Integral Equations

Let $x = (x_1, \ldots, x_d)$ denote points in d-dim. euclid. space $\mathbb{R}^d$, $d \geq 2$

First order partial derivative operators are denoted $\partial = (\partial_1, \ldots, \partial_d)$.

Multiindices are denoted $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with orders $|\alpha| = \alpha_1 + \ldots + \alpha_d$

so that $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_d^{\alpha_d}$ is a monomial of degree $|\alpha|$ and

$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d}$ is a differential operator of order $|\alpha|$.

In this course, we are generally interested in constant coefficient elliptic operators that are homogenous in the order, i.e.

$$L = L(\partial) = \sum_{|\alpha| = \ell} a_\alpha \partial^\alpha \quad \text{for} \quad a_\alpha \in \mathbb{C}.$$

s.t. $$L(\xi) = \sum_{|\alpha| = \ell} a_\alpha \xi^\alpha \neq 0 \text{ for all } 0 \neq \xi \in \mathbb{R}^d.$$

(Symbol of L)

L is strongly elliptic if $\exists$ a number c s.t.

$\mathrm{Re}(c\, L(\xi)) > 0$, $0 \neq \xi \in \mathbb{R}^d$. Thus, a strongly elliptic operator is of even order $\ell = 2m$, $m \in \mathbb{N}$.

An elliptic operator with real coefficients may always be taken to be strongly elliptic and thus of even order.

A classical argument using Rouché's Theorem shows that any elliptic operator when $d \geq 3$ must be of even order (S. Agmon p. 33-34).

The Cauchy-Riemann operator $\partial_{\bar{z}} = \frac{\partial_1 + i\partial_2}{2}$ is elliptic.

We are mostly interested in real coefficient elliptic operators, the fundamental example being the Laplacian $\Delta = \partial_1^2 + \dots + \partial_d^2$. (Strongly elliptic)

The Laplacian

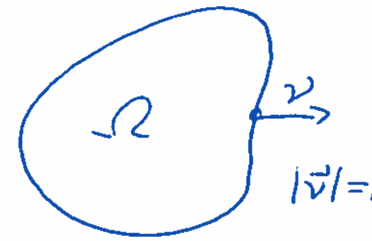
Given a function f, the inhomogeneous equation $\Delta u = f$ is known as Poisson's equation. The homogeneous version $\Delta u = 0$ is Laplace's equation. Solution to Laplace's equation are called "harmonic functions".

Let $\Omega \subset \mathbb{R}^d$ denote a domain (open & connected set) and $\partial\Omega$ its boundary. Given a function g define on the boundary the Dirichlet problem for Laplace's equation $\Delta u = 0$ in Ω is to find a solution u, which when restricted or defined on the boundary in some continuous way equals g; or is to find a "continuous" extension u of g into Ω which solves $\Delta u = 0$ in Ω. g is called the Dirichlet data.

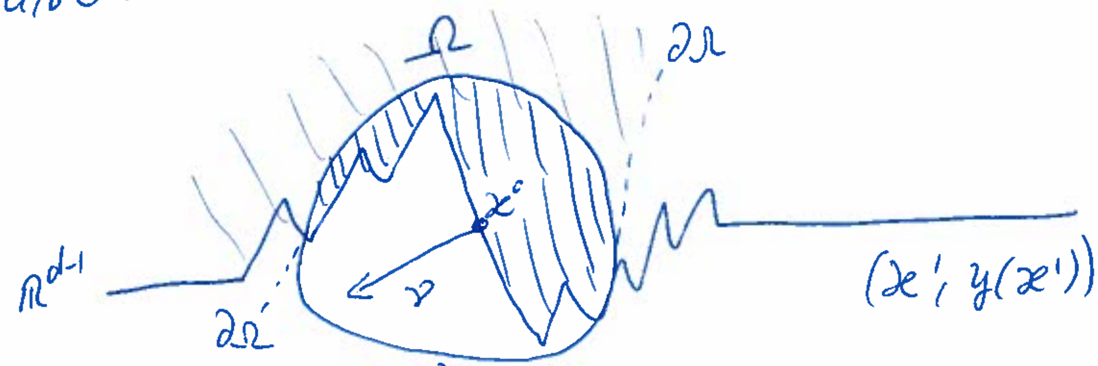
The Neumann problem is to find a solution u so that the normal derivative $\partial_\nu u = \nu u = \sum_{j=1}^{d} \nu_j \partial_j u = g$ on $\partial\Omega$.

Here ν is a differential operator with variable coefficients $\vec{\nu} = (\nu_1, \dots, \nu_d)$ that form the unit outer normal vector at each point on $\partial\Omega$.

g is called the Neumann data. In both problems g is used to represent the boundary values we are trying to achieve.



Our domains will generally be bounded, Lipschitz domains. This means that for each point $x^0 \in \partial\Omega$ after a suitable rotation of the rectangular coordinate system $\exists$ Ball $B_r(x^0)$ open ~~ste~~ and a real-valued function $y(x_1, \dots, x_{d-1})$ satisfying a Lipschitz condition $|y(a) - y(b)| \leq M|a-b|$ $\forall a, b \in \mathbb{R}^{d-1}$ such that $\Omega \cap B_r(x^0) = \{x \mid x_d > y(x_1, \dots, x_{d-1})\} \cap B_r(x^0)$



Put $x' = (x_1, \dots, x_{d-1})$

Surface measure ds on $\partial\Omega$ is given locally as $ds = \sqrt{1 + |\nabla y(x')|^2}\, dx'$

and the outer unit normal vector satisfies $\vec{\nu}\, ds = (\nabla y, -1)\, dx'$ a.e. dx'

(Denjoy, Rademacher, Stepanov - Thm; E. Stein: Singular Integrals and differentiability properties of functions pp. 250, 270)

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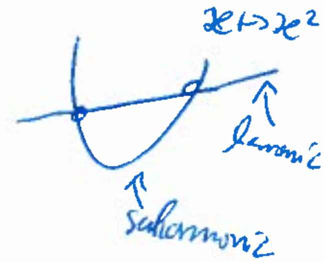
Since $\partial\Omega$ is compact, a finite number of such balls covers the boundary and the largest M in this cover can be taken as the Lipschitz constant for Ω. The Lipschitz constant together with the number of "comparable" (e.g. the same size) balls can be called the Lipschitz nature of the domain of Ω.

A function u is called subharmonic in Ω if $\Delta u \geq 0$ throughout Ω.



If Ω is a $\mathcal{C}^1$-domain. ($\vec{\nu}$ varies continuously on $\partial\Omega$) and $u \in \mathcal{C}^2(\overline{\Omega})$ then the Gauss-divergence theorem applies to yield $\int_\Omega \Delta u\,dx = \int_\Omega \operatorname{div}(\nabla u)\,dx = \int_{\partial\Omega} \vec{\nu}\cdot\nabla u\,ds = \int_{\partial\Omega} \partial_\nu u\,ds.$

Suppose the origin $O \in \Omega$ and take all balls around the origin $B_r(0)$ with $B_r(0) \subset \Omega$ and suppose $u \in \mathcal{C}^2(\Omega)$ is subharmonic in Ω.

Introduce polar coordinates $x = r\theta$ where $\theta = \frac{x}{|x|} \in \mathbb{S}^{n-1} = \partial B_1(0)$ and $r = |x|$ for $x \neq 0$.

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The relationship between surface area of the sphere

$|\partial B_r(0)| = r^{d-1}|\partial B_1(0)| = r^{d-1}|\mathbb{S}^{d-1}|$ generalizes immediately to subsets $ds_r(r\theta) = r^{d-1}\,ds_1(\theta)$ where ds_r is the surface measure on $\partial B_r(0)$. Spherical averages are indicated by $\fint_{\partial B_r(0)} u\,ds_r$

$$\fint_{\partial B_r(0)} u(r\theta)\,dS_r = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} u(r\theta)\,dS_r(r\theta)$$

$$= \frac{1}{r^{d-1}|S^{d-1}|}\int_{S^{d-1}} u(r\theta)\,r^{d-1}\,dS_1(\theta)$$

$$= \fint_{S^{d-1}} u(r\theta)\,dS_1(\theta).$$ Differentiate wrt. r.

$$\frac{\partial}{\partial r}\fint_{\partial B_r(0)} u(r\theta)\,dS_r = \fint_{S^{d-1}} \theta\cdot\nabla u(r\theta)\,dS_1(\theta) = \fint_{\partial B_r(0)} \partial_r u^{(r\theta)}\,dS_r(r\theta)$$

Apply Gauss $\Rightarrow$ $\cancel{\frac{\partial}{\partial r}\fint_{\partial B_r(0)} u}$ $\frac{\partial}{\partial r}\fint_{\partial B_r(0)} u\,dS_r = \frac{1}{r^{d-1}|S^{d-1}|}\int_{B_r(0)} \Delta u\,dx$

Assume $\Delta u \geq 0 \Rightarrow \frac{\partial}{\partial r}\fint_{\partial B_r(0)} u\,dS_r \geq 0$

Since $\lim_{r\to 0}\fint_{\partial B_r(0)} u\,dS_r = u(0)$ and the averages are nondecreasing, it follows

for sub-harmonic functions $u(0) \leq \fint_{\partial B_r(0)} u\,dx \quad \forall\, B_r(0)\subset\Omega$.

For harmonic u, one obtains equality. By translations one obtains

the <u>mean value theorems</u>

(8.1) $\quad u(x) \leq \fint_{\partial B_r(x)} u\,dS_r, \quad u$ subharmonic in Ω.

and $u(x) = \fint_{\partial B_r(0)} u \, ds_r \quad \forall x \in \Omega, \; B_r(x) \subset \Omega$.

Note that v subharmonic and u harmonic with $v = u$ on some

sphere $\partial B_r(x) \subset \Omega \Rightarrow v(x) \le \fint_{\partial B_r(x)} u \, ds = u(x)$

$dx = ds_r(r\theta)\, dr = r^{d-1} ds_1(\theta)\, dr$ and integrating (8.1) wrt r.

We get $\int_0^R r^{d-1} |S^{d-1}| u(x) dr \le \int_0^R \int_{\partial B_r(x)} u \, ds_r \, dr$

$$= \int_{B_R(x)} u(y)\, dy$$

One obtains <u>solid mean value theorems</u>

$$u(x) \le \fint_{B_R(x)} u(y)\, dy \text{ with "$=$" for harmonic } u.$$

Note that $|B_R(x)| = \frac{1}{d} R^d |S^{d-1}|$

<u>The strong maximum principle</u> for subharmonic u in Ω

says that u is either constant in Ω or never achieves its $\sup_{\Omega} u$.

in Ω.

If $u(x_0^\circ) = \sup_\Omega u$, $x^\circ \in \Omega$, then $u(x_0) \leq \fint_{B(x_0^\circ)} u \, dx \leq u(x_0^\circ)$ 01/22

By continuity $u(x) = u(x_0)$ $\forall x \in B(x^\circ)$. By arguing like this, the nonempty set $S = \{x \in \Omega \mid u(x) = u(x^\circ)\}$ is open, but it is also closed since u is continuous. Thus, by connectedness of Ω, $S = \Omega$.

Otherwise $u(x_0) = \sup_\Omega u$ for no point.

If now Ω is bounded, then $\bar{\Omega}$ is compact and if $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is subharmonic, then by the strong maximum principle, u achieves its maximum somewhere on the boundary, i.e. $\sup_\Omega u = \max_{\partial\Omega} u$.

If $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is harmonic then u and $-u$ are both subharmonic and it follows that $\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u$ $\forall x \in \Omega$.

which by itself is called the weak maximum principle for harmonic functions.

From this, one obtains a uniqueness theorem for the Dirichlet-Problem posed with continuous data for Laplace's equation and also Poisson's equation

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Thm If $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $\Delta u = \Delta v$ in Ω

and $u = v$ on $\partial\Omega$, then $u = v$ in $\bar{\Omega}$. (Gilbarg-Trudinger, Elliptic PDE's of 2nd order)

Note that if $\Delta u = 0$, $\Delta v \geq 0$ with $u = v$ on $\partial\Omega$

$\Rightarrow \Delta(v-u) \geq 0$, so $v(x) - u(x) \leq 0 \;\forall x$ in Ω

i.e $v(x) \leq u(x)$ in $\bar{\Omega}$ (sub-harmonic)

What about existence?

Given a continuous function on $\partial\Omega$ is there a solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$

s.t. $\Delta u = 0$ and $u = g$ on $\partial\Omega$.

Two ways to go about this could be:

(1) Find solutions $\Delta u = 0$ with Dirichlet boundary values tha approximate g.

(2) Find extensions of g into Ω so that the resulting Δg approximate 0 in Ω.

An example of (1) is using separation of variables in polar coordinates to obtain $\boxed{d=2}$ separated solutions $r^{|n|} e^{in\theta}$, $n \in \mathbb{Z}$ i.e. in the complex variable $z = re^{i\theta}$, solutions can be expressed as $z^n, 1, \bar{z}^n$ $n \in \mathbb{N}$.

In the open unit disk $\mathbb{D} = B_1(0)$ one considers the series of separated solutions

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(12.1) $\left\{ u(z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \right.$ Choosing $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) e^{-int}\,dt$

the fourier coefficients of $g \in \mathcal{C}(\partial \mathbb{D})$ one approximates g by Fourier-partial-sums and sees that u and all derivatives of all orders converge uniformly on $B_R(0)$ $\forall\, 0<R<1$.

Thus $\Delta u = 0$ in $\mathbb{D}$, but classical example of continuous g show that $u(e^{i\theta}) = g(\theta)$ almost everywhere on $\partial\mathbb{D}$, so that u would not be in $\mathcal{C}(\overline{\mathbb{D}})$. Examples (A. Zygmund, trigonometric series)

However, when $r<1$ the sum and integral in (12.1) may be interchanged and the series computed explicitely, yielding the following $(z = re^{i\theta})$

(13.1)
$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,\frac{1-|z|^2}{|e^{it}-z|^2}\,dt$$

$\forall z \in \mathbb{D}$. If now $u(e^{i\theta})$ is defined to be $g(\theta)$, it can be proved that $u \in \mathcal{C}(\overline{\mathbb{D}})$. (Rudin R+C, pp. 233-234.)

(13.1) Is the Poisson-Integral of g for the disk. Note that any $g \in L^1(dt)$ in (13.1) will produce a harmonic Poisson integral on $\mathbb{D}$. But now

$\lim_{z \to e^{i\theta}} u(z)$ is not so clear.

Another way to take sums of harmonic functions in Ω so that 01/22

g on the boundary $\partial\Omega$ is to sum translations of the fundamental

solution $\Gamma(x)$, $x \in \mathbb{R}^d \setminus \{0\}$ for Δ.

Here $\Gamma(x) = \begin{cases} \dfrac{1}{(2-d)\,|S^{d-1}|} \cdot \dfrac{1}{|x|^{d-2}}, & d \geq 3 \\ \dfrac{1}{2\pi} \log |x|, & d = 2 \end{cases}$

Using $\partial_j |x|^p = p\, x_j\, |x|^{p-2}$ $1 \leq j \leq d$ it follows that Γ is harmonic outside the origin.

Define translations of Γ, Γ^x by $\Gamma^x(y) = \Gamma(x-y)$, $y \in \mathbb{R}^d \setminus \{x\}$ 01/24

The Γ^x may be considered to be distributions with the property that

$\Delta \Gamma^x = \delta_x$, where δ_x is the Dirac-δ-distribution at x, i.e.

let $C_0^\infty(\mathbb{R}^d)$ denote the C^∞-functions with compact support, ~~topolo~~

Those are denoted $\mathcal{D}(\mathbb{R}^d)$ and topologized and called test functions.

Then δ_x is the bounded linear functional that acts as

$$\langle \delta_x, \varphi \rangle = \varphi(x) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Any locally integrable function $f \in L^1_{loc}(\mathbb{R}^d)$ acts as a distribution by $\langle f,\varphi\rangle = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx$

The derivatives of f as a distribution may be defined now by

$$\langle \partial_j f,\varphi\rangle = -\int_{\mathbb{R}^d} f\,\partial_j\varphi\,dx$$

(Yoshida, Functional Analysis pp. 46-52)

Note that Γ and its first derivatives are locally integrable as seen in polar coordinates.

$d\geq 3$
$$\int_{|x|<R}|\Gamma(x)|\,dx = \frac{1}{(d-2)|S^{d-1}|}\int_0^R\int_{S^{d-1}}\frac{1}{r^{d-2}}\,r^{d-1}\,d\theta\,dr$$

where $d\theta$ is $dS_1(\theta)$, the surface measure of S^{d-1}! $\approx \int_0^R r\,dr<\infty.$

Thus
$$\langle \Delta\Gamma^x,\varphi\rangle = \int_{\mathbb{R}^d}\Gamma^x\Delta\varphi\,dy = -\int_{\mathbb{R}^d}\nabla\Gamma^x\cdot\nabla\varphi\,dy$$

$$= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus B_\varepsilon(x)}\nabla\Gamma^x\cdot\nabla\varphi\,dy = +\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)}\partial_\nu\Gamma^x\varphi\,dS + 0$$

$$= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)}\frac{(y-x)\cdot\nu_y}{|S^{d-1}|\,|y-x|^d}\varphi(y)\,dS(y) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{d-1}|S^{d-1}|}\int_{\partial B_\varepsilon(x)}\varphi(y)\,dS(y)$$

$\nu_y = \frac{y-x}{|y-x|}$, $|y-x|=\varepsilon$

$= \varphi(x)$ $\quad\varphi$ cont., so $\Delta\Gamma^x = \delta_x$ in the sense of distributions (s.d.)

Suppose now $u,v\in C^2(\bar\Omega)$ and Ω is a domain that permits Gauss-divergence Theorem $\left(\int_\Omega \operatorname{div}\vec{w}\,dx = \int_{\partial\Omega}\nu\cdot\vec{w}\,dS\right)$

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Then $\int_\Omega \Delta\bar{u}\cdot v\,dx = -\int_\Omega \nabla\bar{u}\cdot\nabla v\,dx + \int_{\partial\Omega} \underbrace{\nabla\bar{u}\cdot\nu}_{=\partial_\nu\bar{u}}\, v\,ds$

is <u>Green's First Identity</u>. <u>Green's Second identity</u> follows by interchanging $\bar{u}$ and v and subtracting

$$(16.1)\quad \int_\Omega (\Delta\bar{u}\,v - \bar{u}\,\Delta v)\,dx = \int_{\partial\Omega} (\partial_\nu\bar{u}\,v - \bar{u}\,\partial_\nu v)\,ds$$

Let $x \in \Omega$ and $u = \Gamma^x$. By rewriting (16.1) over $\Omega \setminus B_\varepsilon(x)$ or by writing v as a sum of functions, s.t. one supported outside $B_\varepsilon(x)$ and using $\Delta\Gamma^x = \delta_x$, one obtains <u>Green's representation formula</u>

$$(17.1)\quad v(x) = \int_\Omega \Gamma^x \Delta v\,dy + \int_{\partial\Omega} \partial_\nu\Gamma^x v\,ds - \int_{\partial\Omega} \Gamma^x \partial_\nu v\,ds \quad \text{for } x \in \Omega$$

If v is harmonic, then v is represented in terms of both its Dirichlet and Neumann-data!

$$v(x) = \int_{\partial\Omega} \partial_\nu\Gamma^x v\,ds - \int_{\partial\Omega} \Gamma^x \partial_\nu v\,ds$$

If $x \notin \bar{\Omega}$, then one obtains (17.1) with 0 on the left side!

17.h Moreover, if in addition $\Delta v = 0$, then one obtains a special case of ~~the divergence theorem~~ integration by parts

$$\int_{\partial\Omega} h\,\partial_\nu v\,ds = \int_{\partial\Omega} \partial_\nu h\,v\,ds$$

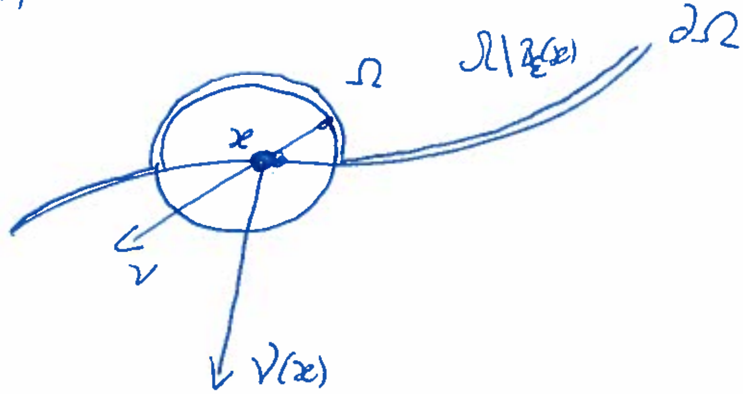
if h, v <u>both</u> harmonic!

When $x \in \partial\Omega$ and $\vec{\nu}(x)$ exists, then (17.1) becomes

$$(18.1)\quad \frac{1}{2} v(\mathring{x}) = \int_{\Omega} \Gamma^{\mathring{x}} \Delta v \, dy + \lim_{\varepsilon \to 0} \int_{\partial\Omega \setminus B_\varepsilon(\mathring{x})} \partial_\nu \Gamma^{\mathring{x}} v \, ds - \cancel{\lim_{\varepsilon\to 0}} \int_{\partial\Omega} \Gamma^{\mathring{x}} \partial_\nu v \, ds$$

Which follows from the $x \notin \bar{\Omega}$ version of (17.1)

$$0 = \int_{\Omega \setminus B_\varepsilon(x)} \Gamma^x \Delta v \, dy + \cancel{\int_{\partial\Omega} \partial_\nu \Gamma^x v \, ds - \int_{\partial\Omega \setminus B_\varepsilon(x)} \Gamma^x \partial_\nu v \, ds}$$
$$+ \int_{\partial(\Omega \setminus B_\varepsilon(x))} \left(\partial_\nu \Gamma^x v - \Gamma^x \partial_\nu v \right) ds$$



Since $\nu(x)$ exists note that $\displaystyle\int_{\partial B_\varepsilon(x) \cap \Omega} ds \cdot \frac{1}{|\partial B_\varepsilon(x)|} \longrightarrow \frac{1}{2}$ as $\varepsilon \to 0$.

In local coordinates take $x = 0$. Orient coordinates so $\exists\, y(x')$ such that $(0', y(0')) = 0$. By differentiability of y at $0'$ $\quad y(x') = \underbrace{y(0')}_{=0} + \nabla y(0') \cdot x' + o(|x'|)$ $\quad \forall x' \in \mathbb{R}^{d-1}$

i.e. $\displaystyle \frac{y(x') - \nabla y(0') x'}{|x'|} \longrightarrow 0$ as $|x'| \to 0$.

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Fix $x^0 \in \partial\Omega$ such that $\nu(x^0)$ exists. Take $y \in \partial\Omega$ and $x \in \mathbb{R}^d$ and observe that $|\Gamma^x(y)| \approx \frac{1}{|x-y|^{d-2}} \leq \frac{1}{|x'-y'|^{d-2}} \in L^1_{loc}(dy; \mathbb{R}^{d-1})$ in local coordinats

Therefore, it ~~is~~ is not hard to justify $\lim_{x \to x^0} \int_{\partial\Omega} \Gamma^x \partial_\nu v \, ds = \int_{\partial\Omega} \Gamma^{x_0} \partial_\nu v \, ds$

Thus, when $\Delta v = 0$ in Ω, (18.1) ~~becomes~~ implies for $x^0 \in \partial\Omega$, $x \in \Omega$

$$\Rightarrow \lim_{x \to x^0} \int_{\partial\Omega} \Gamma^x \partial_\nu v \, ds = \frac{1}{2} v(x^0) + \lim_{\varepsilon \to 0} \int_{\partial\Omega \setminus B_\varepsilon(x^0)} \partial_\nu \Gamma^{x^0} v \, ds$$

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Review: (17.2) $v(x) = \int_{\partial\Omega} \partial_\nu \Gamma^x v \, ds - \int_{\partial\Omega} \Gamma^x \partial_\nu v \, ds$, $x \in \Omega$, $\Delta v = 0$

If $x \notin \bar{\Omega}$, then we get zero on the LHS.

(18.0) $\int_{\partial\Omega} h \partial_\nu v \, ds = \int_{\partial\Omega} \partial_\nu h \, v \, ds$ h, v harmonic

(18.1) $\frac{1}{2} v(x^0) = \int_{\Omega} \Gamma^{x^0} \Delta v \, dy + \lim_{\varepsilon \to 0} \int_{\partial\Omega \setminus B_\varepsilon(x^0)} \partial_\nu \Gamma^{x^0} v \, ds - \int_{\partial\Omega} \Gamma^{x_0} \partial_\nu v \, ds$, $x^0 \in \partial\Omega$

(19.1) $\lim_{x \to x^0} \int_{\partial\Omega} \Gamma^x \partial_\nu v \, ds = \int_{\partial\Omega} \Gamma^{x^0} \partial_\nu v \, ds$

When $\Delta v = 0$ in Ω, (18.1) implies

$$\lim_{x \to x^0} \int_{\partial\Omega} \Gamma^x \partial_\nu v \, ds = -\frac{1}{2} v(x^0) + \lim_{\varepsilon \to 0} \int_{\partial\Omega \setminus B_\varepsilon(x^0)} \partial_\nu \Gamma^{x_0} v \, ds$$

From (17.2) and its exterior version, we get

$$\int_{\partial\Omega} \partial_\nu \Gamma^x v \, ds = \int_{\partial\Omega} \Gamma^x \partial_\nu v \, ds + \begin{cases} v(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}$$

Thus

$$\lim_{x \to x^0 \in \partial\Omega} \int_{\partial\Omega} \partial_\nu \Gamma^x v \, ds = \begin{cases} \frac{1}{2} v(x^0) + \lim_{\varepsilon \to 0} \int_{\partial\Omega \setminus B_\varepsilon(x^0)} \partial_\nu \Gamma^{x_0} v \, ds, & x \in \Omega \\ -\frac{1}{2} v(x^0) + \lim_{\varepsilon \to 0} \int_{\partial\Omega \setminus B_\varepsilon(x^0)} \partial_\nu \Gamma^{x_0} v \, ds, & x \notin \bar{\Omega} \end{cases} \tag{20.1}$$

Generally, $Sf(x) = \int_{\partial\Omega} \Gamma^x f \, ds$, $x \in \mathbb{R}^d \setminus \partial\Omega \ni x$

is called <u>single layer potential</u>. It is harmonic in Ω and $\bar{\Omega}^c$ and extends continuously to a.e. point $x \in \partial\Omega$ as seen in (19.1)

$\mathcal{D} f(x) = \int_{\partial\Omega} \partial_\nu \Gamma^x f \, ds$ $\quad x \in \mathbb{R}^d \setminus \partial\Omega$ (<u>double layer potential</u>)

It is harmonic and obeys the "jump relations" as in (20.1) a.e. on $\partial\Omega$

The integral operator on the right of (20.1) mapping functions on $\partial\Omega$ to functions on $\partial\Omega$ is an example of a <u>Calderon-Zygmund singular integral</u>.

a subclass of principal value operators. (p.v.).

Write points on $\partial\Omega$ as $x = (x', w(x'))$ locally. ~~then~~

It can be shown by use of the Hardy–Littlewood maximal function (Fabes, Jodeit, Rivière Acta. Math (1978)) that for a.e. $x \in \partial\Omega$ (writing $f(y) = f(y')$) the limit

$$\mathring{\mathcal{D}} f(x) = \lim_{\varepsilon\to 0} \int_{\partial\Omega \setminus B_\varepsilon(x)} \partial_\nu \Gamma^x f \, ds = \lim_{\varepsilon \to 0} \frac{1}{|S^{d-1}|} \int_{|y'-x'|>\varepsilon} \frac{\nabla w(y')\cdot(y'-x') - (w(y') - w(x'))}{\left[|y'-x'|^2 + (w(y')-w(x'))^2\right]^{d/2}} \cdot f(y')\,dy'$$

$$= \lim_{\varepsilon\to 0} \frac{1}{|S^{d-1}|} \int_{|y'-x'|>\varepsilon} \frac{1}{|y'-x'|^{d-1}} \frac{\nabla w(y') \frac{y'-x'}{|y'-x'|} - \frac{w(y')-w(x')}{|y'-x'|}}{\left[1 + \left(\frac{w(y')-w(x')}{|y'-x'|}\right)^2\right]^{d/2}} f(y')\,dy'$$

Simpler examples of Singular integrals

$$H f(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x-y} f(y)\,dy, \quad x \in \mathbb{R},$$ the Hilbert transform.

Riesz transforms:

$$R_j f(x) = \text{p.v.} \frac{2}{|S^{d}|} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\,dy, \quad 1 \le j \le d \quad x \in \mathbb{R}^d$$

(Stein pp. 54–60)

- For a.e. y' the limit

$$\lim_{x'\to y'} \frac{w(x'+w(y')-Dw(y')(x'-y')}{|x'-y'|} = 0 \text{ by differentiability.}$$

But there is no uniform rate for Lipschitz nor even $\mathcal{C}^1 w$.

- (Coifman, McIntosh, Meyer Annals of Math (1982))

$\mathring{\mathcal{D}} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is bounded linear operator for $1<p<\infty$

on Lipschitz boundaries.

- When $\partial\Omega$ is a smoother domain $\mathcal{C}^{1,\alpha}$ $\left(|\nabla w(x') - Dw(y')| \le c|x'-y'|^{\alpha}, \alpha>0 \right)$

The singularity is like $\frac{1}{|y'-x'|^{d-1-\alpha}} \in L^1_{loc}(\mathbb{R}^{d-1})$ and it can be shown, that $\mathring{\mathcal{D}} : L^p \to L^p$ $\forall\, 1\le p\le\infty$ and moreover

$\mathring{\mathcal{D}} : \mathcal{C}(\partial\Omega) \to \mathcal{C}(\partial\Omega)$, so $\mathcal{D}f(x)$ extends continuously from Ω to $\bar{\Omega}$ and continuously from $\bar{\Omega}^c$ to Ω^c (but extensions are different!) See (Folland: Introduction to PDE)

- When Ω is $\mathcal{C}^{1,\alpha}$ and $f \in \mathcal{C}(\partial\Omega)$ then $\mathcal{D}f$ ~~~~ is harmonic in Ω and continuous on $\bar{\Omega}$ with $\mathcal{D}f|_{\partial\Omega} = \frac{1}{2}f + \mathring{\mathcal{D}}f$.

by (20.1)

Thus, given $g \in \mathcal{C}(\partial\Omega)$, we are led to solving the <u>integral equation</u> in $\mathcal{C}(\partial\Omega)$ $\quad \frac{1}{2}f + \dot{\mathcal{D}} f = g$

for the unknown f (see <u>Folland</u>). This is called the <u>method of layer potentials.</u>

A generalization of the Poisson integral also emerges from the harmonic Green's representation (17.2). $u(x) = \mathcal{D}u(x) - \int_{\partial\Omega} \Gamma^x \partial_\nu u \, ds$

$x \in \Omega$, $\Delta u = 0$. Suppose for each x there is a harmonic function $h^x(y)$, $y \in \Omega$ s.t. on $\partial\Omega$ $h^x = \Gamma^x$. Then

$$\int_{\partial\Omega} \Gamma^x \partial_\nu u \, ds = \int_{\partial\Omega} \partial_\nu h^x \, u \, ds \quad \text{so that}$$

(23.1) $\quad u(x) = \int_{\partial\Omega} (\partial_\nu \Gamma^x - \partial_\nu h^x) u \, ds$, $x \in \Omega$ (we assume $h \in \mathcal{C}^1(\bar{\Omega})$)

The function $G(x,y) = \Gamma(x-y) - h^x(y)$, $x \in \Omega, y \in \bar{\Omega}, x \neq y$ is the <u>Green function</u> for the domain Ω and $\partial_\nu(y)$

$\partial_{\nu(y)} G(x,y)$ is the <u>Poisson-kernel</u> for the domain Ω

If it can be established that $G(x,y)$ actually exists (i.e. $h^x(y)$) 01/29

~~then~~ and has enough regularity, then the Poisson integral of g

$$P_g(x) = \int_{\partial\Omega} \partial_\nu G(x,\cdot)\, g\, ds.$$ from (23.1) is harmonic in Ω

with Dirichlet boundary value g.

Facts about G: (1) $x \in \Omega$, $y \in \partial\Omega \Rightarrow G(x,y) = 0$

(2) $G(x,y) = G(y,x)$

Delete ε-balls about x, y $x \neq y$ in

two different ways

$-\int_\Omega \nabla_w G(x,w) \cdot \nabla_w G(y,w)\, dw$ & integrate by parts and let $\varepsilon \to 0$

01/31

Mainly for dim ≥ 3 since $\log|x|$ takes both $\pm$ values (but conditions (1), (6), (7) still true for $d=2$). (3) By maximum principle

$h^x(y) < 0$ so $G(x,y) = \cancel{h^x(y)} \Gamma(x-y) - h^x(y) > \Gamma(x-y)$ $x \in \Omega$, $y \in \bar\Omega$

(4) Deleting $B_\varepsilon(x)$ and using max'm. principle $\Rightarrow$ $0 > G(x,y)$

for $x \in \Omega$, $y \in \Omega$.

(5) Thus $|G(x,y)| < |\Gamma(x-y)|$ $x, y \in \bar\Omega$

(6) By (1) & (4) $\partial_{\nu(y)} G(x,y) \geq 0$.

(7) $\int_{\partial\Omega} \partial_{\nu(y)} G(x,y)\, ds(y) = 1 \quad \forall x \in \Omega$

$\left(u(x) = \int_{\partial\Omega} (\partial_\nu \Gamma^x - \partial_\nu h^x) u \, ds \right)$

Note that $\mathcal{D}(1)(x) = 1$, $x \in \Omega$ so by $\mathcal{D}f|_{\partial\Omega} = \frac{1}{2} f + \mathcal{D}^\circ f$ from the interior $\Rightarrow$ $\mathcal{D}^\circ(1) = \frac{1}{2}$.

Explicit green function for $B_R(0)$ (Gilbarg+Trudinger, Folland)

$d \geq 3$ $\quad G(x,y) = \begin{cases} \Gamma^x(y) - \frac{1}{R^{d-2}} \Gamma\left(\frac{x}{|x|} - \frac{|x| y}{R^2}\right) & x \neq 0,\ x, y \in B_R(0) \\ \Gamma(y) + \frac{1}{R^{d-2} |S^{d-1}| (d-2)} & x = 0 \end{cases}$

$d = 2$ $\quad \begin{cases} \log|x-y| \frac{1}{2\pi} - \frac{1}{2\pi} \log\left| \frac{R x}{|x|} - \frac{|x| y}{R} \right| & x, y \in B_R(0),\ x \neq 0 \\ \frac{1}{2\pi} \log\left|\frac{y}{R}\right| , & x = 0 \end{cases}$

Here $h^x \in C^1(\overline{B_R})$ are harmonic in y while

$$\left| \frac{x}{|x|} - \frac{|x| y}{R^2} \right|^2 = 1 - \frac{2 x \cdot y}{R^2} + \frac{|x|^2 |y|^2}{R^4} = \left| \frac{y}{|y|} - \frac{|y| x}{R^2} \right|^2 \quad \text{so}$$

$h^x(y) = h^y(x)$.

A computation yields the Poisson kernel

$$\partial_{\nu(y)} G(x,y) = \frac{1}{R|S^{d-1}|}\,\frac{R^2-|x|^2}{|y-x|^d}, \quad x\in B_R(0),\ |y|=R$$

~~$x\in B_R(0)$~~, generalising the unit disc example.

Note $x=0 \Rightarrow \partial_\nu G(x,y) = \frac{1}{R^{d-1}|S^{d-1}|} = \frac{1}{|\partial B_R(0)|}$.

Suppose now $g\in \mathcal{C}(\partial B_R(0))$ and consider its poisson-integral

$$P_g(x) = \int_{\partial B_R} \partial_\nu G^x g\, ds, \quad x\in B_R.$$

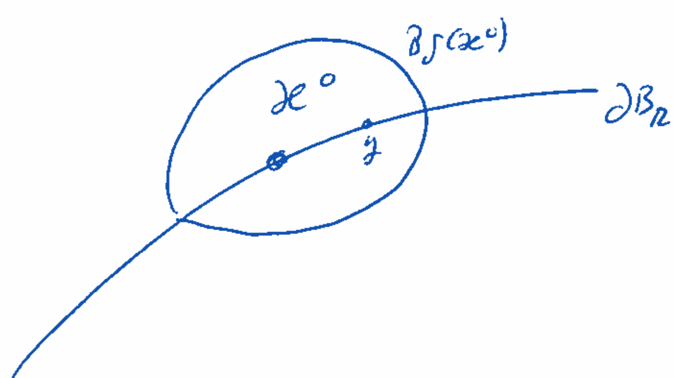
Fix any $x^0\in\partial B_R(0)$.

We show $\lim\limits_{\substack{x\to x^0\\ x\in B_R(0)}} P_g(x) = g(x^0)$. Thus, ~~$P_g(x)$ $x\in B_R$~~ and

$$\tilde g(x) \begin{cases} P_g(x), & x\in B_R(0) \\ g(x), & x\in\partial B_R(0) \end{cases} \in \mathcal{C}(\overline{B_R(0)})$$ and solves the classical Dirichlet-Problem.

Given $\varepsilon>0$ $\exists\,\delta>0$ s.t. $y\in B_\delta(x^0)\cap\partial B_R$ $|g(y)-g(x^0)|<\varepsilon$

using facts (6)-(7) about G:



$|P_g(x) - g(x^0)|$

$$= \left|\int_{\partial B_R} \partial_\nu G^x(y)\,(g(y)-g(x^0))\, ds(y)\right| \le \int_{\partial B_R\cap B_\delta(x^0)} \partial_\nu G^x(y)\,|g(y)-g(x^0)|\, ds(y) +$$

$$+\int_{y\in\partial B_R:\,|y-x^0|>\delta} \partial_\nu G^x(y)\,|g(y)-g(x^0)|\,ds(y) < \varepsilon + 2\max_{\partial B_R}|g| \int_{|y-x^0|>\delta} \frac{1}{R\,|S^{d-1}|}\frac{R^2-|x|^2}{|y-x|^d}\,ds(y) \quad (31)$$

By L.D.C. $\lim_{x\to x^0}$ of integral is zero. (as $x^0\in\partial B_R(0)$ so $R^2-|x|^2\to 0$)

Thus, $\limsup_{x\to x^0} |P_g(x)-g(x^0)| \le \varepsilon$.

Classical Dirichlet Problem in more general domains (including Lipschitz) by

Perron's method of subharmonic functions

- Generalize subharmonic functions:

<u>Def:</u> $u\in C(\Omega)$ is subharmonic in Ω if $\forall$ Ball $\overline{B}\subset\Omega$ and $\forall$ harmonic function h in B with $h\ge u$ on ∂B it follows $h\ge u$ in B.

- u subharmonic in $\Omega \Rightarrow u$ obeys strong maximum principle. (same proof ($u(x)<\sup_\Omega u \Rightarrow u\equiv\text{const.}$). $\left[\text{Use } \int_{\partial B(x^0)} u\,ds = P_{B(x^0)}u(x^0)\right]$

- $v\in C(\Omega)$ can similarly be defined as superharmonic.

Equivalently v is superharmonic iff $-v$ is subharmonic in Ω

- u_1, u_2 subh. in $\Omega \Rightarrow u_1+u_2$ subharmonic.

- u subharmonic in Ω and v superharmonic in Ω, Ω bounded and $u\le v$ in $\partial\Omega \Rightarrow$ either $u<v$ throughout Ω or $u(x)=v(x)\ \forall x\in\Omega$.

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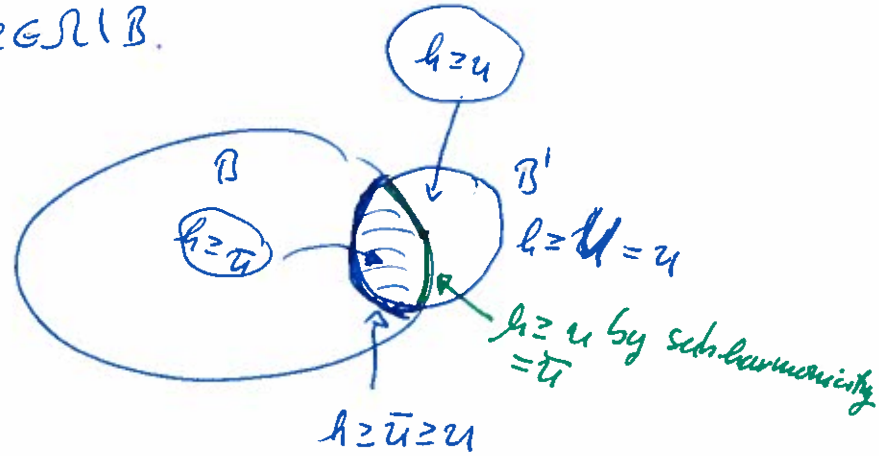
• **Define harmonic lifting**. Take u subharmonic in Ω and $\bar{B}\subset\Omega$. Let $\bar{u} = P_B u$ and define

$$U(x) = \begin{cases} \bar{u}(x), & x\in B \\ u(x), & x\in\Omega\setminus B. \end{cases}$$



• U is subharmonic:

Take h harmonic in B'.

with $h\geq U$ on $\partial B'$.

$\Rightarrow h\geq U$ in B'.

• Facts: $u_1,\dots,u_N$ subharmonic in $\Omega\Rightarrow\max\{u_1,\dots,u_N\}$ subharmonic.

Def: If $u\in C(\bar{\Omega})$ is subharmonic in Ω with $u\leq\varphi$ on $\partial\Omega$ then u is called a subfunction relative to φ. Similarly define v a superfunction relative to φ.

And the maximum principle will imply $u\leq v$ in $\bar{\Omega}$.

Def: $S_\varphi := \{u \text{ subfunction relative to } \varphi\}$. It is nonempty, since the constant function $\leq \inf_{\partial\Omega}\varphi$ is in S_φ.

Thm: [G, T p. 24]. The function $u(x) = \sup_{v\in S_\varphi} v(x)$ is harmonic in Ω.

• Reference: F. John

~~A bounded~~, ~~φ bounded~~ on ∂A

Interior estimates of derivatives of harmonic functions let u be harmonic in Ω, $\overline{B_R(x)} \subseteq \Omega$ then $\partial_j u$ is also harmonic. By the mean value theorem

$$\partial_j u(x) = \fint_{B_R(x)} \partial_j u \, dy = \frac{1}{|B_R|} \int_{\partial B_R} \nu_j u \, ds \quad \text{so that}$$

$$|\partial_j u(x)| \leq \frac{d}{R|\partial B_R|} \int_{\partial B_R} |u| \, ds \leq \frac{d}{R} \max \sup_{\Omega} |u| \quad 1 \leq j \leq d$$

(30.a.1) Suppose $\{u_k\}$ is a uniformly bounded sequence of harmonic functions in Ω. let $\overline{\Omega'} \subset \Omega$ be a compactly contained subdomain, then the above estimate would show that $\{\nabla u_k\}$ is uniformly bonded in $\overline{\Omega'}$. Consequently $\{u_k\}$ would be equicontinuous on $\overline{\Omega'}$. (30.a.2)

By Arzela-Ascoli $\exists$ subsequence $u_{k'} \to u$ uniformly on Ω'.

Furthermore, u is harmonic $\because$ $\forall$ ball $B \subset \Omega'$, we get

$$u(x) = \lim_{k' \to \infty} u_{k'}(x) = \lim_{k' \to \infty} P_B u_{k'}(x) = P_B u(x). \quad \forall x \in B.$$

$\Rightarrow \Delta u = 0$ in B and also in Ω'.

The u in the Theorem on $\boxed{23}$ is called the ~~per~~ Perron solution for the Dirichlet data φ.

In general, it is not true that u (the Perron solution) will actually attain the Dirichlet boundary values φ.

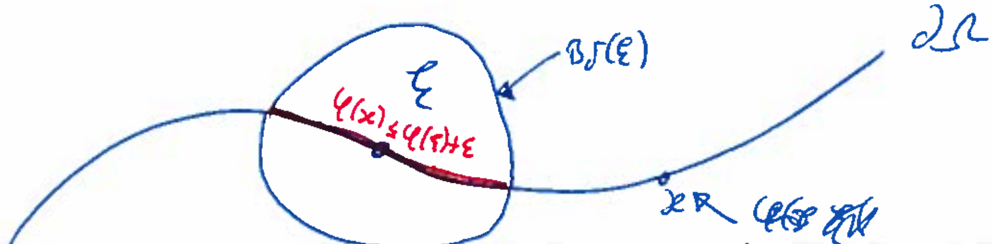
However, if the Dirichlet problem is solved by a (harmonic) function w, then $w = u$ has to hold, because we will have $w \in S_\varphi$ with $w \geq v$ $\forall v \in S_\varphi$ by the maximum principle. ($\Rightarrow w = u$) We need to look at the geometry of the boundary. Let $\xi \in \partial\Omega$.

Def: A function $w = w_\xi \in \mathcal{C}(\bar\Omega)$ is a barrier at ξ relative to Ω if $w(\xi) = 0$, $w > 0$ in $\bar\Omega \setminus \{\xi\}$ and is superharmonic in Ω.

Lemma [G+T, p. 25] Let u be the Perron solution for φ. If w, a barrier at ξ, exists and φ is continuous at ξ, then $\lim\limits_{\substack{x \to \xi \\ x \in \Omega}} u(x) = \varphi(\xi)$.

Proof: Put $M = \sup\limits_{\partial\Omega} |\varphi| < \infty$ and let $\varepsilon > 0$. Then $\exists\, \delta > 0$ s.t. $|x - \xi| < \delta$ implies $|\varphi(x) - \varphi(\xi)| < \varepsilon$. Choose k s.t. $k\, w(x) \geq 2M$ for all x $|x - \xi| \geq \delta$.

Then $\varphi(\xi) + \varepsilon + k\, w(x)$ is a superharmonic (superfunction) function relative to φ.



$\varphi(\xi) + \varepsilon + k\, w(x) \geq \varepsilon + M \geq \varepsilon + \varphi(x)$

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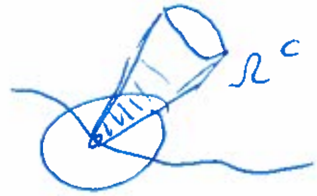
Likewise $\varphi(\xi)-\varepsilon-Kw(x)$ is a subfunction relative to φ.

Thus $\varphi(\xi)-\varepsilon-Kw(x) \le u(x) \underset{\text{max. principle}}{\le} \varphi(\xi)+\varepsilon+Kw(x)$ $\forall x\in\Omega$

i.e. $|u(x)-\varphi(\xi)| \le \varepsilon + Kw(x)$ $x\in\Omega$

$\Rightarrow \overline{\lim_{\substack{x\to\xi \\ x\in\Omega}}} |u(x)-\varphi(\xi)| \le \varepsilon$

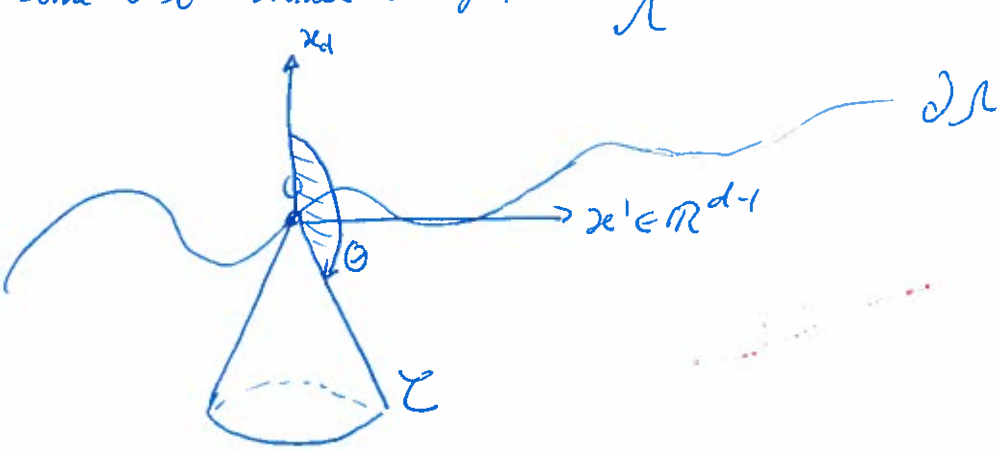


Let $\mathcal{C}$ be an open right circular cone with vertex $\xi\in\partial\Omega$ s.t. $\mathcal{C}\cap B_R(\xi)\subseteq\Omega^c$.
Then, we say Ω satisfies the exterior cone condition at ξ and we claim a barrier at ξ relative to Ω may be constructed. A Lipschitz domain will have this property at every boundary point.

(Jenson-Kenig, "Boundary Value Problems ...." Studies in PDE pp. 1–68 MAA studies in Mathematics Vol. 23, Math. Association America (1982))

In local coordinates, let $\xi = 0\in\partial\Omega$ with Ω outside $C=\left\{x \mid \frac{x_d}{|x|} < \cos(\pi-\delta)\right\}$

for some $\delta>0$ small enough.



let $\theta = \arccos\left(\frac{x^d}{|x|}\right)$, $0 \le \theta < \pi - \delta$

Define $\varphi(\theta) = \int_0^\theta \left[\sin^{2-d} t \int_0^t \sin^{d-2} \tau \, d\tau\right] dt \quad 0 \le \theta < \pi.$

Satisfies $\varphi''(\theta) + (d-2)(\cot\theta)\varphi'(\theta) = 1$. (33.1) There exists a constant

$0 < A = A_\delta$ s.t. $A - \varphi(\theta) > 1 \quad 0 \le \theta < \pi - \delta$.

For $\alpha > 0$ define $w(x) = |x|^\alpha (A - \varphi(\theta)) > 0$ in $\Omega \setminus \{0\}$.

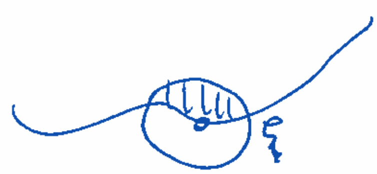
Then $\Delta w(x) = \Delta |x|^\alpha (A - \varphi(\theta)) + |x|^\alpha \Delta(A - \varphi(\theta))$. $\quad \Delta(fg) = \Delta f\, g + 2\nabla f \nabla g + f\Delta g$. here 0

$\Delta w = \alpha(\alpha + d - 2)|x|^{\alpha-2}(A - \varphi(\theta)) - |x|^\alpha \Delta_x \varphi(\theta)$.

and a computation will yield (33.1) and $-|x|^\alpha \Delta_x \varphi(\theta) = -|x|^{\alpha-2}$.

and α small implies $\Delta w < 0$.

Existence of a barrier is a local property of the boundary $\partial\Omega$, i.e. a function w_ξ that is a barrier to $\xi \in \partial\Omega$ relative to $\Omega \cap B(\xi)$ for a small enough ball may be extended to a barrier relative to Ω (e.g. [G+T] p. 25).



Domains that have barriers at each boundary point are domains for which the classical Dirichlet Problem is solved! Conversely, if the Dirichlet problem is solved for Ω and if a boundary point ξ,

02/07

then $|\xi - x|$ is continuous on $\partial\Omega$ and thus it is Dirichlet data for which $\exists$ a positive harmonic function in Ω and thus a barrier at ξ.

For the punctured disc. $\Omega = \mathbb{D} \setminus \{0\}$ there is no barrier at $0 \in \partial\Omega$.

If there ~~was~~ was a barrier one could solve for

$$g = \begin{cases} 0, & \partial\mathbb{D} \\ 1, & \text{at } 0. \end{cases}$$

Then the maximum principle would compare the solution u by $0 \le u < \frac{\log(r)}{\log(\varepsilon)}$ in every $\mathbb{D} \setminus B_\varepsilon(0)$. ($r = |x|$)

$\log(r)$ harmonic! Fix $r > 0$ and let $\varepsilon \to 0 \Rightarrow u \equiv 0$. ↯

"Removable Singularities" (Folland p. 111)

On the other hand the harmonic function $u(z) = -\operatorname{Re} \frac{1}{\log z}$

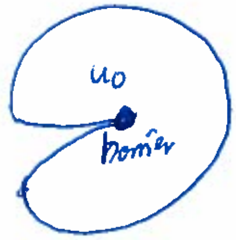
$$= -\frac{\log r}{\log^2 r + \theta^2}, \quad -\pi \le \theta \le \theta$$ is a barrier at 0 for

the slit disc

It serves by translation for all points on the slit.

By conformal mapping, boundary points which are connected to the exterior of $\Omega \subset \mathbb{R}^2$ by a simple arc have barriers.

In higher dimensions, there is the Lebesgue spine example in $\mathbb{R}^3$.



In the study of constant real coefficient homogeneous elliptic operators, the classical Dirichlet Problem is now equally understood for second order operators all of which can be written $L(\partial) = \sum_{j,k=1}^{d} a_{jk}\,\partial_j \partial_k$ with $a_{jk} = a_{kj}$ and the matrix ~~A~~ $a = [a_{jk}]$ is positive definite by the ellipticity condition: $\sum_{j,k} a_{jk}\,\xi^j \xi^k = \xi \cdot a\xi > 0$ for $\xi \neq 0$, $\xi \in \mathbb{R}^d$.

---

a has a positive definite square root $\sqrt{a}$ with $\overset{\sqrt{a}^T\sqrt{a}}{\cancel{(\sqrt{a})^2}} = a$. writing $Lu = 0$ in the sense of distributions:

$$0 = \sum_{jk} \int \partial_j u(x)\, a_{jk}\, \partial_k \varphi(x)\,dx$$

$$= \int \nabla u \cdot a (\nabla \varphi)^T dx$$

Now change variables $x = \sqrt{a}\,y$, $\tilde{u}(y) = u(\sqrt{a}\,y) = u(x)$.

$$\partial \tilde{u}(y) = \partial_y\, u(\sqrt{a}\,y) = \partial u(x) \cdot \sqrt{a}$$

$$dx = \det(\sqrt{a})\,dy.$$ So, finally

$$0 = \det\sqrt{a} \int \partial\tilde{u}\, (\partial\tilde{\varphi})^T dy$$ so $\Delta \tilde{u} = 0$ s.d.

Linear change of variables maps Lip. domains to Lip domains.

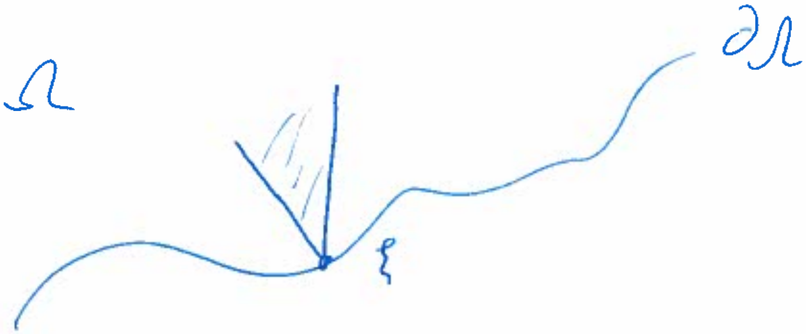
However, the equation $Lu=0$ for 4th- and higher order ~~operators~~ operators fails to put a (direct) condition on the second derivative which seems to be necessary for obtaining the maximum principle upon which the Perron method relies.

The same is true for general second order elliptic systems.

As mentioned, both the Poisson kernel and layer potential representations yield harmonic functions when L^p functions are used in the representations. Suggesting that boundary values in $L^p(\partial\Omega)$ be considered for 2^{nd} order and then higher order operators.

- Boundary values will be taken a.e. necessarily on $\partial\Omega$
- tangential approaches to $\partial\Omega$ must be avoided. We can use "nontangential cones": locally for all $\xi\in\partial\Omega$ and for a Lipschitz constant M,

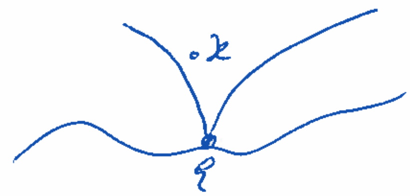
$$\mathcal{C}(\xi)=\left\{x\in\Omega \;\middle|\; \frac{|x_d-\xi_d|}{|x'-\xi'|}\geq M\right\}$$



or more flexible "nontangential approach regions"

$$\Gamma_\beta(\xi) = \{ x \in \Omega \mid \beta\, \mathrm{dist}(x, \partial\Omega) > |x - \xi| \}$$

for $\beta > 1$ sufficiently large depending on M.

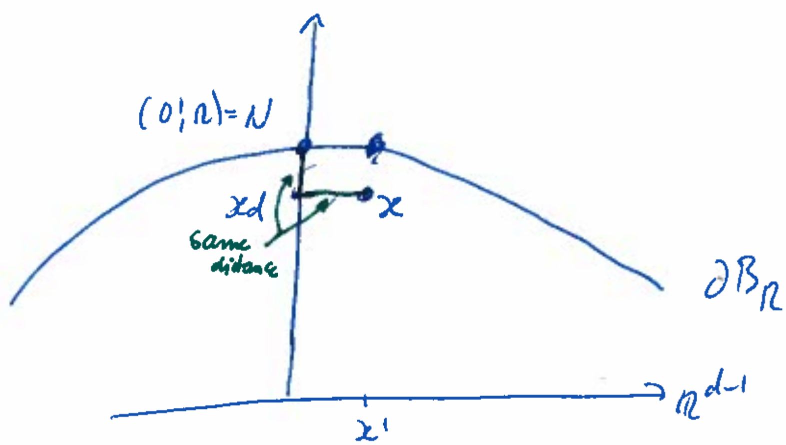


- An appropriate substitute for the maximum principle is needed for uniqueness:

In $B_R(0)$ let $u(x) = \dfrac{R^2 - |x|^2}{|x - (0', R)|^d}$ (North pole)

is the poisson kernel with boundary point $y = (0'; R) \in \partial B_R(0)$. u is harmonic in $B_R(0)$ with zero boundary values at every boundary point except $(0', R)$,

i.e. $u|_{\partial B_R} = 0$ a.e. (ds).



Consider points $x = (x', x_d) \in B_R(0)$ with $|x'| = R - x_d$

then $R^2 - |x|^2 = R^2 - (R - x_d)^2 - x_d^2 = 2 x_d R - 2 x_d^2 = 2 x_d |x'|$

and $|x - (0', R)|^2 = 2|x'|^2$. Thus $u(x) = 2^{\frac{2-d}{2}} \dfrac{x_d}{|x'|^{d-1}} \simeq \dfrac{1}{|x'|^{d-1}} \notin L^1_{loc}$

Wishing to elucidate this harmonic function from the Dirichlet Problem, 02/07

the following Def^n is ~~so~~ suggested.

<u>Def:</u> Let u be a function defined on Ω and $\{\mathcal{C}(\xi) \mid \xi \in \partial\Omega\}$ be a family of cones or approach regions as above. The <u>nontangential maxl function of u</u> is then defined as

$$\eta(u)(\xi) = \sup_{x \in \mathcal{C}(\xi)} |u(x)|.$$ In the ball, note that $\eta(u) \notin L^p(\partial\Omega)$

for any $p \in [1, \infty]$. We state the Dirichlet ~~pro~~ problem

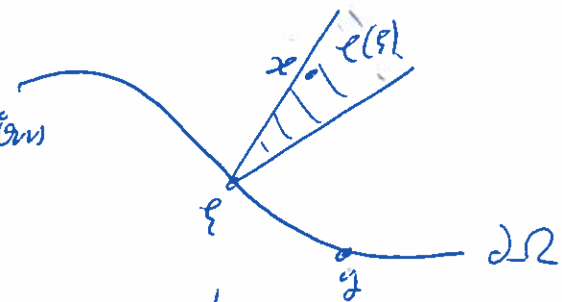
D_p: Given $g \in L^p(\partial\Omega)$, $p \in [1, \infty]$. Find a unique harmonic function in Ω s.t. (i) the <u>nontangential limits</u>

$$\lim_{\substack{x \to \xi \\ x \in \mathcal{C}(\xi)}} u(x) = g(\xi) \quad a.e. \ (ds)$$

(ii) $\eta(u) \in L^p(\partial\Omega)$.

Consider $\mathcal{D} f(x) = \frac{1}{|S^{d-1}|} \int_{\partial\Omega} \frac{\nu_y (y-x)}{|y-x|^d} f(y)\, ds(y)$, $f \in C^0(\partial\Omega)$

$x \in \mathcal{C}(\xi)$. If we take $|y-\xi| > 2|x-\xi|$ (42.1)
$\overset{\shortparallel}{\mathcal{C}_\beta(\xi)}$



the mean value theorem for diff'ble functions

gives

$$(42.2) \quad \left| \frac{\nu_y (y-x)}{|y-x|^d} - \frac{\nu_y (y-\xi)}{|y-\xi|^d} \right| \leq C_d |x-\xi| \cdot \frac{1}{|y-x^*|^d} \quad x^* \text{ s.t.}$$

$|\nabla(\text{function})|$ maximal

for some x^* on the segment $[\xi, x]$ ($C_d = d+1$ works). Then

$$|y-\xi| \leq |y-x^*| + \underbrace{|x^* - \xi|}_{\leq |x-\xi|} \leq |y-x^*| + \tfrac{1}{2}|y-\xi|$$

by (42.1). (42.2) is bounded by

$$\frac{2^d C_d |x-\xi|}{|y-\xi|^d}.$$ On the other hand, we always have

$$|x-y| \geq \operatorname{dist}(x, \partial\Omega) \overset{\text{Def. cone}}{\geq} \frac{1}{\beta}|x-\xi| \quad \text{so that}$$

$$\left| \int_{|y-\xi| \leq 2|x-\xi|} \frac{\nu_y (y-x)}{|y-x|^d} f(y)\, ds(y) \right| \leq \beta^{d-1} \int_{|y-\xi| \leq 2|x-\xi|} \frac{|f(y)|}{|x-\xi|^{d-1}}\, ds(y)$$

Thus (locally) $|\mathcal{D} f(x)| \leq C_d \Big(\beta^{d-1} \fint_{|y-\xi| \leq 2|x-\xi|} f(y)\, ds(y) +$ (43.1)

new const.

02/12

$$+ |x-\xi| \int_{|y-\xi| > 2|x-\xi|} \frac{|f(y)|}{|y-\xi|^d}\, ds(y) + \left| \int_{|y-\xi| > 2|x-\xi|} \frac{\nu_y \cdot (y-\xi)}{|y-\xi|^d} f(y)\, ds(y) \right| \Bigg)$$

The integral average (1st term on right) is dominated by the Hardy-Littlewood Max'l function of $f(y') = f(y', y_d(y'))$

(We are working locally. Note that $\int_{y \in S} |f|^r ds \approx \int_{\{y' : (y', y_d(y')) \in S\}} |f(y')|^r dy'$

$ds = \sqrt{1 + |\nabla y_d|^2}\, dy'$) at ξ'.

$$M f(\xi') = \sup_{B(\xi') \subset \mathbb{R}^{d-1}} \left\{ \frac{1}{|B(\xi')|} \int_{B(\xi')} |f|\, dy' \right.$$

This is also true for the second term.

$$2^{nd} \text{ term} = |x-\xi| \sum_{n=1}^{\infty} \int_{2^{n+1}|x-\xi| > |y-\xi| > 2^n |x-\xi|} \frac{|f(y)|}{|y-\xi|^d}\, ds(y)$$

$$\leq \frac{1}{|x-\xi|^{d-1}} \sum_{n=1}^{\infty} 2^{-nd} \int_{2^{n+1}|x-\xi| > |y-\xi|} |f(y)|\, ds(y)$$

$$= \sum_{n=1}^{\infty} 2^{-nd} \frac{2^{(n+1)(d-1)}}{(2^{n+1}|x-\xi|)^{d-1}} \int_{2^{n+1}|x-\xi|>|y-\xi|} |f| \, ds(y)$$

average

depends on lip. const.

$$\leq C_M \sum_{n=1}^{\infty} 2^{-n} 2^{d-1} \underset{\text{in } \mathbb{R}^{d-1}}{m} f(\xi') \leq C_{M,d} \, m_f(\xi')$$

Recall Principal Value operator $\mathcal{D}^{\circ} f(\xi) = \lim_{\varepsilon \to 0} \int_{|y-\xi|>\varepsilon} \frac{\nu_y \cdot (y-\xi)}{|y-\xi|^d} f(y) \, ds(y)$ for $\xi \in \partial\Omega$. We associate to such operators a corresponding max'l operator which here is

$$\mathcal{D}^{\circ}_{*} f(\xi) = \sup_{\varepsilon>0} \left| \int_{|y-\xi|>\varepsilon} \frac{\nu_y (y-\xi)}{|y-\xi|^d} f(y) \, ds(y) \right|$$

Thus, taking the sup. ~~over~~ over all $x \in \mathcal{C}(\xi)$ of (4.3.1), we get;

$$\eta(\mathcal{D}f)(\xi) \leq C_{M,d} \, m_f(\xi') + C_d \, \mathcal{D}_{*} f(\xi).$$

When $f \in L^p(\partial\Omega)$ the harmonic function $u = \mathcal{D}f$ will satisfy (ii) of D_p if $Mf \in L^p(\partial\Omega)$ and $\mathcal{D}_{*} f(\xi)$. The first one is foundational to the theory and is true for $1 < p \leq \infty$. The latter is true for $1 < p < \infty$ on Lipschitz boundaries, but is difficult.

The H-L max'l function in $\mathbb{R}^d$. Define

$$m f(x) = \sup_{B(x)} \frac{1}{|B(x)|} \int_{B(x)} |f| dy,$$

the sup over all balls centered at x. If $f \in L^1(\mathbb{R}^d)$, then $\forall \alpha > 0$ we have

$$(45.1) \quad |\{x : m f(x) > \alpha\}| \leq \frac{A_d}{\alpha} \|f\|_1$$

Uses Vitali covering lemma (Stein pp. 4-10). Also (Rudin's R+C pp. 137-13?) or (Wheeden - Zygmund, Measure and Integral: An introduction to real analysis) pp. 105-106).

- If $f \in L^\infty$, then $\|m f\|_\infty \leq \|f\|_\infty$.
- $1 < p < \infty$. If $f \in L^p(\mathbb{R}^d)$ then $\|m f\|_p \leq A_{p,d} \|f\|_p$

Proof: We use the device of writing the L^p-norm of a function g in terms of its distribution function $\lambda(\alpha) = |\{x : |g(x)| > \alpha\}|$

(45.1) estimates the distribution function of $m f$. Hence

$$(46.1) \quad \int_{\mathbb{R}^d} |g|^p dx = \int_{\mathbb{R}^d} p \int_0^{|g(x)|} \alpha^{p-1} d\alpha \overset{\text{Fubini}}{=} p \int_0^\infty \alpha^{p-1} \left(\int_{\{|g| > \alpha\}} dx \right) d\alpha$$

$$= p \int_0^\infty \alpha^{p-1} \lambda(\alpha) \, d\alpha.$$

Next, for each $\alpha > 0$ define

$$f^{\alpha}(x) = \begin{cases} f(x), & \text{if } |f(x)| > \alpha/2 \\ 0, & \text{otherwise} \end{cases}$$ and

$f_{\alpha}(x) = f(x) - f^{\alpha}(x)$. We have $\|f_{\alpha}\|_{\infty} \le \frac{\alpha}{2}$. Then

$\mathcal{M}f(x) \le \mathcal{M}f^{\alpha}(x) + \frac{\alpha}{2}$ $\forall x$. Thus,

$\{\mathcal{M}f > \alpha\} \subseteq \{\mathcal{M}f^{\alpha} > \frac{\alpha}{2}\}$ and therefore using (45.1) we get

$$|\{\mathcal{M}f > \alpha\}| \le \frac{2A}{\alpha} \|f^{\alpha}\|_1 = \frac{2A}{\alpha} \int_{|f| > \frac{\alpha}{2}} |f| \, dx$$

Thus by (46.1) we get

$$\int_{\mathbb{R}^d} |\mathcal{M}f|^p \, dx = p \int_0^{\infty} \alpha^{p-1} |\{\mathcal{M}f > \alpha\}| \, d\alpha \le 2Ap \int_0^{\infty} \alpha^{p-2} \int_{|f| > \alpha/2} |f| \, dx \, d\alpha$$

$$= 2Ap \int_{\mathbb{R}^d} |f(x)| \left(\int_0^{2|f(x)|} \alpha^{p-2} \, d\alpha \right) dx = \frac{2Ap}{p-1} \int_{\mathbb{R}^d} |f(x)| \, (2|f(x)|)^{p-1} \, dx$$

$$= \frac{2^p p A_p}{p-1} \|f\|_{L^p}^p.$$

By combining arguments of [CMM: 1982] and [FJR: 1978] 02/12

We know that (i) $\mathcal{D}f(\xi)$ exists for a.e. $\xi \in \partial\Omega$ and

(ii) $\|\mathcal{D}_* f\|_{L^p(\partial\Omega)} \le C_{p,\Omega} \|f\|_{L^p(\partial\Omega)}$, $1<p<\infty$

Thus $\|\eta(\mathcal{D}f)\|_{L^p(\partial\Omega)} \le C_{p,\Omega}\|f\|_p$, $f \in L^p(\partial\Omega)$, $1\le p<\infty$.

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$x \in \mathcal{C}(\xi)$ (42.25) $|y-\xi| \le |y-x^*| + |x^*-\xi| < |y-x^*| + \frac{1}{2}|y-\xi|$

When $|y-\xi| > 2|x-\xi|$. May take $x = x^*$.

(42.3)

To show that $\mathcal{D}f$ has nontangential limits a.e. First consider f to be a restriction of a $\mathcal{C}^1(\mathbb{R}^d)$ function to $\partial\Omega$. Re call $\mathcal{D}1(x) = 1$, $x \in \Omega$

Then $$\mathcal{D}f(x) = \frac{1}{|S^{d-1}|}\int_{\partial\Omega} \frac{\nu_y(y-x)}{|y-x|^d}[f(y)-f(\xi)]\,ds(y) + f(\xi)$$

For $x \in \mathcal{C}(\xi)$, (42.3) with $f(y)-f(\xi)$ in place of f. Shows

$$\lim_{\substack{x\to\xi \\ x\in\mathcal{C}(\xi)}} \int_{|y-\xi|\le 2|x-\xi|} \frac{\nu_y(y-x)}{|y-x|^d}[f(y)-f(\xi)]\,ds(y) = 0$$ by the continuity of f. for all $\xi \in \partial\Omega$.

By (42.25) $|y-\xi| > 2|x-\xi| \Rightarrow (x^*=x) \Rightarrow$

$$\left|\frac{\nu_y(y-x)}{|y-x|^d}[f(y)-f(\xi)]\right| \le C_d \frac{|f(y)-f(\xi)|}{|y-\xi|^{d-1}} \le C_d \frac{\|\nabla f\|_\infty}{|y-\xi|^{d-2}} \in L^1(\partial\Omega)$$ in ξ

which is then a dominating function for the pointwise limits

$$\lim_{x\to\xi} \frac{\nu_y\cdot(y-x)}{|y-x|^d}\left[f(y)-f(\xi)\right]\Big|_{\{y\,|\,|y-\xi|>2|x-\xi|\}} = \frac{\nu_y(y-\xi)}{|y-\xi|^d}\left(f(y)-f(\xi)\right)$$

Together with the previous $\lim\limits_{\substack{x\to\xi\\ x\in\mathcal{C}(\xi)}} = 0$ we get

$$\lim_{\substack{x\to\xi\\ x\in\mathcal{C}(\xi)}} \mathcal{D}f(x) - f(\xi) = \frac{1}{|S^{d-1}|}\int_{\partial\Omega}\frac{\nu_y\cdot(y-\xi)}{|y-\xi|^d}\left[f(y)-f(\xi)\right]ds(y)$$

$$= \lim_{\varepsilon\to 0}\frac{1}{|S^{d-1}|}\int_{|y-\xi|>\varepsilon}\frac{\nu_y(y-\xi)}{|y-\xi|^d}\left[f(y)-f(\xi)\right]ds\,y$$

$= \mathring{\mathcal{D}}f(\xi) - \frac{1}{2}f(\xi)$ where $\mathring{\mathcal{D}}(1)(\xi) = \frac{1}{2}$ as shown before for a.e. ξ in $\partial\Omega$

depending only on $\partial\Omega$. Thus we write $\mathcal{D}f \xrightarrow[\text{limits (NT)}]{\text{nontangential}} \mathring{\mathcal{D}}f + \frac{1}{2}f$

for $f\in\mathcal{C}^1(\mathbb{R}^d)$. For general $f\in L^p(\partial\Omega)$ we use the classical argument

[Stein] p.8 put $\Delta f(\xi) := \overline{\lim\limits_{\substack{x\to\xi\\ x\in\mathcal{C}(\xi)}}}\,\mathcal{D}f(x) - \underline{\lim\limits_{\substack{x\to\xi\\ x\in\mathcal{C}(\xi)}}}\,\mathcal{D}f(x)$ (44.1)

$= \Delta(f-g)(\xi)$ for any $g\in\mathcal{C}^1(\mathbb{R}^d)$ and a.e. ξ (independent of g)

Recall the estimate on the distribution function.

$$|\{x : F(x) > \alpha\}| \leq \int_{\{F>\alpha\}} \frac{F^p}{\alpha^p}\,dx \leq \frac{\|F\|_p^p}{\alpha^p}, \quad 1 \leq p < \infty$$

Note that by

$$\Delta f(\xi) = \Delta(f-g)(\xi) \leq 2\,\mathcal{N}(\mathcal{D}(f-g))(\xi). \quad \text{Thus}$$

$$|\{\Delta f > \alpha\}| \leq |\{\mathcal{N}(\mathcal{D}(f-g)) > \tfrac{\alpha}{2}\}| \leq \left(\tfrac{2}{\alpha}\right)^p \|\mathcal{N}(\mathcal{D}(f-g))\|^p_{L^p(\partial\Omega)}$$

$$\leq \frac{1}{\alpha^p}\, C(p,\Omega)\, \|f-g\|^p_{L^p(\partial\Omega)}$$ Since it can be shown

that restrictions of $g \in \mathcal{C}'(\mathbb{R}^d)$ are dense in $L^p(\partial\Omega)$, $1 \leq p < \infty$ we see that

$|\{\Delta f > \alpha\}| = 0 \ \forall \alpha > 0$. and then $\Delta f = 0$ a.e. on $\partial\Omega$, i.e

$\lim_{\substack{x\to\xi \\ x\in\mathcal{C}(\xi)}} \mathcal{D} f(x)$ exists a.e. when $f \in L^p(\partial\Omega)$, ~~1<p<~~ $1<p<\infty$.

Moreover the $\lim_{n.t.} \mathcal{D} f(\xi) - \dot{\mathcal{D}} f(\xi) - \frac{1}{2} f(\xi) = \lim_{n.t.} \mathcal{D}(f-g)(\xi)$

$+ \dot{\mathcal{D}}(g-f)(\xi) + \frac{1}{2}(g(\xi) - f(\xi))$ and each term for $g \in \mathcal{C}'(\mathbb{R}^d)$

is in $L^p(\partial\Omega)$ with small norm. by [CMM] nontangential max-function

estimate, [CMM] and by hypothesizing $\|g-f\|_p$ small.

Then $\lim_{n.t} \mathcal{D}f = \mathring{\mathcal{S}} f + \frac{1}{2} f$ for a.e. q, $f \in L^p(\partial\Omega)$, $1<p<\infty$. 02/14

(limit exists and equals RHS in L^p).

[Stein] + applied to layer potentials [Fabes, Jodeit, Rivière] (1978).

The fact that restrictions of $\mathcal{C}^1(\mathbb{R}^d)$ are dense in $L^p(\partial\Omega)$, $1 \le p < \infty$ to something else we have been assuming, namely the Gauss-Divergence Theorem holding in Lipschitz domains Ω. The key idea is that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. To see this, take a function $0 \le \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with integral $\int_{\mathbb{R}^d} \varphi \, dx = 1$. Put $\varphi_\delta(x) = \delta^{-d} \varphi\left(\frac{x}{\delta}\right)$ so that $\int_{\mathbb{R}^d} \varphi_\delta \, dx = 1 \;\; \forall \, \delta > 0$. One can show that the convolutions $\varphi_\delta * f(x) = \int_{\mathbb{R}^d} \varphi_\delta(x-y) f(y) \, dy = f * \varphi_\delta(x)$ are $\mathcal{C}^\infty(\mathbb{R}^d)$ functions. By Jensen's inequality (+ Fubini)

$$\int_{\mathbb{R}^d} |\varphi_\delta * f(x) - f(x)|^p \, dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \varphi_\delta(y) \big(f(x-y) - f(x)\big) dy \right) dx$$

$$\overset{\text{Jensen}}{\le} \int_{\mathbb{R}^d} \varphi_\delta(y) \int_{\mathbb{R}^d} |f(x-y) - f(x)|^p \, dy \, dx = \int_{\mathbb{R}^d} \varphi(y) \, dy \int_{\mathbb{R}^d} |f(x-\delta y) - f(x)|^p \, dx$$

We know that $\mathcal{C}_0(\mathbb{R}^d) \ni g$ are dense in $L^p(\mathbb{R}^d)$ so that

$F\delta(y)$ "

$$\overline{\lim_{\delta\to 0}}\int_{\mathbb{R}^d} |f(x-\delta y)-f(x)|^p\,dx \le \varepsilon + \lim_{\delta\to 0}\int_{\mathbb{R}^d} |g(x-\delta y)-g(x)|^p\,dx = \varepsilon$$

by unif. continuity of g (compact support of φ)

$\Rightarrow$ For each y $\quad F_\delta(y) = \int_{\mathbb{R}^d} |f(x-\delta y)-f(x)|^p\,dx \to 0$ as $\delta\to 0$

Also $|F_\delta(y)| \le 2^p\|f\|_p^p$. Then, by L.D.C ($\varphi$ compact support)

$$\|\varphi_\delta * f - f\|_{L^p} \to 0, \ \delta\to 0.$$

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When f is continuous with compact support one can show ~~…~~ $\varphi_\delta * f \to f$ uniformly. The φ_δ's are examples of approximations to the identity. Consequently, knowing $\mathcal{C}(\partial\Omega)$ is dense in $L^p(\partial\Omega)$ and that any cont. ~~…~~ function ~~…~~ on $\partial\Omega$ extends to $\mathcal{C}_0(\mathbb{R}^d)$ we have $\mathcal{C}_0^\infty(\mathbb{R}^d)$ functions converging uniformly to $g\in\mathcal{C}(\partial\Omega)$. and thus they are dense in $L^p(\partial\Omega)$.

Moreover, we get the following useful approximation of Lipschitz domains by smooth domains:

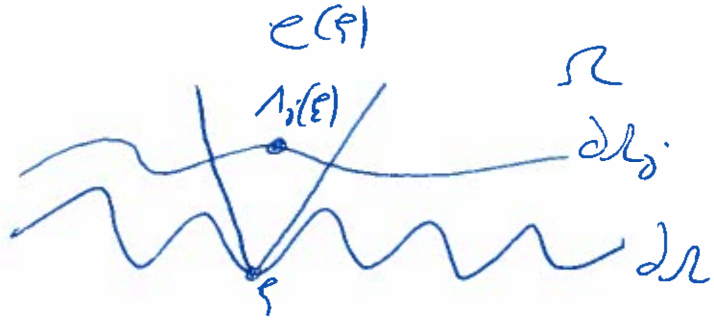
i) Given a bounded Lip. domain Ω, there is a sequence of $\mathcal{C}^\infty$-domains.

$\Omega_j\subset\Omega$ and homeomorphisms Λ_j mapping $\partial\Omega\to\partial\Omega_j$.

s.t. $\sup_{\xi\in\partial\Omega} |\xi-\Lambda_j(\xi)| \to 0$ as $j\to\infty$.

Also, it may be arranged so that $\Lambda_j(\xi) \in \mathcal{C}(\xi)$

$\forall j$ and $\forall \xi \in \partial\Omega$.



ii) The outer unit normal vector

$\nu_{\Lambda_j(\xi)} \longrightarrow \nu_\xi$ pointwise a.e. on $\partial\Omega$,

and in every $L^q(\partial\Omega)$, $1 \le q < \infty$.

iii) there exist positive functions ω_j uniformly bounded above and below away from zero s.t. $\displaystyle\int_{E\subset\partial\Omega} \omega_j \, ds = \int_{\Lambda_j(E)\subset\partial\Omega_j} ds_j$ $\forall$ measurable E

and s.t. $\omega_j \to 1$ a.e. and in every L^q-norm $1 \le q < \infty$, $j \to \infty$.

See [V: "Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains", JFA (1984)]

for a more precise statement.

That this works ~~locally~~ may be seen locally:

Let $\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ be a ~~Lip~~ compactly supported Lipschitz function.

Then, as above, (regularizing)

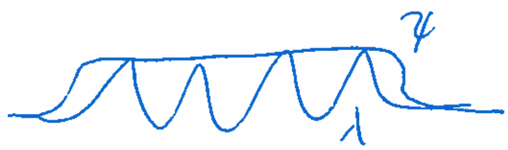
$\varphi_\delta * \lambda \longrightarrow \lambda$ uniformly, in fact

$$|\varphi_\delta * \lambda(x) - \lambda(x)| \le \int_{\mathbb{R}^{d-1}} \varphi_\delta(y)\,|\lambda(x-y) - \lambda(x)|\,dy$$

$$\le \|\nabla\lambda\|_\infty \int_{\mathbb{R}^{d-1}} \varphi_\delta(y)\,|y|\,dy \underset{\substack{\text{change of}\\ \text{variables}}}{=} \|\nabla\lambda\|_\infty\, \delta \int_{\mathbb{R}^{d-1}} \varphi(y)\,|y|\,dy$$

So letting $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1})$ with $\psi \equiv 1$ in a nbhd of supp λ, there is a constant C_λ s.t. $\lambda \le \varphi_\delta * \lambda + \delta C_\lambda \psi \to \lambda$ uniformly as $\delta \to 0$.

Moreover by the above density of $\mathcal{C}^\infty$ convolutions in L^q



$$\nabla(\varphi_\delta * \lambda + \delta C_\lambda \psi) = \varphi_\delta * \nabla\lambda + \delta C_\lambda \nabla\psi \to \nabla\lambda \text{ in every } L^q(\mathbb{R}^{d-1})$$

$1 \le q < \infty$. This last then gives the same for the normal vectors and the surface measures ds_δ on $(x, \varphi_\delta * \lambda(x) + \delta C_\lambda \psi(x))$.

Approximating Ω with $\mathcal{C}^\infty$ domains $\Omega_j \subset \Omega$ as described will be denoted $\Omega_j \uparrow \Omega$. Same idea for $\Omega_j \supset \Omega$, denoted $\Omega_j \downarrow \Omega$.

Assuming $\vec{w} \in \mathcal{C}(\Omega) \cap L^1(\Omega)$ one can then write

$$\int_\Omega \operatorname{div} \vec{w}\,dx = \lim_{\Omega_j \uparrow \Omega} \int_{\Omega_j} \operatorname{div} \vec{w}\,dx = \lim_{\Omega_j \uparrow \Omega} \int_{\partial\Omega_j} \nu_j \cdot \vec{w}\,ds_j$$

$$= \lim_j \int_{\partial\Omega} \nu \circ \Lambda_j \cdot \vec{w} \circ \Lambda_j \; \omega_j\,ds$$

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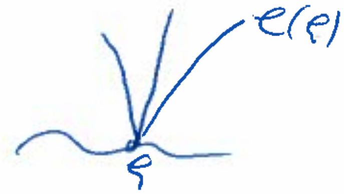
If then $\eta(\vec{w}) \in L^1(ds)$ and and $\vec{w}$ has nontangential limits a.e. (ds), then the pointwise limits of ν_j and w_j and their uniform boundedness justifies L.D.C. so that

$$\int_\Omega \operatorname{div} \vec{w}\, dx = \int_{\partial\Omega} \nu \cdot \vec{w}\, ds.$$

<u>Example</u> $u = Sf$, $f \in L^2(\partial\Omega)$. Then $\eta(\nabla u) \in L^2(\partial\Omega)$.

Single Layer potential

Thus $\nabla u \in L^2(\Omega)$



and $$\int_\Omega |\nabla u|^2 dx = \lim_{j\to\infty} \int_{\partial\Omega_j} \partial_\nu u\, u\, ds_j$$

$$= \lim_j \int_{\partial\Omega} \nu(\Lambda_j(\xi)) \cdot \nabla u(\Lambda_j(\xi))\, u(\Lambda_j(\xi))\, \omega_j(\xi)\, ds(\xi)$$ We have

$\Lambda_j(\xi) \in e(\xi)$ and for any $x \in e(\xi)$, $y \in \partial\Omega$, $|x-y| \geq \operatorname{dis}(x, \partial\Omega) > \frac{1}{\beta}|x-\xi|$

as before and $|y-\xi| \leq |x-y| + |x-\xi| < (\beta+1)\,|x-y|$

So $$|u(x)| = \frac{1}{(d-2)|S^{d-1}|} \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} |f(y)|\, ds(y)$$

$$\leq C_\Omega \int_{\partial\Omega} \frac{1}{|y-\xi|^{d-2}} |f(y)|\, ds(y) \in L^2(\partial\Omega, ds(\xi))$$ when $x = \Lambda_j(\xi)$,

Thus $\eta(\nabla u)\,|Sf| \in L^1(\partial\Omega)$ is a dominating function for the C.D.C.

As a consequence $\int_{\Omega} |Du|^2 dx = \int_{\partial\Omega} \partial_\nu u \, u \, ds$

Given $g \in L^p$ $1 < p < \infty$ we wish to solve

$(\frac{1}{2} I + \mathcal{S}^*) f = g$ for $f \in L^p(\partial\Omega)$ and thus solve the

Dirichlet Problem D_p. I.e. is $\frac{1}{2} I + \mathcal{S}^* : L^p \to L^p$ onto?

In the classical theory for $C^{1,\alpha}$ domains, the operator $\mathcal{S}^*$ is seen to be a compact operator and also by some potential theory the onto question is answered.

Def: For Banach spaces X, Y ~~and~~ a bounded linear operator $K : X \to Y$ is compact if every bounded sequence $\{x_j\}$ in X contains a subsequence s.t. $\{K x_{j_k}\}$ converges in Y.

Theorem If a sequence of compact operators $K_j : X \to Y$ converges in operator-norm i.e. $\|K_j - K\| \to 0$, $j \to \infty$ then K is compact.

The proof uses the standard diagonalization argument of passing to ~~subseqs.~~ successive subsequences $\{x_k\} \supseteq \{x_{k_1}\} \supseteq \{x_{k_2}\} \supseteq \dots$ as operators $K_1, K_2, \dots$ are considered and then ~~using the~~ ~~using the~~ using the "diagonal subsequence". See Yoshida, p. 228.

Other references [G+T], [Folland].

As mentioned earlier, if $\partial\Omega$ is $C^{1,\alpha}$, then for all $\xi, y \in \partial\Omega$.

$|\nu_y \cdot (y-\xi)| \le C_\Omega |y-\xi|^{1+\alpha}$. Locally if $\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ is $C^{1,\alpha}$, then Taylor's Theorem:

$$\lambda(y') = \lambda(\xi') + \nabla\lambda(\xi' + t(y'-\xi'))\cdot(y'-\xi'), \quad t \in (0,1)$$

$$\left|(\nabla\lambda(y'), -1)\cdot(y'-\xi', \lambda(y')-\lambda(\xi'))\right| \le C_\lambda |(1-t)(y'-\xi')|^\alpha |y'-\xi'| \le C_\Omega |y-\xi|^{\alpha+1}$$

Let $\partial\Omega$ be $C^{1,\alpha}$, $\alpha > 0$ and put for $x, y \in \partial\Omega$ $k(x,y) = \frac{1}{|S^{d-1}|}\frac{\nu_y\cdot(y-x)}{|y-x|^d}$. Then

$|k(x,y)| \le C'_\Omega \frac{1}{|x-y|^{d-1-\alpha}}$. and $\int_{\partial\Omega} |k(x,y)|\, ds(y) \le C_\Omega$ independent on $x \in \partial\Omega$.

Similarly $\int_{\partial\Omega} |k(x,y)|\, ds(x) \le C_\Omega$. Then, Jensen:

$$\int_{\partial\Omega} \left| \int_{\partial\Omega} k(x,y) f(y)\, ds(y) \right|^p ds(x) \le C_\Omega^p \cdot \|f\|^p_{L^p(\partial\Omega)}$$

Moreover putting $\mathcal{D}_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x,y) f(y)\, ds(y)$

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$= \int_{\partial\Omega} K_\varepsilon(x,y) f(y) ds(y)$ we can see that

$$\int_{\partial\Omega} |K(x,y) - K_\varepsilon(x,y)|\, ds(y) \leq C_\Omega\, \varepsilon^{\alpha}.$$

So that $\mathring{\mathcal{D}}_\varepsilon \to \mathring{\mathcal{D}}$ in operator norm. as $\varepsilon \to 0$.

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as an operator $L^p(\partial\Omega) \to L^p(\partial\Omega)$, $1 \leq p < \infty$

Further, the $\mathring{\mathcal{D}}_\varepsilon$ are compact. A way to see this quickly for $1 < p < \infty$ is to invoke ~~Abo~~

<u>Thm</u> (Yoshida, p.126) Let X be a reflexive Banach space and let $\{x_n\}$ be a bounded sequence in X, Then there exists a subsequence $\{x_{n'}\}$ which converges <u>weakly</u> to an element $x \in X$, i.e. <u>for every bounded linear functional</u> ~~A~~: $\Lambda \in X'$ we have $\langle \Lambda, x_{n'} \rangle \to \langle \Lambda, x \rangle$ for $n' \to \infty$. ($x_{n'} \rightharpoonup x$)

Thus given a bounded sequence in $L^p(\partial\Omega)$, $1 < p < \infty$ one passes to a subsequence f_j converging weakly to $f \in L^p(\partial\Omega)$.

Write $f_{n_j} \rightharpoonup f$. Since $K_\varepsilon(x,y) \in L^{p'}(ds(y))$ $\forall x$,

$$\mathring{\mathcal{D}}_\varepsilon f_j(x) = \int_{\partial\Omega} K_\varepsilon(x,y) f_j(y) ds(y) \longrightarrow \int_{\partial\Omega} K_\varepsilon(x,y) f(y) ds(y). = \mathring{\mathcal{D}}_\varepsilon f(x) \quad \forall x \in \partial\Omega$$

(61.0)

But the $\mathring{\mathcal{D}}_\varepsilon f_j(x)$ are uniformly bounded functions in j:

(61.1) $\|\mathring{\mathcal{D}}_\varepsilon f_j\|_\infty \leq C_\Omega \frac{1}{\varepsilon^{d-1-\alpha}} \|f_j\|_p$

Thus by L.D.C. on our compact boundary

$$\|\mathring{\mathcal{D}}_\varepsilon f_j - \mathring{\mathcal{D}}_\varepsilon f\|_{L^p} \to 0 \quad \text{as } j \to \infty.$$

Thus $\mathring{\mathcal{D}}_\varepsilon$ is <u>compact</u>, and consequently so is $\mathring{\mathcal{D}} : L^p(\partial\Omega) \to L^p(\partial\Omega)$

$1 < p < \infty$, when $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$.

<u>Remark</u> (61.0) and (61.1) are true on Lipschitz boundaries. Moreover
$\mathring{\mathcal{D}}_\varepsilon \to \mathring{\mathcal{D}}$ in $L^p(\partial\Omega)$ operator norm, $1 \leq p < \infty$, is in fact
true on $\mathcal{C}^1$ boundaries [FJR]. But this is a consequence of A.P. Calderón's
(1977) Preliminary version for <u>small</u> Lipschitz constants
(locally, $\mathcal{C}^1$ boundaries have arbitrarily small Lipschitz constants) of
the [CMM] result. Thus $\mathring{\mathcal{D}}$ is compact on $\mathcal{C}^1$ boundaries.
However, it is known that $\mathring{\mathcal{D}}$ is <u>not</u> compact on Lipschitz boundaries.

We use the following small portion of the "Fredholm Theory" or
"Riesz-Schauder Theory". See Yoshida pp. 282-285

Let $\underline{X}, \underline{Y}$ Banach spaces, $K: \underline{X} \to \underline{Y}$ linear

<u>Thm</u> [Y, p. 282] (Schauder) K is compact if and only if its dual operator $K': \underline{Y}' \to \underline{X}'$ is compact.

<u>Lemma</u> [Y, p. 283] Let $K: \underline{X} \to \underline{X}$ be compact and let $0 \neq \lambda \in \mathbb{C}$. Then $(\lambda I - K)(\underline{X})$ is a <u>closed</u> subspace of $\underline{X}$.

<u>Thm 1 [Y, p. 283]</u> Let $K: \underline{X} \to \underline{X}$ compact, $0 \neq \lambda \in \mathbb{C}$. If $\lambda I - K$ is 1-1 then $\lambda I - K: \underline{X} \to \underline{X}$ is <u>onto</u>.

By Banach's open mapping Theorem $(\lambda I - K)^{-1}$ is a bonded linear operator.

<u>Recall</u> the maximal operator $\mathcal{D}_* f(x) = \sup_{\varepsilon > 0} \{ |\mathcal{D}_\varepsilon f(x)|, x \in \partial\Omega \}$. maps: $L^p(\partial\Omega) \to L^p(\partial\Omega)$ $1 < p < \infty$ by [CMM]. Thus by L.D.C when $g \in L^{p'}(\partial\Omega)$

(max. operator bonded)

$$\int_{\partial\Omega} g(x)\, \mathcal{D} f(x)\, ds(x) = \lim_{\varepsilon \to 0} \int_{\partial\Omega} g(x)\, \mathcal{D}_\varepsilon f(x)\, ds(x)$$

$$\overset{\text{Fub}}{=} \lim_{\varepsilon \to 0} \int_{\partial\Omega} f(y) \left(\int_{\partial\Omega} K_\varepsilon(x,y)\, g(x)\, ds(x) \right) ds(y)$$

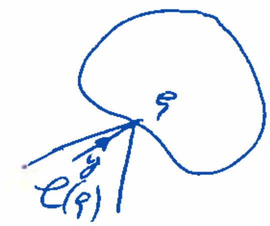
$$= \int_{\partial\Omega} f(y)\, \mathcal{D}' g(y)\, ds(y)$$

because the pointwise limits and corresponding max'l operator results are also true for $\lim\limits_{\varepsilon\to 0}\int_{\partial\Omega} u_\varepsilon(x,y)\, g(x)\, ds(x)$

$$= p.v. \frac{1}{|S^{d-1}|}\int_{\partial\Omega} \frac{\nu_y\cdot(y-x)}{|y-x|^d}\, g(x)\, ds(x)$$ which exists for a.e. $y\in\partial\Omega$

By Analysis similar to the above, the dual operator $\mathcal{D}'$ arises from the Neumann data of the single layer potential

Let ${}_c\mathcal{C}(\xi)$ denote cones for the complementary domain Ω^c.



Then $\lim\limits_{\substack{y\to\xi \\ y\in {}_c\mathcal{C}(\xi)}} \nu(\xi)\cdot\nabla S g(y)$

$$= \frac{1}{2} g(\xi) + p.v. \frac{1}{|S^{d-1}|}\int_{\partial\Omega} \frac{\nu(\xi)(\xi-x)}{|\xi-x|^d}\, g(x)\, ds(x) = \left(\frac{1}{2} I + \mathcal{D}'\right) g(\xi)$$

Denote this normal derivative by ${}_c\nu S_g(\xi)$, $\xi\in\partial\Omega$.

The normal derivative of S_g using the interior cones of Ω is

$\partial\Omega\ni\xi$, $\nu S_g(\xi) = -\frac{1}{2} g(\xi) + \mathcal{D}' g(\xi)$. Note that in both cases (inside and outside) $\vec{\nu}(\xi)$ is the outer normal to Ω.

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To show $\frac{1}{2}I + \mathring{\mathcal{K}}$ is onto, let us first consider only $p=2$, but on Lipschitz domains. For all balls $B_R = B_R(0)$ large enough so that $\overline{\Omega} \subset B_R$

$$(65.1)\quad \int_{B_R \setminus \overline{\Omega}} |\nabla S f|^2 dx = -\int_{\partial\Omega} \partial_\nu S f \cdot S f \, ds + \int_{\partial B_R} \nu S f \cdot S f \, ds.$$

For $x \in \partial B_R$, $\left| \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} f(y)\, ds(y) \right| = O(R^{2-d})$, $R \to \infty$

Likewise $|\nu S f(x)| = O(R^{1-d})$. Since $|\partial B_R| \approx R^{d-1}$

The ∂B_R integral vanish as $R \to \infty$ if $d \geq 3$. (slight modification will yield the $d=2$ results). Consequently $\boxed{\text{if } (65.2)\ \nu_c S f = (\tfrac{1}{2}I + \mathring{\mathcal{K}}')f = 0 \text{ a.e. } (ds)}$

then (65.1) shows $Sf(x) = \text{constant}$ on Ω^c.

By the "continuity across the boundary" of Sf $\lim_{n.t.} Sf(\xi) = \text{const.}$

from the interior. Thus $\int_\Omega |\nabla S f|^2 dx = \int_{\partial\Omega} \nu S f \, S f \, ds = \text{const.} \underbrace{\int_{\partial\Omega} \nu S f\, ds}_{=0.}$

by Gauss and $\Delta S f = 0$. Thus $Sf(x) = \text{const.}\ \forall x \in \Omega$.

Thus $\nu S f = (-\frac{1}{2}I + \mathring{\mathcal{K}}')f = 0$ a.e. $\partial\Omega$

Combine this with $(65.2) \Rightarrow f = 0$ a.e. $\partial\Omega$

In other words $(\frac{1}{2}I + \mathring{\mathcal{D}}')$ is 1-1 on $L^2(\partial\Omega)$. 02/21

By Thm [1, Y] $\Rightarrow$ $\frac{1}{2}I + \mathring{\mathcal{D}}' : L^2 \to L^2$ is onto, thus invertible.

i.e. an isomorphism if $\partial\Omega$ is $\mathcal{C}^1$, since $\mathring{\mathcal{D}}'$ is compact by Schauder, +[F∂R].

If $\partial\Omega$ is $\mathcal{C}^1$ $\frac{1}{2}I + \mathring{\mathcal{D}}' , L^2(\partial\Omega) \to L^2(\partial\Omega)$ isomorphism 02/26

By Banach's closed range thm ([Y], p.205) then $\frac{1}{2}I + \mathring{\mathcal{D}} : L^2 \to L^2$ is an isom. for $\partial\Omega \in \mathcal{C}^1$. Now $L^p(\partial\Omega) \subset L^2(\partial\Omega)$ for $p \geq 2$.

Thus $\frac{1}{2}I + \mathring{\mathcal{D}}'$ is 1-1 on these spaces. By Thm 1 [Y], p. 283

$K : X \to X$ compact, $0 \neq \lambda$ $\lambda I - K$ 1-1 $\Rightarrow$ $\lambda I - K$ onto.

Since $\mathring{\mathcal{D}}$ is compact for $p < \infty$ on $\mathcal{C}^1$ boundaries, $\frac{1}{2}I + \mathring{\mathcal{D}}' : L^p \to L^p$ is an isomorphism, $2 \leq p < \infty$. The same is true for $\frac{1}{2}I + \mathring{\mathcal{D}}$. But then by duality (Banach closed range) both are isomorphisms for $1 < p \leq 2$.

Closed Range Thm. for bounded linear operators: X, Y Banach, $T : X \to Y$.

The following are equivalent:

(1) range $T \subseteq Y$ is closed

(2) range $(T') \subseteq X'$ is closed

(3) Let $\mathrm{Null}(T')^{\perp} = \{y \in Y \mid \langle y', y \rangle = 0 \ \forall y' \in \mathrm{Null}(T')\}$.
then range $(T) = \mathrm{Null}(T')^{\perp}$

range(T')=

(4) $\mathrm{Null}(T)^{\perp} = \{x' \in X' \mid \langle x', x\rangle = 0 \ \forall x \in \mathrm{Null}\,T\}$

then $\mathrm{range}(T') = \mathrm{Null}(T)^{\perp}$.

Applied to the layer potentials (on $\mathcal{C}^1$-domains)

(i) $\frac{1}{2}I + \mathcal{K}'$ is 1-1 for $2 \le p' < \infty$. (also true on Lipschitz)

(ii) Compactness of $\mathcal{K}'$ and Thm.1 [4] $\Rightarrow$ $\frac{1}{2}I + \mathcal{K}'$ is onto, $2 \le p' < \infty$.

(iii) (2) holds in closed range Thm: $\frac{1}{2}I + \mathcal{K}'$ has closed range $= L^{p'}(\partial\Omega)$ $2 \le p' < \infty$.

(iv) (4) holds $\Rightarrow$ $\frac{1}{2}I + \mathcal{K}$ is 1-1 for $1 < p \le 2$.

v) Compactness + Thm 1 $\Rightarrow$ $\frac{1}{2}I + \mathcal{K}$ is onto. $1 < p \le 2$

- By Compactness of $\partial\Omega$ and iv), $\frac{1}{2}I + \mathcal{K}$ is 1-1 for $1 < p < \infty$.

This shows existence for D_p, $1 < p < \infty$, on $\mathcal{C}^1$-domains Ω. by first showing existence for the Neumann Problems N_p, $1 < p < \infty$. for ~~the~~ $\bar{\Omega}^c$. (N_p) for domain Ω: Given $g \in L^p(\partial\Omega)$ find a unique harmonic u on Ω s.t.

i) $\int_\Omega u\,dx = 0$ ii) $\lim_{\substack{x \to \xi \\ x \in \mathcal{C}(\xi)}} \nu(\xi)\cdot\nabla u(x) = g(\xi)$ a.e. (ds)

iii) $\eta(\nabla u) \in L^p(\partial\Omega)$.

Here $\dot{L}^p(\partial\Omega)$ denotes functions with mean value 0, i.e.

$$\int_{\partial\Omega} g\,ds = 0$$

Left open are uniqueness results: for D_p, suppose $\Delta u = 0$ in Ω and $u \xrightarrow{n.t.} 0$ a.e. and $\eta(u) \in L^p(\partial\Omega)$. Why is $u \equiv 0$ in Ω?

Example Define Lipschitz domains in the plane $\Omega_q = \{ z \in \mathbb{C} \; z = re^{i\theta}, 0<r<1, 0<\theta<q\pi \}$ for $0<q<2$. In each domain, consider the harmonic function $u_q = \operatorname{Im}(z^{1/q} + z^{-1/q})$

$= (r^{1/q} - r^{-1/q}) \sin\frac{\theta}{q}$. u_q extends continuously to 0 everywhere on $\partial\Omega_q \setminus \{0\}$. Introduce suitable n.t. approach regions $\eta(u_q)(z) \approx |z|^{-1/q} = r^{-1/q}$ on the radial part of $\partial\Omega_q$.

Thus $\eta(u_q) \in L^p(\partial\Omega_q)$ for $p<q$. But u_q is not zero in Ω_q.

Uniqueness fails for D_p for every $p<2$ by choosing $p<q<2$.

Without compactness of $\mathcal{K}$, $\mathcal{K}'$ on Lipschitz boundaries, the closed range of $\frac{1}{2}I + \mathcal{K}'$ must be obtained in a different way. If it is, then the fact that $\text{Null}(\frac{1}{2}I + \mathcal{K}') = \{0\}$, $2 \le p' < \infty$.

Gives by (3) of closed range thm, that $\frac{1}{2}I + \mathring{K}$ is <u>onto</u>

for any $1<p\le 2$ for which range $(\frac{1}{2}I + \mathring{K}') \subseteq L^{p'}$ is closed.

At least existence for D_p follows. This different way is the main result of the article [V] JFA 1984 and is based on an identity previously used by a number of authors (see, for example, [J-K] paper; Harmonic Analysis techniques for 2^{nd} order elliptic boundary value problems], Kenig, 1991, p.112) and introduced into this Lipschitz-domain theory by Jerison – Kenig: Bull. Am.M Soc. pp 203-207 (1981). Expressed locally, let Ω be the region below the graph of a compactly supported Lipschitz function. $\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ and suppose $u \in C^1(\overline{\Omega})$ is harmonic in Ω, $\nabla u \in L^2(\Omega)$ with sufficient decay at ∞. Then

$$(70.1) \quad \int_{\partial\Omega} |\nabla u|^2 \nu_d \, ds = 2\int_{\partial\Omega} \nu\cdot\nabla u \, \partial_d u \, ds.$$

$$\nu = \frac{(-\nabla\lambda, 1)}{\sqrt{1+|\nabla\lambda|^2}}$$

<u>Pf</u>: $L.S. = 2\int_{\Omega} \nabla u \cdot \nabla \partial_d u \, dx = 2\int_{\partial\Omega} \partial_\nu u \, \partial_d u \, ds.$ $\square$

"<u>Rellich identities</u>" observe

$$\nu_d \ge \frac{1}{\sqrt{1+\|\nabla\lambda\|_\infty^2}} >> 0$$

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$$\frac{1}{\sqrt{1+\|\nabla\lambda\|_\infty^2}}\|\nabla u\|^2_{L^2(\partial\Omega)} \le 2\|\partial_\nu u\|_{L^2(\partial\Omega)}\|\partial_d u\|_{L^2(\partial\Omega)}$$

Since each of the norms on the right is dominated by the norm on the left,

locally there is C_Ω depending on the Lip. constant for Ω s.t.

$$(71.1)\quad \|\nabla u\|_{L^2(\partial\Omega)} \le C_\Omega \|\partial_\nu u\|_{L^2(\partial\Omega)}. \quad \text{(Rellich inequality)}$$

Moreover, [JK, Bulletin AMS] show that (70.1) can be rearranged, let e_d denote the dth unit basis vector in x_d-direction. Then $e_d = (e_d\cdot\nu)\cdot\nu + (e_d - (e_d\cdot\nu)\nu)$

and denote the second vector by τ, then $\nu\cdot\tau = 0$, $|\tau|^2 = 1-\nu_d^2 \le 1$.

Likewise $\nabla u = \partial_\nu u\,\vec\nu + (\nabla u - \partial_\nu u\,\vec\nu) = \partial_\nu u\,\vec\nu + \nabla_T u$

then $|\nabla u|^2 = |\partial_\nu u|^2 + |\nabla_T u|^2$.

$$2\int_{\partial\Omega}\partial_\nu u\,\partial_d u\,ds = 2\int_{\partial\Omega}(\partial_\nu u)^2\nu_d\,ds + 2\int_{\partial\Omega}\partial_\nu u\,\tau\cdot\nabla u\,ds$$

$$= \int_{\partial\Omega}(\partial_\nu u)^2\nu_d\,ds + \int_{\partial\Omega}|\nabla_T u|^2\nu_d\,ds$$

use — and ⌴

subtracting $2\int_{\partial\Omega}\partial_\nu u\,\tau\cdot\nabla u\,ds$ and using Young's inequality:

$$2AB = 2(\sqrt{\delta}A)\left(\frac{B}{\sqrt{\delta}}\right) \le \delta A^2 + \frac{B^2}{\delta}.\quad \forall \delta, A, B > 0$$

We set (if $\delta = \frac{1}{2}\nu_d$)

$$2\int_{\partial\Omega} (\partial_\nu u)^2 \nu_d \, ds \le \frac{3}{2}\int_{\partial\Omega} (\partial_\nu u)^2 \nu_d \, ds + 2\int_{\partial\Omega} (z\cdot\nabla u)^2 \nu_d^{-1} \, ds$$

$$+ \int_{\Omega} |\nabla_T u|^2 \nu_d \, ds$$

Consequently a 3rd Rellich inequality is

$$(73.1) \qquad \|\partial_\nu u\|_{L^2(\partial\Omega)} \le C_\Omega \|\nabla_T u\|_{L^2(\partial\Omega)}$$

(71.1) and (73.1) can be obtained globally and for harmonic functions on either Ω or $\overline{\Omega}^c$, i.e. $\partial_\nu u$ = to either ∂u or $\nu_c u$ as long as one requires $\int_{\partial\Omega} u \, ds = 0$ for (73.1) in the exterior case.

Note that (73.1) when followed by (71.1) allows to control ∂Sf by ${}_c\partial Sf$ and vice versa: (73.1) from the interior permits any $f \in L^2(\partial\Omega)$ as does (71.1)

so that $\|(-\frac{1}{2}I + \mathcal{K}')f\|_2 \overset{(73.1)}{\le} C_\Omega \|\nabla_T Sf\|_2 \overset{(71.1)}{\le} C_\Omega^2 \|{}_c\partial Sf\|_2$

$= C_\Omega^2 \|(\frac{1}{2}I + \mathcal{K}')f\|_2$. We have used "continuity across the boundary of Sf"

By the triangle inequality

$$(74.1) \quad \|f\|_2 \le (1 + C_\Omega^2) \, \|(\tfrac{1}{2} I + \mathring{\mathcal{D}}')f\|_2$$

This shows $\frac{1}{2} I + \mathring{\mathcal{D}}'$ is 1-1 on $L^2(\partial\Omega)$ which we already knew, but also shows $\frac{1}{2} I + \mathring{\mathcal{D}}'$ has closed range for $p' = 2$

$(\frac{1}{2} I + \mathring{\mathcal{D}}')f_j \xrightarrow{L^2} g$. By (74.1) $\{f_j\}$ is Cauchy. Thus $f_j \xrightarrow{L^2} f$ & by continuity of $\frac{1}{2} I + \mathring{\mathcal{D}}'$, $(\frac{1}{2} I + \mathring{\mathcal{D}}')f_j \to (\frac{1}{2} I + \mathring{\mathcal{D}}')f$

$\Rightarrow f = g$. Thus $\frac{1}{2} I + \mathring{\mathcal{D}}'$ has closed range in L^2.

This is enough to show that $\frac{1}{2} I + \mathring{\mathcal{D}}$ is onto. However, the operator $\frac{1}{2} I + \mathring{\mathcal{D}}$ is actually 1-1, which follows b/c $\frac{1}{2} I + \mathring{\mathcal{D}}'$ is onto. This latter follows & in different ways, the nicest uses the method of continuity [GT, p.75]. We will state this in a more general way and use the following singular integral operator result again due to [CMM], see p. 54 Kenig's book

Thm [CMM]: Let λ_j be compactly supported Lipschitz functions converging in Lip-norm to λ, that is $\|\lambda_j - \lambda\|_\infty + \|\nabla\lambda_j - \nabla\lambda\|_\infty \to 0$.

Let $\mathring{\mathcal{D}}_j$ and $\mathring{\mathcal{D}}$ be the corresponding potential operators.
singular integral

Then in operator norms

$$\| \mathcal{D}_\delta' - \mathcal{D}' \|_{L^p \to L^p} \to 0, \quad 1 < p < \infty$$

Note: The smooth fn $\varphi_\varepsilon * \lambda \not\to \lambda$ in L^p-norm, i.e.

$\mathcal{D}'$ is not shown to be compact

Method of continuity: Let $\bar{X}$ be a Banach space and V a normed linear space and let the map ~~$\mathbb{R} \ni t \mapsto L_t \in [0,1] \to$~~ be

~~the~~ $t \mapsto$ ~~the~~ L_t, $L_t : \bar{X} \to V$ bounded linear operator be cont. i.e.

(75.09) $t \to s \Rightarrow \| L_t - L_s \|_{\bar{X} \to V} \to 0$. Suppose $\exists$ a constant C_0 s.t.

$$\begin{cases} t \mapsto L_t \\ [0,1] \to \mathcal{L}(\bar{X}, V). \end{cases}$$

(75.1) $\|x\| \leq C \| L_t x \|$ $\forall x \in \bar{X}$, $t \in [0,1]$

Then $\boxed{L_1 \text{ is onto } V \text{ iff } L_0 \text{ is onto } V}$

Pf: Suppose L_s onto for some $s \in [0,1]$. Then by assumption (75.1) L_s is 1-1 and so L_s^{-1} exists with $\| L_s^{-1} v \| \leq C \|v\|$ $\forall v \in V$ i.e.

(76.05) $\| L_s^{-1} \|_{V \to \bar{X}} \leq C$. Fix $v \in V$ and suppose we wish to solve

$L_t x = v$. This is the same as

(76.1) $L_t x = L_s x + (L_t - L_s) x = v$ or

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(76.2) $x + L_s^{-1}(L_t - L_s)x = L_s^{-1}v.$

But for all t in a nbhd of s

(76.3) $\| L_s^{-1}(L_t - L_s) \|_{\overline{X} \to \overline{X}} < \frac{1}{2}$ by (75.09) Thus

by expanding in a Neumann series:

$\left[I + L_s^{-1}(L_t - L_s)\right]^{-1}$ is well defined and therefore a unique x_t solves (76.2) and also (76.1). Note that (76.3) is independent of v and hence so is t. Thus $S = \{ s \in [0,1] \mid L_s \text{ is onto} \}$ is open

Now suppose t is a limit point of S. Once again (76.1), (76.2) and because of the uniform bound (76.05) on L_s^{-1} (76.3) holds for some s in S, since t is a limit point. Thus L_t is onto and S is hence closed.

$\square$

Thus $S = \phi$ or $S = [0,1]$.

Locally To show $\frac{1}{2}I + \mathcal{D}'$ is onto:

- $\mathcal{D}'$ is associated with a compactly supp $\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$.
- Let $\mathcal{D}_t'$ be associated with $t\lambda$, $0 \le t \le 1$.

The map $t \mapsto t\lambda$ is cont. : $[0,1] \to \text{Lip}$.

- By [CMM] the map $t \mapsto \mathcal{D}_t'$ is continuous

$$t \mapsto \frac{1}{|S^{d-1}|} \int_{\mathbb{R}^{d-1}} \frac{t\nabla\lambda(y')(y'-x') - t(\lambda(y')-\lambda(x'))}{\left[|y'-x'|^2 + t^2(\lambda(y')-\lambda(x'))^2\right]^{d/2}} f(x')dx'$$ 02/28

is not linear.

The bound (79.1) $\|f\|_2 \leq (1+C_\Omega)^2 \|(\tfrac{1}{2}I + \mathcal{K})_t' f\|_2$

unif. in $t \in [0,1]$. The Method of continuity gives the result

~~Sin~~ Since $\frac{1}{2}I + \mathcal{K}_0' = \frac{1}{2}I$ is onto.

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Rellich identities lead to an equivalence $\|\partial_\nu u\|_2 \simeq \|\nabla_T u\|_2$

(71.1) & (73.1). That overcomes the lack of compactness of the layer

potential boundary integral operators for $L^2(\partial\Omega)$ when $\partial\Omega$ is Lipschitz.

One even obtains invertibility, in particular $\frac{1}{2}I + \mathcal{K} : L^2(\partial\Omega) \to L^2(\partial\Omega)$

and $-\frac{1}{2}I + \mathcal{K}' : \mathring{L}^2(\partial\Omega) \to \mathring{L}^2(\partial\Omega)$ (mean value zero) are

isomorphisms, proving existence for interior Dirichlet and Neumann problems

Uniqueness for the Neumann Problem is straightforward.

Suppose $N(\nabla u) \in L^2(\partial\Omega)$, and $\partial_\nu u \overset{n.t.}{=} 0$ a.e. Then

$$\int_\Omega |\nabla u|^2 dx = \lim_{\Omega_j \nearrow \Omega} \int_{\partial\Omega_j} \partial_\nu u \, u \, ds = 0 \text{ by L.D.C. if}$$

we can have $N(u) \in L^2(\partial\Omega)$ also.

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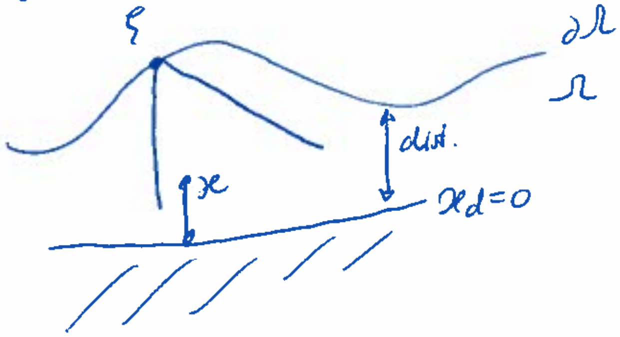
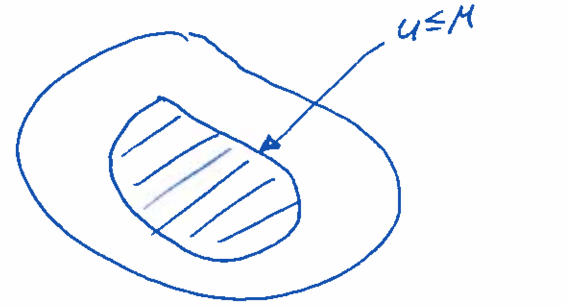
Locally:





$$u(x) = \int_0^{x_d} \frac{d}{dt} u(x', t)\,dt + u(x', 0)$$

So $|u(x)| \le \underset{\text{diam}(D)}{\text{dist.}}\ N(\nabla u)(\xi) + M. \quad \in L^2(\partial\Omega).$

There exists fundamental solution for all constant coefficient linear PDE-operators. $Lu = \delta_0$ s.d. in $\mathbb{R}^d$.

(Yoshida, pp. 182 - ...) (Malgrange Ehrenpreis-Theorem)

In particular for the elliptic homogeneous operators L of order $2m$, $m \ge 1$ there are fundamental solutions $\Gamma_L(x)$ homogeneous of degree $2m - d$ in $\mathbb{R}^d$, $d > 2m$, i.e. $\Gamma_L(tx) = t^{2m-d}\,\Gamma_L(x)$, $t > 0$, $x \in \mathbb{R}^d$.

See pp. 65-69 Fritz John, Plane waves and Spherical Means (1955).

When L is real (real coefficients), then so is Γ_L. We will assume that L is real. For example the operators Δ^m, $m \in \mathbb{N}$.

are elliptic and have fundamental solutions $\Gamma_m = C_{d,m} \dfrac{1}{|x|^{d-2m}}$.

$d > 2m$ in $\mathbb{R}^d$ where $C_{d,m}$ is determined by $\Delta^m \Gamma_m = \delta_0$ s.d.

The Dirichlet Problem for

$$(80.05) \quad L = \sum_{|\alpha| = 2m} a_\alpha \partial^\alpha$$

prescribes values for the normal derivatives up to order $m-1$ of a solution u on $\partial\Omega$.

$$(80.1) \quad u, \partial_\nu u, \ldots, \partial_\nu^{m-1} u.$$

Here $\partial_\nu^k u = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha \partial^\alpha u$.

Where $\alpha! = \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_d!$. For example this definition is

$$\partial_\nu^2 u = \sum_{j,k=1}^{d} \nu_j \nu_k \partial_j \partial_k u, \quad \partial_\nu^3 u = \sum_{j,k,\ell=1}^{d} \nu_j \nu_k \nu_\ell \partial_j \partial_k \partial_\ell u$$

In local coordinates, prescribing u also prescribes

$$(81.1) \quad \frac{\partial}{\partial x_j} u(x', \lambda(x')) = \partial_j u + \frac{\partial \lambda}{\partial x_j} \partial_d u \quad \text{for } 1 \le j \le d-1.$$

Prescribing the normal derivatives determines $\Leftarrow \partial_\nu u = \nabla u \cdot \nu$, $\nu = (\nabla' \lambda, -1)$

$$(81.2) \quad \nabla' \lambda \cdot \nabla' u - \partial_d u$$

$\nabla' = (\partial_1, \ldots, \partial_{d-1})$.

$(81.1) \Rightarrow \nabla'\lambda \cdot \nabla' u + |\nabla'\lambda|^2 \partial_d u$ is determined

So with (81.2) $\nabla u|\partial\Omega$ is completely determined. ☒

More generally, the boundary values (80.1) determine the array of boundary values $(\partial^\alpha u)_{|\alpha| \leq m-1}$ on $\partial\Omega$.

Prescribing (80.1) is in fact equivalent to prescribing $(\partial^\alpha u)_{|\alpha| \leq m-1}$ $= (g_\alpha)_{|\alpha| \leq m-1}$ on $\partial\Omega$ if we define that compatibility conditions like (81.1) are met. Precisely, in local coordinates $(g_\alpha)_{|\alpha| \leq m-1}$ is an array if

$$\frac{\partial}{\partial x_j} g_\alpha(x', \lambda(x')) = g_{\alpha+e_j}(x', \lambda(x')) + \frac{\partial\lambda}{\partial x_j}(x') \cdot g_{\alpha+e_d}(x', \lambda(x'))$$

$\forall\ 1 \leq j \leq d-1,\ 0 \leq |\alpha| \leq m-2$. One can see that our definition of Dirichlet Data at (80.1) is correct in that it implies uniqueness at least for the polyharmonic operators $\Delta^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \partial^{2\alpha}$ when solutions are smooth up to $\partial\Omega$:

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If all derivatives of order $|\alpha| \le m-1$ vanish on $\partial\Omega$ for a solution

$\Delta^m u = 0$, then

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega |\partial^\alpha u|^2 dx = 0$$ by Gauss, so that

$\partial^\alpha u = \text{const.}$ in Ω for all $|\alpha| = m-1$, but then const. $=0$. by

$\partial^\alpha u = 0$ on $\partial\Omega$, $|\alpha| = m-1$. So $\partial^\alpha u = \text{const.}$ $\forall |\alpha| = m-2 \Rightarrow \dots$ etc.

On the other hand, for the strongly elliptic $L = \partial_1^4 + \partial_2^4$

(symbol $\xi_1^4 + \xi_2^4 > 0$ if $\xi \neq 0$)

$$\int_\Omega |\partial_1^2 u|^2 + |\partial_2^2 u|^2 \, dx = 0$$ yields no <u>direct</u> information about

$\partial_1 \partial_2 u$. This hints at our main problem.

Writing (80.05) $\displaystyle L = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha \partial^\beta$ for some choice of

coefficients $a_{\alpha\beta}$, define an <u>associated sesquilinear form</u>

$$A[u,v] = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_\Omega \partial^\alpha \bar{u} \, \partial^\beta v \, dx.$$

letting $u = \Gamma_L^x(y) = \Gamma_L(x-y)$ and using Gauss to obtain $L\Gamma^x = \delta_x$, the resulting boundary integrals pair derivatives of order γ of v $\gamma = 0, \dots, m-1$ with order $2m-1-\gamma$ derivatives of Γ_L. By the degree of homogeneity of $\Gamma_L(y) = O\left(\frac{|y|^{2m}}{|y|^d}\right)$,

$$D^{2m-1-\gamma}\Gamma_L(y) = O\left(\frac{|y|^\gamma}{|y|^{d-1}}\right)$$ we obtain a Green's representation

that formally looks like

$$(83.1)\quad A[\Gamma_L^x, v] = \sum_{\gamma=0}^{m-1} \int_{\partial\Omega} O\left(\frac{|x-y|^\gamma}{|x-y|^{d-1}}\right) D^\gamma v(y)\, ds(y)$$

$$+ (-1)^m v(x)$$

When v is a solution, the left side will become the analog of the single layer potential while the integrals on the right place the role of the double layer acting on the Dirichlet-Data of v.
In principle, this double layer is as before. Thinking of the highest order data $D^{m-1}v \in C^\alpha(\partial\Omega)$ and computing D_x^{m-1} from (83.1)

the $\gamma = m-1$ integral becomes 03/05

$$\int_{\partial\Omega} G\left(\frac{1}{|x-y|^{d-1}}\right) D^{m-1} v\, ds.$$ and [CMM] will again yield

pointwise limits and maximal functions estimates on Lipschitz boundaries, ~~etc~~.

For the $\gamma < m-1$ terms the integral kernels become "hyper-singular"

after applying D_x^{m-1} but there are ways to justify transfering

$m-1-\gamma$ derivatives from the kernel to $D^\gamma v|_{\partial\Omega}$ so that these terms also

are well behaved.

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<u>Recall</u> (83.1) $$A[\Gamma_L^x, v] = \sum_{\gamma=0}^{m-1} \int_{\partial\Omega} G\left(\frac{(x-y)^\gamma}{|x-y|^{d-1}}\right) D^\gamma v(y)\, ds(y)$$

$$+ (-1)^m v(x)$$

When these double layer terms are taken on their own,

as with $\mathcal{D}f$, $f \in L^p(\partial\Omega)$, one substitutes for the $D^\gamma v|_{\partial\Omega}$

an array $(f_\alpha)_{\gamma=|\alpha|\le m-1}$, with $f_\alpha \in L^p(\partial\Omega)$ when $|\alpha| = m-1$

and the compatibility (82.1) is used in any transfer of derivatives to the lower order f_α in order to reduce the hypersingular kernels to singular integral kernels. Before we get to the remaining obstacle to inverting the double layer potentials arising in this higher order theory, it needs to be said, that the $L^2(\partial\Omega)$ Dirichlet Problem can be solved without inverting the double layer potentials.

D_2 (D_p, $2-\varepsilon < p \le \infty$) for Δ was solved by Dahlberg, Studia Math Vol. 66, 1979 pp. 13–24, by a careful study of the Poisson integral for Lipschitz domains. The Green function $G(x,y)$ exists by Perron-Method but without the $C^1(\bar{\Omega}\setminus\{x\})$ regularity we assumed above. An abstract Poisson kernel called harmonic measure $d\omega^x$ exists by the Riesz-Representation theorem: $g \mapsto Hg(x) : C(\partial\Omega) \to \mathbb{R}$ where Hg is the Perron-solution for g and $x \in \Omega$ is a positive linear functional on $C(\partial\Omega)$ by the maximum principle. Thus $Hg(x) = \int_{\partial\Omega} g \, d\omega^x \quad g \in C(\partial\Omega), x \in \Omega$ where ω^x is the unique positive Borel measure guaranteed by Riesz. The total variation $|\omega^x| = 1$ again because of the maximum principle. (b.c. $u = 1$ unique solut. ~~to per~~ Perron)

Dahlberg identifies $d\omega^x$ with $\partial_{\nu(y)} G(x,y)\, ds(y)$ showing that the harmonic measure is absolutely cont. w.r.t. the surface (Lebesgue) measure and $\partial_\nu G^x \in L^2(\partial\Omega)\ \forall x \in \Omega$. Then the Poisson integral for $g \in L^2(\partial\Omega)$ at least is defined, i.e.

$u(x) = \int_{\partial\Omega} \partial_\nu G^x g\, ds$. That $N(u) \in L^2(\partial\Omega)$ requires additional use of the maximum principle, harmonic measure and positive harmonic functions. These tools do not seem available for higher order operators and systems. [Jerison-Kenig] (1980) Bull. of Am. Math. Soc. pp. 307-322 (see also [JK] 1982) ~~a~~ rediscovered the original Rellich-Identity's for the Green function and ~~its~~ its normal derivative, giving another proof of Dahlberg's results, in particular $\partial_\nu G \in L^2(\partial\Omega)$, still with reliance on positivity and the maximum principle. However, in [JK] (1982), they also extend to Lipschitz domains (see [FJR]) another boundary value problem, known as regularity for the Dirichlet problem.

R_2: let $g \in W^{1,2}(\partial\Omega)$, the Sobolev space of 03/07

L^2-functions with distributional derivatives $\frac{\partial}{\partial x_j} g(x, \lambda(x)) \in L^2(\partial\Omega)$.

Find a unique harmonic u in Ω s.t.

(1) $N(\nabla u) \in L^2(\partial\Omega)$

(2) $u \xrightarrow{n.t.} g$ a.e. $(\partial\Omega)$

(3) ∇u has nontangential limits a.e. $(\partial\Omega)$ s.t.

$$\frac{\partial}{\partial x_j} g \overset{n.t.}{=} \partial_j u + \frac{\partial\lambda}{\partial x_j} \partial_d u \quad \text{a.e.}$$

The solution of R_2 gives an easy proof that $\partial_\nu G^x \in L^2(\partial\Omega)$

which can also be extended easily to higher order operators.

Recall $G(x,y) = \Gamma(x-y) - h^x(y)$ with $G(x,y) = 0$ for $y \in \partial\Omega$

so that for each $x \in \Omega$

$$(88.1) \quad N(\nabla h^x) \in L^2(\partial\Omega)$$

Here h^x solution of R_2 with Dirichlet data $\Gamma^x|_{\partial\Omega}$ which has $L^2(\partial\Omega)$ tangential derivatives.

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The Poisson integral with Dahlberg's regularity properties on the Green function answers the uniqueness question for D_p, $2-\varepsilon < p \le \infty$. This follows also from (88.1) by imitating an argument of [FJR]. One can construct $C_0^\infty(\Omega)$ functions ψ_ε s.t. $\psi_\varepsilon(x) = 1$ for x s.t. $\mathrm{dist}(x,\partial\Omega) > C_\Omega \varepsilon$, $\varepsilon > 0$

<u>Example</u> locally with Ω below graph of a Lip. function $\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$

Let $\varphi \in C_0^\infty(B_1(0))$ so that $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\frac{x}{\varepsilon})$ is an approximation of the identity. Let χ_Ω be the characteristic

function of Ω. Then $\chi_\Omega * \varphi_\varepsilon$ will have a <u>gradient supported</u> in a neighborhood of $\partial\Omega$ with points no more than a vertical distance of $\varepsilon \cdot C_\lambda$ from $\partial\Omega$. So define $\psi_\varepsilon(x) = \chi_\Omega * \varphi_\varepsilon(x + C_\lambda e_d)$.

Note that $\|\nabla \psi_\varepsilon\|_\infty = \|\nabla \chi_\Omega * \varphi_\varepsilon\|_\infty \le \frac{1}{\varepsilon}\|\nabla\varphi\|_1$

Suppose Δu, $N(u) \in L^2(\partial\Omega)$ and $u \xrightarrow{n.t.} 0$ a.e. Show $u = 0$ in Ω. For <u>any fixed</u> $x \in \Omega$, Once $\varepsilon > 0$ is small enough.

$$u(x) = \cancel{x}\, \psi_\varepsilon(x)\, u(x)$$

$$= \int_\Omega G^x(y)\, \Delta_y(\cancel{x}\,\psi_\varepsilon u)(y)\, dy$$

$$(90.1)\ \begin{cases} = \int_\Omega G^x(\Delta \psi_\varepsilon u + 2\nabla\psi_\varepsilon \cdot \nabla u)\, dy \\ = -\int_\Omega \nabla G^x \cdot \nabla \psi_\varepsilon\, u\, dy + \int_\Omega G^x \nabla\psi_\varepsilon \cdot \nabla u\, dy \\ = -\int_\Omega G^x \Delta \psi_\varepsilon\, u\, dy - 2\int_\Omega \nabla G^x\, \nabla\psi_\varepsilon\, u\, dy. \end{cases}$$

(90.1) is $\quad I + II = III + II = I + III$. each of which

vanishes of $\varepsilon \to 0$:

In all three, integration ~~is~~ is over S_ε and $|S_\varepsilon| \approx \varepsilon|\partial\Omega|$ ← $d-1$-dimensional

i.e. $\frac{1}{\varepsilon}|S_\varepsilon| \approx |\partial\Omega|$

$I \quad \|\Delta \psi_\varepsilon\| \leq \varepsilon^{-2} C$. Since $G^x \xrightarrow{u.t.} 0$ a.e.

$$y \in S_\varepsilon \Rightarrow G^x(y) = -\int_{g_d}^{\lambda(y')} \frac{\partial}{\partial t} G^x(y', t)\, dt \quad \text{a.e. } (y').$$

and $|G^x(y)| \leq C\varepsilon\, N(\nabla G^x)(y')$.

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Thus, $I \lesssim \frac{1}{\varepsilon} \int_{S_\varepsilon} \eta(\nabla G^x) |u| \, dy \to 0$ by L.D.C.

$(\eta(u) \in C^2(\partial\Omega),\ u \xrightarrow{a.e.} 0)$

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So, $u(x) = 0$

$\| \partial^\alpha \psi_\varepsilon \|_\infty \leq C \varepsilon^{-|\alpha|}$

Fritz John, 1955, p. 155 $\quad Lu = 0$ and $\operatorname{dist}(x, \partial\Omega) = \varepsilon$, then

$$|\partial^\alpha u(x)| \leq C_L \varepsilon^{-|\alpha|} \max_{|y-x| < \varepsilon/2} |u(y)|$$ "Interior estimates"

If we assume solutions to the Dirichlet Problem on Smooth domains then we can solve R_2, by the method of a priori estimates. Let $\Omega_j \downarrow \Omega$ and take $\varphi \in C_0^\infty(\mathbb{R}^d)$. Let u_j solve the Dirichlet problem in Ω_j for data $\varphi|_{\partial\Omega_j}$. The classical layer potential method will show $u_j \in C^\infty(\overline{\Omega_j})$ (Folland, [JK] 1982)

Recall the 3rd Rellich inequality (73.1)

$$\|\nabla u\|_{L^2(\partial\Omega)} \le C\,\|\nabla_T u\|_{L^2(\partial\Omega)}, \quad u \in \mathcal{C}^1(\bar{\Omega}),\ \Delta u = 0.$$

This is an a priori estimate for "nice solutions."

Because the Lipschitz constants for the boundaries $\partial\Omega_j$, $\partial\Omega$ are uniform, Dahlberg's result implies for each harmonic function $\partial_k u_j$, $1 \le k \le d$ in Ω_j that $\|N(\partial_k u_j)\|_{L^2(\partial\Omega_j)} \le C\,\|\partial_k u_j\|_{L^2(\partial\Omega_j)}$ with C independent of j. Thus, $\|N(\nabla u_j)\|_{L^2(\partial\Omega)} \le C\,\|\nabla u_j\|_{L^2(\partial\Omega_j)}$ (92.1)

On the other hand, there is a way to get this last result by using the uniform boundedness [CMM] of the layer potential operators with respect to the uniform Lipschitz constants by using Green's representation formula. The point is that this method generalizes to higher order operators easily.

For a smooth domain ω and harmonic $h \in C^2(\bar{\omega})$ 03/19

For $\nabla h(x) = \nabla h(x) - \int \partial_\nu \nabla h(x)$

However: (Here: summation convention: $\nu_\ell \partial_\ell$ repeated indices

means $\sum_{\ell=1}^{d} \nu_\ell \partial_\ell$)

$$\nu_\ell \partial_\ell \partial_k h = \nu_\ell \partial_k \partial_\ell h - \nu_k \partial_\ell \partial_\ell h$$

$$= \underbrace{(\nu_\ell \partial_k - \nu_k \partial_\ell)}_{\partial_{\tau_{ij}}} \partial_\ell h \quad \text{and } \nu_\ell \partial_k - \nu_k \partial_\ell \text{ is}$$

a <u>tangential derivative</u> that obeys:

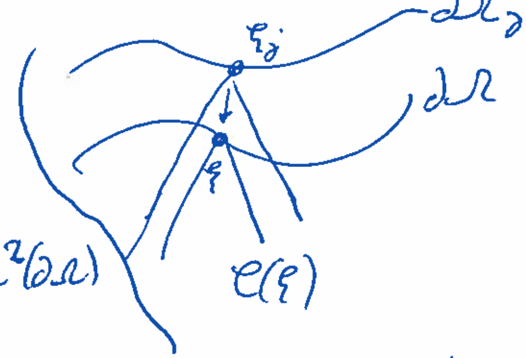
$$\int_{\partial\omega} \partial_\tau F\, G\, ds = -\int_{\partial\omega} F\, \partial_\tau G\, ds$$ independently of any

"interior" values of F and G. (Use Gauss even on Lip. when justified). Thus, by transferring the tangential derivatives to the single layer kernel. [CMM] gives (92.1)

The 3rd method also gives a uniform constant, so continuing (92.1) we obtain:

(94.05) $\|N(\nabla u)\|_{L^2(\partial\Omega_j)} \le C \|\nabla_T \varphi\|_{L^2(\partial\Omega_j)}$

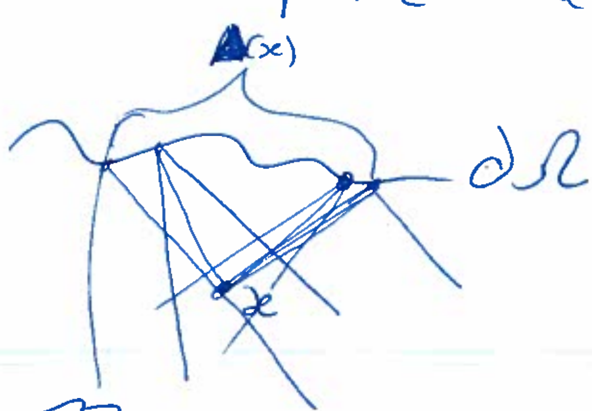
For all j large enough, $\partial\Omega$ may replace $\partial\Omega_j$ on the right (as long as $\nabla_T\varphi|_{\partial\Omega} \neq 0$). Also on the left because the approach regions $\partial\Omega_j$ can be arranged to contain those of the Lipschitz domain $\partial\Omega$.



(94.1) $\|N(\nabla u_j)\|_{L^2(\partial\Omega_j)} \le C\|\nabla_T\varphi\|_{L^2(\partial\Omega)}$

uniformly in j. When one has control in $L^r(\partial\Omega)$ of a nontangential max'l function $N(F)$ then one has a uniform bound on F over compactly contained subsets of Ω: Fix $x \in \Omega$. By the uniformity of the n.t. approach regions, there is a surface disk $\Delta(x) = \{y \in \partial\Omega$ s.t. $e(y) \ni x\}$ and "the radius" of this disk will be comparable to the distance

dist $=$ dist$(x, \partial\Omega)$.



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Thus $|F(x)|^p \leq \left(\fint_{\Delta(x)} \eta(F)\,ds \right)^p$

$\leq C_\Omega \operatorname{dist}^{1-d} \|\eta(F)\|^p_{L^p(\partial\Omega)}$. Thus for $\overline{\Omega'} \subset \Omega$

compactly contained $(94.1) \Rightarrow \|\nabla u_j\|_{L^\infty(\Omega')} \leq C_\Omega \operatorname{dist}(\Omega', \partial\Omega)^{\frac{1-d}{2}}$

$\cdot \|\nabla_T \varphi\|_{L^2(\partial\Omega)}$

By the John interior estimates

(91.1) <u>any</u> derivative of the u_j $\forall j$ can now be uniformly controlled on compacts. All of this works for <u>solutions</u> to higher order $Lu = 0$.

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One obtains equicont. sequences $\{u_j\}$, $\{\nabla u_j\}$ etc. with therefore uniformly convergent subsequences to solutions on $\overline{\Omega'} \subset \Omega$ by the Arzela-Ascoli Theorem. Further, one may take $\Omega_k' \uparrow \Omega$ and use the diagonalization argument to obtain a subsequence $u_j \to u$ uniformly on each Ω_k' with u harmonic in all Ω.

and u corresponding to the data $\varphi|_{\partial\Omega}$. If one puts

$$\eta_k(\nabla u)(y) = \sup_{\mathcal{C}(y)\cap\Omega'_k} |\nabla u| \,, y \in \partial\Omega.$$ one obtains

from (94.1) $\|\eta(\nabla u_j)\|_{L^2(\partial\Omega)} \leq C\|\nabla_T \varphi\|_{L^2(\partial\Omega)}$

that $\|\eta_k(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|\nabla_T\varphi\|_{L^2(\partial\Omega)}$,

and by monotone convergence

(96.1) $\|\eta(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|\nabla_T\varphi\|_{L^2(\partial\Omega)}$

Now, pointwise convergence at the boundary for ∇u will follow by using the Green's representation in Ω for the smooth $\partial_k u_j$

(97.1) $$\partial_k u_j(x) = \mathcal{D}(\partial_k u_j)(x) + \frac{1}{(2-d)|S^{d-1}|}\int_{\partial\Omega}\left(\nu_e(y)\frac{\partial}{\partial y_k} - \nu_k(y)\frac{\partial}{\partial y_e}\right)\frac{1}{|x-y|^{d-2}} \cdot \partial_e u_j(y)\,ds(y)$$

$\forall\, 1 \leq k \leq d$, all j.

By the uniform $L^2(\partial\Omega)$ bound (94.1) $\forall j, k$ $\exists$ subsequence

~~$\partial_k u_j$~~ $\partial_k u_j \longrightarrow f_k \in L^2(\partial\Omega)$ for each k.

Thus, $\partial_k u(x) = \mathcal{D} f_k(x) + \sum_\ell \mathcal{D}^{\ell k} f_\ell(x)$, $x \in \Omega$

where the $\mathcal{D}^{\ell k}$ are [CMM] potentials from (97.1), and

therefore $\lim\limits_{\substack{n.t.\\ x \to \partial\Omega}} \nabla u(x)$ exists a.e. in $L^2(\partial\Omega)$.

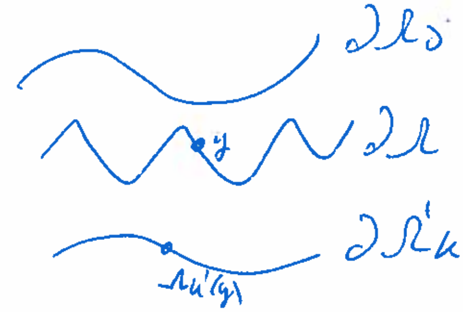
Similarly, one obtains uniform control of $\|N(u_j)\|_{L^2(\partial\Omega_j)}$ etc.

and $u \xrightarrow{n.t.}$ a.e $(\partial\Omega)$. That $\lim\limits_{n.t.} u = \varphi$ a.e. $(\partial\Omega)$

is seen as follows: let $\Omega'_k \uparrow \Omega$ and given $\varepsilon > 0$

choose k large enough so that

$$\max_{y \in \partial\Omega} |y - \Lambda'_k(y)| < \varepsilon$$



Then

(98.1) $$\int_{\partial\Omega} \left|\lim_{n.t.} u - u \circ \Lambda'_k\right|^2 ds \le \int_{\partial\Omega} ds(y) \left(\int_y^{\Lambda'_k(y)} |\partial_t u| \, dt \right)^2 \le \varepsilon^2 \int_{\partial\Omega} \cancel{ds(y)} \, |N(\nabla u)|^2 ds = \varepsilon^2 C^2 \|\nabla_T \varphi\|_2^2$$

For all j large enough, by the uniform convergence on compacts

$$\int_{\partial\Omega} |u \circ \Lambda'_k - u_j \circ \Lambda'_k|^2 ds < \varepsilon^2$$

Recalling that $u_{\gamma_i} \circ \Lambda_{\gamma_i} = \varphi \circ \Lambda_{\gamma_i}$ and arguing

as (98.1) we can compare

$$\int_{\partial\Omega} |\varphi\circ\Lambda_{\gamma_i} - u_{\gamma_i}\circ\Lambda_{u}|^2 ds \leq 2\varepsilon^2 C^2 \|\nabla_T \varphi\|_2^2$$

for all γ_i large enough, this time using (94.05)

$$\left(\|\eta(\nabla u_{\gamma_i})\|_{L^2(\Omega_{\gamma_i})} \leq C \|\nabla_T \varphi\|_{L^2(\partial\Omega_{\gamma_i})} \right) \text{ etc.}$$

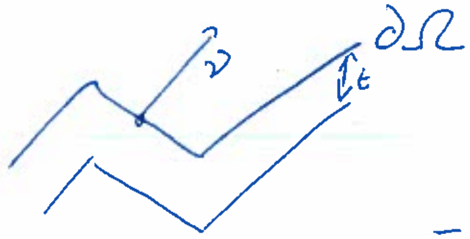
Finally $\int_{\partial\Omega} |\varphi\circ\Lambda_{\gamma_i} - \varphi|^2 ds < \varepsilon^2$ by continuity of φ for

γ_i large enough. Altogether $\|\lim_{n.t.} u - \varphi\|_{L^2(\partial\Omega)} \leq O(\varepsilon)$.

Thus, for smooth $g = \varphi$ it remains to prove (iii) of R_2.

Work locally and let $\psi \in C_0^\infty(\mathbb{R}^{d-1})$. Translate the values of u on

a parallel boundary inside Ω by $u_t(y) = u(y', y_d - t)$ $y \in \bar{\Omega}$, $t > 0$



$$\int_{\partial\Omega} (\nu_d\, \partial_j u_\epsilon - \nu_j \partial_d u_\epsilon)\, \psi(y')\, ds(y)$$

$$= \int_{\mathbb{R}^{d-1}} \frac{\partial}{\partial y_j} u_\epsilon(y', \lambda(y'))\, \psi(y')\, dy'$$

$$= -\int_{\mathbb{R}^{d-1}} u_\epsilon(y', \lambda(y')) \frac{\partial}{\partial y_j} \psi(y')\, dy' \to -\int_{\mathbb{R}^{d-1}} \varphi(y', \lambda(y')) \frac{\partial}{\partial y_j} \psi(y')\, dy'$$

as $\epsilon \downarrow 0$. ~~Now~~ $\cancel{\lim_{\epsilon\to 0} \nu_d \partial_j u_\epsilon - \partial_j \nu_d}$

Now, $\lim_{\epsilon\to 0} \nu_d\, \partial_j u_\epsilon - \nu_j \partial_d u_\epsilon = \nu_d\, \partial_j u - \nu_j \partial_d u$ exists a.e.

and the very last integral is also $\int_{\mathbb{R}^{d-1}} \frac{\partial}{\partial y_j} \varphi(y', \lambda(y'))\, \psi(y')\, dy'$.

Since this is true for all $\psi \in C_c^\infty(\mathbb{R}^{d-1})$ (dense in L^r)

$$\frac{\partial}{\partial y_j} \varphi(y', \lambda(y')) = \partial_j f u(y) + \frac{\partial \lambda}{\partial y_j} \partial_d u(y) \quad \text{a.e. } y \in \partial\Omega.$$

$g \in W^{1,r}(\partial\Omega)$ is by definition ~~and~~ an $L^r(\partial\Omega)$ functions

for which there exists $L^r(\partial\Omega)$ functions <u>denoted</u> $\frac{\partial}{\partial y_j} g$ s.t.

$$\int_{\mathbb{R}^{d-1}} \frac{\partial}{\partial y_j} g(y', \lambda(y')) \psi(y')\, dy'$$

$$= - \int_{\mathbb{R}^{d-1}} g(y', \lambda(y')) \frac{\partial}{\partial y_j} \psi(y')\, dy' \quad \forall \psi. \quad j = 1, -, d-1 \ (\text{locally})$$

i.e. $\frac{\partial}{\partial y_j} g$ exists. S.o.d.

In this way the last argument follows by defn. of $W^{1,2}(\partial\Omega)$ as long as u with Dirichlet data g has n.t. limits for its ∇u.

Obtaining such a solution entails another approximation argument. Now for solutions u_j in Ω with smooth data $\varphi_j \to g$ in $W^{1,2}(\partial\Omega)$

Here, as in $\mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$, the Sobolev spaces $W^{1,r}(\partial\Omega)$ of functions g with weak derivatives $\frac{\partial}{\partial y_j} g$ are also the completion of $\mathcal{C}^\infty$ functions under the Sobolev norm $\|\varphi\|_{L^p(\partial\Omega)} + \|\nabla_T \varphi\|_{L^r(\partial\Omega)}$.

The n.t. limits will now again be proved with the [Stein] p.8 lim sup - lim inf argument, since $\varphi|_{\partial\Omega}$ are good data dense in $W^{1,2}(\partial\Omega)$. See pp.30-32 [CPV, 1995] Annals of Math. for a complete solution to R_2. 03/21

Thus, assuming solutions on smooth domains with smooth data, the a priori Rellich estimate and Green's representation lead to the solution in Lipschitz domains with Lebesgue-integrable data.

This can also be done for the Dirichlet Problem D_2 using the second (Rellich) inequality of (71.1)

$$\|\nabla u\|_{L^2(\partial\Omega)} \le C \|\partial_d u\|_{L^2(\partial\Omega)}$$

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$$(71.1) \quad \|\nabla u\|_{L^2(\partial\Omega)} \le C \|\partial_d u\|_{L^2(\partial\Omega)}$$

The idea is to obtain a-priori-estimates on solutions $u(x)$ under the assumption of sufficient decay as $x_d \downarrow -\infty$.

by defining $v(x) = \int_{-\infty}^{x_d} u(x', t)\,dt$ ("primitive")

So that $u = \partial_d v$. and v is also harmonic:

$$(\partial_1^2 + \dots + \partial_{d-1}^2) v = \int_{-\infty}^{x_d} -\partial_t^2 u \, dt = -\partial_d u = -\partial_d^2 v.$$

By the Green's representation

(97.1) and (97.2): $\boxed{\partial_n u(x) = \mathcal{D} f_u(x) + \sum_e \mathcal{D}^{ek} f_e(x)}$

$$u = \partial_d v = \Phi^{dd}\partial_d v + \sum_{1\le j\le d-1} \Phi^{jd}\partial_j v.$$

Now [CMM] + Rellich yield (71.1)

$$\|N(u)\|_{L^2} = \|N(\partial_d v)\|_{L^2} \le C\,\|\nabla v\|_{L^2(\partial\Omega)} \le C'\,\|\partial_d v\|_{L^2(\partial\Omega)}$$
$$= C'\,\|u\|_{L^2(\partial\Omega)}.$$

Note: $\partial\Omega = \mathbb{R}^{d-1}$ then $\partial_j v|_{\mathbb{R}^{d-1}}$, $1\le j\le d-1$ these are the classical

~~Riesz~~ Riesz-transforms of $u|_{\partial\Omega}$.

With one more idea, this scheme will solve D_2 and R_2 for all of our higher order operators — [PV, 1995].

Let $L = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\,\partial^{\alpha+\beta}$, $Lu = 0$ and consider the Dirichlet-Problem.

Notation let $\nabla^{m-1}u$ denote all $(m-1)$-order derivatives. To use Green's representation and [CMM] to estimate $N(\nabla^{m-1}u)$, let $u = \partial_d^m v$ with decay at $-\infty$, so $Lv = 0$. Now, for all $|\gamma| = m-1$ the transfer of tangential derivatives may be accomplished by integrating in by parts the bilinear form

$$\sum_{\alpha,\beta} a_{\alpha\beta} \int_\Omega \partial^\alpha \partial^\gamma \Pi_L^x \partial^\beta \partial_d^m v \, dy$$ in 2 distinct ways.

Top: $(-1)^m \partial^\alpha \partial_d^m v(x) + \sum_d \int_{\partial\Omega} O\left(\frac{1}{|x-y|^{d-1}}\right) D^{m-1} \partial_d^m v \, ds(y)$

Bottom

$= 0 + \sum \int_{\partial\Omega} O\left(\frac{1}{|x-y|^{d-1}}\right) \cancel{\partial^{m-1} \partial_d^m v \, ds(y)} \, D^{2m-1} v \, ds(y)$ $\quad$ dem $\partial_d^m v = u$ solution!

To return to the Dirichlet-Data of u "the bottom" requires

$$\| D^{2m-1} v \|_{L^2(\partial\Omega)} \leq C \| D^{m-1} \partial_d^m v \|_{L^2(\partial\Omega)}$$

A Rellich Identity with any $|\gamma| = m-1$ is obtained by

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\partial\Omega} \partial^\alpha (\partial^\gamma v) \, \partial^\beta (\partial^\gamma v) \underset{\substack{\uparrow \\ \text{normal}}}{N_d} \, ds$$

$$= \sum_{\alpha,\beta} a_{\alpha\beta} \int_\Omega \partial^\alpha (\partial^\gamma v) \, \partial^\beta (\partial^\gamma \partial_d v) \, dx + \text{similar term}$$

$$= \sum \int_{\partial\Omega} D^{2m-1} v \, D^{m-1} \partial^\gamma \partial_d v \, ds \quad \text{and then}$$

$$\le C \|\nabla^{2m-1} v\|_{L^2(\partial\Omega)} \|\nabla^{m-1}\partial^\gamma \partial_d v\|_{L^2(\partial\Omega)}$$

Thus, if the left side (the form) dominates $c\|\nabla^m \partial^\gamma v\|^2_{L^2(\partial\Omega)}$ for some $c>0$ depending only on Lipschitz constant etc., we get

$$(104.1) \quad \|\nabla^m \partial^\gamma v\|^2 \le C' \|\nabla^{2m-1} v\| \; \|\nabla^{m-1}\partial^\gamma \partial_d v\| \quad \forall |\gamma| = m-1.$$

Since this holds for all γ, we get:

$$(104.2) \quad \|\nabla^{2m-1} v\| \le C'' \|\nabla^{2m-2}\partial_d v\| \quad (\text{Cancel one } \|\nabla^{2m-1} v\|)$$

Now, replace $\partial^\gamma v$ in (104.1) with

$\partial^\beta \partial_d v$ for any β of order $m-2$.

$$\Rightarrow \|\nabla^m \partial^\beta \partial_d v\|^2 \lesssim \|\nabla^{2m-1} v\| \|\nabla^{m-1}\partial^\beta \partial_d^2 v\| \overset{(104.2)}{\lesssim} \|\nabla^{2m-2}\partial_d v\| \cdot \|\nabla^{m-1}\partial^\beta \partial_d^2 v\|$$

Since for all $|\beta| = m-2$, we get $\|\nabla^{2m-2}\partial_d v\| \lesssim \|\nabla^{2m-3}\partial_d^2 v\|$

Further, $(104.2) \Rightarrow \|\nabla^{2m-1} v\| \lesssim \|\nabla^{2m-3}\partial_d^2 v\|$

This continues until $\|\nabla^{2m-1} v\| \lesssim \|\nabla^{m-1}\partial_d^m v\|$. Then (104.1) no longer works.

In this way, the estimate $\|K(\nabla^{m-1}u)\|_2 \lesssim C\|\nabla^{m-1}u\|_2$ 03/26

is obtained apriori and D_2 is solved by an approximation scheme as above for the R_2-Laplacian. The layer potentials here are not shown to be invertible.

R_2 is also solvable with $u = \partial_d^{m-1}V$. and $\|K(\nabla^m u)\|_{L^2(\partial\Omega)} \leq C\|\nabla_T \nabla^{m-1} u\|_{L^2(\partial\Omega)}$ with a rewriting of the Rellich identity similar to that leading to the 3rd Rellich inequality, pp. 13-14 [PV: 1995]. But recall this all depends on the following coercive estimate on the boundary quadratic form associated to L:

$$(106.1) \quad \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\partial\Omega} \partial^\alpha u\, \partial^\beta u\, N_d\, ds \geq c\|\nabla^m u\|^2_{L^2(\partial\Omega)}.$$

And this is not true for every L as soon as $m \geq 2$

- $m=1$ ellipticity is $a_{jk}\xi_j\xi_k \geq c|\xi|^2$, $c>0$

  $\Rightarrow a_{jk}\partial_j u\, \partial_k u \geq c|\nabla u|^2$, pointwise

- $L = \partial_1^4 + \partial_2^4$ has symbol $L(\xi) = \xi_1^4 + \xi_2^4 \geq c|\xi|^4$.

The pointwise form

$(\partial_1^2 u)^2 + (\partial_2^2 u)^2$ would not bound from above $c(\partial_1\partial_2 u)^2$.

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<u>Notation</u> $\partial_j u = u_j$, $\partial_j \partial_k u = u_{jk}$. etc. and also $\partial^\alpha u = u_\alpha$.

On the other hand $\partial_1^4 + \partial_2^4 = (1-a)\partial_1^4 + (1-a)\partial_2^4 + a(\partial_1^2 - \partial_2^2)^2 + 2a(\partial_1\partial_2)^2$

for $0<a<1$. This suggests the pointwise form for L

$$(107.1) \qquad (1-a)(u_{11})^2 + (1-a)u_{22}^2 + a(u_{11} - u_{22})^2 + 2a\, u_{12}^2.$$

$$\geq C_a \left(u_{11}^2 + u_{12}^2 + u_{22}^2 \right).$$

to give (106.1).

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The "difference" between the pointwise quadratic forms (107.1)

and (106.2) (i.e. $a=0$) is a multiple of $u_{12}^2 - u_{11}u_{22}$.

This "difference" made into a symmetric integral-differential-bilinear form

over Ω would be

$$(107.2) \qquad N[u,v] = \int_\Omega 2u_{12}v_{12} - u_{11}v_{22} - u_{22}v_{11}\, dx.$$

Note that N is completely determined by the boundary values of u and

v, since Gauss implies

$$N[u,v] = \int_{\partial\Omega} \partial_2 u_1 v_2 - \partial_\tau u_2 v_1 \, ds, \quad \tau = (-\nu_2, \nu_1)$$

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$N[u,v]$ is called a null-form, since the differential operator over Ω is associated with $L=0$.

Thus every higher order operator (L) has alternative associated bilinear form obtained by adding null form to any given associated form. And all associated forms to L differs by null forms.

This is also true for the Laplacian, when associated sesquilinear forms

$$a[u,v] = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_\Omega \partial^\alpha \bar{u}\, \partial^\beta v\, dx$$ with complex coefficients

are considered. The form

$$4\int_\Omega \bar\partial u\, \partial \bar v - \partial u\, \bar\partial \bar v \, dx \qquad \bar\partial = \tfrac{1}{2}(\partial_1 + i\partial_2)$$

is associated to $L=0$ $\quad 4\partial\bar\partial = \Delta$

is such a null form. An associated form is called formally positive, when

(109.1)
$$\begin{cases} A[u,v] = \displaystyle\sum_{\gamma=1}^{N} \int_\Omega \overline{P_\gamma(\partial)\, u}\; P_\gamma(\partial)\, v\, dx & \text{and} \\ L = \displaystyle\sum_{\gamma=1}^{N} |P_\gamma(\partial)|^2 & \text{for homogeneous polynomials} \end{cases}$$

$P_1, \dots, P_N$ of order m. ((107.1) is formally positive)

If an L were associated to some formally positive form, we could hope to obtain the boundary coercive estimate (106.1) by a pointwise inequality like (107.1).

Such an L would also necessarily be a sum of squares (SOS) of m-th order operators.

Equivalently, the positive definite (pd) polynomial

$L(\xi) > 0$, $\xi \neq 0$. would be a SOS.

But not every homogeneous positive definite polynomial of degree $2m$, $m \geq 2$ is a SOS. (Hilbert 1888 verifying a statement of Minkowski.

It's easier to see this at first for positive semidefinite polynomials (psd).

Let w, x, y, z be real variables and consider

$$M = w^4 + (xy)^2 + (yz)^2 + (zx)^2 - 4wxyz. \text{ in } \mathbb{R}^4.$$

Motzkin Proc. Symp. Wright-Patterson Air Force Base, 1965 pp. 205–224.

By the arithmetic geom. mean inequality:

$$wxyz = w\sqrt{xy}\sqrt{yz}\sqrt{zx} \overset{!}{\leq} \frac{w^4 + (xy)^2 + (yz)^2 + (zx)^2}{4}$$

$\Rightarrow M \geq 0$

Suppose: $M = \sum_{\gamma=1}^{N} \left(P_\gamma(w,x,y,z)\right)^2$

For homogeneous of quadratic P_γ's.

Possible monomials:

(111.0)

$$\begin{array}{cccc} & & & z^2 \\ & & y^2 & yz \\ & x^2 & xy & xz \\ w^2 & \boxed{wx} & \boxed{wy} & \boxed{wz} \end{array}$$

(Choi + Lam) First, ① none of the polynomial contains x^2, y^2, z^2 since

there exists no $x^4 = (x^2)^2$ etc. in M

Second, any $(wx)^2$ term can only be cancelled by $-w^2x^2$ which cannot exist

by ①.

Third w^2, xy, xz, yz cannot produce a $wxyz$ term. $\Rightarrow$ M is not a SOS.

Now $M_\varepsilon = M + \varepsilon(x^4+y^4+z^4)$, $\varepsilon>0$. is positive definite, since

$M_\varepsilon = 0 \Rightarrow 0 \geq \varepsilon(x^4+y^4+z^4) \Rightarrow x,y,z=0 \Rightarrow M_\varepsilon = M = w^4 = 0 \Rightarrow w=0.$

Suppose M_ε is a SOS for all $\varepsilon>0$.

Any SOS for M_ε may be rewritten as a sum of at most 10 squares.

To see this we introduce the idea of a ~~gram~~ Gram-matrix for SOS.

For a coleumn vector $0 \neq a \in \mathbb{R}^{10}$ the symmetric matrix

$a a^t$ is a 10×10 matrix and has rank 1. It is positive semidefinite

with eigenvalues $|a|^2 > 0$ and 0 with multiplicity 9.

By linearly ordering the monomials $w^2 < x^2 < y^2 < \ldots$ in some way let

m be the column vector of monomials $m_1, \ldots, m_{10}$. Then, any quadratic

polynomial in w, x, y, z is given by $a^t m$, some a. It's square is

$m^t (a a^t) m$. A sum of such squares would be

$m^t \left(\sum_{j=1}^{N} a_j a_j^t \right) m$ with the 10×10 matrix. p.s.d. and symmetric.

But conversely if $A_{10 \times 10}$ is psd and symmetric, then

$A = U D U^T$ where D is the diagonal matrix of eigenvalues. and

the column vectors of U are the corresponding ~~___~~ orthonormal eigenvectors.

(Herstein, Topics in Algebra p. 346)

And $m^t A m$ is a sum of at most 10 squares.

A symmetric psd matrix representing a sum of squares is known as a

Gram-matrix.

Note that when $\sum_{j=1}^{N} a_j a_j^t = \mathcal{U} D \mathcal{U}^t$ each a_j is a linear combination of only the columns of $\mathcal{U}$ that correspond to nonzero eigenvalues.

If u is a column of $\mathcal{U}$ with zero eigenvalue, then

$$u^t \sum_{j=1}^{N} a_j a_j^t u = \sum_{j=1}^{N} (a_j \cdot u)^2 = u^t \mathcal{U} D \mathcal{U}^t u = 0$$

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(113.1) $$\sum_{j=1}^{N} a_j a_j^T = \mathcal{U} D \mathcal{U}^t \quad a_j \text{ lin. comb. of only the columns}$$

of $\mathcal{U}$ corresponding to the nonzero eigenvalues. In this way we will see that SOS representations are "different" only when the gram matrices differ. E.g. $p_1^2 + p_2^2 = (\gamma p_1 + \sigma p_2)^2 + (\sigma p_1 - \gamma p_2)^2$

$$= L \quad \text{for } p_j \text{ m-th degree}$$

are not "different" representations for L with only L. (Here $\gamma = \cos\theta, \sigma = \sin\theta$)

The Gram matrices are the same. On the other hand, the sum of squares representations seen in the quadratic form (102.1) for $x^4 + y^4$ are

$$[x^2 y^2 xy] \begin{bmatrix} 1 & -a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 2a \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}$$ where the matrix is gram for $0 \le 1 \le 1$

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The ~~gr~~ Gram-matrices change by a change matrix:

$$\begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 2a \end{bmatrix}$$ which represents the zero polynomial.

Apply to $\begin{bmatrix} u_{11} \\ u_{22} \\ u_{12} \end{bmatrix}$, we get the null form from (107.2)

$a \, N[u,u]$. The SOS representation (107.1) has 4 squares, not 3.

Compute $$\begin{bmatrix} 1 & -a & \cdot \\ -a & 1 & \cdot \\ \cdot & \cdot & 2a \end{bmatrix} = \begin{bmatrix} 1 & -a & \cdot \\ -a & a^2 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & 1-a^2 & \cdot \\ \cdot & \cdot & 2a \end{bmatrix}$$

So $x^4 + y^4 = (x^2 - ay^2)^2 + (1-a^2)y^4 + 2a(xy)^2$

Note $$\begin{bmatrix} a & b & \cdots \\ b & c & \ddots \\ \vdots & & \ddots \end{bmatrix} = a \begin{bmatrix} 1 & b/a & \cdots \\ b/a & c/a & \ddots \\ \vdots & & \ddots \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & b/a & \cdots \\ b/a & b^2/a^2 & \\ \vdots & & \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots \\ 0 & c - \frac{b^2}{a^2} & \cdots \\ 0 & \vdots & \end{bmatrix}$$

We will see that coerciveness for formally positive forms requires there be no nonzero complex roots shared in common

(116.0) by the m-th order polynomials being squared. (Aronzajn's Condition) $^{09}/_{a}$

For $x^2 - ay^2 = \sqrt{1-a^2}\, y^2 = \sqrt{2a}\, xy = 0$. there is only the zero root in $\mathbb{C}^2$, $0 \le a \le 1$.

However, for Δ^2 the gram matrices are (in 2D)

$$\begin{bmatrix} 1 & 1-a & \cdot \\ 1-a & 1 & \cdot \\ \cdot & \cdot & 2a \end{bmatrix}, \quad 0 \le a \le 2$$

and $(x^2+y^2)^2 = (x^2 + (1-a)y^2)^2 + (2a-a^2)y^4 + 2a(xy)^2$.

and $(1, i) \in \mathbb{C}^2$ is a root, when $a = 0$. The quadratic form $\int_\Omega |\Delta u|^2\, dx$ or $\oint_{\partial\Omega} |\Delta u|^2 \,\nu_2\, ds$ vanishes for all harmonic u. (harmonic $\Rightarrow \Delta^2 u = 0$). But $0 < a \le 2 \xrightarrow{\text{Aronzajn}}$ coercive.

Thus, when we write $\sum_{\gamma=1}^{N} a_\gamma a_\gamma^T = U D U^t$ so that

$$(117.1) \quad \sum_{\gamma=1}^{N} (a_\gamma \cdot m)^2 = \sum_{k=1}^{10} \lambda_k (u_k \cdot m)^2 \text{ where } U = [u_1, \ldots, u_{10}]$$

and $\lambda_k \ge 0$ are eigenvalues, statements (116.0) + (113.1) mean that the coerciveness properties of the two SOS (117.1) are the same.

Back to the perturbed Motzkin polynomials:

M_ε, assumed to be SOS for all $\varepsilon > 0$, the Gram matrix eigenvalues must be uniformly bounded from above.

For example, use the norm in $\mathbb{R}^4$

$\|(w, x, y, z)\| = \max\{\cancel{|w|}\,|x|, |y|, |z|\}$. Then for any unit vector

$u \in \mathbb{R}^{10}$, $\max\limits_{\|(w,x,y,z)\|=1} |u \cdot m(w,x,y,z)| \geq \frac{1}{\sqrt{10}}$

Thus $\frac{1}{10} \lambda_j^{(\varepsilon)} \leq \max\limits_{\|(w,x,y,z)\|=1} \sum_{k=1}^{10} \lambda_k^{(\varepsilon)} (u_k^\varepsilon \cdot m)^2$

$= \max\limits_{\|(w,\ldots)\|=1} (w^4 + (xy)^2 + \ldots + 4wxyz + \varepsilon(x^4 + \ldots)) \leq 9, \quad 1 \leq j \leq 10$

Since all the components of $U^{(\varepsilon)}$, $D^{(\varepsilon)}$ are unif. bounded, it is possible to pass to a subsequence as $\varepsilon \searrow 0$ s.t.

$$M = \lim_{\varepsilon \to 0} M_\varepsilon = \lim_{\varepsilon \to 0} U^{(\varepsilon)} D^{(\varepsilon)} (U^{(\varepsilon)})^t = U D U^t$$ with U

unitary, D diagonal. Thus for $\varepsilon > 0$ small enough none of the pd M_ε are sums of squares.

i.e. $M_\xi(\partial)$ are elliptic, homogeneous 4-th order in $\mathbb{R}^4$ with <u>no</u> formally positive (quadratic) forms.

On the other hand, the desired boundary coercive estimate (106.1),

$$(119.1) \quad \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\partial\Omega} \partial^\alpha_t u \, \partial^\beta u \, \nu_d \, ds \geq c \, \| D^m u \|^2_{L^2(\partial\Omega)}$$

(i) is for <u>solution</u> (e.g. $M_\xi(\partial) u = 0$ in Ω)

(ii) is an integral estimate, not a pointwise one.

Neither (i) nor (ii) has been used yet. We wish to show that, using (i) and (ii), (119.1) will still not be true for our constant coefficient operators L, at least if we insist that the form coefficients $a_{\alpha\beta}$ <u>remain constant also</u>.

The <u>classical coercive estimate</u> over Ω is

($A[v] = A[v,v]$ for quadratic form).

(120.0)

$$A[v] \geq c \, \| D^m v \|^2_{L^2(\Omega)} - C_0 \int_\Omega |v|^2 dx. \quad \text{for } W^{m,2}(\Omega),$$

the Sobolev space of $L^2(\Omega)$ functions with distributional derivatives ~~all~~ in $L^2(\Omega)$ of orders up to $|\alpha| = m$. ~~a~~

04/02

<u>Claim</u> Boundary coercivity for <u>solutions</u> (119.1)

$$\Rightarrow (120.0) \text{ for } v \in W^{m,2}(\Omega).$$

Thinking of Ω ~~as a~~ as a domain above the graph of $\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$

let $\Omega_t = \Omega + t e_d$, $t > 0$. Then

$$c \int_{\Omega \setminus \Omega_1} |D^m u|^2 dx = c \int_{\mathbb{R}^{d-1}} dx' \int_{\lambda(x')}^{\lambda(x')+1} |D^m u(x', x_d)|^2 dx_d$$

$$= c \int_{\mathbb{R}^{d-1}} dx' \int_0^1 |D^m u(x', \lambda(x')+t)|^2 dt$$

$$= c \int_0^1 dt \int_{\partial\Omega_t} |D^m u|^2 (-\nu_d)\, ds_t$$

$$\overset{(119.1)}{\leq} \int_0^1 dt \sum_{|\alpha|+|\beta| = m} a_{\alpha\beta} \int_{\partial\Omega_t} \partial^\alpha u\, \partial^\beta u\, (-\nu_d)\, ds_t$$

$$= \sum_{\alpha,\beta} \int_{\Omega \setminus \Omega_1} a_{\alpha\beta}\, \partial^\alpha u\, \partial^\beta u\, dx$$

This local version may be done ~~to~~ globally for a bounded domain. 04/02

Classical coercive inequality 04/04

$$A[v] \geq c\|D^m v\|^2_{L^2(\Omega)} - c_0 \int_\Omega |u|^2\,dx \quad \forall v \in W^{m,2}(\Omega) \tag{120.1}$$

Boundary coerciveness for solutions $\Rightarrow$ $c\int_{\Omega\setminus\Omega_1} |D^m u|^2 dx \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\Omega\setminus\Omega_1} \partial^\alpha u\, \partial^\beta u\, dx$

If u solution, i.e. $Lu = 0$ in Ω.

By the interior estimates ([John, 1955]) both $\int_{\Omega_1} |D^m u|^2 dx$ and

$\sum a_{\alpha\beta} \int_{\Omega_1} \partial^\alpha u\, \partial^\beta u\, dx$ may be controlled by $\int_\Omega |u|^2 dx$. The classical

Coercive inequality (120.1) follows for solutions in place of $v \in W^{m,2}(\Omega)$.

Next, let $\varphi \in C_0^\infty(\Omega)$ (or $C_0^m(\Omega)$). Use the Fourier transform.

$$\hat{\varphi}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \varphi(x)\, e^{-ix\cdot\xi}\, dx \qquad \text{(Agmon p.53)}$$

Integration by parts shows $\widehat{\partial^\alpha \varphi}(\xi) = i^{|\alpha|} \xi^\alpha \hat{\varphi}(\xi)$

The parseval identity yields

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_\Omega \overline{\partial^\alpha \varphi}\, \partial^\beta \varphi \, dx$$

$$= \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\mathbb{R}^d} \xi^\alpha \overline{\hat{\varphi}}\, \xi^\beta \hat{\varphi}\, d\xi = \cancel{\sum} \int_{\mathbb{R}^d} L(\xi)\, |\hat{\varphi}|^2 \, d\xi$$

$$\geq C_L \int_{\mathbb{R}^d} |\xi|^{2m} |\hat{\varphi}|^2 d\xi = \cancel{C_L \sum_{|\alpha|=|\beta|=m}}$$

$$(122.0) \quad = C_L \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^d} \xi^{2\alpha} |\hat{\varphi}|^2 d\xi = C_L \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega |\partial^\alpha \varphi|^2 dx$$

We have the strong coercive estimate

$$(122.1) \quad A[\varphi] \geq C_L |\varphi|_m^2 , \quad \varphi \in C_0^\infty(\Omega), \text{ where}$$

A is any real associated form to L, $C_L > 0$ is the ellipticity constant for L and we define the seminorms

$|\varphi|_m$ of all m-th order derivatives by $|\varphi|_m^2 = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|\partial^\alpha \varphi\|^2_{L^2(\Omega)}$

<u>Note</u> that $|\varphi|_m^2$ is a quadratic form associated to Δ^m.

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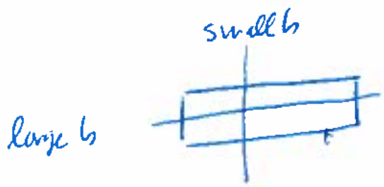
Now, if $Lu = 0$, $\varphi \in C_0^\infty(\Omega)$, then

$A[u, \varphi] = 0$ by integr. by parts. Thus $A[u+\varphi] = A[u] + A[\varphi]$.

We also will use a Poincaré Inequality (Agmon, p. 54)

Def: Ω has bounded width $\leq b$ iff. $\exists$ a line ℓ s.t. each line parallel to ℓ intersects Ω in a set of diameter $\leq b$.



(123.05) Poincaré Inequality for $C_0^m(\Omega)$

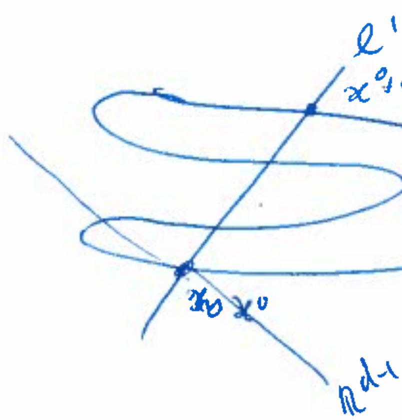
Let Ω have bounded width $\leq b$. Then for $0 \leq j \leq m-1$

$$|\varphi|_j \leq \gamma\, b^{m-j} |\varphi|_m \quad \forall \varphi \in C_0^m(\Omega) \text{ where } \gamma = \gamma(m,d).$$

Proof For ℓ' parallel to ℓ, let

x^0 and $x^0 + q \in \partial\Omega$ s.t. $\ell' \cap \partial\Omega$ is contained in the

line segment $[x^0, x^0+q]$.



Put $f(t) = \varphi\left(x_0 + t\frac{q}{|q|}\right)$ Then $f' = \frac{q}{|q|}\cdot\nabla\varphi$ (123.1)

Now $f(t) = \int_0^t f'(\tau)\,d\tau$ and using C-S.

$$|f(t)|^2 \le t\int_0^t |f'(\tau)|^2 d\tau \le b\int_{-\infty}^{\infty} |f'(\tau)|^2 d\tau$$

Then, (124.0) $\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_0^b |f(t)|^2 dt \le b^2 \int_{-\infty}^{\infty} |f'(t)|^2 dt$

$= b^2 \int_0^b |f'(t)|^2 dt$ Thinking of ℓ perpendicular to $\mathbb{R}^{d-1}$ doing the above for every ℓ' and writing $f(t) = f(x^{0\prime}, t)$

$= \varphi(x^{0\prime}, x_d^0 + t)$. Fubini gives

$$\int_\Omega |\varphi|^2 dx = \int_{\mathbb{R}^{d-1}} dx^{0\prime} \int_{-\infty}^{\infty} |\varphi(x^{0\prime}, x_d^0 + t)|^2 dt$$

$$\overset{(124.0)}{\le} b^2 \int_{\mathbb{R}^{d-1}} dx^{0\prime} \int_0^b |\underset{(f(x_0', t)')}{f'}(t)|^2 dt \le b^2 \int_\Omega |\nabla\varphi|^2 dx \quad (123.1).$$

i.e. $|\varphi|_0 \le b|\varphi|_1$. But also $|\partial^\alpha\varphi|_0 \le b\,|\partial^\alpha\varphi|_1$

and the stated inequality follows $\square$

By the classical (120.1) ~~fot~~ for solutions u with

constants c, c_0 depending on the Lipschitz nature of Ω, and

by the strong (122.1) for $\varphi \in \mathcal{C}_0^\infty(\Omega)$ it follows that

$$c\,|u+\varphi|_m^2 \leq 2c\,|u|_m^2 + 2c\,|\varphi|_m^2 \leq 2\,A[u] + c_0\int_\Omega |u|^2 dx$$

$$+ \frac{2c}{c_L} A[\varphi] \leq 2\,A[u] + \frac{2c}{c_L} A[\varphi] + 2c_0 \int_\Omega |u+\varphi|^2 dx + 2c_0 \int_\Omega |\varphi|^2 dx$$

Poincaré

$$\leq 2\,A[u] + \frac{2c}{c_L} A[\varphi] + 2c_0\,\gamma^2 \delta^{2m} |\varphi|_m^2 + 2c_0\,|u+\varphi|_0^2$$

(122.1)

$$\leq 2\,A[u] + 2\,\frac{c + c_0\,\gamma^2\delta^{2m}}{c_L} A[\varphi] + 2c_0\,|u+\varphi|_0^2$$

$$\leq \underset{L,\Omega}{C}\, A[u+\varphi] + c_0'\,|u+\varphi|_0^2.$$ establishing (120.0) for $v = u+\varphi$.

A norm for $W^{m,2}(\Omega)$ is

$$\|v\|_m = \left(\sum_{\gamma=0}^{m} |v|_\gamma^2\right)^{1/2}$$

04/04

An <u>inner product</u> is

$$(u,v)_{W^{m,2}(\Omega)} = \sum_{|\alpha|\le m} \frac{|\alpha|!}{\alpha!} \int_\Omega \overline{\partial^\alpha u}\, \partial^\alpha v \, dx .$$

<u>Classically</u> (Agmon, p.3) one defines the Hilbert space $H^m(\Omega)$ as the <u>completion</u> w.r.t. the norm $\|\cdot\|_m$ of the $\mathcal{C}^m(\Omega)$ functions (or $\mathcal{C}^\infty(\Omega)$) v with $\|v\|_m < \infty$. To say that $\{v_k\} \subset W^{m,2}(\Omega)$ is Cauchy is to say that $\partial^\alpha v_k \longrightarrow v^\alpha$ in $L^2(\Omega)$ for each $0 \le |\alpha| \le m$, where $\partial^\alpha v_k$ are the derivatives of v_k <u>s.d.</u> But then by def'n of <u>s.d.</u>

$$\int_\Omega \partial^\alpha v_k \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v_k \partial^\alpha \varphi \, dx \qquad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\int_\Omega v^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v^0 \partial^\alpha \varphi \, dx$$

So that $v^\alpha = \partial^\alpha v^0$. <u>s.d.</u> Therefor $W^{m,2}(\Omega)$ is <u>complete</u>, i.e. it is a Hilbert space, too. Since $\mathcal{C}^m(\Omega)$ with finite norm $\|\cdot\|_m$ is also a subset of $W^{m,2}(\Omega)$ we get $H^m(\Omega) \subseteq W^{m,2}(\Omega)$ by the last argument.

It is a Theorem of Meyers-Serrin that says 04/04

$$H^m(\Omega) = W^{m,2}(\Omega)$$

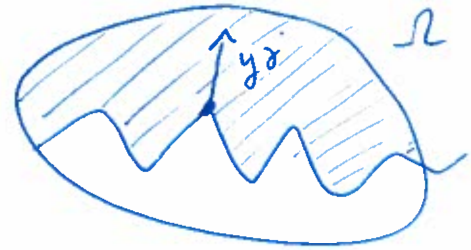
(Agmon, p. 13). and this is true without restrictions on the (bounded?) domain Ω. [pp. 154-155, G.T.].

04/09

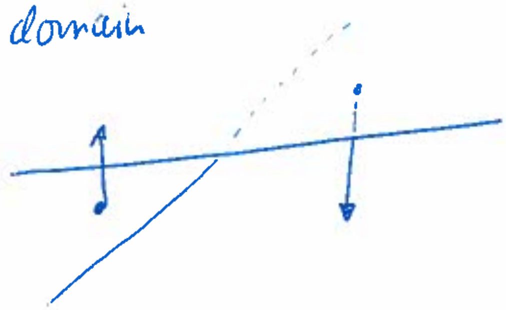
Theorem [Agmon, p. 11] If Ω has the segment property and if $v \in W^{m,2}(\Omega)$, then $\exists \{\varphi_k\} \subset C_0^\infty(\mathbb{R}^d)$ s.t. $\varphi_k \to v$ in $W^{m,2}(\Omega)$. ~~Sometimes~~

Sometimes this is stated saying $C^\infty(\bar{\Omega})$ are dense in $W^{m,2}(\Omega)$. [G.T.]

The segment property is that $\partial\Omega$ has a (locally) finite open ~~cover~~ (in $\mathbb{R}^d$) cover $\{O_\gamma\}$ and corresponding nonzero vectors $\{y^\gamma\}$ so that if $x \in O_\gamma \cap \bar{\Omega} \Rightarrow x + t y^\gamma \in \Omega$ for all $0<t<1$. This is satisfied by a Lipschitz domain. But not by



two bricks domain



Let Ω have the segment property.

Def: $W_0^{m,2} = H_0^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev-Norm.

Def: (Generalized-Dirichlet-Problem) ~~(GDP)~~ (gDp) for the homogeneous equation $Lu=0$. Let L have an associated form A. Given $g \in W^{m,2}(\Omega)$ find $u \in W^{m,2}(\Omega)$ s.t.

i) $A[u,\varphi]=0 \ \forall \varphi \in C_0^\infty(\Omega)$ and

ii) $u-g \in W_0^{m,2}(\Omega)$.

To solve this, first note that $W_0^{m,2}(\Omega)$ is also a Hilbert space, when the inner product $(u,v)_m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega \partial^\alpha \bar{u}\, \partial^\alpha v \, dx$ is used and the seminorm $|v|_m$ is used as the norm! This is a norm by the Poincaré inequality (123.05) which shows that $|v|_m=0$ $\Rightarrow |v|_\gamma=0$, $0 \le \gamma \le m-1$ when $v \in W_0^{m,2}(\Omega)$, since the latter space is the completion of $C_0^\infty(\Omega)$.

Moreover, the strong coercive estimate (122.1) extends to

(129.1) $A[v] \geq C_L |v|_m^2, \quad v \in W_0^{m,2}(\Omega)$

by this completion. Consequently $A[u,v] = \sum a_{\alpha\beta} \int_\Omega \partial^\alpha \bar{u}\, \partial^\beta v \, dx$

is also an inner product for $W_0^{m,2}(\Omega)$, with $(A[v])^{1/2}$ a <u>norm</u>.

With this, <u>existence</u> for the gDp will follow by an application of the Riesz Representation Theorem for Hilbert spaces:

Given "data" $g \in W^{m,2}(\Omega)$, the map

$W_0^{m,2}(\Omega) \ni v \longmapsto A[g,v] \in \mathbb{C}$ is a bounded linear functional

by Cauchy-Schwarz: $|A[g,v]| \leq C_{A,g} |v|_m$.

Since A is ~~the~~ an inner product for the Hilbert space $W_0^{m,2}(\Omega)$, $\exists!$

$v_0 \in W_0^{m,2}(\Omega)$ s.t. $A[g,v] = A[v_0,v], \quad v \in W_0^{m,2}(\Omega)$.

Putting ~~g~~ $u = g - v_0$ it follows that

i) $A[u,\varphi] = 0, \quad \varphi \in C_0^\infty(\Omega)$ and

ii) $u - g \in W_0^{m,2}(\Omega)$,

Solving the gDp with a <u>weak solution</u> to $Lu = 0$.

04/09

Uniqueness in g Dp reduces to assuming i) and ii)

with $g=0$. Then $u \in W_0^{m,2}(\Omega)$ and by the density of φ

we get $A[u]=0$, i.e. $|u|_m \overset{(129.1)}{=} 0$, so $u=0$.

One can also obtain the estimate

$$(130.1) \quad |\underbrace{u_g}_{\text{sol'n with data } g}|_m \le C_A |g|_m$$ for the solution u_g with data g

By the strong $W_0^{m,2}(\Omega)$-coerciveness (129.1), we get $u_g - g \in W_0^{m,2}(\Omega)$

so $|u_g - g|_m^2 \le \frac{1}{C_L} A[u_g-g] \overset{(i)}{=} \frac{1}{C_L} A[-g, u_g-g]$

$\le C_{L,A} |g|_m |u_g - g|_m$. Consequently:

$|u_g|_m \le |u_g - g|_m + |g|_m \le (1 + C_{L,A}) |g|_m$.

Here $|g|_m = 0 \Rightarrow u_g$ is a polynomial of degree $m-1$.

Note that the uniqueness will also improve (130.1) to

$$|u_g|_m \le C_A \inf_{v \in W_0^{m,2}(\Omega)} |g+v|_m$$

Above, we have shown, that boundary coerciveness for <u>solutions</u>

(119.1) yields the classical coercive estimate over $W^{m,2}(\Omega)$

(cf. (120.0)) which was 04/09

$$A[u+\varphi] \ge c|u+\varphi|_m^2 - c_0|u+\varphi|_0^2$$ a priori for

classically differentiable solutions u and $\varphi \in C_0^\infty(\Omega)$.

By defn. of $W_0^{m,2}(\Omega)$ (120.0) holds for φ ~~∈~~ replaced by any $v_0 \in W_0^{m,2}(\Omega)$. By solution of gD_p, every $v \in W^{m,2}(\Omega)$ decomposes uniquely as $v = u_v + v_0$ for $Lu=0$ s.d., thus the claim is proved if weak solutions u are classically differentiable solutions.

boundary conditions $\Rightarrow$ classical conditions

This requires a technique often referred to as Weyl's lemma. ([Y, pp. 177-178], [Agmon, Chapter 6])
(Friedrich's Thm. proof pp. 178-182)

because we are dealing with constant coefficients and (therefore) classically diff'ble fundamental solutions Γ_L [John], ~~so~~ Weyl's lemma is simple.

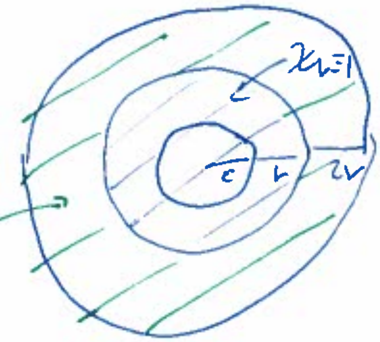
04/11

Weak solutions are classically differentiable

John's interior estimate (91.9) for classically diff'ble solutions. Let ~~χ(x)~~

$$\chi(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>2 \end{cases} \in C_0^\infty(\mathbb{R}^d)$$ and put $\chi_r(x) = \chi\left(\frac{x}{r}\right)$

let $B_\varepsilon(0) = B_\varepsilon$, $\varepsilon < r$



(133.0)
$$0 = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon} (\Gamma_L \chi_r)\, Lu \, dx =$$

$D^\gamma \chi_r \neq 0$ (maybe)

$$= \lim_{\varepsilon \to 0} \sum_{\gamma=1}^{2m-1} (-1)^\gamma \int_{\partial B_\varepsilon} D^{2m-1-\gamma} \Gamma_L \, D^\gamma u \cdot \nu \, ds + u(0)$$

some Derivative operator of order γ depend. on L.

$$+ \sum_{\gamma=0}^{2m-1} \int_{r<|x|<2r} D^\gamma \Gamma_L^0 D^{2m-\gamma} \chi_r^0 \, u \, dx$$

(133.01) Since $\Gamma_L(x) = O\left(\frac{|x|^{2m}}{|x|^d}\right)$ we have

$$D^{2m-1-\gamma} \Gamma_L = O\left(\frac{|x|^\gamma}{|x|^{d-1}}\right), \gamma \geq 1 \qquad \text{So } \lim_{\varepsilon\to 0} \text{ term} = 0$$

For $r < |x| < 2r$

$$D^\gamma \Gamma_L D^{2m-\gamma} \chi_r = O\left(\frac{|r|^{2m-\gamma}}{|r|^d}\right) \cdot O\left(r^{\gamma-2m}\right) = O\left(\frac{1}{|r|^d}\right).$$

for $0 \leq \gamma \leq 2m-1$.

Thus $|u(0)| \leq C_L \fint_{r<|x|<2r} |u| \, dx$.

By writing (133.0) for x_0 in a nbd. of $x_0 = 0$ ← replace $\begin{cases} \Gamma_L^0 \text{ by } \Gamma_L^{x_0} \\ \tau_r^0 \text{ by } \tau_r^{x_0} \end{cases}$

and differentiating w.r.t. x_0. One gets John's estimates.

Additionally one can get.

(133.1) $\max\limits_{B_r} |\partial^\alpha u| \leq C'_{L,\alpha}\, r^{-|\alpha|} \int\limits_{4B_r \setminus 2B_r} |u|\, dx \quad \forall \alpha.$

If now $u \in W^{m,2}(\Omega)$ a weak solution of gDp it may be regularized by the approximation to the identity argument. $u_\varepsilon(x) \stackrel{\text{def}}{=} \psi_\varepsilon * u(x)$ (134.1)

for $\operatorname{dist}(x, \partial\Omega) > \varepsilon$ for example, with $u_\varepsilon \to u$ in $W^{m,2}(\Omega')$

for any $\Omega' \subset\subset \Omega$, so $u_\varepsilon \in C^\infty$ and

$$L u_\varepsilon = ((L\psi_\varepsilon) * u \stackrel{\boxed{116P}}{=} (-1)^m A[\psi_\varepsilon, u] = 0$$
$$\uparrow \in C_0^\infty(\Omega).$$

But: (133.1), (134.1) conv. uniformly on compacts to $\partial^\alpha u \in C^\infty$, i.e.

u is a classically diff'ble solution in Ω.

This means that the assumed boundary coerciveness of the Claim:

(119.1) ⇒ (120.1) does yield

$$c\int_{\Omega_\epsilon\setminus\Omega_1} |\nabla^m u|^2\,dx \le \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\int_{\Omega_\epsilon\setminus\Omega_1} \partial^\alpha u\,\partial^\beta u\,dx \quad \text{for } \epsilon>0$$

for a $\{D_p\}$ solution u. and now $u\in W^{m,2}$ justifies monotone convergence as $\epsilon\searrow 0$, i.e. $\Omega_\epsilon\nearrow\Omega$.

We have shown that boundary coerciveness for solutions implies the classical coercive

$$(135.1)\quad A[v]\ge c|v|_m^2 - c_0|v|_0^2 \text{ for } v\in W^{m,2}(\Omega)$$

and $c>0$.

Coerciveness for $L=\Delta^2$. From the Gram matrix (p. 116) ..., formally positive forms are:

$$\int_\Omega |v_{11}+(1-a)v_{22}|^2 + a(2-a)|v_{22}|^2 + 2a|v_{12}|^2\,dx$$

$$=\int_\Omega |v_{11}+v_{22}|^2 + \underbrace{2a\left(|v_{12}|^2-\operatorname{Re}\overline{v_{11}}v_{22}\right)}_{\text{Null form.}}\,dx,\qquad 0\le a\le 2$$

1.) $a=0$ $\int_\Omega |\Delta v|^2 dx$ not coercive

2.) $a=1$ $\int_\Omega |v_{11}|^2 + 2|v_{12}|^2 + |v_{22}|^2 dx = |v|_2^2$. Coercive

3.) $0<a<2 \Rightarrow$ coercive by arithmetic geometric mean inequality applied to $2(1-a)\overline{v_{11} v_{22}}$.

4.) $a=2$ $\int_\Omega |v_{11} - v_{22}|^2 + 4|v_{12}|^2 dx$ is coercive, but not obviously.

5.) $2<a<4 \Rightarrow$ not formally positive, but coercive (in smooth domains)

6.) $a \leq 0$ or $a \geq 4 \Rightarrow$ not coercive

Comments on 1.) – 6.) and (135.1)

- (135.1) with the seminorm $|v|_m$ is equivalent to $A[v] \geq c' \|v\|_m^2 - c_0' |v|_0^2$

This will follow from the following interpolation theorem.

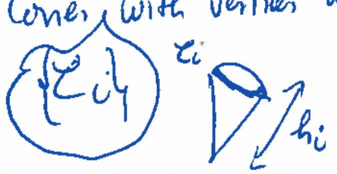
Thm (Agmon pp. 19–20) Let Ω be a bounded open set in $\mathbb{R}^d$. 04/11

If Ω has the restricted cone property, then $\exists$ a constant $\gamma = \gamma(\Omega)$

s.t. all $v \in W^{2,2}(\Omega)$ satisfy.

$$|v|_1^2 \leq \gamma\left(\varepsilon^2 |v|_2^2 + \varepsilon^{-2} |v|_0^2 \right) \text{ for all } 0 < \varepsilon \leq 1.$$

Restricted cone property ; $\partial\Omega$ has a finite open cover O_i and corresponding cones with vertices at 0, s.t. $x \in \Omega \cap O_i \Rightarrow x + C_i \subseteq \Omega$.



Lemma (Agmon p. 17) If $f \in C^2(0,1) \Rightarrow \int_0^1 (f')^2 dx \leq 54 \int_0^1 |f|^2 dx + 2\int_0^1 |f''|^2 dx$. 04/16

Pf of Lemma: let $0 < \alpha < 1/2$ and choose any $x_1 \in (0,\alpha)$, $x_2 \in (1-\alpha, 1)$

Then $\exists\, \eta \in (x_1, x_2)$ s.t. $f'(\eta) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$,

So $|f'(\eta)| \leq \dfrac{|f(x_1)| + |f(x_2)|}{1 - 2\alpha}$. Then for $x \in (0,1)$

$$f'(x) = f'(\eta) + \int_\eta^x f''(\xi)\, d\xi \quad \text{So that}$$

$$|f'(x)| \leq \frac{|f(x_1)| + |f(x_2)|}{1-2\alpha} + \int_0^1 |f''(\xi)|\, d\xi$$

Integrate

$$\int_{1-\alpha}^{1} dx_2 \int_0^{\alpha} dx_1 ,$$ we get:

$$\left| \int_{\xi}^{x} f''(\xi)\,d\xi \right|$$

$$\alpha^2 |f'(x)| \leq \frac{\alpha}{1-2\alpha}\left(\int_0^{\alpha} |f|\,d\xi + \int_{1-\alpha}^{1} |f|\,d\xi\right) + \alpha^2 \int_0^1 |f''(\xi)|\,d\xi$$

Squaring and using $|A+B|^2 \leq 2A^2 + 2B^2$.

$$|f'(x)|^2 \leq 2\,\frac{4}{4\alpha^2(1-2\alpha)^2}\left[\left(\int_0^{\alpha} + \int_{1-\alpha}^{1}\right)|f(\xi)|\,d\xi\right]^2$$

$\int_\Omega \cdot = 1$ Jensen (measure $= \dfrac{\left(\int_0^{\alpha} + \int_{1-\alpha}^{1}\right) 1_A\,dx}{2\alpha}$)

$$+\,2\left[\int_0^1 |f''|\,d\xi\right]^2$$

$$\leq \frac{4}{\alpha(1-2\alpha)^2}\int_0^1 |f|^2\,d\xi + 2\int_0^1 |f''|^2\,d\xi .$$

$\square$ (lemma)

$\alpha = \frac{1}{6}$ minimizes the coefficient. Now integrate $\int_0^1 dx$.

For $f(y)$, $0<y<b$ put $y = bx$, $0<x<1$ and

$g(x) = f(bx) = f(y)$, $0<x<1$. Then

$$g'(x) = b\, f'(bx) = b\, f'(y)$$

$$g''(x) = b^2 f''(bx) = b^2 f''(y) \quad \text{and} \quad \frac{dy}{b} = dx$$

Use the lemma with g in place of f. + change

variables: $$\int_0^b |b\, f'(y)|^2 \frac{dy}{b} \le 54 \int_0^b |f(y)|^2 \frac{dy}{b} + 2\int_0^b |b^2 f''(y)|^2 \frac{dy}{b}$$

So that

$$\int_0^b |f'|^2\, dy \le \frac{54}{b^2} \int_0^b |f|^2 dy \quad + \quad 2b^2 \int_0^1 |f''|^2 dy$$

Translate to any compact interval I with coefficients depending only on the length $|I|$ of I. Finally there is a constant

$\gamma = 216 \max\{|I|^2, |I|^{-2}\}$. So that for every $0 < \varepsilon \le 1$

(139.1) $$\int_I |f'|^2 dx \le \gamma \left[\varepsilon^{-2} \int_I |f|^2 dx + \varepsilon^2 \int_I |f''|^2 dx \right]$$

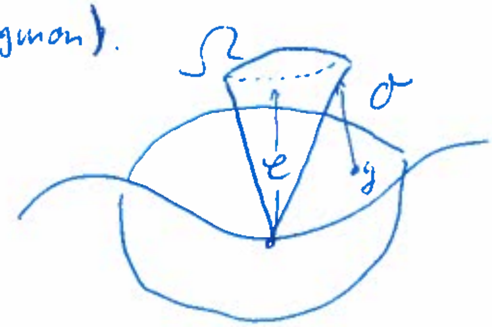
(Agmon, p. 18 (3.3)).

Simply partition I into $\{I_\gamma\}$ with $|I_\gamma| < \varepsilon |I|$ and $|I_\gamma| > \frac{\varepsilon}{2} |I|$ and apply previous lemma.

$$\int_I |f'|^2 dx = \sum_\gamma \int_{I_\gamma} |f'|^2 dx \leq \sum_\gamma \left[\frac{4.54}{|I|^2 \varepsilon^2} \int_{I_\gamma} |f|^2 dx + 2\varepsilon^2 |I|^2 \int_{I_\gamma} |f''|^2 dx \right]$$

$$\leq \gamma \left[\varepsilon^{-2} \int_I |f|^2 dx + \varepsilon^2 \int_I |f''|^2 dx \right]$$

To prove interpolation thm: It can be arranged that if $\mathcal{O}$ is in the open cover of $\partial\Omega$ and if $\mathcal{C}$ is its associated cone, then diam $\mathcal{O}$ < height of $\mathcal{C} = h_{\mathcal{C}}$. (p. 19 Agmon).



Now, whenever ξ is a unit vector that is a positive multiple of a vector in $\mathcal{C}$

define $\Omega_{\mathcal{O}}^{\xi} = \{ x \in \mathbb{R}^d, x = y + t\xi, y \in \Omega \cap \mathcal{O}$ and $0 \leq t \leq h_{\mathcal{C}} \}$

then $\Omega \cap \mathcal{O} \subset \Omega_{\mathcal{O}}^{\xi} \subset \Omega$. It will be shown that

04/16

$$(140.1) \quad \int_{\Omega^{\xi}_{\mathcal{O}}} |\partial_\xi v|^2 dx \le \gamma \left[\varepsilon^2 \int_{\Omega} |\partial_\xi^2 v|^2 dx + \varepsilon^{-2} \int_{\Omega} |v|^2 dx \right]$$

where γ is from (139.1), and ξ is the directional derivative.

But there are d vectors ξ in $\mathcal{C}$ that will span $\mathbb{R}^d$. By taking linear combinations + using $\Omega \cap \mathcal{O} \subseteq \Omega^{\xi}_{\mathcal{O}}$ for each such ξ one obtains

$$\int_{\Omega \cap \mathcal{O}} |\nabla v|^2 dx \le \gamma' \left[\varepsilon^2 |v|_2^2 + \varepsilon^{-2} |v|_0^2 \right].$$ Now

Seen over finite covering $\{\mathcal{O}_\gamma\}$. One need only obtain a similar estimate over a compact $K \subseteq \Omega$ s.t. ~~~~

$\Omega \setminus K \subseteq \Omega \cap (\bigcup_\gamma \mathcal{O}_\gamma)$. K may be considered the finite union of cubes. The estimate over K will then use (139.1), in the same way we use it for (140.1). Since $\operatorname{diam} \mathcal{O} < h_{\mathcal{C}}$ the intersection of any line L parallel to ξ with $\Omega^{\xi}_{\mathcal{O}}$ is of length ℓ,

$h_{\mathcal{C}} \le \ell \le 2 h_{\mathcal{C}}$ or is empty.

Let $L = L_y$ pass through a point y. By (139.1) 04/16

$$\int_{L_y \cap \Omega_0^\xi} |\partial_\xi v|^2 \le \gamma_{L_\xi} \left[\varepsilon^2 \int_{L_y \cap \Omega_0^\xi} |\partial_\xi^2 v|^2 + \varepsilon^{-2} \int_{L_y \cap \Omega_0^\xi} |v|^2 \right]$$

Integrate in y over all $(d-1)$ planes orthogonal to $\xi \Rightarrow$ (140.1) $\square$

More generally, 04/18

<u>Theorem</u> Ω bounded with C_p.-boundary, $0 < \varepsilon \le 1$, $m \ge 2$. Then $v \in W^{m,2}(\Omega)$ and $0 \le \gamma \le m-1$, then

$$|v|_\gamma^2 \le \gamma(\Omega, m) \cdot \left(\varepsilon^{2m-2\gamma} |v|_m^2 + \varepsilon^{-2\gamma} |v|_0^2 \right).$$

Moreover $\exists\, C_\varepsilon = C_{\varepsilon,\Omega,m}$ s.t.

$$(142.1) \quad \|v\|_{m-1}^2 \le \varepsilon^2 |v|_m^2 + C_\varepsilon |v|_0^2$$

Consequently, we have if

$$A[v] \ge c|v|_m^2 - c_0|v|_0^2 \overset{(142.1)}{\ge} (c - \varepsilon^2)|v|_m^2 + \|v\|_{m-1}^2 - C_\varepsilon |v_0|^2 + c_0 |v|_0^2$$

$$\ge (c - \varepsilon^2) \|v\|_m^2 - (C_\varepsilon + c_0) |v_0|^2.$$

That $\int_\Omega |\Delta v|^2\,dx$ is <u>not</u> coercive for $L = \Delta^2$ follows by <u>contradiction</u> using harmonic function $h \in W^{2,2}(\Omega)$ in the coercive inequality, yielding $|h|_2^2 \le \frac{c_0}{c} |h|_0^2$. $\forall$ such h.

That this cannot hold is seen by considering the sequence

$h_n(x) = (x_1 + i x_2)^n$ restricted to $\Omega \subseteq \mathbb{R}^d$.

More generally, when Ω is bounded and

i) A is coercive over $W^{m,2}(\Omega)$

ii) $|A[v]| \le C \|v\|^2_{m-1}(\Omega)$ $\forall v \in S$ subspace of $W^{m,2}(\Omega)$

$\rightarrow$ then S is finite dimensional.

I.e. if (ii) and S infinite dimensional hold $\Rightarrow$ A is <u>not</u> coercive.

(Agmon, p. 112)

To see this let $\bar{S}$ be the closure of S in $W^{m,2}(\Omega)$. Then (ii) holds for $\bar{S}$. If $\bar{S}_{m-1}$ is the closure of S in $W^{m-1,2}(\Omega)$ one always has $\bar{S} \subseteq \bar{S}_{m-1}$. But for $v \in S$ (i) and (ii) yield

(143.1)
$$c\|v\|_m^2 \le C\|v\|_{m-1}^2 + c_0|v|_0^2 \le (C+|c_0|)\|v\|_{m-1}^2.$$

So that $\bar{S}_{m-1} \subseteq \bar{S}$, i.e. $\bar{S}$ closed in both Sobolev spaces.

If now, $\{v_n\} \subset \bar{S}$ is bounded in $W^{m-1,2}(\Omega)$, then by (143.1) it is bounded in $W^{m,2}(\Omega)$. Thus by the Rellich-Compactness Theorem, there is a subsequence that converges in $W^{m-1,2}(\Omega)$-norm. i.e. the unit sphere of $\bar{S}$ in $W^{m-1,2}(\Omega)$-norm is compact. Then $\bar{S}$ has to be finite dimensional (and hence so is $S=\bar{S}$).

Rellich Compactness Theorem (Agmon, p.248 for Ω with segment property). Let Ω be a bounded ~~domain~~ Lipschitz domain. If $\gamma < m$, then every bounded sequence in $W^{m,2}(\Omega)$ has a a subsequence which converges in $W^{\gamma,2}(\Omega)$. If this is stated instead for $W_0^{m,2}(\Omega)$ for any bounded domain Ω, it resembles the

Rellich-Gårding Thm, [Y. p.281]: Let Ω be a bounded domain, let $T: W_0^{m,2}(\Omega) \to W_0^{m,2}(\Omega)$ be bounded and linear and

Satisfy: $\| Tv \|_m \leq C \|v\|_j$ for some $j < m$.

Then, T is a <u>compact operator</u> on $W_0^{m,2}(\Omega)$.

In fact by using classical Calderón-Zygmund Singular integrals, Nellich-Gårding implies Nellich-Compactness for the $W_0^{m,2}$ Case.

<u>Proof of Nellich-Gårding:</u> Suffices to show for $\varphi_n \in C_0^\infty(\Omega)$ with $\|\varphi_n\|_m \leq 1$ that $\exists$ a subsequence with $T\varphi_n$ converging in $W_0^{m,2}(\Omega)$.

Using Schwarz and $\|\varphi_n\|_m \leq 1$ on

$$\hat{\varphi}_n(\xi) = (2\pi)^{-d/2} \int_\Omega \varphi_n(x) e^{-ix\cdot\xi} dx \quad \text{one gets}$$

$|\hat{\varphi}_n(\xi)|^2 \leq (2\pi)^{-d} |\Omega| \quad \forall n, \forall \xi \in \mathbb{R}^d$ Since φ_n are uniformly bounded in $L^2(\Omega)$ $\exists$ subsequence $\varphi_{n'}$ converging <u>weakly</u> in $L^2(\Omega)$ and thus $\lim_{n' \to \infty} \hat{\varphi}_{n'}(\xi)$ exists at every ξ.

Using Parseval $\|T\varphi_{n'} - T\varphi_{k'}\|_m^2 \leq C^2 \|\varphi_{n'} - \varphi_{k'}\|_j^2$

$$\leq c'^2 \sum_{|\alpha| \leq \gamma} \cancel{4\pi^2} \, \| \partial^\alpha (\varphi_{n'} - \varphi_{k'}) \|_0^2$$

$$= c'^2 \sum_{|\alpha| \leq \gamma} \int_{\mathbb{R}^d = \{|\xi| \leq R\} \cup \{|\xi| > R\}} (\xi^\alpha)^2 \, \underbrace{|\hat{\varphi}_{n'}(\xi) - \hat{\varphi}_{k'}(\xi)|^2}_{\to (n', k' \to \infty)} \, d\xi$$

$$\leq 0 \text{ as } n', k' \to \infty + c'^2 \sum_{|\alpha| \leq \gamma} \int_{|\xi| > R} |\xi|^{2\gamma} \, |\hat{\varphi}_{n'}(\xi) - \hat{\varphi}_{k'}(\xi)|^2 \, d\xi .$$

and
$$\int_{|\xi| > R} |\xi|^{2\gamma} \, |\hat{\varphi}_{n'}(\xi) - \hat{\varphi}_{k'}(\xi)|^2 \, d\xi \leq R^{2\gamma - 2m} \int_{|\xi| > R} |\xi|^{2m} \, |\hat{\varphi}_{n'}(\xi) - \hat{\varphi}_{k'}(\xi)| \, d\xi$$

$$\overset{\text{Parseval}}{\leq} C \, R^{2\gamma - 2m} \sum_{|\alpha| = m} \int_\Omega |\partial^\alpha (\varphi_{n'} - \varphi_{k'})|^2 \, dx \leq 4 \cdot C \cdot \underbrace{R^{2\gamma - 2m}}_{\to 0, \; R \to \infty} .$$

$\square$

Def For a measurable function k on $\mathbb{R}^d \setminus \{0\}$ to be a classical <u>Calderón–Zygmund kernel</u> it needs to be homogeneous of degree $-d$ $\left(k(tx) = t^{-d} k(x) \right)$.

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and have mean value zero on S^{d-1}.

The fundamental solutions Γ_L have $|\alpha| = 2m-1$ derivatives $\partial^\alpha \Gamma_L$ that are all homogeneous of degree $1-d$.

One can see (Agmon, p. 107. Stein) that this property (and some smoothness) alone implies that $\partial_\gamma \partial^\alpha \Gamma_L$, $1 \le \gamma \le d$ are all Calderón-Zygmund kernels. A part of the classical C-Z theory for k is that the operator

$$Kf(x) = \lim_{\substack{\varepsilon \searrow 0 \\ R \to \infty}} \int_{R > |x-y| > \varepsilon} k(x-y)\, f(y)\, dy \quad \text{exists in } L^2(\mathbb{R}^d)$$

and is bounded in L^2. (See Stein for more). Moreover, putting

$h = \partial^\alpha \Gamma_L$ for example and $k^\gamma = \partial_\gamma h$ and computing

$c_\gamma = \int_{S^{d-1}} h\, \nu_\gamma\, ds$ one has for $f \in L^2(\Omega)$ that

$h * f$ has derivatives $\underline{\text{s.d.}}$ $\partial_\gamma (h * f) = k^\gamma f + c_\gamma f$. in $\mathbb{R}^d$

Use compact support properties of $\varphi \in C_0^\infty$ and f + Fubini 04/23

$$\int_{\mathbb{R}^d} \partial_j \varphi(x) \Big(\int_{|x-y|>\varepsilon} h(x-y) f(y)\, dy \Big)\, dx$$

$$= \int_\Omega f(y) \Big(\int_{|x-y|>\varepsilon} \partial_j \varphi(x) h(x-y)\, dx \Big)\, dy$$

$$\overset{IBP}{=} \int_\Omega f(y) \Big[-\int_{|x-y|>\varepsilon} \varphi(x) h^j(x-y)\, dy - \int_{|x-y|=\varepsilon} \nu_j(x) \varphi(x) h(x-y)\, ds(x) \Big]\, dy$$

$$= -\int_{\mathbb{R}^d} \varphi(x) \Big(\int_{|x-y|} h^j(x-y) f(y)\, dy \Big)\, dx - \int_\Omega \Big[\int_{|x-y|=\varepsilon} \nu_j(x) h(x-y) \varphi(x)\, ds(x) \Big] f(y)\, dy$$

Recalling the classical C-Z max'l operator:

K_*^j, the lim $\varepsilon \to 0$ is justified on both sides and

$$-\int_{\mathbb{R}^d} \partial_j \varphi \; h * f\, dx = \int_{\mathbb{R}^d} \varphi\, K^j f\, dx + \int_\Omega c_j \varphi(y) f(y)\, dy.$$

Compare with Agmon pp. 108–110.

To show: Rellich–Gårding $\Rightarrow$ Rellich Compactness

in the $W_0^{m,2}$ – case.

Suppose $\{v_n\} \subseteq W_0^{1,2}(\Omega)$ is <u>bounded</u>. Need to show $\exists$ subsequence $v_{n'}$ converging in $L^2(\Omega)$. Take ball $B \supseteq \bar{\Omega}$ and let $\chi \in C_0^\infty(B)$, $\chi \equiv 1$ on Ω. Define linear operators.

$T^k : W_0^{1,2}(B) \to W_0^{1,2}(B)$ by

fund. soltn of Laplacian

$$(T^k w)(x) = \chi(x) \int_B \partial_k \Gamma^x(y)\, w(y)\, dy$$

Then $\partial_j T^k w(x) = \partial_j \chi(x) \int_B \partial_k \Gamma^x(y)\, w(y)\, dy$

$$- \chi(x) \; p.v. \int_B \partial_j \partial_k \Gamma^x(y)\, w(y)\, dy$$

$$+ \chi(x)\, c_{jk} w(x) \quad \text{where}$$

$$c_{jk} = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \nu_j \frac{y_k}{|y|^d}\, ds(y). = \delta_{jk} \cdot \frac{1}{d}$$

By C-Z, $\|T^k w\|_{1,B} \le C \|w\|_{0,B}$ $\forall w \in W_0^{1,2}(B)$.

By Rellich - Gårding, each T^k is compact on $W_0^{1,2}(B)$.

Thus $\exists\, v_{n'}$ s.t. $T^k v_{n'}$ conv. in $W^{1,2}(B)$, $1 \le k \le d$.

But $\sum_{j=1}^{d} \partial_j T^j v_{n'} \big|_\Omega = v_{n'}$. So $v_{n'}$ ~~conv.~~ in $L^2(\Omega)$ □

To complete Rellich compactness, let $\{v_k\} \subseteq W^{m,2}(\Omega)$, $\|v_k\|_m \le 1$ and take $\Omega_j \uparrow \Omega$ and defining $\chi_j \in C_0^\infty(\Omega)$ s.t. $\chi_j \equiv 1$ on Ω_j. By the $W_0^{m,2}$-compactness $\chi_1 v_k$ has a subsequence $\chi_1 v_{k'}$ conv. in $W^{m-1,2}(\Omega)$. But then $\chi_2 v_{k'}$ has a subsequence etc. Use diagonalization and get subsequence $v_{k''}$ that converges in $W^{m-1,2}(\Omega')$ for every $\Omega' \subset\subset \Omega$. This follows b/c given Ω', $\chi_j \equiv 1$ $\forall j$ large enough. We claim that this also converges in $W^{m-1,2}(\Omega)$. Recall $C^\infty(\bar\Omega)$ is dense in $W^{m,2}(\Omega)$.

So the key lemma can come from an inequality like

$$\int_0^1 |f|^2 dx \le 2\varepsilon^2 \int_0^1 |f'|^2 dx + 3 \int_\varepsilon^1 |f|^2 dx \,, \quad \varepsilon > 0$$

Pf: $f(x) = -\int_x^{x+\varepsilon} f'(t)\,dt + f(x+\varepsilon)$

$$\Rightarrow \int_0^{\varepsilon} |f|^2 dx \le 2\int_0^{\varepsilon} dx \left|\int_x^{x+\varepsilon} f'(t)\,dt\right|^2 + 2\int_0^{\varepsilon} |f(x+\varepsilon)|^2 dx$$

$$\le 2\int_0^{\varepsilon} dx \; \varepsilon \int_x^{x+\varepsilon} |f'(t)|^2 dt + 2\int_{\varepsilon}^{2\varepsilon} |f|^2 dx$$

$$\le 2\varepsilon^2 \int_0^1 |f'|^2 dt + 2\int_{\varepsilon}^{2\varepsilon} |f|^2 dx \qquad \Big| + \int_{\varepsilon}^1 |f|^2 dx$$

$$\Rightarrow \int_0^1 |f|^2 dx \le 2\varepsilon^2 \int_0^1 |f'|^2 dx + 3\int_{\varepsilon}^1 |f|^2 dx. \quad \square$$

This is used to assert that for each $\varepsilon > 0 \; \exists \; \Omega_\varepsilon \subseteq \Omega$

$$\max_{x \in \partial\Omega_\varepsilon} \operatorname{dist}(x, \partial\Omega) \lesssim \varepsilon. \quad \text{s.t. } \|v\|_{m-1} \le C\,\varepsilon \|v\|_m + C\|v\|_{W^{m-1,2}(\Omega_\varepsilon)}$$

(Agmon, p.24)

Recalling $\|v_{k''}\|_m \le 1$ and $v_{k''}$ converges in $W^{m-1,2}(\Omega_\varepsilon)$

$$\lim_{k'', n'' \to \infty} \|v_{k''} - v_{n''}\|_{m-1} \le 2C\varepsilon \cdot 1. \quad \forall \varepsilon > 0,$$

i.e. $v_{k''}$ is Cauchy in $W^{m-1,2}(\Omega)$.

Rellich compactness $\Rightarrow$ $T: W^{m,2}(\Omega) \to W^{m,2}(\Omega)$

linear with $\|Tv\|_m \le C\|v\|_\gamma$ some $\gamma < m$

$\Rightarrow$ T is compact on $W^{m,2}(\Omega)$. (Ω Lipschitz?, bounded)

It suffices to show that $A[v] \le 0$ on an infinite dimensional space in order to ~~sta~~ establish non coerciveness over $W^{m,2}(\Omega)$

<u>Lemma</u> (Agmon, p. 108) h a C-Z kernel

Put $k_{\varepsilon,R}(x) = \begin{cases} h, & \varepsilon < |x| < R \\ 0, & \text{otherwise} \end{cases}$ $\Rightarrow$ $\exists\, C$ <u>independent of ε and R</u> s.t. $\forall\, \xi \in \mathbb{R}^d$ we have

$$\left|\widehat{k_{\varepsilon,R}}(\xi)\right| \le C. \quad \text{and} \quad m_h(\xi) = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \widehat{k_{\varepsilon,R}}(\xi)$$

exists for all $\xi \ne 0$. See Stein pp. 40–41 (But $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(x)\,dx$)

When $d=1$

$$\int_{R>|x|>\varepsilon} \frac{1}{x} e^{-ix\xi}\,dx = \int_{R|\xi|>|x|>\varepsilon|\xi|} e^{\pm ix}\frac{dx}{x} = \pm\int_{\varepsilon'}^{R'} 2i\frac{\sin x}{x}\,dx$$

$$\to \pm\pi i.$$

<u>Thm</u> $\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} k_{\varepsilon,R} * f = Kf$ exists in $L^2(\mathbb{R}^d)$

and $\|Kf\|_{L^2(\mathbb{R}^d)} \le (2\pi)^{d/2} \|m_K\|_\infty \|f\|_{L^2(\mathbb{R}^d)}$.

<u>Pf</u>: $\widehat{k_{\varepsilon,R} * f} = (2\pi)^{d/2} \widehat{k_{\varepsilon,R}} \cdot \hat{f}$

Parseval $\Rightarrow \|k_{\varepsilon,R} * f - k_{\varepsilon',R'} * f\|_2^2 =$

$= (2\pi)^d \int_{\mathbb{R}^d} |\widehat{k_{\varepsilon,R}} - \widehat{k_{\varepsilon',R'}}|\,|\hat{f}|^2 \, d\xi$. By the lemma, the

Integrand $\to 0$ a.e. and dominated by $4C^2|\hat{f}|^2 \in L^1$, so $\to 0$ by

L.D.C. Thus $k_{\varepsilon,R} * f \xrightarrow{L^2} Kf$. By Parseval $\blacksquare$

$$\begin{array}{l} \|k_{\varepsilon,R} * f\|_2 \le (2\pi)^{d/2} \|\hat{k}_{\varepsilon,R} \hat{f}\|_2 \le (2\pi)^{d/2} C \cdot \|f\|_2 \\ \quad \downarrow \\ \|Kf\|_2 \end{array}$$

Smith's Thm Let $p_1(\xi), \ldots, p_N(\xi)$ be homogeneous ~~degree~~ polynomials of degree m with constant complex coefficients. Suppose the p_j have no common nonzero complex zeros. (in $\mathbb{C}^d$). Suppose Ω is bounded Lipschitz. Then the formally positive form $A[u,v] = \sum_{k=1}^{N} \int_\Omega \overline{P_k(\partial)u} \cdot P_k v \, dx$ is coercive over $W^{m,2}(\Omega)$., i.e.

$\exists\, C$ s.t. $\forall v \in W^{m,2}(\Omega)$ ~~$|A[v,v]| \geq C \|v\|_m^2$~~

$$\|v\|_m^2 \leq C^2 \left(\sum_{k=1}^{N} \int_\Omega |P_k(\partial) v|^2 \, dx + \int_\Omega |u|^2 dx \right)$$

Equivalently $\|v\|_m^2 \leq c \left(\sum \|P_k(\partial) v\|_0 + \|v\|_0 \right)$.

For example, $\int_\Omega |\partial_1^2 v - \partial_2^2 v|^2 + 4 |\partial_1 \partial_2 v|^2 \, dx$

$P_1 = \xi_1^2 - \xi_2^2$, $P_2 = \xi_1 \xi_2$, $P_2 = 0 \Rightarrow \xi_1 = 0$ (say) ~~$\xi_2 = 0$~~
$P_1 = 0 \Rightarrow \xi_2 = 0 \Rightarrow 0$ only common complex root.

The "no common nonzero root" condition is <u>necessary</u> for coerciveness:

Let $0 \neq z \in \mathbb{C}^d$ be a common ~~zero~~ root of $P_1, \dots, P_N$.

Then let $S \subseteq W^{m,2}(\Omega)$ be the subspace spanned by $u_\lambda(x) = e^{\lambda z \cdot x}$, $\lambda \in \mathbb{C}$. ($u_\lambda \in W^{m,2}(\Omega)$, Ω bdd).

Then $P_k(\partial) u_\lambda(x) = P_k(\lambda z) u_\lambda(x). = \lambda^n P_k(z) u_\lambda(x) = 0$

for all k, and the same is true for any linear combination.

Thus, $A[u] = 0$ for $u \in S$. and since S is infinite dimensional,

A is not coercive. (Arenz–jon (?) condition)

Proof of Smith's Thm: If $P(\xi)$ is any polynomial which vanishes at all common complex zeros of any given collection of polynomials $P_1, \dots, P_N$, then $\exists$ $A_1(\xi), \dots, A_N(\xi)$ polynomials and a natural $\rho \in \mathbb{N}$ s.t. $P^\rho = \sum_{\gamma=1}^{N} A_\gamma P_\gamma$. This is Hilbert's Nullstellensatz.

Since our P_γ only have zero $0 \in \mathbb{C}^d$ as the common root,

$\exists \rho \in \mathbb{N}$ s.t. $\xi_\gamma^\rho = \sum_{k=1}^{N} A_{\gamma k}(\xi) P_k(\xi)$, $\gamma = 1, \dots, d$.

If now $|\alpha| \geq \rho d$, then $\exists \alpha_\gamma \geq \rho$. so that there are polynomials $A_{\alpha,k}$ s.t.

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(157.1) $\xi^\alpha = \sum_{k=1}^{N} A_{\alpha,k}(\xi) P_k(\xi)$. and we suppose

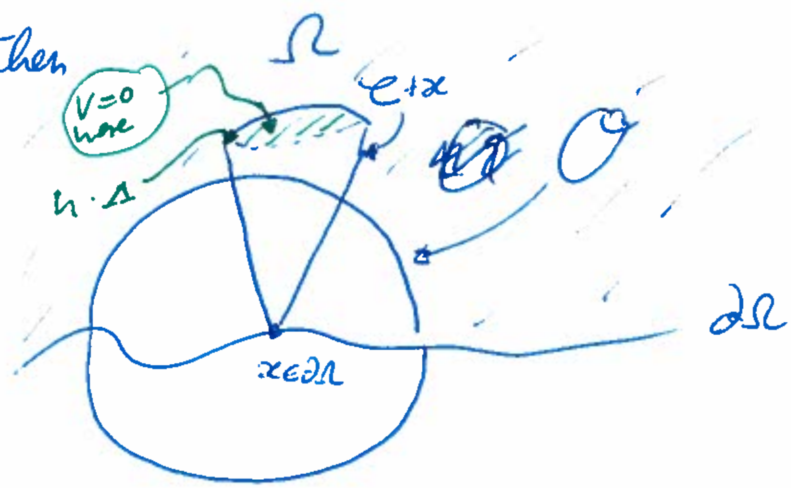
all $|\alpha| = m' \geq m$. (for m' big enough).

It may further be argued that the $A_{\alpha,k}$ are all homogeneous of degree $m'-m$.

Again by density we will take $v \in \mathcal{C}^\infty(\mathbb{R}^d)$ ~~$\mathcal{O}$~~ ~~$\mathcal{O}$~~
$\mathcal{O}$ will be from an open cover of $\partial\Omega$ with associated cone $\mathcal{C}$ s.t. $\operatorname{diam}\mathcal{O} <$ height(radius) h of the cone. We have for any $x \in \Omega \cap \mathcal{O}$ that $x + \mathcal{C} \subset \mathcal{O}$. Here $\mathcal{C}$

$$\mathcal{C} = \{\xi \in \mathbb{R}^d : |\xi| < h \text{ with } \xi \cdot \xi^0 > |\xi| \cos\theta\}$$

for some unit vector ξ^0 and fixed angle $\theta > 0$. Now, suppose first that $\operatorname{supp} v \subset \mathcal{O}$, Then together with the Sobolev-representation formula:



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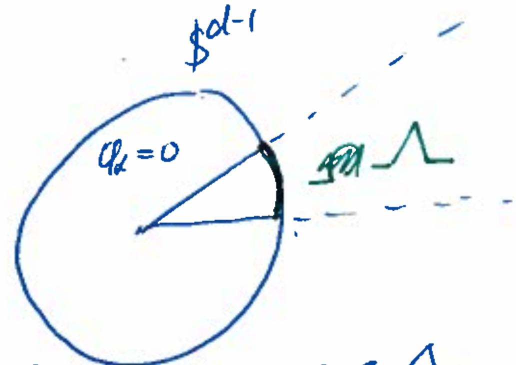
$$V(x) = \sum_{|\alpha|=m'} \int_{\mathcal{C}} \varphi_\alpha(y)\,\partial^\alpha v(x+y)\,dy.$$

works for all orders

$x\in\Omega\cap\mathcal{O}$. (then $x+y\in\Omega \ \forall y\in\mathcal{C}$).

where $\varphi_\alpha\in\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$ homogeneous of degree $m'-d$ and

supp $\varphi_\alpha\cap S^{d-1}\subseteq\Lambda$ where Λ is the portion of the sphere S^{d-1} subtended by $\mathcal{C}$.



<u>Sobolev representation</u> Given a cone $\mathcal{C}, h, \theta, \xi_0, \Lambda$ as above, fix $\varphi\in\mathcal{C}^\infty(S^{d-1})$, supp $\varphi\subseteq\Lambda$.

s.t. $\displaystyle\int_{S^{d-1}}\varphi\,ds = \frac{(-1)^m}{(m-1)!}$ for some $m\in\mathbb{N}$ ($m=m'$ later).

Let $v\in\mathcal{C}^m(\bar{\mathcal{C}})$ with $v(x)=0$ for $|x|\ge(h-\delta)$ for some $\delta>0$. Now

$$v(0)=\sum_{|\alpha|=m}\frac{m!}{\alpha!}\underbrace{\int_{\mathcal{C}}\overbrace{\varphi\left(\frac{y}{|y|}\right)\frac{y^\alpha}{|y|^d}}^{\text{degree } m-d}}_{\varphi_\alpha \text{ with } m' \text{ replacing } m}\partial^\alpha v(y)\,dy$$

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Pf of Eq.: let $\sigma \in S^{d-1}$ s.t. $\sigma \cdot \xi^0 > \cos\theta$

and apply Taylor with integral form of remainder to

the function $t \mapsto v(t\sigma)$ expanding about $t = h$, i.e. where $v=0$.

in a nbhd. of h_Λ.

$$v(0) = \frac{(-1)^m}{(m-1)!} \int_0^h t^{m-1} \frac{d^m}{dt^m} (v(t\sigma))\, dt$$

Now multiply by $\varphi(\sigma)$ and integrate over Λ in σ obtaining

$$v(0) \cdot 1 \, \frac{(-1)^m}{(m-1)!} = \frac{(-1)^m}{(m-1)!} \int_\Lambda \varphi(\sigma) \left[\int_0^h t^{m-1} \frac{d^m}{dt^m} v(t\sigma)\, dt \right] d\sigma$$

Now: Induction yields $t^{m-1} \frac{d^m}{dt^m} v(t\sigma) = t^{-1} \sum_{|\alpha|=m} \frac{m!}{\alpha!} t^m \sigma^\alpha (\partial^\alpha v)(t\sigma)$

Put $y = t\sigma$, Then (not a change of coordinates)

$$v(0) = \int_\Lambda \varphi\left(\frac{y}{|y|}\right) \left(\int_0^h \sum_{|\alpha|=m} \frac{m!}{\alpha!} y^\alpha (\partial^\alpha v)(y)\, t^{-1}\, dt \right) ds(y)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \underbrace{\int_C \varphi\left(\frac{y}{|y|}\right) \frac{y^\alpha}{|y|^d} \partial^\alpha v(y)\, dy}_{\psi_\alpha} \quad \text{since } dy = t^{d-1}\, dt\, ds$$

replace v with $v(x+\cdot)$ $\rightsquigarrow$ (158.1) $\square$

(157.1) $\xi^\alpha = \sum_k A_{\alpha,k}(\xi) P_k(\xi)$, $|\alpha| = m'$ (Plug in ∂ for ξ) 04/25

By this, (158.1) becomes $(m = m')$

$$V(x) = \sum_{|\alpha|=m'} \sum_{k=1}^{N} \int_e \overset{m'-d}{\varphi_\alpha(y)}\, \overset{m'-m}{A_{\alpha,k}(\partial)}\, \overset{m}{P_k(\partial)}\, v(x+y)\, dy$$

$$= \sum_{|\alpha|=m} \sum_{k=1}^{N} (-1)^{m'-m} \int_e A_{\alpha,k}(\partial)\varphi_\alpha(y)\, P_k(\partial)\, v(x+y)\, dy.$$

for $x \in \mathcal{O}\cap\Omega$ since φ_α vanishes at sides of e and $v(x+\cdot)$ vanishes on the spherical part. Put $\psi_k(y) = (-1)^{m'-m} \sum_{|\alpha|=m'} A_{\alpha,k}(\partial)\varphi_\alpha(y)$

is homogeneous of degree $m'-d \overset{\text{take derivatives}}{-} (m'-m) = m-d > -d$.

(so ψ_k integrable at the vertex). $\square$

04/30

<u>Motzkin polynomial (due to Choi + Lam) in $\mathbb{R}^4$</u>

and the <u>elliptic</u> operator $M_\varepsilon(\partial) = \partial_1^2\partial_2^2 + \partial_2^2\partial_3^2 + \partial_3^2\partial_1^2 + \partial_4^4 - 4\,\partial_1\partial_2\partial_3\partial_4 + \varepsilon(\partial_1^4 + \partial_2^4 + \partial_3^4 + \partial_4^4)$.

When $\varepsilon > 0$ is small enough, M_ε <u>not</u> a SOS. (showed that)

When ε is large, M_ε will be SOS, i.e. it will have a psd. Gram matrix.

There must be (by cont. of EV) $\varepsilon_0 > 0$ s.t. M_{ε_0} is a s.o.s. M_{ε_0} might be useful when answering the

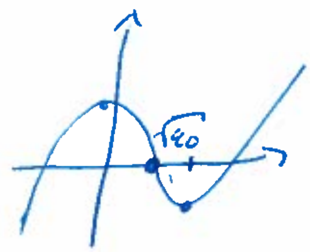
(163.1) question: Are there elliptic operators with formally pos. forms so that none of these forms are coercive?

The answer is yes in contrast to, for example, the bilaplacian Δ^2.

It turns out that $\sqrt{\varepsilon_0}$ is the smallest positive root of $x^3 - \frac{1}{2}x + \frac{1}{9} = 0$

$$M_{\varepsilon_0} = \left(w^2 - \sqrt{\varepsilon_0}(x^2+y^2+z^2)\right)^2 +$$

$$+ \frac{2}{9\sqrt{\varepsilon_0}}\Big[(3\sqrt{\varepsilon_0}wx - yz)^2 + (3\sqrt{\varepsilon_0}wy - zx)^2 + (3\sqrt{\varepsilon_0}wz - xy)^2\Big].$$



By rescaling $\mathbb{R}^4$ in w:

M_ε ends up as a particular case of

$$(164.1) \quad a\left(w^2 - \gamma(x^2+y^2+z^2)\right)^2 + (wx - yz)^2 + (wy - zx)^2 + (wz - xy)^2$$

with $a > 0$, $0 < \gamma < 1/3$, each of which has a unique Green matrix, i.e. each elliptic operator

has exactly one formally positive associated form.

Applying Smith to exercise weirdness means solving the system $w^2 - \gamma(x^2+y^2+z^2) = wx - yz = wy - zx = wz - xy = 0$ in $\mathbb{C}^4$.

1st consider solutions with $w=0$ $\quad yz = zx = xy = 0$

$\Rightarrow x = y = z = w = 0$

2nd consider solutions with $x=0$ . . . .

3rd assume none are zero: $wx^2 - xyz = wy^2 - xyz = wz^2 - xyz = 0$

$\Rightarrow |x| = |y| = |z| = |w|$ & $x^2 = y^2 = z^2$

So $w^2 - \gamma(x^2+y^2+z^2) = 0$

So $w^2 - 3\gamma x^2 = 0 \Rightarrow |w|^2 = 3\gamma|x|^2 \Rightarrow 1 = 3\gamma$ But $\gamma \neq 1/3$.

So: only the trivial solution $w = x = y = z = 0$.

Let $a = 1$ in (164.1). The (unique!) formally positive form for each operator L'_γ is coercive. It is not known if there are any ~~forth~~ 4-th order elliptic operators in dim 4 and 5 that answer (163.1) affirmatively. However, in 6 dimensions the operators

$$(166.1)\quad L_\gamma(\partial) = (\partial_1^2 + \partial_2^2 - \partial_3^2)^2 + \cancel{k'(\gamma)}\, L'_\gamma(\partial_3, \partial_4, \partial_5, \partial_6)$$

$0 < \gamma < 1/3$. ~~these~~ inherit the same uniqueness of formally positive forms and now $\xi_3 = \xi_4 = \xi_5 = \xi_6 = 0$, so

and $\xi_1^2 + \xi_2^2 - \xi_3^2 = 0 \Rightarrow \xi_1 = 1, \xi_2 = i$ provide nontrivial solutions.

$(1, i, 0, 0, 0, 0) \in \mathbb{C}^6 \Rightarrow$ no formally positive forms for ~~these ops~~ L_γ are coercive (But L_γ are elliptic!)

[V., J. of Pure and Applied Algebra, 2010]

In [V, Communications in PDE 37] it is shown that any ~~fo~~ 4-th order elliptic operator, that has at least one ~~p~~ formally positive form will have a coercive form ~~iff~~ if Ω is a $\mathcal{C}^2$-domain.

The L_γ have coercive <u>indefinite forms</u>. (in $\mathcal{C}^2$ domains)

In [V, 2014, JEMS] convex polyhedra (essentially) are constructed for each L_γ in which each "change matrix" (from a 105-dimensional space) is shown to be accompanied by an infinite dimensional subspace of $W^{2,2}(\Omega)$ on which the changed forms A all satisfy $A[v] \leq 0$.

The noncoerciveness of all forms for L_γ over $W^{2,2}(\Omega)$ raises the question of how to formulate and solve a Neumann problem for the L_γ.

Recall for Δ, the invertibility of the double layer potential $\mathcal{D}$ was a consequence of solving the Neumann-Problem and, conversely, the dual of $\mathcal{D}$ solved the Neumann problem.

Classical Hilbert space approach to the Neumann problem

Another Poincaré Inequality is useful. Let $\dot{W}^{1,2}(\Omega)$ denote $W^{1,2}$ functions with mean value $\int_\Omega v\,dx = 0$.

Then $\dot{W}^{1,2}$ can be normed by $|v|_1$ when Ω is Lipschitz because $\exists\, C_\Omega$ s.t. $v \in \dot{W}^{1,2}$ implies $|v|_0 \le C_\Omega |v|_1$.

Suppose $|v_n|_0 = 1$ and $|v_n|_0 \ge n |v_n|_1$, $v_n \in \dot{W}^{1,2}(\Omega)$.

Then $\|v_n\|_1 \le 2$ and by Rellich $\exists\, v_{n'} \to f$ ~~a.e.~~ in $L^2(\Omega)$.

But ~~also~~ $\nabla v_{n'} \longrightarrow \vec{0}$ in $L^2(\Omega)$. But f has derivatives $= 0$ s.d.

$$\int_\Omega \nabla\varphi\, f = \lim_{n'\to\infty} \int_\Omega \nabla\varphi\, v_{n'} = -\lim_{n\to\infty'} \int_\Omega \varphi\, \nabla v_{n'} = 0. \quad (\varphi \in C_0^\infty(\Omega))$$

$\Rightarrow$ One may conclude $f \equiv$ const a.e. But then $\int_\Omega f\,dx = \lim \int_\Omega v_{n'}\,dx = 0$

$\Rightarrow f \equiv 0$ But $|v_n|_0 = 1$, a contradiction.

More concrete: Grisvard-Thuchinger pp. 162-163 (for convex domains).

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05/18

<u>Recall</u> $A[v]$ associated with real L

- boundary coercivity $W^{m,2}(\partial\Omega)$ for classical diff'ble

Solutions $\Rightarrow$ classical coerciveness over $W^{m,2}(\Omega)$

$\mathrm{Re}\, A[v] \geq c|v|_m - c_0 |v|_0^2$ ~~for~~

Fails when $m=2$ for $L_\gamma = (\partial_1^2 + \partial_2^2 - \partial_3^2)^2 + (\partial_3^2 - \gamma(\partial_4^2 + -))^2$
$+ \cancel{(\partial_3 \ldots)^2} + (—)^2 + (-)^2$
$(\partial_3\partial_4 - \partial_5\partial_6)^2$

and $0 < \gamma < \frac{1}{3}$ $\qquad$ L_γ in $\mathbb{R}^4$

In certain bounded convex domains of $\mathbb{R}^6$

no matter what constant coefficient associated forms A are used.

- In <u>smooth domains</u>, the unique formally positive form associated with L_γ may be perturbed to yield coercive ~~but~~ <u>indefinite forms</u>

by using a characterization of coerciveness due Agmon.

The formally pos. form remains noncoercive by Aronszajn - Smith. [V. paper]

The L_γ may be perturbed to elliptic operators with no formally pos. forms. & no coercive forms in the Lipschitz setting.

- In the <u>smooth setting</u>, is there a (perhaps large) elliptic perturbation of L_γ with <u>no coercive forms</u> ?

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See Agmon: "The coerciveness problem for integral differential forms" J. Analyse Math. Vol. 6 (1958).

or "Remarks on self-adjoint and semibounded ..." (1961)

Jerusalem Academic Press. ~~The~~ Pergamon, Oxford.

$\overset{\circ}{W}^{m,2}(\Omega) \subseteq W^{m,2}(\Omega)$ defined as functions with all derivatives up to order $m-1$ having mean value zero. ($\int_\Omega v\,dx = 0$ etc.)

Poincaré-inequality for $v \in \overset{\circ}{W}^{m,2}(\Omega)$

$\|v\|^2_{m-1} \leq C\,|v|_m \Rightarrow |v|_m$ serves as a norm for $\overset{\circ}{W}^{m,2}(\Omega)$.

Poincaré can be proved using ~~Rellich~~ Rellich compactness or Riesz potentials

Dirichlet data $g \in \overset{\circ}{W}^{m,2}(\Omega)$ can be normed ~~by~~ by

$$\|\operatorname{tr} g\|_{m-1/2} = \inf_{g-v \in \overset{\circ}{W}_0^{m,2}(\Omega)} |v|_m \quad , \; g \in \overset{\circ}{W}^{m,2}(\Omega).$$

the "natural norm".

$\frac{1}{2} \sim 1$

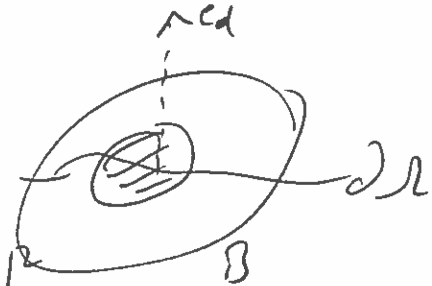
$\operatorname{tr} g \in W^{1/2,2}$

$\frac{1}{2} - \frac{d}{2} \geq 0 - \frac{d}{2}$

tr g has a more concrete understanding, known as the 05/18

trace of $g \in L^2(\partial\Omega)$ if $g \in W^{1,2}(\Omega)$.

Localize by smooth characteristics χ



$$\int_{\partial\Omega\cap B} |\varphi(x)|^2 \, ds(x) \le \int_{\partial\Omega\cap B} ds(x) \left| \int_x \partial_d |\varphi(z)|^2 \, dt \right|$$

$$\le C \int_\Omega |D\varphi|^2 + |\varphi|^2 \, dx \quad \text{for smooth } \varphi.$$

Poincaré $\int_{\partial\Omega} |\varphi|^2 ds \le C |\varphi|_1^2 \quad \varphi \in \mathring{C}^\infty(\bar\Omega)$.

$\mathrm{tr}\,\varphi = \varphi|_{\partial\Omega}$, φ smooth, $\varphi_j \xrightarrow{W^{1,2}} g \Rightarrow \{\mathrm{tr}\,\varphi_j\} \subseteq L^2(\partial\Omega)$

Cauchy $\Rightarrow$ $\mathrm{tr}\, g = \lim_{j\to\infty} \mathrm{tr}\,\varphi_j \in L^2(\partial\Omega)$.

For $g \in W^{m,2}(\Omega)$ $\quad \mathrm{tr}\, g = (\mathrm{tr}\,\partial^\alpha g)_{|\alpha| \le m-1}$

an array of $L^2(\partial\Omega)$ functions. $\Rightarrow H^{m-1/2}(\partial\Omega)$

($\mathrm{tr}\, g = 0$ a.e. $\Rightarrow g \in W_0^{1,2}(\Omega)$)

Rellich compactness, $A[v]$ coercive over $W^{m,2}(\Omega)$

$\Rightarrow$ strong coerciveness: $A[v] \ge C |v|_m^2$ over finite

co dim. subspace $S \subseteq \mathring{W}^{m,2}(\Omega)$. so that S contains $\mathring{W}_0^{m,2}(\Omega)$.

Consequently if $\lambda \in (\mathrm{tr}\, S)'$ will by Riesz Rep. be 05/18

uniquely represented by $u_\lambda \in S$ s.t. $\underbrace{\langle \lambda, \mathrm{tr}\, v\rangle}_{(\lambda \in \vec{S}')} = A[u_\lambda, v]$
$v \in S$

For test functions $\varphi \in \mathring{W}^{m,2}(\Omega)$

$$0 = A[u_\lambda, \varphi] \;\&\; \text{if } \lambda \text{ satisfies } \langle \lambda, \mathrm{tr}(\deg m\text{-}1 \text{ polys})\rangle = 0 \overset{?}{\Rightarrow} L u_\lambda = 0.$$

$\partial^{N}p$ can be done in particular for Δ^m, giving solutions

$$\Delta^m h_\lambda = 0 \text{ with } \langle \lambda, \mathrm{tr}\, v\rangle = (h_\lambda, v)_m := \cancel{\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int \partial^\alpha h_\lambda} \quad (176.1)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega \overline{\partial^\alpha h_\lambda}\, \partial^\alpha v \, dx \qquad v \in \mathring{W}^{m,2}(\Omega)$$

where we have used the inner product (for $\mathring{W}^{m,2}(\Omega)$) that coincides with the form for Δ^m

This can be used to define the single layer potential of λ for L by

$$S_L \lambda(x) = \langle \lambda, \mathrm{tr}\, \Gamma_L^x\rangle = (h_\lambda, \Gamma_L^x)_m, \quad x \in \mathbb{R}^d \setminus \partial\Omega.$$

$d > 2m$. Then $\Gamma_L^x \notin W^{m,2}(\Omega)$

$L S_L \lambda = 0$ in $\mathbb{R}^d \setminus \partial\Omega$.

By C-Z,

$$D^m S_L \lambda \in L^2(\mathbb{R}^d)$$

By (176.1) & def'n $\|\lambda\|_{1/2-m} = \sup_{\|tr\, v\|_{m-1/2}=1} |\langle \lambda, tr\, v\rangle|$

one gets $\|\lambda\|_{1/2-m} \approx |h_\lambda|_m$

So $\quad \|tr\, S_L \lambda\|_{m-1/2} \leq |S_L \lambda|_m \overset{\text{C-Z}}{\leq} C_L |h_\lambda| \approx C_L \|\lambda\|_{1/2-m}$

So $tr\, S_L : H^{1/2-m} \to H^{m-1/2}$ linear & bdd.

By mimicing almost exactly an argument of <u>Nédélec + Planchard (1973)</u> <u>MR 0424022 for Δ</u> & $d = 2, 3$? one can show that

$tr\, S_L$ is onto the space of Dirichlet data $H^{m-1/2}$.

In fact $tr\, S_L$ is invertible.

<u>$S_L \lambda$</u> solves the gDP.

$\circledast \leftarrow$ Adjoint.

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$$\mathcal{D}_A \operatorname{tr} g(x) = - \langle {}_c\nabla_A \Gamma_L^x , \operatorname{tr} g \rangle$$

$\Rightarrow$ Greens:

$$\mathcal{D}_A \operatorname{tr} g(x) = - S_L ({}_c\nabla_A u_g)(x)$$

with $L u_g = 0$.