Definitions...

1. Left R-module - Let R be a ring. A left R-module is an additive abelian group A together with a function $R \times A \longrightarrow A \ni (r,a) \longrightarrow ra$ and $\forall r,s \in R$ and $a,b \in A$
(i) $r(a+b) = ra + rb$
(ii) $(r+s)a = ra + sa$
(iii) $r(sa) = (rs)a$
(iv) If R has identity, $1_R a = a$
2. Unitary - If R has identity, then an R-module A is unitary
3. Bimodule - If A is both a right and left R-module, A is a bimodule, denoted ${}_R A_R$
4. R-module homomorphism - Let A, B be modules over a ring R. A function $f: A \longrightarrow B$ is an R-module homomorphism provided that $\forall a,c \in A$, $r \in R$, $f(a+c) = f(a) + f(c)$; $f(ra) = rf(a)$
5. Submodule - Let R be a ring, A an R-module, and B a nonempty subset of A. B is a submodule of A provided that B is an additive subgroup of A and $rb \in B$ $\forall r \in R, b \in B$. Equivalently, B is a module under same operations
6. Epimorphism - A surjective homomorphism is an epimorphism
7. Monomorphism - An injective homomorphism is a monomorphism
8. Product - Let $\mathcal{C}$ be a category and $\{A_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$. A product for the family $\{A_i\}_{i \in I}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \rightarrow A_i\}_{i \in I}$ $\ni$ for any object B and family of morphisms $\{\varphi_i : B \rightarrow A_i\}_{i \in I}$ $\exists!$ morphism $\varphi : B \rightarrow P \ni$ diagram commutes $\forall i \in I$:

$$\begin{tikzcd} \prod_{i \in I} A_i =: P \arrow[r, "\pi_i"] & A_i \\ & B \arrow[ul, dotted, "\exists \varphi"] \arrow[u, "\varphi_i"'] \end{tikzcd}$$

9. Coproduct - A coproduct for a family $\{A_i\}_{i \in I}$ of objects in a category $\mathcal{C}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \rightarrow S\}_{i \in I}$ $\ni$ for any object B and any family of morphisms $\{\psi_i : A_i \rightarrow B\}_{i \in I}$ $\exists!$ morphism $\psi : S \rightarrow B$

$\ni$ diagram commutes. $\forall i \in I$:

$$\begin{tikzcd} A_i \arrow[r, "\iota_i"] \arrow[d, "\psi_i"'] & S := \bigoplus_{i \in I} A_i \arrow[dl, dashed, "\exists \psi"] \\ B & \end{tikzcd}$$

10. Exact – A pair of module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $\operatorname{Im} f = \operatorname{Ker} g$. More generally, $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n$ is exact if $\operatorname{Im} f_{i-1} = \operatorname{Ker} f_i$ $\forall i$. An infinite sequence is exact if $\operatorname{Im} f_{i-1} = \operatorname{Ker} f_i$ $\forall i$.

11. SES – An exact sequence of the form $0 \to A \to B \to C \to 0$ is a short exact sequence (SES)

12. Isomorphism of SES – Two SES's are isomorphic if

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

f, g, h are R-module isomorphisms and the diagram commutes

13. Idempotent – Let R be a ring. An element $e \in R$ is idempotent if $e^2 = e$

14. Indecomposable – A module M is indecomposable if we cannot write M as a direct sum of two proper submodules.

15. Orthogonal – Two idempotents e_1, e_2 in a ring R are orthogonal if $e_1 e_2 = e_2 e_1 = 0$

16. Split Exact – A SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split if $\exists$ homomorphism $i : C \to B$ with $gi = 1_C$. Equivalently $\exists$ homomorphism $\rho : B \to A$ with $\rho f = 1_A$

17. Kernel – Let $B \xrightarrow{f} C$ be a morphism in $\mathcal{C}$. A kernel of f is a morphism $A \xrightarrow{i} B \ni$

(i) $fi = 0$

(ii) $\forall A' \xrightarrow{i'} B$ with $fi' = 0$ $\exists! \theta \ni$ diagram commutes:

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "f"] & C \\ & A' \arrow[ul, dashed, "\exists! \theta"] \arrow[u, "i'"'] \arrow[ur, bend right, "0"'] & \end{tikzcd}$$

18. <u>Cokernel</u> - Let $B \xrightarrow{f} C$ be a morphism in $\mathcal{C}$. A <u>cokernel</u> of f is a morphism $C \xrightarrow{p} D$ $\ni$

(i) $pf = 0$

(ii) $\forall\, C \xrightarrow{p'} D'$ with $p'f = 0$ $\exists!\theta \ni$ diagram commutes:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, bend right, "0"'] & C \arrow[r, "p"] \arrow[d, "p'"] & D \arrow[dl, dashed, "\exists!\theta"] \\ & D' & \end{tikzcd}$$

19. <u>Projective</u> - An R-module P is <u>projective</u> if $\forall$ surjective homomorphisms $B \xrightarrow{g} C$ and homomorphism $P \xrightarrow{f} C$, $\exists$ homomorphism $P \xrightarrow{h} B \ni$ diagram commutes:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

20. <u>Injective</u> - An R-module I is <u>injective</u> if $\forall$ injective homomorphisms $A \xrightarrow{f} B$ and homomorphisms $A \xrightarrow{h} I$ $\exists$ homomorphism $B \xrightarrow{g} I \ni$ diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "\exists g"] \\ & I & \end{tikzcd}$$

21. <u>Divisible</u> - An abelian group D is <u>divisible</u> if $\forall y \in D$ and $\forall\, 0 \neq n \in \mathbb{Z}$, $\exists x \in D \ni y = nx$

22. <u>Biadditive</u> - Let R be a ring and A_R, ${}_RB$ modules. Let G be a $\mathbb{Z}$-module. A <u>biadditive</u> R-function is a function $A \times B \xrightarrow{f} G \ni \forall a, a_1, a_2 \in A, \forall b, b_1, b_2 \in B, \forall r \in R$:

(i) $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$, $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$

(ii) $f(ar, b) = f(a, rb)$

23. <u>Tensor Product</u> - A <u>tensor product</u> over R of A_R with ${}_RB$ is an abelian group $A \otimes_R B$ together with an additive function $A \times B \xrightarrow{f} A \otimes_R B \ni \forall$ biadditive functions $g: A \times B \longrightarrow G$ where G is an abelian group, $\exists!\, h: A \otimes_R B \longrightarrow G$ homomorphism of abelian groups $\ni$ diagram commutes:

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "g"'] & A \otimes_R B \arrow[dl, dashed, "\exists ! h"] \\ G & \end{tikzcd}$$

24. <u>Flat</u> - A right R-module M is <u>flat</u> if $M \otimes_R -$ is exact

25. <u>Multiplicative</u> - Let R be a commutative ring. A subset $\emptyset \neq S \subseteq R$ is <u>multiplicative</u> if
(i) $1 \in S$, $0 \notin S$
(ii) $a, b \in S \Rightarrow ab \in S$

26. <u>Left Noetherian</u> - A ring R is <u>left noetherian</u> if it satisfies the ascending chain condition (ACC) on left ideals ie given any chain of left ideals $I_1 \subseteq I_2 \subseteq \ldots$, $\exists n \ni I_n = I_{n+1} = \ldots$ it stabilizes

27. <u>Noetherian</u> - An R-module M is <u>noetherian</u> if it satisfies the ACC on submodules

28. <u>Artinian</u> - An R-module M is <u>artinian</u> if it satisfies the descending chain condition (DCC) on submodules ie given a chain of submodules $M_1 \supseteq M_2 \supseteq \ldots$, $\exists n \ni M_n = M_{n+1} = \ldots$

29. <u>Left Artinian</u> - A ring R is <u>left artinian</u> if ${}_RR$ is left artinian

30. <u>Simple</u> - Let R be a ring. An R-module $S \neq 0$ is <u>simple</u> if it has no nonzero proper submodules (S is also called <u>irreducible</u>)

31. <u>Semisimple</u> - An R-module M is <u>semisimple</u> if each of its submodules is a direct summand ie $\forall L \subseteq M$ submodule $\exists X \subseteq M$ submodule with $M = L \oplus X$ (M is also called <u>completely reducible</u>)

32. <u>Series</u> - Let M be an R-module. A chain of submodules of M: $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ is a <u>series</u> for M.

33. <u>Length</u> - Given a series $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$, the <u>length</u> of the series is n.

34. <u>Factors</u> - Given a series $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$, the <u>factors</u> of the series are the modules $M_1/M_0, M_2/M_1, \ldots, M_n/M_{n-1}$

35. <u>Equivalent</u> - Two series of a module M are <u>equivalent</u> if they

have the same length and isomorphic factors in some order

36. Refinement - A series $0=N_0 \subsetneq \ldots \subsetneq N_m = M$ is a refinement of $0=M_0 \subsetneq \ldots \subsetneq M_n = M$ if each M_i is one of the N_j's.

37. Composition series - Let M be an R-module. A composition series (if it exists) of M is a series of the form $0=M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$ and $\forall i=0,\ldots,n-1$, M_{i+1}/M_i is simple

38. Length - Assume that M has a composition series $0=M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$. The length of M, $\ell(M)$, is the length of the composition series, n

39. Composition Factors - Assume that M has a composition series $0=M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$. The composition factors of M are $\{M_1/M_0, \ldots, M_n/M_{n-1}\}$ (which are simple by def)

40. Infinite Length - If M has no composition series, then the length of M is infinite, $\ell(M) = \infty$

41. Left Wedderburn - A ring R is a left Wedderburn ring if it is left Artinian and has no nonzero nilpotent left ideals ie $I^n = 0 \Rightarrow I = 0$

42. Minimal - A minimal left ideal in R is a left ideal that is simple when viewed as a module over R

43. Semisimple - A ring R is left semisimple if $_RR$ is semisimple

44. Nil - Let R be a ring. A left ideal of R is nil if each element of I is nilpotent ie $\forall x \in I \ \exists n_x \ni x^{n_x} = 0$

45. Nilradical - Let R be a ring. It's nilradical, $Nil(R)$, is the sum of all the two-sided nil ideals of R ie $Nil(R)$ is the largest two-sided nil ideal of R

46. Simple - A ring R is simple if whenever $I \triangleleft R$, then $I=0$ or $I=R$

47. Annihilator - Let M be a left R-module. The annihilator of M in R is $ann_R M = \{r \in R \mid rm = 0 \ \forall m \in M\}$

48. Faithful - A module M is faithful if $ann_R M = 0$

49. Torsion - If R is an integral domain and M is an R-module, then the torsion submodule of M is $t(M) = \{m \in M \mid \exists 0 \neq r \in R \ni$

$rm = 0\}$

50. Torsion Free - A module M is torsion free if $t(M) = 0$

51. Left Primitive - A ring is left primitive if it has a simple left faithful module

52. Radical - Let R be a left Artinian ring and let $I_1, \dots, I_t$ be all the maximal two-sided ideals of R. The radical of R is $\operatorname{rad} R = I_1 \cap \dots \cap I_t$

53. Jacobson Radical - Let R be a ring. The Jacobson radical is $J(R) = \bigcap$ all maximal left ideals of R

54. Group Ring - Let G be a group and let K be a field. The group ring KG is spanned by the elements of G as a vector space over K. It's elements are finite formal sums $\sum_{g \in G} a_g g$, $a_g \in K$

55. Local - A ring R is local if $R/J(R)$ is a division ring

56. Essential Extension - Let R be a ring and $M \subseteq E$ an inclusion (extension) of R-modules. The extension $M \subseteq E$ essential, denoted $M \overset{ess}{\subseteq} E$, if $\forall 0 \neq L \leq E$, $L \cap M \neq (0)$

57. Socle - Let M be a module. The socle of M, denoted $\operatorname{soc} M$, is the unique largest semisimple submodule of M (if it exists). If M has no semisimple submodule $\operatorname{soc} M = 0$

58. Maximal Essential extension - E is a maximal essential extension of M if no proper extension of E is an essential extension of M

59. Injective Envelope - Let M be an R-module. An injective envelope (hull) of M is an extension $M \overset{ess}{\subseteq} E$ where E is injective, denoted $E(M)$

60. IBN - A ring R has invariant basis number (IBN) if given any free R-module F, then any two basis of F have the same cardinality

61. Left-minimal - $M \xrightarrow{f} E$ is left-minimal if $\forall$ diagrams

$$\begin{tikzcd} & E \arrow[dd, "h"] \\ M \arrow[ur, "f"] \arrow[dr, "f"'] \arrow[r, phantom, "G"] & \\ & E \end{tikzcd}$$

h is an isomorphism

62. Right-Minimal - $P \xrightarrow{f} M$ is right-minimal if $\forall$ diagrams

$$\begin{tikzcd} P \arrow[dr, "f"] \arrow[dd, "h"'] & \\ \arrow[r, phantom, "G"] & M \\ P \arrow[ur, "f"'] & \end{tikzcd}$$

h is an isomorphism

63. Projective Cover - Let M be a module. A projective cover of M is a projective module P mapping onto M, $P \xrightarrow{f} M \longrightarrow 0$ $\ni$ f is right minimal

64. Flat Cover - Let M be a module. A flat cover of M is a flat module X mapping onto M, $X \xrightarrow{f} M \longrightarrow 0$ $\ni$ f is right minimal

65. Uniform Dimension - Let M be an R-module. The uniform dimension of M, denoted udim M, is the largest $k \in \mathbb{Z}$ $\ni \exists$ inclusion $L_1 \oplus \ldots \oplus L_k \subseteq M$ where $0 \neq L_i \leq M$. If no such k exists, udim $M = \infty$.

66. Uniform - An R-module M is uniform if udim $M = 1$

67. Irreducible - Let R be a ring. A proper left ideal I is (meet) irreducible if whenever $I = A \cap B$ for some left ideals A, B, we have $A = I$ or $B = I$

68. Primitive idempotent - $e \in R$ is a primitive idempotent if we cannot write $e = e_1 + e_2$ where e_1, e_2 are orthogonal idempotents

69. Minimal Prime - Let R be a commutative ring. An ideal $P \triangleleft R$ is a minimal prime ideal if it is prime and if $P' \subseteq P$ is a prime ideal $\Rightarrow P' = P$

70. Incomparable - ideals P_1, P_2 are incomparable if P_1, P_2 are both minimal primes hence $P_1 \not\subseteq P_2$, $P_2 \not\subseteq P_1$

71. Primary - A proper ideal $Q \triangleleft R$ is primary if whenever $ab \in Q$

then $a \in Q$ or $b^n \in Q$ for some $n \geq 1$

72. Minimal Prime over I - Let $I \triangleleft R$. A prime ideal $P \supseteq I$ is a minimal prime over I (or covering I) if $\nexists$ prime ideal $\ni$ $P \supsetneq P' \supseteq I$

73. $\sqrt{I}$ - If $I \triangleleft R$, $\sqrt{I} = \{x \in R \mid x^n \in I$ for some $n \geq 1\} \triangleleft R$

74. P-primary - A primary ideal Q is P-primary if $P = \sqrt{Q}$

75. Spec R - Spec R $= \{$prime ideals of $R\}$

76. Cokerφ - If $M \xrightarrow{\varphi} N$, coker $\varphi = N/\text{Im}\,\varphi$

77. Category - A category $\mathcal{C}$ consists of
(i) obj $\mathcal{C}$ = class or collection of objects
(ii) mor $\mathcal{C}$ = set of morphisms between objects
with (1) an identity morphism $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$ $\forall A \in \text{obj}\,\mathcal{C}$
(2) a composition operation
$$\text{Hom}_{\mathcal{C}}(A,B) \times \text{Hom}_{\mathcal{C}}(B,C) \longrightarrow \text{Hom}_{\mathcal{C}}(A,C)$$
$$(f,g) \longrightarrow g\circ f, \text{ denoted } gf$$
$\ni$ (a) $(hg)f = h(gf)$ $\forall A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$
(b) $1_B \circ f = f = f \circ 1_A$ $\forall A \xrightarrow{f} B$

78. Monic - Let $B \xrightarrow{f} C$ be a morphism. f is monic if $\forall A \underset{e_2}{\overset{e_1}{\rightrightarrows}} B$ morphisms with $fe_1 = fe_2$, then $e_1 = e_2$

79. Epi - Let $B \xrightarrow{f} C$ be a morphism. f is epi if $\forall C \underset{g_2}{\overset{g_1}{\rightrightarrows}} D$ morphisms with $g_1 f = g_2 f$, then $g_1 = g_2$

80. Isomorphism - Let $B \xrightarrow{f} C$ be a morphism. f is an isomorphism if it has an inverse ie $\exists C \xrightarrow{g} B \ni$ $gf = 1_B$ and $fg = 1_C$

81. Subcategory - A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is a collection of some of the objects and some of the morphisms $\ni$ $\mathcal{B}$ forms a category

82. Full Subcategory - A subcategory $\mathcal{B}$ is a full subcategory if mor $\mathcal{B}$ includes all morphisms between objects of $\mathcal{B}$

83. Opposite Category - The opposite category of $\mathcal{C}$, denoted $\mathcal{C}^{op}$, is the category consisting of the same objects

as in $\mathcal{C}$ but all morphisms/arrows are reversed (composition: $g^{op} \circ f^{op} = (f \circ g)^{op}$)

84. Covariant functor - Let $\mathcal{C}, \mathcal{D}$ be categories. $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor if it associates:

$$c \in \mathcal{C} \longrightarrow F(c) \in \mathcal{D}$$
$$(c_1 \xrightarrow{f} c_2) \longrightarrow (F(c_1) \xrightarrow{F(f)} F(c_2))$$

$\ni$ (1) $F(1_c) = 1_{F(c)}$
(2) $F(gf) = F(g)F(f)$ ie F preserves compositions
Equivalently, F preserves commutative diagrams

85. Contravariant functor - Let $\mathcal{C}, \mathcal{D}$ be categories. $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a contravariant functor if it associates:

$$\text{ob}\,\mathcal{C} \xrightarrow{F} \text{ob}\,\mathcal{D}$$
$$\text{Hom}_{\mathcal{C}}(c_1, c_2) \xrightarrow{F} \text{Hom}_{\mathcal{D}}(F(c_2), F(c_1))$$

$\ni$ (1) $F(1_c) = 1_c$
(2) $F(fg) = F(f)F(g)$

86. Faithful - A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is faithful if maps on Hom sets are injective ie $f_1 \neq f_2 \Rightarrow F(f_1) \neq F(f_2)$

87. Full - A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is full if maps on Hom sets are surjective ie $\forall g \in \text{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$, $\exists f \ni g = F(f)$

88. Fully faithful - A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is fully faithful if it is both faithful and full ie $\text{Hom}_{\mathcal{C}}(c_1, c_2) \xrightarrow{F} \text{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$ $\ni f \longrightarrow F(f)$ is a bijection

89. Natural Transformation - Let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ be two functors. A natural transformation $\eta: F \longrightarrow G$ associates:
(1) $\forall c \in \mathcal{C}$, a morphism $F(c) \xrightarrow{\eta_c} G(c)$ in $\mathcal{D}$
(2) $\forall$ morphisms $c \xrightarrow{f} c'$ in $\mathcal{C}$ the diagram commutes:

$$\begin{tikzcd} F(c) \arrow[r, "\eta_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\eta_{c'}"] & G(c') \end{tikzcd}$$

90. Natural Isomorphism - If $\eta: F \longrightarrow G$ is a natural transformation $\ni$ each η_c is an isomorphism in $\mathcal{D}$, then η is a natural isomorphism, denoted $\eta: F \xrightarrow{\simeq} G$

91. Equivalence of Categories - A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence of categories if $\exists$ functor $G: \mathcal{D} \longrightarrow \mathcal{C}$ $\ni$ $GF \xrightarrow{\cong} Id_{\mathcal{C}}$, $FG \xrightarrow{\cong} Id_{\mathcal{D}}$ are natural isomorphisms

92. Initial Object - An initial object I is an object $\ni \forall C \in \mathcal{C}$, $\exists!$ morphism $I \longrightarrow C$

93. Terminal Object - A terminal object T is an object $\ni$ $\forall C \in \mathcal{C}$, $\exists!$ morphism $C \longrightarrow T$

94. Zero Object - The zero object is the unique object $\ni$ it is both initial and terminal, denoted O

95. Additive - A category $\mathcal{A}$ is additive if

(1) Every Hom set is given an abelian group structure $(+)$ $\ni$ composition distributes over addition:

$h(g+g') = hg + hg'$, $(g+g')f = gf + g'f$

(2) $\mathcal{A}$ has a zero object

(3) $\forall A, A' \in \mathcal{A}$, $\exists$ product $A \times A'$

96. Abelian - A category $\mathcal{A}$ is abelian if $\mathcal{A}$ is an additive category and if

(1) Every map has a kernel and cokernel

(2) Every monic is the kernel (of its cokernel)

(3) Every epi is the cokernel (of its kernel)

97. Image - Let $B \xrightarrow{f} C$. The image of f is $Im f = ker(coker f)$

98. Exact - A sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $Im f = ker g$

99. Additive - Let $\mathcal{A}, \mathcal{B}$ be additive categories. A functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is additive if F preserves addition ie $F(f \pm g) = F(f) \pm F(g)$ ie F is an abelian group homomorphism

100. Left Exact - Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between additive categories. F is left-exact if $\forall$ SES in $\mathcal{A}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact in $\mathcal{B}$.

101. Exact - Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. F is exact

if it is both left and right exact

102. Exact - A contravariant functor $F: \mathcal{A} \to \mathcal{B}$ is left/right/exact if $F': \mathcal{A}^{op} \to \mathcal{B}$ is left/right/exact

103. Functor Category - Given categories $I, \mathcal{A}$ the functor category $\mathcal{A}^I$ has objects: functors $I \xrightarrow{F} \mathcal{A}$, and morphisms: natural transformations

104. Yoneda Embedding - The Yoneda Embedding of a category I is the functor:

$$\begin{aligned} I &\longrightarrow \text{SETS}^{I^{op}} \\ i &\longrightarrow \text{the functor } I^{op} \longrightarrow \text{SETS} = \text{Hom}(-, i) \\ C &\longrightarrow \text{Hom}(-, C) \end{aligned}$$

105. Adjoint Functors - Let $\mathcal{A}, \mathcal{B}$ be categories. A pair of functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ are adjoint if $\forall A \in \mathcal{A}, \forall B \in \mathcal{B}$ $\exists$ bijection $\text{Hom}_{\mathcal{B}}(F(A), B) \xrightarrow[\sim]{\tau_{AB}} \text{Hom}_{\mathcal{A}}(A, G(B))$ that is natural in A, B. Say F is left adjoint of G, G is right adjoint of F, or (F, G) is an adjoint pair

106. System - A (direct) system of objects in $\mathcal{C}$ is a functor $F: I \to \mathcal{C}$ from a poset I to $\mathcal{C}$ ie given a category $\mathcal{C}$ and poset I, a system is a collection $\{M_i\}_{i \in I}$ of objects with maps $M_i \xrightarrow{\varphi_j^i} M_j$ wherever $i \leq j$ $\ni$ if $i \leq j \leq k$, $\varphi_k^j \varphi_j^i = \varphi_k^i$

107. Colimit - Given a system in $\mathcal{C}$, the colimit (direct limit), denoted $\text{colim}\, M_i$ or $\varinjlim M_i$, is an object L equipped with maps $M_i \xrightarrow{\tau_i} L$ compatible with the system, ie

$$\begin{tikzcd} M_i \arrow[d, "\varphi_j^i"'] \arrow[dr, "\tau_i"] & \\ M_j \arrow[r, "\tau_j"'] & L \end{tikzcd}$$

commutes, $\ni$ for any collection of maps $\{M_i \xrightarrow{f_i} T\}$ compatible with the system, ie diagram commutes

$$\begin{tikzcd} M_i \arrow[d, "\varphi_j^i"'] \arrow[dr, "f_i"] & \\ M_j \arrow[r, "f_j"'] & T \end{tikzcd}$$

$\exists! L \xrightarrow{f} T \ni f\tau_i = f_i \; \forall i$

Equivalently, the diagram below commutes $\forall i, j$:

$$\begin{tikzcd} M_i \arrow[d, "\varphi^i_j"'] \arrow[r, "\varepsilon_i"] \arrow[rr, bend left, "\forall f_i"] & L \arrow[r, dashed, "\exists! f"] & T \\ M_j \arrow[ur, "\varepsilon_j"'] \arrow[urr, bend right, "\forall f_j"'] & & \end{tikzcd}$$

108. __Pushout__ - The colimit of the system of the form

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \\ B & \end{tikzcd}$$

is the __pushout__ of the system ie the __pushout__ is the object L with maps i, j $\ni$

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "i"] \arrow[ddr, bend left, "\forall f"] & \\ B \arrow[r, "j"'] \arrow[drr, bend right, "\forall g"'] & L \arrow[dr, dashed, "\exists!\theta"] & \\ & & T \end{tikzcd}$$

$i\alpha = j\beta$, $\forall f, \forall g$ with $f\alpha = g\beta$ $\exists! \theta \ni \theta i = f$, $\theta j = g$

109. __Inverse System__ - An __inverse system__ is a contravariant functor $F: I \to \mathcal{C}$ from a poset I to a category $\mathcal{C}$ ie a collection $\{M_i\}_{i \in I}$ of objects and maps $M_j \xrightarrow{\varphi^j_i} M_i$ $\forall i \le j$ $\ni$ $\forall i \le j \le k$ $\varphi^j_i \varphi^k_j = \varphi^k_i$. Equivalently a direct system over I^{op}

110. __Limit__ - Given an inverse system $\{M_i\}_{i \in I}$, the __limit__ (__inverse limit__), denoted $\lim M_i$ or $\varprojlim M_i$, is an object L with maps $L \xrightarrow{\pi_i} M_i$ $\forall i$ compatible with the system $\ni$ for any collection of maps $\{T \xrightarrow{f_i} M_i\}_{i \in I}$ compatible with the system, $\exists!$ $T \xrightarrow{f} L \ni f_i = \pi_i f$ ie the diagram commutes

$$\begin{tikzcd} & & M_i \\ \forall T \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "\forall f_i"] \arrow[drr, bend right, "\forall f_j"'] & L \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\ & & M_j \arrow[uu, "\varphi^j_i"'] \end{tikzcd}$$

111. __Map of Systems__ - Let $\{M_i\}, \{N_i\}$ be systems over a poset I in $\mathcal{C}$. A __map__ $\{M_i\} \longrightarrow \{N_i\}$ is a natural transformation of the functors $I \to \mathcal{C}$. Equivalently, maps $M_i \longrightarrow N_i$ $\forall i \le j$

$\ni$
$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & N_i \arrow[d] \\ M_j \arrow[r] & N_j \end{tikzcd}$$
commutes

112. <u>Exact</u> - A sequence $\{M_i\} \to \{N_i\} \to \{W_i\}$ is <u>exact</u> if it is exact at each i ie $M_i \to N_i \to W_i$ exact $\forall i$

113. <u>Directed</u> - A poset I is <u>directed</u> if $\forall i,j \in I$, $\exists$ arrows $\begin{matrix} i \searrow \\ j \nearrow \end{matrix} k$ for some k (can have $k = i,j$)

114. <u>Filtered</u> - A category I is <u>filtered</u> if it is directed and if $\forall\, i \underset{\beta}{\overset{\alpha}{\rightrightarrows}} j \xrightarrow{\exists \gamma} l \ni \gamma\alpha = \gamma\beta$

115. <u>Directed limit</u> - A colimit over a directed set is a <u>directed limit (directed colimit, filtered colimit)</u>, denoted $\operatorname{colim} M_i$ or $\varinjlim M_i$

116. <u>Flat</u> - A left R-module M is <u>flat</u> if $- \otimes_R M$ is exact

117. <u>Chain complex</u> - A <u>(chain) complex</u> $C.$ is a sequence of R-modules and maps

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

$\ni$ each composition $d_n d_{n+1} = 0$ $\forall n$ ie $d^2 = 0$

118. <u>Differentials</u> - The <u>differentials</u> of a complex are the maps d_n

119. <u>Cycles</u> - The <u>cycles</u> of a complex are $Z_n = \ker d_n$

120. <u>Boundaries</u> - The <u>boundaries</u> of a complex are $B_n = \operatorname{Im} d_{n+1}$

121. <u>Homology</u> - The nth <u>homology</u> module is $H_n = Z_n / B_n$

122. <u>Chain Map</u> - A <u>chain map (map of complexes)</u> $f: C. \to D.$ is a family of maps $f_n: C_n \to D_n \ni$ all squares commute:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

123. <u>Kernel of Chain map</u> - For a chain map $f: C. \to D.$, <u>Ker f</u> is the complex: $\cdots \longrightarrow \operatorname{Ker} f_{n+1} \xrightarrow{d_{n+1}} \operatorname{Ker} f_n \xrightarrow{d_n} \operatorname{Ker} f_{n-1} \longrightarrow \cdots$

124. <u>Cokernel of chain map</u> - For a chain map $f: C. \to D.$, <u>Coker f</u> is the complex: $\cdots \longrightarrow \operatorname{Coker} f_{n+1} \xrightarrow{\bar{d}_{n+1}} \operatorname{Coker} f_n \xrightarrow{\bar{d}_n} \operatorname{Coker} f_{n-1} \longrightarrow \cdots$

125. <u>Subcomplex</u> - $B_\cdot$ is a <u>subcomplex</u> of $C_\cdot$ if $B_n \subseteq C_n\ \forall n$ and if $d_n^B = d_n^C|_{B_n}$ where $B_\cdot : \cdots \longrightarrow B_{n+1} \xrightarrow{d_{n+1}^B} B_n \xrightarrow{d_n^B} B_{n-1} \longrightarrow \cdots$

126. <u>Quotient complex</u> - If $B_\cdot$ is a subcomplex of $C_\cdot$, then the <u>quotient complex</u> is $C_\cdot/B_\cdot : \cdots \longrightarrow C_{n+1}/B_{n+1} \xrightarrow{\bar d} C_n/B_n \longrightarrow \cdots$ where $\bar d(\bar x) = \overline{d(x)}$

127. <u>Image of Chain map</u> - Given a chain map $f: C_\cdot \longrightarrow D_\cdot$, <u>Im f</u> is the complex: $\cdots \longrightarrow \operatorname{Im} f_{n+1} \xrightarrow{d_{n+1}^D} \operatorname{Im} f_n \xrightarrow{d_n^D} \operatorname{Im} f_{n-1} \xrightarrow{d_{n-1}^D} \cdots$

128. <u>Cochain Complex</u> - A <u>cochain complex</u> $C^\cdot$ is a sequence of maps of R-modules:

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^2 = 0$

129. <u>Cocycle</u> - The <u>cocycles</u> of cochain complex are $Z^n = \ker d^n$

130. <u>Coboundary</u> - The coboundaries of cochain complex are $B^n = \operatorname{Im} d^{n-1}$

131. <u>Cohomology</u> - The <u>cohomology</u> of a <u>cochain</u> complex is $H^n = Z^n/B^n$

132. <u>Quasi-isomorphism</u> - A chain map $f: C_\cdot \longrightarrow D_\cdot$ is a <u>quasi-isomorphism</u> if it induces an isomorphism on homology: $H_n(C_\cdot) \xrightarrow[\cong]{H_n(f)} H_n(D_\cdot)$

133. <u>Exact sequence of Chain complexes</u> - The sequence of chain complexes: $0 \longrightarrow A_\cdot \longrightarrow B_\cdot \longrightarrow C_\cdot \longrightarrow 0$ is <u>exact</u> if each $0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0$ is exact

134. <u>Split Exact</u> - A complex $C_\cdot$ is <u>split exact</u> if it is exact and $\forall n$ $0 \longrightarrow Z_n \hookrightarrow C_n \xrightarrow{d_n} B_{n-1} \longrightarrow 0$ is split

135. <u>Null Homotopic</u> - A chain map $f: C_\cdot \longrightarrow D_\cdot$ is <u>null homotopic</u>, denoted $f \simeq 0$, if $\exists$ maps $s_n : C_n \longrightarrow D_{n+1} \ni f = ds + sd$. The map s is called a <u>(null) homotopy</u>. ie have <u>diagram</u>:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d"] \arrow[d, "f_{n+1}"] \arrow[dl, dashed, "s"'] & C_n \arrow[r, "d"] \arrow[d, "f_n"] \arrow[dl, dashed, "s"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, dashed, "s"'] & \cdots \arrow[dl, dashed, "s"'] \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

136. <u>Homotopic</u> - Two chain maps $f, g: C_\cdot \longrightarrow D_\cdot$ are <u>homotopic</u>, denoted $f \simeq g$, if $f - g \simeq 0$

137. <u>Homotopy Equivalence</u> - A chain map $f: C_. \to D_.$ is a <u>homotopy equivalence</u> if f has an inverse up to homotopy ie $\exists g: D_. \to C_.$
$\ni fg \simeq 1_D$, $gf \simeq 1_C$

138. <u>Bicomplex</u> - A <u>bicomplex</u> (<u>double complex</u>) $C_{..}$ is a diagram of R-modules and maps

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d'"'] & C_{p,q} \arrow[l, "d"'] \arrow[d, "d'"] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

$\ni d^2 = 0$, $(d')^2 = 0$ and $dd' = d'd$ ie squares commute

139. <u>Bounded</u> - A bicomplex $C_{..}$ is <u>bounded</u> if along each diagonal $p+q=n$ there are only finitely many nonzero terms

140. <u>Total Complex</u> - Given a bicomplex $C_{..}$, the <u>total complexes</u>, denoted $\mathrm{Tot}(C_{..})$, are $(\mathrm{Tot}^{\oplus}(C))_n = \bigoplus_{p+q=n} C_{p,q}$ and $(\mathrm{Tot}^{\Pi}(C))_n = \prod_{p+q=n} C_{p,q}$ with differentials $d^{Tot} = d + (-1)^p d'$

141. <u>Tensor Product</u> - Given two complexes $P_.: \dots \to P_1 \to P_0 \to 0$ and $Q_.: \dots \to Q_1 \to Q_0 \to 0$, the <u>tensor product</u> $P_. \otimes_R Q_.$ is the bicomplex:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes Q_2 \arrow[d, "1\otimes d"] & P_1 \otimes Q_2 \arrow[l] \arrow[d, "1\otimes d"] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_1 \arrow[d, "1\otimes d"] & P_1 \otimes Q_1 \arrow[l, "d\otimes 1"'] \arrow[d, "1\otimes d"] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l, "d\otimes 1"'] & P_2 \otimes Q_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Or its Tot: $(P_. \otimes Q_.)_n = \bigoplus_{p+q=n} P_p \otimes_R Q_q$ with $d = d\otimes 1 + (-1)^p(1 \otimes d)$

142. <u>Hom of complexes</u> - Given complexes $P_.: \dots \to P_1 \to P_0 \to 0$ and $Q^\cdot: 0 \to Q^0 \to Q^1 \to \dots$, the <u>Hom</u> $\mathrm{Hom}(P_., Q^\cdot)$

is the bicomplex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
\mathrm{Hom}(P_0, Q^2) \arrow[d, "d_*"] & \mathrm{Hom}(P_1, Q^2) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathrm{Hom}(P_0, Q^1) \arrow[d, "d_*"] & \mathrm{Hom}(P_1, Q^1) \arrow[l, "d^*"'] \arrow[d] & \cdots \arrow[l] \\
\mathrm{Hom}(P_0, Q^0) & \mathrm{Hom}(P_1, Q^0) \arrow[l, "d^*"'] & \cdots \arrow[l]
\end{tikzcd}$$

or its Tot: $(\mathrm{Hom}(P_\cdot, Q^\cdot))_n = \bigoplus_{p+q=n} \mathrm{Hom}(P_p, Q^q)$ with $d = d^* + (-1)^p d_*$

143. Mapping cone – Let $f: B_\cdot \longrightarrow C_\cdot$ be a chain map. The mapping cone of f is $C(f) = \mathrm{Tot}(C \xleftarrow{f} B)$ ie $(C(f))_n = C_n \oplus B_{n-1}$ with differential $\begin{bmatrix} d_n^C & f_{n-1} \\ 0 & -d_{n-1}^{B} \end{bmatrix}$

144. Mapping Cylinder – Let $f: B_\cdot \longrightarrow C_\cdot$ be a chain map. The mapping cylinder of f is $Cyl(f) = C(\underline{C(f)[1]} \xrightarrow{p_2} B)$.

Equivalently $Cyl(f) = \mathrm{Tot}(C \xleftarrow{f} B \xrightarrow{=} B)$

$$= \mathrm{Tot}\left(\begin{tikzcd}
C_2 \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[d] & C_2 \arrow[d] \\
C_1 \arrow[d] & B_1 \arrow[l] \arrow[r] \arrow[d] & B_1 \arrow[d] \\
C_0 & B_0 \arrow[l] \arrow[r] & B_0
\end{tikzcd}\right)$$

ie $(Cyl(f))_n = C_n \oplus B_{n-1} \oplus B_n$ with differential

145. Presentation – A presentation of an R-module M is an expression of it as $M \cong F/K$, F free

146. Projective Resolution – A projective resolution of an R-module M is a complex $P_\cdot : \cdots \to P_2 \to P_1 \to P_0 \to 0$ with each P_i projective, together with a map $P_0 \xrightarrow{\epsilon} M$ s.t. the augmented complex $\cdots \to P_1 \to P_0 \to M \to 0$ is exact. Equivalently, we get a quasi-iso:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

147. <u>Injective Resolution</u> - Let M be an R-module. An <u>injective resolution</u> of M is an exact sequence:

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

with each I^n an injective R-module

148. <u>Left Derived Functors</u> - Let $F: \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. If $\mathcal{A}$ has projective resolutions, the <u>left derived functors</u> of F, denoted L_nF, are defined for $M \in \mathcal{A}$ as $L_nF(M) = H_n(F(P_\cdot))$ where $P_\cdot \to M$ is a projective resolution

149. <u>δ-Functor</u> - Any collection of functors $\{T_n\}_{n \geq 0}$ $\ni$ for any SES $0 \to M' \to M \to M'' \to 0$ $\exists$ LES:

$$\cdots \to T_{n+1}(M'') \xrightarrow{\delta_{n+1}} T_n(M') \longrightarrow T_n(M) \longrightarrow T_n(M'') \xrightarrow{\delta_n} T_{n-1}(M') \to \cdots$$

with δ_n's natural, is a <u>(homological) δ-functor</u>

150. <u>Universal δ-Functor</u> - A δ-functor $\{T_n\}$ is <u>universal</u> if given any other δ-functor $\{S_n\}$ and natural transformation $S_0 \to T_0$, $\exists!$ map of δ-functors $\{S_n\} \xrightarrow{\{f_n\}} \{T_n\}$ that commute with δ's

151. <u>Tor</u> - The left derived functor of $F(-) = \_ \otimes_R N$ is <u>$\mathrm{Tor}_n^R(-,N)$</u> defined as $\mathrm{Tor}_n^R(M,N) = L_nF(M) = H_n(F(P_\cdot))$ where $P_\cdot \to M$ is a projective resolution

152. <u>Right Derived Functors</u> - Let $F: \mathcal{A} \to \mathcal{B}$ be an additive left exact functor between abelian categories. If $\mathcal{A}$ has injective resolutions, the <u>right derived functors</u> of F, denoted R^nF, are defined for $N \in \mathcal{A}$ as $R^nF(N) = H^n(F(I^\cdot))$ where $N \to I^\cdot$ is an injective resolution

153. <u>Ext</u> - The right derived functor of $F(-) = \mathrm{Hom}_R(M,-)$ is <u>$\mathrm{Ext}_R^n(M,-)$</u> defined as $\mathrm{Ext}_R^n(M,N) = R^nF(N) = H^n(F(I^\cdot))$ where $N \to I^\cdot$ is an injective resolution

154. <u>Hereditary</u> - R is a <u>hereditary</u> ring if submodules of projective modules are projective. Equivalently every module has projective dimension ≤ 1

155. <u>Extension</u> - An extension of C by A is an exact <u>sequence</u>

$$\xi: 0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$$

156. Equivalent Extensions - Two extensions ξ, ξ' are equivalent if $\exists$ map $\varphi \ni$

$$\begin{tikzcd}
\xi: 0 \arrow[r] & A \arrow[r] \arrow[d, "\parallel"'] & X \arrow[r] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "\parallel"] & 0 \\
\xi': 0 \arrow[r] & A \arrow[r] & X' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

commutes

157. Classic Ext - $e(C,A) = \{[\xi] \mid \xi$ is extension of C by $A\}$

158. Baer Sum - Let $\xi: 0 \longrightarrow A \xrightarrow{i} X \xrightarrow{p} C \longrightarrow 0$

$$\xi': 0 \longrightarrow A \xrightarrow{i'} X' \xrightarrow{p'} C \longrightarrow 0$$

be extensions, X'' be the pullback of p, p' ie
$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X' \arrow[d] \\
X \arrow[r] & C
\end{tikzcd}$$
ie $X'' = \{(x, x') \in X \times X' \mid p(x) = p(x) = p'(x')$ in $C\}$ and $Y = X''/\{(-i(a), i'(a))\}_{a \in A}$. Then the Baersum $\xi + \xi'$ is exact sequence $0 \longrightarrow A \longrightarrow Y \longrightarrow C \longrightarrow 0$ with maps $a \longmapsto (i(a), 0) = (0, i'(a))$, $(x, x') \longmapsto p(x)$

159. Derived Category - The derived category $\mathcal{D}(\mathcal{A})$ is a category equipped with a functor $Ch(\mathcal{A}) \xrightarrow{i} \mathcal{D}(\mathcal{A}) \ni i(f)$ is an iso $\forall f$ quasi-iso satisfying the UMP: Any functor $Ch(\mathcal{A}) \xrightarrow{F} \mathcal{C}$ transforming quasi-isos to isos factors uniquely through $\mathcal{D}(\mathcal{A})$ ie $\exists!$ functor $G \ni$ diagram commutes:

$$\begin{tikzcd}
Ch(\mathcal{A}) \arrow[r, "F"] \arrow[dr, "i"'] & \mathcal{C} \\
& \mathcal{D}(\mathcal{A}) \arrow[u, dashed, "\exists! G"']
\end{tikzcd}$$

160. Localization of a Category - Let $\mathcal{B}$ be a category and S be a collection of morphisms. Then the localization of $\mathcal{B}$, denoted $S^{-1}\mathcal{B}$, is the category with $ob(S^{-1}\mathcal{B}) = ob(\mathcal{B})$ and morphisms $f_1 s_1^{-1} f_2 s_2^{-1} \ldots f_k s_k^{-1}$, $f_i \in mor(\mathcal{B})$, $s_i \in S$

161. Localizing - A collection $S \subseteq mor\,\mathcal{B}$ is localizing if in $\mathcal{B}$:

(i) All $1_X \in S$ and $s, t \in S \Rightarrow st \in S$

(ii) $\forall f, s$ with $s \in S$ with same codomain, $\exists g, t$ with $t \in S$ $\ni$ diagram commutes:

$$\begin{tikzcd}
W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and $\forall g,t$ as above $\exists s,f$ as above ie $\forall f,s,\ \exists g,t \ni$
$ft = sg \Rightarrow s^{-1}f = gt^{-1}$ in $S^{-1}\mathfrak{B}$

(iii) For any f,g, $\exists s \in S \ni sf = sg$ iff $\exists t \in S \ni ft = gt$

162. Roofs – A roof is a map from $X \longrightarrow Y$ given by
$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
written (f,s) in $S^{-1}\mathfrak{B}$

163. Homotopy Category – The homotopy category $K(\mathcal{A})$ is the category consisting of $ob(K(\mathcal{A})) = ob(Ch(\mathcal{A}))$ = cochain complexes and $mor(K(\mathcal{A}))$ = cochain maps mod homotopy

164. Triangle – A triangle, Δ, in a category of complexes is a diagram $K^\cdot \xrightarrow{u} L^\cdot \xrightarrow{v} M^\cdot \xrightarrow{w} K^\cdot[1]$

165. Morphism of Δ's – A morphism of Δ's is a diagram commutative upto homotopy:
$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & L \arrow[r] \arrow[d, "g"'] & M \arrow[r] \arrow[d, "h"'] & K[1] \arrow[d, "f[1]"'] \\ K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & K'[1] \end{tikzcd}$$

166. Isomorphism of Δ's – A morphism of Δ's is an isomorphism if f,g,h are isomorphisms in that category (ie homotopy equivalence in $K(\mathcal{A})$ or quasi-iso in $\mathcal{D}(\mathcal{A})$)

167. Distinguished Δ – A distinguished Δ is a triangle isomorphic to $K \xrightarrow{\tilde{f}} Cyl(f) \xrightarrow{\pi} C(f) \xrightarrow{p} K[1]$ for some $K \xrightarrow{f} L$ or equivalently a triangle isomorphic to $K \xrightarrow{f} L \xrightarrow{i} C(f) \xrightarrow{p} K[1]$

168. $X[-i]$ – For an object $X \in \mathcal{A}$ category, $X[-i]$ is the complex
$\cdots \longrightarrow 0 \longrightarrow \underset{(i)}{X} \longrightarrow 0 \longrightarrow \cdots$

169. $Ext^i_{\mathcal{A}}(X,Y)$ – For $X,Y \in \mathcal{A}$ a category, $Ext^i_{\mathcal{A}}(X,Y) = Hom_{\mathcal{D}(\mathcal{A})}(X[0], Y[i])$

170. Triangulated Category – A Δ'd category is an additive category $\mathcal{T}$ with translation T and a collection of distinguished Δ's $\ni$ the following axioms hold:

TR1. (a) $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow X[1]$ is a distinguished Δ

(b) Any Δ iso to a distinguished Δ is distinguished

(c) Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished Δ, (u,v,w)

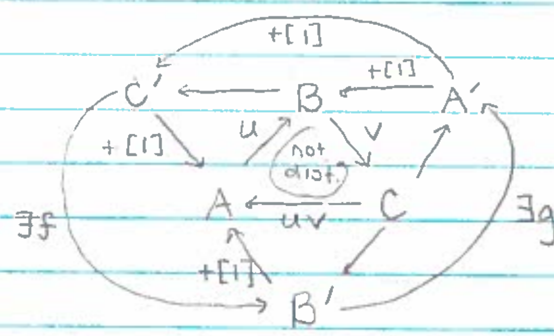
TR2. (Rotation) If (u,v,w) is a distinguished Δ, then its rotations $(v,w,-u[1])$, $(-w[-1],u,v)$ are distinguished Δ's

TR3. Given distinguished Δ's and morphisms f,g $\ni$ the first square commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, dashed, "\exists h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

$\exists h$ completing the diagram to a map of Δ's

TR4. (octahedral) Given morphisms $A \xrightarrow{u} B$, $B \xrightarrow{v} C$ and distinguished Δ's (u,j,∂), (v,x,ε), (vu,y,δ) on ABC', BCA', ACB', $\exists$ distinguished Δ $(f,g,j[1]\circ\varepsilon)$ on $C'B'A'$ ie $C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{j[1]\varepsilon} C'[1]$ $\ni$ any diagram from these commutes:



171. Cohomological Functor - An additive functor $H: \mathcal{T} \longrightarrow \mathcal{A}$ from a Δ'd category $\mathcal{T}$ to an abelian category $\mathcal{A}$ is a cohomological functor if for any distinguished Δ

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

the sequence

$$\cdots \xrightarrow{H(w[-1])} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(X[1]) \xrightarrow{H(u[1])} \cdots$$

is exact

172. Cone - Let $\mathcal{T}$ be a Δ'd category and $X \xrightarrow{u} Y$ be a morphism. The cone of u, denoted $C(u)$, is Z together with adjoining maps v,w where u extends to a dist. Δ: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$

173. Spectral Sequence - A spectral sequence in an abelian

category $\mathcal{A}$ is a family of pages $E^r = \{E^r_{pq}\}$ of objects $\forall p,q \in \mathbb{Z}$, $r \geq r_0$ $\ni$

(i) in each page E^r, have maps $d^r : E^r_{pq} \longrightarrow E^r_{p-r,q+(r-1)}$ with $(d^r)^2 = 0$

(ii) the objects in page E^{r+1} satisfy $E^{r+1}_{p,q} \cong H_{p,q}(E^r, d^r)$

174. Bounded Spectral Sequence – A spectral sequence is bounded if for each n, the diagonal $p+q=n$ has finitely many nonzero terms

175. Convergence – A spectral sequence converges to objects $\{H_n\}$ if for each n the objects $\{E^\infty_{p,q} \mid p+q=n\}$ give factors in some filtration of H_n, denoted $E^{r_0}_{p,q} \Rightarrow H_{p+q}$ or $E^{r_0}_{p,q} \Rightarrow H_n$

176. Cohomology Spectral Sequence – A (cohomology) spectral sequence a family of pages $E_r = \{E_r^{pq}\}$ $\forall p,q \in \mathbb{Z}$, $\forall r \geq r_0$ $\ni$

(i) on each page have maps $d_r : E_r^{pq} \longrightarrow E_r^{p+r,q-(r-1)}$ with $(d_r)^2 = 0$

(ii) $E_{r+1}^{pq} = H^{pq}(E_r^{pq}, d_r)$

177. Stable value – If a spectral sequence is bounded, the stable value, denoted E_∞^{pq}, is the page $\ni$ $E_r^{pq} = E_{r+1}^{pq} = \dots$

178. Filtration – A filtration of a chain complex C, is a chain of subcomplexes of C: $\dots \subseteq F_{p-1}C \subseteq F_pC \subseteq \dots$

179. Bounded Filtration – A filtration is bounded if for each n, only finitely many $(F_pC)_n \neq 0$

180. Exhaustive – A filtration is exhaustive if $\bigcup_p F_pC = C$

Theorems...

1. $\varphi: R \to S$ ring homomorphism $\Rightarrow \varphi(1_R) = 1_S$
2. $B \leq A$ submodule $\Rightarrow A/B$ R-module via $r(a+B) = ra+B$ $\forall r \in R, a \in A$
3. $\pi: A \to A/B \ni \pi(a) = a+B \Rightarrow \pi$ epimorphism
4. 1st Iso Thm $f: A \to B$ homomorphism $\Rightarrow A/\ker f \cong \operatorname{Im} f$
5. 2nd Iso Thm $B, C \leq A$ submodules $\Rightarrow B/B\cap C \cong B+C/C$
6. 3rd Iso Thm $B, C \leq A$ submodules, $C \subseteq B \Rightarrow B/C \leq A/C$ submodule and $A/C / B/C \cong A/B$
7. $(P, \{\pi_i\}), (Q, \{\psi_i\})$ both products of $\{A_i\}_{i \in I} \Rightarrow P \cong Q$
8. $(S, \{\iota_i\}), (T, \{\lambda_i\})$ both coproducts of $\{A_i\}_{i \in I} \Rightarrow S \cong T$
9. Short 5 Lemma R ring, diagram commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

(i) α, γ monomorphisms $\Rightarrow \beta$ monomorphism
(ii) α, γ epimorphisms $\Rightarrow \beta$ epimorphism
(iii) α, γ isomorphisms $\Rightarrow \beta$ isomorphism

10. R ring, $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ SES. TFAE:
(i) sequence split exact
(ii) $\exists h: A_2 \to B$ homomorphism $\ni gh = 1_{A_2}$
(iii) $\exists k: B \to A_1$ homomorphism $\ni kf = 1_{A_1}$
(iv) sequence isomorphic to $0 \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0$
In particular $B \cong A_1 \oplus A_2$ and $h = \iota_2, k = \pi_1$
11. M decomposable $\Rightarrow \operatorname{End} M$ has nontrivial idempotents
12. M R-module, $\operatorname{End} M$ has no nontrivial idempotents $\Rightarrow M$ indecomposable
13. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits with $g\iota = 1_C$ and $\rho f = 1_A \Rightarrow$ ιg and $f\rho$ idempotents
14. M R-module, $e: M \to M$ idempotent $\Rightarrow M = \ker e \oplus \operatorname{Im} e$
15. $A \xrightarrow{f} B$ R-module homomorphism $\Rightarrow$ induced homomorphism of abelian groups $f_*: \operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B) \ni f_*(g) = fg$ for $g \in \operatorname{Hom}_R(M, A)$ and induced homomorphism

$f^* : \mathrm{Hom}_R(B,M) \longrightarrow \mathrm{Hom}_R(A,M) \ni f^*(g) = gf$

16. R ring, $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ SES, M R-module $\Rightarrow$ induced SES $0 \longrightarrow \mathrm{Hom}_R(M,A) \xrightarrow{f_*} \mathrm{Hom}_R(M,B) \xrightarrow{g_*} \mathrm{Hom}_R(M,C) \longrightarrow 0$
ie $\mathrm{Hom}_R(M,-)$ left exact

17. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ split exact sequence, M R-module $\Rightarrow$
$0 \longrightarrow \mathrm{Hom}_R(M,A) \longrightarrow \mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C) \longrightarrow 0$ exact
ie $\mathrm{Hom}_R(M,-)$ exact

18. P R-module. TFAE:
(i) P projective
(ii) $\mathrm{Hom}_R(P,-)$ exact
(iii) $\forall\, M \xrightarrow{f} N$ surjective, $\mathrm{Hom}_R(P,M) \xrightarrow{f_*} \mathrm{Hom}_R(P,N)$ surjective
(iv) Any surjective $M \xrightarrow{f} P$ splits ie $\exists h \ni fh = 1_P$
(v) P direct summand of free module

19. Every free module is projective

20. $\{P_i\}_{i\in I}$ modules $\Rightarrow \bigoplus_{i\in I} P_i$ projective iff P_i projective $\forall i$

21. M R-module $\Rightarrow$ M quotient of some projective module
(If M finitely generated, we can choose projective module to be finitely generated)

22. Diagram commutative with exact rows:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed, "\exists! h"] & 0 \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow \exists! h : C \longrightarrow C'$ commuting the diagram

23. Digram commutative with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, dashed, "\exists! h"] & B \arrow[r] \arrow[d] & C \arrow[d] \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$

$\Rightarrow \exists! h : A \longrightarrow A'$ commuting the diagram

24. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ SES $\Rightarrow$
$0 \longrightarrow \mathrm{Hom}_R(C,M) \xrightarrow{g^*} \mathrm{Hom}_R(B,M) \xrightarrow{f^*} \mathrm{Hom}_R(A,M)$ exact
ie $\mathrm{Hom}_R(-,M)$ left exact $\forall M$ R-module

25. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ SES split exact, M R-module $\Rightarrow$

$0 \to \mathrm{Hom}_R(C,M) \to \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M) \to 0$ exact

ie $\mathrm{Hom}_R(\_,M)$ exact

26. J R-module, TFAE

(i) J injective

(ii) $\mathrm{Hom}_R(\_,J)$ exact

(iii) $\forall\, L \to M$ injective, $\mathrm{Hom}_R(M,J) \to \mathrm{Hom}_R(L,J)$ surjective

(iv) Any injective $J \to M$ splits, hence J isomorphic to direct summand of M

27. Baer's Criterion R ring, E left R-module $\Rightarrow$ E injective iff $\forall$ left ideals I of R and maps $f: I \to E$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists s"] \\ & E & \end{tikzcd}$$

$\exists s: R \to E \ni s|_I = f$

28. $\{E_i\}_{i\in I}$ R-modules $\Rightarrow$ E_i injective $\forall i$ iff $\prod_{i\in I} E_i$ injective

29. Finite direct sum of injective modules injective

30. Each summand of an injective module is injective

31. D abelian group (ie $\overset{(PID)}{\mathbb{Z}}$-module) $\Rightarrow$ D divisible iff D injective

32. A direct sum of divisible modules is divisible

33. Over $\mathbb{Z}$, direct sum of injective modules is injective

34. A quotient of a divisible module is divisible

35. Snake Lemma Diagram commutative with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow \exists$ exact sequence:

$0 \to \ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0$

36. D divisible $\mathbb{Z}$-module, R ring $\Rightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(R,D)$ injective R-module

37. A right R-module, B left R-module $\Rightarrow$ $A \otimes_R B$ exists and is unique up to isomorphism

38. $(a_1+a_2)\otimes b = (a_1\otimes b)+(a_2\otimes b)$, $a\otimes(b_1+b_2) = (a\otimes b_1)+(a\otimes b_2)$

$ar\otimes b = a\otimes rb$

39. Elements of $A \otimes_R B$ are of form $\sum_{i=1}^{m} a_i \otimes b_i$, $a_i \in A$, $b_i \in B$

40. R commutative ring, A, B R-modules $\Rightarrow A \otimes_R B$ R-module

41. R commutative ring, A free with basis $\{e_i\}_{i\in I}$, B free with basis $\{b_j\}_{j\in J}$ $\Rightarrow A \otimes_R B$ free with basis $\{e_i \otimes b_j\}_{i\in I}^{j\in J}$

42. F field, V vectorspace with basis $\{e_i\}_{i\in I}$, W vectorspace with basis $\{f_j\}_{j\in J}$ $\Rightarrow V \otimes_R W$ vectorspace with basis $\{e_i \otimes f_j\}_{i,j}$

43. M left R-module $\Rightarrow R \otimes_R M \cong M$ as R-modules

44. M right R-module $\Rightarrow M \otimes_R R \cong M$ as R-modules

45. R commutative ring, A, B R-modules $\Rightarrow A \otimes_R B \cong B \otimes_R A$ as R-modules

46. M left R-module, $I \triangleleft R \Rightarrow R/I \otimes_R M$ left R-module via $r(\bar{s} \otimes m) = \overline{rs} \otimes m$ and left R/I-module via $\bar{r}(\bar{s} \otimes m) = \overline{rs} \otimes m$

47. $I \triangleleft R$, M left R-module $\Rightarrow R/I \otimes_R M \cong M/IM$ as R-modules (or as R/I-modules) where $IM = \{\sum a_i m_i \mid a_i \in I, m_i \in M\}$

48. A, B right R-modules, A', B' left R-modules, $A \xrightarrow{f} B$, $A' \xrightarrow{f'} B'$ $\Rightarrow \exists$ induced homomorphism of abelian groups $A \otimes_R A' \xrightarrow{f \otimes f'} B \otimes_R B'$ $\ni (f \otimes f')(a \otimes a') = f(a) \otimes f'(a')$

49. M right R-module, A, B left R modules, $A \xrightarrow{f} B \Rightarrow \exists$ induced homomorphism of abelian groups $M \otimes_R A \xrightarrow{1 \otimes f} M \otimes_R B \ni (1 \otimes f)(m \otimes a) = m \otimes f(a)$ (similar, we get $f \otimes 1$)

50. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ exact, M right R-module $\Rightarrow$ sequence of abelian groups $M \otimes_R A \xrightarrow{1\otimes f} M \otimes_R B \xrightarrow{1 \otimes g} M \otimes_R C \longrightarrow 0$ exact ie $M \otimes_R -$ right exact (Also $- \otimes_R M$ right exact)

51. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ split exact $\Rightarrow$ $0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$ split exact ie $M \otimes_R -$ split exact (Also, $- \otimes_R M$ split exact)

52. R ring, M left R-module $\Rightarrow \mathrm{Hom}_R(R, M)$ left R-module via $(rf)(s) = f(sr)$ $\forall s \in R$

53. $\mathrm{Hom}_R(R, M) \cong M$

54. Adjoint Isomorphism Thm R, S rings, A_R, ${}_RB_S$, C_S $\Rightarrow \exists$ natural iso in A, B, C of abelian groups $\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B$

55. $R \otimes_R M \cong M$ naturally in M ie $\forall$ R-modules N and $\forall M \xrightarrow{g} N$, diagram

commutes:

$$\begin{tikzcd} R \otimes_R M \arrow[r, "\phi_M", "\sim"'] \arrow[d, "1\otimes g"'] & M \arrow[d, "g"] \\ R \otimes_R N \arrow[r, "\phi_N", "\sim"'] & N \end{tikzcd}$$

56. M left R-module $\Rightarrow \exists$ injective R-module $E \ni 0 \longrightarrow M \longrightarrow E$

57. Every $\mathbb{Z}$-module can be embedded in an injective module

58. $M \cong N$ R-modules, A R-module $\Rightarrow \mathrm{Hom}_R(M,A) \cong \mathrm{Hom}_R(N,A)$ and $\mathrm{Hom}_R(A,M) \cong \mathrm{Hom}_R(A,N)$

59. $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ sequence of R-modules. TFAE:

(i) $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ exact

(ii) g cokernel of f

(iii) $\forall$ R-modules X have induced exact sequence of abelian groups: $0 \longrightarrow \mathrm{Hom}_R(M_3,X) \xrightarrow{g^*} \mathrm{Hom}_R(M_2,X) \xrightarrow{f^*} \mathrm{Hom}_R(M_1,X)$

60. $\{A_i\}_{i\in I}$ right R-modules, B left R-module $\Rightarrow$ $(\bigoplus_{i\in I} A_i) \otimes_R B \cong \bigoplus_{i\in I}(A_i \otimes_R B)$ as abelian groups (as R-modules if R comm)

61. A right R-module, $\{B_i\}_{i\in I}$ left R-modules $\Rightarrow$ $A \otimes_R (\bigoplus_{i\in I} B_i) \cong \bigoplus_{i\in I}(A \otimes_R B_i)$ as abelian groups (as R-mods if R comm)

62. ${}_S A_R$, ${}_R B \Rightarrow A \otimes_R B$ left S-module via $s(a\otimes b) = sa \otimes b$

63. A_R, ${}_R B_S \Rightarrow A \otimes_R B$ right S-module

64. M right R-module $\Rightarrow$ M flat iff $\forall$ monomorphisms $A \xrightarrow{f} B$, $M \otimes_R A \xrightarrow{1\otimes f} M \otimes_R B$ monomorphism

65. R_R flat

66. $\{M_i\}_{i\in I}$ right R-modules $\Rightarrow \bigoplus_{i\in I} M_i$ flat iff M_i flat $\forall i$

67. $x_i \xrightarrow{h_i} y_i$ monomorphisms $\forall i \Rightarrow \bigoplus_i x_i \xrightarrow{h} \bigoplus_i y_i$ where $h = \bigoplus_i h_i$ ie $h((x_i)_{i\in I}) = (h_i(x_i))_{i\in I}$ monomorphism

68. $M = L \oplus K$ flat $\Longleftrightarrow L, K$ flat

69. Every free module is flat

70. Every projective module is flat

71. Free $\Rightarrow$ Projective $\Rightarrow$ Flat

72. R PID, P projective $\Rightarrow P$ free

73. R Noetherian, M finitely generated $\Rightarrow M$ flat iff M projective

74. $S \subseteq R$ multiplicative, $(a,s) \sim (b,t)$ iff $\exists u \in S \ni u(at-bs) = 0 \Rightarrow$

$R_S = {}^{R\times S}/_{\sim}$ commutative ring with identity via $\frac{a}{s}+\frac{b}{t}=\frac{ta+sb}{st}$
and $\frac{a}{s}\cdot\frac{b}{t}=\frac{ab}{st}$, $\frac{0}{s}=0_{R_S}$, $\frac{1}{1}=1_{R_S}$ (where R commutative)

75. M R-module $\Rightarrow$ M_S R_S-module via $\frac{r}{t}\cdot\frac{m}{s}=\frac{rm}{ts}$

76. $M \xrightarrow{f} N$ homomorphism $\Rightarrow$ $\exists$ induced homomorphism of R_S-mods
$f_S : M_S \longrightarrow N_S \ni f_S\left(\frac{m}{s}\right)=\frac{f(m)}{s}$

77. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ SES R-modules, $S \subseteq R$ multiplicative
$\Rightarrow$ $0 \to A_S \xrightarrow{f_S} B_S \xrightarrow{g_S} C_S \longrightarrow 0$ SES R_S-modules

78. R commutative ring, $S \subseteq R$ multiplicative $\Rightarrow$ $\exists$ natural isomorphism in M of R_S-modules $R_S \otimes_R M \cong M_S$

79. R_S R-module via $r\left(\frac{r'}{s}\right)=\frac{rr'}{s}$

80. Restriction of Scalars $R \xrightarrow{\alpha} R'$ ring map, M R'-module $\Rightarrow$
M R-module via $rm=\alpha(r)m$

81. $S \subseteq R$ multiplicative $\Rightarrow$ R_S flat R-module

82. R ring. TFAE:
(i) R left noetherian
(ii) Every nonempty set of left ideals of R has a maximal element
(iii) Every left ideal of R is finitely generated

83. PID $\Rightarrow$ Noetherian

84. M left R-module, TFAE
(i) M left Noetherian
(ii) Every nonempty set of submodules of M has a maximal element
(iii) Every submodule of M is finitely generated

85. R noetherian ring, $I \trianglelefteq R$ $\Rightarrow$ R/I Noetherian

86. R ring, $0 \to A \longrightarrow B \longrightarrow C \longrightarrow 0$ SES $\Rightarrow$ B noetherian iff A, C noetherian

87. $0 \to L \longrightarrow M \longrightarrow N \longrightarrow 0$ SES, L, N finitely generated $\Rightarrow$
M finitely generated

88. $A_1, \ldots, A_n$ R-modules $\Rightarrow$ $A_1 \oplus \ldots \oplus A_n$ noetherian iff A_i noetherian $\forall i$

89. R noetherian ring, M finitely generated R-module $\Rightarrow$ M noetherian

90. M left R-module. TFAE:
(i) M left Artinian
(ii) Every nonempty set of submodules of M has a minimal element
91. D division ring $\Rightarrow$ D left and right Artinian
92. $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ SES $\Rightarrow$ B Artinian iff A, C artinian
93. A finite direct sum of modules is artinian iff each summand artinian
94. R artinian ring, M finitely generated R-modules $\Rightarrow$ M artinian
95. R noetherian $\Rightarrow$ $R[x_1, \dots, x_n]$ noetherian
96. K field $\Rightarrow$ $_K K$ simple
97. Every module over a field is semisimple
98. Every simple module is semisimple
99. S simple $\Rightarrow$ $S = \langle x \rangle$ $\forall\, 0 \neq x \in S$ ie S cyclic generated by all nonzero
100. S simple R-module $\Rightarrow$ $\mathrm{End}_R S$ division ring
101. Submodules and homomorphic images of semisimple modules are semisimple
102. Modular Law $A, B, C \leq M$ submodules, $B \subseteq A \Rightarrow$
(i) $A \cap (B + C) = B + A \cap C$
(ii) $A + C = B + C$ and $A \cap C = A \cap B \Rightarrow A = B$
103. M module, $\{S_i\}_{i \in I}$ simple modules, $M = \sum_{i \in I} S_i$, $L \leq M \Rightarrow \exists J \subseteq I$ $\ni M = L \oplus (\bigoplus_{j \in J} S_j)$. In particular, M semisimple.
104. M R-module, TFAE:
(i) M sum of simple modules
(ii) M direct sum of simple modules
(iii) M semisimple
105. Schreier-Zassenhaus Lemma Any two series of a module have equivalent refinements
106. Jordan-Hölder Thm Any two composition series of a module are equivalent. Further, if a module M has a composition series and $0 = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_t = M$ is a series with nonzero factors then the series can be refined to a composition series.

107. $0 \to A \to B \to C \to 0$ SES $\Rightarrow \ell(B) = \ell(A) + \ell(C)$

108. M R-module

(i) $L \leq M$ submodule $\Rightarrow$ M has composition series iff both L and M/L have composition series. If so, then $\ell(M) = \ell(L) + \ell(M/L)$

(ii) $M = \bigoplus_{i=1}^{n} S_i$ direct sum of simple modules $\Rightarrow$ M has composition series of length n ie $\ell(M) = n$

109. Artinian modules have simple submodules

110. M R-module $\Rightarrow$ M has composition series iff M both artinian and noetherian

111. Every simple module is both artinian and noetherian

112. I left ideal of R

(i) $I = Re$ for some idempotent $e \in R$ iff I is a direct summand of R

(ii) I minimal left ideal $\Rightarrow$ either $I^2 = 0$ or $I = Re$ for some idempotent e

113. R ring. TFAE:

(i) Every left R-module is projective

(ii) Every left R-module is injective

(iii) Every left R-module is semisimple

(iv) $_RR$ semisimple

(v) R left wedderburn ring

(vi) R left semisimple ring

114. R left semisimple ring, M finitely generated left R-module $\Rightarrow$ M has composition series ie $\ell(M) < \infty$. In particular, M both noetherian and artinian.

115. I left nilpotent ideal of R $\Rightarrow$ I nil

116. Every left/right nilpotent ideal is contained in a two sided nilpotent ideal

117. $x \in R$ nilpotent $\Rightarrow 1 - x$ invertible

118. $I \triangleleft R$ nil, J/I left nil ideal of R/I $\Rightarrow$ J left nil ideal of R

119. $I, J \triangleleft R$ nil $\Rightarrow I + J$ nil. In fact arbitrary sum of 2 sided nil ideals is nil

120. $Nil(R/Nil(R)) = (0)$. Equivalently $R/Nil(R)$ has no nonzero two-sided nil ideals

121. R left artinian ring, $N = Nil(R) \Rightarrow R/N$ left wedderburn and N unique largest two sided nilpotent ideal of R

122. Variation of Nakayama R ring, I nilpotent left ideal of R, M left R-module, $L \leq M$, $M = L + IM \Rightarrow M = L$. In particular, if $M = IM$, then $M = 0$.

123. S simple left R-module, I left nilpotent ideal of $R \Rightarrow IS = 0$

124. R left artinian ring, $N = Nil(R) \Rightarrow \forall$ simple modules S, $NS = 0$. In particular, $\forall$ semisimple modules M, $NM = 0$

125. R left artinian ring $\Rightarrow$ up to isomorphism there are finitely many simple left R-modules and every semisimple R-module can be written up to isomorphism in a unique way as a direct sum of simple modules

126. M semisimple, finitely generated $\Rightarrow M$ finite direct sum of simple modules

127. Hopkins-Levitsky Thm R left artinian ring $\Rightarrow$ every finitely generated R-module has finite length ie has a composition series. In particular, R is also left Noetherian

128. R left artinian $\Rightarrow R$ left Noetherian, R right artinian $\Rightarrow R$ right noetherian where R ring

129. Artin-Wedderburn Thm R left wedderburn ring $\Leftrightarrow$ $R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ where $M_{n_i}(D_i)$ are the $n_i \times n_i$ matrices with entries in a division ring $D_i \Leftrightarrow R$ right wedderburn ring

130. R ring, M, N left R-modules $\Rightarrow Hom_R(M,N)$ left $End_R M$-module and right $End_R M$-module via $f * g = gf$ for $f \in End_R M$, $g \in Hom_R(M,N)$ and $h * g = gh$ for $h \in End_R N$, $g \in Hom_R(M,N)$ In fact $Hom_R(M,N)$ is $End_R M - End_R N$ bimodule

131. Multiplication in $End_R M$ given by $f * g = gf$ composition

132. ${}_RM = M_1 \oplus \dots \oplus M_n \Rightarrow \forall {}_RN$, $Hom_R(M,N) \cong Hom_R(M_1,N) \oplus \dots \oplus Hom_R(M_n,N)$ as right $End_R(N)$-modules

133. $N = N_1 \oplus \dots \oplus N_s \Rightarrow \forall M$, $Hom_R(M,N) \cong Hom_R(M,N_1) \oplus \dots \oplus Hom_R(M,N_s)$ as left $End_R M$ modules

134. $M = M_1 \oplus \dots \oplus M_n$, $Hom_R(M_i, M_j) = 0 \ \forall i \neq j \Rightarrow End_R M \cong End_R M_1 \times \dots \times End_R M$ as rings

135. $M = M_1 \oplus \dots \oplus M_n$, $M_i \cong L$ for some module $L \Rightarrow End_R M \cong M_n(End_R L)$

136. $e \in R$ idempotent $\Rightarrow End_R Re \cong eRe$ as rings. In particular, if $e=1$, $End_R R \cong R$

137. $F = R \oplus \dots \oplus R$ ie F free of rank $n \Rightarrow End_R F \cong M_n(End_R R) \cong M_n R$ as rings

138. Schurs Lemma S, T simple R-modules, $S \not\cong T \Rightarrow End_R S$ division ring and $Hom_R(S,T) = 0$

139. Extension of Artin-wedderburn R left wedderburn ring $\Leftrightarrow$ $R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ and in this case:

(i) There are precisely t nonisomorphic simple R-modules $S_1, \dots, S_t$

(ii) n_k is multiplicity of S_k as a composition factor of $_RR$

(iii) For every k, $D_k = End_R(S_k)$

And the product is unique

140. R simple, artinian ring $\Leftrightarrow R \cong M_n(D)$, D division ring, $n \geq 1$

141. R_1, R_2 artinian/noethenan/wedderburn $\Leftrightarrow R_1 \times R_2$ artinian/noethenan/wedderburn

142. R left semi-simple ring $\Leftrightarrow$ R right semi-simple ring

143. R semisimple $\Rightarrow R$ artinian

144. R semisimple $\Leftrightarrow$ R product of simple artinian rings

145. $ann_R M \triangleleft R$

146. Torsionfree $\Rightarrow$ Faithful

147. Every free module is faithful

148. R left artinian ring $\Rightarrow$ R simple iff R has faithful simple module ie R simple iff R primitive

149. R left artinian ring

(i) S simple R-module $\Rightarrow ann_R S$ maximal two sided ideal of R

(ii) $S \longleftrightarrow ann_R S$ induces 1-1 correspondence

$\left\{\begin{matrix}\text{iso classes of} \\ \text{simple } R\text{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{maximal two} \\ \text{sided ideals}\end{matrix}\right\}$ and these sets are finite

(iii) $\{I_1, \dots, I_t\}$ all maximal two sided ideals of R $\Rightarrow$

$R/I_1 \cap \dots \cap I_t \cong R/I_1 \times \dots \times R/I_t$ semisimple

150. $R/\mathrm{rad}R$ semisimple and $\mathrm{rad}R$ smallest two sided ideal J of $R \ni R/J$ semisimple

151. R semisimple ring $\Rightarrow \mathrm{rad}R = 0$

152. R semisimple ring $\Leftrightarrow$ R left artinian and $\mathrm{rad}R = 0$ $\Leftrightarrow$ R right artinian and $\mathrm{rad}R = 0$

153. R left artinian ring, $J \triangleleft R$, R/J simple $\Rightarrow$ J intersection of maximal two sided ideals. In particular $\mathrm{rad}R \subseteq J$.

154. $J(R)$ left ideal of R

155. R left artinian ring, $I \triangleleft R$ maximal $\Rightarrow$ I intersection of maximal left ideals

156. R left artinian ring, every maximal left ideal contains a maximal two sided ideal

157. R left artinian ring $\Rightarrow J(R) = \mathrm{rad}R$

158. R ring, $x \in R$. TFAE

(i) $x \in J(R) = \bigcap$ all maximal left ideals of R

(ii) $\forall r \in R$, $1 - rx$ has left inverse

(iii) $\forall$ simple left modules S, $xS = 0$

159. R local, commutative ring with unique maximal ideal $\underline{m}$ $\Rightarrow J(R) = \underline{m}$

160. KG group ring $\Rightarrow$ KG ring via $(\alpha g)(\beta h) = \alpha\beta \cdot gh$ and extend by linearity ie $\sum_{g\in G}\alpha_g g + \sum_{g\in G}\beta_g g = \sum_{g\in G}(\alpha_g + \beta_g)g$, $\alpha\sum_{g\in G}\alpha_g g = \sum_{g\in G}\alpha\alpha_g g$
$\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in G}\beta_h h\right) = \sum_{\ell\in G}\left(\sum_{gh=\ell}\alpha_g\beta_h\right)\ell$

161. Maschke's Thm KG semisimple $\Leftrightarrow$ G finite and $\mathrm{char}K \nmid |G|$

162. $|G| < \infty \Rightarrow KG$ artinian ring hence noetherian (left and right)

163. G abelian $\Leftrightarrow KG$ commutative

164. $\mathrm{Char}K = 0 \Rightarrow KG$ semisimple

165. Nakayama's Lemma R ring, M finitely generated R-module, $J(R)M = M \Rightarrow M = 0$

166. R ring, $I \triangleleft R$ nil $\Rightarrow I \subseteq J(R)$

167. R left artinian ring $\Rightarrow J(R)$ nilpotent. In particular $J(R) = Nil(R)$

168. R left artinian ring $\Rightarrow J(R)$ largest two sided nilpotent ideal of R

169. $J(R) = \bigcap_S ann_R S$, S simple left R-modules, Hence $J(R) \triangleleft R$

170. R ring $\Rightarrow J(R) = \{x \mid 1+rxs \text{ invertible } \forall r,s \in R\}$

171. R ring $\Rightarrow J(R) = \bigcap$ all maximal right ideals $= \{x \mid 1+xs$ has right inverse $\forall s \in R\}$

172. Another Version Nakayama M finitely generated R-module, $N \leq M$, $J = J(R)$, $M = N + JM \Rightarrow M = N$

173. M finitely generated, $\{x_1, \ldots, x_n\}$ minimal set of generators of M $\Rightarrow \{x_1, \ldots, x_n\} \subseteq M - JM$

174. $x \in R$ ring. TFAE

(i) $x \in J(R) = \bigcap$ all maximal left ideals

(ii) $1+rx$ has left inverse $\forall r \in R$

(iii) $1+rxs$ has left inverse $\forall r,s \in R$

175. $x \in R$ ring. TFAE

(i) $x \in J(R) = \bigcap$ all maximal right ideals

(ii) $1+xs$ has right inverse $\forall s \in R$

(iii) $1+rxs$ has right inverse $\forall r,s \in R$

176. $J(R/J(R)) = 0$

177. R ring, $J = J(R)$, P finitely generated projective module

(i) M finitely generated module, $f: M \longrightarrow P$ homomorphism $\ni$ induced homomorphism $\bar{f}: M/JM \longrightarrow P/JP$ isomorphism $\Rightarrow$ f isomorphism

(ii) P/JP free R/J-module $\Rightarrow$ P free R-module

178. R ring. TFAE:

(i) R local

(ii) $J(R)$ maximal left ideal. Equivalently $\exists!$ maximal left ideal.

(iii) $J(R)$ maximal right ideal. Equivalently $\exists!$ maximal right ideal

(iv) $x \in R$, $x \notin J(R) \Rightarrow x$ invertible

179. R local ring, $J = J(R)$, $K = R/J$, M finitely generated over R, $\{x_1 + JM, \dots, x_d + JM\}$ basis of M over $K \Rightarrow \{x_1, \dots, x_d\}$ minimal set of generators of M as R-module

180. R local ring, $J = J(R) \Rightarrow$ Every finitely generated projective module is free (True even if not finitely generated!)

181. R local ring, $J = J(R)$, $(a_{ij}) \in M_n(R) \Rightarrow A$ invertible

182 R left Noetherian $\Leftrightarrow$ Every direct sum of injective modules is injective

183. $0 \neq M$ artinian module $\Rightarrow$ $\operatorname{Soc} M \neq 0$

184. M artinian $\Rightarrow$ $\operatorname{Soc} M \overset{ess}{\subseteq} M$

185. $\operatorname{Soc} M$ is the sum of all simple modules of M (direct)

186. $K \overset{ess}{\subseteq} L \overset{ess}{\subseteq} M \Rightarrow K \overset{ess}{\subseteq} M$

187. R ring, E left R-module $\Rightarrow$ E injective iff E has no proper essential extensions

188. $e \neq 0$ idempotent, M left R-module $\Rightarrow \exists$ natural isomorphism of abelian groups $eM \xrightarrow{\Phi_M} \operatorname{Hom}_R(Re, M)$. In fact, isomorphism of eRe-modules.

189. $M \subseteq E$ extension of R-modules $\Rightarrow M \overset{ess}{\subseteq} E$ iff $\forall\, 0 \neq x \in E$, $\exists r \in R \ni 0 \neq rx \in M$

190. $M \subseteq E$, $\{E_i\}_{i \in I}$ chain of submodules of E, $M \overset{ess}{\subseteq} E_i \subseteq E \ \forall i \Rightarrow M \overset{ess}{\subseteq} E$

191. $M \subseteq E_1 \subseteq E_2$, $M \overset{ess}{\subseteq} E_2 \Rightarrow E_1 \overset{ess}{\subseteq} E_2$

192. $M \overset{ess}{\subseteq} E$, $f: E \to X$ homomorphism $\ni f|_M$ injective $\Rightarrow f$ injective

193. $M \subseteq E$ extension. TFAE:
(i) E maximal essential extension of M
(ii) $M \overset{ess}{\subseteq} E$ and E injective
(iii) E injective and $\nexists$ injective module $\bar{E} \ni M \subseteq \bar{E} \subsetneq E$
Moreover, given a module $\Rightarrow$ an extension E as above exists

194. Any two injective envelopes of M are isomorphic

195. R left artinian with no nonzero nilpotents and no nontrivial central idempotents $\Rightarrow R$ division ring

196. R ring $\ni$ every left R-module is free $\Rightarrow R$ division ring

197. F R-module $\Rightarrow$ F free iff $F \cong \bigoplus_{c \in I} R$

198. F finitely generated, free $\Rightarrow$ Every basis of F is finite

199. F free, not finitely generated $\Rightarrow$ Any two bases of F have same cardinality

200. R ring has IBN $\Leftrightarrow$ whenever $R^n \cong R^m$ as R-modules, $n = m$

201. Every division ring has IBN

202. Every commutative ring has IBN

203. R local $\Rightarrow$ R has IBN

204. R, S rings, $\exists$ ring homomorphism $R \xrightarrow{\phi} S$, S has IBN $\Rightarrow$ R has IBN

205. M noetherian, $\phi: M \longrightarrow M$ surjective $\Rightarrow$ ϕ isomorphism

206. R left noetherian ring $\Rightarrow$ R has IBN

207. Every left artinian ring has IBN, hence every semisimple ring has IBN

208. Modified Baer's Criterion E R-module $\Rightarrow$ E injective iff $\forall$ left ideal I or $R \ni I \overset{cas}{\subseteq} R$, every homomorphism $I \longrightarrow E$ can be extended to R ie

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "G"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists g"] \\ & E & \end{tikzcd}$$

209. R integral domain, F field of fractions $\Rightarrow$ $F = E(R)$

210. R ring, M module $\Rightarrow$ $\exists$ injective envelope of M. Hence $\exists$ injective module E and injection $M \xrightarrow{f} E$ and E is minimal with this property ie if $\exists M \xrightarrow{f'} E'$ injection with E' injective, then diagram commutes:

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f'"] & E \arrow[dl, dashed, "\exists h \text{ injective}"] \\ & E' & \end{tikzcd}$$

ie f left minimal

211. P projective, $P \xrightarrow{f} M \longrightarrow 0$ is a projective cover if $\forall Q \xrightarrow{g} M \longrightarrow 0$ with Q projective $\exists$ surjection $h: Q \longrightarrow P \ni$

$$\begin{tikzcd} P \arrow[dr] & & 0 \\ & M \arrow[ur] \arrow[dr] & \\ Q \arrow[uu, "h"] \arrow[ur] & & 0 \end{tikzcd}$$

commutes

212. If a projective cover of M exists, then it is unique up to iso.

213. R ring. TFAE:

(i) Every module has a projective cover

(ii) Every flat module is projective

214. Flat covers exist

215. R left artinian ring, M finitely generated $\Rightarrow$ M has projective cover

216. R ring, M R-module, $K, L \leq M$, $K \overset{ess}{\subseteq} M \Rightarrow K \cap L \overset{ess}{\subseteq} L$

217. $\{L_i\}_{i=1,\dots,n}$ submodules of M, $L_i \overset{ess}{\subseteq} M$ $\forall i \Rightarrow L_1 \cap \dots \cap L_n \overset{ess}{\subseteq} M$

218. $L_i \overset{ess}{\subseteq} M_i$, $i = 1, \dots, n \Rightarrow L_1 \oplus \dots \oplus L_n \overset{ess}{\subseteq} M_1 \oplus \dots \oplus M_n$

219. $K, L \leq M$, $L \overset{ess}{\subseteq} M \Rightarrow$

(i) $E(L) = E(M)$

(ii) $L = L_1 \oplus \dots \oplus L_n \Rightarrow E(L) = E(L_1) \oplus \dots \oplus E(L_n)$

(iii) $\exists X \leq M \ni K \oplus X \overset{ess}{\subseteq} M$.

In particular, $E(M) = E(K) \oplus E(X)$, hence $E(K)$ direct summand of $E(M)$

220. R noetherian $\Rightarrow$ Every nonzero injective module direct sum of indecomposable injective modules (uniquely up to iso)

221. $\text{udim}\, M = 0 \Longleftrightarrow M = 0$

222. S simple module $\Rightarrow \text{udim}\, S = 1$

223. A uniform module is always indecomposable

224. $0 \neq M$ module. TFAE:

(i) M uniform

(ii) $\forall Y \neq 0 \neq X$ submodules of M, $X \cap Y \neq 0$

(iii) $\forall 0 \neq L \leq M$, $L \overset{ess}{\subseteq} M$

225. I meet irreducible $\Leftrightarrow$ 0 is not the intersection of two nonzero left ideals in R/I $\Leftrightarrow$ R/I uniform

226. $0 \neq E$ injective R-module. TFAE:

(i) E uniform

(ii) E is injective envelope of a uniform module

(iii) $E \cong E(R/I)$ where $I \lhd R$ is meet irreducible

(iv) $E \cong$ injective envelope of each of its nonzero submodules

(v) E indecomposable

227. R left artinian ring, $S_1, \dots, S_n$ complete set of simple R-module $\Rightarrow$ Every injective R-module is uniquely up to isomorphism a direct sum of $E(S_i)$ $i = 1, \dots, n$. Hence, there are finitely many nonisomorphic indecomposable injective modules.

228. R left artinian ring $\Rightarrow$ Each indecomposable projective R-module is of form Re where e is primitive idempotent of R (up to iso)

229. # nonisomorphic indecomposable projective modules = # nonisomorphic simple modules = # nonisomorphic indecomposable injective modules

230. Matlis' Thm R commutative noetherian ring, $\operatorname{Spec} R = \{\text{prime ideals of } R\}$ $\Rightarrow$ $\exists$ bijection between $\operatorname{Spec} R$ and $\{$iso classes of indecomposable injective R-modules$\}$ given by $P \longleftrightarrow E(R/P)$, P prime

231. P prime $\Rightarrow$ P irreducible

232. P prime ideal of R $\Rightarrow$ P contains a minimal prime ideal

233. $P_1, \dots, P_n$ incomparable prime ideals of R, $P_1 \dots P_n = 0$ $\Rightarrow$ $\{P_1, \dots, P_n\}$ is the set of minimal prime ideals of R

234. Every prime ideal is primary

235. R commutative, noetherian $\Rightarrow$

(1) Every ideal of R is finite intersection of irreducible ideals

(2) Every irreducible ideal is primary

(3) Every ideal is finite intersection of primary ideals

236. R noetherian, $Q \triangleleft R$ primary, $P = \sqrt{Q}$ $\Rightarrow$

(i) P prime ideal containing Q and $P^m \subseteq Q \subseteq P$ for some $m \geq 1$

(ii) P unique minimal prime over Q

237. R commutative, noetherian, $P \triangleleft R$ prime $\Rightarrow$

(i) A finite intersection of P-primary ideals is P-primary

(ii) Q P-primary, $0 \neq X \leq R/Q$ $\Rightarrow$ $\exists\, 0 \neq Y \leq X \ni \forall\, 0 \neq y \in Y$,

$\mathrm{ann}_R\, y = P$

238. M noetherian module $\Rightarrow$ M contains a uniform submodule

239. $\bigoplus_{i\in I} M_i = \{$ordered tuples $(m_i)_{i\in I}$ with each $m_i \in M_i \ni$ all but finitely many m_i are $0\}$ satisfies coproduct UMP via $\varphi((m_i)_{i\in I}) = \sum_{i\in I} \varphi_i(m_i) \in M$

240. $M_1 \oplus \dots \oplus M_n = \{(m_1, \dots, m_n) \mid m_i \in M_i\}$

241. $\prod_{i\in I} M_i = \{$ordered tuples $(m_i)_{i\in I}\}$ satisfies product UMP

242. $\forall$ R-modules M, $\exists$ F free module mapping onto it ie $F \to M \to 0$ and $\exists J$ injective $\ni 0 \to M \to J$

243. $\mathrm{Hom}_R(K,-)$, $\mathrm{Hom}_R(-,K)$ left exact, $- \otimes_R K$, $K \otimes_R -$ right exact

244. $\mathrm{Hom}_R({}_RM, {}_RN)$ abelian group

245. $\mathrm{Hom}_R({}_RM, {}_RN_S)$ right S-module

246. $\mathrm{Hom}_R({}_RM_S, {}_RN)$ left S-module

247. 1st component of Hom contravarient, 2nd component of Hom covariant

248. η natural isomorphism $\Rightarrow \exists$ natural transformation $\epsilon: G \to F$ $\ni \eta\epsilon = 1_G$ and $\epsilon\eta = 1_F$

249. Any kernel is monic

250. Any two kernels are isomorphic

251. Any cokernel is epi

252. Any two cokernels isomorphic

253. In category, if monic is a kernel, then its a kernel of its cokernel

254. In category, if epi is a cokernel, then its a cokernel of its kernel

255. In abelian category, every map $B \xrightarrow{f} C$ factors as $B \xrightarrow{e} \mathrm{Im} f \xrightarrow{m} C$ with e epi, m monic

256. R-MOD abelian category with Ker, CoKer, Im same as old notions and monic, epi same as injective, surjective

257. F exact $\Leftrightarrow$ F preserves any exact sequence

258. $\mathcal{A}^I$ is a category and if $\mathcal{A}$ is abelian, $\mathcal{A}^I$ is abelian

259. Yoneda embedding is fully faithful

260. $\mathcal{I}$ additive category $\Rightarrow$ $\exists$ fully faithful embedding of $\mathcal{I}$ into an abelian category

261. Freyd-Mitchell Embedding Thm $\mathcal{A}$ abelian category, $\mathcal{A}$ small $\Rightarrow$ $\exists$ exact, fully faithful $\mathcal{A} \longrightarrow$ R-MOD for some ring R

262. $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ functors $\Rightarrow$ (F,G) adjoint pair iff $\exists$ natural transformations $1_{\mathcal{A}} \xrightarrow{\eta} GF$, $FG \xrightarrow{\epsilon} 1_{\mathcal{B}}$ $\ni$ compositions $F \xrightarrow{F(\eta)} FGF \xrightarrow{\epsilon(F)} F$ are identity natural transformations

263. $\mathcal{A}, \mathcal{B}$ abelian categories, (F,G) adjoint pair $\Rightarrow$ F right exact and G left exact

264. Hom left exact for any abelian category

265. The colimit of a system is unique up to isomorphism

266. In R-MOD, the pushout of

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \\ B & \end{tikzcd}$$

is $A \oplus B / \{\alpha(c), -\beta(c)\}_{c \in C}$

267. In abelian category $\mathcal{A}$, the pushout of above is $\operatorname{coker}(C \xrightarrow{(\alpha,\beta)} A \oplus B)$

268. Coproduct in category of commutative R-algebras is $A \otimes_R B$

269. Coproduct in category of commutative rings is $A \otimes_{\mathbb{Z}} B$ and the product is $A \times B$

270. $\mathcal{A}$ abelian category $\Rightarrow$ colimits exist iff coproducts exist

271. limit is colimit of $F^{op} : \mathcal{I}^{op} \longrightarrow \mathcal{C}^{op}$

272. In abelian category, pullback of

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

is $\operatorname{Ker}(A \times B \xrightarrow{(f,-g)} C)$

273. In R-MOD, pullback of above is $\{(a,b) \mid f(a) = g(b)\} \subseteq A \times B$

274. $\mathcal{A}$ abelian $\Rightarrow$ limits exist iff products exist

275. In R-MOD, all limits over posets exists

276. In abelian category $\mathcal{A}$ $\Rightarrow$
(i) colim right exact
(ii) lim left exact

277. W left R-module $\Rightarrow$
(i) $- \otimes_R W$ preserves colimits
(ii) $\mathrm{Hom}_R(W,-)$ preserves limits
(iii) $\mathrm{Hom}_R(-,W)$ converts colimits to limits

278. (F,G) adjoint pair $\Rightarrow$
(i) F preserves colims
(ii) G preserves lims

279. Adjoints unique

280. $\{M_i\}$ directed system in R-MOD $\Rightarrow$
(i) Every $m \in \varinjlim M_i \ni m = \iota_i(m_i)$ for some $m_i \in M_i$ for some M_i
(ii) For any i, $\mathrm{Ker}(M_i \longrightarrow \varinjlim M_i) = \bigcup_{j \geq i} \mathrm{Ker}(M_i \longrightarrow M_j)$
ie $\forall m \in \varinjlim M_i$, $m = 0$ iff $m_i \longrightarrow 0$ under some map $M_i \longrightarrow M_j$ in system

281. $\varinjlim M_i = \dot{\bigcup} M_i/\sim$ where $m_i \sim m_j$ if $\varphi^i_l(m_i) = \varphi^j_l(m_j)$ for some $l \geq i,j$

282. Directed limits are exact in R-MOD

283. In R-MOD, directed limits commute with homology ie
$H_n(\varinjlim M_i) = \varinjlim H_n(M_i)$

284. $\{M_i\}_{i \in I}$ directed system of R-modules, M_i flat $\forall i \Rightarrow \varinjlim M_i$ flat

285. M left R-module $\Rightarrow$ M flat iff M is directed limit of finitely generated free R-modules

286. $f : C_\cdot \longrightarrow D_\cdot$ chain map sends cycles to cycles ie $f_n(Z_n(C_\cdot)) \subseteq Z_n(D_\cdot)$, boundaries to boundaries ie $f_n(B_n) \subseteq B_n$. And get induced map $H_n(C_\cdot) \xrightarrow{H_n(f)} H_n(D_\cdot)$

287. $B_\cdot \longrightarrow C_\cdot \longrightarrow C_\cdot/B_\cdot$ chain map where $B_\cdot$ subcomplex of $C_\cdot$

288. $f : C_\cdot \longrightarrow D_\cdot$ chain map $\Rightarrow$ Ker f subcomplex of $C_\cdot$, coKer f

quotient complex, and Im f subcomplex of D.

289. Complexes Ker f, coKer f, Im f coincide with categorical definitions of kernel, cokernel, image in category ch(R-MOD) with objects chain complexes and morphisms chain maps

290. Ch(R-MOD) abelian

291. $\mathcal{A}$ abelian $\Rightarrow$ ch($\mathcal{A}$) abelian

292. $H_n = 0 \iff C.$ exact at spot n

293. C. exact $\Rightarrow$ $0 \longrightarrow C.$ quasi-iso

294. $0 \longrightarrow A. \xrightarrow{f} B. \xrightarrow{g} C. \longrightarrow 0$ SES chain complexes $\Rightarrow \exists$ natural connecting homomorphisms $\delta: H_n(C) \longrightarrow H_{n-1}(A)$ $\forall n \ni \ldots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \ldots$ LES

295. $0 \longrightarrow A. \xrightarrow{f} B. \xrightarrow{g} C. \longrightarrow 0$ SES cochain complexes $\Rightarrow$ $\exists \delta: H^n(C) \longrightarrow H^{n+1}(A) \ni$ LES:

$\ldots \longrightarrow H^{n-1}(C) \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow H^{n+1}(A) \longrightarrow \ldots$

296. Connecting map/Snake Lemma commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "u"'] & Y \arrow[r, "\beta"] \arrow[d, "\pi"'] & Z \arrow[r] \arrow[d, "\epsilon"'] & 0 \\ 0 \arrow[r] & X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ connecting map $\delta: \mathrm{Ker}\,\epsilon \longrightarrow \mathrm{CoKer}\,u$ defined as $\delta(z) = \bar{x}$ where $\pi(y) = \alpha'(x)$ and $\beta(y) = z$

297. Naturality of LES connecting map δ natural ie for any commutative diagram of complexes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

the diagram below commutes

$$\begin{tikzcd} \ldots \arrow[r] & H_n(B) \arrow[r] \arrow[d, "H_n(\beta)"'] & H_n(C) \arrow[r, "\delta"] \arrow[d, "H_n(\gamma)"'] & H_{n-1}(A) \arrow[r] \arrow[d, "H_n(\alpha)"'] & \ldots \\ \ldots \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\delta"] & H_{n-1}(A') \arrow[r] & \ldots \end{tikzcd}$$

298. C. split exact with splitting map $s: B_{n-1} \longrightarrow C_n$ $\Rightarrow$ $C_n = Z_n \oplus s(B_{n-1})$

hence $C_\cdot : \dots \to Z_{n+1} \oplus X_{n+1} \xrightarrow{d} Z_n \oplus X_n \to \dots$ where $X_n = S(B_{n-1})$

299. $C_\cdot$ exact complex over field (or semisimple ring) $\Rightarrow C_\cdot$ split exact

300. $f \simeq 0 \Rightarrow H(f) = 0$

301. $f \simeq g \Rightarrow H(f) = H(g)$

302. $C_\cdot$ split exact $\Leftrightarrow 1_C \simeq 0$

303. Bicomplex is complex of complexes

304. $C_\cdot$ bounded $\Rightarrow \mathrm{Tot}(C) = \mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}^{\Pi}(C)$

305. $\mathrm{Tot}(C_\cdot)$ is a complex

306. $0 \to A_{\cdot\cdot} \to B_{\cdot\cdot} \to C_{\cdot\cdot} \to 0$ SES of bounded bicomplexes
$\Rightarrow \exists$ SES $0 \to \mathrm{Tot}(A) \to \mathrm{Tot}(B) \to \mathrm{Tot}(C) \to 0$

307. $\exists$ SES of complexes: $0 \to C \xrightarrow{i_1} C(f) \xrightarrow{p_2} B[-1] \to 0$
where $f : B_\cdot \to C_\cdot$ ie for each n, $\exists$ SES:
$0 \to C_n \to C_n \oplus B_{n-1} \to B_{n-1} \to 0$

308. LES of homology for above SES:
$\dots \to H_{n+1}(B[-1]) \xrightarrow{\delta} H_n(C) \to H_n(C(f)) \to H_n(B[-1]) \xrightarrow{\delta} \dots$
ie $\dots \to H_n(B) \xrightarrow{\delta} H_n(C) \to H_n(C(f)) \to H_{n-1}(B) \xrightarrow{\delta} \dots$
has connecting maps: $\delta = H_n(f) : H_n(B) \to H_n(C)$

309. $f : B_\cdot \to C_\cdot$ quasi-isomorphism $\Leftrightarrow C(f)$ exact

310. $\exists$ SES $0 \to C \xrightarrow{i_1} \mathrm{Cyl}(f) \to C(1_B)[-1] \to 0$ and i_1 is quasi-iso. In fact, i_1 homotopy equivalence

311. $C(1_B)$ split exact

312. $\exists$ commutative diagram:

$$\begin{tikzcd}
 & B \arrow[r, "f"] \arrow[d, equal] & C \arrow[r, "i"] \arrow[d, shift left, "j"] & C(f) \arrow[d, equal] & \\
0 \arrow[r] & B \arrow[r, "i_3"] & \mathrm{Cyl}(f) \arrow[r] \arrow[u, shift left, "i"] \arrow[u, phantom, "\simeq"] & C(f) \arrow[r] & 0
\end{tikzcd}$$

where $j(c, b, b') = c + f(b)$. And bottom row is SES.

313. In R-MOD, every module has a projective resolution. In fact, this is true in any category in which all objects have projectives mapping onto them.

314. Comparison Thm $f' : M \to N$, complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "f_2"] & P_1 \arrow[r] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

with P_i projective $\forall i$, bottom row exact $\Rightarrow$ f' can be lifted to a chain map $P_\cdot \xrightarrow{f} Q_\cdot$ and any two lifts are homotopic

315. A projective resolution of a module is unique up to homotopy, ie any two projective resolutions for a module are homotopy equivalent

316. Horseshoe Lemma SES of R-modules: $0 \to M' \to M \to M'' \to 0$, $P_\cdot' \to M'$, $P_\cdot'' \to M''$ projective resolutions $\Rightarrow \exists$ projective resolution $P_\cdot \to M \ni P_n = P_n' \oplus P_n''$ and differential d is of form

$$\begin{matrix} P_n' \\ \oplus \\ P_n'' \end{matrix} \xrightarrow{\begin{bmatrix} d' & * \\ 0 & d'' \end{bmatrix}} \begin{matrix} P_{n-1}' \\ \oplus \\ P_{n-1}'' \end{matrix}$$

Hence, get SES: $0 \to P_\cdot' \xrightarrow{\iota_1} P_\cdot \xrightarrow{\pi_2} P_\cdot'' \to 0$

317. Injective resolutions exist in R-MOD

318. Comparison Thm $f': M \to N$, complexes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f'"] & J^0 \arrow[r] \arrow[d, dashed, "f_0"] & J^1 \arrow[r] \arrow[d, dashed, "f_1"] & J^2 \arrow[r] \arrow[d, dashed, "f_2"] & \cdots \\
0 \arrow[r] & N \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

with top row exact and each I^n injective $\Rightarrow$ f' can be lifted to a cochain map $J^\cdot \xrightarrow{f} I^\cdot$ and any two lifts are homotopic

319. Horseshoe Lemma SES of R-modules: $0 \to M' \to M \to M'' \to 0$, $M' \to I^{\cdot\prime}$, $M'' \to I^{\cdot\prime\prime}$ injective resolutions $\Rightarrow \exists$ injective resolution $M \to I^\cdot \ni I^n = I^{n\prime} \oplus I^{n\prime\prime}$ and differential

$$\begin{matrix} I^{n\prime} \\ \oplus \\ I^{n\prime\prime} \end{matrix} \xrightarrow{\begin{bmatrix} d' & * \\ 0 & d'' \end{bmatrix}} \begin{matrix} I^{n+1\prime} \\ \oplus \\ I^{n+1\prime\prime} \end{matrix}$$

Hence, get SES: $0 \to I^{\cdot\prime} \xrightarrow{\iota_1} I^\cdot \xrightarrow{\pi_2} I^{\cdot\prime\prime} \to 0$

320. L_nF well defined ie independent of choice of projective

resolution

321. L_nF functor

322. $L_0F = F$

323. SES $0 \to M' \to M \to M'' \to 0 \Rightarrow \exists$ connecting maps δ_n and LES: $\ldots \to L_{n+1}F(M'') \to L_nF(M') \to L_nF(M) \to L_nF(M'') \to \ldots$ $\to L_1F(M'') \to F(M') \to F(M) \to F(M'') \to 0$ and δ_n are natural ie for any commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K' \arrow[r] & K \arrow[r] & K'' \arrow[r] & 0 \end{tikzcd}$$

the diagram below commutes:

$$\begin{tikzcd} \ldots \arrow[r] & L_{n+1}F(M'') \arrow[r, "\delta"] \arrow[d] & L_nF(M') \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & L_{n+1}F(K'') \arrow[r, "\delta"] & L_nF(K') \arrow[r] & \ldots \end{tikzcd}$$

324. F right exact functor, F additive between abelian categories with projective resolutions $\Rightarrow$ F exact iff $L_nF = 0 \; \forall n > 0$ iff $L_1F = 0$

325. Any two universal δ-functors $\{T_n\}, \{T_n'\}$ with $T_0 \cong T_0'$, are isomorphic ie $T_n \cong T_n' \; \forall n$ commuting with δ's

326. F right exact functor $\Rightarrow \{L_nF\}_{n \geq 0}$ universal δ-functors

327. $P. \xrightarrow{\varepsilon} M$, $Q. \xrightarrow{\eta} N$ projective resolutions $\Rightarrow$ $\operatorname{Tor}_n^R(M,N) = H_n(P. \otimes N) \cong H_n(M \otimes Q.)$ ie

328. $A.. \xrightarrow{f} B..$ map of 1st quadrant bicomplexes $\ni$ map on each row or column is a quasi-iso $\Rightarrow \operatorname{Tot}(A..) \xrightarrow{\operatorname{Tot}(f)} \operatorname{Tot}(B..)$ quasi-iso

329. $C..$ 1st quadrant bicomplex with exact rows or columns $\Rightarrow \operatorname{Tot}(C..)$ exact

330. $A. \xrightarrow{f} B.$ chain map, f quasi-iso, P flat $\Rightarrow P \otimes A. \xrightarrow{1 \otimes f} P \otimes B.$ quasi-iso

331. LES for Tor $0 \to A \to B \to C \to 0$ exact $\Rightarrow$ LES:
$\ldots \to \operatorname{Tor}_n^R(A,N) \to \operatorname{Tor}_n^R(B,N) \to \operatorname{Tor}_n^R(C,N) \to \ldots$
$\to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ or LES:

$$\cdots \to Tor_n^R(M,A) \to Tor_n^R(M,B) \to Tor_n^R(M,C) \to \cdots$$
$$\to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

332. F left exact $\iff$ F^{op} right exact

333. injectives in $\mathcal{A}$ $\iff$ projectives in $\mathcal{A}^{op}$

334. right resolution in $\mathcal{A}$ $\iff$ left resolution in $\mathcal{A}^{op}$

335. R^nF well defined ie independent of injective resolution chosen

336. R^nF functor

337. $R^0F = F$

338. SES: $0 \to A \to B \to C \to 0 \Rightarrow$ LES:
$0 \to F(A) \to F(B) \to F(C) \xrightarrow{\delta} R^1F(A) \to \cdots \to R^nF(B) \to R^nF(C) \xrightarrow{\delta} R^{n+1}F(A) \to \cdots$ and connecting maps δ are natural

339. $N \to I^\cdot$ injective resolution, $P_\cdot \xrightarrow{\epsilon} M$ projective resolution
$\Rightarrow Ext_R^n(M,N) = H^n(Hom_R(M, I_\cdot)) \cong H^n(Hom_R(P_\cdot, N))$

340. LES for Ext SES: $0 \to A \to B \to C \to 0$, M R-module $\Rightarrow$
LES: $0 \to Hom_R(M,A) \to Hom_R(M,B) \to Hom_R(M,C) \to \cdots$
$\to Ext_R^n(M,C) \to Ext_R^{n+1}(M,A) \to Ext_R^{n+1}(M,B) \to \cdots$
or LES: $0 \to Hom_R(C,M) \to Hom_R(B,M) \to Hom_R(A,M) \to$
$Ext_R^n(A,M) \to Ext_R^{n+1}(C,M) \to Ext_R^{n+1}(B,M) \to \cdots$

341. R commutative, M, N R-modules, $M \xrightarrow{r} M$ multiplication by $r \Rightarrow$ induced maps on $M \otimes_R N$, $Hom_R(M,N)$, $Hom_R(N,M)$, $Tor_n^R(M,N) \cong Tor_n^R(N,M)$, $Ext_R^n(M,N)$, $Ext_R^n(N,M)$ are multiplication by r

342. R commutative $\Rightarrow Tor_n^R(M,N) \cong Tor_n^R(N,M)$.

343. $Tor_n^R(M,N) \cong Tor_n^{R^{op}}(N,M)$

344. F R-module, TFAE:
(i) F flat
(ii) $Tor_n^R(F,N) = 0 \ \forall N, \forall n > 0$
(iii) $Tor_1^R(F,N) = 0 \ \forall N$

345. Tor can be computed with flat resolutions instead of projective resolutions

346. P R-module. TFAE:

(i) P projective

(ii) $\mathrm{Ext}_R^n(P,-)=0 \ \forall n>0$

(iii) $\mathrm{Ext}_R^1(P,-)=0$

347. I R-module. TFAE:

(i) I injective

(ii) $\mathrm{Ext}_R^n(-,I)=0 \ \forall n>0$

(iii) $\mathrm{Ext}_R^1(-,I)=0$

348. R commutative, $S\subseteq R$ multiplicative $\Rightarrow$

(i) $S^{-1}\mathrm{Tor}_n^R(M,N)\cong \mathrm{Tor}_n^{S^{-1}R}(S^{-1}M,S^{-1}N)$

(ii) M finitely presented $\Rightarrow S^{-1}\mathrm{Ext}_R^n(M,N)\cong \mathrm{Ext}_{S^{-1}R}^n(S^{-1}M,S^{-1}N)$

349. $\exists R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0 \Rightarrow M$ finitely presented

350. R noetherian, M finitely generated $\Rightarrow M$ finitely presented

351. Künneth Formula $P.$ chain complex of right R-modules $\ni$

(1) each P_n flat

(2) each $B_{n-1}=d(P_n)$ flat

$\Rightarrow$ For each n and each left R-module M, $\exists$ SES

$0\longrightarrow H_n(P.)\otimes_R M\longrightarrow H_n(P.\otimes_R M)\longrightarrow \mathrm{Tor}_1^R(H_{n-1}(P.),M)\longrightarrow 0$

352. R hereditary, $P.$ projective $\Rightarrow$ Künneth/UCT satisfied

353. Universal Coefficient Thm $P.$ chain complex of right R-modules $\ni$

(1) each P_n projective

(2) each $B_{n-1}=d(P_n)$ projective

$\Rightarrow$ For each n and each left R-module M $\exists$ SES:

$0\longrightarrow H_n(P.)\otimes_R M\longrightarrow H_n(P.\otimes_R M)\longrightarrow \mathrm{Tor}_1^R(H_{n-1}(P.),M)\longrightarrow 0$

and sequence splits

354. Künneth Formula for Complexes $P., Q.$ complexes of right, left R-modules respectively $\ni$

(1) Each P_n flat

(2) Each $B_{n-1}=d(P_n)$ flat

$\Rightarrow$ For each n, get SES:

$0\to \bigoplus_{p+q=n} H_p(P.)\otimes_R H_q(Q.)\longrightarrow H_n(\mathrm{Tot}(P.\otimes_R Q.))\to \bigoplus_{p+q=n-1}\mathrm{Tor}_1^R(H_p(P.),H_q(Q.))\to 0$

And if each P_n, B_{n-1} projective, sequence splits

355. UCT (cohomology) $P_\cdot$ chain complex, M left R-module $\ni$
(1) each P_n projective
(2) each $B_{n-1} = d(P_n)$ projective
$\Rightarrow \forall n$ get SES:
$0 \longrightarrow \mathrm{Ext}^1_R(H_{n-1}(P_\cdot), M) \longrightarrow H^n(\mathrm{Hom}(P_\cdot, M)) \longrightarrow \mathrm{Hom}(H_n(P_\cdot), M) \longrightarrow 0$
and it splits
356. Any split extension is equivalent to $0 \longrightarrow A \xrightarrow{i_1} A \oplus C \xrightarrow{\pi_2} C \longrightarrow 0$
357. $\exists$ bijection, $e(C,A) \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} \mathrm{Ext}^1_R(C,A) \ni$ trivial split extension above corresponds to $0 \in \mathrm{Ext}^1_R(C,A)$
358. $\mathrm{Ext}^1_R(C,A) = 0 \Longleftrightarrow$ Every extension of C by A splits
359. Baer sum in $e(C,A)$ agrees with (under Θ) usual + in $\mathrm{Ext}^1_R(C,A)$ ie $\Theta(\xi + \xi') = \Theta(\xi) + \Theta(\xi')$
360. Baer sum well defined on equivalence classes $[\xi]$
361. Under Baer sum, $e(C,A)$ group
362. $\mathrm{Ext}^n_R(C,A) \cong \{$extensions $\xi: 0 \longrightarrow A \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow C \longrightarrow 0$
exact mod equivalence:

$$\begin{tikzcd} \xi: 0 \arrow[r] & A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "G"] & X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & \dots \arrow[r] \arrow[dr, phantom, "G"] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & C \arrow[r] \arrow[d, equal] & 0 \\ \xi': 0 \arrow[r] & A \arrow[r] & X_n' \arrow[r] & \dots \arrow[r] & X_1' \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \}$$

as groups with usual addition in $\mathrm{Ext}^n_R(C,A)$ and addition of extensions $\xi + \xi'$:
$0 \longrightarrow A \longrightarrow Y_n \longrightarrow X_{n-1} \oplus X_{n-1}' \longrightarrow \dots \longrightarrow X_2 \oplus X_2' \longrightarrow X_1'' \longrightarrow C \longrightarrow 0$
where X_1'' = pullback of X_1, X_1' over C and Y_n = pushout of X_n, X_n' over A $= X_1 \times X_1'' / \{(-a, a)\}_{a \in A}$
363. $0 \longrightarrow A \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow A \longrightarrow 0 \in \mathrm{Ext}^n_R(A,A)$,
$0 \longrightarrow A \longrightarrow Z_m \longrightarrow \dots \longrightarrow Z_1 \longrightarrow A \longrightarrow 0 \in \mathrm{Ext}^m_R(A,A) \Rightarrow$
product in $\mathrm{Ext}^{n+m}_R(A,A)$:
$0 \longrightarrow A \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow Z_m \longrightarrow \dots \longrightarrow Z_1 \longrightarrow A \longrightarrow 0$
364. $\bigoplus_{i=0} \mathrm{Ext}^i_R(A,A)$ ring
365. $\mathcal{D}(\mathcal{A})$ unique up to equivalence of categories
366. $\mathcal{B}$ category, S collection of morphisms $\Rightarrow S^{-1}\mathcal{B}$ category
367. $i: \mathcal{B} \longrightarrow S^{-1}\mathcal{B}$, ob: $x \longmapsto x$, mor: $f \longmapsto f$ functor that satisfies

UMP: $\forall$ functors $\mathcal{B} \xrightarrow{F} \mathcal{C}$ transforming morphisms in S into isos in $\mathcal{C}$, $\exists!$ functor $G \ni$ $\mathcal{B} \xrightarrow{F} \mathcal{C}$ commutes

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "F"] \arrow[dr, "i"'] & & \mathcal{C} \\ & S^{-1}\mathcal{B} \arrow[ur, "G"'] & \end{tikzcd}$$

368. $\mathcal{D}(\mathcal{A}) = S^{-1}(Ch(\mathcal{A}))$ where $S = \{$quasi-isos$\}$

369. S localizing collection of morphisms $\Rightarrow$ $S^{-1}\mathcal{B}$ defined as

(i) $ob(S^{-1}\mathcal{B}) = ob\,\mathcal{B}$

(ii) morphisms $X \longrightarrow Y$ in $S^{-1}\mathcal{B}$ are equivalence classes of roofs (s,f) with $s \in S$ and $(s,f) \sim (t,g)$ iff $\exists$ commutative diagram

$$\begin{tikzcd} & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X' \arrow[d, "s"'] \arrow[drr, "f" near start] & & X'' \arrow[d, "g"] \arrow[dll, "t"' near start] \\ X & & Y \end{tikzcd}$$

with $sr \in S$

(iii) identity morphism on X is roof $(1_X, 1_X)$

(iv) composition of roofs (s,f) and (t,g) is (st', gf'):

$$\begin{tikzcd} & & X'' \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "f'"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

(v) $i: \mathcal{B} \longrightarrow S^{-1}\mathcal{B} \ni$ obj: $X \longrightarrow X$, mor: $f \longrightarrow (1, f)$

370. $\sim$ equivalence relation

371. functor i factors through $\mathcal{K}(\mathcal{A})$:

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[rr, "i"] \arrow[dr, "\pi"'] & & \mathcal{D}(\mathcal{A}) \\ & \mathcal{K}(\mathcal{A}) \arrow[ur, dashed] & \end{tikzcd}$$

372. $\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{K}(\mathcal{A})$ with $S = \{$equivalence classes of quasi-isos$\}$

373. $f \simeq g$, f quasi-iso $\Rightarrow$ g quasi-iso

374. S localizing class of $\mathcal{K}(\mathcal{A})$

375. $\mathcal{D}(\mathcal{A})$ has morphisms: roofs of maps mod equivalence of roofs

376. $K \xrightarrow{f} L$ cochain map $\Rightarrow$ have commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\alpha"] & C(f) \arrow[r, "p"] \arrow[d, equal] & K[1] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, "\tilde{f}"] \arrow[d, equal] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\
 & K \arrow[r, "f"] & L & & &
\end{tikzcd}$$

where $\beta(l,k,k') = l + f(k)$ and other maps are inclusions and projections $\ni$ α, β homotopy equivalences ie $\alpha\beta = 1_L$ and $\beta\alpha \simeq 1_{Cyl(f)}$, hence quasi-isos, hence isos in $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ and diagram is natural in f

377. $f, g : K \to L$, $f \simeq g \Rightarrow i(f) = i(g)$ in $\mathcal{D}(\mathcal{A})$

378. $K \xrightarrow{f} L$ in $Ch(\mathcal{A}) \Rightarrow f = 0$ in $\mathcal{D}(\mathcal{A})$ iff $\exists$ quasi-iso $L \xrightarrow{s} M \ni$ $sf \simeq 0$ in $Ch(\mathcal{A})$ ie $f = 0$ in $\mathcal{K}(\mathcal{A})$

379. $K \xrightarrow{f} L \Rightarrow$ get isomorphism of Δ's:

$$\begin{tikzcd}
K \arrow[r, "\tilde{f}"] \arrow[d, equal] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] \arrow[d, equal] & K[1] \arrow[d, equal] \\
K \arrow[r, "f"] & L \arrow[r, "i"] & C(f) \arrow[r, "p"] & K[1]
\end{tikzcd}$$

where $\tilde{f}$ inclusion

380. Exact sequence $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ in $Ch(\mathcal{A})$ is iso to a SES that can be extended to a distinguished Δ in $\mathcal{D}(\mathcal{A})$ (and in $\mathcal{K}(\mathcal{A})$). Conversely, every distinguished Δ is iso to one that is an extension of a SES in $Ch(\mathcal{A})$

Namely,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] & L \arrow[r, "g"] & M \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u, equal] & Cyl(f) \arrow[r] \arrow[u, "\beta"] & C(f) \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where $\beta(l,k,k') = l + f(k)$, $\gamma(l,k) = g(l)$ are iso's in $\mathcal{D}(\mathcal{A})$ and $K \to Cyl(f) \to C(f) \to K[1]$ dist. Δ

381. $K \xrightarrow{u} L \xrightarrow{v} M \xrightarrow{w} K[1]$ distinguished Δ in $\mathcal{D}(\mathcal{A}) \Rightarrow$ get LES: $\ldots \to H^i(M[-1]) \to H^i(K) \to H^i(L) \to H^i(M) \to H^i(K[1]) \to \ldots$

ie $\ldots \to H^{i-1}(M) \to H^i(K) \to H^i(L) \to H^i(M) \to H^{i+1}(K) \to \ldots$

382. A rotation of a Δ is a Δ. In fact, a rotation of a distinguished Δ is a distinguished Δ

383. Have functor: $\mathcal{A} \xrightarrow{i} \mathcal{D}(\mathcal{A}) \ni$ obj: $X \longrightarrow X[0]$ ie $X \longrightarrow$ complex $\cdots \longrightarrow \overset{(-1)}{0} \longrightarrow \overset{(0)}{X} \longrightarrow \overset{(1)}{0} \longrightarrow \cdots$

384. K complex, $H^i(K) = 0 \ \forall i \neq 0 \Rightarrow K$ H^0-complex

385. functor $i : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ gives equivalence of categories of $\mathcal{A}$ with full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of all H^0-complexes

386. $Ext^0_{\mathcal{A}}(X,Y) = Hom_{\mathcal{A}}(X,Y)$, in particular, if $\mathcal{A} = R\text{-MOD}$, $Ext^0_{\mathcal{A}}(X,Y) = Hom_R(X,Y)$

387. $Ext^i_{\mathcal{A}}(X,Y) \cong Hom_{\mathcal{D}(\mathcal{A})}(X[k], Y[i+k])$ for any $k \in \mathbb{Z}$

388. Composition: $Ext^i_{\mathcal{A}}(X,Y) \times Ext^j_{\mathcal{A}}(Y,Z) \longrightarrow Ext^{i+j}_{\mathcal{A}}(X,Z)$. Hence $\oplus Ext^i_{\mathcal{A}}(X,X)$ graded ring

389. $Ext^i_{\mathcal{A}}(X,Y) = 0$ for $i < 0$

390. An element of Yoneda Ext, ie exact sequence: $0 \longrightarrow Y \longrightarrow K^{-i+1} \longrightarrow \cdots \longrightarrow K^0 \xrightarrow{\epsilon} X \longrightarrow 0$ gives a roof:

$$\begin{tikzcd} & \tilde{K} \arrow[dl, "\epsilon"'] \arrow[dr] & \\ X[0] & & Y[i] \end{tikzcd}$$

which is an element of $Ext^i_{\mathcal{A}}(X,Y)$. In fact, every element of $Ext^i_{\mathcal{A}}(X,Y)$ are of this form.

391. I full subcategory of $\mathcal{A}$ of injective objects $\Rightarrow \mathcal{K}^+(I)$ is bounded below complexes of injective objects with morphisms being cochain maps mod homotopy

392. objects in $\mathcal{A}$ have injective resolutions $\Rightarrow$ functor $\mathcal{K}^+(I) \longrightarrow \mathcal{D}^+(\mathcal{A})$ is equivalence of categories

393. objects in $\mathcal{A}$ have projective resolutions $\Rightarrow$ functor $\mathcal{K}^-(P) \longrightarrow \mathcal{D}^+(\mathcal{A})$ is equivalence of categories

394. $Ext^i_{\mathcal{A}}(X,Y) = Hom_{Ch(\mathcal{A})}(P_\cdot, Y[i])/\text{homotopy}$ where $P_\cdot \longrightarrow X$ projective resolution

$$= H^i(Hom(P_\cdot, Y))$$
$$= Ext^i(X,Y)$$

Hence categorical $Ext^i_{\mathcal{A}}(X,Y) = Ext^i(X,Y)$ classic

395. Right derived functor, RF, of Hom is $R\mathrm{Hom}(X,Y) = \mathrm{Hom}(P,Y)$
hence $H^i(R\mathrm{Hom}(X,Y)) = \mathrm{Ext}^i(X,Y)$

396. Left derived functor, LF, of $\otimes$ is $L\otimes(X,Y) = P.\otimes Y$
hence $H_i(L\otimes(X,Y)) = \mathrm{Tor}_i(X,Y)$

397. $K(\mathcal{A}), \mathcal{D}(\mathcal{A})$ Δ'd categories

398. A distinguished Δ is determined up to iso by any one of its maps. Hence, in TR1, the extension of u is unique up to iso

399. 5-lemma for Δ'd categories. In TR3, if f, g isos in $\mathcal{T}$, then h is an iso in $\mathcal{T}$

400. $\mathcal{T}$ Δ'd category, $u \in \mathcal{T} \Rightarrow \mathrm{Hom}_{\mathcal{T}}(u,-)$ cohomological functor and $\mathrm{Hom}_{\mathcal{T}}(-,u)$ contravariant cohomological functor

401. $\mathcal{K}(\mathcal{A})$ with translation being shift $[1]$ of complex and distinguished Δ's being one's iso to

$$X \xrightarrow{u} Y \longrightarrow C(u) \longrightarrow X[1]$$

is a Δ'd category

402. $\mathcal{K}^+(\mathcal{A}), \mathcal{K}^-(\mathcal{A}), \mathcal{K}^b(\mathcal{A})$ Δ'd categories

403. $C(vu) \cong C(C(v)[-1] \xrightarrow{j[1]i} C(u))$ where have dist. Δ:

$$C(v) \longrightarrow C(u)[1] \longrightarrow C(vu)[1] \longrightarrow C(v)[1]$$

(from octahedral axiom)

404. $\mathcal{T}$ Δ'd category, S localizing class of morphisms, $\forall s \in S$ $T(s) \in S$ in TR3 if $f, g \in S$, h can be chosen in $S \Rightarrow S^{-1}\mathcal{T}$ with translation same as for $\mathcal{T}$ and dist. Δ's being those iso to images of dist Δ's from $\mathcal{T}$ under functor $\mathcal{T} \longrightarrow S^{-1}\mathcal{T} \ni$ $x \longrightarrow x$, $f \longrightarrow (1,f)$ is Δ'd category

405. $\mathcal{D}(\mathcal{A}), \mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A}), \mathcal{D}^b(\mathcal{A})$ Δ'd categories

406. $E^{r+1}_{p,q}$ subquotient of $E^r_{p,q}$

407. Bounded spectral sequence $\Rightarrow$ differentials d^r will be 0-maps for r large enough

408 filtration on C bounded $\Rightarrow$ spectral sequence converges to homology of C ie $E^1_{pq} \Rightarrow H_{p+q}(C)$. In fact, E^∞ diagonals

$p+q=n$ are factors in filtration of $H_n(C)$

409. $C_{..}$ bicomplex $\Rightarrow$ Tot(C.) has two filtrations:

(1) $E^2_{p,q} = H^h_p(H^v_q(C_{..})) \Rightarrow H_n(\mathrm{Tot}(C_{..}))$

(2) $E^2_{p,q} = H^v_q(H^h_p(C_{..})) \Rightarrow H_n(\mathrm{Tot}(C_{..}))$

410. $\exists$ filtration of $H_n(C)$ whose factors are iso to E^∞_{pq}, $p+q=n$, terms

Examples...

1. A bimodule $\not\Rightarrow$ $ar = ra$ $\forall a \in A, \forall r \in R$
2. S ring, $R = M_n(S)$
   $M_i = \{ M \in R \mid M$ is 0 everywhere except possibly ith row $\}$
   M_i right ideal of R, hence right R-module
   $N_i = \{ N \in R \mid N$ is 0 everywhere except possibly ith column $\}$
   N_i left ideal of R, hence left R-module
3. $I \trianglelefteq S$, say $S = \mathbb{Z}$, $I = 2\mathbb{Z}$
   $M(I) = \{ M \in R \mid$ entries in $I \}$
   $\therefore M(I) \trianglelefteq R$, hence bimodule
4. R ring, $f: A \longrightarrow B$ R-module homomorphism $\Rightarrow$
   $\therefore \ker f \leq A$, $\operatorname{Im} f \leq B$ submodules
   And if $C \leq B$, then $f^{-1}(C) \leq A$
5. SES: $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$
6. $0, 1$ idempotents
7. $R = \begin{bmatrix} \mathbb{R} & 0 & 0 \\ \mathbb{R} & \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{R} & \mathbb{R} \end{bmatrix}$ ring
   $e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ idempotents
8. $\operatorname{Hom}(M, -)$ not exact:
   $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$
   $0 \longrightarrow 2\mathbb{Z} \xhookrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ exact
   $0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, 2\mathbb{Z}) \xrightarrow{i_*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$
   $0 \longrightarrow 0 \xrightarrow{i_*} 0 \xrightarrow{\pi_*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$
   π_* not surjective since $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq 0$
9. Commutative diagram with exact rows:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, dashed, "\exists h"] & 0 \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

Show $\exists h: C \longrightarrow C'$ commuting diagram
$g: B \longrightarrow C$ cokernel of $A \xrightarrow{f} B$
And $g'\beta f = g'f'\alpha = 0\alpha = 0$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "0"'] & B \arrow[r, "g"] \arrow[d, "g'\beta"] & C \arrow[dl, dashed, "\exists h"] \\
& C' &
\end{tikzcd}$$

10. Infinitely many injective modules $E_i \not\Rightarrow \bigoplus_{i\in I} E_i$ injective

11. Infinitely many projective modules $P_i \not\Rightarrow \prod_{i\in I} P_i$ projective

12 $_{\mathbb{Z}}\mathbb{Q}$ divisible:

Let $y\in\mathbb{Q}$ and $0\neq n\in\mathbb{Z}$

Then $y=\frac{a}{b}$

Take $x=\frac{a}{bn}\in\mathbb{Q}$

And $nx=\frac{a}{b}=y$

13. $_{\mathbb{Z}}\mathbb{Q}$ injective

14. $x\in A\otimes_R B \not\Rightarrow x=a\otimes b$

15. $A_R, {}_RB$, $A'_R \le A_R$, ${}_RB' \le {}_RB$ submodules

$\not\Rightarrow A'_R\otimes_R {}_RB' \le A_R\otimes_R {}_RB$

Because $a\otimes b=0$ in $A\otimes_R B$ but $a\otimes b\neq 0$ in $A'\otimes_R B'$ possible

For instance: $R=\mathbb{Z}$, $A=\mathbb{Z}$, $B=\mathbb{Z}/3\mathbb{Z}$, $A'=3\mathbb{Z}$, $B'=\mathbb{Z}/3\mathbb{Z}$

$0\neq 3\otimes\bar{1}\in 3\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}=A'\otimes_R B'$

But $3\otimes 1=1\otimes 0=0$ in $A\otimes_R B=\mathbb{Z}\otimes_R\mathbb{Z}/3\mathbb{Z}$

16. $\_\otimes_R M$ not exact:

$0\longrightarrow\mathbb{Z}\xrightarrow{3}\mathbb{Z}\xrightarrow{\pi}\mathbb{Z}/3\mathbb{Z}\longrightarrow 0$ SES

$\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}\xrightarrow{3}\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}\longrightarrow\mathbb{Z}/3\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}\longrightarrow 0$

$\mathbb{Z}/3\mathbb{Z}\xrightarrow{3}\mathbb{Z}/3\mathbb{Z}\longrightarrow\mathbb{Z}/3\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}\longrightarrow 0$

But multiplication by 3 here is 0 map, which is not injective

17. R_R flat: Let $A\xrightarrow{f}B$ monomorphism

Then
$$\begin{tikzcd}
R\otimes_R A \arrow[r, "1\otimes f"] \arrow[d, "\phi_A", "\wr"'] \arrow[dr, phantom, "G"] & R\otimes_R B \arrow[d, "\wr"', "\phi_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

$f\phi_A$ mono since f, ϕ_A both mono

$\Rightarrow \phi_B(1\otimes f)$ mono $\Longrightarrow 1\otimes f$ mono since ϕ_B mono

18. $_{\mathbb{Z}}\mathbb{Q}$ flat but not projective

19. R integral domain, $S=R\setminus\{0\}$ multiplicative

20. $P\lhd R$ prime $\Rightarrow R\setminus P$ multiplicative

21. $R=\mathbb{Q}[x_i]_{i\in\mathbb{N}}$, $I=\langle\{x_i\}_{i\in\mathbb{N}}\rangle$ not finitely generated

$\therefore R$ not noetherian

22. F field, $R=F[x,y]/\langle x^2,xy,y^2\rangle$ 3-dim vector space over F with

basis $\{\bar{1}, \bar{x}, \bar{y}\}$

$\Rightarrow$ All ideals of R are subspaces of dim 0, 1, 2, 3

$\therefore$ R Noetherian

23. $R = F[x,y]/\langle x^2, xy, y^2\rangle$ artinian ring

24. $\mathbb{Z}$ is not artinian ring because $\langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \langle 8\rangle \supsetneq \langle 16\rangle \supsetneq \dots$

25. Over a field, every module semisimple: K field

Let M module of K ie vector space over K

Let $L \leq M$ submodule ie subspace.

Pick basis B of L

B can be extended to $\bar{B}$ a basis of M $\ni \bar{B} = B' \dot{\cup} B$

Take $X = \langle B' \rangle$

$\therefore M = L \oplus X$

26. $M = S + T$, S, T simple submodules

$= S \oplus T$

Composition series: $0 \subseteq S \subseteq M$ $\qquad$ $0 \subseteq T \subseteq M$

Composition Factors: $S, T = M/S$ $\qquad$ $T, S = M/T$

$\therefore$ composition series equivalent + have same composition factors

27. Not all modules have composition series:

${}_{\mathbb{Z}}\mathbb{Z}$ has no simple submodules since simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$, but $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$, hence $\mathbb{Z}/p\mathbb{Z}$ not a submodule of $\mathbb{Z}$

$\therefore {}_{\mathbb{Z}}\mathbb{Z}$ has no composition series

28. Artinian module $\not\Rightarrow$ Noetherian module

29. ${}_RR \leq {}_RM \Rightarrow {}_RM$ faithful

For instance ${}_{\mathbb{Z}}\mathbb{Z} \leq {}_{\mathbb{Z}}\mathbb{Q} \Rightarrow {}_{\mathbb{Z}}\mathbb{Q}$ faithful

30. $I \trianglelefteq R$, M R-module $\Rightarrow IM \leq M$

And $I(M/IM) = 0$ since $i(m+IM) = im + IM = IM = 0_{M/IM}$

$\therefore I \subseteq \mathrm{ann}_R M/IM$

$\therefore M/IM$ R/I-module via $(r+I)(m+IM) = rm + IM$

31. R semisimple $\Rightarrow$ rad $R = 0$: $R = D_1 \times D_2 \times D_3$, D_i division rings

$\Rightarrow$ R semisimple

Maximal 2-sided ideals: $I_1 = D_1 \times D_2 \times 0$, $I_2 = D_1 \times 0 \times D_3$,
$I_3 = 0 \times D_2 \times D_3$

$\therefore I_1 \cap I_2 \cap I_3 = 0$

$\therefore \operatorname{rad} R = 0$

32. $R = M_n(D)$, D division ring

$S_i = \begin{bmatrix} 0 & D & 0 \\ & \vdots & \\ & D & \end{bmatrix}$, $m_i = \begin{bmatrix} * & 0 & * \\ & \vdots & \\ & 0 & \end{bmatrix}$ (column (i))

$\Rightarrow R = m_i \oplus S_i$

Each m_i maximal left ideal

$J(R) = \bigcap m_i = 0$

$\therefore$ In this case $J(R) = \operatorname{rad} R$ (Not Always)

33. $R = \mathbb{Z}$, $J(\mathbb{Z}) = \bigcap_{p \text{ prime}} \langle p \rangle = 0$

34. R local, commutative ring $\Rightarrow$ R has unique maximal ideal M

$\therefore J(R) = M$

For instance: R commutative ring, $P \triangleleft R$ prime

$S = R \setminus P$

$\Rightarrow S^{-1}R$ local with max ideal $\{\frac{x}{s} \mid x \in P, s \notin P\} = P_S$

$\therefore J(R) = P_S$

35. Finitely generated Required in Nakayama's lemma:

$R = \{\frac{a}{b} \in \mathbb{Q} \mid b \text{ odd}\}$, $P = \langle 2 \rangle$

$\bar{P} = \{\frac{2a}{b} \in \mathbb{Q} \mid b \text{ odd}\}$ unique maximal ideal of R because

$\bar{P} = PR$

$\therefore J(R) = \bar{P}$

$\bar{P}\mathbb{Q} = \mathbb{Q}$

This doesn't contradict Nakayama since $\mathbb{Q}$ not finitely generated

36. $\mathbb{Z} \overset{ess}{\subseteq} \mathbb{Q}$ over $\mathbb{Z}$ because if $0 \neq L \leq \mathbb{Q}$, $\frac{a}{b} \in L \Rightarrow a = b \cdot \frac{a}{b} \in L \cap \mathbb{Z}$

37. $M = S_1 \oplus S_2$ semisimple

$S_1 \overset{ess}{\nsubseteq} S_2$ because $S_2 \cap S_1 = 0$

38. $\operatorname{Soc}_{\mathbb{Z}} \mathbb{Z} = 0$ because only simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \nsubseteq \mathbb{Z}$

39. $E(\mathbb{Z}) = \mathbb{Q}$ because $\mathbb{Z} \overset{ess}{\subseteq} \mathbb{Q}$ and $\mathbb{Q}$ injective since divisible (as $\mathbb{Z}$-modules)

40. $\nexists$ projective cover for $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}$-module:

$\mathbb{Z}$ is only candidate (${}_{\mathbb{Z}}\mathbb{Z}$ projective)

But
$$\begin{tikzcd} \mathbb{Z} \arrow[d, "3"'] \arrow[dr, "\pi"] & & 0 \\ \mathbb{Z} \arrow[r, "\pi"'] & \mathbb{Z}/2\mathbb{Z} \arrow[ur] \arrow[r] & 0 \end{tikzcd}$$
3 is not an iso

Hence π not right minimal, hence $\mathbb{Z}$ not projective cover

41. $\text{udim}_{\mathbb{R}}\,\mathbb{Q} = \infty$: $\{e_i\}_{i\in I}$ basis for $\mathbb{Q}$ over $\mathbb{R} \Rightarrow \{e_i\}_{i\in I}$ infinite

And $\mathbb{Q} = \bigoplus_{i\in I} \langle e_i \rangle$

42. M artinian module $\Rightarrow 0 \neq \text{soc}M = \oplus$ all simple submodules of M

Say $\text{soc}M = S_1 \oplus \dots \oplus S_k$ (k finite since $\text{soc}M$ artinian)

clearly $\text{udim}\,M \geq k$

suppose $\exists m > k \ni L_1 \oplus \dots \oplus L_m \leq M$

Each L_i artinian, hence each L_i has simple submodules

$\therefore$ M has at least m simple submodules

contradiction since $\text{soc}M$ is sum of only k simple submodules

$\therefore$ $\text{udim}\,M = k$

43. Indecomposable $\not\Rightarrow$ uniform

44. $\langle 0 \rangle \triangleleft \mathbb{Z}$ minimal prime

45. $R = \mathbb{Z}$, $Q = \langle 4 \rangle$

Q primary:

Let $ab \in Q \Rightarrow ab = 4k \Rightarrow 2 | ab \Rightarrow 2|a$ or $2|b$ since 2 prime

if $2|a$, $4 | a^2 \Rightarrow a^2 \in Q$

if $2|b$, $4 | b^2 \Rightarrow b^2 \in Q$

46. $R = \mathbb{Z}$, $Q \triangleleft \mathbb{Z}$

Q primary iff $Q = \langle p^n \rangle$ for some prime $\# p$

47. primary $\not\Rightarrow$ irreducible

48. Categories: SETS: ob: sets

mor: functions

AB: ob: abelian groups

mor: group homomorphisms

VEC_K: ob: finite dimensional vector spaces over K

mor: K-linear tranoformations

GROUPS: ob: groups

mor: grouphomomorphisms

RINGS: ob: rings

mor: ring homomorphisms

TOP: ob: topological spaces

mor: continuous maps

49. Poset: ob: points — category

mor: arrows $p \longrightarrow q$ iff $p \leq q$

50. In R-MOD, epi and monic ≡ isomorphism, but not true in general category

51. Isomorphisms: SET: bijections

R-MOD: isomorphism of R-modules

TOP: homeomorphisms

52. AB full subcategory of GROUPS

53. Identity Functor: $id: \mathcal{C} \longrightarrow \mathcal{C} \ni id(C) = C$, $id(f) = f$

54. $F(-) = M \otimes_R -$ functor $R\text{-MOD} \xrightarrow{F} AB \ni F(N) = M \otimes_R N$,

$F(f) = 1 \otimes f$

55. $F(-) = Hom_R(M, -)$ functor $R\text{-MOD} \xrightarrow{F} AB \ni F(N) = Hom_R(M,N)$,

$F(f) = f_*$

56. Forgetful Functor: $R\text{-MOD} \xrightarrow{u} AB \xrightarrow{u} SETS$

$$M \longrightarrow M \longrightarrow M$$

$$f \longrightarrow f \longrightarrow f$$

u forgets module structure, then group structure

57. $F(-) = Hom_R(-, M)$ contravariant functor $\ni F(N) = Hom_R(N,M)$

$F(f) = f^*$, $F: R\text{-MOD} \rightarrow AB$

58. $F: \mathcal{C} \longrightarrow$ skeleton of $\mathcal{C}$, skeleton of $\mathcal{C}$ = 1 representative for each iso class

F fully faithful

59. Initial/Terminal/Zero objects: SETS: ϕ initial

$\{*\}$ terminal (singleton)

∴ No zero object

AB, R-MOD: 0 is zero object

60. ~~Kernel~~ $\not\Rightarrow$ monic in GROUPS

61. SETS $\underset{G}{\overset{F}{\leftrightarrows}}$ VEC$_k$

$F(x)$ = vectorspace with basis x

$G(V) = U(V) = V$ forgetfull functor

(F,G) adjoint pair

62. Colimits: coproducts, cokernels, unions, pushouts

63. Limits: Products, kernels, intersections, pullbacks

64. Find colim $\left(\begin{array}{c} M_1 \\ \downarrow \varphi_2^1 \\ M_2 \end{array} \right)$

$$\begin{tikzcd}
M_1 \arrow[d, "\varphi"'] \arrow[dr, "\varphi"] \arrow[drr, bend left, "\forall f_1"] & & \\
M_2 \arrow[r, "1_{M_2}"'] \arrow[rr, bend right, "\forall f_2"'] & M_2 \arrow[r, dashed, "f_2"] & T
\end{tikzcd}$$

$\therefore$ colim$(\quad) = M_2$

65. Find colim $(U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \ldots)$

$$\begin{tikzcd}
U_1 \arrow[d, hook] \arrow[dr, hook] \arrow[drr, bend left, "\forall f_1"] & & \\
U_2 \arrow[d, hook] \arrow[r, hook] \arrow[rr, bend right=60, "\forall f_2"'] & \bigcup_{n=1}^{\infty} U_n \arrow[r, dashed, "f"] & T \\
U_3 \arrow[d, hook] \arrow[ur, hook] \arrow[urr, bend right, "\forall f_3"'] & & \\
\vdots & &
\end{tikzcd}$$

$f(u_n) = f_n(u_n)$

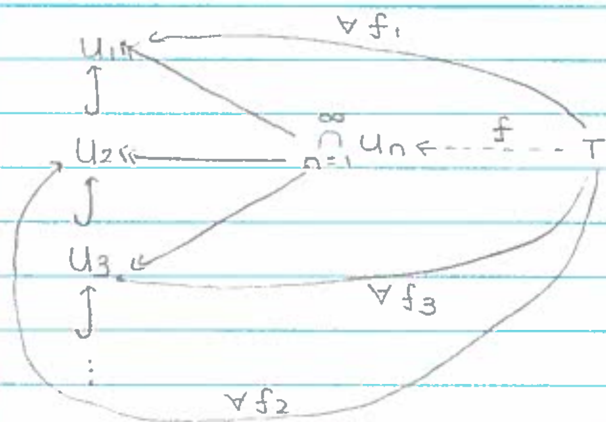
$\therefore$ colim$(\quad) = \bigcup_{n=1}^{\infty} U_n$

66. k, A, B commutative rings $\Rightarrow$ pushout $\left(\begin{tikzcd} k \arrow[r] \arrow[d] & A \\ B & \end{tikzcd} \right) = A \otimes_k B$

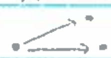
67. RINGS not abelian

68. $\{M_i\}_{i \in I}$ system with no maps between M_i's $\Rightarrow$ limit is $\prod_{i \in I} M_i$

69. Find lim $(U_1 \longleftrightarrow U_2 \longleftrightarrow U_3 \longleftrightarrow \ldots)$



$\therefore \lim(\quad) = \bigcap_{n=1}^{\infty} U_n$

70.  NOT DIRECTED — DIRECTED 

$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$

DIRECTED

71. $f \in R$, $R_f = \varinjlim \left(R \xrightarrow{f} R \xrightarrow{f} R \longrightarrow \cdots \right)$ Directed limit

72. $S^{-1}R$ flat:

Let $0 \to M \to N \to L \to 0$ SES

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_R S^{-1}R \arrow[r] \arrow[d, "\cong"] & N \otimes_R S^{-1}R \arrow[r] \arrow[d, "\cong"] & L \otimes_R S^{-1}R \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & S^{-1}M \arrow[r] & S^{-1}N \arrow[r] & S^{-1}L \arrow[r] & 0
\end{tikzcd}$$

commutes and bottom row exact $\Rightarrow$ top row exact

73. flat $\not\Rightarrow$ free, projective: For instance, $S^{-1}R$

74. A SES is a complex

75. $\cdots \longrightarrow \mathbb{Z}/8\mathbb{Z} \xrightarrow{4} \mathbb{Z}/8\mathbb{Z} \xrightarrow{4} \mathbb{Z}/8\mathbb{Z} \longrightarrow \cdots$ complex because

$d^2 = 4^2 = 16 = 0$ $\mathbb{Z}/8\mathbb{Z}$

76. Compute homology: $\cdots \xrightarrow{4} \mathbb{Z}/12\mathbb{Z} \xrightarrow{3} \mathbb{Z}/12\mathbb{Z} \xrightarrow{4} \mathbb{Z}/12\mathbb{Z} \xrightarrow{3} \mathbb{Z}/12\mathbb{Z} \to 0$

$H_0(C) = \mathbb{Z}/12\mathbb{Z} \big/ 3\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$

$H_1(C) = 4\mathbb{Z}/12\mathbb{Z} \big/ 4\mathbb{Z}/12\mathbb{Z} \cong 0$

$H_2(C) = 3\mathbb{Z}/12\mathbb{Z} \big/ 3\mathbb{Z}/12\mathbb{Z} \cong 0$

$\vdots$

77. Quasi-iso:

$$\begin{tikzcd}
\cdots \arrow[r, "3"] & \mathbb{Z}/12\mathbb{Z} \arrow[r, "4"] \arrow[d] & \mathbb{Z}/12\mathbb{Z} \arrow[r, "3"] \arrow[d] & \mathbb{Z}/12\mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

chain map since squares commute

Sequences have isomorphic homologies

And $\mathbb{Z}/3\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z}$ isomorphism

78. Not every complex is quasi-iso to its homology:

For instance, $0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

have isomorphic homologies but not quasi-iso ie $\not\exists\, f: C \to D$ inducing isomorphism on homologies

79. $R = \mathbb{Z}[x,y]/(xy)$

$C_\cdot: \dots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow 0$

$C_\cdot \xrightarrow{x} C_\cdot$

$\underline{x \simeq 0}$:

$$\begin{tikzcd}
\dots \arrow[r, "y"] & R \arrow[r, "x"] \arrow[d, "x"] \arrow[dl, dashed, "s=1"'] & R \arrow[r, "y"] \arrow[d, "x"] \arrow[dl, dashed, "s=1"'] & R \arrow[r, "x"] \arrow[d, "x"] \arrow[dl, dashed, "s=0"'] & R \arrow[r] \arrow[d, "x"] \arrow[dl, dashed, "s=1"'] & 0 \arrow[dl, dashed, "s=0"'] \\
\dots \arrow[r, "y"] & R \arrow[r, "x"] & R \arrow[r, "y"] & R \arrow[r, "x"] & R \arrow[r] & 0
\end{tikzcd}$$

$x = sd + ds \ \forall n$ where $s = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

or $s = x \ \forall n$ works

$\therefore$ Homotopy not unique!

80. $R = k[x,y]$, $M = R/(x,y) \cong k$

Compute projective resolution of M:

$$\begin{tikzcd}
 & {\begin{bmatrix} y \\ -x \end{bmatrix}} \arrow[dr] & & & & \\
0 \arrow[r] & R \arrow[r, "{\begin{bmatrix} y \\ -x \end{bmatrix}}"] \arrow[u] & R^2 \arrow[r, "{[x\ y]}"] \arrow[dr] & R \arrow[r, "\pi"] & M \arrow[r] & 0 \\
 & & & (x,y) \arrow[u] & &
\end{tikzcd}$$

$[x\ y]\begin{bmatrix} a \\ b \end{bmatrix} = 0$ iff $xa + yb = 0$ iff $a = y$ and $b = -x$ iff $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$

$\begin{bmatrix} y \\ -x \end{bmatrix} a = 0$ iff $\begin{bmatrix} ya \\ -xa \end{bmatrix} = 0$ iff $a = 0$

81. $R = k[x,y]/(x,y)$, $M = R/(x)$

Compute projective resolution of M:

$$\begin{tikzcd}
 & & & (y) \arrow[dr] & & & & \\
\dots \arrow[r, "y"] & R \arrow[r, "x"] \arrow[dr] & R \arrow[r, "y"] \arrow[ur] & R \arrow[r, "x"] \arrow[dr] & R \arrow[r, "\pi"] & M \arrow[r] & 0 \\
 & & (x) \arrow[u] & & (x) \arrow[u] & &
\end{tikzcd}$$

82. R ring, f nonzerodivisor in R, $M = R/(f^2)$, $N = R/(f)$

compute $\mathrm{Tor}_n^R(M,N)$

$P_\cdot: \quad 0 \longrightarrow R \xrightarrow{f^2} R \xrightarrow{\pi} M \longrightarrow 0$

$P_\cdot \otimes_R N: \quad 0 \longrightarrow R \otimes_R N \xrightarrow{f^2} R \otimes_R N \longrightarrow 0$

$0 \longrightarrow N \xrightarrow{f^2} N \longrightarrow 0$

$0 \longrightarrow N \xrightarrow{0} N \longrightarrow 0$.

$\mathrm{Tor}_0^R(M,N) = H_0(P_\cdot \otimes_R N) = N/0 \cong N = R/(f)$

$\mathrm{Tor}_1^R(M,N) = H_1(P_\cdot \otimes_R N) = N/0 \cong N = R/(f)$

$\mathrm{Tor}_n^R(M,N) = 0 \ \forall n \geq 2$

83. Compute $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z})$, $k \geq 2$:

Note over PID, injective $\Longleftrightarrow$ divisible (Always $\Rightarrow$)

$I^\cdot: \quad 0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, I^\cdot): \quad 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Q}) \xrightarrow{\pi} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to$

$0 \longrightarrow 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Q}) = \{\text{maps } \mathbb{Z} \longrightarrow \mathbb{Q} \text{ with } k\mathbb{Z} \subseteq \ker\}$

$= 0$

$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}) = H^0(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, I^\cdot)) = 0/0 \cong 0$

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}) = H^1(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, I^\cdot)) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Q}/\mathbb{Z})/0$

$\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

84. $F(-) = \mathrm{Hom}_R(-,N)$ left exact, contravariant

$\therefore R_nF(M) = H^n(\mathrm{Hom}_R(P_\cdot, N))$ where $P_\cdot \longrightarrow M$ projective resolution

85. Compute $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z})$, $k \geq 2$:

$P_\cdot: \quad 0 \longrightarrow \underset{(1)}{\mathbb{Z}} \xrightarrow{k} \underset{(0)}{\mathbb{Z}} \xrightarrow{\pi} \mathbb{Z}/k\mathbb{Z} \longrightarrow 0$

$\mathrm{Hom}_{\mathbb{Z}}(P_\cdot, \mathbb{Z}): \quad 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{k^*} \underset{(1)}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})} \longrightarrow 0$

$0 \longrightarrow \underset{(0)}{\mathbb{Z}} \xrightarrow{k} \underset{(1)}{\mathbb{Z}} \longrightarrow 0$

$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}) = H^0(\mathrm{Hom}_{\mathbb{Z}}(P_\cdot, \mathbb{Z})) = 0/0 \cong 0$

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$

86. Hereditary rings: $\mathbb{Z}$, PID's, finite dimensional k-algebras where k field

87. $\mathcal{D}(\mathcal{A})$ almost never abelian

88. Cohomological Functor: zeroth homology $H^0: K(\mathcal{A}) \longrightarrow \mathcal{A}$, $H^0: \mathcal{D}(\mathcal{A}) \to \mathcal{A}$

since $H^0(W[i]) = H^i(W)$ gives LES

89. $E = Y_\cdot \xleftarrow{f} X_\cdot$

$\mathrm{Tot}(E) = C(f)$

SES: $0 \longrightarrow Y \longrightarrow C(f) \longrightarrow X[-1] \longrightarrow 0$

Get LES: $\cdots \longrightarrow H_n(X) \xrightarrow[f]{H_n(f)} H_n(Y) \longrightarrow H_n(C(f)) \longrightarrow H_{n-1}(X) \xrightarrow[f]{H_{n-1}(f)} \cdots$

so $0 \longrightarrow \underbrace{\mathrm{coker}(H_n(f))}_{A} \longrightarrow H_n(\mathrm{Tot}E) \longrightarrow \underbrace{\ker(H_{n-1}(f))}_{B} \longrightarrow 0$ exact

Filtration:
$$\begin{array}{c} H_n(\mathrm{Tot}E) \\ \cup\!| \\ A \\ \cup\!| \\ 0 \end{array} \quad \begin{array}{l} \Big] \cong B \\ \\ \Big] \cong A \end{array}$$

Can track this data with spectral sequences:

$E_0 = E$

$E_1 =$ Taking homology on verticals:
$$\begin{array}{ccccc} \vdots & & \vdots & & \\ H_1(Y) & \longleftarrow & H_1(X) & \longleftarrow & 0 \\ & & & & \\ H_0(Y) & \longleftarrow & H_0(X) & \longleftarrow & 0 \end{array}$$

$E_2 =$ Taking homology on horizontals:
$$\begin{array}{ccc} \mathrm{coker}\,H_2(f) & & 0 \\ \mathrm{coker}\,H_1(f) & \ker H_1(f) & 0 \\ & & \\ \mathrm{coker}\,H_0(f) & \ker H_0(f) & 0 \end{array}$$

Continuing, homology gives same maps each time

so $E_2 = E_3 = \cdots = E_\infty$

Hence $H_n(\mathrm{Tot}E)$ has filtration with factors A, B as above

Algebra Qualifying Examination
Rings and Modules Part - MAT 731

February 2015

Instructions: Do as many problems as possible in the time allotted. In what follows, R is an associative ring with unity, all R-modules are unitary left modules, and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of R-modules.

1. We say that a homomorphisms $h : A \to M$ of R-modules *can be extended to* B if there exists a homomorphism $\hat{h} : B \to M$ satisfying $h = \hat{h}f$ (draw a diagram).

For an R-module M prove that the following statements are logically equivalent. You may use a long exact sequence for the functor Ext.

(a) Every homomorphism $A \to M$ can be extended to B.

(b) The sequence $0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{\mathrm{Hom}_R(g,M)} \mathrm{Hom}_R(B, M) \xrightarrow{\mathrm{Hom}_R(f,M)} \mathrm{Hom}_R(A, M) \longrightarrow 0$ of abelian groups is exact.

(c) The map $\mathrm{Ext}^1_R(C, M) \xrightarrow{\mathrm{Ext}^1_R(g,M)} \mathrm{Ext}^1_R(B, M)$ is a monomorphism of abelian groups.

Recall that a monomorphism $u : L \to M$ of R-modules is *essential* if, for all homomorphisms $v : M \to N$ of R-modules, vu is a monomorphism if and only if v is a monomorphism. You may use the fact that if L is a submodule of M, then the inclusion $L \to M$ is an essential monomorphism if and only if $L \cap X \neq 0$ for all nonzero submodules X of M.

2. If $X \neq 0$ is a submodule of an indecomposable injective R-module I, prove that the inclusion $i : X \to I$ is an essential monomorphism. You may use the existence of an *injective envelope* of X, i.e., of an essential monomorphism $j : X \to J$ where J is an injective R-module. *Hint:* what can you say about a homomorphism $k : J \to I$ satisfying $i = kj$?

3. Let I be an indecomposable injective R-module and let u and v be R-endomorphisms of I that are not automorphisms. Using Problem 2, prove that $u + v$ is not an automorphism of I.

4. For a positive integer m, set $\mathbb{Z}_m = \mathbb{Z}/\mathbb{Z}m$ and consider the exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{u} \mathbb{Z} \xrightarrow{v} \mathbb{Z}_m \to 0$ where u is multiplication by m and v is the natural projection. For a positive integer n, denote by d the greatest common divisor of m and n.

(a) Compute the kernel of the homomorphism $v \otimes 1_{\mathbb{Z}_n} : \mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \to \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$ of abelian groups.

(b) Prove that $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_d$.

5. Suppose the ring R is left artinian and denote by J the Jacobson radical of R.

(a) Show that J^k/J^{k+1} is a semisimple module of finite length for all $k \geq 0$. (By definition $J^0 = R$.)

(b) Use the fact that J is nilpotent to show that R must be left Noetherian.

February 2015 - 731

1. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a SES of R-modules.
For an R-module M, prove that TFAE:
(a) Every homomorphism $A \longrightarrow M$ can be extended to B
(b) The sequence
$0 \rightarrow Hom_R(C,M) \xrightarrow{g^*} Hom_R(B,M) \xrightarrow{f^*} Hom_R(A,M) \longrightarrow 0$
of abelian groups is exact.
(c) $Ext_R^1(C,M) \longrightarrow Ext_R^1(B,M)$ is a monomorphism

(a)$\Rightarrow$(b) Assume every homomorphism $A \xrightarrow{h} M$ can be extended to B
Then we have the commutative diagram $\forall h$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "\exists j"] \\ & M & \end{tikzcd}$$

$\therefore$ M is injective
$\therefore Hom_R(\_, M)$ is exact
$\therefore$ We have SES:
$0 \longrightarrow Hom_R(C,M) \longrightarrow Hom_R(B,M) \longrightarrow Hom_R(A,M) \longrightarrow 0$
(b)$\Rightarrow$(c) Assume $Hom_R(\_,M)$ is exact
Then we have LES:
$0 \rightarrow Hom_R(C,M) \xrightarrow{g^*} Hom_R(B,M) \xrightarrow{f^*} Hom_R(A,M) \xrightarrow{\beta} Ext_R^1(C,M) \xrightarrow{\alpha} Ext_R^1(B,M) \rightarrow$
Since $Hom_R(\_,M)$ exact, f^* must be surjective
So $Hom_R(A,M) = Im f^* = Ker \beta$ by exactness
But then β must be the zero map
And $Ker \alpha = Im \beta = 0$
$\therefore \alpha$ injective
$\therefore Ext_R^1(C,M) \longrightarrow Ext_R^1(B,M)$ monomorphism
(c)$\Rightarrow$(a) Assume $Ext_R^1(C,M) \longrightarrow Ext_R^1(B,M)$ monomorphism
Using the LES above, $0 = Ker \alpha = Im \beta$
So $Im f^* = Ker \beta = Hom_R(A,M)$
$\therefore f^*$ surjective

$\therefore\ 0 \longrightarrow Hom_R(C,M) \longrightarrow Hom_R(B,M) \longrightarrow Hom_R(A,M) \longrightarrow 0$
is exact

$\therefore\ Hom_R(\_,M)$ is exact

$\therefore$ M is injective

$\therefore$ Every homomorphism $A \longrightarrow M$ can be extended to B

2. If $X \neq 0$ is a submodule of an indecomposable injective R-module I, prove that the inclusion $i : X \longrightarrow I$ is an essential monomorphism.

Show that $I = E(x)$

First note that injective envelopes exist, so let $J = E(X)$

Then we have an essential monomorphism $X \xrightarrow{j} J$

Then we have the diagram:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "j"] \arrow[d, "i"'] & J \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

Since I is injective, $\exists h : J \longrightarrow I \ni hj = i$

But i is a monomorphism, so hj is a monomorphism

And since j is an essential monomorphism, h must be a monomorphism by definition of essential monomorphism

So we have a monomorphism $J \xrightarrow{h} I$

Then since J is injective, the map splits, hence J is a direct summand of I

$\therefore\ I \cong J \oplus Y$ for some $Y \leq I$

But I is indecomposable, so $Y = 0$

$\therefore\ I \cong J = E(x)$

$\therefore\ I = E(x)$

$\therefore\ i : x \longrightarrow I$ is an essential monomorphism

3. Let I be an indecomposable injective R-module and let u, v be R-endomorphisms of I which are not automorphisms. Prove that $u+v$ is not an automorphism of I.

4. For $m>0$, consider the exact sequence of abelian groups
$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$. For $n>0$, let $d=(m,n)$.

a. Compute $\operatorname{Ker}(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\pi \otimes 1} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$

First note that $\_ \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is right exact, so we get the exact sequence:

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{m \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\pi \otimes 1} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

So $\operatorname{Ker}(\pi \otimes 1) = \operatorname{Im}(m \otimes 1) \cong \operatorname{Im}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z})$ since $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$

And $\operatorname{Im}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}) = m\,\mathbb{Z}/n\mathbb{Z}$

So $\operatorname{Ker}(\pi \otimes 1) \cong m\,\mathbb{Z}/n\mathbb{Z}$

b. Prove that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$

Since $\_ \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is right exact, we get that $\pi \otimes 1$ is surjective

So by 1st Iso Thm, $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \big/ \operatorname{Ker}(\pi \otimes 1)$

$\cong \mathbb{Z}/n\mathbb{Z} \big/ m\,\mathbb{Z}/n\mathbb{Z}$ by (a)

$\cong \mathbb{Z}/(n,m)$

But we know that the ideal $(n,m) = d\mathbb{Z}$

$\therefore \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$

5. Suppose the ring R is left artinian and $J = J(R)$.

a. Show that J^k/J^{k+1} is a semisimple module of finite length $\forall k \geq 0$

Note $J(J^k/J^{k+1}) = J^{k+1}/J^{k+1} \cong 0$

So $J \subseteq \mathrm{ann}_R J^k/J^{k+1}$, hence J^k/J^{k+1} is R/J-module

But R left artinian $\Rightarrow R/J$ is semisimple ring

Then every left R/J-module is semisimple

In particular, J^k/J^{k+1} is a semisimple module $\forall k \geq 0$

But since R is left artinian, R has only finitely many nonisomorphic simple left modules

So $J^k/J^{k+1} = S_1 \oplus \dots \oplus S_t$ is a finite direct sum of simple modules

Then $0 \subsetneq S_1 \subsetneq S_1 \oplus S_2 \subsetneq \dots \subsetneq S_1 \oplus \dots \oplus S_t = J^k/J^{k+1}$ is a composition series since each factor $S_1 \oplus \dots \oplus S_i / S_1 \oplus \dots \oplus S_{i-1} \cong S_i$ which is a simple module

$\therefore J^k/J^{k+1}$ has finite length

b. Show that R is left Noetherian.

By (a) J^k/J^{k+1} has finite length, so J^k/J^{k+1} is both artinian and noetherian $\forall k \geq 0$

Note that since R is left artinian, J is nilpotent, say $J^m = 0$

Then $J^{m-1} \cong J^{m-1}/J^m$

$\therefore J^{m-1}$ is both artinian and noetherian

And we have the SES: $0 \to J^{m-1} \hookrightarrow J^{m-2} \to J^{m-2}/J^{m-1} \to 0$

But J^{m-1}, J^{m-2}/J^{m-1} are both noetherian, so J^{m-2} must also be noetherian

And we have the SES: $0 \to J^{m-2} \hookrightarrow J^{m-3} \to J^{m-3}/J^{m-2} \to 0$

with J^{m-2}, J^{m-3}/J^{m-2} both noetherian

Hence J^{m-3} noetherian

Continuing in this way, we get the SES:

$$0 \longrightarrow J^2 \hookrightarrow J \longrightarrow J/J^2 \longrightarrow 0$$

with J^2, J/J^2 both noetherian

$\therefore$ J is noetherian

And we have the SES: $0 \longrightarrow J \hookrightarrow R \longrightarrow R/J \longrightarrow 0$

And $R/J = J^0/J^1$ noetherian, and J noetherian

$\therefore$ R is noetherian left R-module

$\therefore$ R is a left noetherian ring

ALGEBRA QUALIFYING EXAM
MAT 732 PART

FEBRUARY 2015

Solve any four of the five problems.

Assume that rings are associative and have identity elements. Modules are left modules unless otherwise stated.

1. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of modules over a ring R. Prove that there exist short exact sequences
$$0 \longrightarrow B \longrightarrow Q \longrightarrow D \longrightarrow 0,$$
$$0 \longrightarrow A \longrightarrow Q \longrightarrow E \longrightarrow 0,$$
$$0 \longrightarrow C \longrightarrow E \longrightarrow D \longrightarrow 0$$
where D, E and Q are R-modules with Q injective.
2. Let R be a ring.
   (a) Define the *pushout* of a pair of homomorphisms of R-modules, say $f\colon A \longrightarrow B$ and $\phi\colon A \longrightarrow M$.
   (b) Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R-modules, and $\phi\colon A \longrightarrow M$ is homomorphism of R-modules. Show that there is a short exact sequence $0 \longrightarrow M \xrightarrow{p} P \xrightarrow{q} C \longrightarrow 0$, where P is the pushout of f and ϕ.
3. Let R be a commutative ring and M a flat R-module. Prove that the following are equivalent:
   (a) M is faithfully flat, that is, $X \otimes_R M \neq 0$ for every non-zero R-module X.
   (b) $X \otimes_R M \neq 0$ for every non-zero *cyclic* R-module X.
   (c) $M \neq \mathfrak{m}M$ for every maximal ideal $\mathfrak{m}$ of R.
4. Let k be a commutative ring and R a k-algebra, not necessarily commutative, which is flat as a k-module. Let M be a k-module and N an R-module (and therefore also a k-module). Prove that $\operatorname{Tor}_n^R(M \otimes_k R,\ N) \cong \operatorname{Tor}_n^k(M,\ N)$ for every $n \geq 0$.
5. Let R be a ring and $f\colon C \longrightarrow D$ a chain map between chain complexes of R-modules.
   (a) Show that f preserves cycles and boundaries and hence induces a homomorphism $H_n(f)\colon H_n(C) \longrightarrow H_n(D)$ for every n.
   (b) Recall that f is a *quasi-isomorphism* if $H_n(f)$ is an isomorphism for every n. Also recall that $g\colon D \longrightarrow C$ is a *quasi-inverse* for f if, for every n, $H_n(g)$ is inverse to $H_n(f)$. Give an explicit example of a quasi-isomorphism of chain complexes of abelian groups which does not admit a quasi-inverse (provide a proof).

February 2015 - 732

1. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a SES of R-modules.
Prove that $\exists$ SES's

$$0 \longrightarrow B \longrightarrow Q \longrightarrow D \longrightarrow 0$$
$$0 \longrightarrow A \longrightarrow Q \longrightarrow E \longrightarrow 0$$
$$0 \longrightarrow C \longrightarrow E \longrightarrow D \longrightarrow 0$$

where D, E, Q are R-modules with Q injective

First note that $\exists Q$ injective $\ni 0 \longrightarrow B \xrightarrow{i} Q$ injective
Take $D = \operatorname{Coker} i$
Then $0 \longrightarrow B \xrightarrow{i} Q \xrightarrow{\pi} D \longrightarrow 0$ SES
Now $0 \longrightarrow A \xrightarrow{if} Q$ injective since both i, f injective
And take $E = \operatorname{coker} if$
Then $0 \longrightarrow A \xrightarrow{if} Q \xrightarrow{\pi'} E \longrightarrow 0$ SES
So we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"] & B \arrow[r, "g"] \arrow[d, "i"] & C \arrow[r] \arrow[d, dashed, "\exists h"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r, "if"] \arrow[d, "f"] & Q \arrow[r, "\pi'"] \arrow[d, "1_Q"] & E \arrow[r] \arrow[d, dashed, "\exists j"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & B \arrow[r, "i"] & Q \arrow[r, "\pi"] & D \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Then $\exists h: C \longrightarrow E$, $j: E \longrightarrow D$ commuting the diagram
And h is injective by the 5-Lemma since i injective
And j is surjective by 5-Lemma since 1_Q is surjective and f is injective
Now $jhg = \pi 1_Q i = \pi i = 0$ by exactness
$\therefore jh = 0$
$\therefore \operatorname{Im} h \subseteq \operatorname{Ker} j$
And let $x \in \operatorname{Ker} j \Rightarrow 0 = j(x) = j(\pi'(q))$ since π' is surjective
$= \pi(1_Q(q)) = \pi(q)$
$\therefore q \in \operatorname{Ker} \pi = \operatorname{Im} i \Rightarrow q = i(b) \Rightarrow x = \pi'(q) = \pi'(i(b)) = h(g(b)) \in \operatorname{Im} h$
$\therefore \operatorname{Ker} j = \operatorname{Im} h$, hence $0 \longrightarrow C \xrightarrow{h} E \xrightarrow{j} D \longrightarrow 0$ SES

2. Let R be a ring.

a. Define the pushout of a pair of homomorphisms of R-modules, say $f: A \longrightarrow B$ and $\phi: A \longrightarrow M$

The pushout of f, ϕ is the colimit of the system

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \\ M & \end{tikzcd}$$

ie an object L with maps $i, j \ni if = j\phi \ni \forall g, h$ with $gf = h\phi$, $\exists! \theta: L \longrightarrow T$ commuting the diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "i"] \arrow[ddr, bend left, "\forall g"] & \\ M \arrow[r, "j"] \arrow[drr, bend right, "\forall h"'] & L \arrow[dr, dashed, "\exists! \theta"] & \\ & & T \end{tikzcd}$$

In R-MOD, the pushout $L = B \oplus M / \{(f(a), -\phi(a))\}_{a \in A}$

b. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a SES of R-modules and $\phi: A \longrightarrow M$ a homomorphism of R-modules. Show that $\exists$ SES of R-modules $0 \longrightarrow M \longrightarrow P \longrightarrow C \longrightarrow 0$ where P is the pushout of f, ϕ.

Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[r, "g"] \arrow[d, "i"] \arrow[ddr, bend left, "g"] & C \arrow[r] & 0 \\ & M \arrow[r, "j"] \arrow[drr, bend right, "0"'] & P \arrow[dr, dashed, "\exists! \theta"] & & \\ & & & C & \end{tikzcd}$$

Since $gf = 0 = 0\phi$, $\exists! \theta: P \longrightarrow C$ by definition of pushout commuting the diagram

Now j is injective since f is injective and θ is surjective since g surj $\Rightarrow \theta i$ surjective $\Rightarrow \theta$ surjective

And $\theta j = 0$ by commutativity, hence $\operatorname{Im} j \subseteq \operatorname{Ker}\theta$

Now let $x \in \operatorname{Ker}\theta \Rightarrow \theta(x) = 0$

But $x \in P \Rightarrow x = (b, m) + \{(f(a), -\phi(a))\}_{a\in A}$

Then $0 = \theta(x) = g(b)$ since g surjective and since $\theta c = g \Rightarrow$

$b \xrightarrow{c} x \xrightarrow{\theta} \theta(x)$ and $b \xrightarrow{g} g(b)$

so $b \in \operatorname{Ker} g = \operatorname{Im} f$

$\therefore b = f(a)$, $a \in A$

$\therefore x = (b, m) + \{(f(a), -\phi(a))\}_{a\in A} = (f(a), m) + \{(f(a), -\phi(a))\}_{a\in A}$

$= (f(a), -\phi(a)) + (0, \phi(a) + m) + \{(f(a), -\phi(a))\} = (0, \phi(a) + m) + \{\}$

$= j(\phi(a) + m)$

$\therefore \operatorname{Ker}\theta \subseteq \operatorname{Im} j$

$\therefore \operatorname{Im} j = \operatorname{Ker}\theta$

$\therefore 0 \longrightarrow M \xrightarrow{j} P \xrightarrow{\theta} C \longrightarrow 0$ SES

3. Let R be a commutative ring and M a flat R-module.

Prove that TFAE:

(a) M faithfully flat

(b) $X \otimes_R M \neq 0$ $\forall X \neq 0$ cyclic R-module

(c) $M \neq mM$ $\forall$ maximal ideal m of R

(a)$\Rightarrow$(b) Assume M faithfully flat

Then $X \otimes_R M \neq 0$ $\forall X \neq 0$ R-module

Hence $X \otimes_R M \neq 0$ $\forall X \neq 0$ cyclic R-module

(b)$\Rightarrow$(c) Assume $X \otimes_R M \neq 0$ $\forall X \neq 0$ cyclic R-module

Suppose that $M = mM$ for some $m \triangleleft R$ maximal

Then $0 = M/mM \cong R/m \otimes_R M$

But R/m is a field, hence a simple module over itself

$\therefore R/m$ cyclic

But $R/m \neq 0$ since m maximal, so $m \neq R$

contradiction to assumption

$\therefore M \neq mM$ $\forall m \triangleleft R$ maximal

(c)$\Rightarrow$(a) Assume $M \neq mM$ $\forall m \triangleleft R$ maximal

Let $0 \neq X$ be an R-module

Then $\exists 0 \neq x \in X$

Let $I = \operatorname{ann}_R x$

Then $R/I \cong Rx$

Let $m \triangleleft R$ maximal $\ni I \subseteq m$

Then $IM \subseteq mM \neq M$

So $M/IM \neq 0$

So $R/I \otimes_R M \neq 0 \Rightarrow Rx \otimes_R M \neq 0$

And $0 \longrightarrow Rx \hookrightarrow X$ monomorphism

And M is flat, so $0 \longrightarrow Rx \otimes_R M \hookrightarrow X \otimes_R M$ is a monomorphism

$\therefore X \otimes_R M \neq 0$ because if $X \otimes_R M = 0$ then $Rx \otimes_R M = 0$ since the map is a monomorphism but this would be a contradiction

$\therefore X \otimes_R M \neq 0 \quad \forall X \neq 0$ R-module

$\therefore M$ is faithfully flat

4. Let k be a commutative ring and R a k-algebra which is a flat k-module. Let M be a k-module and N an R-module. Prove that $\operatorname{Tor}_n^R(M \otimes_k R, N) \cong \operatorname{Tor}_n^k(M, N) \quad \forall n \geq 0$

First note that it suffices to use flat resolutions when computing Tor

Let $F_\bullet : \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be a flat resolution of M

Then the sequence is exact, hence the following sequence is exact since R is a flat k-module:

$\tilde{F}_\bullet : \ldots \longrightarrow F_1 \otimes_k R \longrightarrow F_0 \otimes_k R \longrightarrow M \otimes_k R \longrightarrow 0$

Now let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a SES of R-modules

Then we have the natural isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \\ 0 & \longrightarrow & R \otimes_R A & \longrightarrow & R \otimes_R B & \longrightarrow & R \otimes_R C & \longrightarrow & 0 \end{array}$$

So the diagram commutes and the 1st row is exact

Hence the 2nd row is exact

Now since F_i flat k-module, we have SES:

$$\begin{tikzcd}
0 \arrow[r] & F_i \otimes_k (R \otimes_R A) \arrow[r] \arrow[d, "\cong"] & F_i \otimes_k (R \otimes_R B) \arrow[r] \arrow[d, "\cong"] & F_i \otimes_k (R \otimes_R C) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (F_i \otimes_k R) \otimes_R A \arrow[r] & (F_i \otimes_k R) \otimes_R B \arrow[r] & (F_i \otimes_k R) \otimes_R C \arrow[r] & 0
\end{tikzcd}$$

And the isomorphisms are natural so the diagram commutes

Hence the 2nd row is exact

$\therefore$ $F_i \otimes_k R$ is a flat R-module for each i

$\therefore$ $\tilde{F}_\cdot$ is a flat resolution for $M \otimes_k R$

Now $F_\cdot \otimes_k N$: $\cdots \longrightarrow F_1 \otimes_k N \longrightarrow F_0 \otimes_k N \longrightarrow 0$

And $\tilde{F}_\cdot \otimes_R N$: $\cdots \longrightarrow (F_1 \otimes_k R) \otimes_R N \longrightarrow (F_0 \otimes_k R) \otimes_R N \longrightarrow 0$

And we have the natural isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & (F_1 \otimes_k R) \otimes_R N \arrow[r] \arrow[d, "\cong"] & (F_0 \otimes_k R) \otimes_R N \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & F_1 \otimes_k (R \otimes_R N) \arrow[r] \arrow[d, "\cong"] & F_0 \otimes_k (R \otimes_R N) \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & F_1 \otimes_k N \arrow[r] & F_0 \otimes_k N \arrow[r] & 0
\end{tikzcd}$$

So $H_n(F_\cdot \otimes_k N) \cong H_n(\tilde{F}_\cdot \otimes_R N)$ $\forall n \geq 0$

$\therefore$ $\mathrm{Tor}_n^R(M \otimes_k R, N) \cong \mathrm{Tor}_n^k(M, N)$ $\forall n \geq 0$

5. Let R be a ring and $f: C. \longrightarrow D.$ a chain map between complexes of R-modules.

a. Show that f preserves cycles and boundaries and hence induces a homomorphism $H_n(f): H_n(C) \longrightarrow H_n(D)$ $\forall n$

Consider the commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

First show $f_n(Z_n(C)) \subseteq Z_n(D)$

Let $x \in f_n(Z_n(C)) \Rightarrow x = f_n(y)$ for $y \in Z_n(C) = \ker d_n$

Then $d_n(y) = 0$

And $\partial_n(x) = \partial_n(f_n(y)) = f_{n-1}(d_n(y))$ since f chain map

$= f_{n-1}(0) = 0$

$\therefore x \in \ker \partial_n = Z_n(D)$

$\therefore f_n(Z_n(C)) \subseteq Z_n(D)$ $\forall n$

$\therefore f$ preserves cycles

Now show $f_n(B_n(C)) \subseteq B_n(D)$

Let $x \in f_n(B_n(C)) \Rightarrow x = f_n(y)$, $y \in B_n(C) = \operatorname{Im} d_{n+1}$

So $y = d_{n+1}(z)$ for $z \in C_{n+1}$

Then $x = f_n(y) = f_n(d_{n+1}(z)) = \partial_{n+1}(f_{n+1}(z)) \in \operatorname{Im}\partial_{n+1} = B_n(D)$

$\therefore f_n(B_n(C)) \subseteq B_n(D)$

$\therefore f$ preserves boundaries

Finally, define $H_n(f): H_n(C) \longrightarrow H_n(D) \ni$

$H_n(f)(z + B_n(C)) = f_n(z) + B_n(D)$

$z_1 + B_n(C) = z_2 + B_n(C) \Rightarrow z_1 - z_2 \in B_n(C) \Rightarrow f_n(z_1 - z_2) \in B_n(D)$

since f preserves boundaries

So $f_n(z_1) + B_n(D) = f_n(z_2) + B_n(D) \Rightarrow H_n(f)(z_1 + B_n(C)) = H_n(f)(z_2 + B_n(C$

$\therefore H_n(f)$ well defined group homomorphism $\forall n$

b. Give an example of a quasi-isomorphism of complexes of abelian groups which does not admit a quasi-inverse.

Consider $f: C_\cdot \to D_\cdot$ given by:

$$\begin{tikzcd} C_\cdot: \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d] & \cdots \\ D_\cdot: \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Note that f is a chain map since each square commutes

Hence the induced map $H_n(f): H_n(C) \to H_n(D)$ is given by:

$$\begin{tikzcd} H_n(C): \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "1_{\mathbb{Z}/2\mathbb{Z}}"] & 0 \arrow[r] \arrow[d] & \cdots \\ H_n(D): \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

$1_{\mathbb{Z}/2\mathbb{Z}}$ is an isomorphism, so $H_n(f)$ is an isomorphism $\forall n$

$\therefore$ f is a quasi-isomorphism

But the only map $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}$ is the zero map because

$2 \cdot \bar{1} = \bar{0}$ in $\mathbb{Z}/2\mathbb{Z} \Rightarrow 2 \cdot \varphi(\bar{1}) = 0$ in $\mathbb{Z} \Rightarrow \varphi(\bar{1}) = 0$

So the only chain map $g: D_\cdot \to C_\cdot$ is the zero-map

$\therefore H_n(g)$ is the zero map

But 1, 0 are not inverse maps

January 9, 2015.

Algebra Qualifying Examination
MAT 731 Part

Solve the following 5 problems. Support your answers with sound reasons.

1. Let k be a field, $k[x]$ the polynomial ring and (x) the principal ideal of $k[x]$ generated by x. Standard properties of modules and quotient modules give us the following short exact sequence of $k[x]$ modules.

$$0 \to (x) \to k[x] \to k[x]/(x) \to 0$$

Prove whether or not this sequence splits.

2. Let R be a ring with identity and let P a left R-module.

(a) State the definition given in terms of a lifting property of maps, of what it means for P to be a projective module.
(b) Prove that the definition given in (a) is equivalent to the following. The module P is projective if and only if it is a summand of a free module.

3. Prove that if R is a left artinian ring then $J(R)$ (the Jacobson radical of R) is a nilpotent ideal.

4. Let m and n be two not necessarily distinct integers both greater than or equal to 2. Consider the short exact sequence of $\mathbb{Z}$ modules

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by m. We may tensor over $\mathbb{Z}$ this sequence with $\mathbb{Z}/n\mathbb{Z}$ to obtain a new sequence.

$$(*) \quad 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to 0$$

(a) For which pairs of integers (m, n) is the sequence $(*)$ left exact? Prove it.
(b) For which pairs of integers (m, n) is $(*)$ right exact? Prove it.

5. Let R be a commutative ring with identity and $S \subset R$ a multiplicatively closed subset. There is an obvious natural homomorphism $f : R \to S^{-1}R$. State and prove a short easy to state condition on S that is necessary and sufficient for f to be injective.

January 2015 - 731

1. Let k be a field. Prove whether or not the following SES of $k[x]$-modules splits:

$$0 \longrightarrow (x) \longrightarrow k[x] \longrightarrow k[x]/(x) \longrightarrow 0$$

2. Let R be a ring with identity and let P be a left R-module.

a. State the definition of P being projective

P is projective if $\forall$ surjective maps $B \xrightarrow{g} C \longrightarrow 0$ and map $P \xrightarrow{f} C$, $\exists$ map $P \xrightarrow{h} B \ni$ the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

b. Prove that P is projective iff it is a summand of a free module.

($\Rightarrow$) Assume P projective

$\exists F$ free mapping onto P ie $F \xrightarrow{g} P \longrightarrow 0$

Then since P projective, we have commutative diagram

$$\begin{tikzcd} F \arrow[r, "g"] & P \arrow[r] & 0 \\ & P \arrow[u, "1_P"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

Consider the SES:

$$0 \longrightarrow \operatorname{Ker} g \hookrightarrow F \xrightarrow{g} P \longrightarrow 0$$

It splits since $\exists h : P \longrightarrow F \ni gh = 1_P$

$\therefore F \cong \operatorname{Ker} g \oplus P$

$\therefore P$ is a direct summand of a free module

($\Leftarrow$) Assume P is a direct summand of a free module

Then $\exists F$ free $\ni F \cong P \oplus X$ for some $X \leq F$

Let $B \xrightarrow{g} C \longrightarrow 0$ and $P \xrightarrow{f} C$

F free $\Rightarrow F$ projective

So we get the following commutative diagram:

$$
\begin{tikzcd}
B \arrow[r, "g"] & C \arrow[r] & 0 \\
& P \arrow[u, "f"'] \arrow[ul, dashed, "j"] \arrow[d, shift right, "i_1"'] & \\
& F \cong P \oplus X \arrow[u, shift right, "\pi_1"'] \arrow[uul, dashed, "\exists h"] &
\end{tikzcd}
$$

So $\exists h : F \to B \ni gh = f\pi_1$

Define $j : P \to B \ni j = h i_1$

Then $gj = g h i_1 = f \pi_1 i_1 = f$

$\therefore P$ projective

3. Prove that if R is a left Artinian ring, then $J(R)$ is a nilpotent ideal

R left Artinian $\Rightarrow$ R left Noetherian

But $J(R)$ is left ideal of R, so $J(R)$ is finitely generated

Consider $J \supseteq J^2 \supseteq \dots$

Then $\exists n \ni J^n = J^{n+1} = \dots$ since R is left Artinian

But then $J^n = J^{n+1} = JJ^n$, so $J^n = JJ^n$

And since J is finitely generated, J^n finitely generated

So by Nakayama's Lemma, $J^n = 0$

$\therefore J$ nilpotent

4. Let $m, n \in \mathbb{Z} \ni m, n \geq 2$. Consider the SES of $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

Applying the tensor product of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}$ gives the sequence

$$(*) \quad 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to 0$$

(a) For which pairs (m, n) is $(*)$ left exact?

Note that $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is right exact and $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$, so it suffices to find the pairs $(m, n) \ni \mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}$ is injective

But $\bar{0} \ni \bar{a} \longrightarrow \bar{0}$ iff $n | ma$

So (*) left exact iff m injective iff $\mathrm{Ker}\, m = 0$ iff $n \nmid ma$, $a = 1, 2, \ldots, n-1$

Claim $n \nmid ma$, $a = 1, 2, \ldots, n-1$ iff $(n, m) = 1$

($\Leftarrow$) Assume $(n, m) = 1$

Clear that $n \nmid ma$ since the only common factors of n, ma are numbers less than n

($\Rightarrow$) Assume $n \nmid ma$, $a = 1, 2, \ldots, n-1$

Suppose $(n, m) = b > 1$

Then $n = bc$ for some $c < n$

Then $n | mc$

Contradiction

$\therefore (n, m) = 1$

So the claim is true

$\therefore$ (*) is left exact $\forall (m, n) \ni m, n$ relatively prime

b. For which pairs (m, n) is (*) right exact?

(*) is right exact $\forall (m, n)$ because $\forall M$ R-modules, $\otimes_R M$ is right exact

5. Let R be a commutative ring with identity and $S \subseteq R$ a multiplicatively closed subset. State and prove a condition on S that is necessary and sufficient for $f: R \longrightarrow S^{-1}R$ to be injective

Note that $f: R \longrightarrow S^{-1}R \ni f(r) = \frac{r}{1}$

And $\mathrm{Ker} f = \{ r \in R \mid f(r) = 0_{S^{-1}R} \} = \{ r \in R \mid \frac{r}{1} = \frac{0}{1} \}$

But $\frac{r}{1} = \frac{0}{1}$ iff $\exists u \in S \ni (r \cdot 1 - 1 \cdot 0)u = 0$ iff $\exists u \in S$ $\ni ru = 0$

$\therefore \mathrm{Ker} f = \{ r \in R \mid \exists u \in S \ni ru = 0 \}$

Claim f injective iff S has no zero divisors

($\Rightarrow$) Assume f injective

Then $\ker f = 0$

So for $r \neq 0$, $ru \neq 0 \ \forall u \in S$

Then S has no zero divisors

($\Leftarrow$) Assume S has no zero divisors

So if $ru = 0$, then $r = 0$ since $0 \notin S$

Then $\ker f = 0$

$\therefore f$ injective

January 9, 2015.

Algebra Qualifying Examination
Homological Algebra Part

Solve the following 5 problems. Support your answers with sound reasons.

1. Prove Schanuel's lemma: Given the two short exact sequences of R-modules $0 \to K \to P \to M \to 0$ and $0 \to L \to Q \to M \to 0$ where P, Q are projective R-modules, prove that there is an isomorphism $P \oplus L \cong Q \oplus K$.

2. Let R be a ring. Consider the following commutative diagram of R-modules with exact rows and columns.

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Assume that the morphisms $A_3 \to B_3$ and $B_1 \to B_2$ are injections. Prove that $C_1 \to C_2$ is also an injection. Conclude that if the third column and the second row are are short exact sequences, then the third row is also a short exact sequence.

3. Let $0 \to K \to P \to M \to 0$ be a short exact sequence of right R-modules where P is a projective module. Let A be a left R-module. Prove that for every $n \geq 1$, we have $\mathrm{Tor}_{n+1}(M, A) \cong \mathrm{Tor}_n(K, A)$.

4. Let R be a Noetherian ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules. Assume that $\mathrm{pd}\, A < \mathrm{pd}\, B$. Prove that $\mathrm{pd}\, C = \mathrm{pd}\, B$.

5. Let $\mathcal{T}$ be a triangulated category. Recall that one of the axioms says that if we have two distinguished triangles and two morphisms f, g as in the diagram

below,

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

then the diagram can be completed (not necessarily in a unique way) by a morphism $h\colon Z \to Z'$. Prove that if f and g are isomorphisms, then so is h.

January 2015 - 732

1. Given two SES's of R-modules:

$$0 \longrightarrow K \xrightarrow{\alpha_1} P \xrightarrow{\beta_1} M \longrightarrow 0$$
$$0 \longrightarrow L \xrightarrow{\alpha_2} Q \xrightarrow{\beta_2} M \longrightarrow 0$$

where P, Q are projective R-modules, prove that $P \oplus L \cong Q \oplus K$.

Since P is projective, we get the following commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "\beta_2"] & M \arrow[r] & 0 \\ & P \arrow[u, "\beta_1"] \arrow[ul, dashed, "\exists \gamma"] & \end{tikzcd}$$

So we get the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha_1"] \arrow[d, "\delta"'] & P \arrow[r, "\beta_1"] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\ 0 \arrow[r] & L \arrow[r, "\alpha_2"] & Q \arrow[r, "\beta_2"] & M \arrow[r] & 0 \end{tikzcd}$$

where δ is defined as follows:

Let $x \in K$

$(\beta_2 \gamma \alpha_1)(x) = (1_M \beta_1 \alpha_1)(x) = (\beta_1 \alpha_1)(x) = 0$ by exactness

$\Rightarrow (\gamma \alpha_1)(x) \in \ker \beta_2 = \operatorname{Im} \alpha_2$

$\Rightarrow (\gamma\alpha_1)(x) = \alpha_2(y)$ for a unique $y \in L$ since α_2 injective

Define $\delta : K \longrightarrow L \ni \delta(x) = y$

δ is an R-module homomorphism:

$(\gamma\alpha_1)(x_1) = \alpha_2(y_1)$, $(\gamma\alpha_1)(x_2) = \alpha_2(y_2)$

$\Rightarrow (\gamma\alpha_1)(x_1 + x_2) = (\gamma\alpha_1)(x_1) + (\gamma\alpha_1)(x_2) = \alpha_2(y_1) + \alpha_2(y_2)$
$= \alpha_2(y_1 + y_2)$

$\Rightarrow \delta(x_1 + x_2) = y_1 + y_2 = \delta(x_1) + \delta(x_2)$

And $(\gamma\alpha_1)(rx) = r(\gamma\alpha_1)(x) = r\alpha_2(y) = \alpha_2(ry)$

$\Rightarrow \delta(rx) = ry = r\delta(x)$

And it is easy to see the diagram commutes

Now define $\epsilon : K \longrightarrow P \oplus L \ni \epsilon(k) = (\alpha_1(k), \delta(k))$

And $\tau : P \oplus L \longrightarrow Q \ni \tau(p, l) = \gamma(p) - \alpha_2(l)$

Then $\tau\epsilon(k) = \tau(\alpha_1(k), \delta(k)) = (\gamma\alpha_1)(k) - (\alpha_2\delta)(k) = 0$ by commutativity

$\therefore \operatorname{Im}\epsilon \subseteq \ker\tau$

Let $(p, q) \in \ker\tau \Rightarrow 0 = \tau(p, q) = \gamma(p) - \alpha_2(q) \Rightarrow \gamma(p) = \alpha_2(q)$

$\Rightarrow (\beta_2\gamma)(p) = (\beta_2\alpha_2)(q) = 0$ by exactness

$\Rightarrow \beta_1(p) = 0$ by commutativity

$\Rightarrow p \in \ker\beta_1 = \operatorname{Im}\alpha_1 \Rightarrow p = \alpha_1(k)$ for some $k \in K$

And $\alpha_2(q) = \gamma(p) = (\gamma\alpha_1)(k) = (\alpha_2\delta)(k)$ by commutativity

$\Rightarrow q = \delta(k)$ since α_2 is injective

$\therefore (p, q) = (\alpha_1(k), \delta(k)) \in \operatorname{Im}\epsilon$

$\therefore \ker\tau \subseteq \operatorname{Im}\epsilon$

$\therefore \operatorname{Im}\epsilon = \ker\tau$

Now let $x \in \ker\epsilon \Rightarrow 0 = \epsilon(x) = (\alpha_1(x), \delta(x)) \Rightarrow \alpha_1(x) = 0$

$\Rightarrow x \in \ker\alpha_1 = 0$ since α_1 injective

$\therefore \ker\epsilon = 0$

$\therefore \epsilon$ injective

Now let $x \in Q \Rightarrow \beta_2(x) = \beta_1(y)$ for some $y \in P$ since β_1 is surjective

$\Rightarrow \beta_2(x) = (\beta_2\gamma)(y)$

$\Rightarrow x - \gamma(y) \in \ker\beta_2 = \operatorname{Im}\alpha_2$

$\Rightarrow x - \gamma(y) = \alpha_2(z)$ for some $z \in L$

$\Rightarrow x = \alpha_2(z) + \gamma(y) = \gamma(y) - \alpha_2(-z) \in \operatorname{Im}\tau$

$\therefore \tau$ surjective

So we have the following SES:

$$0 \longrightarrow K \longrightarrow P \oplus L \longrightarrow Q \longrightarrow 0$$

And Q is projective, so it splits

$\therefore$ The sequence is split exact

$\therefore P \oplus L \cong Q \oplus K$

2. Let R be a ring. Consider the following commutative diagram of R-modules with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "g_2"] & B_3 \arrow[r] \arrow[d, "g_3"] & 0 \\
 & C_1 \arrow[r, "\gamma_1"] \arrow[d] & C_2 \arrow[r, "\gamma_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Prove that the 3rd row is a SES.

Let $x \in \ker \gamma_1 \Rightarrow 0 = \gamma_1(x)$

But g_1 surjective $\Rightarrow x = g_1(b_1)$ for some $b_1 \in B_1$

$\Rightarrow 0 = \gamma_1(g_1(b_1)) = g_2(\beta_1(b_1)) \Rightarrow \beta_1(b_1) \in \ker g_2 = \operatorname{Im} f_2$

$\Rightarrow \beta_1(b_1) = f_2(a_2)$ for some $a_2 \in A_2$

$\Rightarrow 0 = \beta_2(\beta_1(b_1)) = \beta_2(f_2(a_2)) = f_3(\alpha_2(a_2))$

$\Rightarrow \alpha_2(a_2) \in \ker f_3 = 0$

$\Rightarrow a_2 \in \ker \alpha_2 = \operatorname{Im} \alpha_1 \Rightarrow a_2 = \alpha_1(a_1)$ for some $a_1 \in A_1$

$\Rightarrow \beta_1(b_1) = f_2(\alpha_1(a_1)) = \beta_1(f_1(a_1)) \Rightarrow b_1 = f_1(a_1)$ since β_1 is injective

$\Rightarrow x = g_1(f_1(a_1)) = 0$ by exactness

$\therefore \ker \gamma_1 = 0$

$\therefore \gamma_1$ injective

$\therefore$ 3rd row is SES

3. Let $0 \to K \to P \to M \to 0$ be a SES of right R-modules where P is a projective module. Let A be a left R-module. Prove that $\forall n \geq 1$, $\operatorname{Tor}^R_{n+1}(M, A) \cong \operatorname{Tor}^R_n(K, A)$.

We get LES:

$$\cdots \to \operatorname{Tor}^R_{n+1}(P, A) \to \operatorname{Tor}^R_{n+1}(M, A) \to \operatorname{Tor}^R_n(K, A) \to \operatorname{Tor}^R_n(P, A) \to \cdots$$

Now

$P_\cdot : \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \xrightarrow{1_P} P \longrightarrow 0$

is a projective resolution for P since P is projective

$P_\cdot \otimes_R A : \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \otimes_R A \longrightarrow 0$

$\therefore \operatorname{Tor}_n^R(P, A) = H_n(P_\cdot \otimes_R A) = 0 \quad \forall n \geq 1$

So LES becomes:

$\ldots \longrightarrow 0 \longrightarrow \operatorname{Tor}_{n+1}^R(M, A) \longrightarrow \operatorname{Tor}_n^R(K, A) \longrightarrow 0 \longrightarrow \ldots$

$\therefore \operatorname{Tor}_{n+1}^R(M, A) \cong \operatorname{Tor}_n^R(K, A) \quad \forall n \geq 1$

4. Let R be a Noetherian ring and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a SES of finitely generated R-modules. Assume that $\operatorname{pd} A < \operatorname{pd} B$. Prove that $\operatorname{pd} C = \operatorname{pd} B$.

Let $\operatorname{pd} B = n$

Then $\operatorname{pd} A \leq n-1$

Note we have that $\operatorname{pd} M \leq m$ iff $\operatorname{Ext}_R^i(M, -) = 0 \quad \forall i > m \quad \forall M$ R-modules

So $\operatorname{Ext}_R^i(A, -) = 0 \quad \forall i > n-1$

Let X R-module

Get LES:

$\ldots \rightarrow \operatorname{Ext}_R^n(C, X) \longrightarrow \operatorname{Ext}_R^n(B, X) \longrightarrow \operatorname{Ext}_R^n(A, X) \longrightarrow \operatorname{Ext}_R^{n+1}(C, X)$
$\longrightarrow \operatorname{Ext}_R^{n+1}(B, X) \longrightarrow \operatorname{Ext}_R^{n+1}(A, X) \longrightarrow \ldots$

And LES becomes:

$\ldots \longrightarrow \operatorname{Ext}_R^n(C, X) \longrightarrow \operatorname{Ext}_R^n(B, X) \longrightarrow 0 \longrightarrow \operatorname{Ext}_R^{n+1}(C, X) \longrightarrow \operatorname{Ext}_R^{n+1}(B, X)$
$\longrightarrow 0 \longrightarrow \ldots$

$\therefore \operatorname{Ext}_R^i(C, X) \cong \operatorname{Ext}_R^i(B, X) \quad \forall i \geq n+1$

Now $\operatorname{pd} B = n \Rightarrow \operatorname{pd} B \leq n \Rightarrow \operatorname{Ext}_R^i(B, X) = 0 \quad \forall i > n$

$\therefore \operatorname{Ext}_R^i(C, X) = 0 \quad \forall i \geq n+1$ ie $\forall i > n$

$\therefore \operatorname{pd} C \leq n = \operatorname{pd} B$

Now let $\operatorname{pd} C = m \Rightarrow \operatorname{pd} C \leq m \Rightarrow \operatorname{Ext}_R^i(C, X) = 0 \quad \forall i > m$

But $\operatorname{Ext}_R^i(C, X) \cong \operatorname{Ext}_R^i(B, X) \quad \forall i \geq n+1$

$\Rightarrow \operatorname{Ext}_R^i(B, X) = 0 \quad \forall i > m$

$\therefore \operatorname{pd} B \leq m = \operatorname{pd} C$

$\therefore \operatorname{pd} C = \operatorname{pd} B$

5. Let T be a Δ'd category. Let f, g be two morphisms $\ni$ their completion to a map of distinguished Δ's is shown in diagram below:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Prove that if f, g are isomorphisms, then h is an isomorphism.

Apply $\operatorname{Hom}(Z', -)$:

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Hom}(Z',X) \arrow[r] \arrow[d, "f_*"] & \operatorname{Hom}(Z',Y) \arrow[r] \arrow[d, "g_*"] & \operatorname{Hom}(Z',Z) \arrow[r] \arrow[d, "h_*"] & \operatorname{Hom}(Z',X[1]) \arrow[r] \arrow[d, "f[1]_*"] & \operatorname{Hom}(Z',Y[1]) \arrow[d, "g[1]_*"] \\ \cdots \arrow[r] & \operatorname{Hom}(Z',X') \arrow[r] & \operatorname{Hom}(Z',Y') \arrow[r] & \operatorname{Hom}(Z',Z') \arrow[r] & \operatorname{Hom}(Z',X'[1]) \arrow[r] & \operatorname{Hom}(Z',Y'[1]) \end{tikzcd}$$

The diagram commutes since $\operatorname{Hom}(Z', \_)$ functor

And the rows are exact since $\operatorname{Hom}(Z', \_)$ cohomological functor

Now f, g isomorphisms $\Rightarrow f[1], g[1]$ isomorphisms

$\Rightarrow f_*, g_*, f[1]_*, g[1]_*$ isomorphisms since $\operatorname{Hom}(Z', -)$ functor

Then h_* isomorphism by classic 5-lemma

Hence h_* surjective

$1_{Z'} \in \operatorname{Hom}(Z', Z') \Rightarrow \exists j \in \operatorname{Hom}(Z', Z) \ni 1_{Z'} = h_*(j) = hj$

Now we repeat argument with $\operatorname{Hom}(-, Z)$:

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Hom}(Y[1],Z) \arrow[r] & \operatorname{Hom}(X[1],Z) \arrow[r] & \operatorname{Hom}(Z,Z) \arrow[r] & \operatorname{Hom}(Y,Z) \arrow[r] & \operatorname{Hom}(X,Z) \arrow[r] & \cdots \\ \cdots \arrow[r] & \operatorname{Hom}(Y'[1],Z) \arrow[r] \arrow[u, "g[1]^*"] & \operatorname{Hom}(X'[1],Z) \arrow[r] \arrow[u, "f[1]^*"] & \operatorname{Hom}(Z',Z) \arrow[r] \arrow[u, "h^*"] & \operatorname{Hom}(Y',Z) \arrow[r] \arrow[u, "g^*"] & \operatorname{Hom}(X',Z) \arrow[r] \arrow[u, "f^*"] & \cdots \end{tikzcd}$$

commutative with exact rows

We get h^* isomorphism by classic 5-Lemma

$1_Z \in \operatorname{Hom}(Z, Z) \Rightarrow \exists k \in \operatorname{Hom}(Z', Z) \ni 1_Z = h^*(k) = kh$

$\therefore j = khj = k$

$\therefore$ h is isomorphism with inverse j

Algebra Qualifying Examination – MAT 731/732

August 2014

Instructions: Do as many problems as possible in the time allotted. Assume that rings have an identity element and that modules are unitary left modules.

1. Given a commutative diagram of R-modules with exact rows

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

show that there exists a unique R-homomorphism $h\colon C \to Z$ such that the resulting diagram commutes.

2. If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules with C projective and M is a left R-module, show that

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is exact **without using the theory of Tor-groups.**

3. Prove that a left R-module M is injective if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$ for all left ideals I.

4. Let R be a ring with left ideals

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = R$$

such that L_i/L_{i-1} is a simple left R-module for all $i = 1, \ldots, n$.

(a) If $\mathfrak{m}$ is a maximal left ideal, show that $R/\mathfrak{m} \cong L_i/L_{i-1}$ for some i.

(b) Show that $J^n = 0$ where J is the Jacobson radical of R.

(c) If the modules L_i/L_{i-1} are pairwise nonisomorphic, show that $J = 0$.

(d) If the modules L_i/L_{i-1} are pairwise nonisomorphic, show that R is a direct product of division rings.

(If you choose to do these directly, here are some hints: (a) Choose i minimal such that L_i is not contained in $\mathfrak{m}$. (c) Choose i minimal such that $J(R) \cap L_i \neq 0$. If you do not, you may equally well ignore these hints.)

5. Let M be a left R-module, and assume that M has a submodule N that is maximal with respect to being noetherian.

(a) Show that M/N has no nonzero noetherian or artinian submodules.

(b) If R is left noetherian, show that M must be noetherian.

over please ⟶

6. Let $R = k[x]/(x^p - 1)$ where p is prime and k is a field of characteristic p.

(a) Let $t = 1+\overline{x}+\cdots+\overline{x}^{p-1}$. Show that $\overline{x}t = t$ and conclude that t spans a 1-dimensional ideal.

(b) Let $I = \{\sum_{i=0}^{p-1} a_i \overline{x}^i \mid \sum_i a_i = 0\}$. Show that I is an ideal of R generated by $1 - \overline{x}$.

(c) Show that there exist two short exact sequences

$$0 \to kt \to R \to I \to 0$$
$$0 \to I \to R \to kt \to 0$$

of R-modules.

(d) Show that the second exact sequence does not split by showing that kt is the only subspace of R on which $\overline{x}$ acts as the identity and that $kt \subseteq I$. Explain.

(e) Conclude that the global dimension of R is infinite.

7. Let $\mathcal{A}$ be the category of abelian groups and $\mathrm{Kom}(\mathcal{A})$ be the category of cocomplexes over $\mathcal{A}$. Recall that, given a morphism $f\colon K^\bullet \to L^\bullet$ in $\mathrm{Kom}(\mathcal{A})$, the mapping cylinder is

$$\mathrm{Cyl}(f) = K^\bullet \oplus K^\bullet[1] \oplus L^\bullet$$

with coboundary map

$$d_{\mathrm{Cyl}(f)}(k^i, k^{i+1}, \ell^i) = \left(d_K^i(k^i) - k^{i+1}, -d_K^{i+1}(k^{i+1}), f^{i+1}(k^{i+1}) + d_L^i(\ell^i)\right)$$

Consider the maps

$$\begin{array}{ll} \alpha\colon L^\bullet \longrightarrow \mathrm{Cyl}(f) & \beta\colon \mathrm{Cyl}(f) \longrightarrow L^\bullet \\ \ell^i \mapsto (0,0,\ell^i) & (k^i, k^{i+1}, \ell^i) \mapsto f(k^i) + \ell^i \end{array}$$

(a) Show that α and β are morphisms in $\mathrm{Kom}(\mathcal{A})$.

(b) Show that $\alpha\beta$ is homotopic to $1_{\mathrm{Cyl}(f)}$. (Construct a degree -1 morphism and show it gives a homotopy.)

8. Recall that a poset is called *directed* if given $i, j \in I$ there exists a $k \in I$ such that $i \le k, j \le k$. Let $\left((M_i)_{i\in I}, (\psi_j^i\colon M_i \to M_j)_{i\le j}\right)$ be a direct system of R-modules, where I is a directed poset. Define a relation $\sim$ on the disjoint union $\sqcup_{i\in I} M_i$ by

$$m_i \sim m_j \text{ whenever } m_i \in M_i, m_j \in M_j, \text{and } \exists k \ge i, k \ge j \text{ s.t. } \psi_k^i(m_i) = \psi_k^j(m_j).$$

(a) Show that $\sim$ is an equivalence relation.

(b) If $L = \sqcup_{i\in I} M_i / \sim$ is the set of equivalence classes, show that L becomes an R-module via $[m_r] + [m_s] = [\psi_k^r(m_r) + \psi_k^s(m_s)]$ for $m_r \in M_r, m_s \in M_s$ and $k \ge r, s$.
(Here $[m]$ denotes the class of m in L.)

(c) Show that L, with $\alpha_i\colon M_i \to L$ defined by $\alpha_i(m_i) = [m_i]$, is the direct limit of the given system. That is, show that L satisfies the relevant universal property.

August 2014

1. Given a commutative diagram of R-modules with exact rows

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] & 0 \\ X \arrow[r, "\gamma"] & Y \arrow[r, "\delta"] & Z \arrow[r] & 0 \end{tikzcd}$$

show that $\exists! h : C \longrightarrow Z$ an R-module homomorphism $\ni$ the diagram commutes.

Note that by exactness, β is a cokernel of α

So we have the following diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, bend right, "0"'] & B \arrow[r, "\beta"] \arrow[d, "\delta g"] & C \arrow[dl, dashed, "\exists! h"] \\ & Z & \end{tikzcd}$$

$\delta g \alpha = \delta \gamma f = 0 \cdot f = 0$ by commutativity and exactness

$\therefore \exists! h : C \longrightarrow Z \ni h\beta = \delta g$

$\therefore$ The given diagram commutes

2. If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a SES of left R-modules with C projective and M is a left R-module, show that $0 \longrightarrow A \underset{R}{\otimes} M \xrightarrow{f \otimes 1} B \underset{R}{\otimes} M \xrightarrow{g \otimes 1} C \underset{R}{\otimes} M \longrightarrow 0$ is a SES.

Note that $- \underset{R}{\otimes} M$ is right exact, so it suffices to show that $0 \longrightarrow A \underset{R}{\otimes} M \xrightarrow{f \otimes 1} B \underset{R}{\otimes} M$ is injective.

Since C is projective, the sequence splits, ie $\exists h : C \longrightarrow B$ $\ni gh = 1_C$. Equivalently, $\exists j : B \longrightarrow A \ni jf = 1_A$

Let $\Sigma(a_i \otimes m_i) \in \ker(f \otimes 1)$

Then $0 = (f \otimes 1)(\Sigma(a_i \otimes m_i)) = \Sigma(f \otimes 1)(a_i \otimes m_i) = \Sigma f(a_i) \otimes m_i$

So $0 = (j \otimes 1)(\Sigma f(a_i) \otimes m_i) = \Sigma(j \otimes 1)(f(a_i) \otimes m_i)$

$= \Sigma j(f(a_i)) \otimes m_i = \Sigma(a_i \otimes m_i)$

$\therefore \ker(f \otimes 1) = 0$

$\therefore f \otimes 1$ is injective

$\therefore$ The sequence is a SES

3\. Prove that a left R-module M is injective iff $\mathrm{Ext}_R^1(R/I, M) = 0$ $\forall$ left ideals I.

($\Rightarrow$) Assume that M is injective

Since M is injective, $\mathrm{Hom}_R(\_, M)$ is exact

Let $P_\cdot \to R/I$ be a projective resolution

Then $P_\cdot$ is exact for $n>0$

So $\mathrm{Hom}_R(P_\cdot, M)$ is exact $\forall n>0$

$\therefore$ $\mathrm{Ext}_R^n(R/I, M) = H^n(\mathrm{Hom}_R(P_\cdot, M)) = 0$ $\forall n>0$

In particular, $\mathrm{Ext}_R^1(R/I, M) = 0$ $\forall$ left ideals I

($\Leftarrow$) Assume $\mathrm{Ext}_R^1(R/I, M) = 0$ $\forall$ left ideals I

Let I be a left ideal of R and let $f: I \to M$

Then we have the SES: $0 \to I \xhookrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0$

which gives the LES:

$0 \to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M) \to \mathrm{Ext}_R^1(R/I, M) \to \ldots$

But $\mathrm{Ext}_R^1(R/I, M) = 0$

So we get SES: $0 \to \mathrm{Hom}_R(R/I, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{\iota^*} \mathrm{Hom}_R(I, M) \to 0$

In particular, ι^* is surjective

So $\exists g: R \to M \ni f = \iota^*(g) = g\iota$

Then the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & M & \end{tikzcd}$$

$\therefore$ M is injective by Baer's Criterion

4. Let R be a ring with left ideals $0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n = R \ni$ L_i/L_{i-1} is a simple left R-module $\forall i = 1, \dots, n$

a. If m is a maximal left ideal, show that $R/m \cong L_i/L_{i-1}$ for some i

Since L_i/L_{i-1} is simple $\forall i$, $0 = L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_n = R$ is a composition series of R

Consider the series $0 \subsetneq m \subsetneq R$

The factors $m/0 \cong m$, R/m are nonzero since m is maximal hence $0 \neq m \neq R$.

Then by Jordan-Hölder, the series can be refined to a composition series

But m maximal so $\nexists \hat{m} \ni m \subsetneq \hat{m} \subsetneq R$

Hence we have a composition series:

$$0 \subsetneq m_1 \subsetneq \dots \subsetneq m_p \subsetneq m \subsetneq R$$

And this is equivalent to the given composition series by Jordan-Hölder

$\therefore$ $R/m \cong L_i/L_{i-1}$ for some i

b. Show that $J^n = 0$ where $J = J(R)$

Since R has a composition series, R is both artinian and noetherian as an R-module

But since R is artinian, J is nilpotent ie $J^m = 0$ for some $m > 0$

Consider the series $0 = J^m \subsetneq J^{m-1} \subsetneq \dots \subsetneq J^2 \subsetneq J \subsetneq R$ with nonzero factors

Then again by Jordan-Hölder, it can be refined to a composition series equivalent to the given composition series

Then $m \leq n$, hence $J^n = J^m J^{n-m} = 0$

$\therefore$ $J^n = 0$

c. If the modules L_i/L_{i-1} are pairwise nonisomorphic, show that $J = 0$

d. If the modules L_i/L_{i-1} are pairwise nonisomorphic, show that R is a direct product of division rings

Since L_i/L_{i-1} are pairwise nonisomorphic, $J=0$ by (c)
Hence since R is artinian, R is semisimple
So $R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ by Artin-Wedderburn
And each n_i is the multiplicity of S_i as a composition factor of R
But since L_i/L_{i-1} are pairwise nonisomorphic, each $n_i = 1$
$\therefore R \cong D_1 \times \dots \times D_t$
$\therefore$ R is a direct product of division rings

5. Let M be a left R-module and assume that M has a submodule N that is maximal with respect to being noetherian.

a. Show that M/N has no nonzero noetherian or artinian submodules.

Suppose $0 \neq X \leq M/N$ is a noetherian submodule
Then $X = P/N$ where $N \lneq P \leq M$
So we have the SES: $0 \to N \hookrightarrow P \to P/N \to 0$
But N and P/N are noetherian, so P is noetherian
Contradiction to maximality of N
$\therefore M/N$ has no nonzero noetherian submodules
Suppose $0 \neq X \leq M/N$ is an artinian submodule
$\therefore X$ has a simple submodule $S = P/N \ni N \subsetneq P \subseteq M$
Then S is both noetherian and artinian
Again we have the SES: $0 \to N \hookrightarrow P \to P/N \to 0$
with N and P/N noetherian, hence P is noetherian
Contradiction to maximality of N
$\therefore M/N$ has no nonzero artinian submodules

b. If R is left noetherian, show that M must be noetherian s:

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Suppose $N \subsetneq M$

Then $\exists x \in M \ni x \notin N$

show that M/N has a nonzero noetherian submodule

Since $x \notin N$, $x+N \neq 0_{M/N}$, hence $0 \neq (x+N)$ is a finitely generated R-module

Hence $(x+N)$ is noetherian since R is noetherian

Contradiction to (a)

$\therefore N = M$

$\therefore M$ is noetherian

6. Let $R = k[x]/(x^p - 1)$ where p is prime and k is a field of characteristic p.

a. Let $t = 1 + \bar{x} + \dots + \bar{x}^{p-1}$, show that $\bar{x}t = t$ and conclude that t spans a 1-dimensional ideal hat $0 \longrightarrow I \longrightarrow R \longrightarrow kt \longrightarrow 0$ does not

y showing that kt is the only subspace of

hich $\bar{x}$ acts as the identity and that $kt \subseteq I$.

$I = (1-\bar{x}) \triangleleft R$.

e. Conclude that the global dimension of R is infinite

7. Let $\mathcal{A}$ be the category of abelian groups and $Kom(\mathcal{A})$ the category of cocomplexes over $\mathcal{A}$. Let $f: K^{\cdot} \longrightarrow L^{\cdot}$ be a morphism and consider the maps $\alpha: L^{\cdot} \longrightarrow Cyl(f) \ni \alpha(l^i) = (0,0,l^i)$ and $\beta: Cyl(f) \longrightarrow L^{\cdot} \ni \beta(k^i, k^{i+1}, l^i) = f(k^i)+$

a. Show that α and β are morphisms in $Kom(\mathcal{A})$

Consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & L^{i-1} \arrow[r, "d_L^{i-1}"] \arrow[d, "\alpha^{i-1}"'] & L^i \arrow[r, "d_L^i"] \arrow[d, "\alpha^i"'] & L^{i+1} \arrow[r] \arrow[d, "\alpha^{i+1}"'] & \cdots \\
\cdots \arrow[r] & Cyl(f)^{i-1} \arrow[r, "d_{cyl}^{i-1}"] & Cyl(f)^i \arrow[r, "d_{cyl}^i"] & Cyl(f)^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$\alpha^{i+1}(d_L^i(l^i)) = (0,0,d_L^i(l^i))$

$d_{cyl}^i(\alpha^i(l^i)) = d_{cyl}^i(0,0,l^i) = (d_K^i(0)-0, -d_K^{i+1}(0), f^{i+1}(0)+d_L^i(l^i$

$= (0,0,d_L^i(l^i))$

$\therefore$ The diagram commutes

$\therefore$ α is a cochain map, hence a morphism of $Kom(A)$

Now consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & Cyl(f)^{i-1} \arrow[r] \arrow[d, "\beta^{i-1}"'] & Cyl(f)^i \arrow[r] \arrow[d, "\beta^i"'] & Cyl(f)^{i+1} \arrow[r] \arrow[d, "\beta^{i+1}"'] & \cdots \\
\cdots \arrow[r] & L^{i-1} \arrow[r] & L^i \arrow[r] & L^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$\beta^{i+1}(d_{cyl}^i(k^i, k^{i+1}, l^i)) =$

$= \beta^{i+1}(d_K^i(k^i) - k^{i+1}, -d_K^{i+1}(k^{i+1}), f^{i+1}(k^{i+1}) + d_L^i(l^i))$

$= f^{i+1}(d_K^i(k^i)) - f^{i+1}(k^{i+1}) + f^{i+1}(k^{i+1}) + d_L^i(l^i)$

$= f^{i+1}(d_K^i(k^i)) + d_L^i(l^i)$

$d_L^i(\beta^i(k^i, k^{i+1}, l^i)) = d_L^i(f(k^i) + l^i) = d_L^i(f(k^i)) + d_L^i(l^i)$

$= f^{i+1}(d_K^i(k^i)) + d_L^i(l^i)$ by commutativity

since f is a cochain map

$\therefore$ The diagram commutes

$\therefore$ β is a cochain map, hence a morphism of $Kom(A)$

b. Show that $\alpha\beta \simeq 1_{Cyl(f)}$

Define $s^i : Cyl(f)^i \longrightarrow Cyl(f)^{i-1} \ni s^i(k^i, k^{i+1}, l^i) = (0, k^i, 0)$

Then we have the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & Cyl(f)^{i-1} \arrow[r, "d^{i-1}_{cyl}"] \arrow[d] & Cyl(f)^i \arrow[r, "d^i_{cyl}"] \arrow[d, "\alpha\beta - 1"'] \arrow[dl, dashed, "s^i"'] & Cyl(f)^{i+1} \arrow[r] \arrow[d] \arrow[dl, dashed, "s^{i+1}"'] & \cdots \\
\cdots \arrow[r] & Cyl(f)^{i-1} \arrow[r, "d^{i-1}_{cyl}"] & Cyl(f)^i \arrow[r, "d^i_{cyl}"] & Cyl(f)^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{aligned}
(\alpha\beta - 1_{Cyl(f)})(k^i, k^{i+1}, l^i) &= \alpha(\beta(k^i, k^{i+1}, l^i)) - 1_{Cyl(f)}(k^i, k^{i+1}, l^i) \\
&= \alpha(f^i(k^i) + l^i) - (k^i, k^{i+1}, l^i) \\
&= (0, 0, f^i(k^i) + l^i) - (k^i, k^{i+1}, l^i) \\
&= (-k^i, -k^{i+1}, f^i(k^i))
\end{aligned}$$

$$(s^{i+1} d^i_{cyl} + d^{i-1}_{cyl} s^i)(k^i, k^{i+1}, l^i) = s^{i+1}(d^i_{cyl}(k^i, k^{i+1}, l^i)) + d^{i-1}_{cyl}(s^i(k^i, k^{i+1}, l^i))$$

$$\begin{aligned}
&= s^{i+1}(d^i_K(k^i) - k^{i+1}, -d^{i+1}_K(k^{i+1}), f^{i+1}(k^{i+1}) + d^i_L(l^i)) + d^{i-1}_{cyl}(0, k^i, 0) \\
&= (0, d^i_K(k^i) - k^{i+1}, 0) + (d^{i-1}_K(0) - k^i, -d^i_K(k^i), f^i(k^i) + d^{i-1}_L(0)) \\
&= (0, d^i_K(k^i) - k^{i+1}, 0) + (-k^i, -d^i_K(k^i), f^i(k^i)) \\
&= (-k^i, -k^{i+1}, f^i(k^i))
\end{aligned}$$

$\therefore \alpha\beta - 1_{Cyl(f)} = s^{i+1} d^i_{cyl} + d^{i-1}_{cyl} s^i$

$\therefore \alpha\beta - 1_{Cyl(f)} \simeq 0$

$\therefore \alpha\beta \simeq 1_{Cyl(f)}$

8. Let $\{(M_i)_{i\in I}, (\psi_j^i : M_i \to M_j)_{i\le j}\}$ be a direct system of R-modules, where I is a directed poset. Define $\sim$ on $\dot{\bigcup}_{i\in I} M_i$ by $m_i \sim m_j$ whenever $m_i \in M_i$, $m_j \in M_j$ and $\exists k \geq i,j \ni \psi_k^i(m_i) = \psi_k^j(m_j)$

a. show that $\sim$ is an equivalence relation

$m_i \in M_i$ and $\psi_i^i(m_i) = \psi_i^i(m_i)$

$\therefore m_i \sim m_i$

$\therefore \sim$ is Reflexive

Let $m_i \sim m_j$

Then $m_i \in M_i$, $m_j \in M_j$ and $\exists k \geq i,j \ni \psi_k^i(m_i) = \psi_k^j(m_j)$

Hence $\psi_k^j(m_j) = \psi_k^i(m_i)$

$\therefore m_j \sim m_i$

$\therefore \sim$ is Symmetric

Let $m_i \sim m_j$ and $m_j \sim m_k$

Then $m_i \in M_i$, $m_j \in M_j$, $m_k \in M_k$ and $\exists \ell \geq i,j$, $\exists m \geq j,k \ni$
$\psi_\ell^i(m_i) = \psi_\ell^j(m_j)$, $\psi_m^j(m_j) = \psi_m^k(m_k)$

But I is a directed poset, so $\exists n \geq \ell, m$, hence $n \geq i, k$

And $\psi_n^i(m_i) = \psi_n^\ell \psi_\ell^i(m_i) = \psi_n^\ell \psi_\ell^j(m_j) = \psi_n^j(m_j)$

$\psi_n^k(m_k) = \psi_n^m \psi_m^k(m_k) = \psi_n^m \psi_m^j(m_j) = \psi_n^j(m_j)$

$\therefore \psi_n^i(m_i) = \psi_n^k(m_k)$

$\therefore m_i \sim m_k$

$\therefore \sim$ is Transitive

$\therefore \sim$ is an equivalence relation

b. If $L = \dot{\bigcup}_{i\in I} M_i/\sim$, show that L is an R-module via
$[m_r] + [m_s] = [\psi_k^r(m_r) + \psi_k^s(m_s)]$

C. Show that L with $\alpha_i : M_i \longrightarrow L \ni \alpha_i(m_i) = [m_i]$ is the direct limit of the given system.

Algebra Qualifying Examination
Rings and Modules Part – MAT 731

January 2014

Assume that rings have an identity element and that modules are unitary left modules.

Solve 4 out of the following 6 problems.

1. Prove directly from the definition (the mapping property) that the direct sum $J \oplus K$ of two injective R-modules J and K is again injective.

2. Let R be a commutative ring, and let I be an ideal of R.

   Use the universal property for tensor products to prove that there is an isomorphism $R/I \otimes_R R/I \cong R/I$ of R-modules.

3. Let R be a commutative ring and M and N be R-modules.

   Prove that, if M can be *generated by* n elements, then $\mathrm{Hom}_R(M, N)$ is isomorphic to a *submodule* of a direct sum of n copies of N.

   Hint: You may want to use exact sequences.

4. Let R be a commutative ring and M and N be R-modules.

   Prove that, if M and N each have finite length, then the R-module $M \otimes_R N$ has finite length as well.

   Hint: You may want to use exact sequences.

5. (a) Let R be a ring. Prove that any simple left R-module is of the form $R/\mathfrak{m}$ for some left maximal ideal $\mathfrak{m}$ of R.
   Explain why this also implies that the Jacobson radical J annihilates every simple R-module.

   (b) Use results to argue why an Artinian ring with no nonzero nilpotent elements must be a direct product of simple rings.

6. (a) Let k be an algebraically closed field and R be a noncommutative semisimple k-algebra. Suppose that $\dim_k R = 7$. Find the vector space dimensions of the simple components of R. Justify.
   How many (nonisomorphic) simple R-modules are there?

   (b) What if R is assumed to be commutative instead?

January 2014 - 731

1. Prove that if J, K are injective R-modules, then $J \oplus K$ is injective.

Let $0 \longrightarrow A \xrightarrow{f} B$ be injective and let $h: A \longrightarrow J \oplus K$

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "\exists \psi_2"] \arrow[ddl, dashed, bend left, "\exists \psi_1"] \\
& J \oplus K \arrow[d, shift left, "\pi_2"] \arrow[dd, bend right, "\pi_1"'] & \\
& K \arrow[u, shift left, "i_2"] & \\
& J \arrow[uu, bend right=20, "i_1"'] &
\end{tikzcd}$$

Since J, K are injective, so $\exists \psi_1 : B \longrightarrow J$, $\psi_2 : B \longrightarrow K$ $\ni$ $\psi_1 f = \pi_1 h$, $\psi_2 f = \pi_2 h$

Define $\psi : B \longrightarrow J \oplus K \ni \psi = i_1 \psi_1 + i_2 \psi_2$

Then $\psi(f(a)) = (i_1 \psi_1 + i_2 \psi_2)(f(a)) = (i_1 \psi_1 f + i_2 \psi_2 f)(a)$

$= (i_1 \pi_1 h + i_2 \pi_2 h)(a) = (i_1 \pi_1 + i_2 \pi_2)(h(a))$

$= 1_{J \oplus K}(h(a)) = h(a)$

$\therefore \psi f = h$

$\therefore J \oplus K$ is injective

2. Let R be a commutative ring and let $I \triangleleft R$. Prove that $R/I \otimes_R R/I \cong R/I$ as R-modules

Define $g: R/I \times R/I \longrightarrow R/I \ni g(r+I, r'+I) = rr'+I$

$g(r_1+I + r_2+I, r'+I) = g(r_1+r_2+I, r'+I) = (r_1+r_2)r'+I$

$= r_1r' + r_2r' + I = r_1r'+I + r_2r'+I$

$= g(r_1+I, r'+I) + g(r_2+I, r'+I)$

$g(r+I, r_1'+I + r_2'+I) = g(r+I, r_1'+r_2'+I) = r(r_1'+r_2') + I$

$= rr_1' + rr_2' + I = rr_1'+I + rr_2'+I$

$= g(r+I, r_1'+I) + g(r+I, r_2'+I)$

$g((r+I)s, r'+I) = g(rs+I, r'+I) = rsr'+I = g(r+I, sr'+I)$

$= g(r+I, s(r'+I))$

$\therefore g$ is biadditive

Then by UMP $\otimes$, $\exists h: R/I \otimes_R R/I \longrightarrow R/I \ni$ $h(r+I \otimes r'+I) = rr'+I$ homomorphism of abelian groups

And $h(s(r+I \otimes r'+I)) = h(sr+I \otimes r'+I) = srr'+I$

$= s(rr'+I) = sh(r+I \otimes r'+I)$

$\therefore h$ R-module homomorphism

Define $j: R/I \longrightarrow R/I \otimes_R R/I \ni j(r+I) = r+I \otimes 1+I$

$j(r_1+I+r_2+I) = j(r_1+r_2+I) = r_1+r_2+I \otimes 1+I$

$= r_1+I \otimes 1+I + r_2+I \otimes 1+I = j(r_1+I) + j(r_2+I)$

$j(s(r+I)) = j(sr+I) = sr+I \otimes 1+I = s(r+I \otimes 1+I)$

$= sj(r+I)$

$\therefore j$ R-module homomorphism

And $hj(r+I) = h(r+I \otimes 1+I) = r+I$

$jh(r+I \otimes r'+I) = j(rr'+I) = rr'+I \otimes 1+I = r+I \otimes r'+I$

$\therefore j = h^{-1}$

$\therefore h$ isomorphism of R-modules

$\therefore R/I \otimes_R R/I \cong R/I$ as R-modules

3. Let R be a commutative ring and M, N R-modules. Prove that if M can be generated by n elements, then $Hom_R(M,N)$ is isomorphic to a submodule of a direct sum of n copies of n.

First note that $\exists F$ free $\ni F \longrightarrow M \longrightarrow 0$ surjective
And since M finitely generated, we can choose F to be finitely generated
So $F = R^n$, and $R^n \longrightarrow M \longrightarrow 0$ surjective
But $Hom_R(-,N)$ is left exact, so we have that
$0 \longrightarrow Hom_R(M,N) \longrightarrow Hom_R(R^n,N)$ is injective
But $Hom_R(R^n,N) \cong (Hom_R(R,N))^n \cong N^n$
So $0 \longrightarrow Hom_R(M,N) \longrightarrow N^n$ is injective
$\therefore Hom_R(M,N) \cong Im(Hom_R(M,N) \longrightarrow N^n)$ which is a submodule of N^n

4. Let R be a commutative ring and M, N R-modules. Prove that if M, N have finite length, then the R-module $M \otimes_R N$ has finite length.

Since $\ell(N) < \infty$, N has a composition series

Hence N is both artinian and noetherian

But then N is finitely generated since it is Noetherian

Then $\exists F$ free, finitely generated R-module mapping onto N

So $R^n \longrightarrow N \longrightarrow 0$ surjective for some n

But $M \otimes_R -$ is right exact, so $M \otimes_R R^n \longrightarrow M \otimes_R N \longrightarrow 0$ is also surjective

And $M \otimes_R R^n \cong M^n$

So $M^n \longrightarrow M \otimes_R N \longrightarrow 0$ is surjective

And $\ell(M) < \infty \Rightarrow \ell(M^n) < \infty$

Thus $\ell(M \otimes_R N) \leq \ell(M^n) < \infty$

$\therefore \ell(M \otimes_R N) < \infty$

5. a. Let R be a ring. Prove that any simple left R-module is of the form R/m for some left maximal ideal m of R. Explain why this implies that $J(R)$ annihilates every simple R-module.

Let S be a simple left R-module

Then S is cyclic, thus $S = Rs$ for $0 \neq s \in S$.

Define $\varphi : R \longrightarrow S \ni \varphi(r) = rs$

Then φ is a surjective R-module homomorphism

So by the 1st Iso Thm, $R/\ker\varphi \cong S$

$\ker\varphi$ is a left ideal of R

Suppose $\ker\varphi$ is not maximal

Then $\exists J$ a left ideal of $R \ni \ker\varphi \subsetneq J \subsetneq R$

So $0 \neq J/\ker\varphi$ is a proper left ideal of $R/\ker\varphi$

Contradiction since $R/\ker\varphi$ is simple

$\therefore \ker\varphi$ is a maximal left ideal

$\therefore S \cong R/m$ for m maximal left ideal of R

Now $J(R) = \bigcap$ maximal left ideals of R

Let $x \in J(R) \subseteq m$

Then $xS \cong x\,R/m$

But $\forall y \in x\,R/m$, $y = x(r+m) = xr + m = m$ since $x \in m$

$\therefore y = 0_{R/m}$

$\therefore xS = 0$

$\therefore J$ annihilates simple R-modules

b. Why must an Artinian ring with no nonzero nilpotent elements be a direct product of simple rings.

Since R is artinian, $J(R)$ must be nilpotent, hence $J(R)$ is nil

So every element of $J(R)$ is nilpotent

But R has no nonzero nilpotent elements, so $J(R) = 0$

And R artinian with $J(R) = 0 \Rightarrow$ R semisimple

$\therefore$ R is a direct product of simple rings

6. a. Let k be an algebraically closed field and R a noncommutative semisimple k-algebra. Suppose that $\dim_k R = 7$. Find the vector space dimensions of the simple components of R. How many nonisomorphic simple R-modules are there?

R semisimple $\Rightarrow$ R is a direct sum of simple modules

But also by Artin-Wedderburn Thm, $R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ where each D_i is a division ring, there are precisely t nonisomorphic simple R-modules, and each n_i is the multiplicity of the simple module S_i as a direct summand of R

In fact, $R \cong M_{n_1}(k) \times \dots \times M_{n_t}(k)$ since k algebraically closed

Now $\dim_k R = \dim_k M_{n_1}(k) + \dots + \dim_k M_{n_t}(k)$

Then $7 = n_1^2 + \dots + n_t^2$

So either $n_1 = \dots = n_7 = 1$ or $n_1 = 2, n_2 = n_3 = n_4 = 1$

If $n_1 = \dots = n_7 = 1$, then $R \cong k \times k \times k \times k \times k \times k \times k$ which is commutative since k is a field

contradiction since R noncommutative

$\therefore n_1 = 2, n_2 = n_3 = n_4 = 1$

So the vector space dimensions of the simple components are 2,1,1,1 and there are 4 nonisomorphic simple R-modules

b. What if R is commutative?

Then $n_1 = \ldots = n_7 = 1$
Hence the vectorspace dimensions of the simple components are 1,1,1,1,1,1,1 and there are 7 nonisomorphic simple R-modules

January 2014

Algebra Qualifying Examination
Homological Algebra Part

Solve 4 out of the following 6 problems:

1. Assume that R is a ring such that $\operatorname{pd}_R S \leq n$ for every simple R-module S. Show that $\operatorname{pd}_R M \leq n$ for every module M having a finite composition series. Here pd denotes the projective dimension.

2. (a) Assume that $\xi\colon 0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, and that $A = A_1 \oplus I$ where I is an injective module. Show that ξ can be written as the direct sum of two short exact sequences, one of them being $0 \to I \to I \to 0 \to 0$ where the map $I \to I$ is an isomorphism.

(b) Assume that R is a ring where all the projective modules are injective. Prove that if M is an R-module, then either $\operatorname{pd}_R M = 0$ or $\operatorname{pd}_R M = \infty$.

3. Assume $M_1 \subset M_2 \subset M_3 \subset \ldots$ is an ascending chain of submodules of a module M. Prove that $\varinjlim M_i = \bigcup_{i=1}^{\infty} M_i$.

4. A ring R is called *left hereditary* if every submodule of a projective left module is again projective. For example, the ring of integers $\mathbb{Z}$ and the ring $K[t]$ of polynomials in one indeterminate over a field K are both left (and right) hereditary.

(a) Assume that R is a left and right hereditary ring. Show that for every two R-modules M and N we have $\operatorname{Ext}^i_R(M, N) = 0$ for each $i \geq 2$. Show also that for each right module A and left module B we have $\operatorname{Tor}_i^R(A, B) = 0$ for all $i \geq 2$.

(b) Give an example of a ring R that *is neither* left nor right hereditary.

5. Let $\mathcal{T}$ be a triangulated category and let $u\colon A \to B$ be a monomorphism in $\mathcal{T}$. Prove that u is an isomorphism from A to a direct summand of B (that is u splits).

6. Let $\mathcal{A}$ be an abelian category and assume that the following diagram commutes and that the vertical arrows are isomorphisms.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

Prove that the bottom row is exact if and only if the top row is exact.

January 2014 - 732

1. Assume that R is a ring $\ni pd_R S \le n$ $\forall$ simple R-modules S. Show that $pd_R M \le n$ $\forall$ modules $M \ni \ell(M) < \infty$.

Let M be an R-module $\ni \ell(M) < \infty$

Then M has a composition series $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_m = M$

If $m = 1$, then $0 = M_0 \subsetneq M_1 = M$ is a composition series

Hence $M \cong M/0$ is simple

$\therefore pd_R M \le n$ by assumption

Assume the result $\forall$ modules with length less than m

Consider the SES: $0 \longrightarrow M_{m-1} \longrightarrow M \longrightarrow M/M_{m-1} \longrightarrow 0$

Note that $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_{m-1}$ is a composition series for M_{m-1}, hence $\ell(M_{m-1}) = m-1 < m$

Then by induction, $pd_R M_{m-1} \le n$

And M/M_{m-1} is simple, so $pd_R M/M_{m-1} \le n$ by assumption

But $pd_R M \le \max\{pd_R M_{m-1}, pd_R M/M_{m-1}\} \le n$

$\therefore pd_R M \le n$

2. a. Assume that $\mathcal{E}: 0 \to A \to B \to C \to 0$ is a SES of R-modules and that $A = A_1 \oplus I$ where I is an injective module. Show that $\mathcal{E}$ can be written as the direct sum of two SES's one of them being $0 \to I \to I \to 0 \to 0$ where $I \to I$ is an isomorphism.

Note that we have $0 \to A_1 \oplus I \xrightarrow{f} B$ is injective
Hence $f|_I$ is injective ie $0 \to I \xrightarrow{f} B$ is injective
But since I is injective, $0 \to I \xrightarrow{f} B$ splits
Hence I is a direct summand of B, say $B \cong I \oplus B_1$
Claim $0 \to A \to B \to C \to 0$ is the direct sum of $0 \to I \to I \to 0 \to 0$ and
$0 \to A_1 \to B_1 \to C \to 0$

b. Assume that R is a ring where all the projectives are injectives. Prove that if M is an R-module, then either $pd_R M = 0$ or $pd_R M = \infty$.

Note that if M is projective, then $0 \longrightarrow M \xrightarrow{1_M} M \longrightarrow 0$ is a projective resolution, hence $pd_R M = 0$

So assume M is not projective

Suppose $pd_R M = n < \infty$

Then $\exists P_\cdot$: $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ a projective resolution where n is the smallest integer for which such a projective resolution exists

Then P_i are injective $\forall i$ by assumption

So $0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1}$ splits

And have split SES: $0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \operatorname{coker} d_n \longrightarrow 0$

$\therefore P_{n-1} \cong P_n \oplus \operatorname{Coker} d_n$

But since P_{n-1} is projective, so is $\operatorname{coker} d_n$

And $\operatorname{coker} d_n = P_{n-1}/\operatorname{Im} d_n = P_{n-1}/\ker d_{n-1} \cong \operatorname{Im} d_{n-1} = \ker d_{n-2}$

Then $\tilde{P}_\cdot$: $0 \longrightarrow \ker d_{n-2} \hookrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$

is a projective resolution

Contradiction to minimality of n

$\therefore pd_R M = \infty$

3. Assume $M_1 \subseteq M_2 \subseteq \dots$ is an ascending chain of submodules of a module M. Prove that $\varinjlim M_i = \bigcup_{i=1}^{\infty} M_i$.

Note that the maps of the system are $\varphi^i_{i+1} : M_i \longrightarrow M_{i+1} \ni$ $\varphi^i_{i+1}(m_i) = m_i$ inclusion maps

Hence $\varphi^i_j = \varphi^i_{i+1}\varphi^{i+1}_{i+2}\cdots\varphi^{j-1}_j$

Define $\tau_i : M_i \longrightarrow \bigcup_{i=1}^{\infty} M_i \ni \tau_i(m_i) = m_i$ inclusion map

$\tau_j(\varphi^i_j(m_i)) = \tau_j(m_i) = m_i = \tau_i(m_i) \quad \forall m_i \in M_i$

$\therefore \tau_j \varphi^i_j = \tau_i \quad \forall i < j$

$\therefore$ τ_i's are compatible with the system

Let $f_i : M_i \longrightarrow T \ni f_i = f_j \varphi^i_j \quad \forall i < j$

Consider the following diagram:

$$
\begin{tikzcd}
M_i \arrow[d, "\varphi^i_j"'] \arrow[dr, "\tau_i"] \arrow[drr, bend left, "\forall f_i"] & & \\
 & \bigcup_{i=1}^{\infty} M_i \arrow[r, dashed, "f"] & T \\
M_j \arrow[ur, "\tau_j"'] \arrow[urr, bend right, "\forall f_j"'] & &
\end{tikzcd}
$$

Define $f : \bigcup_{i=1}^{\infty} M_i \longrightarrow T \ni f(x) = f_i(x)$ for $x \in M_i$

If $x \in M_i$ and $x \in M_j$ for $i < j$

Then $f_i(x) = f_j(\varphi^i_j(x)) = f_j(x)$, hence f is well defined

And $f(\tau_i(m_i)) = f(m_i) = f_i(m_i)$

$\therefore f\tau_i = f_i$

And similary $f_j = f\tau_j$

And we have that f is unique by construction

$\therefore \bigcup_{i=1}^{\infty} M_i = \varinjlim M_i$

4. a. Assume that R is a left and right hereditary ring. Show that for any R-modules M, N, $\text{Ext}_R^i(M,N) = 0$ for each $i \geq 2$, show also that for each right module A and left module B, $\text{Tor}_i^R(A,B) = 0 \ \forall i \geq 2$.

Note that $\exists P$ projective $\ni P \xrightarrow{\epsilon} M \longrightarrow 0$

So $P_\cdot : 0 \longrightarrow \ker \epsilon \hookrightarrow P \xrightarrow{\epsilon} M \longrightarrow 0$ is a projective resolution since it is exact and since $\ker \epsilon \leq P$ hence it is projective since R is hereditary

$\therefore \text{Hom}_R(P_\cdot, N) : 0 \longrightarrow \text{Hom}_R(P,N) \longrightarrow \text{Hom}_R(\ker\epsilon, N) \longrightarrow 0 \longrightarrow \ldots$

$\therefore \text{Ext}_R^i(M,N) = H^i(\text{Hom}_R(P_\cdot, N)) = 0 \ \forall i \geq 2$

Similarly $\exists P$ projective $\ni P \xrightarrow{\epsilon} B \longrightarrow 0$

So $P_\cdot : 0 \longrightarrow \ker\epsilon \hookrightarrow P \xrightarrow{\epsilon} B \longrightarrow 0$ is a projective resolution as before since R is left hereditary

$\therefore A \otimes_R P_\cdot : \ldots \longrightarrow 0 \longrightarrow A \otimes_R \ker\epsilon \longrightarrow A \otimes_R P \longrightarrow 0$

$\therefore \text{Tor}_i^R(A,B) = H_i(A \otimes_R P_\cdot) = 0 \ \forall i \geq 2$

b. Give an example of a ring that is neither left nor right hereditary

Take $R = \mathbb{Z}/4\mathbb{Z}$

Then R is a projective R-module

But take $M = \mathbb{Z}/2\mathbb{Z}$

Then M is a submodule of R

But M is not projective since the SES:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Does not split

$\therefore R$ is not hereditary

5. Let $\mathcal{T}$ be a Δ'd category and let $u: A \longrightarrow B$ be a monomorphism in $\mathcal{T}$. Prove that u is an isomorphism from A to a direct summand of B.

Note that u can be completed to a distinguished Δ:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

And $uw[-1] = 0$ since any two adjacent maps in a Δ compose to 0

So $\operatorname{Im} w[-1] \subseteq \ker u = 0$ since u is a monomorphism

$\therefore w[-1] = 0$, hence $w = 0$

Now $A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow A[1]$ is a distinguished Δ

And $0 \longrightarrow C \xrightarrow{1_C} C \longrightarrow 0$ is a distinguished Δ since it is the rotation of a distinguished Δ

Then $A \xrightarrow{i_1} A \oplus C \xrightarrow{p_2} C \xrightarrow{0} A[1]$ is a distinguished Δ since it is the direct sum of distinguished Δ's

And we have the following diagram:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & B \arrow[r, "v"] \arrow[d, dashed] & C \arrow[r, "0"] \arrow[d, equal] & A[1] \arrow[d, equal] \\
A \arrow[r, "i_1"] & A \oplus C \arrow[r, "p_2"] & C \arrow[r, "0"] & A[1]
\end{tikzcd}$$

Note that the diagram commutes and the vertical maps are isomorphisms

$\therefore \exists$ isomorphism $B \cong A \oplus C$ by 5-Lemma for Δ'd category

$\therefore u$ splits

$\therefore u$ is an isomorphism from A to a direct summand of B

6. Let $\mathcal{A}$ be an abelian category and assume that the following diagram commutes and that the vertical maps are isomorphisms

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[d, "f'"'] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & B' \arrow[r, "\beta'"] & B \arrow[r, "\beta"] & B'' \arrow[r] & 0 \end{tikzcd}$$

Prove that the bottom row is exact iff the top row is exact.

($\Rightarrow$) Assume the top row is exact

Let $x \in \ker \beta' \Rightarrow \beta'(x) = 0$

But f' surjective, so $x = f'(a') \Rightarrow 0 = \beta'(f'(a')) = f(\alpha'(a'))$

$\Rightarrow \alpha'(a') \in \ker f = 0 \Rightarrow a' \in \ker \alpha' = 0 \Rightarrow x = f'(0) = 0$

$\therefore \ker \beta' = 0$

$\therefore \beta'$ injective

Now let $x \in \operatorname{Im} \beta' \Rightarrow x = \beta'(b')$

But f' surjective $\Rightarrow b' = f'(a') \Rightarrow x = \beta'(f'(a')) = f(\alpha'(a'))$

$\Rightarrow \beta(x) = \beta(f(\alpha'(a'))) = f''(\alpha(\alpha'(a'))) = f''(0) = 0$

$\therefore x \in \ker \beta$

$\therefore \operatorname{Im} \beta' \subseteq \ker \beta$

Now let $x \in \ker \beta \Rightarrow \beta(x) = 0$

But f surjective $\Rightarrow x = f(a) \Rightarrow 0 = \beta(f(a)) = f''(\alpha(a))$

$\Rightarrow \alpha(a) \in \ker f'' = 0 \Rightarrow a \in \ker \alpha = \operatorname{Im} \alpha' \Rightarrow a = \alpha'(a')$

$\Rightarrow x = f(\alpha'(a')) = \beta'(f'(a')) \in \operatorname{Im} \beta'$

$\therefore \ker \beta \subseteq \operatorname{Im} \beta'$

$\therefore \operatorname{Im} \beta' = \ker \beta$

Finally let $x \in B'' \Rightarrow x = f''(a'')$ since f'' surjective

But α surjective $\Rightarrow a'' = \alpha(a) \Rightarrow x = f''(\alpha(a)) = \beta(f(a)) \in \operatorname{Im} \beta$

$\therefore \beta$ surjective

$\therefore$ Bottom row exact

($\Leftarrow$) Assume the bottom row is exact

Since f, f', f'' are isomorphisms, so are $f^{-1}, f'^{-1}, f''^{-1}$

Then we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle f'^{-1}} & & \downarrow {\scriptstyle f^{-1}} & & \downarrow {\scriptstyle f''^{-1}} & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0
\end{array}$$

Then by 1st part the bottom row is exact

$\therefore$ The top row of the given diagram is exact

Qualifying Examination

August 2013

Algebra Part

- **Please do all five questions.**
- **Assume throughout that rings have an identity element and that modules are unitary left modules.**

1. Give an example to show that $\mathrm{Hom}_R(-, N)$ does not necessarily preserve exact sequences. That is, give a ring R, module N, and exact sequence $0 \to A \to B \to C \to 0$ of R-modules such that applying $\mathrm{Hom}_R(-, N)$ does not give an exact sequence.
   Justify completely.

2. (a) If M and N are nonzero finitely generated R-modules with M projective, prove that $M \otimes_R N$ is nonzero.

   (b) Show by example that this does not necessarily hold if M is not projective. Justify your claim completely.

3. (a) Let $0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ be a sequence (not necessarily exact) of R-modules of finite length (denoted by ℓ) and such that $g \circ f \equiv 0$. For each $i = 1, 2, 3$, define the module

$$H_i = K_i/I_i$$

   where K_i is the kernel of the map going out of A_i and I_i is the image of the map coming into A_i. Prove that

$$\ell(H_1) - \ell(H_2) + \ell(H_3) = \ell(A_1) - \ell(A_2) + \ell(A_3)$$

   You may use certain facts about length with respect to short *exact* sequences.

   (b) Now assume in addition that R is commutative. Prove that if P is a projective module of finite length, then

$$\ell(\mathrm{Hom}(P, H_1)) - \ell(\mathrm{Hom}(P, H_2)) + \ell(\mathrm{Hom}(P, H_3)) = \\ \ell(\mathrm{Hom}(P, A_1)) - \ell(\mathrm{Hom}(P, A_2)) + \ell(\mathrm{Hom}(P, A_3))$$

   You may assume the fact that for any two modules V and W of finite length over a commutative ring R, the module $\mathrm{Hom}_R(V, W)$ has finite length.

4. Let R be a ring and I be an ideal of R.

   (a) Prove that for any R-module M, one has an isomorphism of left R-modules

$$M \otimes_R R/I \simeq M/IM$$

   (b) Suppose I is contained in the Jacobson radical of R and M and N are finitely generated R-modules. If $\phi\colon M \to N$ is an R-module homomorphism such that the induced map $\overline{\phi}\colon M/IM \to N/IN$ is surjective, prove that ϕ is surjective.
   Hint: Exact sequences.

5. Let k be a field.

(a) Prove that any semi-simple ring that is a k-algebra of dimension less than or equal to 3 (as a vector space over k) is commutative.

(b) Does the result of (ii) remain true if you omit the hypothesis of semi-simplicity? Justify your claim completely.

(c) Find the Jacobson radical of the ring you gave as example in part (b) above.

August 2013 - 731

1. Give an example to show that $\mathrm{Hom}_R(\_\_, N)$ does not necessarily preserve exact sequences.

Consider the SES of $\mathbb{Z}$-modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

Note that $\mathrm{Hom}_{\mathbb{Z}}(\_\_, \mathbb{Z}/3\mathbb{Z})$ is left exact, so it suffices to show that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{3^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ is not surjective.

But we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \arrow[r, "3^*"] \arrow[d, "S"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \arrow[d, "?"] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r, "3"] & \mathbb{Z}/3\mathbb{Z}
\end{tikzcd}$$

with vertical maps being isomorphisms

Note that the diagram commutes because the isomorphisms are natural

But $\mathrm{Im}\, 3 = \bar{0}$, so 3 is not surjective

$\therefore$ 3^* is also not surjective

$\therefore$ $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \longrightarrow 0$

is not exact

2. a. If $M, N \neq 0$ are finitely generated R-modules with M projective, prove that $M \otimes_R N \neq 0$

Counterexample: Take $R = R_1 \times R_2$ where R_1, R_2 rings
Also take $M = R_1 \times (0)$, $N = (0) \times R_2$
Note that $M, N \neq 0$ are both finitely generated by $(1,0), (0,1)$ respectively
And R is projective over itself, hence R_1, R_2 are both projective since they are direct summands of a projective module
Hence M is projective
But $(r_1, 0) \otimes (0, r_2) = (1,0)(r_1, 0) \otimes (0, r_2) = (1,0) \otimes (0,0) = 0$
$\therefore M \otimes_R N = 0$

b. Show by example that this does not necessarily hold if M is not projective.

Let $\bar{a} \otimes \bar{b} \in \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$
Then $\bar{a} \otimes \bar{b} = \bar{a} \cdot 1 \otimes \bar{b} = \bar{a}(3-2) \otimes \bar{b} = \bar{a} \cdot 3 \otimes \bar{b} - \bar{a} \cdot 2 \otimes \bar{b}$
$= \bar{a} \otimes 3\bar{b} - 2\bar{a} \otimes \bar{b} = \bar{a} \otimes \bar{0} - \bar{0} \otimes b = 0 - 0 = 0$
$\therefore \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$
But $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \neq 0$ are finitely generated $\mathbb{Z}$-modules
Consider the SES:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$\bar{0} \longrightarrow \bar{0}$ $\bar{0} \longrightarrow \bar{0}$
$\bar{1} \longrightarrow \bar{2}$ $\bar{1} \longrightarrow \bar{1}$
$\bar{2} \longrightarrow \bar{0}$
$\bar{3} \longrightarrow \bar{1}$

It is exact since $\operatorname{Im} 2 = \{\bar{0}, \bar{2}\} = \ker \pi$
But it does not split since $\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
Since $|\bar{1}| = 4$ but $|(\bar{0},\bar{0})| = 1$, $|(\bar{0},\bar{1})| = |(\bar{1},\bar{0})| = |(\bar{1},\bar{1})| = 2$
$\therefore \mathbb{Z}/2\mathbb{Z}$ is not projective

3. a. Let $0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$ be a sequence of R-modules of finite length $\ni gf \equiv 0$. Define $H_i = K_i / I_i$ where $K_i = \ker(A_i \longrightarrow)$ and $I_i = \operatorname{Im}(\longrightarrow A_i)$. Prove that
$\ell(H_1) - \ell(H_2) + \ell(H_3) = \ell(A_1) - \ell(A_2) + \ell(A_3)$

First note that $0 \to I_i \to K_i \to K_i/I_i \to 0$ is a SES,
so $\ell(K_i) = \ell(I_i) + \ell(H_i)$ $\forall i$

Now $\ell(H_1) - \ell(H_2) + \ell(H_3) = \ell(K_1/I_1) - \ell(K_2/I_2) + \ell(K_3/I_3)$
$= \ell(K_1) - \ell(I_1) - \ell(K_2) + \ell(I_2) + \ell(K_3) - \ell(I_3)$
$= \ell(\ker f) - \ell(0) - \ell(\ker g) + \ell(\operatorname{Im} f) + \ell(A_3) - \ell(\operatorname{Im} g)$
$= \ell(\ker f) - \ell(\ker g) + \ell(\operatorname{Im} f) - \ell(\operatorname{Im} g) + \ell(A_3)$
So it suffices to show $\ell(\ker f) - \ell(\ker g) + \ell(\operatorname{Im} f) - \ell(\operatorname{Im} g) = \ell(A_1) - \ell(A_2)$
But we have SES's:

$$0 \longrightarrow \ker g \hookrightarrow A_2 \longrightarrow \operatorname{Im} g \longrightarrow 0$$
$$0 \longrightarrow \ker f \hookrightarrow A_1 \longrightarrow \operatorname{Im} f \longrightarrow 0$$

So $\ell(A_1) = \ell(\ker f) + \ell(\operatorname{Im} f)$
And $\ell(A_2) = \ell(\ker g) + \ell(\operatorname{Im} g)$
$\therefore \ell(A_1) - \ell(A_2) = \ell(\ker f) + \ell(\operatorname{Im} f) - \ell(\ker g) - \ell(\operatorname{Im} g)$
$\therefore \ell(H_1) - \ell(H_2) + \ell(H_3) = \ell(A_1) - \ell(A_2) + \ell(A_3)$

b. Now assume R is commutative. Prove that if P is a projective module of finite length, then:
$\ell(\operatorname{Hom}(P,H_1)) - \ell(\operatorname{Hom}(P,H_2)) + \ell(\operatorname{Hom}(P,H_3)) = \ell(\operatorname{Hom}(P,A_1)) - \ell(\operatorname{Hom}(P,A_2)) + \ell(\operatorname{Hom}(P,A_3))$

Again we have the SES's:

$$0 \longrightarrow I_i \longrightarrow K_i \longrightarrow K_i/I_i \longrightarrow 0$$
$$0 \longrightarrow \ker f \hookrightarrow A_1 \longrightarrow \operatorname{Im} f \longrightarrow 0$$
$$0 \longrightarrow \ker g \hookrightarrow A_2 \longrightarrow \operatorname{Im} g \longrightarrow 0$$

But P is projective, so $\operatorname{Hom}(P,-)$ is exact, so we get SES's:

$$0 \to \operatorname{Hom}(P,I_i) \to \operatorname{Hom}(P,K_i) \to \operatorname{Hom}(P,H_i) \to 0$$
$$0 \to \operatorname{Hom}(P,\ker f) \to \operatorname{Hom}(P,A_1) \to \operatorname{Hom}(P,\operatorname{Im} f) \to 0$$
$$0 \to \operatorname{Hom}(P,\ker g) \to \operatorname{Hom}(P,A_2) \to \operatorname{Hom}(P,\operatorname{Im} g) \to 0$$

So $\ell(\mathrm{Hom}(P,K_i)) = \ell(\mathrm{Hom}(P,I_i)) + \ell(\mathrm{Hom}(P,H_i))$ $\forall i$

$\ell(\mathrm{Hom}(P,A_1)) = \ell(\mathrm{Hom}(P,\ker f)) + \ell(\mathrm{Hom}(P,\mathrm{Im} f))$

$\ell(\mathrm{Hom}(P,A_2)) = \ell(\mathrm{Hom}(P,\ker g)) + \ell(\mathrm{Hom}(P,\mathrm{Im} g))$

Then $\ell(\mathrm{Hom}(P,H_1)) - \ell(\mathrm{Hom}(P,H_2)) + \ell(\mathrm{Hom}(P,H_3))$

$= \ell(\mathrm{Hom}(P,K_1)) - \ell(\mathrm{Hom}(P,I_1)) - \ell(\mathrm{Hom}(P,K_2)) + \ell(\mathrm{Hom}(P,I_2))$
$+ \ell(\mathrm{Hom}(P,K_3)) - \ell(\mathrm{Hom}(P,I_3))$

$= \ell(\mathrm{Hom}(P,\ker f)) - \ell(\mathrm{Hom}(P,0)) - \ell(\mathrm{Hom}(P,\ker g)) + \ell(\mathrm{Hom}(P,\mathrm{Im} f))$
$+ \ell(\mathrm{Hom}(P,A_3)) - \ell(\mathrm{Hom}(P,\mathrm{Im} g))$

$= \ell(\mathrm{Hom}(P,A_1)) - \ell(\mathrm{Hom}(P,A_2)) + \ell(\mathrm{Hom}(P,A_3))$

$\therefore \ell(\mathrm{Hom}(P,H_1)) - \ell(\mathrm{Hom}(P,H_2)) + \ell(\mathrm{Hom}(P,H_3)) = \ell(\mathrm{Hom}(P,A_1)) - \ell(\mathrm{Hom}(P,A_2))$
$+ \ell(\mathrm{Hom}(P,A_3))$

4. Let R be a ring and let $I \triangleleft R$.

a. Prove that for any R-module M, $M \otimes_R R/I \cong M/IM$ as R-modules

Define $\varphi: M \times R/I \longrightarrow M/IM \ni \varphi((m, r+I)) = mr + IM$

$\varphi((m_1+m_2, r+I)) = r(m_1+m_2) + IM = m_1 r + IM + m_2 r + IM$
$= \varphi((m_1, r+I)) + \varphi((m_2, r+I))$

$\varphi((m, r_1+I+r_2+I)) = \varphi((m, r_1+r_2+I)) = m(r_1+r_2) + IM$
$= mr_1 + IM + mr_2 + IM = \varphi(m, r_1+I) + \varphi(m, r_2+I)$

$\varphi((ms, r+I)) = msr + IM = \varphi((m, sr+IM))$

$\therefore \varphi$ bioadditive

Then by UMP of $\otimes$, $\exists \psi: M \otimes_R R/I \longrightarrow M/IM \ni \psi(m \otimes r+I) = mr + IM$

is a homomorphism of abelian groups

And $\psi(s(m \otimes r+I)) = \psi(m \otimes (r+I)s) = \psi(m \otimes rs+I) = mrs + IM$
$= (mr + IM)s = \psi(m \otimes r+I)s$

$\therefore \psi$ R-module homomorphism

Define $\tau: M/IM \longrightarrow M \otimes_R R/I \ni \tau(m+IM) = m \otimes 1+I$

$\tau(m_1 + IM + m_2 + IM) = \tau(m_1 + m_2 + IM) = (m_1+m_2) \otimes 1+I$
$= m_1 \otimes 1+I + m_2 \otimes 1+I = \tau(m_1+IM) + \tau(m_2+IM)$

$\tau((m+IM)r) = \tau(mr+IM) = mr \otimes 1+I = m \otimes r(1+I) = m \otimes (1+I)r$

$= (m \otimes 1 + I)r = \tau(m + IM)r$

$\therefore$ τ R-module homomorphism

And $\Psi\tau(m+IM) = \Psi(m \otimes 1 + I) = m + IM$

$\tau\Psi(m \otimes r + I) = \tau(mr + IM) = mr \otimes 1 + I = m \otimes r + I$

$\therefore \tau = \Psi^{-1}$

$\therefore$ Ψ R-module isomorphism

$\therefore M \otimes_R R/I \cong M/IM$ as R-modules

b. suppose $I \subseteq J(R)$ and M, N are finitely generated modules. If $\phi : M \longrightarrow N$ is an R-module homomorphism $\ni$ the induced map $\bar{\phi} : M/IM \longrightarrow N/IN$ is surjective, prove that ϕ is surjective.

we have the following exact sequence:

$$M \xrightarrow{\phi} N \xrightarrow{\pi} \operatorname{Coker}\phi \longrightarrow 0$$

But $\_ \otimes_R R/I$ is right exact, so the following sequence is also exact:

$$\begin{tikzcd}
M \otimes_R R/I \arrow[r, "\phi \otimes 1"] \arrow[d, "\wr"'] & N \otimes_R R/I \arrow[r, "\pi \otimes 1"] \arrow[d, "\wr"] & \operatorname{Coker}\phi \otimes_R R/I \arrow[r] \arrow[d, "\wr"] & 0 \\
M/IM \arrow[r, "\bar{\phi}"] & N/IN \arrow[r, "\bar{\pi}"] & \operatorname{Coker}\phi / I\operatorname{Coker}\phi \arrow[r] & 0
\end{tikzcd}$$

And the vertical maps are natural isomorphisms, so the diagram commutes

Hence the bottom row is also exact

But by assumption, $\bar{\phi}$ is surjective, so $\operatorname{Im}\bar{\phi} = N/IN$

And exactness gives $N/IN = \operatorname{Im}\bar{\phi} = \operatorname{Ker}\bar{\pi}$

Hence $\operatorname{Im}\bar{\pi} = 0$

But $\bar{\pi}$ is surjective, so $\operatorname{Coker}\phi / I\operatorname{Coker}\phi = 0$

Then $\operatorname{Coker}\phi = I\operatorname{Coker}\phi$

And $\operatorname{Coker}\phi$ finitely generated since N is finitely generated

Then since $I \subseteq J(R)$, $\operatorname{Coker}\phi = 0$ by Nakayama's Lemma

$\therefore N = \operatorname{Im}\phi$, hence ϕ is surjective

5. Let k be a field

a. Prove that any semisimple ring that is a k-algebra of dimension less than or equal to 3 is commutative.

Since R is semisimple, $R = S_1 \oplus \dots \oplus S_t$, S_i simple

But also $R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ by Artin-Wedderburn

where n_i is the multiplicity of S_i in above direct sum

Now, if k is algebraically closed, $D_i = k$ $\forall i$

$\therefore R \cong M_{n_1}(k) \times \dots \times M_{n_t}(k)$

Now $\dim_k R \leq 3$

But $\dim_k \left(M_{n_1}(k) \times \dots \times M_{n_t}(k) \right) = n_1^2 + \dots + n_t^2$

Hence $n_1^2 + \dots + n_t^2 \leq 3$

So each $n_i = 1$

$\therefore R \cong k \oplus \dots \oplus k$

$\therefore R$ commutative since k is a field

b. Is (a) true if the hypothesis of semi-simplicity is omitted?

Take $R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, k field

R is noncommutative since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

And R is a k-algebra of dimension 3

Suppose R semisimple

Then $J(R) = 0$

But $\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$ is a nonzero nilpotent ideal, hence a nonzero nil ideal

$\therefore \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} \subseteq J(R)$

Contradiction since $J(R) = 0$

$\therefore R$ is not semisimple

$\therefore$ (a) is false without semisimplicity

c. Find $J(R)$ of example given in (b)

Note that the only possible maximal left ideals are

$J_1 = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}$, $J_2 = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$, $J_3 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ae & af \\ 0 & 0 \end{pmatrix} \in J_1$, hence J_1 left maximal ideal in R

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} 0 & ae+bf \\ 0 & cf \end{pmatrix} \in J_2$, hence J_2 left maximal ideal in R

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} ae & bf \\ 0 & cf \end{pmatrix} \notin J_3$, hence J_3 is not a left ideal

$\therefore J(R) = J_1 \cap J_2 = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$

August 2013

Algebra Qualifying Examination
Homological Algebra Part

Solve 4 out of the following 6 problems:

1. Let K be a field and let $R = K[x]/\langle x^2 \rangle$. Set $\overline{x} = x + \langle x^2 \rangle$ and let $S = R/R\overline{x}$ be a simple R-module. Find a projective resolution of S where the the n-th term $P^n = R$ for each $n \geq 0$. For each $n \geq 0$ compute the K-dimension of $\mathrm{Ext}^n_R(S, S)$. What is the projective dimension of S?

2. Consider the following commutative diagram of R-modules where R is a ring.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Assume that the columns are exact, and that the first two rows are also exact. Prove that the last row is also exact.

3. Let $\mathcal{T}$ be a triangulated category and let $L \xrightarrow{u} M \xrightarrow{v} N \to L[1]$ be a distinguished triangle. Prove that the composition $vu = 0$.

4. Let R be a Noetherian ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules. Assume that $\mathrm{pd}\, A > \mathrm{pd}\, B$. Prove that $\mathrm{pd}\, C = \mathrm{pd}\, A + 1$.

5. Let $(\mathbf{C}, \mathbf{d})$ be a complex of R-modules and assume that there is a contracting homotopy s, that is a family of maps $s_n \colon C_n \to C_{n+1}$ satisfying

$$d_{n+1}s_n + s_{n-1}d_n = 1_{C_n}.$$

Prove that the complex $(\mathbf{C}, \mathbf{d})$ is exact.

6. Let R be a commutative ring and let L, M, N be three finitely generated R-modules. Prove that there is a natural isomorphism

$$\tau_{L,M,N} : \operatorname{Hom}_R(L \otimes_R M, N) \cong \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$$

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1. Let K be a field and let $R = K[x]/(x^2)$. Set $\bar{x} = x + (x^2)$ and let $S = R/R\bar{x}$ be a simple R-module. Find a projective resolution of S where $P^n = R$ for each $n \geq 0$. For each $n \geq 0$ compute the K-dimension of $\mathrm{Ext}^n_R(S,S)$. What is the projective dimension of S?

$$P_\cdot : \cdots \xrightarrow{\bar{x}} \underset{(3)}{R} \xrightarrow{\bar{x}} \underset{(2)}{R} \xrightarrow{\bar{x}} \underset{(1)}{R} \xrightarrow{\bar{x}} \underset{(0)}{R} \xrightarrow{\pi} S \longrightarrow 0$$

$P_\cdot$ is a projective resolution of S since $\bar{x}(a+(x^2)) = 0$ iff $(x+(x^2))(a+(x^2)) = 0$ iff $ax + (x^2) = (x^2)$ iff $ax \in (x^2)$ iff $a \in (x)$

$$\mathrm{Hom}_R(P_\cdot, S) : 0 \longrightarrow \underset{(0)}{\mathrm{Hom}_R(R,S)} \longrightarrow \underset{(1)}{\mathrm{Hom}_R(R,S)} \longrightarrow \underset{(2)}{\mathrm{Hom}_R(R,S)} \rightarrow \cdots$$

$$\Rightarrow \mathrm{Hom}_R(P_\cdot, S) : 0 \longrightarrow \underset{(0)}{S} \xrightarrow{\bar{x}} \underset{(1)}{S} \xrightarrow{\bar{x}} \underset{(2)}{S} \xrightarrow{\bar{x}} \underset{(3)}{S} \longrightarrow \cdots$$

$$\Rightarrow \mathrm{Hom}_R(P_\cdot, S) : 0 \longrightarrow \underset{(0)}{S} \xrightarrow{0} \underset{(1)}{S} \xrightarrow{0} \underset{(2)}{S} \xrightarrow{0} \underset{(3)}{S} \longrightarrow \cdots$$

$\therefore \mathrm{Ext}^n_R(S,S) = H^n(\mathrm{Hom}_R(P_\cdot, S)) = S/0 \cong S \quad \forall n \geq 0$

And $\dim_K R = 2$ with basis $\{1+(x^2), x+(x^2)\}$

$\therefore \dim_K \mathrm{Ext}^n_R(S,S) = \dim_K S = 1$ with basis $\{1+(x^2)+(\bar{x})\}$

Note that $\mathrm{pd}_R S \leq n$ iff $\mathrm{Ext}^n_R(S,N) = 0$ $\forall R$-modules N, $\forall e > n$

But $\mathrm{Ext}^n_R(S,S) \neq 0$ $\forall n$

$\therefore \mathrm{pd}_R S = \infty$

2. Consider the following commutative diagram of R-modules where R is a ring.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "g_2"] & B_3 \arrow[r] \arrow[d, "g_3"] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "\gamma_1"] \arrow[d] & C_2 \arrow[r, "\gamma_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Assume that the columns are exact and that the first two rows are also exact. Prove that the last row is also exact.

Let $x \in \ker\gamma_1 \Rightarrow \gamma_1(x) = 0$

But g_1 surjective $\Rightarrow x = g_1(b_1)$, $b_1 \in B_1$

Then $0 = \gamma_1(g_1(b_1)) = g_2(\beta_1(b_1)) \Rightarrow \beta_1(b_1) \in \ker g_2 = \operatorname{Im} f_2$

Then $\beta_1(b_1) = f_2(a_2)$, $a_2 \in A_2$

$\Rightarrow 0 = \beta_2(\beta_1(b_1)) = \beta_2(f_2(a_2)) = f_3(\alpha_2(a_2)) \Rightarrow \alpha_2(a_2) \in \ker f_3 = 0$

$\Rightarrow a_2 \in \ker\alpha_2 = \operatorname{Im}\alpha_1 \Rightarrow a_2 = \alpha_1(a_1)$, $a_1 \in A_1$

$\Rightarrow \beta_1(b_1) = f_2(\alpha_1(a_1)) = \beta_1(f_1(a_1)) \Rightarrow b_1 = f_1(a_1)$ since β_1 injectiv

$\therefore x = g_1(f_1(a_1)) = 0$ by exactness

$\therefore \ker\gamma_1 = 0$

$\therefore \gamma_1$ injective

Now let $x \in \operatorname{Im}\gamma_1 \Rightarrow x = \gamma_1(c_1)$, $c_1 \in C_1$

But g_1 surjective $\Rightarrow c_1 = g_1(b_1)$, $b_1 \in B_1$

$\Rightarrow x = \gamma_1(g_1(b_1)) = g_2(\beta_1(b_1)) \Rightarrow \gamma_2(x) = \gamma_2(g_2(\beta_1(b_1)))$

$= g_3(\beta_2(\beta_1(b_1))) = g_3(0) = 0$

$\therefore x \in \ker\gamma_2$

$\therefore \operatorname{Im}\gamma_1 \subseteq \ker\gamma_2$

Now let $x \in \ker\gamma_2 \Rightarrow \gamma_2(x) = 0$

But g_2 surjective $\Rightarrow x = g_2(b_2)$, $b_2 \in B_2$

$\Rightarrow 0 = \gamma_2(g_2(b_2)) = g_3(\beta_2(b_2)) \Rightarrow \beta_2(b_2) \in \ker g_3 = \operatorname{Im} f_3$

$\Rightarrow \beta_2(b_2) = f_3(a_3)$, $a_3 \in A_3$

And α_2 surjective $\Rightarrow a_3 = \alpha_2(a_2)$, $a_2 \in A_2$

$\Rightarrow \beta_2(b_2) = f_3(\alpha_2(a_2)) = \beta_2(f_2(a_2)) \Rightarrow b_2 - f_2(a_2) \in \ker\beta_2 = \operatorname{Im}\beta_1$

$\Rightarrow b_2 - f_2(a_2) = \beta_1(b_1) \Rightarrow b_2 = \beta_1(b_1) + f_2(a_2)$

$\Rightarrow x = g_2(\beta_1(b_1) + f_2(a_2)) = g_2(\beta_1(b_1)) + g_2(f_2(a_2))$

$= \gamma_1(g_1(b_1))$

$\therefore x \in \operatorname{Im}\gamma_1$

$\therefore \ker\gamma_2 \subseteq \operatorname{Im}\gamma_1$

$\therefore \operatorname{Im}\gamma_1 = \ker\gamma_2$

Finally let $x \in C_3 \Rightarrow x = g_3(b_3)$ since g_3 surjective

β_2 surjective $\Rightarrow b_3 = \beta_2(b_2)$, $b_2 \in B_2$

$\Rightarrow x = g_3(\beta_2(b_2)) = \gamma_2(g_2(b_2))$

$\therefore \gamma_2$ surjective

$\therefore$ Third row exact

3. Let $\mathcal{T}$ be a Δ'd category and let $L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{\omega} L[1]$ be a distinguished Δ. Prove that $vu = 0$.

Consider the following diagram:

$$\begin{tikzcd} L \arrow[r, "1_L"] \arrow[d, "1_L"'] & L \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dotted] & L[1] \arrow[d, "1_{L[1]}"] \\ L \arrow[r, "u"] & M \arrow[r, "v"] & N \arrow[r, "\omega"] & L[1] \end{tikzcd}$$

Note that the 1st square commutes

Then $\exists$ map $0 \longrightarrow N$ completing the diagram to a map of Δ's

But this map must be the zero map

$\therefore vu1_L = 0 \cdot 0 \cdot 1_L$

$\therefore vu = 0$

4. Let R be a Noetherian ring and let $0 \to A \to B \to C \to 0$ be a SES of finitely generated R-modules. Assume that $\operatorname{pd} A > \operatorname{pd} B$. Prove that $\operatorname{pd} C = \operatorname{pd} A + 1$

First assume that $\operatorname{pd} A = \infty$ and let X be an R-module.

Then $\operatorname{pd} B < \infty$, say $\operatorname{pd} B = n \Longrightarrow \mathrm{Ext}_R^i(B,X) = 0 \ \forall i > n$

Suppose $\operatorname{pd} C < \infty$, say $\operatorname{pd} C = m \Rightarrow \mathrm{Ext}_R^i(C,X) = 0 \ \forall i > m$

Take $p = \max\{n,m\}$.

LES: $\ldots \longrightarrow \mathrm{Ext}_R^{p+1}(C,X) \longrightarrow \mathrm{Ext}_R^{p+1}(B,X) \longrightarrow \mathrm{Ext}_R^{p+1}(A,X) \longrightarrow \mathrm{Ext}_R^{p+2}(\ldots$

$\Rightarrow 0 \longrightarrow \mathrm{Ext}_R^i(A,X) \longrightarrow 0$ exact $\forall i \geq p+1$

$\therefore \mathrm{Ext}_R^i(A,X) = 0 \ \forall i > p$

$\therefore \operatorname{pd} A \leq p$

Contradiction since $\operatorname{pd} A = \infty$

$\therefore \operatorname{pd} C = \infty$

$\therefore \operatorname{pd} C = \operatorname{pd} A + 1$

Now assume that $\operatorname{pd} A < \infty$, say $\operatorname{pd} A = n$

Then $\operatorname{pd} B \leq n-1 \Rightarrow \mathrm{Ext}_R^i(B,X) = 0 \ \forall i > n-1$

LES: $\ldots \to \mathrm{Ext}_R^n(B,X) \to \mathrm{Ext}_R^n(A,X) \longrightarrow \mathrm{Ext}_R^{n+1}(C,X) \longrightarrow \mathrm{Ext}_R^{n+1}(B,X) \longrightarrow \ldots$

$\Rightarrow 0 \longrightarrow Ext^i_R(A,X) \longrightarrow Ext^{i+1}_R(C,X) \longrightarrow 0$ exact $\forall i \geq n$

$\therefore Ext^i_R(A,X) \cong Ext^{i+1}_R(C,X)$ $\forall i \geq n$

And $pd A = n \Rightarrow pd A \leq n \Rightarrow Ext^i_R(A,X) = 0$ $\forall i > n$

$\therefore Ext^i_R(C,X) = 0$ $\forall i > n+1$

$\therefore pd C \leq n+1 = pd A + 1$

Let $pd C = m \Rightarrow pd C \leq m \Rightarrow Ext^i_R(C,X) = 0$ $\forall i > m$

$\therefore Ext^i_R(A,X) \cong Ext^{i+1}_R(C,X) = 0$ $\forall i > m-1$

$\therefore Ext^i_R(A,X) = 0$ $\forall i > m-1$

$\therefore pd A \leq m-1 = pd C - 1$

$\therefore pd C = pd A + 1$

5. Let $C_\cdot$ be a complex of R-modules and assume that $\exists s$ contracting homotopy $\ni s_n : C_n \longrightarrow C_{n+1}$ with $d_{n+1}s_n + s_{n-1}d_n = 1_{C_n}$. Prove that $C_\cdot$ is exact.

Contracting homotopy $\Rightarrow 1_{C_n} \simeq 0$

Then $H_n(1_{C_n}) = H_n(0) = 0$

But H_n is a functor so $H_n(1_{C_n}) = 1_{H_n(C)}$ is identity map

Hence the identity map $1_{H_n(C)} : H_n(C) \longrightarrow H_n(C)$ is also the zeromap

$\therefore H_n(C) = 0$ $\forall n$

$\therefore C_\cdot$ exact

In fact, $C_\cdot$ split exact (see extra sheet for proof)

6. Let R be a commutative ring and let L, M, N be finitely generated R-modules. Prove that $\exists$ natural isomorphism $\mathrm{Hom}_R(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$.

Define $\Phi : \mathrm{Hom}_R(L \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N)) \ni$
$\Phi(f)(l)(m) = f(l \otimes m)$ where $f : L \otimes_R M \longrightarrow N$ is an R-module homomorphism

$\Phi(f)(l)(m_1 + m_2) = f(l \otimes (m_1 + m_2)) = f(l \otimes m_1 + l \otimes m_2)$
$= f(l \otimes m_1) + f(l \otimes m_2) = \Phi(f)(l)(m_1) + \Phi(f)(l)(m_2$

$\Phi(f)(l)(rm) = f(l \otimes rm) = f(r(l \otimes m)) = r f(l \otimes m) = r\Phi(f)(l)(m)$

$\therefore \Phi(f)(l)$ R-module homomorphism

$\Phi(f)(l_1 + l_2)(m) = f((l_1 + l_2) \otimes m) = f(l_1 \otimes m + l_2 \otimes m)$
$= f(l_1 \otimes m) + f(l_2 \otimes m) = \Phi(f)(l_1)(m) + \Phi(f)(l_2)(m)$

$\Phi(f)(rl)(m) = f(rl \otimes m) = f(r(l \otimes m)) = r f(l \otimes m)$
$= r\Phi(f)(l)(m)$

$\therefore \Phi(f)$ R-module homomorphism

$\Phi(f_1 + f_2)(l)(m) = (f_1 + f_2)(l \otimes m) = f_1(l \otimes m) + f_2(l \otimes m)$
$= \Phi(f_1)(l)(m) + \Phi(f_2)(l)(m)$

$\Phi(rf)(l)(m) = (rf)(l \otimes m) = r f(l \otimes m) = r\Phi(f)(l)(m)$

$\therefore \Phi$ R-module homomorphism

Define $\Psi : \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N)) \longrightarrow \mathrm{Hom}_R(L \otimes_R M, N) \ni$
$\Psi(g) = \Psi_g$ where $\Psi_g : L \otimes M \longrightarrow N \ni \Psi_g(l \otimes m) = g(l)(m)$
ie $\Psi(g)(l \otimes m) = g(l)(m)$

Consider the diagram

$$\begin{tikzcd} L \times M \arrow[r] \arrow[d, "f"'] & L \otimes M \arrow[dl, dashed, "\exists!\, \Psi_g"] \\ N & \end{tikzcd}$$

where $f(l, m) = g(l)(m)$

$f(l_1 + l_2, m) = g(l_1 + l_2)(m) = g(l_1)(m) + g(l_2)(m)$
$= f(l_1, m) + f(l_2, m)$

$f(l, m_1 + m_2) = g(l)(m_1 + m_2) = g(l)(m_1) + g(l)(m_2) = f(l, m_1) + f(l, m$

$f(lr, m) = g(lr)(m) = rg(l)(m) = g(l)(rm) = f(l, rm)$

$\therefore f$ biadditive

$\therefore \exists!\, \Psi_g : L \underset{R}{\otimes} M \longrightarrow N \ni \Psi_g(l \otimes m) = g(l)(m)$ homomorphism of abelian groups

$$\Psi(g_1+g_2)(l\otimes m) = (g_1+g_2)(l)(m) = g_1(l)(m) + g_2(l)(m)$$
$$= \Psi(g_1)(l\otimes m) + \Psi(g_2)(l\otimes m)$$

$$\Psi(rg)(l\otimes m) = (rg)(l)(m) = rg(l)(m) = r\Psi(g)(l\otimes m)$$

$\therefore \Psi$ R-module homomorphism

And $\Psi\Phi(f)(l)(m) = \Psi(f(l\otimes m)) = f(l)(m)$

$\Phi\Psi(g)(l\otimes m) = \Phi(g(l)(m)) = g(l\otimes m)$

$\therefore \Psi = \Phi^{-1}$

$\therefore \Phi$ isomorphism of R-modules

Now let $f: L \longrightarrow L'$

$$\begin{tikzcd}
\mathrm{Hom}_R(L\underset{R}{\otimes} M, N) \arrow[r, "\Phi_L"] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N)) \\
\mathrm{Hom}_R(L'\otimes M, N) \arrow[r, "\Phi_{L'}"] \arrow[u, "(f\otimes 1)^*"] & \mathrm{Hom}_R(L', \mathrm{Hom}_R(M,N)) \arrow[u, "f^*"']
\end{tikzcd}$$

Let $g \in \mathrm{Hom}_R(L'\otimes M, N)$

$$(\Phi_L(f\otimes 1)^*(g))(l)(m) = \Phi_L(g(f\otimes 1))(l)(m) = g(f\otimes 1)(l)(m)$$
$$= g(f(l)\otimes m)$$

$$(f^*\Phi_{L'}(g))(l)(m) = \Phi_{L'}(g)(f(l)(m)) = g(f(l)\otimes m)$$

$\therefore$ Diagram commutes

Now let $f: M \longrightarrow M'$

$$\begin{tikzcd}
\mathrm{Hom}_R(L\otimes M, N) \arrow[r, "\Phi_M"] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N)) \\
\mathrm{Hom}_R(L\otimes M', N) \arrow[r, "\Phi_{M'}"] \arrow[u, "(1\otimes f)^*"] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M',N)) \arrow[u, "(f^*)_*"']
\end{tikzcd}$$

Let $g \in \mathrm{Hom}_R(L\otimes M', N)$

$$(\Phi_M(1\otimes f)^*(g))(l)(m) = \Phi_M(g(1\otimes f))(l)(m) = g(1\otimes f)(l\otimes m)$$
$$= g(l\otimes f(m))$$

$$((f^*)_*\Phi_{M'}(g))(l)(m) = (f^*\Phi_{M'}(g))(l)(m) = \Phi_{M'}(g)(l)f(m)$$
$$= g(l\otimes f(m))$$

$\therefore$ Diagram commutes

Finally let $f: N \longrightarrow N'$

$$\begin{tikzcd}
\mathrm{Hom}_R(L \otimes_R M, N) \arrow[r, "\Phi_L"] \arrow[d, "f_*"'] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N)) \arrow[d, "(f_*)_*"] \\
\mathrm{Hom}_R(L \otimes_R M, N') \arrow[r, "\Phi_{L'}"] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N'))
\end{tikzcd}$$

Let $g \in \mathrm{Hom}_R(L \otimes_R M, N)$

$((f_*)_* \Phi_L(g))(l)(m) = (f \Phi_L(g))(l)(m) = f(g(l \otimes m))$

$(\Phi_{L'} f_*(g))(l)(m) = \Phi_{L'}(fg)(l)(m) = f(g(l \otimes m))$

$\therefore$ Diagram commutes

$\therefore \Phi$ natural in L, M, N

Algebra Part of Qualifying Examination, January 13, 2013

Instructions: Do all questions, justify your answers with the necessary proofs. All rings are associative (not necessarily commutative) with identity and all modules are left unitary modules. We denote by $\mathbb{Q}, \mathbb{R}$ the fields of rational and real numbers, respectively.

1. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules; such a sequence is called a *short exact sequence*. Prove the following statements. You may use the First Isomorphism Theorem.

(a) (5 points) For any homomorphism $u : X \to B$ of R-modules satisfying $gu = 0$, there exists a unique homomorphism $v : X \to A$ satisfying $u = fv$.

(b) (5 points) For any homomorphism $w : B \to Y$ satisfying $wf = 0$, there exists a unique homomorphism $v : C \to Y$ satisfying $w = vg$.

2. (15 points) Prove that the following statements are logically equivalent for the short exact sequence of Problem 1.

(a) There exists a homomorphism $s : B \to A$ of R-modules satisfying $1_A = sf$.

(b) There exists a homomorphism $t : C \to B$ of R-modules satisfying $1_C = gt$.

(c) There exist homomorphisms $s : B \to A$ and $t : C \to B$ of R-modules satisfying $1_A = sf$, $1_C = gt$, and $1_B = fs + tg$.

3. The short exact sequence of Problem 1 is called a *split short exact sequence* if it satisfies any of the equivalent conditions of Problem 2 .

(a) (5 points) Give an example (with proof) of a short exact sequence that is not split.

(b) (5 points) Give an example (with proof) of a split short exact sequence.

4. In this problem you may use that every module is a homomorphic image of a projective module and a submodule of an injective module; that a direct summand of a projective (resp., injective) module is projective (resp., injective); and that functor Hom is left exact.

(a) (15 points) Prove that the following two statements are logically equivalent for an R-module C.

(i) For each exact sequence $0 \to L \xrightarrow{s} M \xrightarrow{t} N \to 0$ of R-modules, the sequence of abelian groups

$$0 \to \operatorname{Hom}_R(C, L) \xrightarrow{\operatorname{Hom}_R(C,s)} \operatorname{Hom}_R(C, M) \xrightarrow{\operatorname{Hom}_R(C,t)} \operatorname{Hom}_R(C, N) \to 0$$

is exact.

(ii) Every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split short exact.

4. (continued)

(b) (15 points) Prove that the following two statements are logically equivalent for an R-module A.

(i) For each exact sequence $0 \to L \xrightarrow{s} M \xrightarrow{t} N \to 0$ of R-modules, the sequence of abelian groups

$$0 \to \mathrm{Hom}_R(N, A) \xrightarrow{\mathrm{Hom}_R(t,A)} \mathrm{Hom}_R(M, A) \xrightarrow{\mathrm{Hom}_R(s,A)} \mathrm{Hom}_R(L, A) \to 0$$

is exact.

(ii) Every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split short exact.

5. Consider the ring $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$ of all 2×2 matrices $A = (a_{ij})$ satisfying $a_{11}, a_{21} \in \mathbb{R}$, $a_{22} \in \mathbb{Q}$, and $a_{12} = 0$, with the usual operations of matrix addition and multiplication.

(a) (5 points) Find the center of R, $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$.

(b) (5 points) Is the ring R left artinian?

(c) (5 points) Is the ring R left noetherian?

(d) (5 points) Is the ring R right artinian?

(e) (5 points) Is the ring R right noetherian?

(f) (5 points) Find the radical of R, $J(R)$, and describe the ring structure of $R/J(R)$ in terms of $\mathbb{Q}$ and $\mathbb{R}$.

(g) (5 points) Describe the nonisomorphic simple left R-modules by indicating their underlying abelian group and R-action.

January 2013

1. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of R-modules. Prove the following:

a. For any homomorphism $u: X \longrightarrow B$ of R-modules $\ni$ $gu = 0$, $\exists!$ homomorphism $v: X \longrightarrow A \ni u = fv$.

Let $x \in X$

Then $gu(x) = 0$, so $u(x) \in \ker g = \operatorname{Im} f$ by exactness

Then $u(x) = f(a)$ for some $a \in A$

And that a is uniquely determined since f is injective

Hence $v: X \longrightarrow A \ni v(x) = a$ is well defined

Let $u(x_1) = f(a_1)$ and $u(x_2) = f(a_2) \Rightarrow u(x_1 + x_2) = f(a_1 + a_2)$

Then $v(x_1 + x_2) = a_1 + a_2 = v(x_1) + v(x_2)$

And let $u(x) = f(a) \Rightarrow u(rx) = f(ra)$

Then $v(rx) = ra = rv(x)$

$\therefore$ v is an R-module homomorphism

And $fv(x) = f(a) = u(x)$ $\forall x$

$\therefore fv = u$

Let $v': X \longrightarrow A \ni u = fv'$

Then $fv = fv'$, so $fv - fv' = 0 \Rightarrow f(v - v') = 0$

so $v - v' \in \ker f = 0$ since f injective

$\therefore v = v'$

$\therefore \exists! v: X \longrightarrow A$ homomorphism $\ni u = fv$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
 & & X \arrow[ul, dashed, "\exists! v"] \arrow[u, "u"] \arrow[ur, bend right, "0"'] & &
\end{tikzcd}$$

Hence f is a kernel of g

b. For any homomorphism $\omega : B \longrightarrow Y \ni wf = 0$,
$\exists!$ homomorphism $v : C \longrightarrow Y \ni w = vg$.

First note that since g is surjective, we have
$B/\ker g \cong C$ by 1st Isomorphism Thm
But by exactness, $\ker g = \operatorname{Im} f$
Hence $C \cong B/\operatorname{Im} f$
Let $x \in C \Rightarrow x = b + \operatorname{Im} f = \pi(b)$ for some $b \in B$
Define $v : C \longrightarrow Y \ni v(x) = w(b)$
Then if $x_1 = x_2$, $b_1 + \operatorname{Im} f = b_2 + \operatorname{Im} f \Rightarrow b_1 - b_2 \in \operatorname{Im} f$
$\Rightarrow b_1 - b_2 = f(a) \Rightarrow w(b_1 - b_2) = w(f(a)) = 0$
$\therefore w(b_1) = w(b_2)$
$\therefore v(x_1) = v(x_2)$
$\therefore v$ is well defined
Now $x_1 + x_2 = b_1 + \operatorname{Im} f + b_2 + \operatorname{Im} f = b_1 + b_2 + \operatorname{Im} f = \pi(b_1 + b_2)$
So $v(x_1 + x_2) = w(b_1 + b_2) = w(b_1) + w(b_2) = v(x_1) + v(x_2)$
And $rx = r(b + \operatorname{Im} f) = r\pi(b) = \pi(rb)$
So $v(rx) = w(rb) = rw(b) = rv(x)$
$\therefore v$ is an R-module homomorphism
And $v\pi(b) = v(b + \operatorname{Im} f) = w(b)$
$\therefore v\pi = w$
And we have uniqueness by construction
$\therefore \exists!$ homomorphism $v : C \longrightarrow Y \ni w = vg$

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[dr, bend right, "0"'] & B \arrow[r, "g"] \arrow[d, "\omega"] & C \arrow[r] \arrow[dl, dashed, "\exists! v"] & 0 \\
 & & Y & &
\end{tikzcd}
$$

Hence g is a cokernel of f

2. Prove that for the SES $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$,
TFAE:

(a) $\exists$ homomorphism $s: B \longrightarrow A$ of R-modules $\ni 1_A = sf$

(b) $\exists$ homomorphism $t: C \longrightarrow B$ of R-modules $\ni 1_C = gt$

(c) $\exists$ homomorphisms $s: B \longrightarrow A$ and $t: C \longrightarrow B$ of R-modules $\ni 1_A = sf$, $1_C = gt$, and $1_B = fs + tg$

(a)$\Rightarrow$(b) Assume $\exists s: B \longrightarrow A \ni 1_A = sf$

Define $\varphi: B \longrightarrow A \oplus C \ni \varphi(b) = s(b) + g(b)$

Then we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "g"] \arrow[d, "\varphi"'] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_1"] & A \oplus C \arrow[r, "p_2"] & C \arrow[r] & 0
\end{tikzcd}$$

where i_1, p_2 are the natural injection, projection respectively

Now $\varphi(f(a)) = s(f(a)) + g(f(a)) = 1_A(a)$ since $gf = 0$
$= a \quad \forall a \in A$

And $i_1(1_A(a)) = i_1(a) = a \quad \forall a \in A$

$\therefore \varphi f = i_1 1_A$

Now $1_C(g(b)) = g(b) \quad \forall b \in B$

And $p_2(\varphi(b)) = p_2(s(b) + g(b)) = g(b) \quad \forall b \in B$

$\therefore 1_C g = p_2 \varphi$

$\therefore$ The diagram commutes

And both rows are exact

So since $1_A, 1_C$ are isomorphisms, φ is also an isomorphism by the 5-Lemma

Now define $t: C \longrightarrow B \ni t = \varphi^{-1} i_2$

Then $gt = p_2 \varphi \varphi^{-1} i_2 = p_2 i_2 = 1_C$

$\therefore \exists$ homomorphism $t: C \longrightarrow B$ of R-modules $\ni 1_C = gt$

(b)$\Rightarrow$(a) Assume $\exists t: C \longrightarrow B \ni 1_C = gt$

Define $\psi: A \oplus C \longrightarrow B \ni \psi(a + c) = f(a) + t(c)$

Then we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i_1} & A\oplus C & \xrightarrow{p_2} & C & \longrightarrow & 0 \\ & & 1_A\downarrow & & \psi\downarrow & & 1_C\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Now $\psi(i_1(a)) = \psi(a) = f(a) = f(1_A(a)) \quad \forall a \in A$

$\therefore \psi i_1 = f 1_A$

And $1_C(p_2(a+c)) = 1_C(c) = c \quad \forall c \in C$

And $g(\psi(a+c)) = g(f(a)+t(c)) = g(f(a)) + g(t(c)) = c \quad \forall c \in C$

$\therefore 1_C p_2 = g\psi$

$\therefore$ The diagram commutes

And we have that ψ is an isomorphism by 5-Lemma

Define $s: B \longrightarrow A \ni s = p_1 \psi^{-1}$

Then $sf = p_1 \varphi^{-1} \varphi i_1 = p_1 i_1 = 1_A$

$\therefore \exists s: B \longrightarrow A$ homomorphism $\ni sf = 1_A$

(a),(b) $\Rightarrow$ (c) Assume $\exists\, s: B \longrightarrow A,\ t: C \longrightarrow B \ni$

$1_A = sf$ and $1_C = gt$

$\varphi\psi(a+c) = \varphi(f(a)+t(c)) = s(f(a)) + s(t(c)) + g(f(a)) + g(t(c))$

$= a + c$

$\therefore \varphi\psi = 1_{A\oplus C}$

But $\varphi\varphi^{-1} = 1_{A\oplus C}$, so $\varphi\psi = \varphi\varphi^{-1} \Rightarrow \varphi(\psi - \varphi^{-1}) = 0$

$\Rightarrow \psi - \varphi^{-1} \in \ker\varphi = 0$

$\therefore \psi = \varphi^{-1}$

$\therefore 1_B = \psi\varphi$

$\therefore 1_B(b) = \psi\varphi(b) = \psi(s(b) + g(b)) = f(s(b)) + t(g(b))$

$\therefore 1_B = fs + tg$

(c) $\Rightarrow$ (a),(b) clear

$\therefore$ The statements are equivalent

3. a. Give an example of a SES that is not split.

Consider the sequence:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$\bar{0} \longrightarrow \bar{0}$ $\qquad$ $\bar{0} \longrightarrow \bar{0}$

$\bar{1} \longrightarrow \bar{2}$ $\qquad$ $\bar{1} \longrightarrow \bar{1}$

$\bar{2} \longrightarrow \bar{0}$

$\bar{3} \longrightarrow \bar{1}$

So $\mathrm{Im}\left(\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z}\right) = \langle \bar{2} \rangle = \ker\left(\mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}\right)$

$\therefore$ The sequence is exact

But $\mathbb{Z}/4\mathbb{Z}$ is cyclic with generator $\bar{1}$ since $|\bar{1}| = 4$

And $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is not cyclic since $|(\bar{0},\bar{0})| = 1$, $|(\bar{0},\bar{1})| = |(\bar{1},\bar{0})| = |(\bar{1},\bar{1})| = 2$

$\therefore \mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Hence sequence is not split

b. Give an example of a split SES.

Consider the sequence:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{p_2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$\bar{0} \longrightarrow (\bar{0},\bar{0})$ $\qquad$ $(\bar{0},\bar{0}) \longrightarrow \bar{0}$

$\bar{1} \longrightarrow (\bar{1},\bar{0})$ $\qquad$ $(\bar{0},\bar{1}) \longrightarrow \bar{1}$

$(\bar{1},\bar{0}) \longrightarrow \bar{0}$

$(\bar{1},\bar{1}) \longrightarrow \bar{1}$

$\therefore \mathrm{Im}\, i_1 = \langle(\bar{1},\bar{0})\rangle = \ker p_2$

$\therefore$ The sequence is exact

And $p_2 i_2(\bar{0}) = p_2(\bar{0},\bar{0}) = \bar{0}$, $p_2 i_2(\bar{1}) = p_2(\bar{0},\bar{1}) = \bar{1}$

$\therefore p_2 i_2 = 1_{\mathbb{Z}/2\mathbb{Z}}$

$\therefore$ sequence splits

4. a. Prove that for an R-module C, TFAE:

(i) For each SES $0 \to L \xrightarrow{s} M \xrightarrow{t} N \to 0$ of R-modules,

$$0 \to Hom_R(C,L) \to Hom_R(C,M) \to Hom_R(C,N) \to 0$$

is a SES of abelian groups

(ii) Every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits

(i) $\Rightarrow$ (ii) Assume $Hom_R(C,-)$ exact

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a SES

Then this $Hom_R(C,-)$ is exact, we have the SES:

$$0 \to Hom_R(C,A) \to Hom_R(C,B) \to Hom_R(C,C) \to 0$$

In particular, $Hom_R(C,B) \xrightarrow{g_*} Hom_R(C,C)$ is surjective

But $1_C \in Hom_R(C,C)$, so $\exists h \in Hom_R(C,B) \ni 1_C = g_*(h)$

$\therefore 1_C = gh$

$\therefore 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits

(ii) $\Rightarrow$ (i) Assume every SES $0 \to A \to B \to C \to 0$ splits

Note that $\exists F$ free R-module mapping onto C ie $F \xrightarrow{g} C \to C$

Then we have the SES: $0 \to \ker g \xhookrightarrow{i} F \xrightarrow{g} C \to 0$

And the sequence splits by hypothesis

$\therefore F \cong C \oplus \ker g$

But F free $\Rightarrow$ F projective $\Rightarrow$ C is projective since it is a direct summand of a projective module

Then $\exists j: C \to M \ni$ the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "t"] & N \arrow[r] & 0 \\ & C \arrow[ul, dashed, "\exists j"] \arrow[u, "h"'] & \end{tikzcd}$$

$\therefore tj = h \Rightarrow h = t_*(j)$

$\therefore t_*$ is surjective

$\therefore Hom_R(C,-)$ exact

b. Prove that for an R-module A, TFAE:

(i) For each SES $0 \to L \xrightarrow{s} M \xrightarrow{t} N \to 0$ of R-modules, the sequence of abelian groups

$0 \to Hom_R(N,A) \to Hom_R(M,A) \to Hom_R(L,A) \to 0$

is exact

(ii) Every SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits

(i)$\Rightarrow$(ii) Assume $Hom_R(-,A)$ exact

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a SES

Then we have the following SES:

$0 \to Hom_R(C,A) \to Hom_R(B,A) \to Hom_R(A,A) \to 0$

In particular, $Hom_R(B,A) \xrightarrow{f^*} Hom_R(A,A)$ surjective

But $1_A \in Hom_R(A,A)$, so $\exists h \in Hom_R(B,A) \ni 1_A = f^*(h) = hf$

$\therefore$ $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits

(ii)$\Rightarrow$(i) Assume every SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits

Let $0 \to L \xrightarrow{s} M \xrightarrow{t} N \to 0$ be a SES

It suffices to show $Hom_R(M,A) \xrightarrow{s^*} Hom_R(L,A)$ is surjective since $Hom_R(-,A)$ is left exact

Let $u \in Hom_R(L,A)$

Note that $A \leq E$ injective

So we have the split SES:

$0 \to A \overset{i}{\hookrightarrow} E \to E/A \to 0$

Then $E \cong E/A \oplus A$

$\therefore$ A is injective since its a direct summand of an injective module

So $\exists v : M \to A \ni$ the diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "s"] \arrow[d, "u"'] & M \arrow[dl, dashed, "\exists v"] \\ & A & \end{tikzcd}$$

$\therefore$ $vs = u \Rightarrow u = s^*(v)$

$\therefore$ s^* is surjective

$\therefore$ $Hom_R(-,A)$ is exact

5. Consider the ring $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$

a. Find $Z(R)$

Note that $\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's+qs' & qp \end{pmatrix}$

And $\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix}\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix} = \begin{pmatrix} sr & 0 \\ s'r+pr' & pq \end{pmatrix}$

So $\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix} \in Z(R)$ iff $s'r + pr' = r's + qs'$ $\forall r, r' \in \mathbb{R}$ and $\forall q \in \mathbb{Q}$

In particular, $s' + p = s + s' \Rightarrow p = s$

And $p = s + s' \Rightarrow s' = 0$

So $\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$

And $\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}$ $\forall r, r', q \Leftrightarrow \begin{pmatrix} rp & 0 \\ r'p & qp \end{pmatrix} = \begin{pmatrix} pr & 0 \\ pr' & pq \end{pmatrix}$ $\forall r, r', q$

$\Longleftrightarrow$ $rp = pr$ $\forall r \in \mathbb{R}$ and $qp = pq$ $\forall q \in \mathbb{Q}$

But $\mathbb{R}, \mathbb{Q}$ commutative, so $Z(R) = \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \middle| \, p \in \mathbb{Q} \right\}$

b. Is the ring R left artinian?

Let $I = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix}$

Then $\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ qa & 0 \end{pmatrix} \in I$ and $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ r & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ar & 0 \end{pmatrix} \in I$

$\therefore I \triangleleft R$

Also there is a 1-1 correspondence between left submodule of ${}_R I$ and left submodules of ${}_{\mathbb{Q}}\mathbb{R}$ preserving inclusions

And ${}_{\mathbb{Q}}\mathbb{R}$ is not finitely generated

Hence ${}_R I$ is not finitely generated

$\therefore$ R is not left noetherian ring

$\therefore$ R is not left artinian ring

c. Is the ring R left noetherian?

R is not left noetherian as shown above

d. Is the ring R right artinian?

Claim $R/I \cong \mathbb{R} \times \mathbb{Q}$ as rings

Define $\varphi: R \longrightarrow \mathbb{R} \times \mathbb{Q} \ni \varphi\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix} = (r, q)$

$\varphi\left(\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix} + \begin{pmatrix} s & 0 \\ s' & p \end{pmatrix}\right) = \varphi\begin{pmatrix} r+s & 0 \\ r'+s' & q+p \end{pmatrix} = (r+s, q+p) = (r,q) + (s,p)$

$= \varphi\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix} + \varphi\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix}$

$\varphi\left(\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix}\right) = \varphi\begin{pmatrix} rs & 0 \\ r's+qs' & qp \end{pmatrix} = (rs, qp) = (r,q)(s,p)$

$= \varphi\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\varphi\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix}$

$\therefore \varphi$ is a ring homomorphism

And $\operatorname{Ker}\varphi = \{r \in R \mid \varphi(r) = 0_{\mathbb{R}\times\mathbb{Q}}\}$

But $\varphi(r) = 0_{\mathbb{R}\times\mathbb{Q}}$ iff $\varphi\begin{pmatrix} s & 0 \\ s' & p \end{pmatrix} = (0,0)$ iff $(s,p) = (0,0)$

iff $r = \begin{pmatrix} 0 & 0 \\ s' & p \end{pmatrix}$ iff $r \in I$

$\therefore \ker\varphi = I$

And $(r,q) = \varphi\begin{pmatrix} r & 0 \\ 0 & q \end{pmatrix}$ $\forall r, q$

$\therefore \varphi$ surjective

Then by 1st Iso Thm, $R/I \cong \mathbb{R} \times \mathbb{Q}$

So we have the SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$

Note that there is a 1-1 correspondence between right submodules of I_R and $\mathbb{R}_{\mathbb{R}}$ preserving inclusions

But $\mathbb{R}$ field, so its only ideals are (0) and $\mathbb{R}$

Hence the only right submodules of I_R are (0) and I

$\therefore I$ satisfies the DCC

$\therefore I$ right artinian

And R/I is right artinian since $\mathbb{R}, \mathbb{Q}$ are fields hence artinian thus $\mathbb{R} \times \mathbb{Q}$ artinian

$\therefore R$ is right artinian as a module over itself

$\therefore R$ right artinian ring

e. Is the ring R right noetherian?

R is right noethenan ring since R is right artinian ring

f. Find $J(R)$ and describe the ring structure of $R/J(R)$ in terms of $\mathbb{Q}$ and $\mathbb{R}$

Let $J_1 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$, $J_2 = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & 0 \end{pmatrix}$, $J_3 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's & qp \end{pmatrix} \notin J_1 \Rightarrow J_1$ is not a left ideal of R

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ s' & 0 \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's+qs' & 0 \end{pmatrix} \in J_2 \Rightarrow J_2$ is a left ideal of R

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ s & p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ qs & qp \end{pmatrix} \in J_3 \Rightarrow J_3$ is a left ideal of R

In fact, J_2 and J_3 are the only maximal left ideals of R

$\therefore J(R) = J_2 \cap J_3 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix} = I$

$\therefore J(R) = I$

Then $R/J(R) = R/I \cong \mathbb{R} \times \mathbb{Q}$

Hence $R/J(R)$ has the ring structure of $\mathbb{R} \times \mathbb{Q}$

g. Describe the nonisomorphic simple left R-modules by indicating their underlying abelian group and R-action

Note that the nonisomorphic simple left R-modules are exactly the nonisomorphic simple left $R/J(R)$-modules hence the ones isomorphic to the simple $\mathbb{R} \times \mathbb{Q}$-modules

And the simple $\mathbb{R} \times \mathbb{Q}$-modules are $\mathbb{R} \times (0)$ and $(0) \times \mathbb{Q}$

Qualifying Exam, August 2012. Algebra I.

Instructions: Solve 4 of the following 6 problems.

1. Let $N \subseteq M$ be R-modules. We say N is an *essential* submodule of M if N intersects every nonzero submodule of M nontrivially. We denote this by $N \, ess \, M$.

(a) Let $L \subseteq N \subseteq M$ be R-modules with $L \, ess \, N$ and $N \, ess \, M$. Show that $L \, ess \, M$.

(b) Assume $N \, ess \, M$ and $m \in M$ and let $I = \{r \in R \mid rm \in N\}$. Show that $I \, ess \, R$.

2. Assume that R and S are rings and that ${}_R M_S$ is an R-S-bimodule. Let $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$.

(a) Give necessary and sufficient conditions on R, S and M for T to be left artinian.

(b) Give an example of a ring that is left artinian but not right artinian.

3. Let M be an R-module with simple submodules $S_1, S_2, \ldots, S_n$ and assume $M = S_1 \oplus S_2 \oplus \ldots \oplus S_n$.

(a) If U is any simple submodule of M, show that $[S_1 \oplus \ldots \oplus S_{i-1} \oplus S_{i+1} \oplus \ldots \oplus S_n] + U$ intersects S_i nontrivially for some i.

(b) Using the same i as in part (a), show that $M = S_1 \oplus \ldots \oplus S_{i-1} \oplus S_{i+1} \oplus \ldots \oplus S_n \oplus U$.

4. Let $\boldsymbol{Z}$ be the ring of integers and $\boldsymbol{Q}$ the field of rational numbers.

(a) Let $A \subseteq B$ be $\boldsymbol{Z}$-modules. Show that $0 \to \boldsymbol{Q} \otimes_{\boldsymbol{Z}} A \xrightarrow{I \otimes i} \boldsymbol{Q} \otimes_{\boldsymbol{Z}} B$ is exact, where I is the identity map and $i : A \to B$ is the inclusion map.

(b) Show that $\boldsymbol{Q}$ is not free as a $\boldsymbol{Z}$-module.

(c) Conclude that $\boldsymbol{Q}$ is a flat $\boldsymbol{Z}$-module that is not projective.

5. (a) If S is a simple R-module, show that $D = End_R(S)$ is a division ring.

(b) Let $x_1, x_2 \in S$ and let $A_i = \{r \in R \mid rx_i = 0\}$, for i = 1 or 2. Viewing S as a vector space over D, show that x_1, x_2 are linearly independent over D if and only if neither of A_1 and A_2 is contained in the other.

6. (a) If I is a nil left ideal of R, show that $I \subseteq J(R)$, where $J(R)$ denotes the Jacobson radical of R.

(b) Give an example of a ring R where $J(R)$ is not a nil ideal.

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1. Let $N \subseteq M$ be R-modules.

a. Let $L \subseteq N \subseteq M$ be R-modules with $L \overset{ess}{\subseteq} N$ and $N \overset{ess}{\subseteq} M$. Show that $L \overset{ess}{\subseteq} M$.

Let $0 \neq X \leq M$

Then $X \cap N \neq 0$ since $N \overset{ess}{\subseteq} M$

Thus $X \cap N \cap L \neq 0$ since $0 \neq X \cap N \leq N$ and $L \overset{ess}{\subseteq} N$

And $0 \neq X \cap N \cap L = X \cap L$ since $L \subseteq N \Rightarrow L \cap N = L$

$\therefore X \cap L \neq 0$

$\therefore L \overset{ess}{\subseteq} M$

b. Assume $N \overset{ess}{\subseteq} M$ and $m \in M$ and let $I = \{r \in R \mid rm \in N\}$. Show that $I \overset{ess}{\subseteq} R$.

Let $0 \neq X \leq R$

Suppose $X \cap I = 0$

Then $\{x \in X \mid xm \in N\} = 0$

But $0 \neq Xm \leq M$ since $X \neq 0$

And $N \cap Xm = 0$

Contradiction since $N \overset{ess}{\subseteq} M \Rightarrow N \cap Xm \neq 0$

$\therefore X \cap I \neq 0$

$\therefore I \overset{ess}{\subseteq} R$

2. Assume that R and S are rings and that ${}_RM_S$ is an R-S-bimodule. Let $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$.

(a) Give necessary and sufficient conditions on R, S, T for T to be left artinian.

Let $N = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & rn \\ 0 & 0 \end{bmatrix} \in N$

$\therefore$ N is a left T-module

Claim $T/N \cong R\times S$ as rings

Define $\varphi: T \to R\times S \ni \varphi\left(\begin{pmatrix} r & m \\ 0 & s\end{pmatrix}\right) = (r,s)$

$$\varphi\left(\begin{pmatrix} r_1 & m_1 \\ 0 & s_1\end{pmatrix} + \begin{pmatrix} r_2 & m_2 \\ 0 & s_2\end{pmatrix}\right) = \varphi\left(\begin{pmatrix} r_1+r_2 & m_1+m_2 \\ 0 & s_1+s_2\end{pmatrix}\right) = (r_1+r_2, s_1+s_2)$$
$$= (r_1,s_1)+(r_2,s_2)$$
$$= \varphi\left(\begin{pmatrix} r_1 & m_1 \\ 0 & s_1\end{pmatrix}\right) + \varphi\left(\begin{pmatrix} r_2 & m_2 \\ 0 & s_2\end{pmatrix}\right)$$

$$\varphi\left(\begin{pmatrix} r_1 & m_1 \\ 0 & s_1\end{pmatrix}\begin{pmatrix} r_2 & m_2 \\ 0 & s_2\end{pmatrix}\right) = \varphi\left(\begin{pmatrix} r_1r_2 & r_1m_2+m_1s_2 \\ 0 & s_1s_2\end{pmatrix}\right) = (r_1r_2, s_1s_2) = (r_1,s_1)(r_2,s_2)$$
$$= \varphi\left(\begin{pmatrix} r_1 & m_1 \\ 0 & s_1\end{pmatrix}\right)\varphi\left(\begin{pmatrix} r_2 & m_2 \\ 0 & s_2\end{pmatrix}\right)$$

$\therefore \varphi$ ring homomorphism

$\varphi\left(\begin{pmatrix} r & m \\ 0 & s\end{pmatrix}\right) = (0,0)$ iff $(r,s) = (0,0)$ iff $\begin{pmatrix} r & m \\ 0 & s\end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & 0\end{pmatrix} \in N$

$\therefore \ker\varphi = N$

Let $(r,s)\in R\times S$

Then $(r,s) = \varphi\left(\begin{pmatrix} r & 0 \\ 0 & s\end{pmatrix}\right)$

$\therefore \varphi$ surjective

Then by 1st Iso Thm, $T/N \cong R\times S$ as rings

Now consider the SES:

$$0 \longrightarrow N \longrightarrow T \longrightarrow T/N \longrightarrow 0$$

T is left artinian iff N and T/N are left artinian
iff N and $R\times S$ are left artinian
iff N, R, S left artinian

But since $\begin{bmatrix} r & m \\ 0 & s\end{bmatrix}\begin{bmatrix} 0 & n \\ 0 & 0\end{bmatrix} = \begin{bmatrix} 0 & rn \\ 0 & 0\end{bmatrix}$, we get a 1-1 correspondence

$$\{\text{left } T\text{-submodules of } N\} \longleftrightarrow \{\text{left } R\text{-submodules of } M\}$$

preserving inclusions

so N is left artinian iff M is left artinian

$\therefore T$ is left artinian iff M, R, S are left artinian

b. Give an example of a ring that is left artinian but not right artinian

First note that since $\mathbb{R}, \mathbb{Q}$ are fields, hence division rings, $\mathbb{R}, \mathbb{Q}$ are left artinian

Hence $R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}$ is left artinian ring by (a)

Let $I = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}$

$\begin{pmatrix} r & p \\ 0 & q \end{pmatrix}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & rt \\ 0 & 0 \end{pmatrix} \in I$ and $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & p \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & tq \\ 0 & 0 \end{pmatrix} \in I$

$\therefore I \triangleleft R$

And we get 1-1 correspondence

$\{$right R-submodules of $I\} \longleftrightarrow \{$right $\mathbb{Q}$-submodules of $\mathbb{R}\}$

But $\mathbb{R}_{\mathbb{Q}}$ is not finitely generated

So I not finitely generated

$\therefore R$ is not right noetherian ring

$\therefore R$ is not right artinian ring

3. Let M be an R-module with simple submodules $S_1, \dots, S_n$ and assume that $M = S_1 \oplus \dots \oplus S_n$.

a. If U is any simple submodule of M, show that $([S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n] + U) \cap S_i \neq 0$ for some i.

First note that $U \cap S_i \leq U, S_i$ simple $\forall i$

So $U \cap S_i = 0$ or $U \cap S_i = U$ or $U \cap S_i = S_i$ for each i

If $U \cap S_i = U$, then $U \leq S_i$, hence $U = S_i$ since S_i simple

If $U \cap S_i = S_i$, then $U = S_i$ since U is simple

$\therefore U \cap S_i = 0$ or $U = S_i$ for each i

If $U \cap S_i = 0$ $\forall i$, then $U \cap (S_1 \oplus \dots \oplus S_n) = 0$, hence $U \cap M = 0$

Contradiction since $U \leq M$

Then $U = S_i$ for some i

$\therefore M = S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n \oplus U$

$\therefore ([S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n] + U) \cap S_i = M \cap S_i = S_i \neq 0$

$\therefore ([S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n] + U) \cap S_i \neq 0$

b. Show that $M = S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n \oplus U$

we proved the result in (a)

4. a. Let $A \subseteq B$ be $\mathbb{Z}$-modules. Show that $0 \to \mathbb{Q} \otimes_{\mathbb{Z}} A \xrightarrow{1 \otimes i} \mathbb{Q} \otimes_{\mathbb{Z}} B$ is exact where i is the inclusion map.

Let $\sum q_i \otimes a_i \in \ker(1 \otimes i)$

Then $0 = (1 \otimes i)(\sum q_i \otimes a_i) = \sum (1 \otimes i)(q_i \otimes a_i) = \sum q_i \otimes a_i$

$\therefore \sum q_i \otimes a_i = 0$

$\therefore \ker(1 \otimes i) = 0$

$\therefore 1 \otimes i$ injective

$\therefore 0 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} A \xrightarrow{1 \otimes i} \mathbb{Q} \otimes_{\mathbb{Z}} B$ exact

b. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module

Suppose $\mathbb{Q}$ is free over $\mathbb{Z}$

Let $0 \neq q_1, q_2 \in \mathbb{Q} \Rightarrow q_1 = \frac{m_1}{n_1}$, $q_2 = \frac{m_2}{n_2}$, $m_1, m_2, n_1, n_2 \in \mathbb{Z}$

So $q_1 = \frac{m_1 n_2}{n_1 n_2}$ and $q_2 = \frac{m_2 n_1}{n_1 n_2}$

Let $\alpha = n_1 m_2$ and $\beta = m_1 n_2$

Then $\alpha q_1 - \beta q_2 = \frac{m_1 m_2 n_1 n_2 - m_1 m_2 n_1 n_2}{n_1 n_2} = 0$

$\therefore$ Any two rational numbers are linearly dependent

$\therefore$ A basis for $\mathbb{Q}$ must have only one element

Let $\{e\}$ be a basis for $\mathbb{Q} \Rightarrow e = \frac{m}{n}$, $(m,n) = 1$

Note that $\frac{m}{2n} \in \mathbb{Q}$ but $\frac{m}{2n} \neq r\frac{m}{n}$ for any $r \in \mathbb{Z}$

$\therefore \langle e \rangle \neq \mathbb{Q}$

contradiction

$\therefore \mathbb{Q}$ not free $\mathbb{Z}$-module

c. Conclude that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module that is not projective.

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a SES

Then we have an isomorphism of SES's:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "f"', "\wr"] \arrow[dr, phantom, "G"] & B \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "G"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Im} f \arrow[r, hook, "i"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Let $\tilde{A} = \operatorname{Im} f$

Since $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is right exact, it suffices to show that

$0 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} B$ is exact

But we did this in (a), hence $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is exact

$\therefore$ $\mathbb{Q}$ is a flat $\mathbb{Z}$-module

But $\mathbb{Z}$ is a PID, hence $\mathbb{Q}$ is free iff $\mathbb{Q}$ is projective

And $\mathbb{Q}$ is not free by (b)

$\therefore$ $\mathbb{Q}$ is not projective

5. a. If S is a simple R-module, show that $D = \operatorname{End}_R(S)$ is a division ring.

Let $0 \neq f \in D \Rightarrow f: S \longrightarrow S$ is an R-module homomorphism

Now $\ker f \leq S$ simple, so $\ker f = 0$ or $\ker f = S$

If $\ker f = S$, then $f \equiv 0$ which is a contradiction

$\therefore$ $\ker f = 0$

$\therefore$ f injective

Also $\operatorname{Im} f \leq S$ simple, so $\operatorname{Im} f = 0$ or $\operatorname{Im} f = S$

If $\operatorname{Im} f = 0$, then $f \equiv 0$ which is a contradiction

$\therefore$ $\operatorname{Im} f = S$

$\therefore f$ surjective

$\therefore f$ isomorphism

$\therefore \exists f^{-1} \in D \ni ff^{-1} = f^{-1}f = 1_S$

$\therefore D = End_R(S)$ is a division ring

b. Let $x_1, x_2 \in S$ and let $A_i = \{r \in R \mid rx_i = 0\}$, $i = 1, 2$. Viewing S as a vector space over D, show that x_1, x_2 are linearly independent over D iff neither of A_1 and A_2 is contained in the other.

($\Leftarrow$) Assume $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$

Let $f(x_1) + g(x_2) = 0$

Then $r(f(x_1) + g(x_2)) = 0 \quad \forall r \in R$

So $f(rx_1) + g(rx_2) = 0 \quad \forall r \in R$

Take $r \in A_1 \ni r \notin A_2$

Then $f(0) + g(rx_2) = 0$

So $g(rx_2) = 0$

If $g \neq 0$, then $\exists g^{-1}$ by (a), so $g^{-1}g(rx_2) = 0$, hence $rx_2 = 0$

Contradiction since $r \notin A_2$

$\therefore g \equiv 0$

Now $f(rx_1) + g(rx_2) = 0 \quad \forall r \in R$

Take $r \in A_2 \ni r \notin A_1$

So $f(rx_1) + g(0) = 0$

And thus $f(rx_1) = 0$

If $f \neq 0$, as before we get a contradiction

$\therefore f \equiv 0$

$\therefore x_1, x_2$ linearly independent over D

($\Rightarrow$) Assume x_1, x_2 linearly independent over D

6. a. If I is a nil left ideal of R, show that $I \subseteq J(R)$.

Let $x \in I$

Then $rx \in I$ $\forall r \in R$ since I left ideal of R

And since I is nil, rx is nilpotent

Claim If y is nilpotent, then $1-y$ is a unit

y nilpotent $\Rightarrow$ $y^n = 0$ for some $n \geq 0$

Then $(1-y)(y^{n-1} + \dots + 1) = y^{n-1} + \dots + 1 - y^n - \dots - y = 1$

And similarly $(y^{n-1} + \dots + 1)(1-y) = 1$

$\therefore 1-y$ is a unit

Then by claim, $1-rx$ is a unit $\forall r \in R$

In particular, $1-rx$ has a left inverse $\forall r \in R$

$\therefore x \in J(R)$

$\therefore I \subseteq J(R)$

b. Give an example of a ring R where $J(R)$ is not a nil ideal

Take $R = \mathbb{Z}$

Then $J(R) = \bigcap$ maximal ideals

$= \bigcap_{p \text{ prime}} \langle p \rangle$

$J(R)$ is not nil since the only nilpotent elements of $\mathbb{Z}$ are $0, 1$ and since $1 \notin \bigcap_{p \text{ prime}} \langle p \rangle \neq 0$

Take $R = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ odd}\}$ local with max ideal $M = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ odd}, m \text{ even}\}$

$\Rightarrow J(R) = M$

But $\frac{m}{n}$ not nilpotent for any nonzero $\frac{m}{n} \in M$

$\therefore J(R)$ not nil

August 2012

Algebra Qualifying Examination
Homological Algebra Part

Solve 4 out of the following 6 problems:

1. Let $\mathcal{A}$ be an abelian category and let $f\colon B^\bullet \to C^\bullet$ be a morphism of chain complexes of objects in $\mathcal{A}$. Recall that the mapping cone $C(f)$ of f is the complex in $\mathrm{Com}(\mathcal{A})$ defined as follows: for each $n \in \mathbb{Z}$, $C(f)^n = B^{n+1} \oplus C^n$ and the differential $d^n_{C(f)}\colon C(f)^n \to C(f)^{n+1}$ is given by the matrix

$$d^n_{C(f)} = \begin{bmatrix} -d_B^{n+1} & 0 \\ f^{n+1} & d_C^n \end{bmatrix}$$

(i) Show that the sequence $0 \to C \to C(f) \to B[1] \to 0$ is exact in $\mathrm{Com}(\mathcal{A})$.

(ii) Show that f is a quasi-isomorphism if and only if $C(f)$ is exact.

2. Let $0 \to A \to B \to C \to 0$ and $0 \to D \to B \to E \to 0$ be two short exact sequence of R-modules. Prove that there exists a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

3. Let $\mathcal{T}$ be a triangulated category and let $L \xrightarrow{u} M \to 0 \to L[1]$ be a distinguished triangle. Prove that u must be an isomorphism.

4. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Prove that $\operatorname{pd} B \leq \max\{\operatorname{pd} A, \operatorname{pd} C\}$. Give an example when the inequality is strict.

5. Let R be a left artinian ring and let $S_1, \ldots, S_n$ be a complete set of non isomorphic simple R-modules. Prove that the left global dimension of R equals $\max\{\operatorname{pd} S_i | i = 1, \ldots, S_n\}$ where pd denotes the projective dimension.

6. Let R be a commutative noetherian ring and let M and N be two finitely generated R-modules. Let E be a flat module over R. Prove that there is an isomorphism

$$\Phi\colon E \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N \otimes_R E)$$

that is natural in E, M and N.

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1. Let $\mathcal{A}$ be an abelian category and let $f: B^{\cdot} \longrightarrow C^{\cdot}$ be a morphism of chain complexes of objects in $\mathcal{A}$.

a. Show that $0 \longrightarrow C \longrightarrow C(f) \longrightarrow B[1] \longrightarrow 0$ is exact in $Com(\mathcal{A})$

It suffices to show that $0 \longrightarrow C^n \xrightarrow{i_2} B^{n+1} \oplus C^n \xrightarrow{p_1} B^{n+1} \longrightarrow 0$ is a SES $\forall n$

Note that i_2 is injective since it is an inclusion and p_1 is surjective since it is a projection

And $\operatorname{Im} i_2 = C^n$

But $p_1(b^{n+1} + c^n) = 0$ iff $b^{n+1} = 0$, hence $\operatorname{Ker} p_1 = C^n$

$\therefore \operatorname{Im} i_2 = \operatorname{Ker} p_1$

$\therefore$ The sequence is exact

b. Show that f is a quasi-isomorphism iff $C(f)$ is exact

Since $0 \longrightarrow C \longrightarrow C(f) \longrightarrow B[1] \longrightarrow 0$ is a SES by (a), we get the LES:

$$\cdots \xrightarrow{\delta^n} H^n(C) \xrightarrow{H^n(i)} H^n(C(f)) \xrightarrow{H^n(p)} H^{n+1}(B) \xrightarrow{\delta^{n+1}} H^{n+1}(C) \xrightarrow{H^{n+1}(i)} H^{n+1}(C(f)) \xrightarrow{H^{n+1}(p)} H^{n+2}(B) \xrightarrow{\delta^{n+2}} \cdots$$

with connecting maps $\delta^n = H^n(f): H^n(B) \longrightarrow H^n(C)$

($\Rightarrow$) Assume f is a quasi-isomorphism

Then δ^n is an isomorphism $\forall n$ since $\delta^n = H^n(f)$

So $H^n(C) = \operatorname{Im} \delta^n = \operatorname{Ker} H^n(i)$

$\therefore H^n(i) \equiv 0$

And $0 = \operatorname{Ker} \delta^{n+1} = \operatorname{Im} H^n(p)$

$\therefore H^n(p) \equiv 0$

So $0 = \operatorname{Im} H^n(C) = \operatorname{Ker} H^n(p)$

Hence $H^n(p) \equiv 0$ is injective

$\therefore H^n(C(f)) = 0 \quad \forall n$

$\therefore C(f)$ is exact

($\Leftarrow$) Assume $C(f)$ is exact

Then $H^n(C(f)) = 0 \;\forall n$

So from the LES, we get: $0 \to H_n(B) \xrightarrow{H_n(f)} H_n(C) \to 0$ is exact $\forall n$

$\therefore H_n(f)$ is an isomorphism $\forall n$

$\therefore f$ is a quasi-isomorphism

2. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to D \xrightarrow{h} B \xrightarrow{j} E \to 0$ be SES's of R-modules. Prove that $\exists$ a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

First let W be the pullback of A, D, $W = \underleftarrow{\lim}\left(\begin{smallmatrix} & D \\ & \downarrow \\ A \to & B \end{smallmatrix}\right)$

Then by definition of pullback, the square

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[d, "\beta"'] & D \arrow[d, "h"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

commutes

And α, β are both injective since f, h are and parrallel maps in pullbacks have the same properties

Now let $Y = \operatorname{coker}\beta$ and $X = \operatorname{coker}\alpha$

Then we have SES's $0 \to W \to A \to Y \to 0$ and $0 \to W \to D \to X \to 0$

And $\exists$ homomorphisms $\epsilon : X \to C$ and $c : Y \to E$ commuting the diagram

Now let $Z = \operatorname{coker} \epsilon$

Note that τ is injective by the 5-Lemma

Then by the snake Lemma, we have the exact sequence

$0 \to \ker \alpha \to \ker f \to \ker \epsilon \to \operatorname{coker} \alpha \to \operatorname{coker} f \to \operatorname{coker} \epsilon \to 0$

But $\ker \alpha = \ker f = \ker \epsilon = 0$ and $\operatorname{coker} \alpha = X$, $\operatorname{coker} f = C$

So we have that $0 \to X \xrightarrow{\epsilon} C$ is injective

$\therefore$ we have the SES $0 \to X \to C \to Z \to 0$

Applying the snake lemma again, we have the exact sequence:

$0 \to \ker \beta \to \ker h \to \ker \epsilon \to \operatorname{coker} \beta \to \operatorname{coker} h \to \operatorname{coker} \epsilon \to 0$

But $\ker \beta = \ker h = \ker \epsilon$ and $\operatorname{coker} \beta = Y$, $\operatorname{coker} h = E$, $\operatorname{coker} \epsilon = Z$

So we get SES: $0 \to Y \to E \to Z \to 0$

$\therefore$ Every row and column is exact and each square commutes

$\therefore \exists$ commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r, "\alpha"] \arrow[d, "\beta"] & D \arrow[r] \arrow[d, "h"] & X \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r, "\tau"] \arrow[d] & E \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

3. Let $\mathcal{T}$ be a Δ'd category and let $L \xrightarrow{u} M \longrightarrow 0 \longrightarrow L[1]$ be a distinguished Δ. Prove that u must be an isomorphism

Since $L \xrightarrow{u} M \longrightarrow 0 \longrightarrow L[1]$ is a distinguished Δ, its rotation $M[-1] \longrightarrow 0 \longrightarrow L \xrightarrow{u} M$ is also a distinguished Δ

Also $M \xrightarrow{1_M} M \longrightarrow 0 \longrightarrow M[1]$ is a distinguished Δ, so its rotation $M[-1] \longrightarrow 0 \longrightarrow M \xrightarrow{1_M} M$ is also a distinguished Δ

Consider the following diagram:

$$\begin{tikzcd}
M[-1] \arrow[r] \arrow[d, "1_{M[-1]}"'] & 0 \arrow[r] \arrow[d] & L \arrow[r, "u"] \arrow[d, dashed] & M \arrow[d, "1_M"] \\
M[-1] \arrow[r] & 0 \arrow[r] & M \arrow[r, "1_M"] & M
\end{tikzcd}$$

Note that the 1st square commutes, so $\exists$ map $L \longrightarrow M$ commuting the diagram

But by construction, that map must be u in order for the last square to commute

And since the other vertical maps are isomorphisms, u is also an isomorphism by 5-lemma for Δ'd category

4. Let $0 \to A \to B \to C \to 0$ be a SES of R-modules. Prove that $\mathrm{pd}\, B \leq \max\{\mathrm{pd}\, A, \mathrm{pd}\, C\}$. Give an example when the inequality is strict

Let X be an R-module and let $m = \max\{\mathrm{pd}\,A, \mathrm{pd}\,C\}$

Then $\mathrm{pd}\,A \leq m$, so $\mathrm{Ext}_R^i(A, X) = 0 \quad \forall i > m$

And also $\mathrm{pd}\,C \leq m$, so $\mathrm{Ext}_R^i(C, X) = 0 \quad \forall i > m$

Consider the LES:

$$\cdots \to \mathrm{Ext}_R^{m+1}(C,X) \to \mathrm{Ext}_R^{m+1}(B,X) \to \mathrm{Ext}_R^{m+1}(A,X) \to \mathrm{Ext}_R^{m+2}(C,X) \to \cdots$$

Hence $\forall i > m$, we have $0 \to \mathrm{Ext}_R^i(B,X) \to 0$ is exact

$\therefore \mathrm{Ext}_R^i(B,X) = 0 \quad \forall i > m$

$\therefore \mathrm{pd}_R\, B \leq m = \max\{\mathrm{pd}\,A, \mathrm{pd}\,C\}$

Consider the SES of $\mathbb{Z}/4\mathbb{Z}$-modules:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Note that $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, hence the sequence does not split, thus $\mathbb{Z}/2\mathbb{Z}$ is not projective

$\therefore \mathrm{pd}_{\mathbb{Z}/4\mathbb{Z}}\, \mathbb{Z}/2\mathbb{Z} > 0$

But $\mathbb{Z}/4\mathbb{Z}$ is projective as a $\mathbb{Z}/4\mathbb{Z}$-module, so $\mathrm{pd}_{\mathbb{Z}/4\mathbb{Z}}\, \mathbb{Z}/4\mathbb{Z} = 0$

$\therefore \mathrm{pd}_{\mathbb{Z}/4\mathbb{Z}}\, \mathbb{Z}/4\mathbb{Z} < \max\{\mathrm{pd}_{\mathbb{Z}/4\mathbb{Z}}\, \mathbb{Z}/2\mathbb{Z}, \mathrm{pd}_{\mathbb{Z}/4\mathbb{Z}}\, \mathbb{Z}/2\mathbb{Z}\}$

5. Let R be a left artinian ring and let $S_1, \ldots, S_n$ be a complete set of nonisomorphic simple R-modules. Prove that the left global dimension of R is $\max\{pd\, S_i \mid i = 1, \ldots, n\}$

First note that by Auslander, $\ell D(R) = \sup\{pd(R/I) : I \text{ left ideal}\}$

Now each $S_i \cong R/m$ where m is a maximal left ideal of R

Hence $pd\, S_i = pd\, R/m \leq \sup\{pd(R/I) : I \text{ left ideal}\} = \ell D(R)$ $\forall i$

$\therefore \ell D(R) \geq \max\{pd\, S_i \mid i = 1, \ldots, n\}$

Now since R is left artinian ring, R is a left noetherian ring, thus R is both artinian and noetherian as an R-module

But we have the SES for each left ideal I:

$$0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$$

So R/I is both artinian and noetherian as an R-module

$\therefore R/I$ has a composition series

Then since $pd\, S_i \leq \max\{pd\, S_i \mid i = 1, \ldots, n\}$ $\forall i$ and this is a complete set of nonisomorphic simple R-modules, we have that $pd\, R/I \leq \max\{pd\, S_i \mid i = 1, \ldots, n\}$ $\forall$ left ideals I of R by Jan 14 732 #1

$\therefore \ell D(R) = \sup\{pd(R/I) : I \text{ left ideal}\} \leq \max\{pd\, S_i \mid i = 1, \ldots, n\}$

$\therefore \ell D(R) = \max\{pd\, S_i \mid i = 1, \ldots, n\}$

6. Let R be a commutative noetherian ring and let M, N be finitely generated R-modules. Let E be a flat R-module. Prove that $\exists$ a natural isomorphism in E, M, N
$\Phi : E \otimes_R \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M, N \otimes_R E)$

First note that since R is noetherian and M is finitely generated, M is finitely presented.
So $\exists$ an exact sequence $R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$
But $\mathrm{Hom}_R(\_, N)$ is left exact, so we have the exact sequence: $0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(R^m, N) \longrightarrow \mathrm{Hom}_R(R^n, N)$
And E is flat, so $E \otimes_R -$ is exact, hence we have the exact sequence:
$0 \longrightarrow E \otimes_R \mathrm{Hom}_R(M,N) \longrightarrow E \otimes_R \mathrm{Hom}_R(R^m, N) \longrightarrow E \otimes_R \mathrm{Hom}_R(R^n, N)$
But $E \otimes_R \mathrm{Hom}_R(R^m, N) \cong E \otimes_R N^m \cong (E \otimes_R N)^m \cong \mathrm{Hom}_R(R^m, E \otimes_R N)$ are natural isomorphisms
And $\mathrm{Hom}_R(\_, N \otimes_R E)$ is left exact so we have the exact sequence: $0 \to \mathrm{Hom}_R(M, N \otimes_R E) \to \mathrm{Hom}_R(R^m, N \otimes_R E) \to \mathrm{Hom}_R(R^n, N \otimes_R E)$
So we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & E \otimes_R \mathrm{Hom}_R(M,N) \arrow[r] \arrow[d, dashed] & E \otimes_R \mathrm{Hom}_R(R^m, N) \arrow[r] \arrow[d, "\wr"] & E \otimes_R \mathrm{Hom}_R(R^n, N) \arrow[d, "\wr"] \\
0 \arrow[r] & \mathrm{Hom}_R(M, N \otimes_R E) \arrow[r] & \mathrm{Hom}_R(R^m, N \otimes_R E) \arrow[r] & \mathrm{Hom}_R(R^n, N \otimes_R E)
\end{tikzcd}$$

Note the diagram has exact rows, so $\exists$ a homomorphism $\Phi : E \otimes_R \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M, N \otimes_R E)$ commuting the diagram and by the 5-Lemma, Φ is an isomorphism
And by commutativity, we see that $\Phi(e \otimes g)(m) = g(m) \otimes e$
Let $f : E \longrightarrow E'$, then we have diagram:

$$\begin{tikzcd}
E \otimes_R \mathrm{Hom}_R(M,N) \arrow[r, "f \otimes 1"] \arrow[d, "\Phi_E"] & E' \otimes_R \mathrm{Hom}_R(M,N) \arrow[d, "\Phi_{E'}"] \\
\mathrm{Hom}_R(M, N \otimes_R E) \arrow[r, "(1 \otimes f)_*"] & \mathrm{Hom}_R(M, N \otimes_R E')
\end{tikzcd}$$

$\Phi_{E'}(f \otimes 1)(e \otimes g)(m) = \Phi_{E'}(f(e) \otimes g)(m) = g(m) \otimes f(e)$
$(1 \otimes f)_* \Phi_E(e \otimes g)(m) = (1 \otimes f)(g(m) \otimes e) = g(m) \otimes f(e)$
$\therefore$ Diagram commutes, hence Φ is natural in E

Now let $f: M \to M'$, then we have diagram:

$$\begin{tikzcd} E \otimes_R \mathrm{Hom}_R(M,N) \arrow[d, "\Phi_M"'] & E \otimes_R \mathrm{Hom}_R(M',N) \arrow[l, "1\otimes f^*"'] \arrow[d, "\Phi_{M'}"] \\ \mathrm{Hom}_R(M, N \otimes_R E) & \mathrm{Hom}_R(M', N \otimes_R E) \arrow[l, "f^*"'] \end{tikzcd}$$

$\Phi_M(1 \otimes f^*)(e \otimes g)(m) = \Phi_M(e \otimes gf)(m) = g(f(m)) \otimes e$

$f^* \Phi_{M'}(e \otimes g)(m) = \Phi_{M'}(e \otimes g)(f(m)) = g(f(m)) \otimes e$

$\therefore$ Diagram commutes, hence Φ is natural in M

Finally let $f: N \to N'$, then we have:

$$\begin{tikzcd} E \otimes_R \mathrm{Hom}_R(M,N) \arrow[r, "1\otimes f_*"] \arrow[d, "\Phi_N"'] & E \otimes_R \mathrm{Hom}_R(M,N') \arrow[d, "\Phi_{N'}"] \\ \mathrm{Hom}_R(M, N \otimes_R E) \arrow[r, "(f\otimes 1)_*"] & \mathrm{Hom}_R(M, N' \otimes_R E) \end{tikzcd}$$

$\Phi_{N'}(1 \otimes f_*)(e \otimes g)(m) = \Phi_{N'}(e \otimes fg)(m) = f(g(m)) \otimes e$

$(f \otimes 1)_* \Phi_N(e \otimes g)(m) = (f \otimes 1)(g(m) \otimes e) = f(g(m)) \otimes e$

$\therefore$ Diagram commutes, hence Φ is natural in N

Algebra Part of the Qualifying Examination, January 2012

Instructions. *There are 4 questions on two pages worth the total of 100 points. Do all questions, and justify your answers with the necessary proofs. All rings are associative (not necessarily commutative) with identity. All modules are unitary. We denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q}$ and $\mathbb{R}$ the fields of rational and real numbers, respectively.*

1. (a) (5 points) Prove that the map $\Phi : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}) \to \mathbb{Q}$ given by $\Phi(f) = f(1)$, for all $f : \mathbb{Q} \to \mathbb{Q}$, is an isomorphism of rings.

(b) (5 points) Use (a) to show that every nonzero endomorphism of the abelian group $\mathbb{Q}$ is an automorphism.

(c) (5 points) If A is a finite abelian group, characterize the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},A)$ up to isomorphism.

(d) (5 points) Characterize the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})$ up to isomorphism.

2. Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\kappa} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \to 0$ of $\mathbb{Z}$-modules where $\kappa(z) = 4z$, $z \in \mathbb{Z}$, and π is the natural projection. Using that functor Hom is left exact and functor $\otimes$ is right exact, determine whether the following sequences of $\mathbb{Z}$-modules are exact.

(a) (8 points) The sequence obtained by tensoring the above sequence with $\mathbb{Z}/2\mathbb{Z}$,

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\kappa\otimes\mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi\otimes\mathrm{id}} \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

(b) (7 points) The sequence obtained by tensoring the above sequence with $\mathbb{Q}$,

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\kappa\otimes\mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\pi\otimes\mathrm{id}} \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0.$$

(c) (8 points) The sequence obtained by homming the above sequence into $\mathbb{Q}$,

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z},\mathbb{Q}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\pi,\mathbb{Q})} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\kappa,\mathbb{Q})} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) \to 0.$$

(d) (7 points) The sequence obtained by homming $\mathbb{Q}$ into the above sequence,

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\kappa)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\pi)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}/4\mathbb{Z}) \to 0.$$

3. Recall that an element e of a ring R is an *idempotent* if $e^2 = e$.

(a) (8 points) Prove that eRe is a ring with identity e.

(b) (9 points) Let $e \in R$ be an idempotent and let M be a left R-module. Prove that the map $\Phi : \mathrm{Hom}_R(Re, M) \to eM$ given by $\Phi(f) = f(e)$, for all $f : Re \to M$, is an isomorphism of abelian groups.

(c) (8 points) For any ring S, denote by S^{op} the opposite ring of S. Recall that S^{op} has the same underlying abelian group as S with multiplication given by $s \circ t = ts$, for all $s, t \in S$, where juxtaposition denotes multiplication in S. Use (b) to prove that the endomorphism ring of Re, $\mathrm{End}_R(Re) = \mathrm{Hom}_R(Re, Re)$, is isomorphic to the ring $(eRe)^{\mathrm{op}}$.

4. Consider the ring $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$ of all 2×2 matrices $A = (a_{ij})$ for which $a_{11}, a_{21} \in \mathbb{R}$, $a_{22} \in \mathbb{Q}$, and $a_{12} = 0$, with the usual operations of matrix addition and multiplication.

(a) (4 points) Find the center of R, $Z(R) = \{z \in R \,|\, zr = rz \text{ for all } r \in R\}$.

(b) (4 point) Is the ring R left artinian?

(c) (4 points) Is the ring R left noetherian?

(d) (4 point) Is the ring R right artinian?

(e) (3 points) Is the ring R right noetherian?

(f) (3 points) Find the radical of R, $J(R)$, and describe the ring structure of $R/J(R)$ in terms of $\mathbb{Q}$ and $\mathbb{R}$.

(g) (3 points) Describe the nonisomorphic simple right R-modules by indicating their underlying abelian group and R-action.

January 2012

1. a. Prove that $\Phi : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}) \longrightarrow \mathbb{Q} \ni \Phi(f) = f(1)$ is an isomorphism of rings

First note that since $\mathbb{Q}$ is abelian, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})$ is a ring
Now let $x \in \mathbb{Q}$, $x = \frac{m}{n}$ and let $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})$
Then $m\varphi(1) = \varphi(m) = \varphi(nx) = n\varphi(x) = n\varphi(\frac{m}{n})$
$\therefore \varphi(\frac{m}{n}) = \varphi(1)\frac{m}{n} \quad \forall \varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})$
$\Phi(f_1+f_2) = (f_1+f_2)(1) = f_1(1) + f_2(1) = \Phi(f_1) + \Phi(f_2)$
$\Phi(f_1 f_2) = f_1(f_2(1)) = f_1(f_2(1)\cdot 1) = f_2(1) f_1(1)$ since $f_2(1) \in \mathbb{Z}$
$= f_1(1) f_2(1)$ by commutativity
$= \Phi(f_1)\Phi(f_2)$
$\therefore \Phi$ is a ring homomorphism
Let $f \in \ker\Phi \Rightarrow \Phi(f) = 0$
Let $\frac{m}{n} \in \mathbb{Q}$, then $f(\frac{m}{n}) = f(1)\frac{m}{n} = \Phi(f)\frac{m}{n} = 0 \cdot \frac{m}{n} = 0$
$\therefore f \equiv 0$
$\therefore \ker\Phi = 0$
$\therefore \Phi$ injective
Let $\frac{m}{n} \in \mathbb{Q}$ and define $\varphi_{\frac{m}{n}} : \mathbb{Q} \longrightarrow \mathbb{Q} \ni \varphi_{\frac{m}{n}}(x) = \frac{m}{n}x$
Then $\frac{m}{n} = \frac{m}{n} \cdot 1 = \varphi_{\frac{m}{n}}(1) = \Phi(\varphi_{\frac{m}{n}})$, $\varphi_{\frac{m}{n}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})$
$\therefore \Phi$ surjective
$\therefore \Phi$ is a ring isomorphism

b. Show that every nonzero endomorphism of the abelian group $\mathbb{Q}$ is an automorphism

Let $0 \neq f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$
Find $0 \neq g \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) \ni fg = gf = 1_{\mathbb{Q}}$
Since $\mathbb{Q}$ is a field, $f(1)$ has an inverse, q
But Φ is surjective so $q = \Phi(g)$ for some $g \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$
So $1 = f(1)q = \Phi(f)\Phi(g) = \Phi(fg)$ since Φ is a ring homomorphism
$\therefore f(g(1)) = 1$

So $fg = 1_{\mathbb{Q}}$

And similarly $gf = 1_{\mathbb{Q}}$

$\therefore g = f^{-1}$

$\therefore f$ is an automorphism

c. If A is a finite abelian group, characterize the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, A)$ up to isomorphism

Suppose $\exists\, 0 \neq \varphi \in \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q})$

Since A is a finite group, every element has finite order

Hence $\forall a \in A\ \exists\, 0 \neq n \ni na = 0$

Then $0 = \varphi(0) = \varphi(na) = n\varphi(a)$

$\therefore \varphi(a) = 0 \quad \forall a \in A$

Contradiction

$\therefore \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}) = 0$

Suppose $\exists\, 0 \neq \varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, A)$

Then again $\forall \frac{m}{n} \in \mathbb{Q}$, $\exists\, 0 \neq p \ni p\varphi(\frac{m}{n}) = 0$

So $pm\varphi(\frac{1}{n}) = 0 \Rightarrow \varphi(\frac{1}{n}) = 0 \quad \forall n \in \mathbb{Z}$

Then $\forall \frac{m}{n} \in \mathbb{Q}$, $\varphi(\frac{m}{n}) = m\varphi(\frac{1}{n}) = 0$

Contradiction

$\therefore \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) \cong 0$

d. Characterize the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ up to isomorphism

Define $\Phi : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \to \mathbb{Q}$ $\quad \Phi(f) = f(1)$

$\Phi(f_1 + f_2) = (f_1 + f_2)(1) = f_1(1) + f_2(1) = \Phi(f_1) + \Phi(f_2)$

$\Phi(zf) = (zf)(1) = zf(1) = z\Phi(f)$

$\therefore \Phi$ is a $\mathbb{Z}$-module homomorphism

Let $f \in \ker\Phi$ and let $z \in \mathbb{Z}$

Then $f(z) = zf(1) = z\Phi(f) = z \cdot 0 = 0$

$\therefore f = 0$

$\therefore \operatorname{Ker}\Phi = 0$

$\therefore \Phi$ is injective

Let $q \in \mathbb{Q}$ and define $\varphi_q : \mathbb{Z} \longrightarrow \mathbb{Q} \ni \varphi_q(z) = zq$ which is a $\mathbb{Z}$-module homomorphism

Then $q = 1 \cdot q = \varphi_q(1) = \Phi(\varphi_q)$

$\therefore \Phi$ is surjective

$\therefore \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Q}$ as $\mathbb{Z}$-modules

Now suppose $\exists\, 0 \neq \varphi \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$

Then take $n \in \mathbb{Z}$ to be the smallest positive integer in $\operatorname{Im}\varphi$

Say $\varphi\left(\frac{a}{b}\right) = n$

Then $n = \varphi\left(\frac{a}{b}\right) = \varphi\left(\frac{na}{nb}\right) = n\varphi\left(\frac{a}{nb}\right)$

$\therefore 1 = \varphi\left(\frac{a}{nb}\right)$

Contradiction to minimality of n

$\therefore \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$

2. Consider the SES $0 \longrightarrow \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$ of $\mathbb{Z}$-modules. Determine whether the following sequences of $\mathbb{Z}$-modules are exact.

a. $0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{4 \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi \otimes 1} \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$

First note that $\_ \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is right exact, so it suffices to check whether $4 \otimes 1$ is injective

And $\exists$ natural isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ so we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r, "4 \otimes 1"] \arrow[d, "\phi"', "\wr"] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[d, "\wr"', "\phi"] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r, "4"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

Suppose $4 \otimes 1$ is injective

Then $\phi(4 \otimes 1)$ is injective since ϕ is an isomorphism

Hence $4 \circ \phi$ is injective

$\therefore 4$ is injective since ϕ is an isomorphism

Contradiction since $\mathbb{Z}/2\mathbb{Z} \xrightarrow{4} \mathbb{Z}/2\mathbb{Z}$ is the 0-map

$\therefore$ $4\otimes 1$ is not injective

$\therefore$ The sequence is not exact

b. $0 \longrightarrow \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{4\otimes 1} \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\pi\otimes 1} \mathbb{Z}/4\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow 0$

Again $-\otimes_{\mathbb{Z}}\mathbb{Q}$ is right exact, so it suffices to check whether $4\otimes 1$ is injective

But $\exists$ natural isomorphism $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}$, so we have the commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q} & \xrightarrow{4\otimes 1} & \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q} \\ \phi\downarrow\wr & & \wr\downarrow\phi \\ \mathbb{Q} & \xrightarrow{4} & \mathbb{Q} \end{array}$$

And 4 is injective, so $4\circ\phi$ is injective since ϕ is an isomorphism

Hence $\phi\circ(4\otimes 1)$ is injective by commutativity

$\therefore$ $4\otimes 1$ is injective

$\therefore$ The sequence is exact

c. $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z},\mathbb{Q}) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) \xrightarrow{4^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) \longrightarrow 0$

First note that $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q})$ is left exact, so it suffices to show that 4^* is surjective

But $\exists$ natural isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q})\cong\mathbb{Q}$ so we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) & \xrightarrow{4^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) \\ \phi\downarrow\wr & & \wr\downarrow\phi \\ \mathbb{Q} & \xrightarrow{4} & \mathbb{Q} \end{array}$$

Let $\frac{m}{n}\in\mathbb{Q}$

Then $\frac{m}{n} = 4\left(\frac{m}{4n}\right)\in \mathrm{Im}\,4$

$\therefore$ 4 is surjective

$\therefore$ $4\circ\phi$ is surjective since ϕ is an isomorphism

$\therefore$ $\phi\circ 4^*$ is surjective

$\therefore 4^*$ is surjective since ϕ is an isomorphism
$\therefore$ The sequence is exact

d. $0 \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \xrightarrow{4*} Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \xrightarrow{\pi*} Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/4\mathbb{Z}) \longrightarrow 0$

Again it suffices to check whether π_* is surjective since $Hom_{\mathbb{Z}}(\mathbb{Q}, -)$ is left exact
But by 1c, $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/4\mathbb{Z}) \cong 0$ and $Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \cong 0$, so π_* is surjective
$\therefore$ The sequence is exact

3. a. Prove that eRe is a ring with identity e

$er_1e + er_2e = e(r_1+r_2)e \in eRe$
$0 = e0e \in R$
$ere + e(-r)e = e(r-r)e = e0e = 0$
$\therefore \forall ere \in eRe, \exists e(-r)e \in eRe$ additive inverse
And associativity is clear
$er_1e + er_2e = e(r_1+r_2)e = e(r_2+r_1)e = er_2e + er_1e$
$\therefore (eRe, +)$ is an abelian group
$er_1e\,er_2e = er_1e^2r_2e = er_1er_2e \in eRe$
Again associativity is clear
$(er_1e + er_2e)er_3e = e(r_1+r_2)eer_3e = e(r_1+r_2)er_3e$
And $er_1eer_3e + er_2eer_3e = er_1er_3e + er_2er_3e = e(r_1+r_2)er_3e$
$\therefore$ Distributivity holds
$eere = e^2re = ere = ere^2 = eree$
$\therefore 1_{eRe} = e$ is the multiplicative identity
$\therefore eRe$ is a ring with identity e

b. Let $e \in R$ be idempotent and let M be a left R-module. Prove that $\Phi : \mathrm{Hom}_R(Re, M) \longrightarrow eM \ni \Phi(f) = f(e)$ is an isomorphism of abelian groups

$\Phi(f_1+f_2) = (f_1+f_2)(e) = f_1(e) + f_2(e) = \Phi(f_1) + \Phi(f_2)$

$\therefore$ Φ is a homomorphism of abelian groups

Let $f \in \ker \Phi$ and let $re \in Re$

Then $f(re) = r f(e) = r\Phi(f) = r \cdot 0 = 0$

$\therefore f \equiv 0$

$\therefore \ker \Phi = 0$

$\therefore \Phi$ is injective

Let $em \in eM$ and define $f_m : Re \longrightarrow M \ni f_m(re) = rem$

which is a R-module homomorphism

Then $em = f_m(e) = \Phi(f_m)$

$\therefore \Phi$ is surjective

$\therefore \Phi$ is an isomorphism of abelian groups

c. Prove that $\mathrm{End}_R(Re) \cong (eRe)^{op}$ as rings

First note that Re is a left R-module so

$\mathrm{End}_R(Re) = \mathrm{Hom}_R(Re, Re) \cong eRe$ as abelian groups via $\Phi(f) = f(e)$ by (b)

And $\Phi(f_1 f_2) = f_1(f_2(e)) = f_1(f_2(e^2)) = f_1(e \cdot f_2(e)) = f_1(f_2(e)e)$

$= f_2(e) f_1(e) = f_1(e) \cdot f_2(e) = \Phi(f_1) \cdot \Phi(f_2)$

$\therefore \Phi$ is a ring homomorphism

$\therefore \Phi$ is an isomorphism of rings by (b)

$\therefore \mathrm{End}_R(Re) \cong (eRe)^{op}$ as rings

4. Consider the ring $R=\begin{pmatrix}\mathbb{R} & 0\\ \mathbb{R} & \mathbb{Q}\end{pmatrix}$

a. Find $Z(R)$

Note that $\begin{pmatrix}r & 0\\ r' & q\end{pmatrix}\begin{pmatrix}s & 0\\ s' & p\end{pmatrix}=\begin{pmatrix}rs & 0\\ r's+qs' & qp\end{pmatrix}$

And $\begin{pmatrix}s & 0\\ s' & p\end{pmatrix}\begin{pmatrix}r & 0\\ r' & q\end{pmatrix}=\begin{pmatrix}sr & 0\\ s'r+pr' & pq\end{pmatrix}$

So $\begin{pmatrix}s & 0\\ s' & p\end{pmatrix}\in Z(R)$ iff $s'r+pr'=r's+qs'$ $\forall r,r'\in\mathbb{R}$, $\forall q\in\mathbb{Q}$

In particular, $s'+p=s+s' \Rightarrow p=s$

And $p=s+s' \Rightarrow s'=0$

$\therefore \begin{pmatrix}s & 0\\ s' & p\end{pmatrix}=\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}$

$\therefore Z(R)=\left\{\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix} \middle| p\in\mathbb{Q}\right\}$

b. Is the ring R left artinian?

Let $I=\begin{pmatrix}0 & 0\\ \mathbb{R} & 0\end{pmatrix}$

Then $\begin{pmatrix}r & 0\\ r' & q\end{pmatrix}\begin{pmatrix}0 & 0\\ a & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ qa & 0\end{pmatrix}\in I$, $\begin{pmatrix}0 & 0\\ a & 0\end{pmatrix}\begin{pmatrix}r & 0\\ r' & q\end{pmatrix}=\begin{pmatrix}0 & 0\\ ar & 0\end{pmatrix}\in I$

$\therefore I \triangleleft R$

Also there is a 1-1 correspondence between left submodules of I and left submodules of ${}_{\mathbb{Q}}\mathbb{R}$ preserving inclusions

And ${}_{\mathbb{Q}}\mathbb{R}$ is not finitely generated

$\therefore {}_R I$ is not finitely generated

$\therefore$ R is not left noetherian ring

$\therefore$ R is not left artinian ring

c. Is the ring R left noetherian?

R is not left noetherian ring as shown above

d. Is the ring R right artinian?

First note that $R/I \cong \mathbb{R} \times \mathbb{Q}$

And we have the SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$

Also note that there is a 1-1 correspondence between right submodules of I_R and $\mathbb{R}_{\mathbb{R}}$ preserving inclusions

And $\mathbb{R}$ is a field, hence artinian as an $\mathbb{R}$-module

$\therefore$ I is a right artinian R-module

And R/I is right artinian since $\mathbb{R}\times\mathbb{Q}$ is artinian because $\mathbb{R}, \mathbb{Q}$ are fields, hence artinian

$\therefore$ R is right artinian as an R-module

$\therefore$ R is right artinian ring

e. Is the ring R right noetherian

R is right noetherian ring since it is right artinian ring

f. Find $J(R)$ and describe the ring structure of $R/J(R)$ interms of $\mathbb{Q}, \mathbb{R}$

Let $J_1 = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$, $J_3 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$

Note that these are the only possible maximal left ideals so it suffices to check that they are ideals

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ s' & 0 \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's + qs' & 0 \end{pmatrix} \in J_1$, hence J_1 is a maximal left ideal

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ s & p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ qs & qp \end{pmatrix} \in J_2$, hence J_2 is a maximal left ideal

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's & qp \end{pmatrix} \notin J_3$, hence J_3 is not a left ideal

$\therefore J(R) = J_1 \cap J_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix} = I$

$\therefore R/J(R) = R/I \cong \mathbb{R} \times \mathbb{Q}$

9. Describe the nonisomorphic simple right R-modules by indicating their underlying abelian group and R-action

Note that the nonisomorphic simple right R-modules are precisely the nonisomorphic simple left $R/J(R)$-modules hence the simple $\mathbb{R}\times\mathbb{Q}$-modules

But the simple $\mathbb{R}\times\mathbb{Q}$-modules are $\mathbb{R}\times(0)$, $(0)\times\mathbb{Q}$

And these are all of them because since R is right artinian, there is a 1-1 correspondence between the nonisomorphic simple right R-modules and the maximal two sided ideals

But there are only 2 two sided maximal ideals, namely J_1, J_2

$\therefore$ The nonisomorphic simple right R-modules are the ones isomorphic to $\mathbb{R}\times(0)$ and $(0)\times\mathbb{Q}$

Qualifying Exam
August 2011

Algebra Part

Instructions: Complete as many of the following 6 problems as you can in the time allowed. All rings have an identity and all modules are unitary. Each problem is worth 10 points.

1. (a) (8 Points.) If R is a left artinian ring, show that R has finitely many simple left modules up to isomorphism.

(b) (2 points.) Give an example of a ring with infinitely many non-isomorphic simple modules.

2. An R-module M is called *divisible* if given any left regular element $x \in R$ and $m \in M$, there exists $n \in M$ such that $xn = m$.

(a) (5 Points.) If E is an injective R-module, show that E is divisible.

(b) (5 Points.) If Q is the set of rational numbers and Z is the set of integers, show that $\frac{Q}{Z}$ is an injective Z-module.

3. (a) (8 points.) Show that any projective module must be a direct summand of a free module.

(b) (2 points.) Give an example of a projective module that is not free.

4. (a) (5 points.) Let A be a finitely generated Z-module, where Z is the ring of integers. If Q is the set of rational numbers, describe the abelian group $Q \otimes_Z A$ up to isomorphism.

(b) (5 points.) Let m and n be positive integers. Describe the abelian group $Hom_Z(Z_n, Z_m)$ up to isomorphism.

5. Let S be a ring. Show that the Jacobson radical of the ring $R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ is $\begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix}$.

6. Let $R = C[0, 1]$ be the ring of continuous real valued functions on the unit interval $[0, 1]$.

(a) (6 points.) If I is a proper ideal of R, show that there exists t, with $0 \le t \le 1$, with $f(t) = 0$ for all $f \in I$.

(b) (4 points.) Find all maximal ideals of R.

August 2011

1. a. R is a left artinian ring, show that R has finitely many simple left modules up to isomorphism.

Note that if S is a simple left R-module, then $J(R)S = 0$ since R left artinian $\Rightarrow J(R) = Nil(R)$ and $Nil(R)S = 0$

Hence $J \subseteq \mathrm{Ann}_R S$, so S is a simple left $R/J(R)$-module

And also, if S is a simple left $R/J(R)$-module then S is a simple left R-module via restriction of scalars: since $\varphi: R \to R/J \ni \varphi(r) = r + J$ is a ring map and S is an $R/J(R)$-module, S is an R-module via $r \cdot s = \varphi(r)s = (r+J)s$

$\therefore$ The simple left R-modules are precisely the simple left $R/J(R)$-modules

Hence it suffices to show that $R/J(R)$ has finitely many simple left modules up to isomorphism

But $R/J(R)$ is a semisimple ring since R left artinian

So $R/J(R)$ is a finite direct sum of simple modules, say $R/J(R) = S_1 \oplus \dots \oplus S_t$, S_i simple left $R/J(R)$-modules

Suppose $\exists T$ another simple left $R/J(R)$-module

Then $T \cong {}^{R/J(R)}/_A$ where A is a maximal ideal of $R/J(R)$

$A \leq R/J(R)$ semisimple $\Rightarrow$ A semisimple

so $A = T_1 \oplus \dots \oplus T_m$, T_i simple

But then $0 \neq T_1 \subsetneq T_1 \oplus T_2 \subsetneq \dots \subsetneq A \subsetneq R/J(R)$ is a composition series

Also $0 \neq S_i \subsetneq \dots \subsetneq R/J(R)$ is a composition series for each i

So by Jordan-Hölder, the two series are equivalent

Then since A maximal, $A \cong S_1 \oplus \dots \oplus S_{i-1} \oplus S_i \oplus \dots \oplus S_t$ for some i

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "?"] & R/J(R) \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d, dashed, "\exists \phi"] & 0 \\
0 \arrow[r] & S_1\oplus\dots\oplus S_{i-1}\oplus S_i\oplus\dots\oplus S_t \arrow[r] & S_1\oplus\dots\oplus S_t \arrow[r] & S_i \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact rows so $\exists \phi$ commuting the diagram and by 5-Lemma, ϕ is an isomorphism

$\therefore T \cong S_i$ for some i

$\therefore R/J(R)$ has finitely many simple left modules up to iso, namely $S_1, \dots, S_t$

$\therefore R$ has finitely many simple left modules up to iso

b. Give an example of a ring with infinitely many non-isomorphic simple modules.

Take $R = \mathbb{Z}$

Then each $\mathbb{Z}/p_k\mathbb{Z}$, $p_1, p_2, \dots$ primes, are nonisomorphic simple modules

Hence $\mathbb{Z}$ has infinitely many non-isomorphic simple modules

2. a. If E is an injective R-module, show that E is divisible.

Let $y \in E$ and $0 \neq r \in R$ be a left regular element

Define $f: (r) \longrightarrow E \ni f(sr) = sy$

Since $r \neq 0$ is left regular, $sr \neq 0$, so it makes sense to define f this way

And clearly f is an R-module homomorphism

Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & (r) \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & E & \end{tikzcd}$$

Since E is injective, $\exists g: R \longrightarrow E \ni g(sr) = f(sr) = sy$

Let $x = g(1)$

Then $rx = rg(1) = g(r) = y$

$\therefore E$ is divisible

b. Show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

Claim If R is a PID, and D is a divisible R-module, then D is injective

Let $0 \neq I \triangleleft R$ and let $f: I \longrightarrow D$

Since R is a PID, $I = \langle r \rangle$ for some $r \in R$

We have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \\ & D & \end{tikzcd}$$

Let $y = f(r)$

Then since D divisible, $\exists x \in D \ni y = rx$

Define $g: R \longrightarrow D \ni g(s) = sx$

Then $g(c) = g(sr) = srx = sy = sf(r) = f(sr) = f(c) \quad \forall c \in I$

$\therefore$ D is injective by Baer's Criterion

Now $\mathbb{Z}$ is a PID, so it suffices to show $\mathbb{Q}/\mathbb{Z}$ is divisible

So let $\bar{y} \in \mathbb{Q}/\mathbb{Z}$ and $n \in \mathbb{Z}$

Then $\bar{y} = \frac{a}{b} + \mathbb{Z}$ for $a, b \in \mathbb{Z}$

Take $\bar{x} = \frac{a}{nb} + \mathbb{Z}$

Then $n\bar{x} = n\left(\frac{a}{nb} + \mathbb{Z}\right) = \frac{a}{b} + \mathbb{Z} = \bar{y}$

$\therefore$ $\mathbb{Q}/\mathbb{Z}$ is divisible

$\therefore$ $\mathbb{Q}/\mathbb{Z}$ is injective by claim

3. a. Show that any projective module must be a direct summand of a free module

Let P be a projective R-module
Then $\exists F$ a free R-module mapping onto P ie $F \xrightarrow{f} P \longrightarrow 0$
Then we have the SES:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r, hook, "i"] & F \arrow[r, "f"] & P \arrow[r] & 0 \\ & & & P \arrow[u, "1_P"] \arrow[ul, dashed, "\exists g"] & \end{tikzcd}$$

But since P is projective $\exists g : P \longrightarrow F \ni fg = 1_P$
Hence the SES splits
$\therefore F \cong \operatorname{Ker} f \oplus P$
$\therefore P$ is a direct summand of a free module

b. Give an example of a projective module which is not free

Claim If P is a direct summand of a free module, then P is projective
Let F be free $\ni F = P \oplus X$ for some $X \leq F$
And let $B \xrightarrow{g} C \longrightarrow 0$ and $P \xrightarrow{f} C$
Then we have the diagram:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u, "f"] \arrow[d, shift right, "i_1"'] & \\ & F \arrow[u, shift right, "\pi_1"'] \arrow[uul, dashed, "\exists h"] & \end{tikzcd}$$

F free $\Rightarrow F$ projective $\Rightarrow \exists h : F \longrightarrow B \ni gh = f\pi_1$
Define $j : P \longrightarrow B \ni j = hi_1$
Then $gj = ghi_1 = f\pi_1 i_1 = f$
$\therefore P$ projective
Now note that $\mathbb{Z}/6\mathbb{Z}$ is free $\mathbb{Z}/6\mathbb{Z}$-module with basis $\{\bar{1}\}$

And we have the sequence:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{3} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

$$\begin{array}{lll} \bar{0} \longrightarrow \bar{0} & \quad & \bar{0} \longrightarrow \bar{0} \\ \bar{1} \longrightarrow \bar{3} & & \bar{1} \longrightarrow \bar{1} \\ & & \bar{2} \longrightarrow \bar{2} \\ & & \bar{3} \longrightarrow \bar{0} \\ & & \bar{4} \longrightarrow \bar{1} \\ & & \bar{5} \longrightarrow \bar{2} \end{array}$$

Hence $\operatorname{Im} 3 = \{\bar{0}, \bar{3}\} = \operatorname{Ker} \pi$

$\therefore$ Sequence is exact

If $\varphi : \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}$ homomorphism then $3 \cdot \bar{1} = 0$ in $\mathbb{Z}/3\mathbb{Z}$

$\Rightarrow 3 \cdot \varphi(\bar{1}) = 0$ in $\mathbb{Z}/6\mathbb{Z} \Rightarrow 6 \mid 3\varphi(\bar{1}) \Rightarrow 2 \mid \varphi(\bar{1})$

$\Rightarrow \varphi(\bar{1}) = \bar{0}, \bar{2}, \bar{4}$

Define $\varphi : \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z} \ni \varphi(\bar{1}) = \bar{4}$

Then $\pi\varphi = 1_{\mathbb{Z}/3\mathbb{Z}}$

$\therefore$ Sequence splits

$\therefore \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

$\therefore \mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module since it is the direct summand of a free module

But suppose $\mathbb{Z}/2\mathbb{Z}$ is a free $\mathbb{Z}/6\mathbb{Z}$-module

Then its basis must be $\{\bar{1}\}$ since its the only nonzero element

But $\bar{2} \cdot \bar{1} = \bar{0}$, hence $\bar{1}$ is linearly dependent

Contradiction

$\therefore \mathbb{Z}/2\mathbb{Z}$ is not a free $\mathbb{Z}/6\mathbb{Z}$ - module

4. a. Let A be a finitely generated $\mathbb{Z}$-module. Describe the abelian group $\mathbb{Q}\otimes_{\mathbb{Z}} A$ up to isomorphism.

A finitely generated $\mathbb{Z}$-module $\Rightarrow$ A finitely generated abelian group

Then the Fundamental Theorem of Finitely Generated Abelian Groups gives that $A \cong \mathbb{Z}^n \oplus \bigoplus_{c\in I} \mathbb{Z}/c\mathbb{Z}$, I finite

So $\mathbb{Q}\otimes_{\mathbb{Z}} A \cong \mathbb{Q}\otimes_{\mathbb{Z}}(\mathbb{Z}^n \oplus \bigoplus_{c\in I}\mathbb{Z}/c\mathbb{Z}) \cong \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^n \oplus \mathbb{Q}\otimes_{\mathbb{Z}}\bigoplus_{c\in I}\mathbb{Z}/c\mathbb{Z}$

$\cong \mathbb{Q}^n \oplus (\bigoplus_{c\in I} \mathbb{Q}\otimes_{\mathbb{Z}} \mathbb{Z}/c\mathbb{Z})$

Let $q\otimes\bar{a} \in \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/c\mathbb{Z}$

Then $q\otimes\bar{a} = \frac{m}{n}\otimes\bar{a} = \frac{mc}{nc}\otimes\bar{a} = \frac{m}{nc}\otimes c\bar{a} = \frac{m}{nc}\otimes\bar{0} = 0$

$\therefore \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/c\mathbb{Z} = 0$

$\therefore \mathbb{Q}\otimes_{\mathbb{Z}} A \cong \mathbb{Q}^n \oplus (\bigoplus_{c\in I}\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/c\mathbb{Z}) = \mathbb{Q}^n \oplus (\bigoplus_{c\in I} 0) = \mathbb{Q}^n$

$\therefore \mathbb{Q}\otimes_{\mathbb{Z}} A \cong \mathbb{Q}^n$

b. Let $m,n>0$. Describe the abelian group $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ up to isomorphism

Consider the SES:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

$\mathrm{Hom}_{\mathbb{Z}}(\_\_, \mathbb{Z}/n\mathbb{Z})$ is left exact, so the following sequence is exact:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{m^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$$

In particular, π^* is injective

So $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Im}\,\pi^* = \ker m^*$

But $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$

So $\ker m^* \cong \ker(\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z})$

But $m(\bar{a}) = \bar{0}$ iff $n \mid ma$ iff $\frac{n}{d} \mid \frac{m}{d} a$ where $d = (m,n)$

iff $\frac{n}{d} \mid a$ iff $n \mid da$ iff $d\bar{a} = \bar{0}$ iff $\bar{a} \in \mathbb{Z}/d\mathbb{Z}$

$\therefore \ker m^* \cong \mathbb{Z}/d\mathbb{Z}$

$\therefore \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$

5. Let S be a ring. Show that $J\left(\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}\right) = \begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix}$.

Let $A \in \begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix} \Rightarrow A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$, $a_{11}, a_{22} \in J(S)$, $a_{12} \in S$

Let $M \in R \Rightarrow M = \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix}$, $m_{11}, m_{12}, m_{22} \in S$

Show $I+MA$ has left inverse

$I+MA = \begin{bmatrix} 1+m_{11}a_{11} & m_{11}a_{12}+m_{12}a_{22} \\ 0 & 1+m_{22}a_{22} \end{bmatrix}$

But $a_{11}, a_{22} \in J(S) \Rightarrow 1+m_{11}a_{11}, 1+m_{22}a_{22}$ have left inverses, say n_{11}, n_{22} respectively

Take $N = \begin{bmatrix} n_{11} & 0 \\ 0 & n_{22} \end{bmatrix}$

$N(I+MA) = \begin{bmatrix} 1 & n_{11}(m_{11}a_{12}+m_{12}a_{22}) \\ 0 & 1 \end{bmatrix}$

Take $P = \begin{bmatrix} 1 & -n_{11}(m_{11}a_{12}+m_{12}a_{22}) \\ 0 & 1 \end{bmatrix}$

Then $PN(I+MA) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore I+MA$ has a left inverse, PN

$\therefore A \in J\left(\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}\right)$

$\therefore \begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix} \subseteq J\left(\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}\right)$

Now let $A \in J\left(\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}\right)$

Then $I+MA$ has a left inverse $\forall M \in \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ say N

So $N(I+MA) = I \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n_{11} & n_{12} \\ 0 & n_{22} \end{bmatrix}\begin{bmatrix} 1+m_{11}a_{11} & m_{11}a_{12}+m_{12}a_{22} \\ 0 & 1+m_{22}a_{22} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} n_{11}(1+m_{11}a_{11}) & n_{11}(m_{11}a_{12}+m_{12}a_{22})+n_{12}(1+m_{22}a_{22}) \\ 0 & n_{22}(1+m_{22}a_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow n_{11}(1+m_{11}a_{11}) = 1$ and $n_{22}(1+m_{22}a_{22}) = 1 \quad \forall m_{11}, m_{22} \in S$

$\therefore a_{11}, a_{22} \in J(S)$

$\therefore A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \in \begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix}$

$\therefore J\left(\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}\right) \subseteq \begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix}$

$\therefore J\left(\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}\right) = \begin{bmatrix} J(S) & S \\ 0 & J(S) \end{bmatrix}$

6. Let $R = \mathcal{C}[0,1]$ be the ring of continuous real valued functions on $[0,1]$.

a. If $I \triangleleft R$ is a proper ideal, show that $\exists t \in [0,1] \ni f(t) = 0 \ \forall f \in I$

Suppose $\exists f \in I \ni \forall t \in [0,1], \ f(t) \neq 0$

Then $\frac{1}{f}$ is continuous on $[0,1]$

Hence $\frac{1}{f} \in R$

So $1_R = \frac{1}{f} \cdot f \in I$ since $I \triangleleft R$

$\therefore I = R$

Contradiction since I proper

$\therefore \exists t \in [0,1] \ni f(t) = 0 \ \forall f \in I$

b. Find all maximal ideals of R.

Let $I_t = \{ f \in R \mid f(t) = 0 \}$

Define $\varphi : R \to \mathbb{R} \ni \varphi(f) = f(t)$ is a surjective ring homom.

$\therefore$ By 1st Iso Thm, $R/\ker\varphi \cong \mathbb{R}$ which is a field

Hence $\ker\varphi \triangleleft R$ maximal

And $\ker\varphi = \{ f \in R \mid \varphi(f) = 0_{\mathbb{R}} \}$

But $\varphi(f) = 0_{\mathbb{R}}$ iff $f(t) = 0$ iff $f \in I_t$

$\therefore \ker\varphi = I_t$

$\therefore I_t \triangleleft R$ maximal

Now let $J \triangleleft R$ be a maximal ideal

Then in particular, J is proper

So by (a) $\exists t \in [0,1] \ni f(t) = 0 \ \forall f \in J$

Then $\forall f \in J, \ f \in I_t$

$\therefore J \subseteq I_t$

$\therefore J = I_t$ since J, I_t are maximal

$\therefore$ The maximal ideals are $I_t, \ t \in [0,1]$

Qualifying Exam
January 2011

Algebra Part

Instructions: Complete as many of the following 6 problems as you can in the time allowed. All rings have an identity and all modules are unitary.

1. Let R be a left artinian ring with Jacobson radical $J = J(R)$.

(a) Show that J^n / J^{n+1} is a semisimple module of finite length for $n \geq 0$. (By definition $J^0 = R$.)

(b) Use the fact that J is nilpotent to show that R must be left Noetherian.

2. Let m and n be positive integers and let Z_m and Z_n denote the integers module m and n respectively. Identify the group $Z_m \otimes_Z Z_n$ up to isomorphism and justify your answer.

3. Let R be a ring and let $J = J(R)$ be the Jacobson radical of R.

(a) Show that J contains every nil ideal of R.

(b) If M is a finitely generated R-module with $JM = M$, show that $M = 0$. (This is known as Nakayama's lemma.)

4. Let M be an R-module with submodules X, A and B, such that $X + A = X + B$ and $A \cap X = B \cap X$. Must A and B be equal? Show this or give a counterexample.

5. Let $V = k[x]$ be the vector space of polynomials over a field k of characteristic zero. Let $X, D \in End_k(V)$ be given by

$$X(f) = xf \quad \text{for all } f \in V,$$

$$D(x^n) = nx^{n-1} \text{ for } n \geq 1 \text{ and}$$

$$D(1) = 0.$$

(a) Show that $[D, X] = DX - XD = I$, the identity transformation on V. (Here we used the "commutator notation" where $[a, b]$ denotes $ab - ba$.)

(b) By induction, show that $[D, X^n] = nX^{n-1}$.

(c) Show that V is a simple module over $R = k\langle X, D\rangle$, the subring of $End_k(V)$ generated by X and D.

6. (a) Is Z_P, the integers module a prime integer p, a projective module over the integers Z? Explain your answer.

(b) Give an example of a ring which contains a copy of the integers as a subring over which Z_P is a projective module.

January 2011

1. Let R be a left artinian ring with $J = J(R)$.

a. Show that J^n/J^{n+1} is a semisimple module of finite length for $n \geq 0$.

Note that $J(J^n/J^{n+1}) = J^{n+1}/J^{n+1} \cong 0 \quad \forall n \geq 0$

Hence $J \subseteq \mathrm{ann}_R J^n/J^{n+1}$, thus J^n/J^{n+1} is an R/J-module

But since R is artinian, R/J is semisimple

So every left R/J-module is semisimple

$\therefore J^n/J^{n+1}$ is semisimple module $\forall n \geq 0$

But since R is left artinian, R has finitely many nonisomorphic simple modules

So $J^n/J^{n+1} = S_1 \oplus \ldots \oplus S_t$ is a finite direct sum of simple modules

Then $0 \subsetneq S_1 \subsetneq S_1 \oplus S_2 \subsetneq \ldots \subsetneq S_1 \oplus \ldots \oplus S_t = J^n/J^{n+1}$ is a composition series

$\therefore J^n/J^{n+1}$ has finite length

b. Show that R is left noetherian.

Since R is left artinian, J is nilpotent, say $J^m = 0$

And J^n/J^{n+1} has finite length $\forall n \geq 0$, so J^n/J^{n+1} is both noetherian and artinian $\forall n \geq 0$

Then $J^{m-1} \cong J^{m-1}/J^m$ is noetherian

But we have the SES: $0 \longrightarrow J^{m-1} \hookrightarrow J^{m-2} \longrightarrow J^{m-2}/J^{m-1} \longrightarrow 0$

with both J^{m-1}, J^{m-2}/J^{m-1} noetherian

Hence J^{m-2} is noetherian

But we have the SES: $0 \longrightarrow J^{m-2} \hookrightarrow J^{m-3} \longrightarrow J^{m-3}/J^{m-2} \longrightarrow 0$

Hence J^{m-3} is noetherian

Continuing in this way, we get that J is noetherian

But also $R/J = J^0/J^1$ is noetherian

Then the SES: $0 \longrightarrow J \hookrightarrow R \longrightarrow R/J \longrightarrow 0$ gives that

R is noetherian as an R-module

$\therefore$ R is a left noetherian ring

2. Let $m,n>0$. Identify the group $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ up to isomorphism.

Let $\Sigma(a_i \otimes b_i) \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

Then $\Sigma(a_i \otimes b_i) = \Sigma(\bar{1} \otimes a_i b_i \cdot \bar{1}) = (\Sigma a_i b_i)(\bar{1} \otimes \bar{1})$

$\therefore \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is an abelian group generated by $\bar{1} \otimes \bar{1}$

Now let $d=(m,n)$, so $d = xm+yn$, $x,y \in \mathbb{Z}$

Then $d(\bar{1} \otimes \bar{1}) = (xm+yn)(\bar{1} \otimes \bar{1}) = x(m \cdot \bar{1} \otimes \bar{1}) + y(\bar{1} \otimes n \cdot \bar{1})$

$= x(\bar{0} \otimes \bar{1}) + y(\bar{1} \otimes 0) = 0$

$\therefore |\bar{1} \otimes \bar{1}| \mid d$

Define $\varphi : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \ni \varphi(\bar{a},\bar{b}) = \overline{ab}$

$\varphi(\bar{a}_1 + \bar{a}_2, \bar{b}) = \varphi(\overline{a_1+a_2}, b) = \overline{(a_1+a_2)b} = \overline{a_1b + a_2b}$

$= \overline{a_1b} + \overline{a_2b} = \varphi(\bar{a}_1, b) + \varphi(\bar{a}_2, b)$

$\varphi(\bar{a}, \bar{b}_1 + \bar{b}_2) = \varphi(\bar{a}, \overline{b_1+b_2}) = \overline{a(b_1+b_2)} = \overline{ab_1 + ab_2}$

$= \overline{ab_1} + \overline{ab_2} = \varphi(\bar{a}, \bar{b}_1) + \varphi(\bar{a}, \bar{b}_2)$

$\varphi(\bar{a}z, \bar{b}) = \varphi(\overline{az}, \bar{b}) = \overline{azb} = \bar{a} \cdot \overline{zb} = \varphi(\bar{a}, \overline{zb}) = \varphi(\bar{a}, z\bar{b})$

$\therefore \varphi$ is biadditive

Then by UMP of $\otimes$, $\exists \psi : \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$ homomorphism of abelian groups $\ni \psi(\bar{a} \otimes \bar{b}) = \overline{ab}$

So $\psi(\bar{1} \otimes \bar{1}) = \bar{1}$

And $|\bar{1}| = d$, so $d \mid |\bar{1} \otimes \bar{1}|$

$\therefore |\bar{1} \otimes \bar{1}| = d$

$\therefore |\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}| = d$

$\therefore \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ as abelian groups

3. Let R be a ring and let $J = J(R)$

a. Show that J contains every nil ideal of R

Let $I \trianglelefteq R$ be a nil ideal and let $x \in I$

Then $rx \in I$ $\forall r \in R$

So rx is nilpotent $\forall r \in R$

$\therefore$ $1 - rx$ is invertible $\forall r \in R$

$\therefore$ $x \in J(R)$

$\therefore$ $I \subseteq J(R)$

b. If M is a finitely generated R-module with $JM = M$, show that $M = 0$.

Proceed by induction on the number of generators, n, of M

If $n = 1$, then $M = Rm$ for some m

So since $M = JM$, we have $Rm = JRm$, hence $m = xm$ for some $x \in J$

Then $(1-x)m = 0$

But since $x \in J$, $1-x$ has a left inverse

So $m = 0$

$\therefore$ $M = 0$

Now assume the result for modules with $n-1$ generators

If M has n generators, $M = Rm_1 + \dots + Rm_n$

Then since $M = JM$, $m_1 = x_1 m_1 + \dots + x_n m_n$, $x_i \in J$

So $(1 - x_1) m_1 = x_2 m_2 + \dots + x_n m_n$

But again since $x_1 \in J$, $1 - x_1$ has a left inverse, y

So $m_1 = y x_2 m_2 + \dots + y x_n m_n$

$\therefore$ M has $n-1$ generators

$\therefore$ $M = 0$ by induction

4. Let M be an R-module with submodules $X, A, B \ni$ $X + A = X + B$ and $A \cap X = B \cap X$. Must $A = B$?

No, take M to be a semisimple module with $M = X \oplus A = X \oplus B$
Then $X + A = X + B$ and $A \cap X = 0 = B \cap X$
$\therefore A \cong B$ but not necessarily $A = B$ since semisimple modules have unique decomposition as a direct sum of simple modules up to isomorphism

$R = \mathbb{Z}$
$M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$X = \{(0,0), (0,1)\}$
$A = \{(0,0), (1,0)\}$
$B = \{(0,0), (1,1)\}$

5. Let $V = k[x]$ where k is a field of characteristic zero. Let $X, D \in \mathrm{End}_k V \ni X(f) = xf \ \forall f \in V$, $D(x^n) = nx^{n-1}$ for $n \geq 1$, and $D(1) = 0$.

a. Show that $[D,X] := DX - XD = I$.

$[D,X](f) = (DX - XD)(f) = D(X(f)) - X(D(f)) = D(xf) - X(D(f))$

Say $f = a_n x^n + \dots + a_0$

Then $D(f) = D(a_n x^n + \dots + a_0) = a_n D(x^n) + \dots + a_0 = n a_n x^{n-1} + \dots + a_1$

So $X(D(f)) = x(n a_n x^{n-1} + \dots + a_1) = n a_n x^n + \dots + a_1 x$

And $D(xf) = D(a_n x^{n+1} + \dots + a_0 x) = (n+1) a_n x^n + \dots + a_0$

$\therefore [D,X](f) = (n+1) a_n x^n + \dots + a_0 - n a_n x^n - \dots - a_1 x$

$= a_n x^n + \dots + a_0 = f$

$\therefore [D,X] = I$

b. Show that $[D, X^n] = n X^{n-1}$

If $n = 1$, $[D, X^1] = [D, X] = I = X^0$, hence the result holds

Assume the result holds for $n-1$

Then $[D, X^n] = DX^n - X^n D = (DX^{n-1})X - X(X^{n-1}D)$

$= (DX^{n-1})X - (X^{n-1}D)X + (X^{n-1}D)X - X(X^{n-1}D)$

$= (DX^{n-1} - X^{n-1}D)X + X^{n-1}DX - X^n D$

$= ((n-1)X^{n-2})X + X^{n-1}(DX - XD)$ by induction

$= (n-1)X^{n-1} + X^{n-1} I$ by (a)

$= (n-1)X^{n-1} + X^{n-1}$

$= nX^{n-1}$

$\therefore [D, X^n] = nX^{n-1}$

C. Show that V is a simple module over $R = k\langle x, D \rangle$, the subring of $\mathrm{End}_k(V)$ generated by X and D.

6. a. Is $\mathbb{Z}/p\mathbb{Z}$ a projective $\mathbb{Z}$-module?

Note that since $\mathbb{Z}$ is a PID, $\mathbb{Z}/p\mathbb{Z}$ is projective iff $\mathbb{Z}/p\mathbb{Z}$ is free

Suppose $\mathbb{Z}/p\mathbb{Z}$ is projective

Then $\mathbb{Z}/p\mathbb{Z}$ is free, hence $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}^n$ for some n since $\mathbb{Z}/p\mathbb{Z}$ is also finitely generated

Contradiction since $\mathbb{Z}/p\mathbb{Z}$ is finite and $\mathbb{Z}^n$ is infinite

$\therefore$ $\mathbb{Z}/p\mathbb{Z}$ is not projective as a $\mathbb{Z}$-module

b. Give an example of a ring which contains a copy of $\mathbb{Z}$ as a subring over which $\mathbb{Z}/p\mathbb{Z}$ is a projective module

Take $R = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ which is a ring

And $\mathbb{Z}$ is a subring of R

Note that R is projective over itself

$\therefore$ $\mathbb{Z}/p\mathbb{Z}$ is projective as an R-module since it is the direct summand of a projective R-module, namely R

Algebra Part of Qualifying Examination, August 23, 2010

Instructions: Do all questions, justify your answers with the necessary proofs. All rings are associative (not necessarily commutative) with identity, and all modules are left unitary modules. We denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{R}, \mathbb{Q}$ the fields of real and rational numbers, respectively.

1. Given a ring R, the *opposite* ring R^{op} has the same underlying abelian group as R and a new multiplication $\circ$ defined by $a \circ b = ba$, for all $a, b \in R$, where juxstaposition denotes the original multiplication in R. Let $M_{m,n}(R)$ be the set of $m \times n$ matrices with entries from R. The set $M_n(R) = M_{n,n}(R)$ is a ring with respect to the usual addition and multiplication of matrices. The sets $M_n(R)$ and $M_n(R^{op})$ coincide as abelian groups, but have different ring structures. For all $A \in M_{m,n}(R), a \in R$, we write $a \circ A = Aa$ and $A \circ a = aA$ for the scalar multiplication. For all $B \in M_{n,p}(R)$, we write AB for the product of A and B over R, and $A \circ B$ for their product over R^{op}.

(a) Consider R as an R-module and prove the following.
- (i) (1 point) For any $a \in R$, the map $\phi_a : R \to R$ given by $\phi_a(r) = a \circ r$, for all $r \in R$, is an endomorphism of R (as an R-module).
- (ii) (2 points) For any $\phi \in \mathrm{End}_R(R)$, there is a unique $a \in R$ satisfying $\phi(r) = a \circ r$, for all $r \in R$.
- (iii) (2 points) The map $R^{op} \to \mathrm{End}_R(R)$ given by $a \mapsto \phi_a$ is an isomorphism of rings.

(b) Consider $L = M_{n,1}(R)$ as an R-module and prove the following.
- (i) (1 point) For any $A \in M_n(R)$, the map $\phi_A : L \to L$ given by $\phi_A(X) = A \circ X$, for all $X \in L$, is an endomorphism of L.
- (ii) (2 points) For any $\phi \in \mathrm{End}_R(L)$, there is a unique $A \in M_n(R)$ satisfying $\phi(X) = A \circ X$, for all $X \in L$.
- (iii) (2 points) The map $M_n(R^{op}) \to \mathrm{End}_R(L)$ given by $A \mapsto \phi_A$ is an isomorphism of rings.

(c) Consider L as an $M_n(R)$-module and prove the following.
- (i) (2 points) For any $a \in R$, the map $\phi_a : L \to L$ given by $\phi_a(X) = a \circ X$, for all $X \in L$, is an endomorphism of L.
- (ii) (4 points) For any $\phi \in \mathrm{End}_{M_n(R)}(L)$, there is a unique $a \in R$ satisfying $\phi(X) = a \circ X$, for all $X \in L$.
- (iii) (2 points) The map $R^{op} \to \mathrm{End}_{M_n(R)}(L)$ given by $a \mapsto \phi_a$ is an isomorphism of rings.

(d) (2 points) Consider $N = M_{n,p}(R)$ as an $M_n(R)$-module. State, but DO NOT prove the analogs of (i)–(iii) of part (c).

2. Let R be an arbitrary ring.

(a) (2 points) Give the definition of when an R-module P is *projective*.

(b) (2 points) Do projective R-modules exist? You may quote an appropriate theorem.

2. (continued)

(c) (4 points) If P is a projective R-module, prove that an exact sequence of R-modules of the form $0 \to X \to Y \to P \to 0$ splits. Begin with a definition of a split exact sequence.

(d) (5 points) Prove that an R-module A is projective if and only if for each exact sequence $0 \to X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \to 0$ of R-modules, the sequence

$$0 \to \operatorname{Hom}_R(A, X) \xrightarrow{\operatorname{Hom}_R(A,\phi)} \operatorname{Hom}_R(A, Y) \xrightarrow{\operatorname{Hom}_R(A,\psi)} \operatorname{Hom}_R(A, Z) \to 0$$

of abelian groups is exact. You may use the definition of a projective module and the left exactness of the functor Hom.

3. Let R be an arbitrary ring.

(a) (2 points) Give the definition of when an R-module I is *injective.*

(b) (2 points) State Baer's criterion for when an R-module I is injective.

(c) (6 points) Using Baer's criterion, prove that if R is a (commutative) principal ideal domain (PID), then an R-module A is injective if and only if it is divisible. First give a definition of a divisible module.

(d) (2 points) Do divisible modules over a PID exist? Explain.

(e) (2 points) Explain how one can construct an injective module over an arbitrary ring R using a divisible $\mathbb{Z}$-module. Quote an appropriate statement.

4. Consider the ring $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$ of all 2×2 matrices $A = (a_{ij})_{1 \le i,j \le 2}$ satisfying $a_{11}, a_{21} \in \mathbb{R}, a_{22} \in \mathbb{Q}$, and $a_{12} = 0$, with the usual operations of matrix addition and multiplication.

(a) (1 points) Is the ring R left artinian?

(b) (1 points) Is the ring R left noetherian?

(c) (1 points) Is the ring R right artinian?

(d) (1 points) Is the ring R right noetherian?

(e) (5 points) Find the radical of R, $J(R)$, and describe the ring structure of $R/J(R)$ in terms of $\mathbb{R}$ and $\mathbb{Q}$.

(f) (4 points) Describe the nonisomorphic simple left R-modules by indicating their underlying abelian group and R-action.

August 2010

1. a. Consider R as an R-module and prove the following:

(i) For any $a \in R$, $\phi_a : R \longrightarrow R \ni \phi_a(r) = a \circ r \ \ \forall r \in R$ is an endomorphism of R

$\phi_a(r_1+r_2) = a \circ (r_1+r_2) = (r_1+r_2)a = r_1 a + r_2 a = a \circ r_1 + a \circ r_2$
$= \phi_a(r_1) + \phi_a(r_2)$

$\phi_a(sr) = a \circ (sr) = (sr)a = s(ra) = s(a \circ r) = s\phi_a(r)$

$\therefore \phi_a \in End_R(R)$

(ii) For any $\phi \in End_R(R)$, $\exists! a \in R \ni \phi(r) = a \circ r \ \forall r \in R$

Note that $\phi(1) = a$ for some $a \in R$

Then $\forall r \in R$, $\phi(r) = \phi(r \cdot 1) = r\phi(1) = ra = a \circ r$

And a is unique because ϕ is a well defined map

$\therefore \exists! a \in R \ni \phi(r) = a \circ r \ \forall r \in R$

(iii) $f : R^{op} \longrightarrow End_R(R) \ni f(a) = \phi_a$ is a ring isomorphism

$f(a_1+a_2)(r) = \phi_{a_1+a_2}(r) = (a_1+a_2) \circ r = r(a_1+a_2) = ra_1 + ra_2$
$= a_1 \circ r + a_2 \circ r = \phi_{a_1}(r) + \phi_{a_2}(r) = f(a_1)(r) + f(a_2)(r)$

$f(a_1 \circ a_2)(r) = \phi_{a_1 \circ a_2}(r) = (a_1 \circ a_2) \circ r = r(a_1 \circ a_2) = r(a_2 a_1)$

$f(a_1)f(a_2)(r) = \phi_{a_1}(\phi_{a_2}(r)) = \phi_{a_1}(a_2 \circ r) = a_1 \circ (a_2 \circ r) = a_1 \circ (ra_2)$
$= (ra_2)a_1 = r(a_2 a_1)$

$\therefore f$ ring homomorphism

Let $f(a_1) = f(a_2) \Rightarrow f(a_1)(r) = f(a_2)(r) \ \forall r \in R$

Let $0 \neq r \in R$ be a nonzero divisor

Then $f(a_1)(r) = f(a_2)(r) \Rightarrow \phi_{a_1}(r) = \phi_{a_2}(r) \Rightarrow a_1 \circ r = a_2 \circ r$
$\Rightarrow ra_1 = ra_2 \Rightarrow r(a_1 - a_2) = 0$
$\Rightarrow a_1 - a_2 = 0$ since $0 \neq r$ nonzero divisor
$\Rightarrow a_1 = a_2$

$\therefore f$ injective

Let $\phi \in End_R(R)$

Then by (ii), $\phi(r) = a \circ r \quad \forall r \in R$, for some $a \in R$

$= \phi_a(r) = f(a)(r)$

$\therefore f$ surjective

$\therefore f$ ring isomorphism

(b) Consider $L = M_{n,1}(R)$ as an R-module and prove the following:

(i) For any $A \in M_n(R)$, $\phi_A : L \longrightarrow L \ni \phi_A(X) = A \circ X \quad \forall X \in L$ is an endomorphism of L

$\phi_A(X_1 + X_2) = A \circ (X_1 + X_2) = (X_1 + X_2)A = X_1 A + X_2 A$

$= A \circ X_1 + A \circ X_2 = \phi_A(X_1) + \phi_A(X_2)$

$\phi_A(SX) = A \circ (SX) = (SX)A = S(XA) = S(A \circ X)$

$= S\phi_A(X)$

$\therefore \phi_A \in End_R(L)$

(ii) For any $\phi \in End_R(L)$, $\exists! A \in M_n(R) \ni \phi(X) = A \circ X \quad \forall X \in L$

Note that $\phi(e_i) = A_i$ for some $A_i \in L$, $i = 1, \dots, n$

Then $\forall X \in L$, $\phi(X) = \phi(r_1 e_1 + \dots + r_n e_n) = r_1 \phi(e_1) + \dots + r_n \phi(e_n)$

$= r_1 A_1 + \dots + r_n A_n = A_1 \circ r_1 + \dots + A_n \circ r_n$

$= \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} \circ \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} \circ X$

$\therefore \exists! A = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} \in M_n(R) \ni \phi(X) = A \circ X \quad \forall X \in L$

(iii) $f : M_n(R^{op}) \longrightarrow End_R(L) \ni f(A) = \phi_A$ is a ring isomorphism

$f(A_1 + A_2)(X) = \phi_{A_1 + A_2}(X) = (A_1 + A_2) \circ X = X(A_1 + A_2) = XA_1 + XA_2$

$= A_1 \circ X + A_2 \circ X = \phi_{A_1}(X) + \phi_{A_2}(X) = f(A_1)(X) + f(A_2)(X)$

$f(A_1) f(A_2)(X) = \phi_{A_1}(\phi_{A_2}(X)) = \phi_{A_1}(A_2 \circ X) = A_1 \circ (A_2 \circ X)$

$= A_1 \circ (XA_2) = XA_2 A_1$

$f(A_1 \circ A_2)(x) = \phi_{A_1 \circ A_2}(x) = (A_1 \circ A_2) \circ X = (A_2 A_1) \circ X = X A_2 A_1$

$\therefore$ f ring homomorphism

Let $f(A_1) = f(A_2) \Rightarrow f(A_1)(x) = f(A_2)(x) \quad \forall X \in L$

$\Rightarrow \phi_{A_1}(x) = \phi_{A_2}(x) \Rightarrow A_1 \circ X = A_2 \circ X \Rightarrow XA_1 = XA_2$

$\Rightarrow X(A_1 - A_2) = 0 \quad \forall X \in L$

Take X invertible, then $A_1 - A_2 = 0 \Rightarrow A_1 = A_2$

$\therefore$ f injective

Let $\phi \in End_R(L) \Rightarrow \phi(x) = A \circ X$ for some $A \in M_n(R)$, $\forall X \in L$

$= \phi_A(x) = f(A)(x)$

$\therefore$ f surjective

$\therefore$ f ring isomorphism

C. Consider L as an $M_n(R)$-module and prove the following:

(i) For any $a \in R$, $\phi_a : L \longrightarrow L \ni \phi_a(x) = a \circ X \quad \forall X \in L$ is an endomorphism of L

$\phi_a(x_1 + x_2) = a \circ (x_1 + x_2) = (x_1 + x_2)a = X_1 a + X_2 a = a \circ X_1 + a \circ X_2$

$= \phi_a(x_1) + \phi_a(x_2)$

$\phi_a(AX) = a \circ (AX) = AXa = A(a \circ X) = A\phi_A(x)$

$\therefore \phi_a \in End_{M_n(R)}(L)$

(ii) For any $\phi \in End_{M_n(R)}(L)$, $\exists! a \in R \ni \phi(x) = a \circ X \quad \forall X \in L$

Note that $\phi(e_1) = a_1$ for some $a_1 \in L$

Then $\forall X \in L$, $\phi(x) = \phi(\tilde{x} e_1)$ where $\tilde{x} = \begin{bmatrix} \vdots & \\ x & 0 \\ \vdots & \end{bmatrix}$

$= \tilde{x}\phi(e_1) = \tilde{x} a_1 = \begin{bmatrix} \vdots & \\ x & 0 \\ \vdots & \end{bmatrix} \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} = Xa_{11} = a_{11} \circ X$

$\therefore \exists! a = a_{11} \in R \ni \phi(x) = a_{11} \circ X \quad \forall X \in L$

(iii) $f : R^{op} \longrightarrow End_{M_n(R)}(L) \ni f(a) = \phi_a$ is a ring isomorphism

$f(a_1 + a_2)(x) = \phi_{a_1 + a_2}(x) = (a_1 + a_2) \circ X = X(a_1 + a_2) = Xa_1 + Xa_2$

$= a_1 \circ X + a_2 \circ X = \phi_{a_1}(x) + \phi_{a_2}(x) = f(a_1)(x) + f(a_2)(x)$

$f(a_1 \circ a_2)(x) = \phi_{a_1 \circ a_2}(x) = (a_1 \circ a_2) \circ X = X(a_1 \circ a_2) = Xa_2a_1$

$f(a_1)f(a_2)(x) = \phi_{a_1}(\phi_{a_2}(x)) = \phi_{a_1}(a_2 \circ X) = a_1 \circ (a_2 \circ X)$

$= a_1 \circ (Xa_2) = Xa_2a_1$

$\therefore$ f ring homomorphism

Let $f(a_1) = f(a_2) \Rightarrow f(a_1)(x) = f(a_2)(x) \quad \forall X \in L$

$\Rightarrow \phi_{a_1}(x) = \phi_{a_2}(x) \Rightarrow a_1 \circ X = a_2 \circ X$

$\Rightarrow Xa_1 = Xa_2 \Rightarrow X(a_1 - a_2) = 0 \quad \forall X \in L$

Take X invertible, then $a_1 - a_2 = 0 \Rightarrow a_1 = a_2$

$\therefore$ f injective

Let $\phi \in End_{M_n(R)}(L)$

Then $\phi(x) = a \circ X$ for some $a \in R$, $\forall X \in L$ by (ii)

$= \phi_a(X) = f(a)(X)$

$\therefore$ f surjective

$\therefore$ f ring isomorphism

d. Consider $N = M_{n,p}(R)$ as an $M_n(R)$-module. State the analogs of (c)

(i) $\forall a \in R$, $\phi_a : N \longrightarrow N \ni \phi_a(x) = a \circ X \quad \forall X \in N$ is an endomorphism of N

(ii) For any $\phi \in End_{M_n(R)}(N)$, $\exists! a \in R \ni \phi(x) = a \circ X \quad \forall X \in N$

(iii) $f : R^{op} \longrightarrow End_{M_n(R)}(N) \ni a \longrightarrow \phi_a$ is a ring isomorphism

2. Let R be a ring.

a. Give the definition of an R-module P being projective.

P is projective if $\forall$ surjections $B \xrightarrow{g} C \longrightarrow 0$ and maps $P \xrightarrow{f} C$, $\exists$ $P \xrightarrow{h} B$ $\ni$ the diagram commutes:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

ie $gh = f$

b. Do projective R-modules exist?

Yes, every free module is projective, and free modules exist, namely R^n, $n \geq 1$

c. If P is a projective R-module, prove that a SES of R-modules $0 \longrightarrow X \longrightarrow Y \longrightarrow P \longrightarrow 0$ splits. Begin with a definition of a split exact sequence.

A sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} P \longrightarrow 0$ is split exact if it is exact and if $\exists h: P \longrightarrow Y \ni gh = 1_P$. Equivalently, $\exists j: Y \longrightarrow X \ni jf = 1_X$

Now assume P is projective

Then we have the following diagram:

$$\begin{tikzcd} Y \arrow[r, "g"] & P \arrow[r] & 0 \\ & P \arrow[u, "1_P"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

Since P projective, $\exists h: P \longrightarrow Y \ni gh = 1_P$

$\therefore$ The sequence splits

d. Prove that an R-module A is projective iff for each SES $0 \longrightarrow X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \longrightarrow 0$ of R-modules, the sequence:

$$0 \longrightarrow \mathrm{Hom}_R(A,X) \xrightarrow{\phi_*} \mathrm{Hom}_R(A,Y) \xrightarrow{\psi_*} \mathrm{Hom}_R(A,Z) \longrightarrow 0$$

of abelian groups is exact.

($\Longrightarrow$) Assume A is projective

Note that $\mathrm{Hom}_R(A,-)$ is left exact, so it suffices to show that ψ_* is surjective

Let $f \in \mathrm{Hom}_R(A,Z)$

Then we have the diagram:

$$\begin{tikzcd} Y \arrow[r, "\psi"] & Z \arrow[r] & 0 \\ & A \arrow[ul, dashed, "\exists g"] \arrow[u, "f"'] & \end{tikzcd}$$

Since A is projective, $\exists g: A \longrightarrow Y \ni f = \psi g = \psi_*(g)$

$\therefore \psi_*$ is surjective

$\therefore 0 \longrightarrow \mathrm{Hom}_R(A,X) \longrightarrow \mathrm{Hom}_R(A,Y) \longrightarrow \mathrm{Hom}_R(A,Z) \longrightarrow 0$ is exact

($\Longleftarrow$) Assume $\mathrm{Hom}_R(A,-)$ exact

Let $Y \xrightarrow{\psi} Z \longrightarrow 0$ and $A \xrightarrow{f} Z$

Then $0 \longrightarrow \ker\psi \xhookrightarrow{i} Y \xrightarrow{\psi} Z \longrightarrow 0$ is a SES

So since $\mathrm{Hom}_R(A,-)$ is exact, we get the SES:

$$0 \longrightarrow \mathrm{Hom}_R(A,\ker\psi) \xrightarrow{i_*} \mathrm{Hom}_R(A,Y) \xrightarrow{\psi_*} \mathrm{Hom}_R(A,Z) \longrightarrow 0$$

In particular, ψ_* is surjective

Then since $f \in \mathrm{Hom}_R(A,Z)$, $\exists g \in \mathrm{Hom}_R(A,Y) \ni f = \psi_*(g) = \psi g$

So we have the commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "\psi"] & Z \arrow[r] & 0 \\ & A \arrow[ul, dashed, "\exists g"] \arrow[u, "f"'] & \end{tikzcd}$$

$\therefore A$ is projective

3. Let R be a ring

a. Give the definition of an R-module I being injective

I is injective if $\forall$ injections $0 \longrightarrow A \xrightarrow{f} B$ and maps $A \xrightarrow{g} I$, $\exists h: B \longrightarrow I \ni$ the diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

ie $hf = g$

b. State Baer's criterion

Baer's criterion gives that I is injective iff $\forall J \triangleleft R$ and maps $f: J \longrightarrow I$, $\exists g: R \longrightarrow I \ni$ the diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & I & \end{tikzcd}$$

ie $g|_J = f$

c. Prove that if R is a commutative PID, then an R-module A is injective iff it is divisible. First give a definition of a divisible module.

A is divisible if $\forall y \in A$ and $\forall 0 \neq r \in R$ a left regular element, $\exists x \in A \ni y = rx$

($\Rightarrow$) Assume A is injective

Let $y \in A$ and $0 \neq r \in R$ a left regular element

Define $f: \langle r \rangle \longrightarrow A \ni f(sr) = sy$

Then we have the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \langle r \rangle \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & A & \end{tikzcd}$$

Since A is injective, $\exists g: R \longrightarrow A \ni g|_{\langle r \rangle} = f$

So $g(sr) = f(sr) = sy$, hence $g(r) = y$

Take $x = g(1)$

Then $rx = rg(1) = g(r) = y$

$\therefore A$ is divisible

$(\Longleftarrow)$ Assume A is divisible

Let $I \triangleleft R$ and $f: I \longrightarrow A$

But R is a PID, so $I = (r)$ for some $r \in R$

And since A is divisible, $\exists x \in A \ni rx = f(r)$

Define $g: R \longrightarrow A \ni g(s) = sx$

Then $g(sr) = (sr)x = s(rx) = sf(r) = f(sr)$

$\therefore g|_I = f$

So we have the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & A & \end{tikzcd}$$

$\therefore A$ is injective by Baer's criterion

d. Do divisible modules over a PID exist?

Yes, $\forall$ R-modules M, M can be embedded in an injective module E, ie $0 \longrightarrow M \longrightarrow E$ injective

But since R is a PID, E is injective iff E is divisible

$\therefore \exists E$ a divisible R-module

e. Explain how to construct an injective module over a ring R using a divisible $\mathbb{Z}$-module.

Let D be a divisible $\mathbb{Z}$-module

Then $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module

4\. Consider the ring $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$

a. Is the ring R left artinian?

Let $I = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix}$

Note that $\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ qa & 0 \end{pmatrix} \in I$, $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ar & 0 \end{pmatrix} \in I$

$\therefore I \trianglelefteq R$

we also se from the left action above that there is a 1-1 correspondence between left submodules of ${}_R I$ and left submodules of ${}_{\mathbb{Q}}\mathbb{R}$ preserving inclusions

And ${}_{\mathbb{Q}}\mathbb{R}$ is not finitely generated

Hence ${}_R I$ is not finitely generated

$\therefore$ R is not a left noetherian ring

$\therefore$ R is not a left artinian ring

b. Is the ring R left noetherian?

No, R is not a left noetherian ring by the proof above

c. Is the ring R right artinian?

First note that $R/I \cong \mathbb{R} \times \mathbb{Q}$ by the 1st Iso Thm via the map

$\varphi: R \longrightarrow \mathbb{R} \times \mathbb{Q} \ni \varphi\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix} = (r, q)$

Consider the SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$

Note that from the right action above, we have a 1-1 correspondence between right submodules of I_R and right submodules of $\mathbb{R}_{\mathbb{R}}$ preserving inclusions

But $\mathbb{R}$ is a field, hence artinian, so I must also be a right artinian module

And R/I is artinian since $R/I \cong \mathbb{R} \times \mathbb{Q}$ since $\mathbb{R}, \mathbb{Q}$ both fields, hence artinian

$\therefore R_R$ is an artinian module

$\therefore$ R is a right artinian ring

d. Is the ring R right noetherian?

Yes, R is right noetherian ring because it is a right artinian ring

e. Find $J(R)$ and describe the ring structure of $R/J(R)$ in terms of $\mathbb{R}$ and $\mathbb{Q}$

Note that $J_1 = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$, $J_3 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$ are the only possible maximal left ideals, but we must check that they are ideals

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ s' & 0 \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's+qs' & 0 \end{pmatrix} \in J_1$, hence J_1 maximal left ideal

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ s & p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ qs & qp \end{pmatrix} \in J_2$, hence J_2 maximal left ideal

$\begin{pmatrix} r & 0 \\ r' & q \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} rs & 0 \\ r's & qp \end{pmatrix} \notin J_3$, hence J_3 not ideal of R

So $J(R) = J_1 \cap J_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix} = I$

$\therefore R/J(R) = R/I \cong \mathbb{R} \times \mathbb{Q}$ as rings

f. Describe the nonisomorphic simple left R-modules by indicating their underlying abelian group and R-action

Note that the nonisomorphic simple left R-modules are precisely the nonisomorphic simple left $R/J(R)$-module

But since $R/J(R) \cong \mathbb{R} \times \mathbb{Q}$, it suffices to find the nonisomorphic simple left $\mathbb{R} \times \mathbb{Q}$-modules

$\mathbb{R} \times (0)$ and $(0) \times \mathbb{Q}$ are nonisomorphic simple left $\mathbb{R} \times \mathbb{Q}$-modules

And these are all of them because $\exists$ 1-1 correspondence between nonisomorphic simple left R-modules and maximal two sided ideals of R since R is artinian, and there are 2 maximal two sided ideals J_1, J_2, hence $\mathbb{R} \times (0)$ and $(0) \times \mathbb{Q}$ are the only nonisomorphic left simple modules of R

Algebra Part of Qualifying Examination, August 25, 2009

Instructions: Do all questions, justify your answers with the necessary proofs. All rings are associative (not necessarily commutative) with identity and all modules are left unitary modules. We denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{R}, \mathbb{C}$ the fields of real and complex numbers, respectively.

1. If M is a module over a ring R, set $\operatorname{ann} M = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. If I is a left ideal of R, denote by $t(I)$ the sum of all two-sided ideals of R contained in I. You may assume that $\operatorname{ann} M$ is a two-sided ideal of R, and $t(I)$ is the largest two-sided ideal of R contained in I.

(a) (3 points) If I is a left ideal of R, prove that $\operatorname{ann} R/I = t(I)$.

(b) (3 points) Let I_1, I_2 be left ideals of R. If the R-modules R/I_1 and R/I_2 are isomorphic, prove that $t(I_1) = t(I_2)$.

(c) (4 points) Let H be a two-sided ideal of R. Prove that there exists a simple R-module S for which $H \subseteq \operatorname{ann} S$, and if H is a maximal two-sided ideal, then $H = \operatorname{ann} S$. Hint: you may use the fact that every left ideal is contained in a maximal left ideal.

2. Let R be a ring, let M be an R-module, and let N be a submodule of M. Assume that $M = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is an internal direct sum where, for all i, S_i is a simple submodule of M. For all i, set $M_i' = \bigoplus_{j \neq i} S_j$ so that $M = S_i \oplus M_i'$.

(a) (3 points) Prove that, for all i, M_i' is a maximal submodule of M, and $\bigcap_{i=1}^{n} M_i' = 0$.

(b) (3 points) If N is a maximal submodule of M, prove that, for some j, $M = N \oplus S_j$ and $N \cong M_j'$.

(c) (4 points) If N is a simple submodule of M, prove that, for some j, $M = N \oplus M_j'$ and $N \cong S_j$.
Hint: examine $N \cap S_j$, $N \cap M_j'$.

3. Consider the ring $R = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{R} \end{pmatrix}$ of all 2×2 matrices $A = (a_{ij})$ satisfying $a_{11}, a_{21} \in \mathbb{C}, a_{22} \in \mathbb{R}$ and $a_{12} = 0$, with the usual operations of matrix addition and multiplication.

(a) (2 points) Find the center of R, $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$.

(b) (1 point) Is the ring R left artinian?

(c) (1 points) Is the ring R left noetherian?

(d) (3 points) Find the radical of R, $J(R)$, and describe the ring structure of $R/J(R)$ in terms of $\mathbb{R}$ and $\mathbb{C}$.

(e) (3 points) Describe the nonisomorphic simple left R-modules by indicating their underlying abelian group and R-action.

4. Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\kappa} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \to 0$ of $\mathbb{Z}$-modules where $\kappa(z) = 3z$ for all $z \in \mathbb{Z}$. Using that functor Hom is left exact and functor $\otimes$ is right exact, determine whether the following sequences of $\mathbb{Z}$-modules are exact.

(a) (10 points) The sequence obtained by tensoring the above sequence with $\mathbb{Z}/3\mathbb{Z}$,

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \xrightarrow{\kappa \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \xrightarrow{\pi \otimes \mathrm{id}} \mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \to 0.$$

(b) (5 points) The sequence obtained by tensoring the above sequence with $\mathbb{Z}$,

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\kappa \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\pi \otimes \mathrm{id}} \mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to 0.$$

(c) (10 points) The sequence obtained by homming the above sequence into $\mathbb{Z}$,

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\pi,\mathbb{Z})} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\kappa,\mathbb{Z})} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to 0.$$

(d) (5 points) The sequence obtained by homming $\mathbb{Z}$ into the above sequence,

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\kappa)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\pi)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to 0.$$

August 2009

1. If I is a left ideal of R, let $t(I)$ be the sum of all two sided ideals of R contained in I.

a. If I is a left ideal of R, prove that $\operatorname{ann} R/I = t(I)$

We first note that R/I is an R-module, hence $\operatorname{ann} R/I \triangleleft R$

Let $r \in \operatorname{ann} R/I$

Then $r(s+I) = 0_{R/I} \ \forall s \in R \Rightarrow rs + I = I \ \forall s \in R$

so $rs \in I \ \forall s \in R$

In particular, $r = r \cdot 1 \in I$ since $1 \in R$

$\therefore \operatorname{ann} R/I \subseteq I$

so $\operatorname{ann} R/I \subseteq \sum_i I_i = t(I)$

$\therefore \operatorname{ann} R/I \subseteq t(I)$

Now let $a \in t(I) \subseteq I$ and let $r + I \in R/I$

Then $a(r+I) = ar + I = I$ since $a \in I$

$\therefore a(r+I) = 0_{R/I}$

$\therefore a \in \operatorname{ann} R/I$

$\therefore t(I) \subseteq \operatorname{ann} R/I$

$\therefore \operatorname{ann} R/I = t(I)$

b. Let I_1, I_2 be left ideals of R. If the R-modules R/I_1 and R/I_2 are isomorphic, prove that $t(I_1) = t(I_2)$

It suffices to show $\operatorname{ann} R/I_1 = \operatorname{ann} R/I_2$ since $\operatorname{ann} R/I = t(I)$ by (a)

Since $R/I_1 \cong R/I_2$, $\exists \varphi : R/I_1 \longrightarrow R/I_2$ an R-module isomorphism

Let $r \in \operatorname{ann} R/I_1$

Then $r(s+I_1) = I_1 \ \forall s \in R$, hence $rs \in I_1 \ \forall s \in R$

But $\varphi(1+I_1) = 1 + I_2$ since $R/I_1 = (1+I_1)$ and $R/I_2 = (1+I_2)$ and since φ is an isomorphism

Let $s + I_2 \in R/I_2$

Then $r(s+I_2) = rs(1+I_2) = rs\varphi(1+I_1) = \varphi(rs+I_1) = \varphi(I_1) = I_2$

$\therefore r \in \operatorname{ann} R/I_2$

$\therefore \operatorname{ann} R/I_1 \subseteq \operatorname{ann} R/I_2$

Similarly, let $r \in \operatorname{ann} R/I_2$ and let $s + I_1 \in R/I_1$

Then $rs \in I_2 \quad \forall s \in R$

And $\varphi^{-1}(1 + I_2) = 1 + I_1$

Then $r(s + I_1) = rs(1 + I_1) = rs\varphi^{-1}(1 + I_2) = \varphi^{-1}(rs + I_2)$

$= \varphi^{-1}(I_2) = I_1$

$\therefore r \in \operatorname{ann} R/I_1$

$\therefore \operatorname{ann} R/I_2 \subseteq \operatorname{ann} R/I_1$

$\therefore \operatorname{ann} R/I_1 = \operatorname{ann} R/I_2$

$\therefore t(I_1) = t(I_2)$

C. Let $H \triangleleft R$. Prove that $\exists S$ a simple R-module $\ni H \subseteq \operatorname{ann} S$ and if H is maximal, then $H = \operatorname{ann} S$

Since $H \triangleleft R$, $\exists I \triangleleft R$ maximal $\ni H \subseteq I$

Then $H \subseteq t(I) = \operatorname{ann} R/I$

But since $I \triangleleft R$ maximal, $R/I = S$ is simple R-module

$\therefore H \subseteq \operatorname{ann} R/I = \operatorname{ann} S$

$\therefore H \subseteq \operatorname{ann} S$ where S is a simple R-module

Now assume that H is maximal

Then $H = I$

So $\operatorname{ann} S = \operatorname{ann} R/I = \operatorname{ann} R/H = t(H) \subseteq H$

$\therefore \operatorname{ann} S = H$

2. Let R be a ring, M an R-module, and N a submodule of M. Assume that $M = S_1 \oplus \dots \oplus S_n$ where $\forall i$ S_i is a simple submodule of M. Set $M_i' = \bigoplus_{j \neq i} S_j$ $\forall i$ $\ni$ $M = S_i \oplus M_i'$.

a. Prove that $\forall i$ M_i' is a maximal submodule of M and $\bigcap_{i=1}^{n} M_i' = 0$.

First note that $M/M_i' = S_1 \oplus \dots \oplus S_n / \bigoplus_{j\neq i} S_j \cong S_i$ simple

Let L be a submodule of M $\ni$ $M_i' \subseteq L \subseteq M$

Then $L/M_i' \leq M/M_i'$ which is simple

so $L/M_i' = 0$ or $L/M_i' = M/M_i'$

Hence $L = M_i'$ or $L = M$

$\therefore$ M_i' is maximal

Now let $x \in \bigcap_{i=1}^{n} M_i' \subseteq M_i'$ $\forall i$

so $x = m_1 + \dots + m_{i-1} + m_{i+1} + \dots + m_n$, $m_i \in S_i$ for each i

But also $x = x_1 + \dots + x_n$, $x_i \in S_i$ since $M = \oplus S_i$

so $x_1 + \dots + x_n = m_1 + \dots + m_{i-1} + m_{i+1} + \dots + m_n$

$\Rightarrow x_i = m_1 - x_1 + \dots + m_{i-1} - x_{i-1} + m_{i+1} - x_{i+1} + \dots + m_n - x_n$

$\in S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n = M_i'$ $\forall i$

$\therefore$ $x_i \in M_i'$ $\forall i$ but also $x_i \in S_i$ $\forall i$

so $x_i \in S_i \cap M_i' = 0$ since $M = S_i \oplus M_i'$

$\therefore x_i = 0$ $\forall i$

$\therefore x = 0$

$\therefore \bigcap_{i=1}^{n} M_i' = 0$

b. If N is a maximal submodule of M, prove that for some j, $M = N \oplus S_j$ and $N \cong M_j'$.

First note that since M is semisimple and since $N \leq M$, $M = N \oplus X$ for some $X \leq M$

And $M/N \cong X$ simple since N is maximal

Show that $X = S_j$ for some j

$X \cap S_j \leq X$ simple $\forall j$, so $X \cap S_j = 0$ or $X \cap S_j = X$ for each i

If $X \cap S_j = 0$ $\forall i$, $X \cap (S_1 + \dots + S_n) = 0 \Rightarrow X \oplus S_1 \oplus \dots \oplus S_n = M$

Hence M has a composition series of length $n+1$

but $\ell(M) = n$

Contradiction

So $\exists j \ni X \cap S_j = X \Rightarrow X \leq S_j$ simple $\Rightarrow X = 0$ or $X = S_j$

But $X \neq 0$ since it is simple

$\therefore X = S_j$ for some j

$\therefore M = N \oplus S_j$ for some j

Now $N \cong M/S_j = S_1 \oplus \dots \oplus S_n / S_j \cong M_j'$

$\therefore N \cong M_j'$

c. If N is a simple submodule of M, prove that for some j $M = N \oplus M_j'$ and $N \cong S_j$.

Again $M = N \oplus X$ for some $X \leq M$

So $N \cong M/X$

And since N is simple, X is maximal

Show that $X = M_j'$ for some j

Consider $X + M_j'$

$X \leq X + M_j'$ and X is maximal, so $X + M_j' = X$ or $X + M_j' = M$ for each j

Suppose $X + M_j' = M$ $\forall j$

Then $S_j \cong M/M_j' \cong X/X \cap M_j'$ by 2nd iso thm

Then $X \cap M_j'$ is maximal since S_j is simple

But $X \cap M_j' \leq M_j'$ maximal

$\therefore X \cap M_j' = M_j'$ $\forall j$

$\therefore M_j' \subseteq X$ $\forall j$

$\therefore X = M_j'$ $\forall j$ since both X, M_j' are maximal

$\therefore M_i' = M_j'$ $\forall i, j$

Contradiction since $\bigcap_{i=1}^{n} M_i' = 0$

$\therefore X + M_j' = X$ for some j

$\therefore M_j' \subseteq X$ for some j, hence $M_j' = X$ for some j

$\therefore M = N \oplus M_j'$ for some j

And $N \cong M/M_j' \cong S_j$

$\therefore N \cong S_j$

3. Consider the ring $R = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{R} \end{pmatrix}$.

a. Find $Z(R)$

$\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix}\begin{pmatrix} d & 0 \\ d' & s \end{pmatrix} = \begin{pmatrix} cd & 0 \\ c'd + rd' & rs \end{pmatrix}$

$\begin{pmatrix} d & 0 \\ d' & s \end{pmatrix}\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix} = \begin{pmatrix} dc & 0 \\ d'c + sc' & sr \end{pmatrix}$

So $\begin{pmatrix} d & 0 \\ d' & s \end{pmatrix} \in Z(R)$ iff $c'd + rd' = d'c + sc'$ $\forall c, c' \in \mathbb{C}$, $\forall r \in \mathbb{R}$

In particular, $d + d' = d' + s$, hence $d = s$

And $d = d' + s$, hence $d' = 0$

$\therefore Z(R) = \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s \in \mathbb{R} \right\}$

b. Is the ring R left artinian?

Let $I = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$

$\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix}\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ rd & 0 \end{pmatrix} \in I$, $\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dc & 0 \end{pmatrix} \in I$

$\therefore I \triangleleft R$

And $R/I \cong \mathbb{C} \times \mathbb{R}$ as rings

We also have the SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$

And we have a 1-1 correspondence between left submodules of $_R I$ and left submodules of $_{\mathbb{R}}\mathbb{C}$ preserving inclusions

And $\mathbb{R}$ is artinian with $_{\mathbb{R}}\mathbb{C}$ finitely generated, namely by $1, i$, hence $_{\mathbb{R}}\mathbb{C}$ is an artinian module

$\therefore$ $_R I$ is an artinian module

But also $\mathbb{R}, \mathbb{C}$ are fields hence artinian, hence R/I is artinian since $R/I \cong \mathbb{C} \times \mathbb{R}$

$\therefore R$ is a left artinian ring

c. Is the ring R left noetherian?

R is a left noetherian ring since it is a left artinian ring

d. Find $J(R)$ and describe the ring structure of $R/J(R)$ in terms of $\mathbb{R}$ and $\mathbb{C}$

Let $J_1 = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{R} \end{pmatrix}$, $J_3 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{R} \end{pmatrix}$

Note these are the only possible maximal left ideals, so we must check that they are ideals

$\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix}\begin{pmatrix} d & 0 \\ d' & 0 \end{pmatrix} = \begin{pmatrix} cd & 0 \\ c'd+rd' & 0 \end{pmatrix} \in J_1$, hence J_1 is a maximal left ideal of R

$\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix}\begin{pmatrix} 0 & 0 \\ d & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ rd & rs \end{pmatrix} \in J_2$, hence J_2 is a maximal left ideal of R

$\begin{pmatrix} c & 0 \\ c' & r \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} cd & 0 \\ c'd & rs \end{pmatrix} \notin J_3$, hence J_3 is not a left ideal of R

$\therefore J(R) = J_1 \cap J_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix} = I$

$\therefore R/J(R) = R/I \cong \mathbb{C} \times \mathbb{R}$ as rings

e. Describe the nonisomorphic simple left R-modules by indicating their underlying abelian group and R-action

Note that the nonisomorphic simple left R-modules are precisely the nonisomorphic simple left $R/J(R)$-modules, hence the nonisomorphic simple left $\mathbb{C} \times \mathbb{R}$-modules

$\therefore \mathbb{C} \times (0)$, $(0) \times \mathbb{R}$ are the nonisomorphic simple $\mathbb{C} \times \mathbb{R}$-modules

And these are all of them since R is left artinian hence there is a 1-1 correspondence between nonisomorphic simple left R-modules and maximal two sided ideals of R and there are only 2 maximal two sided ideals of R, namely J_1, J_2

4. Consider the SES: $0 \longrightarrow \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$ of $\mathbb{Z}$-modules. Determine whether the following sequences of $\mathbb{Z}$-modules are exact:

a. $0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \xrightarrow{3\otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \xrightarrow{\pi\otimes 1} \mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$

Note that $\_ \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ is right exact, so it suffices to check whether $3\otimes 1$ is injective

But $\exists$ natural isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$, so we have the commutative diagram:

$$\begin{tikzcd} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \arrow[r, "3\otimes 1"] \arrow[d, "\phi"', "\cong"] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \arrow[d, "\cong"', "\phi"] \\ \mathbb{Z}/3\mathbb{Z} \arrow[r, "3"] & \mathbb{Z}/3\mathbb{Z} \end{tikzcd}$$

Suppose $3\otimes 1$ is injective

Then $\phi(3\otimes 1)$ is injective, hence $3\circ\phi$ is injective by commutativity

But then 3 is injective since ϕ is an isomorphism

Contradiction since 3 is the 0-map, hence not injective

$\therefore$ $3\otimes 1$ is not injective

$\therefore$ The sequence is not exact

b. $0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{3\otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\pi\otimes 1} \mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow 0$

Note that since $\mathbb{Z}$ is noetherian $\mathbb{Z}$-module, $\mathbb{Z}$ is flat iff $\mathbb{Z}$ is projective

But $\mathbb{Z}$ is a projective $\mathbb{Z}$-module

So $\mathbb{Z}$ is a flat $\mathbb{Z}$-module

$\therefore$ $\_ \otimes_{\mathbb{Z}} \mathbb{Z}$ is exact

$\therefore$ The sequence is exact

c. $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{3^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$

Note that $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is left exact, so it suffices to check whether or not 3^* is surjective

But $\exists$ natural isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, so we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \arrow[r, "3^*"] \arrow[d, "\phi"', "\cong"] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \arrow[d, "\cong"', "\phi"] \\
\mathbb{Z} \arrow[r, "3"] & \mathbb{Z}
\end{tikzcd}$$

Suppose 3^* is surjective

Then $\phi \circ 3^*$ is surjective, hence $3 \circ \phi$ is surjective by commutativity

Then 3 is surjective

Contradiction since $\mathrm{Im}\, 3 = 3\mathbb{Z} \neq \mathbb{Z}$

$\therefore$ 3^* is not surjective

$\therefore$ The sequence is not exact

d. $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{3_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \longrightarrow 0$

Note that $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, hence $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$ is exact

$\therefore$ The sequence is exact

Qualifying Exam – January 2009 Algebra Part

Instructions: Complete as many questions as possible. Answers should be justified with the necessary proofs. All rings are assumed to be noncommutative unless stated otherwise. All rings have an identity element and all modules are unitary.

1. (a) Let I be a finitely generated left ideal of the ring R and assume $S = \{m_\alpha \mid \alpha \in X\} \subset I$ generates I as a left ideal. Show that S contains a finite subset that generates I as a left ideal of R.

   (b) Give an example of a ring with a left ideal that can be generated by one element but also has a minimal set of generators containing two elements.

2. Recall that an ideal of a ring R is called *left primitive* if it is the annihilator of a simple left R-module.

   (a) Show that a left primitive ideal must be prime.

   (b) Give an example of a prime ideal that is not left primitive.

3. An element of a ring R is called *left regular* if its left annihilator is zero and a left R-module is called *divisible* if given a left regular element $a \in R$ and $m \in M$, we can find $n \in M$ such that $an = m$. Show that any injective left R-module is divisible.

4. Let Z be the ring of integers. For any Z-module A, let $F(A) = Hom_Z(A, Z)$ and for any Z-homomorphism $\phi : A \to B$ let $\phi^* : F(B) \to F(A)$ be given by $\phi^*(f) = f \circ \phi$.

   Consider the short exact sequence $0 \to Z \xrightarrow{\alpha} Z \xrightarrow{\pi} Z_2 \to 0$ where $\alpha(n) = 2n$ for all n in Z. Is the sequence $0 \to F(Z_2) \xrightarrow{\pi^*} F(Z) \xrightarrow{\alpha^*} F(Z) \to 0$ exact? Justify your answer.

5. Let R and S be rings, assume P is a projective left S-module and M is an R-S-bimodule that is projective as a left R-module. Show that $M \otimes_S P$ is projective as a left R-module.

6. (a) Suppose $J + I = R$, where I is a left ideal of R and $J = J(R)$ is the Jacobson radical of R. Show that $I = R$.

   (b) Let L be a left ideal of R such that $L + I = R$ for any left ideal I of R implies $I = R$. Show that $L \subseteq J(R)$.

January 2009

1. a. Let I be a finitely generated left ideal of the ring R and assume $S=\{m_\alpha \mid \alpha \in X\} \subseteq I$ generates I as a left ideal. Show that S contains a finite subset that generates I as a left ideal of R.

Since I is finitely generated, $I=(x_1,\dots,x_n)$

Let $r \in I$

Then $r=r_1x_1+\dots+r_nx_n$, $r_i \in R$

But each $x_i \in I$, so $x_i=\sum_{j=1}^{k} s_{ij} m_{\alpha_{ij}}$ for some $k>0$, $s_{ij} \in R$

Then $r=\sum_{i=1}^{n} r_i \sum_{j=1}^{k} s_{ij} m_{\alpha_{ij}}$

$\therefore \{m_{\alpha_{ij}}\}_{\substack{i=1,\dots,n\\ j=1,\dots,k}}$ generates I and $\{m_{\alpha_{ij}}\} \subseteq S$ finite

so S contains a finite subset that generates I

b. Give an example of a ring with a left ideal that can be generated by one element but also has a minimal set of generators containing two elements.

Take $R=\mathrm{End}_{\mathbb{R}}(\mathbb{R}^\infty)$

Note that R is a left ideal of itself

Define $f,g:\mathbb{R}^\infty \to \mathbb{R}^\infty \ni f(e_{2n})=e_n$, $f(e_{2n+1})=0$, $g(e_{2n})=0$, $g(e_{2n+1})=e_n$ where $\{e_i\}_{i=1,2,\dots}$ is the standard basis for $\mathbb{R}^\infty$

Let $h \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^\infty)$

Then $h=1_R \circ h$

$\therefore \mathrm{End}_{\mathbb{R}}(\mathbb{R}^\infty)$ is generated by 1_R

Let $j,k \in R \ni jf+kg \equiv 0$

Then $0=jf(e_{2n})+kg(e_{2n})=j(e_n)$, hence $j\equiv 0$

And $0=jf(e_{2n+1})+kg(e_{2n+1})=k(e_n)$, hence $k \equiv 0$

$\therefore \{f,g\}$ is linearly independent

Now let $\ell,m:\mathbb{R}^\infty \to \mathbb{R}^\infty \ni \ell(e_n)=e_{2n}$, $m(e_n)=e_{2n+1}$

Then $\ell f(e_{2n})+mg(e_{2n})=\ell(e_n)+m(0)=e_{2n}$

And $\ell f(e_{2n+1}) + mg(e_{2n+1}) = \ell(0) + m(e_n) = e_{2n+1}$

$\therefore \ell f + mg = 1_R$

$\therefore R$ is generated by $\{f, g\}$

$\therefore \{f, g\}$ is a basis for R

$\therefore \{f, g\}$ is a minimal set of generators

2. a. Show that a left primitive ideal must be prime

Let I be a left primitive ideal of R

Then $I = \operatorname{ann}_R S$ for some simple left R-module, S

Let $ab \in I = \operatorname{ann}_R S$

Then $abS = 0$

If $b \in I$, then we are done

So assume $b \notin I = \operatorname{ann}_R S$

Then $bS \neq 0$

But $bS \leq S$ simple, so $bS = S$

$\therefore 0 = abS = aS$

$\therefore a \in \operatorname{ann}_R S = I$

$\therefore I$ prime

b. Give an example of a prime ideal that is not left primitive

Take $R = \mathbb{Z}$

Then (0) is a prime ideal of R since $\mathbb{Z}$ is an integral domain thus $ab \in (0) \Rightarrow a \in (0)$ or $b \in (0)$

Now the simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$, p prime

And $\operatorname{ann}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = \{0, p, p^2, \ldots\} \neq (0)$ for each p

$\therefore (0)$ is not the annihilator of any simple $\mathbb{Z}$-module

$\therefore (0)$ is not left primitive

3. Show that any injective left R-module is divisible.

Let M be an injective left R-module

Let $y \in M$ and $0 \neq r \in R$ a left regular element

Then $(r) \neq (0)$

Define $f : (r) \longrightarrow M \ni f(sr) = sy$ which is an R-module homomorphism

Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & (r) \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & M & \end{tikzcd}$$

Since M is injective $\exists g : R \longrightarrow M \ni g|_{(r)} = f$

Take $x = g(1) \in M$

Then $rx = rg(1) = g(r) = f(r) = y$

$\therefore M$ is divisible

4. For any $\mathbb{Z}$-module A, let $F(A) = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$. Consider the SES $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$. Is the sequence $0 \longrightarrow F(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\pi^*} F(\mathbb{Z}) \xrightarrow{2^*} F(\mathbb{Z}) \longrightarrow 0$ exact?

Note that F is left exact, so it suffices to check whether or not 2^* is surjective.

Consider the following diagram:

$$\begin{array}{ccc} F(\mathbb{Z}) & \xrightarrow{2^*} & F(\mathbb{Z}) \\ \| & & \| \\ \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) & \xrightarrow{2^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \\ ?\downarrow & & \downarrow ? \\ \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \end{array}$$

Note that the diagram commutes since the vertical maps are natural isomorphisms.

Now $\mathrm{Im}\, 2 = 2\mathbb{Z} \neq \mathbb{Z}$

$\therefore$ 2 is not surjective

$\therefore$ 2^* not surjective

$\therefore$ $0 \longrightarrow F(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\pi^*} F(\mathbb{Z}) \xrightarrow{2^*} F(\mathbb{Z}) \longrightarrow 0$ is not a SES

5. Let R, S be rings, assume P is a projective left S-module and M is an R-S-bimodule that is projective as a left R-module. Show that $M \otimes_S P$ is projective as a left R-module.

First note that ${}_RM_S \otimes_S {}_SP$ is in fact a left R-module so show $\mathrm{Hom}_R(M \otimes_S P, -)$ is exact

Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a SES of left R-modules

Then since M is projective as a left R-module, we have a SES of left S-modules:

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

And since P is a projective left S-module, we have a SES of abelian groups:

$$0 \to \mathrm{Hom}_S(P, \mathrm{Hom}_R(M, A)) \to \mathrm{Hom}_S(P, \mathrm{Hom}_R(M, B)) \to \mathrm{Hom}_S(P, \mathrm{Hom}_R(M, C)) \to 0$$

But by the adjoint isomorphism Thm, we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(P, \mathrm{Hom}_R(M, A)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_S(P, \mathrm{Hom}_R(M, B)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_S(P, \mathrm{Hom}_R(M, C)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(M \otimes_S P, A) \arrow[r] & \mathrm{Hom}_R(M \otimes_S P, B) \arrow[r] & \mathrm{Hom}_R(M \otimes_S P, C) \arrow[r] & 0
\end{tikzcd}$$

And since the top row is exact, the vertical maps are isomorphisms, and the diagram commutes, we have that the bottom row is exact

$\therefore \mathrm{Hom}_R(M \otimes_S P, -)$ is exact

$\therefore M \otimes_S P$ is projective as a left R-module

6. a. Suppose $J+I=R$ where I is a left ideal of R and $J=J(R)$. Show that $I=R$.

We first note that $1 \in R$, so $1 = j + i$ for some $j \in J$, $i \in I$

Then $j = j^2 + ji \Rightarrow j - j^2 \in ji \Rightarrow (1-j)j \in ji$

But $j \in J = J(R)$, so $1-j$ has a left inverse, say k

So $k(1-j)j = kji \Rightarrow j = kji \in I$ since I is a left ideal

$\therefore 1 = j + i \in I$

$\therefore I = R$

b. Let L be a left ideal of $R \ni L+I=R$ for any left ideal I of R we have $I=R$. Show that $L \subseteq J(R)$

Let m be a maximal left ideal of R

If $L \subseteq m \ \forall m$, then $L \subseteq \bigcap$ left maximal ideals of R

$= J(R) = J$, hence we are done

So assume $\exists m$ maximal left ideal of $R \ni L \nsubseteq m$

Then $L + m \neq m$

But $m \subseteq L + m$

And m is maximal, so $L + m = R$

But then by assumption, $m = R$

Contradiction since m is maximal

$\therefore L \subseteq m \ \forall m$ maximal left ideals

$\therefore L \subseteq J(R)$

August 2008

Qualifying Examination
Algebra Part

There are only 6 questions. Do them all.

1. Let A be a finite abelian group. Prove that A is not a projective $\mathbb{Z}$-module and also that it is not an injective $\mathbb{Z}$-module.

2. Prove that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$

3. Let I be an ideal of a commutative ring R and let the *radical* of I be defined as

$$\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n \geq 0\}$$

(a) Prove that $\sqrt{I}$ is an ideal of R containing I, and that if I is a prime ideal, then $\sqrt{I} = I$.
(b) An ideal Q of R is called *primary* if whenever $ab \in Q$ and $a \notin Q$, then $b^n \in Q$ for some $n \geq 1$. Prove that the primary ideals of $\mathbb{Z}$ are 0 and (p^n) where p is a prime number and n is a positive integer. Prove also that if Q is a primary ideal of a ring R, then $\sqrt{Q}$ is a prime ideal in R.

4. Let R be an integral domain and let Q be its field of fractions. Prove that tensoring with Q over R is exact. Does Q have to be projective too as an R-module? (This could be a little tricky). Prove it, or give a counterexample.

5. Let R be an artinian ring. Prove that the following are equivalent:
(a) Every R-module is projective.
(b) Every R-module is injective.
(c) R is a semisimple ring.

6. Let $k[x, y]$ denote the polynomial ring in two variables over a field k. Prove that every finitely generated $k[x, y]$ is noetherian.

August 2008

1. Let A be a finite abelian group. Prove that A is not a projective $\mathbb{Z}$-module and also that it is not an injective $\mathbb{Z}$-module.

Note that $\mathbb{Z}$ is a PID, so A is projective iff it is free and A is injective iff it is divisible

Suppose A is projective

Then A is free, thus $A \cong \mathbb{Z}^n$ for some $n > 0$

$\therefore$ A has infinitely elements

contradiction since A is finite

$\therefore$ A is not projective

Now suppose A is injective

Then A is divisible

Since A is finite, $A = \{a_1, \dots, a_m\}$ for some $m > 0$ where each a_i has finite order

So for each i, $\exists\, 0 \neq n_i \in \mathbb{Z} \ni n_i a_i = 0$

Consider $0 \neq a_i \in A$ and $0 \neq \prod_{j=1}^{m} n_j \in \mathbb{Z}$

Then since A is divisible, $\exists a_k \in A \ni a_i = \prod_{j=1}^{m} n_j a_k = \prod_{j \neq k}^{m} n_j \cdot n_k a_k = 0$

Contradiction since $a_i \neq 0$

$\therefore$ A is not injective

2. Prove that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$

Let $\sum [(q_i + \mathbb{Z}) \otimes (q_i' + \mathbb{Z})] \in \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$

Then for each i, $(q_i + \mathbb{Z}) \otimes (q_i' + \mathbb{Z}) = (\frac{m_i}{n_i} + \mathbb{Z}) \otimes (\frac{m_i'}{n_i'} + \mathbb{Z})$

$= (\frac{m_i n_i'}{n_i n_i'} + \mathbb{Z}) \otimes (\frac{m_i'}{n_i'} + \mathbb{Z}) = (\frac{m_i}{n_i n_i'} + \mathbb{Z}) \otimes (\frac{n_i' m_i'}{n_i'} + \mathbb{Z})$

$= (\frac{m_i}{n_i n_i'} + \mathbb{Z}) \otimes (m_i' + \mathbb{Z}) = (\frac{m_i}{n_i n_i'} + \mathbb{Z}) \otimes 0_{\mathbb{Q}/\mathbb{Z}} = 0$

$\therefore \sum [(q_i + \mathbb{Z}) \otimes (q_i' + \mathbb{Z})] = 0$

$\therefore \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$

3. Let $I \trianglelefteq R$ where R is a commutative ring.

a. Prove that $\sqrt{I} \trianglelefteq R$ and $I \subseteq \sqrt{I}$. Show that if I is a prime ideal then $\sqrt{I} = I$.

Let $r_1, r_2 \in \sqrt{I}$

Then $r_1^n, r_2^m \in I$ for some $n, m \geq 0$

So $(r_1 + r_2)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} r_1^i r_2^{n+m-i}$

For $i \geq n$, $r_1^i = r_1^n r_1^{i-n} \in I$, so $\binom{n+m}{i} r_1^i r_2^{n+m-i} \in I$

And for $i < n$, $n+m-i > m$, so $r_2^{n+m-i} = r_2^m r_2^{n-i} \in I \Rightarrow$

$\binom{n+m}{i} r_1^i r_2^{n+m-i} \in I$

$\therefore \sum_{i=0}^{n+m} \binom{n+m}{i} r_1^i r_2^{n+m-i} \in I$

$\therefore (r_1 + r_2)^{n+m} \in I$

$\therefore r_1 + r_2 \in \sqrt{I}$

Now let $r \in \sqrt{I}$ and let $s \in R$

Then $r^n \in I$ for some $n \geq 0$

So $(sr)^n = s^n r^n$ since R is commutative

$\in I$ since I is an ideal

$\therefore sr \in \sqrt{I}$

$\therefore \sqrt{I} \trianglelefteq R$

And let $i \in I$

Then $i^1 = i \in I$, so $i \in \sqrt{I}$

$\therefore I \subseteq \sqrt{I}$

Now assume I is prime

Let $r \in \sqrt{I}$

Then $r^n \in I$ for some $n \geq 0$

Choose $n > 1$ minimal $\ni r^n \in I$

So $r r^{n-1} = r^n \in I$

But I is prime, so $r \in I$ or $r^{n-1} \in I$

But $r^{n-1} \notin I$ since n is minimal

$\therefore r \in I$

$\therefore \sqrt{I} \subseteq I$

$\therefore I = \sqrt{I}$

b. Prove that the primary ideals of $\mathbb{Z}$ are (0) and (p^n) where p is prime and $n \geq 0$. Prove also that if Q is a primary ideal of a ring R, then $\sqrt{Q}$ is a prime ideal in R.

We first show that (0) is primary

Let $ab \in (0)$ with $a \notin (0)$

Then $ab = 0$ and $a \neq 0$

So $b = 0$ since $\mathbb{Z}$ is an integral domain

$\therefore b^1 = b \in (0)$

$\therefore (0)$ is primary

Now show that (p^n) is primary

Let $ab \in (p^n)$ with $a \notin (p^n)$

Then $p^n | ab$ and $p^n \nmid a$

Thus $p | b \Rightarrow p^n | b^n \Rightarrow b^n \in (p^n)$

$\therefore (p^n)$ is primary

Now let $n \in \mathbb{Z} \ni n \neq p^n$ for any p prime, $n \geq 0$

$\mathbb{Z}$ is a PID, so it suffices to show (n) is not primary

Note since $n \neq p^n$, $\exists p_1 \neq p_2$ primes $\ni p_1 p_2 | n \Rightarrow a p_1 p_2 = n \in (n)$

But $a p_1 < n$, so $n \nmid a p_1$, hence $a p_1 \notin (n)$

Suppose $p_2{}^k = bn$ for some $b \in \mathbb{Z}$

But $p_1 | n \Rightarrow p_1 | bn \Rightarrow p_1 | p_2{}^k$

Contradiction
$\therefore p_2^k \notin (n)\ \forall k \geq 1$
$\therefore (n)$ is not primary
$\therefore$ The primary ideals of $\mathbb{Z}$ are (0) and (p^n)
Finally assume Q is primary
Let $ab \in \sqrt{Q} \Rightarrow (ab)^n \in Q$ for some $n \geq 0 \Rightarrow a^n b^n \in Q$
If $a \in \sqrt{Q}$, we are done
so assume $a \notin \sqrt{Q}$
Then $a^n \notin Q$
But Q primary, so $(b^n)^m \in Q \Rightarrow b^{nm} \in Q$
$\therefore b \in \sqrt{Q}$
$\therefore \sqrt{Q}$ is prime

4. Let R be an integral domain and let Q be its field of fractions. Prove that $Q \otimes_R -$ is exact. Is Q projective as an R-module?

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a SES
Note that $Q \otimes_R -$ is right exact so it suffices to show that
$0 \to Q \otimes_R A \xrightarrow{1 \otimes f} Q \otimes_R B$ is injective
Consider the following diagram:

$$\begin{tikzcd}
Q \otimes_R A \arrow[r, "1\otimes f"] \arrow[d, equal] & Q \otimes_R B \arrow[d, equal] \\
S^{-1}R \otimes_R A \arrow[r, "1\otimes f"] \arrow[d, "\phi_A \wr"'] & S^{-1}R \otimes_R B \arrow[d, "\wr\, ?\, \phi_B"] \\
S^{-1}A \arrow[r, "f_S"] & S^{-1}B
\end{tikzcd}$$

where f induces the map $f_S : S^{-1}A \to S^{-1}B \ni f_S\left(\frac{a}{s}\right) = \frac{f(a)}{s}$
Note that the vertical maps are natural isomorphisms,
so the diagram commutes
Show f_S is injective
Let $\frac{a}{s} \in \ker f_S \Rightarrow \frac{0}{1} = f_S\left(\frac{a}{s}\right) = \frac{f(a)}{s} \Rightarrow \exists u \in S \ni uf(a) = 0 \Rightarrow f(ua) = C$
$\Rightarrow ua = 0$ since f injective $\Rightarrow \frac{a}{s} = \frac{0}{1} = 0_{S^{-1}A}$

$\therefore \ker f_S = 0$

$\therefore f_S$ is injective

$\therefore 1 \otimes f$ is injective

$\therefore Q \underset{R}{\otimes} -$ is exact

But Q need not be projective as an R-module

Take $R = \mathbb{Z}$ which is an integral domain

Then $Q = \mathbb{Q}$ is its field of fractions

But $\mathbb{Z}$ is a PID so $\mathbb{Q}$ is projective iff $\mathbb{Q}$ free as a $\mathbb{Z}$-module

Let $0 \neq \frac{m_1}{n_1}, \frac{m_2}{n_2} \in \mathbb{Q}$

Then $n_1 m_2 \frac{m_1}{n_1} - n_2 m_1 \frac{m_2}{n_2} = n_1 m_2 \frac{m_1 n_2}{n_1 n_2} - n_2 m_1 \frac{m_2 n_1}{n_2 n_1} = 0$

Hence any two rationals are linearly dependent

So if $\mathbb{Q}$ has a basis, it must have only one element

Suppose $\{\frac{m}{n}\}$ is a basis for $\mathbb{Q}$

Then $\frac{m}{2n} \in \mathbb{Q}$ but $\frac{m}{2n} \neq r\frac{m}{n}$ for any $r \in \mathbb{Z}$

$\therefore \mathbb{Q} \neq (\frac{m}{n})$

$\therefore \mathbb{Q}$ has no basis

$\therefore \mathbb{Q}$ not free

$\therefore \mathbb{Q}$ not projective

5. Let R be an Artinian ring. Prove that TFAE:

(a) Every R-module is projective

(b) Every R-module is injective

(c) R is a semisimple ring

(a)$\Rightarrow$(b) Assume every R-module is projective

Let M be an R-module

Then M can be embedded into an injective module ie $\exists E$ injective $\ni 0 \longrightarrow M \xrightarrow{i} E$ injective

Then we have the SES: $0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} \operatorname{coker} i \longrightarrow 0$

But $\operatorname{coker} i$ is projective by assumption

So the sequence splits

$\therefore E \cong M \oplus \operatorname{coker} i$

$\therefore M$ is injective since its a direct summand of an injective module

$\therefore$ Every R-module is injective

(b)$\Rightarrow$(a) Assume every R-module is injective

Let M be an R-module

Then $\exists P$ projective mapping onto M ie $P \xrightarrow{\pi} M \longrightarrow 0$ surjective

Then we have the SES: $0 \longrightarrow \operatorname{Ker}\pi \xhookrightarrow{i} P \longrightarrow M \longrightarrow 0$

But $\operatorname{Ker}\pi$ is injective by assumption

So the sequence splits

$\therefore P \cong \operatorname{Ker}\pi \oplus M$

$\therefore M$ is projective since it is a direct summand of a projective module

$\therefore$ Every R-module is projective

(b)$\Rightarrow$(c) Assume every R-module is injective

Let M be an R-module and let $L \leq M$

So $0 \longrightarrow L \xhookrightarrow{i} M$ injective, and L is injective by assumption

Then L is a direct summand of M

$\therefore \exists K \leq M \ni M = L \oplus K$

$\therefore$ M is semisimple by definition

$\therefore$ Every R-module is semisimple

In particular, R is a semisimple R-module

$\therefore$ R semisimple ring

<u>(c)$\Rightarrow$(a)</u> Assume R is a semisimple ring

Then R is a direct sum of simple submodules

But every free module $F \cong R \oplus R \oplus \dots$

So every free module is a direct sum of simple submodules

$\therefore$ Every free R-module is semisimple

Let M be an R-module

Then $\exists F$ free $\ni F \xrightarrow{\pi} M \longrightarrow 0$ surjective

And we get the SES: $0 \longrightarrow \ker\pi \xhookrightarrow{i} F \xrightarrow{\pi} M \longrightarrow 0$

Note that $\ker\pi \leq F$ and F is semisimple

So $F = \ker\pi \oplus X$ for some $X \leq F$

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker\pi \arrow[r, hook, "i"] \arrow[d, equal] & F \arrow[r, "\pi"] \arrow[d, equal] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \ker\pi \arrow[r, hook, "i"] & \ker\pi \oplus X \arrow[r, "p_2"] & X \arrow[r] & 0
\end{tikzcd}$$

Note that both sequences are exact and the diagram

commutes, so $\exists$ a map $M \longrightarrow X$ commuting the

diagram and that map is an isomorphism by the 5-Lemma

$\therefore M \cong X$ which is a direct summand of a free module

$\therefore$ M is projective

$\therefore$ Every R-module is projective

6. Let k be a field. Prove that every finitely generated $k[x,y]$-module is noetherian.

Note that since k is a field, k is a commutative noetherian ring, hence $k[x]$ is noetherian by the Hilbert Basis Thm
So $k[x]$ is a commutative noetherian ring, thus $k[x,y]$ is noetherian by the Hilbert Basis Thm again
Now let M be a finitely generated $k[x,y]$-module
Then $\exists$ a finitely generated free module, F mapping onto M
ie $F \xrightarrow{\pi} M \longrightarrow 0$ surjective
But $F \cong (k[x,y])^n$ is a finite direct sum of noetherian modules
so F is noetherian
And we have the SES: $0 \longrightarrow \ker\pi \overset{\iota}{\hookrightarrow} F \xrightarrow{\pi} M \longrightarrow 0$
so since F is noetherian, both $\ker\pi$, M are noetherian
In particular, M is noetherian

Qualifying Examination

January 10, 2008

Algebra Part

- **Please do all five questions.**
- **Problem #5 is worth twice as much as each of the others**

We will always assume that rings have an identity element and that modules are unitary left modules.

1. Let I be an ideal of a ring R and M an R-module. Prove that there is an isomorphism of left R-modules
$$\mathrm{Hom}_R(R/I, M) \cong \{m \in M \mid Im = 0\}$$
(You may assume that the subset $\{m \in M \mid Im = 0\}$ is an R-submodule of M.)

2. Prove directly from the definition that an R-module P is projective if and only if $\mathrm{Hom}_R(P, -)$ is exact. That is, prove that applying $\mathrm{Hom}_R(P, -)$ to any short exact sequence produces a short exact sequence.
(You may assume that Hom is always half-exact.)

3. Use exact sequences to show that for any integers $m, n > 0$, one has an isomorphism
$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m, n)$$
(You do not need to show that the ideal (m, n) is the principal ideal $(\gcd(m, n))$.)

4. Prove that an Artinian ring is isomorphic to a direct product of finitely many division rings if and only if it contains no nonzero nilpotent element. Hint: $J(R)$.

5. Justify your answers completely in each part below.

   a) Is $\mathbb{Z}/6\mathbb{Z}$ a semisimple $\mathbb{Z}$-module?

   b) Give a short exact sequence of $\mathbb{Z}$-modules in which the outside terms are both semisimple modules, but the middle term is not semisimple.

   c) Is $\mathbb{Z}/2\mathbb{Z}$ a projective $\mathbb{Z}/6\mathbb{Z}$-module (via the obvious structure)?

   d) Does $\mathbb{Z}$ have finite length (as a $\mathbb{Z}$-module)?

January 2008

1. Let $I \trianglelefteq R$ and M an R-module. Prove that there is an isomorphism of left R-modules $Hom_R(R/I, M) \cong \{m \in M \mid Im = 0\}$

First we note that $\{m \in M \mid Im = 0\}$ is in fact a left R-module
Consider the SES: $0 \longrightarrow I \xhookrightarrow{i} R \xrightarrow{\pi} R/I \longrightarrow 0$
Since $Hom_R(\_, M)$ is left exact, we get the exact sequence:
$0 \longrightarrow Hom_R(R/I, M) \xrightarrow{\pi^*} Hom_R(R, M) \xrightarrow{i^*} Hom_R(I, M)$
In particular, π^* is injective, hence $Hom_R(R/I, M) \cong Im\,\pi^*$
But $Im\,\pi^* = Ker\, i^*$ by exactness
And $Hom_R(R, M) \cong M$ via $f \longmapsto f(1)$
So $Ker\, i^* = \{f \in Hom_R(R, M) \mid i^*(f) = 0_{Hom_R(I,M)}\}$
$\cong \{m \in M \mid m = f(1) \text{ for some } f: R \to M \text{ and } i^*(f) = 0_{Hom_R(I,M)}\}$
But $i^*(f) = 0_{Hom_R(I,M)}$ iff $f \circ i \equiv 0$ iff $f(i(r)) = 0 \ \forall r \in I$
iff $f(r) = 0 \ \forall r \in I$ iff $r f(1) = 0 \ \forall r \in I$ iff $rm = 0 \ \forall r \in I$
iff $Im = 0$
$\therefore Ker\, i^* = \{m \in M \mid Im = 0\}$
$\therefore Hom_R(R/I, M) \cong \{m \in M \mid Im = 0\}$ as left R-modules

2. Prove that an R-module P is projective iff $Hom_R(P,-)$ is exact.

($\Rightarrow$) Assume P is projective.

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a SES.

Note that $Hom_R(P,-)$ is left exact, so we get the exact sequence $0 \longrightarrow Hom_R(P,A) \xrightarrow{f_*} Hom_R(P,B) \xrightarrow{g_*} Hom_R(P,C)$

So it suffices to show that g_* is surjective

Let $h \in Hom_R(P,C)$

Then we have the following diagram:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u, "h"'] \arrow[ul, dashed, "\exists j"] & \end{tikzcd}$$

Since P is projective, $\exists j : P \longrightarrow B \ni h = gj = g_*(j)$

$\therefore g_*$ is surjective

So we have the SES:

$0 \to Hom_R(P,A) \xrightarrow{f_*} Hom_R(P,B) \xrightarrow{g_*} Hom_R(P,C) \longrightarrow 0$

$\therefore Hom_R(P,-)$ is exact

($\Leftarrow$) Assume $Hom_R(P,-)$ is exact

Let $g : B \longrightarrow C$ be surjective and $h : P \longrightarrow C$

Then we have the SES: $0 \longrightarrow \ker g \xhookrightarrow{i} B \xrightarrow{g} C \longrightarrow 0$

And since $Hom_R(P,-)$ is exact, we have the SES:

$0 \to Hom_R(P, \ker g) \xrightarrow{i_*} Hom_R(P,B) \xrightarrow{g_*} Hom_R(P,C) \longrightarrow 0$

In particular, g_* is surjective, so $\exists j \in Hom_R(P,B) \ni$
$h = g_*(j) = gj$

So we have the commutative diagram:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & P \arrow[u, "h"'] \arrow[ul, dashed, "\exists j"] & \end{tikzcd}$$

$\therefore P$ is projective

3. Show that for any $m,n>0$, $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)$

Consider the SES: $0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$

Since $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is right exact, we have the exact sequence:

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{m\otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\pi\otimes 1} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

In particular $\pi \otimes 1$ is surjective, so by the 1st Iso Thm

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \Big/ \ker(\pi\otimes 1) \cong \mathbb{Z}/n\mathbb{Z} \Big/ \operatorname{im}(m\otimes 1)$$

Consider the following diagram:

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \arrow[r, "m\otimes 1"] \arrow[d, "\cong"'] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z}/n\mathbb{Z} \arrow[r, "m"] & \mathbb{Z}/n\mathbb{Z}
\end{tikzcd}$$

Note that the vertical maps are natural isomorphisms so the diagram commutes

So $\operatorname{Im}(m\otimes 1) \cong \operatorname{Im}(m) = m\,\mathbb{Z}/n\mathbb{Z}$

$\therefore \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \big/ m\,\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)$

$\therefore \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)$

4. Prove that an artinian ring is isomorphic to a direct product of finitely many division rings iff it contains no nonzero nilpotent elements.

Let R be an artinian ring

($\Rightarrow$) Assume $R \cong D_1 \times \dots \times D_n$, D_i division rings

Since R is artinian, $J(R)$ is nilpotent, hence $J(R)$ is nil

so every element of $J(R)$ is nilpotent

In fact, since R is artinian, $J(R) = Nil(R)$ hence $J(R)$ contains all nilpotent elements of R

And $R \cong D_1 \times \dots \times D_n$ is semisimple by Artin-Wedderburn

But then $J(R) = 0$ since R is semisimple

$\therefore$ R contains no nonzero nilpotent elements

($\Leftarrow$) Assume R contains no nonzero nilpotent elements

Again since R is artinian, every element of $J(R)$ is nilpotent

But then by assumption $J(R) = 0$

So since R is artinian and $J(R) = 0$, R is semisimple

$\therefore$ $R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ by Artin-Wedderburn Thm

It remains to show that each $n_i = 1$

Suppose $\exists n_i \ni n_i > 1$

Then $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}^2 = 0$, hence $0 \neq \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \end{bmatrix}$ is nilpotent

Contradiction to assumption

$\therefore n_i = 1 \ \forall i$

$\therefore R \cong D_1 \times \dots \times D_t$, D_i division rings

5. a. Is $\mathbb{Z}/6\mathbb{Z}$ a semisimple $\mathbb{Z}$-module?

Consider the SES: $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{3} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$

$\bar{0} \longrightarrow \bar{0}$ $\qquad$ $\bar{0},\bar{3} \longrightarrow \bar{0}$

$\bar{1} \longrightarrow \bar{3}$ $\qquad$ $\bar{1},\bar{4} \longrightarrow \bar{1}$

$\bar{2},\bar{5} \longrightarrow \bar{2}$

Note it is exact since $\text{Im}\, 3 = \{\bar{0},\bar{3}\} = \ker\pi$

Now let $\varphi: \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}$ be a $\mathbb{Z}$-module homomorphism

Then since $3\cdot\bar{1} = \bar{0}$ in $\mathbb{Z}/3\mathbb{Z}$, $3\cdot\varphi(\bar{1}) = \bar{0}$ in $\mathbb{Z}/6\mathbb{Z}$

So $6 \mid 3\varphi(\bar{1}) \Rightarrow 2 \mid \varphi(\bar{1})$

Hence the $\mathbb{Z}$-module homomorphisms $\mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}$ are $\bar{1} \longrightarrow \bar{0},\bar{2},\bar{4}$

Consider $\tau: \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z} \ni \tau(\bar{1}) = \bar{4}$

Then $\pi\tau = 1_{\mathbb{Z}/3\mathbb{Z}}$

Hence the sequence splits

So $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

And $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ are simple $\mathbb{Z}$-modules since 2, 3 prime

$\therefore$ $\mathbb{Z}/6\mathbb{Z}$ is semisimple

b. Give a SES of $\mathbb{Z}$-modules $\ni$ the outside terms are both semisimple modules, but the middle term is not semisimple.

Consider the SES: $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$

$\bar{0} \longrightarrow \bar{0}$ $\qquad$ $\bar{0},\bar{2} \longrightarrow \bar{0}$

$\bar{1} \longrightarrow \bar{2}$ $\qquad$ $\bar{1},\bar{3} \longrightarrow \bar{1}$

Note it is exact since $\text{Im}\, 2 = \{\bar{0},\bar{2}\} = \ker\pi$

And $\mathbb{Z}/2\mathbb{Z}$ is simple, hence semisimple

Now let $\varphi: \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ be a $\mathbb{Z}$-module homomorphism

Then since $2\cdot\bar{1} = \bar{0}$ in $\mathbb{Z}/2\mathbb{Z}$, $2\cdot\varphi(\bar{1}) = \bar{0}$ in $\mathbb{Z}/4\mathbb{Z}$

So $4 \mid 2\varphi(\bar{1}) \Rightarrow 2 \mid \varphi(\bar{1})$

Hence the $\mathbb{Z}$-module homomorphisms are $\bar{1} \longrightarrow \bar{0},\bar{2}$

But neither of these are splitting maps

Hence $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\therefore$ $\mathbb{Z}/4\mathbb{Z}$ is not semisimple

C. Is $\mathbb{Z}/2\mathbb{Z}$ a projective $\mathbb{Z}/6\mathbb{Z}$ - module?

Note that $\mathbb{Z}/6\mathbb{Z}$ is a semisimple $\mathbb{Z}/6\mathbb{Z}$-module since $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and since $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ are simple $\mathbb{Z}/6\mathbb{Z}$-modules

$\therefore$ $\mathbb{Z}/6\mathbb{Z}$ is a semisimple ring

$\therefore$ Every $\mathbb{Z}/6\mathbb{Z}$-module is projective

$\therefore$ $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module

d. Does $\mathbb{Z}$ have finite length as a $\mathbb{Z}$-module

Note $\mathbb{Z}$ has finite length iff $\mathbb{Z}$ is both artinian and noetherian

But $\langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \dots$ does not stabilize

$\therefore$ $\mathbb{Z}$ is not artinian

$\therefore$ $\mathbb{Z}$ does not have finite length

Qualifying Examination

January 10, 2007

Algebra Part

We will always assume that rings have an identity element and, unless stated otherwise, that modules are unitary left modules.

1. Prove directly from the definition that an R-module M is projective if and only if it is a direct summand of a free R-module.

2. Consider the exact sequence

$$0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0$$

Is the following induced sequence (obtained by tensoring the one above by $\mathbb{Z}/2\mathbb{Z}$) exact? Justify (prove) your answer completely.

$$0 \to 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i \otimes_{\mathbb{Z}} \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{p \otimes_{\mathbb{Z}} \mathrm{id}} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0$$

3. Justify your answers in each part below.

   a) Are all modules over $\mathbb{Z}_4$ semisimple? Why?

   b) Does $\mathbb{Z}_4$ have finite length (as a $\mathbb{Z}_4$-module)?

   c) Is $\mathbb{Z}_2$ a projective $\mathbb{Z}_4$-module?

   d) Is $\mathbb{Q}/\mathbb{Z}$ a Noetherian $\mathbb{Z}$-module?

4. Let R be commutative, and let M be a finitely generated R-module. Let $\mathfrak{p}$ be a prime ideal of R. Show that if the localization $M_{\mathfrak{p}}$ equals 0, then $\operatorname{ann} M \not\subseteq \mathfrak{p}$.

   (Recall that the annihilator of M is defined to be the ideal $\operatorname{ann} M = \{r \in R \mid rM = 0\}$.)

5. Argue that a left Artinian ring R has the same number of nonisomorphic simple left R-modules as it has nonisomorphic simple right R-modules.

6. **Choose only one of the following two problems to do. If you show work for both, please indicate clearly which one you have chosen to be graded.**

   a) Let R be a commutative local ring with maximal ideal $\mathfrak{m}$. Show that if an R-module M has finite length then $\mathfrak{m}^n M = 0$ for some $n > 0$.

   b) Let R be a left Artinian ring. Show that a left R-module M is finitely generated if and only if M has finite length.

January 2007

1. Prove that an R-module M is projective iff it is a direct summand of a free R-module.

($\Rightarrow$) Assume M is projective

Then $\exists F$ free R-module mapping onto M ie $F \xrightarrow{\pi} M \longrightarrow 0$ is surjective

So we have the SES: $0 \longrightarrow \ker\pi \xrightarrow{i} F \xrightarrow{\pi} M \longrightarrow 0$

But consider the following diagram:

$$\begin{tikzcd} F \arrow[r, "\pi"] & M \arrow[r] & 0 \\ & M \arrow[u, "1_M"'] \arrow[ul, dashed, "\exists f"] & \end{tikzcd}$$

Since M is projective, $\exists f: M \longrightarrow F \ni 1_M = \pi f$

$\therefore$ The sequence splits

$\therefore F \cong \ker\pi \oplus M$

$\therefore$ M is a direct summand of a free module

($\Leftarrow$) Assume M is a direct summand of a free module

Then $\exists F$ free $\ni F = M \oplus X$ for some $X \leq F$

Let $B \xrightarrow{g} C \longrightarrow 0$ be surjective and $P \xrightarrow{f} C$

Consider the following diagram:

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & M \arrow[u, "f"'] \arrow[d, shift left, "i_1"] & \\ & F \arrow[u, shift left, "p_1"] \arrow[uul, dashed, "\exists h"] & \end{tikzcd}$$

Since F is free, F is projective, thus $\exists h: F \longrightarrow B \ni gh = fp_1$

Define $j: M \longrightarrow B \ni j = hi_1$

Then $gj = ghi_1 = fp_1i_1 = f$

$\therefore$ M is projective

2. Consider the exact sequence $0 \longrightarrow 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$.
Is $0 \longrightarrow 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ a SES?

Note that $\_ \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is right exact, so it suffices to check whether or not $i \otimes 1$ is injective
$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ and $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong 2\mathbb{Z}/(2\mathbb{Z})^2 \cong 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ are natural isomorphisms, so the diagram below commutes:

$$\begin{tikzcd}
2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r, "i \otimes 1"] \arrow[d, "\wr"'] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[d, "\wr"] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r, "1"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

And $1_{\mathbb{Z}/2\mathbb{Z}}$ is injective
$\therefore$ $i \otimes 1$ is injective
$\therefore$ The sequence is a SES

3. a. Are all modules over $\mathbb{Z}/4\mathbb{Z}$ semisimple?

Note that $\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ since $|\bar{1}| = 4$ but $|(\bar{0}, \bar{0})| = 1$
$|(\bar{0}, \bar{1})| = |(\bar{1}, \bar{0})| = |(\bar{1}, \bar{1})| = 2$
But $\mathbb{Z}/2\mathbb{Z} \leq \mathbb{Z}/4\mathbb{Z}$ and $\nexists\, X \leq \mathbb{Z}/4\mathbb{Z} \ni \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus X$
$\therefore$ $\mathbb{Z}/4\mathbb{Z}$ is not semisimple $\mathbb{Z}/4\mathbb{Z}$-module
$\therefore$ Not all modules over $\mathbb{Z}/4\mathbb{Z}$ are semisimple

b. Does $\mathbb{Z}/4\mathbb{Z}$ have finite length as a $\mathbb{Z}/4\mathbb{Z}$-module?

Note that the only submodules of $\mathbb{Z}/4\mathbb{Z}$ are (0) and $\mathbb{Z}/2\mathbb{Z}$
So $\mathbb{Z}/4\mathbb{Z}$ is noetherian and artinian as a $\mathbb{Z}/4\mathbb{Z}$-module
$\therefore$ $\mathbb{Z}/4\mathbb{Z}$ has finite length

c. Is $\mathbb{Z}/2\mathbb{Z}$ a projective $\mathbb{Z}/4\mathbb{Z}$-module?

Note that $\mathbb{Z}/4\mathbb{Z}$ is a PID, so $\mathbb{Z}/2\mathbb{Z}$ is projective iff $\mathbb{Z}/2\mathbb{Z}$ is free

If $\mathbb{Z}/2\mathbb{Z}$ is free, then the only possible basis is $\{\bar{1}\}$ since it is the only nonzero element

But $\bar{2}\cdot\bar{1}=\bar{0}$

$\therefore \{\bar{1}\}$ is not linearly independent

$\therefore \{\bar{1}\}$ is not a basis

$\therefore \mathbb{Z}/2\mathbb{Z}$ has no basis

$\therefore \mathbb{Z}/2\mathbb{Z}$ is not free

$\therefore \mathbb{Z}/2\mathbb{Z}$ is not projective as a $\mathbb{Z}/4\mathbb{Z}$-module

d. Is $\mathbb{Q}/\mathbb{Z}$ a noetherian $\mathbb{Z}$-module

Consider the ascending chain of $\mathbb{Q}/\mathbb{Z}$ submodules:

$\left(\frac{1}{p}\right) \subsetneq \left(\frac{1}{p^2}\right) \subsetneq \left(\frac{1}{p^3}\right) \subsetneq \ldots$ where p is prime

Note that this chain does not stabilize

$\therefore \mathbb{Q}/\mathbb{Z}$ is not noetherian as a $\mathbb{Z}$-module

However, $\mathbb{Q}/\mathbb{Z}$ is artinian as a $\mathbb{Z}$-module (see separate sheet for proof)

4. Let R be commutative and let M be a finitely generated R-module. Let $p \triangleleft R$ be prime. Show that if the localization $M_p = 0$, then $\operatorname{ann}_R M \not\subseteq p$.

First note that $M_p = S^{-1}R$ where $S = R \setminus p$

since M is finitely generated, $M = (m_1, \dots, m_n)$

$M_p = 0 \Rightarrow \frac{m_i}{1} = \frac{0}{1}$ for each i

So $\exists u_i \notin p \ni u_i m_i = 0$ for each i

And since $u_1, \dots, u_n \notin p$ and p is prime, we have that $u_1 \dots u_n \notin p$

Let $m \in M$

Then $m = r_1 m_1 + \dots + r_n m_n$

So $(u_1 \dots u_n) m = (u_1 \dots u_n)(r_1 m_1 + \dots + r_n m_n)$

$= \left(\prod_{i \neq 1} u_i\right) r_1 u_1 m_1 + \dots + \left(\prod_{i \neq n} u_i\right) r_n u_n m_n$ since R is commutative

But then $(u_1 \dots u_n) m = 0$

$\therefore u_1 \dots u_n \in \operatorname{ann}_R M$ but $u_1 \dots u_n \notin p$

$\therefore \operatorname{ann}_R M \not\subseteq p$

5. Argue that a left Artinian ring R has the same number of nonisomorphic simple left R-modules as it has nonisomorphic simple right R-modules

Note that since R is left artinian, $R/J(R)$ is left semisimple

But we have that left semisimple is equivalent to right semisimple

And thus $R/J(R)$ has the same number of nonisomorphic left simple modules as nonisomorphic right simple modules

But the nonisomorphic simple $R/J(R)$-modules are precisely the nonisomorphic simple R-modules

$\therefore$ R has the same number of nonisomorphic simple left R-modules as it has nonisomorphic simple right R-modules

6\. a. Let R be commutative local ring with maximal ideal m. Show that if an R-module M has finite length then $m^n M = 0$ for some $n > 0$.

First note that since $\ell(M) < \infty$, M is both artinian and noetherian

Consider the descending chain of submodules of M:
$M \supseteq mM \supseteq m^2 M \supseteq \dots$

Since M is artinian, the chain must stabilize, ie $\exists n > 0 \ni$
$m^n M = m^{n+1} M = \dots$

Also, since R is a commutative, local ring, $J(R) = m$

so $J(R) m^n M = m m^n M = m^{n+1} M = m^n M$

Then $J(R) m^n M = m^n M$

And $m^n M \leq M$ noetherian, hence $m^n M$ is finitely generated

$\therefore m^n M = 0$ by Nakayama's Lemma

b. Let R be a left artinian ring. Show that a left R-module M is finitely generated iff M has finite length.

($\Rightarrow$) Assume M is finitely generated

Since R is a left artinian ring, it is also left noetherian

Hence since M is finitely generated, M is both noetherian and artinian as an R-module

$\therefore \ell(M) < \infty$

($\Leftarrow$) Assume $\ell(M) < \infty$

Then M is both artinian and noetherian

And since M is noetherian, M is finitely generated

August 21, 2006

NAME: Marjan Perrin

Qualifying Examination.
Algebra Part.

You should do problems 1 and 4, and any 3 problems from among the remaining ones (2,3,5 and 6). All answers are to be supported by proofs ~~and~~/or reasoning.

1. Recall that a commutative ring R is *local* if it has a unique maximal ideal.
   (a) Let R be a commutative ring and let P be an ideal of R. Prove that P is a prime ideal if and only if whenever $P \supseteq IJ$ for some ideals I and J, then $P \supseteq I$ or $P \supseteq J$.
   (b) Let R be a commutative ring and let J be a maximal ideal of R. Show that R/J^k is a local ring for each $k \geq 1$. What is the unique maximal ideal of R/J^k ?
   (c) Let R be a commutative ring and let P be a prime ideal. Let $S = R - P$ (the complement of P in R). Prove that S is a multiplicative subset of R, and that the localization $S^{-1}R$ is a local ring. Describe the unique maximal ideal of $S^{-1}R$.

2. Prove that if m and n are coprime, then $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$.

3. Let R be a commutative ring and let P and Q be two projective R-modules. Prove that $P \otimes_R Q$ is also a projective R-module.

4. Let M be a module over a ring R. Prove that M is both noetherian and artinian if and only if M has finite length. (Recall that a module has finite length iff it has a composition series.)

5. Let R be a PID (principal ideal domain) and let M be a finitely generated R-module. Prove that M is a noetherian module. (more more statements?)

6. Let R be the ring of 2×2 lower triangular matrices with entries in $\mathbb{C}$.
   (a) Is R noetherian ? Is it artinian? Explain.
   (b) Find the Jacobson radical J of R. Describe R/J.
   (c) Find (up to isomorphism) all the simple R-modules.

August 2006

1. a. Let R be a commutative ring and let $P \triangleleft R$. Prove that P is prime iff whenever $IJ \subseteq P$ for some $I, J \triangleleft R$, then $I \subseteq P$ or $J \subseteq P$.

($\Rightarrow$) Assume P is prime

Let $I, J \triangleleft R \ni IJ \subseteq P$

If $I \subseteq P$, then we are done

So assume $I \nsubseteq P$

Then $\exists i \in I \ni i \notin P$

Let $j \in J$

Then $ij \in IJ \subseteq P$

But P is prime so $i \in P$ or $j \in P$

But $i \notin P$, so $j \in P$

$\therefore J \subseteq P$

($\Leftarrow$) Assume whenever $IJ \subseteq P$ for $I, J \triangleleft R$, $I \subseteq P$ or $J \subseteq P$

Let $ab \in P$

Let $\sum a_i b_i \in (a)(b)$

Then $a_i = r_i a$, $b_i = s_i b \Rightarrow \sum a_i b_i = \sum r_i a s_i b = \sum r_i s_i ab \in P$

$\therefore (a)(b) \subseteq P$

So by assumption $(a) \subseteq P$ or $(b) \subseteq P$

$\therefore a \in P$ or $b \in P$

$\therefore P$ prime

b. Let R be a commutative ring and let $J \triangleleft R$ be maximal. Show that R/J^k is a local ring for each $k \geq 1$. What is the unique maximal ideal of R/J^k.

First note that $I \triangleleft R/J^k$ is maximal iff $I = M/J^k$ where $M \triangleleft R$ is maximal and $J^k \subseteq M$

But M is maximal, hence M is prime

If $k = 1$, then $J \subseteq M$

If $k>1$, $JJ^{k-1}=J^k \subseteq M$, hence $J\subseteq M$ by (a)

$\therefore J\subseteq M$

But both J, M are maximal

$\therefore J=M$

$\therefore J/J^k$ is the unique maximal ideal of R/J^k

Hence R/J^k is a local ring $\forall k\geq 1$

C. Let R be a commutative ring and let $P\triangleleft R$ be prime. Let $S=R\setminus P$. Prove that $S\subseteq R$ is multiplicative and that $S^{-1}R$ is a local ring. Describe the unique maximal ideal of $S^{-1}R$.

Note that $0\in P$ since $P\triangleleft R$, hence $0\notin S$

And $1\notin P$ since P prime, hence $P\neq R$

$\therefore 1\in S$

Let $a,b\in S$

Then $a,b\notin P$, hence $ab\notin P$ since P is prime

$\therefore ab\in S$

$\therefore S\subseteq R$ is multiplicative

Now consider $P_S=\{\frac{p}{s} \mid p\in P, s\in S\}\triangleleft R$ since $P\triangleleft R$

Note that P_S is proper since $\frac{1}{1}\in S^{-1}R$ but $\frac{1}{1}\notin P_S$ because if $\frac{1}{1}=\frac{p}{s}$ then $\exists u\in S \ni u(s-p)=0 \Rightarrow us=up\in P$, but also $us\in S$ which is a contradiction since $S=R\setminus P$

Now let $\frac{r}{s}\in S^{-1}R \ni \frac{r}{s}\notin P_S$

Then $r\notin P$, so $r\in S$

$\therefore \frac{s}{r}\in S^{-1}R$ and $\frac{s}{r}\frac{r}{s}=\frac{1}{1}=\frac{r}{s}\frac{s}{r}$

$\therefore \frac{r}{s}$ invertible

$\therefore S^{-1}R$ local with unique maximal ideal P_S

2. Prove that if $(m,n)=1$, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

First note that since $(m,n)=1$, $1 = mx+ny$ for $x,y \in \mathbb{Z}$

Let $\sum \overline{a_i} \otimes \overline{b_i} \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

Then for each i,
$$\begin{aligned} \overline{a_i} \otimes \overline{b_i} &= 1(\overline{a_i} \otimes \overline{b_i}) = (mx+ny)(\overline{a_i} \otimes \overline{b_i}) \\ &= x(m\overline{a_i} \otimes \overline{b_i}) + y(\overline{a_i} \otimes n\overline{b_i}) \\ &= x(\overline{0} \otimes \overline{b_i}) + y(\overline{a_i} \otimes \overline{0}) \\ &= x \cdot 0 + y \cdot 0 = 0 \end{aligned}$$

$\therefore \sum \overline{a_i} \otimes \overline{b_i} = \sum 0 = 0$

$\therefore \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$

3. Let R be a commutative ring and let P, Q be projective R-modules. Prove that $P \otimes_R Q$ is a projective R-module.

Let $0 \to A \to B \to C \to 0$ be a SES

Since Q is projective, $\mathrm{Hom}_R(Q,-)$ exact

So $0 \to \mathrm{Hom}_R(Q,A) \to \mathrm{Hom}_R(Q,B) \to \mathrm{Hom}_R(Q,C) \to 0$ is a SES

But also P is projective, so $\mathrm{Hom}_R(P,-)$ is exact

Thus the top row of the following diagram is exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(P,\mathrm{Hom}_R(Q,A)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(P,\mathrm{Hom}_R(Q,B)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(P,\mathrm{Hom}_R(Q,C)) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_R(P \otimes_R Q, A) \arrow[r] & \mathrm{Hom}_R(P \otimes_R Q, B) \arrow[r] & \mathrm{Hom}_R(P \otimes_R Q, C) \arrow[r] & 0 \end{tikzcd}$$

And the vertical maps are natural isomorphisms by the adjoint iso thm

Hence, the diagram commutes

$\therefore$ The bottom row is exact

$\therefore \mathrm{Hom}_R(P \otimes_R Q, -)$ is exact

$\therefore P \otimes_R Q$ is projective R-module

4. Let M be an R-module. Prove that M is both noetherian and artinian iff $\ell(M) < \infty$.

($\Rightarrow$) Assume M is both noetherian and artinian
Since M is artinian, $\exists M_1 \leq M$ simple submodule
And we have the SES: $0 \longrightarrow M_1 \hookrightarrow M \longrightarrow M/M_1 \longrightarrow 0$
So since M is artinian, $M_1, M/M_1$ are both artinian
Then M/M_1 has a simple submodule M_2/M_1
Thus $M_2/M_1 \neq 0$, so $M_1 \subsetneq M_2$
And we have the SES: $0 \longrightarrow M_2 \hookrightarrow M \longrightarrow M/M_2 \longrightarrow 0$
So since M is artinian, $M_2, M/M_2$ are both artinian
Then M/M_2 has a simple submodule M_3/M_2
Thus $M_3/M_2 \neq 0$, so $M_2 \subsetneq M_3$
Continuing this way, we get the ascending chain of submodules of M: $0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots$
But since M is noetherian, the chain stabilizes
And the inclusions are proper, so $M_n = M$ for some $n > 0$
So we have the series: $0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$ with $M_1, M_{i+1}/M_i$ simple $\forall i$
Hence, M has a composition series
$\therefore \ell(M) < \infty$

($\Leftarrow$) Assume $\ell(M) < \infty$
Say $\ell(M) = n$
If $n = 1$, $0 \subsetneq M$ is a composition series
Hence $M \cong M/0$ is simple
Then M is both artinian and noetherian, so we are done
Now assume the result $\forall$ modules of length less than n
Then if $\ell(M) = n$
Let S be a simple submodule of M, note that one exists since M has a composition series
And consider the SES: $0 \longrightarrow S \longrightarrow M \longrightarrow M/S \longrightarrow 0$
Then $\ell(M) = \ell(S) + \ell(M/S) \Rightarrow \ell(M/S) = n-1$

Then M/S is both noetherian and artinian by induction
And S is simple, so it is both artinian and noetherian
$\therefore$ M is both noetherian and artinian

5. Let R be a PID and let M be a finitely generated R-module.
Prove that M is a noetherian module

Since R is a PID, every ideal is finitely generated
Hence R is a noetherian ring
Then since M is finitely generated, M is a noetherian R-module

6. Let $R = \begin{bmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix}$.
a. Is R noetherian? Is R artinian?

Let $I = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{bmatrix}$
$\begin{bmatrix} c & 0 \\ c' & c'' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c''d & 0 \end{bmatrix} \in I$, $\begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ c' & c'' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ dc & 0 \end{bmatrix} \in I$
$\therefore I \triangleleft R$
And we see that we have a 1-1 correspondence:
$\{$left/right submodules of $I\} \longleftrightarrow \{$left/right submodules of $\mathbb{C}\}$
But $\mathbb{C}$ is a field, so its only submodules are (0) and $\mathbb{C}$
Hence the only submodules of I are (0) and I
And $R/I \cong \mathbb{C} \times \mathbb{C}$ as rings
So we have the SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$
And I is both artinian and noetherian R-module since its only submodules are (0) and I
But also R/I is both artinian and noetherian since $\mathbb{C}$ is a field hence $\mathbb{C}$ is artinian and noetherian, thus $\mathbb{C} \times \mathbb{C}$ is as well
$\therefore R$ is both artinian and noetherian as an R-module
$\therefore R$ is an artinian and noetherian ring

b. Find $J(R)$ and describe $R/J(R)$

Let $J_1 = \begin{bmatrix} \mathbb{C} & 0 \\ \mathbb{C} & 0 \end{bmatrix}$, $J_2 = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix}$, $J_3 = \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix}$

It is easy to see that these are the only possible maximal left ideals, but we must check that they are ideals

$\begin{bmatrix} c & 0 \\ c' & c'' \end{bmatrix}\begin{bmatrix} d & 0 \\ d' & 0 \end{bmatrix} = \begin{bmatrix} cd & 0 \\ c'd + c''d' & 0 \end{bmatrix} \in J_1$, hence J_1 is a maximal left ideal

$\begin{bmatrix} c & 0 \\ c' & c'' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ d & d' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c''d & c''d' \end{bmatrix} \in J_2$, hence J_2 is a maximal left ideal

$\begin{bmatrix} c & 0 \\ c' & c'' \end{bmatrix}\begin{bmatrix} d & 0 \\ 0 & d' \end{bmatrix} = \begin{bmatrix} cd & 0 \\ c'd & c''d' \end{bmatrix} \notin J_3$, hence J_3 is not a left ideal

$\therefore J(R) = J_1 \cap J_2 = I$

$\therefore R/J(R) = R/I \cong \mathbb{C} \times \mathbb{C}$

c. Find up to isomorphism, all the simple R-modules

Note that the simple R-modules are precisely the simple $R/J(R)$-modules which are isomorphic to the simple $\mathbb{C} \times \mathbb{C}$ modules

$\mathbb{C} \times (0)$ and $(0) \times \mathbb{C}$ are both simple $\mathbb{C} \times \mathbb{C}$ modules

And there are no others because there is a 1-1 correspondence:

$\left\{ \begin{matrix} \text{iso classes of} \\ \text{simple } R\text{-modules} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{maximal 2} \\ \text{sided ideals of } R \end{matrix} \right\}$

since R is artinian

And the only maximal 2 sided ideals of R are J_1, J_2

Hence there are only 2 nonisomorphic simple R-modules namely $\mathbb{C} \times (0)$ and $(0) \times \mathbb{C}$

Weibel

1.1.5. Show that TFAE $\forall$ complexes $C.$:

1. $C.$ is exact
2. $H_n(C.) = 0 \quad \forall n$
3. $0. \to C.$ is a quasi-isomorphism

(1)$\Rightarrow$(2) Assume $C.$ is exact

So $\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$ is exact

Then $\operatorname{Im} d_{n+1} = \ker d_n \quad \forall n$

But $H_n(C.) = \ker d_n / \operatorname{Im} d_{n+1} = \ker d_n / \ker d_n = 0$

$\therefore H_n(C.) = 0 \quad \forall n$

(2)$\Rightarrow$(1) Assume $H_n(C.) = 0 \quad \forall n$

Then $0 = H_n(C.) = \ker d_n / \operatorname{Im} d_{n+1} \quad \forall n$

so $\ker d_n = \operatorname{Im} d_{n+1} \quad \forall n$

$\therefore C.$ is exact

(2)$\Rightarrow$(3) Assume $H_n(C.) = 0 \quad \forall n$

Consider $0. \to C.$:

$$\begin{tikzcd}
0. : \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
C. : \cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Taking homology, we have:

$$\begin{tikzcd}
H_n(0.) : \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
H_n(C.) : \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

since $H_n(C.) = 0 \quad \forall n$

$\therefore$ The induced map $H_n(0)$ is an isomorphism $\forall n$

$\therefore 0. \to C.$ is a quasi-isomorphism

(3)$\Rightarrow$(2) Assume $0. \to C.$ is a quasi-isomorphism

Then $H_n(C.) \cong H_n(0.) = 0 \quad \forall n$

$\therefore H_n(C.) = 0 \quad \forall n$

1.2.1. Show that the direct sum commutes with homology.

For each i, consider the complex:

$$\cdots \longrightarrow A_i^{n+1} \xrightarrow{d_i^{n+1}} A_i^n \xrightarrow{d_i^n} A_i^{n-1} \longrightarrow \cdots$$

And the direct sum:

$$\cdots \longrightarrow \oplus A_i^{n+1} \xrightarrow{d^{n+1}} \oplus A_i^n \xrightarrow{d^n} \oplus A_i^{n-1} \longrightarrow \cdots$$

where $d^n = \oplus d_i^n$

Define $\phi : H_n(\oplus A_i) \longrightarrow \oplus H_n(A_i) \ni \phi(\overline{(a_i)}) = (\overline{a_i})$

And $\psi : \oplus H_n(A_i) \longrightarrow H_n(\oplus A_i) \ni \psi(\overline{a_i}) = \overline{(a_i)}$

Now $\overline{(a_i)} = \overline{(a_i')}$ iff $\overline{(a_i)} - \overline{(a_i')} = \overline{0}$ iff $\overline{(a_i - a_i')} = \overline{0}$

iff $(a_i - a_i') \in \operatorname{Im} d$ iff $(a_i - a_i') = d((b_i)) = (d_i(b_i))$

iff $a_i - a_i' = d_i(b_i) \in \operatorname{Im} d_i \ \forall i$ iff $\overline{a_i - a_i'} = \overline{0} \ \forall i$

iff $(\overline{a_i - a_i'}) = (\overline{0})$ iff $(\overline{a_i}) = (\overline{a_i'})$

$\therefore \phi$ is well defined homomorphism of abelian groups

And the same argument reversed gives that ψ is well defined homomorphism of abelian groups

It is also clear that $\phi\psi = 1 = \psi\phi$

$\therefore H_n(\oplus A_i) \cong \oplus H_n(A_i)$

1.3.1. Let $0 \longrightarrow A. \xrightarrow{f} B. \xrightarrow{g} C. \longrightarrow 0$ be a SES of complexes. Show that if two of the complexes are exact, then so is the third.

From the given SES, we get the LES:

$\cdots \longrightarrow H_{n+1}(C.) \xrightarrow{\delta^{n+1}} H_n(A.) \xrightarrow{H_n(f)} H_n(B.) \xrightarrow{H_n(g)} H_n(C.) \xrightarrow{\delta^n} H_{n-1}(A.) \longrightarrow \cdots$

WLOG assume that $A., B.$ are exact

Then $H_n(A.) = H_n(B.) = 0 \quad \forall n$

So for each n, $0 \xrightarrow{H_n(g)} H_n(C.) \xrightarrow{\delta^n} 0$ is exact

Then $0 = \operatorname{Im} H_n(g) = \ker \delta^n$

But δ^n is the 0-map, so $H_n(C.) = \ker \delta^n = 0$

$\therefore H_n(C.) = 0 \quad \forall n$

$\therefore C.$ is exact

1.4.1. a. Show that exact bounded below complexes of free R-modules are always split exact

Let $C.: \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$ be exact with each C_i a free R-module

Show that $0 \longrightarrow Z_n \xhookrightarrow{i} C_n \xrightarrow{d_n} B_{n-1} \longrightarrow 0$ splits $\forall n \geq 1$

Proceed by induction on n:

We have SES: $0 \longrightarrow \ker d_1 \hookrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$ which is precisely the SES: $0 \longrightarrow Z_1 \xhookrightarrow{i} C_1 \xrightarrow{d_1} B_0 \longrightarrow 0$

And C_0 is a free module, hence C_0 is projective

Thus the sequence splits

Now assume $0 \longrightarrow Z_n \xhookrightarrow{i} C_n \xrightarrow{d_n} B_{n-1} \longrightarrow 0$ splits

Since $C.$ is exact, $Z_n = \ker d_n = \operatorname{Im} d_{n+1} = B_n$

And since the above sequence splits, Z_n is a direct summand of C_n which is free

Hence Z_n is projective, thus B_n is projective

Then the SES: $0 \longrightarrow Z_{n+1} \xhookrightarrow{i} C_{n+1} \xrightarrow{d_{n+1}} B_n \longrightarrow 0$ splits

$\therefore 0 \longrightarrow Z_n \xhookrightarrow{i} C_n \xrightarrow{d_n} B_{n-1} \longrightarrow 0$ splits $\forall n \geq 1$

$\therefore$ $C.$ is split exact

b. Show that an exact complex of finitely generated free abelian groups is always split exact.

Let $C. : \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \dots$ be exact with each C_i a finitely generated free abelian group

So each C_i is a $\mathbb{Z}$-module, and $\mathbb{Z}$ is a PID

hence every submodule of C_i is a free module

Consider the SES: $0 \longrightarrow Z_n \hookrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$

Note that $B_{n-1} \leq C_{n-1}$ hence it is free

$\therefore$ B_{n-1} is projective

$\therefore$ The sequence splits for each n

$\therefore$ $C.$ is split exact

1.4.1. Show that if a complex $C.$ is split exact then there is a homotopy equivalence between $C.$ and $H_*(C.)$ where $H_*(C.)$ is the complex of homologies of $C.$

Since $C.$ is split exact, each $0 \longrightarrow Z_n \hookrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$ splits, thus $C_n \cong Z_n \oplus B_{n-1} = B_n \oplus B_{n-1}$ by exactness

And the differentials are $d_n : C_n \longrightarrow C_{n-1} \ni d_n(b_n, b_{n-1}) = (b_{n-1}, 0$

Define $s_n : C_n \longrightarrow C_{n+1} \ni s_n(b_n, b_{n-1}) = (0, b_n)$

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & C_n \arrow[r, "d_n"] \arrow[d, "1_{C_n}"] \arrow[dl, dashed, "s_n"'] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, dashed, "s_{n-1}"'] & \dots \\
\dots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Then $(d_{n+1} s_n + s_{n-1} d_n)(b_n, b_{n-1}) = d_{n+1}(s_n(b_n, b_{n-1})) + s_{n-1}(d_n(b_n, b_n$

$= d_{n+1}(0, b_n) + s_{n-1}(b_{n-1}, 0) = (b_n, 0) + (0, b_{n-1}) = (b_n, b_{n-1})$

$\therefore$ $d_{n+1} s_n + s_{n-1} d_n = 1_{C_n}$

$\therefore$ $1_{C_n} \simeq 0$

And since $H_n(C.) = 0 \ \forall n$, $1_{H_n(C.)} \simeq 0$ where the homotop is the 0-map, hence there is a homotopy equivalence

2.3.1. Let $R = \mathbb{Z}/m\mathbb{Z}$. Prove that R is an injective R-module.

Let $0 \neq I \trianglelefteq R$ and let $f: I \to R$

Then $I = (\bar{d})$ for some $d \mid m$

Thus $\frac{m}{d} \cdot \bar{d} = \bar{0}$ in $\mathbb{Z}/m\mathbb{Z}$, so $\frac{m}{d} \cdot f(\bar{d}) = 0$ in $\mathbb{Z}/m\mathbb{Z}$

Hence $m \mid \frac{m}{d} f(\bar{d}) \Rightarrow \bar{1} \mid \frac{f(\bar{d})}{d} \Rightarrow d \mid f(\bar{d})$

Define $g: R \to R \ni g(\bar{1}) = \frac{f(\bar{d})}{d}$

Then we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & R & \end{tikzcd}$$

And $g(\bar{d}) = g(\bar{d} \cdot \bar{1}) = \bar{d}\, g(\bar{1}) = \bar{d} \cdot \frac{f(\bar{d})}{d} = f(\bar{d})$

$\therefore g|_I = f$

$\therefore$ R is an injective R-module by Baer's criterion

2.4.2 If $U: \mathcal{B} \to \mathcal{C}$ is an exact functor, show that $U(L_iF) \cong L_i(UF)$.

Note that $L_iF(A) = H_i(F(P.))$ where $P. \to A$ is a projective resolution

So $U(L_iF(A)) = U(H_i(F(P.)))$

And $L_i(UF) = H_i(U(F(P.)))$

But $F(P.)$ is a complex, so it suffices to show that exact functors preserve the homology of a complex

Let $C.$ be a complex and show that $U(H_i(C.)) \cong H_i(U(C.))$

Consider $C.: \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$

And $U(C.): \cdots \to U(C_{n+1}) \xrightarrow{U(d_{n+1})} U(C_n) \xrightarrow{U(d_n)} U(C_{n-1}) \to \cdots$

First show that U preserves kernels

Let $g: B \to C$

Then we get the exact sequence: $0 \to \ker g \xhookrightarrow{i} B \xrightarrow{g} C$

Since U is exact, $0 \to U(\ker g) \hookrightarrow U(B) \xrightarrow{U(g)} U(C)$ is exact

Hence $U(\ker g) \cong \ker U(g)$

Also show that U preserves cokernels

Let $f: A \to B$

Then we get the exact sequence: $A \xrightarrow{f} B \xrightarrow{\pi} \operatorname{coker} f \to 0$

But since U is exact, $U(A) \xrightarrow{U(f)} U(B) \to U(\operatorname{coker} f) \to 0$ is exact

$\therefore U(\operatorname{coker} f) \cong \operatorname{coker} U(f)$

Finally show that U preserves images

Let $g: B \to C$

Then we get the SES: $0 \to \operatorname{Ker} g \xrightarrow{i} B \xrightarrow{g} \operatorname{Im} g \to 0$

But U is exact, so $0 \to U(\operatorname{Ker} g) \to U(B) \xrightarrow{U(g)} U(\operatorname{Im} g) \to 0$ is exact, hence $U(g)$ is surjective

$\therefore U(\operatorname{Im} g) = \operatorname{Im} U(g)$

Now since $U(C_\cdot)$ is a complex, we get the SES:

$$0 \to \operatorname{Im} U(d_{i+1}) \hookrightarrow \operatorname{Ker} U(d_i) \to H_i(U(C_\cdot)) \to 0$$

Thus $H_i(U(C_\cdot)) \cong \operatorname{coker}(\operatorname{Im} U(d_{i+1}) \hookrightarrow \operatorname{Ker} U(d_i))$

$\cong \operatorname{coker}(U(\operatorname{Im} d_{i+1}) \xrightarrow{U(i)} U(\operatorname{Ker} d_i))$

where $0 \to \operatorname{Im} d_{i+1} \xrightarrow{i} \operatorname{Ker} d_i \to H_i(C_\cdot) \to 0$ is a SES since $C_\cdot$ is a complex

we can do this since U preserves kernels and images

But U also preserves cokernels, so $H_i(U(C_\cdot)) \cong U(\operatorname{coker} i)$

$\cong U(H_i(C_\cdot))$

$\therefore H_i(U(C_\cdot)) \cong U(H_i(C_\cdot))$ $\forall$ complexes $C_\cdot$

$\therefore H_i(U(F(P_\cdot))) \cong U(H_i(F(P_\cdot)))$

$\therefore U(L_iF(A)) \cong L_i(UF(A))$ $\forall A$

3.1.2. Suppose that R is a commutative domain with field of fractions F. Show that $\mathrm{Tor}_1^R(F/R, B)$ is the torsion submodule $\{b \in B \mid \exists r \neq 0 \ni rb = 0\}$ of B for every R-module B.

Let B be an R-module.

We have the SES: $0 \longrightarrow R \xhookrightarrow{i} F \xrightarrow{\pi} F/R \longrightarrow 0$

Then we get the LES:

$\ldots \longrightarrow \mathrm{Tor}_1^R(F, B) \longrightarrow \mathrm{Tor}_1^R(F/R, B) \xrightarrow{\delta} R \otimes_R B \xrightarrow{i \otimes 1} F \otimes_R B \longrightarrow \ldots$

First note that since F is a flat R-module, $\mathrm{Tor}_1^R(F, B) = 0$

So we have that $0 \longrightarrow \mathrm{Tor}_1^R(F/R, B) \xrightarrow{\delta} R \otimes_R B$ is exact, hence δ is injective

$\therefore \mathrm{Tor}_1^R(F/R, B) \cong \mathrm{Im}\,\delta = \ker(i \otimes 1)$ by exactness

But $R \otimes_R B \cong B$, so $\ker(i \otimes 1) \cong \ker \varphi$ where $\varphi : B \longrightarrow F \otimes_R B$ $\ni \varphi(b) = \frac{1}{1} \otimes b$

So $\varphi(b) = 0$ iff $\frac{1}{1} \otimes b = 0$ iff $\frac{r}{r} \otimes b = 0 \ \forall r$ iff $\frac{1}{r} \otimes rb = 0$ iff $\exists r \neq 0 \ni rb = 0$

$\therefore \ker \varphi = \{b \in B \mid \exists r \neq 0 \ni rb = 0\}$

$\therefore \mathrm{Tor}_1^R(F/R, B) \cong \{b \in B \mid \exists r \neq 0 \ni rb = 0\}$

3.2.2. Show that if $0 \to A \to B \to C \to 0$ is exact and both B and C are flat, then A is also flat.

First show that TFAE:

(1) F flat R-module

(2) $Tor_n^R(F,X) = 0 \ \forall n \geq 1 \ \forall$ R-modules X

(3) $Tor_1^R(F,X) = 0 \ \forall$ R-modules X

(1)⇒(2) Assume F flat

Let X be an R-module

Then $Tor_n^R(F,X) = H_n(F \otimes_R P.)$ where $P. \to X$ is a projective resolution

So P. is exact, hence $F \otimes_R P.$ is exact $\forall n \geq 1$ since F is flat

$\therefore H_n(F \otimes_R P.) = 0 \ \forall n \geq 1$

$\therefore Tor_n^R(F,X) = 0 \ \forall n \geq 1$

(2)⇒(3) Assume $Tor_n^R(F,X) = 0 \ \forall n \geq 1 \ \forall X$

Then $Tor_1^R(F,X) = 0 \ \forall n \geq 1 \ \forall X$

(3)⇒(1) Assume $Tor_1^R(F,X) = 0 \ \forall n \geq 1 \ \forall X$

Let $0 \to A \to B \to C \to 0$ SES

Then we get the LES: $\dots \to Tor_1^R(F,C) \to F \otimes_R A \to F \otimes_R B \to F \otimes_R C \to 0$

But then $Tor_1^R(F,C) = 0$, so we have the SES:

$$0 \to F \otimes_R A \to F \otimes_R B \to F \otimes_R C \to 0$$

$\therefore$ F flat

Now from the given SES, we get the LES:

$$\dots \to Tor_2^R(C,X) \to Tor_1^R(A,X) \to Tor_1^R(B,X) \to \dots$$

But $Tor_2^R(C,X) = Tor_1^R(B,X) = 0$ since B, C flat

$\therefore \ 0 \to Tor_1^R(A,X) \to 0$ is exact

$\therefore Tor_1^R(A,X) = 0 \ \forall X$

$\therefore$ A is flat

3.2.5. Suppose that $\varphi: F \to M$ is any surjection, where F is finitely generated and M is finitely presented. Use the snake lemma to show that $\ker\varphi$ is finitely generated.

First note that since M is finitely presented, there is a SES: $0 \to \ker\psi \hookrightarrow R^n \xrightarrow{\psi} M \to 0$ with $\ker\psi$ finitely generated

And since F is finitely generated, $\exists\, R^m \xrightarrow{\pi} F \to 0$

We have the following diagram:

$$\begin{tikzcd} F \arrow[r, "\varphi"] & M \arrow[r] & 0 \\ & R^n \arrow[ul, dashed, "\exists f"] \arrow[u, "\psi"'] & \end{tikzcd}$$

Since R^n is free, it is also projective, thus $\exists f: R^n \to F$ $\ni \psi = \varphi f$

And also we have

$$\begin{tikzcd} R^n \arrow[r, "\psi"] & M \arrow[r] & 0 \\ & R^m \arrow[ul, dashed, "\exists g"] \arrow[u, "\varphi\pi"'] & \end{tikzcd}$$

Since R^m free, hence projective $\exists\, g: R^m \to R^n \ni \varphi\pi = \psi g$

Consider the sequence $0 \to R^m \xrightarrow{h} R^m \oplus R^n \xrightarrow{j} R^n \to 0$

where $h(x) = (x, -g(x))$ and $j(x,y) = g(x) + y$

Note h injective, j surjective, and $\ker j = \operatorname{Im} h$

$\therefore$ The sequence is exact

Then we have the diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & R^m \arrow[r, "h"] \arrow[d, dashed, "\exists\gamma"'] & R^m \oplus R^n \arrow[r, "j"] \arrow[d, "\alpha"'] & R^n \arrow[r] \arrow[d, "\psi"'] & 0 \\ 0 \arrow[r] & \ker\varphi \arrow[r] & F \arrow[r, "\varphi"] & M \arrow[r] & 0 \end{tikzcd}$$

where $\alpha(x,y) = \pi(x) + f(y)$

Note that $\psi(j(x,y)) = \psi(g(x) + y) = \varphi(\pi(x)) + \psi(y)$

And $\varphi(\alpha(x,y)) = \varphi(\pi(x) + f(y)) = \varphi(\pi(x)) + \psi(y)$

Hence the diagram commutes

So $\exists\, \gamma: R^m \to \ker\varphi$ commuting the diagram

And by the snake Lemma, we get the exact sequence:

$0 \to \ker\gamma \to \ker\alpha \to \ker\psi \to \operatorname{coker}\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\psi \to 0$

Hence $\ker\psi \to \operatorname{coker}\gamma \to 0$ is exact since α is surjective

Then since $\ker\psi$ is finitely generated, $\operatorname{coker}\gamma$ is also finitely generated

But then we have the exact sequence:

$R^m \to \ker\psi \to \operatorname{coker}\gamma \to 0$

with R^m, $\operatorname{coker}\gamma$ finitely generated

$\therefore \ker\psi$ is finitely generated

A.1.5 Let $\{\alpha_i : A_i \to C_i\}$ be a family of maps in $\mathcal{C}$. Show that

a. If $\prod A_i$ and $\prod C_i$ exist, $\exists!\, \alpha : \prod A_i \to \prod C_i \ni$

$\pi_i \alpha = \alpha_i \pi_i \quad \forall i$. If every α_i is monic, so is α.

Consider the following diagram:

$$\begin{tikzcd} \prod C_i \arrow[r, "\pi_i"] & C_i \\ & A_i \arrow[u, "\alpha_i"] \\ & \prod A_i \arrow[u, "\tilde{\pi}_i"] \arrow[uul, dashed, "\exists!\alpha"] \end{tikzcd}$$

Since $\prod C_i$ exists, $\exists!\, \alpha : \prod A_i \to \prod C_i \ni \pi_i \alpha = \alpha_i \tilde{\pi}_i$ by the universal property of products

Now assume every α_i is monic

Let $A \underset{e_2}{\overset{e_1}{\rightrightarrows}} \prod A_i \xrightarrow{\alpha} \prod C_i \ni \alpha e_1 = \alpha e_2$

Then $\pi_i \alpha e_1 = \pi_i \alpha e_2$, so $\alpha_i \tilde{\pi}_i e_1 = \alpha_i \tilde{\pi}_i e_2$

But then $\tilde{\pi}_i e_1 = \tilde{\pi}_i e_2$ since α_i is monic

So we have the diagram:

$$\begin{tikzcd} \prod A_i \arrow[r, "\tilde{\pi}_i"] & A_i \\ & A \arrow[u, "\tilde{\pi}_i e_1"'] \arrow[ul, shift left, "e_2"] \arrow[ul, shift right, "e_1"'] \end{tikzcd}$$

But since $\prod A_i$ exists, $\exists!\, \theta : A \to \prod A_i$ commuting the diagram

And e_1, e_2 both commute the diagram, hence $\theta = e_1 = e_2$

$\therefore$ α is monic

b. If $\oplus A_i$ and $\oplus C_i$ exist, $\exists!\, \alpha : \oplus A_i \longrightarrow \oplus C_i \ni \tilde{\tau}_i \alpha_i = \alpha \tau_i \ \forall i$.

If every α_i is an epi, so is α.

Consider the following diagram:

$$\begin{tikzcd}
A_i \arrow[r, "\tau_i"] \arrow[d, "\alpha_i"'] & \oplus A_i \arrow[ddl, dashed, "\exists!\, \alpha"] \\
C_i \arrow[d, "\tilde{\tau}_i"'] & \\
\oplus C_i &
\end{tikzcd}$$

Since $\oplus A_i$ exists, $\exists!\, \alpha : \oplus A_i \longrightarrow \oplus C_i \ni \tilde{\tau}_i \alpha_i = \alpha \tau_i \ \forall i$

by the universal property of coproducts

Now assume every α_i is epi

Let $\oplus A_i \xrightarrow{\alpha} \oplus C_i \underset{g_2}{\overset{g_1}{\rightrightarrows}} D \ni g_1 \alpha = g_2 \alpha$

Then $g_1 \tilde{\tau}_i \alpha_i = g_1 \alpha \tau_i = g_2 \alpha \tau_i = g_2 \tilde{\tau}_i \alpha_i$

And since α_i is epi, $g_1 \tilde{\tau}_i = g_2 \tilde{\tau}_i$

So we have the diagram:

$$\begin{tikzcd}
C_i \arrow[r, "\tilde{\tau}_i"] \arrow[d, "g_1 \tilde{\tau}_i"'] & \oplus C_i \arrow[dl, shift right, "g_1"'] \arrow[dl, shift left, "g_2"] \\
D &
\end{tikzcd}$$

But since $\oplus C_i$ exists, $\exists!\, \theta : \oplus C_i \longrightarrow D$ commuting the diagram

And g_1, g_2 both commute the diagram, hence $\theta = g_1 = g_2$

$\therefore$ α is epi

Rotman (732)

2.2. Give an example of a left R-module $M = S \oplus T$ having a submodule $N \ni N \neq (N \cap S) \oplus (N \cap T)$

Take $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, $N = \{(z,z) \mid z \in \mathbb{Z}\} \leq M$
Note that $N \cap S = 0 = N \cap T$ since $(z,z) \in \mathbb{Z} = \mathbb{Z} \oplus 0$ iff $z = 0$
$\therefore N \neq 0 = (N \cap S) \oplus (N \cap T)$

2.5 Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a SES of left R-modules. If M is any left R-module, prove that there are exact sequences $0 \longrightarrow A \oplus M \longrightarrow B \oplus M \longrightarrow C \longrightarrow 0$ and $0 \longrightarrow A \longrightarrow B \oplus M \longrightarrow C \oplus M \longrightarrow 0$.

Define $\tilde{f} : A \oplus M \longrightarrow B \oplus M \ni \tilde{f}(a,m) = (f(a), m)$ and
$\tilde{g} : B \oplus M \longrightarrow M \ni \tilde{g}(b,m) = g(b)$
Then $\tilde{f}(a,m) = 0$ iff $(f(a), m) = (0,0)$ iff $f(a) = 0$ and $m = 0$
iff $a = 0$ and $m = 0$ since f injective
$\therefore \ker \tilde{f} = 0$
$\therefore \tilde{f}$ is injective
Let $x \in C \implies x = g(b) = \tilde{g}(b,m)$
$\therefore \tilde{g}$ is surjective
Now let $x \in \operatorname{Im} \tilde{f} \implies x = \tilde{f}(a,m) = (f(a), m)$
Then $\tilde{g}(x) = \tilde{g}(f(a), m) = g(f(a)) = 0$ by exactness
$\therefore \operatorname{Im} \tilde{f} \subseteq \ker \tilde{g}$
Finally let $x \in \ker \tilde{g} \implies 0 = \tilde{g}(x) = \tilde{g}(b,m) = g(b)$
Then $b \in \ker g = \operatorname{Im} f \implies b = f(a) \implies x = (f(a), m) = \tilde{f}(a,m)$
$\therefore \ker \tilde{g} \subseteq \operatorname{Im} \tilde{f}$
$\therefore \operatorname{Im} \tilde{f} = \ker \tilde{g}$
$\therefore \quad 0 \longrightarrow A \oplus M \xrightarrow{\tilde{f}} B \oplus M \xrightarrow{\tilde{g}} C \longrightarrow 0$ SES
Now define $\tilde{\tilde{f}} : A \longrightarrow B \oplus M \ni \tilde{\tilde{f}}(a) = (f(a), 0)$ and
$\tilde{\tilde{g}} : B \oplus M \longrightarrow C \oplus M \ni \tilde{\tilde{g}}(b,m) = (g(b), m)$
First let $x \in \ker \tilde{\tilde{f}} \implies 0 = \tilde{\tilde{f}}(x) = (f(x), 0) \implies f(x) = 0$

So $x \in \ker f = 0$

$\therefore \ker \tilde{\tilde{f}} = 0$

$\therefore \tilde{\tilde{f}}$ injective

Now let $(c,m) \in C \oplus M \Rightarrow (c,m) = (g(b), m) = \tilde{\tilde{g}}(b,m)$

$\therefore \tilde{\tilde{g}}$ surjective

Let $(b,m) \in \operatorname{Im} \tilde{\tilde{f}} \Rightarrow (b,m) = \tilde{\tilde{f}}(a) = (f(a), 0)$

Then $\tilde{\tilde{g}}(b,m) = \tilde{\tilde{g}}(f(a), 0) = (g(f(a)), 0) = (0,0)$

$\therefore (b,m) \in \ker \tilde{\tilde{g}}$

$\therefore \operatorname{Im} \tilde{\tilde{f}} \subseteq \ker \tilde{\tilde{g}}$

Now let $(b,m) \in \ker \tilde{\tilde{g}} \Rightarrow 0 = \tilde{\tilde{g}}(b,m) = (g(b), m) \Rightarrow$

$g(b) = 0$ and $m = 0 \Rightarrow b \in \ker g = \operatorname{Im} f$ and $m = 0 \Rightarrow$

$b = f(a)$ and $m = 0 \Rightarrow (b,m) = (f(a), 0) = \tilde{\tilde{f}}(a) \in \operatorname{Im} \tilde{\tilde{f}}$

$\therefore \ker \tilde{\tilde{g}} \subseteq \operatorname{Im} \tilde{\tilde{f}}$

$\therefore \operatorname{Im} \tilde{\tilde{f}} = \ker \tilde{\tilde{g}}$

$\therefore 0 \longrightarrow A \xrightarrow{\tilde{\tilde{f}}} B \oplus M \xrightarrow{\tilde{\tilde{g}}} C \oplus M \longrightarrow 0$ SES

2.11. Prove that if $f: M \longrightarrow N$ is an R-module homomorphism and $K \leq M$ where M is a left R-module with $K \subseteq \ker f$, then f induces an R-module homomorphism $\hat{f}: M/K \longrightarrow N$ $\ni$ $\hat{f}(m+K) = f(m)$

If $m_1+K = m_2+K$, then $m_1 - m_2 \in K \subseteq \ker f \Rightarrow f(m_1-m_2) = 0$
so $f(m_1) - f(m_2) = 0 \Rightarrow f(m_1) = f(m_2) \Rightarrow \hat{f}(m_1+K) = \hat{f}(m_2+K)$
$\therefore \hat{f}$ is well defined
$\hat{f}(m_1+K+m_2+K) = \hat{f}(m_1+m_2+K) = f(m_1+m_2) = f(m_1)+f(m_2)$
$= \hat{f}(m_1+K) + \hat{f}(m_2+K)$
$\hat{f}(r(m+K)) = \hat{f}(rm+K) = f(rm) = rf(m) = r\hat{f}(m+K)$
$\therefore \hat{f}$ is an R-module homomorphism

2.14. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of R-module maps. Prove that $gf = 0$ iff $\operatorname{Im} f \subseteq \ker g$. Give an example of such a sequence that is not exact.

($\Rightarrow$) Assume that $gf = 0$
Let $x \in \operatorname{Im} f \Rightarrow x = f(y) \Rightarrow g(x) = g(f(y)) = 0$
$\therefore x \in \ker g$
$\therefore \operatorname{Im} f \subseteq \ker g$
($\Leftarrow$) Assume that $\operatorname{Im} f \subseteq \ker g$
Let $x \in A \Rightarrow f(x) \in \operatorname{Im} f \subseteq \ker g \Rightarrow g(f(x)) = 0$
$\therefore gf = 0$
Now consider $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$
$\operatorname{Im} 2 = 2\mathbb{Z}$ and $\ker 0 = \mathbb{Z}$
Then $\operatorname{Im} 2 \subseteq \ker 0$ but $\ker 0 \nsubseteq \operatorname{Im} 2$
$\therefore \operatorname{Im} 2 \neq \ker 0$
$\therefore$ The sequence is not exact

2.17. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ is exact, prove that $\exists$SES:
$0 \longrightarrow \operatorname{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \longrightarrow 0$ where $\alpha(b+\operatorname{Im} f) = g(b$
and $\beta(c) = h(c)$

Let $b+\operatorname{Im} f \in \ker \alpha \Rightarrow 0 = \alpha(b+\operatorname{Im} f) = g(b) \Rightarrow b \in \ker g = \operatorname{Im} f$
$\Rightarrow b = f(a) \Rightarrow b + \operatorname{Im} f = f(a) + \operatorname{Im} f = \operatorname{Im} f = 0$ coker f
$\therefore \ker \alpha = 0$
$\therefore \alpha$ is injective
Let $d \in \ker k = \operatorname{Im} h \Rightarrow d = h(c) = \beta(c) \in \operatorname{Im} \beta$
$\therefore \beta$ is surjective
Let $c \in \operatorname{Im} \alpha \Rightarrow c = \alpha(b + \operatorname{Im} f) = g(b) \Rightarrow \beta(c) = \beta(g(b)) = h(g(b)$
$= 0$ by exactness
$\therefore c \in \ker \beta$
$\therefore \operatorname{Im} f \subseteq \ker \beta$
Now let $c \in \ker \beta \Rightarrow 0 = \beta(c) = h(c) \Rightarrow c \in \ker h = \operatorname{Im} g$
$\Rightarrow c = g(b) = \alpha(b + \operatorname{Im} f) \in \operatorname{Im} \alpha$
$\therefore \ker \beta \subseteq \operatorname{Im} \alpha$
$\therefore \operatorname{Im} \alpha = \ker \beta$
$\therefore$ The sequence is exact

2.28. Let R be an integral domain with field of fractions Q. If A is an R-module, prove that every element in $Q \otimes_R A$ has the form $q \otimes a$ for $q \in Q$ and $a \in A$.

Let $\sum_{i=1}^{n} (q_i \otimes a_i) \in Q \otimes_R A$

Then $\sum_{i=1}^{n} (q_i \otimes a_i) = \sum_{i=1}^{n} \left(\frac{r_i}{s_i} \otimes a_i \right) = \sum_{i=1}^{n} \left(\frac{1}{s_i} \otimes r_i a_i \right)$

$= \sum_{i=1}^{n} \left(\frac{s_1 \cdots s_{i-1} s_i \cdots s_n}{s_1 \cdots s_n} \otimes r_i a_i \right)$

$= \sum_{i=1}^{n} \left(\frac{1}{s} \otimes s_1 \cdots s_{i-1} s_{i+1} \cdots s_n r_i a_i \right)$

$= \frac{1}{s} \otimes \left(\sum_{i=1}^{n} s_1 \cdots s_{i-1} s_{i+1} \cdots s_n r_i a_i \right)$

$= \frac{1}{s} \otimes a$ for some $\frac{1}{s} \in Q$, $a \in A$

2.34. Give an example of a commutative diagram with exact rows and vertical maps isomorphisms

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

$\ni \nexists\, A_3 \longrightarrow B_3$ commuting the diagram.

2.37. Assume that a ring R has IBN ie if $R^m \cong R^n$ as left R-modules, then $m=n$. Prove that if $R^m \cong R^n$ as right R-modules, then $m=n$.

Assume that $R^m \cong R^n$ as right R-modules
Then $\mathrm{Hom}_R(R^m, R) \cong \mathrm{Hom}_R(R^n, R)$ as left R-modules
since ${}_R R_R$
But then $R^m \cong R^n$ as left R-modules
$\therefore m=n$ since R has IBN

3.2. Let R be a ring and let $0 \neq S \leq F$ where F is a free right R-module. Prove that if $a \in R$ is not a right zero divisor, then $Sa \neq \{0\}$.

Suppose that $Sa = \{0\}$
Then $sa = 0 \quad \forall s \in S$
And since $S \neq 0$, $\exists\, 0 \neq s \in S \ni sa = 0$
But F is free, so $F \cong \bigoplus R$
And $S \leq F$, so $s \in \bigoplus R \Rightarrow s = \bigoplus_{i \in I} r_i$ where $r_j \neq 0$ for some j since $s \neq 0$
Then $0 = sa = \bigoplus_{i \in I} r_i a \Rightarrow r_j a = 0 \Rightarrow a$ is a right zero divisor since $r_j \neq 0$
Contradiction
$\therefore Sa \neq \{0\}$

3.5. Prove that every projective left R-module P has a free complement ie $\exists F$ a free left R-module $\ni P\oplus F$ is free.

Since P is projective, P is a direct summand of a free module, F

So $F \cong P \oplus X$ for some X

Then $\bigoplus_{i=1}^{\infty} F$ is free since F is free

Then $P \oplus \bigoplus_{i=1}^{\infty} F \cong P \oplus \bigoplus_{i=1}^{\infty} (X \oplus P) \cong P \oplus (X\oplus P) \oplus (X \oplus P) \oplus \dots$

$\cong (P\oplus X) \oplus (P \oplus X) \oplus \dots$

$\cong \bigoplus_{i=1}^{\infty} F$ which is free

$\therefore$ P has a free complement, $\bigoplus_{i=1}^{\infty} F$

3.8. Let $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ be a ring. Prove that R is left noetherian, but is not right noetherian.

Let $I = \begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$

$\begin{pmatrix} z & 0 \\ q & q' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ q'r & 0 \end{pmatrix} \in I$, $\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ q & q' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ rz & 0 \end{pmatrix} \in I$

$\therefore I \triangleleft R$

But also we have 1-1 correspondences preserving inclusions:

$\{\text{left submodules of } {}_R I\} \longleftrightarrow \{\text{left submodules of } {}_{\mathbb{Q}}\mathbb{Q}\}$

$\{\text{right submodules of } I_R\} \longleftrightarrow \{\text{right submodules of } \mathbb{Q}_{\mathbb{Z}}\}$

And $R/I \cong \mathbb{Z} \times \mathbb{Q}$ as rings

with SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$

${}_R I$ is left noetherian since ${}_{\mathbb{Q}}\mathbb{Q}$ is noetherian because $\mathbb{Q}$ is a field

And R/I is left noetherian since $\mathbb{Z}$ is a PID hence noetherian and $\mathbb{Q}$ is noetherian, thus $\mathbb{Z}\times\mathbb{Q}$ is noetherian

$\therefore$ R is left noetherian

But $\mathbb{Q}_{\mathbb{Z}}$ is not finitely generated, hence I_R is not finitely generated

$\therefore$ R is not right noetherian
$\therefore$ R is also not right artinian
Suppose that R is left artinian
Then R/I is left artinian, hence $\mathbb{Z} \times \mathbb{Q}$ is left artinian
$\therefore \mathbb{Z}$ is left artinian
contradiction since $(2) \supsetneq (4) \supsetneq \ldots$ does not stabilize
$\therefore$ R is not left artinian

3.11. Prove that $\mathrm{Hom}_R(P,R) \neq \{0\}$ if $P \neq 0$ is a projective left R-module.

Suppose that $\mathrm{Hom}_R(P,R) = 0$
Then since P is projective, $\exists F$ free $\ni F \cong P \oplus X$ for some X
And since F is free, $F = \bigoplus R$
we have the SES: $0 \longrightarrow P \longrightarrow P \oplus X \longrightarrow X \longrightarrow 0$
Then since P is projective, $\mathrm{Hom}_R(P,-)$ is exact, so
we get the SES: $0 \to \mathrm{Hom}_R(P,P) \to \mathrm{Hom}_R(P, P \oplus X) \to \mathrm{Hom}_R(P,X) \to 0$
But $\mathrm{Hom}_R(P, P\oplus X) \cong \mathrm{Hom}_R(P, \bigoplus R) \cong \bigoplus \mathrm{Hom}_R(P,R) = \bigoplus 0 = 0$
So $0 \longrightarrow \mathrm{Hom}_R(P,P) \longrightarrow 0$ is exact
$\therefore \mathrm{Hom}_R(P,P) = 0$
$\therefore P = 0$ because if not $0 \neq 1_P \in \mathrm{Hom}_R(P,P)$
Contradiction since $P \neq 0$
$\therefore \mathrm{Hom}_R(P,R) \neq 0$

3.18. (i) Prove that if a domain R is an injective R-module, then R is a field.

Let $0 \neq r \in R$

Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "f"] \arrow[d, "1_R"'] & R \arrow[dl, dashed, "\exists g"] \\ & R & \end{tikzcd}$$

where $f(s) = sr \ \forall s \in R$

Note that f is an R-module homomorphism and

$s \in \ker f$ iff $0 = f(s) = sr$

But R is an integral domain, so $s = 0$

$\therefore \ker f = 0$

$\therefore f$ is injective

Then since R is an injective R-module, $\exists g: R \longrightarrow R \ni$

$gf = 1_R$

so $1 = g(f(1)) = g(r) = rg(1)$

$\therefore$ $1 = rg(1) = g(1)r$ since R is commutative

$\therefore$ $g(1) = r^{-1}$

$\therefore$ R is a field

(ii) Prove that $\mathbb{Z}/6\mathbb{Z}$ is both an injective and projective $\mathbb{Z}/6\mathbb{Z}$-module.

Note that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

And $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ are simple $\mathbb{Z}/6\mathbb{Z}$-modules

$\therefore$ $\mathbb{Z}/6\mathbb{Z}$ is a semisimple $\mathbb{Z}/6\mathbb{Z}$-module

$\therefore$ $\mathbb{Z}/6\mathbb{Z}$ is a semisimple ring

$\therefore$ Every $\mathbb{Z}/6\mathbb{Z}$-module is both projective and injective

In particular, $\mathbb{Z}/6\mathbb{Z}$ is both projective and injective $\mathbb{Z}/6\mathbb{Z}$-module

(iii) Let R be an integral domain that is not a field, and let M be an R-module that is both injective and projective. Prove that $M=0$.

Suppose $M \neq 0$
Let $0 \neq r \in R$
Then we have that $0 \longrightarrow R \xrightarrow{r} R$ is exact since $rs=0$ iff $s=0$ because R is an integral domain
so since M is injective, $Hom_R(\_, M)$ is exact, so
$Hom_R(R, M) \longrightarrow Hom_R(R, M) \longrightarrow 0$ is exact, hence
$M \xrightarrow{r} M \longrightarrow 0$ is exact
$\therefore rM = M \quad \forall 0 \neq r \in R$
Let $\varphi: M \longrightarrow R$ be a nonzero R-module homomorphism
since $M \neq 0$
Then $\exists m \in M \ni \varphi(m) \neq 0$
so $\varphi(m)M = M \Rightarrow \exists m' \in M \ni m = \varphi(m)m'$
so $\varphi(m) = \varphi(\varphi(m)m') = \varphi(m)\varphi(m') \Rightarrow \varphi(m)(1-\varphi(m')) = 0$
But $\varphi(m) \neq 0$ and R is an integral domain, so $1-\varphi(m')=0$
$\therefore 1 = \varphi(m'), \ m' \in M$
$\therefore \varphi$ is surjective
$\therefore$ Every nonzero R-module homomorphism $M \longrightarrow R$ is surjective
Now since M is projective, M is a direct summand of a free module $F = \oplus R$
so we have $M \xrightarrow{i} F \xrightarrow{\pi} R \xrightarrow{r} R$
Since $M \neq 0$, $r\pi i$ is a nonzero R-module homomorphism $\forall 0 \neq r \in R$
$\therefore r\pi i$ is surjective
$\therefore 1 = r\pi i(m)$ for some $m \in M$
$\therefore 1 = r\pi(m) \quad \forall 0 \neq r \in R$
$\therefore R$ is a field
Contradiction, hence $M=0$

3.21. (i) Let $M \subseteq E$ be left R-modules. Prove that $M \overset{ess}{\subseteq} E$ iff $\forall 0 \neq e \in E$, $\exists r \in R \ni 0 \neq re \in M$.

($\Rightarrow$) Assume $M \overset{ess}{\subseteq} E$

Let $0 \neq e \in E$

Then $0 \neq (e) \leq E$

So since $M \overset{ess}{\subseteq}$, $(e) \cap M \neq (0)$

Then $\exists 0 \neq x \in (e) \cap M$

$\therefore 0 \neq x = re \in M$

$\therefore \forall 0 \neq e \in E$, $\exists r \in R \ni 0 \neq re \in M$

($\Leftarrow$) Assume $\forall 0 \neq e \in E$, $\exists r \in R \ni 0 \neq re \in M$

Let $0 \neq L \leq E$

Then $\exists 0 \neq e \in L \leq E$

So $\exists r \in R \ni 0 \neq re \in M$

But also $re \in L$ since L is a module

$\therefore 0 \neq re \in L \cap M$

$\therefore L \cap M \neq 0$

$\therefore M \overset{ess}{\subseteq} E$

(ii) Let $M \leq E$ be left R-modules and let $\mathcal{S}$ be a chain of intermediate submodules. If $M \overset{ess}{\subseteq} S$ for each $S \in \mathcal{S}$, prove that $M \overset{ess}{\subseteq} \bigcup_{S \in \mathcal{S}} S$

Let $0 \neq x \in \bigcup_{S \in \mathcal{S}} S$

So $x \in S$ for some $S \in \mathcal{S}$

But $M \overset{ess}{\subseteq} E$, so $\exists r \in R \ni 0 \neq rx \in M$ by (i)

$\therefore M \overset{ess}{\subseteq} \bigcup_{S \in \mathcal{S}} S$ by (i)

3.24. If R is an integral domain, prove that $Q = E(R)$ where Q is the field of fractions of R

First show $R \overset{ess}{\subseteq} Q$

Let $0 \neq L \leq Q$

Then $\exists\, 0 \neq \frac{a}{b} \in L$, hence $0 \neq a \in R$

And $a = b \cdot \frac{a}{b} \in L$ since L is an R-submodule

So $0 \neq a \in L \cap R$

$\therefore L \cap R \neq 0$

$\therefore R \overset{ess}{\subseteq} Q$

Now show that Q is injective

Let $I \trianglelefteq R$ and let $I \xrightarrow{f} Q$ be an R-module homomorphism

Then $\forall\, 0 \neq a, b \in I$, $af(b) = f(ab) = bf(a)$

So $af(b) - bf(a) = 0$, hence $\frac{f(a)}{a} = \frac{f(b)}{b}$ $\forall\, 0 \neq a, b \in I$

Define $g : R \longrightarrow Q \ni g(r) = r\frac{f(a)}{a}$

Then we have the diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & Q & \end{tikzcd}$$

And $g(a) = a\frac{f(a)}{a} = f(a)$ $\forall\, 0 \neq a \in I$

$\therefore g|_I = f$

$\therefore Q$ is injective by Baer's criterion

$\therefore R \overset{ess}{\subseteq} Q$

3.34. Let B and C be left R-modules and define $v: \mathrm{Hom}_R(B,R) \otimes_R C \longrightarrow \mathrm{Hom}_R(B,C) \ni v(f\otimes c)(b) = f(b)c$ $\forall b \in B$, $\forall c \in C$. (i) Prove that v is natural in B

$$v(f\otimes c)(b_1+b_2) = f(b_1+b_2)\,c = (f(b_1)+f(b_2))\,c = f(b_1)c + f(b_2)c$$
$$= v(f\otimes c)(b_1) + v(f\otimes c)(b_2)$$
$$v(f\otimes c)(rb) = f(rb)c = r f(b) c = r\, v(f\otimes c)(b)$$

$\therefore$ $v(f\otimes c)$ is an R-module homomorphism

Now define $\varphi: \mathrm{Hom}_R(B,R) \times C \longrightarrow \mathrm{Hom}_R(B,C) \ni$
$\varphi(f,c)(b) = f(b)c$

Then $\varphi(f_1+f_2,c)(b) = (f_1+f_2)(b)c = (f_1(b)+f_2(b))c$
$$= f_1(b)c + f_2(b)c = \varphi(f_1,c)(b) + \varphi(f_2,c)(b)$$
$$\varphi(f, c_1+c_2)(b) = f(b)(c_1+c_2) = f(b)c_1 + f(b)c_2$$
$$= \varphi(f,c_1)(b) + \varphi(f,c_2)(b)$$
$$\varphi(fr,c)(b) = (fr)(b)c = f(b)rc = \varphi(f,rc)(b)$$

$\therefore$ φ is biadditive

So $\exists!\, v: \mathrm{Hom}_R(B,R) \otimes_R C \longrightarrow \mathrm{Hom}_R(B,C) \ni v(f\otimes c)(b) = f(b)c$

a homomorphism of abelian groups

Now let $g: B \longrightarrow B'$

Then we have the diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(B,R)\otimes_R C \arrow[r, "v_B"] & \mathrm{Hom}_R(B,C) \\
\mathrm{Hom}_R(B',R)\otimes_R C \arrow[r, "v_{B'}"] \arrow[u, "g^*\otimes 1"] & \mathrm{Hom}_R(B',C) \arrow[u, "g^*"']
\end{tikzcd}$$

Let $h \otimes c \in \mathrm{Hom}_R(B',R) \otimes_R C$

$$v_B(g^*\otimes 1)(h\otimes c)(b) = v_B(g^*(h)\otimes c)(b) = v_B(hg \otimes c)(b)$$
$$= h(g(b))c$$
$$g^*(v_{B'}(h\otimes c))(b) = v_{B'}(h\otimes c)(g(b)) = h(g(b))c$$

$\therefore$ The diagram commutes

$\therefore$ v is natural in B

(ii) Prove that v is an isomorphism if B is finitely generated free

Note that since B is finitely generated free, $B \cong R^n$ for some n

So $\mathrm{Hom}_R(B,R) \otimes_R C \cong \mathrm{Hom}_R(R^n,R) \otimes_R C \cong R^n \otimes_R C \cong C^n$
$\cong \mathrm{Hom}_R(R^n, C) \cong \mathrm{Hom}_R(B,C)$

where $(f \otimes_R c)(b) \longrightarrow f(b)c$

$\therefore$ v is an isomorphism

(iii) If B is finitely presented left R-module and C is a flat left R-module, prove that v is an isomorphism

Since B is finitely presented, $\exists\, R^m \longrightarrow R^n \longrightarrow B \longrightarrow 0$ exact

And since $\mathrm{Hom}_R(-,R)$ is left exact, we have the exact sequence: $0 \longrightarrow \mathrm{Hom}_R(B,R) \longrightarrow \mathrm{Hom}_R(R^n,R) \rightarrow \mathrm{Hom}_R(R^m,R)$

And since C is flat, $- \otimes_R C$ is exact, so we have the exact sequence:

$0 \rightarrow \mathrm{Hom}_R(B,R) \otimes_R C \longrightarrow \mathrm{Hom}_R(R^n,R) \otimes_R C \longrightarrow \mathrm{Hom}_R(R^m,R) \otimes_R C$

But $\mathrm{Hom}_R(R^n,R) \otimes_R C \cong \mathrm{Hom}_R(R^n,C)$ by (ii) and similarly $\mathrm{Hom}_R(R^m,R) \otimes_R C \cong \mathrm{Hom}_R(R^n,C)$ and these are natural isomorphisms

And since $\mathrm{Hom}_R(-,C)$ is left exact, we have the exact sequence $0 \longrightarrow \mathrm{Hom}_R(B,C) \longrightarrow \mathrm{Hom}_R(R^n,C) \longrightarrow \mathrm{Hom}_R(R^m,C)$

So we have the commutative diagram with exact rows and vertical maps isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(B,R) \otimes_R C \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_R(R^n,R) \otimes_R C \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_R(R^m,R) \otimes_R C \arrow[d, "\wr"] \\
0 \arrow[r] & \mathrm{Hom}_R(B,C) \arrow[r] & \mathrm{Hom}_R(R^n,C) \arrow[r] & \mathrm{Hom}_R(R^m,C)
\end{tikzcd}$$

Then $\exists$ an isomorphism $\mathrm{Hom}_R(B,R) \otimes_R C \cong \mathrm{Hom}_R(B,C)$ by the 5-lemma commuting the diagram

And by commutativity the isomorphism must send
$(f \otimes c)(b) \longmapsto f(b)c$
$\therefore$ v is an isomorphism

5.8. Give an example of a covariant functor that does not preserve coproducts.

5.24. Let the following be a pullback diagram in AB:

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

If $\exists\, c \in C,\ b \in B \ni g(c) = f(b)$, prove that $\exists\, d \in D \ni c = \alpha(d)$ and $b = \beta(d)$.

Note that $D = \varprojlim \left(B \xrightarrow{f} A \xleftarrow{g} C \right)$

Define $p : \mathbb{Z} \longrightarrow C \ni p(n) = nc$ and $q : \mathbb{Z} \longrightarrow B \ni q(n) = nb$

Then we have the following diagram:

$$\begin{tikzcd} & & C \arrow[d, "g"] \\ \mathbb{Z} \arrow[r, dashed, "\exists!\theta"] \arrow[urr, bend left, "p"] \arrow[drr, bend right, "q"'] & D \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & A \\ & & B \arrow[u, "f"'] \end{tikzcd}$$

Then $g(p(n)) = g(nc) = ng(c)$ since g is a $\mathbb{Z}$-module homomorphism

But then $g(p(n)) = ng(c) = n f(b) = f(nb) = f(q(n)) \quad \forall n \in \mathbb{Z}$

$\therefore\ gp = fq$

Hence p, q are compatible with the given system

Then since D is the limit of the system, $\exists!\, \theta : \mathbb{Z} \longrightarrow D$ commuting the diagram

So $\alpha\theta = p$ and $\beta\theta = q$

Define $d = \theta(1)$

Then $\alpha(d) = \alpha(\theta(1)) = p(1) = c$

And $\beta(d) = \beta(\theta(1)) = q(1) = b$

$\therefore\ \exists\, d \in D \ni c = \alpha(d)$ and $b = \beta(d)$

5.52. If $\mathcal{C}$ is an additive category and $C \in \operatorname{obj}\mathcal{C}$, prove that $\operatorname{Hom}(C,C)$ is a ring with composition as a product

First note that $\operatorname{Hom}(\mathcal{C},\mathcal{C})$ is an abelian group under addition since $\mathcal{C}$ is additive
$\operatorname{Hom}(C,C)$ is closed under compositions since $\mathcal{C}$ is a category hence $\operatorname{Hom}(C,C)$ is closed under multiplication
And multiplication is associative since compositions are associative because $\mathcal{C}$ is a category
$1_C \in \operatorname{Hom}(C,C) \ni 1_C f = f = f 1_C \quad \forall f \in \operatorname{Hom}(C,C)$ since $\mathcal{C}$ is a category
And the distributive property holds since $\mathcal{C}$ is additive
$\therefore \operatorname{Hom}(C,C)$ is a ring with composition as a product

5.64. (i) Prove that the category $\mathcal{T}$ of all torsion abelian groups is an abelian category having no nonzero projective objects

First note that $\mathcal{T}$ is a full subcategory of AB which is an abelian category
Let $A, B \in \operatorname{ob}(\mathcal{T})$ and let $f: A \to B$ be a morphism
Now the zero object in AB, 0, is also an object in $\mathcal{T}$ since any $0 \neq z \in \mathbb{Z} \ni z \cdot 0 = 0$, so 0 is torsion group
Let $(a,b) \in A \oplus B$
Then $\exists\, 0 \neq z, z' \in \mathbb{Z} \ni za = 0,\ zb = 0$ since $A, B \in \mathcal{T}$
Then $zz'(a,b) = (z'za, zz'b) = (0,0)$
$\therefore A \oplus B$ is a torsion abelian group
$\therefore A \oplus B \in \operatorname{ob}(\mathcal{T})$
And $\ker f \leq A$, so $\ker f$ is also torsion, hence $\ker f \in \operatorname{ob}(\mathcal{T})$
Let $b + \operatorname{Im} f \in \operatorname{Coker} f$
Then since $b \in B$, $\exists\, 0 \neq z \in \mathbb{Z} \ni zb = 0$, hence
$z(b + \operatorname{Im} f) = zb + \operatorname{Im} f = 0 + \operatorname{Im} f = 0$, thus $\operatorname{coker} f \in \operatorname{ob}(\mathcal{T})$

$\therefore$ $\mathcal{T}$ is an abelian category

Now suppose $\exists\, 0 \neq P \in ob(\mathcal{T})$ projective

Then P is a projective $\mathbb{Z}$-module

So P is a direct summand of a free $\mathbb{Z}$-module, F

Then $P \leq F$ and F has basis $\{x_i\}_{i \in I}$

Let $0 \neq x \in F$ and $z \in \mathbb{Z} \ni zx = 0$

Then $0 = z(\sum z_i x_i) = \sum z z_i x_i$

But the x_i's are linearly independent so $zz_i = 0 \; \forall i$

And since $x \neq 0$, $z_j \neq 0$ for some j

Then in particular, $zz_j = 0$, so $z = 0$ since $\mathbb{Z}$ is an integral domain

$\therefore$ x is torsion free

$\therefore$ F is torsion free

contradiction since $P \leq F$

$\therefore$ $\mathcal{T}$ has no nonzero projective objects

6.2. Prove that isomorphic complexes have the same homology.

Let $C_\cdot \cong D_\cdot$

Then $\exists f : C_\cdot \longrightarrow D_\cdot$ a chain map:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\ni$ each f_n is an isomorphism

we get the induced map $H_n(f) : H_n(C_\cdot) \longrightarrow H_n(D_\cdot) \ni$

$H_n(f)(z + \operatorname{Im} d_{n+1}) = f_n(z) + \operatorname{Im} \partial_{n+1}$

Show that $H_n(f)$ is an isomorphism

Let $z + \operatorname{Im} d_{n+1} \in \ker H_n(f)$

Then $0 = H_n(f)(z + \operatorname{Im} d_{n+1}) = f_n(z) + \operatorname{Im} \partial_{n+1} \Rightarrow f_n(z) \in \operatorname{Im} \partial_{n+1}$

So $f_n(z) = \partial_{n+1}(d_{n+1}) = \partial_{n+1}(f_{n+1}(c_{n+1}))$ since f_{n+1} is surjective

Then $f_n(z) = f_n(d_{n+1}(c_{n+1}))$ since f is a chain map

And since f_n is injective, $z = d_{n+1}(c_{n+1}) \in \operatorname{Im} d_{n+1}$

$\therefore z + \operatorname{Im} d_{n+1} = \operatorname{Im} d_{n+1} = 0_{H_n(C_\cdot)}$

$\therefore H_n(f)$ is surjective

Let $z + \operatorname{Im} \partial_{n+1} \in H_n(D_\cdot)$

Then $z \in \ker \partial_n \subseteq D_n$, so $z = f_n(c_n)$ since f_n is surjective

So $z + \operatorname{Im} \partial_{n+1} = f_n(c_n) + \operatorname{Im} \partial_{n+1}$

Show $c_n \in \ker d_n$

Since $z \in \ker \partial_n$, $0 = \partial_n(z) = \partial_n(f_n(c_n)) = f_{n-1}(d_n(c_n))$

so $d_n(c_n) \in \ker f_{n-1} = 0$

$\therefore c_n \in \ker d_n$

$\therefore z + \operatorname{Im} \partial_{n+1} = f_n(c_n) + \operatorname{Im} \partial_{n+1} = H_n(f)(c_n + \operatorname{Im} d_{n+1})$

$\therefore H_n(f)$ is surjective

$\therefore H_n(f)$ is an isomorphism $\forall n$

$\therefore H_n(C_\cdot) \cong H_n(D_\cdot) \quad \forall n$

6.14. Consider the commutative diagram with exact row

$$\begin{tikzcd} B' \arrow[r, "j"] \arrow[dr, "i"'] & C \arrow[r, "q"] \arrow[d, shift left, "\ell"] & B'' \\ & B \arrow[u, shift left, "k"] \arrow[ur, "p"'] & \end{tikzcd}$$

If k is an isomorphism with inverse ℓ, prove that
$B' \xrightarrow{i} B \xrightarrow{p} B''$ is exact

Let $x \in \operatorname{Im} i$
Then $x = i(b')$, so $p(x) = p(i(b')) = q(j(b')) = 0$ since the row is exact
$\therefore x \in \ker p$
$\therefore \operatorname{Im} i \subseteq \ker p$
Now let $x \in \ker p$
Then $0 = p(x) = p(\ell(c))$ since ℓ is surjective
But then $0 = p(\ell(c)) = q(c)$, hence $c \in \ker q = \operatorname{Im} j$
Then $c = j(b')$, which gives $x = \ell(c) = \ell(j(b')) = i(b') \in \operatorname{Im} i$
$\therefore \ker p \subseteq \operatorname{Im} i$
$\therefore \operatorname{Im} i = \ker p$
$\therefore$ The sequence is exact

6.17. Let R be a semisimple ring.

(i) Prove that $\forall n \geq 1$, $\mathrm{Tor}_n^R(A,B) = 0$ $\forall$ right R-modules A and $\forall$ left R-modules B

First note that since R is semisimple ring, every R-module is both injective and projective.
In particular, B is projective
so $P_\cdot : 0 \longrightarrow B \xrightarrow{1_B} B \longrightarrow 0$ is a projective resolution
Then $A \otimes_R P_\cdot : \quad 0 \longrightarrow A \otimes_R B \longrightarrow 0$
$\therefore \mathrm{Tor}_n^R(A,B) = H_n(A \otimes_R P_\cdot) = 0 \quad \forall n \geq 1$

(ii) Prove that $\forall n \geq 1$, $\mathrm{Ext}_R^n(A,B) = 0$ $\forall$ left R-modules A, B

Since R is semisimple, A is projective
So $P_\cdot : 0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0$ is a projective resolution
And $\mathrm{Hom}_R(P_\cdot, B) : \quad 0 \longrightarrow \mathrm{Hom}_R(A,B) \longrightarrow 0$
$\therefore \mathrm{Ext}_R^n(A,B) = H^n(\mathrm{Hom}_R(P_\cdot, B)) = 0 \quad \forall n \geq 1$

6.20. Let R be an integral domain with field of fractions Q.
(i) If $0 \neq r \in R$ and A is an R-module $\ni rA = 0$, prove that $\mathrm{Ext}^n_R(Q,A) = 0 = \mathrm{Tor}^R_n(Q,A)$ $\forall n \geq 0$.

First show that $Q = S^{-1}R$ is flat

Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a SES

Then we have the following diagram with the top row a SES:

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}A \arrow[r] \arrow[d, "\cong"] & S^{-1}B \arrow[r] \arrow[d, "\cong"] & S^{-1}C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & S^{-1}R \otimes_R A \arrow[r] & S^{-1}R \otimes_R B \arrow[r] & S^{-1}R \otimes_R C \arrow[r] & 0
\end{tikzcd}$$

And since the vertical maps are natural isomorphisms, the diagram commutes, hence the bottom row is exact

$\therefore$ $0 \longrightarrow Q \otimes_R A \longrightarrow Q \otimes_R B \longrightarrow Q \otimes_R C \longrightarrow 0$ is a SES

$\therefore$ Q is a flat R-module

Then $\mathrm{Tor}^R_n(Q,A) = 0$ $\forall n \geq 1$

And $\mathrm{Tor}^R_0(Q,A) = Q \otimes_R A$

Let $q \otimes a \in Q \otimes_R A$

Then $q \otimes a = \frac{t}{s} \otimes a = \frac{tr}{sr} \otimes a = \frac{t}{sr} \otimes ra = \frac{t}{sr} \otimes 0 = 0$

$\therefore$ $\mathrm{Tor}^R_0(Q,A) = 0$

$\therefore$ $\mathrm{Tor}^R_n(Q,A) = 0$ $\forall n \geq 0$

Now consider $Q \xrightarrow{r} Q$

Let $q \in \ker r \Rightarrow 0 = rq = r\frac{m}{n} = \frac{rm}{n} \Rightarrow \frac{rm}{n} = \frac{0}{1} \Rightarrow \exists u \in S \ni urm = 0$

But since $r, u \neq 0$ and R is an integral domain, $m = 0$

$\therefore$ $q = \frac{m}{n} = \frac{0}{n} = 0$

$\therefore$ $\ker r = 0$

$\therefore$ r is injective

Let $q \in Q \Rightarrow q = \frac{m}{n} = \frac{rm}{rn} = r\left(\frac{m}{rn}\right) \in \mathrm{Im}\, r$

$\therefore$ r is surjective

Then we have the SES: $0 \longrightarrow Q \xrightarrow{r} Q \longrightarrow 0 \longrightarrow 0$

And we get the LES:

$$\cdots \longrightarrow \mathrm{Ext}^n_R(0,A) \longrightarrow \mathrm{Ext}^n_R(Q,A) \xrightarrow{r} \mathrm{Ext}^n_R(Q,A) \longrightarrow \mathrm{Ext}^n_R(0,A) \longrightarrow \cdots$$

Hence $0 \longrightarrow Ext^n_R(Q,A) \xrightarrow{r} Ext^n_R(Q,A) \longrightarrow 0$ is exact $\forall n \geq 0$, thus r is an isomorphism $\forall n \geq 0$

Now $r \in Ann_R A \subseteq Ann_R A \cup Ann_R Q \subseteq Ann_R(Ext^n_R(Q,A))$

so $Ext^n_R(Q,A) = Im\, r = r\, Ext^n_R(Q,A) = 0 \quad \forall n \geq 0$

$\therefore Ext^n_R(Q,A) = 0 = Tor^R_n(Q,A) \quad \forall n \geq 0$

(ii) Prove that $Ext^n_R(V,A) = 0 = Tor^R_n(V,A)$ $\forall n \geq 0$ where V is a vectorspace over Q and A is an R-module $\ni$ $rA = 0$ for some $0 \neq r \in R$.

First note that V is flat since it is a vectorspace over the field Q, so $Tor^R_n(V,A) = 0 \quad \forall n \geq 1$

And $\forall v \otimes a \in V \otimes_R A$, $v \otimes a = v \cdot 1 \otimes a = v \cdot \frac{r}{r} \otimes a = v \cdot \frac{1}{r} \otimes ra$
$= v \cdot \frac{1}{r} \otimes 0 = 0$

$\therefore Tor^R_0(V,A) = V \otimes_R A = 0$

$\therefore Tor^R_n(V,A) = 0 \quad \forall n \geq 0$

Now consider $V \xrightarrow{r} V$

Let $v \in Ker\, r$, then $0 = rv = r(r_1 v_1 + \dots + r_n v_n)$ where $\{v_i\}_{i \in I}$ is a basis for V over Q

Then $0 = rr_1 v_1 + \dots + rr_n v_n$, but $\{v_i\}_{i \in I}$ are linearly independent, so $rr_i = 0 \; \forall i$, hence $r_i = 0 \; \forall i$ since R is an integral domain

$\therefore v = 0$

$\therefore Ker\, r = 0$

$\therefore r$ is injective

Let $v \in V$, then $v = 1 \cdot v = \frac{r}{r} \cdot v = r(\frac{1}{r} v) \in Im\, r$

$\therefore r$ is surjective

$\therefore 0 \longrightarrow V \xrightarrow{r} V \longrightarrow 0 \longrightarrow 0$ is a SES of R-modules since V is an R-module via restriction of scalars:
$R \longrightarrow Q \ni r \longmapsto \frac{r}{1}$

Then, similarly to (i), $Ext^n_R(V,A) = 0 \quad \forall n \geq 0$

$\therefore Ext^n_R(V,A) = 0 = Tor^R_n(V,A) \quad \forall n \geq 0$

7.15. (i) For any ring R, prove that a left R-module is injective iff $\operatorname{Ext}^1_R(R/I, B) = 0$ $\forall$ left ideals I.

($\Rightarrow$) Assume B is injective.
Then $\operatorname{Ext}^1_R(X, B) = 0$ $\forall$ R-modules X
In particular, $\operatorname{Ext}^1_R(R/I, B) = 0$ $\forall$ left ideals I
($\Leftarrow$) Assume $\operatorname{Ext}^1_R(R/I, B) = 0$ $\forall$ left ideals I
Let I be a left ideal of R and let $f: I \longrightarrow B$
Then we have the SES: $0 \longrightarrow I \xhookrightarrow{i} R \xrightarrow{\pi} R/I \longrightarrow 0$
which gives the LES:
$0 \to \operatorname{Hom}_R(R/I, B) \longrightarrow \operatorname{Hom}_R(R, B) \longrightarrow \operatorname{Hom}_R(I, B) \longrightarrow \operatorname{Ext}^1_R(R/I, B) \to$
But $\operatorname{Ext}^1_R(R/I, B) = 0$, so we have the SES:
$0 \longrightarrow \operatorname{Hom}_R(R/I, B) \xrightarrow{\pi^*} \operatorname{Hom}_R(R, B) \xrightarrow{i^*} \operatorname{Hom}_R(I, B) \longrightarrow 0$
In particular, i^* is surjective
Then $\exists g \in \operatorname{Hom}_R(R, B) \ni f = i^*(g) = gi$
So the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists g"] \\ & B & \end{tikzcd}$$

$\therefore$ B is injective by Baer's criterion

(ii) If D is abelian and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D) = 0$, prove that D is divisible. Does this hold if we replace $\mathbb{Z}$ by a domain R and $\mathbb{Q}/\mathbb{Z}$ by Q/R where Q is the field of fractions of R.

We will prove the general result:
Assume D is an R-module and $\operatorname{Ext}^1_R(Q/R, D) = 0$
Then we have the SES: $0 \longrightarrow R \xhookrightarrow{i} Q \xrightarrow{\pi} Q/R \longrightarrow 0$
Hence the LES:
$0 \to \operatorname{Hom}_R(Q/R, D) \to \operatorname{Hom}_R(Q, D) \to \operatorname{Hom}_R(R, D) \longrightarrow \operatorname{Ext}^1_R(Q/R, D) \longrightarrow \cdots$
But $\operatorname{Ext}^1_R(Q/R, D) = 0$, so we have the SES:

$$0 \longrightarrow \mathrm{Hom}_R(Q/R, D) \xrightarrow{\pi^*} \mathrm{Hom}_R(Q, D) \xrightarrow{i^*} \mathrm{Hom}_R(R, D) \longrightarrow 0$$

In particular, i^* is surjective

Let $y \in D$ and $r \in R$ and define $f: R \longrightarrow D \ni f(1) = y$

Then $\exists g \in \mathrm{Hom}_R(Q, D) \ni f = i^*(g) = gi$, Hence $g|_R = f$

Now $y = f(1) = g(1) = g(r \cdot \frac{1}{r}) = rg(\frac{1}{r})$

$\therefore$ D is divisible

10.13. Let $C_{..}$ be a 1st quadrant bicomplex all of whose rows (or columns) are exact. Prove that $\mathrm{Tot}(C_{..})$ is exact.

Consider $C_{..} = E^0$ as the first page of a spectral sequence

Taking horizontal homologies, we obtain:

$$E^1 = \begin{array}{|ccccc} \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \\ 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\ \downarrow & & & & \\ 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\ \hline \end{array}$$

since each row is exact

And the pages stabilize here, so $E^1 = E^\infty$

Now $E^\infty_{p,q}$ are the factors in a filtration for $H_n(\mathrm{Tot}(C_{..}))$

So for each n, we have the filtration:

Hence $H_n(\mathrm{Tot}(C_{..})) = 0 \;\; \forall n$

$\therefore \mathrm{Tot}(C_{..})$ is exact

$$\begin{array}{cc} & H_n(\mathrm{Tot}(C_{..})) \\ 0 & \cup\!\mathrm{I} \\ & * \\ 0 & \cup\!\mathrm{I} \\ & * \\ & \cup\!\mathrm{I} \\ & \vdots \\ & \cup\!\mathrm{I} \\ & * \\ 0 & \cup\!\mathrm{I} \\ & 0 \end{array}$$

Rotman (731)

5.2. Prove that if G is a p-primary abelian group, then $S \subseteq G$ is a pure subgroup iff $S \cap p^n G = p^n S$ $\forall n \geq 0$.

($\Rightarrow$) Assume $S \subseteq G$ is pure

Then $\forall m \in \mathbb{Z}$, $S \cap mG = mS$

So in particular, $S \cap p^n G = p^n S$ $\forall n \geq 0$

($\Leftarrow$) Assume $S \cap p^n G = p^n S$ $\forall n \geq 0$

Let $m \in \mathbb{Z}$ and let $x \in S \cap mG$

Then $x \in S$ and $x \in mG \Rightarrow x = p_1^{k_1} \cdots p_t^{k_t} g$, $k_i \geq 0$

$\in p_i^{k_i} G$ $\forall i = 1, \ldots, t$

$\Rightarrow x \in S \cap p_i^{k_i} G$ $\forall i = 1, \ldots, t$

$= p_i^{k_i} S$ $\forall i = 1, \ldots, t$

$\Rightarrow x = p_1^{k_1} \cdots p_t^{k_t} s$ for some $s \in S$

$\in mS$

$\therefore S \cap mG \subseteq mS$

Now let $x \in mS$

Then $x = p_1^{k_1} \cdots p_t^{k_t} s \in p_i^{k_i} S$ $\forall i = 1, \ldots, t$

$= S \cap p_i^{k_i} G$ $\forall i = 1, \ldots, t$

So $x = p_1^{k_1} \cdots p_t^{k_t} g$ for some $g \in G$ and $x \in S$

$\in S \cap mG$

$\therefore mS \subseteq S \cap mG$

$\therefore S \cap mG = mS$

$\therefore S$ is a pure subgroup

5.5. Let G be a cyclic group of finite order m. Prove that G/nG is a cyclic group of order $d=(m,n)$

First note that $G \cong \mathbb{Z}/m\mathbb{Z}$

So it suffices to show that $\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z})$

Let $\bar{a}+n(\mathbb{Z}/m\mathbb{Z}) \in \mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z})$

Then $\bar{a}+n(\mathbb{Z}/m\mathbb{Z}) = a(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))$

$\therefore \mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z})$ is cyclic, generated by $\bar{1}+n(\mathbb{Z}/m\mathbb{Z})$

Now $n(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))=0$, so $|\bar{1}+n(\mathbb{Z}/m\mathbb{Z})| \mid n$

But also $m(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))=0$, so $|\bar{1}+n(\mathbb{Z}/m\mathbb{Z})| \mid m$

$\therefore |\bar{1}+n(\mathbb{Z}/m\mathbb{Z})| \leq d$ since $d=(m,n)$

Define $\varphi: \mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/d\mathbb{Z} \ni \varphi(\bar{a}+n(\mathbb{Z}/m\mathbb{Z})) = \bar{a}$

If $\bar{a}_1+n(\mathbb{Z}/m\mathbb{Z}) = \bar{a}_2+n(\mathbb{Z}/m\mathbb{Z})$, then $\overline{a_1-a_2} \in n(\mathbb{Z}/m\mathbb{Z})$

$\Rightarrow m \mid a_1-a_2$ or $n \mid a_1-a_2 \Rightarrow d \mid a_1-a_2 \Rightarrow \overline{a_1-a_2}=\bar{0}$

$\Rightarrow \bar{a}_1=\bar{a}_2 \Rightarrow \varphi(\bar{a}_1+n(\mathbb{Z}/m\mathbb{Z})) = \varphi(\bar{a}_2+n(\mathbb{Z}/m\mathbb{Z}))$

$\therefore \varphi$ well defined $\mathbb{Z}$-module homomorphism

And $\varphi(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))=\bar{1}$ which is of order d

$\therefore d \mid |\bar{1}+n(\mathbb{Z}/m\mathbb{Z})|$

$\therefore d \leq |\bar{1}+n(\mathbb{Z}/m\mathbb{Z})|$

$\therefore |\bar{1}+n(\mathbb{Z}/m\mathbb{Z})| = d$

$\therefore |\mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z})| = d$

$\therefore \mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$

$\therefore G/nG$ is a cyclic group of order d

6.33. (i) Give an example of a noetherian ring R containing a subring that is not noetherian

Take $R = k[x,y]$

R is noetherian by the Hilbert Basis Thm since k is a field, hence commutative and noetherian

But consider the subring $k[x, xy, xy^2, \dots]$

$k[x, xy, xy^2, \dots]$ is not finitely generated as an ideal over itself, so it is not noetherian

(ii) Give an example of a commutative ring R containing proper ideals $I \subsetneq J \subsetneq R$ with J finitely generated, but I not finitely generated

Take $R = k[x_1, x_2, \dots]$

Then $J = (x_1)$ is a finitely generated proper ideal

But $I = (x_1 x_2, x_1 x_3, \dots) \subsetneq J$ since $x_1 \notin I$ is a proper ideal, but not finitely generated

6.39. If R and S are noetherian, prove that $R \times S$ is noetherian.

First show that any ideal I of $R \times S$ is of the form $I_1 \times I_2$ where $I_1 \trianglelefteq R$ and $I_2 \trianglelefteq S$

Let $I \trianglelefteq R \times S$

Consider the projections $\pi_1 : R \times S \longrightarrow R$ and $\pi_2 : R \times S \longrightarrow S$

Show that $I = \pi_1(I) \times \pi_2(I)$

Let $(a,b) \in I$

Then $(a,b) = (\pi_1(a,b), \pi_2(a,b)) \in \pi_1(I) \times \pi_2(I)$

$\therefore I \subseteq \pi_1(I) \times \pi_2(I)$

Now let $(a,b) \in \pi_1(I) \times \pi_2(I)$

Then $a = \pi_1(a,c)$, $b = \pi_2(d,b)$ for $(a,c),(d,b) \in I$

So $(a,b) = (1,0)(a,c) + (0,1)(d,b) \in I$ since $I \trianglelefteq R \times S$

$\therefore \pi_1(I) \times \pi_2(I) \subseteq I$

$\therefore I = \pi_1(I) \times \pi_2(I)$

And π_1, π_2 are surjective ring homomorphisms, hence they send ideals to ideals

$\therefore \pi_1(I) \trianglelefteq R$ and $\pi_2(I) \trianglelefteq S$

$\therefore \forall I \trianglelefteq R \times S$, $I = I_1 \times I_2$ for $I_1 \trianglelefteq R, I_2 \trianglelefteq S$

Let $I \trianglelefteq R \times S$

Then $I = I_1 \times I_2$ where $I_1 \trianglelefteq R$ and $I_2 \trianglelefteq S$

So since R, S are both noetherian, I_1, I_2 are both finitely generated, hence $I_1 = (x_1, \ldots, x_n)$ and $I_2 = (y_1, \ldots, y_m)$

WLOG assume that $n \leq m$

Let $(\dot{c}_1, \dot{c}_2) \in I$

Then $(\dot{c}_1, c_2) = (r_1x_1 + \ldots + r_nx_n, s_1y_1 + \ldots + s_my_m)$

$= (r_1, s_1)(x_1, y_1) + \ldots + (r_n, s_n)(x_n, y_n) + \ldots + (0, s_m)(0, y_m)$

$\therefore I = ((x_1,y_1), \ldots, (x_n,y_n), (0,y_{n+1}), \ldots, (0,y_m))$ is finitely generated

$\therefore R \times S$ is noetherian

7.2. If $X \subseteq M$ where M is an R-module, prove that $(X) = \bigcap_{X \subseteq S \leq M} S$

Let $y \in (X) = \{x_i\}_{i \in I}$

Then $y = \sum r_i x_i \in S \quad \forall X \subseteq S \leq M$ since each $x_i \in S$ and S is an R-module

$\therefore y \in \bigcap_{X \subseteq S \leq M} S$

$\therefore (X) \subseteq \bigcap_{X \subseteq S \leq M} S$

Now let $y \in \bigcap_{X \subseteq S \leq M} S$

Then $y \in S \quad \forall X \subseteq S \leq M$

In particular, $y \in (X)$ since $X \subseteq (X) \leq M$

$\therefore \bigcap_{X \subseteq S \leq M} S \subseteq (X)$

$\therefore (X) = \bigcap_{X \subseteq S \leq M} S$

7.5. Let M be an R-module. Prove that $\exists \varphi_M : \mathrm{Hom}_R(R, M) \to M$ an R-module isomorphism $\ni \varphi_M(f) = f(1)$

$\varphi_M(f_1 + f_2) = (f_1 + f_2)(1) = f_1(1) + f_2(1) = \varphi_M(f_1) + \varphi_M(f_2)$

$\varphi_M(rf) = (rf)(1) = rf(1) = r\varphi_M(f)$

$\therefore \varphi_M$ is an R-module homomorphism

Let $f \in \ker \varphi_M$ and let $r \in R$

Then $f(r) = f(r \cdot 1) = rf(1) = r\varphi_M(f) = r \cdot 0 = 0$

$\therefore f \equiv 0$

$\therefore \ker \varphi_M = 0$

$\therefore \varphi_M$ is injective

Define $\psi_m : R \longrightarrow M \ni \psi_m(r) = rm$

$\psi_m(r_1 + r_2) = (r_1 + r_2)m = r_1 m + r_2 m = \psi_m(r_1) + \psi_m(r_2)$

$\psi_m(sr) = (sr)m = s(rm) = s\,\psi_m(r)$

$\therefore \psi_m \in \mathrm{Hom}_R(R, M)$

Let $m \in M$

Then $m = 1 \cdot m = \psi_m(1) = \varphi_M(\psi_m)$

$\therefore \varphi_M$ is surjective

$\therefore \varphi_M$ is an R-module isomorphism

7.14. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence, prove that f is surjective iff h is injective.

($\Rightarrow$) Assume that f is surjective
Let $x \in \ker h = \operatorname{Im} g$
So $x = g(b)$ for some $b \in B$
But since f is surjective, $b = f(a)$ for some $a \in A$
$\therefore x = g(f(a)) = 0$ by exactness
$\therefore \ker h = 0$
$\therefore h$ is injective
($\Leftarrow$) Assume that h is injective
Let $b \in B$
Then $h(g(b)) = 0$ by exactness
So $g(b) \in \ker h = 0$ since h is injective
$\therefore g \in \ker g = \operatorname{Im} f$
$\therefore f$ is surjective

7.27. Given a map $\sigma : \prod B_i \longrightarrow \prod C_j$, find a map $\tilde{\sigma}$ commuting the diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(A, \prod B_i) & \xrightarrow{\sigma_*} & \mathrm{Hom}(A, \prod C_j) \\
\tau_B \downarrow \wr & & \wr \downarrow \tau_C \\
\prod \mathrm{Hom}(A, B_i) & \xrightarrow{\tilde{\sigma}} & \prod \mathrm{Hom}(A, C_j)
\end{array}$$

$\ni \tau, \tau'$ are isomorphisms.

First note that $\tau_B(f) = (p_i^B f)$ where $p_i^B : \prod B_i \longrightarrow B_i$ is the natural projection

And similarly $\tau_C(g) = (p_j^C g)$

Define $\tilde{\sigma} : \prod \mathrm{Hom}(A, B_i) \longrightarrow \prod \mathrm{Hom}(A, C_j) \ni \tilde{\sigma}(f_i) = (p_j^C \sigma f)$

where $f = (f_i) \in \mathrm{Hom}(A, \prod B_i)$

Then $\tau_C(\sigma_*(f)) = \tau_C(\sigma f) = (p_j^C \sigma f)$

And $\tilde{\sigma}(\tau_B(f)) = \tilde{\sigma}(p_i^B f) = (p_j^C \sigma f)$

$\therefore$ The diagram commutes

7.30 (i) Prove that (0) is the zero object in RMOD

Let $M \in ob(RMOD)$

Then M is an R-module

So $\exists!$ morphism $(0) \longrightarrow M$, namely the zero map

$\therefore (0)$ is an initial object

And $\exists!$ morphism $M \longrightarrow (0)$, namely the zero map

$\therefore (0)$ is a terminal object

$\therefore (0)$ is the zero object in RMOD

(ii) Prove that φ is an initial object in SETS

Let $X \in ob(SETS)$

Then $\varphi \subseteq X$

So $\exists!$ morphism $\varphi \hookrightarrow X$, namely the inclusion map

$\therefore \varphi$ is an initial object

(iii) Prove that any singleton set is a terminal object in SETS

Let $Y \in ob(SETS)$

Then $\exists!$ morphism $Y \longrightarrow \{x\}$ for any x, namely the map sending every element of Y to x

$\therefore \{x\}$ is a terminal object for any x

(iv) Prove that a zero object does not exist in SETS

First show that initial and terminal objects are unique up to isomorphism

If X, Y are initial objects then $\exists!$ morphism $X \xrightarrow{f} Y$

since X is an initial object and $\exists!$ morphism

$Y \xrightarrow{g} X$ since Y is an initial object

Now $fg, 1_Y : Y \to Y$, but Y is initial so $\exists !$ morphism $Y \to Y$

$\therefore fg = 1_Y$

Similarly $gf = 1_X$

$\therefore f$ is an isomorphism with inverse g

And the same argument shows that terminal objects are unique up to isomorphism

Now $\varnothing$ is not isomorphic to any nonempty set, so $\varnothing$ is the unique initial object in SETS

But $\varnothing$ is not terminal since $\nexists$ morphism $X \to \varnothing$

$\therefore \varnothing$ is not a zero object

$\therefore \nexists$ zero object in SETS

7.41. Prove that if G is an abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong \{g \in G : ng = 0\}$

Consider the SES: $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$

Since $\mathrm{Hom}_{\mathbb{Z}}(\_, G)$ is left exact, we have the exact sequence: $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \xrightarrow{n^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G)$

In particular, π^* is injective

Then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong \mathrm{Im}\,\pi^* = \ker n^*$ by exactness

But $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \cong G$ is a natural isomorphism, hence the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) & \xrightarrow{n^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \\ \downarrow \cong & & \cong \downarrow \\ G & \xrightarrow{n} & G \end{array}$$

$\therefore \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong \ker n^* \cong \ker n = \{g \in G \mid ng = 0\}$

7.44. Prove that every left exact covariant functor $T: {}_R\mathrm{MOD} \to \mathrm{AB}$ preserves pullbacks. Conclude that if B and C are submodules of a module A, then for every module M, we have $\mathrm{Hom}_R(M, B\cap C) = \mathrm{Hom}_R(M,B) \cap \mathrm{Hom}_R(M,C)$.

First note that D is the pullback of f, g
$D = \ker(A\times B \xrightarrow{(f,-g)} C)$ since RMOD is an abelian category

Show that T preserves kernels

Let $h: D \to E$

Then we have the exact sequence: $0 \to \ker h \xhookrightarrow{i} D \xrightarrow{h} E$

But since T is left exact and covariant, we get the exact sequence: $0 \to T(\ker h) \xrightarrow{T(i)} T(D) \xrightarrow{T(h)} T(E)$

Then $T(\ker h) \cong \ker T(h)$

So $T(D) = T(\ker(B\times C \xrightarrow{(f,-g)} A)) \cong \ker(T(B\times C) \xrightarrow{T(f,-g)} T(A))$
$\cong \ker(T(B)\times T(C) \xrightarrow{(T(f),-T(g))} T(A))$ since T is left exact hence it preserves products

But $\ker(T(B)\times T(C) \xrightarrow{(T(f),-T(g))} T(A))$ is just the pullback of $T(B), T(C)$

$\therefore$ T preserves pullbacks

Now note that $\mathrm{Hom}_R(M,-)$ is a left exact covariant functor, so it preserves pullbacks by above

Note that the pullback of the inclusions $B \xhookrightarrow{i_1} A$ and $C \xhookrightarrow{i_2} A$ is $\{(b,c)\in B\times C \mid i_1(b) = i_2(c)\} = \{(b,c)\in B\times C \mid b = c\}$
$= \{(b,b)\in B\times C\} = \{(b,b) \mid b\in B\cap C\} \cong B\cap C$

And the pullback of $\mathrm{Hom}_R(M,B) \xrightarrow{i_{1*}} \mathrm{Hom}_R(M,A)$ and $\mathrm{Hom}_R(M,C) \xrightarrow{i_{2*}} \mathrm{Hom}_R(M,A)$ is $\{(f,g)\in \mathrm{Hom}_R(M,B)\times \mathrm{Hom}_R(M,C) \mid i_{1*}(f) = i_{2*}(g)\} = \{(f,g)\in \mathrm{Hom}_R(M,B)\times\mathrm{Hom}_R(M,C) \mid i_1 f = i_2 g\}$
$= \{(f,g)\in \mathrm{Hom}_R(M,B)\times \mathrm{Hom}_R(M,C) \mid f = g\}$
$= \{(f,f) \mid f\in \mathrm{Hom}_R(M,B)\cap \mathrm{Hom}_R(M,C)\}$
$\cong \mathrm{Hom}_R(M,B)\cap\mathrm{Hom}_R(M,C)$
$\therefore \mathrm{Hom}_R(M,B\cap C) = \mathrm{Hom}_R(M,B)\cap \mathrm{Hom}_R(M,C)$

7.50 Prove that every direct summand of an injective module is injective.

Let I be an injective R-module and let J be a direct summand of I, so $I = J \oplus X$, for some $X \leq I$

Let $0 \longrightarrow A \xrightarrow{f} B$ be injective and let $g : A \longrightarrow J$

Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[ddl, dashed, "\exists h"] \\ & J \arrow[d, shift left, "i_1"] & \\ & I \arrow[u, shift left, "p_1"] & \end{tikzcd}$$

Since I is injective, $\exists h : B \longrightarrow I \ni hf = i_1 g$

Define $j : B \longrightarrow J \ni j = p_1 h$

Then $jf = p_1 h f = p_1 i_1 g = 1_J \cdot g = g$

$\therefore$ J is injective

7.53. (i) If R is a domain and $0 \neq I, J \triangleleft R$, prove that $I \cap J \neq 0$

Suppose that $I \cap J = 0$
Since $I, J \neq 0$, $\exists\, 0 \neq i \in I$ and $0 \neq j \in J$
And $ij \in I \cap J = 0$
But R is an integral domain, so $i = 0$ or $j = 0$
contradiction since $i, j \neq 0$
$\therefore I \cap J \neq 0$

(ii) Let R be a domain and let $I \triangleleft R$ be a free R-module.
Prove that I is a principal ideal.

Since I is a free R-module, I has a basis
But for any $0 \neq x_1, x_2 \in I$, $x_1 \cdot x_2 + (-x_2) \cdot x_1 = 0$
Hence any two nonzero elements of R are linearly dependent
But for any $r \in R$, $0 \neq x \in I$, if $rx = 0$, then $r = 0$ since R is an integral domain
$\therefore x$ is linearly independent
$\therefore I$ has a basis $\{x\}$
$\therefore I = (x)$
$\therefore I$ is a principal ideal

7.56. (i) Prove that every vector space over a field k is an injective k-module.

Let V be a vector space over k

And let $I \triangleleft k$ with $f: I \longrightarrow V$

Then $I = 0$ or $I = k$ since k is a field

If $I = 0$, then $f \equiv 0$, and we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r, hook] \arrow[d] & k \arrow[dl, dashed, "\exists 0"] \\ & V & \end{tikzcd}$$

Then the zero map $k \longrightarrow V$ commutes the diagram

If $I = k$, then the inclusion map is 1_k, and we have the diagram:

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r, "1_k"] \arrow[d, "f"'] & k \arrow[dl, dashed, "\exists f"] \\ & V & \end{tikzcd}$$

And $f: k \longrightarrow V$ commutes the diagram

$\therefore$ V is an injective k-module

(ii) Prove that if $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is an exact sequence of vector spaces, then the corresponding sequence of dual spaces $0 \longrightarrow W^* \longrightarrow V^* \longrightarrow U^* \longrightarrow 0$ is also exact.

Let U, V, W be vector spaces over a field k

Note that k is a vector space over k, so k is an injective k-module by (i)

Then $\mathrm{Ext}_k^1(\_, k) = 0$

So we have the LES:

$0 \longrightarrow \mathrm{Hom}_k(W, k) \longrightarrow \mathrm{Hom}_k(V, k) \longrightarrow \mathrm{Hom}_k(U, k) \longrightarrow \mathrm{Ext}_k^1(W, k) \longrightarrow \cdots$

So since $\mathrm{Ext}_k^1(W, k) = 0$, we have the SES:

$0 \longrightarrow \mathrm{Hom}_k(W, k) \longrightarrow \mathrm{Hom}_k(V, k) \longrightarrow \mathrm{Hom}_k(U, k) \longrightarrow 0$

$\therefore$ $0 \longrightarrow W^* \longrightarrow V^* \longrightarrow U^* \longrightarrow 0$ is a SES

7.72. Let $\{E_i, \varphi^i_j\}$ be a direct system of injective R-modules over a directed index set I. Prove that if R is noetherian, then $\varinjlim E_i$ is an injective module.

Let $I \trianglelefteq R$ and let $f: I \longrightarrow \varinjlim E_i$

But since I is a directed set $\varinjlim E_i = \sum E_i / S$ where $S = (\lambda_j \varphi^i_j m_i - \lambda_i m_i : m_i \in M_i, i \le j)$ and $\lambda_i : M_i \longrightarrow \sum M_i$ is the natural injection

Then f factors as $I \xrightarrow{\bar f} \sum E_i \xrightarrow{\pi} \sum E_i / S$

Now since R is noetherian, I is finitely generated, say $I = (a_1, \dots, a_n)$ and

$\bar f(a_k) \in \sum E_i$ has only finitely many nonzero entries for each k, say $S_k \subseteq I$ is the set of nonzero indices

Then $S = \bigcup_{k=1}^{n} S_k$ is finite

And $\operatorname{Im} \bar f \subseteq \sum_{i \in S} E_i$ which is a finite direct sum of injective modules

$\therefore \sum_{i \in S} E_i$ is injective

So $\exists \bar g : R \longrightarrow \sum_{i \in S} E_i \ni \bar f = \bar g i$

Define $g : R \longrightarrow \sum E_i \ni g = \tilde i \bar g$ where $\sum_{i \in S} E_i \xhookrightarrow{\tilde i} \sum E_i$ is the inclusion map

Then $g i = \tilde i \bar g i = \tilde i \bar f = \bar f$

$\therefore \sum E_i$ is injective

Define $h : R \longrightarrow \sum E_i / S \ni h = \pi g$

Then $h i = \pi g i = \pi \bar f = f$

So the diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists h"] \\ & \varinjlim E_i & \end{tikzcd}$$

$\therefore \varinjlim E_i$ is an injective R-module

7.75. Let (F,G) be an adjoint pair of functors where $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ and let $\tau_{C,D}: \mathrm{Hom}(F(C), D) \to \mathrm{Hom}(C, G(D))$ be the natural bijection.

(c) If $D = F(C)$, $\tau_{C,F(C)}(1_{F(C)}) = \eta_C \in \mathrm{Hom}(C, G(F(C)))$. Prove that $\eta: 1_{\mathcal{C}} \to GF$ is a natural transformation.

First note that $\eta_C(1_{\mathcal{C}}(C)) = \eta_C(C) = G(F(C)) \quad \forall C \in \mathcal{C}$

Let $f: C \to C'$

Then since $\tau_{C,D}$ natural bijection, we have the diagrams:

$$\begin{tikzcd}
\mathrm{Hom}(F(C),F(C)) \arrow[r, "\tau_{C,F(C)}"] \arrow[d, "(F(f))_*"'] & \mathrm{Hom}(C,G(F(C))) \arrow[d, "(G(F(f)))_*"] \\
\mathrm{Hom}(F(C),F(C')) \arrow[r, "\tau_{C,F(C')}"] & \mathrm{Hom}(C,G(F(C')))
\end{tikzcd}$$

with $(G(F(f)))_* \tau_{C,F(C)} = \tau_{C,F(C')}(F(f))_*$, hence

$G(F(f))\tau_{C,F(C)} = \tau_{C,F(C')}F(f)$, and

$$\begin{tikzcd}
\mathrm{Hom}(F(C'),F(C')) \arrow[r, "\tau_{C',F(C')}"] \arrow[d, "(F(f))^*"'] & \mathrm{Hom}(C',G(F(C'))) \arrow[d, "f^*"] \\
\mathrm{Hom}(F(C),F(C')) \arrow[r, "\tau_{C,F(C')}"] & \mathrm{Hom}(C,G(F(C')))
\end{tikzcd}$$

with $f^* \tau_{C',F(C')} = \tau_{C,F(C')}(F(f))^*$

Now consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "f"'] & G(F(C)) \arrow[d, "G(F(f))"] \\
C' \arrow[r, "\eta_{C'}"] & G(F(C'))
\end{tikzcd}$$

Then $G(F(f))\eta_C = G(F(f))\tau_{C,F(C)}(1_{F(C)}) = \tau_{C,F(C')}F(f)(1_{F(C)})$

And $\eta_{C'}f = \tau_{C',F(C')}(1_{F(C')})f = f^*\tau_{C',F(C')}(1_{F(C')})$

$= \tau_{C,F(C')}(F(f))^*(1_{F(C')}) = \tau_{C,F(C')}(1_{F(C')})F(f)$

$= \tau_{C,F(C')}F(f)(1_{F(C)})$

$\therefore$ The diagram commutes

$\therefore$ $\eta: 1_{\mathcal{C}} \to GF$ natural transformation

(ii) If $C = G(D)$, $\exists\, \tau^{-1}_{G(D),D} : Hom(G(D),G(D)) \to Hom(F(G(D)),D)$ a natural transformation with $\tau^{-1}_{G(D),D}(1_D) = \epsilon_D \in Hom(F(G(D)),D)$. Prove that $\epsilon : FG \to 1_{\mathcal{D}}$ is a natural transformation

First note that $\epsilon_D(F(G(D))) = \tau^{-1}(1_D(F(G(D)))) = \tau^{-1}(F(G(D)))$
$= D = 1_{\mathcal{D}}(D)$

Let $g : D \to D'$

Then since $\tau^{-1}_{G(D),D}$ is a natural transformation we have the following commutative diagrams:

$$\begin{tikzcd}
Hom(G(D),G(D)) \arrow[r, "\tau^{-1}_{G(D),D}"] \arrow[d, "(G(g))_*"'] & Hom(F(G(D)),D) \arrow[d, "g_*"] \\
Hom(G(D),G(D')) \arrow[r, "\tau^{-1}_{G(D),D'}"] & Hom(F(G(D)),D')
\end{tikzcd}$$

with $g_* \tau^{-1}_{G(D),D} = \tau^{-1}_{G(D),D'}(G(g))_*$, hence $g\tau^{-1}_{G(D),D} = \tau^{-1}_{G(D),D'}G(g)$, and

$$\begin{tikzcd}
Hom(G(D'),G(D')) \arrow[r, "\tau^{-1}_{G(D'),D'}"] \arrow[d, "(G(g))^*"'] & Hom(F(G(D')),D') \arrow[d, "(F(G(g)))^*"] \\
Hom(G(D),G(D')) \arrow[r, "\tau^{-1}_{G(D),D'}"] & Hom(F(G(D)),D')
\end{tikzcd}$$

with $(F(G(g)))^* \tau^{-1}_{G(D'),D'} = \tau^{-1}_{G(D),D'}(G(g))^*$

Consider the diagram:

$$\begin{tikzcd}
F(G(D)) \arrow[r, "\epsilon_D"] \arrow[d, "F(G(g))"'] & D \arrow[d, "g"] \\
F(G(D')) \arrow[r, "\epsilon_{D'}"] & D'
\end{tikzcd}$$

Then $g\epsilon_D = g\tau^{-1}_{G(D),D}(1_D) = \tau_{G(D),D'}G(g)(1_D)$

And $\epsilon_{D'}(F(G(g))) = \tau^{-1}_{G(D'),D'}(1_{D'})F(G(g))$
$= (F(G(g)))^* \tau^{-1}_{G(D'),D'}(1_{D'})$
$= \tau_{G(D),D'}(G(g))^*(1_{D'})$
$= \tau_{G(D),D'}(1_{D'})G(g) = \tau_{G(D),D'}G(g)(1_D)$

$\therefore$ The diagram commutes

$\therefore$ $\epsilon : FG \to 1_{\mathcal{D}}$ is a natural transformation

7.78. Prove that if $T: RMOD \longrightarrow AB$ is an additive left exact functor preserving products, then T preserves inverse limits.

Let $\{E_i, \varphi_j^i\}_{i\in I}$ be an inverse system of R-modules

Then $\varprojlim E_i = \ker\left(\prod_{i\in I} E_i \longrightarrow \prod_{\varphi_j^i: M_i \to M_j} E_i\right)$

so $T(\varprojlim E_i) = T\left(\ker\left(\prod_{i\in I} E_i \longrightarrow \prod_{\varphi_j^i: M_i \to M_j} E_i\right)\right)$

$\cong \ker\left(T\left(\prod_{i\in I} E_i\right) \longrightarrow T\left(\prod_{\varphi_j^i: M_i \to M_j} E_i\right)\right)$ since T is left exact, hence preserves kernels

Then $T(\varprojlim E_i) \cong \ker\left(\prod_{i\in I} T(E_i) \longrightarrow \prod_{T(\varphi_j^i): T(M_i) \to T(M_j)} T(E_i)\right)$ since T preserves products

$\therefore T(\varprojlim E_i) \cong \varprojlim T(E_i)$

$\therefore T$ preserves limits

8.5. Let $I \triangleleft R$. Prove that an abelian group M is a left R/I-module iff it is a left R-module that is annihilated by I.

($\Rightarrow$) Assume M left R/I-module

Let $m \in M$

Then M is a left R/I-module via the action $(r+I)m = rm$ $\forall r \in R$

so $rm = (r+I)m \in M$ $\forall r \in R$

$\therefore M$ is a left R-module since it is an abelian group

Now $Im = (0+I)m = 0 \cdot m = 0$ $\forall m \in M$

$\therefore I$ annihilates M

($\Leftarrow$) Assume M left R-module annihilated by I

Then $(r+I)m = rm$ is a welldefined action since $Im = 0$

$\therefore (r+I)m = rm \in M$ $\forall r \in R$ $\forall m \in M$

$\therefore M$ is a left R/I-module

8.27. (i) Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a SES of R-modules. Prove that if both A, C have DCC, then B has DCC. Conclude that in this case $A \oplus B$ has DCC.

First note that we have an isomorphism of SES's:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\wr"] \arrow[dr, phantom, "G"] & B \arrow[r, "g"] \arrow[d, "\wr"] \arrow[dr, phantom, "G"] & C \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \operatorname{Im} f \arrow[r, hook, "i"] & B \arrow[r, "\pi"] & B/\operatorname{Im} f \arrow[r] & 0 \end{tikzcd}$$

Hence it suffices to show the result for the SES:

$$0 \longrightarrow D \overset{i}{\hookrightarrow} B \xrightarrow{\pi} B/D \longrightarrow 0$$

($\Leftarrow$) Assume B has DCC

Let $D_0 \supseteq D_1 \supseteq \dots$ be a descending chain of submodules of $D \leq B$, hence submodules of B

$\therefore$ The chain stabilizes since B has DCC

$\therefore$ D has DCC

Now let $C_0 \supseteq C_1 \supseteq \dots$ be a descending chain of submodules of B/D

Then each $C_i = B_i/D$ where $B_i \leq B$ containing D

And $B_0 \supseteq B_1 \supseteq \dots$

But this chain stabilizes since B has DCC

$\therefore C_0 \supseteq C_1 \supseteq \dots$ stabilizes

$\therefore B/D$ has DCC

($\Rightarrow$) Assume $D, E = B/D$ have DCC

Let $B_0 \supseteq B_1 \supseteq \dots$ be a descending chain of submodules of B

Then $D \cap B_0 \supseteq D \cap B_1 \supseteq \dots$ is a descending chain of submodules of D

But D has DCC, so $D \cap B_n = D \cap B_{n+1} = \dots$ for some n

Also $D + B_0 \supseteq D + B_1 \supseteq \dots$ is a descending chain of submodules of B containing D, hence $D+B_0/D \supseteq D+B_1/D \supseteq \dots$ is a descending chain of submodules of $E = B/D$ which has DCC, hence $D+B_m/_D = D+B_{m+1}/_D = \dots$ for some m

so $D+B_m = D+B_{m+1} = \dots$

Choose $k = \max\{n, m\}$

Then $D \cap B_k = D \cap B_{k+1} = \dots$ and $D+B_k = D+B_{k+1} = \dots$

Let $b_k \in B_k$

Then $d+b_k \in D+B_k = D+B_{k+1} \Rightarrow d+b_k = d'+b_{k+1} \Rightarrow d-d' = b_{k+1}-b_k$

so $b_{k+1}-b_k \in D \cap B_k = D \cap B_{k+1}$

$\therefore b_{k+1}-b_k = b_{k+1}' \Rightarrow b_k = b_{k+1}-b_{k+1}' \in B_{k+1}$

$\therefore B_k \subseteq B_{k+1}$

$\therefore B_k = B_{k+1} = \dots$

$\therefore B$ has DCC

Now if A, B have DCC, then the SES:

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

gives that $A \oplus B$ has DCC by above

(ii) Let $0 \to A \to B \to C \to 0$ be a SES of left R-modules. Prove that if both A, C have ACC, then B has ACC. Conclude in this case that $A \oplus B$ has ACC.

Again, it suffices to show the result for the SES

$0 \longrightarrow A \hookrightarrow B \longrightarrow C \longrightarrow 0$ where $C = B/A$

($\Leftarrow$) Assume B has ACC

Similar to (i)

($\Rightarrow$) Assume A, C have ACC

Let $X \leq B$

Then $X \cap A \leq A$

But A has ACC, so A is noetherian, hence $X \cap A$ is finitely generated

Consider the SES: $0 \longrightarrow X \cap A \hookrightarrow X \longrightarrow X/X \cap A \longrightarrow 0$

But $X/X\cap A \cong A+X/A$ by 2nd Iso Thm

And $A+X/A \leq B/A = C$ which has ACC, hence $X/X \cap A$ is finitely generated

Claim: if $0 \to L \to M \to M/L \to 0$ is a SES with $L, M/L$ finitely generated, then M is finitely generated

First note that $L = (x_1, \ldots, x_n)$, $M/L = (y_1 + L, \ldots, y_m + L)$

Let $m \in M$

Then $m + L = r_1(y_1 + L) + \ldots + r_m(y_m + L)$

$= r_1 y_1 + \ldots + r_m y_m + L$

So $m - r_1 y_1 - \ldots - r_m y_m \in L$

$\Rightarrow m - r_1 y_1 - \ldots - r_m y_m = s_1 x_1 + \ldots + s_n x_n$

$\therefore m = r_1 y_1 + \ldots + r_m y_m + s_1 x_1 + \ldots + s_n x_n \in (y_1, \ldots, y_m, x_1, \ldots, x_n)$

$\therefore M$ is finitely generated

$\therefore X$ is finitely generated by claim

$\therefore B$ is noetherian

$\therefore B$ has ACC

Again, if A, B have ACC, the SES:

$0 \to A \to A \oplus B \to B \to 0$ gives that

$A \oplus B$ has ACC by above

(iii) Prove that every semisimple ring is left artinian

Since R is a semisimple ring, $R \cong S_1 \oplus \ldots \oplus S_n$ where each S_i is a simple R-module, hence each S_i is left artinian

And by (i), $S_1 \oplus S_2$ is left artinian

Assume that $S_1 \oplus \ldots \oplus S_{n-1}$ is artinian

Then $S_1 \oplus \ldots \oplus S_{n-1} \oplus S_n$ is left artinian by induction and by (i)

$\therefore R$ is a left artinian R-module

$\therefore R$ is a left artinian ring

8.30. Give an example of a ring $R \ni R \ncong R^{op}$

Take R to be the Klein 4-Ring, $\{0, a, b, c\}$ where the underlying abelian group is the Klein 4-Group and multiplication is given by:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

Note that this ring does not have identity

Suppose that $R \cong R^{op}$ via the isomorphism φ

Note that b is the only nonzero nilpotent since $b^2 = 0$ but $a^2 = a$, $c^2 = c$, hence $\varphi(b) = b$ and $\varphi(a) = a, c$

Then $0 = \varphi(0) = \varphi(ab) = \varphi(b)\varphi(a) = b\varphi(a) = b$ since $ba = bc = b$

Contradiction

$\therefore R \ncong R^{op}$

8.49. Let k be a commutative ring and let P, Q be projective k-modules. Prove that $P \otimes_k Q$ is a projective k-module.

First note that since k is commutative, $P \otimes_k Q$ is in fact a k-module.

Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a SES

Then since Q is projective, we have the SES:

$0 \to \mathrm{Hom}_k(Q,A) \longrightarrow \mathrm{Hom}_k(Q,B) \longrightarrow \mathrm{Hom}_k(Q,C) \longrightarrow 0$

But also P is projective, so we have the SES:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_k(P,\mathrm{Hom}_k(Q,A)) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_k(P,\mathrm{Hom}_k(Q,B)) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_k(P,\mathrm{Hom}_k(Q,C)) \arrow[d, "\wr"] & \\
0 \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, A) \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, B) \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, C) \arrow[r] & 0
\end{tikzcd}$$

Hence the bottom row is exact since the vertical maps are natural isomorphisms, making the diagram commute

$\therefore P \otimes_k Q$ is a projective k-module

8.52. Consider the commutative diagram with exact rows:

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

(i) If h_2, h_4 are surjective and h_5 is injective, prove that h_3 is surjective.

Let $b_3 \in B_3$

Then $g_3(b_3) = h_4(a_4)$ since h_4 is surjective

So $0 = g_4(g_3(b_3)) = g_4(h_4(a_4)) = h_5(f_4(a_4))$

Then $f_4(a_4) \in \ker h_5 = 0$ since h_5 is injective

So $a_4 \in \ker f_4 = \operatorname{Im} f_3 \Rightarrow a_4 = f_3(a_3)$

Then $g_3(b_3) = h_4(f_3(a_3)) = g_3(h_3(a_3))$

So $b_3 - h_3(a_3) \in \ker g_3 = \operatorname{Im} g_2 \Rightarrow b_3 - h_3(a_3) = g_2(b_2)$

$\Rightarrow b_3 = g_2(b_2) + h_3(a_3)$

But $b_2 = h_2(a_2)$ since h_2 is surjective

Then $b_3 = g_2(h_2(a_2)) + h_3(a_3) = h_3(f_2(a_2)) + h_3(a_3)$

$= h_3(f_2(a_2) + a_3) \in \operatorname{Im} h_3$

$\therefore$ h_3 is surjective

(iii) If h_1, h_2, h_4, h_5 are isomorphisms, prove that h_3 is an isomorphism

Since h_1 is surjective and h_2, h_4 are injective, then h_3 is injective by (ii)

And since h_5 is injective and h_2, h_4 are surjective, then h_3 is surjective by (i)

$\therefore$ h_3 is an isomorphism

9.2. Let R be a PID and M an R-module.

(ii) Prove that every direct summand of M is a pure submodule.

Let X be a direct summand of M

Then $M = X \oplus Y$, for some $Y \leq M$

Show $X \cap rM = rX \quad \forall r \in R$

Let $r \in R$

Then $X \cap rM = X \cap r(X \oplus Y) = X \cap (rX \oplus rY) = (X \cap rX) \oplus (X \cap rY)$

$= rX \oplus 0$ since $rX \subseteq X$ and since $X \cap rY \subseteq X \cap Y = ($

$= rX$

$\therefore X \cap rM = rX \quad \forall r \in R$

$\therefore X$ is a pure submodule

(iii) Prove that the torsion submodule $t(M)$ is a pure submodule of M.

Let $r \in R$ and $y \in t(M) \cap rM$

If $r = 0$, the result is trivial, so assume WLOG that $r \neq ($

So $y = rm$ for some $m \in M$ and $\exists\, 0 \neq z \in R \ni zy = 0$

Then $0 = zy = zrm = rzm$ since R PID, hence commutativ

So $m \in t(M)$, hence $y = rm \in rt(M)$

$\therefore t(M) \cap rM \subseteq rt(M)$

Now let $y \in rt(M)$

Then $y = rm$, $m \in t(M)$

But then $y = rm \in t(M)$ since $t(M)$ is an R-module

And $y = rm \in rM$

$\therefore y \in t(M) \cap rM$

$\therefore rt(M) \subseteq t(M) \cap rM$

$\therefore t(M) \cap rM = rt(M) \quad \forall r \in R$

$\therefore t(M)$ is a pure submodule of M

(iv) Prove that if M/S is torsion-free, then S is a pure submodule of M

Let $r \in R$ and let $x \in S \cap rM$
So $x = rm$ for some $m \in M$ and $x \in S$, hence $x + S = 0_{M/S}$
Then $0_{M/S} = x + S = rm + S = r(m+S)$
But M/S is torsionfree, so $r = 0$ or $m \in S$
If $m \in S$, then $x = rm \in rS$
If $r = 0$, then $x = 0 \cdot m = 0 \in 0 \cdot S = rS$
$\therefore S \cap rM \subseteq rS$
Now let $x \in rS$
So $x = rs$ for some $s \in S$
Then $x = rs \in S$ and $x = rs \in rM$ since $S \leq M$
$\therefore rS \subseteq S \cap rM$
$\therefore S \cap rM = rS$
$\therefore S$ is a pure submodule of M

(v) Prove that if $\mathcal{S}$ is a family of pure submodules of M that is a chain under inclusion, then $\bigcup_{S \in \mathcal{S}} S$ is a pure submodule of M.

First show that $\bigcup_{S \in \mathcal{S}} S \leq M$
Let $s_1 + s_2 \in \bigcup_{S \in \mathcal{S}} S$
Then $s_1 \in S_1$, $s_2 \in S_2 \in \mathcal{S}$
But $\bigcup_{S \in \mathcal{S}} S$ is a chain under inclusion, so $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$
WLOG assume that $S_1 \subseteq S_2$
Then $s_1 + s_2 \in S_2 \in \bigcup_{S \in \mathcal{S}} S$
Let $r \in R$ and $s \in \bigcup_{S \in \mathcal{S}} S$
Then $s \in S \in \mathcal{S}$, hence $rs \in S \in \bigcup_{S \in \mathcal{S}} S$
And $s \in S \subseteq M$
$\therefore \bigcup_{S \in \mathcal{S}} S \leq M$
Now let $r \in R$ and $x \in \bigcup_{S \in \mathcal{S}} S \cap rM$

Then $x = rm$ for some $m \in M$ and $x \in S$ for some $S \in \mathcal{S}$

But S is pure, so $x \in S \cap rM = rS \subseteq r \bigcup_{S \in \mathcal{S}} S$

$\therefore \bigcup_{S \in \mathcal{S}} S \cap rM \subseteq r \bigcup_{S \in \mathcal{S}} S$

Now let $x \in r \bigcup_{S \in \mathcal{S}} S$

Then $x \in rS$ for some $S \in \mathcal{S}$

But since S is pure $x \in rS = S \cap rM \subseteq \bigcup_{S \in \mathcal{S}} S \cap rM$

$\therefore r \bigcup_{S \in \mathcal{S}} S \subseteq \bigcup_{S \in \mathcal{S}} S \cap rM$

$\therefore \bigcup_{S \in \mathcal{S}} S \cap rM = r \bigcup_{S \in \mathcal{S}} S$

$\therefore \bigcup_{S \in \mathcal{S}} S$ is a pure submodule of M

(vi) Give an example of a pure submodule that is not a direct summand.

Take $R = \mathbb{Z}$, $M = \prod_p \mathbb{Z}/p\mathbb{Z}$ which is not finitely generated

Then $t(M) = \sum_p \mathbb{Z}/p\mathbb{Z}$ is pure by (iii) but is not a direct summand of M

9.8. (i) Let R be an integral domain, let $r \in R$ and let M be an R-module. If $\mu_r : M \longrightarrow M$ is multiplication by r, prove that for every R-module A, that the induced maps $(\mu_r)_*$, $(\mu_r)^*$ are also multiplication by r.

First note that $(\mu_r)_* : \mathrm{Hom}_R(A,M) \longrightarrow \mathrm{Hom}_R(A,M) \ni (\mu_r)_*(f) = \mu_r f$

so $(\mu_r)_*(f)(a) = \mu_r(f(a)) = rf(a) \quad \forall a \in A$

$\therefore (\mu_r)_*(f) = rf$

$\therefore (\mu_r)_*$ is multiplication by r

And $(\mu_r)^* : \mathrm{Hom}_R(M,A) \longrightarrow \mathrm{Hom}_R(M,A) \ni (\mu_r)^*(f) = f\mu_r$

so $(\mu_r)^*(f)(m) = f(\mu_r(m)) = f(rm) = rf(m) \quad \forall m \in M$

$\therefore (\mu_r)^*(f) = rf$

$\therefore (\mu_r)^*$ is multiplication by r

(ii) Let R be an integral domain with Q the field of fractions of R. Prove that for every R-module M, both $\mathrm{Hom}_R(Q,M)$, $\mathrm{Hom}_R(M,Q)$ are vectorspaces over Q.

First note that $\mathrm{Hom}_R({}_RQ_Q, {}_RM)$, so $\mathrm{Hom}_R(Q,M)$ is a left Q-module

But Q is a field, so $\mathrm{Hom}_R(Q,M)$ is a vectorspace over Q

Similarly, $\mathrm{Hom}_R({}_RM, {}_RQ_Q)$ is a right Q-module, hence a vectorspace over Q

10.25. Consider the commutative diagram with exact rows:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i"] & B' \arrow[r, "g"] & C' & \end{tikzcd}$$

(i) Prove that $\Delta : \ker\gamma \longrightarrow \operatorname{coker}\alpha \ni \Delta(z) = i^{-1}(\beta(p^{-1}(z))) + \operatorname{Im}$ is a well defined homomorphism.

Since p is surjective, $\exists b \in B \ni z = p(b)$

But $z \in \ker\gamma \Rightarrow 0 = \gamma(z) = \gamma(p(b)) = g(\beta(b))$

So $\beta(b) \in \ker g = \operatorname{Im} i \Rightarrow \beta(b) = i(a')$

Define $\Delta(z) = a' + \operatorname{Im}\alpha$

Now if $\exists b_1, b_2 \in B \ni z = p(b_1) = p(b_2) \Rightarrow p(b_1 - b_2) = 0$

$\Rightarrow b_1 - b_2 \in \ker p = \operatorname{Im} f \Rightarrow b_1 - b_2 = f(a)$

$\therefore \beta(b_1 - b_2) = \beta(f(a)) = i(\alpha(a))$

But $\beta(b_1) = i(a_1')$ and $\beta(b_2) = i(a_2')$

So $i(a_1' - a_2') = \beta(b_1 - b_2) = i(\alpha(a)) \Rightarrow a_1' - a_2' = \alpha(a) \in \operatorname{Im}\alpha$

$\therefore a_1' + \operatorname{Im}\alpha = a_2' + \operatorname{Im}\alpha$

$\therefore \Delta$ is welldefined

And if $z_1 = p(b_1)$, $z_2 = p(b_2)$, then $z_1 + z_2 = p(b_1 + b_2)$

And $\beta(b_1) = i(a_1')$, $\beta(b_2) = i(a_2') \Rightarrow \beta(b_1 + b_2) = i(a_1' + a_2')$

So $\Delta(z_1 + z_2) = a_1' + a_2' + \operatorname{Im}\alpha = a_1' + \operatorname{Im}\alpha + a_2' + \operatorname{Im}\alpha = \Delta(z_1) + \Delta(z_2)$

And $rz = rp(b) = p(rb)$, $\beta(rb) = r\beta(b) = ri(a') = i(ra')$

So $\Delta(rz) = ra' + \operatorname{Im}\alpha = r(a' + \operatorname{Im}\alpha) = r\Delta(z)$

$\therefore \Delta$ is an R-module homomorphism

(ii) Prove that $\exists$ exact sequence:

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\Delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$$

First note that $\ker\alpha \hookrightarrow A \xrightarrow{f} B \xrightarrow{\beta} B'$ is the composition $\ker\alpha \hookrightarrow A \xrightarrow{i\alpha} B'$ which is zero

Similarly, the composition $\ker\beta \hookrightarrow B \xrightarrow{\gamma p} C'$ is $\ker\beta \hookrightarrow B \xrightarrow{g\beta} C'$ which is also zero

Then we have the following diagrams:

$$\begin{tikzcd} \ker\beta \arrow[r, hook] & B \arrow[r, "\beta"] & B' \\ & A \arrow[u, "f"] & \\ & \ker\alpha \arrow[u, hook] \arrow[uul, dashed, "\exists \bar f"] & \end{tikzcd} \qquad \begin{tikzcd} \ker\gamma \arrow[r, hook] & C \arrow[r, "\gamma"] & C' \\ & B \arrow[u, "p"] & \\ & \ker\beta \arrow[u, hook] \arrow[uul, dashed, "\exists \bar p"] & \end{tikzcd}$$

By the UMP of kernels, $\exists \bar f, \bar p$ commuting the diagrams so we have the commutative diagram:

$$\begin{tikzcd} & \ker\alpha \arrow[r, "\bar f"] \arrow[d, hook] & \ker\beta \arrow[r, "\bar p"] \arrow[d, hook] & \ker\gamma \arrow[d, hook] & \\ & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i"] & B' \arrow[r, "g"] & C' & \end{tikzcd}$$

And the bottom two rows are exact, so the top row is exact by the 3×3-lemma since the columns exact

Now the composition $A \xrightarrow{i\alpha} B' \xrightarrow{\pi} \operatorname{coker}\beta$ is $\pi\beta f = 0$

And $B \xrightarrow{g\beta} C' \xrightarrow{\pi} \operatorname{coker}\gamma$ is $\pi\gamma p = 0$

Then we have the diagrams:

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A' \arrow[r, "\pi"] \arrow[d, "i"] & \operatorname{coker}\alpha \arrow[ddl, dashed, "\exists h"] \\ & B \arrow[d, "\pi"] & \\ & \operatorname{coker}\beta & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\beta"] & B' \arrow[r, "\pi"] \arrow[d, "g"] & \operatorname{coker}\beta \arrow[ddl, dashed, "\exists j"] \\ & C' \arrow[d, "\pi"] & \\ & \operatorname{coker}\gamma & \end{tikzcd}$$

By the UMP of cokernels, $\exists h, j$ commuting the diagrams so we have the commutative diagram:

$$\begin{tikzcd} & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\ & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma & \end{tikzcd}$$

And the top two rows are exact with exact columns, hence the bottom row is exact by 3×3-lemma

It remains to show exactness at $\ker\gamma$, $\operatorname{coker}\alpha$

Let $c \in \operatorname{Im}\bar{p} \Rightarrow c = \bar{p}(b)$, $b \in \ker\beta$

So $\Delta(c) = \Delta(\bar{p}(b)) = \Delta(p(b)) = a' + \operatorname{Im}\alpha$ where $\beta(b) = i(a')$

But $0 = \beta(b) = i(a') \Rightarrow a \in \ker i = 0$

$\therefore \Delta(c) = 0 + \operatorname{Im}\alpha = \operatorname{Im}\alpha = 0$ $\operatorname{coker}\alpha$

$\therefore c \in \ker\Delta$

$\therefore \operatorname{Im}\bar{p} \subseteq \ker\Delta$

Now let $c \in \ker\Delta \Rightarrow 0_{\operatorname{coker}\alpha} = \Delta(c) = a' + \operatorname{Im}\alpha$ where $c = p(b)$

and $\beta(b) = i(a') \Rightarrow \operatorname{Im}\alpha = a' + \operatorname{Im}\alpha \Rightarrow a' \in \operatorname{Im}\alpha \Rightarrow a' = \alpha(a)$

$\Rightarrow \beta(b) = i(a') = i(\alpha(a)) = \beta(f(a)) \Rightarrow b - f(a) \in \ker\beta$

$\Rightarrow \bar{p}(b - f(a)) = p(b - f(a)) = p(b) - p(f(a)) = p(b) = c$

$\therefore c \in \operatorname{Im}\bar{p}$

$\therefore \ker\Delta \subseteq \operatorname{Im}\bar{p}$

$\therefore \operatorname{Im}\bar{p} = \ker\Delta$

Now let $a' + \operatorname{Im}\alpha \in \operatorname{Im}\Delta \Rightarrow a' + \operatorname{Im}\alpha = \Delta(c)$, $c \in \ker\gamma$

And $i(a') = \beta(b)$ where $c = p(b)$

So $h(a' + \operatorname{Im}\alpha) = i(a') + \operatorname{Im}\beta = \beta(b) + \operatorname{Im}\beta = \operatorname{Im}\beta = 0_{\operatorname{coker}\beta}$

$\therefore a' + \operatorname{Im}\alpha \in \ker h$

$\therefore \operatorname{Im}\Delta \subseteq \ker h$

Now let $a' + \operatorname{Im}\alpha \in \ker h \Rightarrow 0 = h(a' + \operatorname{Im}\alpha) = i(a') + \operatorname{Im}\beta$

$\Rightarrow i(a') \in \operatorname{Im}\beta \Rightarrow i(a') = \beta(b) \Rightarrow 0 = g(i(a')) = g(\beta(b)) = \gamma(p(b$

$\Rightarrow p(b) \in \ker\gamma \Rightarrow \Delta(p(b)) = a' + \operatorname{Im}\alpha$

$\therefore \ker h \subseteq \operatorname{Im}\Delta$

$\therefore \operatorname{Im}\Delta = \ker h$

$\therefore \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\Delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$ exact

(iii) Given a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A'_n \arrow[r, "f_n"] \arrow[d, "d'_n"] & A_n \arrow[r, "g_n"] \arrow[d, "d_n"] & A''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & A'_{n-1} \arrow[r, "f_{n-1}"] & A_{n-1} \arrow[r, "g_{n-1}"] & A''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Prove that the following diagram is commutative and has exact rows:

$$
\begin{tikzcd}
 & A'_n/\mathrm{Im}\, d'_{n+1} \arrow[r, "h"] \arrow[d, "\bar d'_n"] & A_n/\mathrm{Im}\, d_{n+1} \arrow[r, "j"] \arrow[d, "\bar d_n"] & A''_n/\mathrm{Im}\, d''_{n+1} \arrow[r] \arrow[d, "\bar d''_n"] & 0 \\
0 \arrow[r] & \ker d'_{n-1} \arrow[r, "\bar f"] & \ker d_{n-1} \arrow[r, "\bar p"] & \ker d''_{n-1} &
\end{tikzcd}
$$

Applying the snake lemma to the given diagram, we get the exact sequence for each n:

$0 \to \ker d'_n \to \ker d_n \to \ker d''_n \to \operatorname{coker} d'_n \to \operatorname{coker} d_n \to \operatorname{coker} d''_n \to 0$

Hence the rows are exact

Now $\bar d_n(h(a'_n + \mathrm{Im}\, d'_{n+1})) = \bar d_n(f_n(a'_n) + \mathrm{Im}\, d_{n+1}) = d_n(f_n(a'_n))$

$= f_{n-1}(d'_n(a'_n))$

And $\bar f(\bar d'_n(a'_n + \mathrm{Im}\, d'_{n+1})) = \bar f(d'_n(a'_n)) = f_{n-1}(d'_n(a'_n))$

Now $\bar d''_n(j(a_n + \mathrm{Im}\, d_{n+1})) = \bar d''_n(g_n(a_n) + \mathrm{Im}\, d''_{n+1}) = d''_n(g_n(a_n))$

$= g_{n-1}(d_n(a_n))$

And $\bar p(\bar d_n(a_n + \mathrm{Im}\, d_{n+1})) = \bar p(d_n(a_n)) = g_{n-1}(d_n(a_n))$

$\therefore$ The diagram commutes

(iv) Show that $\exists$LES:

Applying the snake lemma to the diagram in (iii), we get the exact sequence for each n:

$$
\begin{tikzcd}[column sep=small]
\ker \bar d'_n \arrow[r] \arrow[d, equal] & \ker \bar d_n \arrow[r] \arrow[d, equal] & \ker \bar d''_n \arrow[r] \arrow[d, equal] & \operatorname{coker} \bar d'_n \arrow[r] \arrow[d, equal] & \operatorname{coker} \bar d_n \arrow[r] \arrow[d, equal] & \operatorname{coker} \bar d''_n \arrow[d, equal] \\
\ker d'_n/\mathrm{Im}\, d'_{n+1} \arrow[r] \arrow[d, equal] & \ker d_n/\mathrm{Im}\, d_{n+1} \arrow[r] \arrow[d, equal] & \ker d''_n/\mathrm{Im}\, d''_{n+1} \arrow[r] \arrow[d, equal] & \ker d'_{n-1}/\mathrm{Im}\, d'_n \arrow[r] \arrow[d, equal] & \ker d_{n-1}/\mathrm{Im}\, d_n \arrow[r] \arrow[d, equal] & \ker d''_{n-1}/\mathrm{Im}\, d''_n \arrow[d, equal] \\
H_n(A') \arrow[r] & H_n(A) \arrow[r] & H_n(A'') \arrow[r] & H_{n-1}(A') \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(A'')
\end{tikzcd}
$$

$\therefore \exists$LES

10.28. Let $C.$ be a complex each of whose differentials are zero maps. Prove that $H_n(C.) \cong C_n \ \forall n$.

$C.: \ \dots \longrightarrow C_{n+1} \xrightarrow{0} C_n \xrightarrow{0} C_{n-1} \longrightarrow \dots$

$H_n(C.) = \ker d_n / \mathrm{Im}\, d_{n+1} = C_n/0 \cong C_n$

$\therefore H_n(C.) \cong C_n \ \forall n$

10.31. (i) Define a direct system of complexes $\{C_\cdot^i, \varphi_j^i\}$, and prove that $\varinjlim C_\cdot^i$ exists

Claim $\varinjlim C_\cdot^i: \ \dots \longrightarrow \varinjlim C_{n+1}^i \longrightarrow \varinjlim C_n^i \longrightarrow \varinjlim C_{n-1}^i \longrightarrow \dots$

And this is a complex since it is:

$\dots \longrightarrow \bigoplus_i C_{n+1}^i/\sim \longrightarrow \bigoplus_i C_n^i/\sim \longrightarrow \bigoplus_i C_{n-1}^i/\sim \longrightarrow \dots$

which is a direct sum of complexes

And by construction it satisfies the UMP

And since each $\varinjlim C_n^i$ exists, the complex exists

$\therefore \varinjlim C_\cdot^i: \dots \varinjlim C_{n+1}^i \longrightarrow \varinjlim C_n^i \longrightarrow \varinjlim C_{n-1}^i \longrightarrow \dots$ exists

(ii) If $\{C_\cdot^i, \varphi_j^i\}$ is a direct system of complexes over a directed index set, prove that $\forall n \geq 0$, $H_n(\varinjlim C_\cdot^i) \cong \varinjlim H_n($

Since $C_\cdot^i$ is a complex, we have the SES:

$0 \longrightarrow \mathrm{Im}\, d_{n+1}^i \hookrightarrow \ker d_n^i \longrightarrow H_n(C_\cdot^i) \longrightarrow 0$

But since the index set is directed, $\varinjlim$ is exact, so we get the SES:

$0 \longrightarrow \varinjlim \mathrm{Im}\, d_{n+1}^i \longrightarrow \varinjlim \ker d_n^i \longrightarrow \varinjlim H_n(C_\cdot^i) \longrightarrow 0$

So $\varinjlim H_n(C_\cdot^i) \cong \mathrm{coker}\left(\varinjlim \mathrm{Im}\, d_{n+1}^i \longrightarrow \varinjlim \ker d_n^i\right)$

$\cong \mathrm{coker}\left(\mathrm{Im} \varinjlim d_{n+1}^i \longrightarrow \ker \varinjlim d_n^i\right)$ since

$\varinjlim$ is exact, hence preserves images, kernels

But then by (i), $\varinjlim H_n(C_\cdot^i) \cong H_n(\varinjlim C_\cdot^i)$

10.37. (i) Let $T: RMOD \to SMOD$ be an exact additive functor, where R, S are rings and suppose that if P is projective then $T(P)$ is projective. If B is a left R-module and P_B is a deleted projective resolution of B, prove that $T(P_B)$ is a deleted projective resolution of $T(B)$

Say $P_\cdot : \ldots \to P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ is a projective resolution of B

And $P_B : \ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$ is a deleted projective resolution of B

Then $T(P_B): \ldots \to T(P_2) \to T(P_1) \longrightarrow T(P_0) \longrightarrow 0$

with $T(P_\cdot): \ldots \longrightarrow T(P_2) \to T(P_1) \longrightarrow T(P_0) \longrightarrow T(B) \longrightarrow 0$ exact since T exact and each $T(P_i)$ projective since each P_i is projective

$\therefore T(P_\cdot)$ is a projective resolution for $T(B)$

$\therefore T(P_B)$ is a deleted projective resolution for $T(B)$

(ii) Let A be an R-algebra, where R is a commutative ring, which is flat as an R-module. Prove that if B is an A-module, then $\mathrm{Tor}_n^R(B, C) \cong \mathrm{Tor}_n^A(B, A \otimes_R C)$ $\forall R$-modules C and $\forall n \geq 0$.

Let $P_\cdot : \ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$ be a projective resolution of R-modules

Then $B \otimes_R P_\cdot : \ldots \longrightarrow B \otimes_R P_2 \longrightarrow B \otimes_R P_1 \longrightarrow B \otimes_R P_0 \longrightarrow 0$

Thus $\mathrm{Tor}_n^R(B, C) = H_n(B \otimes_R P)$

Now $A \otimes_R P_\cdot : \ldots \longrightarrow A \otimes_R P_1 \longrightarrow A \otimes_R P_0 \longrightarrow A \otimes_R C \longrightarrow 0$ is a projective resolution for $A \otimes_R C$ since each $A \otimes_R P_i$ is projective because A is a projective A-module and P_i is a projective R-module and it is exact since $A \otimes_R -$ is exact

And $B \otimes_A (A \otimes_R P) \cong (B \otimes_A A) \otimes_R P \cong B \otimes_R P_\cdot : \ldots \to B \otimes_R P_1 \to B \otimes_R P_0 \longrightarrow 0$

$\therefore Tor_n^A(B, A\otimes_R C) \cong H_n(B\otimes_R P.) = Tor_n^R(B,C)$

$\therefore Tor_n^R(B,C) \cong Tor_n^A(B, A\otimes_R C) \quad \forall n \geq 0$

10.40 Let R be an integral domain and let A be an R-module.

(i) Prove that if the multiplication $\mu_r : A \longrightarrow A$ is injective $\forall\, 0 \neq r \in R$, then A is torsion free

Let $0 \neq a \in A \ni ra = 0$ for $r \in R$

Suppose $r \neq 0$

Then $0 = ra = \mu_r(a) \Rightarrow a \in \ker \mu_r = 0$ since $r \neq 0 \Rightarrow \mu_r$ is injective

Contradiction since $a \neq 0$

$\therefore r = 0$

$\therefore$ A is torsionfree

(ii) Prove that if the multiplication $\mu_r : A \longrightarrow A$ is surjective $\forall\, 0 \neq r \in R$, then A is divisible

Let $y \in A$ and $0 \neq r \in R$ be a left regular element

Then $y = \mu_r(a)$ for some $a \in A$ since μ_r is surjective because $r \neq 0$

So $y = \mu_r(a) = ra$

$\therefore$ A is divisible

(iii) Prove that if the multiplication $\mu_r : A \longrightarrow A$ is an isomorphism $\forall\, 0 \neq r \in R$, then A is a vector space over Q where Q is the field of fractions of R.

Then A is torsion free, divisible by (i), (ii)

Let $\frac{r}{s} \in Q$ and $a \in A$

Then $\frac{r}{s} \cdot a = \frac{r}{s} \cdot sx$ since $s \neq 0$ and A is divisible

$= rx \in A$ and this action is well defined since

A is torsion free

$\therefore A$ is a $\mathbb{Q}$-module via the action $\frac{r}{s} \cdot a = \frac{ra}{s}$

$\therefore A$ is a vectorspace over $\mathbb{Q}$ since $\mathbb{Q}$ is a field

(iv) If either C or A is a vectorspace over $\mathbb{Q}$, prove that $\mathrm{Tor}_n^R(C,A)$, $\mathrm{Ext}_R^n(C,A)$ are vectorspaces over $\mathbb{Q}$

First note that X is a vectorspace over $\mathbb{Q}$ iff X is torsion free and divisible

Now since either C or A is a $\mathbb{Q}$-vectorspace, then either $C \xrightarrow{r} C$ or $A \xrightarrow{r} A$ is an isomorphism as in (iii)

Hence we have the SES: $0 \longrightarrow C \xrightarrow{r} C \longrightarrow 0 \longrightarrow 0$ or the SES: $0 \longrightarrow A \xrightarrow{r} A \longrightarrow 0 \longrightarrow 0$

Then from the LES's, we have $\mathrm{Tor}_n^R(C,A) \xrightarrow{r} \mathrm{Tor}_n^R(C,A)$ and $\mathrm{Ext}_R^n(C,A) \xrightarrow{r} \mathrm{Ext}_R^n(C,A)$ are isomorphisms in either case since Tor, Ext are balanced

$\therefore \mathrm{Tor}_n^R(C,A)$, $\mathrm{Ext}_R^n(C,A)$ are torsion free, divisible

$\therefore \mathrm{Tor}_n^R(C,A)$, $\mathrm{Ext}_R^n(C,A)$ are vectorspaces over $\mathbb{Q}$

10.54. If A, B are finite abelian groups, prove that $\mathrm{Tor}_1^{\mathbb{Z}}(A,B) \cong A \otimes_{\mathbb{Z}} B$

Since A, B are finite abelian groups, $A \cong \prod_{i=1}^{p} \mathbb{Z}/m_i\mathbb{Z}$ and $B \cong \prod_{i=1}^{q} \mathbb{Z}/n_i\mathbb{Z}$

$P_\cdot :\ 0 \longrightarrow \mathbb{Z}^q \xrightarrow{(n_i)} \mathbb{Z}^q \xrightarrow{\pi^q} \prod_{i=1}^{q} \mathbb{Z}/n_i\mathbb{Z} \longrightarrow 0$ is a projective resolution

$A \otimes_{\mathbb{Z}} P_\cdot :\ 0 \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}^q \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}^q \longrightarrow 0$ which is $0 \longrightarrow A^q \xrightarrow{(n_i)} A^q \longrightarrow 0$

$\mathrm{Tor}_1^{\mathbb{Z}}(A,B) = H_1(A \otimes_{\mathbb{Z}} P_\cdot)$

Now consider the SES: $0 \longrightarrow \mathbb{Z}^q \xrightarrow{(n_i)} \mathbb{Z}^q \xrightarrow{\pi^q} \prod_{i=1}^{q} \mathbb{Z}/n_i\mathbb{Z} \longrightarrow 0$

Then $A \otimes_{\mathbb{Z}} \mathbb{Z}^q \xrightarrow{1 \otimes (n_i)} A \otimes_{\mathbb{Z}} \mathbb{Z}^q \xrightarrow{1 \otimes \pi^q} A \otimes_{\mathbb{Z}} \prod_{i=1}^{q} \mathbb{Z}/n_i\mathbb{Z} \longrightarrow 0$ is exact

so $1 \otimes \pi^q$ is surjective, hence by the 1st Iso Thm,
$A \otimes_{\mathbb{Z}} B \cong A \otimes_{\mathbb{Z}} \mathbb{Z}^q / \ker(1 \otimes \pi^q) \cong A^q / \mathrm{Im}(n_i) \cong \mathrm{coker}(n_i)$
$\cong H_1(A \otimes_{\mathbb{Z}} P_\cdot)$

10.57. Let k be a field, let $R = k[x,y]$, and let $I = (x,y)$.
(i) Prove that $x \otimes y - y \otimes x \in I \otimes_R I$ is nonzero.

Define $\varphi : I \times I \longrightarrow k \ni \varphi(f,g) = f_x(0,0)\, g_y(0,0)$

$\varphi(f_1+f_2, g) = (f_1+f_2)_x(0,0)\, g_y(0,0) = (f_{1x}(0,0) + f_{2x}(0,0))\, g_y(0,0)$
$= f_{1x}(0,0)\, g_y(0,0) + f_{2x}(0,0)\, g_y(0,0) = \varphi(f_1,g) + \varphi(f_2,g)$

Similarly, $\varphi(f, g_1+g_2) = \varphi(f,g_1) + \varphi(f,g_2)$

And $\varphi(fr, g) = (fr)_x(0,0)\, g_y(0,0) = f_x(0,0)\, r g_y(0,0)$
$= f_x(0,0)(rg)_y(0,0) = \varphi(f, rg)$

$\therefore \varphi$ is biadditive

$\therefore \exists \Psi : I \otimes_R I \longrightarrow k \ni \Psi(f \otimes g) = f_x(0,0)\, g_y(0,0)$ homomorphism of abelian groups

But $\Psi(x \otimes y) = 1$ and $\Psi(y \otimes x) = 0$

$\therefore x \otimes y \neq y \otimes x$

$\therefore x \otimes y - y \otimes x \neq 0$

(ii) Prove that $x(x \otimes y - y \otimes x) = 0$ and conclude that $I \otimes_R I$ is not torsion free

$x(x \otimes y - y \otimes x) = x \otimes yx - xy \otimes x = xy \otimes x - xy \otimes x = 0$

Hence since $x \neq 0$, $x \otimes y - y \otimes x \neq 0$, $I \otimes_R I$ is not torsion free

11.20 If A is a projective R-module, prove that $S^{-1}A$ is a projective $S^{-1}R$-module.

First note that $S^{-1}A \cong S^{-1}R \otimes_R A$ is a natural isomorphism.

Let $0 \to E \to B \to C \to 0$ be a SES of $S^{-1}R$-modules, hence R-modules by restriction of scalars via: $R \to S^{-1}R \ni r \mapsto \frac{r}{1}$

But since A is a projective R-module, we have the SES:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{Hom}_R(A,E) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(A,B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(A,C) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, E)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, B)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, C)) \arrow[d, "\cong"] & \\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}R}(A \otimes_R S^{-1}R, E) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S^{-1}R}(A \otimes_R S^{-1}R, B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S^{-1}R}(A \otimes_R S^{-1}R, C) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}A, E) \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}A, B) \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}A, C) \arrow[r] & 0
\end{tikzcd}$$

And the vertical maps are natural isomorphisms, so the diagram commutes.

Then since the top row is exact, the bottom row is also exact

$\therefore S^{-1}A$ is a projective $S^{-1}R$-module

11.23. (iii) Prove that $S^{-1}\mathrm{Ext}_R^n(A,B)$ may not be isomorphic to $\mathrm{Ext}^n_{S^{-1}R}(S^{-1}A, S^{-1}B)$ if R is noetherian but A is not finitely generated.

Take $R=\mathbb{Z}$ which is noetherian and $A=\mathbb{Q}$ which is not finitely generated over $\mathbb{Z}$

$S^{-1}\mathrm{Ext}_R^0(A,B) = S^{-1}\mathrm{Ext}_{\mathbb{Z}}^0(\mathbb{Q},\mathbb{Z}) \cong S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = S^{-1}0 = 0$

But $\mathrm{Ext}^0_{S^{-1}R}(S^{-1}A, S^{-1}B) = \mathrm{Ext}^0_{\mathbb{Q}}(\mathbb{Q},\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q},\mathbb{Q}) \cong \mathbb{Q}$

$\therefore\ S^{-1}\mathrm{Ext}_R^0(A,B) \not\cong \mathrm{Ext}^0_{S^{-1}R}(S^{-1}A, S^{-1}B)$

11.69. If $\{M_\alpha\}_{\alpha\in A}$ is a family of left R-modules, prove that $\mathrm{pd}\left(\sum_{\alpha\in A} M_\alpha\right) = \sup_{\alpha\in A}\{\mathrm{pd}(M_\alpha)\}$

Choose M_α maximal with respect to projective dimension

Let $P_\cdot \longrightarrow M_\alpha$ be the smallest projective resolution of M_α

Then the sum of projective resolutions of the M_α's is a projective resolution for $\sum_{\alpha\in A} M_\alpha$

And the length of that projective resolution is the length of $P_\cdot$

$\therefore\ \mathrm{pd}\left(\sum_{\alpha\in A} M_\alpha\right) \leq \sup_{\alpha\in A}\{\mathrm{pd}(M_\alpha)\}$

Now let $P_\cdot : \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} \sum_{\alpha\in A} M_\alpha \longrightarrow 0$ proj. res.

Then $\tilde{P}_\cdot : \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{p_\alpha \epsilon} M_\alpha \longrightarrow 0$ is a projective resolution of M_α, $\forall \alpha\in A$

$\therefore\ \mathrm{pd}(M_\alpha) \leq \mathrm{pd}(\Sigma M_\alpha) \quad \forall \alpha \in A$

$\therefore\ \sup_{\alpha\in A}\{\mathrm{pd}(M_\alpha)\} \leq \mathrm{pd}(\Sigma M_\alpha)$

$\therefore\ \mathrm{pd}(\Sigma M_\alpha) = \sup_{\alpha\in A}\{\mathrm{pd}(M_\alpha)\}$

(See Extra Sheet for Corrected Answer)

3 #5

Hungerford

(a) - Show $\mathbb{Z}$ has ACC but not DCC

$\mathbb{Z}$ PID $\Rightarrow$ $\mathbb{Z}$ Noeth $\Rightarrow$ $\mathbb{Z}$ has ACC

since every ideal f.g.

$(2) \supsetneq (4) \supsetneq \ldots$ Does not stabilize

(b) - Show every f.g ab gp has ACC

A f.g ab gp $\Rightarrow$ A f.g $\mathbb{Z}$-mod $\Rightarrow$ A noeth since $\mathbb{Z}$ noeth & A f.g.

$\Rightarrow$ A has ACC

~~#1 (a) f.g. groups~~

III.4 #2

$S \subseteq R$ mult

R comm ring with 1

$T \subseteq S^{-1}R$ mult

$S_* = \{r \in R \mid \frac{r}{s} \in T \text{ for some } s \in S\}$

- show $S_* \subseteq R$ mult
- Show $S_*^{-1}R \cong T^{-1}(S^{-1}R)$ as rings

Clearly $S_* \subseteq R$

$\frac{0}{s} \notin T$ $\forall s \in S$ since T mult

$\therefore 0 \notin S_*$

$\frac{1}{1} \in T$ since mult $\Rightarrow 1 \in S_*$

Let $a, b \in S_*$ $\Rightarrow$ $\frac{a}{s}, \frac{b}{t} \in T$ for some $s, t \in T$

$\Rightarrow \frac{a}{s}\frac{b}{t} \in T$ since T mult

$\Rightarrow \frac{ab}{st} \in T \Rightarrow ab \in S_*$

$\therefore S_* \subseteq R$ mult

Define $\varphi: S_*^{-1}R \to T^{-1}(S^{-1}R) \ni \varphi\left(\frac{r}{s_*}\right) = \frac{\left(\frac{r}{s}\right)}{s_*}$

$s_* \in S_* \Rightarrow \frac{s_*}{s} \in T$ for some $s \in S$

$$\varphi\left(\frac{r_1}{s_{*1}}+\frac{r_2}{s_{*2}}\right)=\varphi\left(\frac{r_1 s_{*2}+r_2 s_{*1}}{s_{*1}s_{*2}}\right)=\cancel{\frac{\left(\frac{r_1 s_{*2}+r_2 s_{*1}}{s}\right)}{s_{*1}s_{*2}}=\left(\frac{\frac{r_1}{s}}{s_{*1}}\right)+\left(\frac{\frac{r_2}{s}}{s_{*2}}\right)}$$

$\frac{s_{*1}}{s_1}, \frac{s_{*2}}{s_2} \in T$

$\Rightarrow \frac{s_{*1}s_{*2}}{s_1 s_2} \in T$

$$=\frac{\left(\frac{r_1 s_{*2}+r_2 s_{*1}}{s_1 s_2}\right)}{s_{*1}s_{*2}} \overset{?}{=} \frac{\frac{r_1}{s_1}}{s_{*1}}+\frac{\frac{r_2}{s_2}}{s_{*2}}$$

$$\parallel$$

$$\frac{\frac{r_1 s_{*2}}{s_1}+\frac{r_2 s_{*1}}{s_2}}{s_{*1}s_{*2}}$$

#5

R int dom
$F = \mathrm{Frac}(R)$
T int dom $\ni R \subseteq T \subseteq F$
- show $F \cong \mathrm{Frac}(T)$

#8 R comm ring with 1

$I \lhd R$

$\pi : R \to R/I$ can. proj.

(a) $S \subseteq R$ mult

- show $\pi(S) \subseteq R/I$ mult

Clearly $\pi(S) \subseteq R/I$

$$0 \notin S \Rightarrow \underset{\overset{\parallel}{I}}{\pi(0)} \notin \pi(S)$$

$$1 \in S \Rightarrow \underset{\overset{\parallel}{1+I}}{\pi(1)} \in \pi(S)$$

Let $a+I, b+I \in \pi(S) \Rightarrow a+I = \pi(s),\ b+I = \pi(t)$

$\Rightarrow (a+I)(b+I) = \pi(s) \cdot \pi(t) = \pi(st) \in \pi(S)$ since S mult $\Rightarrow st \in S$

$\therefore \pi(S)$ mult

(b) - show $\theta : S^{-1}R \to (\pi(S))^{-1}(R/I) \ni \frac{r}{s} \to \frac{\pi(r)}{\pi(s)}$ well defined

Let $\frac{r_1}{s_1} = \frac{r_2}{s_2} \Rightarrow \exists u \in S \ni u(r_1 s_2 - r_2 s_1) = 0 \Rightarrow u r_1 s_2 = u r_2 s_1$

$\Rightarrow \pi(u r_1 s_2) = \pi(u r_2 s_1)$

$\Rightarrow \pi(u)\left(\pi(r_1)\pi(s_2) - \pi(r_2)\pi(s_1)\right) = 0$

$\Rightarrow \dfrac{\pi(r_1)}{\pi(s_1)} = \dfrac{\pi(r_2)}{\pi(s_2)}$

$\therefore$ wd

(c) - show θ ring epimorphism with kernel $S^{-1}I$

- show $S^{-1}R/S^{-1}I \cong (\pi(S))^{-1}(R/I)$

$$\theta\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}\right) = \theta\left(\frac{r_1 s_2 + r_2 s_1}{s_1 s_2}\right) = \frac{\pi(r_1 s_2 + r_2 s_1)}{\pi(s_1 s_2)} = \frac{\pi(r_1)\pi(s_2) + \pi(r_2)\pi(s_1)}{\pi(s_1)\pi(s_2)}$$

$$= \frac{\pi(r_1)}{\pi(s_1)} + \frac{\pi(r_2)}{\pi(s_2)} = \theta\left(\frac{r_1}{s_1}\right) + \theta\left(\frac{r_2}{s_2}\right)$$

$$\theta\left(\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}\right) = \theta\left(\frac{r_1 r_2}{s_1 s_2}\right) = \frac{\pi(r_1 r_2)}{\pi(s_1 s_2)} = \frac{\pi(r_1)\pi(r_2)}{\pi(s_1)\pi(s_2)} = \frac{\pi(r_1)}{\pi(s_1)} \cdot \frac{\pi(r_2)}{\pi(s_2)}$$

$$= \theta\left(\frac{r_1}{s_1}\right)\theta\left(\frac{r_2}{s_2}\right) \quad \therefore \theta \text{ ring homom}$$

Let $\frac{r+I}{\pi(s)} \in (\pi(S))^{-1}(R/I)$

$\Rightarrow \frac{r+I}{\pi(s)} = \frac{\pi(r)}{\pi(s)} = \theta\left(\frac{r}{s}\right)$

$\therefore \theta$ surj

$\therefore \theta$ epimorphism

$\ker\theta = \{\frac{r}{s} \in S^{-1}R \mid \theta(\frac{r}{s}) = 0_{(\pi(S))^{-1}(R/I)}\}$

But $\theta(\frac{r}{s}) = 0$ iff $\pi(r) = I$ iff $r \in \ker\pi = I$ iff $\frac{r}{s} \in S^{-1}I$

iff $\exists \pi(u) \in \pi(S) \ni \pi(u)\left(\pi(r)\pi(t) - \pi(s)I\right) = I$

iff $\pi(urt) + I = \pi(us+1) + I$

Show $\frac{r}{s} = \frac{i}{t}$ ie $\exists u \in S \ni u(rt - si) = 0$ ie $urt = usi$

ie $\pi(urt) = \pi(usi)$

$\therefore \ker\theta = S^{-1}I$

$\therefore$ By 1st Iso Thm $S^{-1}R/S^{-1}I \cong (\pi(S))^{-1}(R/I)$

#11 p prime

– what can be said about $\mathbb{Z}/p\mathbb{Z}$ and localization $\mathbb{Z}_{(p)}$

$\{\frac{a}{b} \mid p \nmid b\} = S^{-1}\mathbb{Z}$ where $S = \mathbb{Z} \setminus (p)$

Define $\varphi : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}_{(p)} \ni \varphi(\bar{a}) = \frac{\bar{a}}{1}$

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, "\phi"'] \arrow[dr] & \\ \mathbb{Z}_{(p)} \arrow[rr, dashed, "\exists! h"'] & & \mathbb{Z}/p\mathbb{Z} \end{tikzcd}$$

$z \mapsto \frac{z}{1}$ $\qquad$ $z \mapsto \bar{z} \longrightarrow$ unit! since p prime

$h(\frac{z}{1}) = \bar{z}$

h surj

$I = \ker h = \{\frac{a}{b} \mid p \mid a$ but $p \nmid b\}$

(unique max ideal)

1st Iso Thm: $\mathbb{Z}_{(p)}/I \cong \mathbb{Z}/p\mathbb{Z}$

ie $\mathbb{Z}/p\mathbb{Z}$ is iso to a quotient of $\mathbb{Z}_{(p)}$

#14 $M \triangleleft R$ maximal

R comm ring with 1

$n > 0$

- Show R/M^n has unique prime ideal, hence local

$r+M^n$ unit iff $\exists s+M^n \ni (s+M^n)(r+M^n) = 1+M^n$

$sr+M^n = 1+M^n$

iff $\exists s \in R \ni sr-1 \in M^n$

~~M max iff R/M field~~

ideals of R/M^n are $I = J/M^n \ni J \triangleleft R$ containing M^n

IV.1 #2 $f: A \to B$ R-mod homom

(a) - Show f monomorphism iff $\forall$ R-mod homom $g, h: D \to A \ni$ $fg = fh$, have $g = h$

($\Rightarrow$) Assume f monomorphism

Let $D \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B \ni fg = fh$

Let $d \in D$

Then $f(g(d)) = f(h(d)) \Rightarrow g(d) = h(d)$ since f inj

$\therefore g = h$

($\Leftarrow$) Assume $\forall\ D \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B \ni fg = fh$, $g = h$

Take $D = \ker f$ with $D \overset{g}{\hookrightarrow} A$ incl. and $D \xrightarrow{h} A$ zero

Then $fg = 0 = fh$

Then $g = h$ by ass.

So the inclusion map is the zero map

$\therefore \ker f = 0$

$\therefore f$ monomorphism

(b) - Show f epimorph iff $\forall$ R-mod homom $k, t: B \to C \ni$ $kf = tf$, have $k = t$

($\Rightarrow$) Assume f epimorph

Let $A \xrightarrow{f} B \overset{k}{\underset{t}{\rightrightarrows}} C$ with $kf = tf$

Let $b \in B$

Then $b = f(a)$ since f surj

Then $k(b) = k(f(a)) = t(f(a)) = t(b)$

$\therefore k = t$

($\Leftarrow$) Assume $\forall\ A \xrightarrow{f} B \overset{k}{\underset{t}{\rightrightarrows}} C$ with $kf = tf$, have $k = t$

Take $C = B/\operatorname{Im} f$, $B \xrightarrow{k} B/\operatorname{Im} f$, t zero

Then $kf = 0 = tf \Rightarrow k = t \Rightarrow$ proj is zero $\Rightarrow B/\operatorname{Im} f = 0 \Rightarrow B = \operatorname{Im} f$

$\therefore f$ surj.

\# 5 (a) - show every simple R-mod cyclic

Let S simple R-mod

Then $S \neq 0$, so $\exists\, 0 \neq x \in S$

Consider $0 \neq (x) \leq S$ since $x \neq 0$

Then $(x) = S$ since S simple

$\therefore$ S cyclic

(b) A simple

- show every R-mod endomorphism is zero or iso

Let $\varphi : A \longrightarrow A$ R-mod homom.

Then $\ker\varphi \leq A$ simple $\Rightarrow$ $\ker\varphi = 0$ or $\ker\varphi = A$

$\therefore$ φ inj or $\varphi = 0$

Now $\operatorname{Im}\varphi \leq A \Rightarrow \operatorname{Im}\varphi = 0$ or $\operatorname{Im}\varphi = A$

$\therefore$ $\varphi = 0$ or φ surj

$\therefore$ φ iso or $\varphi = 0$

#11 (a) A R-mod
R comm ring
$a \in A$

- show $\mathcal{O}_a = \{r \in R \mid ra = 0\} \lhd R$

Let $r_1, r_2 \in \mathcal{O}_a \Rightarrow r_1 a = r_2 a = 0 \Rightarrow (r_1 + r_2)a = r_1 a + r_2 a = 0$

$\therefore r_1 + r_2 \in \mathcal{O}_a$

Let $s \in R$, $r \in \mathcal{O}_a \Rightarrow ra = 0 \Rightarrow (sr)a = s(ra) = s \cdot 0 = 0$

$\therefore sr \in \mathcal{O}_a$

$\therefore \mathcal{O}_a \lhd R$

(b) R int dom
- show $t(A) \leq A$

Let $a, b \in t(A) \Rightarrow \exists\, r, s \in R$ $\ni$ $ra = sb = 0$ ($\overset{0}{\neq}$)

Then $rs(a+b) = rsa + rsb = sra + rsb = s \cdot 0 + r \cdot 0 = 0$

And since $r, s \neq 0$, $rs \neq 0$ since R int dom.

$\therefore a + b \in t(A)$

Let $r \in R$, $a \in t(A) \Rightarrow \exists\, 0 \neq s \in R \ni sa = 0$

Then $s(ra) = r(sa) = r \cdot 0 = 0$

$\therefore ra \in t(A)$

$\therefore t(A) \leq A$

(c) - show (b) may be false for R comm ring but not int dom.

$R = A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ comm ring

$(1,0)(0,1) = 0 \Rightarrow$ R not int dom

Then $t(A) = \{(0,0), (0,1), (1,0)\}$

But $(0,1) + (1,0) = (1,1) \notin t(A)$

$\therefore t(A) \not\leq A$

R int dom.

(d) $f : A \longrightarrow B$ R-mod homom

- Show $f(T(A)) \subseteq T(B)$
- Show $T(A) \xrightarrow{f|_{T(A)}} T(B)$ R-mod homom.

$$\text{Let } x \in f(T(A)) \Rightarrow x = f(y),\ y \in T(A)$$
$$\Rightarrow \exists\, 0 \neq r \in R \ni ry = 0$$
$$\Rightarrow rx = rf(y) = f(ry) = f(0) = 0$$
$$\Rightarrow x \in T(B)$$

$\therefore f(T(A)) \subseteq T(B)$

$\therefore T(A) \xrightarrow{\quad f \quad} T(B)$ R-mod hom

(e) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ exact R-mods

- Show $0 \longrightarrow T(A) \xrightarrow{f|_{T(A)}} T(B) \xrightarrow{g|_{T(B)}} T(C)$ exact

$$\text{Let } x \in \ker f|_{T(A)} \Rightarrow \cancel{x \in T(A) \text{ and }}\ f(x) = 0$$
$$\cancel{\Rightarrow \exists\, 0 \neq t \in R \ni tx = 0}$$
$$\Rightarrow x \in \ker f = 0 \text{ by exact}$$
$$\Rightarrow \ker f|_{T(A)} = 0$$

$\therefore f|_{T(A)}$ inj

$$\text{Let } x \in \operatorname{Im} f|_{T(A)} \Rightarrow x = f(y),\ y \in T(A)$$
$$\cancel{\Rightarrow \exists\, 0 \neq r \in R \ni ry = 0}$$
$$\Rightarrow \overset{g|_{T(B)}(x) =}{g(x)} = g(f(y)) = 0 \text{ by exact}$$
$$\Rightarrow x \in \ker g|_{T(B)}$$

$$\text{Let } x \in \ker g|_{T(B)} \Rightarrow 0 = g|_{T(B)}(x) = g(x)$$
$$\Rightarrow x \in \ker g = \operatorname{Im} f$$
$$\Rightarrow x = f(a),\ a \in A$$

$$\text{And } x \in T(B) \Rightarrow \exists\, 0 \neq r \in R \ni rx = 0$$
$$\Rightarrow 0 = rx = rf(a) = f(ra)$$
$$\Rightarrow ra = 0 \text{ since } f \text{ inj} \Rightarrow a \in T(A)\ \therefore x = f(a),\ a \in T(A)\ \therefore x = f|_{T(A)}(a)$$

$\therefore$ exact

$\therefore \operatorname{Im} f|_{T(A)} = \ker$

(f) $g: B \to C$ R-mod epimorphism

- Show $g_{T(B)}: T(B) \to T(C)$ need not be epimorphism

Take $R = \mathbb{Z}$

$g: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ nat proj

$T(\mathbb{Z}) = 0$

$T(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

so $g|_{T(\mathbb{Z})}: T(\mathbb{Z}) \to T(\mathbb{Z}/2\mathbb{Z})$ is $0 \to \mathbb{Z}/2\mathbb{Z}$ zero map

which is not surj since $\mathbb{Z}/2\mathbb{Z} \neq 0$

IV.2 #14

$f: V \to V'$ linear trans. of finite dim. vs V, V'

$\dim V = \dim V'$

- show TFAE:

(i) f iso

(ii) f epi

(iii) f mono.

<u>(i)⇒(ii)</u> Assume f iso

Then f epi

<u>(ii)⇒(iii)</u> Assume f epi

Then $V' = \operatorname{Im} f \Rightarrow \dim V' = \dim \operatorname{Im} f$

But $\dim V = \dim \ker f + \dim \operatorname{Im} f$

$\Rightarrow \dim V = \dim \ker f + \dim V'$

But $\dim V = \dim V' \Rightarrow \dim \ker f = 0 \Rightarrow \ker f = 0$

$\therefore f$ mono

<u>(iii)⇒(i)</u> Assume f mono

Then $\ker f = 0 \Rightarrow \dim \ker f = 0$

But $\dim V = \dim \ker f + \dim \operatorname{Im} f = \dim \operatorname{Im} f$

And $\dim V' = \dim V = \dim \operatorname{Im} f$ $\quad \therefore f$ iso

since $\operatorname{Im} f \leq V'$, and $\dim V' = \dim \operatorname{Im} f$, $V' = \operatorname{Im} f$ $\therefore f$ ep

IV.3 #14

D ring with 1
Every unitary D-mod free

- Show D division ring

Every D-mod free $\Rightarrow$ Every D-mod proj

$\Rightarrow$ D semisimple ring

$\Rightarrow D \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$, D_i division rings by Artin-Wed

Then each $M_{n_i}(D_i) \leq M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t) \cong D$

so each $M_{n_i}(D_i)$ free D-mod

But $(r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_t) M_{n_i}(D_i) =$

$(r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_t) \cdot 0 \times \dots \times 0 \times M_{n_i}(D_i) \times 0 \times \dots \times 0$

$= 0 \quad \forall i$

$\Rightarrow$ For any $t \geq 2$, $M_{n_i}(D_i)$ not free
$\Rightarrow t = 1$

$\therefore D \cong M_{n_i}(D_i)$ for some i

Now $D_i^{n_i} \leq M_{n_i}(D_i) \overset{\cong D}{\Rightarrow} D_i^{n_i}$ free
$\Rightarrow D_i^{n_i} \cong \oplus M_{n_i}(D_i)$

But $\dim D_i^{n_i} = n_i$

And $\dim \oplus M_{n_i}(D_i) = (n_i^2)^k$ where k is # summands

contra unless $n_i = 1$

$\therefore n_i = 1$

$\therefore D \cong M_1(D_i) \cong D_i$ division ring

$\therefore D$ division ring

IV.4 #2

A, B ab gps

$m, n \in \mathbb{Z}$

$mA = 0 = nB$

- Show every element of $\mathrm{Hom}(A, B)$ has order dividing (m, n)

Let $\varphi \in \mathrm{Hom}(A, B)$

$mA = 0 \Rightarrow$ Every element of A has order dividing m

$nB = 0 \Rightarrow$ " " B " n

Then $|\varphi(a)| \mid n$ $\forall a \in A$ since $\varphi(a) \in B$

But also $|\varphi(a)| \mid |a| \mid m$

so $|\varphi(a)| \mid m$ $\forall a \in A$

$\therefore |\varphi(a)| \mid (m, n)$ $\forall a \in A$

$\therefore |\varphi| \mid (m, n)$

IV.5 #2   A, B ab gps

(a) $m>0$

- show $A\otimes_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z}\cong A/mA$

SES: $0\longrightarrow \mathbb{Z}\xrightarrow{\;m\;}\mathbb{Z}\xrightarrow{\;\pi\;}\mathbb{Z}/m\mathbb{Z}\longrightarrow 0$

$\Rightarrow A\otimes_{\mathbb{Z}}\mathbb{Z}\xrightarrow{\;1\otimes m\;}A\otimes_{\mathbb{Z}}\mathbb{Z}\xrightarrow{\;1\otimes\pi\;}A\otimes_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z}\longrightarrow 0$ exact

Hence $1\otimes\pi$ surj $\Rightarrow A\otimes_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z}\cong A\otimes_{\mathbb{Z}}\mathbb{Z}/\ker(1\otimes\pi)$ by 1st Iso Thm

$\cong A/\operatorname{Im}(1\otimes m)$

$\cong A/\operatorname{Im} m$

$\cong A/mA$

(c) - Describe $A\otimes_{\mathbb{Z}}B$ where A, B f.g.

$A\cong\mathbb{Z}^n\oplus\bigoplus_{i=1}^{t}\mathbb{Z}/n_i\mathbb{Z}$

$B\cong\mathbb{Z}^m\oplus\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}$ by FundThm F.G Ab GPS.

$$A\otimes_{\mathbb{Z}}B\cong\Big(\mathbb{Z}^n\oplus\bigoplus_{i=1}^{t}\mathbb{Z}/n_i\mathbb{Z}\Big)\otimes_{\mathbb{Z}}\Big(\mathbb{Z}^m\oplus\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}\Big)$$

$$\cong\Big[\mathbb{Z}^n\otimes_{\mathbb{Z}}\Big(\mathbb{Z}^m\oplus\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}\Big)\Big]\oplus\Big[\bigoplus_{i=1}^{t}\mathbb{Z}/n_i\mathbb{Z}\otimes\Big(\mathbb{Z}^m\oplus\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}\Big)$$

$$\cong(\mathbb{Z}^n\otimes_{\mathbb{Z}}\mathbb{Z}^m)\oplus\Big(\mathbb{Z}^n\otimes_{\mathbb{Z}}\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}\Big)\oplus\Big(\bigoplus_{i=1}^{t}\mathbb{Z}/n_i\mathbb{Z}\otimes\mathbb{Z}^m\Big)$$

$$\oplus\Big(\bigoplus_{i=1}^{t}\mathbb{Z}/n_i\mathbb{Z}\otimes\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}\Big)$$

$$\cong\mathbb{Z}^{mn}\oplus\bigoplus_{i=1}^{k}\Big(\mathbb{Z}^n\otimes_{\mathbb{Z}}\mathbb{Z}/m_i\mathbb{Z}\Big)\oplus\bigoplus_{i=1}^{t}\Big(\mathbb{Z}/n_i\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}^m\Big)$$

$$\oplus\bigoplus_{i=1}^{t}\Big(\mathbb{Z}/n_i\mathbb{Z}\otimes_{\mathbb{Z}}\bigoplus_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}\Big)$$

$$\cong\mathbb{Z}^{mn}\oplus\bigoplus_{i=1}^{k}(\mathbb{Z}/m_i\mathbb{Z})^n\oplus\bigoplus_{i=1}^{t}(\mathbb{Z}/n_i\mathbb{Z})^m\oplus\bigoplus_{i=1}^{t}\Big[\bigoplus_{j=1}^{k}\Big(\mathbb{Z}/n_i\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/m_j$$

$$\cong\mathbb{Z}^{mn}\oplus\bigoplus_{i=1}^{k}(\mathbb{Z}/m_i\mathbb{Z})^n\oplus\bigoplus_{i=1}^{t}(\mathbb{Z}/n_i\mathbb{Z})^m\oplus\bigoplus_{i=1}^{t}\bigoplus_{j=1}^{k}\mathbb{Z}/d_{ij}\mathbb{Z}$$ where $d_{ij}=(n_i,m_j)$

#5

$A' \leq A$ right R-mod

$B' \leq B$ left R-mod

– Show $A/A' \otimes_R B/B' \cong A \otimes_R B / C$ where $C = (a' \otimes b, a \otimes b')_{a \in A,\, b \in B,\, a' \in A',\, b' \in B'}$

SES: $0 \longrightarrow B' \xhookrightarrow{i} B \xrightarrow{\pi} B/B' \longrightarrow 0$

$\Rightarrow A/A' \otimes_R B' \xrightarrow{1 \otimes i} A/A' \otimes_R B \xrightarrow{1 \otimes \pi} A/A' \otimes_R B/B' \longrightarrow 0$

$\Rightarrow 1 \otimes \pi$ surj $\Rightarrow A/A' \otimes_R B/B' \cong A/A' \otimes_R B / \ker(1 \otimes \pi)$

$\cong A/A' \otimes_R B / \operatorname{Im}(1 \otimes i)$

$\cong A/A' \otimes_R B / A/A' \otimes_R B'$

#8 $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ SES left R-mods

D right R-module

(a) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ splits

- show $0 \longrightarrow D \otimes_R A \longrightarrow D \otimes_R B \longrightarrow D \otimes_R C \longrightarrow 0$ SES

$\exists h : B \longrightarrow A \ni hf = 1_A$

Note $D \otimes_R -$ right exact, so it suffices to show

$0 \longrightarrow D \otimes_R A \xrightarrow{1 \otimes f} D \otimes_R B$ inj

Let $\sum(d_i \otimes a_i) \in \ker(1 \otimes f)$

$\Rightarrow 0 = (1 \otimes f)(\sum(d_i \otimes a_i)) = \sum(1 \otimes f)(d_i \otimes a_i)$

$= \sum(d_i \otimes f(a_i))$

$\Rightarrow 0 = (1 \otimes h)(\sum(d_i \otimes f(a_i))) = \sum(1 \otimes h)(d_i \otimes f(a_i))$

$= \sum(d_i \otimes h(f(a_i))) = \sum(d_i \otimes a_i)$

$\therefore \ker(1 \otimes f) = 0$

$\therefore 1 \otimes f$ inj

$\therefore$ SES

(b) Rhao 1

D free

- Show $0 \longrightarrow D \otimes_R A \longrightarrow D \otimes_R B \longrightarrow D \otimes_R C \longrightarrow 0$ SES

D free $\Rightarrow$ D proj $\Rightarrow$ D flat

$\therefore D \otimes_R -$ exact

$\therefore$ SES

(c) Rhao 1

D proj, unitary

D proj $\Rightarrow$ D flat $\Rightarrow D \otimes_R -$ exact $\Rightarrow$ SES

IV.6 #2 - show every free module over int dom with 1 is torsion free

- show converse false $\longrightarrow$ $\mathbb{Q}$ torsion free $\mathbb{Z}$-mod but not free $\mathbb{Z}$-mod

Let F free R-mod where R int dom

Let $0 \neq x \in F$, $r \in R \ni rx = 0$

F free $\Rightarrow$ F has basis $\{x_i\}$

Then $x = r_1x_1 + \dots + r_nx_n$, $r_i \neq 0$ for some i since $x \neq 0$

$\Rightarrow 0 = rx = r(r_1x_1 + \dots + r_nx_n) = rr_1x_1 + \dots + rr_nx_n$

$\Rightarrow rr_1 = \dots = rr_n = 0$ since $\{x_i\}$ lin ind.

$\Rightarrow rr_i = 0$

$\Rightarrow r = 0$ since $r_i \neq 0$, R int dom

$\therefore$ R torsion free

IV.7 #2 K comm ring with 1

A, B unitary K-mods (K-algebras)

- Show $\exists$ K-mod iso $\alpha: A \otimes_K B \longrightarrow B \otimes_K A \ni$
$\alpha(a \otimes b) = b \otimes a$ $\forall a \in A, b \in B$

Define $\varphi: A \times B \longrightarrow B \otimes_K A \ni \varphi(a,b) = b \otimes a$

$\varphi(a_1 + a_2, b) = b \otimes a_1 + a_2 = b \otimes a_1 + b \otimes a_2 = \varphi(a_1, b) + \varphi(a_2, b)$

Similarly $\varphi(a, b_1 + b_2) = \varphi(a, b_1) + \varphi(a, b_2)$

$\varphi(ak, b) = b \otimes ak = b \otimes ka = bk \otimes a = \varphi(a, bk) = \varphi(a, kb)$

$\therefore \varphi$ biadditive

so $\exists!\, \alpha: A \otimes_K B \longrightarrow B \otimes_K A$ K-mod homom $\ni \alpha(a \otimes b) = b \otimes a$

Similarly define $\psi: B \times A \longrightarrow A \otimes_K B \ni \psi(b,a) = a \otimes b$

gives $!\beta: B \otimes_K A \longrightarrow A \otimes_K B \ni \beta(b \otimes a) = a \otimes b$

And $\alpha^{-1} = \beta$

$\therefore \alpha$ iso of K-mods

VIII.1 #2

$0 \neq I \triangleleft R$
R PID
- show R/I Noeth + art.

(~~R PID ⇒ R noeth~~)

Let $J_1 \supseteq J_2 \supseteq \dots$ descending chain of ideals of R/I
Then $J_i = I_i/I$ where $I_i \triangleleft R$ containing I

And $I_1 \supseteq I_2 \supseteq \dots$

~~R PID ⇒ R noeth ⇒ ∃n ∋ I_n = I_R~~

with $I_1 = (a_1)$, $I_2 = (a_2)$, ... since R PID, $I = (a)$

$I \subseteq I_i \; \forall i \Rightarrow a_i | a$ (and $a_1 | a_2 | a_3 | \dots$), but a cant have infinitely many divisors since $a \neq 0$

Then $\exists n \ni a_n = a_{n+1} = \dots$

$\therefore I_n = I_{n+1} \dots$

$\therefore J_n = J_{n+1} = \dots$

$\therefore R/I$ artinian

$\therefore R/I$ noeth

#5 - Show every homomorphic image of a left Noeth/left artinian ring is left noeth/left artinian resp.

Let $\varphi: R \to S$ ring homom where R is left Noeth ring

$\operatorname{Im}\varphi \cong R/\ker\varphi$ by 1st Iso Thm

Claim If R noeth, then R/I noeth $\forall I \lhd R$

Let $J_1 \subseteq J_2 \subseteq \dots$ ascending chain ideals of R/I

Then $J_i = K_i/I$ where $K_i \lhd R$ containing I

And $K_1 \subseteq K_2 \subseteq \dots$

Then since R noeth, $K_n = K_{n+1} = \dots$ for some n

$\therefore J_n = J_{n+1} = \dots$

$\therefore R/I$ noeth

$\therefore \operatorname{Im}\varphi$ noeth.

Similar proof works for artinian.

VIII.2 #8

R comm ring with 1
$I \lhd R$
$J \lhd R$ f.g.
$J \subseteq \operatorname{Rad} I$
$\Rightarrow \exists\, n > 0 \ni J^n \subseteq I$

$\operatorname{Rad} I = \{ r \in R \mid r^n \in I \text{ for some } n > 0 \}$

Let $j \in J \subseteq \operatorname{Rad} I \Rightarrow j^n \in I$ for some $n > 0$

J fg $\Rightarrow J = (j_1, \ldots, j_m)$

$\Rightarrow J^n = \left(\prod_{i=1}^{n} k_i \right.$ where $k_i =$

$J^2 = \ldots = \left(j_i^{(2)} \right) i = 1, \ldots, m$

$\cancel{J^n = \left(j_i^{(n)} \right) i = 1, \ldots, m}$

$\cancel{j \in J^n \Rightarrow} j = \Sigma$

$$J^n = \left(\prod_{i=1}^{m} j_i^{p_i} \;\middle|\; n = p_1 + \ldots + p_m \right)$$

$$j_i^{p_i} \in I \Rightarrow \prod_{i=1}^{m} j_i^{p_i} \in I \text{ since } I \lhd R$$

Each $j_i \in J \subseteq \operatorname{rad} I$ $\therefore J^n \subseteq I$
$\Rightarrow j_i^{n_i} \in I$, some $n_i > 0$
Take $n = m \cdot \max n_i$
$\Rightarrow j_i^{p_i} \in I$ some i for each generator
$\Rightarrow \prod_{i=1}^{m} j_i^{p_i} \in I$
$\Rightarrow J^n \subseteq I$

IX.2 #5

R has 1

(a) - Show $J(R) = \{r \in R \mid 1_R + sr \text{ left invertible } \forall s \in R\}$

Let $r \in J(R)$ and $s \in R$

Suppose $1_R + sr$ has no left inverse

Then $\nexists t \in R \ni 1 = t(1_R + sr) \xrightarrow{\Rightarrow 1_R \notin (1_R+sr)} (1_R + sr) \neq R$

So $\exists M \triangleleft R$ maximal $\ni (1_R + sr) \subseteq M$

$\Rightarrow 1_R + sr = m \Rightarrow 1_R = \underset{M}{\underset{\cap}{m}} - s\underset{M}{\underset{\uparrow}{r}} \in M$ since $r \in J(R) = \bigcap$ max left ideals

$\Rightarrow M = R$

contra since M maximal

$\therefore 1_R + sr$ left invertible

$\therefore r \in \{ \quad \}$

$\therefore J(R) \subseteq \{ \quad \}$

Now let $r \in \{ \quad \}$

Then $1_R + sr$ left invertible $\forall s \in R$

~~$\Rightarrow \exists t \in R \ni t(1_R + r) = 1_R$~~

~~$t + tr = 1_R$~~

~~$R = (t + tr)$~~

~~$\Rightarrow (t + tr)M = M$~~ ~~$\forall$ max ideals M~~

~~$tm + trm = m'$~~

~~Suppose $\exists$ M max ideal $\ni r \notin M$~~

~~$(r) \notin M$~~

~~Let $x \in M \triangleleft R \Rightarrow x = u(t + tr)$~~

Suppose $\exists$ S simple $\ni rS \neq 0$

Then $\exists t \in S \ni rt \neq 0$

So $0 \neq (rt) \leq S$ simple

$\therefore (rt) = S = (t)$ since $t \neq 0$ (If $t = 0$, $rt = 0$)

$\Rightarrow t = urt$, $u \in R$

$\Rightarrow t - urt = 0$

$\Rightarrow (1_R - ur)t = 0$

$\Rightarrow t = 0$ since $1_R - ur$ left invertible

contra

$\therefore rS = 0 \ \forall$ S simple

Let $M \triangleleft^R$ maximal $\Rightarrow R/M$ simple

$\Rightarrow r(R/M) = 0$

$\Rightarrow (r) \subseteq M$

$\Rightarrow r \in M$ $\therefore r \in \bigcap M = J(R)$ $\therefore \{ \quad \} \subseteq J(R)$ $\therefore J(R) = \{ \quad \}$

(b) - Show $J(R)$ largest ideal $K \ni \forall r \in K$, $1_R + r$ unit

Note $J(R)$ is also $\cap$ all right max ideals

hence $J(R) = \{ r \in R \mid 1_R + rs \text{ has right inverse} \}$

so $\forall r \in K$, $1_R + r$ has right + left inverse

$\therefore \forall r \in K$ $1_R + r$ unit

Now let $L \triangleleft R \ni \forall r \in L$, $1_R + r$ unit

Let $r \in L \Rightarrow 1_R + r$ unit

$\Rightarrow \exists s \in R \ni s(1_R + r) = 1_R$

$\Rightarrow s + sr = 1_R$

$1_R + r = u$

$r = u - 1_R$

$\cancel{r^2 = (u - 1_R)(u - 1_R) = u^2 - 2u + 1_R}$

$J(R) \subseteq L$ since $1_R + rs$ $\forall s \Rightarrow 1_R + r$

FALSE?

#8 $R = \begin{pmatrix} D & D & \cdots & D \\ & D & \ddots & D \\ & & & D \end{pmatrix}$, D division ring

- Find $J(R)$
- Show $R/J(R) \cong \underbrace{D \times \cdots \times D}_{n \text{ factors}}$

<u>Claim</u> $J(R) = \begin{pmatrix} J(D) & D & \cdots & D \\ & \ddots & & D \\ 0 & & & J(D) \end{pmatrix}$

Go by induction on n

If $n=2$, have shown $J(R) = \begin{pmatrix} J(D) & D \\ 0 & J(D) \end{pmatrix}$

Assume $J(R') = \begin{pmatrix} J(D) & D & D \\ & \ddots & D \\ & & J(D) \end{pmatrix}$ where R' $n-1 \times n-1$

Then if R $n \times n$, $R = \begin{pmatrix} R' & D \\ 0 & D \end{pmatrix}$

so $J(R) = \begin{pmatrix} J(R') & D \\ 0 & J(D) \end{pmatrix} = \begin{pmatrix} J(D) & D & \cdots & D \\ & \ddots & & D \\ & & & J(D) \end{pmatrix}$

And D division ring, so only left ideals are $(0), D$

so (0) only maximal left ideal

$\therefore J(D) = (0)$

$\therefore J(R) = \begin{pmatrix} 0 & D & \cdots & D \\ & \ddots & \ddots & D \\ & & & 0 \end{pmatrix}$

Define $\varphi : R \longrightarrow D \times \cdots \times D \ni \varphi\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} = (d_1, \ldots, d_n)$

$\varphi\left(\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} + \begin{pmatrix} d_1' & & \\ & \ddots & \\ & & d_n' \end{pmatrix}\right) = \varphi\begin{pmatrix} d_1+d_1' & & \\ & \ddots & \\ & & d_n+d_n' \end{pmatrix} = (d_1+d_1', \ldots, d_n+d_n')$ $\therefore \varphi$ ring homom

$= (d_1, \ldots, d_n) + (d_1', \ldots, d_n')$ $\quad = \varphi\begin{pmatrix} d_1 & \\ & d_n \end{pmatrix} + \varphi\begin{pmatrix} d_1' & \\ & d_n' \end{pmatrix}$

$\varphi\left(\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}\begin{pmatrix} d_1' & & \\ & \ddots & \\ & & d_n' \end{pmatrix}\right) = \varphi\begin{pmatrix} d_1d_1' & & \\ & \ddots & \\ & & d_nd_n' \end{pmatrix} = (d_1d_1', \ldots, d_nd_n') = (d_1, \ldots, d_n)(d_1', \ldots, d_n')$ $\quad \varphi\begin{pmatrix} d_1 & \\ & d_n \end{pmatrix}\varphi\begin{pmatrix} d_1' & \\ & d_n' \end{pmatrix}$

Let $(d_1, \ldots, d_n) \in D \times \cdots \times D$

$\Rightarrow (d_1, \ldots, d_n) = \varphi\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

$\therefore \varphi$ surj

$\operatorname{Ker}\varphi = \{A \in R \mid \varphi(A) = 0_{D \times \cdots \times D}\}$

$\varphi(A) = 0_{D \times \cdots \times D}$ iff $\varphi\begin{pmatrix} d_1 & & * \\ & \ddots & \\ & & d_n \end{pmatrix} = (0, \ldots, 0)$ iff $(d_1, \ldots, d_n) = (0, 0, \ldots, 0)$

iff $A \in J(R)$

$\therefore R/J(R) \cong D \times \cdots \times D$ by 1st Iso Thm

IX.2 #17

Nakayama:

R ring with 1

$J \triangleleft R \ni J \subseteq J(R)$

A f.g. R-mod

$JA = A$

- show $A = 0$

Suppose $A \neq 0$

Then $A = (a_1, \dots, a_n)$ minimal generating set (hence $a_1 \neq \dots \neq a_n$)

$JA = A \implies a_1 = j_1 a_1 + \dots + j_n a_n$

$\implies (1 - j_1) a_1 = j_2 a_2 + \dots + j_n a_n$

But $j_1 \in J \subseteq J(R) \implies 1 - j_1$ has left inverse, u

$\implies a_1 = u j_2 a_2 + \dots + u j_n a_n$ (if $n=1$, $a_1 = 0$ contra otherwise...)

$\implies A = (a_2, \dots, a_n)$

contradiction to minimality

$\therefore A = 0$

IX.3 #5

$a \in R$ regular if $\exists\, x \in R \ni axa = a$

If every elt of R regular, then R regular ring

(a) - show Every division ring regular

Let D division ring

Let $d \in D$

If $d = 0$, then $\forall x \in D$ $dxd = 0x0 = 0 = d$, hence d regular

so assume $d \neq 0$

Then $\exists\, d^{-1} \in D$ since D division ring.

And $dd^{-1}d = d$

$\therefore$ d regular

$\therefore$ D regular

(b) - show finite direct product of regular rings is regular

Let $R_1, \ldots, R_n$ regular rings

Let $r = (r_1, \ldots, r_n) \in R_1 \times \ldots \times R_n$

Then for each i, $\exists\, x_i \in R_i \ni r_i x_i r_i = r_i$

Take $x = (x_1, \ldots, x_n)$

Then $rxr = (r_1, \ldots, r_n)(x_1, \ldots, x_n)(r_1, \ldots, r_n) = (r_1 x_1 r_1, \ldots, r_n x_n r_n)$

$= (r_1, \ldots, r_n) = r$

$\therefore$ r regular

$\therefore R_1 \times \ldots \times R_n$ regular

(c) - show every regular ring is semisimple

FALSE

$R = \mathbb{Z}/4\mathbb{Z}$

$0 \cdot 0 \cdot 0 = 0$

$1 \cdot 1 \cdot 1 = 1$

~~$2 \cdot 2 \cdot 2 = 2$~~ oops $\therefore$ R regular

$3 \cdot 3 \cdot 3 = 3$

But R not semisimple since SES of $\mathbb{Z}/4\mathbb{Z}$-mods $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to$

does not split, hence $\mathbb{Z}/2\mathbb{Z}$ not proj

But in fact every semisimple ring is regular:

Let R semisimple

Then $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_t}(D_t)$

~~Note that $M_n(D) \cong \mathrm{End}_D D^n$ and D^n is vector space over D~~

~~Claim If D division ring, $\mathrm{End}_D V$ regular where V vs over D~~

$axa=a \quad bzb=b$

$dyd=d \quad cwc=c$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax & by \\ cx & dy \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} axa + byc & \\ & \end{bmatrix}$

$A = E_1 \ldots E_n$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & w \\ z & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} axa + bz & \\ & \end{bmatrix}$

Note that if $A \in M_n(D)$ is of rank r, then $\exists$ invertible matrices $P, Q \ni PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$$\text{So } A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

$$\text{Take } X = QP$$

$$\begin{aligned} \text{Then } AXA &= P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}QPP^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q^{-1} \\ &= P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q^{-1} \\ &= P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q^{-1} \\ &= A \end{aligned}$$

$\therefore M_n(D)$ regular

$\therefore R$ finite direct product of regular rings

$\therefore R$ regular by (b)

#8

A R-module

R left artinian ring

$Ra \neq 0 \ \forall 0 \neq a \in A$

$J = J(R)$

- show $JA = 0$ iff A semisimple

($\Rightarrow$) Assume $JA = 0$

$JA = 0 \Rightarrow J \subseteq \mathrm{ann}_R A \Longrightarrow A$ R/J-mod

But R left artinian $\Rightarrow$ R/J semisimple

$\Rightarrow$ All R/J-mods are semisimple

$\therefore$ A is semisimple

($\Leftarrow$) Assume A semisimple

Then $A \cong S_1 \oplus \dots \oplus S_n$, S_i simple

So $JA = J(S_1 \oplus \dots \oplus S_n) = JS_1 \oplus \dots \oplus JS_n$

$= 0 \oplus \dots \oplus 0$

$= 0$

R comm

$I, J \triangleleft R$

- show $R/I \underset{R}{\otimes} R/J \cong R/I+J$ as R-mods

First note that:

$$(r+I)\otimes(s+J) = (1+I)r\otimes(s+J) = (1+I)\otimes r(s+J)$$
$$= (1+I)\otimes(rs+J)$$

so $\forall$ $\sum[(r_i+I)\otimes(s_i+J)] = \sum[(1+I)\otimes(r_is_i+J)]$

$$= (1+I)\otimes\sum(r_is_i+J)$$
$$= (1+I)\otimes(\sum r_is_i+J)$$
$$= (1+I)\otimes(r+J) \text{ for some } r\in R$$

Define $\varphi: R/I \times R/J \longrightarrow R/I+J \ni \varphi(r+I, s+J) = rs+(I+J)$

$(r_1+I, s_1+J) = (r_2+I, s_2+J) \Rightarrow r_1+I = r_2+I,\ s_1+J = s_2+J$

$\Rightarrow r_1-r_2\in I,\ s_1-s_2\in J$

$\Rightarrow s_1(r_1-r_2)\in I,\ r_2(s_1-s_2)\in J$

$\Rightarrow s_1r_1 - r_2s_2 \in I+J$

$\Rightarrow s_1r_1+(I+J) = r_2s_2+(I+J)$

$\therefore \varphi$ wd

$$\varphi(r_1+I+r_2+I, s+J) = \varphi(r_1+r_2+I, s+J)$$
$$= (r_1+r_2)s+(I+J)$$
$$= r_1s+(I+J)+r_2s+(I+J)$$
$$= \varphi(r_1+I, s+J)+\varphi(r_2+I, s+J)$$

Similarly $\varphi(r+I, s_1+J+s_2+J) = \varphi(r+I, s_1+J)+\varphi(r+I, s_2+J)$

$\varphi((r+I)t, s+J) = \varphi(rt+I, s+J) = rts+(I+J) = \varphi(r+I, ts+J)$

$\therefore \varphi$ biadditive

$\therefore \exists \psi: R/I \underset{R}{\otimes} R/J \longrightarrow R/I+J$ homom. of ab gps $\ni \psi((r+I)\otimes(s+J)) = rs+(I$

$\psi(t(r+I)\otimes(s+J)) = \psi((tr+I)\otimes(s+J)) = trs+(I+J) = t(rs+(I+J))$

$= t\psi(r+I, s+J)$

$\therefore \psi$ R-mod homom

Let $(1+I)\otimes(r+J)\in\ker\psi \Rightarrow 0 = \psi((1+I)\otimes(r+J)) = r+(I+J) \Rightarrow r\in I+J \Rightarrow r = i+$

$$\Rightarrow (1+I)\otimes(r+J) = (1+I)\otimes(i+j+J)$$
$$= (1+I)\otimes(i+J) + (1+I)\otimes(j+J)$$
$$= (i+I)\otimes(1+J) + (1+I)\otimes(j+J)$$
$$= 0\otimes(1+J) + (1+I)\otimes 0$$
$$= 0+0 = 0$$

$\therefore \ker\Psi = 0$

$\therefore \Psi$ injective

Let $r+(I+J) \in R/_{I+J} \Rightarrow r+(I+J) = \Psi((1+I)\otimes(r+J))$

$\therefore \Psi$ surj

$\therefore R/I \otimes_R R/J \cong R/_{I+J}$ as R-mods

R comm ring

B, C R-mods

A flat R-algebra

$\Rightarrow \mathrm{Tor}_n^A(A\otimes_R B, A\otimes_R C) \cong A\otimes_R \mathrm{Tor}_n^R(B,C)$

$P_\cdot : \quad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$ proj res hence exact

$A\otimes_R P_\cdot : \quad \cdots \longrightarrow A\otimes_R P_1 \longrightarrow A\otimes_R P_0 \longrightarrow A\otimes_R C \longrightarrow 0$ exact since A flat

And each $A\otimes_R P_i$ proj A-mod since A proj A-mod, P_i proj R-mod

$\therefore A\otimes_R P_\cdot$ proj res for $A\otimes_R C$

$(A\otimes_R B)\otimes_A (A\otimes_R P_\cdot) \cong A\otimes_R (B\otimes_A A)\otimes_R P_\cdot \cong A\otimes_R (B\otimes_R P_\cdot)$

$\mathrm{Tor}_n^A(A\otimes_R B, A\otimes_R C) \cong H_n(A\otimes_R (B\otimes_R P_\cdot)) \cong A\otimes_R H_n(B\otimes_R P_\cdot)$ since A flat, hence $A\otimes_R -$ exact, thus preserves homology

$A\otimes_R \mathrm{Tor}_n^R(B,C) \cong A\otimes_R H_n(B\otimes_R P_\cdot)$

$\therefore \mathrm{Tor}_n^A(A\otimes_R B, A\otimes_R C) \cong A\otimes_R \mathrm{Tor}_n^R(B,C)$

Proofs of Thms

(1) M R-mod, $e: M \to M$ idempotent $\Rightarrow M = \ker e \oplus \operatorname{Im} e$

<u>Proof</u>

First show $\operatorname{Im}(1-e) = \ker e$

Let $x \in \ker e \Rightarrow x = x - e(x) = (1-e)(x) \in \operatorname{Im}(1-e)$

$\therefore \ker e \subseteq \operatorname{Im}(1-e)$

Let $x \in \operatorname{Im}(1-e) \Rightarrow x = (1-e)(y) \Rightarrow e(x) = e(1-e)(y) = (e-e^2)(y)$

$= 0$ since e idempotent

$\therefore x \in \ker e$

$\therefore \operatorname{Im}(1-e) \subseteq \ker e$

$\therefore \ker e = \operatorname{Im} e$

Now show $M = \operatorname{Im}(1-e) + \operatorname{Im} e$

Let $x \in M \Rightarrow x = (1-e)(x) + e(x) \in \operatorname{Im}(1-e) + \operatorname{Im} e$

$\therefore M = \operatorname{Im}(1-e) + \operatorname{Im} e$

Let $x \in \ker e \cap \operatorname{Im} e \Rightarrow x = e(y)$ and $0 = e(x) = e(e(y)) = e^2(y) = \underset{= x}{e(y)}$

$\therefore M = \operatorname{Im}(1-e) \oplus \operatorname{Im} e$

---

(2) $\operatorname{Hom}_R(M, -)$ left exact

<u>Proof</u>

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ SES

Show $0 \to \operatorname{Hom}_R(M,A) \xrightarrow{f_*} \operatorname{Hom}_R(M,B) \xrightarrow{g_*} \operatorname{Hom}_R(M,C)$ exact

Let $h \in \ker f_* \Rightarrow 0 = f_*(h) = fh \Rightarrow \operatorname{Im} h \subseteq \ker f = 0 \Rightarrow h = 0 \Rightarrow \ker f_* = 0$

$\therefore f_*$ inj

$g_* f_*(h) = g(f(h)) = 0$

$\therefore \operatorname{Im} f_* \subseteq \ker g_*$

Let $h \in \ker g_* \Rightarrow 0 = g_*(h) = gh$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & M \arrow[ul, dashed, "\exists j"] \arrow[u, "h"] \arrow[ur, bend right, "0"'] & \end{tikzcd}$$

f kernel of $g \Rightarrow \exists j: M \to A \ni h = fj = f_*(j) \in \operatorname{Im} f_*$

$\therefore \ker g_* \subseteq \operatorname{Im} f_*$

$\therefore \operatorname{Im} f_* = \ker g_*$

$\therefore$ Exact

3

$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ split exact $\Rightarrow$ $Hom_R(M,-)$ exact

proof

$Hom_R(M,-)$ left exact, so suffices to show

$Hom_R(M,B) \xrightarrow{g_*} Hom_R(M,C)$ surj

Let $h \in Hom_R(M,C)$

$$\begin{tikzcd} B \arrow[r, shift left, "g"] & C \arrow[l, dashed, shift left, "j"] \\ & M \arrow[u, "h"'] \end{tikzcd}$$

$\exists j : C \longrightarrow B \ni gj = 1_C$

Define $k : M \longrightarrow B \ni k = jh$

Then $g_*(k) = gjh = 1_C h = h$

$\therefore$ g_* surj

$\therefore$ exact

4

Every free module is proj

Proof

Let F free R-mod, $B \xrightarrow{g} C \longrightarrow 0$, $F \xrightarrow{f} C$

F has basis $\{b_i\}_{i \in I}$

g surj $\Rightarrow$ $f(b_i) = g(\overline{b_i})$, $\overline{b_i} \in B$

Define $h : F \longrightarrow B \ni h(b_i) = \overline{b_i}$ ie $h(\Sigma r_i b_i) = \Sigma r_i \overline{b_i}$

$\therefore f = gh$

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & F \arrow[ul, dashed, "\exists h"] \arrow[u, "f"'] & \end{tikzcd}$$

$\therefore F$ proj

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comm dgm, exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

$\Rightarrow \exists \alpha : A \to A$ comm. dgm

Proof

f' kernel of g' $\Rightarrow$

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \\ & & A \arrow[ul, dashed, "\exists \alpha"] \arrow[u, "\beta f"'] & \end{tikzcd}$$

$= \gamma g f = 0$

⑥ $R \underset{R}{\otimes} M \cong M$ as left R-mods

<u>Proof</u> ${}_RR_R \underset{R}{\otimes} {}_RM$ left R-mod

Define $\varphi: R \times M \longrightarrow M \ni \varphi(r,m) = rm$

$\varphi(r_1+r_2, m) = (r_1+r_2)m = r_1m + r_2m = \varphi(r_1,m) + \varphi(r_2,m)$

$\varphi(r, m_1+m_2) = r(m_1+m_2) = rm_1 + rm_2 = \varphi(r,m_1) + \varphi(r,m_2)$

$\varphi(rs,m) = (rs)m = r(sm) = \varphi(r,sm)$

$\therefore \varphi$ biadditive

$\therefore \exists \psi: R \underset{R}{\otimes} M \longrightarrow M$ homom of ab gps $\ni \psi(r\otimes m) = rm$

And $\psi(s(r\otimes m)) = \psi(sr \otimes m) = srm = s\psi(r\otimes m)$

$\therefore \psi$ R-mod homom

Define $\phi: M \longrightarrow R \underset{R}{\otimes} M \ni \phi(m) = 1 \otimes m$

$\phi(m_1+m_2) = 1\otimes(m_1+m_2) = 1\otimes m_1 + 1 \otimes m_2 = \phi(m_1) + \phi(m_2)$

$\phi(rm) = 1 \otimes rm = r \otimes m = r(1\otimes m) = r\phi(m)$

$\therefore \phi$ R-mod homom

And $\psi\phi(m) = \psi(1\otimes m) = m$, $\phi\psi(r\otimes m) = \phi(rm) = r \otimes m$

$\therefore \phi = \psi^{-1}$

$\therefore \psi$ iso

$\therefore R \underset{R}{\otimes} M \cong M$ R-mods

---

⑦ ${}_{\mathbb{Z}}D$ divisible $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(R,D)$ left inj R-mod

<u>Proof</u>

$\mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R, {}_{\mathbb{Z}}D)$ left R-mod

Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ SES left R-mods

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(C,\mathrm{Hom}_{\mathbb{Z}}(R,D)) \arrow[r] \arrow[d] \arrow[dr, phantom, "G^{nat}"] & \mathrm{Hom}_R(B,\mathrm{Hom}_{\mathbb{Z}}(R,D)) \arrow[r] \arrow[d] \arrow[dr, phantom, "G^{nat}"] & \mathrm{Hom}_R(A,\mathrm{Hom}_{\mathbb{Z}}(R,D)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(R\underset{R}{\otimes} C, D) \arrow[r] \arrow[d] \arrow[dr, phantom, "G^{nat}"] & \mathrm{Hom}_{\mathbb{Z}}(R\underset{R}{\otimes} B, D) \arrow[r] \arrow[d] \arrow[dr, phantom, "G^{nat}"] & \mathrm{Hom}_{\mathbb{Z}}(R\underset{R}{\otimes} A, D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(C,D) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(B,D) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(A,D) \arrow[r] & 0
\end{tikzcd}$$

Bottom exact since D divisible hence inj $\mathbb{Z}$-mod $\Rightarrow$ top exact $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(R,D)$ inj

(8) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ SES $\overset{R\text{-mods}}{\wedge}$, $S \subseteq R$ mult $\Rightarrow$

$0 \longrightarrow A_S \xrightarrow{f_S} B_S \xrightarrow{g_S} C_S \longrightarrow 0$ SES R_S-mods

Proof

Let $\frac{a}{s} \in \ker f_S \Rightarrow \frac{0}{1} = f_S\left(\frac{a}{s}\right) = \frac{f(a)}{s} \Rightarrow \exists u \in S \ni 0 = uf(a) = f(ua)$

$\Rightarrow ua = 0 \Rightarrow \frac{a}{s} = \frac{ua}{us} = \frac{0}{us} = 0$

$\therefore f_S$ inj

$g_S\left(f_S\left(\frac{a}{s}\right)\right) = g_S\left(\frac{f(a)}{s}\right) = \frac{g(f(a))}{s} = \frac{0}{s} = 0$

$\therefore \operatorname{Im} f_S \subseteq \ker g_S$

Let $\frac{b}{s} \in \ker g_S \Rightarrow \frac{0}{1} = g_S\left(\frac{b}{s}\right) = \frac{g(b)}{s} \Rightarrow \exists u \in S \Rightarrow 0 = ug(b) = g(ub)$

$\Rightarrow ub \in \ker g = \operatorname{Im} f \Rightarrow ub = f(a) \Rightarrow \frac{b}{s} = \frac{ub}{us} = \frac{f(a)}{us} = f_S\left(\frac{a}{us}\right) \in \operatorname{Im} f_S$

$\therefore \ker g_S \subseteq \operatorname{Im} f_S$

$\therefore \operatorname{Im} f_S = \ker g_S$

Let $\frac{c}{s} \in C_S \Rightarrow \frac{c}{s} = \frac{g(b)}{s} = g_S\left(\frac{b}{s}\right)$

$\therefore g_S$ surj

$\therefore$ SES

---

(9) submodules + homomorphic images of semisimple modules are semisimple

Proof

Let M semisimple, let $L \leq M$ and let $X \leq L$

Then $X \leq M \Rightarrow M = X \oplus Y$, $Y \leq M$

$\Rightarrow L = L \cap M = L \cap (X \oplus Y) = (L \cap X) \oplus (L \cap Y)$ by Modular law

$= X \oplus (L \cap Y)$

$\therefore L$ semisimple

Now let $\varphi : M \longrightarrow L$ R-mod homom

show $\operatorname{Im} \varphi$ semisimple

$\operatorname{Im} \varphi \cong M / \ker \varphi$ by 1st iso Thm

$\cong \ker \varphi \oplus X / \ker \varphi$, $X \leq M$

$\cong X / \ker \varphi \cap X = X/0 \cong X$ semisimple by above since $X \leq M$

$\therefore \operatorname{Im} \varphi$ semisimple

(10) $\ell(M)<\infty \Longrightarrow \ell(L), \ell(M/L)<\infty$ and $\ell(M)=\ell(L)+\ell(M/L)$

Proof Assume $M \neq L \neq 0$ WLOG

series $0 \subsetneq L \subsetneq M$

If comp series, then $0 \subsetneq L$ comp series for L, hence $\ell(L)<\infty$ And same for M/L

If not, refine it to comp series: $0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k \subsetneq \ldots \subsetneq M$, $M_k = L$

$\Rightarrow 0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k = L$ comp series

$\therefore \ell(L)<\infty$

And $0 = M_k/L \subsetneq M_{k+1}/L \subsetneq \ldots \subsetneq M_n/L = M/L$ comp series

$\therefore \ell(M/L)<\infty$

And $\ell(M)=\ell(L)+\ell(M/L)$ by const.

---

(11) R semisimple ring, M f.g R-mod $\Rightarrow M$ has comp series

Proof M f.g $\Rightarrow \exists R^n \longrightarrow M \longrightarrow 0$

And R semisimple $\Rightarrow R$ artinian + noetherian $\Rightarrow \ell(R)<\infty$

$\Rightarrow \ell(R^n)<\infty \Rightarrow \ell(M)<\ell(R^n)<\infty$

$\therefore \ell(M)<\infty$

$\therefore M$ has comp series

---

(12) $I \triangleleft R$ nil, $J/I \triangleleft R/I$ nil $\Rightarrow J$ nil

Proof

Let $x \in J \Rightarrow x+I \in J/I$ nil $\Rightarrow 0=(x+I)^n = x^n+I \Rightarrow x^n \in I$

But I nil, so $(x^n)^m=0 \Rightarrow x^{nm}=0$

$\therefore x$ nilpotent

$\therefore J$ nil

(13) $I, J \triangleleft R$ nil $\Rightarrow I+J$ nil

<u>Proof</u>

$I+J/I \cong J/I\cap J$

Let $x \in J/I\cap J \Rightarrow x = j + I\cap J$

J nil $\Rightarrow j^n = 0$ for some $n \Rightarrow x^n = (j+I\cap J)^n = j^n + I\cap J$

$$= 0 + I\cap J$$
$$= 0$$

$\therefore x$ nilpotent

$\therefore I+J/I$ nil

Since I nil, $I+J/I$ nil, have $I+J$ nil

~~(14) R ring, I nilpotent left ideal of R, M left R-mod, L ≤ M,~~
~~$M = L+IM \Rightarrow M = L$~~

~~$M = L+IM = L+I(L+IM) = L+IL+I^2M$~~

(14) R semisimple $\Rightarrow$ rad$R = 0$

<u>Proof</u>

R semisimple $\Rightarrow R \cong M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$, D_i division rings

$\Rightarrow$ rad$R \cong$ rad$\left(M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)\right)$

Maximal ideals of $M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$ of form $I_1 \times \dots \times I_t$ where $I_i \triangleleft M_{n_i}(D_i)$

But each $M_{n_i}(D_i)$ simple, so the only ideals are (0) and itself, hence only maximal ideal is (0) so only max ideal of product is $(0)\times \dots \times (0)$

$\therefore$ rad$R \cong \bigcap$ max ideals of $M_{n_1}(D_1) \times \dots \times M_{n_t}(D_t)$

$$= (0) \times \dots \times (0)$$
$$= 0$$

---

(15) R left noeth, $\{E_i\}_{i \in I}$ family inj. modules
$\Rightarrow \bigoplus_{i \in I} E_i$ inj

Proof

Let
$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, hook] \arrow[d, "f"] & R \\ & \bigoplus_{i \in I} E_i & \end{tikzcd}$$

R noeth $\Rightarrow J$ f.g $\Rightarrow J = (a_1, \ldots, a_n)$
Now only finitely many entries of $f(a_j)$ are nonzero
$\Rightarrow \exists i_1, \ldots, i_t \in I \ni \operatorname{Im} f = f(J) \subseteq E_{i_1} \oplus \ldots \oplus E_{i_t}$ since J f.g.

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, hook] \arrow[d, "f"] & R \arrow[dl, dashed, "g"] \\ & E_{i_1} \oplus \ldots \oplus E_{i_t} \arrow[d, hook, "i"] & \\ & \bigoplus_{i \in I} E_i & \end{tikzcd}$$

$\exists g$ since finite direct sum of injectives is injective

$\therefore \exists\, ig : R \to \bigoplus_{i \in I} E_i$ commuting dgm.

$\therefore \bigoplus_{i \in I} E_i$ inj

---

(16) M artinian $\Rightarrow \operatorname{soc} M \overset{ess}{\subseteq} M$

Proof

Let $0 \neq L \leq M$
$L \leq M$ artinian $\Rightarrow L$ artinian $\Rightarrow \exists S \leq L$ simple submodule
But $S \subseteq \operatorname{Soc} M \Rightarrow S \subseteq \operatorname{Soc} M \cap L \Rightarrow \operatorname{Soc} M \cap L \neq 0$
$\therefore \operatorname{Soc} M \overset{ess}{\subseteq} M$

(17) $M \subseteq E$, $\{E_i\}_{i\in I}$ chain submodules of $E \ni M \overset{ess}{\subseteq} E_i \subseteq E$ $\forall i$
$\Rightarrow M \overset{ess}{\subseteq} \bigcup_{i\in I} E_i$

Proof

Let $0 \neq x \in \bigcup_{i\in I} E_i \Rightarrow \overset{0}{\neq} x \in E_i$ for some i
Since $M \overset{ess}{\subseteq} E_i$, $\exists r \in R \ni 0 \neq rx \in M$
$\therefore M \overset{ess}{\subseteq} \bigcup_{i\in I} E_i$

---

(18) $M \overset{ess}{\subseteq} E$, $f: E \longrightarrow X$ homom, $f|_M$ inj $\Rightarrow f$ inj

Proof Suppose $\ker f \neq 0$
Then $0 \neq \ker f \leq E$
But $M \overset{ess}{\subseteq} E$, so $M \cap \ker f \neq 0$
Contradiction since $f|_M$ inj
$\therefore \ker f = 0$, hence f inj

---

(19) $P_1, \ldots, P_n$ incomparable prime ideals of R, $P_1 \ldots P_n = 0$
$\Rightarrow \{P_1, \ldots, P_n\}$ is the set of minimal prime ideals of R

Proof

Suppose $\exists P$ prime $\ni P_i \supsetneq P$ for some i
Then $P_i \supsetneq P \supseteq 0 = P_1 \ldots P_n$
But P prime, so $\exists j \ni P_j \subseteq P$
Then $P_i \supsetneq P \supseteq P_j$
If $i = j \Rightarrow$ contra $P \neq P_i$
If $i \neq j \Rightarrow P_j \subsetneq P_i$ comparaable $\Rightarrow$ contra
$\therefore$ Each P_i is minimal prime
Now let P minimal prime
Then $P \supseteq 0 = P_1 \ldots P_n \Rightarrow P_i \subseteq P$ for some i since P prime
But P minimal $\Rightarrow P = P_i$ $\therefore \{P_1, \ldots, P_n\}$ set of min primes

(20) R comm noeth $\Rightarrow$ every ideal of R is finite intersection of irreducible ideals

Proof

Suppose not

Since R noeth, $\exists$ maximal counterexample, I

I not irreducible $\Rightarrow$ $I = A \cap B \ni I \subsetneq A$ and $I \subsetneq B$

By maximality, A, B finite intersections of irreducible ideals

$\therefore$ I intersection of finitely many irreducible ideals

Contra

$\therefore$ Every ideal of R is finite intersection of irreducible ideals

---

(21) R comm noeth, $P \triangleleft R$ prime $\Rightarrow$ A finite intersection of P-primary ideals is P-primary

Proof

Let $Q = Q_1 \cap \dots \cap Q_n$, Q_i P-primary $\forall i$

Then $\sqrt{Q_i} = P \ \forall i$

Let $ab \in Q$ and assume $b^n \notin Q \ \forall n \geq 1$

$\Rightarrow \forall Q_i \ni b^n \notin Q_i \ \forall n \geq 1$

But Q_i primary, so $a \in Q_i \ \forall$ since $ab \in Q_i \ \forall i$

$\therefore a \in Q$

$\therefore$ Q primary

Let $x \in P \Rightarrow \forall i, \exists n_i \ni x^{n_i} \in Q_i$ since $\sqrt{Q_i} = P \ \forall i$

Let $k = \max\{n_i\}$

Then $x^k \in Q_i \ \forall i \Rightarrow x^k \in Q \Rightarrow x \in \sqrt{Q}$

$\therefore P \subseteq \sqrt{Q}$

Let $x \in \sqrt{Q} \Rightarrow x^m \in Q$ for some $m \geq 1 \Rightarrow x^m \in Q_i$ for each i

$\Rightarrow x \in \sqrt{Q_i} = P$

$\therefore \sqrt{Q} \subseteq P$

$\therefore P = \sqrt{Q}$ $\quad$ $\therefore$ Q P-primary

22 $\mathcal{A}, \mathcal{B}$ ab cat.
(F, G) adjoint pair
$\Rightarrow$ F right exact, G left exact

<u>Proof</u>

Let $0 \to A' \to A \to A'' \to 0$ SES in $\mathcal{A}$
Then $\forall B \in \mathcal{B}$, have exact seq since Hom left exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A'', G(B)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft \text{nat}"] & \mathrm{Hom}_{\mathcal{A}}(A, G(B)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft \text{nat}"] & \mathrm{Hom}_{\mathcal{A}}(A', G(B)) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(F(A''), B) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(F(A), B) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(F(A'), B)
\end{tikzcd}$$

And diagram commutes since vertical maps are natural

1505
So bottom row exact $\forall B \in \mathcal{B}$
But Hom reflects exact seqs, so $F(A') \to F(A) \to F(A'') \to 0$ exact
$\therefore$ F right exact

Let $0 \to B' \to B \to B'' \to 0$ SES in $\mathcal{B}$
Then $\forall A \in \mathcal{A}$, have exact seq:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(F(A), B') \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(F(A), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(F(A), B'') \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, G(B')) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, G(B)) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, G(B''))
\end{tikzcd}$$

Again, bottom row exact $\forall A \in \mathcal{A}$
so $0 \to G(B') \to G(B) \to G(B'')$ exact
$\therefore$ G left exact

(23) In R-MOD, $- \underset{R}{\otimes} W$ preserves colimits

Proof

First note $- \underset{R}{\otimes} W$ preserves coproducts: $(\oplus M_i) \otimes N \cong \oplus (M_i \otimes N)$

Define $\varphi : (\oplus M_i) \times N \longrightarrow \oplus (M_i \otimes N) \ni \varphi(((m_i), n)) = (m_i \otimes n)$
biadditive

So $\exists\, \Phi : (\oplus M_i) \otimes N \longrightarrow \oplus (M_i \otimes N) \ni \Phi((m_i) \otimes n) = (m_i \otimes n)$

Define $\Psi : M_i \times N \longrightarrow (\oplus M_i) \otimes N \ni \Psi((m_i, n)) = (\dots, 0, m_i, 0, \dots) \otimes n$

Then by UMP of coproduct, $\exists \oplus \Psi : \oplus (M_i \times N) \longrightarrow (\oplus M_i) \otimes N \ni$
$\oplus \Psi(((m_i, n))) = (m_i) \otimes n$ biadditive

Then by UMP of $\otimes$, $\exists\, \Theta : \oplus (M_i \otimes N) \longrightarrow (\oplus M_i) \otimes N \ni$
$\Theta((m_i \otimes n)) = (m_i) \otimes n$

And $\Theta = \Phi^{-1}$

$\therefore (\oplus M_i) \otimes N \cong \oplus (M_i \otimes N)$

And also since $\otimes$ is right exact, $\otimes$ preserves cokernels

Then
$$
\begin{aligned}
(\operatorname{colim}\{M_i\}) \underset{R}{\otimes} W &\cong (\operatorname{coker}(\oplus M_i \longrightarrow \oplus M_i)) \underset{R}{\otimes} W \\
&\cong \operatorname{coker}((\oplus M_i \longrightarrow \oplus M_i) \underset{R}{\otimes} W) \\
&\cong \operatorname{coker}((\oplus M_i) \underset{R}{\otimes} W \longrightarrow (\oplus M_i) \underset{R}{\otimes} W) \\
&\cong \operatorname{coker}(\oplus (M_i \underset{R}{\otimes} W) \longrightarrow \oplus (M_i \underset{R}{\otimes} W)) \\
&\cong \operatorname{colim}\{M_i \underset{R}{\otimes} W\}
\end{aligned}
$$

---

(24) In R-MOD, $\operatorname{Hom}_R(W, -)$ preserves limits

Proof

First note $\operatorname{Hom}_R(W, \prod M_i) \cong \prod \operatorname{Hom}_R(W, M_i)$ equivalent to UMP of products

And $\operatorname{Hom}_R(W, -)$ is left exact, hence preserves kernels

So
$$
\begin{aligned}
\operatorname{Hom}_R(W, \lim\{M_i\}) &\cong \operatorname{Hom}_R(W, \ker(\textstyle\prod M_i \longrightarrow \prod M_i)) \\
&\cong \ker(\operatorname{Hom}_R(W, \textstyle\prod M_i) \longrightarrow \operatorname{Hom}_R(W, \prod M_i)) \\
&\cong \ker(\textstyle\prod \operatorname{Hom}_R(W, M_i) \longrightarrow \prod \operatorname{Hom}_R(W, M_i)) \\
&\cong \lim\{\operatorname{Hom}_R(W, M_i)\}
\end{aligned}
$$

(25) In R-MOD, $Hom_R(-,W)$ converts colimits to limits

Proof

First note that $Hom_R(\oplus M_i, W) \cong \prod Hom_R(M_i, W)$ is equivalent to UMP for coproducts

And $Hom_R(-,W)$ left exact and contravariant, hence converts cokernels to kernels

$$A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0 \text{ exact}$$

$$\Rightarrow 0 \longrightarrow Hom_R(\operatorname{coker} f, W) \longrightarrow Hom_R(B,W) \longrightarrow Hom_R(A,W)$$

exact

$$\Rightarrow Hom_R(\operatorname{coker} f, W) \cong \ker(Hom_R(B,W) \longrightarrow Hom_R(A,W))$$

Then
$$\begin{aligned} Hom_R(\operatorname{colim}\{M_i\}, W) &\cong Hom_R(\operatorname{coker}(\oplus M_i \longrightarrow \oplus M_i), W) \\ &\cong \ker(Hom_R(\oplus M_i, W) \longrightarrow Hom_R(\oplus M_i, W)) \\ &\cong \ker(\prod Hom_R(M_i, W) \longrightarrow \prod Hom_R(M_i, W)) \\ &\cong \lim\{Hom_R(M_i, W)\} \end{aligned}$$

---

(26) Directed limits exact in R-MOD

Proof Let $0 \to \{M_i\} \longrightarrow \{N_i\} \longrightarrow \{L_i\} \longrightarrow 0$ SES directed systems

$\underrightarrow{\lim}(-)$ right exact, so it suffices to show

$$0 \longrightarrow \underrightarrow{\lim} M_i \xrightarrow{\tilde{f}} \underrightarrow{\lim} N_i \text{ inj}$$

Let $m \in \ker \tilde{f} \Rightarrow 0 = \tilde{f}(m)$, $m \in \underrightarrow{\lim} M_i$

Lemma $\Rightarrow m = i_i(m_i)$ for some $m_i \in M_i$

$\Rightarrow 0 = \tilde{f}(m) = \tilde{f}(i_i(m_i)) = i_i(f(m_i))$

since $\underrightarrow{\lim}(-)$ functor:

$$\begin{tikzcd} M_i \arrow[r, "i_i"] \arrow[d, "f"'] \arrow[dr, phantom, "G"] & \underrightarrow{\lim} M_i \arrow[d, "\tilde{f}"] \\ N_i \arrow[r, "i_i"'] & \underrightarrow{\lim} N \end{tikzcd}$$

Lemma $\Rightarrow 0 = \varphi^i_j(f(m_i))$ for some i, j

$= f(\varphi^i_j(m_i))$

But f inj $\Rightarrow \varphi^i_j(m_i) = 0$

Lemma $\Rightarrow 0 = i_i(m_i) = m$

$\therefore \ker \tilde{f} = 0$

$\therefore \underrightarrow{\lim}(-)$ exact

(27) proj resolutions of a module are unique up to homotopy

<u>Proof</u> Let $P. \to M$, $Q. \to M$ proj. resolutions.

$$\begin{array}{ccc} P. & \longrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle 1_M} \\ Q. & \longrightarrow & M \end{array}$$

Comp Thm $\Rightarrow \exists f : P. \to Q.$ chain map

similarly, $\exists g : Q. \to P.$ chain map

$$\begin{array}{ccc} P. & \longrightarrow & M \\ {\scriptstyle 1_{P.}}\downarrow \downarrow {\scriptstyle gf} & & \downarrow {\scriptstyle 1_M} \\ P. & \longrightarrow & M \end{array}$$

Comp thm $\Rightarrow gf \simeq 1_{P.}$

Similarly $fg \simeq 1_{Q.}$

$\therefore P., Q.$ homotopy equivalent

$\therefore$ proj res unique up to homotopy

---

(28) L_nF is well defined

<u>Proof</u> Let $P. \to M$, $Q. \to M$ proj res.

Then they are homotopy equivalent

$\Rightarrow \exists f : P. \to Q.,\ g : Q. \to P.\ \ni\ fg \simeq 1_{Q.},\ gf \simeq 1_{P.}$

Hence $F(fg) \simeq F(1_{Q.}) \Rightarrow F(f)F(g) \simeq 1$

Similarly $F(g)F(f) \simeq 1$

$\Rightarrow H_n(F(f)F(g)) = H_n(1) \Rightarrow H_n(F(f))H_n(F(g)) = 1$

Similarly $H_n(F(g))H_n(F(f)) = 1$

$\therefore H_n(F(g)) = H_n(F(f))^{-1}$

$\therefore\ H_n(F(f)) : H_n(F(P.)) \to H_n(F(Q.))$ iso

$\therefore L_nF$ independent of choice of proj res

$\therefore L_nF$ well defined

(29) LnF functor

Proof Let $M \xrightarrow{f} K$ and let $P. \to M$, $Q. \to K$

$$\begin{tikzcd} P. \arrow[r] \arrow[d, dashed, "\exists \tilde{f}"'] & M \arrow[d, "f"] \\ Q. \arrow[r] & K \end{tikzcd}$$

Comp Thm $\Rightarrow \exists \tilde{f} : P. \to Q.$ chain map

$\Rightarrow F(P.) \xrightarrow{F(\tilde{f})} F(Q.)$ chain map $\Rightarrow$ Get induced map onhomology $H_n(F(P.)) \xrightarrow{H_n(F(\tilde{f}))} H_n(F(Q.))$

$$\begin{array}{ccc} H_n(F(P.)) & \xrightarrow{H_n(F(\tilde{f}))} & H_n(F(Q.)) \\ \| & & \| \\ L_nF(M) & \xrightarrow{L_nF(f)} & L_nF(K) \end{array}$$

$\therefore$ LnF functor

(30) $L_0F = F$

Proof Let $P. \to M$ proj res

$L_0F(M) = H_0(F(P.)) = H_0(\ldots \to F(P_1) \to F(P_0) \to 0)$

$\cong F(P_0) / \mathrm{Im}(F(P_1) \to F(P_0))$

$\cong \mathrm{coker}(F(P_1) \to F(P_0))$

$\cong F(M)$ since $\ldots \to F(P_1) \to F(P_0) \to F(M) \to$ exact since F right exact

$\therefore L_0F \cong F$

(31) For any SES, $\exists$ LES of left derived functor, LnF

Proof

Let $0 \to M' \to M \to M'' \to 0$ SES

Horseshoe lemma $\Rightarrow \exists$ proj resolutions:

$$\begin{tikzcd} 0 \arrow[r] & P.' \arrow[r] \arrow[d] & P. \arrow[r] \arrow[d] & P.'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

comm dgm with split exact top row, exact bottom row

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & F(P.') \arrow[r] \arrow[d] & F(P.) \arrow[r] \arrow[d] & F(P.'') \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F(M') \arrow[r] & F(M) \arrow[r] & F(M'') \arrow[r] & 0 \end{tikzcd}$$

And top row exact since F right exact and sequence split

Then we get LES:

$$\cdots \longrightarrow H_n(F(P.')) \longrightarrow H_n(F(P.)) \longrightarrow H_n(F(P.'')) \longrightarrow H_{n-1}(F(P.')) \longrightarrow$$

which is:

$$\cdots \longrightarrow L_nF(M') \longrightarrow L_nF(M) \longrightarrow L_nF(M'') \longrightarrow L_{n-1}F(M') \longrightarrow \cdots$$

---

32 $K \xrightarrow{u} L \xrightarrow{v} M \xrightarrow{w} K[1]$ dist Δ in $\mathcal{D}(\mathcal{A}) \Rightarrow$

$$\cdots \longrightarrow H^i(M[-1]) \xrightarrow{H(w[-1])} H^i(K) \xrightarrow{H(u)} H^i(L) \xrightarrow{H(v)} H^i(M) \xrightarrow{H(w)} H^i(K[1]) \longrightarrow \cdots$$

<u>Proof</u> since $K \xrightarrow{u} L \xrightarrow{v} M \xrightarrow{w} K[1]$ dist Δ,

it's isomorphic to $K \xrightarrow{u} L \xrightarrow{i} C(u) \xrightarrow{p} K[1]$

Consider SES: $0 \longrightarrow L \xrightarrow{i} C(u) \xrightarrow{p} K[1] \longrightarrow 0$

$\Rightarrow$ LES:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^i(L) \arrow[r, "H^i(i)"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^i(C(u)) \arrow[r, "H^i(p)"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^i(K[1]) = H^{i+1}(K) \arrow[r, "\delta = H^{i+1}(u)"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^{i+1}(L) \arrow[r, "H^{i+1}(i)"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^{i+1}(C(u)) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H^i(L) \arrow[r, "H^i(v)"] & H^i(M) \arrow[r, "H^i(w)"] & H^i(K[1]) \arrow[r, "H^i(u[1])"] & H^{i+1}(L) \arrow[r, "H^{i+1}(v)"] & H^{i+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

Commutes since H^i is a functor

$\therefore$ Bottom row exact

---

33 $\mathcal{T}$ Δ'd cat, $U \in \mathcal{T} \Rightarrow \mathrm{Hom}_{\mathcal{T}}(U,-)$ cohomological functor

<u>Proof</u> Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ dist Δ

Show $\mathrm{Hom}_{\mathcal{T}}(U,X) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{T}}(U,Y) \xrightarrow{v_*} \mathrm{Hom}_{\mathcal{T}}(U,Z)$ exact

Then $0 = vu = v_* u_* \Rightarrow \mathrm{Im}\, u_* \subseteq \ker v_*$

Let $f \in \ker v_* \Rightarrow 0 = v_*(f) = vf$

Now

$$\begin{tikzcd}
U \arrow[r, "1_U"] & U \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d, "0"] & U[1] & \text{dist } \Delta \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] & \text{dist } \Delta
\end{tikzcd}$$

So the square commutes

Rotation:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] & U[1] \arrow[r, "1_{U[1]}"] \arrow[d] & U[1] \arrow[d, "f[1]"] & \text{dist } \Delta \\
Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \arrow[r, "u"] & Y[1] & \text{dist } \Delta
\end{tikzcd}$$

so $\exists h^{[1]}: U[1] \longrightarrow X[1]$, hence $h: U \longrightarrow X$ completing dgm to map of Δ's

so $f = uh$ by comm

$= u_*(h) \in \operatorname{Im} u_*$

$\therefore \ker v_* \subseteq \operatorname{Im} u_*$

$\therefore \operatorname{Im} u_* = \ker v_*$

$\therefore \operatorname{Hom}_{\mathcal{T}}(U,X) \xrightarrow{u_*} \operatorname{Hom}_{\mathcal{T}}(U,Y) \xrightarrow{v_*} \operatorname{Hom}_{\mathcal{T}}(U,Z)$ exact

By rotation, get LES, hence $\operatorname{Hom}_{\mathcal{T}}(U,-)$ cohomological functor

Rotman (732)

~~N.1.7 (c)~~

~~$\mathrm{Obj}(\mathcal{A}) = \{A, B, C, D\}$~~

~~$\mathrm{mor}(\mathcal{A}) = \begin{cases} \mathrm{Hom}_{\mathcal{A}}(A,B) = \{f\} \\ \mathrm{Hom}_{\mathcal{A}}(C,D) = \{g\} \\ \text{Identities} \end{cases}$~~

~~$F : \mathcal{A} \to \text{SETS} \ni F(A) = \{1\}$~~
~~$F(B) = \{2\} = F(C)$~~
~~$F(D) = \{3\}$~~

~~- Show F functor~~
~~- Show Imf not subcategory of SETS~~

2.2

- Give example left R-module $M \ni M = S \oplus T$, $N \leq M \ni$
  $N \neq (N \cap S) \oplus (N \cap T)$

  $(N \cap S) \oplus (N \cap T) \subseteq N$ Always

~~$M = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ $\quad$ $\mathbb{Z}$-modules~~

~~$N = \mathbb{Z}/2\mathbb{Z}$~~

~~$N \cap S = \mathbb{Z}/2\mathbb{Z} \cap \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$~~

~~$N \cap T = \mathbb{Z}/2\mathbb{Z} \cap \mathbb{Z}/3\mathbb{Z}$~~

$R = \mathbb{Z}$

$S = T = \mathbb{Z}$

$M = \mathbb{Z} \oplus \mathbb{Z}$

$N = \{(z, z) \mid z \in \mathbb{Z}\} \leq M$

$N \cap S = 0 = N \cap T$

since $(z,z) \in \mathbb{Z}$ $\overset{= \mathbb{Z} \oplus 0}{}$ iff $z = 0$

So $N \neq (N \cap S) \oplus (N \cap T) = 0 \oplus 0 = 0$

2.5

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{SES left } R\text{-mods}$$

M left R-mod

- Show $\exists$ exact seqs:

$$0 \longrightarrow A \oplus M \xrightarrow{\tilde{f}} B \oplus M \xrightarrow{\tilde{g}} C \longrightarrow 0$$

and

$$0 \longrightarrow A \xrightarrow{\tilde{\tilde{f}}} B \oplus M \xrightarrow{\tilde{\tilde{g}}} C \oplus M \longrightarrow 0$$

$\tilde{f}(a,m) = (f(a), m)$

$\tilde{g}(b,m) = g(b)$

$\tilde{f}(a,m) = 0$ iff $(f(a), m) = (0,0)$ iff $f(a) = 0$ and $m = 0$

iff $a = 0$ and $m = 0$ since f inj

$\therefore \operatorname{Ker} \tilde{f} = (0,0)$

$\therefore \tilde{f}$ inj

Let $c \in C \Rightarrow c = g(b)$, g surj

$= \tilde{g}(b, 0)$

$\therefore \tilde{g}$ surj

Let $(b,m) \in \operatorname{Ker} \tilde{g}$

$\tilde{g}(b,m) = 0 \Rightarrow g(b) = 0 \Rightarrow b \in \operatorname{Ker} g = \operatorname{Im} f \Rightarrow b = f(a)$

$\Rightarrow (b,m) = (f(a), m) = \tilde{f}(a,m) \in \operatorname{Im} \tilde{f}$

$\therefore \operatorname{Ker} \tilde{g} \subseteq \operatorname{Im} \tilde{f}$

Let $(b,m) \in \operatorname{Im} \tilde{f} \Rightarrow (b,m) = \tilde{f}(a,m) = (f(a), m)$

$\tilde{g}(b,m) = \tilde{g}(f(a), m) = g(f(a)) = 0$

$\therefore (b,m) \in \operatorname{Ker} \tilde{g}$

$\therefore \operatorname{Im} \tilde{f} \subseteq \operatorname{Ker} \tilde{g}$

$\therefore$ SES

Let $(b,m) \in \operatorname{Im} \tilde{\tilde{f}}$

$(b,m) = \tilde{\tilde{f}}(a) = (f(a), 0)$

$\tilde{\tilde{g}}(b,m) = \tilde{\tilde{g}}(f(a), 0)$

$= (g(f(a)), 0)$

$= (0,0)$

$\therefore (b,m) \in \operatorname{Ker} \tilde{\tilde{g}}$

$\therefore \operatorname{Im} \tilde{\tilde{f}} \subseteq \operatorname{Ker} \tilde{\tilde{g}}$

Let $(b,m) \in \operatorname{Ker} \tilde{\tilde{g}} \Rightarrow 0 = \tilde{\tilde{g}}(b,m)$

$= (g(b), m) \Rightarrow g(b) = 0$ and $0 = m$

$\Rightarrow b \in \operatorname{Ker} g = \operatorname{Im} f$

$b = f(a)$

$(b,m) = (f(a), 0)$

$= \tilde{\tilde{f}}(a)$

$\in \operatorname{Im} \tilde{\tilde{f}}$

$\therefore \operatorname{Ker} \tilde{\tilde{g}} \subseteq \operatorname{Im} \tilde{\tilde{f}}$

$\therefore =$

$\therefore$ SES

$\tilde{\tilde{f}}(a) = (f(a), 0)$

$\tilde{\tilde{g}}(b,m) = (g(b), m)$

$\tilde{\tilde{f}}(a) = 0$ iff $f(a) = 0$ iff $a = 0$

$\therefore \tilde{\tilde{f}}$ inj

Let $(c,m) \in C \oplus M \Rightarrow (c,m) = (g(b), m)$

$= \tilde{\tilde{g}}(b,m)$

$\therefore \tilde{\tilde{g}}$ surj

2.11

$f: M \to N$ R-map
$K \leq M$ left R-mod
$K \subseteq \ker f$
- Show f induces R-map $\hat{f}: M/K \to N \ni \hat{f}(m+K) = f(m)$

$$\hat{f}(m_1+K+m_2+K) = \hat{f}(m_1+m_2+K) = f(m_1+m_2) = f(m_1)+f(m_2)$$
$$= \hat{f}(m_1+K) + \hat{f}(m_2+K)$$
$$\hat{f}(r(m+K)) = \hat{f}(rm+K) = f(rm) = rf(m) = r\hat{f}(m+K)$$

2.14

$A \xrightarrow{f} B \xrightarrow{g} C$ module maps

- show $gf=0$ iff $\operatorname{Im} f \subseteq \ker g$
- Give example of such sequence not exact

($\Rightarrow$) Assume $gf=0$

Let $x \in \operatorname{Im} f \Rightarrow x = f(y) \Rightarrow g(x) = g(f(y)) = 0$
$\therefore x \in \ker g$
$\therefore \operatorname{Im} f \subseteq \ker g$

($\Leftarrow$) Assume $\operatorname{Im} f \subseteq \ker g$

Let $x \in A$
Then $f(x) \in \operatorname{Im} f \subseteq \ker g$
$\Rightarrow g(f(x)) = 0$
$\therefore gf = 0$

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$\operatorname{Im} 2 = 2\mathbb{Z}$
$\ker 0 = \mathbb{Z}$
But $\mathbb{Z} \notin 2\mathbb{Z}$ hence not exact

2.17 $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ exact

- show $\exists$ SES

$$0 \to \operatorname{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \operatorname{Ker} k \to 0$$

where $\alpha(b+\operatorname{Im} f) = g(b)$ and $\beta(c) = h(c)$

$\alpha(b+\operatorname{Im} f) = 0$ iff $g(b) = 0$ iff $b \in \operatorname{Ker} g = \operatorname{Im} f$ iff $b = f(a)$

iff $b + \operatorname{Im} f = f(a) + \operatorname{Im} f = 0$

$\therefore \alpha$ inj

Let $x \in \operatorname{Ker} k = \operatorname{Im} h \Rightarrow x = h(c) = \beta(c)$

$\therefore \beta$ surj

Let $x \in \operatorname{Im} \alpha \Rightarrow x = \alpha(b+\operatorname{Im} f) = g(b)$

$\Rightarrow \beta(x) = \beta(g(b)) = h(g(b)) = 0$

$\therefore x \in \operatorname{Ker} \beta$

$\therefore \operatorname{Im} \alpha \subseteq \operatorname{Ker} \beta$

Let $x \in \operatorname{Ker} \beta \Rightarrow 0 = \beta(x) = h(x) \Rightarrow x \in \operatorname{Ker} h = \operatorname{Im} g$

$\Rightarrow x = g(b) = \alpha(b+\operatorname{Im} f) \in \operatorname{Im} \alpha$

$\therefore \operatorname{Ker} \beta \subseteq \operatorname{Im} \alpha$

$\therefore \operatorname{Im} g = \operatorname{Ker} \beta$

$\therefore$ SES

~~2.23~~

~~$(S_\alpha)_{\alpha \in I}$ family of submodules~~

~~2.26~~

~~R ring with IBN~~

~~$\exists$ R^∞ free left R-mod having infinite basis~~

~~Show $R \oplus R^\infty \cong$~~

2.28

R int domain

Q field of fractions of R

A R-mod

-show every element in $Q \otimes_R A$ has form $q \otimes a$

Let $\sum (q_i \otimes a_i) \in Q \otimes_R A$

$$\sum (q_i \otimes a_i) = \sum \left(\frac{r_i}{s_i} \otimes a_i \right) = \sum \left(\frac{1}{s_i} \otimes r_i a_i \right)$$

$$= \sum \left(\frac{s_1 \ldots s_{i-1} s_{i+1} \ldots s_n}{s_1 \ldots s_n} \otimes r_i a_i \right)$$

$$= \sum \left(\frac{1}{s} \otimes s_1 \ldots s_{i-1} s_{i+1} \ldots s_n r_i a_i \right)$$

$$= \frac{1}{s} \otimes \sum s_1 \ldots s_{i-1} s_{i+1} \ldots s_n r_i a_i$$

$$= \frac{1}{s} \otimes a$$

2.34

-Give example of comm diagram with exact rows + vertical maps isos

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[r, "\text{not surj}"] \arrow[d] & A_3 \arrow[r] & A_4 \arrow[r] \arrow[d] & A_5 \arrow[d] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r, "\text{not inj}"'] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

$\exists\!\!\!/\ A_3 \to B_3$ commuting diagm

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "0"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z}
\end{tikzcd}$$

oops not isos

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] \arrow[d, "2"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "1"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "1"] \\
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] \arrow[d, "2"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "1"] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "1"] \\
2\mathbb{Z} \arrow[r, "0"] & 2\mathbb{Z} \arrow[r, hook, "\iota"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] \arrow[d, "1"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "1"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "1"] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "1"] \\
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

2.3.7

R ring with IBN ie If $R^m \cong R^n$ as left R-modules, then $n=n$

$R^m \cong R^n$ as right R-modules

- Show $m=n$

$R^m \cong R^n$ as right R-modules

$\Rightarrow \mathrm{Hom}_R(R^m, R) \cong \mathrm{Hom}_R(R^n, R)$ as left R-modules since ${}_R R_R$

$$\begin{array}{ccc} \mathrm{Hom}_R(R^m, R) & \cong & \mathrm{Hom}_R(R^n, R) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ R^m & \cong & R^n \end{array}$$

$\Rightarrow R^m \cong R^n$ as left R-modules

$\therefore m=n$ since R IBN

3.2 R ring

$0 \neq S \leq F$

F free right R-mod

$a \in R$ not a right zero divisor

- show $Sa \neq \{0\}$

If $Sa = 0$, then $sa = 0 \ \forall s \in S$

And $S \neq 0$, so $\exists\, 0 \neq s \in S \ni sa = 0$

But F free so $F \cong \bigoplus R$

$S \leq F \Rightarrow s \in \bigoplus R \Rightarrow s = \bigoplus r_i$, $r_j \neq 0$ for some j

$\Rightarrow 0 = sa = \bigoplus r_i a \Rightarrow r_j a = 0$

$\Rightarrow a$ right zero divisor

Contra

$\therefore Sa \neq 0$

3.5

- Show every proj. left R-mod P has free complement
  ie $\exists F$ free left R-mod $\ni P \oplus F$ free

P proj $\Rightarrow$ P direct summand of free module F
$\Rightarrow F \cong P \oplus X$ free (hence X proj)

Consider
$$X \oplus P \oplus X \oplus P \oplus \cdots \cong \underbrace{X \oplus F \oplus F \oplus \cdots}_{\text{show free}}$$

Then
$$P \oplus \underbrace{\qquad\qquad} \cong F \oplus F \oplus \cdots \quad \text{free}$$

$\bigoplus_{i=1}^{\infty} F$ free since F free

$$P \oplus \bigoplus_{i=1}^{\infty} F \cong P \oplus \bigoplus_{i=1}^{\infty} (X \oplus P) \cong P \oplus (X \oplus P) \oplus (X \oplus P) \oplus \cdots$$
$$\cong (P \oplus X) \oplus (P \oplus X) \oplus \cdots$$
$$\cong F \oplus F \oplus \cdots$$
$$\cong \bigoplus_{i=1}^{\infty} F$$
free

$\therefore$ P has free complement $\bigoplus_{i=1}^{\infty} F$

3.8 $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$

- Show R left noetherian
- Show R is not right noetherian

$I = \begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$

$\begin{pmatrix} z & 0 \\ q & q' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ q'r & 0 \end{pmatrix} \in I \Rightarrow \left\{\begin{matrix}\text{left submodules} \\ \text{of } {}_R I\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{left submodules} \\ \text{of } {}_{\mathbb{Q}}\mathbb{Q}\end{matrix}\right\}$

$\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ q & q' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ rz & 0 \end{pmatrix} \in I \Rightarrow \left\{\begin{matrix}\text{right submodules} \\ \text{of } I_R\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{right submodules} \\ \text{of } \mathbb{Q}_{\mathbb{Z}}\end{matrix}\right\}$

$\therefore I \triangleleft R$

$R/I \cong \mathbb{Z} \times \mathbb{Q}$ as rings

SES: $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$

I: left noeth since ${}_{\mathbb{Q}}\mathbb{Q}$ left noeth since field

R/I: left noeth since $\mathbb{Z}, \mathbb{Q}$ both left noeth ($\mathbb{Z}$ noeth since PID)

$\therefore R$ left noetherian

$\mathbb{Q}_{\mathbb{Z}}$ not finitely generated, so I_R not finitely generated

$\therefore$ R not right noetherian

(Hence not right artinian)

Suppose R left artinian

Then R/I left artinian by SES

so $\mathbb{Z} \times \mathbb{Q}$ left artinian $\Rightarrow$ $\mathbb{Z}$ left artinian

Contra since $(2) \supsetneq (4) \supsetneq \cdots$

3.11 $0 \neq P$ projective left R-mod

- show $\mathrm{Hom}_R(P,R) \neq \{0\}$

Suppose $\mathrm{Hom}_R(P,R) = 0$

SES $0 \to I \xhookrightarrow{i} R \xrightarrow{\pi} R/I \to 0$ $\forall I \triangleleft R$

P proj $\Rightarrow$ $\mathrm{Hom}_R(P,-)$ exact

$$\Rightarrow 0 \to \mathrm{Hom}_R(P,I) \to \underset{\displaystyle 0}{\underset{\parallel}{\mathrm{Hom}_R(P,R)}} \to \mathrm{Hom}_R(P,R/I) \to$$

exact

$\Rightarrow \mathrm{Hom}_R(P,I) = \mathrm{Hom}_R(P,R/I) = 0$ $\forall I$

P proj $\Rightarrow$ $\exists F$ free $\ni$ $F \cong P \oplus X$ for some X

And $F = \oplus R$

SES: $0 \to P \to P \oplus X \to X \to 0$

$$\Rightarrow 0 \to \mathrm{Hom}_R(P,P) \to \mathrm{Hom}_R(P, P\oplus X) \to \mathrm{Hom}_R(P,X) \to 0$$

exact

$$\begin{aligned} \mathrm{Hom}_R(P, P\oplus X) &\cong \mathrm{Hom}_R(P, \oplus R) \\ &\cong \oplus \mathrm{Hom}_R(P,R) \\ &= \oplus 0 \\ &= 0 \end{aligned}$$

$\therefore$ $0 \to \mathrm{Hom}_R(P,P) \to 0$ exact

$\therefore$ $\mathrm{Hom}_R(P,P) = 0$

$\therefore$ $P = 0$ because if not $0 \neq 1_P \in \mathrm{Hom}_R(P,P)$

contra since $P \neq 0$

$\therefore$ $\mathrm{Hom}_R(P,R) \neq 0$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d] & R \\ & M & \end{tikzcd}$$

$\forall 0 \neq r \in R \quad rM = M$

Show R inj. R-mod

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \\ & R & \end{tikzcd}$$

$0 \longrightarrow I \subset R \longrightarrow R/I \longrightarrow 0$ SES

$\Rightarrow 0 \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Hom}_R(I, M) \longrightarrow 0$

$\mathrm{Hom}_R(R,M) \cong M \neq 0$

exact since M inj

If $\mathrm{Hom}_R(I,M) = 0$ then

$\mathrm{Hom}_R(R,M) \longrightarrow \mathrm{Hom}_R(I,M)$ is 0-map

And its surj

Show $\mathrm{Hom}_R(R/I, M) = \mathrm{Hom}_R(I,M) = 0$

Suppose $0 \neq M \cong \mathrm{Hom}_R(R,M)$

And also $\mathrm{Hom}_R(M,R) \neq 0$ by front

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[dd, "f"'] & R \arrow[dl, "g \neq 0"] \\ & M = rM \arrow[d] & \\ & R & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "gi"'] & R \arrow[dl, dashed, "\exists \varphi"] \\ & M & \end{tikzcd}$$

$\varphi|_I = gi$

3.18

(i) R int dom
R injective R-mod
-show R field

Let $0 \neq r \in R$

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "f"] \arrow[d, "1_R"'] & R \arrow[dl, dashed, "\exists g"] \\ & R & \end{tikzcd}$$

$s \in \ker f$ iff $0 = f(s) = sr$
But R int dom $\Rightarrow$ $s = 0$
$\therefore \ker f = 0$
$\therefore$ f inj

Define $f: R \longrightarrow R \ni f(s) = sr$
Then since R inj, $\exists g: R \longrightarrow R \ni gf = 1_R$
So $1 = g(f(1)) = g(r) = rg(1)$
So $1 = rg(1) = g(1)r$ since R int dom hence comm
$\therefore g(1) = r^{-1}$
$\therefore$ R field

(ii) -show $\mathbb{Z}/6\mathbb{Z}$ injective + projective module over itself

Note $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
And $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ both simple $\mathbb{Z}/6\mathbb{Z}$-modules
$\therefore$ $\mathbb{Z}/6\mathbb{Z}$ semisimple $\mathbb{Z}/6\mathbb{Z}$-module
$\therefore$ $\mathbb{Z}/6\mathbb{Z}$ semisimple ring
$\therefore$ Every $\mathbb{Z}/6\mathbb{Z}$-module both proj + inj
In particular $\mathbb{Z}/6\mathbb{Z}$ both proj + inj $\mathbb{Z}/6\mathbb{Z}$-module

(iii) R int dom, not field
M R-mod both inj + proj
-show $M = 0$

Suppose $M \neq 0$
Then by 3.11, since M proj, $Hom_R(M, R) \neq 0$
Then by 2.22 (iii) (contrapositive), since R int dom, $\exists 0 \neq I \trianglelefteq R \ni Hom_R(M, R/I)$
$0 \to Hom_R(M, I) \to Hom_R(M, R) \to Hom$ $\Leftarrow$ $0 \to I \subset R \to R/I \to 0$ SES

M inj $\Rightarrow$ $\forall\, 0 \neq r \in R$ $\quad 0 \longrightarrow R \xrightarrow{r} R$ exact

$$\mathrm{Hom}_R(R,M) \longrightarrow \mathrm{Hom}_R(R,M) \longrightarrow 0$$

$\therefore$ $M \xrightarrow{r} M \longrightarrow 0$ exact

$\therefore$ $M = rM$ $\quad \forall r \neq 0$, $r \in R$

Suppose $\exists \varphi: M \longrightarrow R$ nonzero

Then $\exists z \in M \ni \varphi(z) \neq 0$

set $r = \varphi(z)$

$rM = M \Rightarrow \exists z' \in M \ni z = rz'$

$r = \varphi(z) = \varphi(rz') = r\varphi(z')$

$r(1 - \varphi(z')) = 0$

$\Rightarrow 1 - \varphi(z') = 0$ since R int dom

$\Rightarrow \varphi(z') = 1$ for some $z' \in M$

$\therefore$ φ surj

$\therefore$ Every nonzero homom $\varphi: M \longrightarrow R$ is surj

M proj $\Rightarrow$ $\exists F \cong \bigoplus R \ni$ M direct summand

$M \neq 0 \Rightarrow$

$$M \hookrightarrow F \xrightarrow{\pi} R \xrightarrow{\forall r \neq 0} R$$

nonzero homom $M \longrightarrow R$

$\therefore$ surj

$1 = r\pi(m)$, $m \in M$

$\therefore$ R field

contra

$\therefore M = 0$

Suppose $\exists \psi: M \to M$ nonzero

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\forall \varphi} & R \\ & & \uparrow \psi & \nearrow \exists \alpha & \\ & & M & & \end{array}$$

3.21

(i) $M \subseteq E$ left R-mods

- show $M \overset{ess}{\subseteq} E$ iff $\forall\ 0 \neq e \in E$, $\exists\, r \in R$ with $0 \neq re \in M$

($\Rightarrow$) Assume $M \overset{ess}{\subseteq} E$

Let $0 \neq e \in E$

Then $0 \neq (e) \leq E$

So since $M \overset{ess}{\subseteq} E$, $(e) \cap M \neq (0)$

so $\exists\ 0 \neq x \in (e) \cap M \Rightarrow x = re$ for some $r \in R$

$\therefore\ 0 \neq re \in M$ for some $r \in R$

($\Leftarrow$) $\forall\, 0 \neq e \in E$, $\exists\, r \in R$ with $0 \neq re \in M$

Let $0 \neq L \leq E$

Then $\exists\ 0 \neq e \in L \leq E$

so $\exists\, r \in R \ni 0 \neq re \in M$

But also $re \in L$

$\therefore\ 0 \neq re \in L \cap M$

$\therefore\ L \cap M \neq 0$

$\therefore\ M \overset{ess}{\subseteq} E$

(ii) $M \subseteq E$ left R-mods

$M \subseteq S \subseteq E$ $\forall\, S \in \mathcal{S}$ and if $S, S' \in \mathcal{S}$, then either $S \subseteq S'$ or $S' \subseteq S$

each $S \in \mathcal{S} \ni M \overset{ess}{\subseteq} S$

- show $M \overset{ess}{\subseteq} \bigcup_{S \in \mathcal{S}} S$

Let $0 \neq x \in \bigcup_{S \in \mathcal{S}} S \Rightarrow x \in S$ for some $S \in \mathcal{S}$

And $M \overset{ess}{\subseteq} S$, so $\forall\, 0 \neq y \in S$, $\exists\, r \in R$ with $0 \neq ry \in M$

But $0 \neq x \in S$, so $\exists\, r \in R$ with $0 \neq rx \in M$

$\therefore\ M \overset{ess}{\subseteq} \bigcup_{S \in \mathcal{S}} S$

Name: ____

#3.5 Use the BMI indices of females in the FBODY worksheet to determine the values below:

Mean: _______ Median: _______

St. Dev.: _______ Range: _______

Min: _______ Q_1: _______ Q_2: _______ Q_3: _______ Max: _______

Outliers: ______________________

#3.6 Use the BMI indices of males in the MBODY worksheet to determine the values below:

Mean: _______ Median: _______

St. Dev.: _______ Range: _______

Min: _______ Q_1: _______ Q_2: _______ Q_3: _______ Max: _______

Outliers: ______________________

#3.7 Use the BMI indices for both men and women in the FBODY and MBODY worksheets to construct their boxplots in the same image (you need not draw, nor print the boxplots). Do the boxplots suggest any notable differences in the two sets of sample data? (*Hint*: you may want to construct both boxplots on the same graph).

Example: Binomial Distribution

85% of adults know what Twitter is. If 5 adults are randomly selected, what is the probability that 3 of them know what Twitter is?

- Is this a binomial distribution?

3.34

$_R B$

C left R-mod

$v : \mathrm{Hom}_R(B,R) \otimes_R C \longrightarrow \mathrm{Hom}_R(B,C)$

$v(f \otimes c) \longmapsto \hat{f}$

where $\hat{f}(b) = f(b)c \quad \forall b \in B, c \in C$

(i) - show v natural in B

Let $g : B \longrightarrow B'$

$$\begin{tikzcd}
\mathrm{Hom}_R(B,R)\otimes_R C \arrow[r, "v_B"] & \mathrm{Hom}_R(B,C) \\
\mathrm{Hom}_R(B',R)\otimes_R C \arrow[r, "v_{B'}"] \arrow[u, "g^*\otimes 1"] & \mathrm{Hom}_R(B',C) \arrow[u, "g^*"']
\end{tikzcd}$$

Let $h \otimes c \in \mathrm{Hom}_R(B',R) \otimes_R C$

$$\begin{aligned}
v_B(g^* \otimes 1)(h \otimes c)(b) &= v_B(g^*(h) \otimes c)(b) \\
&= v_B(hg \otimes c)(b) \\
&= \widehat{hg}(b) \\
&= (hg)(b)c = h(g(b))c \checkmark
\end{aligned}$$

$$\begin{aligned}
g^*(v_{B'}(h \otimes c))(b) &= v_{B'}(h \otimes c)\, g(b) \\
&= \hat{h}(g(b)) \\
&= h(g(b))c \checkmark
\end{aligned}$$

$\therefore$ v is natural in B

Probability Distributions

Chapter 5

Exploring Probability Distributions

- Definitions
  - A ***random variable*** is a variable, x, that has a single numerical value, determined by chance, for each outcome of a procedure.
  - A ***probability distribution*** is a description that gives the probability for each value of the random variable.
    - A probability distribution is expressed as a table, formula, or a graph.

Discrete vs. Continuous

- ***Discrete Random Variable-*** a random variable whose collection of possible values is finite or countable
- ***Continuous Random Variable-*** a random variable whose collection of possible values is infinite and not countable

Exploring Probability Distributions

- Requirements of Probability Distributions
  1. The variable, x, is a numerical random variable and its values are associated with probabilities
  2. The sum of the probabilities is 1
  3. Each value of P(x) is between 0 and 1

Binomial Distributions

- A ***binomial distribution*** is a probability distribution such that:
  1. The experiment has a fixed number of trials
  2. The trials are independent
  3. All outcomes of each trial fall into two categories: success or failure
  4. The probabilities are constant for each trial

Binomial Distributions

- Notation
  - S: success
  - F: failure
  - P(S)=p
  - P(F)=q
  - n: number of trials
  - x: number of successes

Example: Binomial Distribution

85% of adults know what Twitter is. If 5 adults are randomly selected, what is the probability that 3 of them know what Twitter is?

- **Is this a binomial distribution?**

3.24

R integral domain

– show $S^{-1}R = E(R)$ ← injective envelope

$S^{-1}R$ ← field of fractions

Show $R \overset{ess}{\subseteq} S^{-1}R$ and $S^{-1}R$ injective

Let $0 \neq L \leq Q$

~~Then $L = \{\frac{m}{s} \mid m \in M, s \in S\}$ where M is an R-module containing S~~

show $L \cap R \neq 0$

$L \neq 0 \Rightarrow$ ~~$\exists \frac{m}{1} \in L \ni \frac{m}{1} \neq \frac{0}{1}$~~

$0 \neq \frac{a}{b} \in L \Rightarrow a \neq 0$

Then $\underset{R}{a} = b \cdot \frac{a}{b} \in L$ since L R-submodule

$\therefore 0 \neq a \in R \cap L$

$\therefore R \cap L \neq 0$

$\therefore R \overset{ess}{\subseteq} Q$

Let $Q = \mathrm{frac}(R)$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "g"] \\ & Q & \end{tikzcd}$$

Let $a, b \in I$, then $af(b) = f(ab) = bf(a)$

$af(b) - bf(a) = 0$

$\Rightarrow \frac{f(a)}{a} = \frac{f(b)}{b}$ in Q

|| Let c

Define $g: R \to Q \ni g(r) = rc$

Let $a \in I$

$g(a) = ac = a\frac{f(a)}{a} = f(a)$

$\therefore g|_I = f$

Baer $\Rightarrow$ Q inj

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6/1/2015

3.34

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- Is this a binomial distribution?

(cc) B f.g., free

– show ψ iso

B f.g. free $\Rightarrow B \cong R^n$ for some n

$$\mathrm{Hom}_R(B,R)\otimes_R C \cong \mathrm{Hom}_R(R^n,R)\otimes_R C \cong R^n \otimes_R C \cong C^n \cong \mathrm{Hom}_R(R^n, C) \cong \mathrm{Hom}_R(B, C)$$

$B \cong R^n$ via $b = r_1e_1+\dots+r_ne_n \longmapsto r_1+\dots+r_n$

$\cong$ via $f\otimes c \longmapsto \hat{f}\otimes c$ where $\hat{f}(r_1,\dots,r_n) = f(r_1e_1+\dots+r_ne_n)$, $\hat{f} = \hat{f}_1+\dots+\hat{f}_n$

$\cong$ via $\hat{f}\otimes c \longmapsto (\hat{f}_1(1)+\dots+\hat{f}_n(1))\otimes c$

via $(\hat{f}_1(1)+\dots+\hat{f}_n(1))\otimes c \longmapsto c+\dots+c \longmapsto \psi,\ \psi(r_1,\dots,r_n) =$

(ccc) B finitely presented left R-mod

C flat left R-mod

– show ψ iso

B finitely presented $\Rightarrow \exists\ R^m \longrightarrow R^n \longrightarrow B \longrightarrow 0$ exact

$\mathrm{Hom}_R(-,C)$ left exact $\Rightarrow$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(B,C) \arrow[r] \arrow[ddd, dashed, "\exists\ \text{iso by 5-Lemma}"'] & \mathrm{Hom}_R(R^n,C) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_R(R^m,C) \arrow[d, "\cong"'] & \text{exact} \\
 & & C^n \arrow[d, "\cong"'] \arrow[r, phantom, "\circlearrowleft"] & C^m \arrow[d, "\cong"'] & \\
 & & R^n \otimes_R C \arrow[d, "\cong"'] & R^m \otimes_R C \arrow[d, "\cong"'] & \\
\text{exact}\quad 0 \arrow[r] & \mathrm{Hom}_R(B,R)\otimes_R C \arrow[r] & \mathrm{Hom}_R(R^n,R)\otimes_R C \arrow[r] & \mathrm{Hom}_R(R^m,R)\otimes_R C &
\end{tikzcd}$$

(or)

since $\mathrm{Hom}_R(-,R)$ left exact

And $-\otimes_R C$ exact since C flat

Probability Distributions

Chapter 5

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  - P(S)=p
  - P(F)=q
  - n: number of trials
  - x: number of successes

5.8

- Give ex. of cov. functor that doesn't preserve coproducts

$\mathrm{Hom}_R(X, -)$

infinite direct sum!

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$

$2 \cdot \bar{1} = \bar{0}$ in $\mathbb{Z}/2\mathbb{Z} \Rightarrow 2 \cdot \varphi(\bar{1}) = 0$ in $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \quad (\bar{a}, \bar{b})$

$\Downarrow$

$2(\bar{a}, \bar{b}) = (\bar{0}, \bar{0})$

$\Rightarrow 2 \mid 2\bar{a}$ and $2 \mid 2\bar{b}$

$\bar{1} \mapsto (\bar{0}, \bar{0})$
$\bar{1} \mapsto (\bar{0}, \bar{1})$
$\bar{1} \mapsto (\bar{1}, \bar{0})$
$\bar{1} \mapsto (\bar{1}, \bar{1})$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$

$\bar{1} \mapsto \bar{0}$ $\qquad$ $\bar{1} \mapsto \bar{0}$
$\bar{1} \mapsto \bar{1}$ $\qquad$ $\bar{1} \mapsto \bar{1}$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$

$2 \cdot \bar{1} = \bar{0}$ in $\mathbb{Z}/2\mathbb{Z} \Rightarrow 2 \cdot \varphi(\bar{1}) = \bar{0}$ in $\mathbb{Z}/6\mathbb{Z}$

$\bar{1} \mapsto \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$

$\Rightarrow 6 \mid 2\varphi(\bar{1})$

$\Rightarrow 3 \mid \varphi(\bar{1})$

$\Rightarrow \bar{1} \mapsto \bar{0}, \bar{3}$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \bigoplus^{\infty} \mathbb{Z}) = 0$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$

$\bar{1} \mapsto \bar{0}, \bar{1}$ $\qquad$ $3 \mid 2\varphi(1)$

$6 \cdot \bar{1} = \bar{0} \Rightarrow 6\varphi(\bar{1}) = \bar{0}$ $\qquad$ $\bar{1} \mapsto \bar{0}, \bar{1}$

$\Rightarrow 2 \mid 6\varphi(\bar{1})$

$\Rightarrow 1 \mid 3\varphi(\bar{1})$

5.24

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$ pullback in AB

$\exists c \in C, b \in B \ni g(c) = f(b)$

— Show $\exists d \in D \ni c = \alpha(d)$ and $b = \beta(d)$

$$D = \varinjlim \left(\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \right) = \{(c,b) \in C \times B \mid g(c) = f(b)\}$$

$AB = \mathbb{Z}\text{-MOD}$

$\exists c \in C, b \in B \ni g(c) = f(b) \Rightarrow (c,b) \in D$

Then $\alpha(c,b) = c$ and $\beta(c,b) = b$

so $\exists d = (c,b) \ni \alpha(d) = c,\ \beta(d) = b$

Define $p: \mathbb{Z} \to C \ni p(n) = nc$

$q: \mathbb{Z} \to B \ni q(n) = nb$

$$\begin{tikzcd} & & C \arrow[d, "g"] \\ \mathbb{Z} \arrow[r, dashed, "\exists!\theta"] \arrow[urr, bend left, "p"] \arrow[drr, bend right, "q"'] & D \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & A \\ & & B \arrow[u, "f"'] \end{tikzcd}$$

$$\begin{aligned} g(p(n)) &= g(nc) \\ &= n g(c) \\ &= n f(b) \\ &= f(nb) \\ &= f(q(n)) \quad \forall n \end{aligned}$$

$\leftarrow$ $\mathbb{Z}$-mod homom

$\therefore gp = fq$

since $D = \varinjlim \left(\begin{tikzcd} & C \arrow[d] \\ B \arrow[r] & A \end{tikzcd} \right)$, $\exists! \theta: \mathbb{Z} \to D \ni \alpha\theta = p$, $\beta\theta = q$

Define $d = \theta(1)$

$\alpha(d) = \alpha(\theta(1)) = p(1) = c$ ✓

$\beta(d) = \beta(\theta(1)) = q(1) = b$ ✓

5.52

$\mathcal{C}$ additive cat.

$C \in obj(\mathcal{C})$

- show $Hom(C,C)$ is a ring with comp. as product.

$Hom(C,C)$ is abelian group under addition since $\mathcal{C}$ additive

$f, g \in Hom(C,C) \implies fg : C \longrightarrow C \therefore fg \in Hom(C,C)$

mult associative since compositions are associative since $\mathcal{C}$ category

$1_C \in Hom(C,C) \ni 1_C f = f = f 1_C \quad \forall f \in Hom(C,C)$ since $\mathcal{C}$ category

Distributive property holds since $\mathcal{C}$ additive

5.64

(i) T cat of torsion abelian groups

- show T abelian cat. having no nonzero projective objects

Note T full subcategory of AB which is an abelian cat.

Let $A, B \in ob(T)$ and let $f : A \longrightarrow B$

zero obj in AB $\rightarrow$ $0 \in ob(T)$ since 0 is torsion because any $0 \neq z \in \mathbb{Z}$ $z \cdot 0 = 0$

Let $(a,b) \in A \oplus B \implies \exists z, z' \in \mathbb{Z} \ni za = 0,\ z'b = 0$ since $A, B \in T$

Then $zz'(a,b) = (z'za, zz'b) = (0,0)$

$\therefore A \oplus B$ is torsion ab. gp

$\therefore A \oplus B \in ob(T)$

$\ker f \leq A \implies \ker f$ is also torsion

$\therefore \ker f \in ob(T)$

similarly $\operatorname{coker} f \leq B$

$\therefore \operatorname{coker} f \in ob(T)$

$z(b + Im f) = zb + Im f = 0 + Im f = 0$

$\therefore T$ abelian category

suppose $\exists\, 0 \neq P \in ob(T)$ projective

Then P is a projective $\mathbb{Z}$-module

so P is a direct summand of a free $\mathbb{Z}$-module, F

so $P \leq F$

Let $0 \neq x \in F \ni zx = 0$

$x = \sum z_i x_i$

$0 = z(\sum z_i x_i)$

$= \sum z z_i x_i$

x_i lin ind $\implies z z_i = 0 \ \forall i$

in part $z z_j = 0$, $\exists z_j \neq 0$

$\mathbb{Z}$ int dom $\implies z = 0$

$\therefore x$ t.f.

$\therefore F$ t.f. contra $P \leq F$ $\therefore \nexists$ nonzero proj.

6.2

$C_\cdot \cong D_\cdot$

– show $H_n(C_\cdot) \cong H_n(D_\cdot)\ \forall n \in \mathbb{Z}$

$C_\cdot \cong D_\cdot \Rightarrow \exists f: C_\cdot \to D_\cdot$ chain map:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "G"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "G"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\ni$ each f_n isomorphism

$$\begin{array}{ccc} & & \ker\partial_n/\operatorname{Im}\partial_{n+1} \\ & & \| \\ \text{Get induced map} & H_n(C_\cdot) \xrightarrow{H_n(f)} & H_n(D_\cdot) \\ & \| & \\ & \ker d_n/\operatorname{Im} d_{n+1} & \end{array}$$

$\ni H_n(f)(z+\operatorname{Im} d_{n+1}) = f_n(z)+\operatorname{Im}\partial_{n+1}$

Let $z+\operatorname{Im} d_{n+1} \in \ker H_n(f) \Rightarrow 0 = H_n(f)(z+\operatorname{Im} d_{n+1})$

$= f_n(z)+\operatorname{Im}\partial_{n+1}$

$\Rightarrow f_n(z) \in \operatorname{Im}\partial_{n+1}$

$\Rightarrow f_n(z) = \partial_{n+1}(d_{n+1})$

$= \partial_{n+1}(f_{n+1}(c_{n+1}))$, f_{n+1} surj

$= f_n(d_{n+1}(c_{n+1}))$

f_n inj $\Rightarrow z = d_{n+1}(c_{n+1}) \in \operatorname{Im} d_{n+1}$

$\therefore z+\operatorname{Im} d_{n+1} = \operatorname{Im} d_{n+1} = 0_{H_n(C_\cdot)}$

$\therefore H_n(f)$ injective

Let $z+\operatorname{Im}\partial_{n+1} \in H_n(D_\cdot)$

So $z \in \ker\partial_n \subseteq D_n$

f_n surj $\Rightarrow z = f_n(c_n)$

$\therefore z+\operatorname{Im}\partial_{n+1} = f_n(c_n)+\operatorname{Im}\partial_{n+1} = H_n(f)(c_n+\operatorname{Im} d_{n+1})$

$$\left\{\begin{array}{l} \text{show } c_n \in \ker d_n \\ z \in \ker\partial_n \Rightarrow 0 = \partial_n(z) = \partial_n(f_n(c_n)) = f_{n-1}(d_n(c_n)) \\ \qquad \Rightarrow d_n(c_n) \in \ker f_{n-1} = 0 \\ \qquad \Rightarrow c_n \in \ker d_n \end{array}\right\}$$

$\therefore H_n(f)$ surj $\quad \therefore H_n(f)$ iso $\quad \therefore H_n(C_\cdot) \cong H_n(D_\cdot)\ \forall n$

6.14

Comm dgm with exact row:

$$\begin{tikzcd} B' \arrow[r, "j"] \arrow[dr, "i"'] & C \arrow[r, "q"] \arrow[d, shift left, "\ell"] & B'' \\ & B \arrow[u, shift left, "k"] \arrow[ur, "p"'] & \end{tikzcd}$$

k iso with inverse ℓ

– show $B' \xrightarrow{i} B \xrightarrow{p} B''$ exact

Let $x \in \operatorname{Im} i \Rightarrow x = i(b')$

Then $p(x) = p(i(b')) = p(\ell(j(b'))) \Rightarrow p(x - \ell(j(b'))) = 0$

$\therefore x - \ell(j(b')) \in \ker p$

$\quad \parallel$

$x - i(b')$

$p(x) = p(i(b')) = q(k(i(b')))$

ℓ surj $\Rightarrow x = i(b') = \ell(c)$

$\Rightarrow p(x) = p(\ell(c)) = q(c)$

Let $x \in \ker p \Rightarrow 0 = p(x) = p(\ell(c)) = q(c)$

$\Rightarrow c \in \ker q = \operatorname{Im} j$

$\Rightarrow c = j(b')$

$\Rightarrow \cancel{0 = p(x) = q(j(b')) = p(i(b'))}$

$x = \ell(c) = \ell(j(b')) = i(b') \in \operatorname{Im} i$

$\therefore \ker p \subseteq \operatorname{Im} i$

Let $x \in \operatorname{Im} i \Rightarrow x = i(b')$

$\Rightarrow p(x) = p(i(b')) = q(j(b')) = 0$

$\therefore x \in \ker p$

$\therefore \operatorname{Im} i \subseteq \ker p \qquad \therefore \operatorname{Im} i = \ker p \qquad \therefore$ exact

6.17

R semisimple ring

(i) - Show $\forall n \geq 1$, $\mathrm{Tor}_n^R(A,B) = \{0\}$ $\forall$ right R-mods A $\forall$ left R-mods B

R semisimple ring $\Rightarrow$ every R-module is both proj and inj

$$P_\cdot : \quad \underset{(1)}{0} \longrightarrow \underset{(0)}{B} \xrightarrow{1_B} B \longrightarrow 0$$

proj res since B proj.

$$A \otimes_R P_\cdot : \quad \underset{(1)}{0} \longrightarrow \underset{(0)}{A \otimes_R B} \longrightarrow 0$$

$\mathrm{Tor}_0^R(A,B) = H_0(A \otimes_R P_\cdot) = A \otimes_R B / 0 \cong A \otimes_R B$

$\mathrm{Tor}_n^R(A,B) = H_n(A \otimes_R P_\cdot) = 0 \quad \forall n \geq 1$

(ii) - show $\forall n \geq 1$, $\mathrm{Ext}_R^n(A,B) = \{0\}$ $\forall$ left R-mods A, B

$$P_\cdot : \quad \underset{(1)}{0} \longrightarrow \underset{(0)}{A} \xrightarrow{1_A} A \longrightarrow 0$$

proj res since A proj

$$\mathrm{Hom}_R(P_\cdot, B) : \quad 0 \longrightarrow \underset{(0)}{\mathrm{Hom}_R(A,B)} \longrightarrow \underset{(1)}{0}$$

$\mathrm{Ext}_R^0(A,B) = H^0(\mathrm{Hom}_R(P_\cdot, B)) = \mathrm{Hom}_R(A,B)/0 \cong \mathrm{Hom}_R(A,B)$

$\mathrm{Ext}_R^n(A,B) = H^n(\mathrm{Hom}_R(P_\cdot, B)) = 0 \quad \forall n \geq 1$

6.20

R int dom

Q field of fractions of R

(c) $0 \neq r \in R$

· A R-mod $\ni rA = 0$

– show $\mathrm{Ext}^n_R(Q,A) = 0 = \mathrm{Tor}^R_n(Q,A)$ $\forall n \geq 0$

$S^{-1}R = Q$ flat $\Rightarrow \mathrm{Tor}^R_n(Q,A) = 0$ $\forall n \geq 1$

Let $0 \to A \to B \to C \to 0$ SES

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & S^{-1}A \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "G^{nat}"] & S^{-1}B \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "G^{nat}"] & S^{-1}C \arrow[r] \arrow[d, "\wr"] & 0 \quad \text{SES} \\ 0 \arrow[r] & \underset{Q}{\overset{\|}{S^{-1}R}} \otimes_R A \arrow[r] & \underset{Q}{\overset{\|}{S^{-1}R}} \otimes_R B \arrow[r] & \underset{Q}{\overset{\|}{S^{-1}R}} \otimes_R C \arrow[r] & 0 \end{tikzcd}$$

$\therefore$ Bottom row SES

$\therefore$ $S^{-1}R = Q$ flat

Now show $\mathrm{Tor}^R_0(Q,A) = 0$

$\mathrm{Tor}^R_0(Q,A) = Q \otimes_R A$

since $r \neq 0$

Let $q \otimes a \in Q \otimes_R A \Rightarrow \overset{q \otimes a =}{\frac{t}{s}} \otimes a = \frac{tr}{sr} \otimes a = \frac{t}{sr} \otimes ra = \frac{t}{sr} \otimes 0 = 0$

$\therefore Q \otimes_R A = 0$ since every simple tensor is 0, hence every element is 0

$\therefore \mathrm{Tor}^R_n(Q,A) = 0$ $\forall n \geq 0$

$0 \to A \xrightarrow{r} A \to A/rA \to 0$ SES

LES: $0 \to \mathrm{Hom}_R(Q,A) \xrightarrow{r} \mathrm{Hom}_R(Q,A) \to \mathrm{Hom}_R(Q, A/rA)$

$\to \mathrm{Ext}^1_R(Q,A) \to$

SES: $0 \to R \xrightarrow{i} Q \xrightarrow{\pi} Q/R \to 0$

LES: $0 \to \mathrm{Hom}_R(Q/R, A) \to \mathrm{Hom}_R(Q,A) \to \mathrm{Hom}_R(R,A) \to \mathrm{Ext}^1_R(Q/R, A)$

$\to \mathrm{Ext}^1_R(Q,A) \to \mathrm{Ext}^1_R(R,A) \to \ldots$

$f: Q \to Q \ni f(q) = rq$ R-modhomom

Let $q \in Q \Rightarrow q = \frac{m}{n} = \frac{rm}{rn} = r \cdot \frac{m}{rn} = f\left(\frac{m}{rn}\right)$ contd on sep pg: $R \longrightarrow Q \longrightarrow 0$

$\therefore f$ surj

$\frac{m}{n} = \frac{rm}{rn} = \frac{1}{r} \frac{rm}{n}$ $\therefore g(q) = \frac{1}{r} q$ surj

$I^\bullet$: $0 \to R \hookrightarrow Q \to Q/R \to 0$ inj res

(0) (1)

$Hom_R($

Let $\mathcal{R}$: $\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} Q \to \cancel{Q/R \to 0}$ 0 proj res

$$\begin{tikzcd} Q \arrow[r, "r"] & Q \arrow[r] & 0 \\ & P_0 \arrow[u, "\epsilon"] \arrow[ul, dashed, "\exists\theta"] & \end{tikzcd}$$

$r\theta = \epsilon$

$$\begin{tikzcd} Q \arrow[r, "f"] & Q \arrow[r] & 0 \\ P_0 \arrow[u, "\epsilon"] \arrow[ur, dashed, "\exists\theta"] & & \end{tikzcd}$$

$\theta = f\epsilon_Q \to 0$

$\downarrow 1/r$

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r, "\epsilon"] \arrow[d, "1"] \arrow[dr, phantom, "G"] & Q \arrow[r] \arrow[d, "r"] & 0 \\ \cdots \arrow[r] & P_1' \arrow[r] & P_0 \arrow[r, "\theta"] & Q \arrow[r] & 0 \end{tikzcd}$$

comp thm $\Rightarrow \exists$ chain map $\therefore$ proj rrs are homotopic

$$\begin{tikzcd} Q \arrow[r, "\frac{1}{r}"] & Q \arrow[r] & 0 \\ & P_0 \arrow[u, "\epsilon"] \arrow[ul, dashed, "\exists\tau"] & \end{tikzcd}$$

$\frac{1}{r}\tau = \epsilon$

$r\theta = \frac{1}{r}\tau$

(ii) – show $Ext^n_Q(V, A) = \{0\} = Tor^R_n(V, A)$ $\forall n \geq 0$

where V vector space over Q and A is same as above

V flat since v.s. over field

so $Tor^R_n(V, A) = 0$ $\forall n \geq 1$

$Tor^R_0(V, A) = V \otimes_R A$

$v \otimes_R a = v \cdot 1 \otimes_R a = v \cdot \frac{r}{r} \otimes_R a = v \cdot \frac{1}{r} \otimes_R ra = v \cdot \frac{1}{r} \otimes_R 0 = 0$

$\therefore Tor^R_n(V, A) = 0$ $\forall n \geq 0$

$V \xrightarrow{\frac{1}{r}} V$

Let $v \in V \Rightarrow v = 1 \cdot v = \frac{r}{r} \cdot v = \frac{1}{r} \cdot \frac{r}{1} v$

$\therefore \frac{1}{r}$ surj

$v \in \ker \frac{1}{r}$ iff $0 = \frac{1}{r} v$ iff $v = 0$

$\therefore \frac{1}{r}$ inj

$\therefore \frac{1}{r}$ iso

Then

$R \xrightarrow{r} \frac{r}{1}$

$R \longrightarrow Q$

V Q-module $\Rightarrow V$ R-mod

so similarly ~~to~~ 1st part $Ext^i_R(V, A) = 0$ $\forall i \geq 0$

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{P}_1 \arrow[r] & P_0 \arrow[r, "\tau"] \arrow[d, "1"] \arrow[dr, phantom, "G"] & Q \arrow[r] \arrow[d, "\frac{1}{r}"] & 0 \\
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\epsilon"] \arrow[dr, phantom, "G"] & Q \arrow[r] & 0 \\
\cdots \arrow[r] & \tilde{P}_1 \arrow[r] & P_0 \arrow[r, "\theta"] \arrow[u, "1"] & Q \arrow[r] \arrow[u, "r"] & 0
\end{tikzcd}$$

mult by r on Q surj

$q \in \ker r \Rightarrow 0 = rq = r\frac{m}{n} = \frac{rm}{n}$

$\Rightarrow \exists u \in S \ni urm = 0$

$\Rightarrow m = 0$

$\Rightarrow q = \frac{m}{n} = 0$ $\therefore r$ inj $\therefore r > 0$

$$\begin{tikzcd}
P_0 \arrow[r, "\tau"] \arrow[d, "1"] & Q \arrow[d, "\frac{1}{r}"] \\
P_0 \arrow[r, "\epsilon"] & Q
\end{tikzcd}$$

SES $0 \to Q \xrightarrow{r} Q \to 0 \to 0$

LES: $0 \to \mathrm{Hom}_R(0,A) \to \mathrm{Hom}_R(Q,A) \xrightarrow{r} \mathrm{Hom}_R(Q,A)$

$\to \mathrm{Ext}^1_R(0,A) \to \mathrm{Ext}^1_R(Q,A) \xrightarrow{r} \mathrm{Ext}^1_R(Q,A) \to \cdots$

$\mathrm{Ext}^1_R(0,A) = 0$

$0 \to \mathrm{Ext}^i_R(Q,A) \xrightarrow{r} \mathrm{Ext}^i_R(Q,A) \to 0$ $\forall i$ exact

$r \in \mathrm{ann}_R A \subseteq \mathrm{ann}_R A \cup \mathrm{ann}_R Q$

$\subseteq \mathrm{ann}_R \mathrm{Ext}^i_R(Q,A)$ $\forall i \geq 0$

$\therefore r\,\mathrm{Ext}^i_R(Q,A) = 0$ $\forall i \geq 0$

$\| $

$\mathrm{Im}\, r$

$\|$

$\mathrm{Ext}^i_R(Q,A)$ $\forall i \geq 0$

7.15

(i) R ring

- show left R-mod B is injective iff $Ext^1_R(R/I, B) = 0 \ \forall$ left ideal I

($\Rightarrow$) Assume B injective

Then $Ext^1_R(R/I, B) = 0 \ \forall$ I left ideal since R/I R-mod

($\Leftarrow$) Assume $Ext^1_R(R/I, B) = 0 \ \forall$ left ideal I

Let I left ideal of R and let $f: I \longrightarrow B$

SES: $0 \longrightarrow I \xhookrightarrow{i} R \xrightarrow{\pi} R/I \longrightarrow 0$

LES: $0 \longrightarrow Hom_R(R/I, B) \longrightarrow Hom_R(R, B) \longrightarrow Hom_R(I, B)$

$\longrightarrow \underset{0}{\underset{\parallel}{Ext^1_R(R/I, B)}} \longrightarrow \cdots$

SES: $0 \longrightarrow Hom_R(R/I, B) \xrightarrow{\pi^*} Hom_R(R, B) \xrightarrow{i^*} Hom_R(I, B) \longrightarrow 0$

i^* surj $\Rightarrow$

Then $f = i^*(g)$ for some $g \in Hom_R(R, B)$
$= gi$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"'] & R \arrow[dl, dashed, "g"] \\ & B & \end{tikzcd}$$

$\therefore$ dgm commutes

$\therefore$ B injective by Baer

SES $0 \longrightarrow R \xrightarrow{i} Q \xrightarrow{\pi} Q/R \longrightarrow 0$

LES $0 \longrightarrow Hom_R(Q/R, D) \longrightarrow Hom_R(Q, D)$

$\longrightarrow Hom_R(R, D) \longrightarrow \underset{0}{\underset{\parallel}{Ext^1_R(Q/R, D)}} \longrightarrow$

SES $0 \longrightarrow Hom_R(Q/R, D) \xrightarrow{\pi^*} Hom_R(Q, D) \xrightarrow{i^*} Hom_R(R, D) \longrightarrow$

Let $y \in D$ and $r \in R$

Define $f: R \longrightarrow D \ni f(1) = y$

$f = i^*(g) = gi, \quad g \in Hom_R(Q, D)$

$\therefore g|_R = f$

$y = f(1) = g(1) = g(r \cdot \frac{1}{r}) = rg(\frac{1}{r}$

(ii) D ab gp

$Ext^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D) = 0$

- show D divisible
- Does this hold if we replace $\mathbb{Z}$ by domain R and $\mathbb{Q}/\mathbb{Z}$ by Frac(R)/R

$Ext^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D) = 0$ iff D injective iff D divisible since $\mathbb{Z}$ PID

Now assume $Ext^1_R(Q/R, D) = 0$ then D injective by (i) $\Rightarrow$ D divisible

10.13

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
C_{0,2} \arrow[d, "d'_{0,2}"] & C_{1,2} \arrow[l, "d_{1,2}"'] \arrow[d, "d'_{1,2}"] & \cdots \arrow[l, "d_{2,2}"'] \\
C_{0,1} \arrow[d, "d'_{0,1}"] & C_{1,1} \arrow[l, "d_{1,1}"'] \arrow[d, "d'_{1,1}"] & \cdots \arrow[l, "d_{2,1}"'] \\
C_{0,0} & C_{1,0} \arrow[l, "d_{1,0}"'] & \cdots \arrow[l, "d_{2,0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C_{i,j+1} \arrow[d, "d'_{i,j+1}"'] & C_{i+1,j+1} \arrow[l, "d_{i+1,j+1}"'] \arrow[d, "d'_{i+1,j+1}"] \\
C_{i,j} & C_{i+1,j} \arrow[l, "d_{i+1,j}"]
\end{tikzcd}$$

exact rows

so $\operatorname{Im} d_{p,q} = \ker d_{p-1,q} \quad \forall p,q$

$(\mathrm{Tot}(C_{..}))_n = \bigoplus_{p+q=n} C_{p,q}$

$D_n = \sum_{p+q=n} d_{p,q} + (-1)^p d'_{p,q}$

$(\mathrm{Tot}(C_{..}))_0 = C_{0,0}$

$(\mathrm{Tot}(C_{..}))_1 = C_{0,1} \oplus C_{1,0}$ $\quad > D_1 = d_{1,0} + d'_{0,1}$

$(\mathrm{Tot}(C_{..}))_2 = C_{0,2} \oplus C_{1,1} \oplus C_{2,0}$ $\quad > D_2 = d_{2,0} - d'_{1,1} + d_{1,1} + d'_{0,2}$

$\operatorname{Im} D_n = \operatorname{Im}\left(\sum_{p+q=n} d_{p,q} + (-1)^p d'_{p,q}\right)$

$= \sum_{p+q=n} \operatorname{Im} d_{p,q} + (-1)^p \operatorname{Im} d'_{p,q}$

$= \sum_{p+q=n} \ker d_{p-1,q}$

$E^0 =$ bicomplex

$$E^1 = \begin{array}{cc} \vdots & \vdots \\ \downarrow & \downarrow \\ 0 & 0 \\ \downarrow & \downarrow \\ 0 & 0 \end{array}$$

$\| $

E^∞

taking horizontal homologies get 0's since each row exact

$E^\infty_{p,q}$ are factors in filtration $H_n(\mathrm{Tot}(C_{..}))$

Now

$$\begin{array}{ll}
0\left[\begin{array}{l} H_n(\mathrm{Tot}(C_{..})) \\ \quad \cup \end{array}\right. & \\
\quad 0\left[\begin{array}{l} * \\ \cup \end{array}\right. & \\
\quad 0\left[\begin{array}{l} * \\ \cup \end{array}\right. & \\
\quad \quad \vdots & \\
\quad 0\left[\begin{array}{l} \cup \\ * \end{array}\right. &
\end{array}$$

$\Longrightarrow H_n(\mathrm{Tot}(C_{..})) = 0 \ \forall n$

$\therefore \mathrm{Tot}(C_{..})$ exact

(same for columns exact, only take vertical homologies)

10.16

$C = C' \oplus C''$ direct sum of complexes   show $\ker d'_n \cong \operatorname{Im} d'_{n+1} \oplus H_n(C'_\cdot)$

C split

– Show C' split

$C'_\cdot: \quad \cdots \to C'_{n+1} \to C'_n \to C'_{n-1} \to \cdots$

$C''_\cdot: \quad \cdots \to C''_{n+1} \to C''_n \to C''_{n-1} \to \cdots$

$C_\cdot: \quad \cdots \to C'_{n+1} \oplus C''_{n+1} \xrightarrow{D_{n+1}} C'_n \oplus C''_n \xrightarrow{D_n} C'_{n-1} \oplus C''_{n-1} \to \cdots$

$D_{n+1} = d'_{n+1} + d''_{n+1}$

$C_\cdot$ splits $\Rightarrow$ $0 \to \operatorname{Im} D_{n+1} \hookrightarrow \ker D_n \to \ker D_n / \operatorname{Im} D_{n+1}$ splits, $\ker D_n / \operatorname{Im} D_{n+1} = H_n(C_\cdot)$

$C_{\cdot\cdot}:$
$= E^0$

$$\begin{array}{ccccccc}
 & & C'_{n+2} & & & & \\
 & & \downarrow & & & & \\
0 & \leftarrow & C'_{n+1} & \leftarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \leftarrow C''_{n-1} & \leftarrow & C'_n \oplus C''_n & \leftarrow & C''_{n+1} & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & C'_{n-1} & \leftarrow & 0 & & \\
 & & \downarrow & & & & \\
 & & \vdots & & & &
\end{array}$$

$H_n(C'_\cdot)$: $H_n(C_\cdot) \supseteq * \supseteq 0$

$H_{n+1}(C'_\cdot)$, $H_{n+1}(C''_\cdot)$: $H_{n+1}(C_\cdot) \supseteq * \supseteq 0$

$H_n(C'_\cdot) \cong H_n(C_\cdot)$

$H_{n+1}(C'_\cdot) \cong H_{n+1}(C_\cdot) / H_{n+1}(C''_\cdot)$

SES: $0 \to H_{n+1}(C''_\cdot) \hookrightarrow H_{n+1}(C_\cdot) \to H_{n+1}(C$

$E^3 = \quad 0 \quad \overset{?}{\sim} \quad 0$

$E^\infty = \quad H_{n-1}(C'_\cdot) \quad H_n(C'_\cdot) \quad H_{n+1}(C''_\cdot)$

$0 \quad \overset{?}{\sim} \quad 0$

(0,-1)
(-1,0)
(-2,1)
(-3,2)

$\operatorname{Tot}(C_{\cdot\cdot}) = C_\cdot$

$E^1 =$

$$\begin{array}{ccccc}
0 & \leftarrow & H_{n+2}(C'_\cdot) & \leftarrow & 0 \\
0 & \leftarrow & H_{n+1}(C'_\cdot) & \leftarrow & 0 \\
C''_{n-1} & \xleftarrow{d''_n} & H_n(C'_\cdot) & \xleftarrow{d''_n} & C''_{n+1} \\
0 & \leftarrow & H_{n-1}(C'_\cdot) & \leftarrow & 0 \\
0 & \leftarrow & H_{n-2}(C'_\cdot) & \leftarrow & 0
\end{array}$$

$E^2 =$

$$\begin{array}{cccccc}
 & 0 & H_{n+2}(C'_\cdot) & 0 & 0 \\
0 & 0 & H_{n+1}(C'_\cdot) & 0 & 0 \\
0 & & & H_{n+1}(C''_\cdot) & H_{n+2}(C''_\cdot) \\
H_{n-2}(C''_\cdot) & H_{n-1}(C''_\cdot) & H_n(C'_\cdot) & & \\
 & & & 0 & 0 \\
0 & 0 & H_{n-1}(C'_\cdot) & & 0 \\
0 & 0 & H_{n-2}(C'_\cdot) & 0 &
\end{array}$$

$\ker d''_{n+2} / \operatorname{Im} d''_{n+1} \to \ker d'_{n+1} / \operatorname{Im} d'$

$$\psi : H_n(C_\cdot) \longrightarrow \ker(d_n' + d_n'')$$
$$H_n(C_\cdot) = \ker(d_n' + d_n'') / \operatorname{Im}(d_n' + d_n'')$$

Define $\varphi : H_n(C_\cdot') \longrightarrow \ker d_n'$

$$H_n(C_\cdot') = \ker d_n' / \operatorname{Im} d_{n+1}'$$

5.2 G ab gp (p-primary)

Rotman (731)

- show $S \subseteq G$ pure iff $S \cap p^nG = p^nS \quad \forall n \geq 0$

($\Rightarrow$) Assume $S \subseteq G$ pure

Then $\forall m \in \mathbb{Z}$, $S \cap mG = mS$

In particular $S \cap p^nG = p^nS \quad \forall n \geq 0$

($\Leftarrow$) Assume $S \cap p^nG = p^nS \quad \forall n \geq 0$

Let $m \in \mathbb{Z}$ and let $x \in S \cap mG$

$m = p_1^{k_1} \cdots p_t^{k_t}$

So $x \in S$ and $x \in mG \Rightarrow x = p_1^{k_1} \cdots p_t^{k_t} g$, $k_i \geq 0$

$\in p_i^{k_i} G \quad \forall i = 1, \ldots, t$

$\Rightarrow x \in S \cap p_i^{k_i} G = p_i^{k_i} S \quad \forall i$

$\Rightarrow x = p_i^{k_i} s_i$ for each i

$\Rightarrow x = p_1^{k_1} \cdots p_t^{k_t} s$ for some $s \in S$

$\in mS$

$\therefore S \cap mG \subseteq mS$

Let $x \in mS \Rightarrow x = p_1^{k_1} \cdots p_t^{k_t} s$

$\in p_i^{k_i} S \quad \forall i$

$= S \cap p_i^{k_i} G \quad \forall i$

$\Rightarrow x = p_1^{k_1} \cdots p_t^{k_t} h$ for some $h \in G$ and $x \in S$

$\in S \cap mG$

$\therefore mS \subseteq S \cap mG$

$\therefore S \cap mG = mS$

$\therefore S$ pure

5.5 $G=(a)$

$|G|=m$

- show G/nG cyclic gp of order $d=(m,n)$

Then $G\cong \mathbb{Z}/m\mathbb{Z}$

$G/nG \cong \mathbb{Z}/m\mathbb{Z} \big/ n(\mathbb{Z}/m\mathbb{Z})$

Show $G/nG \cong \mathbb{Z}/d\mathbb{Z}$

Define $\varphi: \mathbb{Z}/m\mathbb{Z}\big/n(\mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z} \ni \varphi(\bar{a}+n(\mathbb{Z}/m\mathbb{Z}))=\bar{a}$

$\bar{a}_1+n(\mathbb{Z}/m\mathbb{Z})=\bar{a}_2+n(\mathbb{Z}/m\mathbb{Z})$

$\overline{a_1-a_2}\in n(\mathbb{Z}/m\mathbb{Z})$

$\overline{a_1-a_2}+n(\mathbb{Z}/m\mathbb{Z})=n(\mathbb{Z}/m\mathbb{Z})$

$m|a_1-a_2$ or $n|a_1-a_2$

$=d|a_1-a_2$

$\Rightarrow \overline{a_1-a_2}=$

$\Rightarrow \bar{a}_1=\bar{a}_2$

$\therefore$ wd

homom.

$\varphi(\bar{a}+n(\mathbb{Z}/m\mathbb{Z}))=\bar{0}$ iff $\bar{a}=\bar{0}$ iff $d|a$

Let $\bar{a}+n(\mathbb{Z}/m\mathbb{Z})=a(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))$ $\therefore$ generated by

$\bar{a}+n(\mathbb{Z}/m\mathbb{Z})=\bar{0}$ iff $\bar{a}\in n(\mathbb{Z}/m\mathbb{Z})$ or $m|a$

iff

$n(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))=0 \Rightarrow |\bar{1}+n(\mathbb{Z}/m\mathbb{Z})|\,|\,n$ also $m(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))=0$ so $|\_\_|\,|\,m$

$\varphi(\bar{1}+n(\mathbb{Z}/m\mathbb{Z}))=\bar{1}$ order d $\quad |\_\_|\le d$

$\therefore d\,|\,|\bar{1}+n(\mathbb{Z}/m\mathbb{Z})|$

$d\le|\_\_|$ $\quad\therefore |\_\_|=d$

$\therefore |\text{Grp}|=d$

$\varphi(\bar{1})=\bar{1}+n(\mathbb{Z}/m\mathbb{Z})$

$|\bar{1}|=d \qquad |\bar{1}+n(\mathbb{Z}/m\mathbb{Z})|=n$

$\varphi:\mathbb{Z}/d\mathbb{Z}\to \mathbb{Z}/m\mathbb{Z}\big/n(\mathbb{Z}/m\mathbb{Z}) \ni \varphi(\bar{a})=\bar{a}+n(\mathbb{Z}/m\mathbb{Z})$

$\varphi(\bar{a})=0$ iff $\bar{a}+n(\mathbb{Z}/m\mathbb{Z})=0$ iff $\bar{a}+n(\mathbb{Z}/m\mathbb{Z})=n(\mathbb{Z}/m\mathbb{Z})$

iff $a\in n(\mathbb{Z}/m\mathbb{Z})$

Define $\varphi:\mathbb{Z}/m\mathbb{Z}\to\mathbb{Z}/d\mathbb{Z}\ni\varphi(\bar{a})=\bar{a}$ surj homom

$\ker\varphi=\{\bar{a}\in\mathbb{Z}/m\mathbb{Z}\mid\varphi(\bar{a})=\bar{0}\}$

But ~~$\varphi(\bar{a})=\bar{0}$~~ iff $\bar{a}=\bar{0}$ in $\mathbb{Z}/d\mathbb{Z}$ iff $d|a$

$\bar{a}\in n(\mathbb{Z}/m\mathbb{Z})$ iff $n|a$ and $\bar{a}=0$ iff $m|a$

~~5.8~~

~~(i) G finite p primary ab gp~~
~~p prime~~
~~$x \in G$ has largest order~~
~~show $\langle x \rangle$ is direct summand of G~~

6.33

(i) - Give example of noeth ring R containing subring that is not noeth

$k[x,y]$ noetherian ring by Hilbert basis Thm since k comm noetherian ring since field

Subring generated by $\{xy^i : i \geq 0\}$ is not f.g over k hence not noeth ie $k[x, xy, xy^2, \ldots]$

(ii) - Give example of comm ring R containing proper ideals $I \subsetneq J \subsetneq R$ with J fg. but I not f.g.

Take $R = k[x_1, x_2, \ldots]$

$J = (x_1)$ fg, proper ideal

$I = (x_1 x_2, x_1 x_3, x_1 x_4, \ldots) \subsetneq J$ since $x_1 \notin I$

not f.g.

Let $(a,b) \in I$ ~~$(r,s)(a,b) \in I \ \forall (r,s) \in R \times S$ $\Leftrightarrow$ $(ra,sb) \in I \ \forall (r,s) \in R \times S$~~

6.39 R, S noeth

– show $R \times S$ noeth

Let $I \lhd R \times S \Rightarrow I = I_1 \times I_2$ where $I_1 \lhd R$, $I_2 \lhd S$

Then $I_1 = (x_1, \dots, x_n)$ f.g.

And $I_2 = (y_1, \dots, y_m)$ f.g. since R, S noeth (say $n \leq m$)

Let $(z_1, z_2) \in I \Rightarrow I_1 \ni z_1 = r_1 x_1 + \dots + r_n x_n$

$I_2 \ni z_2 = s_1 y_1 + \dots + s_m y_m$

So $(z_1, z_2) = (r_1 x_1 + \dots + r_n x_n, \ s_1 y_1 + \dots + s_m y_m)$

$= (r_1, s_1)(x_1, y_1) + \dots + (r_n, s_n)(x_n, y_n) + (0, s_{n+1})(0, y_{n+1})$
$+ \dots + (0, s_m)(0, y_m)$

$\therefore I = \big((x_1, y_1), \dots, (x_n, y_n), (0, y_{n+1}), \dots, (0, y_m) \big)$ f.g.

$\therefore R \times S$ noeth

$I \lhd R \times S$

$\pi_1 : R \times S \to R$, $\pi_2 : R \times S \to S$ nat. proj.

Show $I = \pi_1(I) \times \pi_2(I)$

Let $(a,b) \in I \Rightarrow (a,b) = (\pi_1(a,b), \pi_2(a,b)) \in \pi_1(I) \times \pi_2(I)$

$\therefore I \subseteq \pi_1(I) \times \pi_2(I)$

Let $(a,b) \in \pi_1(I) \times \pi_2(I)$

$\Rightarrow a = \pi_1(a,c)$ and $b = \pi_2(d,b)$, $(a,c), (d,b) \in I$

$(a,b) = (1,0)(a,c) + (0,1)(d,b) \in I$ since $I \lhd R \times S$

$\therefore \pi_1(I) \times \pi_2(I) \subseteq I$

$\therefore I = \underbrace{\pi_1(I)}_{\text{ideal of } R} \times \underbrace{\pi_2(I)}_{\text{ideal of } S}$ since π_1, π_2 surj ringhomom and surj ring homom send ideals to ideals

7.2 $X \subseteq M$

M module

- show $(X) = \bigcap_{X \subseteq S \leq M} S$

Let $y \in (X) \Rightarrow y = \sum r_i x_i$ where $X = \{x_i\}_{i \in I}$

$\in S \ \forall S$ since each $x_i \in X \subseteq S$ and S module

$\therefore y \in \bigcap_{X \subseteq S \leq M} S$

$\therefore (X) \subseteq \bigcap_{X \subseteq S \leq M} S$

Now let $y \in \bigcap_{X \subseteq S \leq M} S \Rightarrow y \in S \ \forall X \subseteq S \leq M$

In particular $y \in (X)$ since $X \subseteq (X) \leq M$

$\therefore \bigcap_{X \subseteq S \leq M} S \subseteq (X)$

$\therefore (X) = \bigcap_{X \subseteq S \leq M} S$

7.5 M R-mod

- show $\exists$ R-mod iso $\varphi_M : Hom_R(R,M) \longrightarrow M \ni \varphi_M(f) = f(1)$

$\varphi_M(f_1+f_2) = (f_1+f_2)(1) = f_1(1) + f_2(1) = \varphi_M(f_1) + \varphi_M(f_2)$

$\varphi_M(rf) = (rf)(1) = rf(1) = r\varphi_M(1)$

$\therefore \varphi_M$ R-mod homom.

Let $f \in \ker \varphi_M$ and let $r \in R$

Then $f(r) = f(r \cdot 1) = r f(1) = r \varphi_M(f) = r \cdot 0 = 0$

$\therefore f \equiv 0$

$\therefore \ker \varphi_M = 0$

$\therefore \varphi_M$ inj

Define $f : R \to M \ni f(r) = rm$ R-mod homom

$f(r_1+r_2) = (r_1+r_2)m = r_1 m + r_2 m$
$= f(r_1) + f(r_2)$
$f(sr) = (sr)m = s(rm)$
$= s f(r)$ ✓

Let $m \in M$

Then $m = 1 \cdot m = f(1) = \varphi_M(f)$

$\therefore \varphi_M$ surj

$\therefore \varphi_M$ iso R-mods

$\therefore Hom_R(R,M) \cong M$

7.14

$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ exact

- show f surj iff h inj

($\Rightarrow$) Assume f surj

Let $x \in \ker h = \operatorname{Im} g \Rightarrow x = g(b)$

But f surj $\Rightarrow b = f(a)$

So $x = g(f(a)) = 0$ by exactness

$\therefore \ker h = 0$

$\therefore h$ injective

($\Leftarrow$) Assume h inj

Let $b \in B \Rightarrow h(g(b)) = 0$ by exactness

$\Rightarrow g(b) \in \ker h = 0$ since h inj

$\Rightarrow b \in \ker g = \operatorname{Im} f$

$\therefore f$ surj

7.27. $\sigma : \prod B_i \longrightarrow \prod C_j$

- Find $\tilde{\sigma} \ni$ dgm comm:

$$\begin{tikzcd} \operatorname{Hom}(A, \prod B_i) \arrow[r, "\sigma_*"] \arrow[d, "\tau_B"', "\wr"] & \operatorname{Hom}(A, \prod C_j) \arrow[d, "\tau_C", "\wr"'] \\ \prod \operatorname{Hom}(A, B_i) \arrow[r, "\tilde{\sigma}"] & \prod \operatorname{Hom}(A, C_j) \end{tikzcd}$$

$\exists\, \tau, \tau'$ isos

$\tau_B(f) = (p_i^B f)$ where $p_i^B : \prod B_i \longrightarrow B_i$

$\tau_C(f) = (p_j^C f)$ where $p_i^C : \prod C_j \longrightarrow C_j$

Define $\tilde{\sigma} : \prod \operatorname{Hom}(A, B_i) \longrightarrow \prod \operatorname{Hom}(A, C_j) \ni \tilde{\sigma}(f_i) = (p_j^C \sigma f)$

where $f = (f_i) \in \operatorname{Hom}(A, \prod B_i)$

$\tau_C(\sigma_*(f)) = \tau_C(\sigma f) = (p_j^C \sigma f)$ ✓

$\tilde{\sigma}(\tau_B(f)) = \tilde{\sigma}(p_i^B f) = (p_j^C \sigma f)$ ✓

$\therefore$ Dgm comm.

7.30

(i) - show (0) is zero object in RMOD

Let $M \in \text{ob RMOD} \Rightarrow M$ R-mod

$\exists!$ morphism $(0) \longrightarrow M$ namely zero map

$\therefore (0)$ is initial object

$\exists!$ morphism $M \longrightarrow (0)$ namely zero map

$\therefore (0)$ is terminal object

$\therefore (0)$ is zero object

(ii) - show ϕ is initial object in SETS

Let $X \in \text{ob SETS} \Rightarrow \phi \subseteq X$

so $\exists!$ morphism $\phi \hookrightarrow X$ namely inclusion

$\therefore \phi$ initial object

(iii) - show any singleton set is terminal object in SETS $\{x\}$

Let $Y \in \text{ob SETS}$

Then $\exists!$ morphism $Y \longrightarrow \{x\}$ namely the one that sends every element of Y to x

---

(iv) - show $\nexists$ zero object in SETS

First note that initial objects + terminal objects are unique up to iso :

If X, Y are initial objects, then $\exists!$ morphism $X \xrightarrow{f} Y$ since X initial and $\exists!$ morphism $Y \xrightarrow{g} X$ since Y initial

Now fg, $1_Y : Y \longrightarrow Y$ (identity map), but Y initial, so $\exists!$ morphism $Y \longrightarrow Y$

$\therefore fg = 1_Y$

Similarly $gf = 1_X$

$\therefore f$ isomorphism with inverse g

Same argument for terminal objects

Back to SETS: But ϕ is not isomorphic to any nonempty set (since $\nexists$ morphism $X \longrightarrow \phi$ for any X) so ϕ unique initial object, and ϕ not terminal

$\therefore \nexists$ zero object in SETS

7.41 G ab gp

- show $Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong \{g \in G : ng = 0\}$

SES: $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$

$Hom_{\mathbb{Z}}(\_, G)$ left exact $\Rightarrow$

$$0 \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \xrightarrow{\pi^*} Hom_{\mathbb{Z}}(\mathbb{Z}, G) \xrightarrow{n^*} Hom_{\mathbb{Z}}(\mathbb{Z}, G) \text{ exact}$$

in part, π^* inj

so $Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong Im\,\pi^* = ker\, n^*$

But
$$\begin{tikzcd}
Hom_{\mathbb{Z}}(\mathbb{Z}, G) \arrow[r, "n^*"] \arrow[d, "\cong"] \arrow[dr, phantom, "G^{nat}"] & Hom_{\mathbb{Z}}(\mathbb{Z}, G) \arrow[d, "\cong"] \\
G \arrow[r, "n"] & G
\end{tikzcd}$$

so $Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong ker\, n^* \cong ker\, n = \{g \in G : ng = 0\}$

7.44 contd

in abelian category, $D = \text{pullback}\left(\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}\right)$

$= \ker\left(A \times B \xrightarrow{(f,-g)} C\right)$

But since T is left exact, it preserves kernels

$\left(\begin{array}{l} \text{Let } g : B \to C \\ \text{Then } 0 \to \ker g \hookrightarrow B \xrightarrow{g} C \text{ exact} \\ \Rightarrow 0 \to T(\ker g) \to T(B) \xrightarrow{T(g)} T(C) \text{ exact} \\ \Rightarrow T(\ker g) \cong \ker T(g) \end{array}\right)$

So $T(D) = T\left(\ker\left(B \times C \xrightarrow{(f,-g)} A\right)\right)$

$\cong \ker\left(T(B \times C) \xrightarrow{T(f,-g)} T(A)\right)$

$\cong \ker\left(T(B) \times T(C) \xrightarrow{(T(f),-T(g))} T(A)\right)$ since left exact $\Rightarrow$ preserves products

$= \text{pullback}\left(\begin{tikzcd} & T(C) \arrow[d, "T(g)"] \\ T(B) \arrow[r, "T(f)"'] & T(A) \end{tikzcd}\right)$

$\therefore$ T preserves pullbacks

Now:

$\text{Hom}_R(M,-)$ left exact covariant functor

$\text{Hom}_R(M,-)$ preserves pullbacks

Let $D = \text{pullback}\left(\begin{tikzcd} & C \arrow[d, hook, "i_2"] \\ B \arrow[r, hook, "i_1"'] & A \end{tikzcd}\right)$, i_1, i_2 incl.

$D = \{(b,c) \in B \times C \mid i_1(b) = i_2(c)\} = \{(b,c) \in B \times C \mid b = c\} = \{(b,b) \in B \times C\}$

$= \{(b,b) \mid b \in B \cap C\}$.

$\cong B \cap C$

So by above

$\text{Hom}_R(M, B \cap C) = \text{Hom}_R(M,B) \cap \text{Hom}_R(M,C)$

$i_1(f(x)) = i_2(g(x)) \Rightarrow f(x) = g(x)$

$\text{pullback}\left(\begin{tikzcd} & \text{Hom}_R(M,C) \arrow[d, "i_{2*}"] \\ \text{Hom}_R(M,B) \arrow[r, "i_{1*}"'] & \text{Hom}_R(M,A) \end{tikzcd}\right)$

$= \{(f,g) \in \text{Hom}_R(M,B) \times \text{Hom}_R(M,C) \mid i_{1*}(f) = i_{2*}(g)\}$

$= \{(f,g) \in \text{"} \quad \text{"} \mid i_1 f = i_2 g\} = \ldots = \text{Hom}_R(M,B) \cap \text{Hom}_R(M,C)$

7.44

- Show that every left exact covariant functor $T: \mathrm{RMOD} \to \mathrm{AB}$ preserves pullbacks

$B, C \leq A$ R-mod

- Show $\forall M$ R-mod, $\mathrm{Hom}_R(M, B \cap C) = \mathrm{Hom}_R(M, B) \cap \mathrm{Hom}_R(M, C)$

Let $D = $ pullback $\left(\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \right)$

Show $T(D) = $ pullback $\left(\begin{tikzcd} & T(C) \arrow[d, "T(g)"] \\ T(B) \arrow[r, "T(f)"'] & T(A) \end{tikzcd} \right)$

$D \subseteq B \times C \Rightarrow F(D) \subseteq F(B \times C)$

$D = \cancel{\lim} \left(\begin{tikzcd} & C \arrow[d] \\ B \arrow[r] & A \end{tikzcd} \right) \Rightarrow$ since T preserves injections hence subsets

Let $x \in T(D)$

$\Rightarrow x \in T(B \oplus C) \cong T(B) \oplus T(C) \Rightarrow x = (\tilde{b}, \tilde{c})$ $\ni \tilde{b} \in T(B)$, $\tilde{c} \in T(C)$

$\ni T(f)(\tilde{b}) = T(g)(\tilde{c})$

$\therefore x \in \tilde{D}$

$\therefore T(D) \subseteq \tilde{D}$

Let $x \in \tilde{D}$

See extra sheet

$$\begin{tikzcd} X \arrow[r, dashed, "\exists!\theta"] \arrow[rrr, bend left, "\forall h"] \arrow[rrr, bend right, "\forall j"'] & D \arrow[rr, "\beta"] \arrow[rr, "\alpha"'] & & C \arrow[d, "g"] \\ & & G & A \\ & & & B \arrow[u, "f"'] \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[rrr, bend left, "k"] \arrow[rrr, bend right, "l"'] & T(D) \arrow[rr, "T(\beta)"] \arrow[rr, "T(\alpha)"'] & & T(C) \arrow[d, "T(g)"] \\ & & G & T(A) \\ & & & T(B) \arrow[u, "T(f)"'] \end{tikzcd}$$

commutes since T functor

Let $k, l \ni T(g)k = T(f)l$

In RMOD, pullback of $\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$ is $D = \{(b,c) \in B \times C \mid f(b) = g(c)\}$

And pullback of $\begin{tikzcd} & T(C) \arrow[d, "T(g)"] \\ T(B) \arrow[r, "T(f)"'] & T(A) \end{tikzcd}$ is $\{(\tilde{b}, \tilde{c}) \in T(B) \times T(C) \mid T(f)(\tilde{b}) = T(g)(\tilde{c})\} = \tilde{D}$

Show $T(D) = \{(\tilde{b}, \tilde{c}) \in T(B) \times T(C) \mid T(f)(\tilde{b}) = T(g)(\tilde{c})\}$

OES $0 \to B \to B \oplus C \to C \to 0$

T left exact cov. $\Rightarrow 0 \to T(B) \to T(B \oplus C) \to T(C)$ exact

$T(B \oplus C) \cong T(B) \oplus T(C)$

7.50 - Show that every direct summand of an injective module is injective

Let I injective and let J be a direct summand of I

then $I = J \oplus X$ for some $X \leq I$

Let $0 \longrightarrow A \xrightarrow{f} B$ injective and let

$g : A \longrightarrow J$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "j"] \arrow[ddl, dashed, "\exists h"] \\
& J \arrow[d, shift left, "i_1"] & \\
& I \arrow[u, shift left, "p_1"] &
\end{tikzcd}$$

Since I inj, $\exists h : B \longrightarrow I \ni i_1 g = hf$

Define $j : B \longrightarrow J \ni j = p_1 h$

then $jf = p_1 h f = p_1 i_1 g = 1_J g = g$

$\therefore$ J injective

7.53 (i) R int dom

$0 \neq I, J \triangleleft R$

– Show $I \cap J \neq 0$

Suppose $I \cap J = 0$

$I, J \neq 0 \Rightarrow \exists\ 0 \neq i \in I,\ 0 \neq j \in J$

And $ij \in I \cap J = 0$

so $ij = 0$

But R int dom $\Rightarrow$ $i = 0$ or $j = 0$

Contradiction since $i, j \neq 0$

$\therefore I \cap J \neq 0$

(ii) R int dom

$I \triangleleft R$ free R-mod

– Show I principal

Let $i \in I \Rightarrow i = r_1 i_1 + \dots + r_n i_n$ where $\{i_j\}$ basis for I

If $I = (0)$, done, so assume $I \neq (0)$

Then $r_j \neq 0$ for some j

Show $I = (i_j)$

I free $\Rightarrow$ I has basis $\{x_i\}_{i \in I}$

But $x_i \cdot x_j + (-x_j) x_i = 0$

$\Rightarrow$ Any two nonzero elements are lin dep

$\therefore$ I has basis $\{x_1\}$ but if $rx = 0 \Rightarrow r = 0$ since R int dom $\therefore x_1$ lin ind

$\therefore I = (x_1)$

$\therefore$ I principal

7.56 (i) - Show every vectorspace over a field k is an injective k-module

Let V be a vector space over k

Let $0 \longrightarrow A \xrightarrow{f} B$ injective ~~A, B~~ k-modules

and let $A \xrightarrow{g} V$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h"] \\ & V & \end{tikzcd}$$

A has basis $\{a_i\}_{i\in I}$

B " " $\{b_i\}_{i\in I}$

V " " $\{v_i\}_{i\in I}$

for each i $f(a_i) = b_j$ for some j

Define $h: B \longrightarrow V \ni$

$h(b_j) = g(a_i)$

Then $h(f(a_i)) = h(b_j)$
$= g(a_i)$

$\therefore hf = g$

$\therefore V$ inj

Let $I \lhd k$ and let $f: I \longrightarrow V$

Then $I = 0$ or $I = k$ since k field

if $I = 0$:
(then $f = 0$)

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & k \arrow[dl, dashed, "G", "\exists 0"'] \\ & V & \end{tikzcd}$$

if $I = k$:
(then inclusion is 1_k)

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r, "1_k"] \arrow[d, "f"'] & k \arrow[dl, dashed, "G", "\exists f"'] \\ & V & \end{tikzcd}$$

$\therefore V$ injective k-module by Baer

(ii) $0 \to U \to V \to W \to 0$ SES vector spaces

- Show $0 \to W^* \to V^* \to U^* \to 0$ SES

say U, V, W vectorspaces over field k

Show $0 \to \mathrm{Hom}_k(W,k) \to \mathrm{Hom}_k(V,k) \to \mathrm{Hom}_k(U,k) \to 0$ SES

~~since U,V,W vectorspaces over k, U,V,W injective~~

k vectorspace over k $\Rightarrow$ k injective k-module

$\Rightarrow \mathrm{Ext}^1_R(-,k) = 0$

LES:

$$0 \to \mathrm{Hom}_k(W,k) \to \mathrm{Hom}_k(V,k) \to \mathrm{Hom}_k(U,k) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1_R(W,k)} \to \cdots$$

$\therefore$ SES

$$0 \to \mathrm{Hom}_k(W,k) \to \mathrm{Hom}_k(V,k) \to \mathrm{Hom}_k(U,k) \to 0$$

$\therefore$ SES $0 \to W^* \to V^* \to U^* \to 0$

7.72 $\{E_i, \varphi^i_j\}$ direct system of inj R-mods over directed index set I

R noeth

- Show $\varinjlim E_i$ inj

~~Let $0 \to \quad \xrightarrow{f} B$ inj and let $A \xrightarrow{g} \varinjlim E_i$~~

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \\
 & \varinjlim E_i &
\end{tikzcd}$$

R noeth $\overset{E_i \text{ inj}}{\Longrightarrow}$ ΣE_i injective

$$\begin{tikzcd}
E_i \arrow[d, "\varphi^i_j"'] \arrow[dr, "\alpha_i"] \arrow[drr, bend left, "i_j \varphi^i_j"] & & \\
 & \varinjlim E_i \arrow[r, dashed, "\exists ! \theta"] & \Sigma E_i \\
E_j \arrow[ur, "\alpha_j"'] \arrow[urr, bend right, "i_j"'] & &
\end{tikzcd}$$

outside commute?

Let $I \triangleleft R$ and let $I \xrightarrow{f} \varinjlim E_i$

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"] & R \arrow[dl, "j"] \arrow[dd, dashed, "\exists g"] \arrow[ddd, dashed, bend left, "\exists h"] \\
 & \varinjlim E_i \arrow[d, "\theta"] & \\
 & \Sigma E_i \arrow[d, shift left, "p_i"] & \\
 & E_i \arrow[u, shift left, "i_i"] \arrow[uu, bend left=60, "\alpha_i"] &
\end{tikzcd}$$

$f = \alpha_i h i = \alpha_i p_i g i = j i$

ΣE_i inj $\Rightarrow \exists g \ni g i = \theta f$

E_i inj $\Rightarrow \exists h \ni h i = p_i \theta f$

Define $j : R \to \varinjlim E_i \ni j = \alpha_i h \Rightarrow j i = \alpha_i h i = \alpha_i p_i \theta f = \alpha_i p_i g i$

$\varphi_j^i: E_i \xrightarrow{\tau_i} \varinjlim E_i$, $E_i \downarrow G$, $E_j \xrightarrow{\tau_j} \varinjlim E_i$

Let $I \trianglelefteq R$ and let $f: I \longrightarrow \varinjlim E_i$

$\forall r \in I$, $f(r) \in \varinjlim E_i \Rightarrow f(r) = \tau_i(m_i)$ for some i
$\in \operatorname{Im} \tau_i$

$\therefore f(I) \subseteq \operatorname{Im} \tau_i$

R noeth $\trianglelefteq$
I fg $\Rightarrow I = (x_1, \ldots, x_n)$
Let $x \in f(I)$
$\Rightarrow x = f(r_1 x_1 + \ldots + r_n x_n)$
$= r_1 f(x_1) + \ldots + r_n f(x_n)$
$\in (f(x_1), \ldots, f(x_n))$
$\therefore f(I)$ f.g

Let $A \leq E_i$ f.g, $\ni A$ maps onto $f(I)$

Consider SES $0 \longrightarrow \underset{\ker(A \to f(I))}{\overset{\shortparallel}{B}} \hookrightarrow A \longrightarrow f(I) \longrightarrow 0$

A f.g., R noeth $\Rightarrow$ A noeth R-mod
$\Rightarrow$ B noeth R-mod
$\Rightarrow$ B f.g.

inj since R noeth, E_i inj

$\varinjlim E_i = \sum E_i / S$

SES : $0 \longrightarrow S \hookrightarrow \sum E_i \longrightarrow \sum E_i / S \longrightarrow 0$

<u>claim</u> R noeth, E_i inj $\Rightarrow \sum E_i$ inj

Show $\operatorname{Hom}_R(-, \varinjlim E_i)$ exact

Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ SES

Show $0 \longrightarrow \operatorname{Hom}_R(C, \varinjlim E_i) \to \operatorname{Hom}_R(B, \varinjlim E_i) \to \operatorname{Hom}_R(A, \varinjlim E_i) \to 0$ exact

$0 \longrightarrow I \hookrightarrow R$, $f \downarrow$, $\sum E_i / S$

$i_1 = i_2 \Rightarrow$
$f(i_1) = f(i_2)$
$\tilde{i}_1 + S = \tilde{i}_2 + S$
$\tilde{i}_1 - \tilde{i}_2 \in S$
$\tilde{i}_1 - \tilde{i}_2 = \lambda_j \varphi_j^i \lambda_i m_i \quad i < j$

$\bar{f}: I \longrightarrow \sum E_i \ni \bar{f}(i) = \tilde{i}$

Since $\sum E_i$ inj $\exists \bar{g}: R \to \sum E_i$
$\ni \bar{g}|_I = \bar{f}$

$f(i) = \tilde{i} + S$

Define $g: R \to \sum E_i / S$
$\ni g(r) = \bar{g}(r) + S$ Then $g(i) = \bar{g}(i) + S = \bar{f}(i) + S = f(i)$ ✓

$$
\begin{tikzcd}
0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists g"'] \arrow[ddl, dashed, "\exists h"] \arrow[dddl, dashed, bend left, "\exists j"] \\
 & \varinjlim E_i \arrow[d, "\theta"] & \\
 & \Sigma E_i \arrow[d, shift left, "p_j"] \arrow[dd, bend right=60, "p_i"'] & \\
 & E_j \arrow[u, shift left, "i_j"] \arrow[uu, bend left=60, "\alpha_j"] & \\
 & E_i \arrow[u, "\varphi_j^i"] \arrow[uuu, bend left=80, "\alpha_i"] &
\end{tikzcd}
$$

$gi = \theta f$

$hi = p_j \theta f$

$ji = p_i \theta f$

$\alpha_i = \alpha_j \varphi_j^i$

$i_j = \theta \alpha_j$

Define $k : R \longrightarrow \varinjlim E_i \ni k = \cancel{\alpha_i} j$

Then $ki = \alpha_i j i = \alpha_j \varphi_j^i p_i \theta f$

$= \alpha_j p_j \theta f$

$= \alpha_j h i$

7.72

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"'] & R \arrow[dl, "g"] \\ & \Sigma E_i/S & \end{tikzcd}$$

f factors as $I \xrightarrow{\bar f} \Sigma E_i \xrightarrow{\pi} \Sigma E_i/S$?

* ΣE_i inj $\Rightarrow \exists \bar g : R \longrightarrow \Sigma E_i \ni \bar g i = \bar f$

Define $g : R \longrightarrow \Sigma E_i/S \ni g = \pi \bar g$

Then $g i = \pi \bar g i = \pi \bar f = f$

$\therefore \Sigma E_i/S$ inj by Baer

Define $g : R \longrightarrow \Sigma E_i$ $\ni g = \tilde i \bar g$ where $\tilde i$ incl.

$\pi g i = \tilde i \bar g i = \tilde i f = f$

$$\begin{tikzcd} \Sigma E_i \arrow[d] \\ \Sigma E_i/S \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "i"] \arrow[d, "f"'] & R \\ & \Sigma E_i & \end{tikzcd}$$

R noeth $\Rightarrow I$ f.g., $I = (a_1, \ldots, a_n)$

$f(a_k) \in \Sigma E_i$ has only finitely many nonzero entries for each k, say $S_k \subseteq I$ is set of nonzero indices

Then $S = \bigcup_{k=1}^{n} S_k$ finite

And $\operatorname{Im} f \subseteq \sum_{i \in S} E_i$ finite direct sum of inj

$\rightarrow \sum_{i \in S} E_i$ inj $\Rightarrow \exists \bar g : R \rightarrow \sum_S E_i \ni \bar g i = f$

10. For a random sample of 20 American teenagers, the mean amount of time spent on the computer per day is 210.1 minutes with a sample standard deviation of 15.8 minutes. Construct a 95% confidence interval for the mean time spent on a computer per day for all American teenagers.

7.75 (F, G) adjoint pair

$F: \mathcal{C} \to \mathcal{D}$

$G: \mathcal{D} \to \mathcal{C}$

$\tau_{C,D}: \mathrm{Hom}(F(C), D) \to \mathrm{Hom}(C, G(D))$ nat bijection

(on seperate pg.)

(i) $D = F(C)$

$\tau(1_{F(C)}) = \eta_C \quad \in \mathrm{Hom}(C, G(F(C)))$

- show $\eta: 1_{\mathcal{C}} \to GF$ is nat. trans.

$\eta_C(1_{\mathcal{C}}(C)) = \eta_C(C) = G(F(C)) \quad \forall C \in \mathcal{C}$

Let $f: C \to C'$ in $\mathcal{C}$

$$\begin{tikzcd} C = 1_{\mathcal{C}}(C) \arrow[r, "\eta_C"] \arrow[d, "f = 1_{\mathcal{C}}(f)"'] & G(F(C)) \arrow[d, "G(F(f))"] \\ C = 1_{\mathcal{C}}(C') \arrow[r, "\eta_{C'}"] & G(F(C')) \end{tikzcd}$$

$G(F(f))\eta_C = G(F(f))\tau_{C,D}(1_{F(C)})$

$= G\tau_{C',D}(1_{F(C')})(f)$

$\eta_{C'}(f) = \tau_{C',D}(1_{F(C')})(f)$

$$\begin{tikzcd} \mathrm{Hom}(F(C), D) \arrow[r, "\tau_{C,D}"] & \mathrm{Hom}(C, G(D)) \\ \mathrm{Hom}(F(C'), D) \arrow[r, "\tau_{C',D}"] \arrow[u, "F(f)^*"] & \mathrm{Hom}(C', G(D)) \arrow[u, "f^*"'] \end{tikzcd}$$

$\Rightarrow \tau_{C,D} F(f)^* = f^* \tau_{C',D}$

$\tau_{C,D} F(f)^* = F(f)\tau_{C,D}$

$f^* \tau_{C',D} = \tau_{C',D}(f)$

$$\begin{tikzcd} F(G(D)) \arrow[r, "\varepsilon_D"] \arrow[d, "F(G(g))"'] & D \arrow[d, "g"] \\ F(G(D')) \arrow[r, "\varepsilon_{D'}"] & D' \end{tikzcd}$$

$g\,\varepsilon_D = g\,\tau^{-1}_{G(D),D}(1_D) = \tau_{G(D),D'}\, G(g)(1_D)$

$\varepsilon_{D'} F(G(g)) = \tau^{-1}_{G(D'),D'}(1_{D'})\, F(G(g))$

$= (F(G(g)))^* \tau^{-1}_{G(D'),D'}(1_{D'})$

$= \tau_{G(D),D'}(G(g))^*(1_{D'})$

$= \tau_{G(D),D'}(1_{D'})G(g) = \tau_{G(D),D'}\, G(g)(1_D)$

$$\begin{tikzcd}
\mathrm{Hom}(F(C'),F(C')) \arrow[r, "\tau_{C',F(C')}"] \arrow[d, "(F(f))^*"'] & \mathrm{Hom}(C', GF(C')) \arrow[d, "f^*"] \\
\mathrm{Hom}(F(C),F(C')) \arrow[r, "\tau_{C,F(C')}"] & \mathrm{Hom}(C, GF(C'))
\end{tikzcd}$$

$$f^* \tau_{C',F(C')} = \tau_{C,F(C')} (F(f))^*$$

$$\cancel{\tau_{C',F(C')} f = F(f)\, \tau_{C,F(C')}}$$

$$\eta_{C'} f = \tau_{C',F(C')}(1_{F(C')}) f \cancel{= F(f)\, \tau_{C,F(C')}(1_{F(C')})}$$

$F(C) \to F(C')$ $\quad F(C') \to F(C')$ $\quad F(C) \to F(C')$

$$= f^* \tau_{C',F(C')}(1_{F(C')})$$

$$= \tau_{C,F(C')}(F(f))^*(1_{F(C')})$$

$$= \tau_{C,F(C')}(1_{F(C')}) F(f)$$

$$= \tau_{C,F(C')} F(f) (1_{F(C)}) \checkmark$$

$C = G(D)$

$\tau^{-1}_{G(D),D} : \mathrm{Hom}(G(D),G(D)) \to \mathrm{Hom}(F(G(D)),D)$

$\tau^{-1}(1_D) = \epsilon_D \in \mathrm{Hom}(F(G(D)),D)$

- Show $\epsilon : FG \to 1_{\mathcal{D}}$ nat trans

$\epsilon_D(F(G(D))) = \tau^{-1}(1_D(F(G(D))) = \tau^{-1}(F(G(D))) = D = 1_{\mathcal{D}}(D)$

Let $g : D \to D'$

$$\begin{tikzcd}
\mathrm{Hom}(G(D),G(D)) \arrow[r, "\tau^{-1}_{G(D),D}"] \arrow[d, "(G(g))_*"'] & \mathrm{Hom}(F(G(D)),D) \arrow[d, "g_*"] \\
\mathrm{Hom}(G(D),G(D')) \arrow[r, "\tau^{-1}_{G(D),D'}"] & \mathrm{Hom}(F(G(D)),D')
\end{tikzcd}$$

$g_* \tau_{G(D),D} = \tau_{G(D),D'} (G(g))_*$

$g\, \tau_{G(D),D} = \tau_{G(D),D'}\, G(g)$

$$\begin{tikzcd}
\mathrm{Hom}(G(D'),G(D')) \arrow[r, "\tau^{-1}_{G(D'),D'}"] \arrow[d, "(G(g))^*"'] & \mathrm{Hom}(F(G(D')),D') \arrow[d, "(F(G(g)))^*"] \\
\mathrm{Hom}(G(D),G(D')) \arrow[r, "\tau^{-1}_{G(D),D'}"] & \mathrm{Hom}(F(G(D)),D')
\end{tikzcd}$$

$(F(G(g)))^* \tau_{G(D'),D'} =$

$\tau_{G(D),D'} (G(g))^*$

$f : C \to C'$

$$\begin{tikzcd} \mathrm{Hom}(F(C),F(C)) \arrow[r, "\tau_{C,F(C)}"] & \mathrm{Hom}(C, G(F(C))) \\ \mathrm{Hom}(F(C'),F(C)) \arrow[u, "F(f)^*"] \arrow[r, "\tau_{C',F(C)}"] & \mathrm{Hom}(C', G(F(C))) \arrow[u, "f^*"'] \end{tikzcd}$$

$f^* \tau_{C'} = \tau_C F(f)^*$

$\tau_{C'} f = F(f) \tau_C$

$$\begin{tikzcd} \mathrm{Hom}(F(C),F(C)) \arrow[r, "\tau_{C,F(C)}"] \arrow[d, "F(f)_*"'] & \mathrm{Hom}(C, G(F(C))) \arrow[d, "G(F(f))_*"] \\ \mathrm{Hom}(F(C),F(C')) \arrow[r, "\tau_{C,F(C')}"] & \mathrm{Hom}(C, G(F(C'))) \end{tikzcd}$$

$G(F(f))_* \tau_C = \tau_{C'} F(f)$.

$GF(f) \tau_C = \tau_{C'} F(f)$

$\mathrm{Hom}(C, G(F(C'))) \to \mathrm{Hom}(C', G(F(C')))$

$F(C) \to F(C')$

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}"] & GF(C') \end{tikzcd}$$

$\tau_C : \to F(C')$

$F(C) \to F(C')$ $F(C) \to F(C)$

$GF(f)\eta_C = GF(f)\tau_C(1_{F(C)}) = \tau_{C'} F(f)(1_{F(C)})$

$\eta_{C'} f = \tau_{C'}(1_{F(C')}) f = F(f) \tau_C (1_{F(C')})$

$F(C) \to F(C')$ $F(C') \to F(C')$

$F(C) \to F(C')$

$\eta_{C'} = \tau$

$\Rightarrow \cancel{\tau_{C',F(C)} f = \tau_{C,F(C)} F(f)}$

$f^* \tau_{C',F(C)} = \tau_{C,F(C)} F(f)^* \Rightarrow \tau_{C',F(C)} f = F(f) \tau_{C,F(C)}$

$(GF(f))_* \tau_{C,F(C)} = \tau_{C,F(C')} F(f)_* \Rightarrow GF(f) \tau_{C,F(C)} = \tau_{C,F(C')} F(f)$

$F(C) \to F(C')$

$F(C) \to F(C')$ $F(C) \to F(C)$

$GF(f)\eta_C = GF(f)\tau_{C,F(C)}(1_{F(C)}) = \tau_{C,F(C')} F(f)(1_{F(C)})$ ✓

$\eta_{C'} \cancel{f = \tau_{C',F(C)}(1_{F(C')})} f = F(f) \tau_{C,F(C)} \cancel{(1_{F(C')})}$

$F(C) \to F(C')$ $F(C') \to F(C')$

$F(C) \to F(C')$

7.78 $T: RMOD \longrightarrow AB$ additive left exact functor preserving products

- show T preserves inverse limits

Let $\{E_i, \varphi^i_j\}^{i \in I}$ inverse system R-modules

Then $\varprojlim E_i = \ker\left(\prod_{i \in I} E_i \longrightarrow \prod_{\varphi^i_j : M_i \to M_j} E_i\right)$

So $T(\varprojlim E_i) = T\left(\ker\left(\prod_{i \in I} E_i \longrightarrow \prod_{\varphi^i_j : M_i \to M_j} E_i\right)\right)$

$\cong \ker\left(T\left(\prod_{i \in I} E_i\right) \longrightarrow T\left(\prod_{\varphi^i_j : M_i \to M_j} E_i\right)\right)$ since T left exact hence preserves kernels

$\cong \ker\left(\prod_{i \in I} T(E_i) \longrightarrow \prod_{T(\varphi^i_j) : T(M_i) \to T(M_j)} T(E_i)\right)$

since T preserves products

$\cong \varprojlim T(E_i)$

$\therefore T$ preserves limits

8.5 $I \triangleleft R$

M abelian group

- show M left R/I-module iff M left R-module that is annihilated by I

($\Rightarrow$) Assume M left R/I-module

Let $m \in M$

via $(r+I)m = rm$

So $(r+I)m \in M \quad \forall r \in R \Rightarrow rm \in M \quad \forall r \in R$

$\therefore M$ left R-module since it is ab. gp already

Now $Im = (0+I)m = 0 \cdot m = 0 \quad \forall m \in M$

$\therefore I$ annihilates M

($\Leftarrow$) Assume M left R-module annihilated by I

Then via $(r+I)m = rm$ well defined action since $Im = 0$

$\therefore M$ left R/I-module

8.27 (c) $0 \to A \to B \to C \to 0$ SES left R-mods

A, C have DCC (iff)
- Show B has DCC
- Conclude $A \oplus B$ has DCC

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \operatorname{Im} f \arrow[r, hook, "i"] & B \arrow[r, "\pi"] & B/\operatorname{Im} f \arrow[r] & 0
\end{tikzcd}$$

$B/\operatorname{Im} f = \operatorname{coker} f$

Let $D = \operatorname{Im} f$

So SES's are isomorphic

Hence it suffices to show result for $0 \to D \xhookrightarrow{i} B \xrightarrow{\pi} B/D \to 0$ SES

($\Rightarrow$) Assume B has DCC
Let $D_0 \supseteq D_1 \supseteq \dots$ be a descending chain of submodules of $D \leq B$
Then they are submodules of B
$\therefore$ The chain stabilizes since B has DCC
$\therefore$ D has DCC
Let $C_0 \supseteq C_1 \supseteq \dots$ be a DC of submodules of B/D
Then each $C_i = B_i/D$ where $B_i \leq B$ containing D
And $B_0 \supseteq B_1 \supseteq \dots$
But this chain stabilizes since B has DCC
$\therefore$ $C_0 \supseteq C_1 \supseteq \dots$ stabilizes
$\therefore$ B/D has DCC

($\Leftarrow$) Assume A, C have DCC (change A to D)
Let $B_0 \supseteq B_1 \supseteq \dots$ be a DC of submodules of B
Then $A \cap B_0 \supseteq A \cap B_1 \supseteq \dots$ is a DC of submodules of A
But A has DCC, so $\exists n \ni A \cap B_n = A \cap B_{n+1} = \dots$
And $A + B_0 \supseteq A + B_1 \supseteq \dots$ is a DC of submodules of B containing A, hence $A+B_0/A \supseteq A+B_1/A \supseteq \dots$ is a DC of submodules of B/A
But B/A has DCC, so $\exists m \ni$ $A+B_m/A = A+B_{m+1}/A = \dots \Rightarrow A+B_m = A+B$
choose $k = \max\{n, m\} \Rightarrow A \cap B_k = A \cap B_{k+1} = \dots$ & $A + B_k = A + B_{k+1} = \dots$
Have $B_k \supseteq B_{k+1}$ so let $b_k \in B_k \Rightarrow a + b_k \in A + B_k = A + B_{k+1} \Rightarrow a + b_k = a' + b_{k+1} \Rightarrow a - a' = b_k + b_{k+1} \in B_k \Rightarrow b_k + b_{k+1} \in A \cap$

$= A \cap B_{k+1}$
$\Rightarrow b_k + b_{k+1} = b'_{k+1}$
$b_k = b'_{k+1} - b_{k+1}$
$\in B_{k+1}$
$\therefore B_k \leq B_{k+1}$
$\therefore B_k = B_k$
$= \dots$
$\therefore B$ has DC

SES: $0 \longrightarrow A \xrightarrow{i_1} A\oplus B \xrightarrow{p_2} B \longrightarrow 0$

And we assumed A has DCC and we showed B has DCC

$\therefore$ $A\oplus B$ has DCC by above

(ii) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ SES left R-mods

A, C have ACC (iff)

- show B has ACC
- conclude $A\oplus B$ has ACC

($\Rightarrow$) Assume B has ACC

similar to (i)

($\Leftarrow$) Assume A, C have ACC

Let $X \leq B$

Then $X \cap A \leq A$

But A has ACC $\Rightarrow$ A noeth $\Rightarrow$ $X\cap A$ f.g.

SES: $0 \longrightarrow X\cap A \hookrightarrow X \longrightarrow X/X\cap A \longrightarrow 0$

$$X/X\cap A \cong A+X/A \quad \text{2nd Iso Thm}$$

But $A+X/A \leq B/A = C$

But C has ACC, so $A+X/A$ f.g.

Claim SES $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ with L, N f.g, $N = M/L$

$\Rightarrow M$ f.g.

$L = (x_1, \dots, x_n)$, $N = (y_1 + L, \dots, y_m + L)$

Let $m \in M \Rightarrow m + L = r_1(y_1 + L) + \dots + r_m(y_m + L)$

$= r_1 y_1 + \dots r_m y_m + L$

$\Rightarrow m - r_1y_1 - \dots - r_m y_m \in L$

$\Rightarrow m - r_1 y_1 - \dots - r_m y_m = s_1 x_1 + \dots + s_n x_n$

$\Rightarrow m = r_1 y_1 + \dots + r_m y_m + s_1 x_1 + \dots + s_n x_n \in (y_1, \dots, y_m, x_1, \dots, x_n)$

$\therefore M$ f.g.

Hence X is f.g

$\therefore$ B is noeth

$\therefore$ B has ACC

Again $0 \longrightarrow A \longrightarrow A\oplus B \longrightarrow B \longrightarrow 0$ SES with A, B have ACC

$\therefore$ $A\oplus B$ has ACC by above

(iii) - Show every semisimple ring left artinian

R semisimple ring $\Rightarrow R \cong S_1 \oplus \dots \oplus S_n$, S_i simple submodules

And each S_i is left artinian since simple

$\therefore$ R left artinian R-module by (i) + by induction

$\therefore$ R left artinian ring

(i) $\Rightarrow S_1 \oplus S_2$ art

Assume $S_1 \oplus \dots \oplus S_{n-1}$ art

Then $S_1 \oplus \dots \oplus S_{n-1} \oplus S_n$ art by (i)

8.30 - Give example of ring $R \ncong R^{op}$

Klein 4-ring $\{0, a, b, c\}$

$a+a=0$
$a+b=c$
$a+c=b$
$b+a=c$
$b+b=0$
$b+c=a$
$\vdots$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

suppose φ iso

~~If $\exists$ iso, $b \xrightarrow{\varphi} a, b, c$ since zero not iso~~
~~if $b \to a$ than $a \to b, c$~~
~~Then $0 = \varphi(0) = \varphi(ab) = \varphi(b)\varphi(a)$ in R^{op}~~
~~$= a\varphi(a)$~~

b only nilpotent since $b^2 = 0$ but $a^2 = a$, $c^2 = c$

so $b \xrightarrow{\varphi} b \Rightarrow a \to a, c$

Then $0 = \varphi(0) = \varphi(ab) = \varphi(b)\varphi(a) = b\varphi(a) = b$ since $ba = bc$

contra

$\therefore R \ncong R^{op}$

(note this ring does not have 1)

8.49 k comm ring

P, Q proj k-mods

- Show $P \otimes_k Q$ proj k-mod

Since k comm, $P \otimes_k Q$ is a k-mod

Let $0 \to A \to B \to C \to 0$ SES k-mods

Show

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, A) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft^{nat}"] & \mathrm{Hom}_k(P \otimes_k Q, B) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft^{nat}"] & \mathrm{Hom}_k(P \otimes_k Q, C) \arrow[r] \arrow[d, "\cong"] & 0 \text{ SES} \\
0 \arrow[r] & \mathrm{Hom}_k(P, \mathrm{Hom}_k(Q, A)) \arrow[r] & \mathrm{Hom}_k(P, \mathrm{Hom}_k(Q, B)) \arrow[r] & \mathrm{Hom}_k(P, \mathrm{Hom}_k(Q,
\end{tikzcd}$$

by Adjoint iso Thm

Q proj $\Rightarrow$ $0 \to \mathrm{Hom}_k(Q,A) \to \mathrm{Hom}_k(Q,B) \to \mathrm{Hom}_k(Q,C) \to 0$ SES

P proj $\Rightarrow$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{Hom}_k(P, \mathrm{Hom}_k(Q, A)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft^{nat}"] & \mathrm{Hom}_k(P, \mathrm{Hom}_k(Q, B)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft^{nat}"] & \mathrm{Hom}_k(P, \mathrm{Hom}_k(Q, C)) \arrow[r] \arrow[d, "\cong"] & 0 \text{ SES} \\
0 \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, A) \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, B) \arrow[r] & \mathrm{Hom}_k(P \otimes_k Q, C) \arrow[r] & 0
\end{tikzcd}$$

By adjoint iso Thm

$\therefore$ Bottom row SES

$\therefore$ $\mathrm{Hom}_k(P \otimes_k Q, -)$ exact

$\therefore$ $P \otimes_k Q$ proj k-module

8.52 comm dgm with exact rows:

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & A_5 \arrow[d, "h_5"'] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

(c) h_2, h_4 surj
h_5 inj
– Show h_3 surj

Let $b_3 \in B_3$
Then $g_3(b_3) = h_4(a_4)$ since h_4 surj
$\Rightarrow 0 = g_4(g_3(b_3)) = g_4(h_4(a_4)) = h_5(f_4(a_4))$ by comm + exact
$\Rightarrow f_4(a_4) \in \ker h_5 = 0$ since h_5 inj
$\Rightarrow a_4 \in \ker f_4 = \operatorname{Im} f_3$
$\Rightarrow a_4 = f_3(a_3)$
$\Rightarrow g_3(b_3) = h_4(f_3(a_3)) = g_3(h_3(a_3))$
$\Rightarrow b_3 - h_3(a_3) \in \ker g_3 = \operatorname{Im} g_2$
$\Rightarrow b_3 - h_3(a_3) = g_2(b_2)$
$\Rightarrow b_3 = g_2(b_2) + h_3(a_3)$
But $b_2 = h_2(a_2)$ since h_2 surj
$\Rightarrow b_3 = g_2(h_2(a_2)) + h_3(a_3)$
$= h_3(f_2(a_2)) + h_3(a_3)$ by comm
$= h_3(f_2(a_2) + a_3)$
$\in \operatorname{Im} h_3$
$\therefore h_3$ surj

(iii) h_1, h_2, h_4, h_5 isos
– show h_3 iso
h_1 surj, h_2, h_4 inj $\Rightarrow h_3$ inj by (ii)
h_5 inj, h_2, h_4 surj $\Rightarrow h_3$ surj by (i)
$\Rightarrow h_3$ iso

9.2 R PID

M R-mod

(ii) - show every direct summand of M is pure submodule

Let X be direct summand of M

$M = X \oplus Y$, $Y \leq M$

Show $X \cap rM = rX \quad \forall r \in R$

Let $r \in R$

$$X \cap rM = X \cap r(X \oplus Y) = X \cap (rX \oplus rY)$$
$$= (X \cap rX) \oplus (X \cap rY)$$
$$= rX \oplus 0 \quad \text{since } rX \subseteq X \text{ since } X \text{ R-mod}$$
$$\text{and since } X \cap rY \subseteq X \cap Y = 0$$
$$= rX$$

$\therefore X \cap rM = rX \quad \forall r \in R$

$\therefore X$ pure submodule

(iii) - show $t(M)$ torsion submodule of M is pure submodule

~~SES $0 \to t(M) \xrightarrow{i} M \xrightarrow{\pi} M/t(M) \to 0$~~

~~First note that $M/t(M)$ is torsion free~~

Let $r \in R$ and $y \in t(M) \cap rM$ If $r = 0$, result trivial so assume $r \neq 0$

So $y = rm$ for some $m \in M$ and $\exists\, 0 \neq z \in R \ni zy = 0$

$0 = zy = zrm = rzm$

Then $m \in t(M)$

hence $y = rm \in rt(M)$

$\therefore t(M) \cap rM \subseteq rt(M)$

Now let $y \in rt(M) \Rightarrow y = rm$, $m \in t(M)$

Then $y = rm \in t(M)$ since $t(M)$ R-module

And $y = rm \in rM$

$\therefore y \in t(M) \cap rM$

$\therefore rt(M) \subseteq t(M) \cap rM$ $\therefore t(M) \cap rM = rt(M) \ \forall r$

$\therefore t(M)$ pure

(iv) M/S torsion free

- Show S pure submodule of M

Let $r \in R$ and let $x \in S \cap rM$

so $x = rm$ for some $m \in M$ and $x \in S$

$x + S = 0_{M/S}$ since $x \in S$

so $0_{M/S} = x + S = rm + S = r(m+S)$

But M/S torsion free $\Rightarrow$ $r = 0$ or $m \in S$

If $r = 0$ $\therefore x = 0 \cdot m = 0 \in 0 \cdot S = rS$

if $m \in S$, then $x = rm \in rS$

$\therefore S \cap rM \subseteq rS$

Now let $x \in rS \Rightarrow x = rs$, $s \in S$

Then $x = rs \in S$ since S R-mod

and $x = rs \in rM$ since $S \leq M$

$\therefore rS \subseteq S \cap rM$

$\therefore S \cap rM = rS$

$\therefore S$ pure

(v) $\mathcal{S}$ family pure submodules of M that is chain under inclusion

- Show $\bigcup_{S \in \mathcal{S}} S$ pure submodule of M

First show $\bigcup_{S \in \mathcal{S}} S \leq M$

Let $s_1, s_2 \in \bigcup_{S \in \mathcal{S}} S \Rightarrow s_1 \in S_1$, $s_2 \in S_2 \in \mathcal{S}$

But $\bigcup_{S \in \mathcal{S}} S$ chain under inclusion $\Rightarrow$ $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$

WLOG say $S_1 \subseteq S_2$

Then $s_1 + s_2 \in S_2 \in \bigcup_{S \in \mathcal{S}} S$

Let $r \in R$, $s \in \bigcup_{S \in \mathcal{S}} S \Rightarrow s \in S \in \mathcal{S}$

$rs \in S$ since $S \leq M$ R-module

$\in \bigcup_{S \in \mathcal{S}} S$

Let $s \in \bigcup_{S \in \mathcal{S}} S \Rightarrow s \in S$ for some $S \in \mathcal{S} \Rightarrow s \in S \subseteq M$

$\therefore \bigcup_{S \in \mathcal{S}} S \leq M$

Now Let $r \in R$ and $x \in \bigcup_{S \in \mathcal{S}} S \cap rM \Rightarrow x = rm$, $m \in M$ and $x \in S$ for some $S \in \mathcal{S}$

But S pure, so $x \in S \cap rM = rS \Rightarrow x = rs$, $s \in S \subseteq \bigcup_{S \in \mathcal{S}} S$ $\Rightarrow x \in r \bigcup_{S \in \mathcal{S}} S \Rightarrow \bigcup_{S \in \mathcal{S}} S \cap rM = r\bigcup$ $\therefore \bigcup_{S \in \mathcal{S}} S$ pure

(vi) - Give example of pure submodule that is not a direct summand

~~Consider $\mathbb{R}/\mathbb{Z}$ $\mathbb{Z}$-module~~

~~Then $t(\mathbb{R}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$~~

Need M not finitely generated

~~$\mathbb{R}$ field $\Rightarrow$ $\mathbb{R}[x]$ PID~~

~~Look at $M = F$~~

$R = \mathbb{Z}$ PID

$M = \prod_p \mathbb{Z}/p\mathbb{Z}$ (not f.g.)

$t(M) = \sum_p \mathbb{Z}/p\mathbb{Z}$ pure but not direct summand of M

9.5 M torsion R-mod

R int dom

- Show $\mathrm{Hom}_R(M,M) \cong \prod_p \mathrm{Hom}_R(M_p, M_p)$ where M_p is the p-primary component of M

$M_p = \{ m \in M : p^n m = 0 \text{ for some } n \geq 1 \}$

9.8

(i) R int dom

$r \in R$

M R-mod

$\mu_r : M \longrightarrow M \ni \mu_r(m) = rm$

- show $\forall$ A R-mod $(\mu_r)_*$ and $(\mu_r)^*$ also mult by r

$(\mu_r)_* : Hom_R(A, M) \longrightarrow Hom_R(A, M) \ni (\mu_r)_*(f) = \mu_r f$

$(\mu_r)_*(f)(a) = \mu_r(f(a)) = rf(a) \quad \forall a \in A$

so $(\mu_r)_*(f) = rf$

$\therefore (\mu_r)_*$ mult by r

$(\mu_r)^* : Hom_R(M, A) \longrightarrow Hom_R(M, A) \ni (\mu_r)^*(f) = f\mu_r$

$(\mu_r)^*(f)(a) = f(\mu_r(a)) = f(ra) = rf(a) \quad \forall a \in A$

$\therefore (\mu_r)^*(f) = rf$

$\therefore (\mu_r)^*$ mult by r

(ii) R int dom

$Q = Frac(R)$

- show $Hom_R(Q, M)$, $Hom_R(M, Q)$ are vector spaces over Q

$\forall$ M R-mod

$Hom_R({}_RQ_Q, {}_RM) \Rightarrow Hom_R(Q, M)$ left Q-module

But Q field, so $Hom_R(Q, M)$ vectorspace over Q

Similarly $Hom_R({}_RM, {}_RQ_Q)$ is a right Q-module hence vectorspace over Q

9.20 $\{D_i : i \in I\}$ family divisible ab. gps

- Show $\prod_{i \in I} D_i \cong$ direct sum divisible groups

10.25 comm dgm with exact rows:

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] & B' \arrow[r, "g"] & C' &
\end{tikzcd}$$

(c) – show $\Delta : \ker\gamma \to \operatorname{coker}\alpha \ni \Delta(z) = i^{-1}(\beta(p^{-1}(z))) + \operatorname{Im}\alpha$ well defined homom.

p surj $\Rightarrow \exists b \in B \ni z = p(b)$

But $z \in \ker\gamma \Rightarrow 0 = \gamma(z) = \gamma(p(b)) = g(\beta(b))$

$\Rightarrow \beta(b) \in \ker g = \operatorname{Im} i$

$\Rightarrow \beta(b) = i(a')$

Define $\Delta(z) = a' + \operatorname{Im}\alpha$

If $\exists b_1, b_2 \in B \ni z = p(b_1) = p(b_2)$

$\Rightarrow p(b_1) - p(b_2) = 0 \Rightarrow p(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 \in \ker p = \operatorname{Im} f$

$\Rightarrow b_1 - b_2 = f(a)$

$\Rightarrow \beta(b_1 - b_2) = \beta(f(a)) = i(\alpha(a))$

$\cancel{p(b_1) = p(b_2) \Rightarrow \gamma(p(b_1)) = \gamma(p(b_2))}$

$\cancel{\Rightarrow g(\beta(b_1)) = g(\beta(b_2))}$

$\cancel{\Rightarrow g(\beta(b_1) - \beta(b_2)) = 0}$

$\cancel{\Rightarrow \beta(b_1) - \beta(b_2) \in \ker g}$

$\beta(b_1) = i(a_1') \quad \beta(b_2) = i(a_2')$

$i(a_1' - a_2') = \beta(b_1 - b_2) = i(\alpha(a))$

$\Rightarrow a_1' - a_2' = \alpha(a) \in \operatorname{Im}\alpha$

$\Rightarrow a_1' + \operatorname{Im}\alpha = a_2' + \operatorname{Im}\gamma$

$\therefore$ wd

$z_1 = p(b_1),\ z_2 = (b_2) \Rightarrow z_1 + z_2 = p(b_1) + p(b_2) = p(b_1 + b_2)$

$\beta(b_1) = i(a_1'),\ \beta(b_2) = i(a_2') \Rightarrow \beta(b_1 + b_2) = \beta(b_1) + \beta(b_2) = i(a_1') + i(a_2') = i(a_1' + a_2')$

$\therefore \Delta$ R-mod homom

$z = p(b)$

$\beta(b) = i(a')$

$rz = rp(b) = p(rb)$

$\beta(rb) = r\beta(b) = ri(a')$

$= i(ra')$

$\Delta(rz) = ra' + \operatorname{Im}\alpha$

$= r(a' + \operatorname{Im}\alpha)$

$= r\Delta(z)$

$\Delta(z_1 + z_2) = a_1' + a_2' + \operatorname{Im}\gamma$

$= a_1' + \operatorname{Im}\alpha + a_2' + \operatorname{Im}$

$\Uparrow \quad = \Delta(z_1) + \Delta(z_2)$

(ii) - show $\exists$ exact seq:

$$\operatorname{Ker}\alpha \xrightarrow{\bar f} \operatorname{Ker}\beta \xrightarrow{\bar p} \operatorname{Ker}\gamma \xrightarrow{\Delta} \operatorname{coker}\alpha \xrightarrow{h} \operatorname{coker}\beta \xrightarrow{j} \operatorname{coker}\gamma$$

$$\left(\begin{array}{l} h(a'+\operatorname{Im}\alpha) = i(a') + \operatorname{Im}\beta \\ j(b'+\operatorname{Im}\beta) = g(b') + \operatorname{Im}\gamma \end{array} \right)$$

$$\begin{tikzcd}
\operatorname{Ker}\beta \arrow[r, hook] & B \arrow[r, "\beta"] & B' \\
& A \arrow[u, "f"] \arrow[ur, bend right, "i\alpha"'] & \\
& \operatorname{Ker}\alpha \arrow[u, hook] \arrow[uul, dashed, "\exists \bar f"] \arrow[uur, bend right, "0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Ker}\gamma \arrow[r, hook] & C \arrow[r, "\gamma"] & C' \\
& B \arrow[u, "p"] \arrow[ur, bend right, "g\beta"'] & \\
& \operatorname{Ker}\beta \arrow[u, hook] \arrow[uul, dashed, "\exists \bar p"] \arrow[uur, bend right, "0"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \operatorname{Ker}\alpha \arrow[r, "\bar f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker}\beta \arrow[r, "\bar p"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker}\gamma \arrow[d, hook] & \\
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

columns exact, bottom 2 rows exact, commute

$\Rightarrow$ top row exact by 3×3 lemma

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\beta f"'] \arrow[dd, bend right, "0"'] & A' \arrow[r, "\pi"] \arrow[d, "i"] & \operatorname{coker}\alpha \arrow[ddl, dashed, "\exists h"] \\
& B' \arrow[d, "\pi"] & \\
& \operatorname{coker}\beta &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[dr, "\gamma p"'] \arrow[dd, bend right, "0"'] & B' \arrow[r, "\pi"] \arrow[d, "g"] & \operatorname{coker}\beta \arrow[ddl, dashed, "\exists j"] \\
& C' \arrow[d] & \\
& \operatorname{coker}\gamma &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C' \arrow[d] & \\
& \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma &
\end{tikzcd}$$

comm
top two exact
columns exact
$\therefore$ bottom row exact

Let $c \in \operatorname{Im}\bar{p}$

$\Rightarrow c = \bar{p}(b)$, $b \in \ker\beta$

$\Rightarrow \Delta(c) = \Delta(\bar{p}(b)) = \Delta(p(b))$ by comm.

$= a' + \operatorname{Im}\alpha$ where $\beta(b) = i(a')$

But $0 = \beta(b) = i(a') \Rightarrow a' \in \ker i = 0$

$\therefore \Delta(c) = 0 + \operatorname{Im}\alpha = \operatorname{Im}\alpha = 0_{\operatorname{coker}\alpha}$

$\therefore c \in \ker\Delta$

$\therefore \operatorname{Im}\bar{p} \subseteq \ker\Delta$

Now let $c \in \ker\Delta \Rightarrow 0_{\operatorname{coker}\alpha} = \Delta(c) = a' + \operatorname{Im}$ where $c = p(b)$ and $\beta(b) = i(a')$

$\Rightarrow \operatorname{Im}\alpha = a' + \operatorname{Im}\alpha$

$\Rightarrow a' \in \operatorname{Im}\alpha$

$\Rightarrow a' = \alpha(a) \Rightarrow \beta(b) = i(a') = i(\alpha(a)) = \beta(f(a))$

$\Rightarrow \beta(b - f(a)) = 0$

$\Rightarrow b - f(a) \in \ker\beta$

$\Rightarrow \bar{p}(b - f(a)) = p(b - f(a)) = p(b) - p(f(a)) = p(b) = c$

$\therefore c \in \operatorname{Im}\bar{p}$

$\therefore \ker\Delta \subseteq \operatorname{Im}\bar{p}$

$\therefore \operatorname{Im}\bar{p} = \ker\Delta$

Now let $a' + \operatorname{Im}\alpha \in \operatorname{Im}\Delta \Rightarrow a' + \operatorname{Im}\alpha = \Delta(c)$, $c \in \ker\gamma$

And $i(a') = \beta(b)$ where $c = p(b)$

$h(a' + \operatorname{Im}\alpha) = i(a') + \operatorname{Im}\beta = \beta(b) + \operatorname{Im}\beta = \operatorname{Im}\beta = 0_{\operatorname{coker}\beta}$

$\therefore a' + \operatorname{Im}\alpha \in \ker h$

$\therefore \operatorname{Im}\Delta \subseteq \ker h$

Now let $a' + \operatorname{Im}\alpha \in \ker h \Rightarrow 0 = h(a' + \operatorname{Im}\alpha) = i(a') + \operatorname{Im}\beta$

$\Rightarrow i(a') \in \operatorname{Im}\beta$

$\Rightarrow i(a') = \beta(b)$

$\Rightarrow 0 = g(i(a')) = g(\beta(b)) = \gamma(p(b))$

$\Rightarrow p(b) \in \ker\gamma$

$\Rightarrow \Delta(p(b)) = a' + \operatorname{Im}\alpha$

$\therefore a' + \operatorname{Im}\alpha \in \operatorname{Im}\Delta$

$\ker h \subseteq \operatorname{Im}\Delta$

$\therefore \operatorname{Im}\Delta = \ker h$

$\therefore$ Exact

(iii) comm dgm with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A_n' \arrow[r, "f_n"] \arrow[d, "d_n'"] & A_n \arrow[r, "g_n"] \arrow[d, "d_n"] & A_n'' \arrow[r] \arrow[d, "d_n''"] & 0 \\ 0 \arrow[r] & A_{n-1}' \arrow[r, "f_{n-1}"] & A_{n-1} \arrow[r, "g_{n-1}"] & A_{n-1}'' \arrow[r] & 0 \end{tikzcd}$$

- show dgm comm that has exact rows:

$$\begin{tikzcd} & A_n'/\mathrm{Im}\,d_{n+1}' \arrow[r, "h"] \arrow[d, "\bar{d}_n'"] & A_n/\mathrm{Im}\,d_{n+1} \arrow[r, "j"] \arrow[d, "\bar{d}_n"] & A_n''/\mathrm{Im}\,d_{n+1}'' \arrow[r] \arrow[d, "\bar{d}_n''"] & 0 \\ 0 \arrow[r] & \ker d_{n-1}' \arrow[r, "\bar{f}"] & \ker d_{n-1} \arrow[r, "\bar{p}"] & \ker d_{n-1}'' & \end{tikzcd}$$

snake $\Rightarrow$ $0 \to \ker d_n' \to \ker d_n \to \ker d_n'' \to \operatorname{coker} d_n' \to \operatorname{coker} d_n$
$\to \operatorname{coker} d_n'' \to 0$ exact $\forall n$

hence rows exact

$\bar{d}_n'(a_n' + \mathrm{Im}\,d_{n+1}') = \bar{d}_n'(a_n')$ $\quad \ker \bar{d}_n' = \{a_n' + \mathrm{Im}\,d_{n+1}' \mid d_n'(a_n') = 0\} = \ker d_n'$

$\bar{d}_n(h(a_n' + \mathrm{Im}\,d_{n+1}')) = \bar{d}_n(f_n(a_n') + \mathrm{Im}\,d_{n+1}) = d_n(f_n(a_n'))$
$= f_{n-1}(d_n'(a_n'))$ ✓

$\bar{f}(\bar{d}_n'(a_n' + \mathrm{Im}\,d_{n+1}')) = \bar{f}(d_n'(a_n')) = f_{n-1}(d_n'(a_n'))$ ✓

$\bar{d}_n''(j(a_n + \mathrm{Im}\,d_{n+1})) = \bar{d}_n''(g_n(a_n) + \mathrm{Im}\,d_{n+1}'') = d_n''(g_n(a_n))$
$= g_{n-1}(d_n(a_n))$.

$\bar{p}(\bar{d}_n(a_n + \mathrm{Im}\,d_{n+1})) = \bar{p}(d_n(a_n)) = g_{n-1}(d_n(a_n))$ ✓

$\therefore$ comm

(iv) - show $\exists$ LES

snake $\Rightarrow$ $\ker \bar{d}_n' \to \ker \bar{d}_n \to \ker \bar{d}_n'' \to \operatorname{coker} \bar{d}_n' \to \operatorname{coker} \bar{d}_n \to \operatorname{coker} \bar{d}_n''$

$\ker \bar{d}_n' = \ker d_n'/\mathrm{Im}\,d_{n+1}' = H_n(A')$

$\ker \bar{d}_n = H_n(A)$, $\quad \ker \bar{d}_n'' = H_n(A'')$

$\operatorname{coker} \bar{d}_n' = \ker d_{n-1}'/\mathrm{Im}\,d_n' = H_{n-1}(A')$

$\operatorname{coker} \bar{d}_n = H_{n-1}(A)$, $\quad \operatorname{coker} \bar{d}_n'' = H_{n-1}(A'')$ $\quad \forall n$

$\therefore \exists$ LES

10.28

$C.$ complex with $d_n \equiv 0 \;\forall n$

- show $H_n(C.) \cong C_n \;\forall n$

$$\cdots \longrightarrow C_{n+1} \xrightarrow[\;=\,0\;]{d_{n+1}} C_n \xrightarrow[\;=\,0\;]{d_n} C_{n-1} \longrightarrow \cdots$$

$$H_n(C.) = \ker d_n / \operatorname{Im} d_{n+1} = C_n/0 \cong C_n$$

$\therefore H_n(C.) \cong C_n \;\forall n$

10.31

(i) Direct system of complexes $\{C.^i, \varphi^i_j\}$

- show $\varinjlim C.^i$ exists

$$\begin{tikzcd}
\cdots \arrow[r] & C^i_{n+1} \arrow[r] \arrow[d, "(\varphi^i_j)_{n+1}"'] & C^i_n \arrow[r] \arrow[d, "(\varphi^i_j)_n"'] & C^i_{n-1} \arrow[r] \arrow[d, "(\varphi^i_j)_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C^j_{n+1} \arrow[r] & C^j_n \arrow[r] & C^j_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Claim $\varinjlim C.^i$ = complex:

$$i\lambda d^j_{n+1}(\varphi^i_j)_{n+1} \cdots \longrightarrow \varinjlim C^i_{n+1} \longrightarrow \varinjlim C^i_n \longrightarrow \varinjlim C^i_{n-1} \longrightarrow \cdots$$

$$\begin{tikzcd}
C^i_{n+1} \arrow[r, "i^i_{n+1}"] \arrow[d, "(\varphi^i_j)_{n+1}"'] \arrow[rr, bend left, "i\lambda d^j_{n+1}(\varphi^i_j)_{n+1}"] & \varinjlim C^i_{n+1} \arrow[r, dashed, "\exists \theta^i_{n+1}"] & \varinjlim C^i_n \\
C^j_{n+1} \arrow[ur, "i^j_{n+1}"'] \arrow[urr, bend right, "i^j_n d^j_{n+1}"'] & &
\end{tikzcd}$$

$\ni \; \theta^i_{n+1}\, i^j_{n+1} = i^j_n\, d^j_{n+1}$

$\Downarrow$

$\theta^i_n\, \theta_{n+1}\, i^j_{n+1} = i^j_{n-1}\, d^j_n\, i^j_n\, d^j_{n+1}$

$$\begin{tikzcd}
\arrow[r] & \varinjlim C^i_{n+1} \arrow[r] \arrow[d, equal] & \varinjlim C^i_n \arrow[r] \arrow[d, equal] & \varinjlim C^i_{n-1} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \bigoplus_{i \in I} C^i_{n+1}/\sim \arrow[r] & \bigoplus_{i \in I} C^i_n/\sim \arrow[r] & \bigoplus_{i \in I} C^i_{n-1}/\sim \arrow[r] & \cdots
\end{tikzcd}$$

$= \bigoplus_{i \in I} C.^i/\sim$

complex since direct sum of complexes is a complex

+ by construction it satisfies UMP

(ii) $\{C_\cdot^i, \varphi_j^i\}$ direct system complexes over directed index set

– show $H_n(\varinjlim C_\cdot^i) \cong \varinjlim H_n(C_\cdot^i) \quad \forall n \geq 0$

$$H_n(\varinjlim C_\cdot^i) = H_n(\cdots \to \varinjlim C_{n+1}^i \xrightarrow{\varinjlim d_{n+1}^i} \varinjlim C_n^i \xrightarrow{\varinjlim d_n^i} \varinjlim C_{n-1}^i \to \cdots)$$

$$\cancel{H_n(\varinjlim C_\cdot^i) = H_n(\mathrm{coker}(\oplus C_\cdot^i \xrightarrow{\varphi_j^i} \oplus C_\cdot^i))}$$
$$\cancel{= \mathrm{coker}(H_n(\oplus C_\cdot^i \xrightarrow{\varphi_j^i} \oplus C_\cdot^i))}$$

$\varinjlim H_n(C_\cdot^i) =$

$\varinjlim$ exact since directed

SES $0 \to \mathrm{Im}\, d_{n+1}^i \hookrightarrow \ker d_n^i \to H_n(C_\cdot^i) \to 0$

$\Rightarrow 0 \to \varinjlim \mathrm{Im}\, d_{n+1}^i \hookrightarrow \varinjlim \ker d_n^i \to \varinjlim H_n(C_\cdot^i) \to 0$ SES

$\Rightarrow \varinjlim H_n(C_\cdot^i) \cong \varinjlim \ker d_n^i / \varinjlim \mathrm{Im}\, d_{n+1}^i$

$\cong \ker \varinjlim d_n^i / \mathrm{Im} \varinjlim d_{n+1}^i$ since ~~colim is right~~

$0 \to \ker f \hookrightarrow A \xrightarrow{f} B$

$\varinjlim H_n(C_\cdot^i) \cong \mathrm{coker}(\varinjlim \mathrm{Im}\, d_{n+1}^i \hookrightarrow \varinjlim \ker d_n^i) \cong H_n(\varinjlim C_\cdot^i)$

$\cong \varinjlim \mathrm{coker}(\mathrm{Im}\, d_{n+1}^i \to \ker d_n^i)$ since $\varinjlim$ right exact

$\cancel{\cong \varinjlim H_n(C_\cdot^i)}$

10.37

(i) $T: RMOD \to SMOD$ exact additive functor

R, S rings

P proj $\Rightarrow T(P)$ projective

B left R-mod

P_B deleted proj res of B

- show $T(P_B)$ deleted proj res of $T(B)$

$P_\cdot : \quad \cdots \to P_2 \to P_1 \to P_0 \to B \to 0$ exact with P_i proj

$P_B : \quad \cdots \to P_2 \to P_1 \to P_0 \to 0$

$T(P_B) : \quad \cdots \to T(P_2) \to T(P_1) \to T(P_0) \to 0$

And $T(P_\cdot) : \quad \cdots \to T(P_2) \to T(P_1) \to T(P_0) \to T(B) \to 0$

is exact since T is exact and each $T(P_i)$ is projective

so $T(P_\cdot)$ is a projective resolution

$\therefore T(P_B)$ deleted proj res of $T(B)$

(ii) A R-algebra, flat R-module

R comm ring

B A-module (hence R-module) Typo!

- show $A \otimes_R \operatorname{Tor}_n^R(B, C) \cong \operatorname{Tor}_n^A(B, A \otimes_R C)$ $\forall$ R-mods C, $\forall n \geq 0$

Note that $A \otimes_R -$ exact and if P proj A-mod, $A \otimes_R P$ proj A

Claim if X proj R-mod, then $Y \otimes_R X$ is proj

Let $0 \to A \to B \to C \to 0$ SES

$$\begin{array}{ccccccc} 0 \to & \operatorname{Hom}_R(Y \otimes_R X, A) & \to & \operatorname{Hom}_R(Y \otimes_R X, B) & \to & \operatorname{Hom}_R(Y \otimes_R X, C) & \to 0 \\ & \wr\wr & & \wr\wr & & \wr\wr & \\ 0 \to & \operatorname{Hom}_R(Y, \operatorname{Hom}_R(X, A)) & \to & \operatorname{Hom}_R(Y, \operatorname{Hom}_R(X, B)) & \to & \operatorname{Hom}_R(Y, \operatorname{Hom}_R(X, B)) & \end{array}$$

Let $P_\cdot : \ \cdots \to P_2 \to P_1 \to P_0 \to C \to 0$ proj res of R-mods

$B \otimes_R P_\cdot : \ \cdots \to B \otimes_R P_2 \to B \otimes_R P_1 \to B \otimes_R P_0 \to 0$

$\mathrm{Tor}_n^R(B,C) = H_n(B \otimes_R P_\cdot)$

$A \otimes_R \mathrm{Tor}_n^R(B,C) = A \otimes_R H_n(B \otimes_R P_\cdot) \quad \forall n \geq 0$

Then $A \otimes_R P_\cdot : \ \cdots \to A \otimes_R P_2 \to A \otimes_R P_1 \to A \otimes_R P_0 \to A \otimes_R C \to 0$

proj res of A-mods by (i) ~~since A flat R-mod hence exact and $A \otimes_R P_i$ proj for each i since A proj A-module and P_i proj A-module R-mod, hence A-mod.~~

$B \otimes_A (A \otimes_R P_\cdot) : \ \cdots \to B \otimes_A (A \otimes_R P_2) \to B \otimes_A (A \otimes_R P_1) \to B \otimes_A (A \otimes P_0) \to 0$

$\mathrm{Tor}_n^A(B, A \otimes_R C) = H_n(B \otimes_A (A \otimes_R P_\cdot))$

$\cong H_n(B \otimes_R P_\cdot)$

$\cong A \otimes_R H_n(B \otimes_R P_\cdot)$ by claim

$\cong (A \otimes_R P_i) \otimes_A B \cong A \otimes_R (P_i \otimes_A B) \cong A \otimes_R (B \otimes_A P_i) \cong (B \otimes_A P_i) \otimes_R A \cong B \otimes_A (P_i \otimes_R A)$

$\cong (B \otimes_A A) \otimes_R P_i \cong B \otimes_R P_i$

$B \otimes_A (P_i \otimes_R A)$

$\to 0$

$H_n(F(C_\cdot)) \cong \mathrm{coker}(\mathrm{Im}\,F(d_{n+1}) \xhookrightarrow{i} \ker F(d_n))$

$\cong \mathrm{coker}(F(\mathrm{Im}\,d_{n+1}) \xrightarrow{F(i)} F(\ker d_n))$

$\cong F(\mathrm{coker}(\mathrm{Im}\,d_{n+1} \hookrightarrow \ker d_n))$

$\cong F(H_n(C_\cdot))$

$0 \to \mathrm{Im} \to \ker \to H_n(B \otimes_R P_\cdot) \to 0$ exact

$0 \to A \otimes_R \mathrm{Im} \to A \otimes_R \ker \to A \otimes_R H_n(B \otimes_R P) \to 0$ exact since A flat.

$\mathrm{Tor}_n^R(B,C) \cong \mathrm{Tor}_n^A(B, A \otimes_R C)$

$\cong$

$0 \to \mathrm{Im}(A \otimes_R -) \to \ker(A \otimes -) \to \cdots \to B \otimes_A$

since $A \otimes -$ exact

$A \otimes_R H_n(B \otimes_R P) \cong \mathrm{coker}(A \otimes_R \mathrm{Im} \to A \otimes_R \ker)$

$\cong A \otimes_R \mathrm{coker}(\mathrm{Im} \hookrightarrow \ker)$ $A \otimes -$ presrves kernels, images

$\cong$

claim $C_\cdot$ complex $\wedge$ $F(H_n(C_\cdot)) \cong H_n(F(C_\cdot))$ $\forall$ exact functors F

$P_\cdot \to C$ proj res $\Rightarrow$ $P_\cdot$ exact $\Rightarrow F(P_\cdot) \to F(C)$ proj res

SES's: $0 \to \mathrm{Im}\,d_{n+1} \to \ker d_n \to H_n(C_\cdot) \to 0$ $\Rightarrow$ $0 \to F(\mathrm{Im}) \to F(\ker) \to F(M) \to 0$ SES since F exact

$0 \to \mathrm{Im}\,F(d_{n+1}) \hookrightarrow \ker F(d_n) \to H_n(F(C_\cdot)) \to 0$ since $F(C_\cdot)$ complex

10.40 R int dom

A R-mod

(i) $\mu_r : A \to A \ni \mu_r(a) = ra$ injective $\forall r \neq 0$

- show A torsion free

Let $0 \neq a \in A \ni ra = 0$

suppose $r \neq 0$

$\Rightarrow 0 = ra = \mu_r(a) \Rightarrow a \in \ker \mu_r = 0$ since μ_r inj

Contradiction since $a \neq 0$

$\therefore r = 0$

$\therefore A$ torsion free

(ii) $\mu_r : A \to A \ni \mu_r(a) = ra$ surjective $\forall r \neq 0$

- show A divisible

Let $y \in A$ and $0 \neq r \in R$ (left reg)

Then $y = \mu_r(a)$ for some $a \in A$ since μ_r surj $\forall r \neq 0$

$= ra$

$\therefore A$ divisible

(iii) $\mu_r : A \to A$ $\mu_r(a) = ra$ iso $\forall r \neq 0$

- show A vector space over Q where $Q = \mathrm{Frac}(R)$

Then A is torsion free divisible by (i),(ii)

Let $\frac{r}{s} \in Q$ and $a \in A$

Then $\frac{r}{s} a = \frac{r}{s} \cdot sx$ since $s \neq 0$ and A divisible and $sx \neq 0$ since A torsion free

$= rx$

$\in A$ since A R-mod

$\therefore A$ Q-module

$\therefore A$ vector space over Q since Q field

(iv) Either C or A vector space over Q

~ show $\mathrm{Tor}_n^R(C,A), \mathrm{Ext}_R^n(C,A)$ vector spaces over Q

Note that X vs over Q iff X tf and div

So either $C \xrightarrow{r} C$, $A \xrightarrow{r} A$ iso

$\Rightarrow \mathrm{Tor}_n^R(C,A) \xrightarrow[\text{+ sim for Ext}]{r} \mathrm{Tor}_n^R(C,A)$ iso (since $0 \to C \xrightarrow{r} C \to 0 \to 0$ SES LES ...)

$\Rightarrow$ So $Tor_n^R(C,A)$, $Ext_R^n(C,A)$ divisible + t.f. by (i), (ii)

$\therefore$ $Tor_n^R(C,A)$, $Ext_R^n(C,A)$ vector spaces over $\mathbb{Q}$

10.54 A, B finite ab gps

- show $Tor_1^{\mathbb{Z}}(A,B) \cong A \otimes_{\mathbb{Z}} B$

$P_\cdot : \; \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B$ proj res

$A \otimes_{\mathbb{Z}} P_\cdot : \; \cdots \longrightarrow A \otimes_{\mathbb{Z}} P_2 \longrightarrow A \otimes_{\mathbb{Z}} P_1 \longrightarrow A \otimes_{\mathbb{Z}} P_0 \longrightarrow 0$

$H_1(A \otimes_{\mathbb{Z}} P_\cdot) =$

$\mathbb{Z} \xrightarrow{\varphi} B \Rightarrow \varphi(1)$ has finite order

$\Rightarrow n\varphi(1) = \bar{0}$ for some n

$\Rightarrow \varphi(n) = \bar{0}$ for some n

Suppose $\varphi(1) = 0$

$\varphi(a) = a \Rightarrow a\varphi(1) = a$

$\Rightarrow$

$B \cong \prod \mathbb{Z}/n_i\mathbb{Z}$ for some n, $A \cong \prod \mathbb{Z}/m_i\mathbb{Z}$, m

$0 \longrightarrow \operatorname{Im}(n_c) \hookrightarrow A^{n} \longrightarrow A^{n}/\operatorname{Im}(n_c) \longrightarrow$

$A^n/\operatorname{Im}(n_c) \cong \operatorname{coker}(\operatorname{Im}(n_c) \hookrightarrow$

$0 \xrightarrow{d^{n+1}} D_n \xrightarrow{d_n} D_{n-1}$

$0 \longrightarrow \operatorname{Im} d^{n+1} \longrightarrow \ker d_n \longrightarrow H_n(\ldots$

$P_\cdot :$ $\quad 0 \longrightarrow \prod \mathbb{Z} \xrightarrow{\prod n} \prod \mathbb{Z} \xrightarrow{\prod} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$

$A \otimes_{\mathbb{Z}} P_\cdot :$ $\quad 0 \longrightarrow A \otimes_{\mathbb{Z}} \prod \mathbb{Z} \longrightarrow A \otimes_{\mathbb{Z}} \prod \mathbb{Z} \longrightarrow 0$

$\prod (A \otimes \mathbb{Z}) \cong \prod A \qquad \prod A$

$0 \longrightarrow \prod A \xrightarrow{\prod n} \prod A \longrightarrow 0$

$(1) \qquad (0)$

$\therefore Tor \cong \mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong A \otimes_{\mathbb{Z}} B$

$\bar{a} \in \ker n$ iff $n\bar{a} = \bar{0}$ iff $m \mid na$

iff $\frac{m}{d} \mid \frac{n}{d} a$ iff $\frac{m}{d} \mid a$

iff $m \mid da$ iff

$d\bar{a} = \bar{0}$ in $\mathbb{Z}/m$

$\therefore \ker n \cong \mathbb{Z}/d$

$Tor_1^{\mathbb{Z}}(A,B) = H_1(A \otimes_{\mathbb{Z}} P_\cdot) = \ker n / 0 = \ker n$

$0 \longrightarrow \prod \mathbb{Z} \xrightarrow{\prod n} \prod \mathbb{Z} \xrightarrow{\prod \pi} \prod \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ SES

$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \prod \mathbb{Z} \xrightarrow{1 \otimes \prod n} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \prod \mathbb{Z} \xrightarrow{1 \otimes \prod \pi} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \prod \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ exact

$\Rightarrow \pi \otimes 1$ surj $\Rightarrow \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \prod \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \prod \mathbb{Z} / \ker(1 \otimes \prod \pi) \cong \prod \mathbb{Z}/m\mathbb{Z} / \operatorname{Im}(1 \otimes \prod n) \cong \prod \mathbb{Z}/m\mathbb{Z} / \prod n\mathbb{Z}/m\mathbb{Z}$

$\cong \prod \mathbb{Z}/d\mathbb{Z}$

$0 \to I^2 \hookrightarrow I \to I/I^2 \to 0$

$0 \to I/I^2 \otimes I^2 \to I/I^2 \otimes I \to I/I^2 \otimes I/I^2 \to$

$\mathrm{Tor}(I/I^2, I^2) \to \ldots$

10.57 k field

$R = k[x,y]$

$I = (x,y)$

(i) Show $x \otimes y - y \otimes x \in I \otimes_R I$ is nonzero

$$xy(x\otimes y - y \otimes x) = xy(x \otimes y) - xy(y \otimes x)$$
$$= xy \otimes xy - xy \otimes xy$$
$$= 0$$

Suppose $x \otimes y - y \otimes x = 0 \Rightarrow x \otimes y = y \otimes x$

$\Rightarrow x\otimes y - y \otimes x + x \otimes y - y \otimes x = 0$

$\Rightarrow 2x \otimes y - 2y \otimes x = 0$

$\Rightarrow$

Then $(x+I^2)\otimes(y+I^2) - (y+I^2)\otimes(x+I^2)$

~~$x \otimes y - y \otimes x = xy \otimes 1 - xy \otimes 1 = 0 \otimes 1$~~

Then $r(x\otimes y - y \otimes x) = 0 \ \forall r \in R$

since Φ wd homom

$x \otimes y \neq y \otimes x$

$\Uparrow$

$\Phi(y \otimes x) = 0$

but

$\Phi(x \otimes y) = 1$

$\Uparrow$

$\therefore \exists$ hom of ab gps

$\Phi(f \otimes g) = f_x(0,0) g_y$

Define $\varphi : I \times I \to k$

$\ni \varphi(f,g) = f_x(0,0)\, g_y(0,0)$

$\varphi(f_1+f_2, g) = (f_1+f_2)_x(0,0)\, g_y(0,0)$

$= (f_{1x}(0,0) + f_{2x}(0,0))(g_y(0,0))$

$= f_{1x}(0,0)g_y(0,0) + f_{2x}(0,0)g_y(0,0)$

$= \varphi(f_1,g) + \varphi(f_2,g)$

sim $\varphi(f, g_1+g_2) = \varphi(f,g_1) + \varphi(f,g_2)$

$\varphi(fr, g) = (fr)_x(0,0)\, g_y(0,0)$

$= f_x(0,0)\, r\, g_y(0,0)$

$= \varphi(f, rg)$

$\therefore \varphi$ bilinear

$I/I^2 \otimes I/I^2$

$i + I^2 \otimes i' + I^2 \Rightarrow i + i^2 + I^2 = i' + i'^2 + I^2$

$\Rightarrow i(1+i$

$0 = 0 \otimes y = x \otimes y - x \otimes y = x \otimes$

$f_x g h_y + h g_y$

$f_x g h_y + f_x h g_y$

$(f h_x + h f_x) g_y$

$f h_x g_y + h f_x g_y$

(ii)

$x(x\otimes y - y \otimes x) = x \otimes xy - xy \otimes x = xy \otimes x - xy \otimes x = 0$

$\therefore I \otimes_R I$ not torsion free since $x \otimes y - y \otimes x \neq 0$, $x \neq 0$ but $x(x \otimes y - y \otimes x) = 0$

11.20 A proj R-mod

— show $S^{-1}A$ proj $S^{-1}R$-mod

$$\begin{tikzcd} \text{Let } 0 \arrow[r] & B \arrow[r] \arrow[d, phantom, "\cong"] & C \arrow[r] \arrow[d, phantom, "\cong"] & D \arrow[r] \arrow[d, phantom, "\cong"] & 0 \quad \text{SES } S^{-1}R\text{-mod} \\ \Rightarrow 0 \arrow[r] & S^{-1}B \arrow[r] & S^{-1}C \arrow[r] & S^{-1}D \arrow[r] & 0 \quad \text{SES} \end{tikzcd}$$

Hom

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ & S^{-1}A \arrow[u, "f"] \arrow[d, shift left, "\varphi"] & \\ & A \arrow[u, shift left, "\psi"] \arrow[uul, dashed, "\exists h"] & \end{tikzcd}$$

$\frac{a}{1} \leftarrow a$ $\qquad \frac{a}{s} \mapsto sa$

B, C $S^{-1}R$-mods hence R-mods via rest. scalars

$R \to S^{-1}R$

$r \mapsto \frac{r}{1}$

A proj $\Rightarrow \exists h: A \to B \ni f\psi = gh$

Define $j: S^{-1}A \to B$

$\ni j = h\varphi$

Then $gj = gh\varphi = f\psi\varphi$

$\varphi\left(\frac{a_1}{s_1} + \frac{a_2}{s_2}\right) = \varphi\left(\frac{a_1s_2 + a_2s_1}{s_1s_2}\right)$

$s_1s_2(a_1s_2 + a_2s_1)$

$= s_1s_2a_1s_2 + s_1s_2a_2s_1$

$= \varphi\left(\frac{a_1\cancel{s_2}}{s_1\cancel{s_2}}\right) + \varphi\left(\frac{a_2\cancel{s_1}}{\cancel{s_1}s_2}\right)$

$\varphi\left(r\cdot\frac{a}{s}\right) = \varphi\left(\frac{ra}{s}\right) = sra$

$= r \cdot sa$ Rcomm

$= r\varphi\left(\frac{a}{s}\right)$

$S^{-1}A \cong S^{-1}R \otimes_R A$

$S^{-1}R$ flat: Let $0 \to E \xrightarrow{f} B \xrightarrow{g} C \to 0$ SES exact

$\Rightarrow S^{-1}R \otimes_R E \xrightarrow{1\otimes f} S^{-1}R \otimes_R B \xrightarrow{1\otimes g} S^{-1}R \otimes_R A \to 0$ exact

Let $\frac{r}{s} \otimes e \in \ker(1\otimes f) \Rightarrow 0 = (1\otimes f)\left(\frac{r}{s}\otimes e\right) = \frac{r}{s} \otimes f(e) = \frac{1}{s} \otimes r f(e)$

$= \frac{1}{s} \otimes f(re)$

$\Rightarrow f(re) = 0 \Rightarrow re = 0 \Rightarrow \frac{r}{s} \otimes e = \frac{1}{s} \otimes re = \frac{1}{s} \otimes 0 = 0$

$\therefore \ker(1\otimes f) = 0$

$\therefore 1\otimes f$ inj

$\therefore 0 \to S^{-1}R \otimes_R E \to \ldots$ SES $\therefore S^{-1}R$ flat

Claim: F flat, P proj $\Rightarrow F \otimes_R P$ proj

Let $0 \to A \to B \to C \to 0$ SES

F flat $\Rightarrow 0 \to F \otimes_R A \to F \otimes_R B \to$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(F \otimes_R P, A) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Hom}_R(F \otimes_R P, B) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Hom}_R(F \otimes_R P, C) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(F, \mathrm{Hom}_R(P, A)) \arrow[r] & \mathrm{Hom}_R(F, \mathrm{Hom}_R(P, B)) \arrow[r] & \mathrm{Hom}_R(F, \mathrm{Hom}_R(P, C)) \arrow[r] &
\end{tikzcd}$$

$0 \to E \to B \to C \to 0$ SES $S^{-1}R$-mod (hence R-mod by rest scalars $R \to S^{-1}R$, $r \mapsto \frac{r}{1}$)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R \otimes_R A, E) \arrow[r] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft\text{nat}"] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R \otimes_R A, B) \arrow[r] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft\text{nat}"] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R \otimes_R A, C) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, E)) \arrow[r] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft\text{nat}"] & \mathrm{Hom}_R(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, B)) \arrow[r] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft\text{nat}"] & \mathrm{Hom}_R(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, C)) \arrow[d, phantom, "\cong"] & \\
0 \arrow[r] & \mathrm{Hom}_R(A, E) \arrow[r] & \mathrm{Hom}_R(A, B) \arrow[r] & \mathrm{Hom}_R(A, C) \arrow[r] & 0
\end{tikzcd}$$

SES since A proj R-mod

$\therefore$ top SES

$\therefore S^{-1}R \otimes_R A$ proj $S^{-1}R$mod $\therefore S^{-1}A$ proj $\cdot$ $S^{-1}R$mod

11.23 (c) - Give example of ab gp $B \ni \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, B) \neq 0$.

~~Take $B = \mathbb{Q}$~~

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \quad \text{SES}$$

LES: $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$
$\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \to \cdots$

or

LES: $\cdots \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \cdots$

If Ext

$I^\bullet: \quad 0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, I^\bullet): \quad 0 \to \mathrm{Hom}_{\mathbb{Z}}(\cancel{\mathbb{Q}, \mathbb{Z}}) \to \underset{(0)}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})} \xrightarrow{\pi_*} \underset{(1)}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})} \to$

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = H^1(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, I^\bullet))$
$= \cancel{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})} \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) / \mathrm{Im}\,\pi_*$

$B = \mathbb{Q}/\mathbb{Z}$

$$\cdots \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})} \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to \underset{\substack{\| \\ 0}}{\cdots}$$

Since $\mathbb{Q}$ inj

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$$

$I^\bullet: \quad 0 \to \mathbb{R} \hookrightarrow \mathbb{C} \longrightarrow \mathbb{C}/\mathbb{R} \longrightarrow 0$

$\mathrm{Hom}_{\mathbb{Z}}$

(iii)

$\mathbb{Z}$ noeth

${}_{\mathbb{Z}}\mathbb{Q}$ not f.g.

$S^{-1}\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = S^{-1}0 = 0$ $\cancel{\not\cong}$

$\mathrm{Ext}^0_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$

11.69 $\{M_\alpha : \alpha \in A\}$ family left R-mods

- show $\mathrm{pd}\left(\sum_{\alpha \in A} M_\alpha\right) = \sup_{\alpha \in A} \{\mathrm{pd}(M_\alpha)\}$

Let $P_\bullet : \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} \sum_{\alpha \in A} M_\alpha \to 0$ proj res (smallest)

$\Rightarrow \tilde{P}_\bullet : \cdots P_2 \to P_1 \to P_0 \xrightarrow{p_\alpha \epsilon} M_\alpha \to 0$ proj res

$\therefore$ $\mathrm{pd}\, M_\alpha \leq \mathrm{pd}\left(\sum_{\alpha \in A} M_\alpha\right)$ $\forall \alpha$

$\therefore$ $\mathrm{pd}\left(\sum_{\alpha \in A} M_\alpha\right)$ is an upperbound for $\{\mathrm{pd}(M_\alpha)\}$

$\therefore$ $\sup\{\mathrm{pd}\, M_\alpha\} \leq \mathrm{pd}\left(\sum_{\alpha \in A} M_\alpha\right)$

$P_\bullet \to M_\alpha$ biggest

$\Rightarrow$

$\mathrm{pd}\left(\sum_{\alpha \in A} M_\alpha\right) \leq \sup_{\alpha \in A}\{\mathrm{pd}(M_\alpha)\}$

~~$\ker \epsilon$~~ $\subseteq$

$\ker p_\alpha \epsilon \subseteq \ker \cancel{p_\alpha}$

11.69 Rotman 731

Say $\sup\{\text{pd } M_\alpha\} = n$

then $\text{pd} M_\alpha \leq n \quad \forall \alpha \in A$

$\Rightarrow \text{Ext}^i_R(M_\alpha, X) = 0 \quad \forall X \quad \forall i > n$

But $\text{Ext}^i_R(\Sigma M_\alpha, X) \cong \prod \text{Ext}^i_R(M_\alpha, X) = 0 \quad \forall i > n$

$\therefore \text{pd}(\Sigma M_\alpha) \leq n = \sup\{\text{pd} M_\alpha\}$

Say $\quad \text{pd}(\Sigma M_\alpha) = n$

$\Rightarrow \text{Ext}^i_R(\Sigma M_\alpha, X) = 0 \quad \forall X \quad \forall i > n.$

$\Rightarrow \prod \text{Ext}^i_R(M_\alpha, X) = 0 \quad \forall X \quad \forall i > n$

$\Rightarrow \text{Ext}^i_R(M_\alpha, X) = 0 \quad \forall X \quad \forall i > n$ for each α

$\therefore \text{pd}(M_\alpha) \leq n = \text{pd}(\Sigma M_\alpha) \quad \forall \alpha$

$\therefore \sup_{\alpha \in A}\{\text{pd}(M_\alpha)\} \leq \text{pd}(\Sigma M_\alpha)$

$\therefore \text{pd}(\Sigma M_\alpha) = \sup_{\alpha \in A}\{\text{pd}(M_\alpha)\}$