then $[c] \frown [\phi] = \underbrace{\phi(c)}_{\langle \phi, c \rangle} \cdot$ gen of H_0

follows as $\underset{\in S_m(X)}{\sigma} \frown \underset{\in C^m(X)}{\phi} = \phi(\sigma)\, \sigma|_{[e_m]}$

Relation with $\smile$:

See Hatcher pp 249

Thm: If $\phi \in C^i(X)$, $\psi \in C^j(X)$, $\sigma \in C_m(X)$, $m = i+j$ then

$$\langle \psi, \underbrace{c \frown \phi \rangle}_{\psi(c \frown \phi)} = \langle \underbrace{\phi \smile \psi, c}_{(\phi \smile \psi)(c)} \rangle$$

Pf: Computation. $\boxtimes$

Cor: For a fixed $\phi \in C^i(X)$, the map $\phi \smile -$ is dual to the map $- \frown \phi$.

Diagrammatically:

$$\begin{tikzcd}
C^{m-i}(X) \arrow[r, "\phi \smile -"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & C^m(X) \arrow[d, equal] \\
\mathrm{Hom}(C_{m-i}(X), \mathbb{Z}) \arrow[r, "(- \frown \phi)^*"'] & \mathrm{Hom}(C_m(X), \mathbb{Z})
\end{tikzcd}$$

Thm: If $\phi \in C^i(X)$, $\psi \in C^j(X)$, $c \in C_m(X)$, $m \geq i+j$, then...

$$(c \frown \phi) \frown \psi = c \frown (\phi \smile \psi)$$

Pf: Computation. $\boxtimes$

Cor: $H_*(X) = \bigoplus_{i \geq 0} H_i(X)$ a right module over $H^*(X)$ with mult. given by $- \frown [\phi]$

Warning: If $\phi \in C^i(X)$, then $- \frown \phi$ lower degree by i.

Remark: If M closed oriented n-manifold, then $H_*(M)$ is a free module $\overset{\text{rank 1}}{\wedge}$ over $H^*(M)$ generated by gen of $H_n(M)$. This will follow from Poincaré duality.

Relative Cap Product

If $A, B \subseteq X$ open, $\exists$ a cap product

$$H_m(X, A \cup B) \otimes H^i(X, A) \to H_{m-i}(X, B)$$

$$[\bar{c}] \otimes [\phi] \mapsto [c \frown \phi]$$

Here c is an m-chain $c \in C_m(X)$, where class $\bar{c}$ in $C_*(X, A \cup B) = C_*(X)/C_*(A \cup B)$ is a cycle $C_*(X, A \cup B)$

Cubical Singular Homology

Idea: Use cubes instead of simplices.

$\square^n$ = standard n-cube $= I^n \subseteq \mathbb{R}^n$

For $1 \leq i \leq n$, define maps $a_i^n, b_i^n : \square^{n-1} \to \square^n$

$\uparrow$ front face $\quad \uparrow$ back face

by

$$a_i^n(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, 0, \ldots, t_{n-1})$$

$$b_i^n(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, 1, \ldots, t_{n-1})$$

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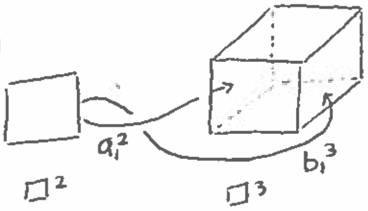
Eg



Let X be a top. space.

Def: A singular n-cube in X is a cont. map $\tau: \square^n \to X$

$S_n^\square(X) = \{\text{sing. } n\text{-cubes in } X\}$

$Q_n(X) = \text{Span}_{\mathbb{Z}} S_n^\square(X)$

For $\tau \in S_n^\square(X)$, let

$A_i\tau = \tau \circ a_i^n \in S_{n-1}^\square(X)$

$B_i\tau = \tau \circ b_i^n \in S_{n-1}^\square(X)$

Define $\partial: Q_n(X) \to Q_{n-1}(X)$ by

$$\partial\tau = \sum_{i=1}^{n} (-1)^i (A_i\tau - B_i\tau)$$

One can check: $\partial^2 = 0$ → $(Q_*(X), \partial)$ is a chain complex

Ex: X = point. → ∃! sing. n-cube τ_n for all $n \geq 0$ - namely the constant n cube at the unique point of X.

Then $Q_*(X)$ looks as follows:

$$\cdots \to \underset{3}{\mathbb{Z}} \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{0} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \xrightarrow{0} 0$$

Moreover, $\partial\tau_n = \sum_{i=1}^{n} (-1)^i (\underbrace{A_i\tau_n}_{\tau_{n-1}} - \underbrace{B_i\tau_n}_{\tau_{n-1}}) = 0$

So $H_n(X) = \mathbb{Z}$ for $n \geq 0$

In particular, $H_n(X) \neq \underbrace{H_n(\text{point})}_{\text{sing. homology}}$

To get singular homology, we have to 'normalize' $Q_*(X)$.

Def: A singular n-cube $\tau: \square^n \to X$ is called degenerate if $\exists i$ such that $\tau(t_1, \dots, t_n)$ does not depend on t_i (constant rgt. to i^{th} coordinate)

$S_{n,d}^\square(X) = \{\text{degenerate } n\text{-cubes in } X\}$

$D_n(X) = \text{Span}_{\mathbb{Z}} S_{n,d}^\square(X) \subseteq Q_n(X) = \text{Span}_{\mathbb{Z}} S_n^\square(X)$

Note: If τ does not depend on t_i, then $A_i\tau = B_i\tau$ so $A_i\tau - B_i\tau = 0$.

$A_j\tau$, $B_j\tau$ are degenerate for $j \neq i$.

Recall $\partial\tau_n = \sum_{i=1}^{n} (-1)^i (A_i\tau_n - B_i\tau_n)$, this is a sum of degenerate cubes so $\partial\tau \in D_{n-1}(X)$ so that $\partial D_n(X) \subseteq D_{n-1}(X)$

Then $D_*(X)$ is a subcomplex of $Q_*(X)$.

Define $C_*^\square(X) \overset{\text{def}}{=} Q_*(X)/D_*(X)$, called the cubical chain complex of X.

The homology of this complex is called the cubical singular homology.

Ex: X = point → ∃! singular n-cube τ_n for $n \geq 0$, namely the constant n-cube. If $n > 0$, τ_n is degenerate so $\equiv 0$ in $C_n^\square(X)$. Then H_n agrees with ord. sing. homology.

Induced Maps

Let $f: X \to Y$ be continuous.

Define $f_\# : C_n^\square(X) \to C_n^\square(Y)$

by $f_\#(\tau) = f \circ \tau$ for $\tau \in S_n^\square(X)$

Note: This is well defined because if τ is degenerate, then so is $f\tau$

Properties:

1) $(f \circ g)_\# = f_\# g_\#$

2) $(1_X)_\# = 1_{C_n^\square(X)}$

Claim: $\partial f_\# = f_\# \partial$

Pf: Let $\tau \in S_n^\square(X)$. Then

$$\partial(f_\#\tau) = \sum_{i=1}^{\wedge} (-1)^i \left(\underbrace{A_i f\tau}_{f\tau a_i^\wedge} - \underbrace{B_i f\tau}_{f\tau b_i^\wedge}\right)$$

$$= f_\#\left(\sum_{i=1}^{\wedge} (-1)^i \left(\underbrace{\tau a_i^\wedge}_{A_i\tau} - \underbrace{\tau b_i^\wedge}_{B_i\tau}\right)\right)$$

$$= f_\#(\partial\tau) \quad ☒$$

Cor: $f_\#$ induces a map

$f_*: H_n^\square(X) \to H_n^\square(Y)$

Cor: $H_n^\square$ is a functor Top → Ab

Thm: If $f, g: X \to Y$ are homotopic then so are $f_\#, g_\#$

Pf: Let $H: X \times I \to Y$ be a homotopy between f & g. For $\tau \in S_n^\square(X)$, define $\tilde\tau \in S_{n+1}^\square(Y)$ by

$$\tilde\tau(t_1, \dots, t_{n+1}) = H(\underset{\in X}{\tau(\vec{t})}, \underset{\in I}{t_{n+1}})$$

then $\tilde\tau = H \circ (\tau \circ 1_I)$

Then $\partial\tilde\tau = \sum_{i=1}^{n+1} (-1)^i (A_i\tilde\tau - B_i\tilde\tau)$

$$= \sum_{i=1}^{\wedge} (-1)^i \left(\underbrace{A_i\tilde\tau}_{\widetilde{A_i\tau}} - \underbrace{B_i\tilde\tau}_{\widetilde{B_i\tau}}\right)$$

$$\widetilde{\partial\tau} \nearrow \quad + (-1)^{n+1}\left(\underbrace{A_{n+1}\tilde\tau - B_{n+1}\tilde\tau}_{f_\#\tau - g_\#\tau}\right)$$

Then $\partial\tilde\tau = \widetilde{\partial\tau} + (-1)^{n+1}(f_\#\tau - g_\#\tau)$

so that....

$(-1)^{n+1}\partial\tilde\tau = (-1)^{n+1}\widetilde{\partial\tau} + f_\#\tau - g_\#\tau$

If we define

$h: C_n^\square(X) \to C_{n+1}^\square(X)$ via

$h(\tau) = (-1)^{n+1}\tilde\tau$ for $\tau \in S_n^\square(X)$ then

$\partial h(\tau) = -h(\partial\tau) + f_\#\tau - g_\#\tau$

Then $\partial h + h\partial = f_\# - g_\#$ so that $f_\#, g_\#$ are homotopic. ☒

*Note that h is well defined as if τ is degenerate, so too is $\tilde\tau$.

Cor: $H_n^\square$ is a functor Top/~ → Ab

~: same homotopy → obj: Top spaces

Morph: Cont maps up to hmpty

Cor: If X, Y are homotopy equiv. then $H_n^\square(X) \cong H_n^\square(Y)$ for all n.

Cor: If $\emptyset \neq X$ is contractible, then
$$H_n^\square(X) \cong H_n^\square(pt) \cong \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n \neq 0 \end{cases}$$

Goal: Show $H_n^\square(X) \cong H_n(X)$ for all X.

Note: $S_0^\square(X) = S_0(X) = \{\text{points in } X\}$.
Furthermore, $S_1^\square(X) = S_1(X) = \{\text{paths in } X\}$.
$\nexists$ degenerate 0-cubes, so $C_0^\square(X) = Q_0(X)$

Then we get a diagram.....

$$\begin{tikzcd}
C_0^\square(X) \arrow[r, equal] & Q_0(X) \arrow[r, equal] & C_0(X) & \Big\}\text{all spanned by points in } X \\
\frac{Q_1(X)}{D_1(X)} = C_1^\square(X) \arrow[u, "\partial"] \arrow[ur, phantom, "\circlearrowleft"] & Q_1(X) \arrow[l, two heads] \arrow[u, "\partial"'] \arrow[r, equal] \arrow[ur, phantom, "\circlearrowleft"] & C(X) \arrow[u, "\partial"'] & \Big\}\text{spanned by paths in } X
\end{tikzcd}$$

In all three theories, $\partial(\text{path})$ = difference of endpoints then the above diagram commutes.

Then there exists canonical isomorphisms (natural)
$$H_0^\square(X) = H_0(Q_*(X)) = H_0(X) \quad \text{for all } X.$$

For $n \geq 0$,
$$Q_n(X) = \text{Span}_{\mathbb{Z}} S_n^\square(X) = \overbrace{\text{Span}_{\mathbb{Z}}(S_n^\square(X) \setminus S_{n,d}^\square(X))}^{\cong C_n^\square(X)} \oplus \underbrace{\text{Span}_{\mathbb{Z}} S_{n,d}^\square(X)}_{D_n(X)}$$

So $C_n^\square(X)$ is isomorphic to a direct summand of $Q_n(X)$.

Moreover, $X \to Q_*(X)$ is free for
$$\mathcal{M} = \{\square^0, \square^1, \square^2, \dots\} \subseteq Ob(Top)$$

Then we could try to apply acyclic models thm to
$F: X \mapsto C_n^\square(X)$
$G: X \mapsto C_*(X)$ to get a $F \to G$

Problem: The above direct sum decomposition isn't natural as the induced map $f_\#$ can send a nondegenerate cube to a degenerate one.

The projection $p: Q_n(X) \to C_n^\square(X) = Q_n(X)/D_n(X)$ is natural. We only need a natural section of the map p, say i.

Def: Define $c_i: \square^n \to \square^n$ by
$$c_i^n(t_1, \dots, t_n) = (t_1, \dots, \underset{i^{th}\text{ spot}}{0}, \dots, t_n)$$

For $\tau \in S_n^\square(X)$, let
$$C_i\tau = \tau c_i^n \in S_{n,d}^\square(X)$$

Properties:

(a) C_i commute with each other

(b) $C_i\tau$ is degenerate as it does not depend on t_i.

(c) If τ does not depend on t_i, then $C_i\tau = \tau$

Define $i: C_n^\square(X) \to Q_n(X)$ by
$$i(\tau) = (1-c_1)(1-c_2)\cdots(1-c_n)(\tau)$$

Lemma:

(1) i is well defined

(2) i is natural

(3) $pi = 1$ for $p: Q_n(X) \to C_n^\square(X)$ the projection

Will get is $Q_n(X) \cong \underbrace{\text{im } i}_{C_n^\square(X)} \oplus \text{Ker } p$

Pf:

(1) Let $\tau \in S^\square_{n,d}(X)$. Then $\exists\, i \ni \tau$ doesn't depend on t_i. By (a), we can reorder the factors in i, so that $1-c_i$ comes last. Then

$$i(\tau) = (\text{other factors})\underbrace{(1-c_i)(\tau)}_{\tau-\tau} = 0$$

(2) Let $f: X \to Y$ be cont. Let $\tau \in S^\square_n(X)$.

$$\begin{aligned} f_\#(c_i\tau) &= f(\tau c_i) \\ &= (f\tau)c_i \\ &= c_i(f_\#\tau) \end{aligned}$$

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(3) Since $c_i\tau$ is degenerate, we have $p c_i \tau = 0$ so that $p c_i = 0$

but then
$$\begin{aligned} p\circ i &= p\circ(1-c_1)(1-c_2)\cdots(1-c_n) \\ &= p\circ 1 \\ &= 1_{C^\square_i(X)} \end{aligned}$$

We are now in a position to prove the thm with acyclic models thm.

Thm: The can. iso $H^\square_0(X) \cong H_0(X)$ extends to a natural homotopy equiv. $C^\square_*(X) \simeq C_*(X)$

Pf: Let $F: X \mapsto C^\square_*(X)$

$G: X \mapsto C_*(X)$

$\mathcal{M} = \{\square^0, \square^1, \square^2, \ldots\} \subseteq \text{Obj Top}$

$\mathcal{M}' = \{\Delta^0, \Delta^1, \Delta^2, \ldots\} \subseteq \text{Obj Top}$

Now....

- F_n is naturally a direct summand of the functor $X \mapsto Q_n(X)$ and $Q_n(X)$ is free for $\mathcal{M}$.
- G is free for $\mathcal{M}'$.
- F, G acyclic for $\mathcal{M}, \mathcal{M}'$. (these are contractible spaces)

By acyclic models thm, the iso. $H^\square_0(X) \cong H_0(X)$ and its inv. extend natural trans. here to $\tau_1: F \to G$ & $\tau_2: G \to F$

Moreover, $\tau_1\tau_2$ and $\tau_2\tau_1$ induce the identity on H_0. The thm gives $\tau_1\tau_2 \simeq 1_G$ & $\tau_2\tau_1 \simeq 1_F$

Cor: $H^\square_n(X)$ & $H_n(X)$ are naturally iso.

Homology Cross Product

$\times: C^\square_i(X) \otimes C^\square_j(Y) \to C^\square_{i+j}(X \times Y)$

$\overset{\sigma}{\in S^\square_i(X)}$ $\quad \overset{\tau}{\in S^\square_j(Y)} \longmapsto$ product map of σ, τ

$$\begin{tikzcd} \square^{i+j} = \square^i \times \square^j \arrow[d, "\sigma\times\tau"] \\ X \times Y \end{tikzcd}$$

$$(\sigma\times\tau)(t_1,\ldots,t_{i+j}) \overset{\text{def}}{=} \sigma(t_1,\ldots,t_i)\tau(t_{i+1},\ldots,t_{i+j})$$

Easy to see well defined & chain map.

Orientations

Recall an n-manifold (w/o boundary) is a 2nd countable Hausdorff space M such that each $x \in M$ has a neigh which is locally homeomorphic to $\mathbb{R}^n$.

Rem: 2nd countable is not always included.

Rem: $\mathcal{U}$ is called a Euclidean neigh. of x.

Any homeo. $\phi: \mathcal{U} \to \mathbb{R}^n$ is called a coord. chart on $\mathcal{U}$.

Closed Manifold: A compact manifold w/o boundary.

Suppose M is an n-manifold and $\mathcal{U}$ Euclidean neigh. of x.

$$H_i(M|x) = H_i(M, M\setminus\{x\}) = H_i(\mathcal{U}|x)$$
$$\cong H_i(\mathcal{U}, \mathcal{U}\setminus\{x\})$$
$$= H_i(\mathbb{R}^n|0)$$
$$= H_i(\mathbb{R}^n, \mathbb{R}^n\setminus 0) = \begin{cases} \mathbb{Z}, & i=n \\ 0, & i\neq n \end{cases}$$

In particular, $H_n(M|x) \cong H_n(\mathbb{R}^n|0) = \mathbb{Z}$ $\Rightarrow \langle 1 \rangle = \langle -1 \rangle \rightsquigarrow H_n(\mathbb{R}^n|0)$ has two possible generators $\rightsquigarrow H_n(M|x)$ has two possible generators.

Def: A local orientation of M at $x \in M$ is a choice of generator $\mu_x \in H_n(M|x)$.

Idea/Motivation: Want preserved under rotation & flipped under reflections. - From $\mathbb{R}^n$ intuition.

$$H_n(\mathbb{R}^n|0) = H_n(\mathbb{R}^n, \mathbb{R}^n\setminus 0) \cong H_{n-1}(S^{n-1})$$

Rotations of S^{n-1} induce id on $H_{n-1}(S^{n-1})$ & reflections of S^{n-1} induce identity on $H_{n-1}(S^{n-1})$

Restriction Maps: Suppose $B \subseteq M$, $x \in M$. Let $H_n(M|B) = H_n(M, M\setminus B)$. Then $\exists$ 'restriction map'

$r: H_n(M|B) \to H_n(M|x)$

induced by the inclusion

$(M, M\setminus B) \hookrightarrow (M, M\setminus\{x\})$

If $\alpha \in H_n(M|B)$, write $\alpha|x$ for image of α under r.

Lem: If $x \in \mathcal{U}$ is a Euclidean neigh and $B \subseteq \mathcal{U} = \mathbb{R}^n$ is an open neigh (n-ball) in $\mathcal{U} = \mathbb{R}^n$ centered at x $\rightsquigarrow$

$r: H_n(M|B) \to H_n(M|x)$

is an iso.

Pf: Follows from long exact seq, using 5 lem & fact that $M\setminus B$ is a def retract of $M\setminus\{x\}$. ☐

Def: An orientation on M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ and satisfying a local orientation cond.

Local orientation cond:

For all $x \in M$, $\exists$ an open n-ball $B \subseteq U = \mathbb{R}^n$ centered at x and a generator $\mu_B \in H_n(M|B)$ such that

$$\mu_y = \mu_B|_y$$

for all $y \in B$

Def: A manifold M is called orientable if it admits an orientation.

Orientation Bundles

Assume M is an n-manifold.

$$M_{\mathbb{Z}} \overset{\text{def}}{=} \coprod \underbrace{H_n(M|x)}_{\cong \mathbb{Z}}$$

$$= \{\alpha_x \mid x \in M \text{ \& } \alpha_x \in H_n(M|x)\}$$

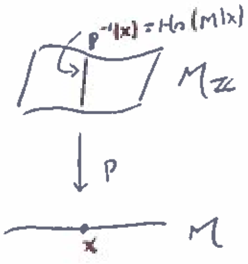
$$\overset{\cup}{\tilde{M}} = \{\mu_x \mid x \in M \text{ \& } \mu_x \text{ loc. orient. of } M \text{ at } x\}$$

ie μ_x is a gen. of $H_n(M|x)$

Define $p: M_{\mathbb{Z}} \to M$ by $p(\alpha_x) = x$

$q: \tilde{M} \to M$ by $q(\mu_x) = x$

so that $q = p|_{\tilde{M}}$.



Given an open n-ball $B \subseteq U = \mathbb{R}^n$ and a $\alpha_B \in H_n(M|B)$, define

$$\mathcal{U}(\alpha_B) = \{\alpha_x \mid x \in B \text{ and } \alpha_x = \alpha_B|_x \in H_n(M|x)\}$$

$$\subseteq M_{\mathbb{Z}}$$

Check:

1) $\mathcal{U}(\alpha_B)$ form a basis for a topology on $M_{\mathbb{Z}}$

2) W.r.t. this topology, map $p: M_{\mathbb{Z}} \to M$ is a covering projection map.

Equip $M_{\mathbb{Z}}$ with this topology and $\tilde{M} \subseteq M_{\mathbb{Z}}$ with the subspace top. Then $M_{\mathbb{Z}}$, $\tilde{M}$ are covering spaces of M. $M_{\mathbb{Z}}$ is an inf. sheeted covering as $p^{-1}(x) = H_n(M|x) \cong \mathbb{Z}$ while $\tilde{M}$ is a two-sheeted covering because $q^{-1}(x) = \{2 \text{ gen. of } H_n(M|x)\}$

Properties:

1) $\tilde{M}$, $M_{\mathbb{Z}}$ are manifolds (covering space of manifold M)

2) The p, q induce iso. on local homology:

$$H_n(M_{\mathbb{Z}}|\alpha_x) \cong H_n(M|x)$$

$$H_n(\tilde{M}|\mu_x) \cong H_n(M|x)$$

(3) $\tilde{M}$ has a canonical orientation given by

$$\mu_x \in \tilde{M} \longmapsto \mu_x \in H_n(M|x) \cong H_n(\tilde{M}|\mu_x)$$

viewed as a local orientation of $\tilde{M}$ at μ_x.

Recall: $\tilde{M} = \{\mu_x \mid x \in M \text{ \& } \mu_x \text{ loc. orient. of } M @ x\}$

Cor: Every manifold M has an orientable two-sheeted covering.

Ex: $M = \mathbb{RP}^2 \to S^2$ is an orientable two-sheeted covering

Def: A (cont.) section of a covering map $p: E \to M$ is a continuous map $s: M \to E$ such that $ps = 1_M$

$\Gamma(E) = \{$(cont.) sections of $E \to M\}$

$\Gamma(M_\mathbb{Z}) = \{$(cont.) sections of $p: M_\mathbb{Z} \to M\}$

$\Gamma(\tilde{M}) = \{$(cont.) sections of $p: \tilde{M} \to M\}$

Rem: $\Gamma(M_\mathbb{Z})$ is an abelian group because each fiber $p^{-1}(x) = H_n(M|x)$ is an ab. group & the "fiberwise" product (sum) of two cont. sections is again cont.

Rem: A $s \in \Gamma(\tilde{M})$ is a function which sends each $x \in \Gamma$ to a local orientation μ_x at x. Continuity of s implies that this function satisfies the local const. condition. Then...

$$\Gamma(\tilde{M}) = \{\text{orientations on } M\}$$

Cor: M is orientable $\Leftrightarrow \Gamma(\tilde{M}) \neq \phi$.

10/27/2016

M n-manifold, $x \in M$

Recall

Def: A local orientation at x is a generator μ_x of $H_n(M|x) \cong \mathbb{Z}$

Orientation on M is a function $x \mapsto \mu_x$ assigning each $x \in M$ a local orientation μ_x at x & local consistency condition.

Thm:

1) M is orientable iff $\tilde{M} \to M$ is trivial.

2) M is connected $\leadsto$ M orientable iff $\tilde{M}$ has two components

3) If M connected & $\pi_1(M)$ has no subgroups of index 2 $\leadsto$ M is orientable.

Pf: By Cor, M orientable $\Leftrightarrow \Gamma(\tilde{M}) \neq \phi$; that is, $\tilde{M} \to M$ has a section. This is iff $\tilde{M} \to M$ is trivial as $\tilde{M}$ is a two-sheeted covering. If M connected, covering is triv. $\Leftrightarrow$ it has 2 components. This proves (1) & (2). Now suppose M is connected & $\pi_1(M)$ has no index 2 subgroups. Then M has no nontriv. 2-sheeted covering - corresponds to index 2. $\tilde{M}$ 2-sheeted so triv.

Cor: If M is connected & simply connected then M is orientable.

Ex: S^n is orientable.

R-orientations

R comm. unital ring

An R-orientation is defined in the same way as an orientation where integer coefficients are replaced by R-coefficients.

Can define a covering $M_R \to M$ by generalizing the def. $M_{\mathbb{Z}} \to M$.

Explicitly,

$$M_R = \{\alpha_x \mid x \in M, \ \alpha_x \in H_n(M|x;R)\}$$

Define projection $p: M_R \to M$ via $\alpha_x \mapsto x$

The top. on M_R is defined mutatis mutandis from before.

$$p^{-1}(x) = H_n(M|x;R)$$

An R-orientation on M is a $\overset{\text{cont.}}{\wedge}$ section $s: M \to M_R$ which sends each $x \in M$ to a generator of $H_n(M|x;R)$.

$$H_n(M|x;R) \underset{\text{UCT}}{\cong} \underbrace{H_n(M|x)}_{\cong \mathbb{Z}} \otimes R$$

$$\cong R$$

So each $\alpha_x \in H_n(M|x;R)$ can be written as $\alpha_x = \mu_x \otimes r$ where μ_x gen. of $H_n(M|x)$, $r \in R$.

For each $r \in R$, we can define a subcovering $M_r \subseteq M_R$ defined as

$$M_r = \{\mu_x \otimes r \mid x \in M, \ \mu_x \text{ gen. } H_n(M|x)\}$$

The fiber of $M_r \to M$ above $x \in M$ consists of $\pm\mu_x \otimes r$, where $\pm\mu_x$ are the two gen. of $H_n(M|x)$. If r has order 2 $\rightsquigarrow$ $M_r \to M$ is a ~~two~~ one sheeted covering, ie $M_r = M$

If r does not have order 2, then $M_r \to M$ is a 2-sheeted covering:

$$M_r \cong \tilde{M}$$

Thm:

1) M orientable $\rightsquigarrow$ M R-orientable

2) If char $R = 2$ $\rightsquigarrow$ M R-orientable

3) If M is not R-orientable, then M_r has a section $\Leftrightarrow$ r has order 2.

Pf:

1) M orientable, then every orientation $x \mapsto \mu_x$ induces an R-orientation $x \mapsto \mu_x \otimes 1$; $1 \in R$.

2) If char $R=2$ then $1 \in R$ has order 2. So $\underset{\wedge M_r}{\tilde{M}}$ 1-sheeted covering. Hence

$$\text{Id}: M \to M_1 \subseteq M_R$$

is an R-orientation for M.

3) M not R-orientable, by (1) M is not orientable by (1) and hence $\tilde{M}$ does not have a section. Then follows from the discussion preceding the thm.

Ex: Every M is $\mathbb{Z}/2\mathbb{Z}$-orientable.

Notations: M n-manifold
$A \subseteq M$ subset

$\Gamma(M_R) = \{\text{sections } M_R \to M\}$

$\Gamma(A\, M_R) = \{\text{sections of } M_R \xrightarrow{p} M \text{ over } A\}$
$= \{s : A \to M_R \mid s \text{ cont.}, ps = 1_A\}$

Recall: $p^{-1}(x) = H_n(M|x; R) \leftarrow$ R-mod.

Then $\Gamma(M_R), \Gamma(A, M_R)$ are R-mod

For $A \subseteq M$, define a map

$$J_A : H_n(M|A; R) \to \Gamma(A, M_R)$$
$\alpha \longmapsto$ the section sends $x \in A$ to $\alpha|_x \in H_n(M|x; R)$

Remark: J_A is a map of R-modules

Lem: (Hatcher, pp 236, Lem 3.27)
Let M be an n-manifold and $A \subseteq M$ compact. Then

(a) $J_A : H_n(M|A; R) \to \Gamma(A, M_R)$ is an isomorphism.

(b) $H_i(M|A; R) = 0$ for $i > n$.

Pf: Later

Cor: If M is a closed (compact w/o boundary) then

(a) $H_n(M; R) \cong \Gamma(M_R)$

(b) $H_i(M; R) = 0$ for $i > n$.

Pf: Since M closed and hence compact, we can apply the lemma to $A = M$. The Cor then follows by noting that
$H_i(M; R) = H_i(M|M; R) = H_i(M, M \setminus M; R)$
& $\Gamma(M_R) = \Gamma(M, M_R)$ ▨

Thm: (Hatcher, 5.25)
Let M be a closed connected n-manifold. Then....

(a) If M R-orientable, then $H_n(M; R) \cong R$

(b) If M is not R-orientable, then $H_n(M; R) \cong \{r \in R \mid 2r = 0\}$

Pf: By Cor, $H_n(M; R) \cong \Gamma(M_R)$ and since M is connected, a section $s \in \Gamma(M_R)$ is uniquely determined by its value at a point $x \in M$. This shows that

$$e : \overset{\Gamma(M_R)}{\overline{H_n(M; R)}} \to H_n(M|x; R) \cong R$$
$$\alpha \longmapsto \alpha|_x$$

is injective for $x \in M$.

If M orientable, one can check that $M_R \to M$ is a trivial covering and hence $\exists$ a section through every point of the fiber $p^{-1}(x) = H_n(M|x;R)$. Then ρ is also surj. If M is not orientable $\rightsquigarrow$ $M_r \subseteq M_R$ has a section iff r has order 2.

$$\operatorname{im} \rho = \{ r \in R \mid 2r = 0 \}$$

☒

Cor: Let M be a closed connected n-manifold. Then

(a) M orientable $\rightsquigarrow$ $H_n(M) \cong \mathbb{Z}$

(b) M not orientable $\rightsquigarrow$ $H_n(M) = 0$

Let M be a n-manifold.

Def: A class $\mu \in H_n(M;R)$ whose local image in $H_n(M|x;R) \cong R$ is a gen. for all x is called a fundamental class for M with coefficients in R.

One can show $\exists$ a fund. class $\mu \in H_n(M;R)$ if and only if M is closed & R-orientable.

Rem: If M closed, connected, & R-orientable then a fund. class is just a gen. μ of $H_n(M;R)$.

Rem: If M has k components, $M_1, \dots, M_k$ then M is R orientable iff each M_i R orientable.

Rem: If M is closed, R-orientable, n-manifold with k components $\rightsquigarrow$ $H_n(M;R) \cong R^k$.

Rem: If M is an orientable n-manifold with k components then $\exists$ 2^k possible orientations on M.

Now for proof of Lem:

Lem: $J_A : H_n(M|A;R) \to \Gamma(A;M_R)$

$\alpha \longmapsto$ section sending $x \in A$ to $\alpha|x \in H_n(M|x;R) = p^{-1}(x)$

(a) Map iso.

(b) $H_i(M|A;R) = 0$ for $i > n$

Pf: Notation: Given an n-manifold & $A \subseteq M$ compact, let $S_M(A)$ be the statement that (a) & (b) hold for $A \subseteq M$.

We prove

1) $S_M(A), S_M(B), S_M(A \cap B) \rightarrow S_M(A \cup B)$

2) $S_{\mathbb{R}^n}(A)$ $\forall$ compact $A \subseteq \mathbb{R}^n$ $\rightsquigarrow$ $S_M(A)$ for all compact $A \subseteq M$

3) $S_{\mathbb{R}^n}(A)$ $\forall$ $A \subseteq \mathbb{R}^n$ fin. union of convex subspaces of $\mathbb{R}^n$

4) $S_{\mathbb{R}^n}(A)$ $\forall$ $A \subseteq \mathbb{R}^n$ compact.

11/01/2016

Recall M n-manifold, $A \subseteq M$

$p: M_R \to M$ covering space with fiber $p^{-1}(x) = H_n(M|x; R)$

$\Gamma(M_R) = \{\text{sections of } p: M_R \to M\}$

$\Gamma(A, M_R) = \{\text{sections of } p: M_R \to M \text{ over } A\}$

$H_n(M|A; R) = H_n(M, M \setminus A; R)$

$\Gamma(M_R), \Gamma(A, M_R)$, & $H_n(M|A; R)$ are R-mod.

$J_A : H_n(M|A; R) \to \Gamma(A, M_R)$

$\alpha \longmapsto$ section sending $x \in A$ to $\alpha|_x \in H_n(M|x; R)$

* J_A is an R-mod. map.

For simplicity, assume $R = \mathbb{Z}$.

M.V. for relative hom.

B/c $= H_n(M, M \setminus (A \cup B))$

(1)

$$\begin{tikzcd}
{=H_{n+1}(M|A\cap B)} \arrow[r, squiggly] & 0 \arrow[r] & H_n(M|A\cup B) \arrow[r] \arrow[d, "J_{A\cap B}"] & H_n(M|A) \oplus H_n(M|B) \arrow[r] \arrow[d, "{(J_A, -J_B)}"] & H_n(M|A\cap B) \arrow[d, "J_{A\cap B}"] \\
 & 0 \arrow[r] & \Gamma(A\cup B, M_{\mathbb{Z}}) \arrow[r, "F"'] & \Gamma(A, M_{\mathbb{Z}}) \oplus \Gamma(B, M_{\mathbb{Z}}) \arrow[r, "G"'] & \Gamma(A\cap B, M_{\mathbb{Z}})
\end{tikzcd}$$

where,

$F(s) = (s|_A, s|_B)$

$G(s_1, s_2) = s_1|_{A\cap B} - s_2|_{A\cap B}$

Check: Second row is exact & diagram commutes

By assumption, J_A, J_B, $J_{A\cap B}$ are iso. By 5-lem, $J_{A\cup B}$ is also an iso. So (a) holds for $A \cup B$.

For $i > n$, a portion of the MV sequence looks like

$$\underset{\overset{\shortparallel}{0}}{H_{i+1}(M|A\cap B)} \to H_i(M|A\cup B) \to \underset{\overset{\shortparallel}{0}}{H_i(M|A) \oplus H_i(M|B)}$$

By assumption.

So (b) holds for $H_i(M|A\cup B)$. This proves (1).

(2) M arb., $A \subseteq M$ compact. We can cover A by fin many open n-balls, $B_i \subseteq \mathcal{U}_i \cong \mathbb{R}^n$. Say m such n-balls. Let $A_i = A \cap \overline{B_i}$. Now A_i is compact. and $A_i \subseteq \mathcal{U}_i \cong \mathbb{R}^n$. By construction, $A = \overset{m}{\bigcup} A_i$. If $m = 1$, $A = A_1 \subseteq \mathcal{U}_1 \cong \mathbb{R}^n$. Lemma holds for A by assumption (and excision.)

$$\underbrace{\text{(and excision.)}}$$

$H_i(M|A) = H_i(\mathcal{U}_1|A)$

If $n > 1$, use induction Lem. holds for $A_1 \cup \cdots \cup A_{m-1}$. Holds for $(A_1 \cup \cdots \cup A_{m-1}) \cap A_m =$

$$\underbrace{(A_1 \cap A_m) \cup \cdots \cup (A_{m-1} \cap A_m)}_{m-1 \text{ terms so holds}}$$

Then holds for union by (1).

$M = \mathbb{R}^n$

(3) Suppose $A \subset \mathbb{R}^n$ is a finite union $A = A_1 \cup \cdots \cup A_m$ where each $A_i \subseteq \mathbb{R}^n$ is compact and convex. If $m = 1$, then $A = A_1$, then A is compact & convex.

$H_n(M|A) \cong H_n(M|\underset{\overset{\uparrow}{\text{pt in } A}}{x})$

$\cong \Gamma(A|M_{\mathbb{Z}})$

$\mathbb{R}^n \setminus A$ is a def retract of $\mathbb{R}^n \setminus x$

$M_{\mathbb{Z}} \to M$ is triv. over A as A convex

Then (a) holds for A.

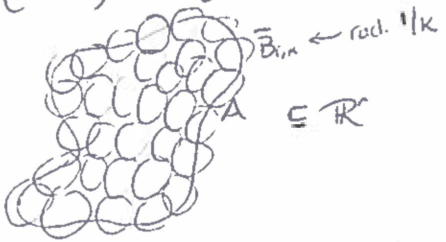
$H_i(M|A) \cong H_i(M|x)$

$= 0$ for $i > n$.

Then (b) holds for A.

If $m>1$, one can use an inductive argument similar to one used to prove (2).

(4) Let $A \subseteq \mathbb{R}^n$ compact. For each $k>0$, let $\overline{B}_{1,k}, \dots, \overline{B}_{m_k,k} \subseteq \mathbb{R}^n$ be a finite collection of closed n-balls centered at points of A and with radius $1/k$ ∋ set $A_k = \overline{B}_{1,k} \cup \dots \cup \overline{B}_{m_k,k}$ contains A & such that $A_k \subseteq A_{k-1}$



Then by construction, $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap A_k = A$. By (3), lem holds for each A_k.

Check: - $\varinjlim_k H_i(M|A_k) \underset{\substack{\uparrow \\ \text{restrict.}}}{\cong} H_i(M|A)$

- $\varinjlim_k \Gamma(A_k, M_{\mathbb{Z}}) \underset{\substack{\uparrow \\ \text{restrict.}}}{\cong} \Gamma(A, M_{\mathbb{Z}})$

- The following diagram commutes

$$\begin{tikzcd}
H_n(M|A_k) \arrow[r, "\text{restrict.}"] \arrow[d, "J_{A_k}"'] \arrow[dr, phantom, "\circlearrowright"] & H_n(M|A_{k+1}) \arrow[d, "J_{A_{k+1}}"] \\
\Gamma(A_k, M_{\mathbb{Z}}) \arrow[r, "\text{restrict.}"] & \Gamma(A_{k+1}, M_{\mathbb{Z}})
\end{tikzcd}$$

↙ Then J induces a map of direct systems which induces a map in the limit.

Since lem. holds for each A_k, each J_{A_k} is an iso. So J_A is an iso → (a) holds for A.

Now if $i > n$, $H_i(M|A) \overset{(i)}{=} \varinjlim_k H_i(M|A_k)$

$= \varinjlim_k 0$

$= 0$

So (b) holds for A. ▨

Recall if M is closed (compact), then one can apply lem. to $A = M$.

Then $J_M : H_n(M;R) \to \Gamma(M_R)$ is an iso.

$\Gamma(M_R) = \{R\text{-orient. on } M\}$

Suppose $\mathcal{O}$ is an R-orientation on M. Then $\mathcal{O}$ corresponds to an $s \in \Gamma(M_R)$ which in turn corresponds to a hom. class $\mu = J_M^{-1}(s) \in H_n(M;R)$. μ is called the fundamental class or orientation class corresponding to the R-orientation $\mathcal{O}$.

Notations: μ_M, $[M]$

Properties: $\mu|_x \in H_n(M|x;R) \cong R$ is a generator of $H_n(M|x;R)$ for $x \in M$.

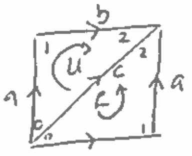
If M_i is a connected component of M then $\mu|_{M_i}$ is a gen. of $H_n(M_i;R) \cong R$

If $R = \mathbb{Z}$ & M is a Δ-complex →

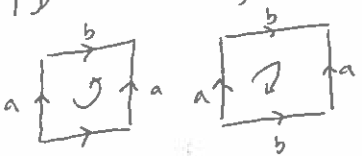
$$\mu = \Big[\sum_\sigma \epsilon_\sigma \sigma\Big]$$

where $\epsilon_\sigma \in \{\pm 1\}$ and where sum is over all n-simplices of the Δ-complex

Ex: $M = S^1 \times S^1 =$



2 possible orientations:



$[M] = [L - U]$ | $[M] = [U - L]$

Let M be a closed R-oriented n-manifold and $[M]$ the fund class

Define a map

$$D_M : H^i(M; R) \longrightarrow H_{n-i}(M; R)$$
$$[\phi] \longmapsto [M] \frown [\phi]$$

Thm: If M is a closed, R-oriented n-manifold $\rightarrow$ D_M is an iso.

$\underbrace{\hphantom{If M is a closed, R-oriented n-manifold}}_{\text{Poincaré Duality}}$

Cor: If M is a closed, R-oriented n-manifold

$\rightarrow H^i(M; R) \cong H_{n-i}(M; R)$

If $R = \mathbb{Z}$, then $H^i(M) \cong H_{n-i}(M)$

One can check if M is a closed n-manifold then $H_i(M)$ is fin. gen. for all i and by

UCT $\quad H^i M \cong \underset{\text{Free}}{\underset{\sim}{FH_iM}} \oplus \underset{\text{torsion}}{\underset{\sim}{TH_{i-1}}}\, M$

Together with Poincaré Duality, this implies:

$FH_iM \cong FH_{n-i}M$

$TH_{i-1}M \cong TH_{n-i}M$

$M = \mathbb{RP}^2$

	$i=3$		↔	$i=0$
FH_iM	$\mathbb{Z}$	0	0	$\mathbb{Z}$
TH_iM	0	0	$\mathbb{Z}_2$	0
			↔	

Cor: If M is a closed n-manifold for n-odd, then $\chi(M) = 0$

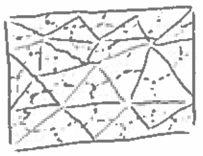
Pf: $\chi(M) = \sum_{i=0}^{n} (-1)^i \underbrace{\operatorname{rank} H_i(M)}_{\operatorname{rank} FH_i(M)}$

$$= \sum_{i=0}^{n-1} \Big((-1)^i \operatorname{rank} FH_iM + (-1)^{n-i} \operatorname{rank} FH_{n-i}M \Big)$$

$= 0$ $\quad\boxtimes$

Idea of Poincaré Duality:

Suppose M is triangulated



triangulation gives a dual cell structure on M

$C_i^{\Delta} M \longrightarrow C_{(n)}^{n-i} M$

11/03/2016

Let $[v_0, \dots, v_i]$ denote the i-simplex of the original triangulation which has vertices $v_0, \dots, v_i$.

To each i-simplex $[v_0, \dots, v_i]$ of the original triangulation, assign a "dual" $(n-i)$-cell as follows:

0-simplex: $v_0 \longmapsto$ n-cell $e(v_0)$ given by closed star of v_0 in barycentric subdiv. of the triang.

Ex

closed star of v_0

1-simplex: $[v_0, v_1] \longmapsto$ the $(n-1)$ cell $e(v_0, v_1)$ given by $e(v_0,v_1) = e(v_0) \cap e(v_1)$

2-simplex: $[v_0, v_1, v_2] \longmapsto$ the $(n-2)$-cell $e(v_0, v_1, v_2)$ given by

$$e(v_0,v_1,v_2) = e(v_0) \cap e(v_1) \cap e(v_2)$$

Define a map:

$$f : C_i^{\Delta}(M; \mathbb{Z}_2) \longrightarrow C^{n-i}_{cw}(M; \mathbb{Z}_2)$$

$$[v_0, \dots, v_i] \longmapsto e(v_0, \dots, v_i)^*$$

- $e(v_0, \dots, v_i) \in C^{cw}_{n-i}(M; \mathbb{Z}_2)$

f $\underset{\text{iso}}{\underline{\text{auto.}}}$ of abelian groups (sends gen. to generators; iso of $\mathbb{Z}_2$ v.s.)

Claim:

$$f \circ \partial^{\Delta} = \partial_{cw} \circ f$$

Pf:

$$\partial^{\Delta}[v_0, \dots, v_i] = \sum_{k=0}^{i} \underbrace{(-1)^k}_{\text{not matter as in } \mathbb{Z}_2 \text{ coeffic. } (-1)=1} [v_0, \dots, \hat{v}_k, \dots, v_i]$$

$$f \partial^{\Delta}[v_0, \dots, v_i] = \sum_{k=0}^{i} e(v_0, \dots, \hat{v}_k, \dots, v_i)$$

construction? By def, $e(v_0, \dots, v_i)$ lies in the boundary of cells $e(v_0, \dots, \hat{v}_k, \dots, v_i)$

Then

$$\delta_{cw}(f([v_0, \dots, v_i])) =$$

$$\delta_{cw}(e(v_0, \dots, v_i)^*)$$

$$e(v_0, \dots, v_i)^* \delta^{cw}$$

$$= \sum_{k=0}^{i} \underbrace{\pm}_{\text{Don't matter} - \mathbb{Z}_2} e(v_0, \dots, \hat{v}_k, \dots, v_i) \qquad \blacksquare$$

So f is a chain map. So iso of chain complexes. Then

$$C_*^{\Delta}(M; \mathbb{Z}_2) \cong C^{n-*}_{cw}(M; \mathbb{Z}_2)$$

Then....

$$H_i(M; \mathbb{Z}_2) \cong H^{n-i}(M; \mathbb{Z}_2)$$

Cohomology with Compact Support

X top. space

$A \subseteq X$ subset

$\phi \in C^i(X)$

Def: ϕ has support in A if $\phi(\sigma) = 0$ for $\sigma \in S_i(X \setminus A)$

Equivalently, this means

$$\phi \in C^i(X, X \setminus A) \subseteq C^i(X)$$
$$\parallel$$
$$\{\phi \in C^i(X) \mid \phi(\sigma) = 0 \text{ for } \sigma \in S_i(X \setminus A)\}$$

Def: ϕ has compact support if $\exists$ a compact $K_\phi \subseteq X$ such that ϕ has support in K_ϕ, ie such that $\phi(\sigma) = 0$ for $\sigma \in S_i(X \setminus K_\phi)$

Let $C^i_c(X) \overset{def}{=} \{\phi \in C^i(X) \mid \phi \text{ has compact support}\}$

$$= \bigcup_{\substack{K \subseteq X \\ \text{compact}}} C^i(X, X \setminus K) = C^i(X/K) \subset C^i(X)$$

Note: $C_c^i(X)$ is a subgroup of $C^i(X)$ as if ϕ has support in a compact $K_\phi \subseteq X$ & ψ has support in a compact $K_\psi \subseteq X$ then $\phi + \psi$ has support in a compact $K_\phi \cup K_\psi \subseteq X$.

$C_c^*(X)$ is a subcomplex of $C^*(X)$ because each $C^*(X|K)$ is a subcomplex

Def: $H^i_c(X) = H^i(C_c^*(X))$

"i^{th} cohomology of X with compact support"

Alternative Def:

$$\mathcal{I} = \{K \subseteq X \mid K \text{ compact}\}$$

Define an order on $\mathcal{I}$ via $K \geq K'$ iff $K \supseteq K'$. $\rightsquigarrow$ $\mathcal{I}$ directed poset. ($K, K' \in \mathcal{I} \to K \cup K' \in \mathcal{I}$ & contains K, K').

Suppose $K \geq K'$. Then $\exists$ an inclusion

$$f_{K',K}: C^*(X|K') \hookrightarrow C^*(X|K)$$

Then $\{C^*(X|K), f_{K',K}\}$ is a direct system over $\mathcal{I}$

Thm: $\varinjlim_{K \in \mathcal{I}} C^*(X|K) = C^*_c(X)$

Pf:

$$\varinjlim_{K \in \mathcal{I}} C^*(X|K) = \frac{\coprod_K C^*(X|K)}{\phi \sim K_{K'K} \text{ for } K \geq K' \in C^*(X|K')}$$

$$C^*_c(X) = \bigcup_K C^*(X|K) \subseteq C^*(X)$$

Define $u: \varinjlim_K C^*(X|K) \to C^*_c(X)$ by sending each $\phi \in C^*(X|K)$ to ϕ viewed as an element of $C^*_c(X)$

Can check: u is a well defined iso. $\boxtimes$

Cor: $\varinjlim_{K\in I} H^i(X|K) = H^i_c(X)$

* Could use this as def of $H^i_c(X)$.

$(I, \geq)$ a directed poset over I...

1) If $J \subseteq I$ is a subset such that $(J, \geq)$ is directed then $\exists$ a map

$$\varinjlim_{j\in J} G_j \to \varinjlim_{i\in I} G_i$$

induced by sending each G_j identically to itself

Def: $\{G_j \mid j\in J\}$ is called a subsystem of $\{G_i \mid i \in I\}$

2) If $J \subseteq I$ is a subset such that for $i \in I \to \exists\, j\in J$ such that $j \geq i \rightsquigarrow (J,\geq)$ directed and

$$\varinjlim_{j\in J} G_j \cong \varinjlim_{i\in I} G_i$$

Def: A subset $J\subseteq I$ with above property is called a cofinal subset.

Ex: $H^i_c(\mathbb{R}^n) = ?$

$I = \{K \subseteq \mathbb{R}^n \mid K \text{ compact}\}$

$\rotatebox{90}{$\subseteq$}$

$J = \{\overline{B_\delta(0)} \mid \delta > 0\}$

Then $J \subseteq I$ is cofinal as every compact $K \subseteq \mathbb{R}^n$ is contained in some $\overline{B_\delta(0)}$

Then $H^i_c(\mathbb{R}^n) =$

$$= \varinjlim H^i(\mathbb{R}^n | K)$$

$$= \varinjlim_{\delta\to 0} H^i(\mathbb{R}^n | \overline{B_\delta(0)})$$

$$= \varinjlim H^i(\underbrace{\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B_\delta(0)}}_{H^i(\mathbb{R}^n, \mathbb{R}^n\setminus(0))})$$

$$= \begin{cases} \mathbb{Z}, & i=n \\ 0, & i\neq n \end{cases}$$

In particular, $H^i_c(X)$ is not invariant under homotopy equivalence

Note also:

If X is compact $\to H^i_c(X) = H^i(X)$

Ex: $H^i_c(\text{closed } n\text{-ball}) = H^i(\text{closed } n\text{-ball})$
$= H^i(\text{point})$

but $H^i_c(\text{open ball}) = \begin{cases} \mathbb{Z}, & i=n \\ 0, & i\neq n \end{cases}$

Thm: Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. If in X^+, the point ∞ has a neighborhood basis consisting of contractible open neighborhoods then $H^i_c(X) \cong H^i(X^+, \infty)$

Recall: Topology on X^+: $\{\text{open sets in } X\}$
$\cup\ \{X^+ \setminus K \mid K \subseteq X \text{ compact}\}$

Ex: $H^i_c(\mathbb{R}^n) \cong H^i(S^n, \infty) = \begin{cases} \mathbb{Z}, & i=n \\ 0, & i\neq n \end{cases}$

Induced Maps

$f: X \to Y$ map

Def: f is proper if $f^{-1}(K)$ is compact for $K \subseteq Y$ compact.

Suppose $f: X \to Y$ is proper and cont. Let $\phi \in C_c^i Y$. Say ϕ has support in K_ϕ, where $K_\phi \subseteq Y$ compact. Then $f^\# \phi = \phi \circ f$ has support in $f^{-1}(K_\phi)$ and $f^{-1}(K_\phi)$ is also compact $\to f^\# \phi \in C_c^i X$ so f induces a map $f^*: C_c^i(Y) \to C_c^i(X)$.

11/08/2016

Cohomology with Compact Support

$$C_c^i(X) = \{ \phi \in C^i(X) \mid \phi \text{ has } \underbrace{\text{compact support}} \}$$

↳ $\exists$ a compact $K_\phi \subseteq X$ such that $\phi(\sigma) = 0$ for $\sigma \in S_i(X \setminus K_\phi)$

We saw $H_c^i(X) = \varinjlim_{\substack{K \subseteq X \\ \text{compact}}} H^i(X|K)$

Recall $f: X \to Y$ is proper if $f^{-1}(U)$ is compact for all compact $U \subseteq Y$

We saw if f is proper and cont. then $\exists$ an induced map $f^*: H_c^i(Y) \to H_c^i(X)$

In particular, if $X \cong Y$ then $H_c^i(X) \cong H_c^i(Y)$.

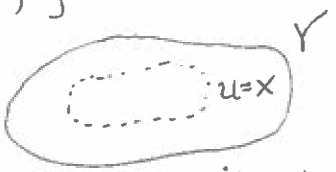
Fact: If f, g are $\underbrace{\text{properly homotopic}}$ then $f^* = g^*$

↳ $\exists$ a homotopy $H: X \times I \to Y$ between f and g which is proper.

Cor: H_c^i is a contravariant functor on the category with objects top spaces and morph. proper cont. maps up to proper homotopy.

Thm: If Y is Hausdorff and $f: X \to Y$ is an open cont. inj. $\to \exists$ an induced map $f_*: H_c^i(X) \to H_c^i(Y)$

Pf: We can assume that $X = U$ for $U \subseteq Y$ and that f is the inclusion $f: U \hookrightarrow Y$.



Then $H_c^i(X) = H_c^i(U) = \varinjlim_{\substack{K \subseteq U \\ \text{compact}}} H^i(U|K)$

excision
K⊆Y closed
as Y Hausdorff

$$\xrightarrow{} \cong \varinjlim_{\substack{K \subseteq U \\ \text{compact}}} H^i(Y|K)$$

So $H_c^i(Y) = \varinjlim_{\substack{K \subseteq Y \\ \text{compact}}} H^i(Y|K)$. Since the $H^i(Y|K)$ for $K \subseteq U$ form a subsystem of the $H^i(Y|K)$ for $K \subseteq Y$ there is a map between the direct limits. ▣

Mayer-Vietoris for $H^i_c(X)$

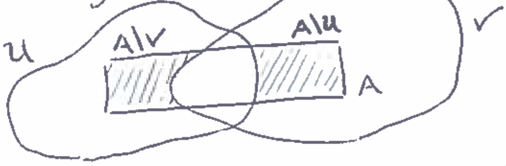
Thm: If X is a metric space, $X = U \cup V$, $U, V \subseteq X$ open, then $\exists$ a l.e.s.

$$\cdots \to H^i_c(U \cap V) \to H^i_c(U) \oplus H^i_c(V) \to H^i_c(X) \to H^{i+1}_c(U \cap V) \to \cdots$$

Lem: If X, U, V are as above, then every compact $A \subseteq X$ can be written as $A = K \cup L$ for $K \subseteq U$, $L \subseteq V$ compact.

Pf: Let $A \subseteq X$ be compact. Define

$\delta = \text{distance}(A \setminus U, A \setminus V) > 0$



Define $K = \{x \in A \mid \text{dist}(x, A \setminus U) \geq \delta/2\} \subseteq U$
$L = \{x \in A \mid \text{dist}(x, A \setminus V) \geq \delta/2\} \subseteq V$

Then K, L are closed. Claim ~~U~~ $K \cup L = A$

Say $\neq$, $\exists\, y \in A \setminus (K \cup L)$. Then

$$\delta = \text{dist}(A \setminus U, A \setminus V) \leq \text{dist}(A \setminus U, y) + \text{dist}(y, A \setminus V) < \delta/2 + \delta/2 = \delta$$

$y \notin K$, $y \notin L$ so

so $\delta < \delta$ $\rightarrow\!\leftarrow$. ∎

Pf (sketch): Let $K \subseteq U$, $L \subseteq V$ be compact. Then $\exists$ a MV sequence for relative cohomology

$$\begin{tikzcd} \cdots \arrow[r] & H^i(X \mid K \cap L) \arrow[r] \arrow[d, equal, "*"] & H^i(X \mid K) \oplus H^i(X \mid L) \arrow[r] \arrow[d, equal, "*"] & H^i(X \mid K \cup L) \arrow[r] & \cdots \\ & H^i(U \cap V \mid K \cap L) & H^i(U \mid K) \oplus H^i(V \mid L) & & \end{tikzcd}$$

$*$: via excision. [Metric so compact $\to$ closed]

Take the direct limit over all compact $K \subseteq U$ and compact $L \subseteq V$.

Direct lim. preserves exactness.

$I = \{(K, L) \mid K \subseteq U, L \subseteq V \text{ compact}\}$

$$\varinjlim_{(K,L)} H^i(U \cap V \mid K \cap L) = \varinjlim_{\substack{A \subseteq U \cap V \\ \text{compact}}} H^i(U \cap V \mid A) = H^i_c(U \cap V)$$

every compact $A \subseteq U \cap V$ can be written $A = K \cap L$; $K \subseteq U$, $L \subseteq V$ compact. Take $K = L = A$

&

$$\varinjlim_{(K,L)} H^i(U \mid K) \oplus H^i(V \mid L) = H^i_c(U) \oplus H^i_c(V)$$

&

$$\varinjlim_{(K,L)} H^i(X \mid K \cup L) = \varinjlim_{\substack{A \subseteq X \\ \text{compact}}} H^i(X \mid A) = H^i_c(X)$$

follows as every compact $A \subseteq X$ can be written as $A = K \cup L$ for $K \subseteq U$, $L \subseteq V$ compact by the Lemma.

Poincaré Duality Manifolds
(without boundary)

M^n n-manifold (possibly non-compact)

Θ R-orientation on M

Let $K \subseteq M$ be compact. We have

$$\left\{\begin{matrix} R \text{ orientations} \\ \text{on } M \end{matrix}\right\} \subseteq \underset{\text{sections of } M_R}{\Gamma(M_R)} \xrightarrow{\text{rest.}} \underset{\text{sections of } M_R \text{ over } K.}{\underbrace{\Gamma(K, M_R)}} \overset{\text{Key lemma}}{\cong} H_n(M|K, R)$$

$$\Theta \longmapsto \alpha_K$$

Define a map $D_{M|K}: H^i(M|K, R) \to H_n(M,R)$ via $[\phi] \mapsto \alpha_K \frown [\phi]$, – the relative cap product.

Claim: If $K, K' \subseteq M$ are compact $K \supseteq K'$, then ....

$$\begin{tikzcd}
H^i(M|K',R) \arrow[dr, "D_{M|K'}"] \arrow[dd, "S_{K'K}"'] & \\
\arrow[r, phantom, "\circlearrowright"] & H_{n-i}(M,R) \\
H^i(M|K,R) \arrow[ur, "D_{M|K}"'] &
\end{tikzcd}$$

[Follows from naturality of $\frown$, see Hatcher pp. 245]

Cor: The maps $D_{M|K}$ induce a map

$$D_M : \varinjlim_{\substack{K \subseteq M \\ \text{compact}}} H^i(M|K,R) \to H_{n-i}(M,R)$$

| Sketch

Thm (3.35 Hatcher) D_M iso. whenever M is an R-oriented n-manifold (without boundary)

Rem: If M compact then $H^i_c(M,R) = H^i(M,R)$ so the thm. implies Poincaré Duality for closed n-manifolds.

Lem A: If $M = U \cup V$, U, V open, & $D_U, D_V, D_{U\cap V}$ iso $\to$ D_M iso.

Lem B: $M = \bigcup\{U_j \mid j \geq 1\}$ $U_1 \subseteq \cdots \subseteq U_n \subseteq \cdots \subseteq M$ open. Then if each D_{U_j} is an iso. $\to$ so is D_M.

For simplicity, assume $R = \mathbb{Z}$

Pf (Lem A): We have ....

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^i_c(U) \oplus H^i_c(V) \arrow[r, "\text{minus}"] \arrow[d, "D_U \oplus -D_V"] & H^i_c(M) \arrow[r] \arrow[d, "D_M"] & H^{i+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-i}(U\cap V) \arrow[r] & H_{n-i}(U) \oplus H_{n-i}(V) \arrow[r] & H_{n-i}(M) \arrow[r] & H_{n-i-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Check: diagram commutes (Hatcher pp 246)

Lem A follows from 5-lemma.

Pf (Lem B): $M = \bigcup\{U_j \mid j \geq 1\}$, nested & open. Since image of $\sigma \in S_{n-i}(M)$ compact & contained in finite union of U_j's and hence one of U_j's. Then

$$\varinjlim_{j \geq 1} C_{n-i}(U_j) = C_{n-i}(M)$$

$$\varinjlim_{j \geq 1} H_{n-i}(U_j) = H_{n-i}(M)$$

Likewise if $\Phi \in C^i(M)$ is supported in a compact $K_\Phi \subseteq M$ then K_Φ is contained in one of the U_j's.

Then $\varinjlim_{j\geq 1} C^i_c(U_j) = C^i_c(M)$

$$\varinjlim_{j\geq 1} H^i_c(U_j) = H^i_c(M)$$

One can check....

$$\begin{tikzcd}
H^i_c(U_j) \arrow[r, "\text{open inclusion}"] \arrow[d, "D_{U_j}"] \arrow[dr, phantom, "\circlearrowright"] & H^i_c(U_{j+1}) \arrow[d, "D_{U_{j+1}}"] \\
H_{n-i}(U_j) \arrow[r] & H_{n-i}(U_{j+1})
\end{tikzcd}$$

commutes. So the D_{U_j} induce a map

$$\underset{\substack{\| \\ D_M}}{\varinjlim D_{U_j}} : \underbrace{\varinjlim_{j\geq 1} H^i_c(U_j)}_{\substack{\| \\ H^i_c(M)}} \to \underset{\substack{\| \\ H_{n-i}(M)}}{\varinjlim_{j\geq 1} H_{n-i}(U_j)}$$

So if each D_{U_j} is an iso so is D_M $\square$

Pf (Sketch 3.31): 3 steps

1) True for $M \cong \mathbb{R}^n$

2) True for $M \cong U$, $U \subseteq \mathbb{R}^n$ open

3) arb. M

We have a comm. diagram

$$\begin{tikzcd}
H^i_c(\mathbb{R}^n) \arrow[r, "D_{\mathbb{R}^n}"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & H_{n-i}(\mathbb{R}^n) \arrow[d, "\wr"] \\
H^i_\Delta(\Delta,\partial\Delta^n) \arrow[r, "\pm[\Delta^n]\frown -"'] & H^\Delta_{n-i}(\Delta^n)
\end{tikzcd}$$

Depends on orientation choice

2 cases:

If $i \neq n$, all groups are 0 so trivial.

If $i = n$, then $H^i_\Delta(\Delta^n, \partial\Delta^n) = H^n_\Delta(\Delta^n, \partial\Delta^n)$ is a generated by $(\Delta^n)^*$. By def of $\frown$,

$$\pm\Delta^n \frown (\Delta^n)^* = (\Delta^n)^*(\Delta^n|_{[e_0,\dots,e_n]})$$

$$\Delta^n|_{[e_n]} = \pm \text{ last } \underbrace{\text{vector of}}_{\text{gen. for } H_0(\Delta^n)} \Delta^n$$

So $\pm\Delta^n \frown \delta -$ sends gen. $(\Delta^n)^*$ to a generator of H_0 then $\pm\Delta^n \frown -$ is an isomorphism. Then $D_{\mathbb{R}^n}$ is an iso. ▨ (1)

11/10/2016

Suppose $U \subseteq \mathbb{R}^n$ open. Write U as countable union $U = \bigcup_i U_i$ of convex open sets $U_i \subseteq \mathbb{R}^n$ - possible as $\mathbb{R}^n$ has a countable basis of convex open sets. Let $V_j = \bigcup_i^{j-1} U_i$. Then $\bigcup V_j = U$ & V_j's open and nested in U. Claim thm true for each V_j. Claim $\to$ have the prop. (D_{V_j} iso). Show by induction: base case simple enough. By def $V_j = \bigcup^{j-1} U_i$

$$V_j \cap U_j = (U_1 \cap U_j) \cup \cdots \cup (U_{j-1} \cap U_j)$$

convex

so $V_j \cap U_j$ union of convex open sets

By induction, can assume D_{V_j} and $D_{V_j \cap U_j}$ are iso. By (1), map D_{U_j} is an iso as U_j is convex open set so $U_j \cong \mathbb{R}^n$. So Lem A implies that $D_{\underbrace{V_j \cup U_j}_{V_{j+1}}}$ is an iso.

So each D_{V_j} is an iso for all indices.

$U = \bigcup V_j$; $V_1 \subseteq V_2 \subseteq \cdots\cdots$

Then Lem B gives D_U is an iso.

Now for (3). Write M as countable union $U_1 \cup \cdots$ where U_i open sets cont. in some coord neigh $\cong \mathbb{R}^n$. Manifold so countable basis. Define $V_j = \bigcup^{j-1} U_i$ and proceed as before. ▨

Intersection Product

M closed oriented n-manifold

Def: The intersection product is the map $\cdot : H_i M \times H_j M \to H_{i+j-n} M$ given by $(a,b) \mapsto a \cdot b$, where

$$a \cdot b \overset{\text{def}}{=} D_M(\underbrace{D_M^{-1}(a)}_{n-i} \smile \underbrace{D_M^{-1}(b)}_{n-j})$$

$$= [M] \frown (D_M^{-1}(a) \smile D_n^{-1}(b))$$

$$= ([M] \frown D_M^{-1}(a)) \frown D_M^{-1}(b)$$

$$= a \frown D_M^{-1}(b)$$

Properties:

(1) $a \cdot b = (-1)^{(n-i)(n-j)} b \cdot a$

(2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Intersection Pairing

M closed oriented n-manifold

Def: The intersection pairing is the map

$\cdot : H_i M \times H_{n-i} M \to \mathbb{Z}$

$(a,b) \mapsto a \cdot b$

Defined by ....

evaluation pairing

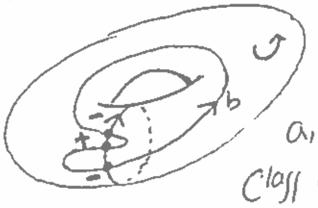
$$a \cdot b = \langle D_M^{-1}(a) \smile D_M^{-1}(b), [M] \rangle$$

$$= \langle D_M^{-1}(b), [M] \frown D_M^{-1}(a) \rangle$$

$$= \langle D_M^{-1}(b), a \rangle$$

$a \cdot b$ is related to intersections of submanifolds

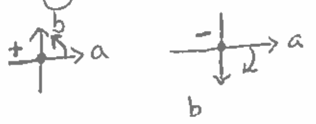
Ex: $M = S^1 \times S^1$



a, b rep. hom. Class in $H_i M$

2 types intersection points.

Uses orientation to distinguish



$a \cdot b$ = signed count of int points

$= -1 + 1 - 1 = -1$

Note:

$a \cdot b = \langle D_M^{-1}(b), a \rangle$ so

The lin. form $\langle D_M^{-1}(b), - \rangle$ is given by taking intersection pairing with b.

M closed oriented n-manifold

Thm: If $\beta \in H^i M$ is non-torsion then $\exists\ \alpha \in H^{n-i} M$ such that $\alpha \smile \beta \neq 0$.

Pf: By UCT, we have a ses

$$0 \to \underbrace{\text{Ext}(\dots)}_{\text{torsion of } H_{n-i}M} \to H^i M \to \text{Hom}(H_i M, \mathbb{Z}) \to 0$$
$$\beta \longmapsto \langle \beta, - \rangle$$

If β is nontorsion, then $\langle \beta, - \rangle \neq 0$

so $\exists$ an $a \in H_i M$ such that $\langle \beta, a \rangle \neq 0$. Define $\alpha = D_M^{-1}(a) \in H^{n-i} M$.

$$\text{Then } \langle \alpha \smile \beta, [M] \rangle = \langle \beta, [M] \frown \alpha \rangle$$
$$= \langle \beta, a \rangle$$
$$\neq 0$$

So that $\alpha \smile \beta \neq 0$. ☒

Let $FH^i M = H^i M$ torsion

Cor: The pairing

$$FH^{n-i} M \times FH^i M \to \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto \langle \alpha \smile \beta, [M] \rangle$$

is non degenerate.

Cor: The pairing

$$FH_i M \times FH_{n-i} M \to \mathbb{Z}$$
$$(a, b) \longmapsto a \cdot b$$

is non degenerate

Rem: Analogous results hold for closed R-oriented n-manifolds.

Manifolds with Boundary

Def: An n-manifold with boundary is a 2nd countable Hausdorff space such that each $x \in M$ has open neigh $\mathcal{U}$ with $\mathcal{U} \cong \mathbb{R}^n$ or $\mathcal{U} \cong \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$

$x \in M$ called int. point if it has neigh $\cong \mathbb{R}^n$ and boundary point otherwise.

Def: The boundary of M is the subspace $\partial M = \{x \in M \mid x \in \text{boundary}\}$

Fact:

1) ∂M $(n-1)$ manifold without boundary.

2) $M \setminus \partial M$ is an n-manifold without boundary

3) $x \in \partial M \to H_n(M|x) \cong \underset{\overset{\parallel}{0}}{H_n(\mathbb{R}^n, \mathbb{R}^n_+ \setminus 0)}$

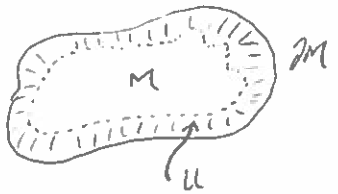
4) $x \notin \partial M \to H_n(M|x) \cong H_n(\mathbb{R}^n|0) \cong \mathbb{Z}$

3,4 give a way to tell if in boundary or not & show $f: M \to N$ takes ∂M to ∂N.

Def: A collar neigh of ∂M is an open neigh $\mathcal{U} \supseteq \partial M$ such that

$$\mathcal{U} \cong \partial M \times [0,1)$$

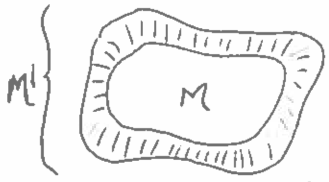
via a homeo. that takes $\partial M \subseteq \mathcal{U}$ to $\partial M \times 0$



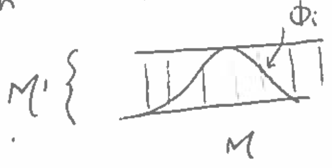
Prop: (Hatcher 3.42) If M is a compact manifold with boundary, then ∂M has a collar neigh.

Pf: (sketch) M' be "external collar"

$$M' = \frac{M \sqcup (\partial M \times [0,1])}{\partial M = (\partial M \times 0)}$$



Define a homeo. $h: M \to M'$ by using suitable functions $\phi_i : \partial M \to [0,1]$ such that $\sum \phi_i(x) = 1$ for $x \in \partial M$



Define $\mathcal{U} = h^{-1}(\partial M \times (0,1])$

Then $\mathcal{U}$ is a collar neigh. of $\partial M \subseteq M$ ▨

Def: R orientation on a manifold with boundary is an R-orientation on $M \setminus \partial M$.

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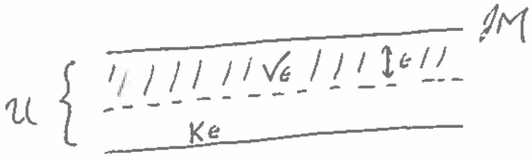
M compact R-oriented n-man. with boundary (assume $R=\mathbb{Z}$)

$\mathcal{U} \cong \partial M \times [0,1)$ collar neigh ∂M

$\underline{\cup}$

$V_\epsilon \cong \partial M \times (0,\epsilon)$; $0<\epsilon<1$



Then $(M, \partial M) \underset{h.e.}{\simeq} (M \setminus \partial M, V_\epsilon)$

In $M\setminus\partial M$, V_ϵ is the complement of a compact subspace $V_\epsilon \subseteq M \setminus \partial M$.

So $H_n(M,\partial M) = H_n(M\setminus\partial M, V_\epsilon)$

$= H_n(M\setminus\partial M, (M\setminus\partial M)\setminus K_\epsilon)$

$= H_n(M\setminus\partial M \,|\, K_\epsilon)$

by key lemma from earlier, $H_n(M\setminus\partial M|K_\epsilon)$ contains a relative fundamental class α_{K_ϵ} corresponding to the given orientation on M.

So we can define a map....

$$D_M : \underset{\overset{\parallel}{H^i(M\setminus\partial M|K_\epsilon)}}{\underline{H^i(M,\partial M)}} \to H_{n-i}(M)$$

$$[\phi] \longmapsto \underset{\overset{\uparrow}{\text{relative cap product}}}{\alpha_{K_\epsilon} \frown [\phi]}$$

Thm: D_M is an isomorphism whenever M is a compact oriented n-manifold with boundary

Pf (sketch): We saw $H^i(M,\partial M) = H^i(M\setminus\partial M \mid K_\epsilon)$ for $0<\epsilon<1$. So we can write $H^i(M,\partial M)$ as

$$H^i(M,\partial M) = \varinjlim_{\epsilon} H^i(M\setminus\partial M \mid K_\epsilon)$$

$$\cong \varinjlim_{\substack{K\subseteq M\setminus\partial M\\ \text{compact}}} H^i(M\setminus\partial M \mid K)$$

$$= H^i_c(M\setminus\partial M)$$

Check: Under these isomorphism, the D_M corresponds to the Poincaré duality map $H^i_c(M\setminus\partial M) \to H_n(M\setminus\partial M) \cong H_{n-i}(M)$, and since the latter map is an isomorphism, so is D_M. □

Generalization

M compact n-manifold, oriented w/ boundary

$A,B \subseteq \partial M$ compact $(n-1)$-submanifolds with boundary $\ni$ $A\cup B = \partial M$, $A\cap B = \partial A = \partial B$

Then $H^i(M,A) \cong H_{n-i}(M,B)$

This is called Poincaré-Lefschetz Duality - see Hatcher pp 254.

2 special cases:

(1) $A = \partial M$, $B = \emptyset \to$
$H^i(M,\partial M) \cong H_{n-i}(M)$

(2) $A = \emptyset$, $B = \partial M \to$
$H^i(M) \cong H_{n-i}(M,\partial M)$

Intersection Pairing

M compact oriented n-manifold with boundary

Def: The int. pairing is the map

$$\cdot : H_i(M,\partial M) \times H_{n-i}(M) \to \mathbb{Z}$$

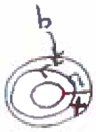
$$(a,b) \mapsto a\cdot b$$

where

$$a\cdot b = \langle D_M^{-1}(a) \smile D_M^{-1}(b), \underset{\overset{\parallel}{\alpha_{K_\epsilon}}}{\overset{\overset{\text{rel. fund class}}{\downarrow}}{[M]}} \rangle$$

$$= \langle D_M^{-1}(b), a \rangle$$

Ex: M = annulus $= S^1 \times I$



$H_1(M,\partial M) = \text{span}\{[a]\} \cong \mathbb{Z}$

$H_1(M) = \text{span}\{[b]\} \cong \mathbb{Z}$

$a\cdot b = +1$

Alexander Duality

Def: X is locally contractible if it has a basis consisting of contractible open subsets
↳ w.r.t. subspace top.

Thm (3.44 Hatcher): If M compact, $\neq \emptyset$, $\neq S^n$, and locally contractible, then

$$\underbrace{\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K)}_{\text{Alexander Duality}}$$

Pf (sketch): Need two facts

1) For $K \subseteq S^n$ arb....

$$H_c^{n-i}(S^n \setminus K) = \varinjlim H^{n-i}(S^n \setminus K, U \setminus K)$$

where limit over all open neigh $U \supseteq K$.

[open neigh U of K are comp. of compact subspaces $K' \subseteq S^n \setminus K$ so...

$$\varinjlim_{\substack{U \supseteq K \\ \text{open neigh}}} H^{n-i}(S^n \setminus K, U \setminus K) = \varinjlim_{\substack{K' \subseteq S^n \setminus K \\ \text{compact}}} H^{n-i}(S^n \setminus K / K')$$]

2) For $K \subseteq S^n$ as in the thm,

$$\tilde{H}^{n-i-1}(K) \cong \varinjlim \tilde{H}^{n-i-1}(U)$$

where limit is taken over all open neigh $U \supseteq K$

This fact is more difficult & uses Thm A.7, Hatcher pp 255.

For simplicity $i \neq 0$, then

$$\tilde{H}_i(S^n \setminus K) = H_i(S^n \setminus K)$$

$\cong H_c^{n-i}(S^n \setminus K)$; Poincaré Duality

$\cong \varinjlim H^{n-i}(S^n \setminus K, U \setminus K)$; (1)

$\cong \varinjlim H^{n-i}(S^n, U)$; excision

$\cong \varinjlim \tilde{H}^{n-i-1}(U)$; les for (S^n, U, pt) uses $i \neq 0, -1$

$\cong \tilde{H}^{n-i-1}(K)$; (2) ∎

Cor: If $K \subseteq \mathbb{R}^n$ is compact & locally contractible $\rightarrow \tilde{H}^i(K)$ is 0 for $i \geq n$ and torsion free for $i \geq n-1$.

Pf: View K as a subspace $K \subseteq S^n = \mathbb{R}^n \cup \{\infty\}$ $\rightarrow$ by Alex. Duality

$$\tilde{H}^i(K) \cong \tilde{H}_{n-i-1}(S^n \setminus K) = \begin{cases} 0, & i \geq n \\ \text{torsion free}, & i \geq n-1 \end{cases}$$ ∎

Ex: M a closed nonorientable surface

Then $H_1(M;\mathbb{Z})$ has 2-torsion →

by UCT $H^2(M;\mathbb{Z})$ has 2-torsion

$$\begin{aligned} &H^2(M;\mathbb{Z}) \\ &\quad \parallel \\ &\tilde{H}^2(M;\mathbb{Z}) \end{aligned}$$

So M cannot be embedded in $\mathbb{R}^3$ (by the cor.). ↓

eg Klein bottle

Ex: $K \subseteq S^2$ a subspace homeomorphic to S^1 (so K a simple closed curve)

Then $\tilde{H}_0(S^2 \setminus K)$

$$\cong \tilde{H}^{2-0-1}(K) \cong \tilde{H}^1(S^1) \cong \mathbb{Z}$$

Then $H_0(S^2\setminus K) \cong \mathbb{Z}^2$ →

Then $S^2 \setminus K$ has two path components.

"Jordan curve thm."

The same proof works for any $K \subseteq S^n$ which is homeomorphic to S^{n-1}.

Duality ~~pairing~~ pairing (for $K \cong S^1 \subseteq S^2$)

$$P : \tilde{H}_1(K) \times \tilde{H}_0(S^2 \setminus K) \to \mathbb{Z}$$

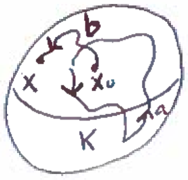
$$(a,x) \longmapsto P(a,x) = \langle \beta, \alpha \rangle$$

"the dual Alex. dual of x"

↳ not canonical.

$x_0 \in S^2 \setminus K$ basepoint

$b \subseteq S^2$ path from x_0 to x



$a \in \tilde{H}_1(K)$

Turns out:

$P(a,x) = \langle \beta, a \rangle$

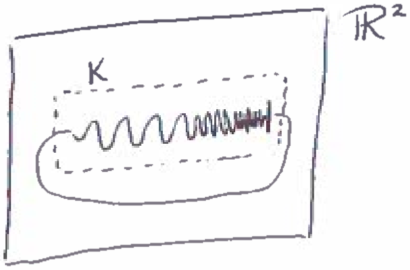
↖ Alex. dual

$= \pm$ "$a \cdot b$"

$= -1$

in ex.

Ex: $K \subseteq \mathbb{R}^2 \subseteq S^2$ the "closed up" closed topologist sine curve



given by....

$\{(x,y) \mid y = \sin(1/x),\ x \in [-1,0)\}$

$\cup \{(x,y) \mid x = 0,\ y \in [-1,1]\}$

$\cup \{(x,y) \mid x \in [0,1],\ y = 0\}$

Can show....

$S^2 \setminus K$ has two path components →

$\tilde{H}_0(S^2\setminus K) \cong \mathbb{Z} \neq \tilde{H}^1(K) = 0$

↖ Not same ↗

So Alex. Duality does not hold for K. Why? K not locally contractible.

Ex: $\underset{\overset{\wr\shortparallel}{S^2}}{K} \subseteq \mathbb{R}^3 \subseteq S^3$

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Ex: $K \subseteq \mathbb{R}^3 \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$

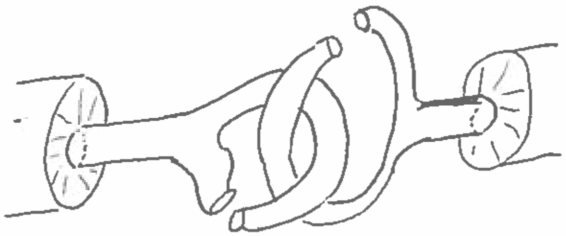
the Alexander horned 2-sphere:

Start with

$\subseteq \mathbb{R}^3 \subseteq S^3$

$S \times I$ ↗ "Horn" ↖ normally flattened to a sphere

Partially fill in holes by attaching horns



And just continue to iterate this process: Eventually get 2-sphere with inf many holes - but these holes become points. So inf. many missing points. These points become a Cantor set. in $\mathbb{R}^3$. After filling in missing points one gets embedded 2-sphere $S \subseteq \mathbb{R}^3 \subseteq S^3$

Alexander Duality still holds.

- $S^3 \setminus S$ has 2 path components
- The "inner component" is $\cong$ to an open ball
- $\tilde{H}_1(S^3 \setminus S) \cong \tilde{H}^{3-1-1}(S) \cong H^1(S^2) = 0$

So "outer component" also has trivial first homology. But $\pi_1(\text{outer component}) \neq 0$: look at loop around one horn - outermost one. loop is ~~non~~zero in H_1 - bounds a genus two surface (oriented). in $S^3 \setminus S$.

Ex: $K \subseteq S^3$ a subspace homeo. to a disjoint union of m circles:

$S^1 \sqcup \underbrace{S^1 \sqcup \cdots}_{m} \sqcup S^1$

Such a K is called an m-component link. If $m=1$, then K is called a knot.

$$\underset{\overset{\shortparallel}{H_1}}{\tilde{H}_1(S^3 \setminus K)} \cong \tilde{H}^{3-1-1}(K) \cong H^1(S^1 \sqcup \cdots \sqcup S^1) \cong \mathbb{Z}^m$$

What are generators of $H_1(S^3 \setminus K)$?

$K = K_1 \cup K_2 \subseteq \mathbb{R}^3 \subseteq S^3$

meridians

μ_1, μ_2 generate $H_1(S^3 \setminus K)$

What is the Alexander pairing?

$$P: H_1(K) \times H_1(S^2 \setminus K) \to \mathbb{Z}$$

$$\mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \qquad \mathbb{Z}\mu_1 \oplus \mathbb{Z}\mu_2$$

λ_1, λ_2 called longitudes

The Alexander pairing is given by:

$$P(\lambda i, \mu_1) = \lambda_i \cdot D \longleftarrow \text{linking number of } \lambda_i \,\&\, \mu_1$$

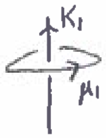
$$= \begin{cases} 1, & i=1 \\ 0, & i=2 \end{cases}$$

$H_1(S^3 \setminus K) \cong \mathbb{Z}^m$

but $\pi_1(S^3 \setminus K)$ can be complicated

Linking Numbers (Informal)

$K = K_1 \cup K_2 \subseteq \mathbb{R}^3 \subseteq S^3$ an oriented 2-component link. Let μ_1 be a meridian for K_1 and orient it as follows



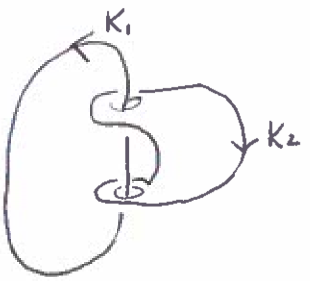
Then $[K_2] = n[\mu_1]$ for some $n \in \mathbb{Z}$

Def: n is called the linking number of K_1 & K_2

Notations:

$lk(K_1, K_2) = n =$ # times K_2 winds around K_1

Ex:



Diagrammatic Definitions:

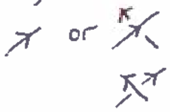
$K = K_1 \cup K_2 \subseteq S^3$

oriented 2-component link sufficiently nice (eg smooth)

represent $K = K_1 \cup K_2$ by a generic picture in the plane

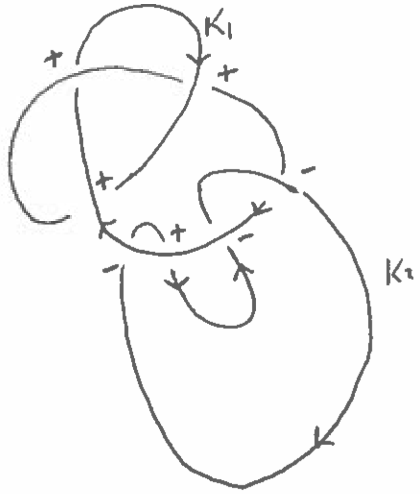
looks like

by 'zooming'

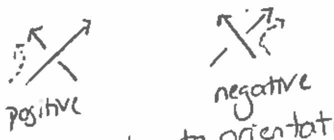


So don't want....

These 'nice' pictures are called a "link diagram"



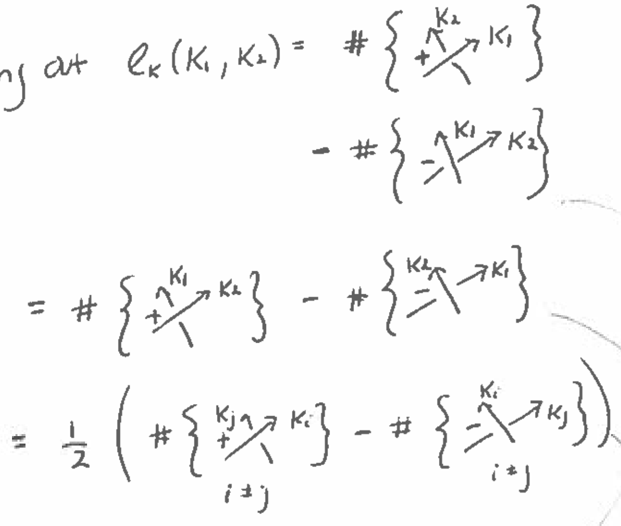
2 types of crossings:



difference due to orientations

"upper level to lower level in direction of orientation - like highway crossings."

Turns out $\ell k(K_1,K_2) =$



In the example $\ell k(K_1,K_2) = 1 - 2$

$= 0 - 1$

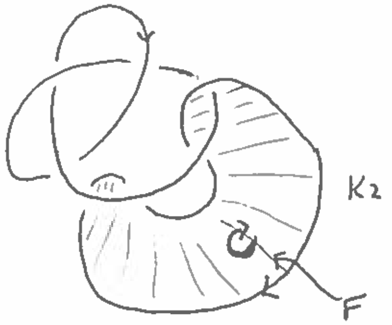
$= \frac{1-3}{2}$

Geometric Definition

Choose an oriented surface $F \subseteq S^3$ whose (oriented) boundary is $\partial F = K_2$

Turns out:

$\ell k(K_1,K_2) = K_1 \cdot F$



$K_1 \cdot F = -1$

Definition in terms of degree

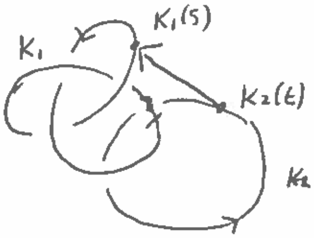
Think of K_1, K_2 as simple closed disjoint curves

$K_1, K_2 : S^1 \to \mathbb{R}^3$

Define a map

$\phi : S^1 \times S^1 \to S^2$

by $\phi(s,t) = \dfrac{K_1(s) - K_2(t)}{\| K_1(s) - K_2(t) \|} \in S^2$



$\ell k(K_1,K_2) = \deg \phi$

Explicitly, $\phi_*([S^1 \times S^1]) \in H_2(S^2)$
$= \mathbb{Z}\langle [S^2] \rangle$

Can be written as $\phi_*([S^1 \times S^1]) = n[S^2]$ for some $n \in \mathbb{Z}$: $\ell k(K_1,K_2) = n$

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$K = K_1 \cup K_2 \subseteq \mathbb{R}^3 \subseteq S^3$

oriented 2-component link

Regard K_1, K_2 as simply connected curves $K_1, K_2 : S^1 \to \mathbb{R}^3$

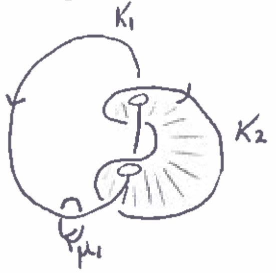
Define $\Phi: S^1 \times S^1 \to S^2$ via

$$\Phi(s,t) = \frac{K_1(s) - K_2(t)}{\| K_1(s) - K_2(t) \|} \in S^2$$

Define $\ell k(K_1, K_2) = \deg \Phi$. More explicitly: consider

$$\Phi_*: \underset{\mathbb{Z}\langle [S^1 \times S^1] \rangle}{\overset{}{\underset{\|}{H_2(S^1 \times S^1)}}} \to \underset{\mathbb{Z}\langle [S^2] \rangle}{\underset{\|}{H_2(S^2)}}$$

Then $\Phi_*([S^1 \times S^1]) = n[S^2]$ and n is the linking number $\ell k(K_1, K_2) = n$.



$\ell k(K_1, K_2) = 2$

Analytic Definition

K_1, K_2 smooth $\subseteq \mathbb{R}^3$

$$\ell k(K_1, K_2) = \frac{1}{4\pi} \int_{S^1} \int_{S^1} \frac{K_1(s) - K_2(t)}{\| K_1(s) - K_2(t) \|^3} \cdot$$

$(dK_1 \times dK_2)$

$$= \frac{1}{4\pi} \iint_{S^1\, S^1} \frac{\det(K_1(s) - K_2(t), K_1'(s), K_2'(t))}{\| K_1(s) - K_2(t) \|^3} \, ds\, dt$$

$\emptyset \neq C \subsetneq S^3$ compact, locally contractible

Then Alexander pairing:

$p: H_1(C) \times H_1(S^3 \setminus C) \to \mathbb{Z}$

Turns out $p(a,b) = \pm \ell k(a,b)$

Smooth Manifolds

M an n-manifold (without boundary)

Recall:

Def: A chart is a homeo. $\Phi: \mathcal{U} \to \mathcal{U}'$ where $\mathcal{U} \underset{\text{open}}{\subseteq} M$ & $\mathcal{U}' \underset{\text{open}}{\subseteq} \mathbb{R}^n$

Def: A smooth atlas on M is a family $\mathcal{A}$ of charts such that

1) Domains $\mathcal{U}_\Phi$ of the $\Phi \in \mathcal{A}$ cover M

2) If $\Phi, \Psi \in \mathcal{A} \to \Psi \circ \Phi^{-1} : \Phi(\mathcal{U}_\Phi \cap \mathcal{U}_\Psi) \to \Psi(\mathcal{U}_\Phi \cap \mathcal{U}_\Psi)$ is smooth, ie C^∞

Remark: If $\mathcal{A}$ is a smooth atlas, then $\mathcal{A}$ can be enlarged to a maximal (wrt to inclusions) by adding all charts $\psi \ni \psi \circ \phi^{-1}$ is smooth for $\phi \in \mathcal{A}$.

Def: A smooth n-manifold is a n-manifold together with a $\underbrace{\text{maximal smooth atlas}}_{\text{called smooth structure}}$ $\mathcal{A}$ on M.

Ex: $M = \mathbb{R}^n$

Let $\mathcal{A} = \left\{ \phi : \mathcal{U} \to \mathcal{U}' \;\middle|\; \begin{array}{l} \mathcal{U}, \mathcal{U}' \subseteq \mathbb{R}^n \text{ open} \\ \phi \text{ diffeo} \end{array} \right\}$

Then $\mathcal{A}$ is a smooth structure on $\mathbb{R}^n$, called the standard smooth structure on $\mathbb{R}^n$.

$\mathcal{B} = \{ id : \mathbb{R}^n \to \mathbb{R}^n \}$ is also a smooth atlas (just not maximal).

Ex: $M = S^n$ $\quad \mathcal{U}_1 = S^n \setminus$ north pole

$\mathcal{U}_2 = S^n \setminus$ south pole

$\Phi_i : \mathcal{U}_i \to \mathbb{R}^n$ stereographic projections

Then $\mathcal{A} = \{ \Phi_1, \Phi_2 \}$ is a smooth structure on S^n, representing the "standard smooth structure".

Smooth Maps: M^m, N^n smooth manifolds

$\mathcal{A}, \mathcal{B}$ smooth structures

Def: A cont. map $f : M \to N$ is called smooth if $\underbrace{\psi \circ f \circ \phi^{-1}}_{\mathbb{R}^m \to \mathbb{R}^n}$ is smooth for all $\phi \in \mathcal{A}, \psi \in \mathcal{B}$

Def: A smooth map $f : M \to N$ is called a diffeo. if it is invertible with smooth inverse.

Ex: M top. n-manifold

$\mathcal{A}$ a smooth structure on M

$f : M \to M$ homeo.

Define $\mathcal{B} = \{ \underbrace{\phi \circ f^{-1}}_{\substack{\text{regarded as chart with} \\ \text{domain } f(\mathcal{U}_\phi)}} \mid \phi \in \mathcal{A} \}$

One can check $\mathcal{B}$ is also a maximal smooth atlas on M. f can be viewed as a diffeo. $f : (M, \mathcal{A}) \to (M, \mathcal{B})$

Q: Does every top. manifold admit a smooth structure?

Q: If it exists, is it unique (up to $\underset{\text{diffeo.}}{\text{equivalence}}$)

Yes to both in dim ≤ 3 and no in dim > 3.

1920s: Radó, every 2 manifold has a unique PL structure.

↳ up to equivalence.

coord. change maps are piecewise lin

∃ triangulation of $\Phi(\mathcal{U}_\phi \cap \mathcal{U}_\psi)$ ∋ restriction of $\psi \circ \phi^{-1}$ to each simplex is affine.

Related result Hatcher 2013
arxiv.org/abs/1312.3518

1940: Whitehead, every smooth manifold has smooth Whitehead compatible PL structure

1950s: Moise, Bing: Every 3 manifold has unique PL structure up to equiv.
Newer proof AJS Hamilton 1974.

1958: Milnor, ∃ exotic smooth structure on S^7. ↖ not diffeo. to standard structure

1962: Stallings, $n \neq 4$ then $\mathbb{R}^n$ admits a unique smooth structure up to diffeo.

1960s: Smale, Stallings, Zeeman
S^n admits a unique PL structure for $n \geq 5$.

Map $\begin{Bmatrix} \text{smooth} \\ \text{manifolds} \end{Bmatrix} \to \begin{Bmatrix} \text{PL} \\ \text{manifolds} \end{Bmatrix}$
is not injective in dim 7

1960s: Munkres, Hirsch-Mazur
In dim ≤ 6, every PL manifold admits unique, up to equiv., Whitehead compatible smooth structure

So PL $\rightleftarrows$ smooth in dim ≤ 6 so

Cor: Every 3 manifold admits a unique smooth structure.

1969: Kirby-Siebenmann, in each dim ≥ 4, ∃ a manifold which does not admit a PL structure (hence no smooth structure)

In fact, in dim ≥ 5 ∃ a single obstruction $K(M) \in H^4(M; \mathbb{Z}_2)$ to the existence of a PL structure on M. If $K(M) = 0 \to \exists$ $|H^3(M; \mathbb{Z}_2)|$ distinct PL structures on M.

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Every 3-manifold has a unique smooth structure up to diffeo.

$\begin{Bmatrix} \text{smooth} \\ \text{manifold} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{PL} \\ \text{manifold} \end{Bmatrix}$

For dim ≤ 6, this is a bijection

For dim ≥ 5, there exist manifolds with no PL structure (hence no smooth structure)

1982: Freedman ∃ a simply connected 4-manifold with no PL structure (hence no smooth structure).

F. Quinn: Every noncompact connected 4-manifold has a smooth structure.

1983: Donaldson ∃ exotic smooth structures on $\mathbb{R}^4$.

1980s: Casson (?) $\exists$ a 4-manifold which cannot be triangulated by a locally finite simplicial complex.

2013: Manolescu $\uparrow$ same true in dim 5 or greater

arXiv.org labs/1303.2354

smooth structures on S^n and on $\mathbb{R}^n$.

n	1	2	3	4	5	6	7	8	9	10	11 .....
S^n	1	1	1	?	1	1	28	2	8	6	992
$\mathbb{R}^n$	1	1	1	∞	1	1	1	1	1	1	1

Smooth 4-dim. Poincaré conjecture: ? = 1 still open. (not even known finite).

$D^4 \subseteq \mathbb{R}^4$ closed 4-ball

$S^3 = \partial D^4$, $K \subseteq S^3$ a smooth knot

Def: K is called smoothly slice if $\exists$ a smoothly embedded disk $D \subseteq D^4 \ni \partial D = K$

Really $f: D \hookrightarrow D^4$ inj., cont., smooth, or homeo onto image, or df inj at all points

Def: K is called topologically slice if $\exists$ a topologically flat topologically embedded disk $D \subseteq D^4 \ni \partial D = K$.

top flat: D has neigh. in D^4 $\cong D \times \mathbb{R}^4$

Rem: Smoothly slice $\rightsquigarrow$ top slice by Tubular neigh. theorem.

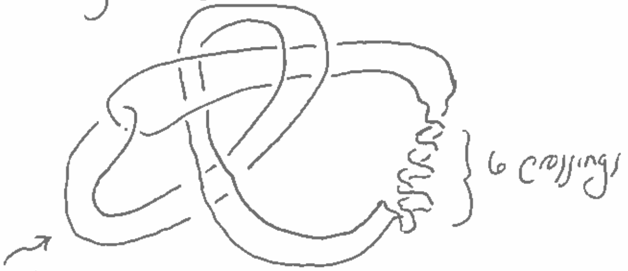
1983: Freedman $\Delta_K(t) = 1$

$\uparrow$ Alexander poly.

then K top. sliced.

1980: Casson, Akbulut Freedman & Donaldson imply existence of knots which are topologically sliced but not smoothly slice

Turns out: This implies that there are exotic smooth structures on $\mathbb{R}^4$.



untwisted positive whitehead double of right-handed trefoil.

2004: Rasmussen comb. obstruction to a knot being smoothly slice

Tangent Vectors

M smooth manifold

$\mathcal{A}$ max smooth atlas

$x \in M$

Def: A tangent vector to M at x is an equivalence class of smooth curves $\gamma: \mathbb{R} \to M \ni \gamma(0) = x$ where $\gamma \sim \eta$ iff

$$(\phi \circ \gamma)'(0) = (\phi \circ \eta)'(0)$$

for any chart $\phi \in \mathcal{A}$ whose domain includes x.

Rem: γ as above; $\phi, \psi \in \mathcal{A}$ with domain cont. x.; $v = (\phi \circ \gamma)'(0)$, $w = (\psi \circ \gamma)'(0)$ $\in \mathbb{R}^n$. Then

$$w_i = \sum_{j=1}^{n} \frac{\partial(\psi \circ \phi^{-1})_i}{\partial x_j}\Big|_{\phi(x)} v_j$$

Can prove using the Chain Rule.

In short, $w = \underbrace{d}_{\text{total differential of } \psi\circ\phi^{-1} \text{ at point } \phi(x)}(\psi \circ \phi^{-1})(v)$ $(*)$

Def: A tangent vector to M at x is an equivalence class of pairs (v, ϕ) where $v \in \mathbb{R}^n$ and $\phi \in \mathcal{A}$ is a chart whose domain contains x and where $(v, \phi) \sim (w, \psi)$ iff $*$ holds

Directional Derivatives

$\gamma: \mathbb{R} \to M$ a smooth curve $\ni \gamma(0) = x$; $f: M \to \mathbb{R}$ smooth function.

Def: Directional derivative of f along γ at x is...

$$D_\gamma(f) \overset{def}{=} (f \circ \gamma)'(0) \in \mathbb{R}$$

"rate of change of f in direction of γ at x."

$$= \underbrace{d}_{\text{total diff. of } f\circ\phi^{-1} \text{ at } \phi(x).}(f \circ \phi^{-1})((\phi \circ \gamma)'(0))$$

Rem: D_γ only depends on tangent vector at x represented by γ.

Note: $f, g: M \to \mathbb{R}$ smooth functions

$$D_\gamma(fg) = (fg \circ \gamma)'(0) = \frac{d}{dt}\Big(f(\gamma(t))\, g(\gamma(t))\Big)\Big|_{t=0}$$

$$= (f\circ\gamma)'(0)\, g(\gamma(0)) + f(\gamma(0))\, (g \circ \gamma)'(0)$$

$$= D_\gamma(f) g(x) + f(x) D_\gamma(g)$$

Fact: Every linear functional $D: C^\infty(M, \mathbb{R}) \to \mathbb{R}$ satisfying

** $D(fg) = D(f)g(x) + f(x)D(g)$

is of the form $D = D_\gamma$ for smooth curve $\gamma: \mathbb{R} \to M \ni \gamma(0) = x$.

Def: A tangent vector to M at x is a linear functional $D: C^\infty(M,\mathbb{R}) \to \mathbb{R} \ni$ ** holds

Def: The tangent space of M at x is the set •:

$$T_xM = \{\text{tangent vectors to } M \text{ at } x\}$$

Properties: M n-manifold

(1) T_xM is an $\mathbb{R}$ vector space

(2) Any chart $\phi \in \mathcal{A}$ whose domain contains x gives rise to an iso. $T_xM \cong \mathbb{R}^n$ basis for T_xM

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \in T_xM$$

where $$\frac{\partial}{\partial x_i}(f) = \underbrace{\frac{\partial f\circ\phi^{-1})}{\partial x_i}}_{\text{partial deriv. at } \phi(x)} \in \mathbb{R}$$

Differentials M, N smooth manifolds

$\Phi: M \to N$ smooth map ; $x \in M$, $y = \Phi(x) \in N$

Def: The differential of Φ at x is the map

$$\Phi_*: T_xM \to T_yN$$

defined to be $$\Phi_*([\gamma]) = [\Phi\circ\gamma]$$

Equivalently, $\Phi_*(D_\gamma) = D_{\Phi\circ\gamma}$

Properties:

(1) If $D \in T_xM$ and $g \in C^\infty(N,\mathbb{R})$ then $\Phi_*(D)(g) = D(g\circ\Phi)$

(2) $\Phi_*: T_xM \to T_yN$ is $\mathbb{R}$-linear

By def $\begin{cases} (3)\ (1_M)_* = 1_{T_xM} \\ (4)\ (\phi_1\circ\phi_2)_* = \phi_{1*}\phi_{2*} \end{cases}$

(5) In local coord., Φ_* is given by

$$\left(\frac{\partial(\psi\circ\Phi\circ\alpha^{-1})_i}{\partial x_j}\right)_{i,j}$$

where α, ψ are smooth charts on M and N whose domain includes x and $y = \Phi(x)$, respectively.

Notation: $\Phi_* = d\Phi$ $(= d\Phi_x)$

Assignment $(M,x) \mapsto T_xM$

$\underset{\text{smooth}}{\Phi: (M,x) \to (N,y)} \mapsto d\Phi_x : T_xM \to T_yN$

is a functor from the category of pointed smooth manifolds to the category of $\mathbb{R}$-vector spaces.

Submanifolds and transversality:

M smooth n-manifold ; $k \le n$

Def A smooth submanifold of M is a subspace $N \subseteq M \ni$ for all $x \in N$ $\exists$ smooth chart $\phi: U \to \mathbb{R}^n \ni x \in U$, $\phi(U) = \mathbb{R}^n$; $\phi(N \cap U) = \mathbb{R}^k \subseteq \mathbb{R}^n$

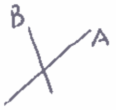
Note: Inclusion $i : N \hookrightarrow M$ induces an inclusion $i_x : T_xN \hookrightarrow T_xM$ for $x \in N$

Def: $A, B \subseteq M$ are transverse if $x \in A \cap B$, $T_xM = \underbrace{T_xA + T_xB}_{\text{sum of subspaces}}$

Note: If $A \cap B = \emptyset \rightsquigarrow A, B$ are transverse.

Ex: $M = \mathbb{R}^2$

$A, B \subseteq M$ smooth s.c. curves



transverse

not transverse

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M smooth manifold $k \le n$

k-submanifold $N \subseteq M$ which locally looks like $\mathbb{R}^k \subseteq \mathbb{R}^n$

$A, B \subseteq M$ submanifold

Def: $A, B \subseteq M$ are transverse if for $x \in A \cap B$

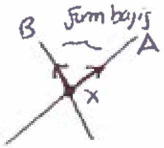
$$T_xM = T_xA + T_xB$$

Rem: $N \subseteq M$ smooth submfd & $x \in N$ $\rightsquigarrow$ inclusion induces an inclusion $T_xN \hookrightarrow T_xM$

Note: If $A \cap B = \emptyset$, $\rightsquigarrow$ A, B transverse.

Ex: $M = \mathbb{R}^2$

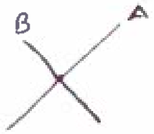
$A, B \subseteq M$ smooth simple closed curves



not transverse

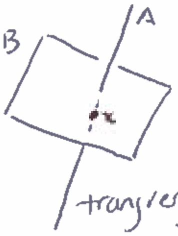
Ex: $M = \mathbb{R}^3$; $A, B \subseteq M$ smooth simple closed curves

transverse



not transverse dim 2 but M dim 3

Ex: $M = \mathbb{R}^3$; $A \subseteq M$ smooth s.c. curve $B \subseteq M$ smooth embedded closed surface



transverse

not transverse

$x \in A \cap B$

Thm: If $A, B \subseteq M$ are transverse then for $x \in A \cap B$, $\exists$ a smooth chart $\phi : U \to \mathbb{R}^n \ni x \in U$, $\phi(U) = \mathbb{R}^n$ & $\phi(U \cap A)$ & $\phi(U \cap B)$ are linear subspaces of $\mathbb{R}^n$.

Pf: Bredon, "Top. & Geo."
Thm 7.7 pp 84 ⊠

Cor: If A, B are transverse then $A \cap B$ is also a submanifold of M.

$\dim(A \cap B) = \underbrace{\dim A + \dim B - \dim M}$

follows from basic lin. alg.

Thm: If $A, B \subseteq M$ are smooth compact submanifolds then $\exists$ a smooth map

$F : A \times [0,1] \to M$

such that

1) $F|_{A \times \{t\}}$ is a smooth embedding

2) $F|_{A \times \{0\}} = 1_A$

3) $F(A \times \{1\})$ is transverse to B

So $A \simeq$ transverse submanifold to B.

Moreover, $F(A \times \{1\})$ can be chosen to be arb. 'close' to B.

$\downarrow$

Bredon: pp 115, Cor 5.4.

M oriented smooth closed n-manifold

$A \subseteq M$ " ________________ " $\dim i$

$B \subseteq M$ " ________________ " $\dim j$

If $A, B \subseteq M$ transverse $\to A \cap B$ oriented smooth closed $(i+j-n)$-submanifold

$\ulcorner$ $i+j<n \to A, B$ disjoint by dim so transverse $\lrcorner$

Thm: Let $[A]$, $[B]$, $[A \cap B]$ be the images of the fundamental classes of A, B, $A \cap B$ in $H_*(M)$. Then

$$[A \cap B] = [A] \cdot [B]$$

where $[A] \cdot [B] = D_M(D_M^{-1}([A]) \smile D_M^{-1}([B]))$

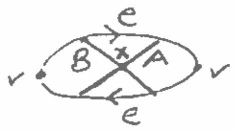
Pf: Bredon Thm 11.9, pp 372 ⊠

Rem: without orientation assumptions, thm holds for $\mathbb{Z}_2$-coefficients

Rem: If $\dim A + \dim B = \dim M \to \dim(A \cap B) = 0$. In particular, if M connected $\to$

$$[A \cap B] \in H_0(M) = \mathbb{Z}$$

Ex: $M = \mathbb{R}P^2$



$a = [A] = [B]$ $\leftarrow$ nonce $B \simeq A$

$\in H_1(M; \mathbb{Z}_2)$

$a \cdot a = [A] \cdot [A]$

$= [A] \cdot [B]$

$\overset{\text{thm}}{=} [A \cap B]$

$= [x] = 1 \in H_0(M; \mathbb{Z}_2) = \mathbb{Z}_2$

x gen of 0^{th} Hom.

So $a \cdot a \neq 0$ so $a \neq 0$. So

$\alpha \smile \alpha \neq 0$ where $\alpha = D_M^{-1}(a)$

Application: Lefschetz Fixed Pt. Thm

M smooth closed oriented connected n-manifold

$f: M \to M$ smooth map

$\Gamma \leq M \times M$ graph of f: $\{(x, f(x)) \mid x \in M\}$

$\Delta \leq M \times M$ = diagonal: $\{(x,x) \mid x \in M\}$

Note Γ, Δ submanifolds of $M \times M$ & diffeo. $\overset{?}{\leftrightarrow}$ just homeo. to M via proj. onto first coordinate.

$(x, f(x)) \in \Gamma \cap \Delta \rightleftarrows x = f(x)$ so x fixed point of f: $\mathrm{Fix}(f)$

A local computation shows:

If $x \in \mathrm{Fix}(f) \to \Gamma$ and Δ intersect transversely at $(x,x) \rightleftarrows \det(\underset{1_{T_xM}}{1} - df_x) \neq 0$

In this case, $x \in \mathrm{Fix}(f)$ is called a nondegenerate fixed point of f.

If $x \in \mathrm{Fix}(f)$ is a nondegenerate fixed point, then sign of int. point $(x,x) \in \Gamma \cap \Delta$ is sign of $\det(1 - df_x)$

$$\left(\begin{array}{c|c} 1_{T_xM} & 1_{T_xM} \\ \hline df_x & 1_{T_xM} \end{array}\right) \sim \left(\begin{array}{c|c} 1 - df_x & 0 \\ \hline 1 & 1 \end{array}\right)$$

form basis for $T_{(x,x)}\Gamma$

form basis for $T_{(x,x)}\Delta$

Cor: If Γ, Δ are transverse then

$$\underbrace{[\Gamma \cap \Delta]}_{[\Gamma] \cdot [\Delta]} = \sum_{x \in \mathrm{Fix}(f)} \mathrm{sign}(\det(1 - df_x))$$

Thm: $[\Gamma] \cdot [\Delta] = \underbrace{\sum_i (-1)^i \operatorname{trace}(f_* : H_i(M;\mathbb{Q}) \to H_i(M;\mathbb{Q}))}_{L(f)}$

not nec. transverse?

Lefschetz number

PF: Bredon pp 379-381 ☒

Cor: Γ, Δ transverse $\to$

$$\sum_{x \in \mathrm{Fix}(f)} \mathrm{sign}(\det(1 - df_x)) = L(f)$$

depends only on hom. class of f.

Cor: If Γ, Δ are transverse then $|\mathrm{Fix}(f)| \geq |L(f)|$

Rem: If $f = 1_M \to L(f) = \chi(M)$

Euler char.

Rem: Above not true for noncompact M.

Higher Homotopy Groups

X top. space, x_0 base point ; $n \geq 0$

Def: $\pi_n(X, x_0) = \left\{ \begin{array}{l} \text{hmty classes of cont.} \\ \text{maps } f: (I^n, \partial I^n) \to (X, x_0) \end{array} \right\}$

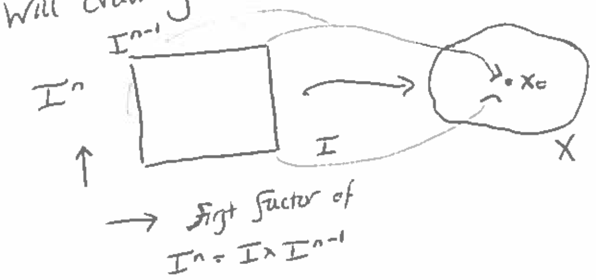
$I^n = [0,1]^n$ unit cube in $\mathbb{R}^n$.

$f: (I^n, \partial I^n) \to (X, x_0)$ means

$f(\partial I^n) \subseteq \{x_0\}$

Homotopies are required to satisfy $f_t(\partial I^n) \subseteq \{x_0\}$ for all t.

Will draw as....

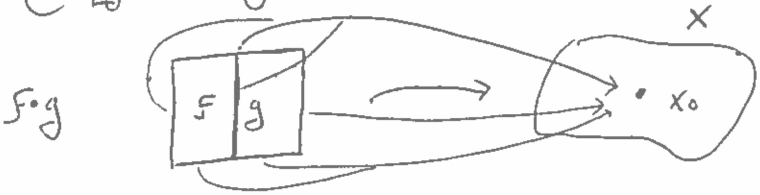


$I^n / \partial I^n \cong S^n$ so gives alt. def

Def: $\pi_n(X, x_0) = \left\{ \begin{array}{l} \text{hmty classes cont.} \\ \text{maps } f: (S^n, p) \to (X, x_0) \end{array} \right\}$

some basepoint

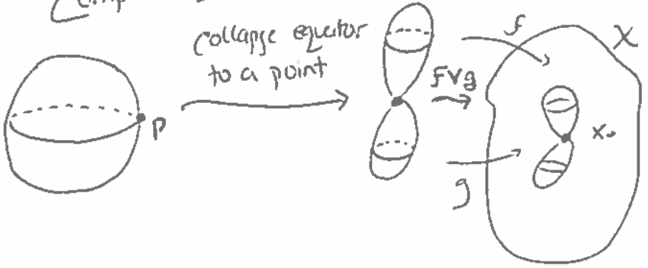
Composition: $f, g: (I^n, \partial I^n) \to (X, x_0)$; $n \geq 1$



Formally,

$$(f \cdot g)\underset{\in I^n}{(t_1, \dots, t_n)} = \begin{cases} f(2t_1, t_2, \dots, t_n), & t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \dots, t_n), & t_1 \geq 1/2 \end{cases}$$

⌈Comp. unique up to hmpty?⌋



Properties of Comp:

1) Compatible with homotopies

2) Comp. associative up to homotopy

3) Constant map at x_0 is an id. element up to hmpty

4) $f(1 - t_1, t_2, \dots, t_n)$ is an inverse of $f(t_1, \dots, t_n)$ up to homotopy

For $n > 0$, $\pi_n(X, x_0)$ is a group under product above.

Thm: If $n > 1 \to \pi_n(X, x_0)$ is abelian.

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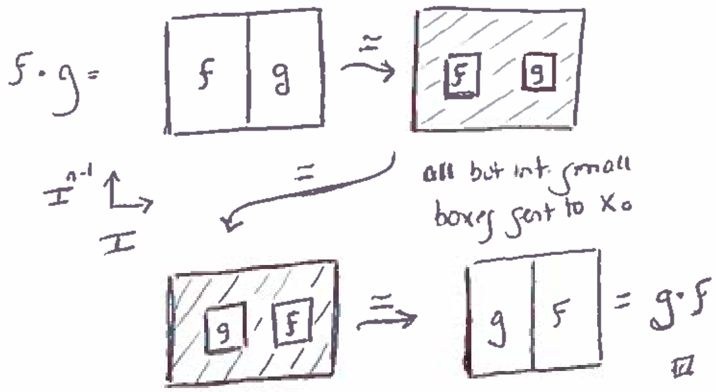
Thm: $\pi_n(X, x_0)$ is abelian for $n > 1$.

"Pf:"



Note: Doesn't work for $n=1$ as "cube" is just a line.

Induced Maps

If $\Phi : X \to Y$ cont., $\exists$ induced map

$$\Phi_* : \pi_n(X, x_0) \to \pi_n(Y, \Phi(x_0))$$

$$[F] \mapsto [\Phi \circ F]$$

Check: Φ_* is a group homomorphism

- $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$
- $(1_X)_* = 1_{\pi_n(X, x_0)}$

Functor on category pointed top spaces ~~ab. groups~~ groups

Thm: Up to isomorphism, $\pi_n(X, x_0)$ is invariant of hmpty type of (X, x_0)

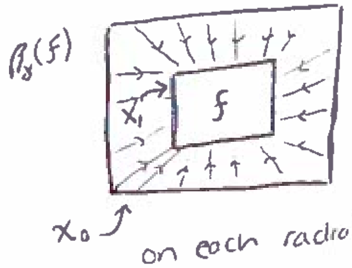
Depends on basepoint?

$x_0, x_1 \in X$

$\gamma \subseteq X$ path from x_0 to x_1.

$$\beta\gamma : \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$$

$$[f] \longmapsto ?$$



on each radial arrow $\beta_\gamma([f])$ is given by γ.

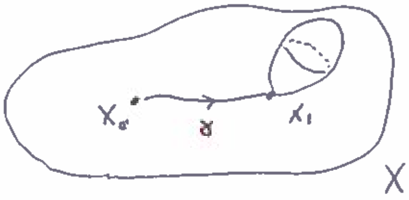
Formal def of $\beta\gamma$:

$$t = (t_1, \dots, t_n) \in I^n$$

$$t_0 = (1/2, \dots, 1/2) \in I^n$$

$$\|t\|_{max} = \max_i |t_i|$$

$$\beta_\gamma(F)(t) = \begin{cases} F(2t - t_0), & \|t - t_0\|_{max} \leq 1/4 \\ \gamma(2 - 4\|t - t_0\|_{max}, & \|t - t_0\|_{max} \geq \frac{1}{4} \end{cases}$$



Properties:

1) β_γ is an isomorphism

2) $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ via $[\gamma] \mapsto \beta_\gamma$

3) $n=1$, action is conjugation of $\pi_1(X, x_0)$ on itself.

Properties of $\pi_n(X, x_0)$

- $\pi_0(X, x_1) = \left\{ \begin{array}{l} \text{hmpty classes of cont.} \\ \text{maps } f: (\underset{\overset{\nearrow}{pt}}{I^0}, \underset{\overset{\shortparallel}{\phi}}{\partial I^0}) \to (X, x_0) \end{array} \right\}$

  $= \{ \text{path components of } X \}$

- $\pi_1(X, x_0) =$ fundamental group

- $\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$

- If $p: \tilde{X} \to X$ is a covering map, then $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$; $x_0 = p(\tilde{x}_0)$ is an isomorphism for $n > 1$. Why? Map $f: (S^n, p) \to (X, x_0)$ can be lifted to a map $\tilde{f}: (S^n, p) \to (\tilde{X}, \tilde{x}_0)$ as $\pi_1(S^n, p) = 0$ if $n > 1$.

Ex: $\pi_n(S^1) = \begin{cases} \mathbb{Z}, & n = 1 \\ 0, & n > 1 \end{cases}$

as $\pi_n(S^1) \underset{\overset{\uparrow}{\text{univ cover}}}{\cong} \pi_n(\mathbb{R}) \underset{\overset{\uparrow}{\text{contractible}}}{\cong} \pi_n(pt) = 0$

Ex:
$\pi_n(S^1 \times S^1) \cong \pi_n(S) \times \pi_n(S) = \begin{cases} \mathbb{Z}^2, & n=1 \\ 0, & n > 1 \end{cases}$

Hurewicz Thms

$x_0 \in X$; $n \geq 0$

Think of $\pi_n(X, x_0)$ as hmpty classes of maps $f: (S^n, p) \to (X, x_0)$

Def: Hurewicz maps breifs map
$h: \pi_n(X, x_0) \to H_n(X)$
Defined by $h([f]) = f_*([S^n])$

Thm: For $n \geq 1$, h is a group homo.

Pf: Recall $f \cdot g$ can be defined as comp

$$\begin{tikzcd} S^n \arrow[r, "c"] \arrow[rr, bend right, "f \cdot g"'] & S^n \vee S^n \arrow[r, "f \vee g"] & X \end{tikzcd}$$

where c is map collapsing equator in S^n to point. Then $(f \cdot g)_*$ is a composition

$$\begin{tikzcd} H_n(S^n) \arrow[r, "c_*"] \arrow[dr, "\text{diagonal embedding}"'] & H_n(S^n \vee S^n) \arrow[r, "(f \vee g)_*"] \arrow[d, phantom, "\wr\!\!\parallel\!\!\wr"] & H_n(X) \\ & H_n(S^n) \oplus H_n(S^n) \arrow[ur, "f \oplus g"'] & \end{tikzcd}$$

Then $(f \cdot g)_* = (f_* \oplus g_*) d = f_* + g_*$

$h([f \cdot g]) = h([f]) + h([g])$ ☒

Thm: For $n=1$, Hurewicz map descends to an isomorphism

$$Ab(\pi_1(X, x_0)) \cong H_1(X)$$

provided X is path connected.

Thm: For $n>1$, the Hurewicz map is an isomorphism

$$\pi_n(X, x_0) \cong H_n(X)$$

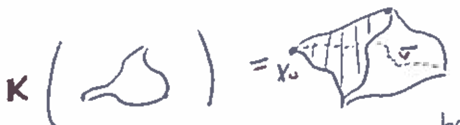
provided X is path connected and $\pi_i(X, x_0) = 0$ for $1 \leq i \leq n$

Cor: The first nonzero homotopy and reduced homology groups of a path connected and simply connected spaces occur in same dimension and are isomorphic.

PF: (idea) Define map

$$K: C_n(X) \to \pi_n(X, x_0)$$

via



sing. n-simplex σ

Connect σ to basepoint so becomes image of an S^n in X.

then take class of image (map of) on right.

Check induces a well-defined map $H_n(X) \to \pi_n(X, x_0)$ which is inverse to h. ☒

Cor: For $n>0$, $\pi_i(S^n)$

$$\cong \begin{cases} 0, & 1 \leq i < n \\ \mathbb{Z}, & i = n \end{cases}$$

Rem: $\pi_i(S^n)$ can be nonzero for $i>n$, e.g.

$$\pi_3(S^2) \cong \mathbb{Z}$$

Rem: $\pi_i(S^n) = 0$ for $i<n$ can also be seen as follows:

if $i<n \to f: S^i \to S^n$ is homotopic to a map $g: S^i \to S^n$ which is not surjective, so misses at least one point. $\to g$ viewed as map $g: S^i \to S^n \setminus pt \cong \mathbb{R}^n$

$\to g$ is nullhomotopic as $\mathbb{R}^n$ is contractible. $\to f$ is null homotopic.

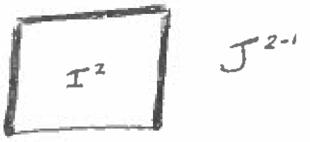
Relative Homotopy Groups

X space, $A \subseteq X$ subspace

$x_0 \in A$ for $n \geq 1$

$$J^{n-1} = \overline{(\partial I^n) - \{t_n = 0\}}$$

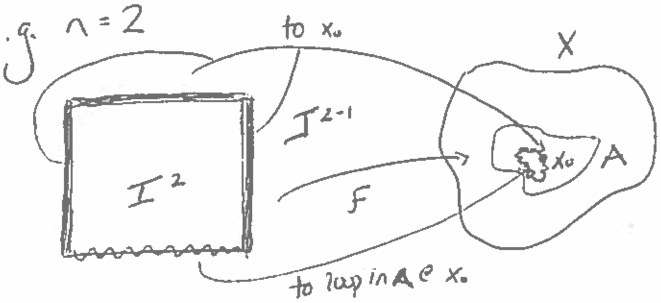
e.g. $n=2$



$n=1$

Def: $\pi_n(X, A, x_0) = \left\{ \begin{array}{l} \text{hmpty classes cont. maps} \\ f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \end{array} \right\}$

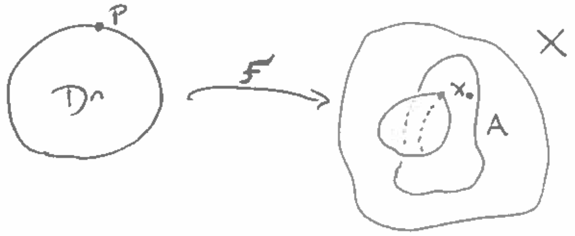
e.g. $n=2$



$$\left(I^n/J^{n-1}, \frac{\partial I^n}{J^{n-1}}, \frac{J^{n-1}}{J^{n-1}} \right) \cong (D^n, \partial D^n, p)$$

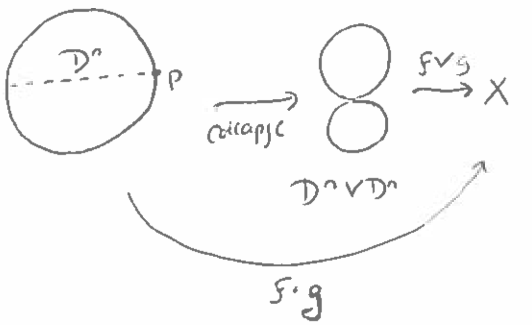
pt on boundary

Def: $\pi_n(X, A, x_0) = \left\{ \begin{array}{l} \text{hmpty classes of cont.} \\ \text{maps } f: (D^n, \partial D^n, p) \to (X, A, x_0) \end{array} \right\}$



Composition: For $n \geq 2$, composition can be defined by same formula as for $\pi_n(X, x_0)$

Thm: For $n \geq 2$, $\pi_n(X, A, x_0)$ is a group and is abelian for $n \geq 3$.



Rem: If $f: (D^n, \partial D^n, p) \to (X, A, x_0)$ has image in A then $[f] = 0 \in \pi_n(X, A, x_0)$ because we can pre-compose f with deformation retraction $D^n \searrow p$.

Induced maps:

$\phi: (X, A, x_0) \to (Y, B, y_0)$ cont.

$\to \exists$ induced map

$\phi_*: \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$

$[f] \mapsto [\phi \circ f]$

Thm: There is a l.e.s. ...

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

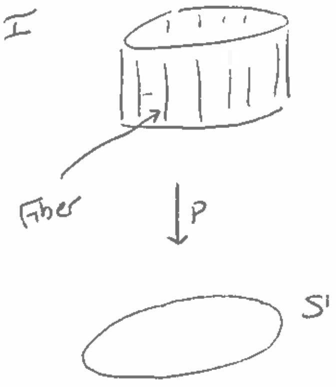
i_* inclusion from $i: (A, x_0) \hookrightarrow (X, x_0)$

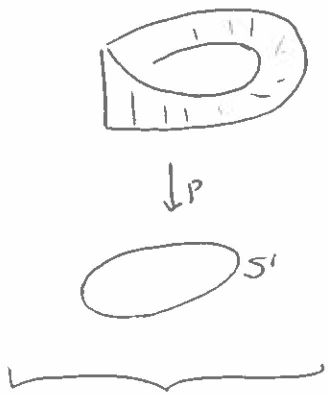
j_* inclusion from $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$

δ comes from restricting map $f: (D^n, \partial D^n, p) \to (X, A, x_0)$ to $\partial D^n = S^{n-1}$

Fiber Bundles

$S^1 \times I$





Fiber bundles over S^1 with fiber $= I$

There is les hmpty sequence that have 'representations' with fiber bundles that are useful for computation.

Used to show $S^3 \to S^2$. nontrivial.