# Differential geometry



**Ex** A two particle system — $P, Q$ particles moving around in $\mathbb{R}^3$

$\circ P \qquad \circ Q$

A "state" of the system is a vector in $\mathbb{R}^{12}$

All possible states form a subset of $\mathbb{R}^{12}$

$\vec{S}_P, \vec{S}_Q$ position

$\vec{V}_P, \vec{V}_Q$ velocity

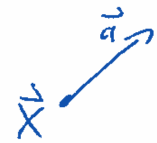
Physical laws $\rightsquigarrow$ equations that must be satisfied

$\rightsquigarrow$ restrict possible states $M \subseteq \mathbb{R}^{12}$

point in $M$ $\Leftrightarrow$ possible state of system.

**Measurement** $f : M \to \mathbb{R} \leftarrow$ Do Calculus?

**Review: Derivatives of Mappings**



$f : \mathbb{R}^n \to \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$, $\vec{a} \in \mathbb{R}^n$

directional derivative of $f$ at $\vec{x}$ in the direction $\vec{a}$

$$df_{\vec{x}}(\vec{a}) = \lim_{s \to 0} \frac{f(\vec{x} + s\vec{a}) - f(\vec{x})}{s}$$

$n=2$ $\quad df_{\vec{x}}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \frac{\partial f}{\partial x_1}(\vec{x})$, $\quad df_{\vec{x}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \frac{\partial f}{\partial x_2}\Big|_{\vec{x}}$

**Example** $\quad f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

all partial derivatives exist at $\vec{0}$ but $f$ not cont. at $(0,0)$.

Can't define derivatives just in terms of partial derivatives!

01/17

**Def:** $f:\mathbb{R}^n \to \mathbb{R}^m$ diff'ble at $\vec{x}\in\mathbb{R}^n$ if $\exists$ lin. transf. $L:\mathbb{R}^n\to\mathbb{R}^m$

s.t. $$\lim_{h\to 0}\frac{|f(\vec{x}+\vec{h})-f(\vec{x})-L(\vec{h})|}{|h|}=0.$$

$\Leftrightarrow$ $\exists$ function $\vec{r}(\vec{h})$ s.t. $\vec{r}(\vec{h})\to 0$ as $|\vec{h}|\to 0$ and

$$f(\vec{x}+\vec{h}) = f(\vec{x}) + L(\vec{h}) + r(\vec{h})|\vec{h}|$$

$f$ is approximated by $L$ to first order near $\vec{x}$.

Facts: 1) If $L$ exists, it is unique and $L$ has to be $df_{\vec{a}}$ as defined above. Write $f(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$,

then $$df_{\vec{a}}\left(\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix}\right) = \underbrace{\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n}\end{pmatrix}}_{m\times n}\begin{pmatrix}h_1\\ \vdots\\ h_n\end{pmatrix}.$$

Jacobian matrix of $f$ at $\vec{a}$.

2) If $f$ is differentiable at $\vec{a}$, then it is continuous.

3) If $f$ and $g$ are differentiable, then $f\circ g$ is and

$$d(f\circ g)_{\vec{a}} = df_{g(\vec{a})}\circ dg_{\vec{a}}.$$

4) If $f$ is diff'ble at $\vec{a}$, then all partial derivatives exist at $\vec{a}$.

5) If $f:\mathbb{R}^n\to\mathbb{R}^m$ & all partials of $f$ exist in a neighborhood of $\vec{a}$, and all partials are continuous ~~at $\vec{a}$~~ in a neighborhood, then $f$ is diff'ble

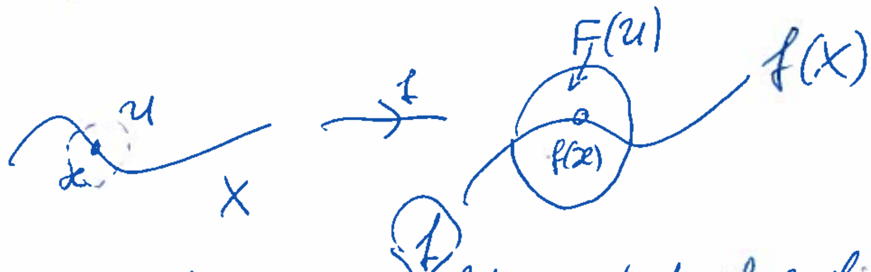
# Submanifolds

01/17

$f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We say $f$ is differentiable if $f$ may be locally extended to a differentiable map from an open set of $\mathbb{R}^n$ to $\mathbb{R}^m$

i.e. $\forall x \in X \ \exists\, \mathcal{U} \ni x$ open in $\mathbb{R}^n$ and a diff'ble map $F: \mathcal{U} \to \mathbb{R}^m$

s.t. $F = f$ on $\mathcal{U} \cap X$.




**Def:** A differentiable map between subsets of euclidean spaces is called a diffeomorphism if it is 1-1, onto, differentiable and $f^{-1}$ is also differentiable.

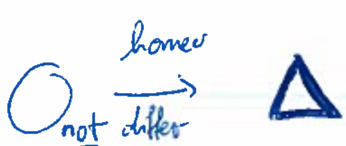
Recall. A homeomorphism is a map that is 1-1, onto, continuous and whose inverse is continuous.

Any diffeo is also a homeo. But $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ homeo, not diffe…

$X, Y$ are diffeomorphic if $\exists$ diffeo between them. $f: X \to Y$.

Exercise: This is an equivalence relation.

Ex:

01/17

Def: Let $M \subset \mathbb{R}^n$, $M$ is a $k$-dim submanifold in $\mathbb{R}^n$ if $\forall p \in M \ \exists \ V \subset M$ (open) which is diffeomorphic to an open subset of $\mathbb{R}^k$.

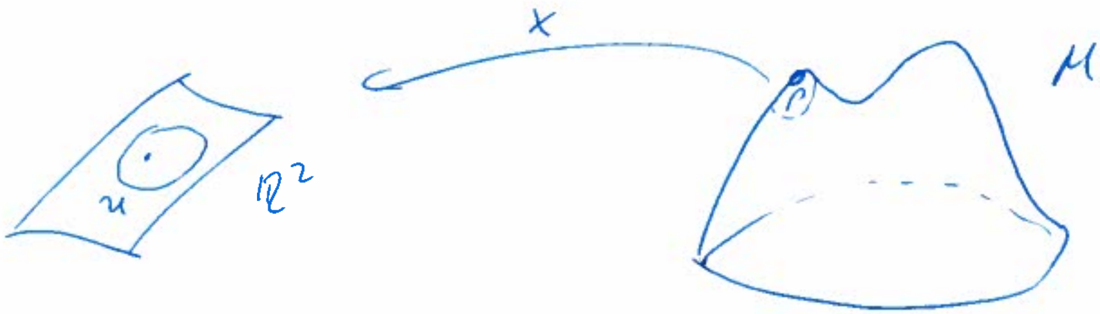
The map $X : U^{\subset \mathbb{R}^k \text{ open}} \longrightarrow V \subset M$ is called a parametrization of $V$.

1-manifold $\Leftrightarrow$ curve

2-manifold $\Leftrightarrow$ surface.

---

01/22

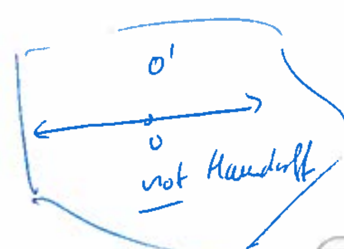



## Abstract Manifold

Def: A topological manifold $M$ of dimension $k$ is a topological space s.t. (1) $\forall p \in M \ \exists \ X : U \to V$ homeom. s.t. $U \subseteq \mathbb{R}^k$ open, $p \in V \subseteq M$ open.

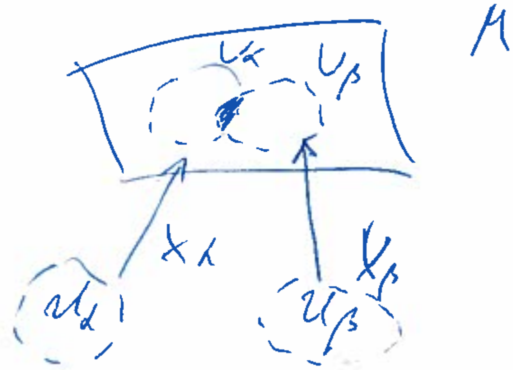
(2) $M$ is Hausdorff

(3) $M$ has a countable basis of open sets.



We need extra structure called differential structure. 01/2

Def: A differential structure on a top. mfld. of dim k. is a family of homeomorphisms $X_\alpha : U_\alpha \to V_\alpha$ (parametrizations) such that $U_\alpha \subseteq \mathbb{R}^k$ open, $V_\alpha$ open in $M$, such that



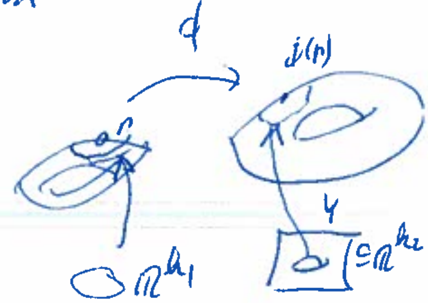

(1) $\bigcup_\alpha V_\alpha = M$

(2) If $V_\alpha \cap V_\beta \neq \phi$ then the map $X_\beta^{-1} \circ X_\alpha : \underbrace{X_\alpha^{-1}(V_\alpha \cap V_\beta)}_{\subseteq \mathbb{R}^k \text{ open}} \to \underbrace{X_\beta^{-1}(V_\alpha \cap V_\beta)}_{\subseteq \mathbb{R}^k \text{ open}}$ is differentiable.

(3) The family $\{(U_\alpha, X_\alpha)\}$ is maximal wrt. to conditions (1) & (2).

Def: A differentiable manifold is a topological mfld. with a ~~differentiable~~ differential structure (differentiable atlas).

Def: Let $M_1, M_2$ differentiable mflds. Then $\phi : M_1 \to M_2$ is differentiable at $p \in M_1$ if $\exists$ parametrizations $X : U_1 \to M_1$, $Y : U_2 \to M_2$ with ~~$p \in U_1$~~ $p \in X(U_1)$, $\phi(p) \in Y(U_2)$ and $Y^{-1} \circ \phi \circ X$ is differentiable at $X^{-1}(p)$.



Definition is <u>well defined</u> (doesn't depend on choice of $X$ & $Y$)

Take $X_1, X_2$ two different param. around $p$, $Y_1, Y_2$ around $\phi(p)$

Then ~~$X_2 \circ \phi \circ Y_2^{-1} = X_2 \circ X_1^{-1} \circ X_1 \circ \phi \circ$~~

$$Y_2 \circ \phi \circ X_2^{-1} = (Y_2 \circ Y_1^{-1}) \circ (Y_1 \circ \phi \circ X_1^{-1}) \circ (X_1 \circ X_2^{-1})$$

is differentiable by the chain rule.

<u>Exercise</u> Show that the comp. of ~~smooth~~ smooth maps is smooth.

<u>Examples of diff. structures</u>

① $S^n = \{\vec{p} \in \mathbb{R}^{n+1} \mid |\vec{p}| = 1\}$ <u>n-sphere</u>

$S^2 \subseteq \mathbb{R}^3$

$X_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1 - \sum_{i=1}^n x_i^2})$ North

$X_2(x_1, \ldots, x_n) = (x_1, \ldots, x_n, -\sqrt{1 - \sum_{i=1}^n x_i^2})$ South

$X_3(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \sqrt{1 - \sum_{i=1}^n x_i^2}, x_n)$ West.

$X_4, X_5, X_6$

$\leadsto$ 6 - charts. In general ~~#~~ $2n+2$ charts for $S^n$.

$M = \{x^3\}$ ~~$x \in (-1,1)$~~ $x \in (-1,1)$

$x \mapsto (x, x^3)$

$U = (-1,1)$

possible to cover $S^n$ by 2 charts:

stereographic projection.

## ② Real projective space



$\mathbb{RP}^n$ = Set of straight lines through origin in $\mathbb{R}^{n+1}$

$= \mathbb{R}^{n+1}\setminus\{0\}/\sim \qquad (x_1,\dots,x_{n+1}) \sim (\lambda x_1,\dots,\lambda x_{n+1}),\ \lambda\in\mathbb{R}\setminus\{0\}$

$\mathbb{RP}^n$ smooth manifold. A point in $\mathbb{RP}^n$ is an equivalence class

$[x_1,\dots,x_{n+1}]$. If $\lambda\neq 0$ $\quad [x_1,\dots,x_{n+1}] = [\frac{x_1}{\lambda},\dots,\frac{x_{n+1}}{\lambda}]$

Let $V_i = \{[x_1,\dots,x_{n+1}] : x_i\neq 0\}$

$X_i : \mathbb{R}^n \to \mathbb{RP}^n$ ~~$(x_1,\dots,x_n)\mapsto$~~

$(y_1,y_2,\dots,y_n) \mapsto [y_1,\dots,y_{i-1},1,y_{i+1},\dots,y_n]$

Image$(X_i) = V_i$, $X_i$ homeomorphisms. $\bigcup_{i=1}^{n+1} X_i(\mathbb{R}^n) = \mathbb{RP}^n$

$\boxed{i>j}$

$X_j^{-1}\circ X_i : \underbrace{X_i^{-1}(V_i\cap V_j)}_{=\{(y_1,\dots,y_n)\mid y_j\neq 0\}\subseteq\mathbb{R}^n} \to$

$X_j^{-1}\circ X_i(y_1,\dots,y_n) = X_j^{-1}[y_1,\dots,y_{i-1},1,y_i,\dots,y_n]$

$= \cancel{X_j^{-1}[\frac{y_1}{y_j},\frac{1}{y_j},\frac{y_n}{y_j}]} = X_j^{-1}[\frac{y_1}{y_j},\dots,\frac{y_{j-1}}{y_j},1,\frac{y_{j+1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j}$

$= \left(\frac{y_1}{y_j},\dots,\frac{y_{j-1}}{y_j},\frac{y_{j+1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_n}{y_j}\right) \qquad \frac{y_i}{y_j},\dots,\frac{y}{y_j}$

01/22

$\therefore$ $\mathbb{RP}^n$ smooth mfld dimension $n$, can be covered using $n+1$ charts.

We know when a mapping $\phi: M \to N$ is differentiable for any smooth manifolds $M, N$. What's the derivative?

$t_p$: Should be a linear map approximating $\phi$ close to $p$.

If $M \subseteq \mathbb{R}^n$ is a $k$-dim submanifold $T_p M$ = tangent space at $p$

$M$ Parametrize:

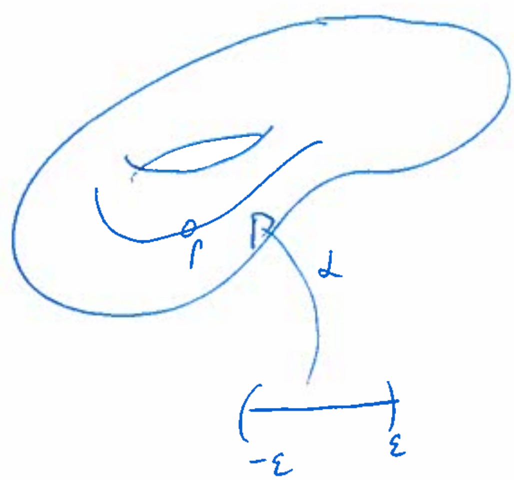
$$d\mathbf{x}_{\vec{0}}: \mathbb{R}^k \to \mathbb{R}^n, \; k < n$$

$$T_p M := \text{Image}(d\mathbf{x}_{\vec{0}})$$

To define $T_p M$ for abstract mflds, we think in terms of directional derivatives.

Let $M$ be smooth, $\alpha: (-\varepsilon, \varepsilon) \to M$ be a diff'ble curve.

Suppose that $\alpha(0) = p$ let $D$ be the set of real valued diff'ble functions on $M$.



01/22

The tangent vector to $\alpha$ at $t=0$ is

$$\alpha'(0) : \mathcal{D} \to \mathbb{R}, \quad \alpha'(0)(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \alpha)$$

The set of all tangent vectors at $p$ is $T_pM$.

---

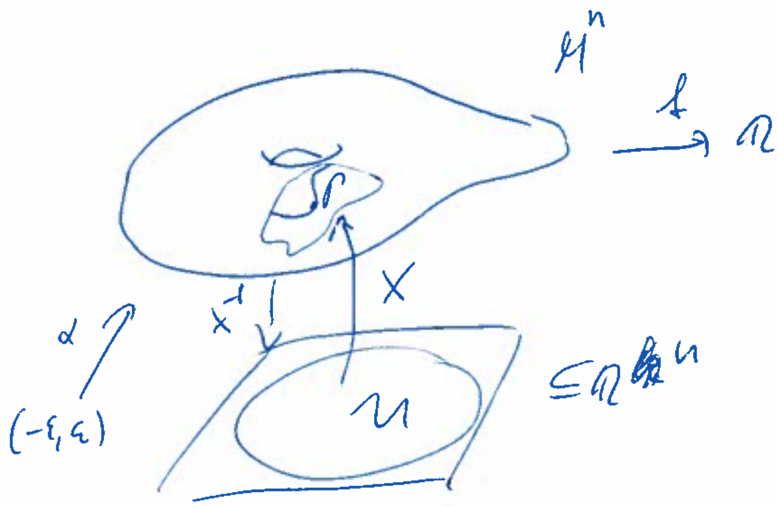
01/24

$T_pM$ <u>tangent space to $M$</u> at $p \in M$. $\mathcal{D} = \{ f : M \to \mathbb{R} \text{ diff'ble} \}$

If $\alpha : (-\varepsilon, \varepsilon) \to M$ diff'ble curve, $\alpha(0) = p$

$$\alpha'(0) : \mathcal{D} \to \mathbb{R} \quad \text{via} \quad \alpha'(0)(f) = \frac{d}{dt} (f \circ \alpha)\Big|_{t=0}$$

Let $X : U \to M$ be some parametrization about $p$.




Write $X^{-1}(q) = (x_1(q), \ldots, x_n(q))$

$\alpha(t) = X(x_1(t), \ldots, x_n(t))$ $\big(t \mapsto x_i(t)$ diff'ble as $X^{-1}$ diff'ble

01/24

Define $\frac{\partial}{\partial x_i} : D \to \mathbb{R}$, $\frac{\partial}{\partial x_i} \in T_pM$ $i$-th partial derivative of $(f \circ X)$, $\frac{\partial}{\partial x_i}(f \circ X) = \frac{\partial f}{\partial x_i}$ $\boxed{\text{wlog } X(\vec{0}) = p}$

$$\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt}\cancel{f(X(x_1(t)}(f \circ X)(x_1(t), \dots, x_n(t))\Big|_{t=0}$$

$$\overset{\text{Chain rule}}{=} \sum_{i=1}^{n} x_i'(0) \frac{\partial f}{\partial x_i} = \left(\sum_{i=1}^{n} x_i'(0) \frac{\partial}{\partial x_i}\Big|_p\right)(f)$$

Consequences ① $\alpha'(0) : D \to \mathbb{R}$ depends only on the first derivative of $\alpha$ rather $(x_1, \dots, x_n)$ in a coordinate chart.

② $T_pM$ V.S. with basis $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$

$\Rightarrow T_pM$ is a $\mathbb{R}$-V.S. of dimension $n$.

Remark $T_pM$ does not depend on the choice of parametrization $X$ but each parametrization gives you a different basis.

It is useful ~~the~~ to write down the change of parametrization formula.

Let $y : V \to M$ be another parametrization about $p$.

$$\alpha'(0)(f) = \sum_{i=1}^{n} \left(x_i'(0) \frac{\partial}{\partial x_i}\Big|_{\vec{0}}\right)(f)$$

$$= \sum_{j=1}^{n} \left(y_j'(0) \frac{\partial}{\partial y_j}\Big|_{\vec{0}}\right)(f)$$

$$\frac{\partial f}{\partial y_j} = \frac{\partial}{\partial y_j}(f \circ Y) = \frac{\partial}{\partial y_j}\left( f \circ X \circ X^{-1} \circ Y \right)$$

$$= d\left( f \circ X \circ X^{-1} \circ Y \right) e_j$$

$$\overset{\text{Chain-}}{\underset{\text{rule}}{=}} d(f \circ X)\, d(X^{-1} \circ Y)\, e_j$$

$$x_i = \pi_i \circ X^{-1}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_{\vec{0}} \frac{\partial x_i}{\partial y_j}\Big|_{\vec{0}}$$

So $$\alpha'(0)(f) = \sum_{j=1}^{n} y_j'(0) \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial y_j}$$

$$= \sum_{j=1}^{n} \left( y_j'(0) \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_j} \frac{\partial}{\partial x_i} \right) f$$

Let $\varphi : M_1^n \to M_2^m$ be diff'ble and let $p \in M_1$. Let $v \in T_pM$

$$d\varphi_p(v) = (\underbrace{\varphi \circ \alpha}_{\text{curve}})'(0) \quad \text{where } \alpha(0) = p \text{ and } \alpha'(0) = v.$$

$$\begin{tikzcd} & M_1 \arrow[r, "\varphi"] \arrow[d, "f"] & M_2 \arrow[d, "g"] \\ {} \arrow[ur, "\alpha"] & \mathbb{R} & \mathbb{R} \end{tikzcd}$$

$\varphi \circ \alpha : (-\varepsilon, \varepsilon) \to M_2$, so $(\varphi \circ \alpha)'(0)$ is an element of $T_{\varphi(p)} M$.

So $\quad d\varphi_p : T_pM \to T_{\varphi(p)}M$.

Prop: This is well-defined, $d\varphi_p(v) \in T_{\varphi(p)} M_2$ does not depend on the choice of $\alpha$.

Proof

$$\begin{array}{ccc} M_1 & \xrightarrow{\varphi} & M_2 \\ {\scriptstyle x}\nearrow & & \uparrow{\scriptstyle y} \end{array}$$

$x(\vec{0}) = p$ $\qquad$ $y(\vec{0}) = \varphi(p)$

Exercise $\quad d\varphi_p(v)(f) = \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i} x_i'(0) \frac{\partial f}{\partial y_j}$

$x^{-1} \circ \alpha(t) = (x_1(t), \dots, x_n(t))$, $\alpha'(0) = \sum_{i=1}^{n} x_i'(0) \frac{\partial f}{\partial x_i}$.

also follows $d\varphi_p : T_p M_1 \to T_{\varphi(p)} M_2$ is linear.

Chain Rule:

Exercise If $\varphi_1 : M_1 \to M_2$, $\varphi_2 : M_2 \to M_3$ diff'ble

then $\quad \varphi_2 \circ \varphi_1 : M_1 \to M_3$ diff'ble and

$$d(\varphi_2 \circ \varphi_1)_p = d{\varphi_2}_{\varphi_1(p)} \circ d{\varphi_1}_p$$

Def: Let $M_1, M_2$ be ~~smooth~~ diff'ble manifolds, then $\varphi : M_1 \to M_2$ is a diffeomorphism

if $\varphi$ is a bijection and $\varphi^{-1}$ is diff'ble.

Note If $\varphi : M_1 \to M_2$ is local diffeomorphism, then

$d\varphi_p : T_p M_1 \to T_p M_2$ is an isomorphism.

$\varphi^{-1} \circ \varphi^{-1} = \mathrm{Id}_{M_1}$

$\varphi \circ \varphi^{-1} = \mathrm{Id}_{M_2}$

$d_p(\varphi^{-1} \circ \varphi^{-1}) = d_p(\mathrm{Id}) = \mathrm{Id}_{T_pM_1}$

$d\varphi_p \circ \cancel{\;} \, d\varphi^{-1}_{\varphi(p)} = \mathrm{Id}_{T_{\varphi(p)}M_2}$

$\|$

$d\varphi^{-1}_{\varphi(p)} \circ d\varphi_p = \mathrm{Id}_{T_pM_1}$

$\Rightarrow \dim(T_pM_1) = \dim(T_{\varphi(p)}M_2)$, so $\dim M_1 = \dim M_2$

$\Rightarrow \mathbb{R}^n$ is not diffeomorphic to $\mathbb{R}^m$ if $n \neq m$.

(True for homeomorphic, but much harder)

<u>Def:</u> $\varphi : M_1 \to M_2$ is called <u>local diffeomorphism at p</u>, if ~~$M_1$~~

$\exists$ $\mathcal{U}$ nbd. of $p$ s.t. $\varphi|_{\mathcal{U}} : \mathcal{U} \to \varphi(\mathcal{U})$ is a diffeomorphism.

<u>Ex:</u> $\mathbb{R} \to S^1$, $t \mapsto (\cos t, \sin t)$ at every $t$.

<u>Inverse Function Theorem</u>

If $\varphi : M_1^{n} \to M_2^{n}$ is a diff'ble mapping and $d\varphi_p : T_pM_1 \to T_pM_2$ is a linear isomorphism, then $\varphi$ is a local diffeomorphism at $p$.

## Mappings of max'l rank



$\varphi : M_1^n \to M_2^m$ diff'ble, $p \in M$, the rank of $\varphi$ at $p$ is rank$(d\varphi_p)$ a linear map.

$$\begin{array}{ccc} M_1 & \xrightarrow{\varphi} & M_2 \\ \uparrow & & \uparrow \\ X & & \psi \end{array}$$

(rank of Jacobian of $(\psi^{-1} \circ \varphi \circ X)$)

Three cases

$n < m$ $\varphi : M_1^n \to M_2^m$ and rank$(d\varphi_p) = n$ (equivalently $d\varphi_p$ is 1-1)
then $\varphi$ is an immersion at $p$.

$n = m$ If $\varphi : M_1^n \to M_2^m$ and rank$(d\varphi_p) = n$
$\Leftrightarrow$ $\varphi$ local diffeom. (Inv. function theorem).

$n > m$ $\varphi : M_1^n \to M_2^m$ and rank$(d\varphi_p) = m$
(equivalently $d\varphi_p$ is surjective)
then $\varphi$ is a submersion at $p$.

Canonical Examples: Canonical Immersion $\mathbb{R}^n \to \mathbb{R}^m$ $m > n$
$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$

Canonical Submersion $\mathbb{R}^n \to \mathbb{R}^m$ $n > m$
$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ projection.

Idea: All immersions and submersions are locally equivalent to the canonical ones.

**Local immersion theorem** $\varphi : M_1^n \to M_2^m$ is an immersion at $p$

then $\exists$ parametrizations $X : \mathcal{U}_1 \to M_1$ about $p$

$\psi : \mathcal{U}_2 \to M_2$ about $\varphi(p)$ such that

$$\psi^{-1} \circ \varphi \circ X(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$$

$$\begin{tikzcd} M_1 \arrow[r, "\varphi"] & M_2 \\ \mathbb{R}^n \supseteq \mathcal{U}_1 \arrow[u, "X"] \arrow[r, "\text{Canonical immersion}"'] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{U}_2 \subseteq \mathbb{R}^m \arrow[u, "\psi"'] \end{tikzcd}$$

**Proof**: Inverse function theorem.

submersion

If $\varphi$ is an immersion at $p$ then $\exists\, \mathcal{U}$ nbhd. of $p$ s.t.

submersion

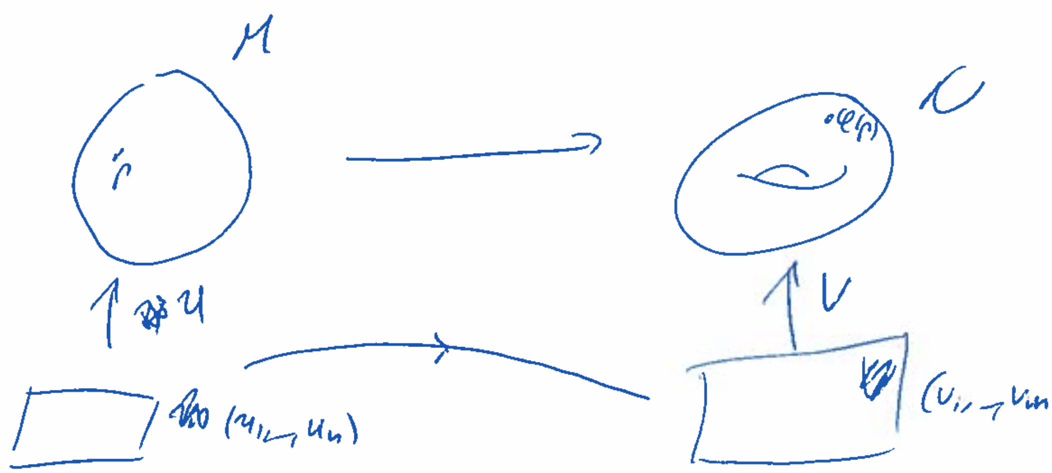
$\varphi|_{\mathcal{U}}$ is an immersion.

---



**Proof** (of local immersion theorem) Chose coordinates $\mathcal{U}$ of $p$

and $V$ of $\varphi(p)$




By permuting $u_i$, $v_i$ we can arrange so that

$$\det\left(\frac{\partial(v_\alpha \circ \varphi)}{\partial u_\beta}\right) \neq 0 \qquad \alpha, \beta = 1, \dots, n$$

let $X_\alpha := V_\alpha \circ \varphi$ $\quad \alpha = 1, \dots, n$

So $X$ is a coordinate system at $p$, by inverse function theorem.

let $X : \underline{\mathcal{U}} \to M$, $q \in \mathcal{U}$, $q = X^{-1}(a_1, \dots, a_n)$

$X(q) = (a_1, \dots, a_n)$, $X_i(q) = a_i$, $V_\alpha \circ \varphi(q) = a_\alpha$

$$V_0^{-1} \varphi \circ X(a_1, \dots, a_n) = (a_1, \dots, a_n, \underbrace{\psi_1, \dots, \psi_{m-n}}_{=?})$$

define $\psi_\alpha^* = V_\alpha^*$, $\alpha = 1, \dots, n$ $\qquad \psi_r^* = V_r - \psi_r \quad r = n+1, \dots, m$

$$= V_r - \psi_r(v_1, \dots, v_n)$$

$$\frac{\partial y_i}{\partial v_j} = \left( \frac{\partial}{\partial v_j} (\psi^{-1} \circ V)_i \right) = \begin{bmatrix} I & 0 \\ * & I \end{bmatrix}$$

<u>Ex.</u> Check $\psi^{-1} \circ \varphi \circ X(a_1, \dots, a_n) = (a_1, \dots, a_n, 0, \dots, 0)$. $\square$

<u>Suppose</u> $\varphi : M \to N$ is an immersion (at every $p \in M$).

Consider $\varphi(M) \subseteq N$

$$\begin{tikzcd} M \arrow[r, "\varphi"] & \varphi(M) \subset N \\ \mathcal{U}_1 \arrow[u, "X"] \arrow[ur, phantom, "\varphi \circ X"] & \end{tikzcd}$$

$\circ\, \varphi(p)$

$\varphi \circ X$ is almost a coordinate chart for $\varphi(M)$ in $\varphi(p)$.

01/29

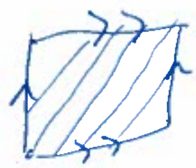
But $\varphi\circ x(U_1)$ ~~may~~ may not be an open set in $\varphi(M)$

Ex. ① $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}^2$

The $\varphi(\mathbb{R})$ not a mfld. with subspace topology

② Topologist's sine curve : Immersion but $\varphi(\mathbb{R})$ not a manifold

③



line w. irrational slope

immersion but $\varphi(\mathbb{R})$ not a 1-mfld. (dense in $T^2$)

Def: An immersion that is a homeomorphism onto its image is called an embedding. A subset $M_1 \subseteq M$ is a submfld if the inclusion map is an embedding.

①, ②, ③ are 1-1 immersions which are not embeddings.

Def: A proper map is a map s.t. the preimage of every compact set is compact.

Thm A 1-1 proper ~~embed~~ immersion is an embedding

01/29

Corollary If $M$ is compact and $\varphi: M \to N$ is a 1-1 immersion, then it is an embedding.

Proof: $\varphi: M \to N$, $M$ cpt. Then $K \subset N$ cpt $\Rightarrow$ $\varphi$ $K$ closed $\Rightarrow$ $\varphi^{-1}(K)$ closed thus cpt. $\square$

More generally than the immersion thm.

Thm. (Constant Rank Thm) If $f: M \to N$ has rank $k$ in a nbhd of $p$ $\exists$ $Y, X$ s.t.

$$Y \circ f \circ X^{-1}(\overbrace{a_1, \ldots, a_k, 0, \ldots, 0}^{n}) = (\overbrace{a_1, \ldots, a_k, 0, \ldots, 0}^{m})$$

Special case $f: M^n \to N^m$ submersion rank $= m$, $m < n$

$$Y \circ f \circ X^{-1}(a_1, \ldots, a_n) = (a_1, \ldots, a_m)$$

Note: in submersion case only need at $p$.

Pf: Similar to immersion theorem.

Pre-image theorem If $f: M^n \to N^m$ which has constant rank $k$ in a nbd. of $f^{-1}(p)$, Then $f^{-1}(p)$ is a submfld of $M$ of dim $n-k$. (or $f^{-1}(p)$ is empty).

<u>Ex</u> $f: \mathbb{R}^{n+1} \to \mathbb{R}$ $\quad f(x) = x_1^2 + \cdots + x_{n+1}^2$

<u>$n=2$</u> $\quad df_x = [2x_1, \cdots, \cancel{\phantom{x}}\ 2x_{n+1}]$, $\text{rank}(df_x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f^{-1}(1) = S^n$ is a $n = n+1-1$ submfld of $\mathbb{R}^{n+1}$.

Similarly $f(x,y,z) = x^2 + y^2 - z^2$

<u>HW:</u> $f^{-1}(a^2)$ is a mfld if $a \neq 0$ $\quad df_{(x,y,z)} = (2x, 2y, -2z)$

<u>More interesting example</u> Let $M(n)$ be the set of (real) $n \times n$ matrices. $M(n)$ diffeomorphic to $\mathbb{R}^{n^2}$. $\cancel{to}$ $O(n) = \{A \in M(n) \mid A^T A$

$f: M(n) \to M(n)$

<u>Define</u> $f(A) = A^T A^T$, then $O(n) = f^{-1}(I)$. $\quad AA^T$ symmetric.

$S(n) = \{B \in M(n) \mid B = B^T\}$. $\quad S(n)$ diffeo to $\mathbb{R}^k$ $\cancel{\phantom{x}}\ \frac{n \cdot (n+1)}{2} = k$

$f: M(n) \to S(n)$

<u>Claim</u>: $f$ is a submersion on $O(n)$ $\quad$ Consequence $O(n)$ mfld of dimensio $\cancel{\phantom{x}}$ $n^2 - \frac{n(n+1)}{2} = \frac{n(n}{2}$

Let $A \in O(n)$, $B \in M(n) \approx T_A M(n)$

$$df_A(B) = \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s} = \lim_{s \to 0} \frac{(A+sB)(A+sB)^t - AA^t}{s}$$

$$= AB^t + BA^t$$

<u>Q.1</u> $df_A(B) : M(n) \to S(n)$ onto?

01/29

let $C \in S^*(n)$ $\exists A^t + AB^t = C$

$\Leftarrow \exists A^t + (BA^t)^t = C$. solve $BA^t = \frac{1}{2}C$.

Define $B := \frac{1}{2}CA$. does it.

01/31

<u>Def</u> • If ~~Dx~~ $\operatorname{rank}_p(f) < \dim N = m$, $p$ is called a <u>critical point</u> of $f$.

• If $\operatorname{rank}_p(f) = m$ then $p$ is ~~also~~ called a <u>regular point</u>.

• $q \in N$ is ~~also~~ called <u>critical value</u>, if $\exists\, p \in f^{-1}(q)$ s.t. $p$ critical point.

• $q \in N$ <u>regular value</u> if it's not a critical value, i.e. every $p \in f^{-1}(q)$ is a regular point.

<u>Sard's Theorem</u> If $f: M \to N$ smooth map then the critical values have measure zero in $N$.

In particular, the regular values of $f$ are <u>dense</u> in $N$.

<u>Corollary</u> $f: M^n \to N^m$ (smooth), $n < m$, then $f(M)$ has measure zero in ~~M~~ $N$.

(There are no smooth "space-filling" curves).

$\exists\, f: \mathbb{R} \to [0,1]^2$, cont. and ~~onto,~~ $f(\mathbb{R})$ dense in $[0,1]\times[0,1]$.

---

two other "constructions" of manifolds

① Tangent bundle

② Quotient of a group action

① $M^n$ mfld. $\mathcal{M} = \{(p,v) \mid p \in M, v \in T_pM\}$, $TM$ diff'ble mfld of dimension $2n$.

<u>Why?</u> Let $(U_\alpha, x_\alpha)$ be charts of a diff'ble structure on $M$

$x_\alpha^{-1} = (x_1^\alpha, \dots, x_n^\alpha)$. Then $\left\{ \frac{\partial}{\partial x_i^\alpha} \mid i = 1,\dots,n \right\}$ basis for $T_pM$.

Ex: $\Psi_\alpha : U_\alpha \times \mathbb{R}^n \to TM$

Then $\Psi_\alpha(x_1^\alpha, \dots, x_n^\alpha, u_1, \dots, u_n) = \Big(\overbrace{X_\alpha(x_1^\alpha, \dots, x_n^\alpha)}^{=p}, \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i^\alpha}\Big)$

$(\Psi_\alpha, U_\alpha \times \mathbb{R}^n)$ forms a diff. structure on $TM$.

Check: $\Psi_\beta^{-1} \circ \Psi_\alpha (q_\alpha, u_1^\alpha, \dots, u_n^\alpha) = \Psi_\beta^{-1}\Big(X_\alpha(q_\alpha), \sum u_i^\alpha \frac{\partial}{\partial x_i^\alpha}\Big)$

$$= \Big(\underbrace{X_\beta^{-1} \circ X_\alpha (q_\alpha)}_{\text{smooth}}, \; dx_{\beta\alpha}^{-1}\big(\underbrace{dx_\alpha}_{\text{smooth}} (u_1, \dots, u_n)\big)\Big)$$

$$= d_q (X_\beta^{-1} \circ X_\alpha)(u_1, \dots, u_n)$$

Topology $U \subset TM$ open if

$TM \xrightarrow{\pi_1} M$ if $\pi_1(U)$ open, $TM \xrightarrow{\pi_p} T_pM$, $\pi_p(U)$ open $\forall p \in M$.
$(p, v) \mapsto p$ $\qquad (p, v) \mapsto v$

Tangent bundle is an example of a vector bundle ~~on~~ over $M$.

$M$ and at each $p \in M$ a v.s. $V_p$ $(= T_pM)$

HW: $TS^1 \cong S^1 \times \mathbb{R}$ But in general $TM$ $TM \not\cong M \times \mathbb{R}^n$.

ex: $TS^2 \not\cong S^2 \times \mathbb{R}^2$, but $TS^3 \cong S^3 \times \mathbb{R}^3$

Idea Any structure on a v.s. can be put to a smooth manifold via the tangent bundle

Ex Orientation If we have two ordered bases $(v_1, \to v_n), (w_1, w_n)$ of $\mathbb{R}^n$ then $(v_1, \to v_n)$ and $(w_1, \to w_n)$ are equivalently oriented if det of change of ~~basis~~ basis matrix between them is positive.

Otherwise, they are oppositely oriented.

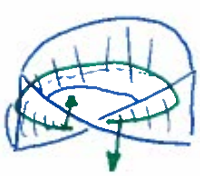
Being equivalently oriented defines eq. rel. on ordered bases of $\mathbb{R}^n$.

An orientation of $\mathbb{R}^n$ is a choice of equivalence class.

A smooth mfld $M$ is orientable if we can choose a smoothly varying orientation of the tangent space.

Def: $M$ orientable if it is possible to choose coordinate charts $(X_\alpha, U_\alpha)$ s.t. $\bigcup_\alpha U_\alpha = M$ and $\det\left(d_p\left(X_\alpha^{-1} \circ X_\beta\right)\right)$ has constant sign $\forall p, \forall \alpha, \beta$

Ex: Möbius band not orientable



② Group actions by diffeomorphisms.

Def: $G$ group $M$ smooth mfld. $G$ acts on $M$ by diffeomorphisms if $\varphi: G \times M \to M$ s.t. (1) $\varphi_g: M \to M$ is a diffeomorphism.

$\varphi_g(p) = \varphi(g, p)$

and $\varphi_e = \mathrm{Id}_M$

If $g_1, g_2 \in G \Rightarrow \varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi_{g_2}$.

Often: In an abuse of notation, we write

$$g(p) = \varphi_g(p).$$

equivalently $g \mapsto \varphi(g,\cdot) : G \to \mathrm{Diff}(M)$ is a group hom.

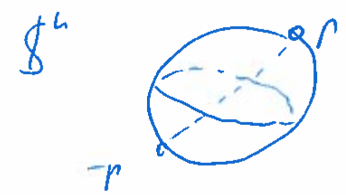
$$M/G = \{ [p] : p \sim g(p) \ \forall g \in G \}.$$

Examples ① $M = S^n$, $G = \mathbb{Z}_2 = \{-1, 1\}$

$\varphi : \mathbb{Z}_2 \times S^n \to S^n$, $\underset{\varphi_1(p) =}{\varphi(1,p)} = p$, $\underset{\varphi_{-1}(p)}{\varphi(-1,p)} = -p$, $p \in S^n$

is an action by diffeomorphisms

$$S^n/\mathbb{Z}_2 = \{ [p] \ p \in S^n, \ p \sim -p \} \cong \mathbb{RP}^n.$$

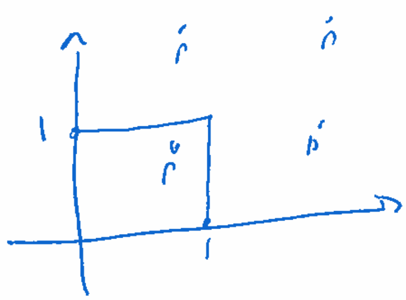



② $M = \mathbb{R}^n$, $G = \mathbb{Z}^n$

$\varphi : \mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(z, p) = p + z$

$\mathbb{R}^n/\mathbb{Z}^n = \{ [x] \mid x \sim x + k, \ k \in \mathbb{Z}^n \}$

$\cong$  $\cong \mathbb{T}^n = S^1 \times \cdots \times S^1$



$G = S^1$, $M = \mathbb{R}^2$ 01/31

$\varphi_\theta(p)$ = rotation counter-clockwise by $\theta$

$\mathbb{R}^2/S^1 \cong [0,\infty)$. not a smooth mfld.

---

A group action is called properly discontinuous if $\forall p \in M$ $\exists$ nbd. $U \ni p$, $U \subseteq M$ s.t. $U \cap g(U) = \phi$ $\forall g \neq e$. (holds for ①, ②, but not ③)

If action is properly discontinuous, then $\pi : M \to M/G$, $\pi(p) = [p]$

~~$\pi : M \to M/G$~~ is a local homeomorphism.

Can put a differential structure on $M/G$ s.t. $\pi : M \to M/G$ is a local diffeo (see p. 23).

<u>Prop:</u> $M/G$ is Hausdorff if and only if $\forall p_1 \neq p_2 \in M$ $\exists$ nbds $U_1, U_2$ of $p_1, p_2$ such that $U_1 \cap g(U_2) = \phi$ $\forall g \in G$.

---

02/0

## Vector fields

<u>Define</u> A vector field $\underline{X}$ on a smooth mfld $M$ is a correspondence that assigns to each $p \in M$ a vector $\underline{X}(p) \in T_pM$ that is smooth as a map $\underline{X} : M \to \underline{TM}$.

means $\ast$ $x : U \to M$ parametrization

$\underline{X}(p) = a_i(p) \dfrac{\partial}{\partial x_i}$ and $a_i : M \to \mathbb{R}$ are smooth functions.

From now on, everything is assumed to be smooth.

<u>Recall:</u> $v \in T_pM$ is a mapping $v : \mathcal{D}(M) \to \mathbb{R}$  $\mathcal{D}(M) = \mathcal{D}$ Smooth function $: M \to \mathbb{R}$.

So we think of $\underline{X} : \mathcal{D} \to \mathcal{D}$. $f \mapsto \underline{X}(f)$
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \shortparallel$
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \underline{X}(p)(f)$.

<u>In coordinates:</u>
$$\underline{X}(f)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p)$$

<u>f:</u> $\underline{X}, \underline{Y} : \mathcal{D} \to \mathcal{D}$  define $\underline{X}(\underline{Y}(f)) = \underline{X} \circ \underline{Y}(f)$ makes sense

But $\underline{X} \circ \underline{Y}$ does <u>not</u> define a vector field

<u>Why</u> ① $x : U \to M$, $\underline{X} = \sum a_i \frac{\partial}{\partial x_i}$, $\underline{Y} = \sum b_j \frac{\partial}{\partial x_j}$

$$X(Y(f)) = X\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j} a_i \frac{\partial}{\partial x_i}\left(b_j \frac{\partial f}{\partial x_j}\right)$$

$$= \sum_{i,j} a_i \left(\frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + b_j \frac{\partial^2 f}{\partial x_j \partial x_i}\right)$$

② If $v \in T_pM$. $v(fg) = v(f)g + f v(g)$

and $(X \circ Y)(fg) = X( Y(f)g + f Y(g)) = X(Y(f))g + Y(f) X(g)$
$\qquad + X(f) Y(g) + f X(Y(g))$
$\qquad \neq (XY)(f)g + f(XY)(g)$.

So, $X \circ Y \notin T_pM$.

**Def:** ~~Let X, Y be vector~~ Let $X, Y$ be vector fields, 02/0

define $[X,Y] = XY - YX$ ~~that~~

Lie bracket of $X$ and $Y$.

**Prop** $[X,Y]$ is a vector field on $M$.

**Proof:** $[X,Y](f) =$

$$(X \circ Y)(f) = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \underline{a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}}$$

$$(Y \circ X)(f) = \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + \underline{a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}}$$

$$[X,Y](f) = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i}$$

$$= \sum_{i,j} \left( a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \cdot \frac{\partial f}{\partial x_j}.$$

**Properties of Lie Bracket** (1) $[X,Y] = -[Y,X]$.

(2) $[aX+bY, Z] = a[X,Z] + b[Y,Z]$ $a,b \in \mathbb{R}$.

(3) (Jacobi Identity) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

(4) $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$. See page 27.

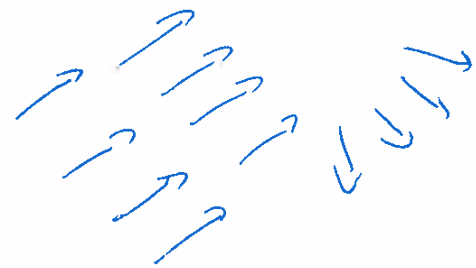
02/05

<u>Integral Curves</u> Let $X$ be a vector field. A curve

$\alpha : (-\delta, \delta) \to M$ is called an <u>integral curve of $X$</u>

(or trajectory of $X$) if it satisfies

$$\frac{d\alpha}{dt} = X(\alpha(t)) \quad \forall t \in (-\delta, \delta)$$
$$= X_{\alpha(t)}$$



If $x : \mathcal{U} \to M$ then $X = a_i \frac{\partial}{\partial x_i}$

$$\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t)), \quad \frac{d\alpha}{dt} = \left(\frac{d\alpha_1}{dt}, \dots, \frac{d\alpha_n}{dt}\right) \quad \cancel{\frac{d\alpha_i}{dt} = a_i \frac{\partial}{\partial x_i}(\alpha(t))}$$

$$\frac{d\alpha_i}{dt} = a_i(\alpha_1(t), \dots, \alpha_n(t)) \quad i = 1, \dots, n.$$

$\leadsto$ Finding integral curves $\Leftrightarrow$ Solving system of ODEs.

We can rewrite the existence and uniqueness theorem for systems of ODEs in the following:

<u>Thm</u> $\underline{X}$ smooth vector field on $M$ and $p \in M$. Then $\exists$ open set $V \subset M$, $\varepsilon > 0$

such that $\exists$ unique diffeomorphism $\varphi_t : V \to \varphi_t(V)$, $|t| < \varepsilon$ s.t.

(1) $\varphi : (-\varepsilon, \varepsilon) \times V \to M$ $\quad \varphi(t, q) \mapsto \varphi_t(q)$ smooth, $\varphi_0 = id$

(2) If $|s+t| < \varepsilon$ and $\varphi_t(q) \in V$, $q \in V$ $\quad \varphi_{s+t}(q) = \varphi_s(q) \circ \varphi_t(q)$

(3) The map $t \mapsto \varphi_t(q)$ is an integral curve of $X$ through $q$.

$\varphi_t$ is called the local 1-parameter group of diffeomorphisms generated by $X$. $\{\text{smooth v.F}\} \Leftrightarrow \{\text{local one parameter groups of diffeos}\}$

<u>Thm</u> If $M$ is <u>compact</u>, then $\varphi_t : M \to M$ exists for all $t \in \mathbb{R}$.

(support $X$ compact is enough).

<u>Lie Derivative</u> • write derivative along the local flow of $X$.

If $X$ smooth vector field and $\varphi_t : V \to \varphi_t(V)$ local flow

$t \mapsto \varphi_t(q)$ is an integral curve of $X$ through $q$

$$X(\varphi_t(q)) = \frac{d\varphi}{dt}\Big|_{t=0} \qquad \cancel{X(f) = \frac{d}{dt}(f\circ\varphi_t) = \lim_{h\to 0} f(\varphi_{t+h}(q))}$$

$$\cancel{X}(f) = \frac{d}{dt}(f\circ\varphi_t)\Big|_{t=0} = \lim_{h\to 0} \frac{(f\circ\varphi_{t+h})(q) - (f\circ\varphi_t)(q)}{h}$$

$X, Y$ two vector fields, $\varphi_t$ local flow for $X$

$$(L_X Y)(q) = \lim_{t\to 0} \frac{Y_{\varphi_t(q)} - (d\varphi_t)_q(Y_q)}{t} \in T_q M. \quad (\text{as a limit in } TM)$$

<u>Thm</u> $(L_X Y)(q) = [X, Y]_q$ <u>Pf:</u> See page 28. $\square$

**Example** $\mathbb{R}^2$ let $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ the standard coordinates 02/05

let $\underline{X} = \frac{\partial}{\partial x}$, $\underline{Y} = \frac{\partial}{\partial y}$

$\varphi_t(a,b) = (a+t, b)$ $\quad d\varphi_t\, \partial a = id$, then $L_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0 = [X, Y]$.

Any coordinates $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$.

**Thm** let $\underline{X}$ be a smooth vector field, then $\forall p \in M\ \exists x : \mathcal{U} \to M$

$x(0) = p$ s.t. $\underline{X} = \frac{\partial}{\partial x^1}$

**Theorem** (Frobenius) let $\underline{X}, \underline{Y}$ two vector fields, then $\exists x : \mathcal{U} \to M$ s.t.

$\frac{\partial}{\partial x_1} = \underline{X}$, $\frac{\partial}{\partial x_2} = \underline{Y}$ if and only if $[\underline{X}, \underline{Y}] \equiv 0$ in $x(\mathcal{U})$.

---

02/07

**Recall** Defn of smooth mfld.

(1) Differential structure

(2) Hausdorff

(3) second countable

(2)+(3) are necessary for the existence of a partition of unity

**Def:** $\mathcal{O}$ open cover of $M$. A collection of smooth functions subordinate to $\mathcal{O}$

$\{\varphi_i : M \to [0,1]\}$ is called partition of unity if

~~$\forall p \in M$ ...~~ (1) the collection is locally finite, i.e

$\forall p \in M\ \exists\, \mathcal{U} \ni p$ s.t. only finitely many $\varphi_i$ are nonzero in $\mathcal{U}$.

(2) For each $p$ $\sum_i \varphi_i(p) = 1$. $\forall p \in M$. 02/07

(3) For each $i$ $\exists\, \mathcal{U} \in \mathcal{O}$ s.t $\operatorname{supp} \varphi_i \subset \mathcal{U}$.

Theorem If $M$ smooth mfld, then any open cover has a smooth partition of unity. [equivalent to (2) and (3) in Def$^n$ of smooth mfld]

Theorem (Embedding Theorem) If $M^n$ is a compact smooth mfld, then there is embedding $F : M \to \mathbb{R}^N$ for some $N \in \mathbb{N}$.

Remarks • Also true in noncompact case

mfld without boundary, not orientable.

• Whitney: $N = 2n$ best you can do $\mathbb{R}P^2 \hookrightarrow \mathbb{R}^4$ but not in $\mathbb{R}^3$

Proof: Since $M$ is compact, any 1-1 immersion is an embedding.

and we can cover $M$ by finitely many coordinate charts.

$$x_i : \underset{\mathbb{R}^n}{\overset{\cap}{\mathcal{U}_i}} \longrightarrow M \quad i = 1, \dots, k \quad \text{and} \quad \bigcup_{i=1}^{k} x_i(\mathcal{U}_i) = M$$

Let $\varphi_i$ be a partition of unity subordinate to $\{x_i(\mathcal{U}_i)\}_{i=1,\dots,k}$.

s.t. $\operatorname{supp} \varphi_i \in x_i(\mathcal{U}_i) \Rightarrow \varphi_i \equiv 0$ on $M \setminus x_i(\mathcal{U}_i)$

Define ($f : M^n \to \mathbb{R}^N$) $f(p) = (\varphi_1 \cdot x_1^{-1}, \varphi_2 \cdot x_2^{-1}, \text{———}, \varphi_k \cdot x_k^{-1}, \varphi_1, \dots, \varphi_k$

$N = k \cdot n + k$.

Claim This is a 1-1 immersion.

02/07

Let $p \in M$ $\exists\, \varphi_i$ s.t. $\varphi_i(p) \neq 0$.

multiplication

$\varphi_i \cdot x_i^{-1} : \cancel{M}\, U \to \mathbb{R}^n$ is a local diffeomorphism

for some $U \ni p$. $\Rightarrow \operatorname{rank}_p(\varphi_i x_i^{-1}) = n$. $\Rightarrow \operatorname{rank}_p(f) \geq n$

So $\operatorname{rank}_p(f) = n$, thus $f$ is an immersion.

If $f(p) = f(q) \Rightarrow \varphi_i(p) = \varphi_i(q)$ $\forall i = 1, \dots, k$. $\exists$ at least one $i_0$ s.t. $\in I$

$\varphi_{i_0}(p) = \varphi_{i_0}(q) \neq 0$. $\quad \underbrace{\varphi_{i_0}}_{\neq 0} \underbrace{x_{i_0}^{-1}}_{\text{diffeo}}(p) = \underbrace{\varphi_{i_0}}_{\neq 0} \underbrace{x_{i_0}^{-1}}_{\text{diffeo}}(q) \rightarrow p = q$. $\quad \square$

---

$p \in M$

$\exists\, U \ni p$

$\psi : U \times \mathbb{R}^n \to \pi^{-1}(U) \in TM$

$\pi : TM \to M$

$\psi|_{U}$

$(\pi \circ \psi)(x, v) = x$

$v \mapsto \psi(x, v)$ v.s. isom

## Chapter 1: Riemannian Metrics

**Def:** $V$ v.s./$\mathbb{R}$. An <u>inner product</u> on $V$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$

~~$\langle x, y \rangle = \langle$~~ (1) $\langle v, w \rangle = \langle w, v \rangle$ $\forall v, w \in V$

(2) $\langle \cdot, \cdot \rangle$ is bilinear

(3) pos. definite $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

**Def:** $M$ smooth mfld. A <u>Riemannian metric</u> on $M$ is a correspondence

$p \mapsto \langle \cdot, \cdot \rangle_p$ where $\langle \cdot, \cdot \rangle_p$ is an inner product on $T_pM$.

which "varies smoothly", i.e. If $V, W$ smooth VF, then

$f(p) = \langle V(p), W(p) \rangle_p : M \to \mathbb{R}$ is smooth $\forall$ $V, W$ smooth VF.

02/6

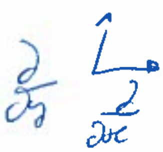
<u>In coordinates:</u> If we have a parametrization $x: U \to M$

$\left\{\frac{\partial}{\partial x_i}\right\}$ basis of $T_pM$, then $g_{ij}(p) = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p$

$g_{ij} = g_{ji}$, $g_{ii} > 0$. $[g_{ij}] = \begin{bmatrix} g_{11} & & \\ & \ddots & \\ & & g_{nn} \end{bmatrix}$ symmetric pos. def.

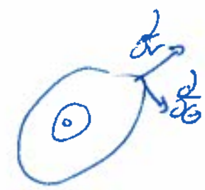
<u>Ex</u> $g_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$ in $\mathbb{R}^n$.

- $\mathbb{R}^2$ polar coordinates instead of standard basis




$(r, \theta)$ polar coordinate

$x = r\cos\theta$
$y = r\sin\theta$




$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

So $\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$. and $\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$

In basis $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$ on $T_pM$

$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1$ $\quad \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle = r^2$.

$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = 0$ $\quad g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$.

<u>Another representation</u>: $\langle\cdot,\cdot\rangle_p : T_pM\times T_pM\to\mathbb{R}$

$\left\{\frac{\partial}{\partial x^i}\right\}$ basis for $T_pM$.

let $dx_i: T_pM\to\mathbb{R}$, dual of $\frac{\partial}{\partial x_i}$ by ~~$\frac{d}{dt}$~~ $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$

<u>Tensor product</u>: $dx_i\otimes dx_j : T_pM\times T_pM\to\mathbb{R}$

$$(dx_i\otimes dx_j)(v,w) = dx_i(v)\cdot dx_j(w).$$

$$\shortparallel$$

$$dx_i dx_j \qquad\qquad dx_i\, dx_i = dx_i^2$$

<u>Exercise</u> $\langle v,w\rangle_p = \sum_{i,j} \cancel{dx}\; g_{ij}\, dx_i\, dx_j\,(v,w).$

~~Dot~~ Dot product in $\mathbb{R}^n$: $\langle\cdot,\cdot\rangle_p = dx_1^2 + \_\_\_ + dx_n^2$.

Polar coordinates in $\mathbb{R}^2$: $\left\{\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right\}$ $\langle\cdot,\cdot\rangle = dr^2 + r^2 d\theta^2$

<u>Why a metric?</u> A Riemannian mfld is a metric space.

Let $\gamma: I\to M$ be a smooth curve, let $[a,b]\in I$, then

we can define the length of $\gamma$ from a ~~to~~ b


as $\ell_a^b(\gamma) = \int_a^b \left\langle \frac{d\gamma}{dt},\frac{d\gamma}{dt}\right\rangle_{\gamma(t)}^{1/2} dt.$

Define a distance between two points $x,y\in M$ as

$$d(x,y) = \inf\{\ell_b^a(\gamma) \mid \gamma \text{ smooth curve } \gamma(a)=x,\ \gamma(b)=y\}$$

d is a metric.

## "Length metric"

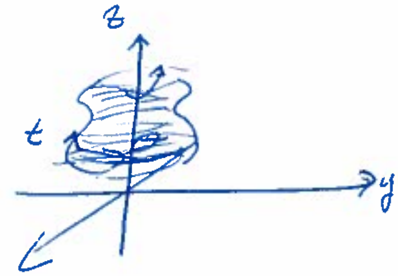
02/07

**Example** Surface of revolution, $r(t)>0$, $t>0$, $t\in(a,b)$

profile curve: $c(t)=(r(t),0,z(t))$ in $xz$-plane $\subseteq \mathbb{R}^3$

rotate about $z$-axis. $\rightsquigarrow S$

$$S \overset{\text{diffeo}}{\cong} (a,b)\times S^1.$$




**Coordinates:** $(t,\theta) = (r(t)\cos\theta,\ r(t)\sin\theta,\ z(t))$ $x$

Give $S$ induced metric by dot product.

dot product: $dx^2+dy^2+dz^2$. $\quad \left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\right\}$ form a basis for $T_pS$.

$$dx = d(r(t)\cos\theta) = \frac{dr}{dt}(t)\cos\theta\, dt - r(t)\sin\theta\, d\theta$$

$$dy = d(r(t)\sin\theta) = \frac{dr}{dt}(t)\sin\theta\, dt + r(t)\cos\theta\, d\theta$$

$$dz = \frac{dz}{dt}dt.$$

$$dx^2 = \left(\frac{dr}{dt}\cos\theta\, dt - r\sin\theta\, d\theta\right)^2 = \left(\frac{dr}{dt}\right)^2\cos^2\theta\, dt^2$$
$$- r\frac{dr}{dt}\sin\theta\cos\theta\, d\theta\, dt - r\frac{dr}{dt}\sin\theta\cos\theta\, dt\, d\theta$$
$$+ r^2\sin^2\theta\, d\theta^2.$$

$$\Rightarrow dx^2+dy^2+dz^2 = \left(\frac{dr}{dt}\right)^2\cos^2\theta\, dt^2 + \left(\frac{dr}{dt}\right)^2\sin^2\theta\, dt^2$$
$$+ r^2\sin^2\theta\, d\theta^2 + r^2\cos^2\theta\, d\theta^2 + \left(\frac{dz}{dt}\right)^2 dt^2.$$

02/12

$$\Rightarrow \boxed{dx^2+dy^2+dz^2 = \left(\left(\frac{dr}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)(dt)^2 + r^2(t)\,d\theta^2}$$

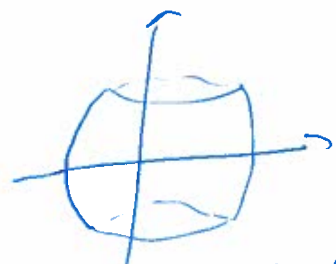
If $c(t)$ is parametrized by unit speed, we get

$$\langle \cdot , \cdot \rangle = dt^2 + r^2(t)\,d\theta^2$$

$r(t) = t \Rightarrow dt^2 + t^2 d\theta^2$ polar coordinates in $\mathbb{R}^2$

unit speed
$r(t) = \sin(t) \Rightarrow z(t) = \cos(t)$

$dt^2 + \sin^2(t)\,d\theta^2.$



piece of a sphere.

More generally, a rotationally symmetric metric on $(a,b)\times S^1$ is any $(t,\theta)$ coordinates metric of the form $a^2(t)\,dt^2 + b^2(t)\,d\theta^2 + 0\cdot d\theta\,dt$ $\left(\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\rangle\right)$

Not all rotational symmetric metrics are surfaces of revolution:

if it is, $\left(\frac{db}{dt}\right)^2 \leq a^2$

Any metric on $(a,b)\times S^1$ will look like $a^2(t,\theta)\,dt^2 + c^2(t,\theta)(dt\,d\theta + d\theta\,dt) + b^2(t,\theta)\,d\theta^2.$

# Lie Groups

A Lie-Grp. $G$ is a smooth manifold ~~with~~ which is a group with $\cdot$ s.t. the maps $G \times G \to G$ $(x,g) \mapsto x \cdot y$ and $G \to G$, $x \mapsto x^{-1}$ are smooth.

Denote $e$ as the identity.

__Examples__ ① $\mathbb{R}^n$ with $\cdot = +$ is a Lie group.

② $\mathbb{R} \setminus \{0\}$ with $\cdot$ multiplication

③ $\mathbb{C} \setminus \{0\}$ —"—

④ $S^1$ is a Lie group

⑤ $GL(n,\mathbb{R}) = \{ n \times n$ real matrices s.t. $\det \neq 0\}$ with matrix multiplication.

⑥ $O(n) = \{ A \in GL(n,\mathbb{R})$ s.t. $AA^T = I\}$ with matrix multiplica

---

$G$ Lie group, $g \in G$ (Left-translation)

$L_g : G \to G$, $L_g x = gx$, $R_g : G \to G$, $x \mapsto xg$ (Right-translation)

and they are __Diffeomorphisms__! $(L_g)^{-1} = L_{g^{-1}}$

A vector field $X$ on $G$ is __called left-invariant__ if

$(dL_g)_h (X_h) = X_{gh}$ $\forall g,h \in G$ (write $dL_g(X) = X_g$)

02/12

A left invariant vector field is completely determined by its value at one point, usually $e \in G$. Conversely, if $v \in T_e G$. let

$X_g := (dL_g)_e(v)$ we get a left invariant vector field with $X_e = v$

$\{\text{left inv. vector fields on } G\} \cong T_e G \cong \mathbb{R}^n$.

$$
\begin{aligned}
d(L_g)_h(X_h) &= (dL_g)_h (dL_h)_e(v) \\
&= d(L_g \circ L_h)_e v \\
&= d(L_{gh})_e v = X_{gh}
\end{aligned}
$$

<u>Remark</u> $S^2$ is not a Lie group

<u>Hairy Ball Theorem</u> If $V$ is a vector field on $S^2$ then $\exists\, p \in S^2$ s.t. $V(p) = 0$ $\Rightarrow$ $TS^2 \neq S^2 \times \mathbb{R}^2$. ~~and~~ and $S^2$ is not a Lie-group.

<u>Lie Bracket</u> $X, Y$ VF; $[X, Y] = XY - YX$.

<u>Prop:</u> If $X, Y$ left invariant VF, then so is $[X, Y]$.

<u>Proof:</u> let $f : M \to \mathbb{R}$. $dL_g([X, Y])(f)$

$$
\begin{aligned}
&= [X, Y](f \circ L_g) = X(Y(f \circ L_g)) - Y(X(f \circ L_g)) \\
&= X(dL_g(Y)(f)) - Y(dL_g(X)(f)) \\
&= XY(f) - YX(f) = [X, Y](f)
\end{aligned}
$$

$\square$

Lie Algebra of $G$ = Vector space of left invariant vector fields.

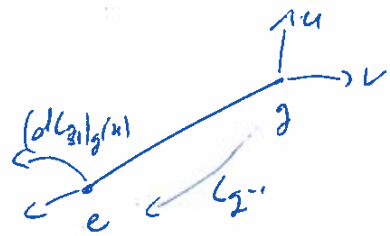
$$\underset{\text{Lie Algebra}}{\mathfrak{g}} \approx (T_e G,\ \underset{\substack{\uparrow \\ \text{multiplication}}}{[\cdot,\cdot]})$$

02/1

If $\langle \cdot, \cdot \rangle_e$ is an inner product on ~~$T_eG$~~ $T_eG$

and $u, v \in T_g G$ define $\langle u, v \rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle_e$

This defines a Riemannian metric on $G$.




$\therefore$ $\langle \cdot, \cdot \rangle_g$ is called a left-invariant metric. in the sense that

$$\langle u, v \rangle_h = \langle (dL_g)_h u, (dL_g)_h v \rangle_{gh} \quad \forall g, h \in G, \; u, v \in T_h G.$$

Each choice of inner product on $T_eG$ gives a left invariant metric on $G$

So $\frac{n(n+1)}{2}$ choices $\langle , \rangle_{\underline{\underline{e}}} = \sum_{i,j} \underline{\underline{g_{ij}}}_{\text{const.}} \, dx_i \otimes dx_j$

Similarly right invariant metrics. replacing $L_g$ with $R_g$.

A metric is <u>biinvariant</u> if it is both left and right invariant.

---

02/1

A metric $\langle \cdot, \cdot \rangle_g$ on $G$ is called left <u>invariant</u> if

$\forall h, g \quad \langle u, v \rangle_h = \langle (dL_g)_h u, (dL_g)_h v \rangle_{gh} \quad \forall u, v \in T_h G$

<u>Right invariant</u> : Replace $L$ with $R$.

Prop Let $\langle \cdot, \cdot \rangle_e$ be an inner product on $T_eG$, then

$u, v \in T_gG$ define $\langle u, v \rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle_e$

Claim: This is a ~~~~ left invariant metric.

Proof: $\langle u, v \rangle_h = \langle \cancel{dL_{g}} (dL_{h^{-1}})_h u, (dL_{h^{-1}})_h v \rangle_e$

$$\|$$

$$\langle (dL_g)_h u, (dL_g)_h v \rangle_{gh} = \langle (dL_{(gh)^{-1}})_{gh} (dL_g)_e u, (dL_{(gh)^{-1}})_{gh} (dL_g)_h v \rangle_e$$

$(gh)^{-1} = h^{-1}g^{-1}$ $\square$

Example $SU(2) = \{ A \in M_{2,2}(\mathbb{C}) : AA^* = I, \det(A) = 1 \}$

$$= \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}$$

$(z, w) \in \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$ $\quad \overset{\text{diffeom.}}{\cong} S^3(1)$

$SU(2)$ with matrix mult. is a Lie group.

Lie Algebra: $T_eG = T_I(SU(2)) = \left\{ \begin{bmatrix} i\alpha & \beta + ic \\ -\beta + ic & -i\alpha \end{bmatrix}, \alpha, \beta, c \in \mathbb{R} \right\}$

Basis for $T_eG$ $\quad x_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Extend $x_1, x_2, x_3$ to left invariant VF on all of $SU(2)$. $\sim X_1, X_2, X_3$

Define a left inv. metric by saying $x_1, x_2, x_3$ are orthonormal

$\{x_1, x_2, x_3\}$ forms a frame field for $SU(2)$

$\langle \cdot , \cdot \rangle = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ , $\sigma_i : T_pM \to \mathbb{R}$ linear

$\sigma_i(x_j) = \delta_{ij}$

usual metric on $S^3$

$\{x_1, x_2, x_3\}_e \to \forall\ X_1(g) = d_e(dL_g)_e(X_1(e))$

On the other hand if we declare $\{X_1, X_2, X_3\}$ to be orthogonal, length $X_1 = \varepsilon$ length $X_2, X_3$ to be 1. Then $\langle \cdot , \cdot \rangle^\varepsilon = \varepsilon^2 \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. (Berger - Sphere

<u>Note</u>: $\{X_1, X_2, X_3\}$ is not a coordinate frame, i.e. there is no choice of coordinates

$x : U \to SU(2)$ s.t. $X_i = \frac{\partial}{\partial x_i}$

Let $G$ be a Lie Group, $h \in G$. The inner automorphism of $G$ by $h$ is

$L_h \circ R_{h^{-1}}$ or $R_{h^{-1}} \circ L_h$

the map: $G \mapsto G$ $\quad g \mapsto hgh^{-1}$ i.e.

is a diffeomorphism of $G$ and $L_h \circ R_{h^{-1}}(e) = e$.

Define $Ad_h : T_eG \to T_eG$ $\quad Ad_h = d(L_h \circ R_{h^{-1}})_e$, $h \mapsto Ad_h \in Gl(T_e$

representation of a Lie group.

<u>Prop</u> Let $X, Y$ left invariant vector fields on $G$

then $[Y, X] = \lim_{t \to 0} \left( \frac{1}{t} (Ad_{X_t^{-1}(e)})(Y(e)) - Y(e) \right)$

where $X_t$ is the one parameter group of diffeomorphisms generated by $X$

<u>Proof</u>: If $X_t$ local flow generated by $X$ then $[Y, X]$

$= \lim_{t \to 0} \frac{1}{t} \left( dX_t \left( Y(X_t^{-1}(e)) \right) - Y \right)$

let $g \in G$, $\alpha_g : t \to g(X_t(e))$ $\quad t \mapsto X_t(e)$ integral curve for $X$.

$\alpha_g(0) = g(X_0(e)) = g \cdot e = g$

$$\frac{d\alpha_g}{dt}\Big|_{t=0} = \frac{d}{dt}(g\, X_t(e))\Big|_{t=0} = (dL_g)_e\left(\frac{dX_t(e)}{dt}\Big|_{t=0}\right) = (dL_g)_e\, X(e)$$
$$= X_g$$

$\alpha_g$ is integral curve of $X$ through $g$

$\overset{\text{uniqueness}}{\Rightarrow}$ $X_t(g) = g\, X_t(e) = R_{X_t(e)}(g) \Rightarrow dX_t = dR_{X_t(e)}$

So
$$[Y,X] = \lim_{t\to 0} \frac{1}{t}\left((dR_{X_t(e)})\left(Y(X_t^{-1}(e))\right) - Y(e)\right)$$
$$= \lim_{t\to 0} \frac{1}{t}\left(\left(dR_{X_t(e)} \circ dL_{X_t^{-1}(e)}\right) Y(e) - Y(e)\right)$$
$$= \lim_{t\to 0} \frac{1}{t}\left(\left(Ad_{X_t^{-1}(e)}\right)(Y(e)) - Y(e)\right)$$

Two Consequences (1) $G = Gl(n,\mathbb{R}), Gl(n,\mathbb{C})$ (open subsets of $\mathbb{R}^{n^2}$, $\mathbb{R}^{(2n)^2}$), $X, Y$ left inv. fields. then $[X,Y]_I = X(I)Y(I) - Y(I)X(I)$ $\quad X(I), Y(I) \in T_I G$.
(matrix multiplication)

$Gl(n,\mathbb{R}) \overset{\text{open subset}}{\approx} \mathbb{R}^{n^2}$, $L_g : Gl(n,\mathbb{R}) \to Gl(n,\mathbb{R})$ is a linear map of $\mathbb{R}^{n^2}$.

$\Rightarrow (dL_g)(u) = gu$, $(dR_g)(u) = ug$.

$$x_t(e) = I + tX + o(t)$$

$$x_t^{-1}(e) = I - tX + o(t)$$

$$dL_{x_t^{-1}(e)}(Y) = (I - tX + o(t))\,Y$$

$$dR_{x_t(e)}\big((dL_{x_t^{-1}(e)})(Y)\big) = \cancel{I}\ \mathrm{Ad}_x\,(I - tX + o(t))\,Y\,(I + tX + o(t))$$

$$= (Y - tXY + o(t))(I + tX + o(t))$$

$$= Y - tXY + tYX + o(t)$$

Formula

$\Rightarrow$ $\lim_{t\to 0}$ $[Y, X] = YX - XY$ matrix multiplication

Remarks Also works for $G \leq Gl(n,\mathbb{R})$ or $Gl(n,\mathbb{C})$

Ex: Berger Spheres: $SU(2)$, $X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 & \\ +i & \end{bmatrix}$

basis for $T_I G$

$$[X_1, X_2] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = 2X_3.$$

Similarly $[X_2, X_3] = 2X_1$, $[X_3, X_1] = 2X_2$

$\Rightarrow \{X_1, X_2, X_3\}$ is not a coordinate field in any open set.

because $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$

Recall A metric on a Lie group is called bi-invariant if it is left invariant and right invariant.

**Prop** If $\langle\, ,\, \rangle$ is biinvariant and $X, Y, Z$ left invariant vector fields, ~~then if~~ $\cancel{[X,Y,Z]}$ then 02/14

$$\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle$$

**Remark:** Actually it is if and only if.

**Ex** Berger sphere: $\langle\cdot,\cdot\rangle = \varepsilon\,\sigma_1^2 + \sigma_2^2 + \sigma_3^2$

$$\langle [X_1,X_2], X_3\rangle = 2\langle X_3,X_3\rangle = 2$$

$$\langle X_1,[X_2,X_3]\rangle = 2\langle X_1,X_1\rangle = 2\varepsilon$$

biinvariant iff. $\varepsilon = 1$.

**Pf of Prop** If biinv. then $\langle X,Z\rangle = \langle dR_g(X), dR_g(Z)\rangle$

let $g_t(e)$ be the 1-param. grp gen by $g$. Then

$$\langle X,Z\rangle = \langle dR_{g_t(e)}(X), dR_{g_t(e)}(Z)\rangle_e \qquad \text{derivative at } t=0$$

$$0 = \langle [X,Y],Z\rangle + \langle X,[Z,Y]\rangle.$$

$$\Rightarrow \langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle \qquad \square$$

<u>Isometries</u> $F: M^n \to N^{n}$   $M, N$ Riemannian manifolds   02/19

If $F$ is a diffeomorphism and $\langle u,v\rangle_p^M = \langle dF_p(u), dF_p(v)\rangle_{F(p)}^N$ (*)
then $F$ is called an isometry. If $F$ is a local diffeo. and satisfies (*)

then $F$ is called a local isometry.

<u>Recall</u> $(X,d_X), (Y,d_Y)$ metric spaces $F: X \to Y$ s.t. $d_X(x_1,x_2) = d_Y(F(x_1), F(x_2))$

Isometry of metric spaces.

<u>Exercise</u> $F$ Riemannian isometry $\Leftrightarrow$ Isometry of metric spaces.
<u>Thm</u>: (Myer's – St.), "$\Leftarrow$"

Isometry forms an equivalence relation on Riemannian mflds. where

$M \underset{\text{isom.}}{\cong} N$ if $\exists F: M \to N$ isometry.

<u>Def</u>: If $F: M^n \to N^{n+k}$ is an immersion, ($N$ Riemannian mfld) then define a Riemannian metric

on $M$ via $\langle u,v\rangle_p = \langle dF_p(u), dF_p(v)\rangle_{F(p)}$   $u,v \in T_pM$.

Metric induced by immersion $F$. (Pullback Metric)

$\leadsto$ "Riemannian" immersion

<u>Submersion</u> $F: M^{n+k} \to N^n$ Riemannian submersion if $\ker(dF_p)^\perp \ni u,v$

then $\langle u,v\rangle_p = \langle dF_p(u), dF_p(v)\rangle_{F(p)}$.

Examples ① $F: \mathbb{R}^n \to \mathbb{R}^n$ ~~Case~~ Euclidean metric.

$F$ isometry $\iff$ ~~$F(x$~~ $F(x) = Ax + b$, $b \in \mathbb{R}^n$, $A \in O(n)$.

② $F: S^n \to S^n$ standard metric.

$F$ isometry $\iff$ $F(x) = Ax$, $A \in O(n+1)$.

③ $S^n \to \mathbb{R}P^n$, $x \mapsto [x]$ $\quad \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong T^n$ are local isometries

major notion

$M$ Riemannian mfld Isometry grp of $M = \{F: M \to M \text{ isometry}\}$

④ Rotationally symmetric metrics

~~Hav~~ $M = I \times S^1 \ni (t,\theta)$ $\langle \cdot, \cdot \rangle \leftarrow$ metric on $M$. Rotationally symmetric

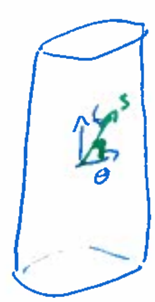
Suppose $F_{\theta_0}: M \to M$, $(t,\theta) \mapsto (t, \theta + \theta_0)$ (Derivative Id but shifted)

Assume $F_{\theta_0}$ are isometries $\forall \theta_0$.

$$\langle \cdot, \cdot \rangle = a^2(t,\theta)\,dt^2 + b(t,\theta)(dt\,d\theta + d\theta\,dt) + c^2(\theta,\theta)\,d\theta^2.$$

then $a(t,\theta) = a(t)$, $b(t,\theta) = b(t)$, $c(t,\theta) = c(t)$.

let $s = A(t)\,t + \overbrace{\frac{b(t)}{a(t)}}^{B(t)}\theta$, $\theta = \theta$. $(t,\theta) \underset{\text{diffeo}}{\mapsto} (s,\theta)$




~~$ds = \frac{dA}{dt}dt + \frac{dB}{dt}d\theta$~~ $\quad \frac{ds}{dt} = A'(t)t + A(t) + B'(t)\theta$

$\frac{ds}{d\theta} = B(t)$ $\quad$ TBC

$d(B(t)\theta) = \frac{\partial}{\partial t}(B(t)\theta)\,dt + \frac{\partial}{\partial \theta}(B(t)\theta)\,d\theta$

Now ~~$\frac{df}{ds}$~~ $\frac{df}{dt}$ = 

<u>Prop:</u> Any diff'ble mfld has a Riemannian metric.

<u>Pf:</u> Cover $M$ by coord. charts $x_\alpha : U_\alpha \to V_\alpha \subset M$, $\bigcup_\alpha V_\alpha = M$

Let $\varphi_\alpha$ be a partition of unity subordinate to $V_\alpha$, i.e. $\{\varphi_\alpha\}$ are locally finite $\sum_\alpha \varphi_\alpha = 1 \ \forall p$, Support$(\varphi_\alpha) \subset V_\alpha$.

<u>Define</u> $\langle u, v \rangle_p^\alpha = \langle d(x_\alpha^{-1})_p(u), d(x_\alpha^{-1})_p(v) \rangle_{x^{-1}(p)}$ ~~$\langle (dx_\alpha)_p^{-1}(u), (dx_\alpha)_p^{-1}(v) \rangle_{x^{-1}(p)}$~~

$\langle \cdot, \cdot \rangle^\alpha$ is a metric on $V^\alpha$.

<u>Define</u> $\langle \cdot, \cdot \rangle_p = \sum_\alpha \varphi_\alpha(p) \langle u, v \rangle_p^\alpha$. Well defined metric on all of $M$. $\square$

<u>Fund. Q. of Riemannian geometry</u> Given $M$ smooth mfld.

What is the "<u>best</u>" metric you can put on it? (up to isometry)

Given two Riemannian metrics, when are they isometric.

<u>Differentiating Vector fields in Riemannian manifolds</u>

Df $\mathcal{X}(M) = \{X : M \to TM \ C^\infty \text{ vector fields}\}$

Parallelism of Vector fields.

<u>In $\mathbb{R}^n$</u> $v \in T_p(\mathbb{R}^n)$, $w \in T_q(\mathbb{R}^n)$ 

In $M$; we need a way to compare vectors at different tangent spaces.

<u>Def:</u> An affine connection is a map $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$(X, Y) \mapsto \nabla_X Y$. s.t.

(1) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$, $f, g \in C^\infty(M)$

(2) $\nabla_X (aY + bZ) = a \nabla_X Y + b \nabla_X Z$. $a, b \in \mathbb{R}$

(3) $\nabla_X (fY) = f \nabla_X Y + X(f) Y$.

<u>Note:</u> $(X, Y) \mapsto [X, Y]$ Does not satisfy (1), so is <u>not</u> an affine connection.

Local coordinates $x: U \to M$, basis $\frac{\partial}{\partial x_i}$ on $T_pM$.

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_j b_j \frac{\partial}{\partial x_j} \qquad a_i, b_j: X(U) \to \mathbb{R}$$

$$\nabla_X Y = \nabla_{\left(\sum_i a_i \frac{\partial}{\partial x_i}\right)} \left(\sum b_j \frac{\partial}{\partial x_j}\right)$$

$$= \sum_{i,j} a_i \nabla_{\frac{\partial}{\partial x_i}} \left(b_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j} a_i \left(\frac{\partial}{\partial x_i}(b_j) \frac{\partial}{\partial x_j} + b_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)$$

Only depends on
- value of $a_i$ at $p$
- value and first derivatives of $b_j$
- coordinate chart.

$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \in \mathcal{X}(M)$ write in Basis $\frac{\partial}{\partial x^k}$

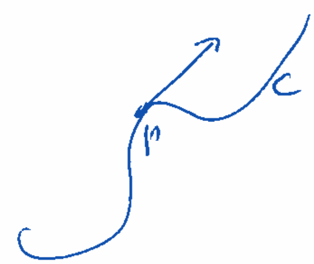
$$\Rightarrow \quad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

christoffel symbols

$$\text{So} \quad \nabla_X Y = \sum_{k} \Big( \sum_{i,j} a_i b_j \Gamma_{ij}^k + \underbrace{X(b_k)}_{= \sum_i a_i \frac{\partial}{\partial x_i}(b_k)} \Big) \frac{\partial}{\partial x^k} \quad (*)$$

Consequences: ① If $X_1, X_2 \in \mathcal{X}(M)$ and $\underbrace{X_1(p) = X_2(p)}_{=v}$, then

$$\nabla_{X_1} Y\big|_p = \nabla_{X_2} Y\big|_p \quad (= \nabla_v Y)$$




② If $c(t)$ is a curve in $M$

$\nabla_{\frac{dc}{dt}} Y\big|_p$ = only depends on the value of

$\frac{dc}{dt}\big|_p$ and the value of $Y$ along $c$.

If $c(t)$ is a curve and $Y$ vector field along $c$ ($c: I \to M$) (i.e $Y: I \to TM$ s.t. $Y(t) \in T_{c(t)}M$

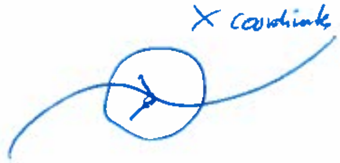
$(*)$ Is well defined for $X = \frac{dc}{dt}$, $Y = Y$.

Define $\frac{D}{dt} Y = (*)$. (covariant derivative of $Y$ along $c$)

$$\left(= \nabla_{\frac{dc}{dt}} Y\right)$$

last time. $\nabla_X Y \in \mathfrak{X}(M)$ derivative of $Y$ along $X$. 02/21

Curve $c(t)$




$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$

$Y$ vector field along $c$, $Y: I \to TM$, s.t. $Y(t) \in T_{c(t)}M$ $\forall t \in I$.

$$\frac{D}{dt}(Y) = \sum_k \left( \frac{db_k}{dt} + \sum_{i,j} \frac{dx_i}{dt} b_j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}, \quad Y = \sum_j b_j(t) \frac{\partial}{\partial x_j}\Big|_{c(t)}$$

$X(c(t)) = (x_1(t), \dots, x_n(t))$. $\frac{D}{dt}(Y) = \nabla_{\frac{dc}{dt}} Y$.

$\frac{D}{dt}(Y)$ vector field along $c(t)$.

Prop. $\frac{D}{dt}$ satisfies

1) $\frac{D}{dt}(Y+Z) = \frac{DY}{dt} + \frac{DZ}{dt}$

2) $\frac{D}{dt}(f(t)Y) = f(t)\frac{DY}{dt} + \frac{df}{dt}Y$

If $Y$ is defined in an open nbhd. of $c$ 3) $\frac{D}{dt} Y = \nabla_{\frac{dc}{dt}} Y$.

Also $\frac{D}{dt}$ is unique with these properties.

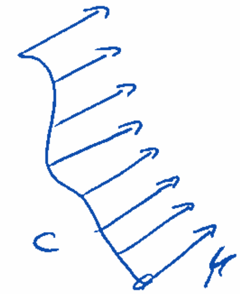
Def: $Y$ is called parallel along a curve $c(t)$ if $\frac{DY}{dt} = 0 \ \forall t$.

Example $X, Y$ vector fields in $\mathbb{R}^n$, def. $\nabla_X Y$ s.t. $\Gamma_{ij}^k = 0 \ \forall i,j,k$

$E_1, \dots, E_n$ standard basis of $\mathbb{R}^n$, i.e. $\nabla_{E_i} E_j = 0$

Pf: $X = \sum a_i E_i$, $Y = \sum b_j E_j$. Then

$$D^{\mathbb{R}^n}_X Y = \sum_j X(b_j) E_j = (X(b_1), \dots, X(b_n)).$$




Parallel: $\nabla_{\frac{dc}{dt}} Y \equiv 0 \Rightarrow b_j(t) =$ constant $\forall j$.

Prop M smooth mfld, $\nabla$ affine connection, $c(t)$ smooth curve, $v \in T_{c(0)}M \Rightarrow \exists!$ vector field $Y$ along $c(t)$ s.t. $\frac{DY}{dt} = 0$ and $Y(0$

Proof iff $\frac{D}{dt}(Y) = 0 \iff \frac{db_k}{dt} + \sum_{i,j} b_j \frac{dx_i}{dt} \Gamma^k_{ij} = 0 \quad \forall k = 1,$

System of linear first order ODE. in $\{b_k\}_{k=1,\dots,n}$.

Solve in any coord. chart, uniqueness $\Rightarrow$ can solve along curve.

The parallel field $Y(t)$ along $c(t)$ is called parallel translation of $v$ along $c(t$

Relationship between connections and metrics

M smooth mfld, $\nabla$ affine connection, $\langle \cdot, \cdot \rangle$ Riemannian metric.

- $\nabla$ is called compatible with $\langle \cdot, \cdot \rangle$ if

$\forall X, Y, Z \in \mathfrak{X}(M)$ if $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X$

02/21

- $\nabla$ is called <u>torsion free</u> if

$$\nabla_X Y - \nabla_Y X = [X,Y] = XY - YX$$

<u>Fund. Theorem of Riemannian Geometry</u> $M$ Riemannian mfld.

$\exists!$ affine connection $\nabla$ which is torsion free and compatible with $\langle\cdot,\cdot\rangle$.

<u>Proof</u>: 

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ + Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ - Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \end{aligned}$$

$$\begin{aligned} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = &\langle [Y,Z], X\rangle \\ &+ \langle [X,Z], Y\rangle \\ &+ \langle \underbrace{[X,Y]}_{\nabla_X Y - \nabla_Y X}, Z\rangle + 2\langle Z, \nabla_Y X\rangle \end{aligned}$$

Koszul's Formula

$$\begin{aligned} 2\langle Z, \nabla_Y X\rangle = &X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &- \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle. \end{aligned}$$

$\Rightarrow$ <u>uniqueness</u> Conversely define $\nabla$ by the formula (exercise: this defines a connection).

$\square$

$\leadsto \nabla$ is called Levi-Civita connection or Riemannian Connection

Examples $\mathbb{R}^n$, $\nabla_X^{\mathbb{R}^n} Y$ defined above is the Levi-Civita connection. 02/2

let $M \subseteq \mathbb{R}^n$ $M$ Riemannian mfld metric induced by $\mathbb{R}^n$.

$X, Y$ vector fields tangent to $M$

$$\nabla_X Y = \left(\nabla_X^{\mathbb{R}^n} Y\right)^T, \quad T_p\mathbb{R}^n \ni V \mapsto V^T \text{ projection onto } T_pM.$$

Then (exercise) $\nabla$ is the Riemannian connection on $M$.

(Torsion free: Frobenius Thm)

Ex: $S^2 \subseteq \mathbb{R}^3$

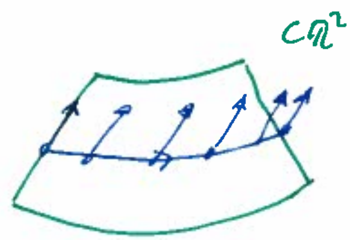

$c(t) = (\cos(t), \sin(t), 0)$

$Y(t) = \frac{\partial}{\partial z}$

$$\nabla_{\frac{dc}{dt}} Y = \Big(\nabla^{\mathbb{R}^n}_{\frac{dc}{dt}} \underset{\substack{\uparrow\\ \text{const.}}}{Y}\Big)^T = 0, \quad \nabla_{\frac{dc}{dt}} \frac{dc}{dt} = \Big(\nabla_{\frac{dc}{dt}} \frac{dc}{dt}\Big)^T$$

$$= \Big(-\overset{\cos t}{\cancel{\sin(t)}}, \overset{-\sin t}{\cancel{\cos(t)}}, 0\Big)^T = 0$$

$c(t)$ latitude line


parallel fields rotate around

<u>Last time</u> $M$ Riem. mfld $\langle\, ,\rangle$

$\exists!$ affine connection $\nabla$ s.t.

(1) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (compatible)

(2) $[X,Y] = \nabla_X Y - \nabla_Y X$. (Torsion free)

We call this the <u>Levi-Civita Connection</u> or Riemannian connection.

Koszul-Formula: $\langle Z, \nabla_Y X\rangle = \frac{1}{2}\Big[X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle$
$- \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle$
$+ \langle [Z,X],Y\rangle\Big]$

---

Levi-Civita in local coordinates: $x : U \to M$

$\frac{\partial}{\partial x_i}$ coordinate vector fields $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle$

$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0 \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$

$$\left\langle \frac{\partial}{\partial x_k}, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right\rangle = \frac{1}{2}\left[\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\right]$$

$$= \left\langle \frac{\partial}{\partial x_k}, \sum_e \Gamma_{ij}^e \frac{\partial}{\partial x_e}\right\rangle$$

$$\Rightarrow \sum_{\ell} \Gamma_{ij}^{\ell} g_{\ell k} = \frac{1}{2}\left[\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\right]$$

$$\underbrace{\qquad\qquad}_{= (g_{\ell k}) \begin{pmatrix} \Gamma_{ij}^{1} \\ \Gamma_{ij}^{2} \\ \vdots \\ \Gamma_{ij}^{n} \end{pmatrix}}$$

$(g_{ij}) \rightsquigarrow$ pos. def. matrix

So it has an inverse $(g^{ij})$.

$$\Rightarrow \sum_{k} g_{\ell k} g^{km} = \delta_{\ell, m}$$

$$\Gamma_{ij}^{m} = \sum_{\ell, k} \Gamma_{ij}^{\ell} \underbrace{g_{\ell k} g^{km}}_{= \delta_{\ell, m}} = \frac{1}{2} \sum_{k} g^{km}\left[\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\right]$$

## Geodesics

**Def:** A curve $c: I \to M$ is a geodesic if $\nabla_{\frac{dc}{dt}} \frac{dc}{dt} = 0$

$\Leftrightarrow \frac{D}{dt}\left(\frac{dc}{dt}\right) = 0.$ $\leftarrow$ "zero acceleration."

Note that: $\frac{d}{dt} \langle \frac{dc}{dt}, \frac{dc}{dt} \rangle = 2 \langle \nabla_{\frac{dc}{dt}} \frac{dc}{dt}, \frac{dc}{dt} \rangle = 0$

So $\langle \frac{dc}{dt}, \frac{dc}{dt} \rangle$ is constant along a geodesic.

Trivial geodesic: $c(t) = \text{constant}$.

03/05

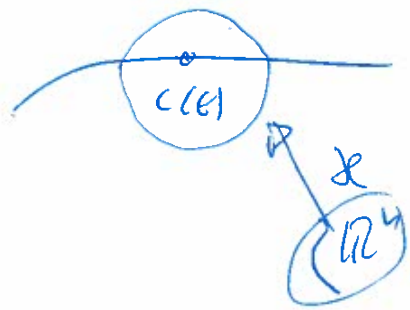
**Note:** Being a geodesic depends on the parametrization.

e.g. In $\mathbb{R}^n$: geodesics are straight lines parametrized to be constant speed.

(a) Existence and Uniqueness of Geodesics $p \in M$, $v \in T_pM$. Then $\exists!$ geodesic $c : I \to M$, $c(0) = p$, & $\frac{dc}{dt}(0) = v$.

Why? Geodesic $\Leftrightarrow$ solving a (nonlinear) $2^{nd}$ order system of ODE's.

In local coordinates:




$x : U \to M$

$x^{-1}(c(t)) = (x_1(t), \dots, x_n(t))$

$V = \sum V_j \frac{\partial}{\partial x_j}$ vector field along $c$

$$\frac{DV}{dt} = \sum_k \left( \frac{dV_k}{dt} + \sum_{i,j} V_j \frac{dx_i}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}$$

For geodesics: $V = \frac{dc}{dt}$, so $V_j = \frac{dx_j}{dt}$, so

Cont

$$0 = \frac{d^2 x_k}{dt^2} + \sum_{i,j} \frac{dx_j}{dt}\frac{dx_i}{dt}\Gamma_{ij}^k \quad \forall k = 1,\dots,n.$$

Introduce new variables $(x_1,\dots,x_n, y_1,\dots,y_n) \leftarrow 2n$-variables

A system of $2n$-first order ODE's (on TM) by

$$y_1 = \frac{dx_1}{dt}, \dots, y_n = \frac{dx_n}{dt}$$

and $$0 = \frac{dy_k}{dt} + \sum_{i,j} y_i y_j \Gamma_{ij}^k \qquad k = 1,\dots,n$$

This Defines a vector field on TM. Geodesics are the trajectories of the vector field.

**Lemma** There is a unique vector field $G$ on $TM$ whose trajectories are of the form $t \mapsto (\gamma(t), \gamma'(t))$ where $\gamma$ is a geodesic.

$\Rightarrow$ $\forall$ each point $(p,v) \in TM$ $\exists!$ geodesic $\gamma(0) = p$, $\gamma'(0) = v$.

**Example** $S^2$

- $c(t) = (\cos(t), \sin(t), 0)$ We saw: $\frac{D}{dt}\left(\frac{dc}{dt}\right) = 0$
- $\Rightarrow$ great circles are geodesics. as a set $C = \pi \cap S^2$, $\pi$ plane through origin. parametrized at constant speed.

On the other Hand: $\forall p \in S^2$, $v \in T_pS^2$ $\exists$ a great circle through $p$ containing $v$ $\Rightarrow$ Great circles are only ____ geodesics on $S^2$.

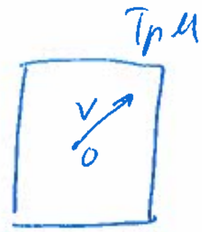
# The exponential map

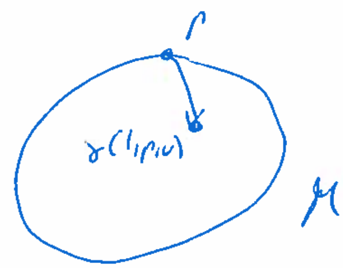
03/05

: let $p \in M, v \in T_pM$  $\gamma(t)$  $\gamma(t,p,v) = \gamma(t)$ where $\gamma(t)$ is the unique geodesic with $\gamma(0)=p$, $\frac{d\gamma}{dt}(0) = v$.

$\exp_p(v) = \gamma(1,p,v)$ (if it exists)  $\exp_p \sim T_p$

$\exp_p : D \subseteq T_pM \to M.$




$\exp_p(0) = p.$

Q: when is $\exp_p$ defined?

Lemma (Homogeneity of geodesics)

$\gamma(at,p,v) = \gamma(t,p,av)$, $a>0$

Proof let $h(t) = \gamma(at,p,v)$, $h(0)=p$, $h'(t) = a\gamma'(at)$, $h'(0) = av$.

$$D_{h'(t)} h'(t) = D_{a\gamma'(at)} a\gamma'(at) = a^2 D_{\gamma'(at)} \gamma'(at) = 0.$$

By uniqueness $h(t) = \gamma(t,p,av)$.

Lemma $\exists\, \varepsilon > 0$ s.t. $\exp_p$ is defined on $B_\varepsilon(0) \subset T_pM$. 03/10

Pf: By local existence Thm for ODE's.

$\gamma(t,p,v)$ is defined for $|t| < \delta_1$, $|v| < \delta_2$

By Homogeneity $\gamma(t, p, \frac{\delta_1}{2} v) = \gamma(\frac{\delta_1}{2} t, p, v)$ is defined for $|t| < 2$
$|v| < \delta_2$.

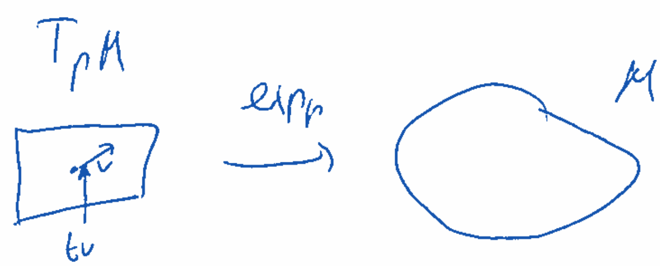
take $\varepsilon < \frac{\delta_1 \delta_2}{2}$, $|w| < \varepsilon$

$$\gamma(t, p, w) = \gamma(t, p, \tfrac{\delta_1}{2}(\tfrac{2}{\delta_1} w)) \qquad |\tfrac{2}{\delta_1} w| < \delta_2$$

Defined for $|t| < 2$. $\Rightarrow$ $\exp_p$ exists on $B_\varepsilon(0)$. $\square$

ODE theory $\Rightarrow$ solutions depend smoothly on initial conditions.

$\Rightarrow$ $\exp_p : B_\varepsilon(0) \to M$ diff'ble. What is the derivative at 0?




$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} (\gamma(1, p, tv))$$

$$= \frac{d}{dt}\Big|_{t=0} (\gamma(t,p,v)) = \gamma'(t,p,v)|_{t=0} = v.$$

Inverse function Thm $\Rightarrow$ $\exp_p$ local diffeom in a nbd of $0$. 03/05

03/07

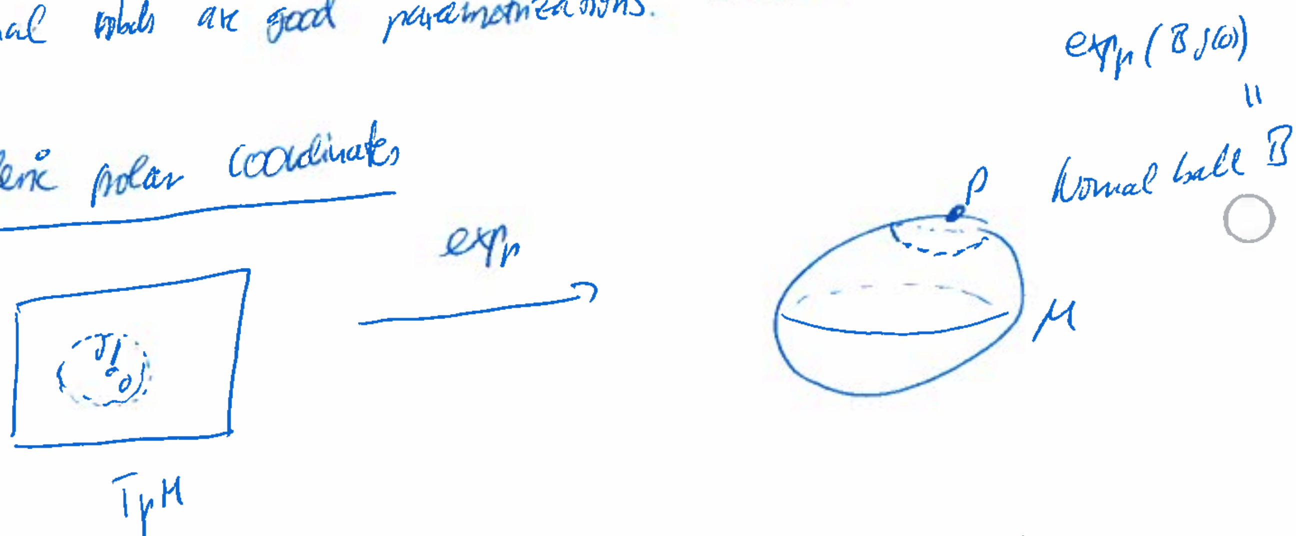
Prop: $\exp_p$ local diffeo at $0$.

So, we can use $\exp_p$ to define coordinates

Def: Consider $\delta>0$ s.t. $\exp_p|_{B_\delta(0)}$ is a ~~local~~ diffeom., then $\exp_p(B_\delta(0)) \subseteq M$ called the "normal ball" in $M$ (normal nbd.)

Normal nbds are good parametrizations.

Geodesic polar coordinates




$B(0,\delta)\setminus\{0\} \overset{\text{diffeo}}{\approx} (0,\delta)\times S^{n-1}$ $(v,\theta)$. Fix $(v_0,\theta_0)$

Let $E_1,\dots,E_{n-1}$ any coordinates of $S^{n-1}$ around $\theta_0$. Geodesic polar coordinate on $B\setminus\{p\}$. $q\in B$, $q=\exp_p(v)$, $v\in B_\delta(0)$.

$$\frac{\partial}{\partial r}\Big|_q = d(\exp_p)_v\left(\frac{v}{|v|}\right)$$

$\frac{\partial}{\partial \theta_i}\Big|_q = d(\exp_p)_v(rE_i)$ define coordinates on $B$. 03/1

Gauss-lemma "The exponential map is a radial isometry"

i.e. $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = 1$, $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i} \rangle = 0$.

$g_{11} \equiv 1$, $g_{1i} \equiv 0$ $\qquad \langle \cdot , \cdot \rangle = dr^2 + g_r \leftarrow$ $g_r$ metric on $S^{n-1}$ that depends on $r$.

Proof: $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle_q = \langle d(\exp_p)_v(\frac{v}{|v|}), d(\exp_p)_v(\frac{v}{|v|}) \rangle$

$= \frac{1}{|v|^2} \langle d(\exp_p)_v(v), d(\exp_p)_v(v) \rangle$

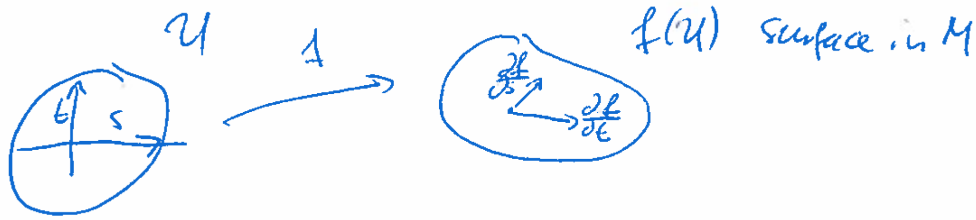
$d(\exp_p)_v(v) = \frac{d}{dt}\Big|_{t=1} (\exp_p)(tv) = \frac{d}{dt}\Big|_{t=1} \gamma(1, p, tv)$

$= \frac{d}{dt}\Big|_{t=1} \gamma(t, p, v) = \gamma'(1, p, v)$ Since geodesics have const.

speed $|\gamma'(t, p, v)| = |\underbrace{\gamma'(0, p, v)}_{=v}| \equiv |v|$

$\Rightarrow \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle_q = \frac{1}{|v|^2} |v|^2 = 1$.

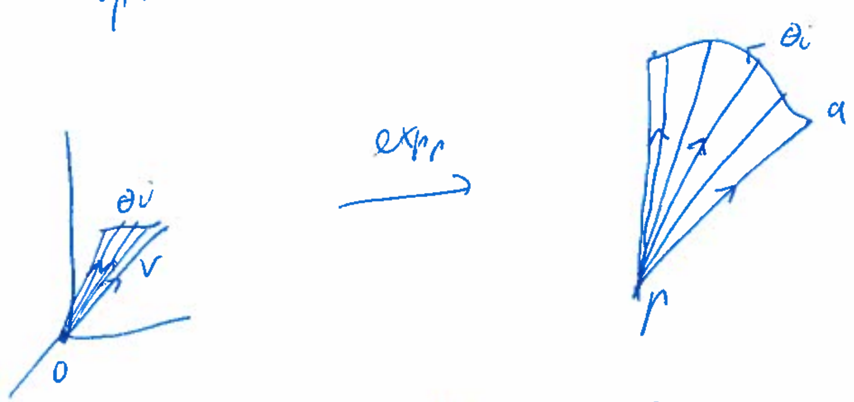
$f : \underset{\text{open}}{U} \subseteq \mathbb{R}^2 \to M^n$ is an ~~diffeom~~ embedding. 03/07




<u>Lemma</u> $\frac{D}{dt}\left(\frac{\partial f}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial f}{\partial t}\right)$.

<u>Proof:</u> ~~Exer~~ Exercise, Local coordinates.

$T_pM \xrightarrow{\exp_p} M$




$f(v, \theta_i)$ for fixed $i \in \{1, \dots, n-1\}$

want $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i} \rangle = 0$

$$\lim_{r\to 0} \frac{\partial}{\partial \theta_i}(v, \theta_0) = \lim_{r \to 0} d(\exp_p)_{r\theta_0}(rE_i) = \lim_{r\to 0} r\,(d\exp_p)_{(v,\theta_0)}(E_i)$$

$= 0$ (b/c. not defined at $r=0$)

$$\frac{\partial}{\partial r}\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i}\rangle = \langle \underset{\substack{| \\ \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0}}{\frac{\partial}{\partial r}}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta_i} \rangle = \langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta_i}} \frac{\partial}{\partial r}\rangle$$

$$= \frac{1}{2} \frac{\partial}{\partial \theta_i} \underbrace{\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\rangle}_{=1} = 0.$$

$\Rightarrow \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i} \rangle = 0.$

## Minimizing property of geodesics

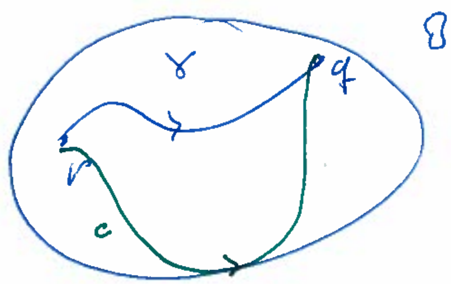
Let $x, y \in M$. $d(x,y) = \inf \{ \ell(c) \mid c(0) = x, c(1) = y$ $c$ piecewise smooth path $\}$

and $\ell(c) = \int_0^1 \langle \frac{dc}{dt}, \frac{dc}{dt} \rangle dt$

A curve is called minimizing on $c|_{[a,b]}$ if
$\ell(c|_{[a,b]}) = d(c(a), c(b))$.

Prop Let $B$ be a normal ball around $p$, $q \in B$. Let $\gamma$ be the unique geodesic with $\gamma(0) = p$, $\gamma(1) = q$ and let $c$ be any piecewise diff'ble curve from $p$ to $q$. Then $\ell(c) \geq \ell(\gamma)$ and
"=" iff $\gamma([0,1]) = c([0,1])$.

Proof:




Assume $c(t)$ stays inside $B$.

$c(t) = (r(t), \theta(t))$ in geodesic polar coordinates.

then $\frac{dc}{dt} = r'(t)\frac{\partial}{\partial r} + \sum_{i=1}^{n-1} \frac{\partial \theta_i}{\partial t}\frac{\partial}{\partial \theta_i}$

$\left|\frac{dc}{dt}\right|^2 = (r'(t))^2 + \dots \geq |r'(t)|^2$

let $\varepsilon > 0$

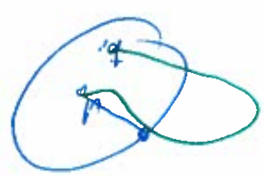
$$\int_\varepsilon^1 |c'(t)| \geq \int_\varepsilon^1 |r'(t)|\,dt \geq \int_\varepsilon^1 r'(t)\,dt = \underbrace{r(1)}_{=\ell(\gamma)} - \underbrace{r(\varepsilon)}_{\to 0}$$

check $\gamma(t) = (tr_0, \theta_0)$ in normal coordinates

If = then $\frac{\partial \theta_i}{\partial t} = 0 \ \forall i. \leadsto$ geodesic

On the other hand if $c$ leaves $B$

$B = \exp_p(B(0))$



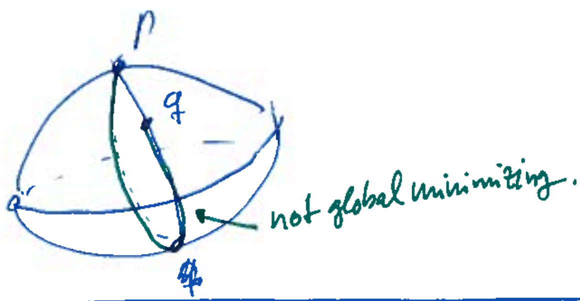

$\ell(\gamma) < \delta \leq \ell(c)$ comparing to geodesic from $p$ to the first point when $c$ leaves $B$.

Cor $B(p,\delta) \subseteq U = \{q : d(p,q) < \delta\}$. Then if $\exp_p$ is a diffeo on $B(0,\delta) \subset T_pM$ then $\exp_p(B(0,\delta)) = B(p,\delta)$

normal balls are metric balls.

Geodesics locally minimize distance.

$S^2$




03/07

03/21

Last time: Geodesics $\quad \nabla_{\frac{dc}{dt}} \frac{dc}{dt} = 0$

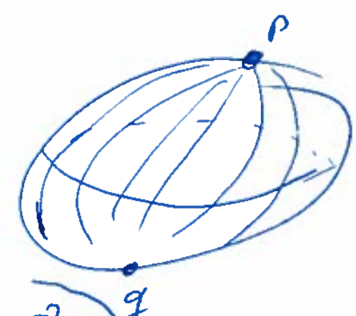
- $\gamma(t,p,v) = \gamma(t)$ the unique Geodesic with $\gamma(0)=p$, $\frac{d\gamma}{dt}(0)=v$.
- $p \in M$: $\quad \exp_p : U \subseteq T_pM \to M \quad \exp_p(v) = \gamma(1,p,v)$
- $\exp_p$ smooth, local diffeom. at $0 \in T_pM$.
- A normal ball $B \subset M$ around $p$ is a set of the form $B = \exp_p(B(0,\delta))$ where $\exp_p$ diffeom. on $B(0,\delta)$

Last time

- let $q \in$ normal ball around $p$, then the geodesic from $p$ to $q$ in $B$ is the shortest path from $p$ to $q$
- Geodesics locally minimize arclength.

Remember: only true locally. (Sphere)

$S^2 \setminus \{q\}$ normal ball around $p$ $(q = -p)$




i.e. $\exp_p$ diffeom. on $B(0,\pi)$, $\exp_p(B(0,\pi)) = S^2 \setminus \{q\}$

03/21

On the other hand, if a curve minimizes arc length, then it is a geodesic.

[$\exists\, W$ open nbd. of $p$]

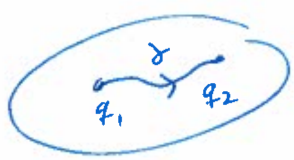
<u>Lemma</u> For any $p \in M$ $\exists\, \delta > 0$ s.t. $\forall q \in W$ $\exp_q$ diffeom. on $B(0,\delta) \subseteq T_qM$ and $W \subseteq \exp_q(B(0,\delta))$ i.e.

$W$ normal nbd. around every $q \in W$. $W$ is called a <u>totally normal neighborhood</u>   <u>Proof</u>. p. 72 $\square$

<u>Ex:</u> $S^2$
- $S^2 \setminus \{q\}$ normal nbd.
- $W$ = Hemisphere

<u>Note:</u> $W$ totally normal, $q_1, q_2 \in W$

$\exists!$ geod. $\gamma$ contained in $W$




Changing $q_1, q_2$ geodesic $\gamma$ changes smoothly.

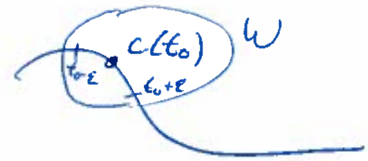
$\int_J \langle \frac{dc}{dt}, \frac{dc}{dt} \rangle_{c(t)}^{\frac{1}{2}} \, dt$

<u>Prop</u> If $c$ is a piecewise diff'ble curve $c : [a,b] \to M$ parametrized proportional to arc length and minimizing

then $c$ is a geodesic

Proof: Suppose $c$ not a geodesic at $c(t_0)$

$W$ totally normal nbd. of $c(t_0)$




$\exists\, \varepsilon > 0$ s.t. $c([t-\varepsilon, t+\varepsilon]) \subseteq W$.

~~then~~ let $q_1 = c(t_0+\varepsilon), q_2 = c(t_0-\varepsilon) \Rightarrow q_1, q_2 \in W$. $W$ normal

nbd. around $q_1$ $\exists!$ geod. $\gamma$ from $q_1$ to $q_2$ in $W$.

$\gamma$ is minimizing from $c_1$ to $c_2$, $c$ not geod $\Rightarrow \gamma \neq c$.

$\Rightarrow \ell(\gamma) < \ell(c|_{[t_0-\varepsilon, t_0+\varepsilon]})$. This contradicts that $c$ minimizes

$\Rightarrow c$ geodesic.

Corollary Minimizing curves are smooth.

Summary Any locally minimizing curve is a geodesic

Any geodesic is locally minimizing.

On the other hand, there is ~~~~ not always a minimizing

geodesic between $x$ and $y$

Ex: $M = \mathbb{R}^2 \setminus \{0,0\}$


No geodesic from $x$ to $y$

03/21

## Curvature:

**Q:1** Are all Riemannian manifolds (of same dimension) locally isometric?

**Recall** ~~M loc. isom. to N if~~ ~~$\varphi: M \to N$ local isometry if $\varphi$ local diffeom.~~

~~Def~~ $M$ loc. isom. to $N$ if $\forall p \in M$ $\exists\, \mathcal{U} \ni p$ open s.t. $\exists\, \varphi: \mathcal{U} \to V \subseteq_{open} N$ diffeo. s.t. $\langle u, v\rangle_q = \langle d\varphi_q(u), d\varphi_q(v)\rangle_{\varphi(q)}$

$\forall q \in \mathcal{U}$.

$\boxed{A}$ No! Yes, if we ask for locally diffeomorphic.

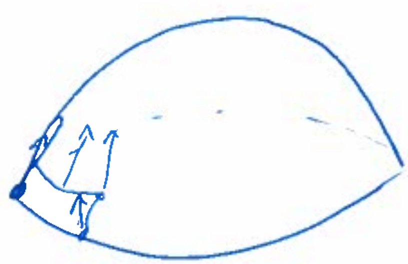
**Example** $S^2$ and $\mathbb{R}^2$ are not locally isometric $\Leftarrow$ Mapmaking.

$\hookrightarrow$ Curvature differ $S^2 > 0$, $\mathbb{R}^2 = 0$ & curv. loc. isometry invariant

**Also** $S^2, \mathbb{R}^2$ not loc. isometric because

in $\mathbb{R}^2$ parallel translation path independent


$S^2$

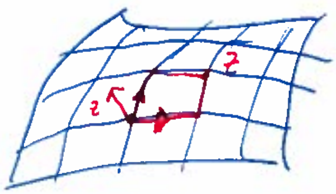


On $S^2$ not path. indep.

& under local isom. parallel fields go to parallel fields.

03/2

If a surface is loc. isometric to $\mathbb{R}^2$, then locally parallel translation is path independent. Take coords. $(x_1, x_2)$ on a small open set where parallel transl. is path independent.




Define $Z$ parallel vector field by taking $Z_p$ and first parallel transl. along $x_1$ then along $x_2$

Independence of path $\Rightarrow$ $Z$ parallel.

$$\nabla_{\frac{\partial}{\partial x_1}} Z = \nabla_{\frac{\partial}{\partial x_2}} Z = 0. \Rightarrow \nabla_{\frac{\partial}{\partial x_2}}\left(\nabla_{\frac{\partial}{\partial x_1}} Z\right) = \nabla_{\frac{\partial}{\partial x_1}}\left(\nabla_{\frac{\partial}{\partial x_2}} Z\right) = 0$$

Curvature measures how far this is from being true

<u>Curv. tensor</u> $X, Y, Z \in \mathcal{X}(M)$

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

---

03/2

<u>Remarks</u> ① Some authors use $(-)$ this definition.

② $M = \mathbb{R}^n$ euclidean space $\Rightarrow R(X,Y)Z = 0 \quad \forall X, Y, Z.$

③ Local coordinates $\{x_i\}$ $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(\nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} - \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}}\right)\left(\frac{\partial}{\partial x_k}\right)$$

03/26

④ If $\varphi: M \to N$ local isometry

$$R^M(X,Y)Z = R^N(d\varphi(X), d\varphi(Y))\, d\varphi(Z)$$

Exercise

⑤ Note the symmetry $R(X,Y)Z = -R(Y,X)Z$.

<u>Prop:</u> $R$ is a tensor, i.e. $\forall f, g: M \to \mathbb{R}$

1) $R(fX_1 + gX_2, Y)Z = fR(X_1,Y)Z + gR(X_2,Y)Z$

2) $R(X, fY_1 + gY_2)Z = fR(X,Y_1)Z + gR(X,Y_2)Z$

3) $R(X,Y)(fZ_1 + gZ_2) = fR(X,Y)Z_1 + gR(X,Y)Z_2$

1, 2) Exercise

<u>Pf of 3)</u> <u>Note:</u> $R(X,Y)(Z_1+Z_2) = R(X,Y)Z_1 + R(X,Y)Z_2$ ✓

<u>ETS:</u> $R(X,Y)(fZ) = fR(X,Y)Z$

$$D_Y D_X(fZ) = D_Y(X(f)Z + fD_XZ)$$

$$= Y(X(f))Z + X(f)D_YZ + Y(f)D_XZ + fD_YD_XZ$$

$$D_X D_Y(fZ) = X(Y(f))Z + Y(f)D_XZ + X(f)D_YZ + fD_XD_YZ.$$

$D_{[X,Y]}(fZ) = [X,Y](f)Z + fD_{[X,Y]}Z$. So: $D_YD_X(fZ) - D_XD_Y(fZ) + D_{[X,Y]}(fZ)$

$= fR(X,Y)Z$

03/2

Corollary $R(X,Y)Z|_p$ depends only on the ~~vec~~ values $X|_p, Y|_p, Z|_p$. (We can write $R(u,v)w$, $u,v,w\in T_pM$)

Proof $E_1,\dots,E_n$ frame around $p$, $X=\sum a_i E_i$, $Y=\sum b_j E_j$

$Z=\sum c_k E_k$ $\quad R(X,Y)Z=\sum_{i,j,k} a_i b_j c_k R(E_i,E_j)E_k$

(Different extension $\bar{X}=\sum a_i' E_i$).

(w. $a_i'(p)=a_i(p)$)

(Historically significant) special case Let $M^n\subseteq\mathbb{R}^{n+1}$ with metric induced by dot product in $\mathbb{R}^{n+1}$

Def: $R(X,Y,Z,W):=\langle R(X,Y)Z,W\rangle$

4-curvature-tensor

~~$V,W\in T_pM$ $\langle V,W\rangle = V\cdot W$~~

$X,Y$ VF on $M$ $\langle X,Y\rangle_M = X\cdot Y$, $N$ unit normal vector.

$\nabla_X^M Y=(\nabla_X Y)^T$ $\qquad \nabla_X Y=(\nabla_X Y)^T+\langle\nabla_X Y,N\rangle N$

euclidean

$\nabla_X^M Y=(\nabla_X Y)^T=\nabla_X Y-\langle\nabla_X Y,N\rangle N.$

03/26

**Def:** The second fundamental form of $M$ at $p$

(dep. on $N$) $\quad \mathrm{II}_N(X,Y) = -\langle \nabla_X Y, N\rangle$, $X,Y$ VF on $M$

**Prop** $\mathrm{II}_N(X,Y) = -\langle \nabla_X Y, N\rangle = \langle Y, \nabla_X N\rangle = -\langle \nabla_Y X, N\rangle$

$= \langle X, \nabla_Y N\rangle \qquad = \mathrm{II}_N(Y,X)$

**Proof:** $X,Y$ tangent fields, $N \perp T_pM$

$$0 = \langle [X,Y]^{\mathbb{R}^{n+1}}, N\rangle \overset{\text{Torsion free}}{=} \langle \nabla_X Y, N\rangle - \langle \nabla_Y X, N\rangle$$

$$\Leftrightarrow \langle \nabla_X Y, N\rangle = \langle \nabla_Y X, N\rangle$$

$$\langle \nabla_X Y, N\rangle \overset{\text{Compatible}}{=} X\underbrace{\langle Y, N\rangle}_{=0} - \langle Y, \nabla_X N\rangle$$

$$\langle \nabla_X Y, N\rangle = -\langle Y, \nabla_X N\rangle \qquad \square$$

$\Rightarrow$ **Corollary** $\mathrm{II}_N(X,Y)$ symmetric in $X,Y$, tensor in both entries,

(only depends on $X|_p$, $Y|_p$)

$\Rightarrow \mathrm{II}_N(X,Y) \rightsquigarrow$ Derivative of $N$, change in normal vector

**Gauss Map** $G : M \longrightarrow S^n$

$p \longmapsto N(p)$

$I\!I_N \rightsquigarrow$ Derivative of $G$.

**Gauss Equation** $M^n \subseteq \mathbb{R}^{n+1}$, $X, Y, Z, W \in T_pM$. Then

$$R(X,Y,Z,W) = I\!I_N(X,Z)\,I\!I_N(Y,W) - I\!I_N(Y,Z)\,I\!I_N(X,W)$$

---

**Example** $n=2$, $M^2 \subseteq \mathbb{R}^3$. $X, Y \in T_pM$, $X \perp Y$ orthonormal basis

$$R(X,Y,X,Y) = I\!I_N(X,X)\,I\!I_N(Y,Y) - I\!I_N(X,Y)^2 = \det A = \det(I\!I_N).$$

$\parallel$ $\lambda_1 \lambda_2$ eigen-values

$$A = \begin{pmatrix} I\!I_N(X,X) & I\!I_N(X,Y) \\ I\!I_N(Y,X) & I\!I_N(Y,Y) \end{pmatrix}$$

$V_1, V_2$ an o.n. basis of eigenvectors. with eigenvalues $\lambda_1$, $\lambda_2$.

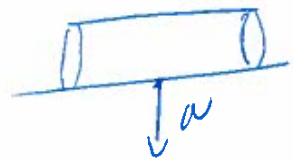
$\lambda_1 = I\!I_N(V_1, V_1)$, $0 = I\!I_N(V_1, V_2)$, $\lambda_2 = I\!I_N(V_2, V_2)$

$= \langle V_1, D_{V_1} N \rangle$ $\quad = \langle V_2, D_{V_1} N \rangle$ $\quad = \langle V_2, D_{V_2} N \rangle$

$\quad = \langle V_1, D_{V_2} N \rangle$

$\det(I\!I_N) > 0 \Rightarrow \lambda_1, \lambda_2$ same sign

$\det(\mathrm{II}_N)=0$, $\lambda_i=0$ 




$\det(\mathrm{II}_N)<0$ $\lambda_1, \lambda_2$ opposite signs

~~Saddle~~

$\boxed{\text{Saddle}}$

geodesic $\gamma$, $\gamma'(0)=V_1$ $\quad \frac{d}{dt}\langle \frac{\partial \gamma}{\partial t}, N\rangle = 0$

<u>Proof:</u> $D_X^M Y = (D_X Y)^T = D_X Y + \mathrm{II}_N(X,Y)N$

$$D_Y^M(D_X^M Z) = D_Y^M(D_X Z + \mathrm{II}_N(X,Z)N)$$

$$= D_Y^M(D_X Z - \langle D_X Z, N\rangle N)$$

$$= D_Y(D_X Z - \langle D_X Z, N\rangle N) - \langle D_Y(D_X Z - \langle D_X Z, N\rangle N), N\rangle N$$

$$= D_Y D_X Z - \langle D_X Z, N\rangle D_Y N + (\cdot\ \cdot)N$$

$$D_X^M(D_Y^M Z) = D_X D_Y Z - \langle D_Y Z, N\rangle D_X N + (\cdot\ \cdot)N$$

$$D^M_{[X,Y]} Z = \nabla_{[X,Y]} Z - \langle D_{[X,Y]} Z, N\rangle N$$

03/28

$\Rightarrow R(X,Y,Z,W) = \langle \cancel{\nabla_X \nabla_Y} \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, W \rangle$

$- \langle \nabla_X Z, N \rangle \langle \nabla_Y N, W \rangle$

$+ \langle \nabla_Y Z, N \rangle \langle \nabla_X N, W \rangle$

$= \mathrm{II}_N(X,Z)\, \mathrm{II}_N(Y,W) - \mathrm{II}_N(Y,Z)\, \mathrm{II}_N(X,W) \qquad \square$

---

03/28

For more general <u>Gauss-Equ.</u> ~~see~~ see p. 130 Thm. 2.5
p. 135 ~~Thm~~ Prop 3.1

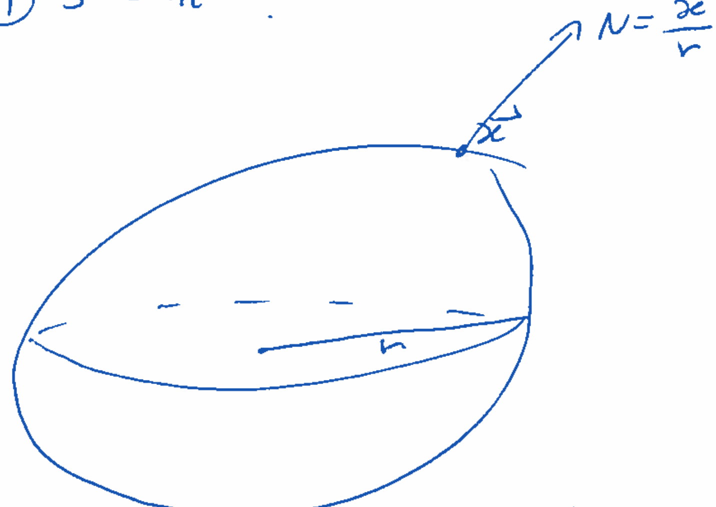
for a more general version

$M^2 \subseteq \mathbb{R}^3$, $X \perp Y$, $|X| = |Y|$ $\langle R(X,Y)X, Y \rangle = \mathrm{II}_N(X,X)\mathrm{II}_N(Y,Y) - \mathrm{II}_N(X$

$=$ Gauss Curvature of $M$.

---

<u>Examples</u> ① $S^n \subseteq \mathbb{R}^{n+1}$.




$\mathrm{II}_N(X,Y) = \langle \nabla_X N, Y \rangle^{\mathbb{R}^{n+1}} = \langle \nabla_X \frac{\vec{x}}{r}, Y \rangle = \frac{1}{r} \langle \nabla_X \vec{x}, Y \rangle$

$X = \frac{\partial}{\partial x_i}$, $\vec{x} = (x_1, \ldots, x_n) \Rightarrow \quad \nabla_{\frac{\partial}{\partial x_i}} (\vec{x}) = \frac{\partial}{\partial x_i}$

By linearity $D_X \vec{x} = X$

$\Rightarrow \mathrm{II}_N(X,Y) = \frac{1}{r}\langle X, Y\rangle$

$X, Y, |X| = |Y| = 1, \quad Y \perp X.$

$$R(X,Y,X,Y) = \mathrm{II}_N(X,X)\,\mathrm{II}_N(Y,Y) - \mathrm{II}_N(X,Y)^2 = \frac{1}{r}\cdot\frac{1}{r} - 0$$

$$= \frac{1}{r^2}.$$

Example ② More generally

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $a$ regular value of $f$ and then $M = f^{-1}(a)$

is a hypersurface $\quad \langle \nabla f, X\rangle = df(X)$

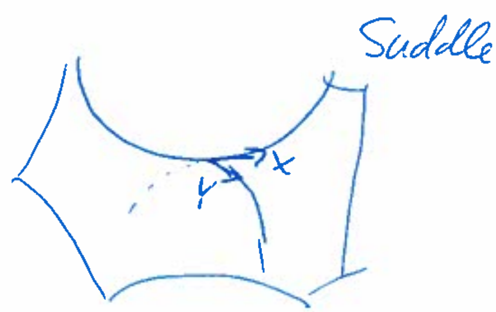
$\nabla f = \sum \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}$ perp. to $M$. $\quad N = \frac{\nabla f}{|\nabla f|}$

If $X, Y \perp \nabla f$. $\quad \mathrm{II}_N(X,Y) = \langle D_X N, Y\rangle$

$$= \langle D_X \frac{\nabla f}{|\nabla f|}, Y\rangle = \langle X\left(\frac{1}{|\nabla f|}\right)\nabla f + \frac{1}{|\nabla f|} D_X \nabla f, Y\rangle$$

$$= \frac{1}{|\nabla f|}\langle D_X \nabla f, Y\rangle$$

~~Saddle~~ Saddle $\quad z = x^2 - y^2 \subseteq \mathbb{R}^3$




$f(x,y,z) = x^2 - y^2 - z$

$$\nabla f = 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$$

$$\cancel{N = \frac{1}{1+\sqrt{4x^2+4y^2}}}$$

$$N = \frac{1}{\sqrt{1+4x^2+4y^2}}\left(2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)$$

let $X = \frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}$

$$\nabla_X \nabla f = \nabla_{\left(\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}\right)}\left(2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)$$

$$= 2\frac{\partial}{\partial x} \qquad (\Gamma_{ij}^k = 0 \ \forall i,j,k)$$

$$\nabla_Y \nabla f = \nabla_{\left(\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}\right)} 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - \frac{\partial}{\partial z} = -2\frac{\partial}{\partial y}.$$

$$\cancel{\langle R(X,Y)X, Y\rangle} = \mathrm{II}_N(X,X) = \frac{1}{|\nabla f|}\langle \nabla_X \nabla f, X\rangle$$

$$= \frac{1}{\sqrt{1+4x^2+4y^2}}\cdot 2$$

$$\mathrm{II}_N(X,Y) = \frac{1}{|\nabla f|}\langle \nabla_X \nabla f, Y\rangle = 0$$

$$\mathrm{II}_N(Y,Y) = \frac{1}{|\nabla f|}\langle \nabla_Y \nabla f, Y\rangle = -2.$$

03/28

So: $R(X,Y,X,Y) = \dfrac{-4}{\sqrt{1+4x^2+4y^2}}$

Q| Why is this enough to know the whole curvature tensor?

In particular Gauss-curvature $< 0$.

Q| Can every compact orientable Riem. n-mfld be isometrically embedded in $\mathbb{R}^{n+1}$?

$(M, \langle\cdot,\cdot\rangle) \xrightarrow{\varphi} (\mathbb{R}^{n+1}, \cdot)$ & $\varphi$ isometric immersion.

A| NO! Gauss equation $\Rightarrow$ obstruction to having such an embedding.

Example Suppose $n \geq 3$ and $M^n$ Riem. mfld with property

$\langle R(X,Y)X, Y\rangle < 0 \quad \forall\ X, Y,\ X \perp Y$ ("negative sectional curvature")

Then $M$ cannot be embedded isometrically in $\mathbb{R}^{n+1}$

Pf: By Gauss equation:

$\mathrm{II}_N(X,Y)$ has eigenvalues $\lambda_1, \dots, \lambda_n$; $v_1, \dots, v_n$ orthogonal eigenvectors

$\langle R(v_1,v_2)v_1, v_2\rangle = \mathrm{II}_N(v_1,v_1)\,\mathrm{II}_N(v_2,v_2) - \mathrm{II}_N(v_1,v_2)^2$

$= \lambda_1\lambda_2 - 0 < 0$

03/2

$R(v_2, v_3, v_2, v_3) = \lambda_2 \lambda_3 < 0$

$R(v_1, v_3, v_1, v_3) = \lambda_1 \lambda_3 < 0$

Contradiction!

~~Ma~~ Back to curvature tensor of a general $M$, ~~g~~ $\langle \cdot, \cdot \rangle$

Symmetries of $R$ $\quad R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,}$

① $R(X,Y,Z,W) = -R(Y,X,Z,W)$.

~~② $R(X,Y,Z,Z) = R(X,Y$~~

② $R(X,Y,Z,Z) = 0$

③ $R(X,Y,Z,W) = -R(X,Y,W,Z)$

Pf: ② $\Rightarrow$ ③ $\quad 0 \overset{②}{=} R(X,Y,Z+W,Z+W) = \cancel{R(X,Y,Z,Z)} + R(X,Y,Z,W) + R(X,Y,W,Z) + \cancel{R(X,Y,W}$

② : $\langle \nabla_Y \nabla_X Z, Z \rangle = Y \langle \nabla_X Z, Z \rangle - \langle \nabla_X Z, \nabla_Y Z \rangle$

$= Y\left(X\left(\frac{1}{2}\langle Z,Z\rangle\right)\right) - \langle \nabla_X Z, \nabla_Y Z \rangle$

$\langle \nabla_X \nabla_Y Z, Z \rangle = X\left(Y\left(\frac{1}{2}\langle Z,Z\rangle\right)\right) - \langle \nabla_Y Z, \nabla_X Z \rangle$.

$\langle \nabla_{[X,Y]} Z, Z \rangle = [X,Y]\left(\frac{1}{2}\langle Z,Z\rangle\right)$. $\Rightarrow$ ②.

03/28

## ④ Bianchi-Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Pf:

$$\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z + \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X$$

$$+ \nabla_{[Y,Z]} X + \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y + \nabla_{[Z,X]} Y$$

$$= \nabla_Y [X,Z] + \nabla_X [Z,Y] + \nabla_Z ([Y,X])$$

$$+ \nabla_{[X,Y]} Z + \nabla_{[Y,Z]} X + \nabla_{[Z,X]} Y$$

$$= [Y,[X,Z]] + [X,[Z,Y]] + [Z,[Y,X]] = 0$$ by Jacobi-Identity.

## ⑤ $R(X,Y,Z,T) = R(Z,T,X,Y)$.

Proof Bianchi: $R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0$

$$R(Y,Z,T,X) + R(Z,T,Y,X) + R(T,Y,Z,X) = 0$$

$$R(Z,T,X,Y) + R(T,X,Z,Y) + R(X,Z,T,Y) = 0$$

$R(T,X,Y,Z) + R(X,Y,T,Z) + R(Y,T,X,Z) = 0$ 03/28

Cancelling by (3)

$\Rightarrow 2R(Z,X,Y,T) + 2R(Y,T,X,Z) = 0$

$\Rightarrow R(Z,X,Y,T) = R(Y,T,Z,X)$ $\square$

Curvature tensor as an operator:

$\Lambda^2(\overset{V}{T_pM}) = \operatorname{span}\{v\wedge w : v,w \in V\}$ $\quad \Lambda : \begin{cases} v\wedge w = -w\wedge v. \\ v\wedge v = 0 \end{cases}$

If $\langle\cdot,\cdot\rangle$ on $V$.

define $|v\wedge w|^2 := |v|^2|w|^2 - \langle v,w\rangle^2$ defines inner product norm

Def: $\mathcal{R} : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ curvature operator

inner prod. on 2-forms

$\Leftarrow R(X,Y,Z,W) = \langle \mathcal{R}(X\wedge Y), Z\wedge W\rangle$

$\langle X\wedge Y, Z\wedge W\rangle = \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y$

antisymmetry in $X,Y$ and $Z,W$ $\Rightarrow$ well defined on $\Lambda^2(T_pM)$

$\Leftarrow |X\wedge Y|^2 = |X|^2|Y|^2 - \langle X,$

$R(X,Y,Z,W) = R(Z,W,X,Y) \Rightarrow \mathcal{R}$ is self-adjoint.

03/28

If $\dim V = 2$, $\dim(\Lambda^2(V)) = 1$

$\textcircled{R}$ $R$: 1-dim $\to$ 1-dim determined by $\langle R(x \wedge y), x \wedge y \rangle$
$= R(x, y, x, y)$.

---

04/02

## Sectional Curvature

Let $\sigma$ be a plane in $T_pM$, and

$X, Y$ linear indep. vectors in $\sigma$.

Def: $$K(\sigma) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y\rangle^2} = \frac{R(X,Y,X,Y)}{|X\wedge Y|^2}$$

Note $K(\sigma)$ does not depend on the choice of basis $X, Y$.

Change basis ⓐ $\{X,Y\} \to \{Y,X\}$ ✓

ⓑ $\{X,Y\} \to \{\lambda X, Y\}$ ✓

ⓒ $\{X,Y\} \to \cancel{\{X, X+Y\}}\ \{X+\lambda Y, Y\}$

$$R(X+\lambda Y, Y, X+\lambda Y, Y) = R(X,Y,X,Y)$$

$(\langle X, Y\rangle + \lambda\langle Y,Y\rangle)^2$

and $|X+\lambda Y|^2|Y|^2 - \langle X+\lambda Y, Y\rangle^2$

$= |X|^2|Y|^2 - \langle X,Y\rangle^2 + \cancel{(2\lambda\langle X,Y\rangle} + \lambda^2|Y|^2)|Y|^2$

✓ $\qquad - \lambda^2|Y|^4 \cancel{-2\lambda\langle X,Y\rangle|Y|^2}$

If $M^2 \subset \mathbb{R}^3$: Gauss curvature $= K(\sigma)$.

Ex: Saddle $z = x^2 - y^2$

$E_1 = \frac{\partial}{\partial x_1} + 2x\frac{\partial}{\partial z}$

$E_2 = \frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}$

$R(E_1, E_2, E_1, E_2) = -\frac{4}{\sqrt{1+4x^2+4y^2}}$

$|E_1|^2|E_2|^2 - \langle E_1, E_2\rangle^2 = \ldots = 1+4x^2+4y^2$.

So $k(\sigma) = \dfrac{-4}{(1+4x^2+4y^2)^{3/2}}$

__Lemma__ If have two Riemannian metrics $\langle \cdot,\cdot \rangle, \langle \cdot,\cdot \rangle'$ (on $M$) on $M$ and if

$R(X,Y,X,Y) = R'(X,Y,X,Y)$ (at $p$), then $R(X,Y,Z,W) = R'(X,Y,Z,W)$ $\forall X,Y,Z,W \in T_pM$.

"__Pf__": $R(X,Y,X,Y) = \langle R(X\wedge Y), X\wedge Y\rangle$

see p. 95.

__Defn__ A Riem. Mfld $M$ has constant sectional curvature $c$ at $p$ if $k(\sigma) = c$ $\forall$ 2-planes $\sigma \subset T_pM$.

__r__: Then ⓐ $R(X,Y,Z,W) = \underbrace{c\big(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\big)}_{= R'(X,Y,Z,W)}$ $\forall X,Y,Z,W \in T_pM$.

ⓑ $k(\sigma) = 0$, then $R(X,Y,Z,W) = 0$ $\forall X,Y,Z,W \in T_pM$ $\forall \sigma \in T_pM$

$(\Rightarrow)$ $R(X\wedge Y) = c(X\wedge Y)$ $\forall X,Y \in T_pM$.

04/0

# Tensors in general

## Tensors in linear Algebra

$V$ v.s. (finite dim.), $(r,s) \in \mathbb{N}_0^2$.

An $(r,s)$-tensor is a map

$$T : \underbrace{(V^* \times \ldots \times V^*)}_{r\text{-times}} \times \underbrace{(V \times \ldots \times V)}_{s\text{-times}} \to \mathbb{R}.$$

contravariant — covariant.

linear in each entry.

$(0,0)$-tensor is a number.

$(0,1)$-tensor $T : V \to \mathbb{R}$ $\in V^*$ linear functional.

$(0,2)$-tensor $T : V \times V \to \mathbb{R}$ bilinear form.

---

If $V$ has an inner product $\langle \cdot , \cdot \rangle$

Isomorphism $V \to V^*$ $v \mapsto \langle v, \cdot \rangle$.

Ex: $T$ $(1,1)$-tensor

$$T : V^* \times V \to \mathbb{R}.$$

$v \in V$ $\quad T(v) : V^* \to \mathbb{R}$, $\quad T(v)(\alpha) = T(\alpha, v)$

$\Rightarrow$ $T(v) \in (V^*)^* = V$, $\quad T : V \to V$.

04/a

Type change: $T : V \times V \to \mathbb{R}$ ~~$\Rightarrow$~~ $\Leftrightarrow$ $(1,1)$-tensor

(0,2)-tensor $\qquad T : V \to V$

$T(v,w) = \langle T(v), w \rangle$

In general, we can type change $(r,s)$ to $(0, r+s)$.

Exe: $R(X,Y)Z$ $\xrightarrow{\text{Type change}}$ $R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$

$\uparrow$ (1,3)-tensor $\qquad \uparrow$ (0,4)-tensor

## Tensors in Diff. Geom.

$\mathcal{D}$ = real valued, smooth functions on $M$.

$\mathfrak{X}(M)$ = Smooth Vector fields on $M$.

(0,s) tensor field is a map $T : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s} \to \mathcal{D}$.

(1,s) tensor field $\quad T : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s} \to \mathfrak{X}(M)$.

linear in each component.

04/02

**Examples** ① $f: M \to \mathbb{R}$

$df: \mathfrak{X}(M) \to \mathcal{D}.$

$df(X) = X(f)$. is a $(0,1)$-tensor. (1-form)

② $\nabla$ affine connection, $X$ vector field

$\nabla X: \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(\nabla X)(Y) = \nabla_Y X$ $(1,1)$-tensor

③ $\nabla_Y: \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla_Y(X) = \nabla_Y X$ not a tensor field

(not $\mathcal{D}$-linear / linear with respect to functions)

(need product rule)

④ If $M, \langle\, ,\, \rangle$ Riemannian mfld $g(X,Y) = \langle X, Y \rangle$ $(0,2)$-tensor field.

**Covariant Derivative of Tensors** $M$ smooth mfld, $\nabla$ affine connecti

Given vector field $X$ $\underbrace{\text{we}}_{(\ ,\ )\text{ Tensor } T}$ want to define

$\nabla_X T$ so that $\nabla_X T$ is a tensor of the same type as $T$.

e.g.: $f$ function $X$ $\nabla_X f = X(f)$.

If $Y, X$ vector fields $\nabla_X Y$ vector field 04/02

Let $T$ be a ~~C∞~~ $(1,1)$ tensor

$$(\nabla_X T)(Y) \overset{def}{:=} \nabla_X (T(Y)) - T(\nabla_X Y)$$

This is a ~~C∞~~ $(1,1)$-tensor! Exercise.

04/04

An $(0,r)$ tensor field is ~~a smo~~ a map

$$T : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{r} \to D \quad \text{linear in each variable}$$

~~An~~ A $(1,r)$ tensor field is a map

$$T : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{r} \to \mathfrak{X}(M) \quad \text{linear in each comp.}$$

We can always change a $(1,r)$ tensor to $(0,r+1)$-tensor and vice versa

$$\langle \underbrace{T(X_1, \dots X_r)}_{(1,r)\ \text{tensor}}, X_{r+1} \rangle \quad (0, r+1)\text{-tensor}.$$

$$\| $$

$$T(X_1, \dots, X_{r+1})$$

Exe: $R(X, Y)Z$ $(1,3)$ ~~curvat tensor~~ curvature-tensor

$$R(X, Y, Z, W) = \langle R(X,Y)Z, W \rangle \quad (0,4)\text{-tensor}$$

# Covariant Derivative of a tensor

**Def.** Given a tensor $T$, $X \in \mathfrak{X}(M)$

$$(\nabla_X T)(Y_1, \dots, Y_r) = \nabla_X \big( T(Y_1, \dots, Y_r) \big) - \sum_{i=1}^{r} T(Y_1, \dots, \nabla_X Y_i, \dots Y_r)$$

Then $\nabla_X T$ is a tensor of the same type.

If $T$ is a $(1,1)$-tensor, i.e. $T : \mathfrak{X}(M) \to \mathfrak{X}(M)$

then $(\nabla_X T)(Y) = \nabla_X (T(Y)) - T(\nabla_X Y)$

$$
\begin{aligned}
(\nabla_X T)(fY) &= \nabla_X (T(fY)) - T(\nabla_X (fY)) \\
&= \nabla_X (f T(Y)) - T(X(f) Y + f \nabla_X Y) \\
&= \cancel{X(f) T(Y)} + f \nabla_X (T(Y)) - \cancel{X(f) T(Y)} \\
&\qquad - f T(\nabla_X Y) \\
&= f \cdot (\nabla_X T)(Y).
\end{aligned}
$$

If $T$ $(s, r)$-tensor

$$(\nabla T)(X, Y_1, \dots, Y_r) = (\nabla_X T)(Y_1, \dots, Y_r)$$

$\uparrow$

$(s, r+1)$-tensor.

04/04

Examples: ① $f: M \to \mathbb{R}$ (0,0)-tensor

$$\nabla_X f = X(f) = df(X).$$

(0,1)-tensor

$(df_p(X) = X(f)|_p.)$

Type change of $df$. Define $\nabla f$ "gradient of $f$" to be the unique vector field s.t. $df(X) = \langle \nabla f, X \rangle = X(f)$

("gradient · vector = directional derivative")

② $df \to (0,1)$ tensor

$\nabla df \to (0,2)$ tensor

$$(\nabla df)(X,Y) = (\nabla_X df)(Y) = \nabla_X (df(Y)) - df(\nabla_X Y)$$

"Hessian of $f$"

$$= X(Y(f)) - df(\nabla_X Y)$$

$$= X(Y(f)) - (\nabla_X Y)(f).$$

We also write. $\mathrm{Hess}\, f(X,Y) = (\nabla_X df)(Y)$

in $\mathbb{R}^n \to E_i$ standard basis, $\mathrm{Hess}\, f(E_i, E_j) = E_i E_j(f)$

$$\nabla_{E_i} E_j = 0$$

Also $\langle \nabla_X \nabla f, Y\rangle = X\langle \nabla f, Y\rangle - \langle \nabla f, \nabla_X Y\rangle$ 04/06

$$= X(Y(f)) - (\nabla_X Y)(f) = \cancel{\text{Hd}}\ \text{Hess}(f)(X,Y).$$

$$= (\nabla df)(X,Y)$$

$$\begin{tikzcd} df \arrow[r, "\text{Type change}"] \arrow[d] & \nabla f \arrow[d, "\nabla"] \\ \nabla df = \text{Hess} f \arrow[r] & \nabla(\nabla f) \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, "\text{Type ch.}"] \arrow[d] \arrow[dr, phantom, "\text{Ca}"] & T^* \arrow[d] \\ \nabla T \arrow[r, "\text{Type}", "\text{change}"'] & \nabla(T^*) \end{tikzcd}$$

so $(\nabla T)^* = \nabla(T$

③ Defn $g(X,Y) = \langle X, Y\rangle$ $(0,2)$-tensor.

$$(\nabla_Z g)(X,Y) = Z\, g(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

$$= Z\langle X, Y\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle =$$

by compatibility. $\Rightarrow$ $\nabla g = 0$ $g$ is parallel wrt. $\nabla$.

④ Let $Z$ be a vector field - $(1,0)$ tensor.

$\nabla Z \leftarrow (1,1)$ tensor

$$(\nabla Z)(Y) = \nabla_Y Z$$

$$(\nabla_Y(\nabla Z))(X) = \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z$$

04/04

$$\nabla(\nabla Z)(Y,X) =: (\nabla^2 Z)(Y,X) = \nabla^2_{Y,X} Z.$$

" $\qquad$ $2^{\text{nd}}$ cov. der.

$$(\nabla_Y(\nabla Z))(X) = \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z$$

$$= \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z$$

$$\Rightarrow R(X,Y)Z = \nabla^2_{Y,X} Z - \nabla^2_{X,Y} Z. \leftarrow \text{"Ricci identity"}$$

$$= \nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z - \nabla_X \nabla_Y Z + \nabla_{\nabla_X Y} Z$$

Torsion-free

$$= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Define curvature of a tensor $X, Y \in \mathfrak{X}(M)$

$$R(X,Y)T = \nabla^2_{Y,X} T - \nabla^2_{X,Y} T.$$

$f$ function:

$$R(X,Y)f = \nabla^2_{Y,X} f - \nabla^2_{X,Y} f = (\nabla_X df)(Y) - (\nabla_Y df)(X)$$

$$= XY(f) - (\nabla_X Y)f - YX(f) + (\nabla_Y X)f$$

$$= [X,Y](f) - [X,Y](f) = 0 \quad \text{(Torsion free)}$$

$$\rightarrow \operatorname{Hess} f(X,Y) = \operatorname{Hess} f(Y,X).$$

$T$ an $(s,r)$-tensor.

$\operatorname{div} T$ is a $(s, r-1)$ tensor. Let $E_i$ be an orthonormal ba
of $T_pM$

$$(\operatorname{div} T)(Y_1, \dots, Y_{r-1}) = \sum_{i=1}^{n} (\nabla_{E_i} T)(Y_1, \dots, Y_{r-1}, E_i).$$

Exe: $\operatorname{div}(df) = \sum_{i=1}^{n} (\nabla_{E_i} df)(E_i) = \sum_{i=1}^{n} \operatorname{Hess} f(E_i, E_i) = \Delta$

"Riemannian Laplacian of $f$".

In $\mathbb{R}^n$, $E_i$ std. basis $\operatorname{Hess} f(E_i, E_i) = \frac{\partial f}{\partial x_i^2}$.
$\partial/\partial x_i, \partial/\partial x_i$

Derivative of curv. Tensor

$2^{nd}$ Bianchi identity

$$(\nabla_X R)(Y, Z, W, U) + (\nabla_Y R)(Z, X, W, U) + (\nabla_Z R)(X, Y, W, U) = 0$$

Pf: HW

Ricci-tensor ~~$R(X,Y,Z)$~~ $R(X,Y)Z$ $(1,3)$-curvature tensor

Consider $Y \longmapsto R(X,Y)Z : T_pM \to T_pM$.
Fix $X, Z$

Ricci-tensor: $\operatorname{Ric}(X,Z) = \operatorname{Trace}(Y \longmapsto R(X,Y)Z)$.

If $E_i$ ONB for $T_pM$

$$Ric(X,Z) = \sum_{i=1}^{n} \langle R(X,E_i)Z, E_i\rangle$$

$$= \sum_{i=1}^{n} R(X,E_i,Z,E_i). \quad (0,2)\text{-tensor, symmetric.}$$

$$Ric(X,Y) = Ric(Y,X)$$

Let $v$ be a unit vector in $T_pM$

Ricci curvature: $Ric(v,v) = \sum_{i=1}^{n} R(v,E_i,v,E_i).$

Ric Pick ONB $E_1 = v, E_2, \ldots E_n$

$$\Rightarrow \cancel{\sum_{i=1}^{n-1}} \frac{1}{n-1} \sum_{i=2}^{n} \underbrace{R(v,E_i,v,E_i)}_{=K(\sigma_i)} \quad \sigma_i = \langle v, E_i\rangle.$$

sometimes

Ricci curvature is average of sectional curvature.

**Def**: An Einstein manifold is a Riem. mfld s.t. all Ricci curvatures ~~are~~ are equal $Ric(v,v) = \lambda \ \forall |v| = 1$

## Scalar curvature

$$Scal = \sum_{j=1}^{n} Ric(E_j, E_j)$$

04/04

so, $\mathrm{Scal} = \sum_{i,j=1}^{n} R(E_i, E_j, E_i, E_j)$

Einstein tensor $G = \mathrm{Ric} - \frac{\mathrm{Scal}}{2} g$.

$\rightsquigarrow$ $\mathrm{div}(G) = 0$. ($2^{nd}$ Bianchi)

---

2nd Bianchi-Identity

04/06

$$(\nabla_X R)(Y, Z, W, U) + (\nabla_Y R)(Z, X, W, U) + (\nabla_Z R)(X, Y, W, U) = 0$$

Ric: $\mathrm{tr}(Y \mapsto R(X,Y)Z)$

Contracted Bianchi-Identities

~~(1) $(\nabla_Y \mathrm{Ric})(X, Z) - (\nabla_Z$~~ Ric(

(1) $(\nabla_Y \mathrm{Ric})(Z, W) - (\nabla_Z \mathrm{Ric})(Y, W) = -(\mathrm{div} R)(Y, Z, W)$

(2) $2 \, \mathrm{div}\, \mathrm{Ric} = d\, \mathrm{Scal}$.

Proof: $0 = \sum_i \Big( (\nabla_{E_i} R)(X, Z, W, E_i) + (\nabla_Y R)(Z, E_i, W, E_i)$
$\qquad + (\nabla_Z R)(E_i, Y, W, E_i) \Big)$

$= (\mathrm{div}\, R)(X, Z, W) + (\nabla_Y \mathrm{Ric})(Z, W) - (\nabla_Z \mathrm{Ric})(Y, W) \Rightarrow ($

Sum over $W, E$

$$0 = \sum_{i,j} (\nabla_{E_i} R)(Y, E_j, E_j, E_i) + (\nabla_Y Ric)(E_j, E_j) - (\nabla_{E_j} Ric)(Y, E_j)$$

$$= -(\operatorname{div} Ric)(Y) + Y(Scal) - (\operatorname{div} R)(Y)$$

Def: The Einstein tensor $G = Ric - \frac{1}{2} Scal \cdot g$, $g(X,Y) = \langle X, Y \rangle$

Prop $\operatorname{div} G = 0$

Einst. Eqn. $G = \text{st.} \cdot T$ — Stress Energy Tensor

$$\operatorname{div} G = \operatorname{div} Ric - \operatorname{div}\left(\tfrac{1}{2} scal\, g\right)$$

$$= \tfrac{1}{2}\, d\, scal$$

let $S = \varphi \cdot g$

$$\operatorname{div} S(X) = \sum_i (\nabla_{E_i} S)(X, E_i) = \sum_i \nabla E_i (S(X, E_i)) - S(\nabla_{E_i} X, E_i) - S(X, \nabla_{E_i} E_i)$$

$$= \sum_i \nabla_{E_i} (\varphi\, g(X, E_i)) - \varphi\, g(\nabla_{E_i} X, E_i) - \varphi\, g(X, \nabla_{E_i} E_i)$$

$$= \sum_i E_i(\varphi)\, g(X, E_i) + \varphi \left[ E_i (g(X, E_i)) - g(\nabla_{E_i} X, E_i) - g(X, \nabla_{E_i} E_i) \right]$$

compatibility $= 0$

$$= \sum_i d\varphi(E_i)\, g(X, E_i)$$

$$= d\varphi\Big(\sum_i \langle X, E_i\rangle E_i\Big) = d\varphi(X).$$

$$\Rightarrow \operatorname{div}\Big(\tfrac{1}{2}\operatorname{scal} g\Big) = \tfrac{1}{2}\, d\operatorname{scal} \Rightarrow \operatorname{div} G = 0.$$

__Cor.__ Schur's Lemma If $M^n$ Riem., connected, $n>2$, then if

$Ric_p = \varphi(p)\, g_p$ for $\varphi : M \to \mathbb{R}$. $\Rightarrow$ $\varphi$ constant. (or loc. const.)

__Cor.__ If $n>2$, ~~$K(\sigma) = \varphi(p)$~~ If $\exists \varphi$ s.t. $K(\sigma_p) = \varphi(p)$, $\sigma_p \subseteq T_pM$

$\Rightarrow$ $M$ has const. sectional curvature.

__Pf of Schur__ If $Ric = \varphi \cdot g$

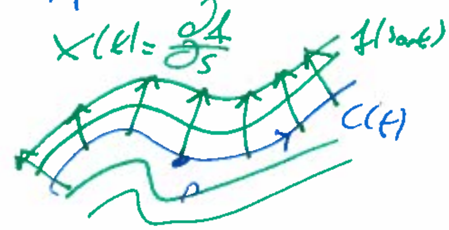
$$\operatorname{Scal} = \sum_{j=1}^{n} Ric(E_j, E_j) = \sum_{j=1}^{n} \varphi = n\varphi.$$

$$\Rightarrow G = \Big(\varphi + \frac{n}{2}\varphi\Big) g = \Big(\frac{2-n}{2}\varphi(p)\Big) g$$

$$\operatorname{div} G = \frac{2-n}{2}\, d\varphi. \Rightarrow d\varphi = 0 \Rightarrow \varphi \text{ const.}$$

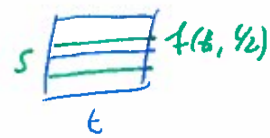
**Jacobi-Field** let $c(t)$ be a curve in $M$ $t \in (-\varepsilon, \varepsilon)$

A variation of $c$ is a small map $f: (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to M$
$f(t,s)$ s.t. $f(t,0) = c(t)$.




Variation field along $c(t)$

$$X(t) = \cancel{\frac{df}{ds}} \frac{\partial f}{\partial s}(t,0)$$




Lemma: Let $V(s,t)$ be a VF along $f$
($V(s,t)$ is a vector field $V(s,t)$ vector at $f(s,t)$)
$\in T_{f(s,t)}M$

$$\frac{D}{dt}\frac{D}{ds} V - \frac{D}{ds}\frac{D}{dt} V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

recall $\frac{D}{ds} V = \nabla_{\frac{\partial f}{\partial s}} V$ $\qquad$ $\frac{\partial f}{\partial s} = df\left(\frac{\partial}{\partial s}\right)$

$\frac{\partial f}{\partial t} = df\left(\frac{\partial}{\partial t}\right)$

Pf: See pp. 98-99 DoCarmo.

$$\left[\nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} V - \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} V + \nabla_{\underbrace{\left[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right]}_{=0}} V \right.$$

$$\left. = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) V\right].$$

$= df\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]$
$= 0$

Variation of geodesics is a variation $f(t,s)$ s.t. 04/

$t \mapsto f(t,s)$ geod. $\forall s$.

$\Leftrightarrow \frac{D}{\partial t}\left(\frac{\partial}{\partial t} f\right) = 0$

$$0 = \frac{D}{\partial s} \frac{\partial D}{\partial t} \frac{\partial f}{\partial t} = \frac{D}{\partial t} \frac{D}{\partial s} \frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t}$$

$$= \frac{D}{\partial t} \frac{D}{\partial t} \frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t}$$

$s=0$ here

Set $J(t) = \frac{\partial f}{\partial s}(t,0)$ variation field of variation by geodesics.

$\gamma(t) = f(t,0)$

$$\frac{D^2}{\partial t^2} J + R(\gamma', J)\gamma' = 0 \quad \text{Jacobi Equation (JE)}$$

**Def** If $\gamma$ geodesic, a Jacobi field along $\gamma$ is a VF along $\gamma$, satisfying the Jacobi-Equation.

**Idea** Jacobi ~~field~~ fields control how fast ~~good~~ geodesics spread.

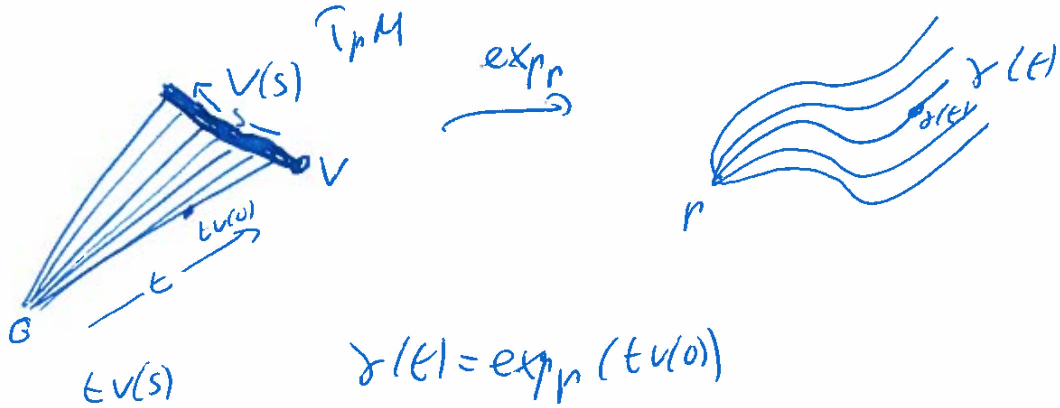
**Rem** Remarks about JE. $2^{nd}$ order linear system of ODE's.

$\Rightarrow$ given $v, w \in T_{\gamma(0)}M \; \exists!\; J(t)$ Jacobi field along $\gamma(t)$

s.t. $J(0) = v$, $\frac{D}{dt} J(0) = w$.

# Jacobi-Fields and Exponential Map

04/06




$$\gamma(t) = \exp_p(tV(0))$$

$$f(s,t) = \exp_p(tV(s))$$

is a variation by geodesics. $V(0) = v$, $V'(0) = w \in T_v(T_pM)$

$$J(t) = \frac{\partial f}{\partial s}(t,0) = d(\exp_p)_{tv}(tw) \quad (*)$$

$$J(0) = d(\exp_p)_{tv}(tw)$$

Every $J$ s.t. $J(0)=0$ looks like $(*)$

$$= \underbrace{d(\exp_p)_0}_{= Id}(0w) = 0$$

$$\frac{DJ}{dt} = \frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{dt}\left(d(\exp_p)_{tv}(tw)\right)$$

$$= \frac{D}{dt}\left(t\, d(\exp_p)_{tv}(w)\right)$$

$$= d(\exp_p)_{tv}(w) + t\,\frac{D}{dt}\left(d(\exp_p)_{tv}(w)\right)$$

at $t=0$ $\quad \frac{DJ}{dt} = d(\exp_p)_0(w) = w$

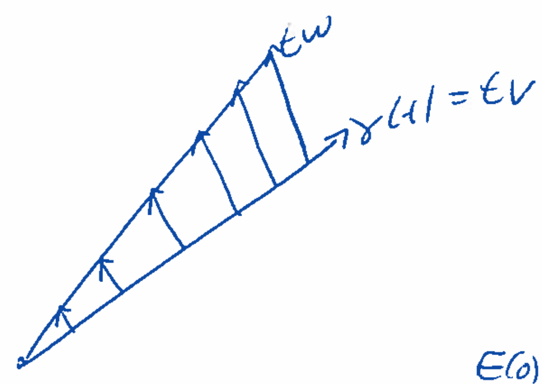
$J(t) \in$ Jacobi field along $\gamma$ geodesic. If $J(0)=0$ 

$(\exists E) \iff J(t) = d(\exp_p)_{tv}(tw)$

<u>Examples</u> ① $\mathbb{R}^n$ $\gamma(t)$ geod through $0 = tv$.

$J(t) = tw$

"linear field" along $\gamma(t)$

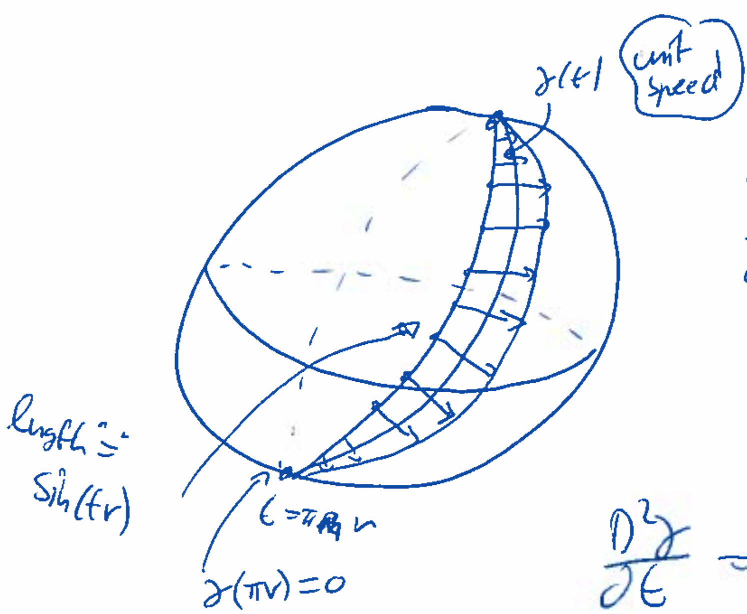



$$\frac{D^2 J}{dt^2} = 0 = -\underbrace{R(\gamma', J)\gamma'}_{=0}.$$

② $S^n(r)$

$E(0)=0$, $E(0) \perp \gamma'(0)$

$J(t) = \phi(t) \cdot E(t)$ $\qquad$ $E(t)$ parallel along $\gamma$




$$\frac{DE}{dt} = 0$$

$$\frac{DJ}{dt} = \frac{D}{dt}\left(\phi(t) E(t)\right) = \frac{d\phi}{dt} E(t) + \phi(t) \underbrace{\frac{DE}{dt}}_{=0}$$

$$\frac{D^2 J}{dt^2} = \frac{D}{dt}\left(\frac{d\phi}{dt} E(t)\right) = \frac{d^2\phi}{dt^2} E(t)$$

$R(\gamma', J)\gamma' = \frac{1}{r^2}$ $\quad$ (as $\langle R(\gamma', J)\gamma', J\rangle = \frac{1}{r^2} |J|^2$

$\langle R(\gamma', J)\gamma', V\rangle = \underbrace{\frac{1}{r^2}\langle J, V\rangle}_{=0}$

$(J E)$ of $\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0$ 04/09

$\Leftrightarrow \left(\frac{d^2\phi}{dt^2} + \frac{1}{r^2}\phi\right) E = 0$

$\Rightarrow \frac{d^2\varphi}{dt^2} = -\frac{1}{r^2}\varphi \quad J(0) = 0, \varphi(0) = 0$

$\varphi(t) = A \cdot \sin\left(\frac{t}{r}\right)$, so $J(t) = A \sin\left(\frac{t}{r}\right) E(t)$.

**Observation** $\gamma$ geodesic

- $J(t) \equiv 0$ is a Jacobi field ✓
- $J(t) = \frac{d\gamma}{dt}$ is a Jacobi field. ✓ $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ and $R(\gamma', \gamma')\gamma' = 0$ $\rightarrow J(0) = \frac{d\gamma}{dt}(0), \frac{DJ}{dt}(0) = 0$.
- $J(t) = t \frac{d\gamma}{dt}$ is a Jacobi field ✓

$J(0) = 0$

$\frac{D}{dt}\left(t \frac{d\gamma}{dt}\right) = \frac{d\gamma}{dt} + t \cdot 0$

$\frac{DJ}{dt}(0) = \frac{d\gamma}{dt}(0)$

$\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0$

$\frac{D^2}{dt^2}(J(t)) = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0.$

$R(\gamma', J)\gamma' = R(\gamma', t\gamma')\gamma' = 0.$

04/2

$\mathcal{J}(0)=0 \rightarrow$ If $\frac{D\mathcal{J}}{dt}(0) \parallel \frac{d\gamma}{dt}(0) \Rightarrow \mathcal{J}(t) = a t \frac{d\gamma}{dt}$

$\rightarrow$ If $\frac{D\mathcal{J}}{dt}(0) \perp \frac{d\gamma}{dt}(0) \rightarrow \mathcal{J}(t) \perp \frac{d\gamma}{dt} \; \forall t.$

(perpendicular Jacobi-Fields).

---

<u>Conjugate point</u> Let $\gamma$ be a geodesic. A point $\gamma(t_0)$ is said to be conjugate along $\gamma$ to $\gamma(0)$ if $\exists$ Jacobi field $\mathcal{J}(t)$ along $\gamma$ s.t. $\mathcal{J}(0)=0=\mathcal{J}(t_0)$ and $\mathcal{J}\not\equiv 0$.

<u>E.g.</u> · <u>In $\mathbb{R}^n$</u> no points,

· <u>In $S^n$</u> only antipodal points are conjugate along any geodesic connecting them

Conjugate points $\Leftrightarrow$ Singular points of derivative of exp. map :

<u>Prop.</u> Let $\gamma(t) = \exp_p(tv)$ geodesic, $\gamma(t_0)$ conjugate to $\gamma(0)$ iff. $t_0 v$ is a critical point of $\exp_p$. (i.e. $d(\exp_p)_{t_0 v}$ is singular)

<u>Pf</u> Suppose $\gamma(0)$ & $\gamma(t_0)$ conj. $\exists\, \mathcal{J}$ along $\gamma$ Jacobi field. $\mathcal{J}(0)=0$ $\mathcal{J}(t_0)=0$. $\mathcal{J}(t) = \cancel{\exp_p(tv)}\, d(\exp_p)_{tv}(tw)$, $\frac{D\mathcal{J}}{dt}(0)=w$

$\overset{\mathcal{J}\neq 0}{\Rightarrow} w\neq 0$. $\quad 0 = \mathcal{J}(t_0) = d(\exp_p)_{t_0 v}(\underbrace{t_0 w}_{\neq 0}) \Rightarrow d(\exp_p)_{t_0 v}$ is singular.

Conversely if $d(\exp_p)_{t_0 v}(z) = 0$ for some $z \neq 0$

let $w = \frac{z}{t_0}$, $J(t) = d(\exp_p)_{tv}(tw)$, $J(t_0) = 0$. $\square$

## Hadamard Theorem

If $M$ is a Riemann. mfld with nonpositive sectional curvature, then $M$ has no conjugate points.

Def: nonpos. sectional curvature if $K_p(\sigma) \le 0 \quad \forall \sigma \subseteq T_pM$ planes.

ex: $\mathbb{R}^n$, not $S^n$

Proof: let $\gamma(t)$ geodesic ($|\gamma'| = 1$), $J(t)$ jacobi field along $\gamma(t)$ ~~$J(t) \perp \frac{d\gamma}{dt}(t)$~~ s.t. $J(0) = 0$ ~~of J parallel~~

~~Ats. $J(t) \neq 0$ $\forall t > 0$~~

let $\varphi(t) = \frac{1}{2}\langle J(t), J(t)\rangle$, $\varphi(0) = 0$

$$\frac{d\varphi}{dt} = \frac{d}{dt}\frac{1}{2}\langle J(t), J(t)\rangle = \frac{2}{2}\langle \frac{DJ}{dt}(t), J(t)\rangle$$

$\frac{d\varphi}{dt}(0) = 0$.

$$\frac{d^2\varphi}{dt^2} = \frac{d}{dt}\langle \frac{DJ}{dt}, J\rangle = \langle \frac{D^2J}{dt^2}, J\rangle + \langle \frac{DJ}{dt}, \frac{DJ}{dt}\rangle$$

$$\overset{\text{by } E}{=} -\langle R(\gamma',\gamma)\gamma',\gamma\rangle + \langle \tfrac{D\gamma}{dt}, \tfrac{D\gamma}{dt}\rangle$$

$$= -K_\sigma(\gamma',\gamma)\cdot|\gamma|^2 + \left|\tfrac{D\gamma}{dt}\right|^2 \geq 0.$$

$\sigma = \text{span}\{\gamma',\gamma\}$

We have $\varphi(t)$, $\varphi(0)=0$, $\frac{d\varphi}{dt}(0)=0$, $\frac{d^2\varphi}{dt^2}\geq 0$

If $\gamma(t_0)=0$ some $t_0 \Rightarrow \frac{d\varphi}{dt}(t_0)=0 \Rightarrow \frac{d\varphi}{dt}\equiv 0 \Rightarrow \varphi\equiv 0$

$\Rightarrow \gamma\equiv 0.$ $\square$

Next Time Completeness:

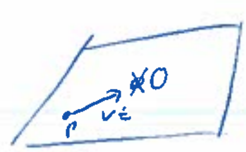
Def: $(M,g)$ geod. complete if geod. exists $\forall$ time $\Leftrightarrow$ $\exp_p$ defined on

## Complete Riemannian Manifolds (Ch. 7)

Always assume $M$ connected

**Def:** $(M,g)$ is geodesically complete if all geodesics can be extended for all time. Equivalently $\forall p\in M$ $\exp_p : T_pM \to M$ defined on all of $T_pM$.

Example $\mathbb{R}^n$, $S^n$  geod. complete

Not geod. complete $\mathbb{R}^n\setminus\{0\}$ 

**Rem.** If $M$ is complete, it is non-extendable, i.e. cannot be isometrically embedded as a proper subset of another mfld. of dim. $n$.

(see p.145)

**Recall:** metric $d(p,q) = \inf \{ \ell(c),\ c$ p.w. smooth path connecting $p$ and $q \}$

$\ell(c) = \int_a^b |\dot{c}(t)|\, dt$, $c(a)=p$, $c(b)=q$

**Gauss Lemma:** $\forall p \in M\ \exists\, \varepsilon > 0$ s.t. $\exp_p(B(0,\varepsilon)) = B(p,\varepsilon) \leftarrow$ w.r.t. $d$

$\Rightarrow$ Manifold topology is equivalent to metric space topology.

$\Rightarrow$ $d: M \times M \to \mathbb{R}$ is continuous.

**Hopf-Rinow Thm** $M$ connected (path-connected.) Riem. mfld, $p \in M$. TFAE:

a) $\exp_p$ is defined on all of $T_pM$

b) The closed and bounded subsets of $M$ are compact

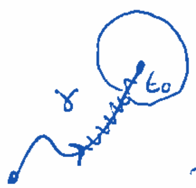
c) $M$ is compact as a metric space

d) $M$ geod. complete.

**Proof:** b) $\Rightarrow$ c) point set topology argument.

d) $\Rightarrow$ a) ✓

c) $\Rightarrow$ d) Assume $M$ complete as a metric space. Suppose $\exists$ geodesic $\gamma : [0, t_0) \to M$ which can not be extended past $t_0$, s.t. $|\gamma'(t)| = 1 \; \forall t$.



Let $s_n \to t_0$, $0 < s_n < t_0$. Consider $\{\gamma(s_n)\}$.

$(s_n > s_m)$

Then $d(\gamma(s_n), \gamma(s_m)) \leq \ell(\gamma|_{[s_n, s_m]}) = |s_n - s_m|$.

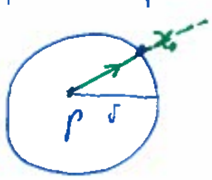
$\Rightarrow \{\gamma(s_n)\}$ Cauchy in $M$.

$\Rightarrow \gamma(s_n) \to p_0 \in M$. Let $W$ be a totally normal nbd. about $p_0$

$\forall n \geq n_0$, $\gamma(s_n) \in W$. So $\gamma$ is the unique geodesic passing through $\gamma(s_n)$, $\gamma(s_m)$. Therefore $\gamma$ can be extended past $t_0$.

**Lemma** a) $\Rightarrow$ $\forall q \in M$ $\exists$ geodesic $\gamma$ s.t. $d(p,q) = \ell(\gamma)$.

**Pf. of Lemma**



$q$

Let $d(p,q) = r$. Let $B(p,\delta)$ be a normal ball around $p$. Let $S(p,\delta)$ be the boundary of $B(p,\delta)$, $S(p,\delta) = \{x \mid d(p,x) = \delta\}$. Let Now

$x \mapsto d(q,x)$ is cont. Since $S$ cpct, $\exists x_0$ s.t. $d(q, x_0) = \inf_{x \in S} d(q,$

$x \in S(p,\delta)$

Idea $\gamma_0$ Should be on the minimal geod. connecting $p$ & $q$. 04/11

Since normal ball

$\gamma_0 = \exp_p(\delta v)$, $v \in T_pM$, $|v|=1$. ~~let $\gamma(t)$ be~~ the

Let $\gamma(t) = \exp_p(tv)$. Claim $\gamma(t)$ minimizes distance from $p$ to $q$.

i.e. to show $\gamma(r) = q$.

Let $A = \{ s \in [0,r],\ d(\gamma(s), q) = r - s \}$. Now $0 \in A$, $A$ is closed.

~~Want~~ Want $A = [0,r] \Rightarrow d(\gamma(r), q) = 0 \Rightarrow \gamma(r) = q$.

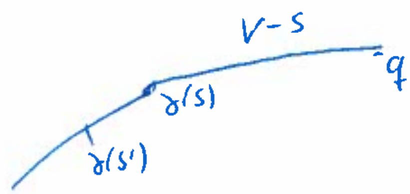
Show $A$ open Let $s_0 \in A$, ~~so~~ $0 < s_0 < r$ $\exists\, \delta_0' > 0$ s.t. $s_0 + \delta_0 \in A$

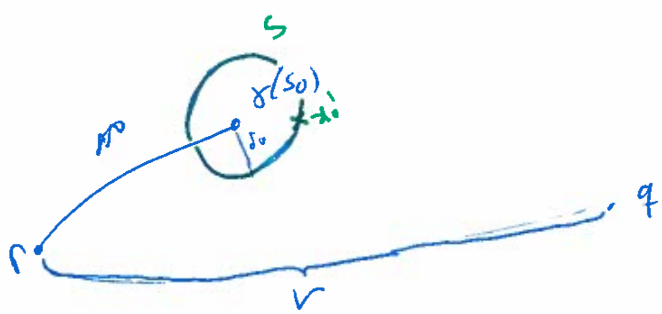
[ Why enough? : If $s \in A$, $s' < s$

$d(\gamma(s'), q) \leq s - s' + r - s = r - s'$

$d \geq d(p,q) - d(p, \gamma(s'))$

$= r - s'$. $\Rightarrow$ If $s_0 + \delta_0 \in A \to (s_0 - \delta_0, s_0 + \delta_0) \subset A$ ]




$d(\gamma(s_0), q) = r - s_0$

04/11

Let $\delta_0 > 0$ s.t. $B(\gamma(s_0), \delta_0)$ normal ball and assume $s_0 + \delta_0 < r$. Let $x_0'$ be a point that minimizes $d(x_0', q)$

$$= \min_{x' \in S(\gamma(s_0), \delta_0) = S} d(x', q)$$

$$r - s_0 = d(\gamma(s_0), q) = \inf\left(\ell(c),\ c \text{ connects } \gamma(s_0), q\right)$$

$$= \delta_0 + d(x_0', q).$$

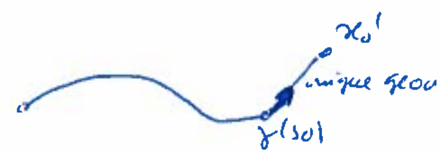
So $\quad d(x_0', q) = r - s_0 - \delta_0 \quad (*)$

So: $\quad d(p, x_0') \geq s_0 + \delta_0$

$$d(p,q) - d(q, x_0) = r - (r - s_0 - \delta_0) = s_0 + \delta_0.$$

On the other hand concatenation of $\gamma$ with the unique geod from $\gamma(s_0)$ to $x_0'$ has length $s_0 + \delta_0$, so the concatenation is a geodesic, thus smooth

$\Rightarrow \gamma(s_0 + \delta_0) = x_0'$. $\Rightarrow$ $(*)$ implies $s_0 + \delta_0 \in A$ $\square$




Proof of a) $\Rightarrow$ b):

Let $A \subseteq M$ closed and bounded. By lemma $\exp_p : T_pM \to M$ onto.

A bounded: $\exists\, R > 0$ s.t. $d(q, p) \leq R \ \forall q \in A$.

$\overset{\text{lemma}}{\Rightarrow}$ $A \subseteq \exp_p(\underbrace{\overline{B(0,R)}}_{\text{cpt.}})$, so $A$ compact. ($M$ Hausdorff). $\square$

$q \in A \Rightarrow \exists\, \gamma : d(p,q) = \ell(\gamma),\ |\dot\gamma| = 1 \Rightarrow \ell(\gamma) = r - 0 \leq R - 0$

$\gamma(r) = q,\ r \leq R$

04/11

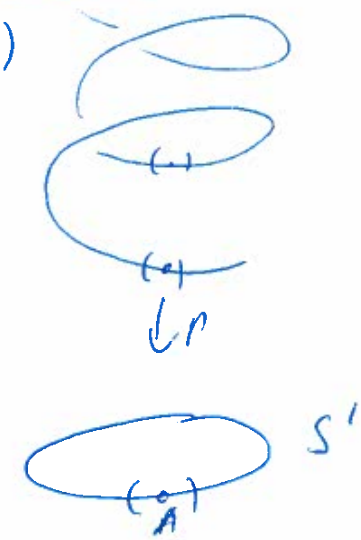
**Corollary** If $M$ is compact (& connected), then $M$ is complete.

$\forall\, p, q \in M$ $\exists$ minimizing geod. from $p$ to $q$.

---

04/13

**Ex:** covering space $\mathbb{R} \to S^1$, $t \to (\cos t, \sin t)$




**Remark:** If $X$ manifold, then we have cov. spaces

$$\pi_1(M, p) = \frac{\{\text{loops based at } p\}}{\text{homotopy}} \quad \text{group}$$

$M$ is simply connected, if $M$ connected and

$\pi_1(M, p) = 0$.

**Def:** The universal cover of $M$ is a cov. space $p : U \to M$ s.t.

$U$ simply connected.

**Rem** If $M$ connected $\exists!$ universal cover and

~~fact~~ $|\pi_1(M)| = \#(p^{-1}(x))$ (Cardinality)

04/11

Let $M$ be a connected mfld. $p: M \to Y$ cov. space.

$\rightsquigarrow Y$ is a mfld. (same dim.) $Y$ can be equipped with a smooth structure

s.t. $p$ is a local diffeom.

If $F: Y \to M$ is a local diffeo, $M$ Riem. mfld. If $v, w \in T_y Y$

Define $\langle v, w \rangle_y = \langle dF_y v, dF_y w \rangle_{F(y)}$

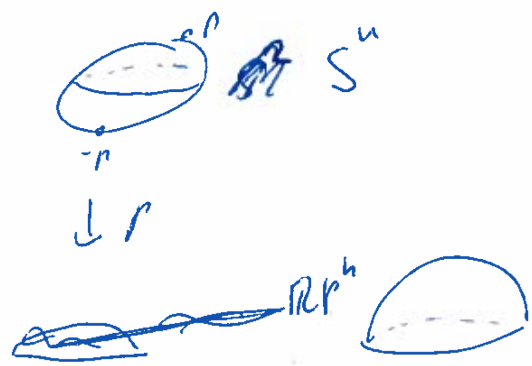
$\Rightarrow F$ is a local isometry.

Def: If $p: Y \to M$ is a cov. space ~~and~~ between Riem. mflds $Y$ and $M$

$p$ is called a Riemannian cov. space if $p$ is a local isometry

Thm: If $F: M \to N$ be a local isometry. If $M$ is complete and $N$ is connected, then $F$ is a Riemannian cov. space.

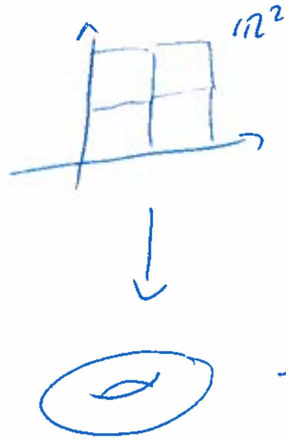
Ex: $\mathbb{R}P^n = S^2 / \sim_{p \sim -p \ \forall p}$




$p$ local Riemannian isometry

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$

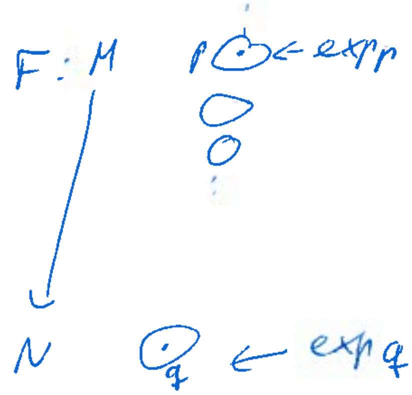
$\mathbb{R}^n/\sim \; = T^n$ $\quad$ $\underline{n=2}$ $(x,y)\sim(x+m,\, y+n)$, $m,n\in\mathbb{Z}$.




$\pi_1(T^n) = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n}$

<u>Proof of Thm</u>: $q\in N$

Let $B(q,\varepsilon)$ be a normal ball around $q$.




Let $p\in F^{-1}(q)$. By completeness $\exp_p$ defined on $T_pM$. , F local isometry

<u>HW 6 #2</u> $\quad F\circ \exp_p(v) = \exp_q(dF_p(v))$, $v\in T_pM$

So $\exp_q\circ dF_p$ is a diffeom. on $B(0,\varepsilon)\subseteq T_pM$.

$\Rightarrow \overbrace{F\circ \exp_p(v)}^{\text{take derivative}}$ is a diffeom. on $B(0,\varepsilon)$. $\exp_p(v)$ : $B(0,\varepsilon)\to B(p,\varepsilon)$

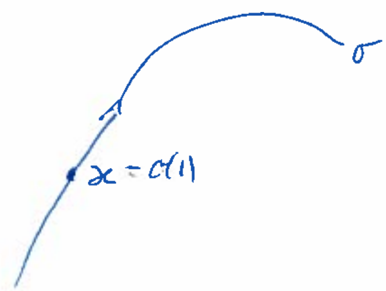
is <u>onto</u> by <u>geodesic completeness</u>. $\quad\searrow$ $\exp_p$ injective on $B(0,\varepsilon)$

and $d(\exp_p)_v$ isom. $\forall v\in B(0,\varepsilon)$.

OC

$\Rightarrow F: B(p,\varepsilon) \to B(q,\varepsilon)$ a diffeo.

<u>Claim</u> $F^{-1}\left(B(q,\varepsilon)\right) = \dot{\bigcup}_{p\in F^{-1}(q)} B(p,\varepsilon)$




Let $x \in F^{-1}(B(q,\varepsilon))$

$F(x) \in B(q,\varepsilon) \leftarrow$ normal ball


$\Rightarrow \exists!$ geod. $\gamma(t)$ from $q$ to $F(x)$, length $= \varepsilon$.

$\gamma(0) = q$, $\gamma(1) = F(x)$

Let $\sigma$ be a geod. $\sigma: [0,1] \to M$, $\sigma(1) = x$,

and $\frac{d\sigma}{dt}(t) = (dF_x)^{-1}\left(\frac{d\gamma}{dt}(1)\right)$.

By geod. completeness, $\sigma$ exists $\forall$ time.

$F\circ\sigma$ is a geodesic and in $M$ and.

$F(\sigma(1)) = F(x)$ $\qquad \frac{d}{dt}(F\circ\sigma)(1) = \frac{d\gamma}{dt}(1)$

$\Rightarrow F\circ\sigma = \gamma \Rightarrow F(\sigma(0)) = \gamma(0) = q \Rightarrow \sigma(0) \in F^{-1}(q)$.

$\Rightarrow d(\sigma(1),\sigma(0)) = d(p,x) < \varepsilon$ $\quad$ + $F$ surjection (open & closed argument

$F(M)$ open & closed.

Thm If $M$ complete Riem. mfld. with nonpositive sectional curvature, then for each $p \in M$ $\exp_p$ is a covering map.

↳ $K \leq 0 \iff K(\sigma_p) \leq 0 \;\forall \sigma_p \subseteq T_pM \;\forall p \in M$.

(Cartan-Hadamard Theorem).

Pf: We proved that $K \leq 0 \Rightarrow \nexists$ no conjugate points, so $\exp_p : T_pM \xrightarrow{\text{onto}} M$ is a local diffeo.

completeness

Define a Riem. metric on $T_pM$ $\langle v, w\rangle = \langle d\exp_p(v), d\exp_p(w)\rangle$

$\Rightarrow \exp_p$ local isometry.

Claim $(T_pM, \langle \cdot, \cdot \rangle)$ complete.

Pf: $\exp_p$ local isometry so geodesics are mapped to geodesics.

$\exp_p$ maps straight lines from $0$ to geod starting at $p$.

$\therefore$ straight lines from origin in $T_pM$ are geodesics.

Hopf-Rinow $\Rightarrow$ all geod. from $0$ can be extended for all time

$\therefore T_pM$ is complete.

By prop.: ($M$ connected) $\exp_p$ cov. map.

04

**Cor.** If $M$ is complete, $K \le 0$, then the universal cover of $M$ is $\exp_p : \mathbb{R}^n \to M$. "comparison thm."

$\Rightarrow$ If $M$ is compact, $K \le 0$

$$\begin{array}{c} \mathbb{R}^n \\ \downarrow \\ (M, p) \end{array} \qquad F^{-1}(p) \text{ has to be inf.}$$

$\Rightarrow$ $|\pi_1(M)| = \infty$.

- $\pi_1(M)$ will be torsion free.

**Ex:** $S^n$ has no metric with $K \le 0$, $n \ge 2$

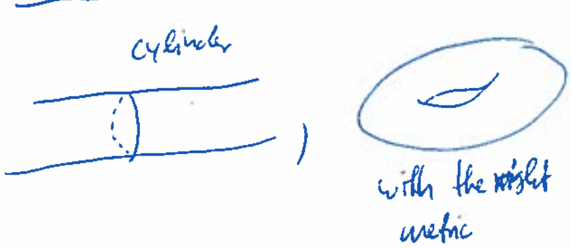
## Spaces of constant curvature

$\mathbb{R}^n$ $K \equiv 0$

$S^n(r)$, has $K \equiv \frac{1}{r^2}$

**Thm:** Let $\bar{M}^n$, $M^n$ be two Riem. manifolds with $\bar{p} \in \bar{M}$, $p \in M$ s.t. both $\bar{p}$ and $p$ have nbds with constant curvature $k$. Then there is a local isometry from an open nbd of $p$ to ~~and~~ an open nbd. of $\bar{p}$.

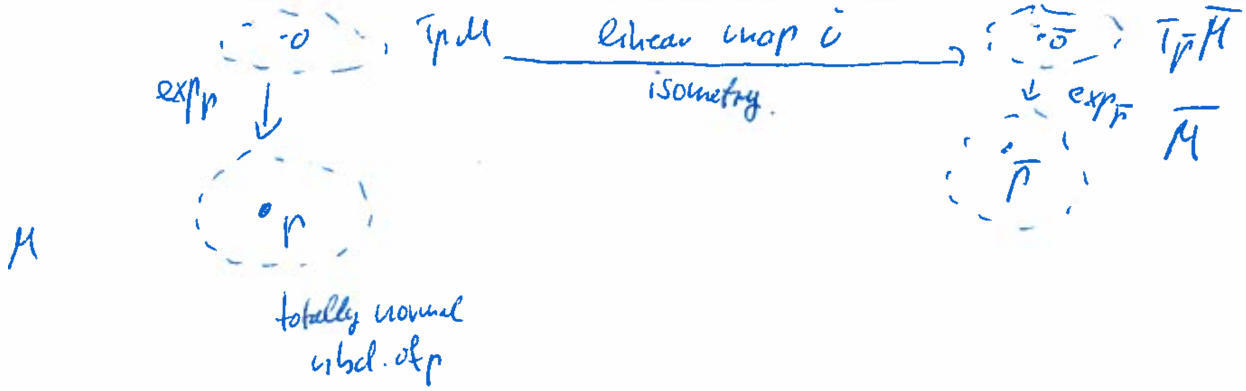
**e.g.** Constant curv. 0




, $\mathbb{R}^2$ } locally isometric

$$\mathbb{R}^2 \xrightarrow[\text{isom.}]{\text{local}} \mathbb{R}^2/\mathbb{Z} = T^2 \quad \underline{\text{flat torus}}$$

04/16




Let $V$ be a totally normal nbd. of $p$., $i: T_pM \to T_{\bar p}\bar M$ be an arbitrary linear isometry. Define $f: V \to \bar M$ by

$$f(q) = \exp_{\bar p} \circ i \circ (\exp_p)^{-1}$$

<u>Claim</u> If const. curvature $= K$ in $V$ and $f(V)$, then $f$ is an isometry.

In fact, more is true even when not const. curvature:

$\forall q \in V$ $\exists!$ geod. $\gamma$ from $p$ to $q$. Let $P_t$ parallel transl. along $\gamma$ from $\gamma(0)$ to $\gamma(t)$

By construction $\bar\gamma = f \circ \gamma$ geod from $\bar p$ to $f(q)$. Let $\bar P_t$ be parallel transl. from $\bar\gamma(0)$ to $\bar\gamma(t)$ along $\bar\gamma = f \circ \gamma$.

<u>Define</u> $\phi_t: T_qM \to T_{f(q)}\bar M$ by

$$\bar P_t \circ i \circ P_t^{-1}$$

Then: If $\forall q \in V$ and $\forall x, y, z, w \in T_qM$

$<$ ~~$R(x,y)z,v) = \bar{R}$~~ $R(x,y,z,w) = \bar{R}(\phi_t(x), \phi_t(y), \phi_t(z), \phi_t(w))$

then $f$ is a local isometry.

Pf: Let $v \in T_qM$. Note $\gamma$ does not have conjugate points.

Let $\tilde{\mathcal{J}} = \{$ space of Jacobi fields along $\gamma$, $J(0)=0\}$.

$\dim(\tilde{\mathcal{J}}) = n$

$\theta : \tilde{\mathcal{J}} \to T_{\gamma(l)}M$, $J \mapsto J(l)$ linear map, injective (no conjugate points)

$\Rightarrow$ $\theta$ isomorphism

So $\forall v \in T_qM \; \exists J(t) \in \tilde{\mathcal{J}}$ s.t. $J(l) = v$, $J(0) = 0$

Let $\{E_i(t)\}$ parallel ONB along $\gamma$. So

$$J(t) = \sum_{i=1}^{n} y_i(t) E_i(t) \qquad \frac{DJ}{dt} = \sum_{i=1}^{n} y_i'(t) E_i(t)$$

parallel along $\gamma$

$$\frac{D^2 J}{dt^2} = \sum_{i=1}^{n} y_i''(t) E_i(t)$$

04/16

Then $\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0$

$(\Leftrightarrow)$ $\quad y_j'' = -\sum_{i=1}^{n} R(\gamma', E_i, \gamma', E_j)\, y_j$

Let $\bar{\gamma}$ geod. in $\bar{M}$, $\bar{\gamma}(0) = \bar{p}$, $\bar{\gamma}'(0) = i(\gamma'(0))$

Let $\bar{J}(t)$ be the field along $\bar{\gamma}$ given by

$\bar{J}(t) = \phi_t(J(t))$.

<u>Claim</u> $\bar{J}$ Jacobi field in $\bar{M}$.

<u>Pf:</u> $\bar{E}_i(t) = \phi_t(E_i(t))$. $\leftarrow$ parallel orthonormal basis along $\bar{\gamma}$.

$\bar{\gamma}'(t) = \phi_t(\gamma'(t))$

$\bar{J}(t) = \phi_t(J(t)) = \phi_t(\sum y_i E_i) = \sum_{i=1}^{n} y_i(t)\, \bar{E}_i(t)$.

By hypoth. $R(\gamma', E_i, \gamma', E_j) = \bar{R}(\bar{\gamma}', \bar{E}_i, \bar{\gamma}', \bar{E}_j)$.

Then ~~$y_j'' + \sum \bar{R}(\bar{\gamma}'$~~

Then $\quad$ d4/16

$$y_\gamma'' + \sum_i \bar{R}(\bar{\gamma}', \bar{E_i}, \bar{\gamma}', \bar{E_\gamma})\, y_\gamma = 0$$

iso

$\bar{J}$ is a Jacobi field.

So $\gamma, \bar{\gamma}$ both Jacobi fields

$$J(t) = d(\exp_p)_{t\gamma'(0)}(t J'(0))$$

$$\bar{J}(t) = d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}(t \bar{J}'(0))$$

Since $\bar{\gamma} = \phi_t(\gamma(t))$, $\bar{J}'(0) = i\, J'(0)$

$$\bar{J}(\ell) = d(\exp_{\bar{p}})_{\ell\bar{\gamma}'(0)}(\ell\, i \cdot J'(0))$$

$$= d(\exp_{\bar{p}})_{\ell\bar{\gamma}'(0)} \circ i \circ (d(\exp_p))^{-1}_{\ell\gamma'(0)}(\gamma(\ell)).$$

$$= df_q(\gamma(\ell)).$$

$$df_q(\gamma(\ell)) = \bar{\gamma}(\ell) = \underset{\text{isometry.}}{\phi_t}(\gamma(\ell))$$

and $\ell = v$ $\quad$ $|df_q(\gamma(v))| = |df_q(v)| = |v|$ $\quad \forall v \Rightarrow$ isometry $\square$

Last time $M, \bar{M}$ constant sectional curvature $k$. 

$\forall p \in M, \bar{p} \in \bar{M}$, $i : T_pM \to T_{\bar{p}}\bar{M}$ lin. isom.

the map $f : \underset{\substack{\uparrow \\ \text{normal nbd. of } p}}{V} \longrightarrow \bar{M}$, $f = \exp_{\bar{p}} \circ i \circ (\exp_p)^{-1}$ isometry onto its image

& $df_p = i$.

---

<u>Model spaces</u>: $k \in \mathbb{R}$, $M_k^n$ = <u>Unique</u> simply connected complete Riem. mfld with constant curvature $k$

- <u>$k=0$</u> $M_k^n = \mathbb{R}^n$ standard metric.

- <u>$k>0$</u> $M_k^n = S^n\left(\frac{1}{\sqrt{k}}\right)$ sphere with radius $\frac{1}{\sqrt{k}}$.

- <u>$k<0$</u> Hyperbolic space $\mathbb{H}^n = \{(x_1, , x_n) \mid x_n > 0\}$. (upper half-space)

metric $\langle \cdot, \cdot \rangle = \frac{1}{\sqrt{-k}} \frac{dx_1^2 + dx_2^2 + \dots + dx_n^2}{x_n^2}$.

$$g_{ij} = \frac{1}{\sqrt{-k}} \cdot \frac{\delta_{ij}}{x_n^2}$$

$\mathbb{H}^n$ complete, ~~sy~~ simply connected, constant curvature $k$.

 (see pp. 160-162).

Thm: Suppose $M^n$ ~~has~~ is complete

Manifold with constant sectional curvature $k$, Then the universal

cover is $M^n_k \to M$ (and the cov. map is a local isometry)

$M^n_k \xrightarrow{p} M$ + cov. theory $\Rightarrow$ $M^n_k$ unique (up to isometry) constant curvature space which is simply connected.

Cor: Most manifolds do not have a metric with const. sectional curvature. $S^1 \times S^2$, $S^n \times T^m$, $\mathbb{C}P^n$.

$\rightsquigarrow$ Reduces problem of classifying constant curvature spaces to Algebraic Topology

Proof of Thm: $M$ complete (connected) Riem. mfld. , const. curvature $k$.

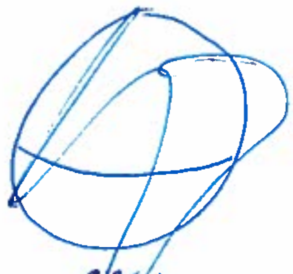
Case 1: $k \leq 0$. Cartan–Hadamard $\exp_p : T_pM \to M$ is the universal cover ($\forall p \in M$). And also $\exp_{\bar{p}} : T_{\bar{p}} M^n_k \to M^n_k$ covering map.

$$f = \exp_p \circ i \circ (\exp_{\bar{p}})^{-1} \text{ ~~local~~ isometry.}$$

$$\begin{tikzcd} T_{\bar{p}} M^n_k \arrow[r, "i"] \arrow[d, "\text{diffeo}"'] & T_p M \arrow[d, "\exp_p"] \\ \bar{p} \in M^n_k & M \end{tikzcd}$$

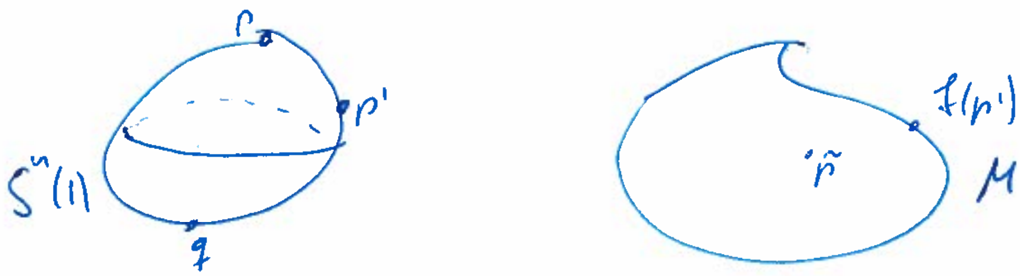
$\Rightarrow$ $i \circ (\exp_{\tilde{p}})^{-1}$ is an isometry.

Case 2: $(k>0)$ $M^n_k = S^n\left(\frac{1}{\sqrt{k}}\right)$, assume $k=1$.



$S^n(1)$

$$\begin{tikzcd} T_p S^n \arrow[r, "i"] \arrow[d, "\exp_p"] & T_{\tilde{p}} M \arrow[d, "\exp_{\tilde{p}}"] \\ \phantom{x} & \phantom{x} \end{tikzcd}$$




Let $q = -p$, $V = S^n \setminus \{q\}$ is a normal nbd. of $p \in S^n$.

$\therefore$ $f : V \to M \Rightarrow f = \exp_{\tilde{p}} \circ i \circ (\exp_p)^{-1}$

is a local isometry. Chose $p' \notin \{p, q\}$.

Let $q' = -p'$, let $\tilde{p}' = f(p')$, let $i' = df_{p'}$

$\leadsto f' : S^n \setminus \{q'\} \to M$

$f' = \exp_{\tilde{p}'} \circ i' \circ (\exp_{p'})^{-1}$ local isometry

Let $W = S^n \setminus \{q, q'\}$ (connected, $n \geq 2$). Then $f, f' : W \to M$

local isometries $f(p') = f'(p')$, $df_{p'} = df'_{p'}$

So by HW #5 $f \equiv f'$ on $W$. (loc. isom. on a connected mfld value & derivative agree at one pt.)

<u>Define</u> $h : S^n \to M$, $h(x) = \begin{cases} f(x), & x \in S^n \setminus \{q\} \\ f'(x), & x \in S^n \setminus \{q'\} \end{cases}$ local isometry.

$S^n$ complete, $M$ connected

$\Rightarrow h : S^n \to M$ Riem. cov. space.

## Variation of Energy

<u>Calculus of variations</u> $c : [0, a] \to M$ curve

functional $L(c) = \int_0^a |\dot{c}(t)| \, dt \leftarrow$ length, $L : \{\text{curves}\} \to \mathbb{R}$.

"Differentiate $L$".

Recall A variation of $c$ is a map $f: (-\underset{s}{\varepsilon, \varepsilon}) \times [\underset{t}{0,a}] \to M$

s.t. $f(0,t) = c(t)$ Variation field $\frac{\partial f}{\partial s}(0,t) = V(t)$.

Vector field along $c(t)$.

Note: Book does this in generality that $f(s,t)$ piecewise diff'ble in $t$.

$0 \le t_1 \le t_2 \le \_\_\_ \le t_n = a$ $\overset{f(s,\cdot)}{\text{smooth}}$ on $[t_i, t_{i+1}]$

$L$~~(f(s,t))~~

Turns out to be more convenient to work with Energy

functional $E(s) = \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt$ ($f$ variation)

$$E(0) = \int_0^a |\frac{\partial f}{\partial t}(0,t)|^2 dt = \int_0^a |c'(t)|^2 dt.$$

$E(c) = \int_0^a |c'(t)|^2 dt$ $L(c) = \int_0^a |c'(t)| dt$

If $f = 1$, $g = |c'(t)|$ $\overset{C\text{-}S}{\leadsto}$ $\left(\int_0^a fg \, dt\right)^2 \le \int_0^a f^2 dt \cdot \int_0^a g^2 dt$

So $(L(c))^2 \le a\, E(c)$. "=" iff $\underset{|\frac{dc}{dt}|}{g} \equiv$ const, i.e. $c$ parametrized proportional to arclength.

We know geodesics are minimizers of length between two fixed points $p, q$. Also true for energy.

Lemma: Let $p,q \in M$, $\gamma : [0,a] \to M$ be a minimizing geodesic joining $p,q$ then for all curves $c : [0,a] \to M$ joining $p,q$

$E(\gamma) \le E(c)$ "=" iff $c$ is also a minimizing geod. from $p$ to $q$

Pf: $\underbrace{L(\gamma)^2}_{a E(\gamma)} \le L(c)^2 \le a E(c) \to E(\gamma) \le E(c)$.

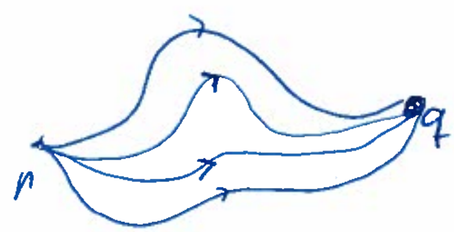
$\gamma$ const. speed

If $E(\gamma) = E(c) \to L(\gamma) = L(c)$, so $c$ is also a minimizing geodesic. $\square$.

Compute first variation of Energy

Variation field $V(t)$ is proper if $V(0) = V(a) = 0$,

i.e. $f(0,s_1) = f(0,s_2)$, $f(a,s_1) = f(a,s_2)$ $\forall s_1, s_2 \in (-\varepsilon,\varepsilon)$




Prop: Let $c : [0,a] \to M$ be diff'ble curve, let $f : (-\varepsilon,\varepsilon) \times [0,a] \to M$ be a variation of $c$.

If $E(s) = \int_0^a \left|\frac{\partial f}{\partial t}(t,s)\right|^2 dt$, then

$$\frac{1}{2}\frac{d}{ds}E(s)\Big|_{s=0} = -\int_0^a \left\langle V, \frac{D}{dt}\frac{dc}{dt} \right\rangle dt + \left\langle V(a), \frac{dc}{dt}(a) \right\rangle - \left\langle V(0), \frac{dc}{dt}(0) \right\rangle$$

04/23

For each $V(t)$ ~~vector~~ $\exists\ \infty$ many $f$ s.t. $\frac{\partial f}{\partial s}(0,t) = V(t)$
vector field along $c$

But we can choose a "nice variation" using the exponential map.

<u>Lemma:</u> Let $c:[0,a]\to M$ be a smooth curve, $V(t)$ smooth VF along $c$. Then $f(s,t) = \exp_{c(t)}(sV(t))$ is a variation of $c$ s.t. $\frac{\partial f}{\partial s}(0,t) = V(t)$ and if $V(0)=V(a)=0$ then $f(s_1,0)=f(s_2,0)$ and $f(s_1,a)=f(s_2,a)$ $\forall s_1,s_2\in(-\varepsilon,\varepsilon)$

<u>Pf:</u> For each $c(t_0)$ ~~$\exists\ \varepsilon>0$ s.t. $\exp(B(c(t_0),\varepsilon))$ is a diffeo~~ $W_{t_0}$ totally normal nbd. of $c(t_0)$ ~> $c(t)$ covered

by finitely many $W$'s So pick uniform $\varepsilon>0$ s.t.

$\exp_{c(t)}$ exists on $B(c(t),\varepsilon)$ $\forall t$.

$f(0,t) = \exp_{c(t)}(0) = c(t)$.

$\frac{\partial f}{\partial s} = d(\exp_{c(t)})_{sV(t)}(V(t))$, so $\frac{\partial f}{\partial s}(0,t) = d(\exp_{c(t)})_0(V(t)) = V(t)$.

$\because$ $V(0)=0$ then $f(s,0) = \exp_{c(0)}(sV(0)) = \exp_{c(0)}(0) = c(0)$. $\forall s$.

$\because$ $V(a)=0$ ~~so~~ then $f(s,a) = \exp_{c(a)}(0) = c(a)$ $\forall s$. $\square$

$\rightsquigarrow$ Choice of $f$ s.t $s \mapsto f(s,t_0)$ is a geodesic $\forall\, t_0 \in [0,a]$.

<u>Prop</u> First variation formula. Let $c:[0,a]\to M$ diff'ble curve $f:(-\varepsilon,\varepsilon)\times[0,a]\to M$ variat. of $c$, then if

$$E(s) = \int_0^a \cancel{\frac{\partial f}{\partial s}} \left|\frac{\partial f}{\partial t}(t,s)\right|^2 dt \quad \text{we get}$$

$$\frac{1}{2}\frac{d}{ds}\Big|_{s=0} E = -\int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt}\rangle dt + \langle V(a), \frac{dc}{dt}(a)\rangle - \langle V(0), \frac{dc}{dt}(0)\rangle$$

<u>Pf:</u> $\frac{1}{2}E(s) = \frac{1}{2}\int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt$

$$\frac{1}{2}\frac{d}{ds}E(s) = \frac{1}{2}\frac{d}{ds}\left(\int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt\right) = \cancel{\int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial^2}{\partial s \partial t} f\rangle dt}$$

$$= \int_0^a \langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle dt$$

$$= \int_0^a \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle dt$$

$$= \int_0^a \left(\frac{d}{dt}\left(\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle\right) - \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle\right) dt$$

$$= \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle\Big|_0^a - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle dt$$

at $s=0$ $= \langle V, \frac{dc}{dt}\rangle\Big|_0^a - \int_0^a \langle V, \frac{D}{dt}\frac{dc}{dt}\rangle dt.$

04/13

**Cor** If $f$ proper variation, $f(s_1,0)=f(s_2,0)$ $\forall s_1, s_2 \in (-\varepsilon,\varepsilon)$, $f(s_1,a)=f(s_2,a)$, then

$$\frac{1}{2}\frac{dE}{ds}(0) = -\int_0^a \left\langle V, \frac{D}{dt}\frac{dc}{dt} \right\rangle dt$$

**Cor** A curve $c$ satisfies $\frac{dE}{ds}(0)=0$ $\forall$ proper variations of $f$ iff $c$ is a geodesic.

**Prop** Let $\gamma(t)$ be a geodesic, $f(s,t)$ is a proper variation of $\gamma$
↳ new proper case in book.

then $$\frac{1}{2}\frac{d^2}{ds^2}E(0) = -\int_0^a \left\langle V, \frac{D^2V}{dt^2} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle dt$$

**Pf:** $$\frac{1}{2}\frac{d}{ds}E(s) = -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt$$

$$\to \frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0} = -\int_0^a \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \underbrace{\cancel{\frac{D}{dt}\frac{\partial f}{\partial t}}}_{=0 \ \gamma \text{ geod.}}\right\rangle \cancel{dt} + \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt$$

$$= -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt$$

**Recall:** $$\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$$

$$= -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\underbrace{\frac{D}{ds}\frac{\partial f}{\partial t}}_{=\frac{D}{dt}\frac{\partial f}{\partial s}} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\right\rangle dt$$

04/12

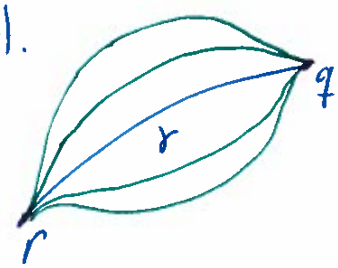
at $s=0$ $\quad \frac{\partial f}{\partial s} = V, \quad \frac{\partial f}{\partial t} = \frac{d\gamma}{dt} = \dot\gamma$

$$= -\int_0^a \left\langle V, \frac{D^2}{dt^2}V + R(\dot\gamma, V)\dot\gamma \right\rangle dt.$$

↗ is zero for proper Jacobi-fields. □

Assume $\gamma$ is a minimizing geodesic: $\gamma : [0,a] \to M$

$\gamma(0) = p$, $\gamma(a) = q$, $\ell(\gamma) = d(p,q)$.




$E(\gamma) \leq E(f_s) \Rightarrow \boxed{\frac{d^2}{ds^2}E(0) \geq 0}$

for all proper variations $f$.

<u>Myer's Thm</u> $M$ Riemannian mfld. Define $\operatorname{diam}(M)$

$= \sup\{ d(p,q), \ p,q \in M\}$.

<u>Thm</u>: $M^n$ complete (& connected) Riemannian mfld and $Ric(v,v) \geq (n-1)k$ $\forall |v|=1$. (where $k>0$). Then $\operatorname{diam} M \leq \frac{\pi}{\sqrt{k}}$

<u>Ex:</u> $S^n(r)$ sphere of radius $r$ in $\mathbb{R}^{n+1}$.

$Ric(v,v) = \frac{n-1}{r^2}$, $\operatorname{diam}(S^n(r)) = \pi r$, $k = \frac{1}{r^2}$

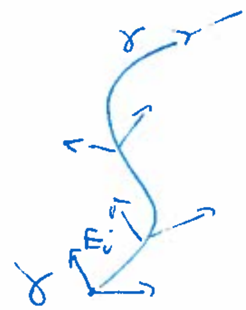
So $Ric = (n-1)k$ & $\operatorname{diam}(S^n(r)) = \frac{\pi}{\sqrt{k}}$

**Cheng's diameter thm** If $Ric \geq (n-1)k$ and $\operatorname{diam} M = \frac{\pi}{\sqrt{k}}$, then $M$ is isometric to $S^n(r)$.

**Proof (of Myers)** Let $p, q \in M$. Since $M$ is complete, $\exists$ geodesic $\gamma$ from $p$ to $q$ s.t. $\ell(\gamma) = d(p,q)$.

Parametrize $\gamma$ s.t. $|\gamma'| = d(p,q)$ $\rightsquigarrow$ $\gamma : [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$.

**WTS** $d(p,q) \leq \frac{\pi}{\sqrt{k}}$. Let $E_1, \ldots, E_{n-1}$ be $n-1$ perpendicular, lin. indep. parallel fields along $\gamma$.




Let $V_i(t) = \sin(\pi t) E_i(t)$. Then

$V_i(0) = V_i(1) = 0 \Rightarrow$ proper variation $\forall i$.

$E_i(s)$ = Energy of variation coming from $V_i$ (via exponential map)

$\frac{1}{2} \frac{d^2}{ds^2}$

$$0 \leq \frac{1}{2} E_i''(0) = -\int_0^1 \left\langle V_i, \frac{D^2}{dt^2} V_i + \cancel{R(\ldots)} + R\left(\frac{dx}{dt}, V\right)\frac{dx}{dt} \right\rangle dt$$

↑ $\gamma$ minimizing.

$$\frac{D^2}{dt^2} V_i = \frac{D}{dt} \frac{D}{dt} \left(\sin(\pi t) \underset{\text{parallel}}{E_i}\right) = -\pi^2 \sin(\pi t) E_i.$$

$$\left\langle R\left(\frac{d\gamma}{dt}, V_i\right)\frac{d\gamma}{dt}, V_i\right\rangle = \underbrace{\left|\frac{d\gamma}{dt}\right|^2 |V_i|^2 K(\sigma_i)}_{= d(p,q)^2 \sin^2(\pi t) K(\sigma_i)}$$

when ~~kata~~ $\sigma_i = \operatorname{span}\left\{\frac{d\gamma}{dt}, E_i\right\}$.

$$\frac{1}{2} E_i''(0) = -\int_0^1 -\pi^2 \sin^2(\pi t) + d(p,q)^2 \sin^2(\pi t) K(\sigma_i)\, dt$$

$$0 \le \frac{1}{2}\sum_{i=1}^{n-1} E_i''(0) = -\int_0^1 (n-1)(-\pi^2 \sin^2(\pi t)) + \cancel{(n-1)}\, d(p,q)^2 \sin^2(\pi t)\, Ric\left(\frac{\frac{d\gamma}{dt}}{\|\cdot\|}, \ldots\right)$$

By Hypothesis $Ric\left(\frac{\frac{d\gamma}{dt}}{|\cdot|}, \frac{\frac{d\gamma}{dt}}{|\cdot|}\right) \ge (n-1)k$ we get

$$0 \le \int_0^1 \sin^2(\pi t)\left[ Ric\left(\frac{\frac{d\gamma}{dt}}{|\cdot|}, \frac{\frac{d\gamma}{dt}}{|\cdot|}\right) d(p,q)^2 - (n-1)\pi \right]$$

$$0 \le \int_0^1 \sin^2(\pi t)\left((n-1)k\, d^2(p,q) - (n-1)\pi\right)$$

$$\to \cancel{d} \quad k\, d^2(p,q) - \pi^2 < 0 \to d(p,q) \le \frac{\pi}{\sqrt{k}}. \quad \square$$

So diam $M \le \frac{\pi}{\sqrt{k}}$.

<u>Cor</u>: $M$ complete $Ric \geq (n-1)k$, $k>0$

$\Rightarrow M$ cpt.

<u>Ex</u>: Paraboloid $z = x^2+y^2$ complete, $K>0$

not compact.

<u>Cor</u>: $M^n$ compact, $Ric>0 \Rightarrow \pi_1(M)$ is finite:

<u>Proof</u>: $M$ cpt, $Ric>0$ $\exists\, k$ s.t. $Ric \geq (n-1)k$ $k>0$

let $\hat{M} \xrightarrow{p} M$ universal cover, $\hat{M}$ covering metric $\Rightarrow Ric_{\hat{M}} \geq (n-1)k$.

$\rightsquigarrow$ <u>lemma</u> $M$ complete $\Leftrightarrow \hat{M}$ complete (path lifting)

$\Rightarrow \hat{M}$ complete $Ric_{\hat{M}} \geq (n-1)k$ & $\hat{M}$ cpt (by Myers)

$\Rightarrow \pi_1(M)$ is <u>finite</u>.

<u>Ex</u>:

$$\begin{array}{c} S^n \\ \downarrow \\ \mathbb{R}P^n \end{array} \qquad \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2.$$

Recall $M$ compact, $M$ non-positive sectional curvature

$\Rightarrow \pi_1(M)$ infinite

Cor. $M^n$ compact mfld, $M$ cannot have both a metric with $Ric > 0$ & some other metric with non-positive sectional curvature

Lohkampf: $M^n$ compact, $n \geq 4$ then $M$ has a metric with $Ric < 0$.

Scalar curvature: $Scal = \sum_{i=1}^{n} Ric(E_i, E_i)$, $E_i$ ONB.

Yamabe problem (Aubin, Schoen)

Every compact mfld. has a metric with constant scalar curvature.

Einstein Equation: Ricci curvature is constant

$n=3 \overset{HW}{\Rightarrow}$ const. sectional curvature.

open Q1 Does every 5-mfld admit an Einstein metric?

# Analysis on Riemannian Manifolds



<u>Riemannian Volume</u> let $M$ be a Riemannian manifold

let $\varphi : \mathcal{U} \to V \subseteq M$ be a coordinate chart

$\frac{\partial}{\partial x_i}$ coordinate vector fields.

let $E_1, \dots, E_n$ be an orthonormal basis for $T_pM$.

$\frac{\partial}{\partial x_i}\Big|_p = \sum_{k=1}^{n} a_{ik} e_k$ The volume of parallelopiped spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is $|\det(a_{ij})|$

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \left\langle \sum_k a_{ik} e_k, \sum_l a_{jl} e_l \right\rangle$$

$$= \sum_{k=1}^{n} a_{ik} a_{jk} \quad \rightsquigarrow \quad [g] = [a] \cdot [a]^T$$

$$\det(g_{ij}) = \det(a_{ij})^2 \Rightarrow \sqrt{\det(g_{ij})} = |\det(a_{ij})|$$

<u>Define</u> Volume of $V$:

$$\mathrm{Vol}(V) = \int_{\mathcal{U}} \sqrt{\det g_{ij}} \circ \varphi^{-1} \cdot dx_1 \dots dx_n.$$

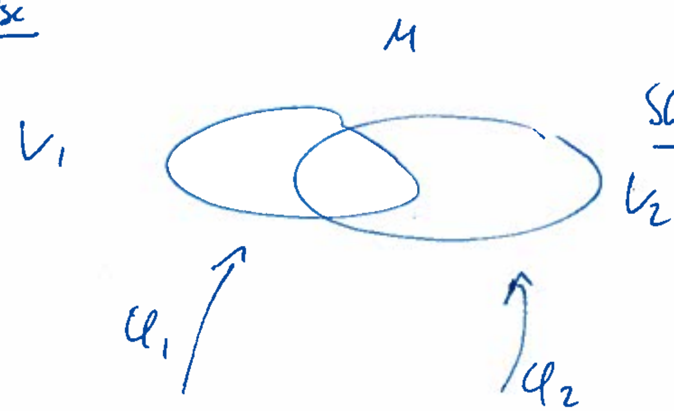
04/2

Recall: $\psi : \underbrace{V \to \mathcal{U}}_{\text{open}}$ diffeo, then
$V \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^n$

*Change of Variables Formula*

$$\int_{\mathcal{U}} dy_1 \ldots dy_n = \int_V \underbrace{|\operatorname{Jac}(\psi)|}_{=|\det(d\psi)|} dx_1 \ldots dx_n$$

Exercise




Show: $\operatorname{Vol}(V_1 \cap V_2)$ the same for $\varphi_1, \varphi_2 \,\#|_{V_1 \cap V_2}$.

i.e. $\displaystyle \int_{V_1 \cap V_2} \sqrt{\det g_{ij}} \circ \varphi_1 \, dx_1 \ldots dx_n = \int_{V_1 \cap V_2} \sqrt{\det \tilde{g}_{ij}} \circ \varphi_2 \, dy_1 \ldots dy_n \qquad \tilde{g}_{ij} = \left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right\rangle.$

Def: let $M$ be a Riem. mfld, ~~let $\varphi_\alpha$ partition of unity for $M$~~ ~~such~~ let $\varphi_\alpha : \mathcal{U}_\alpha \to V_\alpha$ be an atlas for $M$, $f_\alpha$ be a part. of unity subordinate to $\{\mathcal{U}_\alpha\}$.

$$\cancel{\operatorname{Vol}(M) = \sum_\alpha} \quad \operatorname{Vol}(M) = \sum_\alpha \int_{V_\alpha} f_\alpha \, d\operatorname{vol}_g.$$

04/25

Remark: $d\,vol_g$ does define a $n$-form if $M$ is orientable. non-degenerate

$$dvol_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge \dots \wedge dx_n$$

Recall if $f: M \to \mathbb{R}$, the gradient of $f$ $\nabla f$ is the unique vector field on $M$ s.t. $df(X) = \langle \nabla f, X\rangle$ $\forall X \in \mathfrak{X}(M)$.

The Hessian of $f$. $\quad \mathrm{Hess} f(X,Y) = \langle \nabla_X \nabla f, Y\rangle$

( symmetric $(0,2)$-tensor

If $\{E_i\}$ ONB of tangent space $\Delta f = \sum_{i=1}^n \langle \nabla_{E_i} \nabla f, E_i\rangle$.

(Laplace-Beltrami, Riemannian Laplacian)

$$\Delta_g f = \sum_{i=1}^n \langle \nabla_{E_i} \nabla f, E_i\rangle = \sum_{i=1}^n E_i \langle \nabla f, E_i\rangle - \langle \nabla f, \nabla_{E_i} E_i\rangle$$

$$= \sum_{i=1}^n E_i(E_i(f)) - (\nabla_{E_i} E_i)(f).$$

So if $M = \mathbb{R}^n$, $E_i$ standard basis

$$\Delta f = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f \qquad \text{usual Laplacian on } \mathbb{R}^n$$

04/25

Idea $\Delta_g$ has similar properties as $\Delta_{\mathbb{R}^n}$.

Properties ⓪ $\Delta_g (f+h) = \Delta_g f + \Delta_g h$.

① local maximum of $f$ at $p$ $(\nabla f_p = 0)$

$(\Delta f)(p) \le 0$, local minimum $(\Delta f)(p) \ge 0$

Divergence Theorem If $V$ is a vector field on $M$

$$\operatorname{div} V = \sum_{i=1}^{n} \langle \nabla_{E_i} V, E_i \rangle, \quad E_i \text{ ONB}$$

e.g.: $\operatorname{div}(\nabla f) = \Delta_g f$

Theorem If $M$ ~~oriented~~, $V$ has compact support, then

$$\int_M \operatorname{div} V \, d\mathrm{vol}_g = 0.$$

If $M$ is compact $f$ function: $M \to \mathbb{R}$, then $\int_M \Delta_g f \, d\mathrm{vol}_g = 0$.

04/25

**Green's formula** $f_1, f_2 : M \to \mathbb{R}$, $M$ cpt

$$\int_M \Delta f_1 \cdot f_2 \, dvol_g = -\int_M \langle \nabla f_1, \nabla f_2 \rangle \, dvol_g$$

$$= \int_M f_1 \, \Delta f_2 \, dvol_g .$$

**Proof:** $\operatorname{div}(f_1 \nabla f_2) = \langle \nabla f_1, \nabla f_2 \rangle + f_1 \Delta f_2$.

**M compact mfld** $\Delta_g : C^\infty(M) \to C^\infty(M)$

$L^2(M,g) = L^2(M) =$ Function with $\int_M f^2 \, dvol_g < \infty$

$\qquad\qquad\qquad\quad \cup\!\!\!\mid$
$\qquad\qquad\qquad\quad C^\infty(M)$

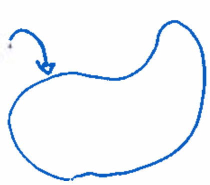
$\hookrightarrow$ Hilbert space

$\langle f, g \rangle = \int_M f \cdot g \, dvol_g$. Green's formula $\Delta$ self-adjoint.

**Spectrum of $\Delta_g$** An eigenfunction of $\Delta$ with eigenvalue $-\lambda$ is a function $f$ s.t. $\Delta f + \lambda f = 0$

**Colloq.** on Friday: Can you hear the shape of a drum?

$f \equiv 0$ on bd.



$\rightsquigarrow$ Spectral geometry

Last time $M$ compact, Riem.

$$\Delta : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$
$$\subseteq L^2(M) \qquad \subseteq L^2(M)$$

If $E_i$ ONB of $T_pM$ $\quad \Delta f|_p = \sum_{i=1}^n \langle \nabla_{E_i} \nabla f, E_i \rangle$

$$= \sum_{i=1}^n E_i E_i f - (\nabla_{E_i} E_i) f.$$

An eigenfunction of $\Delta$ is a function $f$ s.t.

$$\begin{cases} \Delta f + \lambda f = 0 & (\underset{\in \mathbb{R}}{-\lambda} \text{ eigenvalue}) \\ f \not\equiv 0 \end{cases}$$

Prop If $M$ compact then $\lambda = 0$ is an eigenvalue of $\Delta$ and for any other ~~eigenvalue~~ eigfctn. $\lambda \geq 0$.

Pf: $\lambda = 0$ eigenvalue to $f \equiv c$ const. $\in L^2(M)$.

Suppose $\Delta f + \lambda f = 0$

$$\int_M \Delta f \, f \, dvol_g = -\int_M \langle \nabla f, \nabla f \rangle \, dvol_g \Rightarrow \cancel{\lambda} \geq 0. \quad \lambda \geq 0$$

$$\| \qquad \Rightarrow \qquad b/c$$

$$-\lambda \int_M f^2 \, dvol_g \qquad \lambda \int_M f^2 \, dvol_g = \int_M |\nabla f|^2 d\ldots$$

and $\lambda = 0 \iff f \equiv \text{const}$. $\square$

In general, $-\Delta$ has a discrete spectrum

$$0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots\ldots$$

$\lambda_i$ eigenvalues of $-\Delta$ (with multiplicity)

$\lambda_1(M) =$ smallest pos. eigenvalue.

There is a relationship between $\lambda_1(M)$ and curvature

<u>Prop:</u> (Bochner formula) Let $f \in C^3(M)$. Then

$$\frac{1}{2}\Delta |\nabla f|^2 = Ric(\nabla f, \nabla f) + |Hess f|^2 + \langle \nabla\Delta f, \nabla f \rangle$$

Where $|A|^2$ is Euclid. norm of the $(0,2)$ tensor $A$ def'd below.

<u>Lichnerowicz Eigenvalue Comparison</u> If $M^n$ complete Riem. mfld. and $Ric \ge (n-1)k$, $k > 0$ then it holds:

$$\lambda_1(M) \ge nk.$$

$= \lambda_1(S^n$ [illegible]$)$

Norm of (0,2)-tensor: let $V$ be a real V.S. of dim $n$. inner prod. space

~~Euclidean~~ $L: V \to V$ linear operator

$$|L| = \sqrt{tr(L \circ L^*)} \quad \text{where } L^* \text{ adjoint} : \langle Lv, w\rangle = \langle v, L^* w\rangle$$

In euclidean space, standard dot product

$L: V \to V$, matrix for $L$ in standard basis, if $V = \mathbb{R}^2$

$$L = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad L^* = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$L \circ L^* = \begin{pmatrix} a_{11}^2 + a_{12}^2 & * \\ * & a_{21}^2 + a_{22}^2 \end{pmatrix}$$

$$tr(L \circ L^*) = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = \sum_{i,j} a_{ij}^2$$

if $A$ (0,2)-tensor. Define $L$ to be the (type change) dual (1,1) tensor

$$A(x, y) = \langle L(x), y\rangle, \quad L: T_pM \to T_pM$$

<u>Define</u>: $|A| = |L|$ If $A$ symmetric, $A(x,y) = A(y,x)$

$$\Rightarrow \langle L(x), y\rangle = \langle x, L(y)\rangle \Rightarrow L^* = L.$$

$|A| = \sqrt{tr(L^2)}$, $E_i$ ONB for $T_pM$

$\Rightarrow |A| = \sqrt{\sum_{i=1}^{n} \langle L \circ L(E_i), E_i \rangle}$

$= \sqrt{\sum_{i=1} \langle L(E_i), L(E_i) \rangle}$

so If $A = \text{Hess} f$, $\text{Hess} f(X,Y) = \langle \nabla_X \nabla f, Y \rangle$

$L : T_pM \to T_pM$, $X \mapsto \nabla_X \nabla f$.

$|\text{Hess} f|^2 = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla f, \nabla_{E_i} f \rangle$

As $A$ symmetric $(0,2)$-tensor $\exists$ ONB of Eigenvectors of $L$

s.t. $\{E_i\}$ s.t. $L(E_i) = a_i E_i$ $\Rightarrow$ $|A| = \sum_{i=1}^{n} (a_i)^2 \geq \frac{(\sum_{i=1}^{n} a_i)^2}{n}$

Cauchy Schwarz

$= \frac{(trA)^2}{n}$ so: if $A = \text{Hess} f$

$|\text{Hess} f|^2 \geq \frac{(\Delta f)^2}{n}$

Bochner $\Rightarrow$ $\frac{1}{2} \Delta |\nabla f|^2 \geq Ric(\nabla f, \nabla f) + \frac{(\Delta f)^2}{n} + \langle \nabla \Delta f, \nabla f \rangle$

Pf of Lichnerowicz Thm: Suppose that $\Delta f = -\lambda f$, $f \not\equiv const.$

& $Ric(\nabla f, \nabla f) \geq (n-1)k\, |\nabla f|^2$ ← not necessarily $=1$

$$\frac{1}{2}\Delta|\nabla f|^2 \geq (n-1)k|\nabla f|^2 + \frac{\lambda^2 f^2}{n} + \langle \nabla(-\lambda f), \nabla f\rangle$$

$$= (n-1)k|\nabla f|^2 + \frac{\lambda^2 f^2}{n} - \lambda|\nabla f|^2.$$

Integrate both sides $\int_M \cdot\, dvol_g$, then

$$0 \geq \int_M \left((n-1)k|\nabla f|^2 + \frac{\lambda^2 f^2}{n} - \lambda|\nabla f|^2\right) dvol_g$$

$$= \int_M ((n-1)k - \lambda)\, |\nabla f|^2 dvol_g + \underbrace{\int_M \frac{\lambda^2 f^2}{n} dvol_g}_{= \int_M \frac{\lambda}{n} |\nabla f|^2 dvol_g}.$$

But $\lambda \int_M f^2 dvol_g = \int_M |\nabla f|^2 dvol_g$.

$$0 \geq \left((n-1)k - \lambda + \frac{\lambda}{n}\right) \underbrace{\int_M |\nabla f|^2 dvol_g}_{\neq 0}$$

$\Rightarrow (n-1)k - \lambda + \lambda/n \leq 0 \Rightarrow (n-1)k - \frac{n-1}{n}\lambda \leq 0$ $\boxed{\Rightarrow \lambda \geq nk}$

# Proof of Bochner formula

$$\frac{1}{2}\Delta|\nabla f|^2 = \mathrm{Ric}(\nabla f,\nabla f) + |\mathrm{Hess}\, f|^2 + \langle \nabla \Delta f, \nabla f\rangle$$

Pf: Let $p\in M$, pick geodesic normal coordinates at $p$

$E_i$ ONB s.t. $\nabla_{E_i} E_j|_p = 0$.

Then $\frac{1}{2}\Delta|\nabla f|^2 = \frac{1}{2}\sum_{i=1}^n E_i(E_i(|\nabla f|^2)) - 0$

$$= \frac{1}{2}\sum_{i=1}^n E_i(E_i(\langle\nabla f,\nabla f\rangle)) - \frac{1}{2}\sum_{i=1}^n E_i\left(\langle\nabla_{E_i}\nabla f,\nabla f\rangle + \langle\nabla f,\nabla_{E_i}\nabla f\rangle\right)$$

$$= \sum_{i=1}^n E_i(\underbrace{\langle\nabla_{E_i}\nabla f,\nabla f\rangle}) = \sum_{i=1}^n E_i(\langle\nabla_{\nabla f}\nabla f, E_i\rangle)$$

$= \mathrm{Hess}\, f(E_i,\nabla f) = \mathrm{Hess}\, f(\nabla f, E_i)$

$$= \sum_{i=1}^n \langle\nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle + \langle\nabla_{\nabla f}\nabla f, \underbrace{\nabla_{E_i}E_i}_{=0}\rangle$$

$$= \sum_{i=1}^n \langle\nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle = \cancel{\sum_{i=1}^n \langle\nabla_{\nabla f}\nabla_{E_i}\nabla f, E_i\rangle}$$

$$= \sum_{i=1}^n R(\nabla f, E_i, \nabla f, E_i) + \langle\nabla_{\nabla f}\nabla_{E_i}\nabla f, E_i\rangle - \langle\nabla_{[\nabla f, E_i]}\nabla f, E_i\rangle$$

$$① = Ric(\nabla f, \nabla f)$$ 04/30

$$② = \sum_{i=1}^{n} \langle \nabla_{\nabla f} \nabla_{E_i} \nabla f, E_i \rangle = \sum_{i=1}^{n} \nabla f \langle \nabla_{E_i} \nabla f, E_i \rangle - \langle \nabla_{E_i} \nabla f, \underbrace{\nabla_{\nabla f} E_i}_{=0 \text{ at } p} \rangle$$

$$= \nabla f \overbrace{\underbrace{\sum_{i=1}^{n} \langle \nabla_{E_i} \nabla f, E_i \rangle}_{= div(\nabla f)}}^{(\Delta f)} = \langle \nabla \Delta f, \nabla f \rangle$$

$$③ -\sum_{i=1}^{n} \langle \nabla_{[\nabla f, E_i]} \nabla f, E_i \rangle \qquad [\nabla f, E_i] = \underbrace{\nabla_{\nabla f} E_i}_{=0} - \nabla_{E_i} \nabla f$$

$$= \sum_{i=1}^{n} \langle \nabla_{+\nabla_{E_i} \nabla f} \nabla f, E_i \rangle$$

$$= \sum_{i=1}^{n} Hess\, f(\nabla_{E_i} \nabla f, E_i) = \sum_{i=1}^{n} Hess\, f(E_i, \nabla_{E_i} \nabla f)$$

$$= \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f \rangle = |Hess\, f|^2$$