Foland Chapter 1

1.2.1 Rings are closed under finite intersections.

Pf Let $E_1, E_2, \ldots, E_n \in \mathcal{R}$ (the set of rings)
We proceed by induction on n
If $n=2$ then $E_1 \cap E_2 = (E_1 \cup E_2) \setminus ((E_2 \setminus E_1) \cup (E_1 \setminus E_2)) \in \mathcal{R}$
Assume true for $n=k$. Let $E_1 \cap \ldots \cap E_k = \hat{E} \in \mathcal{R}$
$n=k+1 \Rightarrow \bigcap_{i=1}^{k+1} E_i = E_{k+1} \cap \hat{E} = (E_{k+1} \cup \hat{E}) \setminus ((E_{k+1} \setminus \hat{E}) \cup (\hat{E} \setminus E_{k+1})) \in \mathcal{R}$
$\square$

If $\mathcal{R}$ is a ring then $\mathcal{R}$ is an algebra $\Leftrightarrow X \in \mathcal{R}$

Pf Assume $\mathcal{R}$ is an algebra
Let $E \in \mathcal{R}$ then $E \cup E^c = X \in \mathcal{R}$
$\Rightarrow X \in \mathcal{R}$

Assume $X \in \mathcal{R}$
Let $E \in \mathcal{R}$. Then $X \setminus E = E^c \in \mathcal{R}$
$\Rightarrow \mathcal{R}$ is an algebra since we adready have unions

If $\mathcal{R}$ is a σ-ring then $\{E \subset X : X \in \mathcal{R}$ or $E^c \in \mathcal{R}\} = \mathcal{M}$ is a σ-algebra

Pf (i) Let $E \in \mathcal{M}$.
$\Rightarrow E \in \mathcal{R}$ or $E^c \in \mathcal{R}$.
$\Rightarrow E^c \in \mathcal{M}$ since $E^c \in \mathcal{R}$ or $(E^c)^c = E \in \mathcal{R}$

(ii) Let $E_1, E_2, \ldots \in \mathcal{M}$
If $E_i \in \mathcal{R}\ \forall i$ then $\bigcup_1^\infty E_i \in \mathcal{R} \Rightarrow \bigcup_1^\infty E_i \in \mathcal{M}$.
if $E_i \notin \mathcal{R}\ \forall i$ then $(\bigcap_1^\infty E_i^c)^c \in \mathcal{R} \Rightarrow \bigcup_1^\infty E_i \in \mathcal{M}$.
if $\exists E_i \in \mathcal{R} \Rightarrow E_i^c \in \mathcal{R}$
$\Rightarrow$

If $\mathcal{R}$ is a σ ring then $\{E \subset X : E \cap F \in \mathcal{R} \ \forall F \in \mathcal{R}\} = \mathcal{M}$ is a σ-algebra.

Pf 1) Let $E \in \mathcal{M}$

$\Rightarrow E^c \cap F = (E \cup \underbrace{F^c)^c}_{\in \mathcal{R}} \in \mathcal{R}$

$\Rightarrow E^c \in \mathcal{M}$

2) Let $E_1, E_2, \ldots \in \mathcal{M}$

$(\bigcup_1^\infty E_k) \cap F = \bigcup_1^\infty (E_k \cap F) \in \mathcal{R}$ since $E_k \cap F \in \mathcal{R}$

$\Rightarrow \bigcup_1^\infty E_i \in \mathcal{M}$

$\Rightarrow \mathcal{M}$ is a σ-algebra. $\square$

1.2.2 Complete proof of prop 1.2.

Pf $(a,b) \in \mathcal{M}(\mathcal{E}_1) \Rightarrow \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_1)$

$(a,b) = \bigcup_1^\infty (a, b+\frac{1}{n}] \in \mathcal{M}(\mathcal{E}_2)$

$(a,b) = (a,\infty) \setminus \bigcup_1^\infty (b-\frac{1}{n}, \infty) \in \mathcal{M}(\mathcal{E}_5)$

$(a,b) = \bigcup_1^\infty (a-\frac{1}{n}, \infty) \setminus \bigcup_1^\infty (b-\frac{1}{n}, \infty) \in \mathcal{M}(\mathcal{E}_7)$

$\square$

1.2.3 Let $\mathcal{M}$ be an infinite σ-algebra.

a) $\mathcal{M}$ contains infinite seq of disjoint sets

(b) Card $(\mathcal{M}) \geq c$.

Pf a) Let A_n be infinite sequence of sets in $\mathcal{M}$.

Let $E_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ and define $E_1 = A_1$

$\Rightarrow E_n$'s infinite disjoint sets.

b) Let $\{E_n\}$ be infinite disjoint sets as in (a).

Let $\mathcal{N} = \{A \mid A$ is a union of A_i's$\} \in \mathcal{M}$

Define $f: \mathcal{N} \to [0,1]$ s.t. $f(A) = \sum_1^\infty 2^{-n} \chi_{A_n}$

f is surjective

$\Rightarrow$ Card $\mathcal{M} \geq c$

1.2.4 Show an algebra $\mathcal{A}$ is a σ algebra $\Leftrightarrow$ $\mathcal{A}$ is closed under countable increasing unions

Pf Assume $\mathcal{A}$ is a σ-algebra.
$\Rightarrow \mathcal{A}$ is closed under countable unions
$\Rightarrow \mathcal{A}$ is closed under countable increasing unions

Assume $\mathcal{A}$ is closed under countable increasing unions.
Let $A_1, A_2, \ldots \in \mathcal{A}$.
$$\bigcup_1^\infty A_k = \underbrace{A_1 \setminus (\cup_2^\infty A_k) \cup (A_1 \cup A_2) \setminus \cup_3^\infty A_k \cup \ldots}_{\text{increasing unions}}$$
$\in \mathcal{A}$

$\Rightarrow \mathcal{A}$ is an algebra.

$\square$

1.2.5 If $\mathcal{M}$ is σ-algebra generated by $\mathcal{E}$ then $\mathcal{M}$ is union of σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Pf Let $\mathcal{N} = \{\cup \mathcal{M}(\mathcal{F}) : \mathcal{F}$ all countable subsets of $\mathcal{E}\}$
$\mathcal{F} \subset \mathcal{E} \Rightarrow \mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E}) = \mathcal{M}$
$\Rightarrow \mathcal{N} \subset \mathcal{M}$

Now notice $\mathcal{E} \subset \mathcal{N}$. since $\mathcal{E} = \{A_\alpha\}_{\alpha \in J} \Rightarrow \mathcal{E} \subset \cup \mathcal{M}(\{A_\alpha\}) \subset \mathcal{N}$
If we can show $\mathcal{N}$ is a σ-algebra then
$\mathcal{M}(\mathcal{E})$ is smallest σ-algebra containing $\mathcal{E}$
$\Rightarrow \mathcal{M} = \mathcal{M}(\mathcal{E}) \subset \mathcal{N}$.

Let $A \subset \mathcal{N}$
$\Rightarrow A \in \mathcal{M}(\mathcal{F}) \Rightarrow A^c \in \mathcal{M}(\mathcal{F}) \subset \mathcal{N}$

Let $A_1, A_2, \ldots \in \mathcal{N}$
$A_i \in \mathcal{M}(\mathcal{F}_i)$ $\bigcup_1^\infty \mathcal{F}_i$ countable so
$A_i \in \mathcal{M}(\cup \mathcal{F}_i)$
$\Rightarrow \cup A_i \in \mathcal{M}(\cup \mathcal{F}_i) \subset \mathcal{N}$.

$\square$

So $\mathcal{N}$ is a σ-algebra.

1.3.6. Suppose $(X, \mathcal{M}, \mu)$ a m.s. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}$ and $F \subset N$ some $N \in \mathcal{N}\}$
Then $\overline{\mathcal{M}}$ is a σ-algebra and $\exists!$ extension $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$

Pf • $\overline{\mathcal{M}}$ is closed under countable unions since $\mathcal{M}$ & $\mathcal{N}$ are
If $E \cup F \in \overline{\mathcal{M}}$ $(E \in \mathcal{M}, F \subset N \in \mathcal{N}$ then $E \cap N = \emptyset\}$
(otherwise replace F & N w/ $F \setminus E$ and $N \setminus E$)
Then $E \cup F = (E \cup N) \cap (N^c \cup F)$

$\Rightarrow (E \cup F)^c = \underbrace{(E \cup N)^c}_{\subset \mathcal{M}} \cup \underbrace{(N \setminus F)}_{\subset N}$

$\Rightarrow (E \cup F)^c \in \overline{\mathcal{M}}$

$\Rightarrow \overline{\mathcal{M}}$ is a σ algebra.

• If $E \cup F \in \overline{\mathcal{M}}$ as above. Let $\bar{\mu}(E \cup F) = \mu(E)$
If $E_1 \cup F_1 = E_2 \cup F_2$ $F_j \subset N_j \in \mathcal{N}$ $E_1 \subset E_2 \cup N_2$ so
$\mu(E_1) \le \mu(E_2) + \mu(N_2) = \mu(E_2)$ Similarly $\mu(E_2) \le \mu(E_1)$
So $\mu(E_1) = \mu(E_2)$ so $\bar{\mu}(E \cup F) = \mu(E)$ is well defined

• Now WTS $\bar{\mu}$ is a complete measure on $\bar{\mu}$
i.e. WTS $\bar{\mu}$ contains all its null sets.
Let $\mathcal{B}$ be all null sets of $\bar{\mu}$, $\Rightarrow \bar{\mu}(\mathcal{B}) = 0$
WTS $\mathcal{B} \subset \bar{\mu}$.
$B = \emptyset \cup B$, $\emptyset \in \mathcal{M}$, $0 = \bar{\mu}(B) = \bar{\mu}(\emptyset \cup B) = \mu(B)$
$\Rightarrow \mathcal{B} \subset \overline{\mathcal{M}}$ so $\bar{\mu}$ is a complete measure.

• Finally WTS $\bar{\mu}$ is unique.
Let $\bar{\nu}$ be another such measure

$\Rightarrow \bar{\nu}|_{\mathcal{M}} = \mu$

we have $\bar{\mu}|_{\mathcal{M}} = \mu$

1.3.7 If $\mu_1, \dots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \dots, a_n \in [0, \infty)$ then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Pf Let $\hat{\mu} = \sum_1^n a_j \mu_j$.

$\hat{\mu}(\emptyset) = \sum_1^n a_j \mu_j(\emptyset) = \sum a_j \cdot 0 = 0$ since μ_j's are measures.

Let $\{E_k\}_1^\infty$ a sequence of disjoint sets in $\mathcal{M}$

$$\Rightarrow \hat{\mu}(\cup_1^\infty E_k) = \sum_{j=1}^n a_j \mu_j(\cup_1^\infty E_k)$$

$$= \sum_{j=1}^n a_j \sum_1^\infty \mu_j(E_k) \quad \text{since } \mu_j \text{ are measures,}$$

$$= \sum_1^n \sum_1^\infty a_j \mu_j(E_k)$$

$$= \sum_1^n \hat{\mu}(E_k).$$

$\therefore \hat{\mu}$ is a measure. $\square$

1.3.8 If $(X, \mu, \mathcal{M})$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$ then $\mu(\varliminf E_j) \leq \varliminf \mu(E_j)$ Also $\mu(\varlimsup E_j) \geq \varlimsup \mu(E_j)$ provided $\mu(\cup_1^\infty E_j) < \infty$

Pf Notice $\varliminf E_j = \cup_{k=1}^\infty \cap_{n \geq k}^\infty E_n$.

Let $A_k = \cap_{n \geq k}^\infty E_n$ then $A_k \subset A_{k+1}$

$$\Rightarrow \mu(\varliminf E_j) = \mu(\cup_1^\infty A_k)$$

?

$$= \lim_{k \to \infty} \mu(A_k) \quad \text{continuous from below.}$$

$$= \lim_{k \to \infty} \mu(\cap_{n \geq k}^\infty E_n)$$

1.3.9 If $(X, \mathcal{M}, \mu)$ is a m.s and $E, F \in \mathcal{M}$ then

$$\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F)$$

Pf Case 1 $\mu(E) = \infty$ or $\mu(F) = \infty$

$$E \subset E \cup F \Rightarrow \mu(E) \leq \mu(E \cup F) \Rightarrow \mu(E) = \infty$$

$$\Rightarrow \infty = \infty$$

• Case 2 $\mu(E) < \infty$ and $\mu(F) < \infty$

$$\mu(E \cup F) = \mu(E \cup (F \setminus E \cap F))$$

$$= \mu(E) + \mu(F \setminus E \cap F)$$

$$= \mu(E) + \mu(F) - \underbrace{\mu(E \cap F)}_{< \infty}$$

$$\Rightarrow \mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$$

$\square$

1.3.10 Given $(X, \mathcal{M}, \mu)$ $E \in \mathcal{M}$. Define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$.

Then μ_E is a measure.

Pf • $\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0$

Let $E_1, E_2, \ldots$ be disjoint sets in $\mathcal{M}$.

$$\mu_E(\cup_1^\infty E_j) = \mu((\cup_1^\infty E_j) \cap E)$$

$$= \mu(\cup_1^\infty (E_j \cap E))$$

$$= \sum_1^\infty \mu(E_j \cap E)$$

$$= \sum_1^\infty \mu_E(E_j)$$

$\therefore$ μ_E is a measure

$\square$

1.3.11 (a) A finitely additive measure μ is a measure $\Leftrightarrow$ it is continuous from below as in 1.8c.

(b). If $\mu(X)<\infty$ μ is a measure $\Leftrightarrow$ it is continuous as in 1.8.d

Pf (a) Let μ be continuous from below.

Let $\{E_j\}_1^\infty$ be a seq. of disjoint sets in $\mathcal{M}$.

WTS $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$

we know $\mu(\bigcup_1^n E_j) = \sum_1^n \mu(E_j)$ $\forall n$.

and if $E_1 \subset E_2 \subset \dots$ then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$

1.3.12 Let $(X, \mathcal{M}, \mu)$ be a finite measure space

a) If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$ then $\mu(E) = \mu(F)$

b) $E \sim F$ if $\mu(E \Delta F) = 0$ show $\sim$ is an equivalence relation

c) For $E, F \in \mathcal{M}$ define $\rho(E,F) = \mu(E \Delta F)$ Then $\rho(E,G) \leq \rho(E,F) + \rho(F,G)$ hence ρ defines a metric on $\mathcal{M}/\sim$

Pf a) $0 = \mu(E \Delta F) = \mu(E \setminus F \cup F \setminus E) = \mu(E \setminus F) + \mu(F \setminus E)$

$\Rightarrow \mu(F \setminus E) = 0 = \mu(E \setminus F)$ since μ is a positive measure

$\Rightarrow \mu(E) = \mu(E \setminus F) + \mu(E \cap F) = \mu(E \cap F)$

$\mu(F) = \mu(F \setminus E) + \mu(E \cap F) = \mu(E \cap F)$

$\Rightarrow \mu(E) = \mu(F)$

b) $\mu(E \Delta E) = \mu(E \setminus E \cup E \setminus E) = 0 \Rightarrow E \sim E$

$E \sim F \Rightarrow \mu(E \Delta F) = 0 = \mu(F \Delta E) = F \sim E \Rightarrow$ symmetric

$E \sim F,\ F \sim G \Rightarrow \mu(E \Delta F) = 0,\ \mu(F \Delta G) = 0$

$$\begin{aligned}
\mu(E \Delta G) &= \mu(E \setminus G \cup G \setminus E) \\
&= \mu(E \cap G^c) + \mu(G \cap E^c) \\
&= \mu(E \cap G^c \cap F^c) + \mu(E \cap G^c \cap F) + \mu(G \cap E^c \cap F^c) + \mu(G \cap E^c \cap F) \\
&\leq \mu(E \cap F^c) + \mu(E^c \cap F) + \mu(F \cap G^c) + \mu(F^c \cap G) \\
&= \mu(E \Delta F) + \mu(F \Delta G) \\
&= 0
\end{aligned}$$

$\Rightarrow$ transitive.

c)
$$\begin{aligned}
\rho(E,G) &= \mu(E \cap G^c) + \mu(G \cap E^c) \\
&= \mu(E \cap G^c \cap F^c) + \mu(E \cap G^c \cap F) + \mu(G \cap E^c \cap F^c) + \mu(G \cap E^c \cap F) \\
&\leq \mu(E \cap F^c) + \mu(F \cap G^c) + \mu(G \cap F^c) + \mu(E^c \cap F) \\
&= \mu(E \Delta F) + \mu(F \Delta G) \\
&= \rho(E,F) + \mu(F,G)
\end{aligned}$$

$\therefore \rho$ is a metric $\square$

1.3.13 Every σ-finite measure space is semifinite.

Pf Let μ be a σ-finite measure
Let $E \in \mathcal{M}$ s.t. $\mu(E) = \infty$ WTS $\exists F \subset E$ w/ $\mu(F) < \infty$
$\Rightarrow \exists \bigcup_1^\infty E_j = X$ s.t. $\mu(E_j) < \infty$ $\forall j$.
$E \subset X$
If $\exists j$ s.t. $E_j \subset E$ we're done.
If not $\exists E_j$ s.t. $E_j \cap E \neq \emptyset$
$\Rightarrow \mu(E_j \cap E) < \infty$ and $E_j \cap E \subset E$

$\square$

1.3.14 If μ is a semifinite measure and $\mu(E) = \infty$
$\forall c > 0$ $\exists F \subset E$ w/ $c < \mu(F) < \infty$.

Pf Let μ be a semifinite measure with $\mu(E) = \infty$
Let $c > 0$.
Let $\mathcal{F} = \{F \subset E : F$ measurable and $0 < \mu(F) < \infty\}$
$\mathcal{F} \neq \emptyset$ since μ is semifinite
Let $S = \sup\{\mu(F) : F \in \mathcal{F}\}$. WTS $S = \infty$.
Let $\{F_n\} \subset \mathcal{F}$ s.t. $\lim_{n\to\infty} \mu(F_n) = S$.
$\Rightarrow F = \bigcup F_n \subset E$ and $\lim_{n\to\infty} \mu(F_n) = S$
If $S < \infty$ then $\mu(E \setminus F) = \infty$
$\Rightarrow \exists F' \subset E \setminus F$ s.t. $0 < \mu(F') < \infty$
$\Rightarrow F \cup F' \subset E$ and $S < \mu(F \cup F') < \infty$
$\Rightarrow S = \infty$

$\Rightarrow \forall c$ $\exists F \subset E$ w/ $c < \mu(F) < \infty$

$\square$

1.4.17 If μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint measurable sets then $\mu^*(E\cap(\cup_1^\infty A_j)) = \sum_1^\infty \mu^*(E\cap A_j)$ $\forall E\subset X$.

Pf Let $E\subset X$ and $\{A_j\}$ disjoint sets in X.

$\mu^*(E\cap(\cup_1^\infty A_j)) = \mu^*(\cup_1^\infty E\cap A_j) \le \sum_1^\infty \mu^*(E\cap A_j)$

Now WTS

1.4.21 Let μ^* be an outer measure induced from a premeasure and $\bar{\mu}$ the restriction of μ^* to μ^* measurable sets then $\bar{\mu}$ is saturated

Pf Let $E\subset X$ s.t. $E\cap A$ is μ^* measurable $\forall A$ μ^* msble $\mu^*(A)<\infty$

WTS $\forall B\subset X$ $\mu^*(B)\ge\mu^*(E\cap B)+\mu^*(E^c\cap B)$

Inequality holds trivially if $\mu^*(B)=\infty$ so suppose $\mu^*(B)<\infty$
$\forall\varepsilon>0$ $\exists$ a μ^* measurable set A with $B\subset A$ and $\mu^*(A)\le\mu^*(B)+\varepsilon$

$$\begin{aligned}\Rightarrow \mu^*(B) &\ge \mu^*(A)-\varepsilon\\ &=\mu^*(A\cap(E\cap A))+\mu^*(A\cap(E\cap A)^c)-\varepsilon\\ &=\mu^*(A\cap E)+\mu^*(A\cap E^c)-\varepsilon\\ &\ge \mu^*(B\cap E)+\mu^*(B\cap E^c)-\varepsilon.\end{aligned}$$

$\Rightarrow E$ is μ^* msble since it holds $\forall\varepsilon$

$\square$

1.5.27 Prove C is compact, nowhere dense and totally disconnected with no isolated points

i.e show if $x, y \in C$ and $x<y$ the $\exists z \notin C$ s.t. $x<z<y$

Pf Let $x = \sum a_j 3^{-j}$ $\quad a_j \neq 1 \ \forall j$

$y = \sum b_j 3^{-j}$ $\quad b_j \neq 1 \ \forall j$ $\qquad x<y.$

then $0 \leq y - x = \sum (b_j - a_j) 3^{j}$ and $b_j \geq a_j$ for some j

$\Rightarrow \exists b_{j_0} > a_{j_0} \quad \Rightarrow b_{j_0} = 2, \ a_{j_0} = 0$

Let $z = \sum c_j 3^{-j}$ where $c_j = a_j \ \forall j \neq j_0$ $\quad c_{j_0} = 1$

$\Rightarrow z \notin C$ by $x<z<y$.

$\square$

1.5.28 Let F be increasing, right continuous and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$

$\mu_F([a,b)) = F(b-) - F(a-)$ $\quad \mu_F([a,b]) = F(b) - F(a-)$, $\mu_F((a,b)) = F(b-) - F($

Pf $\mu_F(\{a\}) = \mu_F(\bigcap_1^\infty (a - 1/n, a + 1/n])$

$= \lim \mu_F((a - 1/n, a + 1/n])$

$= \lim F(a + 1/n) - F(a - 1/n)$

$= F(a) - F(a-)$ since right continuous

$\mu_F([a,b)) = \mu((a,b] \cup \{a\} \setminus \{b\})$

$= \cancel{F(b)} - \cancel{F(a)} + \cancel{F(a)} - F(a-) - \cancel{F(b)} + F(b-)$

$= F(b-) - F(a-)$

$\mu_F([a,b]) = \mu((a,b] \cup \{a\})$

$= F(b) - F(a) + F(a) - F(a-)$

$= F(b) - F(a-)$

$\mu_F((a,b)) = \mu((a,b] \setminus \{b\})$ $\qquad \square$

$= F(b) - F(a) - F(b) + F(b-)$

$= F(b-) - F(a-)$

1.5.29 Let E be Lebesgue measurable

a) If $E \subset N$ where N is as in §1.1 then $m(E)=0$

b) If $m(E)>0$ then E contains a nonmeasurable set

pf Let $x \sim y$ if $x-y \in \mathbb{Q}$

Let $N = \{$set of one representative from each class$\}$

If $S \subset [0,1)$ let $S_r =$

1.5.30 If $E \in \mathcal{L}$ and $m(E)>0$. $\forall \alpha<1$ $\exists$ interval I s.t.
$m(E\cap I) > \alpha m(I)$

Pf By prop 1.2 $\forall \varepsilon>0$ $\exists A = \bigcup_1^n I_k$ s.t. $\mu(E \triangle A) < \varepsilon$.
$\hookrightarrow$ intervals

$\Rightarrow \mu(E\setminus A) + \mu(A \setminus E) < \varepsilon \mu(E)$

$\mu(E) = \mu(E\cap A) + \mu(E\setminus A) \le \sum \mu(I_j) + \varepsilon\mu(E)$

$\Rightarrow (1-\varepsilon)\mu(E) \le \sum \mu(I_j)$

now $\sum \mu(I_j) = \sum \mu(I_j \cap E) + \sum \mu(I_j \setminus E)$

$< \sum \mu(I_j \cap E) + \varepsilon \mu(E)$

$< \sum \mu(I_j \cap E) + \frac{\varepsilon}{1-\varepsilon} \sum \mu(I_j)$

Choose ε small enough so that $1 - \frac{\varepsilon}{1-\varepsilon} \geq \alpha$

$\Rightarrow \sum \mu(I_j)\left(1 - \frac{\varepsilon}{1-\varepsilon}\right) < \sum \mu(I_j \cap E)$

$\Rightarrow \mu(\sum I_j)\alpha < \mu(\cup I_j \cap E)$

$\Rightarrow \mu(I)\alpha < \mu(I\cap E)$

where $\cup I_j \subset I$. $\square$

1.5.31 If $E \subset \mathcal{L}$ and $m(E) > 0$ Show $E - E = \{x - y \mid x, y \in E\}$ contains an interval centered at 0.

Pf $m(E) > 0$

$\Rightarrow \exists$ interval $I = (a,b)$ s.t. $m(E \cap I) > \alpha m(I)$ $\quad 3/4 < \alpha < 1$

WTS $\exists r > 0$ s.t. $(t + (E \cap I)) \cap (E \cap I) \neq \emptyset \quad \forall |t| < r$

b/c then $(-r, r) \subset (E \cap I) - (E \cap I)$

$\Rightarrow (-r, r) \subset E - E$

Suppose BWOC that $\exists |t| < \frac{m(I)}{4}$ s.t. $(t + (E \cap I)) \cap (E \cap I) = \emptyset$

$\Rightarrow 3/2\, m(I) = m(I) + \frac{m(I)}{2}$

$\geq m(I) + 2|t|$

$\geq m(t + (E \cap I) \cup (E \cap I))$

Since $t + (E \cap I) \cup (E \cap I) \subset (t + I) \cup I \subset (a - |t|, b + |t|)$

Now $3/2\, m(I) \geq m(t + (E \cap I) \cup E \cap I)$

$= m(t + E \cap I) + m(E \cap I)$ since assumed disjoint

$= 2\, m(E \cap I)$ since m is translation invarient

$> 2 \cdot 3/4\, m(I)$

$= 3/2\, m(I)$.

which contradicts since the inequality cannot be strict

$\Rightarrow t + (E \cap I) \cap (E \cap I) \neq \emptyset \quad \forall |t| < \frac{m(I)}{4}$

$\Rightarrow \left(-\frac{m(I)}{4}, \frac{m(I)}{4}\right) \subset E - E$

$\square$

Folland Chapter 2

2.1.1 Let $f: X \to \overline{\mathbb{R}}$ any $Y = f^{-1}(\mathbb{R})$. Then f is measurable $\iff f^{-1}(\{\infty\}) \in \mathcal{M}$, $f^{-1}(\{-\infty\}) \in \mathcal{M}$ and f measurable on Y.

Pf Assume f measurable on Y.

$$\Rightarrow f^{-1}(\{\infty\}) = f^{-1}((a,\infty] \setminus (a,\infty)) \quad \text{for some fixed } a \in \mathbb{R}$$

$$= \underbrace{f^{-1}((a,\infty])}_{\in \mathcal{M}} \cap \underbrace{f^{-1}((a,\infty)^c)}_{\in \mathcal{M}} \quad \text{by definition}$$

$$\in \mathcal{M}$$

$$\Rightarrow f^{-1}(\{-\infty\}) = f^{-1}([-\infty, a) \setminus (-\infty, a))$$

$$= \underbrace{f^{-1}([-\infty,a))}_{\in \mathcal{M}} \cap \underbrace{f^{-1}((-\infty,a)^c)}_{\in \mathcal{M}}$$

$$\in \mathcal{M}.$$

Assume $f^{-1}(\{\pm\infty\}) \in \mathcal{M}$ and f is measurable on Y

Let $E \in \mathcal{B}_{\overline{\mathbb{R}}}$

$$\Rightarrow f^{-1}(E) = f^{-1}((E \cap \mathbb{R}) \cup (E \cap \{\infty\}) \cup (E \cap \{-\infty\}))$$

$$= \underbrace{f^{-1}(E \cap \mathbb{R})}_{\in \mathcal{B}_{\mathbb{R}}} \cup f^{-1}(\underbrace{E \cap \{\infty\}}_{\emptyset \text{ or } \{\infty\}}) \cup f^{-1}(\underbrace{E \cap \{-\infty\}}_{\emptyset \text{ or } \{-\infty\}})$$

$$\in \mathcal{M}$$

$\Rightarrow f$ is measurable. $\square$

To show function msble write f^{-1} as union or intersection of msble sets.

2.1.2 Suppose $f, g : X \to \overline{\mathbb{R}}$ measurable.

a) fg measurable

b) Fix $a \in \overline{\mathbb{R}}$. Let $h(x) = \begin{cases} a & f(x) = -g(x) = \pm\infty \\ f(x)+g(x) & \text{otherwise} \end{cases}$

Show h is measurable.

Pf (a) fg is measurable on $(fg)^{-1}(\mathbb{R})$

b/c $\Phi : X \times X \to \overline{\mathbb{R}}$ is a continuous map

$$(fg)^{-1}(\{\infty\}) = \left\{x \,\middle|\, \begin{matrix} f(x)>0 \\ g(x)=\infty \end{matrix}\right\} \cup \left\{x \,\middle|\, \begin{matrix} f(x)<0 \\ g(x)=-\infty \end{matrix}\right\} \cup \left\{x \,\middle|\, \begin{matrix} f(x)=\infty \\ g(x)>0 \end{matrix}\right\} \cup \left\{x \,\middle|\, \begin{matrix} f(x)=-\infty \\ g(x)<0 \end{matrix}\right\}$$

$$= [f^{-1}((0,\infty]) \cap g^{-1}(\{\infty\})] \cup [f^{-1}([-\infty,0)) \cap g^{-1}(\{-\infty\})] \cup [g^{-1}((0,\infty]) \cap f^{-1}(\{\infty\})] \cup [g^{-1}([-\infty,0) \cap f^{-1}(\{\infty$$

$\in \mathcal{X}$

all terms are measurable by 2.1.1.

Similarly $(fg)^{-1}(\{-\infty\}) \in \mathcal{X}$.

$\Rightarrow fg$ is measurable for all $\overline{\mathbb{R}}$

(b) if $b<a$.

$h^{-1}((-\infty,b]) = (f+g)^{-1}((-\infty,b))$

If $a<b$

$h^{-1}((-\infty,b]) = (f+g)^{-1}((-\infty,b)) \cup (f^{-1}(\{\infty\}) \cap g^{-1}(\{-\infty\})) \cup (f^{-1}(\{-\infty\}) \cap g^{-1}(\{\infty\}))$

$h^{-1}(\{\infty\}) = (f^{-1}(\{\infty\}) \cap g^{-1}(\mathbb{R})) \cup (f^{-1}(\mathbb{R}) \cap g^{-1}(\{\infty\}))$

$h^{-1}(\{-\infty\}) = (f^{-1}(\{-\infty\}) \cap g^{-1}(\mathbb{R})\} \cup (f^{-1}(\mathbb{R}) \cap g^{-1}(\{-\infty\}))$

$\Rightarrow h$ is measurable.

$\square$

2.1.3 If $\{f_n\}$ is a sequence of measurable functions on X then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Pf $\{x : \exists \lim f_n(x)\} = \{x : \overline{\lim}\, f_n(x) = \underline{\lim}\, f_n(x)\}$
$= \{x : \overline{\lim}\, f_n(x) - \underline{\lim}\, f_n(x) = 0\}.$

$\overline{\lim}$ and $\underline{\lim}$ are measurable and addition is continuous.

$\Rightarrow \{x : \exists \lim f_n(x)\}$ is measurable $\square$

2.1.4 If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r,\infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$ then f is measurable.

Pf for $a \in \overline{\mathbb{R}}$

$f^{-1}((a,\infty)) = \bigcup_{r>a} f^{-1}((r,\infty]) \setminus \underbrace{f^{-1}(\{\infty\})}_{\text{measurable by below.}}$ is msble on $f^{-1}(\mathbb{R})$

$f^{-1}(\{\infty\}) = \bigcap_{r \in \mathbb{Q}} f^{-1}((r,\infty]) \in \mathcal{M}$ since countable intersections are in $\mathcal{M}$

$f^{-1}(\{-\infty\}) = \bigcap_{r \in \mathbb{Q}} f^{-1}([-\infty, r]) = \bigcap_{r \in \mathbb{Q}} f^{-1}((r,\infty])^c \in \mathcal{M}$

$\square$

- Need to check (a,∞), $\{\infty\}$, $\{-\infty\}$
- Can approximate $a \in \overline{\mathbb{R}}$ by $\bigcup_{r>a}$

2.1.5 If $X = A \cup B$, $A, B \in \mathcal{M}$. Then f is measurable on X $\iff$ f measurable on A and B.

Pf Assume f is measurable on X.

$\Rightarrow f^{-1}(E) \in \mathcal{M} \quad \forall E \in \mathcal{B}$

$\Rightarrow f^{-1}(E) \cap A \in \mathcal{M}$ since $f^{-1}(E)$ and A are.

$f^{-1}(E) \cap B \in \mathcal{M}$ since $f^{-1}(E)$ and B are.

$\Rightarrow f$ is measurable on A and B.

Assume f is measurable on A and B.

Let $E \in \mathcal{B}$

$$\Rightarrow f^{-1}(E) = f^{-1}(E) \cap X = f^{-1}(E) \cap (A \cup B) = \underbrace{[f^{-1}(E) \cap A]}_{\in \mathcal{M}} \cup \underbrace{[f^{-1}(E) \cap B]}_{\in \mathcal{M}} \text{ by assumption}$$

$\in \mathcal{M}$

$\Rightarrow f$ is measurable on X. $\square$

2.1.6 Show the sup of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable

Pf Let N be a nonmeasurable set such as Vitali set

Define $A = \{\alpha : \alpha \in N\}$ and $f_\alpha = \chi_{\{\alpha\}}$

$\Rightarrow \sup\limits_{\alpha \in A} f_\alpha = \chi_N$ which is not measurable since N isnt

$\square$

$\sup\limits_{\alpha \in N} f_\alpha$ is function in question.

2.1.7 Suppose $\forall \alpha \in \mathbb{R}$ we have set $E_\alpha \in \mathcal{M}$ s.t. $E_\alpha \subset E_\beta$ if $\alpha < \beta$. Let $X = \bigcup_{\alpha \in \mathbb{R}} E_\alpha$ and $\bigcap_{\alpha \in \mathbb{R}} E_\alpha = \phi$. Then $\exists$ measurable $f: X \to \mathbb{R}$ s.t. $f(x) \leq \alpha$ on E_α and $f(x) \geq \alpha$ on E_α^c $\forall \varepsilon$

Pf Consider $f(x) = \inf_{\alpha \in \mathbb{R}} \{\alpha \mid x \in E_\alpha\}$

$\Rightarrow f(x) \leq \alpha$ on E_α and $f(x) \geq \alpha$ on E_α^c

We need to show f is measurable

Let $r \in \mathbb{Q}$

$\Rightarrow f^{-1}((-\infty, r]) = \bigcup_{\alpha < r} E_\alpha$ its enough to consider $\alpha \in \mathbb{Q}$

$= E_r$ which is msble

$\Rightarrow f^{-1}((-\infty, r]) = \bigcup_{\alpha \leq r} \bigcap_{\beta > \alpha} E_\beta = \bigcup_{\substack{\alpha \in \mathbb{Q} \\ \alpha \leq r}} \bigcap_{\substack{\beta \in \mathbb{Q} \\ \beta > \alpha}} E_\beta = \bigcup_{\substack{\alpha \in \mathbb{Q} \\ \alpha < r}} E_\alpha = E_r$

$\square$

2.1.8 If $f: \mathbb{R} \to \mathbb{R}$ is monotone then f is Borel measurable.

Pf WLOG assume f is increasing.

f monotone $\Rightarrow$ f continuous at all but possibly countably many points.

Let x_0 be discontinuity of f.

Let (a,b) be s.t. x_0 is only discontinuity in $f^{-1}((a,b))$

$f^{-1}((a,b)) = \underbrace{f^{-1}((a, f(x_0)))}_{\in \mathcal{B}} \cup \underbrace{f^{-1}(f(x_0))}_{\text{either single pt or interval}} \cup \underbrace{f^{-1}((f(x_0), b))}_{\in \mathcal{B}}$

$\in \mathcal{B}$

$\Rightarrow f$ is Borel measurable

$\square$

2.1.9 $f: [0,1] \to [0,1]$ Cantor function. $g(x) = f(x) + x$.

a) Prove g is bijection from $[0,1] \to [0,2]$, g^{-1} cont.

b) C Cantor set $\Rightarrow m(g(C)) = 1$

c) $g(C)$ contains a Lebesgue nonmsble set A.

$B = g^{-1}(A) \Rightarrow B$ Lebesgue msble but not Borel

d) $\exists$ lebesgue msble F and cont. G s.t. $F \circ G$ not L.M.

Pf a) f increasing, x strictly increasing

$\Rightarrow g$ strictly increasing

$\Rightarrow g$ injective.

f continuous, x continuous

$\Rightarrow g$ continuous

$g(0) = f(0) = 0$, $g(1) = f(1) + 1 = 2$

$\Rightarrow g$ surjective by IVT

$\Rightarrow g$ bijective

$\Rightarrow g^{-1}$ continuous since g is bijective and continuous.

b) Let C be the cantor set.

$\Rightarrow g(C) = [0,2]$

$\Rightarrow m(g(C))$

c) $m(g(C)) > 0 \Rightarrow \exists A \notin \mathcal{L}$ s.t. $A \subset g(C)$ since a set of positive measure always contains a nonmsble set.

Let $B = g^{-1}(A)$.

WTS $B \in \mathcal{L}$ but $B \notin \mathcal{B}$

$B = g^{-1}(A) \subset g^{-1}(g(C)) = C$ and $m(C) = 0$.

$\Rightarrow g^{-1}(A)$ is msble since m is a complete measure

Assume BWOC that $B \in \mathcal{B}$

$\Rightarrow (g^{-1})^{-1}(B) \in \mathcal{B}$ since g^{-1} is continuous

$\Rightarrow g(B) \in \mathcal{B}$

but $g(B) = g(g^{-1}(A)) = A \notin \mathcal{L}$.

However $\mathcal{B} \subset \mathcal{L}$ which contradicts

$\therefore B \in \mathcal{L}$ but $B \notin \mathcal{B}$

(d) Let $F = \chi_{g^{-1}(A)}$ and $G = g$

$\Rightarrow F \circ G^{-1}\{1\}$

2.2.12 If $f \in L^+$ and $\int f < \infty$ then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite

Pf If $m(\{f = \infty\}) > 0$

$\Rightarrow \int f \geq \int_{\{f=\infty\}} f \, dm = \infty$ ↯

$\Rightarrow m(\{f = \infty\}) = 0$

$\Rightarrow \{x : f(x) = \infty\}$ is a null set.

Now to show $\{x : f(x) > 0\}$ is σ-finite.

$\{x \mid f(x) > 0\} = \bigcup_{n \in \mathbb{N}} \{x \mid f(x) > \frac{1}{n}\}$

$\infty > \int f > \int_{\{x \mid f(x) > \frac{1}{n}\}} f > \int_{\{x \mid f(x) > \frac{1}{n}\}} \frac{1}{n} = \frac{1}{n} m(\{x \mid f(x) > \frac{1}{n}\})$

$\Rightarrow m(\{x \mid f(x) > \frac{1}{n}\}) < \infty \quad \forall n \in \mathbb{N}$

$\Rightarrow \{x \mid f(x) > 0\}$ is σ-finite.

$\square$

$m(x : f(x) > 1/n\} < \frac{1}{n} \int f.$

2.2.13 Suppose $\{f_n\} \subset L^+$, $f_n \to f$ ptwise and $\int f = \lim \int f_n \nearrow \infty$

Then $\int_E f = \lim \int_E f_n \quad \forall E \in M$. but need not be

true if $\int f = \lim \int f_n = \infty$

Pf Let $E \in M$.

$$\int_E f = \int f \chi_E = \int \lim f_n \chi_E \underset{\text{fatou}}{\leq} \underline{\lim} \int f_n \chi_E = \underline{\lim} \int_E f_n$$

Now $\int f - \int_E f = \int f - f\chi_E$

$= \int \lim (f_n - f_n \chi_E)$

$\leq \underline{\lim} \left(\int f_n - \int f_n \chi_E \right)$

$= \int f - \underline{\lim} \int_E f_n$

$\Rightarrow \; -\int_E f \leq -\underline{\lim} \int_E f_n$

$\Rightarrow \; \int_E f \geq \overline{\lim} \int_E f_n$

$\Rightarrow \; \int_E f = \lim \int_E f_n. \quad \forall E \in M.$

Now assume $\int f = \lim \int f_n = \infty$

By way of a counterexample let $f_n = \chi_{(-\infty,0)} + \frac{1}{n} \chi_{[0,n)}$

$\Rightarrow f_n \to \chi_{(-\infty,0)}$ as $n \to \infty$

However $\int f = \int \chi_{(-\infty,0)} = \int_{\infty}^{0} 1 = \infty$

$\lim \int f_n = \lim \int \chi_{(-\infty,0} + \frac{1}{n} \chi_{[0,n)}$?

2.2.14 If $f \in L^+$ let $\lambda(E) = \int_E f\,d\mu$, $E \in M$ then λ is a measure and $\int g\,d\lambda = \int gf\,d\mu$

Pf $\cdot$ $\lambda(\phi) = \int_\phi f\,d\mu = \int f \chi_\phi\,d\mu = \int 0\,d\mu = 0$

$\cdot$ Let $(E_n)_{n \in \mathbb{N}}$ disjoint

$\Rightarrow \lambda(\bigcup_1^\infty E_n) = \int_{\bigcup_1^\infty E_n} f\,d\mu$

$= \int f \chi_{\bigcup_1^\infty E_n}\,d\mu$

$= \int \sum_1^\infty f \chi_{E_n}\,d\mu$

$= \sum_1^\infty \int f \chi_{E_n}\,d\mu$

$= \sum_1^\infty \lambda(E_n)$

$\Rightarrow \lambda$ is a measure.

First consider $g = \chi_A$ for $A \in M$

$\Rightarrow \int g\,d\lambda = \int_A d\lambda = \lambda(A) = \int_A f\,d\mu = \int f \chi_A\,d\mu = \int fg\,d\mu$

Claim follows for simple functions by additivity and linearity of integrals.

Now let $g \in L^+$

$\Rightarrow \exists g_n \nearrow g$ g_n simple fcns.

$\Rightarrow \int g\,d\lambda = \int \lim g_n\,d\lambda = \lim \int g_n\,d\lambda = \lim \int g_n f\,d\mu$.

and $g_n f \nearrow gf$ so by MCT $\int g\,d\lambda = \int gf\,d\mu$

$\square$

2.2.15 If $\{f_n\} \subset L^+$, $f_n \searrow f$ pointwise and $\int f_1 < \infty$ then $\int f = \lim \int f_n$

Pf First note $\int f \leq \int f_1 < \infty$

Now $f_1 - f_n \nearrow f_1 - f$

$\Rightarrow \lim \int (f_1 - f_n) = \int \lim (f_1 - f_n) = \int f_1 - f = \int f_1 - \int f$ by MCT.

and $\lim \int (f_1 - f_n) = \int f_1 - \lim \int f_n$

$\Rightarrow \int f_1 - \lim \int f_n = \int f_1 - \int f$

$\Rightarrow \lim \int f_n = \int f$ $\square$

2.2.16 $f \in L^+$ and $\int f < \infty$. $\forall \varepsilon > 0$, $\exists E \in M$ s.t. $\mu(E) < \infty$ and $\int_E f > (\int f) - \varepsilon$

Pf Consider $E_n = \{x \mid f(x) > \frac{1}{n}\}$

$\Rightarrow \mu(E_n) < \infty$

Let $f_n = f \chi_{E_n}$. Then $f_n \nearrow f$

$\Rightarrow \int f_n \to f$ as $n \to \infty$

$\Rightarrow \int_{E_n} f \to \int f$ as $n \to \infty$

$\Rightarrow \forall \varepsilon > 0$ $\exists n_0$ s.t. $|\int_{E_n} f - \int f| < \varepsilon$ $n \geq n_0$

$\Rightarrow \int f - \int_{E_n} f < \varepsilon$

$\Rightarrow \int_{E_n} f > \int f - \varepsilon$. $\square$

2.2.17 Assume Fatous and deduce MCT

Pf Let $\{f_n\} \in L^+$ s.t. $f_n \nearrow f \in L^+$ pointwise.

$\Rightarrow \int f_n \leq \int f$ by Monotonicity $\forall n$

$\Rightarrow \overline{\lim} \int f_n \leq \int f$

Now by Fatou

$\int f = \int \lim f_n \leq \underline{\lim} \int f_n$

$\Rightarrow \int f \leq \underline{\lim} \int f_n \leq \overline{\lim} \int f_n \leq \int f$.

$\Rightarrow \int f = \lim \int f_n$

2.3.19 Suppose $\{f_n\} \subset \mathcal{L}^1(\mu)$ and $f_n \to f$ uniformly

a) If $\mu(X) < \infty$ then $f \in \mathcal{L}^1(\mu)$ and $\int f_n \to \int f$

b) If $\mu(X) = \infty$ then the claim can fail

Pf Let $\varepsilon > 0$

$f_n \to f$ uniformly $\Rightarrow \exists N$ s.t $\forall x, \forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon/2$

$$\Rightarrow \int_X |f| d\mu = \int_X |f - f_n + f_n| d\mu$$
$$\leq \int_X |f_n - f| d\mu + \int_X |f_n| d\mu$$
$$\leq \int_X \varepsilon/2 \, d\mu + \int_X |f_n| d\mu \quad \text{for } n \geq N$$
$$= \varepsilon/2 \, \mu(X) + M\mu(X) \quad (f_n \in \mathcal{L}^1(\mu) \Rightarrow \int_X |f_n| < \infty$$
$$< \infty$$

$\Rightarrow f \in \mathcal{L}^1$

Now $\int |f_n - f| d\mu \leq \int |f - f_N| d\mu + \int |f_n - f_N| d\mu < \frac{\varepsilon \mu(X)}{2} + \frac{\varepsilon \mu(X)}{2}$ for n large enough

$\Rightarrow \int f_n \to \int f$.

b). Let $f_n = \frac{1}{n} \chi_{[0,n)}$

$\Rightarrow f_n \to 0$ as $n \to \infty$

$\int f_n = 1 \;\; \forall n$ but $\int f = \int 0 = 0$

So $\int f_n \not\to \int f$.

$\square$

$\underline{2.3.20.}$ If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e and $|f_n| \le g_n$ and $\int g_n \to \int g$ then $\int f_n \to \int f$.
(Gereralized Dominated convergence thm.

$\underline{Pf}$ $|f_n| \le g_n \Rightarrow -g_n \le f_n \le g_n$

$\Rightarrow f_n + g_n > 0$ and $g_n - f_n > 0$

$$\Rightarrow \int f + \int g = \int f+g \qquad\qquad \text{and} \qquad \int g - \int f = \int g - f$$
$$= \int \lim f_n + \lim g_n \qquad\qquad = \int \lim g_n - \lim f_n$$
$$= \int \lim (f_n + g_n)$$
$$\le \underline{\lim} (\int f_n + \int g_n) \leftarrow \text{by Fatou} \rightarrow \quad \le \underline{\lim} \int g_n - \int f_n$$
$$= \underline{\lim} \int f_n + \int g \qquad\qquad = \int g - \underline{\lim} \int f$$
$$\Rightarrow \int f \le \underline{\lim} \int f_n \qquad\qquad \Rightarrow \int f \ge \overline{\lim} \int f_n$$

$\therefore \int f = \lim \int f_n$ $\square$

$\underline{2.3.21.}$ Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$

$\underline{Pf}$ Assume $\int |f_n - f| \to 0$ Let $\varepsilon > 0$.

$\Rightarrow \exists N$ s.t. $n \ge N \Rightarrow \int |f_n - f| < \varepsilon$ (or use 2.3.20)

$\Rightarrow \int |f_n| - \int |f| = \int |f_n| - |f| \le \int |f_n - f| < \varepsilon$

$\Rightarrow \int |f_n| \to \int |f|$

Assume $\int |f_n| - \int |f|$

$\Rightarrow \exists N$ s.t. $n \ge N \Rightarrow \int |f_n| - \int |f| < \varepsilon$.

Let $g_n = |f_n| + |f|$. $\Rightarrow |f_n - f| \le g_n$ and $\int g_n \to 2\int f \in L^1$

$\Rightarrow$ By GDCT we have. $\lim \int |f_n - f| = \int \lim |f_n - f| = 0$

$\square$

$$\left\{ \begin{aligned} &|f_n| = |f_n - f + f| \le |f_n - f| + |f| \\ &\Rightarrow |f_n| - |f| \le |f_n - f| \end{aligned} \right\}$$

2.3.25 Let $f(x) = \begin{cases} x^{-1/2} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$ Let $\mathbb{Q} = \{r_n\}$

Let $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$

a) Show $g \in L^1(m)$ and $g < \infty$ a.e

b) g is discontinuous at every pt and unbdd on ever interval

c) $g^2 < \infty$ a.e but g^2 is not integrable on any interval

Pf a) $g = \sum_1^\infty f_n$ where $f_n = 2^{-n} f(x - r_n)$

$\Rightarrow \int f_n = \int_{-\infty}^{\infty} 2^{-n} \frac{1}{\sqrt{x - r_n}} \chi_{(0,1)} dx$

$= \int_{r_n}^{r_n+1} \frac{1}{2^n (x - r_n)^{1/2}} dx \qquad u = x - r_n$

$= \int_0^1 \frac{1}{2^n u} du$

$= 2^{-n+1}$

$\int g = \sum \int f_n = \sum_1^\infty 2^{-n+1} = \sum_0^\infty 2^{-n} = 2 < \infty$

$\Rightarrow g \in L_1$

b). Let $M > 0$. Let (a, b) be any interval.

$\mathbb{Q}$ dense $\Rightarrow \exists r_j \in (a, b)$.

$\Rightarrow 2^j \frac{1}{\sqrt{x - r_j}} \geq M$ for some $x \in (a, b)$.

$\Rightarrow g(x) \geq 2^j \frac{1}{\sqrt{x - r_j}} \geq M$

$\Rightarrow g$ is unbounded on any interval

Let $x_0 \in \mathbb{R}$.

Case 1 $g(x_0) = \infty$

$\forall \delta > 0 \; \exists x \in B_\delta(x_0)$ s.t. $g(x)$ is finite since $g \in L^1$

$\Rightarrow g$ not cont at x_0

Case 2 $g(x_0) = C < \infty$

g unbdd on $B_\delta(x_0)$ $\quad \exists x$ s.t. $g(x) > C + 1$

$\Rightarrow |g(x) - g(x_0)| > 1 \Rightarrow g$ not cont

c).

$\Rightarrow \int_a^b |g^2| = \int_a^b g^2 = \int_a^b \left(\sum_1^\infty 2^{-n} f(x-r_n)\right)^2$

$\geq \int_a^b \sum_1^\infty 2^{-n} f^2(x-r_n)$

$= \int_{a-r_n}^{a+r} 2^{-n} \frac{1}{x} \chi_{(0,1)}$

$= \int_0^{\min\{b-r_n,1\}} 2^{-n} \frac{1}{x}\, dx$

$= \infty$

2.3.26 If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^{x} f(t)\,dt \Rightarrow F(x)$ is cont on $\mathbb{R}$

Pf Let $x_n \to x$

Let $f_n = f\chi_{(-\infty, x_n]}$

$\Rightarrow |f_n| \leq |f|$ $\forall n$ and f_n is msble.

$\Rightarrow f_n \to f\chi_{(-\infty, x]}$ a.e.

$\Rightarrow |f| \geq 0$ $\forall x$

$\Rightarrow$ By DCT. $\int f_n \; \int f$ $\Rightarrow \int_{-\infty}^{x_n} f = F(x_n) \nearrow \int_{-\infty}^{x} f = F(x)$

$\Rightarrow F(x)$ is continuous on $\mathbb{R}$

Since $\forall x_n \to x$ $F(x_n) \to F(x)$

2.3.28 a) $\lim \int_0^\infty (1+x/n)^{-n} \sin x/n \, dx$

b) $\lim \int_0^1 (1+nx^2)(1+x^2)^{-n} dx$

c) $\lim \int_0^\infty n \sin x/n \, [x(1+x^2)]^{-1} dx$

d) $\lim \int_a^\infty n(1+n^2x^2)^{-1} dx.$

Pf a) $n \geq 2 \Rightarrow |(1+x/n)^{-n} \sin x/n| \leq (1+x/n)^{-n} \leq (1+x/n)^{-2} \in L^1(0,\infty)$

By DCT $\lim_{n\to\infty} \int_0^\infty (1+x/n)^{-n} \sin x/n = \int_0^\infty \lim\left[(1+\frac{x}{n})^{-n} \sin x/n\right] = 0.$

Note: $\int_0^\infty (1+x/n)^{-2} = \frac{n(1+x/n)^{-1}}{-1}\Big|_0^\infty = -n.$

$\lim (1+\frac{x}{n})^{-n} \sin x/n =$

b) $\frac{1+nx^2}{(1+x^2)^n} \leq 1$ and $\int_0^1 1 = 1$

$\therefore$ By DCT $\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} = \int_0^1 \lim \frac{1+nx^2}{(1+x^2)^n} dx$

$= \int_0^1 \lim \frac{2nx}{n(1+x^2)^{n-1}}$

$= \int_0^1 \lim \frac{2x}{n(n-1)(1+x^2)^{n-2}}$

$= \int_0^1 0 = 0$

c) $\left|\frac{n \sin n x}{x(1+x^2)}\right| \leq \frac{1}{x(1+x^2)} \in L^1(0,\infty)$

By DCT $\lim \int_0^\infty \frac{n \sin n x}{x(1+x^2)} = \int_0^\infty \lim \frac{n \sin n x}{x(1+x^2)}$

$= \int_0^\infty \frac{1}{1+x^2} = \arctan\Big|_0^\infty = \pi/2,$

d) $\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1} = \lim \int_{na}^\infty (1+y^2)^{-1} dy = \lim \arctan(y)\Big|_{na}^\infty = \begin{cases} \pi/2 & a=0 \\ 0 & a>0 \\ \pi & a<0 \end{cases}$

$\square$

2.4.32 Suppose $\mu(X) < \infty$. If f and g are complex valued measurable functions on X. Define $\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}\,d\mu$. Then ρ is a metric and $f_n \to f \iff f_n \xrightarrow{\mu} f$

Pf To show metric

- $\rho(f,g) = \rho(g,f)$ ✓
- $\rho(f,g) \geq 0$ since $\frac{|f-g|}{1+|f-g} \geq 0$
- $\rho(f,g) = 0 \iff 0 = \int \frac{|f-g|}{1+|f-g|}\,d\mu$

$$\iff 0 = |f-g|$$

$$\iff f = g \text{ a.e.}$$

- For triangle inequality.

$$\frac{|f-g|}{1+|f-g|} = \frac{1}{\frac{1}{|f-g|}+1} \leq \frac{1}{\frac{1}{|f-h|+|h-g|}+1} = \frac{|f-h|+|h-g|}{1+|f-h|+|h-g|} \leq \frac{|f-h|}{1+|f-h|} + \frac{|h-g|}{1-|h-g}$$

$\Rightarrow \rho(f,g) \leq \rho(f,h) + \rho(h,g)$

$\Rightarrow \rho$ is a metric

Now assume $f_n \xrightarrow{\mu} f$

Let $E_\varepsilon = \{x \mid |f_n(x) - f(x)| \leq \varepsilon\}$ since $f_n \xrightarrow{\mu} f$ then $\mu(E_\varepsilon{}^c) \to 0$

$$\Rightarrow \rho(f_n, f) = \int_{E_\varepsilon} \frac{1}{\frac{1}{|f_n-f|}+1} + \int_{E_\varepsilon^c} \underbrace{\frac{1}{\frac{1}{|f_n-f|}+1}}_{\leq 1}$$

$$\leq \int_{E_\varepsilon} \frac{\varepsilon}{1+\varepsilon} + \int_{E_\varepsilon{}^c} 1$$

$$= \frac{\varepsilon}{1+\varepsilon} \int_{E_\varepsilon} 1 + \mu(E_\varepsilon{}^c)$$

$$\leq \frac{\varepsilon}{1+\varepsilon}\mu(X) + \mu(E_\varepsilon{}^c) \to 0 \quad \Rightarrow \quad f_n \xrightarrow{\rho} f$$

Now assume $f_n \xrightarrow{\rho} f$

$$0 \leftarrow \rho(f_n - f) \geq \int_{E_\varepsilon{}^c} \frac{1}{\frac{1}{|f_n-f|}+1}$$

$$\geq \int_{E_\varepsilon{}^c} \frac{1}{\frac{1}{\varepsilon}+1}$$

$$= \varepsilon/_{1+\varepsilon}\, \mu(E_\varepsilon{}^c) \to 0 \quad \therefore \quad f_n \xrightarrow{\mu} f. \qquad \square$$

2.4.33 If $f_n \geq 0$ and $f_n \xrightarrow{m} f$ then $\int f \leq \underline{\lim} \int f_n$

Pf Let f_{n_k} be a subsequence of f_n.

$\Rightarrow f_{n_k} \xrightarrow{m} f$

$\Rightarrow \exists f_{n_{k_\ell}} \to f$ a.e.

$\Rightarrow \int f = \int \lim f_{n_{k_\ell}} \leq \underline{\lim} \int f_{n_{k_\ell}}$ by fatou.

2.4.34. Suppose $|f_n| \leq g \in \mathcal{L}^1$, $f_n \xrightarrow{m} f$

a) Show $\int f = \lim \int f_n$

b) $f_n \to f$ in $\mathcal{L}^1$

Pf a) Let f_{n_k} be a subsequence of f_n.

$\Rightarrow \exists f_{n_{k_\ell}} \to f$ a.e

$\Rightarrow \lim \int f_{n_{k_\ell}} = \int f$ by DCT

$\Rightarrow \lim \int f_n = \int f$ since every subseq has conv. sub

b) Consider $\int |f_n - f| d\mu$.

Let $|h_n| = |f_n - f|$ then $|h_n| \leq 2g \in \mathcal{L}^1$

$\Rightarrow \int \lim |h_n| = \lim \int |h_n|$

$\Rightarrow 0 = \lim \int |f_n - f|$

$\Rightarrow f_n \to f$ in $\mathcal{L}$.

$\square$

2.4.35 $f_n \xrightarrow{\mu} f \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

s.t. $\mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\}) < \varepsilon \quad \forall n \in \mathbb{N}$

Pf Assume $f_n \xrightarrow{\mu} f$ and let $\varepsilon > 0$.

$\Rightarrow \mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\}) \to 0$ as $n \to \infty$

$\Rightarrow \exists N > 0$ s.t. $\mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\}) < \varepsilon \quad \forall n \geq N$

Now assume $\mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\}) < \varepsilon$

Let $E_\varepsilon = \{x \mid |f_n(x) - f(x)| \geq \varepsilon\}$.

$\Rightarrow \mu(E_\varepsilon) \leq \mu(E_\delta) < \delta \quad \forall \delta < \varepsilon$

$\Rightarrow f_n \xrightarrow{\mu} f$.

□

2.4.36 If $\mu(E_n) < \infty \ \forall n \in \mathbb{N}$ and $\chi_{E_n} \xrightarrow{L_1} f$ then $f = \chi_E$ for some measurable E a.e.

Pf $\chi_{E_n} \xrightarrow{L_1} f \Rightarrow \exists E_{n_j}$ s.t. $\chi_{E_{n_j}} \to f$ a.e.

$\Rightarrow f$ attains only the values $\{0, 1\}$ up to a set of measure 0.

Let $E = \{x : \chi_{E_{n_j}} \to 1\}$

$\Rightarrow$ if $x \in E$ $\chi_{E_{n_j}} \to f$ a.e so $f = 1$

$\Rightarrow$ if $x \notin E$ $\chi_{E_{n_j}} \not\to f$ a.e so $f = 0$

$\Rightarrow f = \chi_E$.

□

subsubsequence

2.4.37 Suppose f_n, f measurable complex valued fcns, $\phi : \mathbb{C} \to \mathbb{C}$

a) If ϕ is continuous and $f_n \to f$ a.e. then $\phi \circ f_n \to \phi \circ f$

b) If ϕ is u.c and $f_n \to f$ uniformly, almost uniformly or in measur then $\phi \circ f_n \to \phi \circ f$ in same

c) There are counterexamples when continuity assumptions on ϕ are not satisfied.

Pf Let $\varepsilon > 0$.

ϕ cont $\Rightarrow \forall x \ \exists \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |\phi(x) - \phi(y)| < \varepsilon$

$f_n \to f$ ae $\Rightarrow \forall x \ \exists N > 0$ s.t $n \geq N_x \Rightarrow |f_n(x) - f(x)| < \delta$

Now let $n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \delta$

$\Rightarrow |\phi(f_n(x)) - \phi(f(x))| < \varepsilon$

$\Rightarrow \phi \circ f_n \longrightarrow \phi \circ f$ a.e

b) ϕ u.c $\Rightarrow \exists \delta > 0$ s.t $\forall |x_1 - x_2| < \delta \Rightarrow |\phi(x_1) - \phi(x_2) < \varepsilon$

- $f_n \xrightarrow{u} f \Rightarrow \exists M$ s.t. $n \geq M \Rightarrow |f_n(x) - f(x)| < \delta$

  $\Rightarrow |\phi \circ f_n(x) - \phi(f(x))| < \varepsilon$

  $\Rightarrow \phi \circ f_n \xrightarrow{u} \phi \circ f$

- $f_n \xrightarrow{a.u} f \Rightarrow \forall \varepsilon_1, \varepsilon_2 > 0 \ \exists E$ and $M \in \mathbb{N}$ s.t $\mu(E) < \varepsilon_1$ and $n > M \Rightarrow$ for $x \in X - E$ $|f_n(x) - f(x)| < \varepsilon_2$.

  $\Rightarrow |\phi \circ f_n(x) - \phi(f(x))| < \varepsilon$

  $\Rightarrow \phi \circ f_n \xrightarrow{a.u} \phi \circ f$

- $f_n \xrightarrow{\mu} f \Rightarrow \forall \varepsilon > 0$ $\mu(x : |f_n(x) - f| > \delta\} \to 0$.

  Notice $\{x : |\phi \circ f_n - \phi \circ f| > \varepsilon\} \subset \{x \mid |f_n - f| > \delta\}$

  $\Rightarrow \mu(\{x : |\phi \circ f_n - \phi \circ f| > \varepsilon\} \to 0$

  $\Rightarrow \phi \circ f_n \xrightarrow{\mu} \phi \circ f$

c) counter to a: $f_n(x) = 1/n$, $f(x) = 0$, $\phi(x) = \chi_{\{0\}}$

$\phi \circ f_n(x) = 0 \ \forall n$ but $\phi \circ f(x) = 1$

counter to b: $f_n(x) = x + 1/n$ $f(x) = x$ $\phi(x) = x^2$

$\phi \circ f_n(x) = (x + 1/n)^2 = x^2 + 2x/n + 1/n^2$ $\square$

$\phi \circ f(x) = x^2$

$|\phi \circ f_n(x) - \phi(f(x))| = \frac{2x}{n} + \frac{1}{n^2} = 2 + \frac{1}{n^2}$ at $x = n$

2.4.38 Suppose $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$

Show (a) $f_n+g_n \xrightarrow{\mu} f+g$ (b) $f_ng_n \xrightarrow{\mu} fg$ if $\mu(X) < \infty$ maybe not if $\mu(X)$

Pf (a) Let $\varepsilon > 0$.

$f_n \xrightarrow{\mu} f \Rightarrow \mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$

$g_n \xrightarrow{\mu} g \Rightarrow \mu(\{x \mid |g_n(x) - g(x)| > \varepsilon\}) \to 0$

$\{x \mid |f_n+g_n-f-g| > \varepsilon\} \subset \{x \mid |f_n - f| + |g_n - g| > \varepsilon\} = \{x \mid |f_n(x)-f(x)| > \varepsilon\} \cup \{x \mid |g_n - g| >$

$\mu|\{x \mid |f_n+g_n-f-g| > \varepsilon\} < \underbrace{\mu\{x \mid |f_n - f| > \varepsilon\}}_{\to 0} + \underbrace{\mu\{x \mid |g_n-g| > \varepsilon\}}_{\to 0} \to 0$

$\therefore f_n+g_n \xrightarrow{\mu} f+g$

(b) Assume $\mu(X) < \infty$.

$|f_ng_n - fg| > \varepsilon \Rightarrow |f_ng_n - fg_n + fg_n - fg| > \varepsilon$

$\Rightarrow \underbrace{|g_n|}_{\text{bdd since } \mu(X)<\infty} \underbrace{|f_n - f|}_{\to 0} + \underbrace{|f|}_{\text{bdd since } \mu(X)<\infty} \underbrace{|g_n - g|}_{\to 0} > \varepsilon$

$\Rightarrow \mu(\{x \mid |f_ng_n - fg| > \varepsilon\}) \to 0$

As a counter let $\mu(X) = \infty$

Let $f_n(x) = \frac{1}{n+1} \chi_{[0,n+1]} \xrightarrow{\mu} 0$ $\quad g_n(x) = \frac{1}{n} \chi_{[0,n]} \xrightarrow{\mu} 0$

$f_n g_n =$

2.4.39 If $f_n \xrightarrow{au} f$ then $f_n \to f$ a.e and $f_n \xrightarrow{\mu} f$

Pf $f_n \xrightarrow{au} f \Rightarrow \forall \varepsilon_1, \varepsilon_2 > 0$, $\exists E \subset X$ and $M \in \mathbb{N}$ s.t $\mu(E) < \varepsilon_1$ and $n > M$

$\Rightarrow$ for $x \in X - E$ $|f_n(x) - f(x)| < \varepsilon_2$

Consider $\{x \mid f_n \not\to f\}$. We know $f_n(x) \to f(x)$ on $X - E$

So $\{x \mid f_n \not\to f\} \subset E$

$\Rightarrow \mu(\{x \mid f_n \not\to f\} < \varepsilon_1 \to 0$

Consider $\{x \mid |f_n - f| > \varepsilon\} \subset E$

So $\mu\{x \mid |f_n - f| > \varepsilon\} \subset \varepsilon_1 \to 0$

$\square$

2.4.40 In Egoroff's thm the hypothesis $\mu(X)<\infty$ can be replaced by $|f_n|\le g$ $\forall n$ where $g\in L^1(\mu)$.

Pf Let $f_1, f_2, \ldots$ and f be measurable complex valued fcns on X s.t. $f_n\to f$ a.e. and $|f_n|\le g$ $\forall n$ $g\in L^1(\mu)$.

WLOG assume $f_n\to f$ everywhere on X.

Let $E_n(k)=\bigcup_{m=n}^{\infty}\{x \mid |f_m(x)-f(x)|\ge \frac{1}{k}\}$ for $k, N\in\mathbb{N}$

For fixed k $E_n(k)\searrow$ as $n\nearrow$ and $\bigcap E_n(k)=\emptyset$

We need $\mu(E_1(k))<\infty$

Note $|f_n-f|\le 2|g|$

$\Rightarrow E_1(k)\subset A(k):=\{x: 2|g|\ge 1/k\}$

$\Rightarrow \infty> 2\int_X |g|\ge \int_{A(k)} 2|g|\ge \frac{1}{k}\mu(A(k))$

$\Rightarrow \mu(E_1(k))<\infty$

$\Rightarrow \mu(E_n(k))\to 0$ as $n\to\infty$ By continuity of measure

$\Rightarrow$ Given $\varepsilon>0$ and $k\in\mathbb{N}$ $\exists n_k$ s.t $\mu(E_{n_k}(k))<2^{-k}\varepsilon$

Let $E=\bigcup_1^{\infty} E_{n_k}(k)$

$\Rightarrow \mu(E)<\varepsilon$ and $f_n\to f$ uniformly on E^c

2.5.45 If $(X_j, \mathcal{M}_j)$ is msble space for $j=1,2,3$. then $\otimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3)$. Moreover if μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$. then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$

Pf Let $(E_1, E_2, E_3) \in \bigotimes_{j=1}^{3} \mathcal{M}_j$

$((E_1, E_2), E_3) \in (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3 \Rightarrow \otimes_1^3 \mathcal{M}_j \subseteq (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$

Similarly $((E_1, E_2), E_3) = (E_1, E_2, E_3) \in \otimes_1^3 \mathcal{M}_j$
$\Rightarrow (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3) \subseteq \otimes_1^3 \mathcal{M}_j$

$\therefore \otimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$.

Note: $\otimes_1^3 \mathcal{M}_j = \{\pi_j^{-1}(E_j) : E_j \in \mathcal{M}_j \quad j=1,2,3\}$
$= \sigma(\{\pi_1^3 E_j : E_j \in \mathcal{M}_j\})$.

Let $\mathcal{M}_j$ be σ-finite.
$\mu_1 \times \mu_2 \times \mu_3$ on $\otimes_1^3 \mathcal{M}_j$ $\quad A_1 \times A_2 \times A_3 \in \otimes_1^3 \mathcal{M}_j$
$\Rightarrow \mu_1 \times \mu_2 \times \mu_3 (A_1 \times A_2 \times A_3) = \mu_1(A_1)\mu_2(A_2)\mu_3(A_3)$ (unique measure)
$(\mu_1 \times \mu_2) \times \mu_3 ((A_1 \times A_2) \times A_3) = (\mu_1 \times \mu_2)(A_1 \times A_2) \cdot \mu_3(A_3$
$= \mu_1(A_1)\mu_2(A_2)\mu_3(A_3)$

So by uniqueness they agree on generators

2.5.46 Let $X=Y=[0,1]$, $\mathcal{M}=\mathcal{N}=\mathcal{B}_{[0,1]}$ $\mu=$ Lebesgue measure and $\nu=$ counting measure. If $D=\{(x,x)\ x\in[0,1]\}$ is diagonal in $X\times Y$ then $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$ and $\int \chi_D \, d(\mu\times\nu)$ are all unequal.

Pf $\iint \chi_D \, d\mu \, d\nu = \int 0 \, d\nu$

$\iint \chi_D \, d\nu \, d\mu = \int 1 \, d\mu = 1$

$\iint \chi_D \, d(\mu\times\nu) = \mu\times\nu(D) = \infty$ since
$\mu\times\nu(D) = \inf\{\sum_1^\infty \mu(A_j)\times\nu(B_j) \ \ A_j\times B_j \text{ rect}\}$
and $\nu(B_j)=\infty$.

2.5.47 Let $X = Y$ be uncountable linear ordered set s.t. $\forall x \in X$ $\{y \in X;\ y < x\}$ is countable* Then E_x and E^y are msble $\forall x, y$ and $\iint \chi_E \, d\mu \, d\nu$ and $\iint \chi_E \, d\nu \, d\mu$ exist but are not equal

*Let $\mathcal{M} = \mathcal{N}$ be the σ algebra of countable or cocountable sets and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X \mid y < x\}$

2.5.48 Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$ $\mu = \nu =$ counting measure
Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$ and $f(m,n) = 0$ otherwise. Then $\int |f| \, d(\mu \times \nu) = \infty$ and $\exists \int\int f \, d\mu \, d\nu$. $\int\int f \, d\nu \, d\mu$ exist and are unequal.

Pf

$$
\begin{array}{c|c}
 & \overbrace{\qquad\qquad\qquad}^{n} \\
m & \begin{matrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ & 0 & -1 & \cdots \\ & \vdots & 0 & \cdots \\ & & \vdots & \end{matrix}
\end{array}
$$

1) $\int\int f \, d\mu \, d\nu = \int 0 \, d\nu = 0$
2) $\int\int f \, d\nu \, d\mu = \int \chi_{\{m=1\}} \, d\mu = 1$

$$\int |f| \, d(\mu \times \nu) = \int\int |f| \, d\mu \, d\nu \quad \text{by Tonelli}$$
$$= \int 2 \, d\nu$$
$$= \infty$$

3.1.1 If ν is a signed measure on $(X, \mathcal{M})$
If $\{E_j\}$ is an increasing seq in $\mathcal{M}$ then
$\nu(\bigcup_1^\infty E_j) = \lim \nu(E_j)$. If E_j is a decreasing seq in $\mathcal{M}$ and $\nu(E_1)$ finite then $\nu(\bigcap_1^\infty E_j) = \lim \nu(E_j)$

Pf WLOG ν omits ∞ (otherwise consider $-\nu$)
Let E_n be an increasing sequence in $\mathcal{M}$.
Define $F_n = E_n \setminus E_{n-1}$

$\Rightarrow F_n$ disjoint and $\bigcup_1^n E_j = \bigcup_1^n F_j$

$$\begin{aligned}\Rightarrow \nu(E_n) &= \nu(\textstyle\bigcup_1^n E_j) \\ &= \nu(\textstyle\bigcup_1^n F_j) \\ &= \textstyle\sum_1^n \nu(F_j) \quad \text{since disjoint}\end{aligned}$$

$$\begin{aligned}\Rightarrow \lim_{n\to\infty} \nu(E_n) &= \lim_{n\to\infty} \textstyle\sum_1^n \nu(F_j) \\ &= \textstyle\sum_1^\infty \nu(F_j) \\ &= \nu(\textstyle\bigcup_1^\infty F_j) \\ &= \nu(\textstyle\bigcup_1^\infty E_j)\end{aligned}$$

Now let E_n be a decreasing sequence in M.
Define $F_n = E_1 \setminus E_n$.

$\Rightarrow F_1 \subset F_2 \subset \ldots$

$\Rightarrow \bigcup_1^\infty F_j = E_1 \setminus \bigcap_2^\infty E_j$

$\Rightarrow \nu(E_1) = \nu(F_j) + \nu(E_j)$

$= \nu(\bigcap_1^\infty E_j) + \lim \nu(F_j)$

$= \nu(\bigcap_1^\infty E_j) + \lim [\nu(E_1) - \nu(E_j)]$

$\Rightarrow \lim \nu(E_j) = \nu(\bigcap_1^\infty E_j)$

$\square$

3.1.2 If ν is a signed measure. E ν-null $\iff |\nu|(E)=0$
Also if ν, μ signed measures. $\nu \perp \mu \iff |\nu| \perp \mu \iff \nu^+$ and $\nu^- \perp \mu$

Pf Assume E ν-null
$\Rightarrow \forall F \in \mathcal{M}$ with $F \subset E$ $\nu(F)=0$
Let $X = P \cup N$ with $P \cap N = \phi$ be the Hahn decomp.
$\Rightarrow E \cap P \subset E$ and $E \cap N \subset E$
$\Rightarrow \nu(E \cap P) = 0 = \nu(E \cap N)$
$\Rightarrow \nu^+(E) = 0 = \nu^-(E)$
$\Rightarrow |\nu|(E)=0$

Assume $|\nu|(E)=0$
$\Rightarrow \nu^+(E) + \nu^-(E) = 0$ with $\nu^+(E), \nu^-(E) \geq 0$
$\Rightarrow \nu^+(E) = \nu^-(E) = 0$
Let $F \in \mathcal{M}$ with $F \subset E$
$\Rightarrow 0 \leq |\nu|(F) \leq |\nu|(E) = 0$
$\Rightarrow |\nu|(F) = 0$
$\Rightarrow \nu^+(F) = 0 = \nu^-(F)$
$\Rightarrow \nu(F) = 0$
$\Rightarrow E$ is ν null

Let $\nu \perp \mu$
$\Rightarrow \exists E, F$ s.t. $E \cup F = X$, $E \cap F = \not\phi$ and $\nu(E)=0$ $\mu(F)=0$
$\Rightarrow \exists N, P$ s.t. $P \cup N = X$ $P \cap N = \phi$ (Hahn decomp.
WTS E is null for $|\nu|$
Let $T \subset E$ with $T \in \mathcal{M}$
$\Rightarrow \nu^+(T) = \nu(T \cap P) = 0 = \nu(T \cap N) = \nu^-(T)$
$\Rightarrow E$ is null for $|\nu|$
$\Rightarrow |\nu| \perp \mu$

Let $|\nu| \perp \mu$ $\Rightarrow \nu^+ \perp \mu$ and $\nu^- \perp \mu$ by above.

Let $\nu^+ \perp \mu$ and $\nu^- \perp \mu$
$\Rightarrow \nu^+(E) = 0$ and $-\nu^-(E) = 0$
$\Rightarrow \nu(E) = 0$ $\Rightarrow \nu \perp \mu$.

$\square$

3.1.3 Let ν be a signed measure on $(X, \mathcal{M})$

a) $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$

b) If $f \in \mathcal{L}^1(\nu)$ $|\int f d\nu| \leq \int |f| d|\nu|$

c) If $E \in \mathcal{M}$ $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$

Pf a) $f \in \mathcal{L}^1(\nu)$

$\Leftrightarrow \int |f| d\nu^+ < \infty$ and $\int |f| d\nu^- < \infty$

$\Leftrightarrow \int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$

$\Leftrightarrow f \in \mathcal{L}^1(|\nu|)$

b) $|\int f d\nu| = |\int f d\nu^+ - \int f d\nu^-|$

$\leq |\int f d\nu^+| + |\int f d\nu^-|$

$\leq \int |f| d\nu^+ + \int |f| d\nu^-$

$= \int |f| d|\nu|$

c) Let $E \in \mathcal{M}$ and $|f| \leq 1$

$\Rightarrow |\int_E f d\nu| \leq \int_E |f| d|\nu| \leq \int_E 1 d|\nu| = |\nu|(E)$

$\Rightarrow \sup\{|\int_E f d\nu| : |f| \leq 1\} \leq |\nu|(E)$

Now consider $f = \chi_{E \cap P} - \chi_{E \cap N}$ where $E = P \cup N$ is Hahn Decomp.

$\Rightarrow |f| \leq 1$

$\Rightarrow |\int_E f d\nu| = |\int \chi_{E \cap P} - \chi_{E \cap N} d\nu|$

$= |\int \chi_{E \cap P} d\nu^+ + \int \chi_{E \cap N} d\nu^-|$

$= |\int_{E \cap P} d\nu^+ + \int_{E \cap N} d\nu^-|$

$= |\nu^+(E \cap P) + \nu^-(E \cap N)|$

$= |\nu^+(E) + \nu^-(E)|$

$= |\nu|(E)$

$\therefore \sup\{|\int_E f d\nu : |f| \leq 1\} |\nu|(E).$

3.1.4 If ν is a signed measure, λ, μ a positive measure s.t. $\nu = \lambda - \mu$

Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$

Pf Let $E \in \mathcal{M}$.

Let $\nu = \lambda - m$.

Let $X = P \cup N$ the hahn decomposition of X.

$\Rightarrow \nu^+(E) = \nu(E \cap P)$

$= \lambda(E \cap P) - \mu(E \cap P)$

$\leq \lambda(E \cap P)$ since μ positive

$\leq \lambda(E)$ since $E \cap P \subset E$

$\Rightarrow \nu^-(E) = -\nu(E \cap N)$

$= -\lambda(E \cap N) + \mu(E \cap N)$

$\leq \mu(E \cap N)$

$\leq \mu(E)$.

3.1.5 Let ν_1, ν_2 be signed measures omitting $\pm\infty$

Show $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$

Pf

3.1.6 Suppose $\nu(E) = \int f d\mu$ where μ is a positive measure and f an extended μ-integrable function. Describe Hahn decomp. of ν and positive, negative, and total variation of ν in terms of f and μ.

Pf Let $P = \{f \geq 0\}$ and $N = \{f < 0\}$

Then $P \cup N = X$ and $P \cap N = \emptyset$.

P is clearly positive and N negatve.

$\nu^+(E) = \int_{E \cap P} f d\mu = \int_{E \cap P} |f| d\mu$.

$\nu^-(E) = -\int_{E \cap N} f d\mu = \int_{E \cap N} |f| d\mu$

$\therefore |\nu|(E) = \nu^+(E) + \nu^-(E) = \int_E |f| d\mu$.

$\square$

3.1.7 Suppose ν a signed measure on $(X, \mathcal{M})$ $E \in \mathcal{M}$

a) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}\ F \subset E\}$ $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}\ F \subset E\}$

b) $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \dots, E_n \text{ disjoint } \bigcup_1^n E_j = E\}$

Pf (a) Let $X = P \cup N$ be Hahn Decomp of X.

Let $F \in \mathcal{M}$ be s.t. $F \subset E$

$$\begin{aligned}
\Rightarrow \nu(F) &= \nu(F \cap P) + \nu(F \cap N) \\
&= \nu^+(F) - \nu^-(F) \\
&\leq \nu^+(F) \quad \text{since } \nu^- \text{ positive} \\
&\leq \nu^+(E) \quad \text{since } F \subset E
\end{aligned}$$

$\Rightarrow \nu^+(E) \geq \sup\{\ \}$ and $\nu(E \cap P) = \nu^+(E)$ so equality holds

$$\begin{aligned}
\Rightarrow \nu(F) &= \nu^+(F) - \nu^-(F) \\
&\geq -\nu^-(F) \\
&\geq -\nu^-(E)
\end{aligned}$$

$\Rightarrow -\nu^-(E) \leq \inf$ and $\nu(E \cap N) = -\nu^-(E)$ so equality holds

(b) Let $E_1, \dots, E_n$ disjoint be s.t. $\bigcup_1^n E_j = E$.

$$\begin{aligned}
\sum |\nu(E_j)| &= \sum |\nu(E_j \cap P) + \nu(E_j \cap N)| \\
&= \sum |\nu^+(E_j) - \nu^-(E_j)| \\
&\leq \sum |\nu^+(E_j)| + |\nu^-(E_j)| \quad \text{by } \Delta\text{-ineq} \\
&= \sum |\nu|(E_j) \\
&= |\nu|(\bigcup E_j) \\
&= |\nu|(E)
\end{aligned}$$

Equality holds since we can just consider E.

$\sum_1^1 |\nu(E)| = |\nu(E)| = |\nu|(E)$.

$\square$

$\underline{3.2.8}$ $\nu \ll \mu \iff |\nu| \ll \mu \iff \nu^+ \ll \mu$ & $\nu^- \ll \mu$.

$\underline{\text{Pf}}$ Assume $\nu \ll \mu$.

Let $E \in \mathcal{M}$ s.t $\mu(E)=0$

Let $X = P \cup N$ be Hahn Decomp of X

Let $\nu = \nu^+ - \nu^-$ be Jordan Decomp of ν.

$$\begin{aligned} |\nu|(E) &= \nu^+(E) + \nu^-(E) \\ &= \nu(E \cap P) - \nu(E \cap N) \\ &= 0 - 0 \quad \text{since } E \cap P \subset E \text{ and } E \cap N \subset E \\ &= 0 \end{aligned}$$

$\Rightarrow |\nu| \ll \mu$

Assume $|\nu| \ll \mu$

Let $E \in \mathcal{M}$ s.t. $\mu(E)=0$

$0 \leq \nu^+(E) \leq |\nu|(E) = 0$

$0 \leq \nu^-(E) \leq |\nu|(E) = 0$

$\Rightarrow \nu^+ \ll \mu \quad \nu^- \ll \mu$.

Assume $\nu^+ \ll \mu \quad \nu^- \ll \mu$.

Let $E \in \mathcal{M}$ s.t. $\mu(E)=0$

$$\begin{aligned} \nu(E) &= \nu^+(E) - \nu^-(E) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

$\Rightarrow \nu \ll \mu$. $\square$

3.2.9 Suppose $\{v_j\}$ is a seq of positive measures,
If $v_j \perp \mu$ $\forall j$ then show $\sum_1^\infty V_j \perp \mu$
and if $V_j \ll \mu$ then $\sum_1^\infty V_j \ll \mu$

Pf $\forall j$ let $X = E_j \cup F_j$ s.t. μ is null on E_j, v null on F_j.
Let $F = \bigcup_1^\infty F_j$.
Notice $X \setminus F = X \cap F^c = X \cap (\bigcap F_j^c) = \bigcap (X \cap F_j^c) = \bigcap E_j \equiv E$
$\Rightarrow E \cap F = \phi$ $\quad E \cup F = X$
$\Rightarrow E$ null for μ, F null for $\sum v_j$
$\Rightarrow \sum v_j \perp \mu$.

Now assume $\mu(E) = 0$
$(\sum_1^\infty v_j)(E) = \sum_1^\infty v_j(E) = \sum 0 = 0$
$\Rightarrow \sum_1^\infty v_j \ll \mu$.

$\square$

3.2.10 Let v be a finite signed measure and
μ a positive measure on (X, μ)
Then $v \ll \mu \Leftrightarrow \forall \varepsilon \geq 0$ $\exists \delta > 0$ s.t. $|v(E)| < \varepsilon$ if $\mu(E) < \delta$.
show we need v σ-finite for this to hold.

Pf Consider v the counting measure and $\mu(E) = \sum_{n \in E} 2^{-n}$
Let $E \in \mathcal{P}(\mathbb{N})$ with $\mu(E) = 0$
$\Rightarrow E = \phi$
$\Rightarrow v(E) = \phi$
$\Rightarrow \mu \ll v$.

Let $\varepsilon = 1/2$, $\delta > 0$.
$\Rightarrow \exists N$ s.t. $\delta^{-N} < \delta$.
$\Rightarrow \mu(\{N\}) = 2^{-N} < \delta$
but $v(\{N\}) = 1 \nless 1/2$.

$\square$

3.2.11 Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A}\subset L^1(\mu)$ is uniformly integrable if $\forall\, \varepsilon>0$, $\exists\, \delta>0$ s.t. $|\int_E f_\alpha d\mu|<\varepsilon$ $\forall \alpha\in A$ if $\mu(E)<\delta$

a) Any finite subset of $L^1(\mu)$ is uniformly integrable

b) If $\{f_n\}$ is a seq. in $L^1(\mu)$ that converges in L^1 metric to $f\in L^1(\mu)$ then $\{f_n\}$ is uniformly integrable

Pf a) Let $\{f_1,\dots,f_n\}\subset L^1(\mu)$ and $\varepsilon>0$

Let $\nu_i(E)=\int_E f_i\, d\mu$.

$\Rightarrow \nu_i \ll \mu$.

$\Rightarrow \exists\, \delta_i>0$ s.t. $|\nu_i(E)|<\varepsilon$ if $\mu(E)<\delta_i$

$\Rightarrow |\int_E f_i\, d\mu|<\varepsilon$ if $\mu(E)<\delta=\min\{\delta_1,\dots,\delta_n\}$

b). Let $\varepsilon>0$.

Let $\{f_n\}\subset L^1(\mu)$ s.t. $f_n\to f$ L^1

$\Rightarrow \exists\, n_0$ s.t. $|\int f_n\, d\mu|\le|\int f\, d\mu|+\varepsilon/2$ $\forall n\ge n_0$

$\exists\, \delta_0$ s.t $|\int_E f\, d\mu|<\varepsilon/2$ if $\mu(E)<\delta_0$

$\exists\, \delta_i$ s.t $|\int_E f_i\, d\mu|<\varepsilon$ if $\mu(E)<\delta_i$

Let $\delta=\min\{\delta_0,\delta_1,\dots\delta_n\}$

Let $\mu(E)<\delta$

$$\Rightarrow |\int_E f_n\, d\mu| < \begin{cases} \varepsilon & \text{if } 1\le n\le n_0-1 \\ |\int_E f|+\varepsilon/2 < \varepsilon/2+\varepsilon/2=\varepsilon & n\ge n_0 \end{cases}$$

3.2.12 For $j=1,2$. Let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ s.t. $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and $\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)$

Pf (a). Let $E_1 \times E_2 \in X_1 \times X_2$ s.t. $\mu_1 \times \mu_2 (E_1 \times E_2) = 0$

Since $\mu_1 \times \mu_2 (E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$

$\Rightarrow \mu_1(E_1) = 0$ or $\mu_2(E_2) = 0$

WLOG assume $\mu_1(E_1) = 0$

$\Rightarrow \nu_1(E_1) = 0$ since $\nu_1 \ll \mu_1$

$\Rightarrow \nu_1(E_1)\nu_2(E_2) = \nu_1 \times \nu_2 (E_1 \times E_2) = 0$

(since finite disjoint unions of rectangles generate $X_1 \otimes X_2$)

$\Rightarrow \nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$

(b). $\int_{E_1 \times E_2} \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} d(\mu_1 \times \mu_2) = (\nu_1 \times \nu_2)(E_1 \times E_2)$ by (a).

$= \nu_1(E_1)\nu_2(E_2)$

$= \int_{E_1} \frac{d\nu_1}{d\mu_1} d\mu_1 \int_{E_2} \frac{d\nu_2}{d\mu_2} d\mu_2$

$= \int_{E_1}\int_{E_2} \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)\, d\mu_2\, d\mu_1$

$= \int_{E_1 \times E_2} \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)\, d(\mu_1 \times \mu_2)$

$\square$

13.2.13 Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, m = lebesgue measure, μ counting measure.

a) $m \ll \mu$ but $dm \neq f d\mu$ $\forall f$

b) μ has no Lebesgue decomp wrt m.

Pf a) Let $E \in \mathcal{M}$ s.t. $\mu(E) = 0$

$\Rightarrow E = \emptyset$

$\Rightarrow m(E) = 0$

$\Rightarrow m \ll \mu$.

Assume BWOC $\exists f$ s.t. $dm = f d\mu$ and $E_x = \{x\}$

$\Rightarrow m(E) = \int_E f d\mu$

$\Rightarrow 0 = m(E_x) = \int_{E_x} f d\mu = \int_{[0,1]} f \chi_{E_x} d\mu = f(x)\mu(E_x) = f(x)$.

$\Rightarrow 0 \equiv f(x)$

However $m([0,1]) = 1$

$\Rightarrow \int_{[0,1]} f d\mu = 1$ but $f \equiv 0$

$\Rightarrow \nexists f$.

b). Suppose BWOC μ has Lebesgue decomp wrt m

$\Rightarrow \mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$.

$\Rightarrow 1 = \mu(E_x) = \lambda(E_x) + \rho(E_x) = \lambda(E_x)$ since $m(E_x) = 0$

$\Rightarrow \lambda$ defined on $[0,1]$ and $\lambda = \mu$

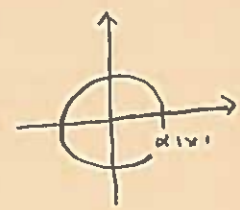
$\Rightarrow \mu \perp m$ which contradicts

$\Rightarrow \nexists$ a decomp

$\square$

3.4.22 If $f \in L^1(\mathbb{R}^n)$ $f \neq 0$ $\exists C, R > 0$ s.t. $Hf(x) \geq c|x|^{-n}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) \geq C/\alpha$ so estimate is essentially sharp.

Pf Let $B = B(0, \alpha|x|)$



$$\Rightarrow Hf(x) \geq 2^{-n} H^*f(x)$$

$$\geq \frac{1}{2^n m(B)} \int_B |f(y)|dy$$

$$= \frac{1}{2^n |x|^n m(B_\alpha(0))} \int_B |f(y)|dy$$

$$\geq \frac{1}{2^n |x|^n m(B_\alpha(0))} \int_{B_\alpha(0)} f(y)dy$$

$$= \frac{C}{|x|^n} \qquad \square$$

3.4.22 A useful variant of $Hf(x)$ is

$$H^*f(x) = \sup\left\{\frac{1}{m(B)} \int_B |f(y)|dy \;\middle|\; B \text{ a ball, } x \in B\right\}$$

Show $Hf \leq H^*f \leq 2^n Hf$.

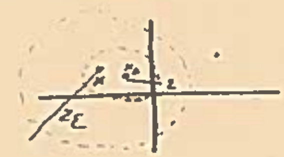
Pf First note $x \in B(r, x) \Rightarrow Hf \leq H^*f$

Now Let $\frac{1}{m(B)} \int_B |f(y)|dy \in H^*f$.

$$= \frac{1}{m(B_\varepsilon(x_0))} \int_{B_\varepsilon(x_0)} |f(y)|dy \quad \text{for some } \varepsilon, x_0.$$

$$= \frac{1}{m(B_\varepsilon(x))} \int_{B_\varepsilon(x_0)} |f(y)|dy \quad \text{since invariant.}$$

$$\leq \frac{1}{m(B_\varepsilon(x))} \int_{B_{2\varepsilon}(x)} |f(y)|dy \quad B_\varepsilon(x_0) \subset B_{2\varepsilon}(x)$$



$$= \frac{2^n}{2^n m(B_\varepsilon(x))} \int_{B_{2\varepsilon}(x)} |f(y)|dy$$

$$= \frac{2^n}{m(B_{2\varepsilon}(x))} \int_{B_{2\varepsilon}(x)} |f(y)|dy$$

$$\leq 2^n Hf(x)$$

$\therefore$ $Hf \leq H^*f \leq 2^n Hf$. $\square$

3.4.24. If $f \in L^1_{loc}$ and f is continuous at x then x is in the $\underbrace{\text{lebesgue set}}$ of f.

$\hookrightarrow \{x : \lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)-f(x)|\,dy = 0\}$.

Pf f cont at x. $\Rightarrow \forall \varepsilon>0\ \exists r_0>0$ s.t. $|f(y)-f(x)|<\varepsilon$ for $y \in B_{r_0}(x)$.

$\Rightarrow \int_{B_r(x)} |f(y)-f(x)|\,dy \leq \int_{B_r(x)} \varepsilon\, dy = \varepsilon\, m(B_r(x))$

$\Rightarrow \lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)-f(x)| = \varepsilon \to 0$ as $\varepsilon \to 0$

$\Rightarrow x$ is in the Lebesgue set.

$\square$

3.4.25 If E is a Borel set in $\mathbb{R}^n$, $D_E(x) = \lim_{r\to 0} \frac{m(E\cap B_r(x))}{m(B_r(x))}$

a) show $D_E(x)=1$ for a.e $x\in E$; $D_E(x)=0$ for a.e $x\notin E$

b) Find E, x s.t. $D_E(x) = \alpha \in (0,1)$ or $D_E(x)$ DNE

Pf a) Define $\nu(F) = m(F\cap E)$

$\Rightarrow d\nu = \chi_E\, dm$ since then $\nu = \int_F \chi_E\, dm = m(E\cap F)$

$\Rightarrow D_E(x) = \lim \frac{m(E\cap B_r(x))}{m(B_r(x))}$

$= \lim \frac{\nu(B_r(x))}{m(B_r(x))}$

$= \chi_E(x)$ a.e (by Thm 3.22)

$\therefore$ the claim holds.

b) Let E be an arc of angle $2\alpha < 2\pi$, $x=0$.

$\Rightarrow m(E\cap B_r(0)) = 2\alpha r$

$m(B_r(0)) = 2r$

$\Rightarrow D_E(x) = \frac{2\alpha r}{2r} = \alpha.$

$\square$

3.4.26 If $\lambda, \mu > 0$ $\lambda \perp \mu$ Borel measures on $\mathbb{R}^n$
If $\lambda + \mu$ is regular show λ, μ are too.

Pf Let K be a compact set.
WTS $\lambda(K), \mu(K) < \infty$
and $\lambda(E) = \inf\{\nu(U) : U \text{ open } E \subset U\}$ $\forall E \in \mathcal{B}_{\mathbb{R}^n}$
Similarly for μ

$\lambda(K) \leq \lambda(K) + \mu(K) < \infty$
$\mu(K) \leq \lambda(K) + \mu(K) < \infty$ ✓

Let $E \in \mathcal{B}_{\mathbb{R}^n}$
$\lambda + \mu$ regular $\Rightarrow \forall \varepsilon > 0$ $\exists U$ open w/ $E \subset U$ s.t. $\lambda + \mu(U \setminus E) < \varepsilon$
$\Rightarrow \lambda(U \setminus E) \leq (\lambda + \mu)(U \setminus E) < \varepsilon$
$\mu(U \setminus E) \leq (\lambda + \mu)(U \setminus E) < \varepsilon$ ✓
$\therefore$ λ + μ are regular.

3.5.27 Verify examples in exercise 25 pg 102
a) $F : \mathbb{R} \to \mathbb{R}$ bdd increasing $\Rightarrow F \in BV$ $T_F(x) = F(x) - F(-\infty)$
b) $F, G \in BV$ $a, b \in \mathbb{C} \Rightarrow aF + bG \in BV$
c) F differentiable on $\mathbb{R}$, F' bdd $\Rightarrow F \in BV([a,b])$
d) $F(x) = \sin(x) \in BV([a,b])$ $\notin BV$
e) $F(x) = \begin{cases} \sin 1/x & x \neq 0 \\ 0 & x = c \end{cases}$ $\Rightarrow F \notin B([a,b])$ $a \leq 0 < b$, $a < 0 \leq b$.

Pf a) $T_F(x) = \sup\{\sum_1^n |F(x_j) - F(x_{j-1})| \mid n \in \mathbb{N}\ -\infty < x_0 < \dots < x_n = x\}$
$= \sup\{F(x_n) - F(x_0)\}$ since increasing.
$= \sup\{F(x) - F(x_0)\}$
$= \underbrace{F(x)}_{\text{bdd}} - \underbrace{\lim_{x_0 \to -\infty} F(x_0)}_{\text{bdd}}$

$= F(x) - F(-\infty) < \infty$

b) $T_{aF+bG} = \sup\{\sum |aF(x_j) + bG(x_j) - aF(x_{j-1}) - bG(x_{j-1})|\}$

$\leq \sup\{\sum |a||F(x_j) - F(x_{j-1})| + |b||G(x_j) - G(x_{j-1})|\}$

$\leq |a| \sup\{\sum |F(x_j) - F(x_{j-1})|\} + |b| \sup\{\sum G(x_j) - G(x_{j-1})\}$

$\leq |a| T_F(x) + |b| T_G(x)$

$\leq |a| T_F(\infty) + |b| T_G(\infty)$

$< \infty \quad \forall x \in \mathbb{R}$

$\Rightarrow aF + bG \in BV.$

c) $T_F(x) = \sup\{\sum |F(x_j) - F(x_{j-1})|\}$

$= \sup\{\sum |F'(x_j^*) \Delta(x_j)|\}$ for some $x_j^* \in (x_{j-1}, x_j)$

$\leq \sup\{M \sum \Delta x_j\}$

$= M|b-a|$

$< \infty.$

d) $F(x) = \sin(x) \in BV[a,b]$ by (c) since $-1 \leq F' \leq 1$

Now $T_F(x) \geq \sum |\sin(x_j) - \sin(x_{j-1})| \quad \forall$ partition

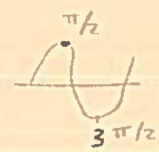
Let $x_n = x + \frac{\pi}{2}$, $x_{n-1} = x - \pi + \frac{\pi}{2}$ $\quad x_{n-k} = x - k\pi + \frac{\pi}{2}$

$\Rightarrow \sum |\sin(x - j\pi + \frac{\pi}{2}) - \sin(x - (j-1)\pi - \frac{\pi}{2})|$

$= 2n \quad \forall n$

$\Rightarrow 2n \to \infty$ as $n \to \infty$

$\Rightarrow F \notin BV.$



e) Pick infinitely many pts at top and bottom of peaks approaching 0.

□

3.5.28 $F \in NBV$. Let $G(x) = |\mu_F|((-\infty, x])$. Prove $|\mu_F| = \mu_{T_F}$ by showing $G = T_F$ via.

a) $T_F \le G$

b) $|\mu_F(E)| \le \mu_{T_F}(E)$ E interval ($\Rightarrow E$ Borel)

c) $|\mu_F| \le \mu_{T_F}$ and hence $G \le T_F$

Pf a) $F \in NBV \Rightarrow F \in BV$, $F(-\infty) = 0$, F right cont

$$T_F(x) = \sup\{\sum |F(x_j) - F(x_{j-1})|\}$$ $F(x) = \mu_F(-\infty, x]$

$$= \sup\{\sum |\mu_F((x_{j-1}, x_j))|$$

$$\le \sup\{\sum |\mu_F|(x_{j-1}, x_j)|$$

$$= \sup\{|\mu_F|(x_0, x])$$

$$= |\mu_F|((-\infty, x])$$

$\therefore F_F \le G$.

b)

?

c) $$|\mu_F|(E) = \sup\{\sum_1^n |\mu_F(a_j, b_j)|\}$$

$$= \sup\{\sum |F(b_j) - F(a_j)|\}$$

$$\le \sup\{\sum T_F(b_j) - T_F(a_j)\}$$

$$= \sup\{\sum \mu_{T_F}(a_j, b_j)\}$$

$$= \mu_{F_j}$$

3.5.34 Suppose $F, G \in NBV$ $-\infty < a < b < \infty$.

a) Show $\int_{[a,b]} \frac{F(x)+F(x-)}{2} dG(x) + \int_{[a,b]} \frac{G(x)+G(x-)}{2} dF(x) = F(b)G(b) - F(a-)G$

b) If $\nexists x \in [a,b]$ s.t. F, G discont then $\int F dG + \int G dF = F(b)G(b) - F(a)G$

Pf WLOG assume F, G increasing.

Let $\Omega = \{(x,y) \mid a \le x \le y \le b\}$

$$\mu_F \times \mu_G(\Omega) = \int_a^b \int_a^y dF(x)\, dG(y)$$
$$= \int_a^b F(y) - F(a-)\, dG(y)$$
$$= \int_a^b F(y) dG(y) - F(a-)(G(b) - G(a-))$$

$$\mu_F \times \mu_G(\Omega) = \int_a^b \int_x^b dG(y)\, dF(x)$$
$$= \int_a^b G(b) - G(x-)\, dF(x)$$
$$= G(b)(F(b) - F(a-)) - \int_{[a,b]} G(x-) dF(x)$$

$$\Rightarrow \int_{[a,b]} F(x) dG(x) + \int_{[a,b]} G(x-) dF(x) + F(a-)G(a-) - G(b)F(b) = 0$$

Let $\Omega' = \{(x,y) \mid a \le y \le x \le b\}$

$$\mu_F \times \mu_G(\Omega') = \int_{[a,b]} - F(y-) dG(y) + F(b)[G(b) - G(a-)]$$
$$\mu_F \times \mu_G(\Omega') = \int_{[a,b]} G(x) dF(x) - G(a-)[F(b) - F(a-)]$$

$$\overset{\Omega, \Omega'}{\Rightarrow} 0 = 2[F(a-)G(a-) - F(b)G(b)] + \int_{[a,b]} G(x) dF(x) + \int_{[a,b]} F(x-) dG(x)$$
$$+ \int_{[a,b]} F(x) dG(x) + \int_{[a,b]} G(x-) dF(x)$$

b) Consider $\{d_i\}$ discuities of F in $[a,b]$

$$\int_{[a,b]} \frac{F(x)+F(x-)}{2} dG(x) = \int_{[a,b] \setminus \cup \{d_i\}} \frac{F(x)+F(x-)}{2} dx + \sum_i^n \underbrace{\int_{\{d_i\}} \frac{F(x)+F(x-)}{2} dG}_{\frac{F(d_n) - F(d_n^-)}{2}[\underbrace{G(d_n) - G(d_n^-)}_{=0}}$$

$\square$

3.5.35 If F, G are absolutely continuous on $[a,b]$ then so is FG and $\int_a^b (FG' + GF')(x)\,dx = F(b)G(b) - F(a)G(a)$

Pf If FG is abs. cont then second claim follows directly from fundamental thm of Calc for Lebesgue integrals

F, G abs cont
$\Rightarrow F, G$ bdd
$\Rightarrow |F| \leq M \quad |G| \leq N$

Let $\varepsilon > 0$. $\exists\, \delta_F$ s.t. $\sum |F(b_j) - F(a_j)| < \varepsilon/2M$ if $\sum b_j - a_j < \delta_F$
$\exists\, \delta_T$ s.t. $\sum |G(b_j) - F(a_j)| < \varepsilon/2N$ if $\sum b_j - a_j < \delta_G$

$$\sum |FG(b_j) - FG(a_j)| = \sum |F(b_j)G(b_j) - F(b_j)G(a_j) + F(b_j)G(a_j) - F(a_j)G(a_j)|$$

$$\leq \sum |F(b_j)|\,|G(b_j) - G(a_j)| + |G(a_j)|\,|F(b_j) - F(a_j)|$$

$$\leq M \underbrace{\sum |G(b_j) - G(a_j)|}_{< \varepsilon/M} + N \sum |F(b_j) - F(a_j)|$$

$$< \varepsilon$$

$\therefore$ FG is abs. cont. $\square$

3.5.29 If $F \in NBV \in \mathbb{R}$ then $\mu_{F^+} = \mu_P$ and $\mu_{F^-} = \mu_N$ where P & N are positive and negative variations of F.

Pf $P = \frac{1}{2}(T_F + F)$, $N = \frac{1}{2}(T_F - F)$

$\Rightarrow \mu_P = \mu_{\frac{1}{2}(T_F+F)}$
$= \frac{1}{2}(|\mu_F| + \mu_F)$
$= \frac{1}{2}(\mu_F^+ + \mu_F^- + \mu_F^+ - \mu_F^-)$
$= \mu_F^+$

$\mu_N = \mu_{\frac{1}{2}(T_F - F)}$
$= \frac{1}{2}(|\mu_F| - \mu_F)$
$= \frac{1}{2}(\mu_F^+ + \mu_F^- - \mu_F^+ + \mu_F^-)$
$= \mu_F^-$

□

3.5.30 Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

Pf Enumerate $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$

Let $f(x) = \sum_{\{n : q_n < x\}} \frac{1}{q_n^2}$

$\Rightarrow f$ is increasing and discontinuous at all $\mathbb{Q}$

□

3.5.31 $F(x) = x^2 \sin 1/x$ $G(x) = x^2 \sin(1/x^2)$ $F(0) = G(0) = 0$

a) Show F, G differentiable everywher.

b) Show $F \in BV([-1,1])$ but $G \notin BV([-1,1])$

Pf a) For $x \neq 0$, $F'(x) = 2x \sin 1/x - \cos 1/x$, $G'(x) = 2x \sin 1/x^2 - \frac{1}{x} \cos 1/x^2$

For $x = 0$ $\frac{F(x) - F(0)}{x - 0} = x \sin 1/x \to 0$ as $x \to 0$

$\frac{G(x) - G(0)}{x - 0} = x \sin 1/x^2 \to 0$ as $x \to 0$

b)

3.5.32 If $\{F_n\} \subset NBV$ $F_j \to F$ ptwise then $T_F \leq \underline{\lim}\, T_{F_j}$

Pf $T_F(x) = |\mu_F|((-\infty, x])$

$= \sup \{ \int_{-\infty}^{\infty} |f| \, d\mu_F , |F| \leq 1 \}$?

$= \sup \{ \sum_1^n |F(x_a) - F(x_{a-1})|$

$\leq \sum_1^n |F(x_j) - F(x_{j-1})| + \varepsilon$

$\forall x_j , \exists k_j$ s.t. $F(x_j) \leq F_{k_j}(x\ ) + \varepsilon/2n$

$\leq \sum_1^n (F_{k_j}(x_j) - F_{k_{j-1}}(x_{j-1})| + \varepsilon/2n) + \varepsilon$

$= \sum_1^n |F_{k_j}(x_j) - F_{k_{j-1}}(x_{j-1})| + 2\varepsilon$

$\leq T_{F_j}(x_k) + 2\varepsilon$

$\leq \underline{\lim}\, T_{F_j}(x_k)$

$\square$

3.5.33 If F is increasing on $\mathbb{R}$ then $F(b) - F(a) \geq \int_a^b F'(t)\,dt$.

Pf $\mu_F = \lambda + f\,dm$ by Radon Nicodym Thm

$\Rightarrow \mu_F = F(b) - F(a) = \lambda((a,b)) + \int_a^b f\,dx$ $\qquad \underbrace{f = F' \text{ a.e.}}_{?}$

$\Rightarrow \lambda((a,b)) = \underbrace{\sum \underbrace{\lambda(\{\delta_x\})}_{\geq 0 \text{ since } f \text{ increasing}}}_{\geq 0}$

$\Rightarrow F(b) - F(a) \geq \int_a^b F'(x)\,dx$

3.5.37 Show Lipschitz cont $\Leftrightarrow$ F AC and $|F'| \le M$.

Pf Assume $\exists M$ s.t. $|F(x) - F(y)| \le M|x-y|$

$\Rightarrow$ if $|x_j - x_{j-1}| < \delta = \varepsilon/M \Rightarrow |F(x_j) - F(x_{j-1})| < M\varepsilon/M = \varepsilon$

$\Rightarrow$ F is AC. ✓

$\Rightarrow$ F' exists a.e

$\Rightarrow |F'(x)| = \left|\frac{F(x+h) - F(x)}{h}\right| \le \left|\frac{M(x+h-x)}{h}\right| = M$

Now assume F AC and $|F'| \le M$

$\Rightarrow |F(x) - F(y)| = \left|\int_y^x F'(t)\,dt\right|$ By FTC for LI

$\le \int_y^x |F'(t)|\,dt$

$\le \int_y^x M\,dt$

$= M|x-y|$ $\square$

3.5.42 A function $F: (a,b) \to \mathbb{R}$ convex if

$F(\lambda s + (1-\lambda)t) \le \lambda F(s) + (1-\lambda)F(t)$ $\forall s,t \in (a,b)$ $\lambda \in (0,1)$

a) F convex $\Leftrightarrow$ $\forall s,t,s',t' \in (a,b)$ $s \le s' \le t$ $s < t \le t'$ $\frac{F(t) - F(s)}{t-s} \le \frac{F(t') - F(s')}{t'-s'}$

b) F convex $\Rightarrow$ F abs cont on every compact subinterval of (a,b) and $F' \nearrow$.

c) If F convex and $t_0 \in (a,b)$ $\exists \beta \in \mathbb{R}$ s.t. $F(t) - F(t_0) \ge \beta(t - t_0)$

d) If $(X, \mathcal{M}, \mu)$ is a measure space w/ $\mu(X) = 1$

$g: X \to (a,b)$ is in $L^1(\mu)$ F convex on (a,b)

$\Rightarrow F(\int g\,d\mu) \le \int F\circ g\,d\mu$

Pf (a) Let $s' = \lambda' s + (1-\lambda')t'$, $t = \lambda s + (1-\lambda)t'$

$\Rightarrow F(t) \le \lambda F(s) + (1-\lambda)F(t')$ $F(s') \le \lambda' F(s) + (1-\lambda')F(t')$

$\Rightarrow \frac{F(t) - F(s)}{t-s} \le \frac{\lambda F(s) + (1-\lambda)F(t') - F(s)}{\lambda s + (1-\lambda)t' - s}$

$= \frac{(1-\lambda)(F(t') - F(s))}{(1-\lambda)(t'-s)} = \frac{F(t') - F(s)}{t'-s}$

$\le \frac{F(t') + \left(\frac{1-\lambda'}{\lambda'}\right)F(t') - \frac{1}{\lambda'}F(s')}{t'-s}$

$= \frac{1}{\lambda'}[F(t') - F(s')] / \frac{1}{\lambda'}(t'-s') \le \frac{F(t') - F(s')}{t'-s'}$

b) Convex ⇒ Lipschitz ⇒ bdd derivative and abs cont by #37.

c)

Foland Chapter 6

6.1.1 When does equality hold in Minkowskis thm?

Pf $\underline{p=1}$

$\int |f+g| = \int |f| + \int |g| = \int |f|+|g|$

$\iff f, g \geq 0$ or $f, g \leq 0$

$\underline{1<p<\infty}$

We proceed by induction

$p=2$. $|f+g|^2 = |f^2 + 2fg + g^2| = |f|^2 + |g|^2 \iff f=0$ or $g=0$

Assume equality holds for $p=k-1$

$$\|f+g\|_{k-1} = \|f\|_{k-1} + \|g\|_{k-1}$$

$\underline{p=\infty}$

$\|f+g\|_\infty = \|f\|_\infty + \|g\|_\infty$

ess sup $|f+g|$ = ess sup $|f|$ + ess sup $|g|$

$\iff f, g \geq 0$

6.1.2 a) If f and g are msble on X then $\|fg\| \le \|f\|_1 \|g\|_\infty$
If $f \in L^1$ and $g \in L^\infty$, $\|fg\|_1 = \|f\|_1 \|g\|_\infty \iff |g(x)| = \|g\|_\infty$ a.e
on set where $f(x) \ne 0$

c) $\|f_n - f\|_\infty \to 0 \iff \exists E \in \mathcal{M}$ s.t. $\mu(E^c) = 0$ and $f_n \xrightarrow{u} f$ on E

Pf a) $\|fg\|_1 = \int |fg|$
$= \int |f||g|$
$\le \int |f| \operatorname{ess\,sup} |g|$
$= \operatorname{ess\,sup} |g| \int |f|$
$= \|g\|_\infty \|f\|$

assume $|g(x)| = \|g\|_\infty$ on $\{x \mid f(x) \ne 0\}$
$\Rightarrow \|fg\|_1 = \|f\| \|g\|_\infty$ Clearly

Now assume $\|fg\|_1 = \|f\|_1 \|g\|_\infty$
$\Rightarrow \int |fg| = \int |f| \cdot \operatorname{ess\,sup} |g|$
$\Rightarrow \int |f||g| = \int |f| \cdot \operatorname{ess\,sup} |g|$
$\Rightarrow |f||g| = |f| \cdot \operatorname{ess\,sup} |g|$ a.e since both positive
$\Rightarrow |g| = \operatorname{ess\,sup} |g|$ when $f(x) \ne 0$
$\Rightarrow |g| = \|g\|_\infty$ when $f(x) \ne 0$

c) Assume $\exists E \in \mathcal{M}$ s.t. $\mu(E^c) = 0$ and $f_n \xrightarrow{u} f$ on E
$\Rightarrow \int |f_n - f| = \int_E |f_n - f| + \int_{E^c} |f_n - f| \to 0$
$\Rightarrow \left(\int |f_n - f|^p\right)^{1/p} \to 0$
$\Rightarrow \|f_n - f\|_p \to 0$ $\forall p$
$\Rightarrow \|f_n - f\|_\infty \to 0$ as $p \to \infty$

Assume $\|f_n - f\|_\infty \to 0$
$\Rightarrow \operatorname{ess\,sup} |f_n - f| \to 0$
$\Rightarrow \inf \{M > 0 : \mu(\{x \mid \underbrace{|f_n(x) - f(x)| > M}_{E^c}\}) = 0\} \to 0$
$|f_N(x) - f(x)| < \frac{1}{N}$ for N sufficiently large $\Rightarrow$
and $\mu(E^c) = 0$

6.1.5 Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ $\Leftrightarrow$ $\mathcal{X}$ contains sets of arbitrarily small positive measure and $L^q \not\subset L^p \Leftrightarrow \mathcal{X}$ contains sets of arbitrarily large finite measure.

Pf Assume $L^p \not\subset L^q$.

Let $E_n = \{x : |f(x)|^p > n\}$

$\Rightarrow E_n = \{x : |f(x)| > n^{1/p}\}$

$\Rightarrow \int_{E_n} |f|^p \geq n\mu(E_n)$

$\Rightarrow \mu(E_n) \leq \frac{1}{n} \underbrace{\int_{E_n} |f|^p}_{=0} \longrightarrow 0$ since $\int |f|^p < \infty$

$\Rightarrow \mu(E_n) \to 0$

$\Rightarrow \mu(E_n) = 0 \ \forall n$

or $\mu(E_n) \to 0$ w/ $\mu(E_n) > 0$

If $\mu(E_n) = 0 \ \forall n \Rightarrow \int_{E_0} |f| \overset{0}{=} \int_X |f|^p = 0$

$\Rightarrow f \equiv 0$ a.e.

$\Rightarrow f \in L^q$.

$\Rightarrow \mu(E_n) > 0 \ \forall n$ and $\mu(E_n) \to 0$

Assume $\mathcal{X}$ contains sets of arbitrarily small measure.

Let E_n be s.t. $0 < \mu(E_n) < \frac{1}{2^n}$

Let $f = \sum \mu(E_n)^{-1/q} \chi_{E_n}$

$$\int |f|^q = \sum \mu(E_n)^{-1} \mu(E_n) = \sum 1 = \infty$$

$$\int |f|^p = \sum \mu(E_n)^{-p/q} \mu(E_n) = \sum \mu(E_n)^{1-p/q} < \sum \frac{1}{2^{\frac{n(q-p)}{q}}} = \sum \left(\frac{1}{2^{1-p/q}}\right)^k < \infty$$

since $\frac{1}{2^{1-p/q}} < 1$ if $p < q$.

$\square$

b) Now suppose $L^q \not\subset L^p$

Assume BWOC that $\exists N$ s.t. $\mu(E) < N \ \forall E \subset X$

$\Rightarrow \mu(X) < N$

$$\Rightarrow \int |f|^p = \int |f|^p \cdot 1$$
$$\leq \| |f|^p \|_{q/p} \, \| 1 \|_{\frac{p-q}{p}}$$
$$= \left(\int |f|^{pq/p}\right)^{p/q} \mu(x)^{\frac{p}{p-q}}$$
$$< \|f\|_q^p \, N^{p/p-q}$$
$$< \infty$$

$\Rightarrow f \in L^p$ which contradicts

Now assume $\mu(E) < N$ or $\mu(E) = \infty$. w/ some $\mu(\bar{E}) = \infty$

$\Rightarrow$ Not poss. ble.

Now if $\mu(E) = \infty \ \forall E$.

$\Rightarrow$ Not possible since $\mu(\emptyset) = 0$.

Now assume X contains sets of arbitrarily large finite measure

Let $\{E_n\}$ be disjoint with $n \leq \mu(E_n) < \infty$

Let $f = \sum_0^\infty n^{-a/p} \mu(E_n)^{-1/p} \chi_{E_n}$ $\quad 0 < a < 1$

$$\Rightarrow \|f\|_p^p = \sum_0^\infty n^{-a} \mu(E_n)^{-1} \mu(E_n)$$
$$= \sum_0^\infty n^{-a}$$
$$= \infty \quad \text{since } a \in (0,1)$$

$$\|f\|_q^q = \sum_0^\infty n^{\frac{-aq}{p}} \mu(E_n)^{-q/p}$$
$$\leq \sum_0^\infty n^{\frac{-aq}{p}} n^{-q/p}$$
$$= \sum_0^\infty \left(\frac{1}{n}^{q/p}\right)^{a+1}$$
$$< \infty \quad \text{since } \underbrace{q/p}_{>1} \underbrace{(a+1)}_{>1} > 1$$

6.1.7 If $f \in L^p \cap L^\infty$ for some $p < \infty$. s.t. $f \in L^q$ $\forall q > p$

then $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$

Pf Case 1 f bounded

$\|f\|_q \leq \|f\|_p^\lambda \|f\|_\infty^{1-\lambda}$ by Littlewoods inequality. $\lambda = 1/p$

$= \|f\|_p^{p/q} \|f\|_\infty^{1-\frac{p}{q}}$

$\Rightarrow \lim_{q \to \infty} \|f\|_q \leq \|f\|_\infty$

Let $E_\varepsilon = \{x \mid |f(x)| > \|f\|_\infty - \varepsilon\}$

$\|f\|_q \geq \|f \chi_{E_\varepsilon}\|_q$

$= (\int |f \chi_{E_\varepsilon}|^q)^{1/q}$

$\geq (\|f\|_\infty - \varepsilon)\, \mu(E_\varepsilon)^{1/q}$

$\Rightarrow \lim_{q \to \infty} \|f\|_q \geq \|f\|_\infty$

$\Rightarrow \lim_{q \to \infty} \|f\|_q = \|f\|_\infty$

Case 2 f unbdd

Let $E_M = \{x \mid |f(x)| > M\}$

$m(E_M) > 0$ since f unbdd

$\|f\|_q \geq \|f \chi_{E_M}\|_q = ($

$= (\int |f \chi_{E_M}|^q)^{1/q}$

$\geq M \mu(E_M)^{1/q}$

$\Rightarrow \lim \|f\|_q \geq M$ $\forall M$

$\Rightarrow \lim \|f\|_q = \|f\|_\infty$ since f unbdd.

$f \in L^p \Rightarrow f$ bdd except on a set of measure 0

$\square$

6.1.8 Suppose $\mu(X)=1$ and $f \in L^p$ for some $p>0$ s.t. $f \in L^q$ $\forall 0<q<p$.

a) $\log \|f\|_q \geq \int \log|f|$

b) $(\int |f|^q - 1)/q \geq \log \|f\|_q$ and $(\int |f|^q - 1)/q \to \int \log|f|$ $q \to 0$

c) $\lim \|f\|_q = \exp(\int \log|f|)$

Pf a) Let $F(t)=e^t$, Let $g = \log|f|$

$\Rightarrow F(\int g) \leq \int F \circ g$ by Jensen's since F is covex.

$\Rightarrow e^{\int \log|f|} \leq \int |f|$

$\Rightarrow e^{\int \log|f|} \leq \|f\| \underset{\text{By Holders}}{\leq} \|f\|_q \|1\|_{q'} = \|f\|_q \, \mu(X)^{1/q'} = \|f\|_q$. since $\mu(X)=1$

$\Rightarrow e^{\int \log|f|} \leq \|f\|_q$

$\Rightarrow \int \log|f| \leq \log \|f\|_q$. ✓

b). WTS $\int |f|^q - 1 \geq q \log \|f\|_q = \log \|f\|_q^q$

$\log \|f\|_q^q \leq \|f\|_q^q - 1 = \int |f|^q - 1$

$\therefore \dfrac{\int |f|^q - 1}{q} \geq \log \|f\|_q$ ✓

Let $h: (0, \infty) \to \mathbb{R}$ be s.t. $h(x) = \dfrac{a^x - 1}{x}$ $a \geq 0$

1) show h monotone increasing.

$a=0 \Rightarrow h(x) = \frac{-1}{x}$ ✓

$a>0 \Rightarrow h'(x) = \dfrac{x(a^x \log a) - (a^x - 1)}{x^2} = \dfrac{a^x \log a^x - a^x + 1}{x^2}$

$h' \geq 0 \iff a^x \log a^x - a^x \geq -1$

Let $w(x) = x \log x - x$.

$w'(x) = \log x + 1 - 1 = \log x = 0 \iff x = 1$

$w''(x) = \frac{1}{x} \geq 0$

$\Rightarrow h'(x) \geq 0$

Now $\lim_{q\to 0} \frac{|f|^q - 1}{q} = \lim_{q\to 0} \frac{|f|^q q \log|f|}{q} = \log|f|$

Since $\frac{|f|^q - 1}{q}$ is monotone increasing then the limit is monotone decreasing

$$\Rightarrow \lim_{q\to 0} \int \frac{1}{q}(|f|^q - 1) = \int \lim \frac{1}{q}|f|^q - 1 \quad \text{By MCT}$$
$$= \int \log|f|$$

c) By (a) we have $\log\|f\|_q \geq \int \log|f|$

$$\Rightarrow \|f\|_q \geq e^{\int \log|f|}$$

By (b) we have $\overline{\lim} \log\|f\|_q \leq \overline{\lim} \int \frac{|f|^q - 1}{q} \to \int \log|f|.$

$$\Rightarrow \overline{\lim} \|f\|_q \leq e^{\int \log|f|}$$

$$\Rightarrow \lim_{q\to 0} \|f\|_q = e^{\int \log|f|}$$

$\square$

6.1.9 Suppose $1 \le p < \infty$. If $\|f_n - f\|_p \to 0$ then $f_n \xrightarrow{\mu} f$ and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \le g \in L^p$ $\forall n$ and $\|f_n - f\|_p \to 0$

Pf Let $E_{\varepsilon,n} = \{x \mid |f_n(x) - f(x)| > \varepsilon^{1/p}\}$.

$$\|f_n - f\|_p^p = \int |f_n - f|^p d\mu$$

$$\ge \int_{E_{n\varepsilon}} |f_n - f|^p d\mu$$

$$\ge \varepsilon \mu(E_{n\varepsilon})$$

$$\Rightarrow \mu(E_{n\varepsilon}) \le \varepsilon^{-1} \int |f_n - f|^p d\mu \to 0$$

Now let $f_n \xrightarrow{\mu} f$ and $|f_n| \le g \in L^p$

Let $|f_{n_k} - f|^p$ be a subseq. of $|f_n - f|^p$

$\Rightarrow \exists |f_{n_{k_\ell}} - f|^p \xrightarrow{a.e.} 0$ since $f_{n_k} \xrightarrow{\mu} f$

$|f_{n_{k_\ell}} - f|^p \le 2^p |g|^p \in L^1$ since $|f_n| \le g \Rightarrow |f| \le g$.

$\Rightarrow$ By DCT $\int |f_{n_{k_\ell}} - f|^p \to 0$.

$\Rightarrow \int |f_n - f|^p \to 0$ since every subsequence has a convergent subseq

$\square$

6.1.10 Suppose $1 \le p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e then $\|f_n - f\|_p \to 0 \iff \|f_n\|_p \to \|f\|_p$

Pf Assume $\|f_n - f\|_p \to 0$

$\Rightarrow \big| \|f_n\|_p - \|f\|_p \big| \le \|f_n - f\|_p \to 0$

$\Rightarrow \|f_n\|_p \to \|f\|_p$.

Assume $\|f_n\|_p \to \|f\|_p$.

Let $g_n = 2^p(|f|^p + |f_n|^p)$ and $g = 2^{p+1}(|f|^p)$

$\Rightarrow g_n \to g$ a.e.

Now $|f_n - f|^p \le g_n$

$\Rightarrow \int \lim |f_n - f|^p = \lim \int |f_n - f|^p$ by GDCT

$\Rightarrow 0 = \lim \int |f_n - f|^p$

$\Rightarrow \|f_n - f\|_p \to 0$

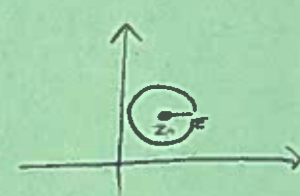
6.1.11 essential range $R_f = \{z \in \mathbb{C} : m(\{x : |f(x) - z| < \varepsilon\}) > 0\}\ \forall \varepsilon > 0$

a. R_f closed b. $f \in L^\infty \Rightarrow R_f$ compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$

Pf a. Let $\{z_n\} \subset R_f$ s.t. $z_n \to z$. WTS $z_n \in R_f$.

$m(\{x : |f(x) - z_n| < \varepsilon\}) > 0 \quad \forall n$.

$z_n \to z \Rightarrow \forall \varepsilon > 0\ \exists N$ s.t. $n \ge N \Rightarrow |z - z_n| < \varepsilon$



WTS $f^{-1}(B_\varepsilon(z)) > 0 \quad \forall \varepsilon \ge 0$.

Let N be s.t. $n > N \Rightarrow z_n \in B_\varepsilon(z)$

$\Rightarrow \exists B_\delta(z_{N+1}) \subset B_\varepsilon(z)$ since $B_\varepsilon(z)$ open and $z_{N+1} \in B_\varepsilon(z)$

$\Rightarrow m(f^{-1}(B_\delta(z_{N+1}))) > 0$ since $z_{N+1} \in R_f$

$\Rightarrow f^{-1}(B_\delta(z_{N+1})) \subset f^{-1}(B_\varepsilon(z))$

$\Rightarrow m(f^{-1}(B_\varepsilon(z))) > 0$

$\Rightarrow z \in R_f$

$\Rightarrow R_f$ closed.

b. Let $f \in L^\infty$.

$\Rightarrow \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\} < \infty$

$\Rightarrow \text{ess sup } |f(x)| < \infty$

WTS R_f is compact.

We know R_f is closed so it remains to show R_f is bdd.

$f \in L^\infty \Rightarrow \exists M$ s.t. $m(\{|f|^{-1}(M, \infty)\}) = 0$

Let $K = \{z \in \mathbb{C} : |z| > M\}$. Let $z \in K$.

$\Rightarrow \exists \varepsilon$ s.t. $V_p = B_\varepsilon(z) \subset K$ (since K open)

$\Rightarrow |f|^{-1}(M, \infty) = f^{-1}(K)$

$\Rightarrow m(f^{-1}(K)) = 0$

$\Rightarrow m(V_p) = 0$

$\Rightarrow R_f \subset K^c$

$\Rightarrow R_f$ bdd.

Now to show $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

$\|f\|_\infty = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}$

$= \inf\{a \geq 0 : \mu(\{|f|^{-1}(a, \infty)\}) = 0\}$

If $|z| > a \Rightarrow \exists \varepsilon$ s.t. $f^{-1}(B_\varepsilon(z)) \subset |f|^{-1}(a, \infty)$

$\Rightarrow \mu(f^{-1}(B_\varepsilon(z))) = 0$

$\Rightarrow z \notin R_f$

$\Rightarrow \|f\|_\infty < \max\{|z| : z \in R_f\}$.

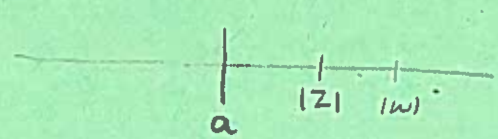
Now if $|w| > \max\{|z| : z \in R_f\}$.

$\Rightarrow w \notin R_f$

$\Rightarrow \exists \varepsilon > 0$ s.t. $m(f^{-1}(B_\varepsilon(w))) = 0$

$\Rightarrow \mu(\{|f|^{-1}(|z|, \infty)\}) = 0$ since holds $\forall |w| > |z|$.

$\Rightarrow \|f\|_\infty \supset \max\{|z| : z \in R_f\}$



6.1.12 If $p \neq 2$ the L^p norm does not arise from on an inner product on L^p except in trivial case when $\dim(L^p) \leq 1$ (hint: show parallelogram law fails)

Pf Parallelogram law

$$\Rightarrow \|x+y\|_p^2 + \|x-y\|_p^2 = 2(\|x\|_p^2 + \|y\|_p^2) \quad *$$

Let $x = (1, 1, 0, 0, \dots 0) \in L^p$ and $y = (1, -1, 0, \dots 0) \in L^p$

$\|x+y\|_p^2 = \|(2, 0, \dots 0)\|_p = 2 = \|x-y\|_p$

$\|x\|_p = 2^{1/p} = \|y\|_p$

So $* = 2^2 + 2^2 = 2(2^{2/p} + 2^{2/p})$

$\Rightarrow 8 = 4(2^{2/p})$

$\Rightarrow p = 2.$

$\square$

6.1.14. If $g \in L^\infty$. The operator T defined by $Tf = fg$ is bdd on L^p, its operator norm is at most $\|g\|_\infty$ w/ equality if μ is semifinite.

Pf $\|Tf\|_p^p = \|fg\|_p^p = \int |f|^p |g|^p \leq \|g\|_\infty^p \int |f|^p = \|g\|_\infty^p \|f\|_p^p < \infty$ for $f \in L^p$

$\Rightarrow \|T\|_p \leq \|g\|_\infty$

Let μ be semifinite

$\Rightarrow \exists A$ st $0 < \mu(A) < \infty$ s.t. $|g(x)| > \|g\|_\infty - \varepsilon.$

6.1.15 The Vitali Convergence Thm

Suppose $1 \le p < \infty$ and $\{f_n\}_1^\infty \subset L^p$ In order for $\{f_n\}$ to be Cauchy in L^p norm it is necessary and sufficient for following 3 conditions to hold.

(i) $\{f_n\}$ Cauchy in measure.

(ii) $\{|f_n|^p\}$ is uniformly integrable

(iii) $\forall \varepsilon > 0$ $\exists E \subset X$ s.t. $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \varepsilon$ $\forall n$

Pf Assume $\{f_n\}$ is Cauchy in L^p norm.

Let $\varepsilon > 0$. and $E_{n,m} = \{x \in X : |f_n(x) - f_m(x)| \ge \varepsilon\}$

$\Rightarrow \mu(E_{nm}) \varepsilon^p = \int_{E_{nm}} \varepsilon^p dx \le \int_{E_{nm}} |f_n(x) - f_m(x)|^p dx \le \|f_n - f_m\|_p^p$

$\Rightarrow \mu(E_{nm}) \le \left(\frac{\|f_n - f_m\|_p}{\varepsilon}\right)^p \to 0$ as $n, m \to \infty$ since f is L_p Cauchy

$\Rightarrow \{f_n\}$ Cauchy in measure.

f_n Cauchy in L_p

$\Rightarrow f_n \to f$ in L^p for some f since L^p is complete.

If we can show $|f_n|^p \to |f|^p$ in L^1 then $\{|f_n|^p\}$ will be uniformly integrable by 3.2.11.

Claim: $|f_n|^p \to |f|^p$

Note for $a, b \in \mathbb{R}$ w/ $a < b$ $\frac{b^p - a^p}{b - a} = pc^{p-1}$ for some $c \in (a,b)$

If we dont know $a < b$ we have.

$|a^p - b^p| \le p|a - b| \max\{|a|^{p-1}, |b|^{p-1}\} \le p|a-b|(|a| + |b|)^{p-1}$

$\Rightarrow \||f_n|^p - |f|^p\| = \int ||f_n|^p - |f|^p|$

$\le p \int |f_n - f| \cdot (|f_n| + |f|)^{p-1}$ by above

$= p \| |f_n - f| (|f_n| + |f|)^{p-1} \|$

$\le p \|f_n - f\|_p \|(|f_n| + |f|)^{p-1}\|_q$ by Hölders. $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q + p = pq$

$\le p \|f_n - f\|_p \left(\int (|f_n| + |f|)^{q(p-1)}\right)^{1/q}$

$\le p \|f_n - f\|_p \left(\int |f_n + f|^p\right)^{1/q}$

$= p \|f_n - f\|_p (\||f_n| + |f|\|_p)^{p/q}$

$= p \|f_n - f\|_p (\|f_n\|_p + \|f\|_p)^{p/q}$

$\le p \|f_n - f\|_p (\|f_n\|_p + \|f\|_p)^{p/q}$

$\to 0$ ✓ pf of claim

Since $|f|^p \in L^1$

$\Rightarrow$ given $\varepsilon > 0$ $\exists E$ s.t. $\mu(E) < \infty$ and $\int_E |f|^p < \varepsilon/2$

We have $\| |f_n|^p - |f|^p \| \to 0$

$\Rightarrow \int_{E^c} |f_n|^p \to \int_{E^c} |f|^p$

$\Rightarrow \exists N$ s.t. $n \geq N \Rightarrow \int_{E^c} |f_n|^p < \varepsilon$,

and $|f_1|^p, \dots, |f_{N-1}|^p \in L^1$ $\exists$

$\Rightarrow \exists E_1, \dots E_{N-1}$ w/ finite measure s.t. $\int_{E_n^c} |f_n|^p < \varepsilon$

Let $D = E_1 \cup \dots \cup E_{N-1} \cup E$

$\Rightarrow \mu(D) < \infty$ and $\int_{D^c} |f_n|^p < \varepsilon$ $\forall n$. ✓.

Now assume (i), (ii), (iii) hold.

(iii) $\Rightarrow$ $\exists E$ w/ $\mu(E)$ s.t. $\forall n$ $\|\chi_{E^c} f_n\|_p < \varepsilon/6$

$\Rightarrow \|\chi_{E^c}(f_m - f_n)\|_p \leq \|\chi_{E^c} f_m\|_p + \|\chi_{E^c} f_n\|_p \leq \varepsilon/3$ $\forall m, n$

Let $A_{m,n} = \{x \in E : |f_m(x) - f_n(x)| > \varepsilon\}$

on $E \setminus A_{mn}$ we have $|f_m - f_n| < \varepsilon^p$

if $g_{m,n} = \chi_{E \setminus A_{mn}} |f_m - f_n|^p$ $\Rightarrow$ ε^p is uniform bd for g_{mn}.

$g_{mn} \xrightarrow{m} 0$ by (i)

$\mu(E) < \infty \Rightarrow$ By DCT $\int g_{mn} \to 0$

$\Rightarrow \exists N_1$, s.t. $m, n \geq N_1 \Rightarrow \|\chi_{E \setminus A_{mn}}(f_m - f_n)\|_p < \varepsilon/3$

(ii) $\Rightarrow \exists \delta > 0$ s.t. $\forall n$ $\mu(A) < \delta \Rightarrow \|\chi_A f_n\|_p < \varepsilon/6$

$\{f_n\}$ Cauchy in measure.

$\Rightarrow \exists N_2$ s.t. $\mu(A_{m,n}) < \delta$ if $n, m \geq N_2$.

$\Rightarrow \|\chi_{A_{mn}}(f_m - f_n)\| \leq \|\chi_{A_{mn}} f_m\|_p + \|\chi_{A_{mn}} f_n\|_p < \varepsilon/3$.

if $m, n > \max\{N_1, N_2\}$

$\Rightarrow \|f_m - f_n\|_p \leq \|\chi_{E^c}(f_m - f_n)\|_p + \|\chi_{E \setminus A_{mn}}(f_m - f_n)\|_p + \|\chi_{A_{mn}}(f_m - f_n)\|_p$

$< \varepsilon$

$\Rightarrow \{f_n\}$ is Cauchy in L^p.

$\square$

6.3.26. Complete proof of 6.18 for $p=1$, $p=\infty$

Pf $\underline{p=1}$ $\|Tf\| = \int\left|\int k(x,y) f(y) dy\right| dx$

$\leq \int \int |k(x,y)||f(y)| dy\, dx$

$= \int \int |k(x,y)||f(y)| dx\, dy$ by fubini

$= \int |f(y)| \int |k(x,y)| dx\, dy$

$= \int |f(y)|\, C\, dy$

$= C \int |f(y)| dy$

$= C\|f\|$

$\underline{p=\infty}$ $\|Tf\|_\infty = \text{ess sup } |Tf|$

$= \text{ess sup } \left|\int k(x,y) f(y) dy\right|$

$\leq \text{ess sup } \int |k(x,y)||f(y)| dy$

$\leq \text{ess sup } \|k(x,y)\| \, \|f\|_\infty$ by Hölders.

$\leq \text{ess sup } C\|f\|_\infty$

$= C\|f\|_\infty$.

$\square$

6.27. Hilberts Inequality

The operator $Tf(x) = \int_0^\infty (x+y)^{-1} f(y)\,dy$ satisfies $\|Tf\|_p \le C_p \|f\|_p$ for $1<p<\infty$ where $C_p = \int_0^\infty x^{-1/p}(x+1)^{-1}$

Pf $\|Tf\|_p = \left[\int_0^\infty \left| \int_0^\infty (x+y)^{-1} f(y)\,dy \right|^p dx\right]^{1/p}$

$\le \int \left[\int |(x+y)^{-1} f(y)|^p dx\right]^{1/p} dy$

$= \int \left[\int |x+y|^{-p} |f(y)|^p dx\right]^{1/p} dy$

$= \int \|f\|_p \left[\int |x+y|^{-p} dx\right]^{1/p} dy$

Measure Theory Exams

Midterm 2013

1. Let $(X, \mathcal{A})$, $(Y, \mathcal{F})$ be 2 measurable spaces and $g: X \to Y$ a measurable fcn. Let μ be a measure on $\mathcal{A}$ and define ν on $\mathcal{F}$ be $\nu(B) = \mu(g^{-1}(B))$

   a) Prove ν a measure

   b) Let $f: Y \to \mathbb{R}$ be nonneg msble. Prove $\int f d\nu = \int (f \circ g) d\mu$

Pf a) $\nu(\phi) = \mu(g^{-1}(\phi)) = \mu(\phi) = 0$ ✓

Let $A_1, A_2, \ldots$ be disjoint.

$\nu(\cup_i^\infty A_i) = \mu(g^{-1}(\cup_i^\infty A_i))$

$= \mu(\cup_i^\infty g^{-1}(A_i))$ ($g^{-1}(A_i)$ disjoint since g msble)

$= \sum_i^\infty \mu(g^{-1}(A_i))$

$= \sum_i^\infty \nu(A_i)$ ✓

b). Let $f = \chi_E$ for msble E

$\Rightarrow \int f d\nu = \int_E d\nu = \nu(E) = \mu(g^{-1}(E)) = \int \chi_{g^{-1}(E)} d\mu = \int \chi_E \circ g d\mu = \int f \circ g d\mu$

Let $f = \sum_i^n b_i \chi_{B_i}$ be a simple fcn in canonical form

$\Rightarrow \int f d\mu = \int \sum_i^n b_i \chi_{B_i} d\nu = \sum_i^n b_i \nu(\chi_{B_i}) d\nu = \sum_i^n b_i \int \chi_{B_i} \circ g = \int \sum b_i \chi_{B_i} \circ g$

Let f be non negative msble fcn

$\Rightarrow \exists f_n$ simple s.t. $f_n \nearrow f$.

$\Rightarrow \int f = \int \lim f_n d\nu$ by MCT

$= \int \lim f_n \circ g d\mu$ by previous case

$= \int f \circ g d\mu$

$\square$

2. a) State Fatous lemma

b) If $\sup_n \int |f_n| < \infty$ and $f_n \to f$ a.e prove $\int |f| d\mu < \infty$

c) If $\sup_n \int |f_n| < \infty$ and $f_n \to f$ a.e and $\int |f_n| d\mu \to \int |f| d\mu$

Show $\forall A \in \mathcal{A}$ $\int_A |f_n| d\mu \to \int_A |f| d\mu$

Pf a) Let $f_n \geq 0$ be measurable: Then $\int \underline{\lim} f_n \leq \underline{\lim} \int f_n$

b). Let $M = \sup_n \int |f_n|$

$f_n \xrightarrow{a.e} f \Rightarrow \mu(\{x \mid f_n(x) \not\to f(x)\}) = 0$

$M < \infty \Rightarrow \forall n \in \mathbb{N}$ $\int |f_n| d\mu < M$.

Now $\int |f| = \int \lim |f_n| \leq \underline{\lim} \int |f_n| < \underline{\lim} M = M < \infty$

c). Consider $f_n \chi_A \Rightarrow |f_n| \chi_A \leq |f_n|$ and $\int |f_n| \to \int f$.

$\Rightarrow \int_A |f_n| d\mu = \int |f_n| \chi_A d\mu \xrightarrow[\text{by GDCT}]{} \int |f| \chi_A d\mu = \int_A |f| d\mu$

$\square$

3) a) State DCT when $f_n \to f$ in measure

b) Show if $f \in L^1$ then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $\forall A \in \mathcal{A}$ if $\mu(A) < \delta$ then $\int_A |f| d\mu < \varepsilon$

Pf a) Let f_n be measurable fcns with $|f_n| \leq g$ $\forall n$

If $f_n \xrightarrow{\mu} f$ and $g \in L^1(\mu)$. Then $f \in L^1(\mu)$ and $\int |f_n - f| \to 0$

b). Let $\varepsilon > 0$. Let $\delta < \varepsilon/2M$, Let M be s.t. $\int |f| \chi_{\{|f| > M\}} < \varepsilon/2$

$$\int_A |f| = \int_{A \cap \{|f| > M\}} |f| + \int_{A \cap \{|f| \leq M\}} |f|$$

$$\leq \int_A |f| \chi_{\{|f| > M\}} + \int_A |f| \chi_{\{|f| \leq M\}}$$

$$\leq \underbrace{\int_A |f| \chi_{\{|f| > M\}}}_{\to 0 \text{ as } M \to \infty \text{ by DCT}} + M\mu(A).$$

$$\leq \varepsilon/2 + M \varepsilon/2M$$

$$= \varepsilon.$$

$\square$

4.) Set $\alpha(x) = \sum_{\{i \geq 1 : 2^{-i} < x\}} 3^{-i}$ and let μ_α be Lebesgue-Stieltjes measure associated w/ α

a) Let $A = \{2^{-i} : i \geq 1\}$ Show $\mu_\alpha(A^c) = 0$

b) Evaluate $\int \frac{1}{x} \mu_\alpha(dx)$

Pf $A = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$

$A^c = (-\infty, \frac{1}{2})(\frac{1}{2}, \frac{1}{4}) \dots = (-\infty, 0] \cup (\frac{1}{2}, \infty) \cup \bigcup_{i=1}^{\infty} (\frac{1}{2^{i+1}}, \frac{1}{2^i})$

$\mu((-\infty, 0]) = \alpha(0) - \alpha(-\infty)$

$\mu((\frac{1}{2}, \infty)) = \alpha(\infty) - \mu(\frac{1}{2})$

$\mu((\frac{1}{2^{i+1}}, \frac{1}{2^i})) = \alpha(\frac{1}{2^i}) - \alpha(\frac{1}{2^{i+1}})$

$\Rightarrow \mu(A^c) = \mu((-\infty, 0]) + \mu((\frac{1}{2}, \infty)) + \sum_{1}^{\infty} (\frac{1}{2^{i+1}}, \frac{1}{2^i})$

$= \underbrace{\alpha(0) - \alpha(-\infty)}_{=0 \text{ since } \nexists i \text{ s.t. } 2^{-i} < 0} + \underbrace{\alpha(\infty) - \alpha(\frac{1}{2})}_{=0} + \sum \underbrace{\alpha(\frac{1}{2^i}) - \alpha(\frac{1}{2^{i+1}})}_{=0 \text{ since in same interval}}$

b). Let $s_n = \sum_1^n 2^i \chi_{\{2^{-i}\}} \Rightarrow s_n \nearrow \frac{1}{x}$ wrt μ_α.

$\int s_n = \sum_1^n 2^{-i} \mu_\alpha(\{2^{-i}\}) = \sum_1^n \frac{2^i}{3^i}$

$\Rightarrow$ By DCT $\lim \int s_n = \int \frac{1}{x} \mu_\alpha(dx)$

$\Rightarrow \lim \int s_n = \frac{1}{1 - 2/3} = 3$

$\square$

Midterm 2012

1. Let $f: \mathbb{R} \to \mathbb{R}$. Define f Lebesgue msble, Borel msble.
If f is Lebesgue msble show $\exists\, g: \mathbb{R} \to \mathbb{R}$ which is Borel msble s.t $f = g$ a.e.

Pf • $f: (\mathbb{R}, \mathcal{M}) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ is Lebesgue msble if $f^{-1}(B) \in \mathcal{M}$ $\forall B \in \bar{\mathcal{B}}$

• $f: (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{B})$ is Borel msble if $f^{-1}(B) \in \mathcal{B}$ $\forall B \in \mathcal{B}$

• Let f be Lebesgue msble
Let $E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$
Since f is Lebesgue msble, E_{nk} is L.msble
$\Rightarrow E_{nk} = B_{nk} + N_{nk}$ where B_{nk} is Borel and N_{nk} is a Lebesgue null set

Let $g_n = \begin{cases} \frac{k}{2^n} & x \in B_{nk} \\ 0 & x \in N_{nk} \end{cases}$

g_n is Borel msble $\Rightarrow g_n \to g$ where g is Borel msble
Now $\mu(\{x \mid f(x) \neq g(x)\}) = \mu(\{N_{nk}\}) = 0$
Since $g_n \to f$ a.e.

2. a) State MCT

b) Prove $f_n \nearrow f$ and $f_1^- \in L^1(\mu)$ then $\int f_n \nearrow \int f$

Pf a) Let $0 \le f_1 \le f_2 \le \dots$ be msble fcns. w/ $f_n \to f$.

Then $\lim \int f_n = \int f$.

b). Let $g_n = f_n + |f_1^-|$

$\Rightarrow g_n$ msble.

$\Rightarrow g_n \nearrow f + |f_1^-| = g$

$\Rightarrow \lim \int g_n = \int g$ by MCT

$\Rightarrow \lim \int f_n + |f_1^-| = \int f + |f_1^-|$

$\Rightarrow (\lim \int f_n) + \int |f_1^-| = \int f + \int |f_1^-|$ by linearity

$\therefore \lim \int f_n = \int f$ since $\int |f_1^-|$ is constant & finite.

$\square$

3. $f \in L^1(m)$. Fix $a \in \mathbb{R}$. Set $F(x) = \int_a^x f(t)\,dt$. Prove F continuous.

Pf Let $x_n \to x \in \mathbb{R}$.

Set $f_n = f \chi_{(a, x_n)} \to f \chi_{ax}$

$\Rightarrow |f_n| \le |f| \; \forall n$ and f_n is msble

$\Rightarrow \int f_n = \int f \chi_{(a, x_n)} = \int_a^{x_n} f = F(x_n)$

DCT $\Rightarrow \int f_n \to \int f$

$\Rightarrow F(x_n) \to F$

4) a) Define $f_n \xrightarrow{\mu} f$

b) Prove or counter:

If $f_n \geq 0$ and $f_n \xrightarrow{a.e} f$ then $\int f \leq \underline{\lim} \int f_n \, d\mu$

Pf a) $f_n \to f \Rightarrow \mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$ $\forall \varepsilon$. as $n \to \infty$

b). Suppose BWOC $\int f > \underline{\lim} \int f_n$

$\Rightarrow \int f - \underline{\lim} \int f_{n_k} d\mu \to i$ for some subseq n_k *

$\Rightarrow \exists n_{k_l}$ s.t. $f_{n_{k_l}} \xrightarrow{a.e} f$ since $f_{n_k} \xrightarrow{\mu} f$

$\Rightarrow \int f \, d\mu \leq \underline{\lim} \int f_{n_{k_l}} d\mu$ by Fato

$\Rightarrow \int f \, d\mu - \underline{\lim} \int f_{n_{k_l}} \leq 0$

which contradicts *

$\Rightarrow \int f \, d\mu \leq \underline{\lim} \int f_n$

$\square$

Final 2013

1. Let α be Cantor fcn. Evaluate following

a. $\int_0^1 \alpha(x)\,dx$

b. $\int_0^1 x\,\alpha(dx)$

c. $\int_0^1 \alpha(x)\,\alpha(dx)$.

Pf a. Note: $\alpha(1-x) = 1-\alpha(x)$

$\Rightarrow \alpha$ invariant under transformation $x \longmapsto 1-x$

$\Rightarrow \int_0^1 \alpha(x)\,dx = \int_0^1 1-\alpha(x)\,dx = 1-\int_0^1 \alpha(x)$

$\Rightarrow \int_0^1 \alpha(x)\,dx = 1/2$

b. $\int_0^1 x\,\alpha(dx) = 1\,\alpha(1) - 0\,\alpha(0) - \int_0^1 \alpha(x)\,dx$

$= 1 - 1/2$

$= 1/2$

$\boxed{\text{IBP}}$

c. $\int_0^1 \alpha(x)\,\alpha(dx) = \alpha(1)\alpha(1) - \alpha(0)\alpha(0) - \int_0^1 \alpha(x)\,\alpha(dx)$

$\Rightarrow 2\int_0^1 \alpha(x)\,\alpha(dx) = 1$

$\Rightarrow \int_0^1 \alpha(x)\,\alpha(dx) = 1/2$.

$\int_a^b \alpha(x)\,\beta(dx) = \alpha(b)\beta(a) - \alpha(b)\beta(a-) - \int_a^b \beta(x)\,\alpha(dx)$

2. Let $f \in L^2([1,\infty), m)$ Prove or counter:

a) If f is cont then $f(x) \to 0$ as $x \to \infty$

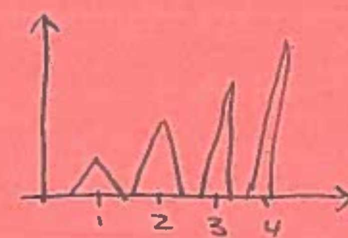
b) $\int_n^{n+1} |f| \, dm \to 0$ as $n \to \infty$

c) $\sqrt{n} \int_n^{n+1} |f| \, dm \to 0$ as $n \to \infty$

d) $\underline{\lim} \sqrt{n} \int_n^{n+1} |f| \, dm = 0$

Pf a) f cont and $f \in L^2([1,\infty), m)$

Let $f = \begin{cases} n & x = n \\ 0 & x \in (n + \frac{1}{n^2}, n+1 - \frac{1}{n^2}) \\ \text{interpolates linearly otherwise} \end{cases}$



$\int_1^\infty |f|^2 = \sum \left(\frac{1}{2} n \left(n + \frac{1}{n^2} - n + \frac{1}{n^2}\right)\right)^2$

$= \sum \left(\frac{1}{n}\right)^2 < \infty \Rightarrow f \in L^2$

but $f \not\to 0$ as $n \to \infty$.

$\boxed{\text{FALSE}}$

b) $\infty > \int_1^\infty |f|^2 = \sum_{n=1}^\infty \int_n^{n+1} |f|^2$

$\geq \sum_1^\infty \left(\int_n^{n+1} |f|\right)^2$ by Hölder. $\boxed{\text{TRUE}}$

$\Rightarrow \int_n^{n+1} |f| \to 0$

c) Let $f = \begin{cases} 2^{-n/2} & x \in [2^n, 2^n + 1] \\ 0 & \text{otherwise} \end{cases}$

$\int_1^\infty |f|^2 = \int_1^\infty \frac{1}{2^n} < \infty \Rightarrow f \in L^2$ $\boxed{\text{FALSE}}$

Let $n = 2^k \Rightarrow \sqrt{n} \int_n^{n+1} |f| \, dm = \sqrt{2^k} \int_{2^k}^{2^k+1} 2^{-k/2}$

$=$

$= \int_{2^k}^{2^k+1} 1$

$= 1 \quad \forall k$

$\Rightarrow \sqrt{n} \int_n^{n+1} |f| \, dm \not\to 0$

d). Suppose not

$\Rightarrow \exists c$ s.t. $\sqrt{n}\int_n^{n+1}|f|dm > c \quad \forall n.$

$\Rightarrow \int_n^{n+1}|f|\,dm > \frac{c}{\sqrt{n}}$

$\Rightarrow \int_1^{\infty}|f|^2 dm > \left(\int_1^{\infty}|f|\,dm\right)^2$

$= \left(\sum_1^{\infty}\int_n^{n+1}|f|dm\right)^2$

$> \left(\sum_1^{\infty}\frac{c}{\sqrt{n}}\right)^2$

$= \infty$

$\boxed{\text{TRUE}}$

which contradicts since $f \in L^2$.

3. Define $f: [0,1] \to [0,1]$ by $f(x) = \begin{cases} 0 & 0 = x \\ \frac{1}{n^2} & \frac{1}{n^2} = x \\ 0 & x = a_n = \frac{\frac{1}{n}+\frac{1}{n+1}}{2} \end{cases}$

f linear otherwise.

$g: [0,1] \to [0,1] \quad g(x) = \sqrt{x}$

Prove or counter

a. g AC

b. f AC

c. $g \circ f$ AC

Pf g is differentiable a.e on $[0,1]$.

$g' = \frac{1}{2\sqrt{x}}$ is integrable on $[0,1]$

and $g(x) - g(0) = g(x) = \int_0^x g'(x)$

$\Rightarrow g$ is AC by FTC

b) No matter how small we choose $\sum |b_j - a_j|$

$$\sum |f(b_j) - f(a_j)| \le \sum \left|\frac{1}{n^2} - 0\right|$$
$$= \sum_1^{\infty} \frac{1}{n^2}$$
$$< \infty$$

$\Rightarrow$ we can choose ε small enough so that $\sum_N^{\infty} \frac{1}{n^2} < \varepsilon$.

c) $g \circ f$ is <u>not</u> AC

$\forall \varepsilon > 0 \quad \sum \left|\sqrt{\frac{1}{n^2}} - 0\right| = \sum \frac{1}{n} = \infty$

So no matter how small ε is

we can choose $\sum \frac{1}{n} > 1$.

$\square$

4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{F}, \nu)$ 2 σ finite msures and $f: X \times Y \to \mathbb{R}$ a product msble fcn.
Prove for $1 \le p < \infty$ $\left(\int_X \left(\int_Y |f(x,y)| \, \nu(dy)\right)^p \mu(dx)\right)^{1/p} \le \int_Y \left(\int_X |f|^p \mu dx\right)^{1/p} \nu dy$

Pf $|f(x,y)| > 0$ so can apply Fubini

We WTS $\left\| \int_Y |f(x,y)| d\nu \right\|_p \le \int_Y \|f\|_p$

Notice $\int_X \int_Y |f(x,y)||g(x)| d\nu d\mu \le \int_X \|g\|_q \left(\int_Y |f(x,y)|^p d\mu\right)^{1/p} d\nu \quad \forall q \in L^q$

$\Rightarrow \int_X \left|\int_Y |f(x,y)||g(x)| d\nu\right)^p) d\mu)^{1/p} \le \int_Y \|g\|_q \left(\int_X |f(x,y)|^p d\mu\right)^{1/p} d\mu$

$\le \|g\|_q \int_X \left(\int_Y |f(x,y)|^p\right)^{1/p} d\nu d\mu$

$\Rightarrow \left\| \int_Y |f(x,y)| d\nu \right\|_p \le \int_X \left(\int_Y |f(x,y)|^p\right)^{1/p}$

since $\int |fg| d\mu \le C\|g\|_q$ $\square$

$\Rightarrow \|f\|_p \le C$

Midterm 2014

1. (a) Give an example of a sequence of functions $f_n \in L^1([0,1])$ s.t. $\forall x \in [0,1]$ $\lim f_n(x) = 0$ and yet f_n does not converge to 0 in $L^1([0,1])$

(b) Suppose $f_n : [0,1] \to \mathbb{R}$ and $f_n \to f$ a.e. in $[0,1]$. If $\|f_n\|_{L^1} \leq 30$ show $f \in L^1([0,1])$

Pf (a) Let $f_n = n \chi_{(0, 1/n)}$

$f_n \to 0$ $\forall x \in [0,1]$

However $\int |f_n| = \int_0^{1/n} n = 1$ $\forall n$

So $f_n \to 1$ in $L^1([0,1])$

(b) $\int |f| = \int \lim |f_n| \leq \underline{\lim} \int |f_n| \leq \underline{\lim}\, 30 = 30$

$\Rightarrow f \in L^1$

$\square$

2. Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^3$.

Prove or counter: $\forall$ Borel sets $E\subset\mathbb{R}$, $|E|=0 \Rightarrow |f(E)|=0$

Pf First note $x^3\in AC$ on any $[a,b]$.

- x^3 is differentiable
- $\frac{d}{dx}x^3=3x^2\in L'[a,b]$
- $f(x)-f(a)=\int_a^x 3y^2dy$

Now to show $f\in AC$ then $|E|=0 \Rightarrow |f(E)|=0$

Let $\varepsilon>0$.

$f\in AC \Rightarrow \exists\delta>0$ s.t. $\sum|b_j-a_j|<\delta \Rightarrow \sum|f(b_j)-f(a_j)|<\varepsilon$

$|E|=0 \Rightarrow \exists U=U_1^n(a_j,b_j)$, $E\subset U$ and $\sum|b_j-a_j|<\delta$

$\Rightarrow \exists$ max, min $y_j, x_j\in(a_j,b_j)$

$\Rightarrow f(E)\subset U(f(x_j),f(y_j))$

and $\sum|y_j-x_j|<\sum|b_j-a_j|<\delta$

Finally $|f(E)|\le|U(f(x_j)-f(y_j))$

$\le\sum|f(x_j)-f(y_j)|$

$<\varepsilon$

$\square$

5. Let $f: \mathbb{R} \to [0, \infty]$ be a Lebesgue integrable function. Prove $\lim_{N \to \infty} \frac{1}{N} \int_{[0,N]} x f(x)\,dx = 0$

Pf $\frac{1}{N} \int_{[0,N]} x f(x)\,dx = \frac{1}{N} \int_{[0,\sqrt{N}]} x f(x)\,dx + \frac{1}{N} \int_{[\sqrt{N},N]} x f(x)\,dx$

$$\leq \frac{\sqrt{N}}{N} \int_0^{\sqrt{N}} f(x)\,dx + \frac{N}{N} \int_{[\sqrt{N},N]} f(x)\,dx$$

$$= \frac{1}{N} \int_0^{\sqrt{N}} f(x)\,dx + \int f(x) \chi_{[\sqrt{N},N]}$$

Notice $|f(x) \chi_{[\sqrt{N},N]}| \leq |f(x)| \in L^1$

and $f(x) \chi_{[\sqrt{N},N]} \longrightarrow 0$ a.e

$\Rightarrow$ By DCT $\int f(x) \chi_{[\sqrt{N},N]} \longrightarrow \int 0 = 0$

Secondly $\int_0^{\sqrt{N}} f(x) < \infty$ since $f \in L^1$

$\Rightarrow \frac{1}{N} \int_0^{\sqrt{N}} f(x)\,dx + \int f(x) \chi_{[\sqrt{N},N]} \longrightarrow 0$ as $N \to \infty$

$\square$

6. Give example of compact set in $\mathbb{R}$ w/ strictly positive Lebesgue measure and empty interior.

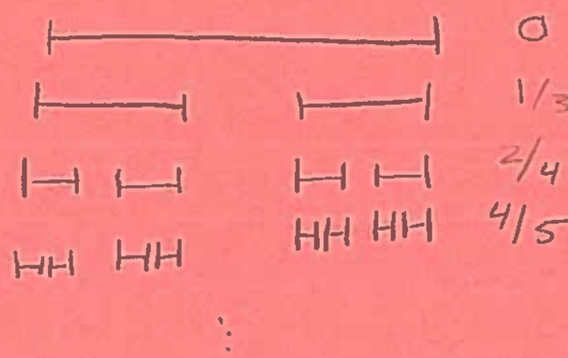
Pf Let $C_0 = [0,1]$.

$C_1 = C_0$ - middle $1/3$

$C_2 = C_1$ - middle $1/4$s

$C_3 = C_1$ - middle $1/5$s

$\vdots$



as

• Let $C = \bigcap_0^\infty C_i$

$|C| = 1 - \sum_{i=3}^{\infty} \frac{2^{i-3}}{i!}$

$= 1 - \frac{1}{8} \sum_{i=3}^{\infty} \frac{2^i}{i!}$

$\geq 1 - \frac{1}{8} \sum_0^\infty \frac{2^i}{i!}$

$= 1 - \frac{1}{8} e^2$

$\Rightarrow 0 < |C| < 1$

• C is compact since closed and bdd.

• C has empty interior as can be shown by same construction as in cantor set

$\square$

MAT 701 Real Variables I

Final Exam

December 10. 2014

Choose 4 out of the following 8 problems. Only 4 problems will be graded.

1. (a) Let $E \subset \mathbb{R}$ have measure zero. Show that $E \times \mathbb{R}$ has measure zero in $\mathbb{R}^2$.

   (b) Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $|A|$.

2. We say that the sets $A, B \subset \mathbb{R}^n$ are congruent if $A = z + B$ for some $z \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be measurable such that $0 < |E| < \infty$. Suppose that there exists a sequence of disjoint sets $\{E_i\}$, $i = 1, \ldots$ such that for all i, j, E_i and E_j are congruent, and $E = \cup_{j=1}^{\infty} E_j$. Prove that all the E_j's are non-measurable.

3. If $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies a Lipschitz condition, then we know that it sends measurable sets to measurable sets. With the same definition of the Lipschitz condition, is it true that if $f: \mathbb{R}^3 \to \mathbb{R}^2$ satisfies a Lipschitz condition, then it sends (three-dimensionally) Lebesgue measurable sets to (2-dimensionally) Lebesgue measurable sets?

4. Let $f \in L^2(0, \infty)$. Prove that

   (a) $\left|\int_0^x f(t)\, dt\right| \leq x^{\frac{1}{2}} ||f||_{L^2(0,\infty)}$ and

   (b) $\lim_{x\to\infty} x^{-\frac{1}{2}} \int_0^x f(t)\, dt = 0$.

5. Suppose that $p > 1$, $E \subset \mathbb{R}^n$ with $|E| < \infty$ and that f is measurable on E. Show that if there exist $C \geq 1$ and $T \geq 1$ such that

$$|\{x \in E \colon |f(x)| > t\}| \leq \frac{C}{t^p} \qquad \text{for all } t \geq T$$

   then $|f| \in L^q(E)$ for any $q \in [1, p)$.

6. Prove that if $0 < \varepsilon < 1$, there is no measurable subset E of $\mathbb{R}$ that satisfies

$$\varepsilon < \frac{|E \cap I|}{|I|} < 1 - \varepsilon$$

   for every interval I in $\mathbb{R}$.

7. If $f \in L^1[0,1]$ and $a > 0$, show that the integral

$$F_a(x) = \int_{[0,x]} (x-t)^{a-1} f(t)\, dt$$

exists for almost every $x \in [0,1]$, and that $F_a \in L^1[0,1]$.

8. Let f be a real valued function on the interval $I = [a,b]$.

   (a) Give the definition of absolute continuity for f on I.

   (b) Suppose f is absolutely continuous on I. **True or False.** If false give a counterexample. Do not prove if true.

      i. f is uniformly continuous on I.

      ii. f is differentiable at every x in the interior of I.

      iii. $f' \in L^1(I)$ and $f(x) - f(a) = \int_{[a,x]} f'(t)\, dt$, for $a \leq x \leq b$.

      iv. $|\{y = f(x) \colon f'(x) = 0\}| = 0$.

Final 2014

1. a) Let E have measure 0 show $E \times \mathbb{R} \subset \mathbb{R}^2$ has measure 0

b) Let $A \subset [0,1]$ which consists of all # which do not have digit 4 in decimal expansion. Find $|A|$

Pf a) Assume $|E| = 0$. and $\varepsilon > 0$ and $M > 0$

$\Rightarrow E \subset \bigcup_1^n (a_k, b_k)$ s.t. $|b_j - a_j| < \varepsilon / 2^{j+1}M$

$\Rightarrow E \times \mathbb{R} \subset \bigcup_1^n (a_k, b_k) \times \mathbb{R}$

$= \bigcup_1^n ((a_k, b_k) \times \mathbb{R})$

$= \lim_{M \to \infty} \bigcup_1^n (a_k, b_k) \times [-M, M]$

$$\Rightarrow \sum_1^n |(a_k, b_k) \times [-M, M]|$$

$$= \sum_1^n |(b_k - a_k) \cdot 2M|$$

$$< \sum_1^\infty |(b_k - a_k) 2M|$$

$$\leq \sum_1^\infty \frac{\varepsilon 2M}{2^{k+1} M}$$

$$= \sum_1^\infty \frac{\varepsilon}{2^k}$$

$$= \varepsilon$$

Let $M \to \infty$

$\Rightarrow |E \times \mathbb{R}| < \varepsilon$

Let $\varepsilon \to 0$

$\Rightarrow |E \times \mathbb{R}| = 0$

$\square$

b. Let $A_0 = [0, 1]$

$A_1 = [0, .4) \cup [.5, 1]$

$A_2 = [0, .04) \cup [.05, .14) \cup [.15, .24) \cup [.25, .34) \cup [.34, .4)$

$\vdots \qquad \cup [.5, .54) \cup [.55, .64) \cup [.65, .74) \cup \dots$

Let $A = \bigcap_1^\infty A_i$

$\Rightarrow |A| = 1 - \sum_1^\infty \frac{9^{i-1}}{10^i}$

$= 1 - \frac{1}{9} \sum_1^\infty \left(\frac{9}{10}\right)^i$

$= 1 - \frac{1}{9}\left(\frac{1}{1 - 9/10} - 1\right)$

$= 1 - \frac{1}{9}\left(\frac{1}{1/10} - 1\right)$

$= 1 - \frac{1}{9}(10 - 1)$

$= 1 - 1$

$= 0$

$\square$

2. We say that the sets $A, B \subset \mathbb{R}^n$ are congruent if $A = x + B$ for some $x \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be removable s.t. $0 < |E| < \infty$. Suppose that $\exists$ a sequence of disjoint sets $\{E_i\}$ s.t. $\forall\, i,j$ E_i and E_j are congruent and $E = \bigcup_1^\infty E_j$. Prove all the E_j's are nonmsble.

Pf Assume $\exists i$ s.t. E_i is msble

$\Rightarrow E_i$ is msble $\forall i$ since E_n is a translation of E_i

$\Rightarrow |E_i| = |E_j|$ since measures are translation invariant

if $|E_i| = 0$ then $|\bigcup_1^\infty E_j| = \sum |E_j| = 0$ ↯

if $|E_i| = c > 0$ then $|\bigcup_1^\infty E_j| = \sum |E_j| = \sum c = \infty$ ↯

$\therefore$ E_i is non msble

$\square$

3. If $f:\mathbb{R}^n \to \mathbb{R}^m$ satisfies a Lipschitz condition then it send msble sets to msble Sets. With same def of Lipschitz. Condition then is it true that if $f:\mathbb{R}^3 \to \mathbb{R}^2$ is Lipschitz then it sends 3D msble sets to 2D Lebesque msble sets.

Pf <u>Not true</u>

Let $N \subset \mathbb{R}^2$ be non measurable.

then $|[0,1] \times N|_3 = 0$. So its measurable

Now consider $f(x,y,z) = (y,z)$

$\Rightarrow f$ satisfies a Lipshits condition.

However $f([0,1] \times N) = N$ which is non msble.

So f sends a measurable set to a non measurable set

$\square$

4. Let $f \in L^2(0,\infty)$. Prove:

(a) $\left| \int_0^X f(t)\,dt \right| \le X^{1/2} \|f\|_2$

(b) $\lim_{X \to \infty} X^{-1/2} \int_0^X f(t)\,dt = 0$.

Pf (a) $\left|\int_0^X f(t)\,dt\right| \le \int_0^X |f(t)|\,dt$

$\le \|f\|_2 \left| \int_0^X 1^2 \right|^{1/2}$

$= \|f\|_2\, X^{1/2}$

(b) Let $\varepsilon > 0$

$f \in L^2 \Rightarrow \exists M < \infty$ s.t. $\int_M^\infty |f|^2 < \varepsilon$

$f \in L^2 \Rightarrow f \in L^1$ on a finite measure space

$\Rightarrow \int_0^M |f(t)|\,dt < C_M < \infty$

$|X^{-1/2} \int_0^X f(t)\,dt| \le X^{-1/2} \int_0^X |f(t)|\,dt$

$= X^{-1/2} \int_0^M |f(t)|\,dt + X^{-1/2} \int_M^X |f(t)|\,dt$

$\le X^{-1/2} C_M + X^{-1/2} \left| \int_M^X |f(t)|^2 \right|^{1/2} X^{1/2}$

$= X^{-1/2} C_M + \varepsilon^{1/2}$

$\to \varepsilon^{1/2}$ as $X \to \infty$

$\Rightarrow \lim_{X \to \infty} X^{-1/2} \int_0^X f(t)\,dt = 0$

$\square$

5. Suppose $p>1$ $E\subset\mathbb{R}^n$ with $|E|<\infty$ and f msble on E Show if $\exists C\geq 1,\ T\geq 1$ s.t. $|\{x\in E : |f(x)|>t\}|\leq \frac{C}{t^p}$ $\forall t\geq T$ then $|f|\in L^q(E)$ $\forall q\in[1,p)$.

pf Let $E_t=\{x\in E: |f(x)|>t\}$

$$\|f\|_q^q=\int_E |f|^q$$

$$=q\int_0^\infty t^{q-1}|E_t|\,dt$$

$$=q\int_0^T t^{q-1}|E_t|\,dt+q\int_T^\infty t^{q-1}|E_t|\,dt$$

$$\leq q\int_0^T t^{q-1}|E_t|\,dt+q\int_T^\infty t^{q-1}\frac{C}{t^p}\,dt$$

$$=q\int_0^T t^{q-1}|E_t|\,dt+Cq\int_T^\infty t^{q-p-1}\,dt$$

$$\leq q|E|\int_0^T t^{q-1}\,dt+Cq\int_T^\infty t^{q-p-1}\,dt$$

$$=q|E|\frac{t^q}{q}\Big|_0^T+Cq\frac{t^{q-p}}{q-p}\Big|_T^\infty$$

$$=|E|T^q-Cq\frac{T^{q-p}}{q-p}\quad \text{since } q-p \quad \lim_{t\to\infty}\frac{t^{q-p}}{q-p}=0$$

$<\infty$

$\Rightarrow f\in L^q$ $\forall q\in[1,p)$.

$\square$

6. Prove that if $0<\varepsilon<1$ there is no measurable subset E of $\mathbb{R}$ that satisfies $\varepsilon < \frac{|E\cap I|}{|I|} < 1-\varepsilon$ $\forall$ interval I in $\mathbb{R}$.

Pf By Lebesgue Differentiation Thm

$$\frac{1}{|Q|}\int_Q |f| \to f(x) \quad \text{as } |Q| \to \{x\} \quad \text{for a.e. } x$$

Let $Q = I$ and $f = \chi_E$

$$\Rightarrow \frac{1}{|Q|}\int_Q |f| = \frac{1}{|I|}\int_I \chi_E$$

$$= \frac{1}{|I|}\int \chi_{E\cap I}$$

$$= \frac{|E\cap I|}{|I|}$$

$$\Rightarrow \lim_{I\to\{x\}} \frac{|E\cap I|}{|I|} = \chi_E(x) = \begin{cases} 1 & x\in E \\ 0 & x\notin E. \end{cases}$$

$\Rightarrow$ No matter what E we choose $\exists I$ s.t. $\frac{|E\cap I|}{|I|} < \varepsilon$ or $\frac{|E\cap I|}{|I|} > 1-\varepsilon$ depending on what $\{x\}$ we choose to shrink to.

$\square$

7 If $f \in L'[0,1]$ and $a > 0$ show the integral

$$F_a(x) = \int_{[0,x]} (x-t)^{a-1} f(t)\, dt$$

exists for almost every $x \in [0,1]$ and $F_a \in L'[0,1]$.

<u>Pf</u> $|F_a(x)| = \left| \int_{[0,x]} (x-t)^{a-1} f(t)\, dt \right|$

$\leq \int_0^x |(x-t)|^{a-1} |f(t)|\, dt$

$\leq \int_0^1 |1-t|^{a-1} |f(t)|\, dt$ since $x \in [0,1]$

$\leq \int_0^1 |f(t)|\, dt$ since $t < 1$

$= \|f\|_1$

$< \infty$ $\Rightarrow$ $F_a(x)$ exist for a.e x.

$\|F_a\| = \int_0^1 \left| \int_0^x (x-t)^{a-1} f(t)\, dt \right| dx$

$\leq \int_0^1 \int_0^x |x-t|^{a-1} |f(t)|\, dt\, dx$

$= \int_0^1 \int_0^t |x-t|^{a-1} |f(t)|\, dx\, dt$ by Fubini

$= \int_0^1 |f(t)| \int_0^t |x-t|^{a-1}\, dx\, dt$

$= \int_0^1 |f(t)| \int_0^t (t-x)^{a-1}\, dx\, dt$

$= \int_0^1 |f(t)| \left. -\frac{(t-x)^a}{a} \right|_0^t dt$

$= \int_0^1 |f(t)| \frac{t^a}{a}\, dt$

$\leq \int_0^1 \frac{f(t)}{a}\, dt$ since $t < 1$

$= \frac{\|f\|}{a} < \infty$

$\square$

8. Let f be a real valued function on $I = [a,b]$

a. Give definition of absolute continuity for f on I

b. Suppose f abs cont on I. True or false.

i. f is uniformly cont.

ii. f is differentiable at every $x \in I^\circ$

iii. $f' \in L^1(I) \Rightarrow f(x) - f(a) = \int_{[a,x]} f'(t)\,dt \quad a \le x \le b$

iv. $|\{y = f(x) : f'(x) = 0\}| = 0$

Pf a) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\sum |b_j - a_j| < \delta \Rightarrow \sum |f(b_j) - f(a_j)| < \varepsilon$

b.) i) True $n=1$ in definition.

ii) Consider $|x|$ on $[-1,1]$.

Let $\varepsilon > 0$, Let $\delta = \varepsilon$ and $\sum |b_j - a_j| < \delta$

$$\Rightarrow \sum |f(b_j) - f(a_j)| = \sum \big||b_j| - |a_j|\big|$$
$$< \sum \big||b_j - a_j + a_j| - |a_j|\big|$$
$$\le \sum \big||b_j - a_j| + |a_j| - |a_j|\big|$$
$$= \sum |b_j - a_j|$$
$$< \varepsilon$$

but $|x|$ is not differentiable at $0 \in [-1,1]$

iii) true

iv) true.

if not in AC consider Cantor function as a counter example.

$\square$

Named Theorem Proofs

Dominated Convergence Thm

Let $\{f_n\}$ be a sequence in $\mathcal{L}^1$ s.t.

(a) $f_n \to f$ a.e

(b) $\exists g \in \mathcal{L}^1$ s.t. $|f_n| \leq g$ a.e $\forall n$

$\Rightarrow f \in \mathcal{L}^1$ and $\int f = \lim \int f_n$.

Pf f is measurable (possibly after redefinition on nullset)

$|f_n| \leq g \Rightarrow -g \leq f_n \leq g$

$\Rightarrow f_n + g \geq 0 \qquad g - f_n \geq 0$

$$\begin{aligned}\int f + \int g &= \int \lim f_n + g \\ &\leq \underline{\lim} \int f_n + g \\ &= \underline{\lim} \int f_n + \int g\end{aligned} \qquad \Rightarrow \int f \leq \underline{\lim} \int f.$$

$$\begin{aligned}\int g - \int f &= \int \lim g - f_n \\ &\leq \underline{\lim} \int g - f_n \\ &= \int g - \underline{\lim} \int f_n\end{aligned} \qquad \Rightarrow -\int f \leq -\overline{\lim} \int f_n \Rightarrow \int f \geq \overline{\lim} \int f_n$$

$\Rightarrow \overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n$.

$\Rightarrow \lim \int f_n = \int f \qquad \square$

Monotone Convergence Thm

If $\{f_n\}$ is a sequence in L^+ s.t. $f_j \le f_{j+1}$ $\forall j$ and $f = \lim f_n$ then $\int f = \lim \int f_n$

Pf $\{f_n\}$ increasing $\Rightarrow$ $\{\int f_n\}$ increasing.

$\int f_n \le \int f$ $\forall n$ $\Rightarrow$ $\lim \int f_n \le \int f$.

Let $\alpha \in (0,1)$.

Let s_j be a simple function s.t $s_j \nearrow f$.

Let $E_{nj} = \{x : f_n(x) \geq \alpha s_j(x)\}$

$\Rightarrow \bigcup_{n=1}^{\infty} E_{nj} = X$ and $\{E_{nj}\}$ is increasing sequence.

$\Rightarrow \int f_n \geq \int_{E_{nj}} f_n \geq \alpha \int_{E_{nj}} s_j \xrightarrow[n\to\infty]{} \alpha \int s_j \xrightarrow[j\to\infty]{} \alpha \int f \xrightarrow[\alpha \to 1]{} \int f.$

$\Rightarrow \lim \int f_n \geq \int f$

$\therefore \lim f_n = \int f$

$\square$

Fatou's Lemma

If $\{f_n\}$ is any sequence in L^+ then $\int \underline{\lim} f_n \leq \underline{\lim} \int f_n$

Pf $\forall k \geq 1$ $\inf_{n \geq k} f_n \leq f_j$ for $j \geq k$.

$\Rightarrow \int \inf_{n \geq k} f_n \leq \int f_j$ for $j \geq k$

$\Rightarrow \int \inf_{n \geq k} f_n \leq \inf_{j \geq k} \int f_j$

$\Rightarrow \int \liminf f_n = \lim_{k \to \infty} \int \inf_{n \geq k} f_n \leq \liminf \int f_n$

By MCT with $k \to \infty$

$\square$

Egoroff's Thm

Suppose $\mu(X) < \infty$ and $f_1, f_2, \ldots$ and f are measurable complex valued functions on X s.t. $f_n \to f$ a.e.
Then $\forall \varepsilon > 0$ $\exists E \subset X$ s.t. $\mu(E) < \varepsilon$ and $f_n \xrightarrow{u} f$ on E^c

Pf WLOG assume $f_n \to f$ everywhere on X

Let $E_n(k) = \bigcup_{m=n}^{\infty} \{x : |f_m(x) - f(x)| \geq \frac{1}{k}\}$

Show $\mu(E_n(k)) \to 0$

For fixed k, $E_n(k) \searrow$ as $n \nearrow$ and $\bigcap_{n=1}^{\infty} E_n(k) = \emptyset$

$\Rightarrow \mu(E_n(k)) \to 0$ as $n \to \infty$ since $\mu(X) < \infty$

Given $\varepsilon > 0$ and $k \in \mathbb{N}$.

Choose n_k s.t. $\mu(E_{n_k}(k)) < \varepsilon 2^{-k}$.

Let $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$

$\Rightarrow \mu(E) = \mu(\bigcup_1^\infty E_{n_k}(k)) \leq \sum \mu(E_{n_k}(k)) < \sum \varepsilon 2^{-k} = \varepsilon$

$\Rightarrow |f_n(x) - f(x)| < k^{-1}$ for $n \geq n_k$ and $x \notin E$

$\Rightarrow f_n \to f$ uniformly on E^c

$\square$

Hölders Inequality

Suppose $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. If f,g msble on X

then $\|fg\|_1 \le \|f\|_p \|g\|_q$ equality $\Leftrightarrow$ $\alpha|f|^p=\beta|g|^q$

Pf The result is trivial if $\|f\|_p=0$ or $\|g\|_q=0$ $(\Rightarrow f$ or $g=0$ ae$)$

Similarly trivial if $\|f\|_p=\infty$ or $\|g\|_q=\infty$

Note: $\|fg\|_1 \le \|f\|_p\|g\|_q$

$\Rightarrow \|af\,bg\|_1 \le \|af\|_p\|bg\|_q$

$\Rightarrow$ it suffices to show claim holds for $\|f\|_p=1=\|g\|_q$

with equality $\Leftrightarrow |f|^p=|g|^q$ a.e.

we have $a^{\lambda}b^{1-\lambda} \le \lambda a+(1-\lambda)b$

$\Rightarrow |f(x)g(x)| \le \frac{|f(x)|^p}{p}+\frac{|g(x)|^q}{q}$ $\quad \lambda=\frac{1}{p}$ $\quad a=|f(x)|^p$, $b=|g(x)|^q$

$\Rightarrow \int|f(x)g(x)| \le \int\frac{|f(x)|^p}{p}+\int\frac{|g(x)|^q}{q}$

$\Rightarrow \|fg\|_1 \le \frac{1}{p}\int 1+\frac{1}{q}\int 1$ $\quad$ since $|f|^p=|g|^q=1$

$\Rightarrow \|fg\| \le \frac{1}{p}+\frac{1}{q}=1=\|f\|_p\|g\|_q$

$\square$

Minkowskis Thm

If $1 \leq p < \infty$ and $f, g \in L^p$ then $\|f+g\|_p \leq \|f\|_p + \|g\|_g$

Pf $\underline{p=1}$ $\Rightarrow$ triangle inequality ✓

$\underline{f+g=0}$ $\Rightarrow$ trivial ✓

$\underline{\text{otherwise}}$ $|f+g|^p = |f+g|\,|f+g|^{p-1}$

$$\leq (|f|+|g|)\,|f+g|^{p-1} \quad \text{by triangle inequality}$$

$$\Rightarrow \int |f+g|^p = \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1}$$

$$\leq \|f\|_p \, \||f+g|^{p-1}\|_q + \|g\|_p \, \||f+g|^{p-1}\|_q$$

$$= (\|f\|_p + \|g\|_p)\, \||f+g|^{p-1}\|_q$$

$$= (\|f\|_p + \|g\|_p) \left(\int |f+g|^{(p-1)q}\right)^{1/q}$$

$$= (\|f\|_p + \|g\|_p) \left(\int |f+g|^p\right)^{1/q}$$

$$\therefore \|f+g\|_p = \left(\int |f+g|^p\right)^{1-1/q} \leq \|f\|_p + \|g\|_p$$

$\square$

Lusins Thm

Suppose $f: E \to \mathbb{R}$ is measurable.
$\Rightarrow \forall \varepsilon > 0$ $\exists$ closed F_ε s.t. $F_\varepsilon \subset E$ and $|E - F_\varepsilon| < \varepsilon$
and $f|_{F_\varepsilon}$ is continuous.

Pf Suppose f is a simple function, $f = \sum_1^N a_j \chi_{E_j}$
$\Rightarrow \exists$ closed sets $F_j \subset E_j$ s.t. $|E_j - F_j| < \varepsilon/N$ with $\chi_{E_j}|_{F_\varepsilon}$ cont.
Let $F_\varepsilon = \bigcup_1^N F_j$
$\Rightarrow F_\varepsilon$ is closed and $|E - F_\varepsilon| \leq \sum_1^N |E_j - F_j| < \varepsilon$

Suppose f is measurable, $F: E \to \infty$ with $|E| < \infty$
$\Rightarrow \exists\, S_n$ simple $S_n \nearrow f$.
$\Rightarrow \exists$ closed sets $A_\varepsilon \subset E$ s.t. $S_n \to f$ uniformly on A_ε
and $|E - A_\varepsilon| < \varepsilon/2$ by Egoroff.
$\Rightarrow \exists$ closed $F_k \subset E$ s.t. $|E - F_k| < \varepsilon/2^k$ w/ $S_k|_{F_k}$ cont by above.
Let $F_\varepsilon = A_\varepsilon \cap (\bigcap_1^\infty F_k)$ closed and $f|_{F_\varepsilon}$ cont
and $|E - F_\varepsilon| \leq \underbrace{|E - A_\varepsilon|}_{< \varepsilon/2} + \sum \underbrace{|E - F_k|}_{< \varepsilon/2^k} < \varepsilon$

Suppose $|E| = \infty$
Write $E = \bigcup_1^\infty \underbrace{E \cap \{x : k-1 \leq |x| < k\}}_{:= E_k}$
$\Rightarrow \exists$ closed $F_k \subset E_k$ s.t. $f|_{F_k}$ is cont. and $|E_k - F_k| < \varepsilon/2^k$
$\Rightarrow f|_F$ is cont where $F = \bigcup_1^\infty F_k$.

$\square$

Minkowski's Inequality for Integrals

If $f \geq 0$, $1 \leq p < \infty$ then $\left[\int \left(\int f(x,y)\, d\nu(x)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y)^p\, d\mu\right]^{1/p} d\nu$

Pf For $p=1$ the claim is merely Torelli/Fubini.

$1 < p < \infty$

$$\int \left(\int f(x,y)\, d\nu(x)\right)^p d\mu(x)$$
$$= \int \left|\int f(x,y)dx\right|^{p-1} \left|\int f(x,y)dx\right| dy$$
$$\leq \int \left|\int f(t,y)dt\right|^{p-1} \left|\int f(x,y)dx\right| dy$$
$$= \int\int \left|\int f(t,y)dt\right|^{p-1} f(x,y)\,dx\, dy$$
$$= \int\int \left|\int f(t,y)dt\right|^{p-1} f(x,y)\,dy\, dx$$
$$\leq \int \left[\int \left|\int f(t,y)dt\right|^{pq-q} dy\right]^{1/q} \left[\int |f(x,y)|^p dy\right]^{1/p} dx$$
$$= \int \left[\int \left|\int f(t,y)dt\right|^{p}\right]^{1/q} \left[\int |f(x,y)|^p dy\right]^{1/p} dx$$
$$\leq \left[\int \left|\int f(t,y)\,dt\right|^p dy\right]^{1/q} \int \left[\int |f(x,y)|^p dy\right]^{1/p} dx$$

$$\Rightarrow \frac{\int \left(\int f(x,y)\, d\nu(x)\right)^p dy}{\left[\int \left|\int f(t,y)\,dt\right|^p dy\right]^{1/q}} \leq \int \left[\int |f(x,y)|^p dy\right]^{1/p} dx$$

$$\Rightarrow \left[\int \left(\int f(x,y)\, d\nu(x)\right)^p\right]^{1/p} \leq \int \left[\int |f(x,y)|^p dy\right]^{1/p} dx \quad \checkmark$$

$f \in L^p$
$\int |fg|\, d\mu \leq C\|g\|_q \;\; \forall g \in L^q$
then $\|f\|_p \leq C$

701 2014 HW 1

1 Prove De Morgan's Laws:

a) $\left(\bigcup_{E\in F} E\right)^c = \bigcap_{E\in F} E^c$

b) $\left(\bigcap_{E\in F} E\right)^c = \bigcup_{E\in F} E^c$

Pf a) $e \in \left(\bigcup_{E\in F} E\right)^c \iff e \notin \bigcup E$
$\iff e \notin E$ for any E
$\iff e \in E^c \;\forall E$
$\iff e \in \bigcap E^c$
$\therefore (\bigcup E)^c = \bigcap E^c$

b) $e \in (\bigcap E)^c \iff e \notin \bigcap E$
$\iff e \notin E$ for some E
$\iff e \in E^c$ for some E
$\iff e \in \bigcup E^c$
$\therefore (\bigcap E)^c = \bigcup E^c$ $\square$

2. $L = \limsup a_k \iff \exists \underbrace{\{a_{k_j}\} \to L}_{(i)}$ and $\underbrace{\text{if } L' > L, \exists k \text{ s.t. } a_k < L'}_{(ii)}$

Pf ($\Rightarrow$) Assume $L = \overline{\lim}\, a_k$
$\Rightarrow L = \lim_{j\to\infty} \{\sup_{k\geq j} a_k\}$
$\Rightarrow \exists a_{k_n}$ s.t. $|a_{k_n} - L| < 1/2^n$
$\Rightarrow \{a_{k_n}\} \subset \{a_k\}$ s.t. $a_{k_n} \to L$

Now assume BWOC that for $L' > L$ $\forall k$ $\exists k' > k$ s.t. $a_k \geq L'$ Then $\sup_{k\geq j} a_k \geq L'$ $\forall j$
$\Rightarrow \overline{\lim}\, a_k \geq L'$ but that contradicts since $\overline{\lim}\, a_k = L$

($\Leftarrow$) Assume $\exists a_{k_j} \to L$ and $L' > L$ $\exists k$ s.t. $a_k < L'$
if (i) holds then $\overline{\lim}\, a_k$ converges since $\{\sup_{k\geq j} a_k\}_j$ is a decreasing sequence
if (ii) holds then $\overline{\lim}\, a_k \leq L$ $\square$

3. Let E be relatively open wrt interval I. Show E can be written as a countable union of non overlapping intervals.

Pf Let E be relatively open wrt I

$\Rightarrow E = I \cap G$ for some open set G

$\Rightarrow E = I \cap (\bigcup_1^\infty I_n)$ since open sets can be written as countable union of disjoint intervals

$\Rightarrow E = \bigcup_1^\infty \underbrace{(I_n \cap I)}_{\text{intervals}}$

$\square$

4. Give example of decreasing sequence of closed sets in $\mathbb{R}^n$ with empty intersection.

Pf Let $F_j = [j, \infty] \times \underbrace{[0,1] \times \cdots \times [0,1]}_{n-1 \text{ times}}$ for $j \in \mathbb{N}$

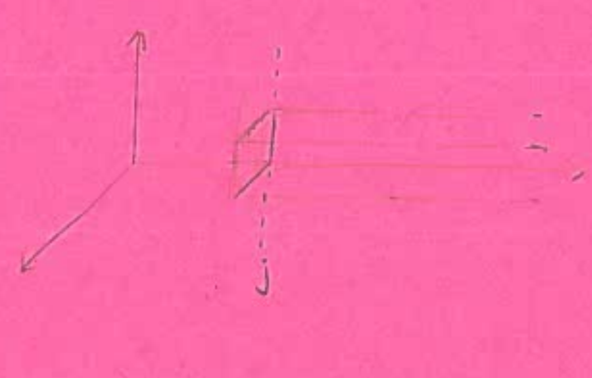
$\Rightarrow$ we get an interval from j to ∞ with side lengths of 1 in other variables

$\Rightarrow F_j$ are nested and decreasing, and as $j \to \infty$ $F_j \to \emptyset$

$\Rightarrow \bigcap F_j \to \emptyset$

Example in $\mathbb{R}^3$

$\square$



3. A sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ Converges <u>almost uniformly</u> to the msble fcn f $\iff$ $\forall \varepsilon > 0$ $\exists$ msble $E \subset \mathbb{R}$ s.t $|E| < \varepsilon$ and $f_n \to f$ uniformly on $\mathbb{R} \setminus E$. Give example of $f_n \to f$ ptwise almost everywhere but $f_n \not\to f$ a.u.

Pf Let $f_n = \chi_{[n,n+1]}$

$\Rightarrow f_n \to 0$ ptwise a.e

However let $\varepsilon > 0$. Let E be s.t. $|E| < \varepsilon$

$\Rightarrow \forall N \in \mathbb{N}, \exists x \in \mathbb{R} \setminus E$ s.t. $|f_N(x)| = 1 > \varepsilon$

$\Rightarrow f_n \overset{u}{\not\to} f$ on $\mathbb{R} \setminus E$

$\square$

4. Let $A, B \subset \mathbb{R}^n$ s.t. A is Lebesgue msble.
If $A \cap B = \phi$ show $|A \cup B|_e = |A| + |B|_e$.

Pf $|A \cup B|_e \le |A|_e + |B|_e = |A| + |B|_e$ always ✓

Now let $\mathcal{C}$ be a family of open sets s.t $A \cup B \subset \mathcal{C}$
Let U, V be open sets s.t. $A \subset U$ and $B \subset V$
Then $\mathcal{C}$ can be subdivided so that sets in $\mathcal{C}$ are either in V or U.

$$\Rightarrow |A \cup B|_e = \inf \{ \Sigma |\mathcal{C}| \}$$
$$= \inf \{ \Sigma |\mathcal{C} \cap V| + |\mathcal{C} \cap U| \}$$
$$\ge \inf \{ \Sigma |\mathcal{C} \cap V| \} + \inf \{ \Sigma |\mathcal{C} \cap U| \}$$
$$= |A|_e + |B|_e$$
$$= |A| + |B|_e$$

$\therefore |A \cup B|_e = |A| + |B|$

$\square$

5. Show a bdd f is Reimann Integrable on I
$\Leftrightarrow$ given $\varepsilon > 0$ $\exists$ a partition Γ of I
s.t. $0 < U_\Gamma - L_\Gamma < \varepsilon$.

Pf Assume f is bdd and Reimann Integrable

$\Rightarrow \exists$ partitions of I, Γ_1, Γ_2 s.t.

$U_{\Gamma_2} - \int f\,dx < \varepsilon/2$ and $\int f\,dx - L_{\Gamma_1} < \varepsilon/2$

Let Γ be the common refinement of Γ_1, Γ_2

$\Rightarrow U_\Gamma \le U_{\Gamma_2} < \int f\,dx + \varepsilon/2 < L_{\Gamma_1} + \varepsilon \le L_\Gamma + \varepsilon$

$\Rightarrow 0 < U_\Gamma - L_\Gamma < \varepsilon$

Now assume for $\varepsilon > 0$, $\exists \Gamma$ s.t. $0 \le U_\Gamma - L_\Gamma < \varepsilon$

$\Rightarrow \underline{\lim}\, U_\Gamma - \overline{\lim}\, L_\Gamma < \varepsilon$.

$\Rightarrow \underline{\lim}\, U_\Gamma = \overline{\lim}\, L_\Gamma$

$\Rightarrow f$ Riemann Integrable

$\square$

6. If $\{f_k\}$ is a sequence of bounded Riemann Integrable functions on interval I which converge Uniformly on I to f

Show $f \in RI$ and $(R)\int_I f_k(x)\,dx \to (R)\int_I f(x)\,dx$

Pf Let $\varepsilon_k = \sup|f_k(x) - f(x)|$ for $x \in I$

$\Rightarrow f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n$

$\Rightarrow \int(f_n - \varepsilon_n) \leq \sup L_P \leq \inf U_P \leq \int_I (f_n + \varepsilon_n)$

$\Rightarrow 0 \leq \inf U_P - \sup L_P \leq 2\varepsilon_n V(I)$

Now since $f_n \xrightarrow{u} f$ then $\varepsilon_n \to 0$ as $n \to \infty$

$\Rightarrow \inf U_P = \sup L_P$

$\Rightarrow f$ Reimann Integrable

Finally $\int_I f_n - \varepsilon_n \leq \int_I f \leq \int_I f_n + \varepsilon_n$

$\Rightarrow \left|\int_I f_n - \int_I f\right| = \left|\int_I (f_n - f)\right|$

$\leq \int_I |f_n - f|$

$\leq \int_I \varepsilon_n$

$< \varepsilon_n V(I)$

$\to 0$ as $n \to \infty$

$\square$

7. Prove Cantor set ^{C} is totally disconnected and perfect

Pf <u>Totally Disconnected</u>

Let $x, y \in C$ s.t. $x \neq y$ WLOG assume $x < y$

$\Rightarrow \exists k$ s.t. $|x-y| > 1/3^k$

Consider C_k as in usual construction of cantor set

$\Rightarrow$ each interval in C^k has length $1/3^k$

$\Rightarrow x, y$ are in distinct intervals

$\Rightarrow \exists z \notin C_k$ s.t. $x < z < y$

$\Rightarrow z \notin C$ and $x < z < y$ since $C = \cap_1^\infty C_k$

$\Rightarrow C$ is totally disconnected.

<u>Perfect</u>

We WTS C has no isolated points

Let $x \in C$ and let I be interval in $\mathbb{R}$ containing x

Let $I_k \subset C_k$ be interval containing x

$\Rightarrow \exists N$ s.t. $k > N \Rightarrow I_k \subset I$

Let x_k be endpt of I_k s.t. $I_k \neq x$ for $k > N$

$\Rightarrow x$ is limit point of x_k

Since each $x_k \in C$ x is not isolated

$\Rightarrow C$ has no isolated points

$\Rightarrow C$ is perfect.

$\square$

8. Note every # in $[0,1]$ has ternary expansion

$$x = \sum_{1}^{\infty} a_k 3^{-k} \quad a_k \in \{0,1,2\}$$

Prove $x \in C \iff x$ has representation above w/ $a_k = 0, 2$.

Pf ($\Rightarrow$) Let $x \in C$

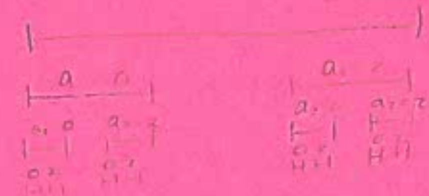
$\Rightarrow x = \sum_1^{\infty} a_k 3^{-k} \quad a_k \in \{0,1,2\}$

If $x \in [0, 1/3]$ let $a_1 = 0$, if $x \in [2/3, 1]$ let $a_1 = 2$

if $x \in [0, 1/9]$, $[2/3, 7/9]$ let $a_2 = 0$

if $x \in [2/9, 1/3]$ $[8/9, 1]$ let $a_2 = 2$

and so on



$$x = \sum_1^{\infty} a_k 3^{-k} \quad a_k \in \{0,2\}$$

($\Leftarrow$) Let x have rep above w/ $a_k \in \{0,2\}$

Use similar construction as above to show if $a_k \neq 1$ $\forall k$ then middle subintervals will always be avoided

$\Rightarrow x \in C$

$\square$

701 2014 HW 2

1 Construct 2-D cantor set in $[0,1]\times[0,1]$ by subdividing squares into 9 equal parts keeping only the 4 corners. Show its perfect, has measure 0 and equals $C\times C$

Pf Let S_0 = unit square, S_k = pts remaining after k steps

$\Rightarrow S = \bigcap_0^\infty S_k$

$\Rightarrow S^c = \bigcup_0^\infty S_k^c$

$\Rightarrow S$ closed since S_k closed

Let S_k^* be corners of squares of S_k.

$\Rightarrow \forall q_k \in S_k^*$ $q_k \in S$ since $S_k \in S_N \ \forall N \geq k$

$\Rightarrow \bigcup_0^\infty S_k^* \subset S$

Let $s \in S$

$\Rightarrow \forall k \geq 0 \ \exists$ square T_k s.t. $s \in T_k$

Let $q_k \in T_k$ s.t. $q_k \in S_k^*$ and $q_k \neq s$

$\Rightarrow$ As $k \to \infty$ $q_k \to s$ for each $q_k \in S$

$\Rightarrow s$ is limit pt in S.

$\Rightarrow S$ is perfect.

Each S_k is msble

$\Rightarrow S$ is msble

$|S_k| = 4^k 9^{-k}$

$\Rightarrow |S_k| = (4/9)^k \to 0$

$\Rightarrow |S| = 0$ since $S \subset S_k \ \forall k$

Finally we WTS $S = C \times C$ i.e. $S_k = S_{k-1} \subseteq S_k \; \forall k$

Proceed by induction on k.

$S_0 = C_0 \times C_0$ ✓.

Now assume it holds for k and show for $k+1$

Let $p = (x, y) \in S_{k+1}$

$\Rightarrow p \in T_{k+1}$ a square of S_k

$\Rightarrow x$ is is 1st or 3rd third of $\pi_x(T_k)$

$\Rightarrow x \in C_k$ ↳ projection onto x-axis

$\Rightarrow x \in C_{k+1}$

Similarly $y \in C_{k+1}$

$\Rightarrow p \in C_{k+1} \times C_{k+1}$

If $p \notin S_{k+1}$ then p is in open cross removed from S_k

$\Rightarrow x$ or y is in middle third of $\pi_x(S_k)$ or $\pi_y(S_k)$

$\Rightarrow p \notin C_{k+1} \times C_{k+1}$

$\Rightarrow S_k = C_k \times C_k \;\; \forall k$

$\Rightarrow S = \bigcap S_k$

$= \bigcap C_k \times C_k$

$\bigcap C_k \times \bigcap C_k$

$= C \times C$

$\square$

3. If $\{E_k\}$ is s.t. $\sum |E_k|_e < \infty$ show $\overline{\lim}\, E_k$ and $\underline{\lim}\, E_k$ have measure 0.

Pf Define $E = \overline{\lim}\, E_k$ and $F_j = \bigcup_j^\infty E_k$

$\Rightarrow F_1 \supset F_2 \supset \ldots$ and $\lim F_j = E$

$\Rightarrow E \subset \bigcap_n^\infty F_j \subset F_n \quad \forall n$

$\Rightarrow |E|_e \leq |F_n|_e$ by monotonicity

$\Rightarrow |F_n|_e = |\bigcup_{k=n}^\infty E_k| \leq \sum_n^\infty |E_k|_e$ by subadditivity

$\sum_1^\infty |E_j|_e < \infty$

$\Rightarrow \forall \varepsilon > 0, \ \exists N$ s.t. $\sum_N^\infty |E_j|_e < \varepsilon$

$\Rightarrow 0 \leq |E|_e \leq |F_N|_e < \varepsilon$

$\Rightarrow |E|_e = 0$

$\Rightarrow \overline{\lim}\, E_k$ msble w/ measure 0

$\underline{\lim}\, E_k \subset \overline{\lim}\, E_k$

$\Rightarrow |\underline{\lim}\, E_k|_e = 0$

$\square$

4. E a set, $\mathcal{O}_n \{x : d(x, E) < 1/n\}$

a) If E compact show $|E|_e = \lim |\mathcal{O}_n|_e$

b) (a) may be false if E closed but unbdd

Pf a) WTS $E = \bigcap_1^\infty \mathcal{O}_n$

$\subseteq$ Since $E \subset \mathcal{O}_n \; \forall n \Rightarrow E \subset \bigcap_1^\infty \mathcal{O}_n$

$\supseteq$ Let $x \in \bigcap_1^\infty \mathcal{O}_n$

$\Rightarrow d(x, E) < 1/n \;\; \forall n$

$\Rightarrow \exists x_n \in E$ with $d(x, x_n) < 1/n$

$\Rightarrow x_n \to x$

$\Rightarrow x \in E$ since E closed.

$\Rightarrow \bigcap_1^\infty \mathcal{O}_n \subset E$

$\therefore E = \bigcap_1^\infty \mathcal{O}_n$

E compact in $\mathbb{R}^n$

$\Rightarrow E$ bdd

$\Rightarrow \mathcal{O}_n$ bdd $\forall n$

$\Rightarrow |\mathcal{O}_n|_e < \infty \;\; \forall n$

$\mathcal{O}_{n+1} \subset \mathcal{O}_n \; \forall n \Rightarrow \lim \mathcal{O}_n = \bigcap_1^\infty \mathcal{O}_n$

$\Rightarrow |E|_e = |\bigcap_1^\infty \mathcal{O}_n|_e$

$= |\lim \mathcal{O}_n|_e$

$= \lim |\mathcal{O}_n|_e$ since $|\mathcal{O}_n|_e < \infty$

b) Let $E = \mathbb{Z}$, E closed + unbdd

$|E| = 0$ since $\mathbb{Z}$ countable

However $|\mathcal{O}_n| = \infty \;\; \forall n$

5. $J_*(E) = \inf\{\sum^N v(I_j) : E \subset \bigcup_1^N I_j\}$

(a) Prove $J_*(E) = J_*(\bar{E})$

(b) Give countable $E \subset [0,1]$ s.t. $J^*(\bar{E}) = 1$, $|E|_e = 0$

Pf a) $E \subset \mathbb{R}$ with $\bar{E} \subset \bigcup_1^N I_j$

$\Rightarrow E \subset \bar{E}$

$\Rightarrow E \subset \bigcup_1^N I_j$

$\Rightarrow J_*(E) \leq J_*(\bar{E})$

Let $E \subset S = \bigcup_1^N I_j$ WTS $\bar{E} \subset \bar{S}$

Suppose BWOC $\bar{E} \not\subset \bar{S}$

$\Rightarrow \exists x \in \bar{E}$ s.t. $x \notin \bar{S}$

$\Rightarrow \forall \varepsilon > 0$ $\exists (x-\varepsilon, x+\varepsilon) \cap E \neq \phi$ since $x \in \bar{E}$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \cap \bar{S} \neq 0$ since $E \subset \bar{S}$

$\Rightarrow x \in \bar{S}'$

$\Rightarrow x \in S$ since $\bar{S}$ is closed

$E \subset S \Rightarrow \bar{E} \subset \bar{S}$

$\Rightarrow J_*(E) \geq J_*(\bar{E})$ since $v(I_j) = v(\bar{I}_j)$

$\therefore J_*(E) = J_*(\bar{E})$

b) Let $E = \mathbb{Q} \cap [0,1]$

E dense in $[0,1]$ $\Rightarrow \bar{E} = [0,1]$

$\Rightarrow J_*(E) = J_*(\bar{E}) = 1$

Let $E = \{q_1, q_2, \ldots q_n\}$

$\Rightarrow |\{q_j\}|_e = 0 \quad \forall j$

$\Rightarrow |E|_e \leq \sum |\{q_j\}|_e = 0$

$\Rightarrow |E|_e = 0$

$\square$

701 2014 Hw 3

1. If E_1, E_2 msble show $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$

Pf If $|E_1 \cap E_2| < \infty$

$E_1 \cup E_2 = (E_1 \setminus (E_1 \cap E_2)) \cup (E_2 \setminus (E_1 \cap E_2)) \cup (E_1 \cap E_2)$

which are all disjoin

$\Rightarrow |E_1 \cup E_2| = |E_1 \setminus (E_1 \cap E_2)| + |E_2 \setminus (E_1 \cap E_2)| + |E_1 \cap E_2|$

$= |E_1| - |E_1 \cap E_2| + |E_2| - |E_1 \cap E_2| + |E_1 \cap E_2|$

$= |E_1| + |E_2| - |E_1 \cap E_2|$

$\Rightarrow |E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$

If $|E_1 \cap E_2| = \infty$

$\Rightarrow |E_1| = |E_2| = |E_1 \cup E_2| = \infty$ by Monotonicity

$\Rightarrow$ Claim holds w/ ∞ on both sides

$\square$

2 Let inner measure be $|E|_i = \sup\{|F| \mid F \subset E \text{ closed}\}$

Show a. $|E|_i \le |E|_e$

b. If $|E|_e < \infty$ then E msble $\iff |E|_i = |E|_e$

Pf a. Let $E \subset \mathbb{R}^n$

Let G be open, F closed s.t. $F \subset E \subset G$

$\Rightarrow |F| \le |G|$ for all such F, G

$\Rightarrow |E|_i \le |E|_e$

b. Assume $|E|_e < \infty$

Let E be msble and $\varepsilon > 0$

$\Rightarrow \exists$ closed F s.t. $|E - F| < \varepsilon$

$\Rightarrow |E|_e = |E \cap F|_e + |E - F|_e < |E \cap F|_e + \varepsilon$

$\Rightarrow |E \cap F|_e > |E|_e - \varepsilon$

$|E|_i \geq F$

$\Rightarrow |E|_i \geq |F| - \underbrace{|F - E|}_{=0} = |E \cap F|_e > |E|_e - \varepsilon$

$\Rightarrow |E|_e \geq |E|_i > |E|_e - \varepsilon$

$\Rightarrow |E|_i = |E|_e$ as $\varepsilon \to 0$

Now let $|E|_i = |E|_e$

$\Rightarrow \exists G$ of type G_δ and F of type F_σ s.t.
$F \subset E \subset G$ and $|F| = |G|$

$\Rightarrow E$ is msble since $E = F \cup (E - F)$
where $|E - F| = 0$ since $E - F \subset G - F$
and $|G - F|_e = 0$

$\square$

3. Prove outer measure is translation invariant
ie if $E_h = \{x+h : x \in E\}$ show $|E_h|_e = |E|_e$
If E msble show E_h msble.

Pf Fix $\varepsilon > 0$

$\exists$ an interval cover of E, $\{I_n\}$ s.t. $|E| + \varepsilon \geq \sum |I_n|$

$\Rightarrow \{(I_n)_h\}$ covers E_h

$\Rightarrow |(I_n)_h| = |I_n| \ \forall n$ since $(I_n)_h$ is also an interval

$\Rightarrow |E| + \varepsilon \geq \sum |I_n| = \sum |(I_n)_h| \geq |E_h|$

$\Rightarrow |E| \geq |E_h|$ as $\varepsilon \to 0$

Now $E = \{(x+h) - h : x \in E\} = (E_h)_{-h}$

$\Rightarrow |E_h| \geq |(E_h)_{-h}| = |E|$

$\therefore |E| = |E_h|$

Now assume E is msble

$\Rightarrow \exists$ open G s.t. $E \subset G$ and $|G - E|_e < \varepsilon$

$\Rightarrow G_h$ is an open set and $E_h \subset G_h$

$\Rightarrow G_h - E_h = (G - E)_h$

$\Rightarrow |G_h - E_h|_e = |(G-E)_h|_e = |G - E|_e < \varepsilon$

$\Rightarrow E_h$ is msble.

$\square$

4) Suppose $E \subset \mathbb{R}^n$ and $|E| < \infty$

Prove $\forall \varepsilon > 0$ $\exists$ compact K w/ $K \subset E$ and $|E - K| < \varepsilon$

Pf Fix $\varepsilon > 0$

E msble

$\Rightarrow \exists$ closed $F \subset E$ w/ $|E - F| < \varepsilon/2$

Case 1 E bdd

$\Rightarrow F$ bdd

$\Rightarrow F$ compact by Heine Borel

Case 2 E unbdd

Let $A_1 = \overline{B_1(0)}$ $A_2 = \overline{B_2(0)}$ ... $A_n = \overline{B_n(0)}$

$\Rightarrow F \cap A_n \to F$ as $n \to \infty$

$|E| < \infty \Rightarrow |F| < \infty$

$\Rightarrow \exists N$ s.t. $|F - F \cap A_N| < \varepsilon/2$

$$\begin{aligned}\Rightarrow |E - (F \cap A_N)| &= |E + F - F - F \cap A_N| \\ &\leq |E - F| + |F - F \cap A_N| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon\end{aligned}$$

$F \cap A_n$ is compact since it is closed and bdd

$\square$

5. Show $\exists$ closed A, B w/ $|A| = |B| = 0$ and $|A+B| > 0$

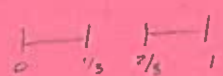
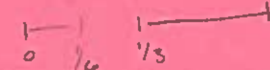
a) In $\mathbb{R}$ let $A = C$, $B = C/2$

b) In $\mathbb{R}^n$ let $A = [0,1] \times \{0\}$ $B = \{0\} \times [0,1]$

Pf a. In $\mathbb{R}$ consider $A = C$, $B = C/2$

$|C| = 0 \Rightarrow |C/2| = 0$

$A + B = \{x + y : x \in A, y \in B\}$

Consider C_1  $C_1/2$ 

$\Rightarrow [0,1] \subset C_1 + C_1/2$

Similarly we see $[0,1] \subset C_k + C_k/2$ $\forall k$.

$\Rightarrow [0,1] \subset C + C/2$

$\Rightarrow |[0,1]| < |C + C/2|$

$\Rightarrow 1 < |C + C/2|$

$\Rightarrow |A+B| > 0$

b. In $\mathbb{R}^n$ let $A = [0,1] \times \{0\}$ $B = \{0\} \times [0,1]$

$|A| = |B| = 0$

Since A can be coveed by interval $[0,1] \times [-\varepsilon, \varepsilon] = I$

$|I| = 2\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly for B.

Let $(x,y) \in [0,1] \times [0,1]$

$\Rightarrow (x,y) = (x,0) + (0,y) \in A + B$

$\Rightarrow [0,1] \times [0,1] \subset A + B$

$\Rightarrow |A+B| > 1 = |[0,1] \times [0,1]|$

$\Rightarrow |A+B| > 0$

$\square$

1. Suppose $A \subset E \subset B$, A, B msble, $|A| = |B| < \infty$. Prove E msble

Pf Fix $\varepsilon > 0$.

B msble $\Rightarrow \exists$ open G with $B \subset G$ and $|G - B| < \varepsilon/2$

A msble $\Rightarrow \exists$ closed F with $F \subset A$ and $|A - F| < \varepsilon/2$.

$\Rightarrow F \subset A \subset E \subset B \subset G$

$\Rightarrow |G - B| = |G| - |B| < \varepsilon/2$ since $|B| < \infty$

$\Rightarrow |A - F| = |A| - |F| < \varepsilon/2$ Since $F \subset A$ and $|A| < \infty$

$\Rightarrow |A| = |B| < \varepsilon/2 + |F|$

$\Rightarrow |G| - \varepsilon/2 < |B| = |A| < \varepsilon/2 + |F|$

$\Rightarrow |G| - |F| < \varepsilon$

$\Rightarrow |G - F| < \varepsilon$

$E \subset G \Rightarrow E - F \subset G - F$

$\Rightarrow |E - F|_e \leq |G - F|_e < \varepsilon$

$\Rightarrow \exists$ closed F w/ $F \subset E$ s.t. $|E - F|_e < \varepsilon$

$\Rightarrow E$ msble.

7. Let E be a measurable subset of $\mathbb{R}$ w/ $|E|>0$. Prove $\forall c<\alpha<1$ $\exists$ open I w/ $|E\cap I| \geq \alpha |I|$

Pf Fix $\varepsilon>0$.

E msble $\Rightarrow$ $\exists$ disjoint intervals $\{I_n\}$ s.t. if $A = \bigcup_1^N I_j$ and $|E-A|+|A-E| < \varepsilon |E|$ since $|E|>0$

$\Rightarrow |E| = |E\cap A| + |E-A| \leq \sum |I_n| + \varepsilon |E|$

$\Rightarrow (1-\varepsilon)|E| \leq \sum |I_n|$

Now $\sum |I_n| = \sum |I_n \cap E| + \sum |I_n - E|$

$< \sum |I_n \cap E| + \varepsilon |E|$

$\Rightarrow \sum |I_n| < \sum |I_n \cap E| + \varepsilon/_{1-\varepsilon} \sum |I_j|$

$\Rightarrow \sum |I_n \cap E| > (1 - \varepsilon/_{1-\varepsilon}) \sum |I_j|$

$\Rightarrow$ Choose ε small enough s.t. $\varepsilon/_{1-\varepsilon} < \alpha$

$\Rightarrow |A\cap E| > \alpha |A|$

Now $\sum |I_n \cap E| \geq \alpha \sum |I_j|$

$\Rightarrow \exists N$ s.t. $|I_N \cap E| \geq \alpha I_N$

Otherwise $|I_n \cap E| < \alpha |I_n|$ $\forall n$

$\Rightarrow \sum |I_n \cap E| < \sum \alpha I_n$ which contradicts

701 2014 HW 4

1. Give example which shows image of measurable set under continuous transformation need not be measurable.

Pf Let f be cantor function and C the cantor set.

$\Rightarrow f(C) = [0,1]$

$\Rightarrow |f(C)| = 1 > 0$

$\Rightarrow \exists$ nonmsble $A \in f(C)$

$\Rightarrow f^{-1}(A) \subset C$

$\Rightarrow 0 \leq |f^{-1}(A)| \leq |C| = 0$

$\Rightarrow f^{-1}(A)$ is msble

However $f(f^{-1}(A)) = A$ since f is continuous

$\Rightarrow f(f^{-1}(A))$ is non msble $\square$

2. Show $\exists$ disjoint $E_1, E_2, \ldots$ s.t. $|\cup E_k|_e < \sum |E_k|_e$

Pf Let $x \sim y$ if $x - y \in \mathbb{Q}$.

Let $r \in \mathbb{Q} \cap [0,1]$

Let E = {one rep from each equivalence class}

$E_r = \{x+r \mid x \in [0, 1-r]\} \cup \{x+r-1 \mid x \in (1-r, 1]\}$

$\Rightarrow |E_r|_e = |E|_e \quad \forall r \in \mathbb{Q}$

Since outer measure is translation invariant

Now $E_r \subset [0,1] \quad \forall r$

$\Rightarrow \cup E_r \subset [0,1]$

$\Rightarrow |\cup E_r|_e \le |[0,1]|_e = 1$

E not measurable

$\Rightarrow |E|_e \neq 0$

$\Rightarrow |E|_e > 0$

$\Rightarrow \sum_r |E_r|_e = \sum_r |E|_e = \infty$

$\therefore |\cup E_r|_e < \sum |E_r|_e$ $\square$

3. Let $Z \subset \mathbb{R}$ be s.t. $|Z| = 0$. Show $|\{x^2 : x \in Z\}| = 0$

$\underline{\text{Pf}}$ Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $f(x) = x^2$

On $[-n, n+1]$, $|f(x) - f(y)| = |x^2 - y^2|$

$= |(x-y)(x+y)|$

$= |n+1-n|\,|x-y$

$= |x-y|$

$\Rightarrow f$ is Lipschitz

$\Rightarrow f$ is msble on $[-n, n+1]$

$\Rightarrow \{x^2 : x \in Z\} = Z^2 \cap [-n, n+1]$ is msble.

$\Rightarrow Z^2 = \bigcup_0^\infty Z^2 \cap [-n, n+1]$ is msble

Since countable unions

Now $f(Z) = Z^2$ and $|f(Z)| = |Z| = 0$ on $[-n, n+1]$

$\Rightarrow |Z^2| = |\bigcup_0^\infty Z^2 \cap [-n, n+1]$

$\leq \sum_0^\infty |Z^2 \cap [-n, n+1]|$

$= \sum_0^\infty 0$

$= 0$

5. Give example of measurable set which is not Borel

Let $g = f(x) + x$ where f is cantor fcn

f strictly increasing

$\Rightarrow g$ strictly increasing

$\Rightarrow g$ 1-1

$g(0) = 0$, $g(1) = 2$

$\Rightarrow g$ onto

$\Rightarrow g$ bijective

$\Rightarrow g^{-1}$ cont

g continuous since f, g are.

Let C be cantor set

$\Rightarrow |g(C)| = 2 > 0$

$\Rightarrow \exists A$ non msble w/ $A \subset g(C)$

$\Rightarrow g^{-1}(A) \subset g^{-1}(g(C)) \subset C$

$\Rightarrow g^{-1}$ is Lebesgue msble since $|C| = 0$.

Assume BWOC $g^{-1}(A)$ is not Borel measurable is Borel measurable

$\Rightarrow (g^{-1})^{-1}(g^{-1}(A)) = g(g^{-1}(A)) = A$ is Borel msble $\Rightarrow$ Lebesgue msble

$\Rightarrow$ contradiction since

$\therefore g^{-1}(A)$ is not Borel msble but is Lebesgue msble

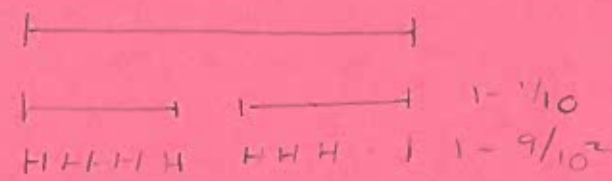
701 2014 HW 5

1. Let A be subset of $[0,1]$ which consists of all #'s w/ no 4 in their decimal expansion. Find $|A|$.

Pf Let $A_0 = [0,1]$

$A_1 = [0,.4) \cup [.5,1]$

$A_2 = [0,.04) \cup [.05,.14) \cup \ldots$



where $A_n = \{x \in [0,1] : \text{first } n \text{ digits are not } 4$

Let $A = \bigcap_0^\infty A_n$

$$|A| = 1 - \sum_1^\infty \frac{9^{k-1}}{10^k}$$

$$= 1 - \frac{1}{9} \sum_1^\infty (9/10)^k$$

$$= 1 - \frac{1}{9}\left(\frac{1}{1-9/10} - 1\right)$$

$$= 1 - \frac{1}{9}(10-1)$$

$$= 0.$$

$\square$

2. Let f be a simple fcn. $f = \sum_1^N a_j \chi_{E_j}$

Show f msble $\iff E_1, \ldots, E_N$ msble

Pf Assume f msble

$\Rightarrow f^{-1}(B)$ msble $\forall B \subset \mathbb{R}$.

$\Rightarrow f^{-1}(f(E_j))$ is msble

$\Rightarrow E_j$ msble.

Assume $E_1, \ldots, E_N$ msble

Let $B \subset \mathbb{R}$.

Let $I = \{i \in \mathbb{N} : a_i \in B\}$

$\Rightarrow f^{-1}(B) = \bigcup_{i \in I} E_i$ $\qquad \Rightarrow I = \{3, 5, 6\}$

$\Rightarrow f^{-1}(B)$ msble since E_i is and countable union.

$\square$

3. Suppose f, g real valued in $\mathbb{R}^n$. Let $F(x) = (f(x), g(x))$

Then F is msble if $F^{-1}(G)$ is msble $\forall$ open $G \subset \mathbb{R}^2$

Prove F msble iff f, g are

Pf Suppose F msble

$\Rightarrow F^{-1}((a,\infty) \times \mathbb{R}) = \{f > a\}$ and $F^{-1}(\mathbb{R} \times (a,\infty)) = \{g > a\}$ msble

$\Rightarrow f, g$ msble.

Suppose f, g msble

$\Rightarrow \forall a, b, c, d \in \mathbb{R}$ $\{a < f < b\}$ & $\{c < g < d\}$ msble.

Let G be open in $\mathbb{R}^2$

$\Rightarrow G$ is a countable union of closed rectangles

$\Rightarrow F^{-1}(G) = F^{-1}(\bigcup_1^\infty [a_k, b_k] \times [c_k, d_k]$

$= \bigcup F^{-1}([a_k, b_k] \times [c_k, d_k])$

$= \bigcup (\{a_k \le f \le b_k\} \cap \{c_k \le g \le d_k\})$

$\Rightarrow F^{-1}(G)$ is msble since its countable union of msble sets

$\Rightarrow F$ is msble $\qquad \square$

4. Let $\{f_k\}$ be msble fcns on msble E w/ $|E|<\infty$
If $|f_k(x)| \le M_x < \infty$ $\forall k$, $\forall x$ Show given $\varepsilon > 0$
$\exists$ a closed $F \subset E$ and $M < \infty$ s.t. $|E-F| < \varepsilon$
and $|f_k(x)| \le M$ $\forall k$ and $\forall x \in F$

Pf Let $E_m = \{f \le m\}$ for each m and $f = \sup f_k(x)$

$\Rightarrow E_m \nearrow E$ since f is finite valued

$\Rightarrow |E_m| \nearrow |E|$

$\Rightarrow |E - E_m| \to 0$ since $|E| < \infty$

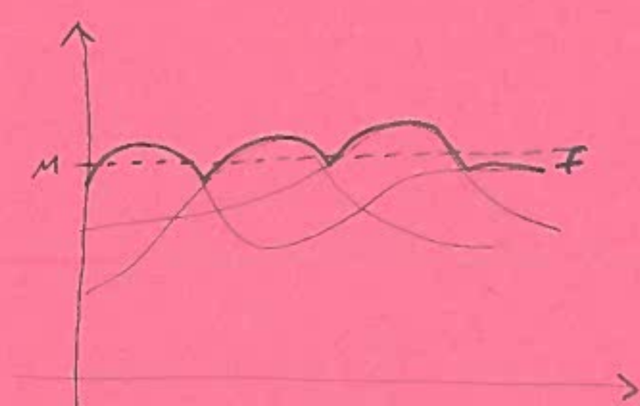
$\Rightarrow \forall \varepsilon > 0$, $\exists M$ s.t. $|E - E_M| < \varepsilon/2$

$\exists F \subset E_M$ s.t. $|E_M - F| < \varepsilon/2$ since E_m msble

$$\begin{aligned}\Rightarrow |E-F| &= |E - E_M + E_M - F| \\ &\le |E - E_M| + |E_M - F| \\ &< \varepsilon\end{aligned}$$

$\Rightarrow F \subset E_m \Rightarrow |f_k(x)| \le M$ $\forall k$, $\forall x$

$\square$



5. Give example to show $\phi(f(x))$ need not be msble even if ϕ and f are.

Pf Let F be Cantor function and $f = F^{-1}$

Let $\phi: \mathbb{R} \to \mathbb{R}$ be $\phi(x) = F(x) + x$

Let χ_A be s.t. A is msble but $\phi(A)$ is not

(one exists since $|\phi(C)| > 0$

φ is continuous, strictly increasing + bijective

$\Rightarrow \varphi^{-1}$ is continuous + strictly increasing

$\Rightarrow \varphi^{-1}$ is msble

$$\text{Now } (\chi_A \circ \varphi^{-1})^{-1}((.5, 1.5)) = (\chi_A \circ \varphi^{-1})^{-1}(\{1\})$$
$$= \varphi(\chi_A^{-1}(\{1\}))$$
$$= \varphi(A) \text{ nonmsble,}$$

$\Rightarrow \chi_A \circ \varphi^{-1}$ is not measurable

□

701 2014 HW 6

1. If f measurable on $[a,b]$ show given $\varepsilon > 0$ $\exists$ cont. g on $[a,b]$ s.t. $|\{x : f(x) \neq g(x)\}| < \varepsilon$

Pf Let $\varepsilon > 0$.
By Lusin's since f is msble $\exists$ closed F_ε s.t. $F_\varepsilon \subset [a,b]$ and $|[a,b] - F_\varepsilon| < \varepsilon$ w/ $f|_{F_\varepsilon}$ is cont

Let g be continuous on $[a,b]$ s.t. $g|_{F_\varepsilon} = f|_{F_\varepsilon}$

$\Rightarrow \{x : f(x) \neq g(x)\} \subset [a,b] - F_\varepsilon$

$\Rightarrow |\{x : f(x) \neq g(x)\}| < |[a,b] - F_\varepsilon| < \varepsilon$

$\square$

3. If f is a simple measurable fcn taking values a_j on E_j show $\int_E f = \sum_1^n a_j |E_j|$

Pf Let $E = \bigcup_k E_k$

$$\Rightarrow \int_E f = \sum_k \int_{E_k} f$$
$$= \sum_k \int_{E_k} a_k$$
$$= \sum_k a_k \int_{E_k} 1$$
$$= \sum_k a_k |E_k|$$

$\square$

4. Let $\{f_k\}$ be nonnegative msble on E. If $f_k \to f$ and $f_k \le f$ a.e show $\int_E f_k \to \int_E f$

Pf f_k nonnegative $\Rightarrow f$ nonnegative
f_k measurable $\Rightarrow \lim f_k = f$ is msble.
$\Rightarrow \int f$ exists since $f: E \to [0, \infty]$

Case 1 $\int f < \infty$
claim holds by DCT

Case 2 $\int f = \infty$
$\infty = \int f = \int \underline{\lim} f_k \le \underline{\lim} \int f_k$ by Fatou
$\Rightarrow \underline{\lim} \int f_k = \infty$
$\Rightarrow \int_E f_k \to \int_E f$ trivially $\square$

5. Let $f : \mathbb{R}^n \to [0, \infty)$ $\quad f(x) = \begin{cases} \frac{1}{|x|^{n+1}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Show $\exists\, C > 0$ s.t. $\forall \varepsilon > 0$ $\quad \int_{|x| \geq \varepsilon} f(x)\,dx \leq C/\varepsilon$

Pf Let $A_k = \{2^k \varepsilon \leq x \leq 2^{k+1}\varepsilon\}$ for fixed $\varepsilon > 0$

$$\Rightarrow \frac{1}{(2^{k+1}\varepsilon)^{n+1}} \chi_{A_k} \leq f \chi_{A_k} \leq \frac{1}{(2^k \varepsilon)^{n+1}} \chi_{A_k}$$

Let $A = B_2(0) - B_1(0)$ and $|A| = C/2$

$$\Rightarrow |A_k| = |2^k \varepsilon A| = (2^k \varepsilon)^n |A| = \frac{(2^k \varepsilon)^n C}{2}$$

Now $\int_{|x| \geq \varepsilon} f(x)\,dx = \int \sum_0^\infty f \chi_{A_k}$

$$\begin{aligned} \int \sum_0^N f \chi_{A_k} &\leq \int \sum_0^N \frac{1}{(2^k \varepsilon)^{n+1}} \chi_{A_k} \\ &= \sum_0^N \int \frac{1}{(2^k \varepsilon)^{n+1}} \chi_{A_k} \\ &= \sum_0^N \frac{1}{(2^k \varepsilon)^{n+1}} |A_k| \\ &= \sum_0^N \frac{1}{2^k \varepsilon} C \\ &= C/\varepsilon \end{aligned}$$

Let $N \to \infty$

6 If $f \in L(0,1)$ show $x^k f(x) \in L(0,1)$ $k = 1, 2, \ldots$
and $\int_0^1 x^k f(x)\,dx \to 0$

Pf Note on $(0,1)$ $f \in L(0,1)$
$\Rightarrow f < \infty$
$\Rightarrow x^k f(x) \to 0$ as $k \to \infty$ for $x \in [0,1]$ a.e.

In addition $|x^k f(x)| < |f(x)| \in L(0,1)$

Thus by DCT $\int_{(0,1)} x^k f(x)\,dx \to \int 0 = 0$

$\square$

7. Use Egorov's to prove Bounded Converger

Pf Let $f_k : E \to [-\infty, \infty]$ be msble w/ $f_k \to f$ ae
Let $\varepsilon > 0$, and $|E| < \infty$ and $\exists M$ s.t. $|f_k(x)| < M$
By Egorov's, $\exists$ closed $A \subset E$ s.t. $|E - A| < \varepsilon/4M$
and $f_k \to f$ uniformly on A.

WTS $\lim \int_E f_k = \int_E f$

- i.e $\exists N \in \mathbb{N}$ s.t $k \geq N \Rightarrow |\int_E (f_k - f)| < \varepsilon$

- $f_k \xrightarrow{u} f \Rightarrow \exists K$ s.t. $k \geq K$
$\Rightarrow \forall x \in A \quad |f_k - f| < \varepsilon/2|A|$

- since $f_k < M \Rightarrow f < M \Rightarrow |f_k - f| < 2M \quad \forall k$

On A

$$\begin{aligned}\lim |\textstyle\int_A f_k - f| &\leq \lim \textstyle\int_A |f_k - f| \\ &= \lim \textstyle\int_A \varepsilon/2|A| \quad \text{for } k \geq N \\ &= \lim \varepsilon/2 \\ &= \varepsilon/2\end{aligned}$$

On $E \setminus A$

$$\begin{aligned}\lim |\textstyle\int_{E-A} f_k - f| &\leq \lim \textstyle\int_{E \setminus A} |f_k - f| \\ &\leq \lim \textstyle\int 2M \chi_{E-A} \\ &= \lim 2M |E \setminus A| \\ &\leq \lim 2M \, \varepsilon/4M \\ &= \varepsilon/2\end{aligned}$$

$\therefore \lim |\int f_k - f| < \varepsilon$

$\square$

701 2014 Hw 7.

1. If $p>0$ and $\int_E |f-f_k|^p \to 0$ as $k \to \infty$
Show $f_k \xrightarrow{m} f$

Pf Let $E_\varepsilon = \{x : |f_k(x) - f(x)| \geq \varepsilon^{1/p}\}$

Then $\int_E |f-f_k|^p \geq \int_{E_\varepsilon} |f-f_k|^p dx$

$$\geq \int_{E_\varepsilon} \varepsilon \, dx$$

$$= \varepsilon |E_\varepsilon|$$

$\Rightarrow \frac{1}{\varepsilon} \underbrace{\int_E |f-f_k|^p}_{\to 0} \geq |E_\varepsilon|$

$\Rightarrow |E_\varepsilon| \to 0$

$\Rightarrow f_k \xrightarrow{m} f$ $\square$

2. If $p>0$, $\int_E |f-f_k|^p \to 0$ and $\int_E |f_k|^p \leq M$
Show $\int_E |f|^p \leq M$

Pf $\int_E |f|^p = \int_E |f - f_k + f_k|^p$

$$\leq \int_E |f-f_k|^p + \int |f_k|^p$$

$$\leq \int_E |f-f_k|^p + M$$

$$\to M \quad \text{as } k \to \infty.$$

$\therefore \int |f|^p \leq M$

$\square$

3. For which $p>0$ does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

Pf $\quad 1/x \in L^p(0,1) \iff \int_0^1 |1/x|^p \, dx < \infty$

$$\iff \int_0^1 1/x^p \, dx < \infty$$

$$\iff \frac{x^{1-p}}{1-p}\Big|_0^1 < \infty$$

$$\iff \frac{1}{1-p} < \infty$$

$$\iff p \neq 1$$

$1/x \in L^p(1,\infty) \iff \int_1^\infty 1/x^p \, dx < \infty$

$$\iff \frac{x^{1-p}}{1-p}\Big|_1^\infty < \infty$$

$$\iff 1-p<0$$

$$\iff 1<p$$

$1/x \in L^p(0,\infty) \iff \int_0^\infty 1/x^p \, dx < \infty$

$$\iff \frac{x^{1-p}}{1-p}\Big|_0^\infty < \infty$$

$$\iff \frac{x^{1-p}}{1-p}\Big|_\infty < \infty \text{ iff } 1<p$$

$$\frac{x^{1-p}}{1-p}\Big|_0 < \infty \text{ iff } p>1$$

$$\text{iff } p=1 \quad \int_0^\infty \frac{1}{x} = \ln x \Big|_0^\infty = \infty$$

So $\frac{1}{x^p} \notin L[0,\infty) \ \forall n$

$\triangleright$

4\. a Give example of bdd continuous f on $(0,\infty)$ s.t. $\lim f(x) = 0$ but $f \notin L^p(0,\infty)$ $\forall p$

b If f is u.c. on $(0,\infty)$ and $f \in L^p(E)$ for $p > 0$ then $\lim f(x) = 0$

Pf a. Consider $f = \begin{cases} \frac{1}{n^{1/p}} & x = 2n \\ 0 & x = 2n+1 \\ \text{interpolates linearly} & \end{cases}$

Then $\int f^p = \sum_1^\infty \left(\frac{1}{n^{1/p}} \cdot \frac{2}{2}\right)^p$

$= \sum_1^\infty \frac{1}{n}$

$= \infty$

f is clearly bdd continuous

b. Fix $\varepsilon > 0$.

f u.c. $\Rightarrow \exists \delta > 0$ s.t. if $|x-y| < \delta$ $|f(x) - f(y)| < \varepsilon/2k$

where k is # of δ intervals in $[0,N]$

$f \in L^p \Rightarrow \int_0^\infty |f|^p < \infty$

$\Rightarrow \exists N > 0$ s.t. $\int_N^\infty |f|^p < \varepsilon$

or $\int_0^N |f|^p > \|f\|_p - \varepsilon$

Consider $[0,N]$ Break into k intervals of length at most δ then $|f(x) - f(y)| < \varepsilon$ on each.

$\Rightarrow |f(N) - f(0)| < k\varepsilon/2k = \varepsilon/2$

$\square$

5. a. Let $\{f_k\}$ msble on E. Show $\sum f_k$ converges a.e on E if $\sum \int_E |f_k| < \infty$

b. If $\{r_k\} = \mathbb{Q} \cap [0,1]$ and a_k satisfies $\sum |a_k| < \infty$

Show $\sum a_k |x - r_k|^{-1/2}$ converges absolutely ae in $[0,1]$

Pf. Assume $\{f_k\}$ msble on E

Assume $\sum \int_E |f_k| < \infty$

$\Rightarrow \int_E |f_k| \to 0$ as $k \to \infty$

6. Suppose $f: \mathbb{R}^n \to \mathbb{R}$ Prove if $f \in L(\mathbb{R}^n)$ and $\int_E f(x) \geq 0$ $\forall$ msble E then $f(x) \geq 0$ a.e on $\mathbb{R}^n$
As a result, if $\int_E f(x) = 0$ $\forall$ msble E then $f(x) = 0$ a.e in $\mathbb{R}^n$

Pf BWOC assume $\exists$ msble E s.t. $|\{x : f(x) < 0\}| \neq 0$

$\Rightarrow \exists \varepsilon$ s.t. $\{x : f(x) < -\varepsilon\} \neq 0$

$\Rightarrow \{x : f(x) < -\varepsilon\} = \bar{E}$ is a msble set

$\Rightarrow \int_{\bar{E}} f \geq 0$ however $\int_{\bar{E}} f < -\varepsilon |\bar{E}| < 0$

which contradicts

If $\int_E f(x) = 0$ $\forall$ msble E.

Let $\tilde{E} = \{x : f(x) \neq 0\}$ Assume BWOC $|\tilde{E}| \neq 0$

Then $\int_{\tilde{E}} f(x) \neq 0$ since $f(x) \neq 0$ and $|\tilde{E}| \neq 0$.

$\square$

7. a. Suppose $f: \mathbb{R}^n \to [0, \infty]$ is msble $E_k = \{x : f(x) > 2^k\}$

and $F_k = \{x : 2^k < f(x) < 2^{k+1}\}$

Prove $f \in L(\mathbb{R}^n) \Leftrightarrow \sum_{-\infty}^{\infty} 2^k |F_k| < \infty \Leftrightarrow \sum_{-\infty}^{\infty} 2^k |E_k| < \infty$

b. Let $g(x) = \begin{cases} |x|^{-b} & |x| > 1 \\ 0 & \text{other} \end{cases}$ $g \in L(\mathbb{R}^n) \Leftrightarrow b > n$

701 2014 HW 8

1. a. Let $E \subset \mathbb{R}^2$ msble s.t. $|\{y \mid (x,y) \in E\}| = 0$ for a.e x

Show $|E| = 0$ and $|\{x \mid (x,y) \in E\}| = 0$ for a.e y

b. Let $f(x,y) \geq 0$ msble Suppose $f(x,y) < \infty$ $\forall x$

Show for a.e $y \in \mathbb{R}$ $f(x,y) < \infty$ for a.e x

Pf a. Let $E_x = \{y \mid (x,y) \in E\}$ $E^y = \{x \mid (x,y) \in E\}$

$\Rightarrow \int \chi_E dy = |E_x| = 0$ a.e x

$\Rightarrow |E| = \iint_{\mathbb{R}^2} \chi_E$

$= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E \, dx \, dy$

$= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E \, dy \, dx$

$= \int_{\mathbb{R}} 0$

$= 0$

Now $|E^y| \geq 0$ and $0 = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E \, dx \, dy = \int |E^y| \, dy$

$\Rightarrow |E^y| = 0$

b. Let $E = \{(x,y) \mid f(x,y) = \infty\}$

$\Rightarrow E$ is msble in $\mathbb{R}^2$

$\Rightarrow |E_x| = 0$ $\forall x$

$\Rightarrow |E^y| = 0$ by a

$\Rightarrow f(x,y) < \infty$ for a.e x

$\square$

2. If f, g msble in $\mathbb{R}^n$. Show $h(x,y) = f(x)g(y)$ msble in $\mathbb{R}^n \times \mathbb{R}^n$. If $E_1, E_2 \subset \mathbb{R}^n$ msble then $E_1 \times E_2$ msble and $|E_1 \times E_2| = |E_1||E_2|$

Pf f, g msble. Let E be msble

$\Rightarrow f^{-1}(E) = E_f$ and $g^{-1}(E) = E_g$ where E_f, E_g msble

3. Let f be msble on $(0,1)$. If $f(x) - f(y)$ is integrable over $0 < x < 1$, $0 \le y \le 1$ show $f \in L(0,1)$

Pf By Fubinis we have for a.e. $y \in (0,1)$ $f(x) - f(y)$ is integrable. In particular for such y $f(y)$ is finite so $f(x)$ is integrable.

$f(x) - f(y) \in L((0,1) \times (0,1))$

$\therefore \underbrace{\int_0^1 \int_0^1 f(x) - f(y)\, dx\, dy}_{<\infty} < \infty$

4. Let f be msble, periodic w/ period 1
Suppose $\exists c$ s.t. $\int_0^1 |f(a+t) - f(b+d)dt \le C$
Show $f \in (0,1)$.

Pf Let $a = x$, $b = -x$

$$\Rightarrow C \geq \int_0^1 \int_0^1 |f(a+t) - f(b+t)| \, dt \, dx$$

$$= \int_0^1 \int_0^1 |f(x+t) - f(-x+t)| \, dt \, x,$$

$$= \frac{1}{2} \int_0^1 \int_{-\xi}^{\xi} |f(\xi) - f(\eta)| \, d\eta \, d\xi + \frac{1}{2} \int_1^2 \int_{\xi-1}^{1-\xi} |f(\xi) - f(\eta)| \, d\eta \, d\xi$$

5 a If $f \geq 0$ is msble on E and $w(y) = |\{x \in E : f(x) > y|$ $y > 0$
Use Tonelly to prove $\int_E f = \int_0^\infty w(y) \, dy$
b. Deduce from this special case $\int_E f^p = p \int_0^\infty y^{p-1} w(y) dy$

Pf a. w is monotone decreasing so has countably many discontinuities

$$\Rightarrow |w(y)| = |\{x \in E : f(x) \geq y\}| \text{ for a.e. } y.$$

$$\Rightarrow \int_0^\infty w(y) dy = \int_0^\infty |\{x \in E : f(x) \geq y| \, dy$$

$$= \int_0^\infty |\{x : (x,y) \in \mathcal{R}(f,E)\}| \, dy$$

$$= \int_0^\infty \int_{\mathcal{R}(f,E)y} dx \, dy$$

$$= \iint \mathcal{R}(f,E)$$

$$= \int_E f$$

b. Let $|w_{f^p}(E)| = |\{x \in E : f^p(x) > y\}|$

$$\Rightarrow \int_E f^p = \int_0^\infty w_{f^p} \, dy = \int_0^\infty w_f(y^{1/p}) \, dy \qquad u = y^{1/p} \quad du = 1/p \, y^{\frac{p-1}{p}} = \frac{1}{p} u^p$$

$$\Rightarrow \int_E f^p = p \int_0^\infty u^{p-1} w_f \, u \, du$$

$\square$

6. For $f \in L(\mathbb{R})$, let $\hat{f} = \int_{-\infty}^{\infty} f(t) e^{-ixt}\, dt$. Show if $f, g \in L(\mathbb{R})$ then $(\widehat{f * g}) = \hat{f}\, \hat{g}$

Pf $\widehat{(f * g)}(x) = \widehat{\int_{\mathbb{R}} f(x-t) g(t)\, dt}$ (convolution defined w/ + z pg 93)

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u-t) g(t) e^{-ixu} dt \right) du$$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u-t) g(t) e^{-ixu} du \right) dt$$

$$\int_{\mathbb{R}} g(t) \int_{\mathbb{R}} f(u-t) e^{-ixu} du\, dt$$

$$\int_{\mathbb{R}} g(t) e^{-ixt} \int_{\mathbb{R}} f(u-t) e^{-ix(u-t)} du\, dt$$

$$= \int_{\mathbb{R}} g(t) e^{-ixt} \int_{\mathbb{R}} f(v) e^{-ixv} dv\, dt$$

$$= \int_{\mathbb{R}} g(t) e^{-ixt} \hat{f}(x)\, dt$$

$$= \hat{f}(x) \int_{\mathbb{R}} g(t) e^{-ixt} dt$$

$$= \hat{f}(x)\, \hat{g}(x)$$

$\square$

7. Let V_n be volume of unit ball in $\mathbb{R}^n$. Show $V_n = 2V_{n-1} \int_0^1 (1-t^2)^{n-1/2}$

Pf Induction:
$n=1 \Rightarrow V_1 = 0 = 2V_0 \int_0^1 (1-t)^0 dt = 0$

Suppose claim holds for $k < n$. Show it holds for $k=n$.

$$V_n = \int \cdots \int_{B_n} 1 = \int \cdots \int_{|x|<1} dx_1 \cdots dx_n$$

$$= \int_{-1}^{1} \int \cdots \int_{x_2^2 + \cdots + x_n^2 < 1 - x_1^2} dx_1 \cdots dx_n$$

Let $y_i = \frac{x_i}{\sqrt{1-x_1^2}}$ for $i = 2, \ldots, n$ $\quad dy_i = \frac{dx_i}{\sqrt{1-x_1^2}}$

$$\Rightarrow V_n = \int_{-1}^{1} \underbrace{\int \cdots \int_{y_2^2 + \cdots + y_n^2 < 1}}_{V_{n-1}} (1-x_1^2)^{\frac{n-1}{2}} dx_1\, dy_2 \cdots dy_n$$

$$= 2V_{n-1} \int_0^1 (1-x_1^2)^{\frac{n-1}{2}} dx$$

701 2014 Hw 9

1. Let f be msble and not zero on some set of positive measure. Show $\exists$ a constant c s.t. $f^*(x) \geq c|x|^{-n}$ for $|x| \geq 1$

Pf Let E be s.t. $|E| > 0$ and $|f(x)| > 0$ $\forall x \in E$

Let $x \in \mathbb{R}^n$ w/ $|x| > 1$

Let Q_x be smallest cube centered at x with $E \subset Q_x$

$\Rightarrow \exists C_x$ s.t. side length of Q_x is $C_x |x|$

Let $c = \frac{1}{C_x^n} \int_E |f|$

$$\Rightarrow f^*(x) = \sup_Q \frac{1}{|Q|} \int_Q |f|$$

$$\geq \sup \frac{1}{|Q|} \int_{Q \cap E} |f|$$

$$\geq \frac{1}{|Q_x|} \int_{Q_x \cap E} |f|$$

$$= \frac{\int_E |f|}{C_x^n |x|^n}$$

$$= c|x|^{-n}$$

$\square$

4. Prove if msble set $E \subset [0,1]$ satisfies $|E \cap I| \geq \alpha |I|$ for some $\alpha > 0$ and all intervals I in $[0,1]$ then $|E| = 1$

Pf By Lebesgue differentiation Thm

$\frac{1}{|Q|} \int_Q f(y)\,dy \to f(x)$ as $Q \to \{x\}$.

Let $Q = I$ and $f(x) = \chi_E$

$\Rightarrow \frac{1}{|I|} \int_I \chi_E = \frac{|E \cap I|}{|I|} \to \chi_E(x) = \begin{cases} 0 & x \notin E \\ 1 & x \in E \end{cases}$

Assume $|E \cap I| \geq \alpha |I|$ for $\alpha > 0$

$\Rightarrow x \in E$ for every $x \in [0,1]$ since we can shrink I to any point

$\Rightarrow [0,1] \subset E \subset [0,1]$

$\Rightarrow |E| = 1$

$\square$

3. Consider $f(x) = \begin{cases} \frac{1}{|x|(\log(1/|x|))^2} & \text{if } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases}$ verify f is integrable, establish inequality $f^*(x) \geq \frac{c}{|x| \log 1/x}$ for some c and $|x| < 1/2$ to conclude maximal fcn f^* is not locally integrable

Pf Notice f can be integrated using improper Riemann integral since f is even, and on $0 < |x| < 1/2$ f is finite, positive and continuous

$$\int_{0<|x|<1/2} = 2\int_0^{1/2} \frac{1}{x(\log 1/x)^2} dx$$

$$= 2\int_\infty^{\log 2} u^{-2}\, du$$

$$= 2 \lim \frac{-1}{u} \Big|_\infty^{\log 2}$$

$$= \frac{2}{\log 2} < \infty$$

$\Rightarrow f$ is integrable

Now let $|x| < 1/2$ and consider $Q = [-x, x]$

$$f^*(x) = \sup \frac{1}{|Q|} \int_Q |f|$$

$$\geq \frac{1}{2|x|} \int_{[-x,x]} \frac{1}{|y|(\log 1/|y|)^2}$$

$$\geq \frac{1}{x} \int_0^x \frac{1}{y \log 1/y^2} dy$$

$$= \frac{1}{x} \frac{1}{\log 1/x} \quad \therefore f^* \text{ is integrable}$$

Now let $\bar{Q} = [-1/2, 1/2]$

$$\int_Q |f^*| = \int_Q \frac{1}{|x| \log 1/x} \geq \int_0^{1/2} \frac{1}{x \log 1/x} dx$$

$$\geq \lim \log u \Big|_n^{\log 2} = \infty$$

$\Rightarrow \int f^*$ isn't finite on $[-1/2, 1/2]$ so f^* is not locally integrable

$\square$