MAT 771: Differential Geometry Spring 2018

Instructor: Will Wylie, Carnegie 206C, email: wwylie@syr.edu

Lecture: Carnegie 110, MoWe 12:45-2:05.

Office Hours: Mo 3:45-4:45, We 10-12, or by appointment. I have an open door policy for quick/short questions.

Text: M. P. Do Carmo, *Riemannian Geometry*, (Translated by Francis Flaherty) , Birkhauser, 1992.

Prerequisites: **MAT 602** (Functions of several variables), **MAT 661** (Point Set Topology), a strong background in linear algebra and multi-variable calculus.

Course Material: The course will cover elements of the theory of abstract smooth manifolds with emphasis on Riemannian Geometry. This will cover topics contained in Chapter 0-8 of the text.

Homework: There will be homework assignments assigned bi-weekly. Four problems from each assignment will be collected and graded. Each solution will be graded as either complete or incomplete. If a solution is graded as incomplete, you may re-do and hand in the assignment as many times as you'd like until it is complete. You are encouraged to work together on the homework assignments.

Grading: Course grade will be based on completion of the homework assignments. Completing at least 75% of the homework problems will be a grade of A, 50% will be a grade of A-, less than 50% and regular class participation/attendance will be a grade of B+.

Students with disabilities: If you believe that you need academic adjustments (accommodations) for a disability, please contact the Office of Disability Services (ODS), located in Room 309 of 804 University Avenue, visit the ODS website- http://disabilityservices.syr.edu, or call (315) 443-4498 or TDD: (315) 443-1371 for an appointment to discuss your needs and the process for requesting academic adjustments. ODS is responsible for coordinating disability-related academic adjustments and will issue students with documented Disabilities Accommodation Authorization Letters, as appropriate. Since academic adjustments may require early planning and generally are not provided retroactively, please contact ODS as soon as possible.

Syracuse University values diversity and inclusion; we are committed to a climate of mutual respect and full participation. My goal is to create learning environments that are useable, equitable, inclusive and welcoming. If there are aspects of the instruction or design of this course that result in barriers to your inclusion or accurate assessment or achievement, I invite any student to meet with me to discuss additional strategies beyond academic adjustments that may be helpful to your success.

Religious observances policy: Syracuse University's Religious Observances Policy recognizes the diversity of faiths represented among the campus community and protects the rights of students, faculty, and staff to observe religious holy days according to their tradition. Under the policy, students are provided an opportunity to make up any examination, study, or work requirements that may be missed due to a religious observance provided they **notify their instructors no later than the end of the second week of classes** for regular session classes and by the submission deadline for flexibly formatted classes. Student deadlines are posted in MySlice under Student Services/Enrollment/My Religious Observances/Add a Notification.

Academic integrity: Syracuse University's Academic Integrity Policy reflects the high value that we, as a university community, place on honesty in academic work. The policy defines our expectations for academic honesty and holds students accountable for the integrity of all work they submit. Students should understand that it is their responsibility to learn about course-specific expectations, as well as about university-wide academic integrity expectations. The policy governs appropriate citation and use of sources, the integrity of work submitted in exams and assignments, and the veracity of signatures on attendance sheets and other verification of participation in class activities. The policy also prohibits students from submitting the same work in more than one class without receiving written authorization in advance from both instructors. Under the policy, students found in violation are subject to grade sanctions determined by the course instructor and non-grade sanctions determined by the School or College where the course is offered as described in the Violation and Sanction Classification Rubric. SU students are required to read an online summary of the Universitys academic integrity expectations and provide an electronic signature agreeing to abide by them twice a year during pre-term check-in on MySlice. http://academicintegrity.syr.edu.

The Violation and Sanction Classification Rubric establishes recommended guidelines for the determination of grade penalties by faculty and instructors, while also giving them discretion to select the grade penalty they believe most suitable, including course failure, regardless of violation level. Any established violation in this course may result in course failure regardless of violation level.

Copying homework solutions from any source, including the Internet, is plagiarism and is considered a violation.

Ally Statement: I have participated in the safer spaces training program through the LGBT center at Syracuse University. Please let me know if you use a different name than the one that shows up on my roster, and also let me know the pronouns that you use. I strive to use gender-neutral language in the classroom (e.g. your classmate, singular they), but I have old habits and I am not always successful. Feel free to correct me if I make a mistake.

Additional References:

Smooth Manifolds:

- J. Lee, *Introduction to smooth manifolds*, Springer.
- M. Spivak, *A comprehensive Intro. to Diff. Geometry, vol. I*, 3rd edition, Publish or Perish.
- W. Boothby, *An Intro. to diff. manifolds and Riem. geometry*, 2nd edition, Academic Press.
- V. Guillemin & A. Pollack, *Differential Topology*, First Edition, AMS Chelsea Publishing, 2010. (Reprint on original Prentice-Hall version, 1974).
- M. Spivak, *Calculus on Manifolds*, Addison-Wesley.
- J. Milnor, *Topology from the differential viewpoint*, Princeton.

Riemannian Geometry:

- J. Lee, *Riemannian manifolds: an introduction to curvature*, Springer.
- P. Petersen, *Riemannian Geometry*, 3rd edition, Springer.

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MAT 771 - Differential Geometry

Ex: A two particle system:

$\dot{P}$ $\dot{Q}$ P, Q particles moving around in $\mathbb{R}^3$

$\left.\begin{array}{l}\vec{s}_P\\ \vec{s}_Q\end{array}\right\}$ position of P&Q

$\left.\begin{array}{l}\vec{v}_P\\ \vec{v}_Q\end{array}\right\}$ velocity of P&Q

A "state" of system is a vector in $\mathbb{R}^{12}$

$$\underset{\in\mathbb{R}^3}{\vec{s}_P},\ \underset{\in\mathbb{R}^3}{\vec{s}_Q},\ \underset{\in\mathbb{R}^3}{\vec{v}_P},\ \underset{\in\mathbb{R}^3}{\vec{v}_Q}$$

All possible states of system form a subset $\overset{M}{\vee}$of $\mathbb{R}^{12}$.

Physical laws $\Leftrightarrow$ equations must be satisfied $\leftarrow$ Restrict possible states

$M\subset\mathbb{R}^{12}$ point in M $\Leftrightarrow$ possible state of system.

Measurement: $f: M\to\mathbb{R}$

$\uparrow$

do calculus?

"Distance in M":

Review: Derivatives of Mappings

In $\mathbb{R}^n$ $f:\mathbb{R}^n\to\mathbb{R}$

Let $\vec{x}\in\mathbb{R}^n$, $\vec{a}\in\mathbb{R}^n$

Directional Derivative of f at $\vec{x}$ in direction of $\vec{a}$

$$df_{\vec{x}}(\vec{a})=\lim_{s\to 0}\frac{f(\vec{x}+s\vec{a})-f(\vec{x})}{s}$$

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n=2: $df_{\vec{x}}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \frac{\partial f}{\partial x_1}\Big|_{\vec{x}}$, $df_{\vec{x}}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{\partial f}{\partial x_2}\Big|_{\vec{x}}$

Example:
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

All partial derivatives exist at $\vec{0}$, but f is not continuous at $\vec{0}$.

Can't define derivative just in terms of partials!

Def: $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a} \in \mathbb{R}^n$ if $\exists$ a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ s.t.
$$\lim_{|\vec{h}|\to 0} \frac{|f(\vec{a}+\vec{h}) - f(\vec{a}) - L(\vec{h})|}{|\vec{h}|} = 0.$$

$\Leftrightarrow$ $\exists$ fct $r(h)$ s.t. $r(h)\to 0$ as $|\vec{h}|\to 0$ & $f(\vec{a}+\vec{h}) = f(\vec{a}) + L(\vec{h}) + r(\vec{h})\,|\vec{h}|$

f is approximated by L to first order near $\vec{a}$.

Facts:

1) If L exists, it is unique, and $L = df_{\vec{a}}$ as defined above.

write $f(x_1, \dots, x_n) = (f_1(x_1,\dots,x_n), f_2(x_1,\dots,x_n), f_m(x_1,\dots,x_m))$

$$df_{\vec{a}}\left(\begin{pmatrix}h_1\\ \vdots \\ h_n\end{pmatrix}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ \vdots \\ h_n \end{pmatrix}$$

↗ Jacobian matrix of f at $\vec{a}$

↑ $m\times n$ matrix

↑ $n\times 1$

n=m=1: $df_{\vec{a}}(h) = f'(a)\,h$

2) If f is differentiable at $\vec{a}$ then f is continuous at $\vec{a}$.

3) **Chain Rule:** If f and g are differentiable then $f\circ g$ is differentiable.

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$d(f\circ g)_{\vec a} = df_{g(\vec a)} \circ dg_{\vec a}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R}^m \arrow[r, "f"] & \mathbb{R}^k
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "dg_{\vec a}"] \arrow[rr, bend right, "d(f\circ g)_{\vec a}"'] & \mathbb{R}^m \arrow[r, "df_{g(\vec a)}"] & \mathbb{R}^k
\end{tikzcd}$$

4) If f is differentiable at $\vec a$, then all partial derivatives exist at $\vec a$

$$\begin{tikzcd}
& & (x_1, \ldots, x_m) \arrow[r, mapsto] & x_3 \\
f: & \mathbb{R}^n \arrow[r] & \mathbb{R}^m \arrow[r, "p"'] & \mathbb{R} \\
& (a_1, a_2, x, a_4, \ldots) & & \\
\mathbb{R} \arrow[uur, "i"] & x \arrow[u, mapsto] & &
\end{tikzcd}$$

$\frac{\partial f_3}{\partial x_4} = d(p\circ f\circ i)$

5) If $f: \mathbb{R}^n \to \mathbb{R}^m$ and all partial derivatives exist in a nbhd of $\vec a$

and all partial der. functions are cont. in a nbhd of $\vec a$

then f is differentiable.

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) = (\cos(y), \sin(x), xy)$

Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix}_{3\times 2} = \begin{pmatrix} 0 & -\sin y \\ \cos(x) & 0 \\ y & x \end{pmatrix}$$

↑

All cont. ∴ f is differentiabl

Submanifolds

$f: X \subset \mathbb{R}^n \to \mathbb{R}^m$
$Y \subset \mathbb{R}^m$

We say f is differentiable if f may be locally extended to a differentiable map from open set of $\mathbb{R}^n$ to $\mathbb{R}^m$

ie. $\forall x \in X$ $\exists$ open $U \subset \mathbb{R}^n$ $x \in U$ and a differentiable map

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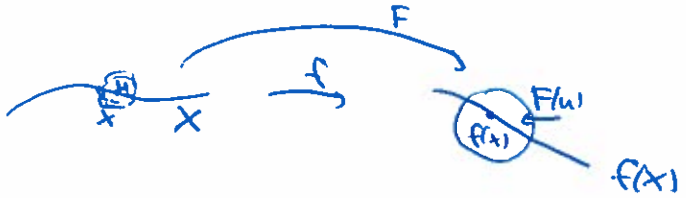
$F: U \to \mathbb{R}^m$ s.t. $F = f$ on $X \cap U$.



<u>Def:</u> A differentiable map between subsets of Euclidean spaces is called a diffeomorphism if it is one-to-one, onto, differentiable and f^{-1} is also differentiable.

<u>Recall:</u> A homeomorphism is a map that is one-to-one, onto, continuous whose inverse is also cont.

Any diffeom. is a homeom.

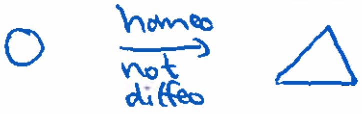
But <u>ex</u> $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^3$ homeo, no diffeo

$f^{-1}(x) = x^{1/3}$ not differentiable at $x=0$.

X and Y are diffeomorphic if $\exists$ a diffeomorphism $f: X \to Y$.

<u>exercise:</u> Being diffeomorphic is an equivalence relation.

<u>Ex:</u>



<u>Def:</u> Let $M \subset \mathbb{R}^n$, M is a smooth k-dimensional submanifold of $\mathbb{R}^n$ if for each point $p \in M$ $\exists$ nbhd $\underset{\text{open}}{V \subset M}$ which is diffeomorphic to an open subset of $\mathbb{R}^k$.

↑ spaces which are "locally" Euclidean

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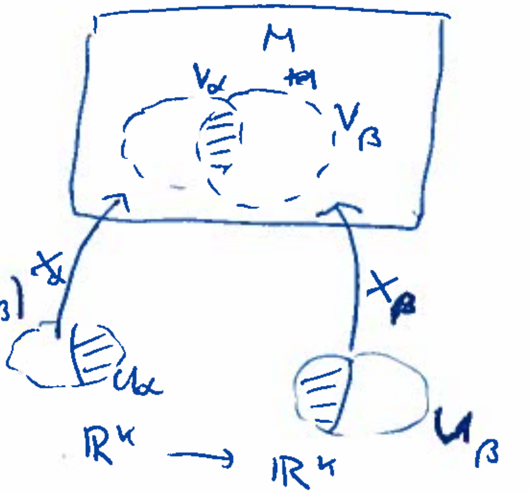
(1) holds
not Hausdorff

(real line w/ two origins)

How do we define derivatives?

We need any extra structure called a <u>differentiable structure</u>.

Def: A differential structure on a topological manifold of dim k is a family of homeomorphisms $X_\alpha : U_\alpha \to V_\alpha$ (parametrizations) where $U_\alpha \underset{open}{\subseteq} \mathbb{R}^k$, $V_\alpha \underset{open}{\subseteq} M$.



s.t.: (1) $\bigcup_\alpha V_\alpha = M$

(2) if $V_\alpha \cap V_\beta \neq \emptyset$ then the map

$$X_\beta^{-1} \circ X_\alpha : \underset{\underset{open}{\subset \mathbb{R}^k}}{X_\alpha^{-1}(V_\alpha \cap V_\beta)} \to \underset{\underset{open}{\subset \mathbb{R}^k}}{X_\beta^{-1}(V_\alpha \cap V_\beta)}$$

is differentiable.

(3) the family $\{(U_\alpha, X_\alpha)\}$ is <u>maximal</u> wrt. conditions (1) and (2).

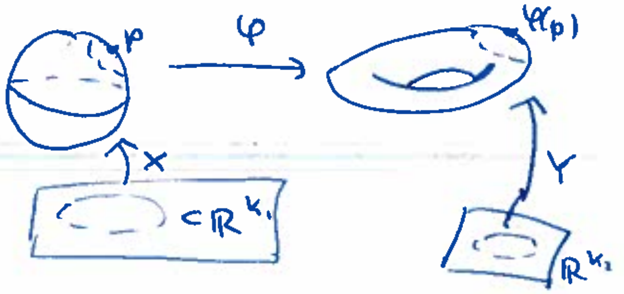
Def: A differentiable manifold is a topological manifold with a differential structure.

Def: Let M_1, M_2 be two differentiable manifolds. A map $\varphi : M_1 \to M_2$ is differentiable at p if $\exists$ parametrizations

$X : U_1 \to M$, $Y : U_2 \to M_2$

with $p \in X(U_1)$ and $\varphi(p) \in Y(U_2)$

s.t. $Y^{-1} \circ \varphi \circ X$ is differentiable at $X^{-1}(p)$.



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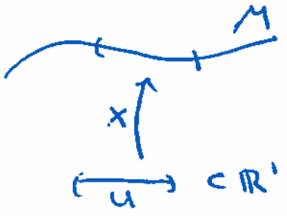
The map $x: U^{\subset \mathbb{R}^k \text{ open}} \longrightarrow V^{\subset M}$ is called a parametrization of V.



1-mfd ↔ curve

2-mfd ↔ surface

__Problem:__ Don't want to think of our spaces as sitting in $\mathbb{R}^n$.

__Next time:__ "Abstract" mfds

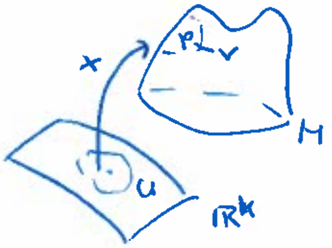
Does not depend on __embedding in $\mathbb{R}^n$.__

__1/22/18__

__Last time:__

$M \subseteq \mathbb{R}^n$ is a __smooth submfd of dimension k in $\mathbb{R}^n$__ if

$\forall p \in M \; \exists \; x: U \to V$ a diffeomorphism s.t. $\underset{\text{open}}{U \subset \mathbb{R}^k}$, ~~$V \subset \mathbb{R}$~~ $\underset{\text{open}}{p \in V \subset M}$.



__"Abstract" mfd__

__Def:__ A topological manifold M of dimension k is a topological space s.t.

locally Euclidean → ! (1) $\forall p \in M \; \exists \; X: U \to V$ a __homeo__ s.t. $\underset{\text{open}}{U \subset \mathbb{R}^k}$, $\underset{\text{open}}{p \in V \subseteq M}$

(2) M is Hausdorff

(3) M has a countable basis of open sets.

} technical we'll discuss later

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Definition is <u>well-defined</u> (doesn't depend on choice of x and y)

Take x_1, x_2 two different param. of p

and y_1, y_2 -"- -"- -"- -"- $\varphi(p)$)

then $y_2^{-1}\circ\varphi\circ x_2 = y_2^{-1}\circ y_1\circ y_1^{-1}\circ\varphi\circ x_1\circ x_1^{-1}\circ x_2$

$$= (y_2^{-1}\circ y_1)\circ y_1^{-1}\circ\varphi\circ x_1\circ(x_1^{-1}\circ x_2)$$

transition maps

is differentiable by the chain rule.

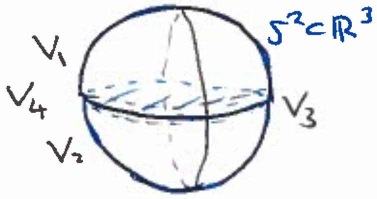
<u>Exercise:</u> Show chain rule works for smooth mfds.

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} S$$

<u>Examples of differential structures</u>

① $S^n = \{ \vec{p} \in \mathbb{R}^{n+1} : |\vec{p}| = 1 \}$

↑
n-sphere



$$X_1 : B_{\vec{0}}(1) \to V_1 \qquad B_{\vec{0}}(1) = \{\vec{x}\in\mathbb{R}^n \mid |\vec{x}|<1\}$$

$$X_1(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_n, \sqrt{1-\sum_{i=1}^{n}(x_i)^2})$$

$$X_2(x_1, \ldots, x_n) = (x_1, \ldots, x_n, -\sqrt{1-\sum_{i=1}^{n}(x_i)^2})$$

$$X_3(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \sqrt{1-\sum_{i=1}^{n}(x_i)^2}, x_n)$$

$$X_4(\ldots) = (\ldots, -\sqrt{\ldots}, \ldots)$$

X_5
X_6
⋮
X_{2n+2} → ~~X_{2n}~~ $2n+2$ <u>charts.</u>

S^2 covered by 6 charts.

<u>HW:</u> It's possible to cover S^n by 2 charts. $S^n - \{pt\} \underset{\text{diffeo}}{\cong} \mathbb{R}^n$.

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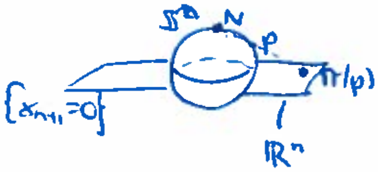
Stereographic projection



$\pi: \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$

② Real projective space

$\mathbb{R}\mathbb{P}^n$ = set of straight lines thru origin in $\mathbb{R}^{n+1}$

$= \mathbb{R}^{n+1}\setminus\{0\}/\sim \qquad (x_1, x_2, \dots, x_{n+1}) \sim (\lambda x_1, \lambda x_2, \dots, \lambda x_{n+1})$

$\lambda \in \mathbb{R}\setminus\{0\}$.

$\mathbb{R}\mathbb{P}^n$ is a smooth mfd, dimension n, can be covered by n+1 charts.

A pt in $\mathbb{R}\mathbb{P}^n$ is an equiv. class which we denote by $[x_1, x_2, \dots, x_{n+1}]$

if $\lambda \neq 0$ $\qquad [x_1, \dots, x_{n+1}] = \left[\frac{x_1}{\lambda}, \frac{x_2}{\lambda}, \dots \frac{x_{n+1}}{\lambda}\right]$.

Let $V_i = \{[x_1, \dots, x_{n+1}] : x_i \neq 0\}$.

$X_i : \mathbb{R}^n \to \mathbb{R}\mathbb{P}^n$

$X_i(y_1, \dots, y_n) = \underbrace{[y_1, y_2, \dots, y_{i-1}, 1, y_i, \dots, y_n]}_{n+1}$

Image $(X_i) = V_i$

$[y_1, \dots y_{n+1}] \qquad y_i \neq 0$

$\shortparallel$

$\left[\frac{y_1}{y_i}, \dots, \frac{y_{i-1}}{y_i}, 1, \frac{y_{i+1}}{y_i}, \dots, \frac{y_{n+1}}{y_i}\right]$

$\bigcup_{i=1}^{n+1} X_i(\mathbb{R}^n) = \mathbb{R}\mathbb{P}^n$.

$i>j \quad X_j^{-1} \circ X_i : X_i^{-1}(V_i \cap V_j) \to$

$\shortparallel$

$\{(y_1, \dots y_n) : y_j \neq 0\} \subset \mathbb{R}^n$

$X_j^{-1} \circ X_i(y_1, \dots y_n) = X_j^{-1}([y_1, \dots y_{i-1}, 1, y_i, \dots, y_n])$

$= X_j^{-1}\left(\left[\frac{y_1}{y_j}, \dots, \underset{\overset{\uparrow}{j}}{1}, \dots \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \dots \frac{y_n}{y_j}\right]\right)$

$= \left(\frac{y_1}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \dots, \frac{y_n}{y_j}\right)$. differentiable

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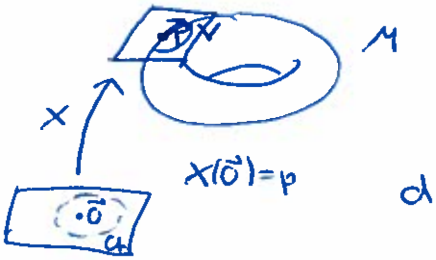
We know when a mapping $\varphi: M \to N$ is differentiable for any smooth mfds M & N.

Q: What is the derivative of φ?

The derivative should be a linear map that approximates φ close to p.

What are the v.s.? $\rightsquigarrow$ Tangent space

If $M \subset \mathbb{R}^n$ is a k-dim submfd. T_pM tangent space at p, $p \in M$



$dx_{\vec{0}}: \mathbb{R}^k \to \mathbb{R}^n$
$k < n.$

$\text{Image}(dx_{\vec{0}}) = T_pM.$

To define T_pM for abstract mfd, think in terms of directional derivatives.

Let M be a smooth mfd, let $\alpha: (-\varepsilon, \varepsilon) \to M$ be a differentiable curve in M.

Suppose that $\alpha(0)=p$.

Let D be the set of real valued differentiable fcts on M
$f: M \to \mathbb{R}$, $f \in D$

The tangent vector to a curve α at $t=0$ is the function

$$\alpha'(0): D \to \mathbb{R}, \quad \alpha'(0)(f) = \left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0}.$$

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The set of all tangent vectors at p is T_pM.

next time: $\alpha'(0)(f) = \sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_0 f.$

Last time: 1/24/18

M smooth mfd, $p \in M$. T_pM the tangent space to M at p.

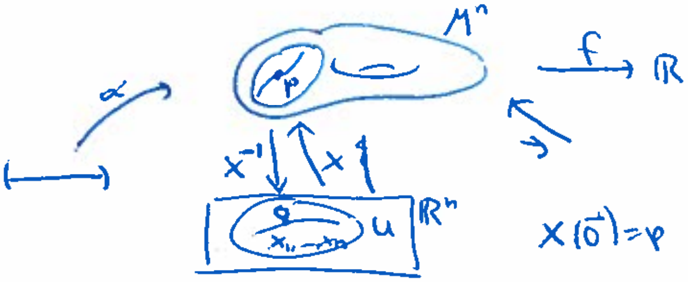
$D = \{ f \mid f: M \to \mathbb{R}$ is differentiable$\}$

$\alpha: (-\varepsilon, \varepsilon) \to M$ diff curve $\alpha(0) = p$, $\alpha'(0): D \to \mathbb{R}$

$f \mapsto \frac{d}{dt}(f\circ\alpha)\big|_{t=0}.$

T_pM = set of all such $\alpha'(0)$.

Let $X: U \to M$ parametrization of p.



$X(\vec{0}) = p$

Let $X^{-1}(q) = (x_1(q), x_2(q), \dots, x_n(q))$

$\alpha(t) = X(x_1(t), x_2(t), \dots, x_n(t))$

Define $\left(\frac{\partial}{\partial x_i}\right)_{\vec{0}} : D \to \mathbb{R}$ $\frac{\partial}{\partial x_i} \in T_pM$

i-th partial derivative of $(f\circ X)$ $\frac{\partial}{\partial x_i}(f\circ X) \overset{\text{ntn}}{=} \frac{\partial f}{\partial x_i}$

$$\alpha'(0)(f) = \frac{d}{dt}(f\circ\alpha)\Big|_{t=0}$$

$$= \frac{d}{dt}\overset{(f\circ x)}{\cancel{f}}(x_1(t), x_2(t), \dots, x_n(t))\Big|_{t=0}$$

$$= \sum_{i=0}^{n} x_i'(0)\frac{\partial f}{\partial x_i} = \left(\sum_{i=1}^{n} x_i'(0)\frac{\partial}{\partial x_i}\Big|_{\vec{0}}\right)(f)$$

Consequences:

① $\alpha'(0): D \to \mathbb{R}$, depends only of the first derivative of α in a coordinate chart.

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② T_pM is a vector space and $\frac{\partial}{\partial x_i}$ $i=1,\ldots,n$ is a basis of T_pM.

$\Rightarrow$ T_pM is a vector space of dimension n.

Remark: T_pM does not depend on choice of parametrization x.

But each parametrization of p gives a different basis for T_pM.

It is useful to write down change of parametrization formula.

If $y: Y \to M$ is another parametrization of p.

$$\alpha'(0)(f) = \sum_{i=1}^{n}\left(x_i'(0)\,\frac{\partial}{\partial x_i}\Big|_{\vec{0}}\right)(f)$$

$$= \sum_{j=1}^{n}\left(y_j'(0)\,\frac{\partial}{\partial y_j}\Big|_{\vec{0}}\right)(f)$$

$$\frac{\partial f}{\partial y_j} = \frac{\partial}{\partial y_j}(f\circ y) = \frac{\partial}{\partial y_j}(f\circ x\circ x^{-1}\circ y)$$

$$= d(f\circ x\circ x^{-1}\circ y)\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\leftarrow j$$

chain rule

$$= d(f\circ x)\,d(x^{-1}\circ y)\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

$$= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\,\frac{\partial x_i}{\partial y_j}$$

$$\alpha'(0)(f) = \sum_{j=1}^{n} y_j'(0)\,\frac{\partial f}{\partial y_j} = \sum_{j=1}^{n} y_j'(0)\left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\,\frac{\partial x_i}{\partial y_j}\right)$$

$$= \sum_{j=1}^{n}\left(y_j'(0)\underbrace{\sum_{i=1}^{n}\frac{\partial x_i}{\partial y_j}}\,\frac{\partial}{\partial x_i}\right)(f)$$

Let $\varphi: M_1^n \to M_2^n$ be differentiable.

Let $p\in M_1$, $v\in T_pM$. $d\varphi_p(v) = (\varphi\circ\alpha)'(0)$ where $\alpha(0)=p$, $\alpha'(0)=v$.

$\uparrow$ defines an element of $T_{\varphi..}M$

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$$\begin{tikzcd} {} \arrow[r, "\alpha"] & M_1 \arrow[r, "\varphi"] \arrow[d, "f"] & M_2 \arrow[d, "g"] \\ & \mathbb{R} & \mathbb{R} \end{tikzcd}$$

$\varphi \circ \alpha : (-\varepsilon, \varepsilon) \to M_2$

$d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2.$

Prop: This is well defined, $d\varphi_p(v) \in T_{\varphi(p)}M_2$ that does not depend on the choice of α.

Pf:

$$\begin{tikzcd} & M_1^n \arrow[r, "\varphi"] & M_2^m \\ (x_1, \dots, x_n) \arrow[ur, "x"] & & (y_1, \dots, y_m) \arrow[u, "y"] \end{tikzcd}$$

$y(\vec{0}) = \varphi(p)$

Exercise: $d\varphi_p(v) = \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i} x_i'(0) \frac{\partial \varphi}{\partial y_j}$

$\alpha(t) \overset{x}{=} (x_1(t), \dots, x_n(t))$

$\alpha'(0) = \sum x_i'(0) \frac{\partial f}{\partial x_i}$

$\Rightarrow d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is a linear map. □

Exercise: If $\varphi : M_1 \to M_2$, $\varphi_2 : M_2 \to M_3$ differentiable, then $\varphi_2 \circ \varphi_1$ is differentable, and $d(\varphi_2 \circ \varphi_1) = d\varphi_2 \circ d\varphi_1$. (Chain Rule)

Def: Let M_1, M_2 be differentable mfds.

A mapping $\varphi : M_1 \to M_2$ is a diffeomorphism if φ is a bijection and φ^{-1} is differentable.

Note: If $\varphi : M_1 \to M_2$ is a local diffeomorphism then $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is a linear isomorphism.

why? $\varphi^{-1} \circ \varphi = id.$ $d(\varphi^{-1} \circ \varphi)_p = d(id)_p \overset{\text{check}}{=} id.$

$\parallel$

$d\varphi^{-1}_{\varphi(p)} \circ d\varphi_p = id$

$\varphi \circ \varphi^{-1} = id.$

$d(\varphi \circ \varphi^{-1}) = d_{\varphi(p)}(id)$

$d\varphi_p \circ d\varphi^{-1}_{\varphi(p)} = id_{T_{\varphi(p)}M}$

$\Rightarrow d\varphi_p$ is an isomorphism with inverse $d\varphi^{-1}_{\varphi(p)}$.

$\Rightarrow \dim(T_pM_1) = \dim(T_{\varphi(p)}M_2) \Rightarrow \dim M_1 = \dim M_2$.

$\Rightarrow \mathbb{R}^n$ is not diffeomorphic to $\mathbb{R}^m$, $m \neq n$.

(True for homeomorphism ( ) but much harder!

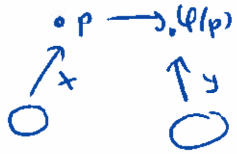
<u>Def:</u> $\varphi : M_1 \to M_2$ is called a local diffeomorphism at p if

~~$\forall p \in M$~~ $\exists U$ nbhd of p s.t. $\varphi|_U : U \to \varphi(U)$ is a diffeomorphism.

<u>Ex:</u> $\mathbb{R} \to S^1$
$t \to (\cos t, \sin t)$

Inverse Function Theorem

If $\varphi : M_1^n \to M_2^n$ is a differentiable map and $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is a linear isomorphism, then φ is a local diffeomorphism at p.



(Follows from statement in $\mathbb{R}^n \to \mathbb{R}^m$ since local defn's)

Mappings of maximal rank

$\varphi : M_1^n \to M_2^m$ differentiable

$p \in M_1$, **rank** of φ at p is $\text{rank}(d\varphi_p)$ as a linear map.

$\hookrightarrow$ rank Jacobian of $x^{-1} \circ \varphi \circ x$.

Three cases

$n < m$ $[\text{rank}(d\varphi_p) \leq n]$. max'l rank: $\text{rank}(d\varphi_p) = n$ $(\Leftrightarrow d\varphi_p$ injective$)$
$\Leftrightarrow$: φ is an immersion at p

$n = m$ If $\varphi : M_1^n \to M_2^n$. $\text{rank}(d\varphi_p) = n \Leftrightarrow$ (Inv. fct. Thm)
φ is a local diffeo at p

$n > m$ $\text{rank}(d\varphi_p) = m \Leftrightarrow$ φ is a submersion at p
$(\Leftrightarrow d\varphi_p$ is surjective$)$

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Canonical examples

Ex: Canonical Immersion $\mathbb{R}^n \to \mathbb{R}^m \quad n < m$

$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$ $\qquad \left(\begin{array}{c} I_n \\ \hline 0 \end{array}\right)$

Canonical submersion $\mathbb{R}^n \to \mathbb{R}^m \quad n > m$

$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ ← projection $\qquad (I_m | 0)$

idea: All immersions & submsersions are locally equivalent to the canonical ones.

Local immersion thm: If $\varphi: M_1^n \to M_2^n$ is an immersion at p

then ∃ parametrizations $X: U_1 \to M_1$ of p
$Y: U_2 \to M_2$ of $\varphi(p)$

s.t. $y^{-1} \circ \varphi \circ x(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, \ldots, 0)$

$$\begin{tikzcd}
M_1^n \arrow[r, "\varphi"] & M_2^m \\
U_1 \subset \mathbb{R}^n \arrow[u, "x"] \arrow[r, "\text{canonical}", "\text{immersion}"'] \arrow[ur, phantom, "\circlearrowright"] & U_2 \subset \mathbb{R}^m \arrow[u, "y"']
\end{tikzcd}$$

Pf: Inverse fct thm.

If φ is an $\overset{\text{subm.}}{\text{immersion}}$ at p then ∃ open set U s.t. φ is an $\overset{\text{subm.}}{\text{immersion}}$ on all of U.

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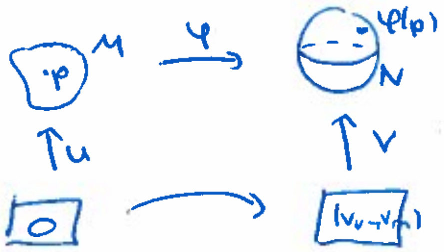
Last time: If $\varphi: M^n \to N^m$ has rank n at p then φ is called an immersion at p.

Pf of Local immersion thm: Choose coords U of p and V of $\varphi(p)$



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By permuting u_i, v_i, we can arrange so that

$$\det\left(\left[\frac{\partial(v_\alpha\circ\varphi)}{\partial u_\beta}\right]_{n\times n}\right)\neq 0 \qquad \alpha,\beta=1,\to n.$$

$\frac{\partial}{\partial u_\beta}(v_\alpha\circ\varphi\circ u_1)$

Let $x_\alpha := v_\alpha\circ\varphi$, $\alpha=1,\dots,n$. So x is a coord. system at p.

(By Inv. Fct. Thm.) Let $x: \underline{U}\to M$. $[x^{-1}(p)=(x_1,\to x_n)]$

Let $q\in U$. $q=x^{-1}(a_1,\dots,a_n)$. $x(q)=(a_1,\to a_n)$, $x_i(q)=a_i$.

$v_\alpha\circ\varphi(q)=a_\alpha$. $\quad v^{-1}\circ\varphi\circ x^{-1}(a_1,\to a_n)=(a_1,\to a_n,\underbrace{\overset{(v_1,\to n)}{\psi_1},\psi_2,\dots,\psi_{m-n}}_{?})$

define $\quad y_\alpha=v_\alpha$, $\alpha=1,\to n$

$y_r=v_r-\psi_r(v_1,\to v_n)$, $r=n+1,\to m$.

$$\left[\frac{\partial y_i}{\partial v_j}\right]=\begin{pmatrix} I & 0\\ (*) & I\end{pmatrix} \qquad \det\left(\left[\frac{\partial y_i}{\partial v_j}\right]\right)\neq 0.$$

$\Rightarrow$ $y^{-1}\circ v$ is a local diffeom. $\Rightarrow$ y is a local diffeo,

so y is a coord. system

<u>Exer:</u> <u>Check:</u> $y^{-1}\circ\varphi\circ x(a_1,\to a_n)=(a_1,\dots,a_n,0,\to 0)$. $\square$

<u>Suppose:</u> $\varphi: M\to N$ is an immersion $(\forall p\in M)$

Consider $\varphi(M)\subset N$.

$$\begin{tikzcd} M \arrow[r, "\varphi"] & \varphi(M)\subset N \\ \boxed{U_1} \arrow[u, "x"] & \bullet\,\varphi(p) \end{tikzcd}$$

$\varphi\circ x$ is <u>almost</u> a coord. chart for $\varphi(M)$ in $\varphi(p)$

But $\varphi(U_1)$ may not be an open set in $\varphi(M)$.

<u>Ex:</u> ① $\mathbb{R}\xrightarrow{\varphi}\mathbb{R}^2$

$\varphi(\mathbb{R})$ is not a mfd with subspace topology.

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② Topologist's sine curve

Immersion, but $\varphi(\mathbb{R})$ not a mfd

③

$\mathbb{R} \xrightarrow{\varphi}$ line with irrational slope.

immersion but $\varphi(\mathbb{R})$ is dense in T^2.

Def: An immersion that is a homeomorphism onto its image is called an embedding

A subset $M_1 \subset M$ is called a submanifold if the inclusion map is an embedding.

Ex: ①, ②, ③ are 1-1, immersions which are not embeddings.

Def: A proper map is a map s.t. the pre-image of every compact set is compact.

Thm: A 1-1 proper immersion is an embedding.

Cor: If M is compact and $\varphi: M \to N$ is a 1-1 immersion, then it is an embedding.

Pf: $\varphi: M \to N$ M compact then φ is proper:

$K \subset N$ cpct then K is closed (N Hausdorff), $\varphi^{-1}(K)$ is closed (φ cont.), so $\varphi^{-1}(K) \subseteq M$ is a closed subset of a compact space $\therefore \varphi^{-1}(K)$ cpct.

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More generally then immersion thm:

Thm: Constant Rank thm:

If $f: M^n \to N^m$ has rank k in a nbhd of p, then $\exists$ coordinates y, x s.t. $y \circ f \circ x^{-1} \underbrace{(a_1, \ldots, a_k, 0, \ldots, 0)}_{n} = \underbrace{(a_1, \ldots, a_k, 0, \ldots, 0)}_{m}$

Special case: $f: M^n \to N^m$ is a submersion (rank $= m$) $m < n$

$$y \circ f \circ x^{-1} (a_1, \ldots, a_n) = (a_1, \ldots, a_m)$$

Note: In submersion case, only need rank $= m$ at p.

Pf is similar to immersion thm.

Pre-image thm

If $f: M^n \to N^m$ which has constant rank k in a nbhd of $f^{-1}(p)$ then $f^{-1}(p)$ is a submfd of M of dimension $n-k$.

(or $f^{-1}(p)$ is empty).

Ex: $f: \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto x_1^2 + \ldots + x_{n+1}^2$

$n=2$: $f: \mathbb{R}^3 \to \mathbb{R}$.

$$df_x = [2x_1 \ \ 2x_2 \ \ \ldots \ \ 2x_{n+1}]$$

$$\text{rank } df_x = \begin{cases} 1 & (x_1, \ldots, x_{n+1}) \neq (0, \ldots, 0) \\ 0 & (0, \ldots, 0) \end{cases}$$

$$f^{-1}(1) = \{(x_1, \ldots, x_{n+1}) \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\} = S^n$$

$\therefore$ S^n is a mfd by Pre-image thm of dimension $n+1-1 = n$.

Similarly $f(x,y,z) = x^2 + y^2 - z^2$

HW problem: $x^2 + y^2 - z^2 = a^2$ is a mfd if $a \neq 0$

$df = [2x \ 2y \ -2z]$

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More interesting example

Let $M(n)$ be set of $n \times n$ matrices with real entries.

$M(n)$ is diffeomorphic to $\mathbb{R}^{n^2}$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & & & \vdots \\ \cdot & - & - & a_{n^2} \end{pmatrix} \xleftarrow{\varphi} \begin{pmatrix} a_1 \\ \vdots \\ a_{n^2} \end{pmatrix}$$

$O(n) = \{ A \in M(n) : A \cdot A^t = I \}$ $\quad A^t$ transpose of A.

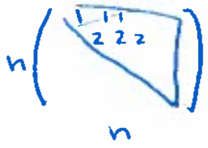
Defn $f(A) = AA^t$, $O(n) = f^{-1}(I)$, $f : M(n) \to M(n)$.

Note that $(AA^t)^t = A A^t$ So AA^t is symmetric.

$S(n) := \{ B \in M(n) : B = B^t \}$

$[\rightsquigarrow f : M(n) \to S(n)$ onto.$]$

$S(n)$ is diffeo to $\mathbb{R}^k$, $k = \frac{n(n+1)}{2}$,



$f : M(n) \to S(n)$
$\mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}$

Claim: f is a submersion at all points in $O(n) = f^{-1}(I)$.

Consequence: $O(n)$ is a mfd of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$

$A \in O(n)$, $B \in M(n) \simeq \underset{\mathbb{R}^k}{T_A M(n)}$ [tang. sp.]

$$df_A(B) = \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s} = \lim_{s \to 0} \frac{(A+sB)(A+sB)^t - AA^t}{s}$$

$$= \ldots = AB^t + BA^t$$

Q: Is $df_A(B) : M(n) \to S(n)$ onto?

Let $C \in S(n)$. $BA^t + AB^t = C \Leftrightarrow BA^t + (BA^t)^t = C$

just solve $BA^t = \frac{1}{2} C$ $(C = C^t)$ $\quad B = \frac{1}{2} CA$

Check $df_A(B) = C$.

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$f : M^n \to N^m$ differentiable

$p \in M$. $\operatorname{rank}_p(f) = \operatorname{rank}(df_p)$

- If $\operatorname{rank}_p(f) = n$, $\forall p$ f is an immersion.
  - A 1-1 immersion, which is a homeo onto $f(M)$ is called an embedding.
  - M cpct then any 1-1 immersion is an embedding

If f embedding, $f(M)$ is a submfd of N.

- $\operatorname{rank}_p(f) = k$ in a nbhd of $f^{-1}(p)$, $p \in N$ then $f^{-1}(p)$ is a submfd of M of dimension $n-k$.

Def: If $\operatorname{rank}_p(f) < m$, p is called a critical pt of f.

If $\operatorname{rank}_p(f) = m$, then p is called a regular pt of f.

$q \in N$ is called a critical value if $\exists p \in f^{-1}(q)$, p is a critical pt.

$q \in N$ is a regular value if it is not a critical value,

i.e. $\forall p \in f^{-1}(q)$, p is a regular pt of f.

Sard's Thm

If $f: M \to N$ is a smooth map then the critical values of f have measure zero in N.

In particular, the regular values of f are dense in N.

[$\forall q \in N$ $\forall$ open sets U cont. q $\exists q' \in U$ s.t. q' is a regular value]

Corollary: $f: M^n \to N^m$ smooth, $n < m$, then $f(M)$ has measure zero in N.

(There are no smooth "space-filling" curves)

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$\exists f : \mathbb{R} \to [0,1]\times[0,1]$ cont. and $f(\mathbb{R})$ is dense in $[0,1]\times[0,1]$.

Two "other constructions" of manifolds

① Tangent bundle

② Quotient of a group action

Tangent Bundle:

$M^n \overset{\text{smooth}}{\text{mfd}}$, $TM = \{(p,v) : p \in M, v \in T_pM\}$

TM is a differential mfd of dimension $2n$.

why? Let (U_α, x_α) be differentable structure on M.

$x_\alpha^{-1} = (x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha)$

$\left\{\frac{\partial}{\partial x_i^\alpha} : i = 1, \dots, n\right\}$ is a basis for T_pM.

$Y_\alpha : U_\alpha \times \mathbb{R}^n \to TM$

$$Y_\alpha(x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha, u_1, \dots, u_n) = \Big(\overset{p}{x_\alpha(x_1^\alpha, x_2^\alpha, \dots x_n^\alpha)}, \overset{v}{\sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha}}\Big)$$

$(Y_\alpha, U_\alpha \times \mathbb{R}^n)$ forms a diff. structure on TM.

Check: $Y_\beta^{-1} \circ Y_\alpha(q_\alpha, u_1^\alpha, \dots, u_n^\alpha) = Y_\beta^{-1}\left(X_\alpha(q_\alpha), \sum u_i^\alpha \frac{\partial}{\partial x_i^\alpha}\right)$

$$= \Big(\underbrace{X_\beta^{-1} \circ X_\alpha (q_\alpha)}_{\text{Smooth}},\ dx_\beta^{-1}(dx_\alpha(u_1, \dots, u_n))\Big)$$

$$d_p(\underset{\substack{\uparrow \\ \text{smooth}}}{x_\beta^{-1} \circ x_\alpha})(u_1, \dots, u_n$$

Topology: $U \subset TM$ open if $\pi_1(U)$ open and $\pi_2(U)$ open

$$TM \xrightarrow{\pi_1} M \qquad\qquad TM \xrightarrow{\pi_p} T_pM$$

$$(p,v) \mapsto p \qquad\qquad (p,v) \mapsto v$$

Tangent bundle is an example of a <u>vector bundle</u> over M.

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M and at each $p \in M$, $\overset{T_pM}{V_p}$ vector space

Hw: $TS^1 \underset{\text{diffeo}}{\cong} S^1 \times \mathbb{R}$ [trivial vector bundle]

But in general $TM \underset{\text{diffeo}}{\not\cong} M \times \mathbb{R}^n$.

ex: $TS^2 \not\cong S^2 \times \mathbb{R}^2$.

$TS^3 \cong S^3 \times \mathbb{R}^3$.

Idea: Any structure we have on a vector space can be put on a smooth mfd via the tangent bundle.

example:

Orientation:

If we have two ordered bases $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_n)$ of $\mathbb{R}^n$ then $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_n)$ are equivalently oriented if det of change of basis matrix between the two bases is positive.

Otherwise, they are oppositely oriented

Being equally oriented defines an equivalence relation of ordered bases of $\mathbb{R}^n$.

An orientation of $\mathbb{R}^n$ is a choice of equivalence class.

M is orientable if we can choose a smoothly varying orientation of the tangent space.

Def: M is orientable if it is possible to choose coordinate charts (X_α, U_α) s.t. $\bigcup_\alpha U_\alpha = M$ and $\det(d(X_\alpha \circ X_\beta^{-1}))$ has constant sign.

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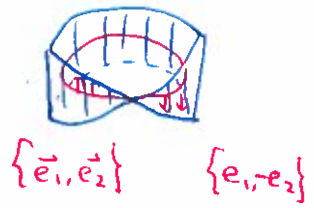
Ex: Möbius band is not orientable.



② Groups acting by diffeomorphisms:

G group, M smooth mfd.

Def: G acts on M by diffeomorphisms if

$\varphi: G\times M\to M$ s.t.

(1) $\varphi_g: M\to M$ is a diffeomorphism, and φ_e = identity.

$\varphi_g(p)=\varphi(g,p)$.

(2) If $g_1, g_2\in G$ $\varphi_{(g_1g_2)}=\varphi_{g_1}\circ\varphi_{g_2}$

Often in abuse of notation, we write: $g(p)=\varphi_g(p)$.

$M/_G=\{[p] : p\sim gp, g\in G\}$

Ex: ① $M=S^n$, $G=\mathbb{Z}_2=\{-1,1\}$

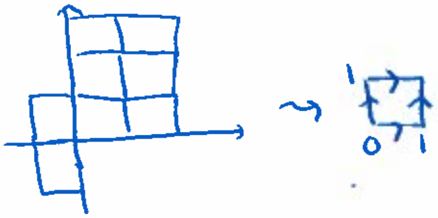
$\varphi:\mathbb{Z}_2\times S^n\to S^n$, $\varphi_1(p)=\varphi(1,p)=p$ $\forall p\in S^n$
$\varphi_{-1}(p)=\varphi(-1,p)=-p$

Is a action of $\mathbb{Z}_2$ on S^n by a diffeom.

$S^n/_{\mathbb{Z}_2}=\{(p) : p\sim -p\}\cong\mathbb{RP}^n$.

② $M=\mathbb{R}^n$, $G=\mathbb{Z}^n$, $\varphi:\mathbb{Z}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $\varphi(z,p)=p+z$

$\mathbb{R}^n/_{\mathbb{Z}^n}=\{[x]: x\sim x+k \quad k\in\mathbb{Z}^n\}$
||
get $T^n\cong S^1\times\dots\times S^1$



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Ex: $G = S^1$, $\mathbb{R}^2 = M$

$\varphi_\theta(p)$ = rotation count clockwise around origin of ang. θ

$\mathbb{R}^2/S^1 \cong [0,\infty)$

↑ not a smooth mfd

A group action is called properly discontinuous if $\forall p \in M$ $\exists$ nbhd $U \subset M$ s.t. $U \cap g(U) = \emptyset$ $\forall g \neq e$.

If action is properly discontinuous, then $\pi : M \to M/G$ is a local homeom
$\pi(p) = [p]$

Can put a diff structure on M/G s.t. $\pi : M \to M/G$ is a local diffeo. (see p.23 of textbook)

Prop: M/G is Hausdorff iff $\forall p_1, p_2 \in M$ $\exists$ nbhds U_1, U_2 of p_1, p_2 resp. s.t. $U_1 \cap g\,U_2 = \emptyset$ $\forall g \in G$.

[Covering space is always mfd. gp action is fund. gp.]
(univ. cover)

Vectorfields

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Def: A vector field X on a smooth mfd M is a correspondence that associates to each $p \in M$ a vector $X(p) \in T_pM$, that is smooth as a map $X : M \to TM$.

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Means: $X: U \to M$ parametrization

$X(p) = \sum a_i(p) \frac{\partial}{\partial x_i}$ and $a_i : M \to \mathbb{R}$ are smooth functions

From now on everything C^∞.

Recall: $v \in T_pM$ is a mapping $v: D \to \mathbb{R}$, $D = \{$smooth fcts $f: M \to \mathbb{R}\}$

So we can think of $\mathbb{X}: D \to D$

$$f \mapsto \mathbb{X}(f)$$

$$\mathbb{X}(f)(p) = \sum a_i(p) \frac{\partial f}{\partial x_i}(p)$$

Two vector fields $\mathbb{X}, \mathbb{Y}: D \to D$

$\mathbb{X} \circ \mathbb{Y}: D \to D$, define $\mathbb{X}(\mathbb{Y}(f)) = X \circ Y(f)$.

$\mathbb{X} \circ \mathbb{Y}$ does <u>not</u> define a new vector field.

Why? $x: U \to M$, $\mathbb{X} = \sum a_i \frac{\partial}{\partial x_i}$, $\mathbb{Y} = \sum b_j \frac{\partial}{\partial x_j}$

$$\mathbb{X}(\mathbb{Y}(f)) = \mathbb{X}\Big(\sum_j b_j \frac{\partial f}{\partial x_j}\Big) = \sum_i a_i \Big(\frac{\partial}{\partial x_i}\Big(b_j \frac{\partial f}{\partial x_j}\Big)\Big)$$

$$= \sum_{i,j} a_i \Big(\frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \underbrace{b_j \frac{\partial^2}{\partial x_i \partial x_j} f}_{\text{cannot be written as } \sum \frac{\partial}{\partial x_k}} \Big).$$

<u>or</u> if $v \in T_pM$ $\quad v(fg) = v(f)g + f v(g)$

and $\mathbb{X}(\mathbb{Y}(fg)) = \mathbb{X}(\mathbb{Y}(f)g + f\,\mathbb{Y}(g))$

$$= \mathbb{X}(\mathbb{Y}(f))g + \mathbb{Y}(f)\mathbb{X}(g) + \mathbb{X}(f)\mathbb{Y}(g) + f\,\mathbb{X}(\mathbb{Y}(g))$$

$$\neq \mathbb{X}\mathbb{Y}(f)g + f\,\mathbb{X}\mathbb{Y}(g).$$

so $\mathbb{X} \circ \mathbb{Y} \neq v$.

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[Note: $v \in T_pM$ linear & satisfies product rule.
In fact, can be used as equiv. def'n.]

Def: Let X, Y be vector fields.

Define $[X, Y] = XY - YX$, the Lie Bracket of X and Y.

Prop: $[X, Y]$ is a vector field on M.

Pf:

$$XY(f) = \sum_{ij} a_i\left(\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + b_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

$$YX(f) = \sum_{ij} b_j\left(\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right)$$

$$\Rightarrow (XY - YX)(f) = \sum_{ij}\left(a_i \frac{\partial b_j}{\partial x_i} \cancel{\frac{\partial f}{\partial x_j}} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial \cancel{f}}{\partial x_j}(f). \qquad \square$$

Properties of the Lie Bracket

(1) $[X, Y] = -[Y, X]$

(2) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ $a, b \in \mathbb{R}$.

(3) Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

(4) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

See p.27 of text. $\square$

Integral Curves

Let X be a smooth vector field.

A curve $\alpha : (-\delta, \delta) \to M$ is called an integral curve of X (or trajectory of X) if $\frac{\partial \alpha}{\partial t} = \cancel{X(\alpha(t))}$ $\forall t \in (-\delta, \delta)$.

$= X_{\alpha(t)}$ $\leftarrow$

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Let $x: U \to M$. $\quad X = a_i \frac{\partial}{\partial x_i}$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$.

$\frac{d\alpha}{dt} = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \ldots, \frac{d\alpha_n}{dt} \right)$ $\qquad$ $\left[\frac{\partial \alpha}{\partial t}(t) = \frac{d}{dt}(x \circ \alpha) \right.$

$\rightsquigarrow$ Want $\frac{d\alpha_i}{dt} = \sum a_i(\alpha_1(t), \ldots, \alpha_n(t)) \frac{\partial}{\partial x_i}$ $\quad \forall i = 1, \ldots, n$.

$\rightarrow$ Finding integral curves $\Leftrightarrow$ Solving system of ODEs.

We can rewrite the existence & uniqueness thm for systems of ODEs in the following way:

Thm: Let X be a smooth vectorfield on M and let $p \in M$. Then $\exists$ open set V and $\varepsilon > 0$ s.t. $\exists$ unique diffeomorphism

$$\underset{|t|<\varepsilon}{\varphi_t : V \to \varphi_t(V)} \qquad \text{s.t.} \qquad \varphi_0 = id$$

(1) $\varphi: (-\varepsilon, \varepsilon) \times V \to M$, $\quad \varphi(t, p) = \varphi_t(p)$ is smooth.

(2) If $|s|, |t|, |s+t| < \varepsilon$ and $q \in V$ s.t. $\varphi_t(q) \in V$:

$$\varphi_{s+t}(q) = \varphi_s \circ \varphi_t(q)$$

(3) The map $t \mapsto \varphi_t(q)$ is an integral curve of X through q. p

φ_t is called the local one parameter group of diffeos generated by X. (local flow)

$\varphi_0(q) = q$

$\begin{Bmatrix} \text{smooth} \\ \text{vector fields} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{local one-parameter} \\ \text{grps of diffeos} \end{Bmatrix}$

Thm: If M compact, then $\varphi_t : M \to M$ exists for all t.

(globally)
↳ not only locally as in prev. thm

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Lie Derivatives

- Write derivative along the local flow of X.

Let X smooth vector field.

Let $\varphi_t : V \to \varphi_t(V)$ local flow.

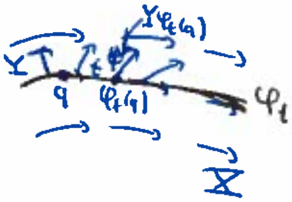
$t \mapsto \varphi_t(q)$ is an integral curve of X through q.

$$X(\varphi_t(q)) = \frac{d\varphi}{dt}\Big|_{t=0}$$

$$X_{\varphi_t(q)}(f) = \frac{d}{dt}(f \circ \varphi_t)\Big|_{t=0} = \lim_{h\to 0} \frac{f(\varphi_h(q)) - f(q)}{h}.$$

X, Y two vector fields

φ_t local flow for X.



$$(L_X Y)(q) = \lim_{t\to 0} \frac{Y_{\varphi_t(q)} - (d\varphi_t)_q(Y_q)}{t} \in T_qM.$$

$[\varphi_t : M \to M \to (d\varphi_t)_q : T_qM \to T_{\varphi_t(q)}M]$

Thm: $(L_X Y)(q) = [X, Y]_q = XY - YX.$

See p. 28 for the proof. $\square$

example: $\mathbb{R}^2$. Let $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ be standard coord's in $\mathbb{R}^2$.

$X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$.

$\varphi_t(a,b) = (a+t, b)$

$d\varphi_t = id.$

$$L_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0$$

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0.$$

Any coordinates: $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$

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Thm: Let X be a smooth vector field.

$\forall p \in M \ \exists \quad x: U \to M$, $x(0) = p$ s.t. $X = \frac{\partial}{\partial x^1}$.

Thm: (Frobenius) Let X, Y two vector fields.

Then $\exists$ coordinate chart $x: U \to M$ s.t.

$\frac{\partial}{\partial x^1} = X$, $\frac{\partial}{\partial x^2} = Y$, iff $[X\,Y] \equiv 0$ in $x(U)$.

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Partition of Unity

Recall: Defn of smooth mfd M,

what we need most → (1) differential structure

(2) M is Hausdorff

(3) M has a count. basis

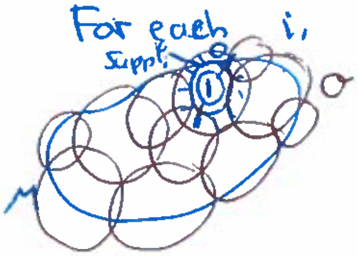
(2) & (3) are necessary for the existence of a partition of unity

Def: Let $\mathcal{O}$ be an open cover of M. A collection of smooth functions $\{\varphi_i : M \to [0,1]\}$ is called a partition of unity (subordinate to $\mathcal{O}$) if

(1) Locally finite: $\forall p \in M \ \exists$ open set U with $p \in U$ s.t. only finitely many of φ_i are nonzero in U.

(2) $\underset{\uparrow \atop \text{finite}}{\sum} \varphi_i(p) = 1 \quad \forall p \in M$

(3) For each i, $\exists\, U \in \mathcal{O}$ s.t. $\text{support}(\varphi_i) \subset U$.



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Thm: If M is a smooth mfd then any open cover has a smooth partition of unity.

[equiv. to (2)+(3) in def of smth mfd]

Thm: (Embedding Thm)

If M^n is compact smooth mfd then there is an embedding $f: M^n \to \mathbb{R}^N$ for some N.

Remarks: • Also true in non-cpct case.

- Whitney: $N = 2n$.

best you can do since $\mathbb{R}P^2 \hookrightarrow \mathbb{R}^4$ but can not be embedded in $\mathbb{R}^3$.
($\mathbb{R}P^2$ non-orient. ; every submfd of $\mathbb{R}^3$ is orient.)

Pf: Since M is cpct any 1-1 immersion is an embedding.

We can cover M by finitely many coordinate charts

$$X_i : U_i \subset \mathbb{R}^n \to M, \quad i=1,\dots,k \qquad \text{s.t.} \ \bigcup_{i=1}^{k} X_i(U_i) = M.$$

Let φ_i be a partition of unity subordinate to $X_i(U_i)$, s.t.

$$\operatorname{supp} \varphi_i \subseteq X_i(U_i) \qquad (\Rightarrow \varphi_i \equiv 0 \text{ on } M \setminus X_i(U_i))$$

Define $f: M^n \to \mathbb{R}^N$,

$$f(p) = (\underbrace{\overset{\text{mult!}}{\varphi_1 X_1^{-1}}}_{\in \mathbb{R}^n}, \varphi_2 X_2^{-1}, \dots, \underbrace{\varphi_k X_k^{-1}}_{\in \mathbb{R}^n}, \varphi_1, \varphi_2, \dots, \varphi_k)(p)$$

$$X_i^{-1} : \underset{\substack{\cap \\ M}}{X_i(U_i)} \to \mathbb{R}^n \qquad N = nk + k$$

Claim: f is a 1-1 immersion.

Let $p \in M$. $\exists \varphi_i$ s.t. $\varphi_i(p) \neq 0$. $\underbrace{\hphantom{xx}}_{}$ $\exists$ open V $p \in V$ s.t. $\varphi_i X_i^{-1} : V \to \mathbb{R}^n$

is a local diffeo on U. $\Rightarrow \operatorname{rank}_p (\varphi_i X_i^{-1}) = n$ $\Rightarrow \operatorname{rank}_p (f) \geq n$

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So, $= n$ $\Rightarrow$ f is an immersion at p.

If $f(p) = f(q) \Rightarrow \varphi_i(p) = \varphi_i(q) \quad \forall i = 1 \dots k.$

$\exists$ at least one φ_{i_0} s.t. $\underset{\varphi_{i_0}(q)}{\overset{\|}{\varphi_{i_0}(p)}} \neq 0$

$$\Rightarrow \quad \varphi_{i_0}(p) x_{i_0}^{-1}(p) = \varphi_{i_0}(p) x_{i_0}^{-1}(q) \quad \Rightarrow x_{i_0}^{-1}(p) = x_{i_0}^{-1}(q) \quad \Rightarrow p = q$$

since $x_{i_0}^{-1}$ diffeo

$\square$

Chapter 1: Riemannian metrics

Def: Let V be a vector space. An inner product on V is a map $\langle\, ,\, \rangle : V \times V \to \mathbb{R}$

s.t. (1) $\langle v, w \rangle = \langle w, v \rangle \quad \forall v, w \in V$

(2) bilinear (linear in each entry)

(3) positive definite $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$

ex: $V = \mathbb{R}^n$ $\langle v, w \rangle = v \cdot w$

dot product in $\mathbb{R}^n$.

recall $v \cdot w = |v|\,|w| \cos\theta$

$= \langle v, v \rangle^{1/2} \langle w, w \rangle^{1/2} \cos\theta$

Def: Let M be a smooth mfd. A ~~Rien~~ Riemannian metric on M is a correspondence $p \mapsto \langle\, ,\, \rangle_p$ where $\langle \cdot, \cdot \rangle_p$ is an inner product on T_pM which "varies smoothly"

i.e. if v, w are smooth vector fields on M, ~~$f: M \to \mathbb{R}$~~

$f: M \to \mathbb{R}$

then $f(p) = \langle v, w \rangle_p$ is smooth.

$M \subseteq \mathbb{R}^n$

$\langle v, w \rangle_p = \vec{v} \cdot \vec{w}$

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In coordinates:

If we have a parametrization $x: U \to M$

$\{\frac{\partial}{\partial x_i}\}$ basis of T_pM then $g_{ij}(p) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p$.

$g_{ij} = g_{ji}$, $g_{ii} > 0$

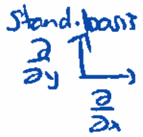
$$[g_{ij}] = \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \\ \vdots & & \ddots \end{pmatrix}$$ $n \times n$ matrix positive definite and symmetric

ex: In $\mathbb{R}^n$, $\{e_i\}$ = standard basis of $T_p\mathbb{R}^n$

$$\langle v, w \rangle_p = v \cdot w$$

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \rightsquigarrow [g_{ij}] = \underset{= I_n}{Id} \qquad \mathbb{R}^2: g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- $\mathbb{R}^2$, use polar coordinates instead of standard basis



$\swarrow$ (omit (0,0))

(r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

$$\longrightarrow \frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

In basis $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\}$ of T_pM

$$g_{11} = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = \cos^2\theta + \sin^2\theta = 1$$

$$g_{21} = g_{12} = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0$$

$$g_{22} = \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle = r^2$$

$$\Rightarrow g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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Another representation:

$$\langle\ ,\ \rangle_p : T_pM \times T_pM \to \mathbb{R}$$

$\left\{ \frac{\partial}{\partial x_i} \right\}$ basis of T_pM.

Let $dx_i : \underset{\text{linear}}{T_pM \to \mathbb{R}}$ dual of $\frac{\partial}{\partial x_i}$

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

"Tensor product" $dx_i \otimes dx_j : T_pM \times T_pM \to \mathbb{R}$

$$\underset{\substack{\| \\ dx_i dx_j}}{(dx_i \otimes dx_j)}(v,w) = dx_i(v)dx_j(w)$$

$dx_i dx_i = dx_i^2$.

ex: $\langle v,w \rangle_p = \sum_{i,j} g_{ij} dx_i\, dx_j\, (v,w)$

[compare both sides on basis element]

① Flat on $\mathbb{R}^n$: $\langle\ ,\ \rangle_p = dx_1^2 + dx_2^2 + \_ + dx_n^2$

② Polar coord's $\left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\}$ $\langle\ ,\ \rangle_p = dr^2 + r^2 d\theta^2$.

Why a metric?

A Riemannian mfd is a metric space.

Let $\gamma : I \to M$ be a smooth curve.

Let $[a,b] \subset I$.

Then we can define the length of γ from a to b as

$$\ell_a^b(\gamma) = \int_a^b \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle_{\gamma(t)}^{1/2} dt$$

Define the distance between two points $x,y \in M$ as

$$d(x,y) = \inf\left\{ \ell_a^b(\gamma) : \gamma \text{ is a smooth path with } \gamma(a) = x \text{ and } \gamma(b) = y \right\}$$

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d is a metric:

(1) $d(x,y) \geq 0$ and " = " 0 iff $x=y$.

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) \leq d(x,z) + d(z,y)$ "<u>Length metric</u>"

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Last time: Riemannian metric $\Leftrightarrow$ smoothly varying inner product on tangent vectors.

dot product in $\mathbb{R}^n$

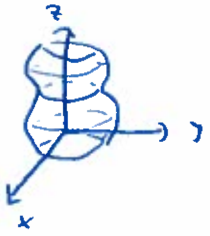
$M \subset \mathbb{R}^n$ $\langle v,w \rangle = v \cdot w$

Examples A surface of revolution

profile curve $c(t) = (r(t), 0, z(t))$

in xz-plane in $\mathbb{R}^3$ $r(t) > 0, t > 0$

rotate around the z-axis $\leftarrow S \underset{\text{diffeo}}{\cong} (a,b) \times S^1$



Coordinates: $(t,\theta) = (\underset{x}{\underset{\parallel}{r(t)\cos\theta}}, \underset{y}{\underset{\parallel}{r(t)\sin\theta}}, \underset{z}{\underset{\parallel}{z(t)}})$

give S induced Riem. metric for dot product.

dot product $dx^2 + dy^2 + dz^2$

$\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right\}$ form a basis for T_pS.

$\underline{dx} = d(r(t)\cos\theta) = \frac{dr}{dt}\cos\theta\, dt - r\sin\theta\, d\theta$

$\underline{dy} = d(r(t)\sin\theta) = \frac{dr}{dt}\sin\theta\, dt + r\cos\theta\, d\theta$

$\underline{dz} = \frac{dz}{dt}\, dt$

$\Rightarrow dx^2 = \left(\frac{dr}{dt}\right)^2 \cos^2\theta\, dt^2 - r\sin\theta\cos\theta \frac{dr}{dt}\, d\theta dt - r\frac{dr}{dt}\sin\theta\cos\theta\, dt d\theta$

$+ r^2 \sin^2\theta\, d\theta^2$

$dy^2 = \ldots$

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$dz^2 = \ldots$

$$\Rightarrow dx^2 + dy^2 + dz^2 = \left(\frac{dr}{dt}\right)^2 \cos^2\theta \, dt^2 + r^2 \sin^2\theta \, d\theta^2$$

$$+ \left(\frac{dr}{dt}\right)^2 \sin^2\theta \, dt^2 + r^2 \cos^2\theta \, d\theta^2$$

$$+ \left(\frac{dz}{dt}\right)^2 dt^2$$

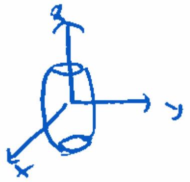
$$\Rightarrow \left(\left(\frac{dr}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right) dt^2 + r^2(t) \, d\theta^2 = \langle \, , \, \rangle_p.$$

If $c(t)$ is parametrized to be unit speed.

$\langle \, , \, \rangle = dt^2 + r^2(t) \, d\theta^2.$

$r(t) = t \Rightarrow dt^2 + t^2 d\theta^2$ ← polar coord's in $\mathbb{R}^2$.

$r(t) = \sin(t)$, $z(t) = \cos(t)$ $\Rightarrow dt^2 + \sin^2(t) \, d\theta^2$



More generally, a rotationally symmetric metric on $(a,b) \times S^1$ is any metric of the form $\langle \, , \, \rangle = a^2(t) \, dt^2 + b^2(t) \, d\theta$ $\quad a, b \in C^{\infty}(\mathbb{R})$, $a, b > 0$

Not all rot. symm. metrics are <u>surfaces of revolution.</u>

e.g. it is necessary cond. $\left(\frac{db}{dt}\right)^2 \leq a^2.$

Any metric on $(a,b) \times S^1$

$$\underset{g_{tt}}{a^2(t,\theta) \, dt^2} + \underset{g_{t\theta}}{C^2(t,\theta)} \, (dt\,d\theta + d\theta\,dt) + \underset{g_{\theta\theta}}{b^2(t,\theta) \, d\theta^2}$$

Lie Groups

A Lie Group, G, is a smooth mfd with a group structure, "$\cdot$", s.t. the maps

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$G \times G \to G$ $\qquad$ $G \to G$
$(x,y) \mapsto x \cdot y$ $\qquad$ $x \to x^{-1}$ $\qquad$ are smooth.

Denote e as identity element of G.

<u>Examples</u>

① $\mathbb{R}^n, \cdot$ coord'wise add. is a Lie group.

② $\mathbb{R}\setminus\{0\}, *$ with mult.

③ $\mathbb{C}\setminus\{0\}$ with complex mult.

④ $S^1 = \{e^{i\theta}, \theta \in [0, 2\pi)\} \subseteq \mathbb{C}$.

⑤ $GL(n,\mathbb{R}) \overset{\subset \mathbb{R}^{n^2}}{=} \{n \times n$ real matrices s.t. $\det \neq 0\}$ w/ matrix mult.

⑥ $O(n) = \{A \in GL(n,\mathbb{R}) : AA^* = I\}$ — " —

G Lie group, $g \in G$

$L_g : G \to G$ "Left translation by g"

$L_g(x) = gx$

$R_g : G \to G,\ x \mapsto xg$ "Right transl. by g"

$$\begin{tikzcd}
x \arrow[r, mapsto] & (g,x) \arrow[r, mapsto] & g \cdot x \\
G \arrow[r, "i_g"] \arrow[rr, bend right, "L_g\ R_g"'] & G \times G \arrow[r] & G
\end{tikzcd}$$
smooth

and $\quad L_g(L_{g^{-1}}(x)) = gg^{-1}x = x$

$L_{g^{-1}}(L_g(x)) = g^{-1}gx = x$

$\Rightarrow (L_g)^{-1} = L_{g^{-1}}$

$\Rightarrow L_g, R_g$ are diffeomorphisms.

A vector field X on G is called left invariant if

$(dL_g)_h(X_h) = X_{gh}$ $\forall g,h$ $\qquad$ $dL_g(X) = X_g$.

$$\begin{tikzcd}
X_h \arrow[r, bend left] & dL_g(X_h) \\
h \arrow[u, leftrightarrow] \arrow[r, bend left, "L_g"] & gh \arrow[u]
\end{tikzcd}$$

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A left invariant vector field is completely determined by its value at one point, usually we pick e.

Conversely, if $v \in T_e G$ let $X_g = (dL_g)_e(v)$, then X_g is a left invariant vector field with $X_e = v$.

$$\left\{ \begin{array}{c} \text{left inv. vector} \\ \text{fields on } G \end{array} \right\} \cong T_e G \simeq \mathbb{R}^n$$

Remark: S^2 is not a Lie Group.

Harry Ball Thm

If V is a vector field on S^2 then $\exists p \in S^2$ s.t. $V(p) = 0$.

$\rightarrow TS^2 \not\cong S^2 \times \mathbb{R}^2$.

Lie Bracket: X, Y $\quad [X,Y] = XY - YX$.

Prop: If X, Y are left invariant v. fields, then so is $[X,Y]$.

Pf Let $f: M \to \mathbb{R}$.

$$\begin{aligned} dL_g([X,Y])(f) &= [X,Y](f \circ L_g) \\ &= X(Y(f \circ L_g)) - Y(X(f \circ L_g)) \\ &= X(dL_g(Y)(f)) - Y(dL_g(X)(f)) \\ &= XY(f) - YX(f) = [X,Y](f). \end{aligned}$$

$\square$

The Lie algebra of G = Vector span of left inv. vector fields with mult. $X, Y \to [X,Y]$

Lie alg. $\rightarrow \mathfrak{g} \approx (T_e G, [\,,\,])$

If $\langle\, , \rangle$ is an inner product on $T_e G$

$u, v \in T_g G$ define $\langle u, v \rangle_g = \langle (dL_{g^{-1}})_g(u), (dL_{g^{-1}})_g(v) \rangle$

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Defines a Riemannian metric on G.

$\langle\, ,\, \rangle_g$ is called a left-invariant metric on G.

In the sense that $\langle u,v\rangle_h = \langle (dL_g)_h(u), (dL_g)_h(v)\rangle_{gh}$

(exercise)

Each choice of inner product on T_eG gives a left-inv. metric on G. So, $\exists$ $\frac{n(n+1)}{2}$ ~~left~~ inv. metrics on G.

$$\langle\, ,\, \rangle_e = \sum g_{ij}\, dx_i \otimes dx_j$$

Similarly, we can define right inv. metrics; replace L_g with R_g everywhere.

A metric is bi-invariant if it is both left & right invariant.

Last time: Lie groups <u>2/14/18</u>

G is smooth mfd that is also a gp

$g \in G$ $\quad L_g : G \to G$, $x \mapsto gx$

$\quad\quad R_g : G \to G$, $x \mapsto xg$

A Riem. metric $\langle\, ,\, \rangle$ on G is called <u>left-invariant</u> if

$\forall\, h, g$ $\quad \langle u,v\rangle_h = \langle (dL_g)_h(u), (dL_g)_h(v)\rangle_{gh}$ $\quad \forall u,v \in T_hG$.

<u>Right-inv.</u> $\Leftrightarrow$ same def. with R instead of L.

<u>Prop:</u> Let $\langle \cdot, \cdot \rangle_e$ be an inner product on T_eG, then

for $u,v \in T_gG$ define $\langle u,v\rangle_g = \langle (dL_{g^{-1}})_g(u), (dL_{g^{-1}})_g(v)\rangle_e$

<u>Claim:</u> $\langle\, ,\, \rangle_g$ is a left-inv. metric.

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Pf: $\langle u,v\rangle_h = \langle (dL_{h^{-1}})_h(u), (dL_{h^{-1}})_h(v)\rangle_e$.

$\langle (dL_g)_h(u), (dL_g)_h(v)\rangle_{gh} = \langle (dL_{(gh)^{-1}})_{gh}((dL_g)_h(u)),$

$(dL_{(gh)^{-1}})_{gh}((dL_g)_h(v))_e \rangle$

$(= \langle u,v\rangle_h)$

$(gh)^{-1} = h^{-1}g^{-1}$

$L_{(gh)^{-1}} = L_{h^{-1}} \circ L_{g^{-1}}$

$dL_{(gh)^{-1}} = dL_{h^{-1}} \circ dL_{g^{-1}}$.

Example: $SU(2) = \{A \in M_{2\times 2}(\mathbb{C}) : AA^* = Id,\ \det(A) = 1\}$

$= \left\{ \begin{bmatrix} z & w \\ -\bar{z} & \bar{w} \end{bmatrix} \middle| \begin{array}{l} z, w \in \mathbb{C}, \\ |z|^2 + |w|^2 = 1 \end{array} \right\}$

$= S^3(1)$

$(z = 1 + 0i$
$w = 0 + 0i]$

$(z,w) \in \underset{\mathbb{R}^4}{\underset{\|\!\wr}{\mathbb{C}\times\mathbb{C}}}$

SO(2) w/ matrix mult. is a Lie group.

$T_e G =$ ~~[illegible]~~ $T_I(SU(2))$

$= \left\{ \begin{bmatrix} i\alpha & \beta + ic \\ -\beta + ic & -i\alpha \end{bmatrix} : \alpha, \beta, c \in \mathbb{R} \right\}$

Basis for $T_e G$: $X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Extend X_1, X_2, X_3 to left-invariant vector fields on all of SU(2).

Define a left-inv. metric, X_1, X_2, X_3 are orthonormal.

$\{X_1, X_2, X_3\}$ forms a frame field for SU(2).

$[\ X_i(g) = (dL_g)_e(X_i(e))\ ]$

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$\langle\, ,\, \rangle = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ ← usual metric on S^3.

where σ_i are duals of X_i, $\sigma_i : T_pM \to \mathbb{R}$ linear, and $\sigma_i(X_j) = \delta_{ij}$.

On the other hand, if we declare $\{X_1, X_2, X_3\}$ to be orthogonal, length X_1 to be ε, length X_2, X_3 to be 1.

→ $\langle\, ,\, \rangle^{\varepsilon} = \varepsilon^2 \sigma_1^2 + \sigma_2^2 + \sigma_3^2$.

Berger sphere

<u>Note:</u> $\{X_1, X_2, X_3\}$ is <u>not</u> a coord. frame.

i.e. there is <u>no choice of coord's</u> $x : U \to SU(2)$ s.t. $X_i = \frac{\partial}{\partial x_i}$.

Let G be a Lie group, $h \in G$.

The inner automorphism of G by h is $G \to G$, $g \mapsto hgh^{-1}$.

i.e. $L_h \circ R_{h^{-1}}$ or $R_{h^{-1}} \circ L_h$

$L_h \circ R_{h^{-1}}$ is a diffeomorphism of G, and $L_h \circ R_{h^{-1}}(e) = heh^{-1} = e$.

Define $Ad_h : T_eG \to T_eG$

$$Ad_h = d(L_h \circ R_{h^{-1}}).$$

[adjoint representation $G \to GL(T_eG)$, $h \mapsto Ad_h$.

<u>Prop:</u> Let X, Y be left-invariants vector fields on G, then

$$[Y, X] = \lim_{t\to 0} \left(\tfrac{1}{t} (Ad_{x_t^{-1}(e)})(Y(e)) - Y(e) \right)$$

where x_t is one parameter gp at diffeos gen'd by X.

<u>Pf:</u> If x_t is local flow gen'd by X,

$$[Y, X] = \lim_{t\to 0} \tfrac{1}{t} \left(d\cancel{x} x_t (Y(x_t^{-1}(e))) - Y \right)$$

Let $y \in G$. $\alpha_y : t \mapsto y(x_t(e))$. $\alpha_y(0) = y(x_0(e)) = y \cdot e = y$

$t \to x_t(e)$ int curve for X

$\frac{d\alpha_y}{dt}\Big|_{t=0} = \frac{d}{dt}(y\, x_t(e)) = (dL_y)(\frac{dx_t}{dt}(e)) = dL_y(X(e)) = X_y$

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α_y is integral curve $\overset{\text{of } X}{}$ thru y.

But so is $t \mapsto x_t(y)$.

$$\text{So } x_t(y) = y\, x_t(e).$$

$$x_t(y) = R_{x_t(e)}(y).$$

$$dx_t = (dR_{x_t(e)}).$$

$$\Rightarrow [Y,X] = \lim_{t\to 0} \frac{1}{t}\left((dR_{x_t(e)})(Y(x_t^{-1}(e))) - Y(e)\right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left((dR_{x_t(e)} \circ dL_{x_t^{-1}(e)})Y(e) - Y(e)\right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left((Ad_{x_t^{-1}(e)})(Y(e)) - Y(e)\right) \quad \square$$

Two consequences:

① $G = GL(n,\mathbb{R})$, $Gl(n,\mathbb{C})$ open subset of $\mathbb{R}^{n^2}, \mathbb{R}^N$

X, Y left inv. fields $X(I), Y(I) \in T_I G$.

$[X,Y]_I = X(I)Y(I) - Y(I)X(I)$ ← matrix mult.

$GL(n,\mathbb{R}) \overset{\text{open subset}}{\approx} \mathbb{R}^{n^2}$ $\quad T_I(GL(n,\mathbb{R})) \approx \mathbb{R}^{n^2} = M_{n\times n}(\mathbb{R})$

$L_g : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is a linear map of $\mathbb{R}^{n^2}$

$$\Rightarrow (dL_g)(u) = gu, \quad (dR_g)(u) = ug.$$

$$x_t(e) = I + tX + o(t)$$

$$x_t^{-1}(e) = I - tX + o(t)$$

$$dL_{x_t^{-1}(e)}(Y) = (I - tX + o(t))(Y)$$

$$dR_{x_t(e)}(dL_{x_t^{-1}(e)}(Y)) = (I - tX + o(t))(Y)(I + tX + o(t))$$

$$= (Y - tXY \cancel{+o(t)} + tYX + o(t))$$

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$\Rightarrow [Y, X] = YX - XY.$ matrix mult.

Prop $\parallel$ $\qquad\qquad\qquad \parallel$

$\lim_{t\to 0} \frac{1}{t} (Y - tXY + tYX + o(t) - Y)$

Remark: Also works for $G \leq GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.

Ex: Berger Spheres

$SU(2)$ $\qquad X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$

basis for $T_I G$.

$$[X_1, X_2] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = 2X_3.$$

Similarly, $[X_2, X_3] = 2X_1$, $[X_3, X_1] = 2X_2$.

$\Rightarrow \{X_1, X_2, X_3\}$ is not a coord. field in any open set.

because $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$

Recall: A metric on a Lie group is called bi-invariant if it is left-invariant and right-invariant

Prop: If $\langle \, , \rangle$ ↙ left inv. metric is bi-invariant and X, Y, Z are left invariant vector fields, then $\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle.$

Rk: Actually is iff.

Ex: Berger Spheres. $\langle \, , \rangle = \varepsilon \sigma_1^2 + \sigma_2^2 + \sigma_3^2.$

$\langle [X_1, X_2], X_3 \rangle = 2 \langle X_3, X_3 \rangle \; 2$

$\langle X_1, [X_2, X_3] \rangle = 2 \langle X_1, X_1 \rangle = 2.$

bi-inv. iff $\varepsilon = 1$

Pf of prop:

bi-inv. $\Rightarrow$ $\langle X, Z \rangle = \langle dR_y(X), dR_y(Z) \rangle$

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Let $y_t(e)$ be the one-param. gp gen'd by y.

$\langle X, Z\rangle = \langle dR_{y_t(e)}(x), dR_{y_t(e)}(z)\rangle_e \quad \forall t \quad$ dev.n at $t=0$

product rule $\rightarrow$ $0 = \langle [X,Y], Z\rangle + \langle X, [Z,Y]\rangle \Rightarrow \langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle$ □

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No class next week.

Isometries

Let $F: M^n \to N^m$, M, N Riemannian mfds

If F is a diffeomorphism and $\langle u, v\rangle_p^M = \langle dF_p(u), dF_p(v)\rangle_{F(p)}^N$, $(*)$

then F is called an isometry.

If F is a local diffeo & $(*)$ then F is called a local isometry.

Recall: (X, d_X), (Y, d_Y) $\quad F: X \to Y$ s.t. $d_X(x_1, x_2) = d_Y(F(x_1), F(x_2))$

$\rightsquigarrow$ F isometry of metric spaces.

exercise: F is Riemannian isometry $\Rightarrow$ F isometry of metric spaces.

$d(x,y) = \inf\{\ell_a^b(\gamma)$ γ smth path $x \to y\}$

Thm: (Myers' -Steenrod -thm) "$\Leftarrow$"

Isometry is an equivalence relation on Riemannian mfds where

$M \underset{\text{ism}}{\approx} N$ if $\exists F: M \to N$ isometry.

Def: If $F: M^n \to N^{n+k}$ is an immersion and N is a Riemannian mfd, define a Riem. ~~metric~~ mfd/metric on M via

$$\langle u, v\rangle_p = \langle dF_p(u), dF_p(v)\rangle, \quad u, v \in T_pM.$$

$\uparrow$ metric on M induced by F.

$\rightarrow$ "Riemannian" Immersion [pullback metric]

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Submersion $F: M^{n+k} \to N^n$ M, N both Riem. mfds

$\rightsquigarrow$ F <u>Riemannian submersion</u> if $u, v \in \operatorname{Ker}(dF_p)^\perp$ then

$\langle u, v\rangle_p = \langle dF_p(u), dF_p(v)\rangle$.

<u>Examples</u>: ① $F: \mathbb{R}^n \to \mathbb{R}^n$ Euclidean metric.

F is an isometry $\Leftrightarrow$ $F(\vec{x}) = A\vec{x} + \vec{b}$, $\vec{b} \in \mathbb{R}^n$, $A \in O(n)$

② $F: S^n \to S^n$ standart metric.

F is an isometry $\Leftrightarrow$ $F(\vec{x}) = A\vec{x}$, $A \in O(n+1)$. $\rightsquigarrow$ origin fixed

③ $S^n \to \mathbb{R}P^n$ $\qquad$ $\mathbb{R}^n \to \mathbb{R}^n/_{\mathbb{Z}^n} \cong T^n$

$x \longmapsto [x]$ $\qquad$ $x \mapsto [x]$

$\rightsquigarrow$ local isometry

↳ is a local isometry

M is Riem. mfd $\rightsquigarrow$ isometry gp of M $\{F: M \to M,$ isometry$\}$

④ Rotationally symmetric metrics.

$M = I \times S^1$. $\qquad$ $\langle\,,\,\rangle$ metric on M, "rotationally symmetric"

$\Downarrow$

(t, θ) (coord's) $\rightsquigarrow \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}$

<u>Suppose</u>: $F_{\theta_0}: I \times S^1 \to I \times S^1$,

$(t, \theta) \mapsto (t, \theta + \theta_0)$

Assume F_{θ_0} are isometries $\forall \theta_0$.

$\langle\,,\,\rangle = a^2(t,\theta)\,dt^2 + b(t,\theta)(dt\,d\theta + d\theta\,dt) + c^2(t,\theta)\,d\theta^2$.

↳ $\Rightarrow a = a(t)$

$b = b(t)$

$c = c(t)$

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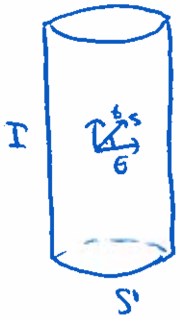


$$b(t) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right\rangle$$

$$(t, \theta) \mapsto (s, \theta).$$

$$s = A(t)\, t + B(t)\, \theta \qquad \theta = \theta.$$

$$ds = \frac{dA}{dt}\, dt + \frac{dB}{dt}\, d\theta \qquad d\theta = d\theta$$

- - -

To be continued.

Prop: Any diff. mfd has a Riem. metric on it.

Pf: Cover M by coordinate charts.

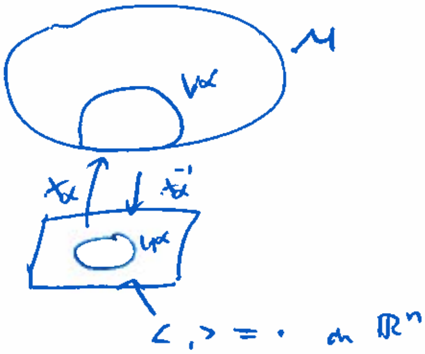
$$x_\alpha : U_\alpha \to V_\alpha \subseteq M \qquad \bigcup_\alpha V_\alpha = M.$$

Let φ_α be a part. of unity subordinate to $\{V_\alpha\}$.

i.e.
- φ_α are locally finite
- $\sum_\alpha \varphi_\alpha(p) = 1 \quad \forall p \in M$
- $\operatorname{supp}(\varphi_\alpha) \subseteq V_\alpha$

Define $$\langle u, v \rangle_p^\alpha = \langle (dx_\alpha)_p^{-1}(u), (dx_\alpha)_p^{-1}(v) \rangle_{x^{-1}(p)}$$

$\langle\, , \rangle^\alpha$ is a Riem. metric on V^α.



Define $$\langle u, v \rangle_p = \sum_\alpha \varphi_\alpha(p) \langle u, v \rangle_p^\alpha$$

↑ is a metric on M.

□

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Fundamental question of Riem geometry:

- Given a smooth mfd M, what is the "best" metric you can put on it?

- Given two Riem. metrics, when are they isometric?

Differentiating vector fields on Riem. mfds

M smooth mfd.

~~$\mathfrak{X}(M)$~~ $\mathfrak{X}(M)$ = set of all C^∞ vector fields on M.

Parallelism: In $\mathbb{R}^n$:

In M, we need a way to compare vectors in different tangent spaces.

Def: An affine connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$(X,Y) \mapsto \nabla_X Y$$

s.t.

(1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ $\quad f,g : M \to \mathbb{R}$ smth

(2) $\nabla_X(aY+bZ) = a\nabla_X Y + b\nabla_X Z$ $\quad a,b \in \mathbb{R}$

(3) $\nabla_X(fY) = f\nabla_X Y + X(f)\,Y.$

Note: $(X,Y) \mapsto [X,Y]$ is not an affine connection.

(doesn't satisfy (1)).

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Local coordinates:

$x: U \to M$, basis $\frac{\partial}{\partial x_i}$ on T_pM

$X = \sum_i a_i \frac{\partial}{\partial x_i}$, $Y = \sum_j b_j \frac{\partial}{\partial x_j}$ $a_i, b_j :$ ~~A~~ $x(U) \to \mathbb{R}$.

$$\nabla_X Y = \nabla_{(\sum_i a_i \frac{\partial}{\partial x_i})} \left(\sum_j b_j \frac{\partial}{\partial x_j} \right)$$

$$\overset{1,2}{=} \sum_i a_i \sum_j \nabla_{\frac{\partial}{\partial x_i}} \left(b_j \frac{\partial}{\partial x_j} \right) \overset{3)}{=} \sum_{i,j} a_i \left(\frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + b_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)$$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \in \mathfrak{X}(M)$$

write in basis $\left\{ \frac{\partial}{\partial x_k} \right\}$ $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \underline{\underline{\Gamma_{ij}^k}} \frac{\partial}{\partial x_k}$

↳ Christoffel symbols

$$\rightsquigarrow \nabla_X Y = \sum_k \left(\sum_{i,j} a_i b_j \Gamma_{ij}^k + \underbrace{X(b_k)}_{\circledast} \right) \frac{\partial}{\partial x_k} \qquad (*)$$

<u>Consequences:</u>

① If $X_1, X_2 \in \mathfrak{X}(M)$ and $X_1(p) = X_2(p) = v \in T_pM$, then

$$\nabla_{X_1(p)} Y = \nabla_{X_2(p)} Y =: \nabla_v Y$$

② If $c(t)$ is a curve in M

$\nabla_{\frac{dc}{dt}} Y$ = only depends on value of $\frac{dc}{dt}$ and the value of Y along c. $\circledast$

If $c(t)$ is a curve and Y is a vector field along c

(i.e. $Y: I \to TM$ s.t. $\overset{Y_{c(t)}}{Y(t)} \in T_{c(t)}M$),

then $(*)$ is well defined for $X = \frac{dc}{dt}$.

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Define $\frac{D}{dt}(Y) = (*)$,

is called covariant derivative of Y along c.

Y is parallel along c if $\frac{D}{dt}(Y) \equiv 0, \ \forall t$.

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<u>Last time:</u> Affine connections

M smooth mfd, ∇ affine connection

$X, Y \in \mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}(M)$ ← defines the derivative of Y along X

Y a vector field along $c(t)$ $Y: I \to TM$ $Y_{c(t)} \in T_{c(t)}M$

$$\frac{D}{dt}(Y) = \sum_k \left(\frac{db_k}{dt} + \sum_{i,j} \frac{dx_i}{dt} b_j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}$$ ← vector field along $c(t)$

$$\parallel$$

$$\nabla_{\frac{dc}{dt}} Y$$

↑ Covariant der. of Y along c

$X(c(t)) = (x_1(t), x_2(t), \ldots, x_n(t))$

$Y = \sum_{j=1}^{n} b_j(t) \frac{\partial}{\partial x_j}\Big|_{d(t)}$

<u>Prop:</u> $\frac{D}{dt}$ satisfies

(1) $\frac{D}{dt}(Y+Z) = \left(\frac{DY}{dt} + \frac{DZ}{dt}\right)$

(2) $\frac{D}{dt}(f(t)Y) = f(t)\frac{DY}{dt} + \frac{df}{dt} Y$

(3) If Y is defined on an open nbhd of C, then $\frac{D}{dt}(Y) = \nabla_{\frac{dc}{dt}} Y$.

and $\{ \frac{D}{dt}$ is unique "thing" (map) that satisfies (1)-(3). $\}$

<u>Example:</u> X, Y vector fields in $\mathbb{R}^n$, $E_1, \ldots, E_n$ standard basis of $\mathbb{R}^n$.

Define $\nabla_X^{\mathbb{R}^n} Y$ s.t. $\Gamma_{ij}^k = 0$ $\forall i, j, k$, i.e. $\nabla_{E_i} E_j = 0$.

<u>then</u> for $X = \sum a_i E_i$, $Y = \sum b_i E_j$, $\nabla_X Y = \sum_j X(b_j) E_j$.

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$D_X^{\mathbb{R}^n} Y$ $Y = (b_1, b_2, \dots, b_n)$

$D_X Y = (X(b_1), X(b_2), \dots X(b_n))$

Parallel $\nabla_{\frac{dc}{dt}} Y \equiv 0$ $\Rightarrow b_j(t) = \text{constant} \ \forall j$

<u>Prop:</u> Let $c(t)$ be a smooth curve, $v \in T_{c(0)} M$ then $\exists!$ vector field Y along $c(t)$ s.t. Y is parallel along $c(t)$ & $Y(0) = v$.

<u>Pf:</u> $\frac{D}{dt} Y \equiv 0 \Leftrightarrow \frac{db_k}{dt} + \sum_{i,j} b_j \frac{\partial x_i}{\partial t} \Gamma_{ij}^k = 0 \quad \forall k = 1, 2, \dots n.$

Linear system of n-~~dimensional system~~ differential equations in the functions b_k.

Solve this in any coordinate chart.

Uniqueness implies that we can solve along the whole curve. □

The parallel field $Y(t)$ along $c(t)$ is called the parallel translation of v along $c(t)$.

<u>Relationship between connections and metrics:</u>

M smooth mfd, ∇ affine connection and $\langle , \rangle$ Riemannian metric.

- ∇ is called compatible with $\langle \cdot , \cdot \rangle$ if $\forall X, Y, Z \in \mathfrak{X}(M)$ $X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$.
- ∇ is called torsion free if $\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX$.

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"Fundamental thm of Riemannian Geometry"

Let M is a Riem. mfd then $\exists!$ affine connection ∇ which is torsion free and compatible with $\langle \ , \ \rangle$.

∇ is called Levi-Civita connection or Riemannian connection

Pf: Supp. we have such a ∇.

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle$$

$$= \langle [Y,Z], X\rangle + \langle [X,Z], Y\rangle + \langle [X,Y], Z\rangle + 2\langle Z, \nabla_Y X\rangle$$

$$\Rightarrow 2\langle \nabla_Y X, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle - \langle [X,Y], Z\rangle.$$

Kozul's Formula

$\Rightarrow \nabla$ determined by $\langle \ , \ \rangle$, $[\cdot,\cdot]$.

$\Rightarrow$ Uniqueness.

Conversely, can define ∇ by formula. (exercise) ◾

(this determines a connection)

□

Examples: $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle \to \nabla_X^{\mathbb{R}^n} Y$ as defined above is the Levi-Civita connection

Let $M \subset \mathbb{R}^n$. M Riem. mfd induced by dot product.

X, Y vector fields tangent to M.

$$\nabla_X Y = (\nabla_X^{\mathbb{R}^n} Y)^T \leftarrow \begin{matrix} v \in T_p\mathbb{R}^n \\ v \mapsto v^T \end{matrix} \text{ projection onto } T_pM.$$

$\to \nabla$ is the Riemannian [illegible] on M

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Ex: $S^2 \subset \mathbb{R}^3$

$c(t) = (\cos t, \sin t, 0)$

$Y(t) = \frac{\partial}{\partial t}$ $\qquad \frac{\partial c}{\partial t} = (-\sin t, \cos t, 0)$

$$\nabla_{\frac{\partial c}{\partial t}} Y = \left(\nabla_{\frac{\partial c}{\partial t}} Y\right)^T = 0.$$

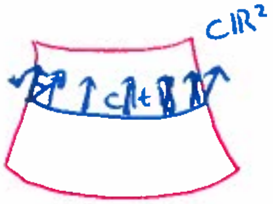
$$\nabla_{\frac{\partial c}{\partial t}} \frac{\partial c}{\partial t} = \left(\nabla_{\frac{\partial c}{\partial t}} \frac{\partial c}{\partial t}\right)^T = \left(\overset{-\cos}{\cancel{-\sin}}\, t, \overset{-\sin}{\cancel{\cos}}\, t, 0\right)^T = 0$$

← proj. onto $T_p S^2$

acceleration

since vector has no component in tangent space

$c(t) =$ lattitude Line



parallel fields rotate around

Next time: Geodesics

$c(t)$ s.t. $\nabla_{\frac{\partial c}{\partial t}} \frac{\partial c}{\partial t} \equiv 0$ ← "curve with no acceleration"

03/05/18

Last time: M Riem. mfd $\langle\, , \rangle$

$\exists!$ affine connection ∇ s.t.

(1) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

(2) $[X, Y] = \nabla_X Y - \nabla_Y X$.

We call this connection the Levi-Civita -connection.

or Riemannian connection.

03/05/18

Note that: $\frac{d}{dt} < \frac{dc}{dt}, \frac{dc}{dt} > = 2 < \underbrace{\nabla_{\frac{dc}{dt}} \frac{dc}{dt}}_{=0}, \frac{dc}{dt} > = 0$

So, $< \frac{dc}{dt}, \frac{dc}{dt} >$ is constant along a geodesic.

$\Rightarrow$ Geodesics have constant speed.

$\rightarrow$ Trivial geodesic, $c(t) =$ constant.

Note: Being a geodesic depends on parametrization

e.g.: In $\mathbb{R}^n$, geodesics are straight lines, parametrized to be constant speed.

Local Existence and Uniqueness of geodesics

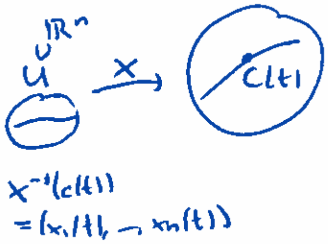
For any $p \in M$, $v \in T_pM$, there is a unique geodesic γ s.t.

$\gamma(0) = p$, $\gamma'(0) = \cancel{p}\, v$.

Why? geodesic $\Leftrightarrow$ solving a second order system of ODEs.

In local coordinates:

$X: U \to M$



$$v = \sum v_j \frac{\partial}{\partial x_j}$$

$$\frac{Dv}{dt} = \sum_k \left(\frac{dv_k}{dt} + \sum_{i,j} v_j \frac{dx_i}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}$$

Geodesics: $V = \frac{dc}{dt}$

$V_j = \frac{dx_j}{dt}$ $\rightsquigarrow$ $0 = \frac{d^2 x_k}{dt} + \sum_{i,j} \frac{dx_j}{dt} \frac{dx_i}{dt} \Gamma_{ij}^k$

for $k = 1, 2, \ldots, n$.

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Kozul Formula:

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}\Big[X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle$$
$$- \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \Big]$$

Levi-Civita in local coordinates:

$X: U \to M$ $\quad \frac{\partial}{\partial x_i}$ coord. vector field

$$g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle$$

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0 \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

$$\langle \frac{\partial}{\partial x_k}, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\rangle = \frac{1}{2}\left[\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\right]$$

[$\to$ in local coord's: bottom row in Koszul form vanishes]

[left inv. fields $\to$ in coord's: top row vanishes, bottom row stays]

$$\parallel$$

$$\langle \frac{\partial}{\partial x_k}, \sum_l \Gamma_{ij}^l \frac{\partial}{\partial x_l}\rangle$$

$$\parallel$$

$$\sum_l \Gamma_{ij}^l g_{lk}$$

$$(g_{lk}) \begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \\ \Gamma_{ij}^3 \\ \Gamma_{ij}^4 \end{pmatrix}$$

$(g_{ij}) \to$ pos. definit symm matrix

so it has an inverse matrix $\underline{g^{ij}}$

$$\sum_k g_{lk} g^{km} = \delta_{l,m}$$

$$\Rightarrow \sum_{lk} \Gamma_{ij}^l g_{lk} g^{km} = \frac{1}{2} \sum_k g^{km}\left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\right)$$

$$\parallel$$

$$\Gamma_{ij}^m$$

Geodesics

Def: A curve $c: I \to M$ is called a geodesic if

$$\nabla_{\frac{\partial c}{\partial t}} \frac{\partial c}{\partial t} = 0 \quad \leftarrow \text{"zero acceleration"}$$

$$\parallel$$

$$\frac{D}{dt}\left(\frac{\partial c}{\partial t}\right)$$

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Introduce new variables $(x_1, \to x_n, y_1, —, y_n)$ $\leftarrow 2n$ variables

A system of $2n$ - first order ODEs on $TM = \{(x,v) \mid x \in M, v \in T_x M\}$.

$$(x_1, \to x_n, y_1, —, y_n)$$
$$y_1 = \frac{\partial}{\partial x_1}, \quad y_n = \frac{\partial}{\partial x_n}$$

$$y_1 = \frac{dx_1}{dt}$$

$$y_2 = \frac{dx_2}{dt}$$

$$\vdots$$

$$y_n = \frac{dx_n}{dt}$$

$$0 = \frac{dy_1}{dt} + \sum_{i,j} y_i y_j \Gamma_{ij}^{\cancel{k}1}$$

$$\vdots$$

$$0 = \frac{dy_n}{dt} + \sum_{i,j} y_i y_j \Gamma_{ij}^{n}$$

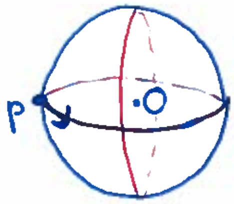
↗

Defines a vector field on TM, geodesics are the trajectories of the vectorfield.

Lemma: There is a unique vector field G on TM whose trajectories are ~~such that~~ of the form $t \mapsto (\gamma(t), \gamma'(t))$ where $\gamma(t)$ is a geodesic ~~on~~ in M.

$\Rightarrow$ At each pt $(p, v) \in TM^*$ $\exists$ unique geodesic with $\gamma(0) = p$, $\gamma'(0) = v$.

Example: S^2



$c(t) = (\cos(t), \sin(t), 0)$

is geodesic

$$(\nabla_{\frac{\partial c}{\partial t}} \frac{\partial c}{\partial t} = 0)$$

$\Rightarrow$ great circles are geodesics (when parametrized for constant speed)

↓

$C = \Pi \cap S^2$ where Π is a plane through the origin.

On the other hand, $\forall p \in S^2$ and $v \in T_p S^2$ $\exists$ a great circle through p ~~that~~ containing $\vec{v}$.

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(take Π = plane containing v_1 and $\vec{0}$)

$\therefore$ Great circles are the only geodesics of S^2.

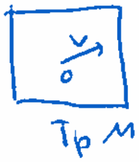
The exponential map

Let $p \in M$, $v \in T_pM$.

Def.: $\gamma(t, p, v) = \gamma(t)$ where $\gamma(t)$ is the geodesic with $\gamma(0) = p$, $\gamma'(0) = v$ exists for $|t| < \varepsilon$.

$\exp_p(v) = \gamma(1, p, v)$ (if it exists)

$\exp_p : D^{\subset T_pM} \to M$



$\boxed{\exp_p(\vec{0}) = p}$

Q: When is $\exp_p$ defined?

Lemma: (Homogeneity of geodesics)

$$\gamma(at, p, v) = \gamma(t, p, av) \quad \forall a > 0$$

Pf. Let $h(t) = \gamma(at, p, v)$. $\qquad h(0) = p$.

$h'(t) = a\gamma'(\overset{a}{}t, p, v) \qquad h'(0) = av$

$$\nabla_{h'(t)} h'(t) = \nabla_{a\gamma'(t, p, v)} a\gamma'(\overset{a}{}t, p, v) = a^2 \nabla_{\gamma'(\overset{a}{}t, p, v)} \gamma'(\overset{a}{}t, p, v) = 0$$

$\Rightarrow h(t)$ is a geodesic with $h(0) = p$, $h'(0) = av$ by uniqueness

$\Rightarrow h(t) = \gamma(t, p, av)$.

Lemma: $\exists \varepsilon > 0$ s.t. $\exp_p$ is defined on $B_\varepsilon(0) \subset T_pM$.

Pf: By local existence thm for ODE,

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Pf. $\gamma(t,p,v)$ is defined for $|t|<\delta_1$, $|v|<\delta_2$

$\gamma(t, p, \frac{\delta_1}{2} v) = \gamma(\frac{\delta_1}{2} t, p, v)$ is def'd for $|t|<2$, $|v|<\delta_2$.

take $\varepsilon < \frac{\delta_1 \delta_2}{2}$, $|w|<\varepsilon$

$\gamma(t,p,w) = \gamma(t, p, \frac{\delta_1}{2}(\frac{2}{\delta_1} w))$ def'd for $|t|<2$.

$|\frac{2}{\delta_1} w| < \delta_2$

$\therefore$ $\exp_p(w) = \gamma(1,p,w)$ exists. □

The exponential map:

ODE theory $\to$ solutions depend smoothly on initial conditions.

$\Rightarrow \exp_p : B_\varepsilon(0) \to M$ is differentiable.

What is the derivative at 0?

$$d(\exp_p)_0(v) = \frac{d}{dx}\Big|_{t=0} (\exp_p(tv))$$

$$= \frac{d}{dt}\Big|_{t=0} (\gamma(1, p, tv)) = \frac{d}{dt}\Big|_{t=0} (\gamma(\cancel{p} t, p, tv))$$

$$= v.$$

Inv. fct. thm $\Rightarrow$ $\exp_p$ is a local diffeo in a nbhd of p.

Last time: Geodesics 3/7/18

Given $p \in M$, $v \in T_pM$ $\exists!$ geodesic $\gamma(t,p,v)$ with

$\gamma(0,p,v)=p$ $\gamma'(0,p,v)=v$

Moreover, for each $p\in M$, $\exists \delta>0$ s.t. $\gamma(1,p,v)$ exists for all $|v|<\delta$.

$\exp_p : U \subset T_pM \to M$, $\exp_p(v) = \gamma(1,p,v)$

$B(0,\delta) \subset U$.

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Prop: $\exp_p$ is a local diffeo at 0.

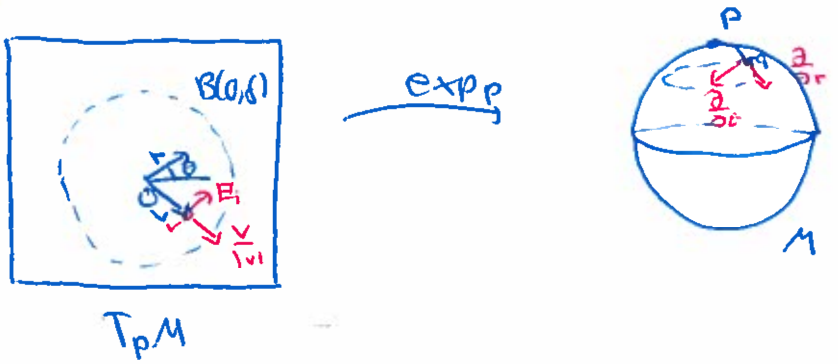
So we can use $\exp_p$ to define coordinates.

Def: Consider δ s.t. $\exp_p|_{B(0,\delta)}$ is a diffeo.

$\exp_p(B(0,\delta))$ is called a normal ball in M (normal nbhd)

Normal nbhds are "good" parametrizations.

Geodesic polar coord's:



Normal ball $\exp_p(B(0,\delta)) = \mathcal{B}$

$B(0,\delta)\setminus\vec{0} \approx (0,\delta)\times S^{n-1}$

$\in$

Fix (r_0,θ_0) (r,θ)

Let $E_1, E_2, \ldots, E_{n-1}$ be coord's around θ_0 on S^{n-1}.

Geodesic polar coord's on $\mathcal{B}\setminus\{p\}$

$q\in\mathcal{B}$ $q=\exp_p(v)$ $v\in B(0,\delta)$

$$\left.\begin{aligned}\frac{\partial}{\partial r}\Big|_q &= d(\exp_p)_v\left(\frac{v}{|v|}\right)\\ \frac{\partial}{\partial \theta}\Big|_q &\cancel{=} d(\exp_p)_v(E_i)\end{aligned}\right\}\text{ Define coord's on }\mathcal{B}$$

$\frac{\partial}{\partial \theta_i}\Big|_q = d(\exp_p)_{\vec{v}}(rE_i)$

Gauss Lemma: "The exp. map is a radial isometry"

$\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\rangle = 1,$ $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i}\rangle = 0.$

$g_{11}\equiv 1$ $g_{1i}=0$

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$\langle\ ,\ \rangle = dr^2 + g_r$

g_r is some metric on S^{n-1} that depends on r.

Pf: $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle_q = \langle d(\exp_p)_v(\frac{v}{|v|}), d(\exp_p)_v(\frac{v}{|v|}) \rangle$

$$= \frac{1}{|v|^2} \langle d(\exp_p)_v(v), d(\exp_p)_v(v) \rangle$$

$$d(\exp_p)_v(v) = \frac{d}{dt}\Big|_{t=1} (\exp_p)\underset{\substack{\uparrow \\ \text{straight line} \\ \text{with}}}{(tv)} = \frac{d}{dt}\Big|_{t=1} \gamma(1, p, tv) = \frac{d}{dt}\Big|_{t=1} \gamma(t, p, v)$$
$$= \gamma'(1, p, v)$$

since geodesics have constant speed,

$$|\gamma'(1, p, v)| = |\gamma'(0, p, v)| = |v|.$$

$$\Rightarrow \frac{1}{|v|^2} |d(\exp_p)_v(v)|^2 = \frac{1}{|v|^2} |v|^2 = 1 \qquad \therefore \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle_q = 1.$$

Lemma: $f: \underset{\text{open}}{U} \subset \mathbb{R}^3 \to M^n$ is embedding. $f(u)$ surface in M

$$\Rightarrow \frac{D}{dt}\left(\frac{\partial f}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial f}{\partial t}\right)$$

Pf: Exer. (just write down in local coord's, see text.)

$$T_pM \xrightarrow{\exp_p} M$$

$f(r, \theta_i)$ for $i = 1, \ldots, n-1$.

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}$$

$$\frac{\partial f}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}$$

Want $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i} \rangle = 0.$

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$$\lim_{r\to 0} \frac{\partial}{\partial \theta_i}(r,\theta) = \lim_{r\to 0} (d\exp_p)_{r\theta_0}(rE_i) = \lim_{r\to 0} r(d\exp_p)_{r\theta_0}(E_i) = 0$$

$$\frac{\partial}{\partial r}\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i}\right\rangle = \left\langle \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i}\right\rangle + \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta_i}\right\rangle$$

$$= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta_i}\right\rangle$$

$$= \underset{\overset{\uparrow}{\text{Lemma}}}{\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta_i}} \frac{\partial}{\partial r}\right\rangle} = \frac{1}{2} \frac{\partial}{\partial \theta_i} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right\rangle = 0$$

$$\left\langle \frac{\partial}{\partial r}, r\frac{\partial}{\partial \theta_i}\right\rangle = r\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i}\right\rangle$$

Minimizing properties of geodesics

Let $x,y \in M$. $d(x,y) = \inf\limits_{\substack{c(0)=x \\ c(1)=y}} \ell(c)$ where c is a piecewise smth path

and $\ell(c) = \int_0^1 \left|\frac{dc}{dt}\right| dt$

A curve is called minimizing on $c|_{[a,b]}$ if $\ell(c|_{[a,b]}) = d(c(a), c(b))$.

Prop: Let B be a normal ball around p, and let $q \in B$. Let γ be the unique geodesic with $\gamma(0)=p$ and $\gamma(1)=q$. Let c be any other piecewise diff'ble curve from p to q, then $\ell(c) \geq \ell(\gamma)$ and "=" if $\gamma([0,1]) = c([0,1])$.

Assume $c(t)$ stays inside B. $c(t) = (r(t), \overset{\theta_0}{\cancel{\theta(t)}})$ in geod. polar coord's.

$$\frac{dc}{dt} = r'(t) \frac{\partial}{\partial r} + \sum_{i=1}^{n-1} \frac{\partial \theta_i}{\partial t} \frac{\partial}{\partial \theta_i}$$

$$\left|\frac{dc}{dt}\right|^2 = |r'(t)|^2 + \dots \geq |r'(t)|^2$$

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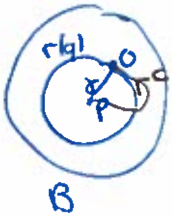
Let $\varepsilon>0$. $\int_\varepsilon^1 |c'(t)|\,dt \geq \int_\varepsilon^1 |r'(t)|dt \geq \int_\varepsilon^1 r'(t)dt = r(1) - r(\varepsilon)$

$\ell(c) = \int_0^1 |c'(t)|dt \geq r(1) = \ell(\gamma)$.
check

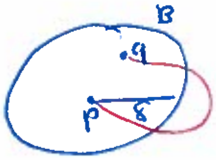
$\gamma(t) = (tr_0, \theta_0)$

[not $\int_0^1$ since polar coord's not def'd at 0]



If "=", $\frac{\partial \theta_i}{\partial t} = 0$ $\forall i$.

On the other hand if c leaves B, $B = \exp_p(B(0,\delta))$



$\ell(\gamma) < \delta \leq \ell(c)$

by comparing to geodesic for p to last pt where c leaves B.

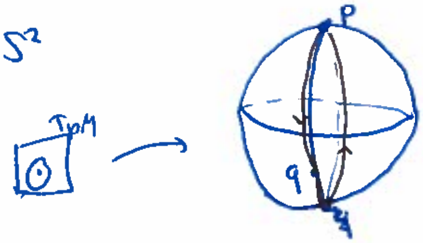
<u>Cor:</u> Let $B(p,\delta) \subset M$
$\qquad\quad \shortparallel$
$\quad \{q \mid d(p,q) < \delta\}$

Then if $\exp_p$ is a diffeo on $B(0,\delta) \subset T_pM$

then $\exp_p(B(0,\delta)) = B(p,\delta)$.

Geodesics locally minimize distance.

<u>Ex.</u> S^2



not globally!

not there, see Fabian's notes

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<u>Last time:</u> Curvature tensor, X, Y, Z v. fields

$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$.

<u>Remarks:</u> ① Some authors use (−) this definition.

3/26/18

② $M =$ Euclidean space $R(X,Y)Z \equiv 0 \quad \forall X, Y, Z$.

③ $\{x_i\}$ local coordinates

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] \equiv 0, \quad R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = D_{\frac{\partial}{\partial x_j}} D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} - D_{\frac{\partial}{\partial x_i}} D_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k}$$

④ If $\varphi: M \to N$ local isometry

$$R^n(X,Y)Z = R^n(d\varphi(X), d\varphi(Y))\, d\varphi(Z) \qquad \text{(exercise)}$$

⑤ Note the symmetry

$$R(X,Y)\{Z\} = -R(Y,X)Z$$

<u>Prop:</u> R is a tensor, i.e. $f, g: M \to \mathbb{R}$ $\quad X_i, Y_i, Z_i$ are v. fields.

1) $R(fX_1 + gX_2, Y)Z = f R(X_1,Y)Z + g R(X_2,Y)Z$

2) $R(X, fY_1 + gY_2)Z = f\,(R(X,Y_1)Z) + g(R(X,Y_2)Z)$

3) $R(X,Y)(fZ_1 + gZ_2) = f R(X,Y)Z_1 + g R(X,Y)Z_2$.

<u>Pf:</u> 1) exer. 2) follows from 1) by symmetry.

3) <u>Note:</u> $R(X,Y)(Z_1 + Z_2) = R(X,Y)Z_1 + R(X,Y)Z_2$ ✓

Show: $R(X,Y)(fZ) = f R(X,Y)Z$

$$D_Y D_X (fZ) = D_Y(X(f)Z + f D_X Z) = \underline{Y(X(f))}Z + \underline{X(f) D_Y Z}$$
$$+ \underline{Y(f) D_X Z} + f D_Y D_X Z$$

$$D_X D_Y(fZ) = -\Big(\underset{=[Y,X](f)}{\underline{X(Y(f))}}Z + \underline{Y(f) D_X Z} + \underline{X(f) D_Y Z} + f D_X D_Y Z\Big)$$

$$D_{[X,Y]}(fZ) = \underline{\underline{[X,Y](f)}}Z + f D_{[X,Y]} Z.$$

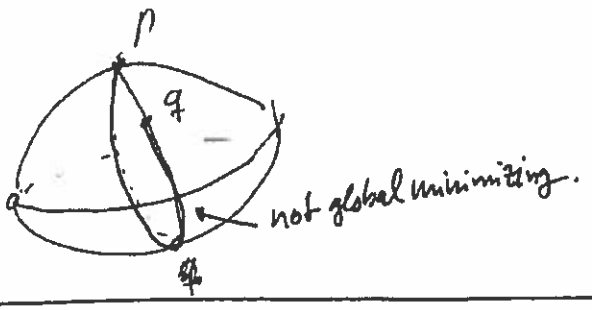
$f R(X,Y)Z$

∎

<u>Cor:</u> $R(X,Y)Z$ at p depends only on value of $X(p), Y(p), Z(p)$

(we can write $R(u,v)w$, $u,v,w \in T_pM$).

S^2 03/07



03/21

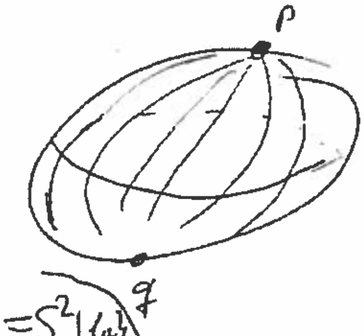
Last time: Geodesics $\nabla_{\frac{dc}{dt}} \frac{dc}{dt} = 0$

- $\gamma(t,p,v) = \gamma(t)$ the unique Geodesic with $\gamma(0)=p$, $\frac{d\gamma}{dt}(0)=v$
- $p \in M$: $\exp_p : \mathcal{U} \subseteq T_pM \to M$ $\exp_p(v) = \gamma(1,p,v)$
- $\exp_p$ smooth, local diffeom at $0 \in T_pM$
- A normal ball $B \subset M$ around p is a set of the form $B = \exp_p(B(0,\delta))$ where $\exp_p$ diffeom. on $B(0,\delta)$

Last time
- Let $q \in$ normal ball around p, then the geodesic from p to q in B is the shortest path from p to q
- Geodesics locally minimize arc length.

Remember: only true locally. (Sphere)

$S^2 \setminus \{q\}$ normal ball around p ($q = -p$)



i.e. $\exp_p$ diffeom. on $B(0,\pi)$, $\exp_p(B(0,\pi)) = S^2 \setminus \{q\}$

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On the other hand, if a curve minimizes arc length, then it is a geodesic.

Lemma For any $p \in M$ [$\exists W$ open nbd. of p] $\exists \delta > 0$ s.t. $\forall q \in W$ $\exp_q$ diffeom. on $B(0,\delta)$ $\subseteq T_qM$ and $W \subseteq \exp_q(B(0,\delta))$ i.e.

W normal nbd around every $q \in W$. W is called a totally normal neighborhood

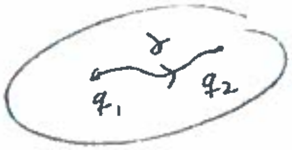
Proof p. 72 $\square$

Exs: S^2

- $S^2 \setminus \{q\}$ normal nbd.
- $W =$ Hemisphere

Note: W totally normal, $q_1, q_2 \in W$

$\exists!$ geod. γ. contained in W



Changing q_1, q_2 geodesic γ changes smoothly.

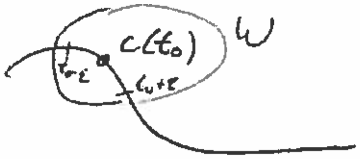
$\int_J (\langle \frac{dc}{dt}, \frac{dc}{dt} \rangle_M)^2 dt$

Prop If c is a piecewise diff'ble curve $c: [a,b] \to M$ parametrized proportional to arc length and minimizing then c is a geodesic

03/21

Proof: Suppose c not a geodesic at $c(t_0)$

W totally normal nbd. of $c(t_0)$



$\exists\, \varepsilon > 0$ s.t. $c([t_0-\varepsilon, t_0+\varepsilon]) \subseteq W$.

then let. $q_1 = c(t_0+\varepsilon), q_2 = c(t_0-\varepsilon) \Rightarrow q_1, q_2 \in W$. W normal

nbd. around q_1 $\exists!$ geod. γ from q_1 to q_2 in W.

γ is minimizing (from c_1 to c_2), c not geod $\Rightarrow \gamma \neq c$.

$\Rightarrow \ell(\gamma) < \ell(c|_{[t_0-\varepsilon, t_0+\varepsilon]})$ this contradicts that c minimizes

$\Rightarrow c$ geodesic.

Corollary Minimizing curves are smooth.

Summary Any (locally) minimizing curve is a geodesic.

Any geodesic is locally minimizing.

On the other hand, there is ~~the geod~~ not always a minimizing geodesic between x and y

Ex: $M = \mathbb{R}^2 \setminus \{0,0\}$

No geodesic from x to y

that

03/21

Curvature:

Q:1 Are all Riemannian manifolds $\underbrace{\text{locally}}_{}$ isometric? (of same dimension)

Recall ~~M loc isom to N if~~ ~~$\varphi: M \to N$ local isometry if φ local diffeom.~~

~~Cont~~ M loc. isom. to N if ~~$\forall$~~ $\forall p \in M$ $\exists$ $\mathcal{U} \ni p$ open

s.t. $\exists$ $\varphi: \mathcal{U} \to V \subseteq_{open} N$ diffeo. s.t. $\langle u, v \rangle_q = \langle d\varphi_q(u), d\varphi_q(v) \rangle_{\varphi(q)}$

$\forall q \in \mathcal{U}$.

A] No! Yes, if we ask for locally diffeomorphic.

Example S^2 and $\mathbb{R}^2$ are not locally isometric $\leftarrow$ Mapmaking

$\hookrightarrow$ Curvature differ $S^2 > 0$ & curv. loc isometry invariant
$\mathbb{R}^2 \equiv 0$

Also $S^2, \mathbb{R}^2$ not loc. isometric because

in $\mathbb{R}^2$ parallel translation path independent

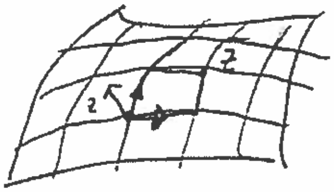
S^2

on S^2 not path. indep.

& under local isom. parallel fields go to parallel fields.

03/21

If a surface is loc. isometric to $\mathbb{R}^2$, then locally parallel translation is path independent. Take coords (x_1, x_2) on a small open set where parallel transl. is path independent.



Define Z parallel vector field by taking Z_p and first parallel transl. along x_1 then along x_2

Independence of path $\Rightarrow$ Z parallel.

$$\nabla_{\frac{\partial}{\partial x_1}} Z = \nabla_{\frac{\partial}{\partial x_2}} Z = 0. \Rightarrow \nabla_{\frac{\partial}{\partial x_2}}\left(\nabla_{\frac{\partial}{\partial x_1}} Z\right) = \nabla_{\frac{\partial}{\partial x_1}}\left(\nabla_{\frac{\partial}{\partial x_2}} Z\right) = 0$$

Curvature measures how far this is from being true

<u>Curv. tensor</u> $X, Y, Z \in \mathfrak{X}(M)$

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

03/26

<u>Remarks</u> ① Some authors use (−) this definition.

② $M = \mathbb{R}^n$ euclidean space $\Rightarrow R(X,Y)Z = 0 \quad \forall X, Y, Z.$

③ Local coordinates $\{x_i\}$ $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(\nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} - \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}}\right)\left(\frac{\partial}{\partial x_k}\right)$$

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④ If $\varphi: M \to N$ local isometry

$$R^M(X,Y)Z\cancel{{}_{Z}} = \cancel{R^N(X,Y)Z}\ R^N(d\varphi(X), d\varphi(Y))\, d\varphi(Z)$$

Exercise

⑤ Note the symmetry $R(X,Y)Z = -R(Y,X)Z$.

Prop: R is a tensor, i.e. $\forall f,g: M \to \mathbb{R}$

1) $R(fX_1 + gX_2, Y)Z = fR(X_1,Y)Z + gR(X_2,Y)Z$

2) $R(X_1, fY_1 + gY_2)Z = \cancel{R(}\ fR(X,Y_1)Z + gR(X,Y_2)Z$

3) $R(X,Y)(fZ_1 + gZ_2) = fR(X,Y)Z_1 + gR(X,Y)Z_2$

1), 2) Exercise

Pf of 3) Note: $R(X,Y)(Z_1+Z_2) = R(X,Y)(Z_1) + R(X,Y)Z_2$ ✓

ETS: $R(X,Y)(fZ) = f R(X,Y)Z$

$$D_Y D_X(fZ) = D_Y(X(f)Z + fD_XZ)$$

$$= \underline{Y(X(f))Z} + \underline{X(f)D_YZ} + \underline{Y(f)D_XZ} + fD_YD_XZ$$

$$D_XD_Y(fZ) = \underline{X(Y(f))Z} + \underline{Y(f)D_XZ} + \underline{X(f)D_YZ} + fD_XD_YZ.$$

$$D_{[X,Y]}(fZ) = \underline{[X,Y](f)Z} + fD_{[X,Y]}Z.$$ So: $D_YD_X(fZ) - D_XD_Y(fZ)$

$\cancel{+} + D_{[X,Y]}(fZ)$

$= fR(X,Y)Z$

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Pf: $E_1, \dots, E_n$ be a frame at p.

$X = \sum a_i E_i$

$Y = \sum b_j E_j$

$Z = \sum c_k E_k$

$$R(X,Y)Z = \sum_{i,j,k} a_i b_j c_k \; R(E_i, E_j) E_k$$

$\square$

Before going into general theory of curvature tensor, look at an (historically sign) special case.

Let $M^n \subsetneq \mathbb{R}^{n+1}$ with metric induced from dot product on $\mathbb{R}^n$.

Def: Notation: $\underset{\substack{\uparrow \\ \text{4-curvature tensor}}}{R}(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$

M X, Y vector fields on M.

$\langle X, Y\rangle_M = X \cdot Y$

→ N be unit normal vector on M
$N \perp v \;\; \forall v \in T_pM$
$|N| = 1$

$D^M_X Y = (\overset{\text{Euclidean}}{D_X} Y)^T$ — proj. onto T_pM

$D_X Y = (D_X Y)^T + \langle D_X Y, N\rangle N$

$D^M_X Y = (D_X Y)^T = D_X Y - \langle D_X Y, N\rangle N$

Def: The second fundamental form of M (depends on N)

$\mathrm{II}_N(X,Y) = -\langle D_X Y, N\rangle$ X, Y vector fields on M.

Prop: $\mathrm{II}_N(X,Y) = -\langle D_X Y, N\rangle = \langle Y, D_X N\rangle = -\langle D_X Y, N\rangle = \langle X, D_Y N\rangle$.

Pf: X, Y tangent to M, $N \perp M$.

$$0 = \langle [X,Y], N\rangle \overset{\text{torsion free}}{=} \langle D_X Y, N\rangle - \langle D_Y X, N\rangle$$

$$\Leftrightarrow \quad \langle D_X Y, N\rangle = \langle D_Y X, N\rangle$$

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$$\langle D_X Y, N\rangle \overset{\text{compatible}}{=\!=} X\underset{0}{\cancel{\langle Y,N\rangle}} - \langle Y, D_X N\rangle$$

$$\langle D_X Y, N\rangle = -\langle Y, D_X N\rangle$$

Cor: $\mathrm{II}_N(X,Y)$ is symmetric in X,Y, and tensor in both entries.

$\uparrow$ only depends on value of X,Y at pt.

$\Rightarrow \mathrm{II}_p(X,Y) \approx$ derivative of N change in normal vector.

Gauss map: $G: M^n \to S^n$
$x \mapsto N_x$ $\qquad \mathrm{II}_N \approx$ derivative of G.

Gauss Equation: $M^n \subseteq \mathbb{R}^{n+1}$ $\quad X,Y,Z,W \in T_pM$.

Then $R(X,Y,Z,W) = \mathrm{II}_N(X,Z)\,\mathrm{II}_N(Y,W) - \mathrm{II}_N(Y,Z)\,\mathrm{II}_N(X,W)$

Examples $n=2$, $M^2 \subseteq \mathbb{R}^3$

$X,Y \in T_pM$, $X \perp Y$

$R(X,Y,X,Y) = \mathrm{II}_N(X,X)\,\mathrm{II}_N(Y,Y) - \mathrm{II}_N(X,Y)^2$ $\leftarrow \det(\mathrm{II}_N) = \lambda_1\lambda_2 \leftarrow$ eigenvalues

X,Y orth. basis for T_pM

$$\begin{pmatrix} \mathrm{II}_N(X,X) & \mathrm{II}_N(X,Y) \\ \mathrm{II}_N(X,Y) & \mathrm{II}_N(Y,Y) \end{pmatrix} = A$$

[symm.]

v_1,v_2 o.n. basis eigenvectors, λ_1, λ_2 eigenval

$\lambda_1 = \mathrm{II}_N(v_1,v_1) = \langle v_1, D_{v_1} N\rangle$ $\qquad$ [change of basis to v_1,v_2 s.t. $\begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix} = A$]

$0 = \mathrm{II}_N(v_1,v_2) = \langle v_2, D_{v_1} N\rangle = \langle v_1, D_{v_2} N\rangle$

$\lambda_2 = \mathrm{II}_N(v_2,v_2) = \langle v_2, D_{v_2} N\rangle$

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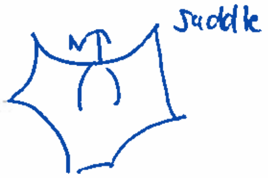
$\det(\mathrm{II}_N) > 0$ λ_1, λ_2 same sign

$\det(\mathrm{II}_N) = 0$ $\lambda_i = 0$

$\det(\mathrm{II}_N) < 0$ λ_1, λ_2 opposite sign



Pf: $D_X^M Y = (D_X Y)^T = D_X Y + \mathrm{II}_N(X,Y) N$

$$D_Y^M(D_X^M Z) = D_Y^M(D_X Z + \mathrm{II}_N(X,Z)N) = D_Y^M(D_X Z - \langle D_X Z, N\rangle N)$$

$$= D_Y(D_X Z - \langle D_X Z, N\rangle N) - \underbrace{\langle D_Y(D_X Z - \langle D_X Z, N\rangle N), N\rangle}_{*} N$$

$$= D_Y D_X Z - Y(\langle D_X Z, N\rangle) N - \langle D_X Z, N\rangle D_Y N - * N$$

$$D_X^M(D_Y^M Z) = D_X D_Y Z - \langle D_Y Z, N\rangle D_X N + (--) N$$

$$D_{[X,Y]}^M Z = D_{[X,Y]} Z - \langle D_{[X,Y]} Z, N\rangle N$$

$R(X,Y)Z = 0$ in $\mathbb{R}^n$

$$R(X,Y,Z,W) = \overset{0}{\overset{\|}{(D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z, W)}} - \overset{-\mathrm{II}_N(\cancel{X},Z)}{\langle D_X Z, N\rangle} \overset{-\mathrm{II}_N(Y,W)}{\langle D_Y N, W\rangle}$$

$$+ \underset{-\mathrm{II}_N(Y,Z)}{\langle D_Y Z, N\rangle} \underset{\mathrm{II}_N(X,W)}{\langle D_X N, W\rangle} + 0$$

□

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Last time:

Curvature tensor: $R(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z$

$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$

Gauss equation: If $M^n \subseteq \mathbb{R}^{n+1}$ then

$$R(X,Y,Z,W) = \mathrm{II}_N(X,Z)\,\mathrm{II}_N(Y,W) - \mathrm{II}_N(X,W)\,\mathrm{II}_N(Y,Z)$$

where $N \perp M$, $|N| = 1$ $\mathrm{II}_N(X,Y) = -\langle D_X Y, N\rangle = \langle Y, D_X N\rangle$

(second fund. form)

(See Thm 2.5, p. 130

or Prop. 3.1, p. 135 in text for more general version)

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$M^2 \subseteq \mathbb{R}^3 \quad X \perp Y, \ |X| = |Y| = 1$

$$\langle R(X,Y)X,Y\rangle = \mathrm{II}_N(X,X)\,\mathrm{II}_N(Y,Y) - \mathrm{II}_N(X,Y)^2 = \text{Gauss curvature of } M.$$

Examples:

① ~~$S^n(r)\,\mathbb{R}^{n+1}$~~

$\{(x_1,\dots,x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = r^2\} = S^n(r) \subseteq \mathbb{R}^{n+1}$

$N(\vec{x}) = \frac{1}{r}\vec{x}$

$$\mathrm{II}_N(X,Y) = \langle \nabla_X N, Y\rangle = \langle \nabla_X (\tfrac{\vec{x}}{r}), Y\rangle$$
$$= \frac{1}{r}\langle \nabla_X \vec{x}, Y\rangle$$

$X = \frac{\partial}{\partial x_i} \qquad \vec{x} = (x_1,\dots,x_m)$

$$\nabla_{\frac{\partial}{\partial x_i}}(\vec{x}) = \frac{\partial}{\partial x_i}, \quad \nabla_X \vec{x} = X$$

$$\mathrm{II}_N(X,Y) = \frac{1}{r}\langle X,Y\rangle$$

Normalization $\rightarrow X,Y, \ |X|,|Y| = 1 \ X \perp Y$

$$R(X,Y,X,Y) = \mathrm{II}_N(X,X)\,\mathrm{II}_N(Y,Y) - \mathrm{II}_N(X,Y)^2$$
$$= \frac{1}{r}\cdot\frac{1}{r} - 0 = \frac{1}{r^2}.$$

② More generally, Let $f: \mathbb{R}^{n+1} \to \mathbb{R}$, ~~be~~ let a be a regular value of f,

$M = f^{-1}(a)$. $\quad \nabla f = \sum \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}$ is perpend. to M.

$N = \frac{\nabla f}{|\nabla f|} \qquad X,Y \perp \nabla f, \qquad \langle \nabla f, X\rangle = df(X)$

$$\mathrm{II}_N(X,Y) = \langle \nabla_X N, Y\rangle = \langle \nabla_X \frac{\nabla f}{|\nabla f|}, Y\rangle = \langle \underbrace{X\left(\frac{1}{|\nabla f|}\right)\nabla f}_{=0} + \frac{1}{|\nabla f|}\nabla_X \nabla f, Y$$
$$= \frac{1}{|\nabla f|}\langle \nabla_X \nabla f, Y\rangle$$

Saddle: $\quad z = x^2 - y^2 \subset \mathbb{R}^3$

$f(x,y,z) = x^2 - y^2 - z$

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$$\nabla f = 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - \frac{\partial}{\partial z}$$

$$N = \frac{1}{\sqrt{1+4x^2+4y^2}} \left(2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right)$$

Let $X = \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial z}$, $\quad Y = \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}$

$$\nabla_X \nabla f = \nabla_{\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}} \; 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - \frac{\partial}{\partial z} = 2 \frac{\partial}{\partial x}$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} \equiv 0$$

$$\nabla_Y \nabla f = \nabla_{\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}} \; 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - \frac{\partial}{\partial z} = -2 \frac{\partial}{\partial y}$$

$$\mathrm{II}_N(X,X) = \frac{1}{|\nabla f|} \langle \nabla_X \nabla f, X \rangle = \frac{1}{\sqrt{1+4x^2+4y^2}} 2$$

$$\mathrm{II}_N(Y,Y) = \frac{-2}{\sqrt{1+4x^2+4y^2}}$$

$$R(X,Y,X,Y) = \frac{-4}{\sqrt{1+4x^2+4y^2}}$$

<u>check:</u> $\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2 = 0$

<u>Q:</u> Why is this enough to know the whole $R(X,Y,Z,W)$?

In particular, Gauss curvature < 0.

<u>Q:</u> Can every n-dim. Riem. mfd Cpct, orientable be isometrically embedded into $\mathbb{R}^{n+1}$?

$$(M, \langle , \rangle) \xrightarrow{\varphi} (\mathbb{R}^{n+1}, \cdot)$$

φ is a isometric immersion

<u>A:</u> No.

Gauss Eqn. ⇒ obstruction to having such an embedding

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Example: Suppose $n \geq 3$ and M^n is a Riem. mfd with property that $\langle R(X,Y)X, Y\rangle < 0 \quad \forall X, Y$ s.t. $X \perp Y$

("negative sectional curvature")

Then M^n can not be isometrically embedded in $\mathbb{R}^{n+1}$!

Pf: By GE $\mathrm{II}_N(X,Y)$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3, -$

$v_1, v_2, v_3, -$ eigenvectors

$$R(v_1, v_2, v_1, v_2) = \mathrm{II}_N(v_1,v_1)\,\mathrm{II}_N(v_2,v_2) - \mathrm{II}_N(v_1,v_2)^2 = \lambda_1\lambda_2 - 0 < 0$$

$$R(v_1, v_3, v_1, v_3) = \lambda_1 \lambda_3 < 0$$

$$R(v_2, v_3, v_2, v_3) = \lambda_2 \lambda_3 < 0$$

Contradiction! □

Back to Curvature tensor of general $(M, \langle\, ,\, \rangle)$

Symmetries of R $\quad R(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z.$

① $R(X,Y,Z,W) = -R(Y,X,Z,W) \Leftarrow R(X,X,Z,W) = 0,$

② $R(X,Y,Z,Z) = 0$

③ $R(X,Y,Z,W) = -R(X,Y,W,Z)$

Pf: ② $\Rightarrow$ ③ $0 = R(X,Y,Z+W,Z+W) = R(X,\overset{0}{\cancel{Y,Z}},Z) + R(X,Y,Z,W)$

$+ R(X,Y,W,Z) + R(X,\overset{0}{\cancel{Y,W}},W)$

Pf of ②: $\langle D_Y D_X Z, Z\rangle = Y(\langle D_X Z, Z\rangle) - \langle D_X Z, D_Y Z\rangle$

$= Y(X(|Z|^2)) - \langle D_X Z \cancel{,} D_Y Z\rangle$

$\langle D_X D_Y Z, Z\rangle = X(Y(|Z|^2)) - \langle D_Y \cancel{Z,} D_X Z\rangle$

$\langle D_{[X,Y]} Z, Z\rangle = \underset{XY-YX}{[X,Y]}(\tfrac{1}{2}|Z|^2)$

$= 0$

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④ Bianchi Identity.

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Pf: $D_Y \underline{D_X Z} - \underline{D_X D_Y Z} + D_{[X,Y]} Z + \underline{\underline{D_Z D_Y X}} - D_Y \underline{D_Z X} + D_{[Y,Z]} X$

$+ \underline{D_X D_Z Y} - \underline{\underline{D_Z D_X Y}} + D_{[Z,X]} Y$

$= \underline{D_Y([X,Z])} + \underline{D_X[Z,Y]} + \underline{\underline{D_Z([Y,X])}} + D_{[X,Y]}Z + D_{[Y,Z]}X + D_{[Z,X]}Y$

$= [Y,[X,Z]] + [X,[Z,Y]] + [Z,[Y,X]] = 0.$

Jacobi identity

⑤ $R(X,Y,Z,T) = R(Z,T,X,Y)$

Pf: By Bianchi:

$\underline{\underline{R(X,Y,Z,T)}} + \underline{R(Y,Z,X,T)} + R(Z,X,Y,T) = 0$

$*\ \underline{R(Y,Z,T,X)} + \underline{R(Z,T,Y,X)} + R(T,Y,Z,X) = 0$

$\underline{R(Z,T,X,Y)} + \underline{\underline{R(T,X,Z,Y)}} + R(X,Z,T,Y) = 0$

$\underline{\underline{R(T,X,Y,Z)}} + \underline{\underline{R(X,Y,T,Z)}} + R(Y,T,X,Z) = 0$

$+$ ______

$2R(Z,X,Y,T) - 2R(Y,T,Z,X) = 0 \Rightarrow R(Z,X,Y,T) = R(Y,T,Z,X)$

Curvature operator

$\Lambda^2(\overset{V}{\cancel{T_pM}}) = \{v \wedge w : v,w \in V\}$

$v \wedge w = -w \wedge v$

$v \wedge v = 0$

define $|v \wedge w| = |v|^2 |w|^2 - \langle v,w \rangle^2$

$\uparrow$

defines an inner product on $\Lambda^2(V)$.

Def: $\mathcal{R}\ R : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ (curvature operator)

$R(X,Y,Z,W) = \langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle$

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antisymmetric in X, Y and Z, W $\Rightarrow$ well defined on $\Lambda^2 T_pM$

$R(X, Y, Z, W) = R(Z, W, X, Y)$

$\Rightarrow \langle R(X\wedge Y), Z\wedge W\rangle = \langle X\wedge Y, R(Z\wedge W)\rangle \leftarrow$ self adjoint

If $\dim(V) = 2 \Rightarrow \dim(\Lambda^2(V)) = 1$

R 1-dim. v. space $\to$ 1-dim. v-space

denoted by $\langle R(X\wedge Y), X\wedge Y\rangle = R(X, Y, X, Y)$.

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Make up lectures:

① April 6 3:30 - 4:50

② April 13th 11:30-12:30

③ April 27th?

Last time: Symmetries of Curv. tensor

Curvature operator: $\mathcal{R}: \Lambda^2 T_pM \to \Lambda^2 T_pM$ * $\{\sum v\wedge w : v\wedge w = -w\wedge v\}$

defined by $\langle \mathcal{R}(X\wedge Y), Z\wedge W\rangle = \mathcal{R}(X, Y, Z, W)$

self-adjoint $\langle \mathcal{R}(X\wedge Y), Z\wedge W\rangle = \langle \mathcal{R}(Z\wedge W), X\wedge Y\rangle$

where, $\langle X\wedge Y, Z\wedge W\rangle = \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle$

$$|X\wedge Y|^2 = |X|^2|Y|^2 - \langle X, Y\rangle^2$$

Sectional Curvature:

Let σ be a plane in T_pM

Let X, Y be linearly independent vectors in σ.

$$\text{Define } K(\sigma) = \frac{R(X, Y, X, Y)}{|X|^2|Y|^2 - \langle X, Y\rangle} = \frac{R(X, Y, X, Y)}{|X\wedge Y|^2}$$

Note: $K(\sigma)$ does not depend on choice of basis X, Y.

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Change basis:

ⓐ $\{X,Y\} \to \{Y,X\}$ ⓑ $\{X,Y\} \to \{\lambda X, Y\}$ ⓒ $\{X,Y\} \to \{X+\lambda Y, Y$

$$R(\lambda X, Y, \lambda X, Y) = \lambda^2 R(X,Y,X,Y)$$

$$|\lambda X|^2 - \langle \lambda X, Y\rangle^2 = \lambda^2(|X|^2|Y|^2 - \langle X,Y\rangle^2)$$

~~$R(X,Y,X,Y)$~~ $R(X+\lambda Y, Y, X+\lambda Y, Y) = R(X,Y,X,Y) + \lambda R(Y,Y,X,Y)$

$+ \lambda R(X,Y,Y,Y) + \lambda^2 R(Y,Y,Y,Y) = 0$

Check: above works for $|X|^2|Y|^2 - \langle X,Y\rangle^2$.

If $M^2 \subseteq \mathbb{R}^3$ Gauss curv. $= K(\sigma)$

ex: Saddle $z = x^2 - y^2$

$$E_1 = \frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}$$

$$E_2 = \frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}$$

$$R(E_1, E_2, E_1, E_2) = \frac{-4}{\sqrt{1+4x^2+4y^2}}$$

$$|E_1|^2|E_2|^2 - \langle E_1, E_2\rangle = (1+4x^2)(1+4y^2) - (-4xy)^2 = 1+4x^2+4y^2$$

$$K(\sigma) = \frac{-4}{(1+4x^2+4y^2)^{3/2}}$$

Lemma: If have two Riem. metrics $\langle,\rangle, \langle,\rangle'$ on same M and if $R(X,Y,X,Y) = R'(X,Y,X,Y)$, then $R(X,Y,Z,W) = R'(X,Y,Z,W)$ $\forall X,Y,Z,W \in T_pM$.

Pf: Hand wavy: $R(X,Y,X,Y) = \langle \mathcal{R}(X\wedge Y), X\wedge Y\rangle$

see p. 95 of text.

Def: M Riem. mfd has constant sectional curvature at p if $K(\sigma) = \mathcal{K}$ $\forall$ 2-planes $\sigma \subset T_pM$.

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Cor: If M has a constant sect. curvature at p then

$$R(X,Y,Z,W) = \mathcal{K}(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle)$$

$\forall X,Y,Z,W \in T_pM.$

$\Updownarrow$

$R(X\wedge Y) = \mathcal{K}(X\wedge Y)$
$\forall X,Y$

$\left[\frac{R'(X,Y,X,Y)}{|X\wedge Y|^2} = \mathcal{K} \right.$ ✓

⑥ $K(\sigma) = 0$, $R(X,Y,Z,W) = 0$ ~~$\forall \sigma \subset T_pM$~~
$\forall \sigma \subset T_pM$ $\forall X,Y,Z,W \in T_pM$

Tensors in linear algebra:

V vector space, (r,s) two nonneg. integers.

An (r,s)-tensor is a map

$$T: \underbrace{V^* \times V^* \times \dots \times V^*}_{r\text{-times}} \times \underbrace{V \times V \times \dots \times V}_{s\text{-times}} \to \mathbb{R}$$

which is linear in each entry, where $V^* =$ dual space of V
$= \{ V \to \mathbb{R} \}$

$(0,0)$-tensor, is a number

$(0,1)$-tensor $T: V \to \mathbb{R}$ linear functional on V.

$(0,2)$-tensor: $T: V\times V \to \mathbb{R}$ bilinear form (ex. inner product)

If V has an inner product, $\langle\,,\rangle$

isomorphism $V \to V^*$
$v \longmapsto \langle v, \cdot \rangle$

ex: T $(1,1)$-tensor. $T: V^* \times V \to \mathbb{R}$

$v \in V$ $T(v): V^* \to \mathbb{R}$, $T(v)(a) = T(a,v)$

$T(v) \in (V^*)^* = V$

$(1,1)$-tensor ~~$T: V \to V$~~ $\to T: V \to V$

$T: V \to V$

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Type change:

$T: V \times V \to \mathbb{R}$

$$\underset{\substack{\uparrow \\ (0,2)\text{ tensor}}}{T(v,w)} = \langle T(v), w \rangle$$

In general, type change any (r,s)-tensor to be $(0,r+s)$.

ex: $\underset{\substack{\uparrow \\ (1,3)\text{-tensor}}}{R(X,Y)Z} \quad \xrightarrow[\text{change}]{\text{type}} \quad \underset{\substack{\uparrow \\ (0,4)\text{-tensor}}}{R(X,Y,Z,W)} = \langle R(X,Y)Z, W \rangle$

Tensors in diff. geometry:

$\mathcal{D}$ = real valued smooth fcts on M into $\mathbb{R}$

$\mathfrak{X}(M)$ = smth vector fields on M

$(0,s)$ tensor field $\quad T: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{s} \to \mathcal{D}$

$(1,s)$ tensor field $\quad T: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{s} \to \mathfrak{X}(M)$

linear in each entry

examples: ① $f: M \to \mathbb{R}$

$df: \mathfrak{X}(M) \to \mathcal{D}$, $\quad df(X) = X(f) \quad$ is a $(0,1)$-tensor (1-form)

② ∇ affine connection, X vector field.

$\nabla X: \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad (\nabla X)(Y) = \nabla_Y X \quad$ is a $(1,1)$-tensor.

③ $\nabla_Y: \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\quad \nabla_Y(X) = \nabla_Y X$.

is not a tensor because not linear w.r.t. fcts in X (product rule)

④ If $M, \langle , \rangle$ is a Riem. mfd $\quad g(X,Y) = \langle X, Y \rangle \quad$ is a $(0,2)$-tensor field

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~~Given vector field X, and~~

Covariant Derivative of Tensors

M, ∇ affine connection

Given vector field X, and tensor T, define $\nabla_X T$ so that

$(\nabla_X T)$ is a tensor of the same type as T.

e.g.: $\underset{\text{fct}}{\underset{\uparrow}{f}}, X, \quad \nabla_X f = \underset{\text{fct}}{X(f)}$

$\underset{\text{v.field}}{Y, X} \qquad \underset{\text{v.field}}{\nabla_X Y}$

Let T be a $(1,1)$-tensor.

Cl.: $\nabla_X T \to (1,1)$-tensor

$$(\nabla_X T Y) = \nabla_X(T(Y)) - T(\nabla_X Y)$$ is a $(1,1)$-tensor.

exercise. $(\nabla_X T)(fY) = f(\nabla_X T)(Y)$.

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Last time: Tensors, M Riem. mfd

Make up class
Fr Apr 6 3:30-4:50 Cam 100
Fr Apr. 13 11:40-12:30 Cam 119

$\mathfrak{X}(M) \to$ smooth vector fields on M

$\mathcal{D} \to$ smth real valued fcts

An $(0,r)$ - tensor field is a map $T: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{r} \to \mathcal{D}$

linear in each component

An $(1,r)$-tensor field is a map $T: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{r} \to \mathfrak{X}(M)$

linear comp.

We can always change $(1,r)$ tensor to a $(0,r+1)$-tensor and vice-versa

$$T(X_1, \dots, X_{r+1}) = \underbrace{\langle \underbrace{T(X_1, \dots, X_r)}_{(1,r)\text{-tensor}}, X_{r+1} \rangle}_{(0,r+1)\text{ tensor}}$$

type change ↗

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<u>ex:</u> $R(X,Y)Z \leftarrow (1,3)$ curvature tensor

$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle \leftarrow (0,4)$-tensor

Covariant Derivative of a Tensor

We can diff. any tensor in the direction of a vector field X.

Def: Given a tensor T, vector field X

$$(\nabla_X T)(Y_1, \dots, Y_r) = \nabla_X (T(Y_1, \dots, Y_r)) - \sum_{i=1}^{r} T(Y_1, \dots, \nabla_X Y_i, \dots, Y_r).$$

Then $\nabla_X T$ is a tensor of the same type as T.

$$\underset{\uparrow}{\nabla_X \circ T} - \underset{\nearrow}{T \circ \nabla_X}$$

product rule

If T is a $(1,1)$-tensor, $T: \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$(\nabla_X T)(Y) = \nabla_X (T(Y)) - T(\nabla_X Y)$$

$$\begin{aligned}
(\nabla_X T)(fY) &= \nabla_X (T(fY)) - T(\nabla_X (fY)) \\
&= \nabla_X (fT(Y)) - T(X(f) Y + f \nabla_X Y) \\
&= \underline{X(f)\ T(Y)} + f \nabla_X T(Y) - \underline{(X(f))T(Y)} - f\, T(\nabla_X Y) \\
&= f \nabla_X T(Y) - f\, T(\nabla_X Y) \quad = f (\nabla_X T)(Y).
\end{aligned}$$

If T is an (s,r)-tensor,

$$(\nabla T)(X, Y_1, \dots, Y_r) = (\nabla_X T)(Y_1, \dots, Y_r)$$

$\uparrow$

$(s, r+1)$-tensor

<u>Examples:</u>

① $f: M \to \mathbb{R}$ $(0,0)$-tensor.

$$\nabla_X f = X(f) = df(X) \quad \leftarrow (0,1)\text{-tensor}$$

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Type change of f.

Define ∇f $\leftarrow$ <u>gradient</u> of f

to be unique vector field s.t.

$X(f) = df(X) = \langle X, \nabla f \rangle$

② $df \to (0,1)$-tensor $\qquad (\nabla_X df) \leftarrow (0,1)$ tensor

$\nabla df \to (0,2)$-tensor

$(\nabla df)(X,Y) = (\nabla_X df)(Y) = \nabla_X df(Y) - df(\nabla_X Y)$

$= X(Y(f)) - (\nabla_X Y)(f)$ $\leftarrow$ Hessian of f

We also write $\operatorname{Hess} f\,(X,Y) = (\nabla_X df)\,(Y)$

In $\mathbb{R}^n$ $\quad E_i \to$ standard basis

$\operatorname{Hess} f\,(E_i, E_j) = E_i\, E_j(f) \qquad \nabla_{E_i} E_j \equiv 0$

Also $\quad \langle \nabla_X \nabla f, Y \rangle = X\langle \nabla f, Y\rangle - \langle \nabla f, \nabla_X Y \rangle = X(Y(f)) - (\nabla_X Y)(f)$

$= \operatorname{Hess} f\,(X,Y) = (\nabla df)\,(X,Y)$

$$\begin{tikzcd}
T \arrow[r, "\text{type change}"] \arrow[d] & T^* \arrow[d] \\
\nabla T \arrow[r, "\text{type}", "\text{change}"'] & \nabla(T^*) \arrow[d, equal] \\
 & (\nabla T)^*
\end{tikzcd}$$

③ Define $g(X,Y) = \langle X, Y \rangle$ $\leftarrow (0,2)$-tensor. $\quad \to (\nabla g) = 0$

$(\nabla_Z g)(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$

$= 0$ $\leftarrow$ compatible

$\uparrow$

g is parallel w.r.t. L.C. connection

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④ Let Z be a vector field. $-(1,0)$-tensor.

$\nabla Z \leftarrow (1,1)$-tensor

$(\nabla Z)(Y) = (\nabla_Y Z)$

$\nabla_Y(\nabla Z)(X) = \nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z$

$$\nabla(\nabla Z)(X,Y) = \nabla^2_{Y,X} Z = \nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z$$

↑

second cov. der.

$$R(X,Y)\{Z\} = \nabla^2_{Y,X} Z - \nabla_{\nabla_Y X} Z - \nabla_X \nabla_Y Z + \nabla_{\nabla_X Y} Z$$

$$= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Define Curv. of a tensor, $X, Y \in \mathfrak{X}(M)$

$$R(X,Y)T = \nabla^2_{Y,X} T - \nabla^2_{X,Y} T$$

$$R(X,Y)f = \nabla^2_{Y,X} f - \nabla^2_{X,Y} f = (\nabla_X df)(Y) - (\nabla_Y df)(X)$$

torsion free

$$= \underline{XY(f)} - \underline{(\nabla_X Y) f} - \underline{YX(f)} + \underline{(\nabla_Y X)(f)} = [X,Y](f) - [X,Y](f) = 0.$$

Hess $f(X,Y)$ = Hess $f(Y,X)$.

T is an (s,r) -tensor.

div T is an $(s,r-1)$ -tensor. Let E_i be an orthonormal basis of T_pM.

$$(\operatorname{div} T)(Y_1, \dots, Y_{r-1}) = \sum_{i=1}^{n} (\nabla_{E_i} T)(Y_1, \dots, Y_{r-1}, E_i)$$

ex $\operatorname{div}(df) = \sum_{i=1}^{n} (\nabla_{E_i} df)(E_i) = \sum_{i=1}^{n} \operatorname{Hess} f(E_i, E_i) =: \Delta f$

↑

Riem. Laplacian of f

In $\mathbb{R}^n$ $\frac{\partial}{\partial x_i}$ standard basis $\rightarrow$ Hess $f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_i^2}$

$\Delta f = \sum^{n} \frac{\partial^2 f}{}$

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Derivative of Curvature Tensor

2nd Bianchi Identity:

$$(\nabla_X R)(Y, Z, W, U) + (\nabla_Y R)(Z, X, W, U) + (\nabla_Z R)(X, Y, W, U) = 0.$$

Pf: HW.

Ricci tensor: $R(X,Y)Z \leftarrow (1,3)$ curv. tensor

Consider Fix X, Z. $T_pM \to T_pM$

$$Y \mapsto R(X,Y)Z$$

$$\mathrm{Ric}(X,Z) \Longrightarrow \mathrm{Trace}\,(Y \mapsto R(X,Y)Z)$$

Let E_i be orthonormal basis of T_pM

$$= \sum_{i=1}^{n} \langle R(X,E_i)Z, E_i \rangle$$

$$= \sum_{i=1}^{n} R(X, E_i, Z, E_i)$$

(0,2)-tensor symmetric

Let v be a unit vector in T_pM

Ricci curvature $\mathrm{Ric}(v,v) = \sum_{i=1}^{n} R(v, E_i, v, E_i)$

- $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$

Pick o.n. basis $E_1 = v, E_2, \to E_n$

$$\left(\frac{1}{n-1}\right) \sum_{i=2}^{n} \underset{K(\sigma_i)}{\underset{\parallel}{R(v, E_i, v, E_i)}}$$

where σ_i is the plan spanned by $\{v, E_i\}$

Ricci curvature is the average of sectional curvatures.

Def: An Einstein mfd is a Riem mfd such that all Ricci curvatures are equal.

$$\mathrm{Ric}(v,v) = \lambda \quad \forall\ |v| = 1.$$

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Scalar curvature

$$\mathrm{Scal} = \sum_{j=1}^{n} \mathrm{Ric}(E_j, E_j)$$

$$= \sum_{i,j=1}^{n} R(E_i, E_j, E_j, E_i)$$

Einstein tensor $\quad G = \mathrm{Ric} - \frac{\mathrm{Scal}}{2} g \qquad \underline{\mathrm{div}(G) = 0}$

[follow from 2nd Bianchi Id]

~~06~~ 4/6/18

Last time: M Riem. mfd

$\{E_i\}$ orthonorm. basis ~~for~~ of T_pM

Ricci tensor $\quad \mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(X, E_i, Y, E_i)$

$$\mathrm{Scal}(p) = \sum_{i,j=1}^{n} R(E_i, E_i, E_j, E_j)$$

Second Bianchi:

$$(\nabla_X R)(Y, Z, W, U) + (\nabla_Y R)(Z, X, W, U) + (\nabla_Z R)(X, Y, W, U) = 0.$$

$\{X_i\} \quad \{X_i\}$

$T_pM \to T_pM$

[Ric : $\mathrm{tr}\, R(X, Y, Z)$

L

$X_i = E_i$ o.n. $\langle L(E_i), E_i \rangle$

$L(X_i) = \sum a_{ij} X_j$

trace $\sum a_{ii}$

$\sum_i \langle L(X_i), X_i \rangle = \sum_{ij} \langle a_{ij} X_i, X_j \rangle$

$= \sum_{ij} a_{ij} g_{ij}$

$\langle L(X_i), Y_i \rangle g^{ij} = \sum_{i=1}^{n} a_{ii}$]

Contracted Bianchi Identities

(1) $\quad (\nabla_Y \mathrm{Ric})(Z, W) - (\nabla_Z \mathrm{Ric})(Y, W) = -(\mathrm{div}\, R)(Y, Z, W)$

(2) $\quad 2\, \mathrm{div}\, \mathrm{Ric} = d\, \mathrm{Scal}$

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Pf: $0 = \sum_i \left[(\nabla_{E_i} R)(Y, Z, W, E_i) + (\nabla_Y R)(Z, E_i, W, E_i) \right.$

$\left. + (\nabla_Z R)(E_i, Y, W, E_i) \right]$

$\overset{\text{exer}}{=} \operatorname{div} R(Y, Z, W) + (\nabla_Y \text{Ric})(Z, W) + (\nabla_Z \text{Ric})(Y, W)$ ✓ (1)

sum over W, Z $\underline{= \sum_i (\nabla_{E_i} \text{Ric})(Y, E_j)}$

$= \sum_{i,j} (\nabla_{E_i} R)(Y, E_j, E_j, E_i) + \underbrace{\sum_j (\nabla_Y \text{Ric})(E_j, E_j)}_{= Y(\text{Scal})}$

$+ \sum_j (\nabla_{E_j} \text{Ric})(Y, E_j)$

$= -(\operatorname{div} \text{Ric})(Y) + Y(\text{Scal}) - \operatorname{div}(\text{Ric})(Y)$

$\Rightarrow Y(\text{Scal}) = 2 \operatorname{div}(\text{Ric})(Y)$

$d(\text{Scal})(Y) = 2 (\operatorname{div} \text{Ric})(Y)$

Def: The Einstein tensor $G = \text{Ric} - \frac{1}{2} \text{Scal}\, g$. $g(X, Y) = \langle X, Y \rangle$

Prop: $\operatorname{div} G = 0$.

Einstein's eqn $G = (\text{—} \}$ stress energy tensor $= 0$

$\operatorname{div} G = \operatorname{div} \text{Ric} - \operatorname{div}\left(\frac{1}{2} \text{Scal}\, g\right)$

$= \frac{1}{2} d\text{Scal}$

Let $S = \varphi g$

$\operatorname{div} S(X) = \sum_i (\nabla_{E_i} S)(X, E_i)$

$= \sum_i \nabla_{E_i} (S(X, E_i)) - S(\nabla_{E_i} X, E_i) - S(X, \nabla_{E_i} E_i)$

$= \sum_i (\nabla_{E_i} (\varphi g(X, E_i)) - \varphi g(\nabla_{E_i} X, E_i) - \varphi g(X, \nabla_{E_i} E_i)$

$= \sum_i [E_i(\varphi) g(X, E_i) + \underline{\varphi E_i(g(X, E_i)) - \varphi g(\nabla_{E_i} X, E_i]}$

$\underline{- \varphi g(X, \nabla_{E_i} E_i)}]$ compatible $\to 0$

$= \sum_i d\varphi(E_i) g(X, E_i) = d\varphi\left(\sum_i \langle X, E_i \rangle\right) = d\varphi(X)$

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$\varphi = \frac{1}{2} \text{Scal}$

$\text{div} \left(\frac{1}{2} \text{Scal}\right) = \frac{1}{2} d\,\text{Scal} \quad \Rightarrow \text{div}\, G \equiv 0.$

Cor: Schur's Lemma.

If M^n Riem. mfd (connected), $n > 2$, then if $\text{Ric}_p = \varphi(p) g$ for some fct φ, then φ is constant.

Cor: If $n > 2$, if $\exists\, \varphi : M \to \mathbb{R}$ s.t. $K(\sigma_p) = \varphi(p)$, σ_p is a plane in T_pM

then M has constant sectional curvature.

Pf of Schur: $\text{Ric} = \varphi(p) g \qquad \text{Scal} = \sum_{j=1}^{n} \text{Ric}(E_j, E_j) = \sum_{j=1}^{n} \varphi(p) = n\varphi$

$$G = \left(\varphi(p) - \frac{n}{2}\varphi(p)\right) g = \left(\frac{2-n}{2}\varphi(p)\right) g$$

$$\underset{\overset{\shortparallel}{0}}{\text{div}\, G} = \frac{2-n}{2} d\varphi \qquad \Rightarrow d\varphi = 0 \Rightarrow \varphi \text{ is constant.}$$

Jacobi Fields

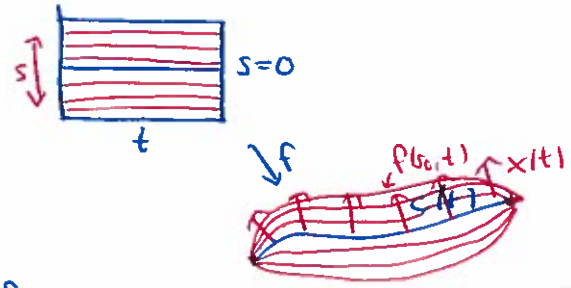
Let $c(t)$ be a curve in M, $t \in (-\varepsilon, \varepsilon)$

A variation of c is a map $f : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to M$
$f(t,s)$

smooth

such that $f(t,0) = c(t)$.

[Usually $f(0,s) = c(0)\ \forall s$]



Variation field along $c(t)$ is

$X(t) = \frac{\partial f}{\partial s}(t, 0).$

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Lemma: Let $V(s,t)$ be a vector field along f.

($V(s,t)$ is a vector field with $V(s,t)$ a vector at $f(s,t)$)

Lemma: $\frac{D}{dt}\frac{D}{ds}V - \frac{D}{ds}\frac{D}{dt}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V$

when $\frac{D}{ds}V = \nabla_{\frac{\partial f}{\partial s}} V$

where $\frac{D}{ds}V = \nabla_{\frac{\partial f}{\partial s}} V$ $\quad \frac{\partial f}{\partial s} = df\left(\frac{\partial}{\partial s}\right)$

see p. 98 & 99 of text $\quad \frac{\partial f}{\partial t} = df\left(\frac{\partial}{\partial t}\right)$

$$\left[\nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} V - \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} V + \nabla_{\underbrace{\cancel{\left[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right]}}_{=0}} V = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V \right]$$

$$= df\left(\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]\right)$$

Variation of geodesics is a variation $F(t,s)$ s.t.

$t \mapsto f(t,s)$ is a geodesic for each s.

$\iff \frac{D}{dt}\frac{\partial f}{\partial t} = 0.$

$$0 = \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}$$
$$= \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$
$$= \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}.$$

Set $J(T) = \frac{\partial f}{\partial s}(t,s)$ variation field of variation by geodesics.

$$\frac{D^2}{dt^2} J + R(\gamma', J)\gamma' = 0$$

Jacobi Equation (JE)

Def: γ geodesic. A Jacobi field along γ is a vector field along γ satisfying (JE).

Idea: Jacobi fields control how fast geodesics are spreading.

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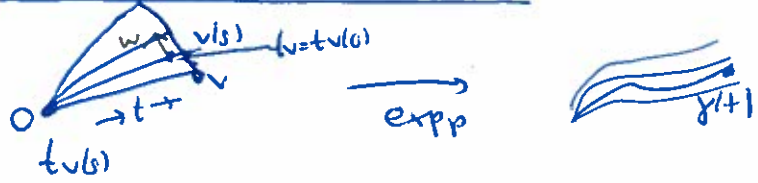
Remarks about (JE):

Second order linear system of ODEs

$\Rightarrow$ given $v, w \in T_{\gamma(0)} M$ $\exists!$ $J(t)$ such that J is a Jacobi field along $\gamma(t)$ $J(0) = v$, $\frac{D}{dt} J(0) = w$.

Jacobi fields & exponential map:



$$\gamma(t) = \exp_p(tv(0))$$

$$f(s,t) = \exp_p(tv(s))$$

$\uparrow$ variation by geodesics

$$v(0) = v \qquad v \in T_pM$$
$$v'(0) = w \qquad w \in T_v(T_pM)$$

$v(t,s) = v(s)$

$w = \frac{\partial v}{\partial s}(0)$

w const ~~along~~ in t

$$J(t) = \frac{\partial f}{\partial s}(t,0) = d(\exp_p)_{tv}(tw)$$

$w(t,s) = w(s)$

$$\left[t\,w(t,0) \in T_{tv}(T_pM) \qquad \frac{\partial v}{\partial s} = w \right]$$

In the end, only depends on value of at 0

Every Jacobi field s.t. $J(0)=0$ looks like

$$J(0) = d(\exp_p)_{0v}(0w) = 0$$

$$\frac{DJ}{dt} = \frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{dt}\left(d(\exp_p)_{tv}\, tw\right) = d(\exp_p)_{tv}(w) + t\frac{D}{dt}(\ldots)$$

$t=0$ $\qquad = w$

$d(\exp)_0 = Id.$

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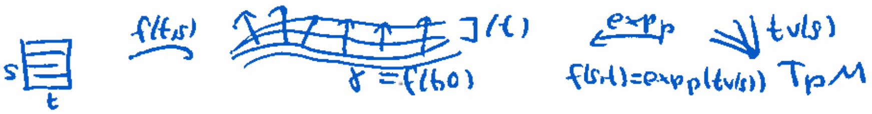
Last time: Jacobi fields

Let $\gamma(t)$ be a geodesic. Let $J(t)$ be a vector field along γ.



$J(t)$ is a Jacobi field if $\frac{D^2J}{dt^2} + R(\gamma', J)\gamma' = 0$ (JE).

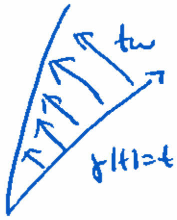
$\Leftrightarrow$ $J(t)$ is a variation field of variation by geodesics.
$f(s,t)$ $\rightarrow f_s(t)$ geodesic
$\frac{\partial f}{\partial s}(t,0) = J(t)$

If $J(0)=0$ $\Leftrightarrow$ $J(t) = d(\exp_p)_{tv}(tw)$

for some v, w where $p=\gamma(0)$, $\gamma(t) = \exp_p(tv)$

examples: ① $\mathbb{R}^n$ $\gamma(t)$ = geodesic through origin $= tv$

$J(t) = tw$
↑
"linear field along $\gamma(t)$"



$\frac{D^2J}{dt} = 0$

$\frac{D}{dt}\left(\frac{DJ}{dt}\right) = \frac{D}{dt}(w) = 0$

$R(\gamma', J)\gamma' \equiv 0$

② $S^n(r)$

$E(0) \neq 0$
$E(0) \perp \gamma(0)$

$J(t) = \varphi(t)E(t)$ where $E(t)$ is parallel along γ.
$\nabla_{\frac{d\partial}{\partial t}} E(t) = 0$
$\shortparallel$
$\frac{DE(t)}{dt} = 0$

$$\frac{DJ}{dt} = \frac{D}{dt}(\varphi(t)E(t))$$

$$= \frac{d\varphi}{dt}E(t) + \varphi(t)\underbrace{\frac{D}{dt}E(t)}_{=0}$$

$$\Rightarrow \frac{D^2J}{dt} = \frac{d^2\varphi}{dt}E(t)$$

$R(\gamma', J)\gamma' = \frac{1}{r^2}J$

$\langle R(\gamma', J)\gamma', J\rangle = \frac{1}{r^2}|J|^2$ $\langle R(\gamma', J)\gamma', v\rangle = \frac{1}{r^2}\langle J, v\rangle = 0$

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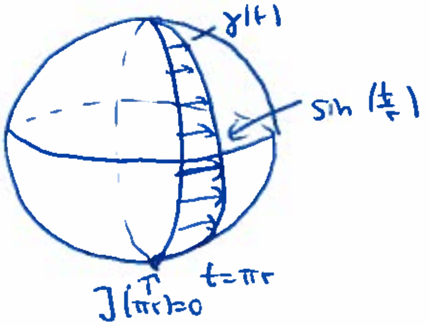
$$\frac{D^2J}{dt^2} + R(\gamma', J)\gamma' = 0$$

$$\left(\frac{d^2\varphi}{dt} + \frac{1}{r^2}\varphi\right)E = 0 \qquad J(0)=0 \quad \varphi(0)=0$$

$$\frac{d^2\varphi}{dt^2} = -\frac{1}{r^2}\varphi(t)$$

$$\varphi(t) = A \sin\left(\frac{t}{r}\right) \qquad J(t) = A\sin\left(\frac{t}{r}\right)$$

$\left[+B\cos\left(\frac{t}{r}\right) \text{ if } J(0)\neq 0\right]$



Observation: γ

$$\frac{D^2J}{dt^2} + R(\gamma', J)\gamma' = 0$$

- $J(t) \equiv 0$ is a Jacobi field ✓

$J(t) = \frac{d\gamma}{dt}$ is — " — — " — ✓ $\rightarrow \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0 \Rightarrow \frac{DJ}{dt} \equiv 0 \Rightarrow \frac{D^2J}{dt^2} = 0$

and $R(\gamma', \gamma')\gamma' = 0$

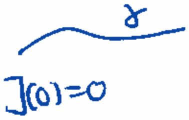
$J(t) = t\frac{d\gamma}{dt}$ is — " — — " —

$$\frac{d\gamma}{dt} = \left\{ \begin{array}{l} \underline{J(0) = \frac{d\gamma}{dt}} \\ \underline{\frac{DJ}{dt}(0) = 0} \end{array} \right.$$

$$J(t) = t\frac{d\gamma}{dt} \Rightarrow \underline{J(0)=0}, \quad \underline{\frac{DJ}{dt}} = \frac{d\gamma}{dt} + t\frac{D}{dt}\left(\frac{\partial\gamma}{\partial t}\right) = \underline{\frac{d\gamma}{dt}}$$

$$\frac{D^2J}{dt^2} = \frac{D}{dt}\left(\frac{\partial\gamma}{\partial t}\right) = 0$$

$$R(\gamma', J)\gamma' = R(\gamma', t\gamma')\gamma' = tR(\gamma', \gamma')\gamma' = 0.$$



$\rightarrow$ If $\frac{DJ}{dt}(0) \parallel \gamma \Rightarrow J(t) = at\frac{d\gamma}{dt} \quad \forall t$

$\rightarrow$ If $\frac{DJ}{dt} \perp \gamma \Rightarrow J(t) \perp \frac{d\gamma}{dt} \quad \forall t$

$\rightarrow$ perpendicular Jacobi fields

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Conjugate points

Let γ be a geodesic.

A point $\gamma(t_0)$ is said to be conjugate along γ to $\gamma(0)$ if there is a Jacobi field along γ s.t. $J(0) = 0 = J(t_0)$ and $J \neq 0$.

[**ex:** $S^n(R)$ N, S (north pole and south pole) are conjugate

$\mathbb{R}^n$ **no** conjugate pts (Jacobi fields increase linearly)]

ex:
- In $S^n(R)$, antipodal points are conjugate along any geodesic connecting the points
- In $\mathbb{R}^n$, no conjugate points

Prop: Conjugate pts $\Leftrightarrow$ singular pts of derivative of $\exp_p$.

Let $\gamma(t) = \exp_p(tv)$ be a geodesic γ.

$\gamma(t_0)$ is conjugate to $\gamma(0)$ iff $t_0 v$ is a critical pt of $\exp_p$.

Recall: $\exp_p : T_pM^n \to M^n$

$\overset{\in}{tv}$ is critical if $d(\exp_p)_{tv}$ is singular.

$(\Leftrightarrow \operatorname{rank}((d\exp_p)_{tv}) \neq n)$.

Pf: Suppose $\gamma(0)$ & $\gamma(t_0)$ are conjugate, then $\exists$ J Jacobi field along γ s.t. $J(0) = 0$ and $J(t_0) = 0$, then $J(t) = d(\exp_p)_{tv}(tw)$

$\frac{DJ}{dt}(0) = w \Rightarrow w \neq 0$ since if $w = 0$, $J(0) = 0$, $\frac{DJ}{dt}(0) = 0 \Rightarrow J \equiv 0$.

$0 = J(t_0) = d(\exp_p)_{t_0 v}(t_0 w)$ $t_0 w \neq 0$ $\Rightarrow d(\exp_p)_{t_0 v}$ is singular.

Conversely, if $d(\exp_p)_{t_0 v}(z) = 0$ for some $z \neq 0$. Let $w = \frac{z}{t_0}$.

$J(t) = d(\exp_p)_{tv}(tw) \Rightarrow J(t_0) = 0$. ∎

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Hadamard thm:

If M is a Riem mfd with non positive sectional curvature then M has no conjugate points.

Def: Non positive sectional curvature $\iff K_p(\sigma) \leq 0 \ \forall p \in M, \forall \sigma \subset T_pM.$

ex: $\mathbb{R}^n$ not S^n.

[If $K(\sigma) \leq 0$

$N = \exp_p(B)$ is not a normal nbhd

$$T_pM \cong \mathbb{R}^n \xrightarrow[\exp_p]{\text{local diffeo}} M \qquad \text{cov. map}$$

$\exp_p$ stops being 1-1

Pf: Let $\gamma(t)$ be a geodesic, $|\gamma'(t)| = 1$

Let $J(t)$ be a Jacobi field along $\gamma(t)$, $J(t) \perp \gamma(t)$,

s.t. $J(0) = 0$. W.t.s. $J(t) \neq 0 \ \forall t > 0$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \frac{1}{2}|J(t)|^2 = \frac{1}{2}\langle J(t), J(t)\rangle$, $\varphi(0) = 0$

$$\frac{d\varphi}{dt} = \frac{1}{2}\frac{d}{dt}\langle J(t), J(t)\rangle = \overset{\frac{1}{2}\cdot 2}{\cancel{\frac{2}{2}}}\left\langle \frac{DJ}{dt}, J(t)\right\rangle$$

$$\frac{d\varphi}{dt}(0) = 0 \qquad \frac{d^2\varphi}{dt^2} = \frac{d}{dt}\left\langle \frac{DJ}{dt}, J(t)\right\rangle = \left\langle \frac{D^2J}{dt^2}, J\right\rangle + \left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle$$

$$(JE) \qquad = \langle -R(\gamma', J)\gamma', J\rangle + \left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle$$

$$\nearrow = \underbrace{-\underbrace{|J|^2}_{\geq 0}\underbrace{K(\sigma)}_{\leq 0}}_{\geq 0} + \underbrace{\left|\frac{DJ}{dt}\right|^2}_{\geq 0} \geq 0$$

σ is plane spanned by γ', J.

$\varphi(t) \quad \varphi(0) = 0, \ \frac{d\varphi}{dt}(0) = 0, \ \frac{d^2\varphi}{dt^2} \geq 0 \quad \forall t.$

If $J(t_0) = 0 \Rightarrow \frac{d\varphi}{dt}(t_0) = 0 \Rightarrow \frac{d\varphi}{dt}(t_0) \equiv 0 \ \cancel{\forall} t \in [0, t_0)$

$\Rightarrow \varphi(t) \equiv 0 \ \forall t \Rightarrow J \equiv 0.$ $\square$

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Next time: Completeness

Def: M is geodesically complete if geodesics exist for all time.

Hopf-Rinow $\Leftrightarrow$ $\exp_p : T_pM \to M$ defined on all of T_pM.

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Fr 11:40-12:30 Carn 119 Make up lecture

Complete Riemannian manifolds (Ch 7 of do Carmo)

Always assume M connected.

Def: A Riemannian mfd is called geodesically complete if all geodesics can be extended for all time.

(equivalently, $\forall p \in M$ $\exp_p$ is defined on all of T_pM).

ex: $\mathbb{R}^n$ geod complete

S^n geod. complete

$\mathbb{R}\setminus\{p\}$ not geod. complete

Rem: If M is complete, it ~~cannot~~ it is not extendible meaning it cannot be isometrically embedded as a proper subset of another mfd of dimension n. (see p. 145 of text.)

Recall: $d(x,y) = \inf(\ell_c)$ where c is a path connecting x and y,

$$\ell_c = \int_a^b |\dot{c}(t)|\,dt \quad \text{where } c(a)=x,\ c(b)=y.$$

Gauss lemma: $\forall p \in M$ $\exists\ \varepsilon$ s.t. $\exp_p(B(0,\varepsilon)) = \underset{\{x : d(x,p)<\varepsilon\}}{\underset{\parallel}{B(p,\varepsilon)}}$

$\Rightarrow$ Mfd topology is equivalent to metric space topology.

$\Rightarrow$ $d : M \times M \to \mathbb{R}$ is continuous

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Hopf-Rinow Thm:

Let M be a connected Riem. mfd, let $p \in M$. TFAE:

(a) $\exp_p$ is defined on all of T_pM.

(b) The closed & bounded subsets of M are compact.

(c) M is complete as a metric space.

(d) M is geodesically complete.

Pf: (b) ⇒ (c) point-set topology exercise using sequential compactness.

(d) ⇒ (a) obvious

(c) ⇒ (d): Assume M is complete as metric space.

Suppose $\exists$ geodesic $\gamma : [0, t_0) \to M$, s.t. $|\gamma'(t)| \equiv 1$ $\forall t$, which cannot be extended past t_0.

Let $s_n \to t_0$ $s_n < t_0$ $\forall n$.

Consider $\{\gamma(s_n)\}$ $\quad d(\gamma(s_n), \gamma(s_m)) \leq \int_{s_n}^{s_m} |\dot{\gamma}(t)|\,dt = |s_m - s_n|$

$\Rightarrow \{\gamma(s_n)\}$ is a Cauchy seq. $\therefore$ $\gamma(s_n) \to p_0 \in M$.

Let W be a totally normal ~~[illegible]~~ nbhd around p_0, and $\forall n \geq n_0$

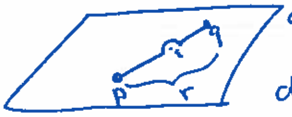
$\gamma(s_n) \in W$, so γ is the unique geodesic passing through $\gamma(s_n), \gamma(s_m)$

$\Rightarrow \gamma$ is a geodesic between two points in W

$\Rightarrow \gamma$ can be extended past γ t_0.

Lemma: (a) $\Rightarrow$ $\forall q \in M$ $\exists$ a geodesic γ s.t. $d(p,q) = \ell_\gamma$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\|}{\inf \ell_c}$

example



← not geod. complete

$d(p,q) > r$

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Pf of lemmas

Let $d(p,q)=r$. Let $B(p,\delta)$ be a normal ball around p.

Let $S(p,\delta) = \partial B(p,\delta) = \{x \mid d(p,x)=\delta\}$ compact set

$x \longmapsto d(q,x)$, $x\in S(p,\delta)$ cont. $S(p,\delta)$ compact $\Rightarrow \exists x_0$ s.t. $d(q,x_0)=\inf_{x\in S} d(q,x)$.

idea: x_0 should be on the minimal geodesic connecting p and q.

$x_0 = \exp_p(\delta v)$, $v\in T_pM$, $|v|=1$.

let $\gamma(t) = \exp_p(tv)$. Cl: $\gamma(t)$ minimizes distance from p to q.

i.e. $\gamma(r)=q$

Let $A = \{s\in[0,r] \mid d(\gamma(s),q)=r-s\}$

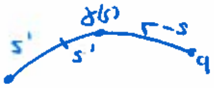
$0\in A$ $[\gamma(0)=p]$, A closed.

Want $A=[0,r]$ $\Rightarrow$ $d(\gamma(r),q)=r-r=0 \to \gamma(r)=q$.

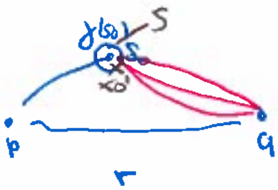
Show A is open, as a subset of $[0,r]$.

If $s_0\in A$, $s_0<r$, then $\exists\, \delta_0$ s.t. $s_0+\delta_0\in A$.

[Note: If $s\in A$, $s'<s$



$r-s' \le d(\gamma(s'),q) \le s-s'+r-s \le r-s'$]



$d(\gamma(s_0),q)=r-s_0$.

Let $\delta_0>0$ s.t. $B(\gamma(s_0),\delta_0)$ is a normal ball and $s_0+\delta_0<r$.

Let x_0' be a point that minimizes $d(x_0',q)$ among all pts in $S(\gamma(s_0),\delta_0)=S$

$$r-s_0 = d(\gamma(s_0),q) = \inf_{\substack{c \text{ connecting} \\ \gamma(s_0) \text{ to } q}} (\ell(c)) = \delta_0 + d(x_0',q)$$

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$s_0 \; d(x_0', q) = r - s_0 - \delta_0$ (*)

Δ ineq. $\Rightarrow$ $d(p, x_0') \geq \delta_0 + \delta_0$

On the other hand, concatenation of γ with unique minimal geodesic from for $\gamma(s_0)$ to x_0' has length $s_0 + \delta_0$.

$\Rightarrow$ The concatenation is a geodesic $\Rightarrow$ $\gamma(s_0 + \delta_0) = x_0'$

$\Rightarrow$ by (*) $s_0 + \delta_0 \in A$ $\square$ Lemma

<u>(a) $\Rightarrow$ (b):</u> Let $A \subseteq M$ be closed & bounded set.

By lemma, $\exp_p : T_pM \to M$ is surjective.

A bounded, $\exists$ R s.t. $d(p,q) \leq R$ $\forall q \in A$

Lemma $\exists$ γ s.t. $\gamma(0) = p$, $\gamma(d(p,q)) = q$ $|\gamma| \equiv 1$.

$\Rightarrow A \subseteq \exp_p(\underset{\substack{\uparrow \\ \text{compact}}}{\overline{B(0, R)}})$ compact (M Hausdorff)

A closed subset of compact set $\Rightarrow$ A compact.

<u>Corollary:</u> If M is compact, M is complete & $\forall \overset{p,q}{p} \in M$ $\exists$ minimizing geodesic from p to q.

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Fundamental Q of Riem. Geometry:

Given a smth mfd M, what is the "best" metric we can put on it?

Recall: Covering space: X top. space. Cov. space of X is (Y, p) where $p: Y \to X$ s.t. p cont. & p is a surjection s.t. $\forall p\ \exists U$ nbhd s.t. $p^{-1}(U)$ is a disjoint union of open sets (in Y) each homeomorphic to U.

ex: $\mathbb{R} \to S^1$

$\{t\} \mapsto (\cos(t), \sin(t), t)$

$\pi_1(S_1) = \mathbb{Z}$

Rem: If X is a mfd, then we always have covering spaces.

$\underset{\substack{\uparrow \\ \text{gp}}}{\pi_1}(M, p) = \{\text{loops based at } p\}/\sim$ ← homotopy

M simply connected if M connected and $\pi_1(M, p) = \{e\}$.

Def: The universal cover of M is a covering space $p: Y \to M$ s.t. Y is simply connected.

Rem: If M connected $\exists!$ universal cover.

and $|\pi_1(M)| = \# p^{-1}(x)$

Let M be a connected mfd, $p: Y \to M$ covering space.

$\Rightarrow$ Y is a mfd. Y is a smth mfd, p is a local diffeo (by pulling back atlas)

If $F: Y \to M$ is a local diffeo, M Riem. mfd.

If $v, w \in T_y Y$, define $\langle v, w \rangle_y = \langle dF_y(v), dF_y(w) \rangle_{F(y)}$

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$\Rightarrow F$ is a local isometry.

Def: $p: Y \to M$ is a covering space between Riem. mfd Y and M, p is called a Riemannian ~~mfd~~ covering space if p is also a local isometry.

Thm: If $F: M \to N$ be a local isometry. If M is complete and N is connected, then F is a Riemannian covering space.

ex: $\underset{P(\mathbb{R}^n)}{\overset{\shortparallel}{\mathbb{R}P^n}} = S^n/\sim$ $\quad p \sim -p$

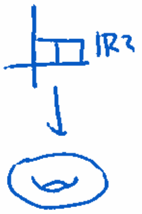
is a local isometry, covering space

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$

$\mathbb{R}^n/\sim \; = T^n$

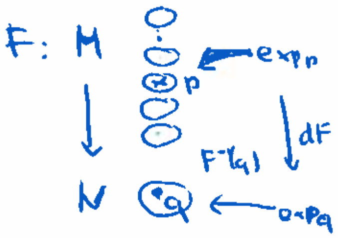
$n=2$

$(x,y) \sim (x+n, y+m) \quad \forall m,n \in \mathbb{Z}$



$\pi_1(T^n) = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}}$

Pf:



Let $q \in N$. Let $B(q,\varepsilon)$ be a normal ball around q.

Let $p \in F^{-1}(q)$. By completeness of M, $\exp_p$ is defined on all of T_pM. F is a local isometry, by HW $\overset{6?}{5}$; #2,

$F \circ \exp_p (v) = \exp_q \circ DF_p(v)$

$(\exp_q \circ DF_p)$ is a diffeo on $B(0,\varepsilon) \subset T_pM$.

$\Rightarrow F \circ \exp_p$ is a diffeo on $B(0,\varepsilon)$. $\exp_p : B(0,\varepsilon) \to B(p,\varepsilon)$ onto by geod. completeness and $\exp_p$ is 1-1 on $B(0,\varepsilon)$ by $F \circ \exp_p$ diffeo.

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$\Rightarrow F: B(p,\varepsilon) \to B(q,\varepsilon)$ is a diffeo.

[$\exp_p$ diffeo since $d(\exp_p)(tv) \neq 0$ since $F \circ \exp_p$ diffeo]

Cl: $F^{-1}(B(q,\varepsilon)) = \bigcup_{p \in F^{-1}(q)} B(p,\varepsilon)$

Pf. Let $x \in F^{-1}(B(q,\varepsilon))$

$F(x) \in B(q,\varepsilon)$

$\Rightarrow \exists!$ geodesic $\gamma(t)$ from q to $F(x)$

of length ε $\gamma(0)=q$, $\gamma(1)=F(x)$

σ be geodesic $\sigma:[0,1]\to M$ s.t. $\sigma(1)=x$

$$\frac{d\sigma}{dt}(1) = (dF_x)^{-1}\left(\frac{d\gamma}{dt}(1)\right)$$

By geodesic completeness of M σ exists on $[0,1]$

$F\circ\sigma$ is a geodesic in M. $(F\circ\sigma)(1) = F(x)$, $\frac{d}{dt}(F\circ\sigma)(1) = \frac{d\gamma}{dt}(1)$

$\Rightarrow F\circ\sigma = \gamma$, $F(\sigma(0)) = \gamma(0) = q$ $\sigma(0) \in F^{-1}(q)$

$\Rightarrow d(\sigma(1),\sigma(0)) = d(p,x) < \varepsilon$.

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Last time:

Prop: Let $F: M \to N$ be a local isometry. If M is complete & N is connected then F is a covering map.

Thm: If M is a complete Riemannian mfd with nonpositive sectional curvature ($K \leq 0 \iff K(\sigma_p) \leq 0$ $\forall \sigma \in T_pM$ $\forall p \in M$) then for each $p \in M$, $\exp_p$ is a covering map. (Cartan-Hadamard Thm)

Pf: We proved that if $K \leq 0$ then there are no conjugate points.

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$\Rightarrow \exp_p : \underset{\text{completeness}}{\underline{T_pM}} \to M$ is a local diffeo & onto

Define a Riem metric on T_pM by $\langle v, w\rangle = \langle d\exp_p(v), d\exp_p(w)\rangle$.

$\Rightarrow \exp_p$ a local isometry.

Claim: $(T_pM, \langle , \rangle)$ is complete.

Pf. $\exp_p$ is a local isometry. So geodesics are mapped to geodesics.

$\exp_p$ maps straight lines through 0 to geodesics through p.

$\Rightarrow$ straight lines through the origin in ~~0~~ T_pM are geodesics.

By Hopf Rinow all geodesics through 0 can be extended for all t.

$\Rightarrow$ $(T_pM, \langle , \rangle)$ is complete. □claim

By prop: $\exp_p$ is a covering map. □

Cor.: If M is complete and $K \le 0$ then the universal cover of M ~~[illegible]~~ is $\mathbb{R}^n$.

$\left[\exp_p : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \text{homeo } \mathbb{R}^n}}{T_pM} \to M \text{ cov. map} \quad \leftarrow \text{simply-conn.} \right]$

$\Rightarrow$
- If M compact, $K \le 0$ $\Rightarrow |\pi_1(M)| = \infty$.
- $\pi_1(M)$ is torsion-free (every element has ∞ group order)

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, "F"] \\ (M, p) \end{tikzcd}$$

$F^{-1}(p)$ is infinite set

ex/cor: S^n has no metric on it with $K \le 0$.
$n \ge 2$

Spaces of constant curvature

ex $\mathbb{R}^n$ $K \equiv 0$ $S^n(r)$ $K \equiv \frac{1}{r^2}$

Thm: Let $\bar{M}^n, M^n$ be two Riem. mflds with $\bar{p} \in \bar{M}$, $p \in M$ s.t.

both have neighborhoods with constant curvature K, then there

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is a local isometry from an open nbhd of p to an open nbhd of $\bar{p}$.

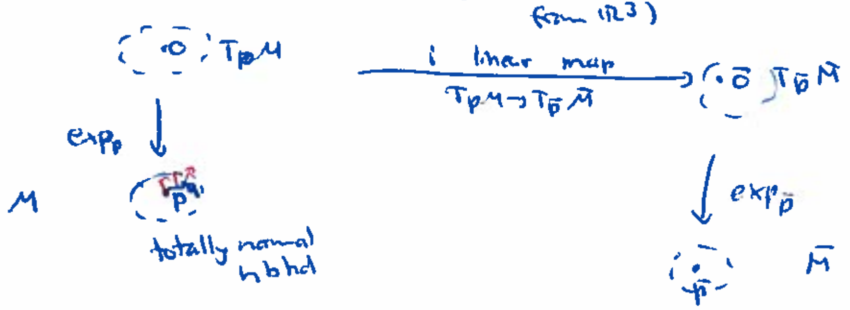
e.g. const. curv. 0

, $\mathbb{R}^2$ are all locally isometric.

$\mathbb{R}^2/\mathbb{Z}$

$\mathbb{R}^2 \to T$

T with right metric (not the one induced from $\mathbb{R}^3$)



Let V be a totally normal nbhd of p, i an arbitrary linear isometry $T_pM \to T_{\bar{p}}\bar{M}$.

Define f: $V \to \bar{M}$, $f(q) = \exp_{\bar{p}} \circ i \circ \exp_p^{-1}$.

Claim: If constant curv. $= k$ in V and $f(V)$ then f is an isometry.

In fact, more true, even when not constant curvature.

$\forall q \in V$, $\exists!$ geod γ from p to q. Let P_t be parallel translation along γ from $\gamma(0)$ to $\gamma(t)$. Then $\bar{\gamma} = f \circ \gamma$ is a geodesic from $\bar{p}$ to $f(q)$.

Let $\bar{P}_t$ ~~be~~ the parallel translation along $\bar{\gamma}$. from $\bar{\gamma}(0)$ to $\bar{\gamma}(t)$.

Define φ_t: $T_qM \to T_{f(q)}\bar{M}$ by

$$\bar{P}_t \circ i \circ P_t^{-1}$$

Thm: If $\forall q \in V$ and $\forall X, Y, U, W \in T_qM$ $R(X, Y, U, W) = \bar{R}(\varphi_t(X), \varphi_t(Y), \varphi_t(U), \varphi_t(W))$ $\forall t$

then f is a local isometry.

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Pf of Thm: Let $v \in T_q M$

Note: γ does not have conjugate pts.

Let $\mathcal{J}$ = space of Jacobi fields along ~~γ~~ γ with $J(0)=0$.

$\dim(\mathcal{J}) = n$. $\quad \theta : \mathcal{J} \to T_{\gamma(r)} M$ ~~exer~~ linear map since no conjugate pts

$J \mapsto J(r)$

If $J(r)=0$ for some r then $J=0$

So θ is injective $\Rightarrow$ θ is an isom.

$$\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0.$$

$\Rightarrow \forall v \in T_q M \ \exists J(t) \in \mathcal{J}$ s.t. $J(r)=v$, $J(0)=0$.

Let $\{E_i(t)\}$ be a parallel orthonormal basis along γ

Write $J(t) = \sum_i y_i(t) E_i(t)$

$$\frac{DJ}{dt} = \sum_i y_i'(t) E_i(t)$$

$$\frac{D^2J}{dt^2} = \sum_i y_i''(t) E_i(t)$$

Then $\frac{D^2J}{dt^2} + R(\gamma', J)\gamma' = 0 \Leftrightarrow \quad y_j'' = -\sum_i R(\gamma', E_i, \gamma', E_j) y_j$

Let $\bar{\gamma}$ be the geodesic in $\bar{M}$ with $\bar{\gamma}(0) = \bar{p}$, $\bar{\gamma}'(0) = i(\gamma'(0))$

Let $\bar{J}(t)$ be the field along $\bar{\gamma}$ given by $\bar{J}(t) = \varphi_t(J)$.

Cl: $\bar{J}$ is a Jacobi field in $\bar{M}$.

Pf: $\quad \bar{E}_i(t) = \varphi_t(E_i)$ ← parallel onto basis along ~~$E(t)$~~ $\bar{\gamma}$

$$\bar{\gamma}'(t) = \varphi_t(\gamma'(t)).$$

$$\bar{J}(t) = \varphi_t(J) = \varphi_t\left(\sum y_i(t) E_i(t)\right) = \sum y_i(t) \bar{E}_i(t)$$

By hypothesis $\quad R(\gamma', E_i, \gamma', E_j) = \bar{R}(\bar{\gamma}', \bar{E}_i, \bar{\gamma}', E_j)$

$$\Rightarrow \quad y_j'' + \sum \bar{R}(\bar{\gamma}', \bar{E}_i, \bar{\gamma}', \bar{E}_j) y_j = y_j'' + \sum R(\gamma', E_i, \gamma', E_j) y_i = 0$$

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$\Rightarrow \bar{J}(t)$ is a Jacobi field

So J and $\bar{J}$ are both Jacobi fields.

$J(t) = d(\exp_p)_{t\gamma'(0)}(t\, J'(0))$ $\qquad J' = \frac{DJ}{dt}$

$\bar{J}(t) = d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}(t\, \bar{J}'(0))$

since $\bar{J} = \varphi_t(J(t))$ $\qquad \bar{J}'(0) = i\, J'(0)$

$\bar{J}(\ell) = d(\exp_{\bar{p}})_{\ell\bar{\gamma}'(0)}(\ell\, i\, J'(0)) = d(\exp_{\bar{p}})_{\ell\bar{\gamma}'(0)} \circ i \circ (d\exp_p)^{-1}_{\ell\gamma'(0)}(J(\ell))$

$= df_q(J(\ell))$

$df_q(J(\ell)) = \bar{J}(\ell) = \underline{\varphi_t}(J(\ell))$

isometry

and $\ell = r$ $\qquad |df_q(J(r))| = |df_q(v)| = |v| \ \forall v \Rightarrow f$ is an isometry. $\square$

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Last time: Let $M, \bar{M}$ have constant sectional curvature k, then $\forall p \in M$, $\bar{p} \in \bar{M}$ and $\forall i: T_pM \to T_{\bar{p}}\bar{M}$ linear isometry, then the map

$f: V \to \bar{M}$ $\qquad f = \exp_{\bar{p}} \circ i \circ (\exp_p)^{-1}$ is an isometry onto $f(V)$

V normal nbhd of p. $\qquad df_p = i$

Model spaces $\quad k \in \mathbb{R}$ $\quad M^n_k =$ (unique) simply-connected Riem. mfd with constant sectional curvature k.

- $k = 0$ $\quad M^n_k = \mathbb{R}^n$ standard metric
- $k > 0$ $\quad M^n_k = S^n\left(\frac{1}{\sqrt{k}}\right)$ ← sphere of radius $\frac{1}{\sqrt{k}}$
- $k < 0$ $\quad$ Hyperbolic space $= \{(x_1, \dots, x_n) : x_n > 0\}$ ← upper half space

metric $\langle , \rangle = \frac{1}{\sqrt{k}} \frac{dx_1^2 + dx_2^2 + \dots + dx_n^2}{x_n^2}$ $\qquad g_{ij} = \frac{1}{\sqrt{-k}} \frac{\delta_{ij}}{x_n^2}$

$\left[n = 2 : \frac{dx^2 + dy^2}{y^2} \right]$

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$\mathbb{H}^n$ complete, simply connected constant curv. k. (see pp. 160-162)

Thm: Suppose M is a complete mfd with constant sectional curvature then the universal cover of M is M^n_k.

(a cov. map is a local isometry).

[$$\begin{tikzcd} \tilde{M}^n_k \arrow[d, "p"'] \\ M \end{tikzcd}$$ + covering theory $\Rightarrow$ M^n_k is <u>unique</u> complete constant curv. space which is simply-connected.]

Cor.: "Most" mfds do not have a metric with constant sectional curvature.

ex: $S^1 \times S^2$, $S^n \times T^m$, $\mathbb{C}P^n$.

Reduce problem of classifying const. curv. space to Algebra Problem.

Pf of Thm: M complete Riem. mfd w/ const. curvature k.

Case 1: $k \le 0$, Cartan-Hadamard thm, $\exp_p : T_pM \to M$ is a covering map

$f = \underbrace{\exp_p \circ i \circ (\exp_{\tilde{p}})^{-1}}_{\text{isometry}}$

$V = M^n_k$

$$\begin{tikzcd} T_{\tilde{p}} M^n_k \arrow[r, "i"] \arrow[d, "\exp_{\tilde{p}}"', "\text{diffeo}"] & T_pM \arrow[d, "\exp_p"] \\ \tilde{p} \in M^n_k & M \end{tikzcd}$$

By C-H $\exp_{\tilde{p}}$ is an isometry $\Rightarrow \exists\ i \circ (\exp_{\tilde{p}})^{-1} : M^n_k \to T_pM$ is an isometry.

Case 2: $k > 0$, $M^n_k = S^n\left(\frac{1}{\sqrt{k}}\right)$. Assume $k = 1$. $S^n(1)$

$$\begin{tikzcd} T_pM \arrow[r, "i"] \arrow[d, "\exp_p"] & T_{\tilde{p}} \tilde{M} \arrow[d, "\exp \tilde{p}"] \\ S^n(1) & M \end{tikzcd}$$

Let $q = -p$.

$V = S^n \setminus \{q\}$ is a normal nbhd of p.

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$f: V \to M$, $f = (\exp_{\tilde{p}}) \circ i \circ (\exp_p^{-1})$ is a local isometry.

Choose $p' \neq p, q$. Let $q' = -p'$.

Let $\tilde{p}' = f(p')$, $i' = df_{p'}$

$f': S^n \setminus \{q'\} \to M$, $f' = \exp_{\tilde{p}'} \circ i' \circ (\exp_{p'})^{-1}$ is also a local isometry.

Let $W = S^n \setminus (\{q\} \cup \{q'\})$ $f, f': W \to M$ local isometries

$f(p') = f'(p')$, $df_{p'} = df'_{p'}$ by construction

$S^n \setminus \{\text{two pts}\} \approx \mathbb{R}^n \setminus \{1 \text{ pt}\}$

$n \geq 2$, $\mathbb{R}^n \setminus \{1 \text{ pt}\}$ is connected

So W is connected. So by HW #5 $f \equiv f'$ on W

(local isometries on connected mfd, whose value & derivative agree at p')

Def. $h: S^n \to M$,
$$h(x) = \begin{cases} f(x) & \text{if } x \in S^n \setminus \{q\} \\ f'(x) & \text{if } x \in S^n \setminus \{q'\} \end{cases}$$

is a local isometry. S^n complete, M connected

$\Rightarrow$ h is a covering map. $\square$

Variation of Energy

Calculus of variations:

Let $c: [0,a] \to M$ be a curve.

$L(c) = \int_0^a |c'(t)| \, dt$ $\leftarrow$ length of c

$L: \{\text{curves in } M\} \to \mathbb{R}^{\geq 0}$ $\leftarrow$ functional

"Differentiate" L.

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Recall: A variation of c is a map $f:(-\varepsilon,\varepsilon)\times[0,a]\to M$

s.t. $f(0,t)=c(t)$.

Variation field: $\frac{\partial f}{\partial s}(0,t) = V(t)$ ← vector field along c.

Note: Book does this in generality that $f(s,t)$ is piecewise different in t

$0 \le t_1 \le t_2 \le \dots \le t_n = a$ smooth in $[t_i, t_{i+1}]$

$\frac{d}{ds} L(f_s) =$?

$f_s(t) = f(s,t)$

Turns out to be more convenient to work with Energy functional

If f is a variation, define $E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right|^2 dt$

$$E(0) = \int_0^a \left| \frac{\partial f}{\partial t}(0,t) \right|^2 dt = \int_0^a |c'(t)|^2 dt$$

(Just like in $\mathbb{R}^n$, d^2 instead of d)

$E(c) = \int_0^a |c'(t)|^2 dt$ $\qquad L(c) = \int_0^a |c'(t)|\,dt$

(If $f=1$, $g = \left|\frac{dc}{dt}\right|$, Cauchy-Schwarz, $\left(\int_0^a fg\,dt\right)^2 \le \left(\int_0^a f^2 dt\right) \cdot \left(\int_0^a g^2 dt\right)$

$$(L(c))^2 = \left(\int_0^a \left|\frac{dc}{dt}\right| dt\right)^2 \le a \int_0^a \left|\frac{dc}{dt}\right|^2 dt = a\,E(c)$$

So $L(c)^2 \le a\,E(c)$ & "=" if and only if $\left|\frac{dc}{dt}\right|$ is constant.

i.e. C is parametrized proportional to arclength.

We know geodesics are minimizers of the length between two fixed points p, q. Also true for Energy.

Lemma: Let $p,q \in M$. Let $\gamma:[0,a]\to M$ be a minimizing geodesic joining p and q then for all curves

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$c:[0,a]\to M \qquad c(0)=p,\ c(a)=q \qquad E(\gamma)\le E(c)$

and equality iff c is also a minimizing geodesic from p to q.

Pf: $$a\,E(\gamma) \underset{\substack{\uparrow\\ \gamma \text{ has}\\ \text{const speed}}}{=} L(\gamma)^2 \le L(c^*)^2 \le a\,E(c)$$

$$\Rightarrow E(\gamma)\le E(c)$$

If $E(\gamma)=E(c)$, then $L(\gamma)=L(c) \Rightarrow c$ is also a minimizing geodesic. □

Compute the first variation of Energy. (derivative)

Recall: A variation field is called proper if $V(0)=V(a)=0$

$\Leftrightarrow \quad f(0,s_1)=f(0,s_2) \qquad f(a,s_1)=f(a,s_2) \qquad \forall s_1,s_2\in(-\varepsilon,\varepsilon).$

Prop: Let $c:[0,a]\to M$ be differentiable curve, let $f:(-\varepsilon,\varepsilon)\times[0,a]\to M$ be a variation of c.

If $E(s)=\int_0^a \left|\frac{\partial f}{\partial t}(t,s)\right|^2 dt$

$V(t)=\frac{\partial f}{\partial s}(0,t)$

then $$\frac{1}{2}\frac{d}{ds}E\Big|_{s=0} = -\int_0^a \left\langle V, \frac{D}{dt}\frac{dc}{dt}\right\rangle dt$$

$$+\left\langle V(a), \frac{dc}{dt}(a)\right\rangle - \left\langle V(0), \frac{dc}{dt}(0)\right\rangle$$

Note: If V is proper, $V(0)=V(a)=0$

so $\frac{dE}{ds}(0)=0 \quad \forall V$ proper $\Leftrightarrow$ c is a geodesic.

Last time: Variations of Energy 4/23/18

$c:[0,a]\to M$ curve, $f:(-\varepsilon,\varepsilon)\times[0,a]\to M$ is a variation of c

if $f(0,t)=c(t)$

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Variation field $V(t) = \frac{\partial f}{\partial s}(0,t)$ ← vector field along c

Energy $E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right|^2 dt$

For each $V(t)$ ← vector field along c

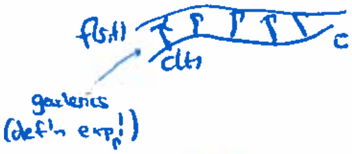
$\exists \infty$ many f s.t. $\frac{\partial f}{\partial s}(0,t) = V$.

But we can choose a "nice" variation using expon. map.

<u>Lemma</u>: Let $c: [0,a] \to M$ smooth curve and $V(t)$ a smooth v.field along c. Then $f(s,t) = \exp_{c(t)}(sV(t))$ is a variation of c s.t. $\frac{\partial f}{\partial s}(0,t) = V(t)$. Moreover, if $V(0) = V(a) = 0$ then

$f(s_1,0) = f(s_2,0)$, $f(s_1,a) = f(s_2,a)$ $\forall s_1, s_2 \in (-\varepsilon, \varepsilon)$.

$\uparrow$
proper



<u>Pf</u>: For each $c(t_0)$ ~~$\exists \varepsilon > 0$ s.t. ex~~ $\exists W_{t_0}$ a totally normal neighborhood at $c(t_0)$.

$c(t)$ covered by finitely many Ws so pick uniform $\varepsilon > 0$ s.t. $\exp_{c(t)}$ is def'd on $B(c(t), \varepsilon)$ $\forall t$.

$f(0,t) = \exp_{c(t)}(0) = c(t)$.

$\frac{\partial f}{\partial s} = d(\exp_{c(t)})_{sV}(V)$

$\frac{\partial f}{\partial s}(0,t) = d(\exp_{c(t)})_0(V) = V$

$V(0) = 0$ $f(s,0) = \exp_{c(0)}(sV(0))$
$= \exp_{c(0)}(0) = c(0)$ $\forall s$

$V(a) = 0$ then $f(s,a) = \exp_{c(a)}(0)$
$= c(a)$ $\forall s$.

Choice of f s.t. $s \mapsto f(s,t_0)$ is a geodesic $\forall t_0$

<u>or</u> $\frac{D}{ds}\frac{\partial f}{\partial s} = 0$.

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Prop: First variation formula

Let $c: [0,a] \to M$ diff curve, $f: (-\varepsilon,\varepsilon) \times [0,a] \to M$ variation of c

If $E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(t,s) \right|^2 dt$, then $\frac{1}{2} \frac{d}{ds}\Big|_{s=0} E = -\int_0^a \langle V(t), \frac{D}{dt}\frac{dc}{dt} \rangle d$

$+ \langle V(a), \frac{dc}{dt}(a) \rangle - \langle V(0), \frac{dc}{dt}(0) \rangle$

$$\frac{1}{2}\frac{d}{ds} E(s) = \frac{1}{2}\int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt \qquad \Rightarrow \frac{1}{2}\frac{d}{ds} E(s) \overset{=\frac{1}{2}\frac{d}{ds}}{(} \int_0^a \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt)$$

$$= \frac{1}{2}\int_0^a \frac{d}{ds} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt = \frac{1}{2}\int_0^a \left(\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial t}, \frac{D}{ds}\frac{\partial f}{\partial t} \rangle \right).$$

$$= \int_0^a \langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle dt = \int_0^a \left(\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle dt = \int_0^a \left(\frac{d}{dt} \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \right.\right.$$

$$\left. - \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle \right) dt = \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \Big|_0^a - \int_0^a \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle dt$$

$$\overset{s=0}{=} \langle V, \frac{dc}{dt} \rangle \Big|_0^a - \int_0^a \langle V, \frac{D}{dt}\frac{dc}{dt} \rangle dt \checkmark$$

Cor.: If f is a proper variation $f(s_1,0) = f(s_2,0)$ $\forall s_1, s_2 \in (-\varepsilon,\varepsilon)$

$f(s_1,a) = f(s_2,a)$

$$\frac{1}{2}\frac{dE}{ds}(0) = -\int_0^a \langle V, \frac{D}{dt}\frac{dc}{dt} \rangle dt$$

Cor.: A curve $c(t)$ has the property that $\frac{dE}{ds}(0) = 0$ $\forall$ proper variation f iff c is a geodesic.

"$\Leftarrow$" ✓ c geod $\Rightarrow \frac{D}{dt}\frac{dc}{dt} = 0$ "$\Rightarrow$": $\frac{D}{dc}\frac{dc}{dt} \neq 0$ at t_0 $\qquad V(t) = a(t) \cdot \frac{D}{dt}\frac{dc}{dt}$ $\Rightarrow \frac{1}{2}\frac{dE}{ds}(0) \neq 0$

Prop: Let $\gamma(t)$ be a geodesic. $f(s,t)$ is a proper variation of γ

then $\frac{1}{2}\frac{d^2}{ds^2} E(0) = -\int_0^a \langle V, \frac{D^2 V}{dt^2} + R(\frac{d\gamma}{dt}, V)\frac{d\gamma}{dt} \rangle dt$

(non proper is also in text.)

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$$\frac{1}{2}\frac{d}{ds}E(s) \overset{\text{proper}}{=} -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$$

$$\frac{1}{2}\frac{d^2}{ds^2}E(0) = -\int_0^a \frac{d}{ds}\left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$$

$$= -\int_0^a \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \overset{\gamma \text{ a geod.}}{\cancel{\frac{D}{dt}\frac{\partial f}{\partial t}}} \right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle$$

$$= -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$$

Recall: $\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$

$$= -\int_0^a \left\langle \frac{\partial f}{\partial s}, \underbrace{\frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t}}_{\frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s}} \right\rangle + \left\langle \frac{\partial f}{\partial s}, R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} \right\rangle dt$$

$$\overset{s=0}{=} -\int_0^a \left\langle V, \frac{D^2V}{dt} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle dt$$

s=0: $\frac{\partial f}{\partial s} = V$

$\frac{\partial f}{\partial t} = \frac{d\gamma}{dt}$

□

Assume γ is a minimizing geodesic. $\gamma : [0,a] \to M$ $\gamma(0) = p$, $\gamma(a) = q$

$\ell(\gamma) = d(p,q)$ $E(\gamma) \le E(f_s)$ $\forall s$

$\forall$ proper variations f $\Rightarrow \frac{d^2}{ds^2}E \ge 0$

Myers' thm

M Riem. mfd

Define diam$(M) = \sup\{ d(p,q) : p, q \in M \}$

Thm: Let M^n be a complete Riem mfd with $Ric(v,v) \ge (n-1)k$

$\forall v$ s.t. $|v| = 1$ for some $k > 0$

then diam$(M) \le \frac{\pi}{\sqrt{k}}$.

Ex: $S^n(r)$ ← sphere of radius r in $\mathbb{R}^{n+1}$

$Ric(v,v) = \frac{(n-1)}{r^2}$ diam$(S^n(r)) = \pi r$

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Set $k = \frac{1}{r^2}$ $\quad Ric \equiv (n-1)k$ $\quad$ & $\quad diam(S^n(r)) = \frac{\pi}{\sqrt{k}}$

<u>Cheng's Maximal diameter thm</u>

If $Ric \geq (n-1)k$ and $diam(M) = \frac{\pi}{\sqrt{k}}$ then M is isometric to $S^n(r)$.

<u>Pf:</u> (Myers)

Let $p, q \in M$. Since M is complete $\exists$ a minimizing geodesic γ from p to q

s.t. $\ell(\gamma) = d(p,q)$.

Parametrize γ s.t. $|\gamma'| = d(p,q)$.

so $\gamma: [0,1] \to M$ $\quad \gamma(0) = p$, $\gamma(1) = q$.

<u>nts:</u> $d(p,q) \leq \frac{\pi}{\sqrt{k}}$

Let $E_1, \dots, E_{n-1}$ be $(n-1)$-perpendicular, & orthonormal linearly indep. parallel fields along γ.

Let $V_i(t) = \sin(\pi t) E_i(t)$ $\qquad V_i(0) = V_i(1) = 0$.

Proper var. for each i.

$E_i(s)$ = energy of variation coming from V_i

$$\frac{1}{2} E_i''(s) = -\int_0^1 \langle V_i, \frac{D^2}{dt^2} V_i + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt} \rangle dt$$

Since γ is minimizing, $E_i''(0) \geq 0$ $\forall i$.

$$\frac{D^2}{dt^2} = \frac{D}{dt}\frac{D}{dt}(\sin(\pi t) E_i) = -\pi^2 \sin(\pi t) E_i$$

$$\langle R\left(\frac{d\gamma}{dt}, V_i\right)\frac{d\gamma}{dt}, V_i \rangle = \left|\frac{d\gamma}{dt}\right|^2 |V_i|^2 K(\sigma_i) = d(p,q)^2 \sin^2(\pi t) K(q)$$

σ_i is the two plane spanned by $\{\frac{D\gamma}{dt}, E_i\}$

$$\frac{1}{2} E_i''(0) = -\int_0^1 -\pi^2 \sin(\pi t) E_i + d(p,q)^2 \sin^2(\pi t) \quad K(\sigma_i)\, dt$$

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$$0 \le \frac{1}{2}\sum_{i=0}^{n-1} E_i(0) = -\int_0^1 -\pi^2 (n-1)\sin^2(\pi t) + d(p,q)^2 \sin^2(\pi t)\, Ric\left(\frac{\frac{d\gamma}{dt}}{|\frac{d\gamma}{dt}|}, \frac{\frac{d\gamma}{dt}}{|\frac{d\gamma}{dt}|}\right)$$

By hypothesis $Ric\left(\frac{\frac{d\gamma}{dt}}{|\frac{d\gamma}{dt}|}, \frac{\frac{d\gamma}{dt}}{|\frac{d\gamma}{dt}|}\right) \ge (n-1)k.$

$$\Rightarrow 0 \ge \int_0^1 \sin^2(\pi t)\left[Ric\left(\frac{\frac{d\gamma}{dt}}{|\frac{d\gamma}{dt}|}, \frac{\frac{d\gamma}{dt}}{|\frac{d\gamma}{dt}|}\right)(d^2(p,q))^2 - (n-1)\pi^2\right]dt$$

$$\ge \int_0^{1} \sin^2(\pi t)\left[(n-1)k\, d^2(p,q) - (n-1)\pi^2\right]dt$$

$$\Rightarrow k\, d^2(p,q) - \pi^2 \le 0 \Rightarrow d^2(p,q) \le \frac{\pi^2}{k} \Rightarrow d(p,q) \le \frac{\pi}{\sqrt{k}}$$

$$\Rightarrow \text{diam}(p,q) \le \frac{\pi}{\sqrt{k}}.$$ □

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Last time: Myers's' thm

Thm: If M^n complete, $Ric \ge (n-1)k$, $k>0$, then $\text{diam}(M) \le \frac{\pi}{\sqrt{k}}$.

Cor: If M^n complete, $Ric \ge (n-1)k$, $k>0$, then M is compact.

ex: Paraboloid $z = x^2+y^2$ complete, $k>0$ but noncompact

Cor: If M^n compact, and $Ric > 0$, then $\pi_1(M)$ is finite.

Pf. M compact, $Ric > 0$ $\exists k$ s.t. $Ric \ge (n-1)k$, $k>0$

Let $\tilde{M}$ be the universal cover of M give $\tilde{M}$ the covering metric.

$$\begin{tikzcd} \tilde{M} \arrow[d, "\pi"] \\ M \end{tikzcd}$$

$\Rightarrow Ric_{\tilde{M}} \ge (n-1)k.$

Lemma M is complete $\Leftrightarrow \tilde{M}$ is complete. (path lifting)

$\Rightarrow \tilde{M}$ is complete $Ric_{\tilde{M}} \ge (n-1)k \overset{\text{Myers'}}{\Rightarrow} \tilde{M}$ is compact $\Rightarrow \pi_1(M)$ is finite □

ex: $$\begin{tikzcd} S^n \arrow[d] \\ \mathbb{RP}^n \end{tikzcd}$$ $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$

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Recall: If M compact and M has nonpositive sectional curvature then $\pi_1(M)$ is infinite.

Cor: If M^n is a compact mfd, M can not have both a metric with $Ric > 0$ and some other metric with nonpos sectional curvature.

Lohkampf: M compact, $\dim(M) \geq 4$ then M has a metric on it with $Ric < 0$.

Scalar curvature: $Scal = \sum_{i=1}^{n} Ric(E_i, E_i)$ E_i o.n. basis

Yamabe problem (Aubin, Schoen) Every compact mfd has a metric with constant scalar curvature.

Einstein Equation: Ricci curv. is const

n=3: Const sectional curv.

Q: Does every comp. 5-mfd admit an Einstein metric?

Analysis on Riemannian manifolds

Riemannian Volume

Let M be a Riem. mfd.

Let $\varphi : U \to V \subseteq M$ be a coordinate chart

$\frac{\partial}{\partial x_i}$ are coordinate vector field.

Let $e_1, \to e_n$ be an o.n. basis of T_pM.

$\frac{\partial}{\partial x_i}(p) = \sum_{k=1}^{n} a_{ik} e_k$ The volume of parallelepiped spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is $|\det(a_{ij})|$

$|\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}|$

$v_1 = a_{11} e_1 + a_{12} e_2$

$v_2 = a_{21} e_1 + a_{22} e_2$

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$$g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \langle \sum_{k=1}^{n} a_{ik} e_k, \sum_{l=1}^{n} a_{jl} e_l \rangle = \sum_{k=1}^{n} a_{ik} a_{jk}$$

$$\det(g_{ij}) = \det(a_{ij})^2 \quad \leftarrow [g] = [a]\cdot[a]^T$$

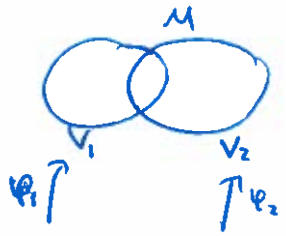
$$|\det(a_{ij})| = \sqrt{\det(g_{ij})}$$

Define <u>volume</u> of V $[\varphi: U \to \underline{V}]$ $\quad \mathrm{Vol}(V) = \int_U \sqrt{\det(g_{ij})} \circ \varphi^{-1} \, dx_1 \ldots dx_n$

<u>Recall:</u> $\psi: V_{\text{open}}^{\subset \mathbb{R}^n} \to U_{\text{open}}^{\subset \mathbb{R}^n}$ diffeo. (Change of variables)

Then $\int_U dy_1 \ldots dy_n = \int_V \underset{|\det(d\psi)|}{|\mathrm{Jac}(\psi)|} \, dx_1 \ldots dx_n$

<u>exer</u>



[$\mathrm{Vol}(V_1 \cap V_2)$ indep. of choice of chart - (φ_1, φ_2)]

Show $\int_{V_1 \cap V_2} \sqrt{\det(g_{ij})} \circ \varphi_1 \, dx_1 \ldots dx_n = \int_{V_1 \cap V_2} \sqrt{\det(\tilde{g}_{ij})} \circ \varphi_2 \, dy_1 \ldots dy_n$

$$\tilde{g}_{ij} = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle$$

<u>Def:</u> Let M be a Riem. mfd.

Let ψ_α be a cover of M by coordinate charts, f_α partition of unity subordinate to M.

$$\mathrm{Vol}(M) = \sum_\alpha \int_{\psi_\alpha} f_\alpha \, d\mathrm{vol}_g$$

$$d\mathrm{vol}_g = \sqrt{\det(g_{ij})} \, dx_1 \ldots dx_n$$

<u>Rem:</u> $d\mathrm{vol}_g$ does define a non-degenerate n-form if M is orientable.

$$d\mathrm{vol}_g = \sqrt{\det(g_{ij})} \, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

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Recall: $f: M \to \mathbb{R}$.

The gradient of f ∇f is unique vector field on M s.t.

$$\langle \nabla f, X \rangle = df(X) = X(f) \quad \forall X$$

The Hessian of f $\quad \mathrm{Hess}([illegible]) f(X,Y) = \langle \nabla_X \nabla f, Y \rangle = \langle \nabla_Y \nabla f, X \rangle$

symmetric (0,2)-tensor

Let $\{E_i\}$ be o.n. basis of T_pM

$$\underset{\substack{\uparrow \\ \text{function}}}{\Delta f} = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla f, E_i \rangle \quad \leftarrow$$

Laplace-Beltrami operator
Riemannian Laplacian

$$\Delta_g f = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla f, E_i \rangle = \sum_{i=1}^{n} E_i \langle \nabla f, E_i \rangle - \langle \nabla f, \nabla_{E_i} E_i \rangle$$

$$= \sum_{i=1}^{n} E_i E_i(f) - (\nabla_{E_i} E_i)(f).$$

So $M = \mathbb{R}^n$, E_i standard basis. $\quad \Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Idea: Δ_g has similar properties to $\Delta_{\mathbb{R}^n}$

⓪ $\Delta_g(f+h) = (\Delta_g f) + (\Delta_g h)$

① Local max of f at p $\quad (\nabla f)_p = 0 \quad (\Delta f)(p) \le 0$

Local min $\quad \nabla f = 0 \quad \Delta f \ge 0$

Divergence Thm: If V is a vector field on M

$$\mathrm{div}\, V = \sum_{i=1}^{n} \langle \nabla_{E_i} V, E_i \rangle \quad E_i \text{ o.n. basis}$$

e.g. $\mathrm{div}(\nabla f) = \Delta f$.

Thm: If M oriented and V has compact support, then

$$\int_M \mathrm{div}\, V \, dvol_g = 0$$

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If M ~~orientable~~ compact, $f : M \to \mathbb{R}$ $\quad \int \Delta f \, dvol_g = 0.$

<u>Pf:</u> Stokes Thm.

<u>Green's formula:</u>

$f_1, f_2 : M \to \mathbb{R}$ $\quad M$ compact

$$\int_M \Delta f_1 \cdot f_2 \, dvol_g = -\int_M \langle \nabla f_1, \nabla f_2 \rangle = \int_M f_1 \cdot \Delta f_2 \, dvol_g$$

<u>Pf:</u> $\operatorname{div}(f_1 \nabla f_2) = \langle \nabla f_1, \nabla f_2 \rangle + f_1 \Delta f_2.$

M cpct mfd

$\Delta : C^\infty(M) \to C^\infty(M)$

$L^2(M) =$ fcts s.t. $\int_M f^2 \, dvol_g < \infty$

$\cup\!\vert$

$C^\infty(M)$

$\downarrow$ Hilbert space $\quad \langle f, g \rangle = \int_M f \cdot g \, dvol_g$

Green's formula $\Longrightarrow$ Δ is self-adjoint

Spectrum of Δ.

<u>Def:</u> An eigenfunction of Δ with eigenvalue $-\lambda$ is a fct f s.t. $\Delta f + \lambda f = 0.$ ← Spectral geometry

<u>Colloq. on Friday:</u> Can you hear the shape of a drum?

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<u>Last time:</u> M compact Riem mfd.

$$\Delta : \overset{L^2(M)}{\overset{\cup\vert}{C^\infty(M)}} \to \overset{L^2(M)}{\overset{\cup\vert}{C^\infty(M)}}$$

If E_i is an o.n. basis of T_pM

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$$\Delta f(p) = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla f, E_i \rangle = \sum_{i=1}^{n} \left(E_i E_i(f) - (\nabla_{E_i} E_i)(f) \right)$$

An eigenfunction is a fct f s.t. $\Delta f + \lambda f = 0$ for some $\lambda \in \mathbb{R}$, $f \neq 0$.

Prop: If M cpct then $\lambda = 0$ is an eigenvalue of Δ and for any other eigenfct $\lambda \geq 0$.

Pf: f = constant $\Delta f = 0$ $\lambda = 0$.

Suppose $\Delta f + \lambda f = 0$.

Then by Green's Formula

$$\int_M \Delta f \, f \, d\,\mathrm{vol}_g = -\int \langle \nabla f, \nabla f \rangle \, d\,\mathrm{vol}_g = -\int |\nabla f|^2 \, d\,\mathrm{vol}_g$$

$$\parallel$$

$$\int_M (-\lambda f) f \, d\,\mathrm{vol}_g$$

$$\parallel$$

$$-\lambda \int f^2 \, d\,\mathrm{vol}_g$$

$$\Rightarrow \lambda \int_M f^2 \, d\,\mathrm{vol}_g = \int |\nabla f|^2 \, d\,\mathrm{vol}_g \quad \Rightarrow \lambda \geq 0 \quad \lambda = 0 \text{ iff } f \text{ is constant.}$$

□

In general, $-\Delta$, discrete spectrum

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

λ_i eigenvalues (possibly with multiplicity)

$\lambda_1(M)$ = smallest possible eigenvalue.

There is a relationship between $\lambda_1(M)$ & curvature.

Prop: (Bochner formula) Let $f \in C^3(M)$. Then

$$\frac{1}{2} \Delta |\nabla f|^2 = \mathrm{Ric}(\nabla f, \nabla f) + |\mathrm{Hess}\, f|^2 + \langle \overset{\nabla\Delta}{\cancel{\Delta\nabla}} f, \nabla f \rangle$$

where $|A|^2$ is Euclidean norm of a $(0,2)$-tensor A.

(Defined below.)

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$$Rc\,(\nabla f, \nabla f) \geq (n-1)\,k\,|\nabla f|^2$$

$\xRightarrow{\text{Bochner}}$

$$\tfrac{1}{2}\Delta|\nabla f|^2 \geq (n-1)k\,|\nabla f|^2 + \frac{\lambda^2 f^2}{n} + \langle \nabla(-\lambda f), \nabla f\rangle$$

$$\geq (n-1)k\,|\nabla f|^2 + \frac{\lambda^2 f^2}{n} - \lambda|\nabla f|^2$$

Integrate both sides w.r.t. $d\,vol_g$

$$\underset{\displaystyle 0}{\underset{\shortparallel}{\int \tfrac{1}{2}\Delta|\nabla f|^2 d\,vol_g}} \geq \int_M \left((n-1)k|\nabla f|^2 + \frac{\lambda^2 f^2}{n} - \lambda|\nabla f|^2\right) d\,vol_g$$

$$\geq \int_M ((n-1)k - \lambda)\,|\nabla f|^2\, d\,vol_g + \underset{\displaystyle \int_M \frac{\lambda}{n}|\nabla f|^2 d\,vol_g}{\underset{\shortparallel}{\int_M \frac{\lambda^2 f^2}{n}\, d\,vol_g}}$$

$$0 \geq \left((n-1)k - \lambda + \frac{\lambda}{n}\right)\int_M |\nabla f|^2 d\,vol_g$$

$$\Rightarrow (n-1)k - \lambda + \frac{\lambda}{n} \leq 0 \quad \Rightarrow (n-1)k - \left(\frac{n-1}{n}\right)\lambda \leq 0 \quad \Rightarrow nk \leq \lambda.$$

□

Pf of Bochner Formula

$$\tfrac{1}{2}\Delta|\nabla f|^2 = Ric(\nabla f, \nabla f) + |\mathrm{Hess}\, f|^2 + \langle \nabla\Delta f, \nabla f\rangle.$$

Pf: Let $p \in M$. Pick geodesic normal coordinates at p.

o.n. basis $\{E_i\}$ s.t. $\nabla_{E_i} E_i = 0$ at p.

Then $\tfrac{1}{2}\Delta|\nabla f|^2 = \tfrac{1}{2}\sum_{i=1}^n E_i(E_i(|\nabla f|^2)) = \tfrac{1}{2}\sum_{i=1}^n E_i(E_i(\langle \nabla f, \nabla f\rangle))$

$$= \tfrac{1}{2}\sum_{i=1}^n E_i\left(\langle \nabla_{E_i}\nabla f, \nabla f\rangle + \langle \nabla f, \nabla_{E_i}\nabla f\rangle\right)$$

$\mathrm{Hess} f(E_i, \nabla f) = \mathrm{Hess}\, f(\nabla f, E_i)$

Hessian is symmetric

$$= \sum_{i=1}^n E_i(\langle \nabla_{E_i}\nabla f, \nabla f\rangle)$$

$$= \sum_{i=1}^n \langle \nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle + \langle \nabla_{E_i}\cancel{\nabla f, \nabla_{E_i}}^{=0} E_i\rangle = \sum_{i=1}^n \langle \nabla_{E_i}\nabla_{\nabla f}\nabla f, E$$

$$= \sum_{i=1}^n \Big(\underset{①}{R(\nabla f, E_i, \nabla f, E_i)} + \underset{②}{\langle \nabla_{\nabla f}\nabla_{E_i}\nabla f, E_i\rangle} - \underset{③}{\langle \nabla_{[\nabla f, E_i]}\nabla f, E_i\rangle}\Big)$$

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① $= Ric(\nabla f, \nabla f)$

② $\sum_{i=1}^{n} < \nabla_f \nabla_{E_i} \nabla f, E_i > = \sum_{i=1}^{n} \nabla f (< \nabla_{E_i} \nabla f, E_i >) - < \nabla_{E_i} \nabla f, \underset{0}{\cancel{\nabla_{\nabla f}} E_i} >$

$= \nabla f (\sum_{i=1}^{n} < \nabla_{E_i} \nabla f, E_i >) = \nabla f (\Delta f)$

$= < \nabla \Delta f, \nabla f >$

③ $-\sum_{i=1}^{n} (\nabla_{[\nabla f, E_i]} \nabla f, E_i)$ $[\nabla f, E_i] = \underset{0}{\cancel{\nabla_{\nabla f} E_i}} - \nabla_{E_i} \nabla f$

$= \sum_{i=1}^{n} < \nabla_{\nabla_{E_i} \nabla f} \nabla f, E_i > = \sum_{i=1}^{n} Hess\, f (\nabla_{E_i} \nabla f, E_i)$

↙symmetry

$= \sum_{i=1}^{n} Hess\, f (E_i, \nabla_{E_i} \nabla f)$

$= \sum_{i=1}^{n} < \nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) = |Hess\, f|^2$

$< L(E_i), L(E_i) >$

$Hess f(X,Y) = < \nabla_X \nabla f, Y >$ $L : X \longrightarrow \nabla_X \nabla f$

$\square$