Sequences & Series:

3.1 Def: $\{p_n\}$ converges to p in (X,d) if $\forall \varepsilon>0\ \exists N: n\geq N \to d(p_n,p)<\varepsilon$. Then $\lim_{n\to\infty} p_n = p$ (unique)

3.6 Thm: (X,d) cpt, $\{p_n\}\subseteq X$, the $\{p_n\}$ has a convergent subsequence.

3.6b: Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

3.8 Def: $\{p_n\}$ Cauchy $\leftrightarrow \forall\varepsilon>0\ \exists N: n,m\geq N \longrightarrow d(p_n,p_m)<\varepsilon$

3.9 Def: $E\subset(X,d)$, then $\operatorname{diam} E = \sup\{d(p,q) \mid p,q\in E\}$

3.11 Thm: (X,d) cpt, $\{p_n\}\subseteq X$; $\{p_n\}$ Cauchy $\Rightarrow \{p_n\}\to p\in X$

3.12 Def: A metric space is complete if every Cauchy sequence converges.

3.13 Def: ($\{s_n\}\subset\mathbb{R}$) monotonically increasing $\rightarrow s_n\leq s_{n+1}\ \forall n$

3.14 Thm: ($\{s_n\}$ monotonic) $\{s_n\}$ converges $\longleftrightarrow \{s_n\}$ is bounded

3.15 Def: If $\forall M\in\mathbb{R}\ \exists N\in\mathbb{N}: n\geq N \to s_n > M$, then $\{s_n\}\to+\infty$

3.17 Def: (Let $s^* = \limsup_{n\to\infty} s_n$) then ⓐ s^* is a limit point (subsequential limit) ⓑ $x>s^* \Rightarrow \exists N: n\geq N \Rightarrow s_n < x$

"Comparison Test" (3.25) ⓐ If $\sum c_n$ converges and $\exists N: n\geq N \Rightarrow |a_n|\leq c_n$, then $\sum a_n$ converges.

ⓑ If $\sum d_n$ diverges and $\exists N: n\geq N \Rightarrow a_n \geq d_n \geq 0$, then $\sum a_n$ diverges.

3.26 Thm Geometric Series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $0\leq r<1$, otherwise diverges.

3.27 Cauchy Condensation Criterion: ($a_n\geq a_{n+1}\ \forall n$) $\sum a_n$ converges $\longleftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1+2a_2+4a_4+\cdots$ converges

3.28 Thm: $\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p\leq 1$

3.29 Thm: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p>1$, diverges if $p\leq 1$

Root Test (stronger): Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then $\alpha<1 \Rightarrow \sum a_n$ converges, $\alpha>1 \Rightarrow \sum a_n$ diverges

Ratio Test: $\sum a_n$ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, diverges if ≥ 1, first n terms don't matter

Power Series: (Hadamard) Given $\sum c_n z^n$, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, then $R = 1/\alpha$, $\sum c_n z^n$ conv if $|z|<R$, div if $|z|>R$

3.42 Thm: (Dirichlet) ⓐ partial sums A_n of $\sum a_n$ form bounded sequence ⓑ $\{b_n\}$ monotonically decreasing ⓒ $\lim_{n\to\infty} b_n = 0$. Then $\sum a_n b_n$ converges

3.43 Alternating: (Leibnitz) ⓐ $|c_1|\geq|c_2|\geq\cdots$ ⓑ alternating sign ⓒ $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n$ converges

3.48 Cauchy Product: Given $\sum a_n$ and $\sum b_n$, there product $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

3.50 Thm: (Mertens) If $\sum a_n \to A$ and $\sum b_n \to B$ converge (at least 1 absolutely), then $\lim_{n\to\infty} \sum_{k=0}^{n} a_k b_{n-k} = AB$

CONTINUITY:

Def: (X, Y Metric Spaces, $E \subset X$, p a limit point of E, $f: X \to Y$, $f(E) \subseteq Y$)

$\lim_{x \to p} f(x) = q \iff \forall \varepsilon > 0 \, \exists \delta > 0 \;\; d_x(x,p) < \delta \Rightarrow d_y(f(x), q) < \varepsilon$

$\iff \lim_{n \to \infty} f(p_n) = q$ for every $\{p_n\}$ in E s.t. $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

4.2, 4.4 Thm: If f has a limit at p, it is unique. Sums, Differences, products, quotients, preserve limits and $(f+g)(x) = A+B$

4.5 Def: If p is a limit point, f cont @ $p \iff \lim_{x \to p} f(x) = q$ and $f(p) = q$

If p is an isolated point, f cont @ $p \iff f$ defined at p.

4.7 Thm: (X, Y, Z Metric Spaces / $E \subset X$ / $f: E \to f(E) \subset Y$ / $g: f(E) \to g(f(E)) \subset Z$ / $h = g(f(x))$)

If f is continuous at p, and g is continuous at $f(p)$, then h is continuous at p.

4.8 Thm: (X, Y Metric) f continuous $\iff f^{-1}(V)$ is open (closed) in X for every open (closed) set $V \subset Y$

4.9 Thm Sums, differences, products, and quotients of continuous (or cont @ x) functions are continuous.

4.10 "Vector Valued" ($\vec{f}(x) = (f_1(x), \ldots, f_k(x))$), $\vec{f}$ continuous at $x \iff$ each component func continuous at x.

If $\vec{f}$ and $\vec{g}$ are continuous mappings into $\mathbb{R}^k$, then $\vec{f} + \vec{g}$ cont into $\mathbb{R}^k$ and $\vec{f} \cdot \vec{g}$ cont into $\mathbb{R}$

4.14 Thm: (Suppose $f: X \to Y$ cont / X compact) Then $f(X)$ is compact in Y.

4.15 Thm: Suppose X compact, $\vec{f}: X \to \mathbb{R}^k$ continuous, then $\vec{f}(X)$ is closed and bounded, thus $\vec{f}$ is bounded.

4.16 Thm: Suppose X compact, $f: X \to Y$ continuous, $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$, then $\exists p, q \in X: f(p) = M, f(q) = m$

4.17 Thm: (X compact, $f: X \to Y$ cont 1-1 mapping) Then $f^{-1}: Y \twoheadrightarrow X$ is continuous

4.18 Def: f is <u>uniformly continuous</u> if $\forall \varepsilon > 0 \, \exists \delta > 0: d_x(p,q) < \delta \Rightarrow d_y(f(p), f(q)) < \varepsilon$

4.19 Thm: If [X, Y metric / X cpt / $f: X \to Y$ cont], then $f: X \to Y$ uniformly continuous.

4.20 Thm: (Let E be a non-compact set in $\mathbb{R}^1$) Then (a) $\exists$ continuous function on E which is not bounded (b) $\exists$ cont bounded func on E w/no max (c) If E also bounded, $\exists$ cont. f that isn't unif cont.

4.22 Thm: If X, Y metric / $E \subset X$ / $f: X \to Y$ cont / E connected, then $f(E)$ connected

4.23 Thm: "Intermediate Value" Let $f: [a,b] \to \mathbb{R}$ continuous. $\forall c \in [f(a), f(b)] \, \exists x \in [a,b]: f(x) = c$

4.25 Defn: [f defined on (a,b), $x \in [a,b)$] $f(x+) = q \iff \forall \{t_n\} \in (x, b)$ ("from the right") s.t. $t_n \to x$: $\lim_{n \to \infty} f(t_n) = q$

4.26 Def: Discontinuity of 1st kind if $f(x-), f(x), f(x+)$ exist, but don't all match.

4.28 Def: f real on (a,b) monotonically increasing iff $a < x_1 < x_2 < b \Rightarrow f(x_1) \leq f(x_2)$

4.29 Thm: (f monotonically increasing on (a,b)) Then $\forall x \in (a,b)$: $f(x-), f(x+)$ exists; $\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$

Cor: (f monotonic) no discontinuities of first kind, at most countable of 2nd.

4.32 Def: $\forall c \in \mathbb{R}$, $(c, +\infty)$ is a neighborhood of $+\infty$

4.33 Def: (f real, defined on $E \subset \mathbb{R}$) $f(t) \to A$ as $t \to x \iff$ For every neighborhood U of A, there exists a nhd V of x s.t. $\forall t \neq x \in V \cap E: f(t) \in U$ (Coincides w/ 4.1 for reals, recast to allow limits of $\pm\infty$)

DIFFERENTIATION:

5.1 Def: Let f be a real valued function defined on $[a,b]$. For any $x \in [a,b]$, define
$$f'(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \quad \text{provided the limit exists}$$

5.2 Thm: (Let f be defined on $[a,b]$, $x \in [a,b]$) f diff @ $x \Rightarrow f$ continuous @ x

5.3 Thm: (Let f,g be defined on $[a,b]$, diff at $x \in [a,b]$) (a) $(f+g)'(x) = f'(x) + g'(x)$
(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (c) $\left(\frac{f}{g}\right)'(x) = [g(x)f'(x) - g'(x)f(x)] \div g^2(x)$

5.5 Thm: Suppose f continuous on $[a,b]$, $f'(x)$ exists at some $x \in [a,b]$, g defined on an interval containing the range of f, g' defined at $f(x)$, Let $h(t) = g(f(t))$ $t \in [a,b]$
Then $h(x) = g(f(x))$ is diff at x, and $h'(x) = g'(f(x))f'(x)$

5.8 Thm: (f defined on $[a,b]$) If [f has a local max(min) at $x \in (a,b)$ ^ $f'(x)$ exists] then $f'(x) = 0$

5.9 Cauchy MVT: (f,g (real) cont on $[a,b]$, diff on (a,b)) then $\exists x \in (a,b)$: $[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$
If denominators are not zero $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

5.10 Lagrange MVT: ($f: \mathbb{R} \to \mathbb{R}$ continuous on $[a,b]$, diff on (a,b)) Then $\exists c \in (a,b)$ at which
$f(b) - f(a) = (b-a)f'(c)$ or if $b \neq a$ $f'(c) = \frac{f(b)-f(a)}{b-a}$

5.11 Thm: (Suppose f diff in (a,b)) (a) $f'(x) \geq 0$ $\forall x \in (a,b) \Rightarrow f$ monotonically increasing (b) $=0 \Rightarrow$ constant (c) ≤ 0 mono. decreasing

5.12 Thm: (f real, diff on $[a,b]$) If $f'(a) < \lambda < f'(b)$, then $\exists x \in (a,b)$: $f'(x) = \lambda$

5.13 L'Hôpital's Rule: Suppose f and g are real and diff in (a,b) $\forall$ $-\infty \leq a < b \leq +\infty$ and suppose $g' \neq 0$ on (a,b), and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = A$ in the extended reals.
If $[(f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a^+) \vee (g(x) \to \pm\infty \text{ as } x \to a^+)]$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = A$
Note: written here as on (a,b) as $x \to a^+$, could be $(a,b]$ as $x \to a^+$, or $[a,b)$ as $x \to b^-$, or (a,b) as $x \to b^-$

5.15 Taylor's Thm: Suppose f is a real function on $[a,b]$, $f^{(n-1)}$ is continuous on $[a,b]$ and f is n times differentiable on (a,b), then
$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(x)}{n!}(b-a)^n \quad \text{for some } x \in (a,b)$$

Ch 6: The Riemann-Stieltjes Integral

6.1 Defn: A <u>partition</u> P is an increasing finite point set $\{x_0, \dots, x_n\}$. $\Delta x_i = x_i - x_{i-1}$

Let $M_i = \sup f(x)$, $m_i = \inf f(x)$, $(x \in [x_{i-1}, x_i])$, Then $U(P,f) = \sum M_i \Delta x_i$, $L(P,f) = \sum m_i \Delta x_i$

$\overline{\int} f\,dx = \inf U(P,f)$, $\underline{\int} f\,dx = \sup L(P,f)$, (f must be bounded), $f \in R$ if $\overline{\int} f\,dx = \underline{\int} f\,dx$, $m \le f(x) \le M$ $\forall x \in [a,b]$

$m(b-a) \le L(P,f) \le \underline{\int} f\,dx$ $(= \int f\,dx =)$ $\overline{\int} f\,dx \le U(P,f) \le M(b-a)$ for $\forall P$.

6.2 Defn: Let α be monotonically increasing on $[a,b]$ (so bounded). $U(P,f,\alpha) = \sum M_i \Delta\alpha_i$, L similar, etc.

6.4 Thm: If P^* is a refinement of P, then $L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha)$

6.6 Thm: $f \in R(\alpha)$ on $[a,b] \leftrightarrow \forall \varepsilon > 0 \, \exists P$: $\boxed{U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon}$ $(*)$

6.7 Thm: (a) If $(*)$ holds for ε and P, then $(*)$ holds (w/same ε) for every refinement P^*.

(b) If $(*)$ holds for P and ε, and $s_i, t_i \in [x_{i-1}, x_i]$ arbitrary, then $\sum |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$

(c) If $f \in R(\alpha)$, $(*)$ holds for P/ε, $s_i, t_i \in [x_{i-1}, x_i]$ arbitrary, then $\left|\sum f(t_i)\Delta\alpha_i - \int f\,d\alpha\right| < \varepsilon$

6.8 Thm: If: f is continuous on $[a,b]$, then: $f \in R(\alpha)$ on $[a,b]$

6.9 Thm: If [f monotonic and continuous on $[a,b]$], then [$f \in R(\alpha)$ on $[a,b]$]

6.9 Cor: f monotonic on $[a,b] \longrightarrow f \in R$ on $[a,b]$

6.10 Thm: If [f on $[a,b]$ is bounded w/finitely many discontinuities <u>and</u> α is continuous at every such discontinuity], then $f \in R(\alpha)$

6.10 Cor: If f is continuous on $[a,b]$ at all but a finite number of points (and bounded?), then $f \in R$.

6.11 Thm: Suppose $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a,b]$. then $h \in R(\alpha)$

6.12 Thm: (a) $f_1, f_2 \in R(\alpha)$, $c \in \mathbb{R} \rightarrow (f_1 + f_2) \in R(\alpha) \wedge \int f_1 + f_2 = \int f_1 + \int f_2 \wedge cf \in R(\alpha) \wedge \int cf = c\int f$

(b) $f_1 \le f_2$ on $[a,b] \rightarrow \int f_1 \le \int f_2$ (c) $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ (if it exists) (d) $|f| < M \rightarrow \int f\,d\alpha \le M[\alpha(b) - \alpha(a)]$

(e) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$, then $f \in R(\alpha_1 + \alpha_2)$ and $\int f\,d(\alpha_1 + \alpha_2) = \int f\,d\alpha_1 + \int f\,d\alpha_2$ (f) $f \in R(\alpha) \rightarrow f \in R(c\alpha) \wedge \int f\,d(c\alpha) = c\int f\,d\alpha$

6.13 Thm: If $f, g \in R(\alpha)$ on $[a,b]$, then $fg \in R(\alpha) \wedge |f| \in R(\alpha) \wedge \left|\int f\,d\alpha\right| \le \int |f|\,d\alpha$

6.14 Def: The heaviside function (unit step) is $H(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1, & \text{if } x > 0 \end{cases}$

6.15 Thm: If [$a < s < b \wedge f$ is bounded on $[a,b] \wedge f$ cont at $s \wedge \alpha = H(x-s)$] then $\int_a^b f\,d\alpha = f(s)$

6.16 Thm: Suppose $c_n \ge 0 \; \forall n$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b), and $\alpha(x) = \sum_{i=1}^{\infty} c_n H(x - s_n)$, f continuous on $[a,b] \longrightarrow \int f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

6.17 Thm: Assume α increases monotonically, $\alpha' \in R$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then: $f \in R(\alpha) \leftrightarrow f\alpha' \in R$. In that case $\int f\,d\alpha = \int f(x)\alpha'(x)\,dx$

6.19 Thm: Suppose $\varphi: [A,B] \rightarrow [a,b]$ strictly increasing/continuous, α increasing monotonically on $[a,b]$, and $f \in R(\alpha)$ on $[a,b]$. Define $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$ on $[A,B]$. Then $g \in R(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$

6.20 Thm: Let $f \in R$ on $[a,b]$. For $x \in [a,b]$, put $F(x) = \int_a^x f(t)\,dt$. Then F is cont. on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is diff at x_0 and $F'(x_0) = f(x_0)$.

6.21 Thm: If $f \in R$ on $[a,b]$, and $\exists$ a diff F on $[a,b]$ s.t. $F' = f$, then $\int f(x)\,dx = F(b) - F(a)$.

6.22 Thm: If $f, g \in R$ on $[a,b]$, F, G diff s.t. $F' = f$, $G' = g$, then $\int Fg\,dx = F(b)G(b) - F(a)G(a) - \int fG\,dx$

6.25 Thm: If $\vec{f}$ maps $[a,b]$ into $\mathbb{R}^k$, and $\vec{f} \in R(\alpha)$ for some <u>monotonically</u> increasing α on $[a,b]$, Then $|\vec{f}| \in R(\alpha)$ and $\left|\int \vec{f}\,d\alpha\right| \le \int |\vec{f}|\,d\alpha$. (i.e. $\sqrt{(\int f_1\,d\alpha)^2 + \dots + (\int f_k\,d\alpha)^2} \le \int \sqrt{f_1^2(x) + \dots + f_k^2(x)}\,d\alpha$

Sequences & Series of Functions - Part I:

7.1 Def: $\{f_n\}$ converges pointwise $\Leftrightarrow \exists f(x): \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$

(Let $S_N(x) = \sum_{n=1}^{N} f_n(x)$) $\sum f_n(x)$ converges pointwise $\Leftrightarrow \exists S(x): \lim_{N\to\infty} S_N(x) = S(x)\ \forall x \in E$

7.7 Def: $\{f_n\} \to f$ uniformly on $E \Leftrightarrow \forall \varepsilon > 0\ \exists N: n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon\quad \forall x \in E$

equivalently; $\{f_n\} \to f$ uniformly on $E \Leftrightarrow \lim_{n\to\infty} M_n = 0$ (Note: $M_n = \sup_{x\in E} |f_n(x) - f(x)|$)

negation: (kiddie) $\exists \varepsilon_0 > 0\ \forall n \in \mathbb{N}: |f_n(x) - f(x)| > \varepsilon_0$ for some $x \in E$ | (full) $\exists \varepsilon_0 > 0\ \forall N\ \exists n \geq N: |f_n(x) - f(x)| > \varepsilon_0$ for some $x \in E$

or show $\limsup_{n\to\infty} M_n \neq 0$, so show $\exists \varepsilon_0 > 0\ \exists \{f_{n_k}\}\ \exists x \in E: |f_{n_k}(x) - f(x)| > \varepsilon_0$ for all k (the subsequence of $\{f_n\}$)

7.8 Cauchy Crit for Unif Conv: $\{f_n\}$ conv. unif on $E \Leftrightarrow \forall \varepsilon > 0\ \exists N: m, n \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon\ \forall x \in E$

7.10 Weierstrass M-test: (Suppose $|f_n(x)| \leq M_n$) $\forall x \in E\ \forall n \in \mathbb{N}$, then $\sum M_n$ conv $\Rightarrow \sum f_n$ conv unif.

7.11 Thm: (Suppose $E \subseteq (X, d)$ / x a limit point of E / $f_n \to f$ unif on E)

Suppose $\lim_{t\to x} f_n(t) = A_n$ for each n, then $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$

In other words, if $\{f_n\} \to f$ unif, then $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$

7.12 Thm: If $\{f_n\}$ is a sequence of cont functions that $\to f$ unif on E, then f cont on E.

contrapositive: If f not cont and f_n's are cont, then $f_n \not\to f$ unif on E

7.13 Thm: Suppose K is compact, then the following 3 conditions imply $f_n \to f$ unif on K.

ⓐ Each f_n is cont on K (implies unif cont) ⓑ $f_n \to f$ pointwise on K ⓒ $f_n(x) \geq f_{n+1}(x)\quad \forall x$ and $\forall n$

7.14 Def: Given (X, d), $\mathscr{C}(X)$ is the set of all complex valued, cont, bdd func w/dom X.

If X is compact, $\mathscr{C}(X)$ is all complex continuous functions on X (since cpt $\Rightarrow$ bdd)

7.15 Thm: $\mathscr{C}(X)$ is a metric space w/ the following metric: Let $\|f\| = \sup_{x\in X} |f(x)|$,

then given $f, g \in \mathscr{C}(X)$, $d(f, g) = \|f - g\| = \sup_{x\in X} |f(x) - g(x)|$

(Restatement 7.9) $\{f_n\} \to f$ w.r.t metric of $\mathscr{C}(X) \Leftrightarrow f_n \to f$ unif on X

$(\mathscr{C}(X), d)$ is a complete metric space every Cauchy sequence converges

7.16 Thm: (α monotonically increasing on $[a,b]$ / Each $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ / $f_n \to f$ unif)

Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$ (Existence of limit is part of conclusion)

Cor: If [$f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ for each n ^ $S(x) = \sum_{n=1}^{\infty} f_n(x)$ ($a \leq x \leq b$) w/ the series converging unif on $[a, b]$]

then, $\int_a^b S\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha$ in other words, the series may be integrated term by term

7.17 Thm: (Suppose f_n diff on $[a,b]$ for each n / $\{f_n(x_0)\}$ conv on $[a,b]$ for some x_0)

If $\{f_n'\}$ conv unif on $[a,b]$, then $\{f_n\} \to f$ unif on $[a,b]$ and $f'(x) = \lim_{n\to\infty} f_n'(x)\ \forall x \in [a,b]$

7.19 Def: $\{f_n\}$ is pointwise bounded on E if $\{f_n(x)\}$ is bdd for $\forall x$ (bound may depend on x, $|f_n(x)| < \phi(x)\ \forall x \forall n$)

$\{f_n\}$ is uniformly bounded on E if $\exists M: |f_n(x)| < M\quad \forall x \forall n$

7.22 Def: A family $\mathcal{F}$ of complex func f is equicontinuous in a metric space (X, d)

iff $\forall \varepsilon > 0\ \exists \delta > 0: d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\quad \forall x, y \in E$ and $\forall f \in \mathcal{F}$ (Clearly each member of an equicontinuous family is uniformly cont)

7.23 Thm: If $\{f_n\}$ is a pointwise bdd seq of $\mathbb{C}$ func on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges pointwise for every $x \in E$.

7.24 Thm: ((K,d) cpt) If each $f_n \in \mathscr{C}(K)$ (so cont, bdd) and $\{f_n\}$ conv unif on K, then $\{f_n\}$ equicontinuous on K.

7.25 Thm: ((K,d) cpt) If each $f_n \in \mathscr{C}(K)$ (so cont, bdd) and $\{f_n\}$ pointwise bdd/equicontinuous on K,

then ⓐ $\{f_n\}$ is uniformly bounded on K ⓑ $\{f_n\}$ contains a uniformly convergent subsequence

SEQUENCES & SERIES OF FUNCTIONS - PART II: Algebras

7.26 Thm: (original from Weierstrass) If f is a continuous complex function on $[a,b]$, then there exists a sequence of polynomials P_n such that $\{P_n(x)\} \to f(x)$ uniformly on $[a,b]$. (If f is real, P_n may be taken real)

7.27 Cor: For every interval $[-a,a]$ there is a sequence of real polynomials P_n such that $P_n(0)=0$ $\forall n$, and $\{P_n(x)\} \to |x|$ uniformly on $[-a,a]$.

Def 7.28: A family $\mathcal{A}$ of complex/real functions defined on a set E is an algebra if (i) $f+g \in \mathcal{A}$, (ii) $fg \in \mathcal{A}$, and (iii) $cf \in \mathcal{A}$, for all $f,g \in \mathcal{A}$ and $c \in \mathbb{C}/\mathbb{R}$

Def: $\mathcal{A}$ is said to be uniformly closed if for every uniformly convergent $\{f_n\} \in \mathcal{A}$, $\lim_{n\to\infty} f_n = f \in \mathcal{A}$

Def: $\mathcal{B}$ is the uniform closure of $\mathcal{A}$ if $\mathcal{B} = \mathcal{A} \cup \{\text{limits of unif conv. seq}\}$ Note: this piece is actually all of $\mathcal{B}$ (Notation: $\mathcal{B} = \bar{\mathcal{A}}$)

Note: The uniform closure of the set of all complex/real polynomials on $[a,b]$ is is set of all continuous complex/real functions on $[a,b]$.

7.29 Thm: The uniform closure of $\mathcal{A}$ is uniformly closed.

7.30 Def: Let $\mathcal{A}$ be an algebra (or really any family of functions) on a set E.

$\mathcal{A}$ seperates points on E if: $\forall x_1 \neq x_2 \in E \to \exists f \in \mathcal{A}$ s.t. $f(x_1) \neq f(x_2)$

$\mathcal{A}$ vanishes at no point of E if: $\exists f \in \mathcal{A}$ such that $f(x) \neq 0$ for each $x \in E$.

7.31 Thm: If $\mathcal{A}$ is an algebra on E that seperates points and vanishes nowhere, then given any $x_1 \neq x_2 \in E$ and any $c_1, c_2 \in \mathbb{C}/\mathbb{R}$, $\exists f \in \mathcal{A}: f(x_1) = c_1, f(x_2) = c_2$

7.32 Thm (Stone's Generalization of 7.26) Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set K. If $\mathcal{A}$ seperates points and vanishes nowhere on K, then the uniform closure of $\mathcal{A}$ consists of all real continuous funcs on K.

Def: A complex algebra $\mathcal{A}$ is said to be self-adjoint if for every $f \in \mathcal{A}$, its complex conjugate $\bar{f}$ is also in $\mathcal{A}$. ($\bar{f}$ is defined by $\bar{f}(x) := \overline{f(x)}$)

7.33 Theorem: Suppose $\mathcal{A}$ is a complex algebra on a ~~complex~~ compact set K that is self-adjoint, seperates points, and vanishes nowhere. Then the uniform closure of $\mathcal{A}$ is all complex continuous functions on K.

Note: If $\mathcal{B}$ is the uniform closure of $\mathcal{A}$, then $\mathcal{A}$ is dense in $\mathcal{B}$.

Functions of Several Variables

Def: Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, the <u>norm</u> $\|A\| = \sup |A\vec{x}|$ over $\forall x \in \mathbb{R}^n : |x| \le 1$

Thm 9.7 ⓐ If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$

ⓑ If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$, c scalar, then $\|A+B\| \le \|A\| + \|B\|$, and $\|cA\| = |c|\,\|A\|$

ⓒ If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then $\|BA\| \le \|B\|\,\|A\|$

Fact: $L(\mathbb{R}^n, \mathbb{R}^m)$ is a complete metric space w/ $d(A,B) = \|A-B\|$

Lemma: Let $\{A_k\} = A_1, A_2, \ldots \in L(\mathbb{R}^n, \mathbb{R}^m)$ ⓐ If $A_k \to A \in L(\mathbb{R}^n, \mathbb{R}^m)$ as $k \to \infty$, then for each $x \in \mathbb{R}^n$, $A_k x \to Ax \in \mathbb{R}^m$ as $k \to \infty$ ⓑ If $\{A_k\} \in L(\mathbb{R}^n, \mathbb{R}^m)$, and $\{A_k\}$ is Cauchy in $L(\mathbb{R}^n, \mathbb{R}^m)$, then $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that $A_n \to A$ in $L(\mathbb{R}^n, \mathbb{R}^m)$

Thm 9.8 ($A, B \in L(\mathbb{R}^n)$) ⓐ If A is invertible, $\exists$ open ball of radius $\frac{1}{\|A^{-1}\|}$ that is invertible

Def 9.11 Suppose $E \subset \mathbb{R}^n$ open, $f: E \to \mathbb{R}^m$, and $x \in E$. If $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that
$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$ for $h \in \mathbb{R}^n$, we say f is diff at x, write $f'(x) = A$.
If f is diff at every $x \in E$, we say f is diff in E.

Thm 9.15 If $F(x) = g(f(x))$ and F is diff at x, then $F'(x) = g'(f(x))f'(x)$

Def 9.16 "Partial Derivatives" Let $E \subset \mathbb{R}^n$ open, $f: E \to \mathbb{R}^m$, std bases for $\mathbb{R}^n, \mathbb{R}^m$ are $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$, $f = (f_1, \ldots, f_m)$ w/each $f_i : \mathbb{R}^n \to \mathbb{R}$. The partial derivative $(D_j f_i)(x)$ w/ $j \le n$, $i \le m$ is the measure of the effect of infinitessimal change in x along the j^{th} stadard basis vector in $\mathbb{R}^n$ on the i^{th} component function, or equivalently the effect on $f(x)$ along the i^{th} basis vector in $\mathbb{R}^m$. $D_j f_i(x) := \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t}$

Thm 9.19 Suppose $E \subset \mathbb{R}^n$ open & convex, and $f: E \to \mathbb{R}^m$ is diff in E.
If $\forall x \in E$, $\|f'(x)\| \le M$, then $\forall a, b \in E$ $|f(b) - f(a)| \le M|b-a|$

Cor: If $\forall x \in E$ $f'(x) = 0$, then f is constant

Def 9.20 Let $E \subset \mathbb{R}^n$ open, $f: E \to \mathbb{R}^m$, f is <u>continuously differentiable</u> (write $f \in \mathscr{C}'(E)$)
if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in E$ $|x - y| < \delta \Rightarrow \|f'(y) - f'(x)\| < \varepsilon$

Thm 9.21 (Suppose $E \subset \mathbb{R}^n$ open, $f: E \to \mathbb{R}^m$) $f \in \mathscr{C}'(E) \Leftrightarrow$ Each $D_j f_i$ exists and is continuous on E.

Thm 9.24 "Inverse Function Theorem" (Suppose $E \subset \mathbb{R}^n$ open, $f: E \to \mathbb{R}^{n}$)
If i) $f \in \mathscr{C}'(E)$, ii) $f'(a)$ inv'tbl for some $a \in E$ and $b = f(a)$, then
ⓐ $\exists$ open sets U and V in $\mathbb{R}^n$ w/ $a \in U$ and $b \in V$ s.t. f is bijective on U and $f(U) = V$. so f^{-1} exists, and is $\in \mathscr{C}'(V)$

Thm 9.27: (Let $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ $A = [A_x | A_y]$) If A_x is invertible, then $\forall k \in \mathbb{R}^m$
$\exists! h \in \mathbb{R}^n$, call $h = g(k)$, such that $A(g(k), k) = A(h, k) = A\begin{bmatrix} h_1 \\ \vdots \\ h_n \\ k_1 \\ \vdots \\ k_m \end{bmatrix} = 0$. $h = g(k) = -(A_x)^{-1} A_y k$

Thm 9.28: "Implicit Function Theorem" (Suppose $E \subset \mathbb{R}^{n+m}$ open, $f: E \to \mathbb{R}^n$
If i) $f \in \mathscr{C}'(E)$, ii) $(a,b) \in E$, $f(a,b) = 0$, iii) $f'(a,b) = A = [A_x | A_y]$ w/ A_x invertible
Then, ⓐ $\exists U \subset \mathbb{R}^{n+m}$, $W \subset \mathbb{R}^m$ s.t. $(a,b) \in U$, $b \in W$, $\forall y \in W\, \exists! x$: $(x,y) \in U$ and $f(x,y) = 0$ say $x = g(y)$
ⓑ $g: W \to \mathbb{R}^n$, $g \in \mathscr{C}'(W)$, $g(b) = a$, so $f(x,y) = f(g(y), y) = 0$ $\forall y \in W$ ⓒ $g'(b) = -(A_x)^{-1} A_y$