Complex Analysis Terms and Theorems

Preparation for Analysis Qualifying Exam
Based on *Complex Analysis* by Theodore W. Gamelin

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Contents

Chapter 1 Complex Plane and Elementary Function

1.1 Complex Numbers

Complex Number A **complex number** is an expression of the form $z = x + iy$, where x and y are real numbers.

Real and Imaginary Parts The component x is called the **real part** of z and y is the **imaginary part** of z. Denote these by:

$$x = \operatorname{Re} z \qquad y = \operatorname{Im} z$$

Complex Plane The set of complex numbers form the **complex plane**. We denote it by $\mathbb{C}$. The correspondence $z = x + iy \leftrightarrow (x, y)$ is one-to-one between the complex numbers and points in $\mathbb{R}^2$. The real numbers correspond to the x-axis. The purely imaginary numbers correspond to the y-axis.

Modulus The **modulus** of a complex number $z = x + iy$ is the length $\sqrt{x^2 + y^2}$ of the corresponding vector (x, y):

$$|z| = \sqrt{x^2 + y^2}$$

This abides by the following properties:

$$|z + w| \leqslant |z| + |w| \qquad |z - w| \geqslant |z| - |w|$$

Complex Conjugate The **complex conjugate** of a complex number $z = x + iy$ is defined to be $\bar{z} = x - iy$. Geometrically this is the reflection of z across the x-axis.

Properties of $\bar{z}$

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\bar{w} \qquad |z| = |\bar{z}| \qquad |z|^2 = z\bar{z}$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2} \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

$$|zw| = |z||w| \qquad \frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

Complex Polynomial of Degree $n \geqslant 0$ A function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

Theorem 1.1.1 (Fundamental Theorem of Algebra). Every copmlex polynomial $p(z)$ of degree $n \geqslant 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1} \ldots (z - z_k)^{m_k}$$

where the z_j's are distinct and $m_j \geqslant 1$. This factorization is unique up to permutation of the factors.

The points z_k are called the **roots** of $p(z)$.

1.2 Polar Representation

Polar Coordinates For a point $z = x + iy \rightarrow (x, y) \neq (0, 0)$ in the complex plane:

$$r = \sqrt{x^2 + y^2} = |z| \qquad x = r\cos\theta \qquad y = r\sin\theta \qquad z = x + iy = r(\cos\theta + i\sin\theta$$

Argument The **argument** of $z \neq 0$ is the angle θ, write:

$$\theta = \arg z$$

The argument is a multivalued function, defined for $z \neq 0$.

The **principal value of** $\arg z$, $\operatorname{Arg} z$ is specifed to be the value of θ such that $-\pi < \theta \leqslant \pi$. Thus:

$$\arg z = \{\operatorname{Arg} z + 2\pi k | k = \pm 1, \pm 2, \ldots\}$$

Polar Representation Since $e^{i\theta} = \cos\theta + i\sin\theta$, we get that the **polar representation** of $z \in \mathbb{C}$ is

$$z = re^{i\theta} \qquad r = |z|, \quad \theta = \arg z$$

Note, since sine and cosine are 2π periodic, different choices of $\arg z$ yield the same value for $e^{i\theta}$.

Properties of Polar Representation

$$|e^{i\theta}| = 1 \qquad \overline{e^{i\theta}} = e^{-i\theta} \qquad \frac{1}{e^{i\theta}} = e^{-i\theta} \qquad e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}$$

These correspond to:

$$\arg \bar{z} = -\arg z \qquad \arg(1/z) = -\arg z \qquad \arg z_1 z_2 = \arg z_1 + \arg z_2$$

nth Root of Unity A complex number z is an **nth root** of w if $z^n = w$. If $w = \rho e^{i\varphi}$, then

$$(re^{i\theta})^n = r^n e^{ni\theta} = \rho e^{i\varphi} \qquad \Longrightarrow \qquad r = \rho^{1/n} \qquad \theta = \frac{\varphi}{n} + \frac{2\pi k}{n}$$

The **nth roos of unity** are the nth roots of 1 given explicitely by:

$$\omega_k = e^{2\pi ik/n} \qquad 0 \leqslant k \leqslant n-1$$

Thus, for any complex number $w \neq 0$, the kth nth root of w can be written

$$z_k = z_0 w_k = \rho^{1/n} e^{i\varphi/n} e^{2\pi ik/n}$$

1.3 Stereographic Projection

Extended Complex Plane The **extended complex plane** is the complex plane together with the point at infinity. $\mathbb{C}^* = \mathbb{C} = \{\infty\}$.

Stereographic Projection The **stereographic projection** of a point $P = (X, Y, Z)$ on the unit sphere from the north pole of the unit sphere $N = (0, 0, 1)$ is the point $z = x + iy \sim (x, y, 0)$ where the straight line meets the coordinate plane $Z = 0$. Explicitly:

$$\begin{cases} X = 2x/(|z|^2 + 1) \\ Y = 2y/(|z|^2 + 1) \\ Z = 1 - 1/t = (|z|^2 - 1)/(|z|^2 + 1) \end{cases}$$

Theorem 1.3.1. Under the stereographic projection, circles on the sphere correspond to circles nad straight lines in the plane.

1.4 The Square and Square Root Functions

Slit / Branch Cut A way to define the inverse function of $w = z^2$. Since $w = z^2$ wraps around the plane twice, in order to define an inverse function we must limit it's domain. To do so we make a **branch cut**, commonly along $(-\infty, 0]$.

Slit Plane This yields the **slit plane**, $\mathbb{C}\backslash(-\infty, 0]$. Every value w in the slit plane corresponds to exactly two z-values. (That is, when we square z we get the same value for two different z-values, one where $\operatorname{Re} z > 0$ and one where $\operatorname{Re} z < 0$.)

Branch As there are two possibilities for the inverse image on the slit plane, the determination of the inverse function is called a **branch** of the inverse.

Principal Branch The function $f_1(w)$ (which maps to values of z such that $\operatorname{Re} z > 0$) is called the **principal branch of $\sqrt{w}$**. It is expressed in terms of he principal branch of the argument function as

$$f_1(w) = |w|^{1/2} e^{i(\operatorname{Arg} w)/2}, \qquad w \in \mathbb{C}\backslash(-\infty, 0]$$

Riemannian Surface The surface construced to represent the unverse function by gluing together the edges where the functions $f_1(w)$ and $f_2(w)$ coincide. The surface is essentially a sphere with two punctures corresponding to 0 and ∞.

1.5 The Exponential Function

Exponential Function We extend the definitino of the exponential function to all complex numbers z by defining:

$$e^z = e^x \cos y + ie^x \sin y = e^x e^{iy} \qquad |e^z| = e^x \qquad \arg e^z = y \qquad e^{z+w} = e^z e^w \qquad \frac{1}{e^z} = e^{-z}$$

Periodic The complex number λ is a **period** of the function $f(z)$ if $f(z+\lambda) = f(z)$ for all z for which $f(z)$ and $f(z+\lambda)$ are defined. The function is called **periodic** if it has a nonzero period.

The exponential function is periodic with a period $2\pi ik$ since $e^{z+2\pi i} = e^z$.

1.6 The Logarithm Function

Logarithm Function For $z \neq 0$ we define $\log z$ to be the multivalued function:

$$\log z = \log|z| + i \arg z = \log|z| + i \operatorname{Arg} z + 2\pi i m$$

Precisely the complex numbers w such that $e^w = z$.

Principal Value of Log The **principal value of** $\log z$ is

$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$$

1.7 Power Functions and Phase Factors

Power Function Let α be an arbitrary complex number. The **power function** z^α is the multivlued function

$$z^\alpha = e^{\alpha \log z} = e^{\alpha[\log|z| + i \operatorname{Arg} z + 2\pi i m]} = e^{\alpha \operatorname{Log} z} e^{2\pi i \alpha m}$$

Note, if α is not an integer, we cannot define z^α continuously on the entire complex plane, so we myst make a branch cut.

Phase Factor If $z^\alpha = r^\alpha e^{i\alpha\theta}$, then $e^{i\alpha\theta}$ is the **phase factor** and comes from the branch cut we made.

Theorem 1.7.1 (Phase Change Lemma). Let $f(z)$ be a (single-valued) function that is defined and continuous near z_0. For any continuously varying branch of $(z-z_0)^\alpha$ the function $f(z) = (z-z_0)^\alpha g(z)$ is multiplied by the phase factor $e^{2\pi i\alpha}$ when z traverses a complete circle about z_0 in the positive direction.

1.8 Trigonometric and Hyperbolic Functions

Trigonometric Functions Since for real θ

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$

we can extend this to the complex numbers:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin\theta = \frac{e^{iz}-e^{-iz}}{2i},$$

which are 2π periodic.

$$\tan z = \frac{\sin z}{\cos z} \qquad \tanh z = \frac{\sinh z}{\cosh z}$$

Properties of Trig Functions

$$\cos(-z) = \cos(z) \qquad \sin(-z) = -sin(z)$$

$$\cos(z+w) = \cos z\cos w - \sin z\sin w \qquad \sin(z+w) = \sin z\cos w + \cos z\sin w$$

Hyperbolic Functions The hyperbolic extension to the complex plane is:

$$\cosh z = \frac{e^{z}+e^{-z}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

which are $2\pi i$ periodic.

Propertires of Hyperbolic Functions

$$\cosh(iz) = \cos z \qquad \cos(iz) = \cosh z \qquad \sinh(iz) = i\sin z \qquad \sin(iz) = i\sinh z$$

$$\cosh(z+w) = \cosh z\cosh w - \sinh z\sinh w \qquad \sinh(z+w) = \sinh z\cosh w + \cosh z\sinh w$$

Cartesian Representation For $z = x+iy$,

$$\sin z = \sin x\cosh y + i\cos x\sinh y \qquad \cos z = \cos x\cosh y - i\sin x\sinh y$$

Trig Moduli

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y$$

Inverse Trig

$$\sin^{-1} z = -i\log\left(iz \pm \sqrt{1-z^2}\right) \qquad \cos^{-1} = -i\log\left(iz \pm \sqrt{z^2-1}\right) \qquad \tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$$

Chapter 2 Analytic Functions

2.1 Review of Basic Analysis

Converges A sequence of complex numbers $\{s_n\}$ **converges to** s if for any $\varepsilon > 0$, there is an integer $N \geqslant 1$ such that $|s_n - s| < \varepsilon$ for all $n \geqslant N$.

Theorem 2.1.1. A convergenet sequence is bounded. Further, if $\{s_n\}$ and $\{t_n\}$ are sequences of complex numbers such that $s_n \to s$ and $t_n \to t$, then

a. $s_n + t_n \to s + t$

b. $s_n t_n \to st$

c. $s_n/t_n \to s/t$

Theorem 2.1.2. If $r_n \leqslant s_n \leqslant t_n$, and if $r_n \to L$ and $t_n \to L$ then $s_n \to L$.

Monotone A sequence of real numbers $\{s_n\}$ is said to be **monotone increasing** if $s_{n+1} \geqslant s_n$ for all n, **monotone decreasing** if $s_{n+1} \leqslant s_n$ for all n, and **monotone** if it is either monotone increasing or decreasing.

Theorem 2.1.3. A bounded monotone sequence of real numbers converges.

Theorem 2.1.4 (Complex Convergence). A sequence $\{s_k\}$ of complex numbers converges if and only if the corresponding sequences of real and imaginary parts of the s_k's converge.

Cauchy Sequence A sequence of complex numbers $\{s_n\}$ is a **Cauchy sequence** if for every $\varepsilon > 0$ there exists an $N > 1$ such that $|s_n - s_m| < \varepsilon$ if $m, n \geqslant N$.

Theorem 2.1.5 (Completeness Axiom Equivalent). A sequence of complex numbers converges if and only if it is a Cauchy sequence.

Functional Complex Limit A complex-valued function $f(z)$ **has limit** L **as** z **tend to** z_0 if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $|z - z_0| < \delta$. That is $\lim_{z \to z_0} f(z) = L$, or $f(z) \to L$ as $z \to z_0$.

Lemma 2.1.1. The complex-valued function $f(z)$ has limit L as $z \to z_0$ if and only if $f(z_n) \to L$ for any sequence $\{z_n\}$ in the domain of $f(z)$ such that $z_n \neq z_0$ and $z_n \to z_0$.

Theorem 2.1.6. If a function has a limit at z_0, then the function is bounded near z_0. Further, if $f(z) \to L$ and $g(z) \to M$ as $z \to z_0$, then $z \to z_0$ we have

a. $f(z) + g(z) \to L + M$

b. $f(z)g(z) \to LM$

c. $f(z)/g(z) \to L/M$, provided that $M \neq 0$

Continuous We say that $f(z)$ is **continuous at** z_0 if $f(z) \to f(z_0)$ as $z \to z_0$. A **continuous function** is a function that is continuous at each point of its domain.

Open A subset U of the complex plane is **open** if whenever $z \in U$, there is a disk centered at z that is contained in U.

Domain A subset D of the complex plane is a **domain** if D is open and if any two points of D can be connected by a broken line segment in D.

Theorem 2.1.7. If $h(x, y)$ is a continuously differentiable function on a domain D such that $\nabla h = 0$ on D, then h is constant.

Convex A set is **convex** if whenever two points belong to the set, then the straight line segment joining the two points is contained in the set. (A punctured disk is a domain, but is not convex.)

Star-shaped A set is **star-shaped with respect to** z_0 if whenever a point belongs to the set, then the straight line segment joining z_0 to the point is contained in the set.

A **star-shaped domain** is a domain that is star-shaped with respect to one of its points. (e.g. $\mathbb{C}\backslash(-\infty, 0])$)

Boundary The **boundary** of a set E consists of points z such that every disk centered at z contains both points in E and points not in E.

Compact A subset of the complex plane that is closed and bounded is said to be **compact**.

Theorem 2.1.8. A continuous real-valued function on a compact set attains its maximum.

Euler's Constant Consider the sequence

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \qquad n \geqslant 1.$$

This sequence decreases to limit γ such that $\frac{1}{2} < \gamma < \frac{3}{5}$, while $a_n = b_n - \frac{1}{n}$ increases to γ. The limit γ is called **Euler's constant**.

2.2 Analytic Functions

Differentiable A complex-valued function $f(z)$ is **differentiable** at z_0 if the differnece quotients

$$\frac{f(z)-f(z_0)}{z-z_0}$$

have limits at $z \to z_0$. The limit is denoted by $f'(z_0)$, or by $\frac{df}{dz}(z_0)$, and we refer to it as the **complex derivative** of $f(z)$ at z_0. Thus

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Theorem 2.2.1. If $f(z)$ if differentiable at z_0, then $f(z)$ is continuous at z_0.

Rules a. $(c)' = 0$

b. $(z^m)' = mz^{m-1}$

c. $(cf)'(z_0) = cf'(z_0)$

d. $(f+g)'(z_0) = f'(z_0) + g(z_0)'$

e. $(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$

f. $\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0)-f(z_0)g'(z_0)}{(g(z_0))^2}$

g. $(f\circ g)'(z_0) = f'(g(z_0))g'(z_0)$

Homework 2 Findings Let f be differentiable at z_0.

a. $\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

b. $\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

c. $\frac{\partial z}{\partial \bar{z}} = 0 \qquad \frac{\partial \bar{z}}{\partial \bar{z}} = 1$

d. $\frac{\partial \bar{z}^m}{\partial \bar{z}} = m\bar{z}^{m-1}$

e. $\frac{\partial (f+g)}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}$

f. $\frac{\partial (fg)}{\partial \bar{z}} = g\frac{\partial f}{\partial \bar{z}} + f\frac{\partial g}{\partial \bar{z}}$

Analytic A function $f(z)$ is **analytic on the open set** U if $f(z)$ is (complex) differentiable at each point of U and the copmlex derivative $f'(z)$ is continuous on U.

2.3 The Cauchy-Riemann Equations

Cauchy-Riemann Equations Suppose $f = u + iv$ and $z = x + iy$. Taking derivatives we see that

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y)$$

Equating real and imaginary parts we get the **Cauchy-Riemann equations** for u and v:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{equivalently} \qquad u_x = v_y \qquad u_y = -v_x$$

Theorem 2.3.1 (Analytic & CR). Let $f = u + iv$ be defined on a domain D in the complex plane, where u and v are real-valued. Then $f(z)$ is analytic on D if and only if $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives that satisfy the Cauchy-Remann equations.

Theorem 2.3.2. If $f(z)$ is analytic on a domain D, and if $f'(z) = 0$ on D, then $f(z)$ is constant.

Theorem 2.3.3. If $f(z)$ is analytic and real-valued on a domain D, then $f(z)$ is constant.

2.4 Inverse Mappings and the Jacobian

Jacobian Matrix Let $f = u + iv$ be analytic on a domain D. The **Jacobian Matrix** of this map is

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Theorem 2.4.1. If $f(x)$ is analytic then it's Jacobian matrix J_f has a determinant

$$\det J_f(z) = |f'(z)|^2$$

Theorem 2.4.2. Suppose $f(z)$ is analytic on a domain D, $z_0 \in D$, and $f'(z_0) \neq 0$. Then there is a (small) disk $U \subset D$ containing z_0 such that $f(z)$ is one-to-one on U, the image $V = f(U)$ of U is open and the inverse function $f^{-1} : V \to U$ is analytic and satisfies

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)} \qquad z \in U$$

Proof. All of the assertions of this theorem are consequences of the inverse function theorem, except for the assertions concerning the analyticity of f^{-1}. To check this, write $g = f^{-1}$ on U and differentiate by hand. Fix $w, w_1 \in U$ with $w \neq w_1$, set $z = g(w)$, $z_1 = g(w_1)$. Then $z \neq z_1$, $f(z) = w$, $f(z_1) = w_1$, and we have:

$$\frac{g(w) - g(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{\left(\frac{f(z)-f(z_1)}{z-z_1}\right)}$$

As w tends to w_1, z tends to z_1, and the right-hand side tends to $1/f'(z_1)$. Thus g is differentiable at w_1, and $g'(w_1) = 1/f'(z_1)$, which required by the thereom. Since $\frac{1}{f'(z)}$ is continuous, $(f^1)'$ is continuous, and thus f^{-1} is analytic.

If we write $w = g(z)$, the identity becomes $\frac{dz}{dw} = \frac{1}{dw/dz}$ ☐

Dirichlet form For smooth functions g and h defined on a bounded domain U, we define the **Dirichlet form** $D_U(g,h)$ by

$$D_U(g,h) = \iint\limits_U \left[\frac{\partial g}{\partial x}\frac{\overline{\partial h}}{\partial x} + \frac{\partial g}{\partial y}\frac{\overline{\partial h}}{\partial y}\right] dx\, dy.$$

2.5 Harmonic Functions

Laplacian The operator

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}$$

is called the **Laplacian**. The equation

$$\Delta u = \frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2} = 0$$

is called **Laplace's equation.**

Hamonic Functions Smooth functions $u(x_1, \ldots, x_n)$ that satisfy Laplace's equation, $\Delta u = 0$, are called **harmonic functions.**

We say a function $u(x,y)$ is **harmonic** if all its first- and second-order partial derivatives exist and are continuous and satisfy Laplace's equation.

Theorem 2.5.1. If $f = u + iv$ is analytics and the functions u and v have continuous second-order partial derivatives, then u and v are harmonic.

Note. We will show in Chapter 4 that an analytic function has continuous partial derivatives of all orders. Thus, we only need analyticity.

Harmonic Conjugate If u is harmonic on a domain D, and v is a harmonic function such that $u+iv$ is analytic, we say v is the **harmonic conjugate** of u. This conjugate is unique up to adding a constant.

Since f is analytic, we know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Thus

$$v = \int \frac{\partial u(x,y)}{\partial x}\, dy = U(x,y) + h(x)$$

Then:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} - h'(x)$$

Solve this equation for $h'(x)$ up to a constant.

Theorem 2.5.2. Let D be an open disk, or an open rectangle with sides parallel to the axes, andlet $u(x,y)$ be a harmonic function on D. Then there is a harmonic function $v(x,y)$ on D such that $u+iv$ is analytic on D. The harmonic conjugate v is unique, up to adding a constant.

2.6 Conformal Mappings

Tangent Vector and Angle Between Curves Let $\gamma(t) = x(t) + iy(t)$, $0 \leqslant t \leqslant 1$, be a smooth parameterized curve terminating at $z_0 = \gamma(0)$. We refer to

$$\gamma'(0) = \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t} = x'(0) + iy'(0)$$

as the **tangent vector**. We define the **angle between two curves** at z_0 to be the angle between their tangent vectors at z_0.

Theorem 2.6.1. If $\gamma(t)$, $0 \leqslant t \leqslant 1$, is a smooth parameterized curve terminating at $z_0 = \gamma(0)$, and $f(z)$ is analytic at z_0, then the tangent to the curve $f(\gamma(t))$ terminating at $f(z_0)$ is

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$$

Conformal A function is **conformal** if it preserves angles. More precisely, we say that a smooth complex-valued function $g(z)$ is **conformal at** z_0 if whenever γ_0 and γ_1 are two curves terminating at z_0 with nonzero tangents, then the curves $g \circ \gamma_0$ and $g \circ \gamma_1$ have nonzero tangents at $g(z_0)$ and the angle from $(g \circ \gamma_0)'(z_0)$ to $(g \circ \gamma_1)'(z_0)$ is the same as the angle from $\gamma_0'(z_0)$ to $\gamma_1'(z_0)$.

Conformal Mapping A **conformal mapping** of one domain D onto another V is a continuously differentiable function that is conformal at each point of D and that maps D one-to-one onto V.

Examples

a. $f(z) = z + b$
b. $g(z) = az$ for $a \neq 0$
c. $w = z^2$ maps $\{\operatorname{Re} z > 0\}$ onto $\mathbb{C}\backslash(-\infty, 0]$.
d. Fix θ_0, $0 < \theta_0 \leqslant \pi$. If $0 < a < \pi/\theta_0$, the function z^a maps the sector $\{|\arg z| < \theta_0\}$ conformally onto the sector $\{|\arg z| < a\theta_0\}$.
e. e^z is conformal at every z. However $\mathbb{C} \to \mathbb{C}\backslash 0$ is not conformal, however $\{|\operatorname{Im} z| < \pi\} \to \mathbb{C}\backslash(-\infty, 0]$ is.
f. Principal branch $\operatorname{Log} z$ is conformal $\mathbb{C}\backslash(-\infty, 0]$ onto $\{|\operatorname{Im} w| < \pi\}$

Theorem 2.6.2. If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

2.7 Fractional Linear Transformations

A **fractional linear transformation** is a function of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

where a, b, c, and d are complex constants satisfying $ad - bc \neq 0$. Fractional linear transformations are also called **Mobius transformations**.

Affine Transformation A function of the form $f(z) = az + b$, where $a \neq 0$, is called an **affine transformation.** These are the factional linear transformations of the above form with $c = 0$. Special cases are **translations** and **dialations.**

Inversion The factional linear transformation $f(z) = \frac{1}{z}$ is called an **inversion.**

Theorem 2.7.1. Given any three distinct points z_0, z_1, z_2 in the extended complex plane, and given any three distinct values w_0, w_1, w_2 in the extended complex plane, there is a unique fractional linear transformation $w = w(z)$ such that $w(z_0) = w_0$, $w(z_1) = w_1$, and $w(z_2) = w_2$.

Example Find the fraction linear transformation mapping -1 to 0, ∞ to 1, and i to ∞.

Since $w(i) = \infty$ place $z - i$ in the denominator. Since $w(-1) = 0$ place $z + 1$ in the numerator. To obtain $w(z) = \frac{a(z+1)}{z-i}$. Since $w(z) \to 1$ as $z \to \infty$, we obtain $a = 1$. Therefore $w(z) = (z + 1)/(z - i)$.

Theorem 2.7.2. Every fractional linear transformation is a composition of dilations, translations, and inversions.

Theorem 2.7.3. A fractional linear transformation maps circles in the extended complex plane to circles.

Note This section involved a lot of sketching. Go back through the homework to see how this works.

Chapter 3 Line Integrals and Harmonic Functions

3.1 Line Integrals and Green's Theorem

Path A **path** in the plane from A to B is a continuous function $t \rightarrow \gamma(t)$ on some parameter interval $a \leqslant t \leqslant b$ such that $\gamma(a) = A$ and $\gamma(b) = B$.

Simple The path is **simple** if $\gamma(s) \neq \gamma(t)$ when $s \neq t$.

Closed The path is **closed** if it starts and ends at the same point, that is $\gamma(a) = \gamma(b)$.

Simple Closed Path A **simple closed path** is a closed path γ such that $\gamma(s) \neq \gamma(t)$ for $a \leqslant s < t \leqslant b$.

Reparametrization If $\gamma(t)$, $a \leqslant t \leqslant b$, is a path from A to B, and if $\varphi(s)$, $\alpha \leqslant s \leqslant \beta$, is a strictly increasing continuous function satisfying $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, then the composition $\gamma(\varphi(s))$ is also a path from A to B. The composition $\gamma \circ \varphi$ is a called **reparametrizatino** of γ and preserves orientation.

Note, we usually regard γ and any of it reparametrizations as being the same path, though it is technically an equivalence class of paths.

Trace The **trace** of the path γ is its image $\gamma([a, b])$, which is a subset of the plane.

Smooth Path A **smooth path** is a path that can be represented in the form $\gamma(t) = (x(t), y(t))$, $a \leqslant t \leqslant b$, where the functions $x(t)$ and $y(t)$ are smooth, that is they have as many derivatives as necessary for whatever is being asserted to be true.

Piecewise Smooth Path A **piecewise smooth path** is a concatenationof smooth paths.

Curve A **curve** is a smooth or piecewise smooth path.

Line Integral Let γ be a path in the plane from A to B, and let $P(x, y)$ and $Q(x, y)$ be continuous complex-valued functions on γ. Consider (x_i, y_i) successive points on the pat and form the sum

$$\sum P(x_j, y_j)(x_{j+1} - x_j) + \sum Q(x_j, y_j)(x_{j+1} - x_j)$$

If these sums have a limit as th distances between successive points on γ tend to 0, we define the limit to be the **line integral of $P\,dx + Q\,dy$ along** γ and we denote it by

$$\int_\gamma P\,dx + Q\,dy = \int_a^b P(x(t), y(t))\frac{dx}{dt}dt + \int_a^b Q(x(t), y(t))\frac{dy}{dt}dt$$

For a curve parameterized by $t \to (x(t), y(t))$.

Theorem 3.1.1 (Green's Theorem). Let D be a bounded domain in the plane whose boundary ∂D consists of a finite number of disjoint piecewise smooth closed curves. Let P and Q be continuously differentiable functions on $D \cup \partial D$. Then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

3.2 Independence of Path

Antiderivative $F(t)$ is an **antiderivative** for $f(t)$ if its derivative is f, that is $F' = f$.

Theorem 3.2.1 (Fundamental Theorem of Calculus). **Part I.** If $F(t)$ is an antiderivative for the continuous function $f(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Part II. If $f(t)$ is a continuous function on $[a, b]$, then the indefinite integral

$$F(t) = \int_a^t f(s)\,ds, \qquad a \leqslant t \leqslant b$$

is an antiderivative for $f(t)$. Further, each antiderivative for $f(t)$ differs from $F(t)$ by a constant.

Differential If $h(x, y)$ is a continuously differentiable complex-alued function, we define the **differential** dh of h by

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

Exact We say that a differential $P\,dx + Q\,dy$ is **exact** if $P\,dx + Q\,dy = dh$ for some function h.

Theorem 3.2.2 (Part I). If γ is a piecewise smooth curve from A to B, and if $h(x, y)$ is continuously differentiable on γ, then

$$\int_\gamma dh = h(B) - h(A).$$

Lemma 3.2.1. Let P and Q be continuous complex-valued functions on a domain D. Then $\int P\,dx + Q\,dy$ is independent of path in D if and only if $P\,dx + Q\,dy$ is exact, that is, there is a continuously differentiable function $h(x, y)$ such that $dh = P\,dx + Q\,dy$. Moreoverm the function h is unique, up to adding a constant.

Closed Let P and Q be continuously differentiable complex-valued functions on a domain D. We say that $P\,dx + Q\,dy$ is **closed** on D if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

By Green's Theorem $\int_{\partial U} P\,dx + Q\,dy = 0$

Lemma 3.2.2. Exact differentials are closed.

Theorem 3.2.3 (Part II). Let P and Q be continuously differentiable complex-valued functions on a domain D. Suppose

a. D is a star-shaped domain (as a disk or rectangle), and

b. the differential $P\,dx + Q\,dy$ is closed on D.

Then $P\,dx + Q\,dy$ is exact on D.

Theorem 3.2.4. Let D be a domain, and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \leqslant t \leqslant b$, be two paths in D from A to B. Suppose that γ_0 can be continuously deformed to γ_1, in the sense that for $0 \leqslant s \leqslant 1$ there are paths $\gamma_s(t)$, $a \leqslant t \leqslant b$, from A to B such that $\gamma_s(t)$ depends continuously on s and t for $0 \leqslant s \leqslant 1$, $a \leqslant t \leqslant b$. Then

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$$

for any closed differential $P\,dx + Q\,dy$ on D.

Theorem 3.2.5. Let D be a domain, and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \leqslant t \leqslant b$, be two closed paths in D. Suppose that γ_0 can be continuously deformed to γ_1, in the sense that $0 \leqslant s \leqslant 1$ there are closed paths $\gamma_s(t)$, $a \leqslant t \leqslant b$, such that $\gamma_s(t)$ depends continuously on s and t for $0 \leqslant s \leqslant 1$, $a \leqslant t \leqslant b$. Then

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$$

for any closed differential $P\,dx + Q\,dy$ on D.

Summary

independent of path $\Leftrightarrow$ exact $\Rightarrow$ closed

For a star-shaped domain:

independent of path $\Leftrightarrow$ exact $\Leftrightarrow$ closed

And that if $P\,dx + Q\,dy$ is a closed differential, then a deformation in the path from A to B does not change the value of the integral of $P\,dx + Q\,dy$ along the path.

3.3 Harmonic Conjugates

Lemma 3.3.1. If $u(x, y)$ is harmoic, then the differential

$$-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

is closed.

Theorem 3.3.1. Any harmonic function $u(x, y)$ on a star-shaped domain D (as a disk or rectangle) has a harmonic conjugate function $v(x, y)$ on D.

3.4 The Mean Value Property

Average Value Let $h(z)$ be a continuous real-valued function on a domain D. Let $z_0 \in D$, and suppose D contains the disk $\{|z - z_0| < \rho\}$. We define the **average value** of $h(z)$ on the circle $\{|z - z_0| = r\}$ to be

$$A(r) = \int_0^{2\pi} h(z_0 + re^{i\theta})\frac{d\theta}{2\pi} \qquad 0 < r < \rho$$

Theorem 3.4.1. If $u(z)$ is a harmonic function on a domain D, and if the disk $\{|z - z_0| < \rho\}$ is contained in D, then

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}, \qquad 0 < r < \rho$$

In other words, the average value of a harmonic function of the boundary circle of any disk contained in D is its value at the center of he disk.

Mean Value Property A continuous function $h(z)$ on a domain D has he **mean value property** if for each point $z_0 \in D$, $h(z_0)$ is the average of its values over any small circle centered at z_0. More formally, for any $z_0 \in D$, there is an $\varepsilon > 0$ such that

$$h(z_0) = \int_0^{2\pi} h(z_0 + re^{i\theta})\frac{d\theta}{2\pi}, \qquad 0 < r < \varepsilon$$

In other words harmonic functions have the mean value property.

Mean Value Property Affine Functions A function $f(t)$ on an interval $I = (a, b)$ has the **mean value property** if

$$f\left(\frac{s+t}{2}\right) = \frac{f(s) + f(t)}{2}$$

The any affine function $f(t) = At + B$ has the mean value property. Furhter, any continuous function on I with the mean value property is affine.

3.5 The Maximum Principle

Theorem 3.5.1 (Strict Maximum Principle (Real Version)). Let $u(z)$ be a real-valued harmonic function on a domain D such that $u(z) \leqslant M$ for all $z \in D$. If $u(z_0) = M$ for some $z_0 \in D$, then $u(z) = M$ for all $z \in D$.

Theorem 3.5.2 (Strict Maximum Principle (Complex Version)). Let h be a bounded complex-valued harmonic function on a domain D. If $|h(z)| \leqslant M$ for all $z \in D$, and $|h(z_0)| = M$ for some $z_0 \in D$, then $h(z)$ is constant on D.

Theorem 3.5.3 (Maximum Principle). Let $h(z)$ be a complex-valued harmonic function on a bounded domain D such that $h(z)$ extends continuously to the boundary ∂D of D. If $|h(z)| \leqslant M$ for all $z \in \partial D$, then $|h(z)| \leqslant M$ for all $z \in D$.

Chapter 4 Complex Integraion and Analyticity

4.1 Complex Line Integrals

This section begins with a number of examples.

Theorem 4.1.1 (ML-estimate). Suppose γ is a piecewise smooth curve. If $h(z)$ is a continuous function on γ, then

$$\left|\int_\gamma h(z)\,\mathrm{d}z\right| \leqslant \int_\gamma |h(z)||\,\mathrm{d}z|$$

Further, if γ has length L, and $|h(z)| \leqslant M$ on γ, then

$$\left|\int_\gamma h(z)\,\mathrm{d}z\right| \leqslant ML$$

Sharp Estimate If equality holds on the estimate, then the estimate is a **sharp estimate**.

4.2 Fundamental Theorem of Calculus for Analytic Functions

(Complex) Primitive Let $f(z)$ be a continuous function on a domain D. A function $F(z)$ on D is a **(complex) primitive** for $f(z)$ if $F(z)$ is analytic and $F'(z) = f(z)$.

Theorem 4.2.1 (Part I). If $f(z)$ is continuous on a domain D, and if $F(z)$ is a primitive for $f(z)$ then

$$\int_A^B f(z)\,\mathrm{d}z = F(B) - F(A)$$

where the integral can be taken over any path in D from A to B.

Theorem 4.2.2 (Part II). Let D be a star-shaped domain, and let $f(z)$ be analytic on D. THen $f(z)$ has a primitive on D, and the primitive is unique up to adding a constant. A primitive for $f(z)$ is given explicitly by

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta, \qquad z \in D$$

where z_0 is any fixed point of D, and where the integral can be taken along any path in D from z_0 to z.

4.3 Cauchy's Theorem

Theorem 4.3.1. A continuously differentiable function $f(z)$ on D is analytic if and only if the differential $f(z)\,\mathrm{d}z$ is closed.

Theorem 4.3.2 (Cauchy's Theorem). Let D be a bounded domain with piecewise smooth boundar. If $f(z)$ is an analytic function on D that extends smoothly to ∂D, then

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0$$

4.4 The Cauchy Integral Formula

Theorem 4.4.1 (Cauchy Integral Formula). Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and $f(z)$ extends smoothly to the boundary of D, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w, \qquad z \in D$$

Theorem 4.4.2. Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to the boundary of D, then $f(z)$ has complex derivatives of all orders on D, which are given by:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,\mathrm{d}w \qquad z \in D,\ m \geqslant 0$$

Theorem 4.4.3. If $f(z)$ is analytic on a domain D, then $f(z)$ is infinitely differentiable, and the successive complex derivatives $f'(z)$, $f''(z)$, ... are all analytic on D.

4.5 Liouville's Theorem

Theorem 4.5.1 (Cauchy Estimates). Suppose $f(z)$ is analytic for $|z-z_0| \leqslant \rho$. If $|f(z)| \leqslant M$ for $|z-z_0| = \rho$, then

$$\left|f^{(m)}(z_0)\right| \leqslant \frac{m!}{\rho^m}M \qquad m \geqslant 0$$

Theorem 4.5.2 (Liouville's Theorem). Let $f(z)$ be an analytic function on the complex plane. If $f(z)$ is bounded, then $f(z)$ is constant.

Entire Function An **entire function** is a function that is analytic on the entire complex plane.

This transform's Liouville's Theorem: A bounded entire function is constant.

4.6 Morera's Theorem

Theorem 4.6.1 (Morera's Theorem). Let $f(z)$ be a continuous function on a domain D. If $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle R contained in D with sides parallel to the coordinate axes, then $f(z)$ is analytic on D.

Theorem 4.6.2. Suppose that $h(t,z)$ is a continuous complex-valued function, defined for $a \leqslant t \leqslant b$ and $z \in D$. If for each fixed t, $h(t,z)$ is an analytic function of $z \in D$, then

$$H(z) = \int_a^b h(t,z)\,dt \qquad z \in D$$

is analytic on D.

Theorem 4.6.3. Suppose that $f(z)$ is a continuous function on a domain D that is analytic on $D \backslash \mathbb{R}$, that is, on the part of D not lying on the real axis. Then $f(z)$ is analytic on D.

4.7 Goursat's Theorem

Theorem 4.7.1 (Goursat's Theorem). If $f(z)$ is a complex-valued function on a domain D such that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists at each point z_0 of D, then $f(z)$ is analytic on D.

4.8 Complex Notation and Pompeiu's Formula

Complex form of the Cauchy-Riemann Equations For $f = u + iv$:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$$

The Cauchy-Riemann equations yield that

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

This is referred to as the **complex form of the Cauchy-Riemann equations.**

Theorem 4.8.1 (Analytic $\frac{\partial f}{\partial \bar{z}} = 0$). Let $f(z)$ be a continuously differentiabe function on a domain D. Then $f(z)$ is analytic if and only if $f(z)$ satisfies the complex form of the Cauchy-Riemann equation:

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

If $f(z)$ is analytic then the derivative of $f(z)$ is given by

$$f'(z) = \frac{\partial f}{\partial z}$$

Theorem 4.8.2. Let $f(z)$ be a continuously differentiable function on a domain D. Suppose that the gradient of $f(z)$ does not vanish at any point of D, and that $f(z)$ is conformal. Then $f(z)$ is analytic on D, and $f'(z) \neq 0$ on D.

Theorem 4.8.3. If D is a bounded domain in the complex plane with piecewise smooth boundary, and if $g(z)$ is a smooth function on $D \cup \partial D$, then

$$\int_{\partial D} g(z)\,dz = 2i \iint\limits_{D} \frac{\partial g}{\partial \bar{z}}\,dx\,dy$$

Theorem 4.8.4 (Pompeiu's Formula). SUppose D is a bounded domain with piecewise smooth boundary. If $g(z)$ is a smooth complex-valued function on $D \cup \partial D$, then

$$g(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(z)}{z-w}\,dz - \frac{1}{\pi} \iint\limits_{D} \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w}\,dx\,dy, \qquad w \in D$$

This equation is also known as the **Cauchy-Green formula**.

Chapter 5 Power Series

5.1 Infinite Series

Converge A series $\sum_{k=0}^{\infty} a_k$ of complex nubers is said to **converge to** S if the sequence of partial sums $\{S_k\}$, defined by $S_k = a_0 + \cdots + a_k$, converges to S.

Theorem 5.1.1 (Comparison Test). If $0 \leqslant a_k \leqslant r_k$ and if $\sum r_k$ converges, then $\sum a_k$ converges, and $\sum a_k \leqslant \sum r_k$.

Theorem 5.1.2. If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

Theorem 5.1.3. If $\sum a_k$ converges absolutely, then $\sum a_k$ converges, and

$$\left|\sum_{k=0}^{\infty} a_k\right| \leqslant \sum_{k=0}^{\infty} |a_k|$$

Cauchy Criterion for Series The series $\sum a_k$ converges if and only if $\sum_{k=m}^{k=n} a_k$ tends to 0 as $m, n \to \infty$.

5.2 Sequences and Series of Functions

Sequential Pointwise Convergence Let $\{f_j\}$ be a sequence of complex-valued functions defined on some set E. We say that the sequence $\{f_j\}$ **converges pointwise** on E if for each point $x \in E$ the sequence of complex numbers $\{f_j(x)\}$ converges. The limit $f(x)$ of $\{f_j(x)\}$ is then a complex-valued function on E.

Sequential Uniform Convergence We say that the sequence $\{f_j\}$ of functions on E **converges uniformly** to f on E if $|f_j(x) - f(x)| \leqslant \varepsilon_j$ for all $x \in E$, where $\varepsilon_j \to 0$ as $j \to \infty$.

We can regard ε_j as a worst-case estimator for the difference $f_j(x) - f(x)$, and usually we take ε_j to e the supremum of $|f_j(x) - f(x)|$ over $x \in E$.

Theorem 5.2.1. Let $\{f_j\}$ be a sequence of complex-valued functions defined on a subset E of the complex plane. If each f_j is continuous on E, and of $\{f_j\}$ converges uniformly to f on E, then f is continuous on E.

Theorem 5.2.2. Let γ be a piecewise smooth curve in the complex plane. If $\{f_j\}$ is a sequence of continuous complex-valued functions on γ, and if $\{f_j\}$ converges uniformly to f on γ, then $\int_\gamma f_j(z)\,dz$ converges to $\int_\gamma f(z)\,dz$.

Convergence of Series of Functions Let $\sum g_j(x)$ be a series of complex valued functions defined on a set E. The partial sums of the series are the functions

$$S_n(x) = \sum_{k=0}^{n} g_j(x) = g_0(x) + g_1(x) + \cdots + g_n(x).$$

We say that the series **converges pointwise** on E if the sequence of partial sums converges pointwise on E, and the series **converges uniformly** on E if the sequence of partial sums converges uniformly on E.

Theorem 5.2.3 (Weierstrass M-Test). Suppose $M_k \geqslant 0$ and $\sum M_k$ converges. If $g_k(x)$ are complex-valued functions on a set E such that $|g_k(x)| \leqslant M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E.

Theorem 5.2.4. If $\{f_k(x)\}$ is a sequence of analytic functions on a domain D that converges uniformly to $f(z)$ on D, then $f(z)$ is analytic on D.

Theorem 5.2.5. Suppose that $f_k(z)$ is analytic for $|z - z_0| \leqslant R$, and suppose that the sequence $\{f_k(z)\}$ converges uniformly to $f(z)$ for $|z - z_0| \leqslant R$. Then for each $r < R$ and for each $m \geqslant 1$, the sequence of mth derivatives $\{f_k^{(m)}(z)\}$ converges uniformly to $f^{(m)}(z)$ for $|z - z_0| \leqslant r$.

Normal Convergence We say that a sequence $\{f_k(z)\}$ of analytic functions on a domain D **converges normally** to the analytic function $f(z)$ on D if it converges uniformly to $f(z)$ on each closed disk ocntained in D.

This occurs if and only if $\{f_k(z)\}$ converges to $f(z)$ uniformly on each bounded subset E of D at a strictly positive distance from the boundary of D.

Theorem 5.2.6. Suppose that $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to the analytic function $f(z)$. Then for each $m \geqslant 1$, the sequence of mth derivatives $\{f_k^{(m)}(z)\}$ converges normally to $f^{(m)}(z)$ on D.

5.3 Power Series

Power Series A **power series** (centered at z_0) is a series of the form $\sum_{k=0}^{\infty} a_k(z - z_0)^k$.

By making a change of variable $w = z - z_0$ we can always reduce to the case of the power series centered at $z = 0$.

Theorem 5.3.1. Let $\sum a_k z^k$ be a power series. Then there is R, $0 \leqslant R \leqslant +\infty$, such taht $\sum a_k z^k$ converges absolutely if $|z| < R$, and $\sum a_k z^k$ does not converge if $|z| > R$. For each fixed r satisfying $r < R$, the series $\sum a_k z^k$ converges uniformly for $|z| \leqslant r$.

Radius of Convergence We call R the **radius of convergence** of the series $\sum a_k z^k$.

Theorem 5.3.2. Suppose $\sum a_k z^k$ is a power series with radius of convergence $R > 0$. Then the function

$$f(z) = \sum_{k=)}^{\infty} a_k z^k \qquad |z| < R$$

is analytic. The derivatives of $f(z)$ are obtained by differentiating the series term by term,

$$f'(z) \sum_{k=1}^{\infty} k a_K z^{k-1} \qquad f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2}, \qquad |z| < R$$

amd similarly for the higher-order derivatives. The coefficients of the series are given by

$$a_k = \frac{1}{k!} f^{(k)}(0), \qquad k \geqslant 0$$

Theorem 5.3.3 (Ratio Test). If $|a_k/a_{k+1}|$ has a limit as $k \to \infty$, either finite of $+\infty$, then the limit is the radius of convergence R of $\sum a_k z^k$,

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

Theorem 5.3.4 (Root Test). If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, either finite of $+\infty$, then the radius of converence of $\sum a_k z^k$ is given by

$$R = \frac{1}{\lim \sqrt[k]{|a_k|}}.$$

More generally, we can use the **Cauchy-Hadamard formula**

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}.$$

5.4 Power Series Expansion of an Analytic Function

Theorem 5.4.1. Suppose that $f(z)$ is analytic for $|z - z_0| < \rho$. Then $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad |z - z_0| < \rho \qquad a_k = \frac{f^{(k)}(z_0)}{k!}, \qquad k \geqslant 0$$

and where the power series has radius of convergence $R \geqslant \rho$. For any fixed f, $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta, \qquad k \geqslant 0$$

Further, if $|f(z)| \leqslant M$ for $|z - z_0| = r$, then

$$|a_k| \leqslant \frac{M}{r^k}, \qquad k \geqslant 0$$

Corollary 5.4.2. Suppose that $f(z)$ and $g(z)$ are analytic for $|z - z_0| < r$. If $f^{(k)}(z_0) = g^{(k)}(z_0)$ for $k \geqslant 0$, then $f(z) = g(z)$ for $|z - z_0| < r$.

Corollary 5.4.3. Suppose that $f(z)$ is analytic at z_0, with power series expansion $f(z) = \sum a_k (z - z_0)^k$ centered at z_0. Then the radius of convergence of the power series is the largest number R such that $f(z)$ extends to be analytic on the disk $\{|z - z_0| < R\}$.

5.5 Power Series Expansion at Infinity

Analytic at Infinity We say that a function $f(z)$ is **analytic at** $z = \infty$ if the function $g(w) = f(1/w)$ is analytic at $w = 0$.

We can make a change of variable $w = 1/z$ and thus study $f(z)$ at $z = \infty$ by studying the behavior of $g(w)$ at $w = 0$.

If $f(z)$ is analytic at ∞, then $g(w) = f(1/w)$ has a power series expansion centered at $w = 0$,

$$g(w) = \sum_{k=0}^{\infty} b_k w^k \qquad |w| < \rho \quad \Longrightarrow \quad f(z) = \sum_{k=0}^{\infty} \frac{b_k}{z^k} \qquad |z| > \frac{1}{\rho}$$

5.6 Manipulation of Power Series

Manipulation of Power Series Consider the following functions:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

a. $f(z) + g(z) = \sum_{k=0}^{\infty} (a_k + b_k) z^k$

b. $cf(z) = \sum_{k=0}^{\infty} c a_k z^k$

c. $f(z)g(z) = \sum_{k=0}^{\infty} c_k z_k \qquad c_k = a_k b_0 + a_{k-1} b_1 + \cdots + a_0 b_k$

d. $\dfrac{1}{g(z)} = \dfrac{1}{1 + \sum_{k=1}^{\infty} b_k z^k} = 1 - \left(\sum_{k=1}^{\infty} b_k z^k\right) + \left(\sum_{k=1}^{\infty} b_k z^k\right)^2 - \left(\sum_{k=1}^{\infty} b_k z^k\right)^3 + \ldots$

5.7 The Zeros of an Analytic Function

Zero of Order N Let $f(z)$ be analytic at z_0 and suppose that $f(z_0) = 0$ but $f(z)$ is not identically zero. We say that $f(z)$ has a **zero of order** N at z_0 if $f(z_0) = f'(z_0) =$

$\dots = f^{(N-1)}(z_0) = 0$, while $f^{(N)}(z_0) \neq 0$.

This occurs if and only if the power series exampsion of $f(z)$ has the form:

$$f(z) = a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \dots = (z - z_0)^N h(z)$$

where $h(z)$ is analytic at z_0 and $h(z_0) = a_N \neq 0$

A zero of order one is called a **simple zero**, and a zero of order two is called a **double zero**.

Zero of Order Infinity If $f(z)$ is analytic at ∞ and $f(\infty) = 0$, we define the order of the zero of $f(z)$ at $z = \infty$ in the usual way, by making the change of variable $w = 1/z$. We say that $f(z)$ has a **zero at** $z = \infty$ **of order** N if $g(w) = f(1/w)$ has a zero at $w = 0$ of order N.

Thus $g(w) = b_N w^N + b_{N+1} w^{N+1} + \dots$ and subsequently

$$f(z) = \frac{b_N}{z^N} + \frac{b_{N+1}}{z^{N+1}} + \dots, \qquad |z| > R$$

Isolated Point We say that a point $z_0 \in E$ is an **isolated point** of the set E if there is a $\rho > 0$ such that $|z - z_0| \geqslant \rho$ for all points $z \in E$ other than z_0.

That is, z_0 is an isolated point of E if z_0 is a positive distance from $E \backslash \{z_0\}$.

Theorem 5.7.1. If D is a domain, and $f(z)$ is an analytic function on D that is not identically zero, then the zeros of $f(z)$ are isolated.

Theorem 5.7.2 (Uniqueness Principle)**.** If $f(z)$ and $g(z)$ are analytic on a domain D, and if $f(z) = g(z)$ for z belonging to a set that has a nonisolated point, then $f(z) = g(z)$ for all $z \in D$.

Theorem 5.7.3. Let D be a domain, and let E be a subset of D that has a nonisolated point. Let $F(z, w)$ be a function defined for $z, w \in D$ such that $F(z, w)$ is analytic in z for each fixed $w \in D$ and analytic in w for each fixed $z \in D$. If $F(z, w) = 0$ whenever z and w both belong to E, then $F(z, w) = 0$ for all $z, w \in D$.

Theorem 5.7.4 (Open Mapping Theorem for Analytic Functions)**.** If $f(z)$ is a nonconstant analytic function on a domain D, then the image under $f(z)$ of any open set is open.

5.8 Analytic Continuation

Lemma 5.8.1. Suppose D is a disk, $f(z)$ is analytic on D, and $R(z_1)$ is the radius of convergence of the power series expansion of $f(z)$ about a point $z_1 \in D$. Then

$$|R(z_1) - R(z_2)| \leqslant |z_1 - z_2|, \qquad z_1, z_2 \in D$$

Analytically Continuable Along We start with a power series $\sum a_n(z-z_0)^n$ that represents a function $f(z)$ near z_0. We are interested in the behavior of $f(z)$ only near z_0, and we say that the power series represents the "germ" of $f(z)$ at z_0. Let $\gamma(t)$, $a \leqslant t \leqslant b$, be a path starting at $z_0 = \gamma(a)$.

We say that $f(z)$ is **analytically continuable along** γ if for each t there is a convergent power series

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z-\gamma(t))^n, \qquad |z-\gamma(t)| < r(t)$$

such that $f_a(z)$ is the power series representing $f(z)$ at z_0, and such that when s is near t, then $f_s(z) = f_t(z)$ for z in the intersection of the disks of convergence.

By the uniqueness principle, the series $f_t(z)$ determines uniquely each of the series $f_s(z)$ for s near t.

Analytic Continuation We refer to $f_b(z)$ as the **analytic continuation** of $f(z)$ along γ, wehere we regard $f_b(z)$ either as a power series or as an analytic function defined near $\gamma(b)$.

Theorem 5.8.1. Suppose $f(z)$ can be continued analytically along the path $\gamma(t)$, $a \leqslant t \leqslant b$. Then the analytic continuation is unique. Further, for each $n \geqslant 0$ the coefficient $a_n(t)$ of the series depends continuously on t, and the radius of convergence of the series depends continuously on t.

Lemma 5.8.2. Suppose $f(z)$ is analytic at z_0 and suppose that $\gamma(t)$, $a \leqslant t \leqslant b$, is a path from $z_0 = \gamma(a)$ to $z_1 = \gamma(b)$ along which $f(z)$ has an analytic continuation $f_t(z)$. The radius of convergence $R(t)$ of the power series varies continuously with t. Hence there is $\delta > 0$ such that $R(t) \geqslant \delta$ for all t, $a \leqslant t \leqslant b$. If $\sigma(t)$ is another path from z_0 to z_1 such that $|\sigma(t) - \gamma(t)| < \delta$, then there is an analytic continuation $g_t(z)$ of $f_t(z)$ along σ, and the terminal series $g_b(z)$ centered at $\sigma(b) = z_1$ coincides with f_b.

(See picture on Page 161 of text)

Theorem 5.8.2 (Monodromy Theorem). Let $f(z)$ be analyticc at z_0. Let $\gamma_0(t)$ and $\gamma_1(t)$, $a \leqslant t \leqslant b$, be two paths from z_0 to z_1 along which $f(z)$ can be continued analytically. Suppose $\gamma_0(t)$ can be deforemed continuously to $\gamma_1(t)$ by paths $\gamma_s(t)$, $0 \leqslant s \leqslant 1$, from z_0 to z_1 such that $f(z)$ can be continued analytically along each path γ_s. Then the analytic continuations of $f(z)$ along γ_0 and along γ_1 coincide at z_1.

Chapter 6 Laurent Series and Isolated Singularities

6.1 The Laurent Decomposition

Theorem 6.1.1 (Laurent Decomposition). Suppose $0 \leqslant \rho < \sigma \leqslant +\infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. Then $f(z)$ can be decomposed as a sum

$$f(z) = f_0(z) + f_1(z)$$

where $f_0(z)$ is analytic for $|z - z_0| < \sigma$, and $f_1(z)$ is analytic for $|z - z_0| > \rho$ and at ∞. If we normalize the decomposition so that $f_1(\infty) = 0$, then the decomposition is unique.

Laurent Series Expansion Suppose that $f(z) = f_0(z) + f_1(z)$ is the Laurent decompositino for a function analytic for $\rho < |z - z_0| < \sigma$. We can express $f_0(z)$ as a power series in $z - z_0$:

$$f_0(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k, \qquad |z - z_0| < \sigma$$

where the series converges absolutely, and for any $s < \sigma$ it converges uniformly for $|z - z_0| \leqslant s$. Further, we can also express $f_1(z)$ as a series of negative powers of $z - z_0$, with zero constant term, since $f_1(z)$ tends to 0 at ∞,

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k(z - z_0)^k, \qquad |z - z_0| > rho$$

This series converges absolutely, and for any $r > \rho$ it converges uniformly for $|z - z_0| \geqslant r$. If we add the two series, we obtain a two-tailed expansion for $f(z)$,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k \qquad \rho < |z - z_0| < \sigma$$

that converges absolutely, and that converges uniformly for $r \leqslant |z - z_0| \leqslant s$. The last series is called the **Laurent series expansion** of $f(z)$ with respect to the annulus $\rho < |z - z_0| < \sigma$.

Theorem 6.1.2 (Laurent Series Expansion). Suppose $0 \leqslant \rho < \sigma \leqslant \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. Then $f(z)$ has a Laurent expansion that converges absolutely at each point of the annulus, and that converges uniformly on each subannulus $r \leqslant |z - z_0| \leqslant s$, where $\rho < r < s < \sigma$. The coefficients are uniquely determined by $f(z)$, and they are given by

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz, \qquad -\infty < n < \infty$$

for any fixed r, $\rho < r < \sigma$.

6.2 Isolated Singularities of an Analytic Function

Isolated Singularity A point z_0 is an **isolated singularity** of $f(z)$ if $f(z)$ is analytic in some punctured disk $\{0 < |z - z_0| < r\}$ centered at z_0.

Theorem 6.2.1. Suppose $f(z)$ has an isolated singularity at z_0. Then $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k. \qquad 0 < |z - z_0| < r$$

Removable Singularity The isolated singularity of $f(z)$ at z_0 is defined to be a **removable singularity** if $a_k = 0$ for all $k < 0$. IN this case the Laurent series becomes a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad 0 < |z - z_0| < r$$

If we define $f(z_0) = a_0$, the function $f(z)$ becomes analytic on the entire disk $\{|z - z_0| < r\}$.

Theorem 6.2.2 (Riemann's Theorem on Removable Singularities). Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_0, then $f(z)$ has a removable singularity at z_0.

Pole The isolated singularity of $f(z)$ at z_0 is defined to be a **pole** if there is $N > 0$ such that $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$.

The integer N is the **order** of the pole.

In this case the Laurent series becomes

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k$$

The sum of the negative powers

$$P(z) = \sum_{k=-N}^{-} 1a_k(z - z_0)^k = f_1(z)$$

is called the **principal part** of $f(z)$ at the pole z_0. Then $f(z) - P(z)$ is analytic at z_0.

A pole of order one is called a **simple pole**, abd a pole of order two is called a **double pole**.

Theorem 6.2.3. Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $f(z) = g(z)/(z - z_0)^N$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Theorem 6.2.4. Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $1/f(z)$is analytic at z_0 and has a zero of order N.

Meromorphic We say that a function $f(z)$ is **meromorphic** on a domain D if $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole.

Theorem 6.2.5. Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Essential Singularity The isolated singularity of $f(z)$ at z_0 is defined to be an **essentail singularity** if $a_k \neq 0$ for infinitely many $k < 0$.

An isolated singularity that is neither removable nor a pole is declared to be essential.

Theorem 6.2.6 (Casorati-Weierstrass Theorem). Suppose z_0 is an essential isolated singularity of $f(z)$. Then for every complex number w_0, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$.

6.3 Isolated Singularity at Infinity

Isolated Singularity at Infinity We say that $f(z)$ has an **isolated singularity at** ∞ if $f(z)$ is analytic outside some bounded set, that is, if there is $R > 0$ such that $f(z)$ is analytic for $|z| > R$. Thus $f(z)$ has an isolated singularity at ∞ if and only if $g(w) = f(1/w)$ has an isolated singularity at $w = 0$.

Removable Singularity Suppose $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \qquad |z| > R$$

The singularity of $f(z)$ at ∞ is **removable** if $b_k = -$ for all $k > 0$, in which case $f(z)$ is analytic at ∞.

Essential Singularity Suppose $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \qquad |z| > R$$

The singularity of $f(z)$ at ∞ is **essential** if $b_k \neq 0$ for infinitely many $k > 0$.

Pole Suppose $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \qquad |z| > R$$

For fixed $N \geqslant 1$, $f(z)$ has a **pole** of order N at ∞ if $b_N \neq 0$ while $b_k = 0$ for $k > N$.

Principal Part of $f(z)$ at Infinity Suppose $f(z)$ has a pole of order N at ∞. The Laurent series expansion of $f(z)$ becomes

$$f(z) = b_N z^N + b_{N-1} z^{N-1} + \cdots + b_0 + \frac{b_{-1}}{z} + \ldots, \qquad |z| > R$$

where $b_N \neq 0$. We define the **principal part of $f(z)$ at ∞** to be the polynomial

$$P(z) = b_N z^N + b_{N-1} z^{N-1} + \cdots + b_1 z + b_0$$

Then $f(z) - P(z)$ is analytic at ∞ and vanishes there.

6.4 Partial Fractions Decomposition

Meromorphic A function $f(z)$ is **meromorphic** on a ddomain D in the extended complex plane $\mathbb{C}^*$ if $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole.

Theorem 6.4.1. A meromorphic function on the extended complex plane $\mathbb{C}^*$ is rational.

Partial Fraction Decomposition Breaking $f(z) = P_\infty(z) + \sum_{j=1}^{m} P_j(z)$ is called the partial fractions decomposition of the rational function $f(z)$.

Theorem 6.4.2. Every rational function has a partial fractions decomposition, expressing it as the sum of a polynomial in z and its principal parts at each of its poles in the finite complex plane.

Chapter 7 The Residue Calculus

7.1 The Residue Theorem

Residue Suppose z_0 is an isolated singularity of $f(z)$ and that $f(z)$ has Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad 0 < |z - z_0| < \rho$$

The **residue** of $f(z)$ at z_0 is the coefficient a_{-1} of $1/(z - z_0)$, that is

$$\text{Res}\,[f(z), z_0] = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\,\mathrm{d}z$$

where r is any fixed radius satisfying $0 < r < \rho$.

Theorem 7.1.1 (Residue Theorem). Let D be a bounded domain in the complex plane with piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$, except for a finite number of isolated singularities $z_1, \ldots, z_m$ in D. Then

$$\int_{\partial D} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{m} \text{Res}\,[f(z), z_j]$$

Rule 1 If $f(z)$ has a simple pole at z_0, then

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

Rule 2 If $f(z)$ has a double pole at z_0, then

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} \frac{d}{dz} \left[(z - z_0)^2 f(z) \right]$$

Rule 3 If $f(z)$ and $g(z)$ are analytic at z_0 and if $g(z)$ has a simple zero at z_0, then

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

Rule 4 If $g(z)$ is analytic and has a simple zero at z_0, then

$$\text{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$$

7.2 Integrals Featuring Rational Functions

Main Method

- Locate poles or singularities
- Define an appropriate contour around those poles such that the poles are within the countour (normally a disk of radius R)
- Calculate residues
- Use Residue Theorem to calculate the integral of the whole contour
- Take the limit of the real component, γ_1, to get I
- Use the ML estimate to show that the line integral of top arc, γ_2.
- Use $\int_{\partial D} = \int_{\gamma_1} + \int_{\gamma_2}$
- Take limits carefully (This is not done well in the subsequent example.

Example: 7.2.4 Using residue theory, show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

Proof. Let $I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$.

Consider $f(z) = \frac{1}{z^4+1}$ where $a > 0$. Note that $f(z)$ has simple poles at $z = e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, $e^{7\pi i/4}$. Let D_R be the upper half disk of radius R and let $D = \lim_{R\to\infty} D_R$. Calculuating the residues using Rule 3:

$$\text{Res}\left[[,f](z), e^{\pi i/4}\right] = \left.\frac{1}{4z^3}\right|_{e^{\pi i/4}} = \frac{1}{4e^{3\pi i/4}} = \frac{\sqrt{2}}{4(-1+i)}$$

$$\text{Res}\left[[,f](z), e^{3\pi i/4}\right] = \left.\frac{1}{4z^3}\right|_{e^{3\pi i/4}} = \frac{1}{4e^{\pi i/4}} = \frac{\sqrt{2}}{4(1+i)}$$

Thus for all R, by the Residue Theorem,

$$\int_{\partial D_R} \frac{dz}{z^4+1} = 2\pi i\left(\frac{\sqrt{2}}{4(-1+i)} + \frac{\sqrt{2}}{4(1+i)}\right) = 2\pi i\left(\frac{-i}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

Examining ∂D we see that we can break it into two pieces, γ_1 the piece along the x-axis and γ_2 be the arc. Let $\gamma_{1,R}$, $\gamma_{2,R}$ be the corresponding pieces of ∂D_R Thus, for all R, $\partial D_R = \gamma_{1,R} \cup \gamma_{2,R}$.

Examining the integral over γ_1:

$$\int_{\gamma_1} \frac{dz}{z^4+1} = \lim_{R\to\infty} \int_{\gamma_{1,R}} \frac{dz}{z^4+1} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = I$$

Examining the integral over γ_2, we see using the ML- estimate:

$$\left|\int_{\gamma_2} \frac{dz}{z^4+1}\right| = \left|\lim_{R\to\infty}\int_{\gamma_{2,R}} \frac{dz}{z^4+1}\right| \leqslant \lim_{R\to\infty}\left(\frac{1}{R^4+1}\cdot \pi R\right) \leqslant \lim_{R\to\infty}\frac{\pi}{R^3} = 0$$

Since $\partial D = \gamma_1 \cup \gamma_2$:

$$\int_D \frac{dz}{z^4+1} = \int_{\gamma_1} \frac{dz}{z^4+1} + \int_{\gamma_2} \frac{dz}{z^4+1} = I + 0 = I$$

Therefore $I = \frac{\pi}{\sqrt{2}}$. □

7.3 Integrals of Trigonometric Functions

Main Method

- Use the unit circle as a contour
- Switch everything to z:
  On the unit circle $|z| = 1$ and $z = e^{i\theta}$.

$$\sin\theta = \frac{z^2-1}{2iz} \qquad \cos\theta = \frac{z^2+1}{2z}$$

  Further $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ and the integral is now over $|z| = 1$.
- Locate poles or singularities
- Calculate residues
- Use Residue Theorem to calculate the integral of the whole contour
- Since we wanted the integral $d\theta$ around the circle

Example: 7.3.2 Show using residue theory that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \qquad a > b > 0$$

Proof. For $a > b > 0$, consider $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$. Since for all $|z| = 1$ $z = e^{i\theta}$:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

Notice that $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ substituting these into the integral we see that:

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \int_{|z|=1} \frac{dz}{iz\left(a + b\frac{z^2-1}{2iz}\right)} = \int_{|z|=1} \frac{dz}{\frac{b}{2}z^2 + aiz - \frac{b}{2}}$$

Examining $f(z) = \frac{1}{\frac{b}{2}z^2+aiz-\frac{b}{2}}$, we see that f has simple poles in $|z| \leqslant 1$ at $z = \frac{-ia+\sqrt{b^2-a^2}}{b}$. Calculating the residues using Rule 3 we see that:

$$\begin{aligned}\operatorname{Res}[f(z),0] &= \frac{1}{bz+ai}\Big|_{z=\frac{-ia+\sqrt{b^2-a^2}}{b}} = \frac{1}{-ia+\sqrt{b^2-a^2}+ia} \\ &= \frac{1}{\sqrt{b^2-a^2}} = \frac{1}{i\sqrt{a^2-b^2}}\end{aligned}$$

Therefore, using residue theory we see that:

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_{|z|=1} \frac{dz}{\frac{b}{2}z^2+aiz-\frac{b}{2}} = 2\pi i\frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

□

7.4 Integrands with Branch Points

Main Method

- Locate poles or singularities
- Define an appropriate contour around those poles such that the poles are within the countour (normally a keyhole contour around pole)
- Calculate residues
- Use Residue Theorem to calculate the integral of the whole contour
- Take the limit of the top real component, γ_1, to get I
- Use the ML estimate to show that the line integral of large arc, γ_2.
- Take the limit of the bottom real component, γ_3, to get $e^{\omega}I$ where ω is some multiple of $i\arg z$.
- Use the ML estimate (or fractional residue in next section) to show that the line integral of small arc, γ_4.
- Use $\int_{\partial D} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}$
- Take limits carefully (This is not done well in the subsequent example.

Example: 7.4.1 By integrating around the keyhole contour, show that

$$\int_0^{\infty} \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin(\pi a)}, \qquad 0 < a < 1.$$

Proof. Let $I = \int_0^{\infty} \frac{x^{-a}}{1+x}\,dx$ and $0 < a < 1$.

Consider

$$f(z) = \frac{z^{-a}}{1+z} = \frac{|z|^{-a}e^{-ai\arg z}}{1+z}$$

Note that $f(z)$ has a simple pole at $z = -1$ and a pole of order a at $z = 0$. Let D_R keyhole contour of radius R (where R is the radius of the outer arc and ε is the radius of the inner arc as is standard) and let $D = \lim_{R\to\infty} D_R$. Calculuating the residues using Rule 1:

$$\operatorname{Res}[[,f](z),-1] = \lim_{z\to -1} |z|^{-a}e^{-ai\arg z} = 1^{-a}e^{-ai\pi} = e^{-ai\pi}$$

Thus for all R, by the Residue Theorem,

$$\int_{\partial D_R} \frac{z^{-a}}{1+z}\,\mathrm{d}z = 2\pi i e^{-ai\pi}$$

Examining ∂D we see that we can break it into four pieces, γ_1 is the limit as $\varepsilon \to 0$ of the piece along the positive x-axis above the x-axis, γ_2 is the large outer arc, γ_3 is the limit as $\varepsilon \to 0$ of the piece along the positive x-axis below the x-axis, and γ_4 is the small inner arc of radius ε. Let $\gamma_{1,R}$, $\gamma_{2,R}$, $\gamma_{3,R}$, and $\gamma_{4,\varepsilon}$ be the corresponding pieces of ∂D_R. Thus, for all R, $\partial D_R = \gamma_{1,R} \cup \gamma_{2,R} \cup \gamma_{3,R} \cup \gamma_{4,\varepsilon}$.

Examining the integral over γ_1:

$$\int_{\gamma_1} \frac{z^{-a}}{1+z}\,\mathrm{d}z = \lim_{R\to\infty}\int_{\gamma_{1,R}} \frac{z^{-a}}{1+z}\,\mathrm{d}z = \lim_{R\to\infty}\int_0^R \frac{x^{-a}}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{x^{-a}}{1+x}\,\mathrm{d}x = I$$

Examining the integral over γ_2, we see using the ML-estimate:

$$\left|\int_{\gamma_2} \frac{z^{-a}}{1+z}\,\mathrm{d}z\right| = \left|\lim_{R\to\infty}\int_{\gamma_{2,R}} \frac{z^{-a}}{1+z}\,\mathrm{d}z\right| \leqslant \lim_{R\to\infty}\left(\frac{R^{-a}}{1+R}\cdot 2\pi R\right) \leqslant \lim_{R\to\infty}\frac{2\pi}{R^a} = 0$$

Examining the integral over γ_3:

$$\begin{aligned}\int_{\gamma_3} \frac{z^{-a}}{1+z}\,\mathrm{d}z &= \lim_{R\to\infty}\int_{\gamma_{3,R}} \frac{|z|^-ae^{-ai\arg z}}{1+z}\,\mathrm{d}z \\ &= \lim_{R\to\infty}\int_R^0 \frac{x^-ae^{-2\pi ia}}{1+x}\,\mathrm{d}x \\ &= e^{-2\pi ia} - \int_0^\infty \frac{x^{-a}}{1+x}\,\mathrm{d}x \\ &= -e^{-2\pi ia}I\end{aligned}$$

Examining the integral over γ_4, we see using the ML-estimate:

$$\left|\int_{\gamma_{4,\varepsilon}} \frac{z^{-a}}{1+z}\,\mathrm{d}z\right| = \left|\lim_{\varepsilon\to 0}\int_{\gamma_{4,\varepsilon}} \frac{z^{-a}}{1+z}\,\mathrm{d}z\right| \leqslant \lim_{\varepsilon\to 0}\frac{\varepsilon^{-a}}{1+\varepsilon}\cdot 2\pi\varepsilon| \leqslant \lim_{\varepsilon\to 0}\frac{2\pi\varepsilon^{1-a}}{1+\varepsilon} = 0$$

Since $\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

$$\begin{aligned}\int_D \frac{z^{-a}}{1+z}\,\mathrm{d}z &= \int_{\gamma_1} \frac{z^{-a}}{1+z}\,\mathrm{d}z + \int_{\gamma_2} \frac{z^{-a}}{1+z}\,\mathrm{d}z + \int_{\gamma_3} \frac{z^{-a}}{1+z}\,\mathrm{d}z + \int_{\gamma_4} \frac{z^{-a}}{1+z}\,\mathrm{d}z \\ &= I + 0 - e^{-2\pi i a} I + 0 \\ &= \left(1 - e^{-2\pi i a}\right) I \\ &= 2\pi i e^{-\pi i a}\end{aligned}$$

Therefore $I = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \frac{\pi}{\sin(\pi a)}$. □

7.5 Fractional Residues

Theorem 7.5.1 (Fractional Residue Theorem). If z_0 is a simple pole of $f(z)$, and C_ε is an arc of the circle $\{|z - z_0| = \varepsilon\}$ of angle α, then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,\mathrm{d}z = \alpha i \,\mathrm{Res}\left[f(z), z_0\right]$$

Main Method

- Replace things using

$$e^{iaz} = \cos az + i \sin az \qquad z^a = e^{a(\log|z| + i \arg z)}$$

- Locate poles or singularities after switches were made
- Define an appropriate contour around those poles such that the poles are within the countour (normally a keyhole contour around pole)
- Calculate residues
- Use Residue Theorem to calculate the integral of the whole contour
- Take integrals of pieces carefully using residues, ML-Theorem, Fractional residue, and tricks (see hw), combine them creastively
- May need to equate real and imaginary parts.

Example: 7.5.2 Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\,\mathrm{d}x = \pi(1 - e^{-a}), \qquad a > 0.$$

Hint. Replace $\sin(az)$ by e^{iaz}, and integrate around the boundary of a half-disk indented at $z = 0$.

Proof. Let $I = \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\,\mathrm{d}x$.

Consider $f(z) = \frac{e^{iaz}}{z(z^2+1)} = \frac{\cos az + i\sin az}{z(z^2+1)}$. Note that $f(z)$ has singularities at $z = 0, \pm i$. Let D_R upper half disk of radius R indented at 0 (where R is the radius of the outer arc and ε is the radius of the inner arc as is standard) and let $D = \lim_{R\to\infty} D_R$. Calculuating the pertinent residues using Rule 1:

$$\operatorname{Res}[[,f](z),0] = \lim_{z\to 0} z\frac{e^{iaz}}{z(z^2+1)} = \lim_{z\to 0}\frac{e^{iaz}}{z^2+1} = \frac{1}{1} = 1$$

$$\operatorname{Res}[[,f](z),i] = \lim_{z\to i}(z-i)\frac{e^{iaz}}{z(z^2+1)} = \lim_{z\to i}\frac{e^{iaz}}{z(z+i)} = \frac{e^{-a}}{-2} = -\frac{1}{2e^a}$$

Thus for all R, by the Residue Theorem,

$$\int_{\partial D_R} = 2\pi i\left(-\frac{1}{2e^a}\right) = \frac{-\pi i}{e^a}$$

Examining ∂D we see that we can break it into four pieces, γ_1 is the limit as $\varepsilon \to 0$ of the piece along the positive x-axis, γ_2 is the large outer arc, γ_3 is the limit as $\varepsilon \to 0$ of the piece along the negative x-axis, and γ_4 is the limit as $\varepsilon \to 0$ of the small inner arc of radius ε around 0. Let $\gamma_{1,R}$, $\gamma_{2,R}$, $\gamma_{3,R}$, and $\gamma_{4,\varepsilon}$ be the corresponding pieces of ∂D_R. Thus, for all R, $\partial D_R = \gamma_{1,R} \cup \gamma_{2,R} \cup \gamma_{3,R} \cup \gamma_{4,\varepsilon}$.

Examining the integral over γ_1 and γ_3:

$$\begin{aligned}
\int_{\gamma_1}\frac{e^{iaz}}{z(z^2+1)}\,\mathrm{d}z + \int_{\gamma_3}\frac{e^{iaz}}{z(z^2+1)}\,\mathrm{d}z &= \lim_{R\to\infty}\left(\int_{\gamma_{1,R}}\frac{e^{iaz}}{z(z^2+1)}\,\mathrm{d}z + \int_{\gamma_{3,R}}\frac{e^{iaz}}{z(z^2+1)}\,\mathrm{d}z\right)\\
&= \lim_{R\to\infty,\,\varepsilon\to 0}\left(\int_{\varepsilon}^{R}\frac{e^{iax}}{x(x^2+1)}\,\mathrm{d}x + \int_{-R}^{-\varepsilon}\frac{e^{iax}}{x(x^2+1)}\,\mathrm{d}x\right)\\
&= \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iax}}{x(x^2+1)}\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\frac{e^{iax}}{x(x^2+1)}\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\frac{\cos(ax) + i\sin(ax)}{x(x^2+1)}\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\frac{\cos(ax)}{x(x^2+1)}\,\mathrm{d}x + iI
\end{aligned}$$

Examining the integral over γ_2, we see using the ML- estimate:

$$\begin{aligned}
\left|\int_{\gamma_2}\frac{e^{iaz}}{z(z^2+1)}\,\mathrm{d}z\right| &= \left|\lim_{R\to\infty}\int_{\gamma_{2,R}}\frac{e^{iaz}}{z(z^2+1)}\,\mathrm{d}z\right|\\
&\leqslant \lim_{R\to\infty}\left(\frac{1}{R(R^2+1)}\,\mathrm{d}z\cdot\pi R\right)\\
&= \lim_{R\to\infty}\frac{\pi}{R^2+1} = 0
\end{aligned}$$

Examining the integral over γ_4, we see using the fractional residue theorem:

$$\int_{\gamma_4} \frac{e^{iaz}}{z(z^2+1)}\, dz = \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{e^{iaz}}{z(z^2+1)}\, dz = (0-\pi)i \operatorname{Res}[[, f](z), 0] = -\pi i$$

Since $\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

$$\begin{aligned}
\int_D \frac{e^{iaz}}{z(z^2+1)}\, dz &= \int_{\gamma_1} \frac{e^{iaz}}{z(z^2+1)}\, dz + \int_{\gamma_2} \frac{e^{iaz}}{z(z^2+1)}\, dz \\
&\quad + \int_{\gamma_3} \frac{e^{iaz}}{z(z^2+1)}\, dz + \int_{\gamma_4} \frac{e^{iaz}}{z(z^2+1)}\, dz \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{x(x^2+1)}\, dx + iI + 0 - \pi i \\
&= \frac{-\pi i}{e^a}
\end{aligned}$$

Equating real and imaginary parts we see that:

$$-\pi e^{-a} = I - \pi$$

Therefore $I = \pi - \pi e^{-a}$. □

7.6 Principal Values

Absolutely Convergent An integral $\int_a^b f(x)\, dx$ **absolutely convergent** if the (proper or imporoper) integral $\int_a^b |f(x)|\, dx$ is finite.

Absolutely Divergent THe integral is **absolutely divergent** if $\int_a^b |f(x)|\, dx = +\infty$.

Principal Value Suppose that $f(x)$ is continuous for $a \leqslant x < x_0$ and for $x_0 < x \leqslant b$. We define the **principal value** of the integral $\int_a^b f(x)\, dx$ to be

$$\mathrm{PV} \int_a^b f(x)\, dx = \lim_{\varepsilon\to 0} \left(\int_a^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^b \right) f(x)\, dx$$

This value coincides with the usual value of the integral if $f(x)$ is absolutely integrable.

Main Method

- Use definition of PV
- Same tricks as before

Example: 7.6.3 By integrating around the boundary of an indented half-disk in the upper half-plane, show that

$$PV \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}\, dx = -\frac{\pi a}{a^2+1}, \qquad -\infty < a < \infty$$

Proof. Let $I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x$ and so

$$PV = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{a-\varepsilon} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x + \int_{a+\varepsilon}^{\infty} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x\right).$$

Consider $f(z) = \frac{1}{(z^2+1)(z-a)}$. Note that $f(z)$ has singularities at $z = \pm i, a$. Let D_R upper half disk of radius R indented at a (where R is the radius of the outer arc and ε is the radius of the inner arc as is standard) and let $D = \lim_{R\to\infty} D_R$. Calculuating the pertinent residues using Rule 1:

$$\mathrm{Res}\,[f(z), i] = \lim_{z\to i} \frac{1}{(z+i)(z-a)} = \frac{1}{2i(i-a)}$$

$$\mathrm{Res}\,[f(z), a] = \lim_{z\to a} \frac{1}{z^2+1} = \frac{1}{a^2+1}$$

Since there is one singularities in D_R:

$$\int_{\partial D_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z = 2\pi i\,\mathrm{Res}\,[f(z), i] = 2\pi i \frac{1}{2i(i-a)} = \frac{\pi}{i-a}$$

Examining ∂D we see that we can break it into four pieces, γ_1 is the limit as $\varepsilon \to 0$ of the piece along the positive x-axis, γ_2 is the large outer arc, γ_3 is the limit as $\varepsilon \to 0$ of the piece along the negative x-axis, and γ_4 is the limit as $\varepsilon \to 0$ of the small inner arc of radius ε around a. Let $\gamma_{1,R}$, $\gamma_{2,R}$, $\gamma_{3,R}$, and $\gamma_{4,\varepsilon}$ be the corresponding pieces of ∂D_R. Thus, for all R, $\partial D_R = \gamma_{1,R} \cup \gamma_{2,R} \cup \gamma_{3,R} \cup \gamma_{4,\varepsilon}$.

Examining the integral over γ_1 and γ_3:

$$\begin{aligned}
&\int_{\gamma_1} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z + \int_{\gamma_3} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z \\
&= \lim_{R\to\infty}\left(\int_{\gamma_{1,R}} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z + \int_{\gamma_{3,R}} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z\right) \\
&= \lim_{R\to\infty,\,\varepsilon\to 0}\left(\int_{a+\varepsilon}^{R} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x\right) + \lim_{R\to\infty,\,\varepsilon\to 0}\left(\int_{-R}^{a-\varepsilon} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x\right) \\
&= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{a-\varepsilon} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x + \int_{a+\varepsilon}^{\infty} \frac{1}{(x^2+1)(x-a)}\,\mathrm{d}x\right) \\
&= PV
\end{aligned}$$

Examining the integral over γ_2, we see using the ML- estimate:

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z\right| &= \left|\lim_{R\to\infty}\int_{\gamma_{2,R}} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z\right| \\
&\leqslant \lim_{R\to\infty}\left(\frac{1}{(R^2+1)(R-a)}\cdot \pi R\right) \\
&\sim \lim_{R\to\infty}\frac{\pi}{R^2} = 0
\end{aligned}$$

Examining the integral over γ_4, we see using the fractional residue theorem:

$$\begin{aligned}\int_{\gamma_4} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z &= \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z \\ &= (0-\pi)i\,\mathrm{Res}\,[[,f]\,(z),a] \\ &= -\pi i\cdot\frac{1}{a^2+1} \\ &= \frac{-\pi i}{a^2+1}\end{aligned}$$

Since $\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

$$\begin{aligned}\int_D \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z &= \int_{\gamma_1} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z \\ &\quad + \int_{\gamma_3} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z + \int_{\gamma_4} \frac{1}{(z^2+1)(z-a)}\,\mathrm{d}z \\ &= PV + 0 - \frac{\pi i}{a^2+1} \\ &= \frac{\pi}{i-a}\end{aligned}$$

Solving for PV:

$$PV = \frac{-\pi}{a-i} + \frac{\pi i}{a^2+1} = \frac{\pi i - \pi(a+i)}{a^2+1} = \frac{-\pi a}{a^2+1}$$

□

7.7 Jordan's Lemma

Theorem 7.7.1 (Jordan's Lemma). If Γ_R is the semicircular contour $z(\theta) = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, in the upper half plane, then

$$\int_{\Gamma_R} |e^{iz}|\,|\mathrm{d}z| < \pi$$

Main Method

- We use Jordan's Lemma in a similar way as we've used the ML Theorem previously.
- Goal is to take absolute values, bound things above, pull things out so that we get $|e^{iz}|\,|\mathrm{d}z|$
- Use Jordan's Lemma to bound everything
- Take limits and get things to go to zero.

Example Show that

$$\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

To do this problem, they throw it to the indented half-disk. The pertinent part is:

$$\int_{\Gamma_R} \frac{e^{iz}}{z}\,dz \leqslant \frac{1}{R}\int_{\Gamma_R} |e^{iz}||\,dz| < \frac{\pi}{R}$$

When we take the limit as $R \to \infty$, we see the integral is 0.

Above, the first inequality comes from the ML Theorem and the second comes from Jordan's Lemma.

7.8 Exterior Domains

Exterior Domain An **exterior domain** is a domain D in the complex plane that includes all large z, that is, D includes all z such that $|z| \geqslant R$ for some R.

The residue theorem is valid also for exterior domains, though the residue formula must take into account the point at ∞.

Theorem 7.8.1. Let D be an exterior domain with piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$, except for a finite number of isolate singularities $z_1, \ldots, z_m$ in D, and let a_{-1} be the coefficient of $1/z$ in the Laurent expansion $f(z) = \sum a_k z^k$ that converges for $|z| > R$. Then

$$\int_{\partial D} f(z)\,dz = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j]$$

Residue of $f(z)$ at Infinity Suppose $f(z)$ is analytic for $|z| \geqslant R$, with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad |z| \geqslant R$$

We define the **residue of $f(z)$ at ∞** to be $\operatorname{Res}[f(z), \infty] = -a_{-1}$.

If D_R is the exterior domain $\{|z| > R\}$, this definition is equivalent to

$$\int_{\partial D_R} f(z)\,dz = 2\pi i \operatorname{Res}[f(z), \infty]$$

The orientation of the circle $\{|z| = R\}$ with respect to D_R is clockwise, and this accounts for the minus sign. With this definition of residue at ∞:

$$\int_{\partial D} f(z)\,dz = 2\pi i \operatorname{Res}[f(z), \infty] + 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j]$$

Chapter 8 Logarighmic Integral

8.1 The Argument Principle

Logarithmic Integral Suppose $f(z)$ is analytic on a domain D. For a curve γ in D such that $f(z) \neq 0$ on γ, we refer to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \mathrm{dlog} f(z)$$

as the **logarithmic integral** of $f(z)$ along γ.

The logarithmic integral measures the change of $\log f(z)$ along the curve γ.

Theorem 8.1.1. Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_0 - N_\infty$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities.

Increase in the Argument of $f(z)$ Along γ For any continuous path γ in D providing there are no zeros or poles on the path, the quantity

$$\int_\gamma \mathrm{darg}(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$$

is referred to as the **increase in the argument of $f(z)$ along γ**.

Increase in the Argument of $f(z)$ Around the Boundary of D We define the **increase in the argument of $f(z)$ around the boundary of D** to be the sum of it's increases around the closed curves in ∂D.

Theorem 8.1.2. Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then the increase in the argument of $f(z)$ around the boundary of D is 2π times the number of zeros minus the number of poles of $f(z)$ in D.

$$\int_{\partial D} \mathrm{darg}(f(z)) = 2\pi(N_0 - N_\infty)$$

8.2 Rouche's Theorem

Theorem 8.2.1 (Rouche's Theorem). Let D be a bounded domain with piecewise smooth boundary ∂D. LEt $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z)+h(z)$ have the same number of zeros in D, counting multiplicities.

Simple Example: 8.2.1 Show that $2z^5 + 6z - 1$ has one root in the interval $0 < x < 1$ and four roots in the annulus $\{1 < |z| < 2\}$.

Proof. Consider the polynomial $p(z) = 2z^5 + 6z - 1$.

Inside $|z| = 1$, we see that we can split $p(z) = f_1(z) + g_1(z)$ where $f_1(z) = 6z$ and $g_1(z) = 2z^5 - 1$. On $|z| = 1$:

$$|f_1(z)| = |6z| = 6 \qquad |g_1(z)| = |2z^5 - 1| \leqslant 2|z|^5 + 1 = 3$$

Since $|f_1(z)| > |g_1(z)|$ on $|z| = 1$ and $f_1(z)$ has one root in $|z| = 1$, we know by Roche's Theorem that p has one root in $|z| = 1$. Since any complex root comes in a conjugate pair, we know that this root must be a real root. Further, we know that $p(0) = -1 \neq 0$. Thus $p(z)$ has one root in $0 < x < 1$.

Inside $|z| = 2$ we see that we can split $p(z) = f_2(z) + g_2(z)$ where $f_2(z) = z^5$ and $g_2(z) = 6z - 1$. On $|z| = 2$:

$$|f_2(z)| = |z|^5 = 32 \qquad |g_2(z)| = |6z - 1| \leqslant 6|z| + 1 = 13$$

Since $|f_2(z)| > |g_2(z)|$ on $|z| = 2$ and $f_2(z)$ has five roots in $|z| = 2$, we know by Roche's Theorem that p has five roots in $|z| = 2$.

Since we know there is pne root in $|z| = 1$, we know four roots must live between $|z| = 1$ and $|z| = 2$. Thus $p(z)$ has four roots in $1 < |z| < 2$. □

8.3 Hurwitz's Theorem

Theorem 8.3.1 (Hurwitz's Theorem). Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$, and suppose that $f(z)$ has a zero of order N at z_0. Then there exists $\rho > 0$ such that for k large, $f_k(z)$ has exactly N zeros in the disk $\{|z - z_0| < \rho\}$, counting multiplicity, and these zeros converge to z_0 as $k \to \infty$.

univalent We say that a function is **univalent** on a domain D if it is analytic and one-to-one on D. That is, they are conformal maps of D to other domains.

Theorem 8.3.2. Suppose $\{f_k(z)\}$ is a sequence of univalent functions on a domain D that converges normally on D to a function $f(z)$. Then either $f(z)$ is univalent or $f(z)$ is constant.

8.4 Open Mapping and Inverse Funcion Theorems

Attains a Value Let $f(z)$ be a meromorphic function on a domain D. We say that $f(z)$ **attains the value** w_0 m **times at** z_0 if $f(z) - w_0$ has a zero of order m at z_0.

We make the usual modifications to cover the cases $z_0 = \infty$ and $w_0 = \infty$, so that $f(z)$ attains a finite value w_0 m times at $z_0 = \infty$ if $f(1/z) - w_0$ has a zero of order m at $z = 0$, and $f(z)$ attains the values ∞ m times at z_0 if z_0 is a pole of $f(z)$ of order m.

Theorem 8.4.1 (Open Mapping Theorem for Analytic Functions). If $f(z)$ is analytic on a domain D, and $f(z)$ is not constant, ten $f(z)$ maps open sets to open sets, that is, $f(U)$ is open for each open subset U of D.

Theorem 8.4.2 (Inverse Function Theorem). Suppose $f(z)$ is analytic for $|z - z_0| \leqslant \rho$ and satisfies $f(z_0) = w_0$, $f'(z_0) \neq 0$, and $f(z) \neq w_0$ for $0 < |z - z_0| \leqslant \rho$. Let $\delta > 0$ be chosen such that $|f(z) - w_0| \geqslant \delta$ for $|z - z_0| = \rho$. Then for each w such that $|w - w_0| < \delta$, there is a unique z satisfying $|z - z_0| < \rho$ and $f(z) = w$. Writting $z = f^{-1}(w)$, we have

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} \, \mathrm{d}\zeta, \qquad |w - w_0| < \delta$$

Chapter 9 The Schwarz Lemma and Hyperbolic Geometry

9.1 The Schwarz Lemma

Theorem 9.1.1 (Schwarz Lemma). Let $f(z)$ be analytic for $|z| < 1$. Suppose $|f(z)| \leqslant 1$ for all $|z| < 1$, and $f(0) = 0$. Then

$$|f(z)| \leqslant |z| \qquad |z| < 1$$

Further, if equality holds at some point $z_0 \neq 0$, then $f(z) = \lambda z$ for some constant λ of unit modulus.

Theorem 9.1.2. Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \leqslant 1$ for $|z| < 1$, and $f(0) = 0$, then $|f'(0)| \leqslant 1$, with equality if and only if $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.

9.2 Conformal Self-Maps of the Unit Disk

Conformal Self-Map of the Unit Disk **A conformal self-map of the unit disk** is an analytic function from $\mathbb{D}$ to itself that is one-to-one and onto.

Lemma 9.2.1. If $g(z)$ is a conformal self-map of the uni disk $\mathbb{D}$ such that $g(0) = 0$, then $g(z)$ is a rotation, that is, $g(z) = e^{i\varphi}z$ for some fixed φ, $0 \leqslant \varphi \leqslant 2\pi$.

Theorem 9.2.1. The conformal self-maps of the open unit disk $\mathbb{D}$ are precisely the fractional linear transformations of the form

$$f(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z} \qquad |z| < 1$$

where a is complex, $|a| < 1$, and $0 \leqslant \varphi \leqslant 2\pi$.

Theorem 9.2.2 (Pick's Lemma). If $f(z)$ is analytic and satisfies $|f(z)| < 1$ for $|z| < 1$, then

$$|f'(z)| \leqslant \frac{1-|f(z)|^2}{1-|z|^2} \qquad |z| < 1$$

If $f(z)$ is a conformal self-map of $\mathbb{D}$, then equality holds, otherwise the inequality is strict for all $|z| < 1$.

Finite Blaschke Product A **finite Blaschke product** is a rational function of the form

$$B(z) = e^{i\varphi}\left(\frac{z-a_1}{1-\overline{a_1}z}\right)\cdots\left(\frac{z-a_n}{1-\overline{a_n}z}\right)$$

where $a_1, \ldots, a_n \in \mathbb{D}$ and $0 \leqslant \varphi \leqslant 2\pi$.

Chapter 10 Harmonic Functions and the Reflection Principle

10.1 The Poisson Integral Formula

Poisson Kernel Function The **Poisson kernel function** is defined by:

$$P_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$$

For each fixed $\rho < 1$, this series converges uniformly for $r \leqslant \rho$ and $-\pi \leqslant \theta \leqslant \pi$. Simplifying this we obtain that:

$$P_r(\theta) = \frac{1 - |z|^2}{|1 - z|^2} = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta} \qquad z = re^{i\theta} \in \mathbb{D}$$

Poisson Integral The **Poisson integral** $\tilde{h}(z)$ of $h(e^{i\theta})$ to be the function on the open unit disk $\mathbb{D}$ given by

$$\tilde{h}(z) = \int_{-\pi}^{\pi} h(e^{i\varphi}) P_r(\theta - \varphi) \frac{d\varphi}{2\pi}, \qquad z = re^{i\theta} \in \mathbb{D}$$

Theorem 10.1.1. Let $h(e^{i\theta})$ be a continuous function on the nit circle. Then the Poisson integral $\tilde{h}(z)$ defined above is a harmonic function on te open unit disk that has boundary values $h(e^{i\theta})$, that is $\tilde{h}(z)$ tends to $h(\zeta)$ as $z \in \mathbb{D}$ tends to $\zeta \in \partial\mathbb{D}$.

Schwarz Formula Suppose that $f(z) = u(z) + iv(z)$ is analytic for $|z| < 1$ and that $u(z)$ extends to be continuous on the closed disk $\{|z| \leqslant 1\}$. The formula

$$f(z) = \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \frac{d\varphi}{2\pi} + iv(0), \qquad |z| < 1$$

is the **Schwarz formula**, expressing an analytic function in terms of the boundary values of its real part.

Radial Limit function $f(z)$, $z \in \mathbb{D}$, is said to have **radial limit** L at $\zeta \in \partial\mathbb{D}$ if $f(r\zeta) \to L$ as r increases to 1.

We know that $\tilde{h}(z)$ has a raidal limit at each $\zeta \in \partial\mathbb{D}$, equal to the average of the limits of $h(e^{i\theta})$ at ζ from each side.

10.2 Characterization of Harmonic Functions

Theorem 10.2.1. Let $h(z)$ be a continuous function on a domain D. Then $h(z)$ is harmonic on D if and only if $h(z)$ has the mean value property on D.

10.3 The Schwarz Reflection Principle

Theorem 10.3.1. Let D be a domain that is symmetric with respect to the real axis, and let $D^+ = D \cap \{\operatorname{Im} z > 0\}$ be the part of D in the open upper half-plane. Let $u(z)$ be a real-valued harmonic function on D^+ such that $u(z) \to 0$ as $z \in D^+$ tends to any point of $D \cap \mathbb{R}$. Then $u(z)$ extends to be harmonic on D, and the extension satisfies $u(\bar{z}) = -u(z)$.

Theorem 10.3.2. Let D be a domain that is symmetric with respect to the real axis, and let $D^+ = D \cap \{\operatorname{Im} z > 0\}$. Let $f(z)$ be an analytic function on D^+ such that $\operatorname{Im} f(z) \to 0$ as $z \in D^+$ tend to $D \cap \mathbb{R}$. Then $f(z)$ extends to be analytic on D, and the extension satisfies $f(\bar{z}) = \overline{f(z)}$.

Analytic Curve We define a curve γ to be an **analytic curve** if every point of γ has an open neighborhood U for which there is a conformal map $\zeta \to z(\zeta)$ of a disk D centered on the real line $\mathbb{R}$ onto U, such that the image of $D \cap \mathbb{R}$ coincides with $U \cap \gamma$. We also refer to such a γ as an **analytic arc**.

Reflection Across γ The map $\zeta \to \bar{\zeta}$ interchanges the top half and bottom half of $D \backslash \mathbb{R}$, which are the two components of $D \backslash \mathbb{R}$, so the map $z \to z^*$ interchanges the two compponents of $U \backslash \gamma$. We refer to these two components as the **neighborhoods of the sides of** γ, and we refer to the map $z \to z^*$ as the **reflection across** γ

Theorem 10.3.3. Let D be a domain, and let γ be a free analytic boundary arc of D. Let $f(z)$ be analytic on D. If $|f(z)| \to 1$ as $z \in D$ tends to γ, then $f(z)$ extends to be analytic in a neighborhood of γ, and the extension satisfies $f(z^*) = 1/\overline{f(z)}$ in a neighborhood of γ, where $z \to z^*$ is the reflection across γ.

Modulus of an Annulus The **modulus of an annulus** $\{a < |z - z_0| < b\}$ is defined to be $(1/2\pi) \log(b/a)$.

Chapter 11 Conformal Mapping

11.1 Mappings to the Unit Disk and Upper Half-Plane

Conformal Map A **conformal map** of a domain D onto a domain V is a analytic function $\varphi(z)$ from D to V that is one-to-one and onto.

Self Maps of the Open Unit Disk From Section 9.2 that the conformal self-maps of the open unit disk have the form:

$$g(z) = \lambda \frac{z-a}{1-\bar{a}z} \qquad z \in D$$

for $|a| < 1$ and $|\lambda| = 1$.

Maps between Upper Half-Plane and Disk

$$w = \frac{z-i}{z+i} \quad \mathbb{H} \to \mathbb{D} \qquad\qquad z = i\frac{1+w}{1-w} \quad \mathbb{D} \to \mathbb{H}$$

Sectors Any sector with vertex at 0 can be rotated by the map $z \to \lambda z$, $|z| = 1$, to a sector of the form $D = \{0 < \arg z < \alpha\}$, where $\alpha \leqslant 2\pi$.

$$\zeta = z^{\pi/\alpha} : S \to \mathbb{H} \qquad w = \frac{\zeta - i}{\zeta + i} : \mathbb{H} \to D \qquad w = \varphi(z) = \frac{z^{\pi/\alpha} - i}{z^{\pi/\alpha} + i} : S \to \mathbb{D}$$

Strips We can map any strip to a horizontal strip by rotation $z \to \lambda z$. The exponential function $e^{\alpha z}$ maps horizontal strips to the half-plane. Another rotation by φ maps this half-plane to the upper half-plane

$$\zeta = \varphi e^{\alpha z} : St \to \mathbb{H}$$

Lunar Domains A **lunar domain** is a domain D with a boundary consisting of two curves, each of which is an arc of a circle or a straight line segmen. Let z_0 and z_1 be the endpoints of the curves. We assum $z_0 \neq z_1$. We can map a lunar domain to a sector as follows:

$$w = \lambda \frac{z - z_0}{z - z_1} : L \to S$$

11.2 The Reimann Mapping Theorem

Theorem 11.2.1 (Riemann Mapping Theorem). If D is a simply connected domain in the complex plane, and D is not the entire complex plane, then there is a conformal map of D onto the open unit disk $\mathbb{D}$.

Conformally Equivalent We say that two domains are **conformally equivalent** if there is a conformal map of one onto the other.

Thus the Riemann Mapping theorem asserts that any simply connected domain in the complex plane $\mathbb{C}$ either coincides with $\mathbb{C}$ or is conformally equivalent to $\mathbb{D}$.

Riemann Map We refer to a conformal map $w = \varphi(z)$ of D onto $\mathbb{D}$ as the **Riemann map of** D onto $\mathbb{D}$. It is unique, up to postcomposing with a conformal selfmap of $\mathbb{D}$.

Corollary 11.2.2. A simply connected domain in the Riemann sphere is either the entire Riemann sphere, or it is conformally equivalent to the complex plane, or it is conformally equivalent to the open unit disk.

Theorem 11.2.3. Let D be a simply connected domain in $\mathbb{C}$, $D \neq \mathbb{C}$. Then the Riemann map $\varphi(z)$ of D onto $\mathbb{D}$ extends analytically across any free analytic boundary arc γ of D, and $\varphi(z)$ maps γ one-to-one onto an arc of $\partial\mathbb{D}$. The extended function satisfies $\varphi'(z) \neq 0$ for $z \in \gamma$, and $\varphi(z^*) = 1/\overline{\varphi(z)}$ for z in a neighborhood of γ, where $z \to z^*$ is a reflection across γ. Disjoint free analytic boundary arcs of D are mapped by $\varphi(z)$ to disjoint arcs of $\partial\mathbb{D}$.

11.3 Compactness of Families of Functions

Equicontinuous Let E be a subset of the complex plane $\mathbb{C}$, and let $\mathscr{F}$ be a family of complex-valued functions on E. We say that $\mathscr{F}$ is **equicontinuous** at a point $z_0 \in E$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $z \in E$ satisfies $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$ for all $f \in \mathscr{F}$.

Uniformly Bounded We say that the family $\mathscr{F}$ is **uniformly bounded** on E if there is a constant $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in E$ and all $f \in \mathscr{F}$.

Theorem 11.3.1 (Arzela- Ascoli Theorem). Let E be a compact subset of $\mathbb{C}$, and let $\mathscr{F}$ be a family of continuous complex-valued functions on E that is uniformly bounded. Then the following are equivalent.

1. The family $\mathscr{F}$ is equicontinuous at each point of E
2. Each sequence of functions in $\mathscr{F}$ has a subsequence that converges uniformly on E.

Spherical Metric Let $\sigma(z, w)$ be the spherical metric, or the spherical distance from z to w aas in Section 9.3.

A sequence of functions $\{f_n\}$ on E **converges uniformly to f in th spherical metric** if $\sigma(f_n(z), f_m(z))$ tends to 0 uniformly for $z \in E$ as $n, m \to \infty$.

A family $\mathscr{F}$ is **equicontinuous with respect to the spherical metric** at $z_0 \in E$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that if $z \in E$ satisfies $|z - z_0| < \delta$, then $\sigma(f(z), f(z_0)) < \varepsilon$ for all $f \in \mathscr{F}$.

Theorem 11.3.2. Let D be a domain in the complex plane, and let $\mathscr{F}$ be a family of continuous functions from D to the extended complex plane $\mathbb{C}^*$. Then the following are equivalent.

1. Any sequence in $\mathscr{F}$ has a subsequence that converges uniformly on compact subsets of D in the spherical metric.
2. The family $\mathscr{F}$ is equicontinuous at each point of D, with respect to the spherical metric.

Theorem 11.3.3. Suppose $\mathscr{F}$ is a family of analytic functions on a domain D such that $\mathscr{F}$ is uniformly bounded on each compmact subset of D. Then every sequence in $\mathscr{F}$ has a subsequence that converges normally on D, that is, uniformly on each compact subset of D.

Extremal Let D be a domain, and fix a point $z_0 \in D$. Let $\mathscr{F}$ be the family of analytic functions $f(z)$ on D such that $|f(z)| \leqslant 1$ on D. The extremal problem is to maximize $|f'(z_0)|$ among all functions $f \in \mathscr{F}$. The **extremal value** for the problem is

$$A = \sup\{|f'(z_0)| : f \in \mathscr{F}\}$$

Since the functions in $\mathscr{F}$ are uniformly bounded on D, their derivatives are uniformly bounded at z_0 and A is finite. A function $G \in \mathscr{F}$ such that $|G'(z_0)| = A$ is an **extremal function** for the problem. (Existence follows from Montel's.)

Theorem 11.3.4. Let D be a domain in the complex plane on which there is a nonconstant bounded analytic function, and let $z_0 \in D$. Then there is an analytic function $G(z)$ on D such that $|G(z)| \leqslant 1$ for $z \in D$, and $|f'(z_0)| \leqslant |G'(z_0)|$ for any analytic function $f(z)$ on D satisfying $|f(z)| \leqslant 1$ on D. Further, $G(z_0) = 0$ and $G'(z_0) \neq 0$.

Ahlfors Function The extremal function $G(z)$ is called the **Ahlfors function** of D and depends on z_0.

The extremal value $A = |G'(z_0)|$ can be regarded as the best constant for which the Schwarz lemma holds with respect to $z_0 \in D$.

11.4 Proof of the Riemann Mapping Theorem

Note I'm not going through the entire proof here, see book for that.

Lemma 11.4.1. Let D be a simply connected domain. Suppose $a \notin D$, and let $h(z)$ be an analytc branch of $\sqrt{z-a}$ in D. Then $h(z)$ is univalent on D, and further, $h(D)$ is disjoint from $-h(D)$.

Lemma 11.4.2. Let D be a simply connected subdomain of $\mathbb{D}$ such that $0 \in D$. If $D \neq \mathbb{D}$, then there is a conformal map $\psi(\zeta)$ of D onto a subdomain of $\mathbb{D}$ such that $\psi(0) = 0$ and $|\psi'(0)| > 1$.

May 8, 2012

Conan's Finals

MAT 712. Final Exam

NAME: **Student ID:**

Instructions: Write your answers and **show all your work** on this test. There are 4 problems on 4 pages, for a total of 80 points. To receive credit, you **must** justify your answers and show all details of your work.

1.(20 points) Find a conformal map f from the domain $D = \{z \in \mathbb{C} : \mathbf{Re}\, z > 0,\ \mathbf{Im}\, z > 0\}$ onto the unit disk, so that $f(1+i) = 0$.

2.(20 points) How many zeros, counted with multiplicity, does the function $f(z) = z^4 + e^z + 2$ have in the domain $D = \{z \in \mathbb{C} : \mathbf{Re}\, z < 0\}$?

3.(20 points) Find all the entire functions f such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$.

4.(20 points) Find $\displaystyle\int_0^{+\infty} \frac{\cos x}{1+x^4}\, dx$.

Coman's Finals

Due Friday May 1, 2009, at 2:00 PM (317 D Carnegie)

MAT 712, Final Exam

NAME:

Instructions: To receive credit, write your answers and **show all your work** on this test. There are 6 problems on 6 pages, for a total of 60 points. Notes or textbooks are not allowed.

1.(10 points) Find a conformal map from $D = \{z \in \mathbb{C} : 0 < \mathbf{Re}\, z < 1\}$ onto the unit disk.

2.(10 points) Let $S \neq \emptyset$ be a subset of the open unit disk. For $c \in S$, define the meromorphic function

$$f_c(z) = \frac{1-c}{z+c}, \quad z \in \mathbb{C}.$$

Prove that the family of meromorphic functions $\mathcal{F} = \{f_c : c \in S\}$ is a normal family on $\mathbb{C}$ if and only if $1 \notin \overline{S}$.

3.(10 points) If $\lambda > 1$, show that the equation $z + e^{-z} = \lambda$ has exactly one solution with positive real part.

4.(10 points) Let $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, be a harmonic function such that $g(z) = zf(z)$ is also harmonic. Prove that f is holomorphic.

5.(10 points) Let $f : \mathbb{C} \to \mathbb{C}$ be a function of class C^1 such that $\int_C f(z)\,dz = 0$ for every **circle** C. Prove that f is entire.

6.(10 points) Let $P(z)$ be a polynomial of degree $n \geq 2$, and let $z_1, \ldots, z_k$ be the distinct zeros of P. Prove that

$$\sum_{j=1}^{k} \operatorname{Res}\left(\frac{1}{P}, z_j\right) = 0.$$

Analytic (holomorphic)

- $u_x = v_y \quad u_y = -v_x$ (Cauchy Riemann eqns)
- f is $\mathbb{C}$ diff at every point z. (Goursat's)
- $df/d\bar{z} = 0$
- $\{|z - z_0| < r\} \subset D$, $f(z) = \sum a_n (z - z_0)^n$
- f is cts and $\int_R f\,dz = 0 \quad \forall R \subset D$, R is closed rectangle (Morera's)
- Any analytic f is harmonic.

Harmonic:

- $\Delta u = \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0$
- harmonic $\Rightarrow$ real part of analytic f
- imaginary part = harmonic conjugate $\rightarrow$! up to constant
  ↳ guaranteed on ☆-shaped domain.

  $V(x,y) = \left(\int u_y\, dx\right) = U(x,y) + h(x)$ ← don't forget this.
- Any cts fn w/ mean value property is harmonic

Max' Principle

(Strict $\mathbb{C}$)

- Let h be bdd $\mathbb{C}$-valued harmonic fn on D. If
  - $|h(z)| \leq M \quad \forall z \in D$
  - $|h(z_0)| = M$ for some $z_0 \in D$

$\Rightarrow h(z) \equiv M \quad \forall z \in D$

(Strict $\mathbb{R}$)

- Let u be a real-valued, harmonic fn on D. If
  - $u(z) \leq M \quad \forall z \in D$
  - $u(z_0) = M$ for some $z_0 \in D$

$\Longrightarrow u(z) = M \quad \forall z \in D$

- Let $h(z)$ be a $\mathbb{C}$-valued harmonic fn on a bdd domain D. If
  - $h(z)$ extends cts'ly to ∂D, & $|h(z)| \leq M \quad \forall z \in \partial D$

Then $|h(z)| \leq M \quad \forall z \in D$

- Extend Continuously:

Various integral formulas that aren't about integrals

Mean value Property. ($\forall z_0 \in D\ \exists\ \varepsilon > 0$ s.t. $h(z_0)$ is the avg value)

$$h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})\, d\theta \qquad 0 < r < \varepsilon$$

Cauchy Integral formula (f analytic on D & extends ~~cts~~ smoothly to ∂D)

$$f = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w - z}\, dw \qquad f^{(m)} = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\, dw$$

Cauchy Estimates. If $|f(z)| \le M$ $\forall$ $|z - z_0| = \rho$

$$|f^{(m)}(z_0)| \le \frac{m!}{\rho^m} M \qquad m \ge 0$$

Pompeiu's g is a smooth $\mathbb{C}$-valued fn on $D \cup \partial D$

$$g(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{g(z)}{z - w} - \frac{1}{\pi}\iint_D \frac{dg}{d\bar{z}} \cdot \frac{1}{z - w}\, dx\, dy$$

Logarithmic integral. (D bdd, plus smooth ∂D, $f(z)$ meromorphic on D extends to be analytic on ∂D)

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\, dz = N_0 - N_\infty$$

Inverse Function Thm

$$z = f^{-1}(w) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w}\, d\zeta$$

Schwarz Formula ($f = u + iv$ analytic for $|z| < 1$, u extends cts on $\bar{\mathbb{D}}$)

$$f(z) = \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\frac{d\varphi}{2\pi} + iv(0) \qquad |z| < 1$$

Poisson integral

$$\tilde{h}(z) = \int_{-\pi}^{\pi} h(e^{i\varphi}) P_r(\theta - \varphi)\frac{d\varphi}{2\pi} \qquad z = re^{i\theta}$$

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

$\hookrightarrow$ harmonic on $\mathbb{D}$ + has bdy values $h(e^{i\theta})$

Sectors to Upper half Plane

$D = \{ z \mid 0 < \arg z < \alpha \}$ ($\alpha < 2\pi$) then $\boxed{w = z^{\pi/\alpha}}$

Rotate anything:

$w(z) = \lambda z$, $|\lambda| = 1$, $\lambda = e^{i\theta}$ → rotate by θ

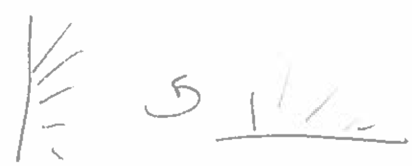
$\boxed{w(z) = e^{i\theta} z}$ is rotation by θ, ccw

Horizontal Strip to Sector.

$\{a < \operatorname{Im} z < b\} \longrightarrow \{a\alpha < \arg z < b\alpha\}$

$w = e^{\alpha z}$ → $\alpha = \pi/2$ normally works

↳ $\{-1 < \operatorname{Im} z < 1\} \longrightarrow \{-\pi/2 < \arg z < \pi/2\} \longrightarrow$



Half plane to $\mathbb{D}$

$$w = \frac{z - i}{z + i}$$

Open Mapping

If $f(z)$ is analytic on a domain D and f is not constant, then $f(z)$ maps open sets to open sets, that is

$f(U)$ is open for each U of D

Inverse Function Theorem

Suppose $f(z)$ is analytic for $|z-z_0| \le \rho$ and satisfies $f(z_0) = w_0$

$f'(z_0) \neq 0$, $f(z) \neq w_0$ (for $0 < |z-z_0| < \rho$). Choose δ s.t. $|f(z)-w_0| \geq \delta$

for $|z-z_0| = \rho$. Then $\forall$ w s.t. $|w-w_0| < \delta$ $\exists!$ z s.t. $|z-z_0| < \rho$, $f(z) = w$.

$$z = f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} \, d\zeta \qquad |w-w_0| < \delta$$

Schwarz

Let $f(z)$ be analytic for $|z| < 1$. Suppose $|f(z)| \le 1$ $\forall$ $|z| < 1$

and $f(0) = 0$. Then $|f(z)| \le |z|$, $|z| < 1$

If equal @ somepoint $z_0 \neq 0$, then $f(z) = \lambda z$ for

λ s.t. $|\lambda| = 1$

· Exact: $P\,dx + Q\,dy$ is exact if $P\,dx + Q\,dy = dh$ for h.

· independence of path ⟷ exact ⟶ closed.
↳ same $\int_\gamma$ for any γ connecting A to B
⟷ ↳ (if D is ☆.)

· Extends continuously to bdry ?

Primative

Principle value

· Singularities @ ∞

Measure Theory Terms and Theorems

Preparation for Analysis Qualifying Exam
Based on *Real and Complex Analysis* by Walter Rudin
and *Measure and Integral* by Richard L. Wheeden and Antoni Zygmund

Erin Griffin

July 12, 2019

Note to the Reader

I began creating this resources using both Rudin and Wheeden & Zygmund. I decided midway through creating this document that my previous course of action was neither necessary nor efficient. Thus, for the remaining sections I focused solely on Rudin's book. To see notes based on Wheeden & Zygmund refer to the course notes typed by Caleb McWhorter on GitHub

Contents

Chapter 1 σ-algebras

1.1 σ-algebras

Rudin

Topology A collection τ of subsets of a set X is said to be a *topology* in X is τ has the following three properties:

a. $\varnothing \in \tau$ and $X \in \tau$

b. If $V_i \in \tau$ for $i = 1, \ldots, n$ then $V_1 \cap V_2 \cap \cdots \cap V_n \in \tau$

c. If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable) then $\bigcup_\alpha V_\alpha \in \tau$.

Topological Space If τ is a topology in X, then X is called a *topological space*, and the members of τ are called the *open sets* in X.

Continuous If X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

σ-algebras A collection $\mathscr{M}$ of subsets of a set X is said to be a *σ-algebra* in X if $\mathscr{M}$ has the following properties:

a. $X \in \mathscr{M}$

b. If $A \in \mathscr{M}$, then $A^c \in \mathscr{M}$, where A^c is the complement of A relative to X

c. If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathscr{M}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathscr{M}$.

Also known as a countably additive family of sets.

Theorem (1.10). If $\mathscr{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathscr{M}^*$ in X such that $\mathscr{F} \subset M^*$

Wheeden & Zygmund

Theorem (WZ, 162). Immediate consequences of the definition. Let Σ be a σ-algebra. Then the following sets belong to Σ:

1. The empty set $\varnothing$
2. $\bigcap E_k$ if $E_k \in \Sigma$, $k = 1, 2, \ldots$.
3. $\limsup E_k$ and $\liminf E_k$ if each $E_k \in \Sigma$
4. $E_1 - E_2$ if $E_1, E_2 \in \Sigma$.

Additive Set Function If Σ is a σ-algebra, then a real-valued function $\varphi(E)$, $E \in \Sigma$, is called an **additive set function** on Σ if

a. $\varphi(E)$ is finite for every $E \in \Sigma$, and

b. $\varphi(\bigcup E_k) = \Sigma\varphi(E_k)$ for every countable family $\{E_k\}$ of disjoint sets in Σ.

Theorem. (10.1) If $\{E_k\}$ is a monotone sequence of sets in Σ and φ is an additive set function, then $\varphi(E) = \lim_{k\to\infty} \varphi(E_k)$.

Note There was a lot more in Wheeden and Zygmund on additive set functions. Page 163-165.

Chapter 2 Measures

2.1 Measures

Rudin

Positive Measure A **positive measure** is a function μ, defined on a σ-algebra $\mathscr{M}$ whose range is in $[0, \infty]$ and which is **countably additive**. THis means that if $\{A_i\}$is a disjoint countable collection of members of $\mathscr{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least on $A \in \mathscr{M}$

Frequently just called measure.

Measure Space A **measure space** is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

Complex measure A **complex measure** is a copmlex-valued countably additive function defined on a σ-algebra.

Theorem (1.19). Let μ be a positive meausre on a σ-algebra $\mathscr{M}$. Then

a. $\mu(\varnothing) = 0$

b. (finite additivity) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ if $A_1, \ldots A_n$ are pairwise disjoint members of $\mathscr{M}$.

c. (monotonicity) $A \subset B$ implies $\mu(A) \leqslant \mu(B)$ if $A \in \mathscr{M}$, $B \in \mathscr{M}$

d. $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathscr{M}$, and $A_1 \subset A_2 \subset A_3 \subset \ldots$

e. $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathscr{M}$, $A_1 \supset A_2 \supset A_3 \supset \ldots$, and $\mu(A_1)$ is finite

2.2 Outer Measures

Wheeden & Zygmund

Lesbegue Outer Measure

Lebesgue Outer Measure Consider an arbitrary subset E of $\mathbb{R}^n$, cover E by a countable collection S of I_k, and let

$$\sigma(S) = \sum_{I_k \in S} \text{vol}(I_k)$$

The **Lebesgue outer measure** (or exterior measure) of E denoted $|E|_e$, is defined by

$$|E|_e = \inf \sigma(S)$$

where the infimum is taken over all such covers S of E. Thus $0 \leqslant |E|_e \leqslant +\infty$.

Theorem (3.2). For an interval I, $|I|_e = \text{vol}(I)$

Theorem (3.3). If $E_1 \subset E_2$, then $|E_1|_e \leqslant |E_2|_e$

Theorem (3.4). If $E = \bigcup E_k$ is a countable union of sets, then $|E|_e \leqslant \sum |E_k|_e$

Theorem. Any set consisting of a single point clearly has outer measure zero, it follows that any countable subset of $\mathbb{R}^n$ has outer measure zero.

Cantor Set The subset of $[0,1]$ which remains after infinitely iterating "removing the inner third" is called the **Cantor set** C, this if C_k denotes the union of the intervals left at the kth stage, then

$$C = \bigcap_{k=1}^{\infty} C_k$$

- C is closed (since each C_k is closed)
- C_k consists of 2^k closed intervals, each of length 3^{-k}
- C contains the enpoints of all the intervals
- Any point of C belongs to every C_k, and is therefore a limit point of the endpoints of the intervals. So C is perfect. That is, C is a closed set each of whose points is a limit point of C. Further, C is a closed set which is dense in itself and also is uncountable.
- $|C|_e \leqslant 2^k 3^{-k}$ thus $|C|_e$

Cantor-Lebesgue Function Let $D_k = [0,1] \backslash C_k$, which consists of $2^k - 1$ intervals I_j^k removed in the first k stages of construction on the Cantor set. Let f_k be the continuous function on $[0,1]$ which satisfies $f_k(0) = 0$, $f_k(1) = 1$, $f_k(x) = j2^{-k}$ for $x \in I_j^k$, $j = 1, \ldots, 2^k - 1$, and which is linear on each interval of C_k. Each f_k is monotone increasing, $f_{k+1} = f_k$ on I_j^k, $j = 1, 2, \ldots, 2^k - 1$, and $|f_k - f_{k+1}| < 2^{-k}$. Hence $\sum(f_k - f_{k+1})$ converges uniformly on $[0,1]$, and therefore, $\{f_k\}$ converges uniformly on $[0,1]$. Let $f = \lim_{k \to \infty} f_k$. Then $f(0) = 0$, $f(1) = 1$, f is monotone increasing and continuous on $[0,1]$, and f is constant on every interval removed in constructing C. This f is called the **Cantor Lebesgue function**.

Carathéodory E meas iff $\forall A$

$|A|_e = |A \cap E|_e + |A \setminus E|_e$

Theorem (3.6). Let $E \subset \mathbb{R}^n$. Then given $\varepsilon > 0$, there exists an open set G such that $E \subset G$ and $|G|_e \leqslant |E|_e + \varepsilon$. Hence $|E|_e = \inf |G|_e$, where the infimum is taken over all open sets G containing E.

Theorem (3.8). IF $E \subset \mathbb{R}^n$, there exists a set H of type G_δ such that $E \subset H$ and $|E|_e = |H|_e$.

Theorem (3.10). $|E|'_e = |E|_e$ for every $E \subset \mathbb{R}^n$.

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Outer Measure A function $\Gamma = \Gamma(A)$ which is defined for every subset A of a space $\mathscr{M}$ is called an **outer measure** if it satisfies the following:

a. $\Gamma(A) \geqslant 0$, $\Gamma(\varnothing) = 0$.

b. $\Gamma(A_1) \leqslant \Gamma(A_2)$ if $A_1 \subseteq A_2$

c. $\Gamma(\bigcup A_k) \leqslant \sum \Gamma(A_k)$ for any countable collection of sets $\{A_k\}$.

Theorem. Given an outer measure Γ we say that a subset E of $\mathscr{M}$ is Γ-measurable, or simply measurable, if

$$\Gamma(A) = \Gamma(A \cap E) + \Gamma(A \backslash E)$$

Equivalently, E is measurable if and only if

$$\Gamma(A_1 \cup A_2) = \Gamma(A_1) + \Gamma(A_2)$$

whenever $A_1 \subseteq E$, $A_2 \subseteq \mathscr{M} \backslash E$.

Theorem (11.2). Let Γ be an outer measure on the subsets of $\mathscr{M}$.

a. The family of Γ-measurable subsets of $\mathscr{M}$ forms a σ-algebra.

b. If $\{E_k\}$ is a countable collection of disjoint measurable sets, then $\Gamma(\bigcup E_k) = \sum \Gamma(E_k)$. More generally, for any A, measurable of not, $\Gamma(A \cap \bigcup E_k) = \sum \Gamma(A \cap E_k)$ and $\Gamma(A) = \sum \Gamma(A \cap E_k) + \Gamma(A - \bigcup E_k)$.

2.3 Borel Measures

Rudin

Borel Sets Let X be a topological space. There exists a smallest σ-algebra $\mathscr{B}$ in X such that every open set in X belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the **Borel sets** of X. (Open sets are normally noted with F.)

Since closed sets are complements of open sets, they are necessarily Borel. (Closed sets are normally noted with G.)

Let F_σ be the countable unions of all closed sets and G_δ be the countable intersections of open sets. (Unions and Intersections are normally noted with σ and δ respectively.)

Since $\mathscr{B}$ is a σ-algebra, $(X, \mathscr{B})$ is a measureable space. If $f : X \to Y$ is continuous, then $f^{-1}(V) \in \mathscr{B}$ for every open set $V \in Y$. Thus, we get the following theorem.

Borel Measure A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a **Borel measure** on X.

Regular If μ is positive, a Borel set $E \subset X$ is **outer regular** or **inner regular**, respectively, if E has property (c) or (d) of the Riesz Representation Theorem. If every Borel set in X is both outer and inner regular, μ is called **regular**

σ-compact A set E in a topological space is called **σ-compact** if E is a countable union of compact sets.

σ-finite Measure A set E in a measure space (with measure μ) is said to have **σ-finite measure** if E is a countable union of sets E_i with $\mu(E_i) < \infty$.

In the situation presented in the RRT, every σ-compact set has a σ-finite measure. Further, if $E \in \mathscr{M}$ and E has a σ-finite measure, then E is inner regular.

Theorem (2.17). Suppose X is a locally compact, σ-compact Hausdorff space. If $\mathscr{M}$ and μ are as described in the statement of the RRT, then $\mathscr{M}$ and μ have the following properties:

a. If $E \in \mathscr{M}$ and $\varepsilon > 0$, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V \backslash F) < \varepsilon$

b. μ is regular Borel measure on X

c. If $E \in \mathscr{M}$, there are sets A and B such that A is an F_σ, B is a G_δ, $A \subset E \subset B$, and $\mu(B \backslash A) = 0$

Theorem (2.18). Let X be a locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then λ is regular.

Chapter 3 Measurable Functions

3.1 Measurable Functions

Rudin

Measurable Function If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$is a measurable set in X for every open set V in Y.

Measurable Space If $\mathscr{M}$ is a σ-algebra in X, then X is called a *measurable space*, and the members of $\mathscr{M}$ are called the *measurable sets* in X.

Theorem. Every continuous mapping is Borel measrable.

Theorem (1.12). Suppose $\mathscr{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.

a. If Ω is te collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathscr{M}$, then Ω is a σ-algebra in Y.
b. If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathscr{M}$.
c. If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathscr{M}$ for every real α, then f is measurable.
d. If f is measurabl, if Z is a topological space, if $g : Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then $h : X \to Z$ is measurable.

Upper and Lower Limit If $\{a_n\}$ is a sequence in $[-\infty, \infty]$ and let $b_k = \sup\{a_k, a_{k+1}, \dots\}$ and $\beta = inf\{b_1, b_2, \dots\}$, then β is the **upper limit** up $\{a_n\}$, $\beta = \limsup_{n\to\infty} a_n$. **Lower limit** is defined analogously.

$$\limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k \qquad \liminf E_k = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} E_k$$

Theorem (1.14). If $f_n : X \to [-\infty, \infty]$ is measurable for $n \in \mathbb{N}$, and $g = \sup f_n$, $h = \limsup f_n$, then g and h are measurable.

Note. Proof uses Thm 1.12.

Corollary.

a. The limit of every pointwise convergent sequence of complex measurable functions is measurable.

b. If f and g are measurable (with range in $[-\infty, \infty]$), then so are $\max f, g$ and $\min f, g$.

Positive and Negative Parts Define the following functions:

$$f^+ = \max\{f, 0\} \qquad f^- = -\min\{f, 0\}$$

Furthermore, notice that:

$$|f| = f^+ + f^- \qquad f = f^+ - f^-$$

Proposition. If $f = g - h$, $g \geqslant 0$, and $h \geqslant 0$, then $f^+ \leqslant g$ and $f^- \leqslant h$.

Simple Function A complex function s on a measurable space X whose range consists of only finitely many points will be called a **simple function**. Among these are the nonnegative simple functions, whose range is a finite subset of $[0, \infty)$. Note that we explicitly exclude ∞ from the values of a simple function.

If $\alpha_1, \ldots, \alpha_n$ are the distinct values of a simple function s, and if we set $A_1 = \{x : s(x) = \alpha_i\}$, then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

wjere χ_{A_i} is the characteristic function of A_i.

Theorem (1.17). Let $f : X \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that:

a. $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant f$

b. $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

Semicontinuous Let f be a real (or extended-real) function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real α, f is said to be **lower semicontinuous**

If $\{x : f(x) < \alpha\}$ is open for every real α, f is said to be **upper semicontinuous**.

Theorem (2.8). a. Characteristic functions of *open* sets are *lower* semicontinuous

b. Characteristic functions of *closed* sets are *upper* semicontinuous

c. The supremum of any collection of lower semicontinuous functions is lower semicontinuous. The infimum of any collection of upper semicontinuous functions is upper semicontinuous.

Support The **support** of a complex function f on a topological space X is the closure of the set $\{x : f(x) \neq 0\}$.

Compact Support $C_c(X)$ The collection of all continuous complex functions on X whose support is compact is denoted by $C_c(X)$.

Notation The notation

$$K \prec f$$

will mean that K is a compact subset of X, that $f \in C_c(X)$, that $0 \leqslant f(x) \leqslant 1$ for all $x \in X$, and that $f(x) = 1$ for all $x \in K$. The notation

$$f \prec V$$

will mean that V is open, that $f \in C_c(X)$, $0 \leqslant f \leqslant 1$, and that the support of f lies in V. The notation

$$K \prec f \prec V$$

will mean that both hold.

Theorem (Urysohn's Lemma). Suppose X is locally compact Hausdorff space, V is open in X, $K \subset V$, and K is copmact. Then there exists an $f \in C_c(X)$ such taht

$$K \prec f \prec V$$

Note. The conclusion asserts the existence of a continuous function f which satisfies the inequalities $\chi_K \leqslant f \leqslant \chi_V$. Note that it is easy to find sermicontinuous functions which do this.

Theorem (2.13). Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and

$$K \subset V_1 \cup \cdots \cup V_n$$

Then there exist functions $h_i \prec V_i$ such that

$$h_1(x) + \cdots + h_n(x) = 1$$

The collection $\{h_1, \ldots, h_n\}$ is called **a partition of unity on** K, subordinate to the cover $\{V_1, \ldots, V_n\}$.

Theorem (Riesz Representation Theorem). Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathscr{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathscr{M}$ which represents Λ in the sense that

a. $\Lambda f = \int_X f \, d\mu$ for every $f \in C_c(X)$

b. $\mu(K) < \infty$ for every compact set $K \subset X$

c. (Outer Regularity) For every $E \in \mathscr{M}$, we have

$$\mu(E) = \int \{\mu(V) : E \subset V, \ V \text{ open}\}$$

d. (Inner Regularity) The relation

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$$

holds for every open set E, and for every $E \in \mathscr{M}$ with $\mu(E) < \infty$

e. If $E \in \mathscr{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathscr{M}$

Theorem (Lusin's Theorem). Suppose f is a complex measurable function on X, $\mu(A) < \infty$, $f(x) = 0$ if $x \notin A$, and $\varepsilon > 0$. Then there exists a $g \in C_c(X)$ such that $\mu(\{x : f(x) \neq g(x)\}) < \varepsilon$. Furthermore, we may arrange it so that $\sup_{x \in X} |g(x)| \leqslant \sup_{x \in X} |f(x)|$.

Corollary. Assume that the hypotheses of Lusin's theorem are satisfied and that $|f| \leqslant 1$. Then there is a sequence $\{g_n\}$ such that $g_n \in C_c(X)$, $|g_n| \leqslant 1$, and $f(x) = \lim_{n\to\infty} g_n(x)$ a.e.

Corollary (Vitali- Caratheodory Theorem). Suppose $f \in L^1(\mu($, f is real-valued, and $\varepsilon > 0$. Then there exist functions u and v on X such that $u \leqslant f \leqslant v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and

$$\int_X (v - u)\, d\mu < \varepsilon$$

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Theorem. (WZ 10.13)

a. If f and g are measurable on a set $E \in \Sigma$, then so are $f + g$, cf for real c, $\varphi(f)$ if φ is continuous on $\mathbb{R}^1$, f^+, f^-, $|f|^p$ for $p > 0$, fg, and $1/f$ if $f \neq 0$ in E.

b. If $\{f_k\}$ are measurable on $E \in \Sigma$, then so are $sup_k f_k$, $\inf_k f_k$, $\limsup_{k\to\infty} f_k$, $\liminf_{k\to\infty} f_k$, and if it exists, $\lim_{k\to\infty} f_k$.

c. If f is a simple function taking values $v_1, \ldots, v_N$ on disjoint sets $E_1, \ldots, E_N$, respectively, then f is measurable if and only if each E_k is measurable. In particular, χ_E is measurable if and only if E is.

d. If f is nonnegative and measurable on $E \in \Sigma$, then there exists nonnegative, simple measurable $f_k \nearrow f$ on E.

3.2 Measure Zero

Rudin

Almost Everywhere If μ is a measure on a σ-algebra $\mathscr{M}$ and if $E \in \mathscr{M}$, the statement "P holds almost everywhere on E" means that there exists an $N \in \mathscr{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds at every point of $E|setminusN$.

If f and g are measureable functions and if

$$\mu(\{x : f(x) \neq g(x)\}) = 0$$

we say that $f = g$ a.e. $[\mu]$ on X and we may write $f \sim g$. (Which is an equivalence relation)

If $f \sim g$, then, for every $E \in \mathscr{M}$

$$\int_E f \, d\mu = \int_E g \, d\mu$$

Theorem (1.36). Let $(X, \mathscr{M}.\mu)$ be a measure space, let $\mathscr{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathscr{M}$ such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Then $\mathscr{M}^*$ is a σ-algebra and μ is a measure on $\mathscr{M}^*$.

Complete The aforementioned extended measure μ is called **complete**, since all subsets of sets of measure 0 are now measurable; the σ-algebra $\mathscr{M}^*$ is called the **μ-completion** of $\mathscr{M}$.

Measurable A function f defined on a set $E \in \mathscr{M}$ **measurable on** X if $\mu(E^c) = 0$ and if $f^{-1}(V) \cap E$ is measurable for every open set V. (If we define $f(x) = 0$ for $x \in E^c$, we obtain a measurable function on X, in the old sense.

Theorem (1.38). Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X such that

$$\sum_{n=1}^{\infty} \infty_X |f_n| \, d\mu < \infty$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for almost all x, $f \in L^1(\mu)$, and

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} f_n \, d\mu$$

Theorem (1.39). a. Suppose $f : X \to [0, \infty]$ is measurable, $E \in \mathscr{M}$, and $\int_E f \, d\mu = 0$. Then $f = 0$ a.e. on E.

b. Suppose $f \in L^1(\mu)$ and $\int_E f \, d\mu = 0$ for every $E \in \mathscr{M}$. Then $f = 0$ a.e. on X.

c. Suppose $f \in L^1(\mu)$ and

$$\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu$$

Then there is a constant α such that $\alpha f = |f|$ a.e. on X.

Theorem (1.40). Suppose $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in the complex plane, and the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f \, d\mu$$

lie in S for every $E \in \mathscr{M}$ with $\mu(E) > 0$, Then $f(x) \in S$ for almost all $x \in X$.

Theorem (1.41). Let $\{E_k\}$ be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

Chapter 4 Lebesgue

4.1 Lebesgue Integration in Abstract Measure Spaces

Rudin

Lebesgue Integral If $s : X \to [0, \infty)$ is a measurable simple function, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s and if $E \in \mathscr{M}$, we define

$$\int_E s \, \mathrm{d}\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

Then convention $0 \cdot \infty = 0$ is use here; it may happen the $\alpha_i = 0$ for some i and that $\mu(A_i \cap E) = \infty$.

If $f : X \to [0, \infty]$ is measurable, and $E \in \mathscr{M}$, we define

$$\int_E f \, \mathrm{d}\mu = \sup \int_E s \, \mathrm{d}\mu$$

the supremum being taken over all simple measurable functinos s such that $0 \leqslant s \leqslant f$.

The left member of the above equality is called the **Lebesgue intgral** of f over E with respect to the measure μ. It is a number $[0, \infty]$.

Lemma (1.24). The following propositions are immediate consequences of the definitions. The functions and sets occurring in them are assumed to be measurable:

a. If $0 \leqslant f \leqslant g$, then $\int E f \, \mathrm{d}\mu \leqslant \int_E g \, \mathrm{d}\mu$
b. If $A \subset B$ and $f \geqslant 0$, then $\int_A f \, \mathrm{d}\mu \leqslant \int_B f \, \mathrm{d}\mu$
c. If $f \geqslant 0$ and c is a constant $0 \leqslant c < \infty$, then

$$\int_E cf \, \mathrm{d}\mu = c \int_E f \, \mathrm{d}\mu$$

d. If $f(x) = 0$ for all $x \in E$, then $\int_E f\,d\mu = 0$, even if $\mu(E) = \infty$

e. If $\mu(E) = 0$, then $\int_E f\,d\mu = 0$, even if $f(x) = \infty$ for every $x \in E$

f. If $f \geqslant 0$, then $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$.

Lemma (1.25). Let s and t be nonnegative measurable simple functions on X. For $E \in \mathscr{M}$, define

$$\varphi(E) = \int_E s\,d\mu$$

Then φ is a measure on $\mathscr{M}$. Also

$$\int_X (s + t)\,d\mu = \int_X s\,d\mu + \int Xt\,d\mu$$

Theorem (Lebesgue's Monotone Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that

a. $0 \leqslant f_1(x) \leqslant f_2(x) \leqslant \cdots \leqslant \infty$ for every $x \in X$

b. $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$

Then f is measurable, and

$$\int_X f_n\,d\mu \to \int_X f\,d\mu \qquad n \to \infty$$

Theorem (1.27). If $f_n : X \to [0, \infty]$ is measurable, for $n = 1, 2, 3, \ldots$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X)$$

then

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu$$

Corollary. If $a_{ij} \geqslant 0$ for i and $j = 1, 2, 3, \ldots$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

Theorem (Fatou's Lemma). If $f_n : X \to [0, \infty]$ is measurable, for each positive integer N, then

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leqslant \liminf_{n\to\infty} \int_X f_n\,d\mu$$

Theorem (1.29). Suppose $f : X \to [0, \infty]$ is measurable, and

$$\varphi(E) = \int_E f \, d\mu \qquad (E \in \mathscr{M})$$

Then φ is a measure on $\mathscr{M}$ and

$$\int_X g \, d\varphi = \int_X gf \, d\mu$$

for every measurable g on X with range in $[0, \infty]$.

Lesbegue Integrable We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which

$$\int_X |f| \, d\mu < \infty$$

Note that the measurability of f implies that of $|f|$. The members of $L^1(\mu)$ are called **Lebesgue integrable** functions (with respect to μ) or **summable functions**.

Complex Integration If $f = u + iv$ where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \int v^+ \, d\mu - i \int v^- \, d\mu$$

for every measurable set E.

Theorem (1.32). Suppose f and $g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g \in L^1(\mu)$, and

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

Theorem (1.33). If $f \in L^1(\mu)$, then

$$\left| \int_X f \, d\mu \right| \leqslant \int_X |f| \, d\mu$$

Theorem (Lebesgue's Dominated Convergence Theorem). Suppose $\{f_n\}$ is a sequence of complex measurable function on X such that

$$f(x) = \lim_{n \to \infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leqslant g(x)$$

then $f \in L^1(\mu)$,

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0 \qquad \text{and} \qquad \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu = \int_X f \, d\mu$$

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Theorem. If $|E| = 0$ or if $f = 0$ a.e. in E, then $\int_E f = 0$.

Theorem (Bounded Convergence Theorem). Let $\{f_k\}$ be a sequence of measurable function on E such that $f_k \to f$ almost everywhere in E. If $|E| < +\infty$ and there is a finite constant M such that $|f_k| \leqslant M$ a.e. in E, then $\int_E f_k \to \int_E f$.

Theorem (Egorov's Theorem). Suppose that $\{f_k\}$ is a sequence of measurable functions which converges almost everywhere in a set E of finite measure to a finite limit f. Then $\varepsilon > 0$, there is a closed subset F of E such that $|E \backslash F| < \varepsilon$ and $\{f_k\}$ converges uniformly to f on F.

4.2 Lebesgue Integration in $\mathbb{R}$

Rudin

Theorem (3.15). If the distance between two continuous functions f and g, with compact supports in $\mathbb{R}$, is defined to be

$$\int_{-\infty}^{\infty} |f(t) - g(t)|\, dt$$

the completion of the resulting metric space consists precisely of the Lebesgue integrable function on $\mathbb{R}^1$, provided we identify any two that are equal almost everywhere.

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Riemann-Stieltjes vs. Lebesgue Integral Consider the function

$$\omega(\alpha) = \omega_{f,E}(\alpha) = |\{x \in E : f(x) > \alpha\}|$$

where f is a measurable function on E and $-\infty < \alpha < +\infty$. We call $\omega_{f,E}$ the **distribution function of f on E.**

If we assume that f is finite a.e. in E , then by (3.26 ii)

$$\lim_{\alpha \to +\infty} \omega(\alpha) = 0$$

unless $\omega(\alpha) \equiv +\infty$. Similarly

$$\lim_{\alpha \to -\infty} \omega(\alpha) = |E|$$

Assuming $|E| < +|infty$, ω is bounded, the first equality holds, and ω is of bounded variation on $(-\infty, +\infty)$ with variation equal to $|E|$.

Lemma (5.38). If $\alpha < \beta$, then $|\{\alpha < f \leqslant \beta\}| = \omega(\alpha) - \omega(\beta)$.

Lemma (5.39). Let $w(\alpha+) = \lim_{\varepsilon \searrow 0} \omega(\alpha + \varepsilon)$ and $w(\alpha-) = \lim_{\varepsilon \searrow 0} \omega(\alpha - \varepsilon)$. Then:

a. $\omega(\alpha+) = \omega(\alpha)$; that is ω is continuous from the right

b. $\omega(\alpha-) = |\{f \geqslant \alpha\}|$

Corollary (5.40). a. $\omega(\alpha-) - \omega(\alpha) = |\{f = \alpha\}|$; in particular, ω is continuous at α if and only if $|\{f = \alpha\}| = 0$

b. ω is constant in an open interval (α, β) if and only if $|\{\alpha < f < \beta\}| = 0$, that is, if and only if f takes almost no values between α and β.

Theorem (5.41). If $a < f(x) < b$ for $x \in E$, then

$$\int_E f = -\int_a^b \alpha \, d\omega(\alpha)$$

Theorem (5.42). Let f be any measurable function E, and let $E_{ab} = \{x \in E : a < f(x) \leqslant b\}$. Then

$$\int_{E_{ab}} f = -\int_a^b \alpha \, d\omega(\alpha)$$

Theorem (5.43). If either $\int_E f$ of $\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha)$ is finite, then the other exists and is finite, and

$$\int_E f = -\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha)$$

Equimeasurable Two measurable functions f and g defined on E are said to be **equimeasurable**, or equidistributed, if

$$\omega_{f,E}(\alpha) = \omega_{g,E}(\alpha) \quad \forall \alpha$$

We might think of these functions as rearrangements.

Corollary (5.44). If f and g are equivmeasurable on E and $f \in L(E)$, then $g \in L(E)$ and

$$\int_E f = \int_E g$$

4.3 Lebesgue Measure

Rudin

Theorem (Lebesgue Measure). There exists a positive complete measure m defined on a σ-algebra $\mathscr{M}$ in $\mathbb{R}^k$, with the following properties:

a. $m(W) = \text{vol}(W)$ for every k-cell W.

b. $\mathscr{M}$ contains all Borel sets in $\mathbb{R}^k$, more precisely, $E \in \mathscr{M}$ if and only if there are sets A and $B \subset \mathbb{R}^k$ such that $A \subset E \subset B$, A is an F_σ, B is a G_δ, and $m(B \backslash A) = 0$. Also m is regular.

c. m is translation-invariant, i.e. $m(E + x) = m(E)$ for every $E \in \mathscr{M}$ and every $x \in \mathbb{R}^k$.

d. If μ is any positive translation-invariant Borel measure on $\mathbb{R}^k$ such that $\mu(K) < \infty$ for every compact set K, then there is a constant c such that $\mu(E) = cm(E)$ for all Borel sets $E \subset \mathbb{R}^k$.

e. To every linear transformation T of $\mathbb{R}^k$ into $\mathbb{R}^k$ corresponds a real number $\Delta(T)$ such that

$$m(T(E)) = \Delta(T)m(E)$$

for every $E \in \mathscr{M}$. In particular, $m(T(E)) = m(E)$ when T is a rotation.

The members of $\mathscr{M}$ are the **Lebesgue measurable** set in $\mathbb{R}^k$; m is the **Lebesgue measure** on $\mathbb{R}^k$.

Theorem. If $A \subset \mathbb{R}^1$ and every subset of A is Lebesgue measurable then $m(A) = 0$

Corollary. Every set of positive measure has nonmeasurable subsets

Wheeden & Zygmund

Lebesgue Measureable A subset E of $\mathbb{R}^n$ is said to be **Lebesgue Measurable**, or simply **measurable**,, if given $\varepsilon > 0$, there exists an open set G such that

$$E \subset G \qquad \text{and} \qquad |G \backslash E|_e < \varepsilon$$

Lebesgue Measure If E is measurable, its outer measure is called its **Lebesgue measure** or simple its **measure**, and denoted $|E|$ where $|E| = |E|_e$ for measurable E.

Corollary. An interval I is measurable, and $|I| = v(I)$.

Lemma (3.22). A set E in $\mathbb{R}^n$ is measurable if and only if given $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $|E \backslash F|_e < \varepsilon$.

Theorem (Caratheodory). E is measurable if and only if for every set A,

$$|A|_e = |A \cap E|_e + |A \backslash E|_e$$

Chapter 5 L^p Spaces

5.1 L^p Spaces

Rudin

L^p-norm If $0 < p < \infty$ and if f si a copmlex measurable function on X, define

$$\|f\|_p = \left\{ \int_X |f|^p \, \mathrm{d}\mu \right\}^{1/p}$$

and let $L^p(\mu)$ consist of all f for which $\|f\|_p < \infty$. We call $\|f\|_p$ the **L^p-norm** of f.

Essential Supremum Suppose $g : X \to [0, \infty[$ is measurable. Let S be the set of all real α such that

$$\mu(g^{-1}((\alpha, \infty])) = 0$$

If $S = \varnothing$, put $\beta = \infty$. If $S \neq \varnothing$, put $\beta = \inf S$. Since

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right)$$

and since the union of a countable collection of sets of measure 0 has measure 0, we see that $\beta \in S$. We call β the **essential supremum** of g.

Essentially Bounded If f is a complex measurable function on X, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and we let $L^\infty(\mu)$ consist of all f for which $\|f\|_\infty < \infty$. The members of $L^\infty(\mu)$ are sometimes called **essentially bounded** measurable functions on X.

Lemma. It follows from this definition that the inequality $|f(x)| \leqslant \lambda$ holds for almost all x if and only if $\lambda \geqslant \|f\|_\infty$

[Conjugate Exponents] If p and q are positive real numbers such that $p + q = pq$ or equivalently

$$\frac{1}{p} + \frac{1}{q} = 1$$

then we call p and q a pair of **conjugate exponents**.

As $p \to 1$, $q \to \infty$, so 1 and ∞ are also regarded as a pair of conjugate exponents.

Theorem (3.8). If p and q are conjugate exponents, $1 \leqslant pleq\infty$ and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$

Theorem (3.9). Suppose $1 \leqslant p \leqslant \infty$, and $f \in L^p(\mu)$, $g \in L^p(\mu)$,Then $f + g \in L^p(\mu)$, and

$$||f + g||_p \leqslant ||f||_p + ||g||_p$$

Lemma (3.10). FIx p, $1 \leqslant p \leqslant \infty$. If $f \in L^p(\mu)$ and α is a complex number it is clear that $\alpha f \in L^p(\mu)$. In fact,

$$||\alpha f||_p = |\alpha|||f||_p$$

Thus, $L^p(\mu)$ is a **complex vector space**.

Lemma. Suppose f, g, and h are in $L^p(\mu)$. Then

$$||f - h||_p \leqslant ||f - g||_p + ||g - h||_p$$

Thus, we can regard $L^p(\mu)$ as a metric space. (It is in fact a complete metric space.)

Convergence in $L^p(\mu)$ If $\{f_n\}$ is a sequence in $L^p(\mu)$, if $f \in L^p(\mu)$, and if $\lim_{n\to\infty} ||f_n - f||_p = 0$, we say that $\{f_n\}$ **converges to f in $L^p(\mu)$** (or that $\{f_n\}$ converges to f in the mean of order p, or that $\{f_n\}$ is L^p-convergent to f).

Cauchy in $L^p(\mu)$ If to every $\varepsilon > 0$ there corresponds an integer N such that $||f_n - f_m||_p < \varepsilon$ as soon as $n, m > N$, we call $\{f_n\}$ a **Cauchy sequence in $L^p(\mu)$**

Theorem (3.11). $L^p(\mu)$ is a complete metric space, for $1 \leqslant p \leqslant \infty$ and for every positive measure μ.

Theorem (3.12). If $1 \leqslant p \leqslant \infty$ and if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$, with limit f, then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to $f(x)$.

Theorem (3.13). Let S be the class of all complex, measurable, simple functions on X such that

$$\mu(\{x : s(x) \neq 0\}) < \infty$$

If $1 \leqslant p < \infty$, then S is dense in $L^p(\mu)$.

5.2 Holder's and Minkowski's Inequalities

Rudin

Convex A real function φ defined on a segment (a, b), where $-\infty \leqslant a < b \leqslant \infty$, is called **convex** if the inequality

$$\varphi((1 - \lambda)x + \lambda y) \leqslant (1 - \lambda)\varphi(x) + \lambda\varphi(y)$$

holds whenever $a < x < b$, $a < y < b$, and $0 \leqslant \lambda \leqslant 1$.

Graphically, the condition is that if $x < t < y$, then the point $(t, \varphi(t))$ should lie below or on the line connecting the points $(x, \varphi(x))$ and $(y, \varphi(y))$ in the plane. Also,

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leqslant \frac{\varphi(u) - \varphi(t)}{u - t}$$

whenever $a < s < t < u < b$.

Lemma. A real differentiable function φ is convex in (a, b) if and only if $a < s < t < b$ implies $\varphi'(s) \leqslant \varphi'(t)$, i.e., if and only if the derivative φ' is a monotonically increasing function.

Theorem (3.2). If φ is convex on (a, b) then φ is continuous on (a, b).

Theorem (Jensen's Inequality). Let μ be a positive measure on a σ-algebra $\mathscr{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then

$$\varphi\left(\int_\Omega f \,\mathrm{d}\mu\right) \leqslant \int_\Omega (\varphi \circ f) \,\mathrm{d}\mu$$

Theorem (Holder's Inequality). Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable function on X, with range in $[0, \infty]$. Then

$$\int_X fg \,\mathrm{d}\mu \leqslant ||f||_p ||g||_q = \left\{\int_X f^p \,\mathrm{d}\mu\right\}^{1/p} \left\{\int_X g^q \,\mathrm{d}\mu\right\}^{1/q}$$

Note. If $p = q = 2$ this is called **Schwarz inequality**.

Theorem (Minkowski's Inequality). Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable function on X, with range in $[0, \infty]$. Then

$$||f + g||_p \leqslant ||f||_p ||g||_p$$

That is:

$$\left\{\int_X (f + g)^p \,\mathrm{d}\mu\right\}^{1/p} = \left\{\int_X f^p \,\mathrm{d}\mu\right\}^{1/p} + \left\{\int_X g^p \,\mathrm{d}\mu\right\}^{1/p}$$

5.3 Approximation by Continuous Functions

Rudin

Theorem (3.14). For $1 \leqslant p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$

Lemma. For every $p \in [1, \infty]$ we have a metric on $C_c(\mathbb{R}^k)$, the distance between f and g is $||f - g||_p$.

Theorem. If two continuous functions on $\mathbb{R}^k$ are not identical, then they differ on some nonempty open set V, and $m(V) > 0$, since V contains a k-cell. Thus if two members of $C_c(\mathbb{R}^k)$ are equal a.e., they are equal.

Lemma. In $C_c(\mathbb{R}^k)$ the esential supremum is the same as the actual supremum: for $f \in C_c(\mathbb{R}^k)$

$$||f||_\infty = \sup_{x \in \mathbb{R}^k} |f(x)|$$

Theorem. $L^p(\mathbb{R}^k)$ is the completion of the metric space which is obtained by endowing $C_c(\mathbb{R}^k)$ with the L^p-metric.

Theorem. (Case of $p = 1$) If the distance between two continuous functions f and g, with compact supports in $\mathbb{R}$, is defined to be

$$\int_{-\infty}^{\infty} |f(t) - g(t)| \, dt$$

the completion of the resulting metric space consists precisely of the Lebesgue integrable function on $\mathbb{R}^1$, provided we identify any two that are equal almost everywhere.

Lemma. Every metric space S has a completion S^* whose elements may be viewed abstractly as equivalence classes of Cauchy sequences

Theorem. The L^∞- completion of $C_c(\mathbb{R}^k)$ is not $L^\infty(\mathbb{R}^k)$, but is $C_0(\mathbb{R}^k)$, the space of all continuous functions of $\mathbb{R}^k$ which "vanish at infinity".

Vanish at Infinity A complex function f on a locally compact Hausdorff space X is said to **vanish at infinity** if to every $\varepsilon > 0$ there exists a copmact set $K \subset X$ such that $|f(x)| < \varepsilon$ for all x not in K.

$C_0(X)$ The class of all continuous f on X which vanish at infinity is called $C_0(X)$.

Clearly $C_c(X) \subset C_0(X)$, and the two classes coincide if X is compact. In that case we write $C(X)$ for either of them.

Theorem (3.17). If X is a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$, relative to the metric defined by the supremum norm

$$||f|| = \sup_{x \in X} |f(x)|$$

5.4 Duality of L^p and L^q

Wheeden & Zygmund

Linear Functional If B s a Banach space over the real numbers, a real-valued **linear functional** l on B is by definition a real-valued function $l(f)$, $f \in B$, which satisties:

$$l(f_1 + f_2) = l(f_1) + l(f_2) \qquad l(\alpha f) = \alpha l(f), \quad -\infty < \alpha < \infty$$

Bounded A linear functional l is said to be **bounded** if tehre is a constant c such that $|l(f)| \leqslant c||f||$ for all $f \in B$. A bounded linear untional l is continuous with respect to the norm in B, by which we mean that if $||f - f_k|| \to 0$ as $k \to \infty$, then $l(f_k) \to l(f)$, since $|l(f) - l(f_k)| = |l(f - f_k)| \leqslant c||f - f_k|| \to 0$.

Norm The **norm** $||l||$ of a bounded linear functional l is defined as

$$||l|| = \sup_{||f|| \leqslant 1} |l(f)|$$

since $f/||f||$ has norm 1 for any $f \neq 0$, and since l is linear, we have $||l|| = \sup |l(f)|/||f||$.

Dual Space The collection of all bounded linear functionals on B is called the **dual space** B' of B. We shall consider the case when $B = L^p = L^p(E, \mathrm{d}\mu) = L^p(\mu)$. The goal is to show that if $1 \leqslant p < \infty$ and μ is σ-finite, then $(L^p)' = L^{p'} (= L^q)$.

Theorem (10.43). Let $1 \leqslant p \leqslant \infty$, $1/p + 1/q = 1$. If $g \in L^q(\mu)$, then the formula

$$l(f) = \int_E fg \, \mathrm{d}\mu$$

defines a bounded linear functional $l \in [L^p(\mu)]'$. Moreover $||l|| \leqslant ||g||_q$

Theorem (10.44). Let $1 \leqslant p < \infty$, p, q be conjugate exponents, μ be σ-finite. If $l \in (L^p(\mu))'$, there is a unique $g \in L^q(\mu)$ such that

$$l(f) = \int_E fg \, \mathrm{d}\mu$$

Moreover, $||l|| = ||g||_q$, so that the correspondence between l and g defines an isometry between $(L^p(\mu))'$ and $L^q(\mu)$.

Chapter 6 Miscellaneous

6.1 Radon-Nikodym Theorem

Rudin

Theorem (Lebesgue- Radon-Nikodym). Let μ be a positive σ-finite measure on a σ-algebra $\mathscr{M}$ in a set X, and let λ be a complex measure on $\mathscr{M}$.

a. There is a unique pair of complex measures λ_a and λ_s on $\mathscr{M}$ such that

$$\lambda = \lambda_a + \lambda_s \qquad \lambda_a \ll \mu \qquad \lambda_s \perp \mu$$

If λ is positive and finite, then so are λ_a and λ_s.

b. (Radon-Nikodym Theorem.) There is a unique $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu$$

for every set $E \in \mathscr{M}$.

Lebesgue Decomposition The pair (λ_a, λ_s) is called the **Lebesgue decomposition** of λ relative to μ. The uniqueness of the decomposition is easily seen, for if (λ_a', λ_s') is another pair which satisfies the first clause of the previous theorem, then

$$\lambda_a' - \lambda_a = \lambda_s - \lambda_s'$$

$\lambda_a' - \lambda_a \ll \mu$ and $\lambda_s - \lambda_s' \perp \mu$, hence both sides must be 0.

Radon- Nikodym Derivative The function h is the previous theorem is calleb the **Radon-Nikodym derivative** of λ_a with respect to μ. We may then express (b) in the form $\mathrm{d}\lambda_a = h \, \mathrm{d}\mu$ or even $h = \mathrm{d}\lambda_a / \mathrm{d}\mu$.

Lemma. Let μ be Lebesgue measure on $(0,1)$, and let λ be the counting measure on the σ-algebra of all Lebesgue measurable sets in $(0,1)$. Then λ has no Lebesgue decomposition relative to μ, and although $\mu \ll \lambda$ and μ is bounded, there is no $h \in L^1(\lambda)$ such that $\mathrm{d}\mu = h \, \mathrm{d}\lambda$.

Theorem (6.11). Suppose μ and λ are measures on a σ-algebra $\mathscr{M}$, μ is positive and λ is complex. Then the following two conditions are equivalent:

a. $\lambda \ll \mu$

b. To every $\varepsilon > 0$ corresponds a $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ for all $E \in \mathscr{M}$ with $\mu(E) < \delta$.

Property (b) is sometimes used as the definition of the absolute continuity. However, (a) does not imply (b) if λ is a positive unbounded measure.

Theorem (6.12). Let μ be a complex measure on a σ- algebra $\mathscr{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that

$$\mathrm{d}\mu = h\,\mathrm{d}|\mu|$$

(This is sometimes referred to as the **polar representation** of μ.)

Theorem (6.13). Suppose μ is a positive measure on $\mathscr{M}$, $g \in L^1(\mu)$, and

$$\lambda(E) = \int_E g\,\mathrm{d}\mu \qquad (E \in \mathscr{M})$$

Then

$$|\lambda|(E) = \int_E |g|\,\mathrm{d}\mu \qquad (E \in \mathscr{M})$$

Theorem (Hahn Decomposition Theorem). Let μ be a real measure on a σ-algebra $\mathscr{M}$ in a set X. Then tehre exist sets A and $B \in \mathscr{M}$ such that $A \cup B = X$, $A \cap B = \varnothing$, and such that the positive and negative variations μ^+ and μ^- of μ satisfy

$$\mu^+(E) = \mu(A \cap E), \qquad u^-(E) = 0\mu(B \cap E) \qquad (E \in \mathscr{M})$$

IN other words, X is the union of two disjoint measurable sets A and B, such that A carries all the positive mass of μ and B carries all the negative mass of μ.

Corollary. If $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measure, then $\lambda_1 \geqslant \mu^+$ and $\lambda_2 \geqslant \mu^-$.

6.2 Lebesgue Points

Rudin

Lebesgue Points If $f \in L^1(\mathbb{R}^k)$, any $x \in \mathbb{R}^k$ for which it is true that

$$\lim_{r\to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}m(y) = 0$$

is called a **Lebesgue point** of f.

Theorem (7.7). If $f \in L^1(\mathbb{R}^k)$, then almost every $x \in \mathbb{R}^k$ is a Lebesgue point of f.

Theorem (7.8). Suppose μ is a complex Borel measure on $\mathbb{R}^k$, and $\mu \ll m$. Let f be the Radon-Nikodym derivative of μ with respect to m. Then $D\mu - f$ a.e.[m], and

$$\mu(E) = \int_E (D\mu)\, \mathrm{d}m$$

for all Borel sets $E \subset \mathbb{R}^k$.

Nicely Shrinking Sets Suppose $x \in \mathbb{R}^k$. A sequence $\{E_i\}$ of Borel sets in $\mathbb{R}^k$ is said to **shrink to x nicely** if there is a number $\alpha > 0$ with the following property. There is a sequence of balls $B(x, r_i)$, with $\lim r_i = 0$, such that $E_i \subset B(x, r_i)$ and

$$m(E_i) \geqslant \alpha \cdot m(B(x, r_i))$$

Theorem (7.10). Associate to each $x \in \mathbb{R}^k$ a sequence $\{E_i(x)\}$ that shrinks to x nicely, and let $f \in L^1(\mathbb{R}^k)$. Then

$$f(x) = \lim_{i \to \infty} \frac{1}{m(E_i(x))} \int_{E_i(x)} f\, \mathrm{d}m$$

at every Lebesgue point of f, hence a.e.$[m]$.

Theorem. If $f \in L^1(\mathbb{R}^1)$ and

$$F(x) = \int_{-\infty}^{x} f\, \mathrm{d}m \qquad (-\infty < x < \infty)$$

then $F'(x) = f(x)$ at every Lebesgue point of f, hence a.e.$[m]$.

6.3 Absolutely Continuous Functions (General)

Rudin

Absolutely Continuous Let μ be a positive measure on a σ-algebra $\mathscr{M}$, and let λ be an arbitrary measure on $\mathscr{M}$; λ may be positive or complex. We say that λ is **absolutely continuous** with respect to μ, and write

$$\lambda \ll \mu$$

if $\lambda(E) = 0$ for every $E \in \mathscr{M}$ for which $\mu(E) = 0$.

Concentrated on A If there is a set $A \in \mathscr{M}$ such that $\lambda(E) = \lambda(A \cap E$ for every $E \in \mathscr{M}$, we say that λ is **concentrated on A**.

This is equivalent to the hypothesis that $\lambda(E) = 0$ whenever $E \cap A = \varnothing$.

Mutually Singular Suppose λ_1 and λ_2 are measures on $\mathscr{M}$, and suppose there exists a pair of disjoint sets A and B such that λ_1 and λ_2 are **mutually singular**, and write $\lambda_1 \perp \lambda_2$.

Lemma (6.8). Suppose μ, λ, λ_1, λ_2 are measures on a σ-algebra $\mathscr{M}$, and μ is positive.

a. If λ is concentrated on A, so is $|\lambda|$
b. If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$
c. If $\lambda_1 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$
d. If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$
e. If $\lambda \ll \mu$, then $|\lambda| \ll \mu$
f. If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$
g. If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

Lemma (6.9). If μ is a positive σ-finite measure on a σ-algebra $\mathscr{M}$ in a set X, then there is a unction $w \in L^1(\mu)$ such that $0 < w(x) < 1$ for every $x \in X$.

6.4 Functions of Bounded Variation (General)

Rudin

Partition Let $\mathscr{M}$ be a σ-algebra in a set X. Call a countable collection $\{E_i\}$ of members of $\mathscr{M}$ a **partition of** E if $E_i \cap E_j = \varnothing$ whenever $i \neq j$, and if $E = \bigcup E_i$

Lemma (6.1). For a complex measure μ on $\mathscr{M}$, μ is then a complex function on $\mathscr{M}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

for every partition $\{E_i\}$ of E. Note, this series is absolutely convergent by Theorem 3.56 and since E_i's are pairwise disjoint.

Total Variation We can define a set function $|\mu|$ on $\mathscr{M}$ by

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \qquad (E \in \mathscr{M})$$

the supremum being taken over all partitions $\{E_i\}$ of E.

Note $|\mu|(E) \geqslant |\mu(E)|$, but the two are generally unequal.

The set function $|\mu|$ is called the **total variation of** μ, or sometimes, to avoid misunderstanding, the **total variation measure**.

Lemma. If μ is a positive measure, then $|\mu| = \mu$.

Bounded Variation If the range of μ lies in the cmoplex plane, then it actually lies in some disc of finite radius. This property (proved in Theorem 6.4) is sometimes expressed by saying that μ is **of bounded variation**.

Theorem (6.2). The total variation $|\mu|$ of a complex measure μ on $\mathscr{M}$ is a positive measure on $\mathscr{M}$.

Lemma (6.3). If $z_1, \dots, z_N$ are complex numbers then there is a subset S of $\{1, \dots, N\}$ for which

$$\left|\sum_{k\in S} z_k\right| \geqslant \frac{1}{\pi}\sum_{k=1}^{N}|z_k|$$

Theorem (6.4). If μ is a complex measure on X, then

$$|\mu|(X) < \infty$$

Lemma (6.5). If μ and λ are complex measures on the same σ-algebra $\mathscr{M}$, we defined $\mu + \lambda$ and $c\mu$ by:

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E) \qquad (c\mu)(E) = c\mu(E) \qquad E \in \mathscr{M}$$

for any scalar c, in the usual maner. Further, these are complex measures. Lastly, if we put $||\mu|| = |\mu|(X)$, it is easy to verify that all axioms of a normed linear spae are satisfied.

Positive and Negative Variation Consider a **real** measure μ on a σ-algebra $\mathscr{M}$. Suc measures are frequently called **signed measures**. Define $|\mu|$ as before, and define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \qquad \mu^- = \frac{1}{2}(|\mu| - \mu)$$

Then both μ^+ and μ^- are positive measures on $\mathscr{M}$ and they are bounded. Also:

$$\mu = \mu^+ - \mu^- \qquad |\mu| = \mu^+ + \mu^-$$

The measures μ^+ and μ^- are called the **positive and negative variations of μ** respectively.

Jordan Decomposition of μ The representation $\mu = \mu^+ - \mu^-$ is known as the **Jordan decomposition of μ.**

6.5 Fundamental Theorem of Calculus

Rudin

Absolutely Continuous A complex function f defined on an interval $I = [a, b]$, is said to be **absolutely continuous** on I (briefly, f is AC on I) if there corresponds to every $\varepsilon > 0$ a $\delta > 0$ so that

$$\sum_{i=1}^{n}|f(\beta_i) - f(\alpha_i)| < \varepsilon$$

for any n and any disjoint collection of segments $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$ in I whose lengths satisfy

$$\sum_{i=1}^{n}(\beta_i - \alpha_i) < \delta$$

Theorem (7.18). Let $I = [a, b]$, let $f : I \to \mathbb{R}^1$ be continuous and nondecreassing. Each of the following three statements about f implies the other two:

a. f is AC on I

b. f maps sets of measure 0 to sets of measure 0.

c. f is differentiable a.e. on I, $f' \in L^1$, and

$$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t \qquad (a \leqslant x \leqslant b)$$

Theorem (7.19). Suppose $f : I \to \mathbb{R}^1$ is AC, $I = [a, b]$. Define

$$F(x) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| \qquad (a \leqslant x \leqslant b)$$

where te supremum is taken over all N and over all choices of $\{t_i\}$ such that $a = t_0 < t_1 < \cdots < t_N = x$.

The functions F, $F + f$, $F - f$ are then nondecreasing and AC on I.

Bounded Variation F is called the **total variation** functino of f. If f is any (complex) functino on I, AC or not, and $F(b) < \infty$, then f is said to have **bounded variation** on I, and $F(b)$ the **total variation** of f on I.

Theorem (Fundamental Theorem of Calculus). If f is a copmlex function that is AC on $I = [a, b]$, then f is differentiable at almost all points of I, $f' \in L^1(m)$, and

$$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t \qquad (a \leqslant x \leqslant b)$$

Theorem (7.21). If $f : [a, b] \to \mathbb{R}^1$ is differentiable at every point of $[a, b]$ and $f' \in L^1$ on $[a, b]$, then

$$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t \qquad (a \leqslant x \leqslant b)$$

Note. This differs from the previous theorem in that we require differentiability holds at every point of $[a, b]$.

6.6 Product Measures

Rudin

Cartesian Product If X and Y are two sets, their **cartesian product** $X \times Y$ is the set of all ordered pairs (x, y), with $x \in X$ and $y \in Y$. If $A \subset X$ and $B \subset Y$, it follows that $A \times B \subset X \times Y$. We call any set of the form $A \times B$ a **rectangle** in $X \times Y$.

Measurable Rectangle Suppose now that $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ are measurable spaces. (Recall, this simply means that $\mathscr{S}$ is a σ-algebra in X and $\mathscr{T}$ is a σ-algebra in Y.

A **measurable rectangle** is any set of the form $A \times B$ where $A \in \mathscr{S}$ and $B \in \mathscr{T}$.

Elementary Sets If $Q = R_1 \cup \cdots \cup R_n$, where each R_i is a measurable rectangle and $R_i \cap R_j = \varnothing$ for $i \neq j$, we say $Q \in \mathscr{E}$, the class of all **elementary sets.**

Lemma. Note $\mathscr{S} \times \mathscr{T}$ is defined to be the smallest σ-algebra in $X \times Y$ which contains every measurable rectangle.

Monotone Class A **monotone class** $\mathscr{M}$ is a collection of sets with the following properties: If $A_i \in \mathscr{M}$, $B_i \in \mathscr{M}$, $A_i \subset A_{i+1}$, $B_i \supset B_{i+1}$, for $i = 1, 2, 3, \ldots$, and if

$$A = \bigcup_{i=1}^{\infty} A_i, \qquad B = \bigcap_{i=1}^{\infty} B_i$$

then $A \in \mathscr{M}$ and $B \in \mathscr{M}$.

x-Section and y-Section If $E \subset X \times Y$, $x \in X$, $y \in Y$, we define

$$E_x = \{y : (x, y) \in E\} \qquad E^y = \{x : (x, y) \in E\}$$

We call E_x and E^y the **x-section** and **y-section**, respectively, of E. Note that $E_x \subset Y$, $E^y \subset X$.

Theorem (8.2). If $E \in \mathscr{S} \times \mathscr{T}$, then $E_x \in \mathscr{T}$ and $E^y \in \mathscr{S}$, for every $x \in X$ and $y \in Y$.

Theorem (8.3). $\mathscr{S} \times \mathscr{T}$ is the smallest monotone class which contians all elementary sets.

Function With each function f on $X \times Y$ and with each $x \in X$, we associate a function f_x defined on Y by $f_x(y) = f(x, y)$.

Similarly, if $y \in Y$, f^y is the function defined on X by $f^y(x) = f(x, y)$.

Since we are now dealing with three σ-algebras, $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{S} \times \mathscr{T}$, we shall, for the sake of clarity, indicate in the sequel to which of these three σ-algebras the word "measurable" refers.

Theorem (8.5). Let f be an $(\mathscr{S} \times \mathscr{T})$-measurable functions on $X \times Y$. Then

a. For each $x \in X$, f_x is a $\mathscr{T}$-measurable function

b. For each $y \in Y$, f^y is an $\mathscr{S}$-measurable function

Theorem (8.6). Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ be σ-finite measure spaces. Suppose $Q \in \mathscr{S} \times \mathscr{T}$. If

$$\varphi(x) = \lambda(Q_x) \qquad \psi(y) = \mu(Q^y)$$

for every $x \in X$ and $y \in Y$, then φ is $\mathscr{S}$-measurable, ψ is $\mathscr{T}$-measurable, and

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda$$

Product of Measures If $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are as in the previous Theorem, and if $Q \in \mathscr{S} \times \mathscr{T}$, we define

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) \, d\mu(x) = \int_Y \mu(Q^y) \, d\lambda(y).$$

The equality of the integrals is the content of the previous theorem. We call $\mu \times \lambda$ the **product** of the measures μ and λ. That $\mu \times \lambda$ is really a measure follows immediately from Theorem 1.27.

6.7 Fubini's theorem

Rudin

Theorem (Fubini's Theorem). Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ be σ-finte measure spaces, and let f be an $(\mathscr{S} \times \mathscr{T})$-measurable function on $X \times Y$.

a. If $0 \leqslant f \leqslant \infty$, and if

$$\varphi(x) = \int_Y f_x \, d\lambda, \qquad \psi(y) = \int_X f^y \, d\mu \qquad (x \in X, y \in Y)$$

then φ is $\mathscr{S}$-measurable, ψ is $\mathscr{T}$-measurable, and

$$\int_X \varphi \, d\mu = \int_{X \times Y} f \, d(\mu \times \lambda) = \int_Y \psi \, d\lambda$$

b. If f is complex and if

$$\varphi^*(x) = \int +Y|f|_x \, d\lambda \qquad \text{and} \qquad \int_X \varphi^* \, d\mu < \infty$$

then $f \in L^1(\mu \times \lambda)$.

c. If $f \in L^1(\mu \times \lambda)$, then $f_x \in L^1(\lambda)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$; the functions φ amd ψ, defined by the first equation in (a) a.e., are in $L^1(\mu)$ and $L^1(\lambda)$, respectively and the consequence of (a) holds.

Corollary. We can rewrite the consequence of (a) in the more usual form of iterated integrals:

$$\int_X d\mu(x) \int_Y f(x,y) \, d\lambda(y) = \int_Y d\lambda(y) \int_X f(x,y) \, d\mu(x)$$

Corollary. The combination of (b) and (c) give the following result. If f is $(\mathscr{S} \times \mathscr{T})$-measurable and if

$$\int_X d\mu(x) \int_Y |f(x, y)| \, d\lambda(y) < \infty$$

then the two iterated integrals are finite and equal.

Summary All this to say, the order of integration may be reversed for $(\mathscr{S} \times \mathscr{T})$-measurable functions f whenever $f \geqslant 0$ and also whenever one of the iterated integrals of $|f|$ is finite.

See Rudin page 166 for Counterexamples.

Miscellaneous

Theorem (Tonelli's Theorem). (6.10 WZ) Let $f(x, y)$ be nonnegative and measurable on an interval $I = I_1 \times I_2$ of $\mathbb{R}^{n+m}$. Then, for almost every $x \in I_i$, $f(x, y)$ is a measurable function of y on I_2. Moreover, as a function of x, $\int_{I_2} f(x, y) \, dy$ is measurable on I_1, and

$$\iint_I f(x, y) \, dx \, dy = \int_{I_1} \left[\int_{I_2} f(x, y) \, dy \right] dx$$

Theorem (Bernoulli's Inequality).

$$(1 + x)^r \geqslant 1 + rx$$

For $r \geqslant 0$ and $x \geqslant -2$.

Nonmeasurable (Zermelo's Axioms) Consider a family of arbitrary nonempty disjoint sets indexed by a set A, $\{E_\alpha : \alpha \in A\}$. Then there exists a set consisting of exactly one element from each E_α, $\alpha \in A$.

Lemma (WZ 3.37). Let E be a measurable subset of $\mathbb{R}^1$ with $|E| > 0$. Then the set of diffecences $\{d : d = x - y, x \in E, y \in E\}$ contained an interval centered at the origin.

Theorem (Vitali's). There exist nonmeasurable sets.

Corollary (WZ 3.39). Any set in $\mathbb{R}^1$ with positive outer measure contains a non-measurable set.

Theorem (Borel-Cantelli Lemma). (Royden p.46) Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Rational Equivalence Relation For $x \sim y$ if $x, y \in \mathbb{Q}$. This was used on a problem on a qual at some point. (Look up later)

Theorem (Theorem 10.33). a. If φ is both absolutely continuous and singular on E with respect to μ , then $\varphi(A) = 0$ for every measurable $A \subset E$.

b. If both ψ and φ are absolutely continuous (singular) on E with respect to μ, then so are $\psi + \varphi$ and $c\varphi$, where c is any real constant.

c. φ is absolutely continuous (singular) on E with respect to μ if and only if its variations $\overline{V}$ and $\underline{V}$ are, or, equivalently, if and only if its total variation is.

d. If $\{\varphi_k\}$ is a sequence of addititve set functions which are absolutely continuous (singular) on E with respect to μ, and if $\varphi(A) = \lim_{k\to\infty} \varphi_k(A)$ exists for every measurable $A \subset E$, then φ is absolutely continuous (singular) on E with respect to μ.

Convolutions If f and g are measurable in $\mathbb{R}^n$, their **convolution** $(f * g)(x)$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-t)g(t)\,\mathrm{d}t$$

Things to note:

- $f * g = g * f$
- $\int_{\mathbb{R}^n} |f * g|\,\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^n} |f|\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^n} |g|\,\mathrm{d}x\right)$
- If f, g are nonnegative and measurable $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f * g\,\mathrm{d}x = \left(\int_{\mathbb{R}^n} f\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^n} g\,\mathrm{d}x\right)$$

Fat Cantor Set The **fat Cantor set** is an example of a set of points on the real line $\mathbb{R}$ that is nowhere dense (contains no intervals), yet has positive measure.

This set is made by starting with $[0,1]$ and removing the middle quarters (like one would do with the Cantor set).

lim sup

sc. $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \longleftrightarrow \{x \in X \mid x \in A_k \text{ for } \infty\text{ly many } k\}$.

$(n \geq k)$

seq. $\lim_{n\to\infty} \sup_{k\geq n} a_k = \inf_{n} \sup_{k \geq n} a_k$

value/func. $\inf_{\delta>0} \left(\sup_{0<|x-a|<\delta} f(x) \right) \longleftrightarrow \limsup_{x\to a} f(x)$

one side $\limsup_{x\to a^+} f(x) = \lim_{k\to\infty} \left(\sup_{(a, a+1/k)} f(x) \right)$

lim inf

$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \longrightarrow \{x \mid x \in A_k \text{ for all but finitely many } k\}$

$(k \geq n)$

$\lim_{n\to\infty} \inf_{k\geq n} a_k = \sup_{n} \inf_{k \geq n} a_k$

$\sup_{\delta>0} \left(\inf_{0<|x-a|<\delta} f(x) \right) \longleftrightarrow \liminf_{x\to a} f(x)$

$\liminf_{x\to a^+} f(x) = \lim_{k\to\infty} \left(\inf_{(a, a+1/k)} f(x) \right)$

Theorems

- $\liminf A_k \subset \limsup A_k$

- $\limsup a_k = -\liminf(-a_k)$

- if $a_n \le b_n \ \forall n \longrightarrow \limsup\{a_n\} \le \liminf\{b_n\}$

- If $B = \liminf_{k\to\infty} E_n$

  $\chi_B = \liminf_{k\to\infty} \chi_{E_n}$

- Both sides of a nonstrict inequality have limits $\longrightarrow$ inequality holds for limits as well.

Measurable Set

- Open / Closed / G_δ / F_σ / Countable $\cap$ $\cup$ of them
- Show $\forall \varepsilon > 0$ $\exists$ open G s.t. $E \subset G$ and $|G \setminus E|_e < \varepsilon$
- $\forall E$ $\exists G$ s.t. $E \subset G$, $|G|_e \leq (1+\varepsilon)|E|_e$, $\exists G_\delta$ set H $|H|_e = |E|_e$
- Show $\forall \varepsilon > 0$ $\exists$ closed F s.t. $F \subset E$ and $|E \setminus F|_e < \varepsilon$
- Show $E = A$ where A is meas
- $|E|_e = 0 \rightarrow$ measurable.
- Show set can be represented as an interval
- E meas iff $E = H \setminus Z$ iff $E = F \cup Z$, H type G_δ + F a F_σ + $|Z| = 0$
- (Caratheodory) $\forall A$ $|A|_e = |A \cap E|_e + |A \setminus E|_e$

  $\forall E$ $\exists$ G_δ set H st. $E \subset H$, $|E|_e = |H|_e$

Measurable function

- Continuous, Borel, Lipschitz (cts)
- $\{f > a\}$ is measurable $\forall |a| < \infty$, $\{f \geq \infty\} = \bigcap_{k \in \mathbb{Z}} \{f > k\}$ $\rightarrow$ Borel forces $\{f > a\}$ borel.
- f meas, $f = g$ a.e. $\rightarrow$ g meas
- f meas $\rightarrow$ $f^{-1}(G)$ is meas $\forall$ open G, borel G
- f finite a.e., $f^{-1}(G)$ is meas $\forall$ open G, $\{f > a\} = f^{-1}(a, \infty)$
- $f \pm g$, $f \cdot g$, f/g, $\lfloor f \rfloor$, $|f|$, $\lceil f \rceil$, sgn f
- χ_E meas iff E meas.
- f meas iff $\forall \varepsilon > 0$ $\exists$ closed $E(\varepsilon)$ st. $f|_{E(\varepsilon)}$ is cts and $|E \setminus E(\varepsilon)| < \varepsilon$
- If $f = \sum f_k$ $\rightarrow$ Show each S_k is meas $\rightarrow$ $S_k \rightarrow f$

- $E \cup Z$ meas., $|Z| = 0 \implies |E \cup Z| = |E|$

  $|E \cup Z| \leq |E| + |Z| = |E|$.

- Show a cts fn maps meas. to meas.

  1. $f(F_\sigma) = F_\sigma$ (use fact. $F_\sigma = \bigcup F$ (closed), $F = \bigcup K$ (compact), $f(K)$ is cmpct)
  2. $|f(Z)| = 0$ for $|Z| = 0$ (use $Z \subset \{I_n\}$ s.t. $\sum |I_n| < \varepsilon$, $f(Z) \subset f(I_n)$)
  3. $E = H \cup Z$ for F_σ - H + $|Z| = 0$ - Z

If f is meas $\rightarrow$ $\{f_k\}$ s.t. $f_k \nearrow f$ simple can be chosen to be meas.

Monotone Convergence Thm

- If $\{f_k\}$ is a sequence of nonnegative, meas fns s.t. $f_k \nearrow f$ on E, then $\int f_k \to \int f$ i.e. $\int \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int f$.
- $\{f_k\}$ is seq. of meas fns on E, $f_k \nearrow f$ a.e on E, $\exists\, g \in L'$ s.t $f_k \geq g$ a.e $\forall k$, $\int f_k \to \int f$
  $f_k \searrow f$ a.e on E, $\exists\, g \in L'$ s.t. $f_k \leq g$ a.e $\forall k$ $\int f_k \to \int f$

Fatou's Lemma

- If $\{f_k\}$ is a sequence of nonneg meas fns on E, then
$$\int \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int f_k$$
- $\{f_k\}$ are meas. If $\exists\, g \in L'(E)$ s.t. $f_k \geq g$ a.e. on E $\forall k$ then $\int \liminf f_k \leq \liminf \int f_k$
  ↳ further if $\exists\, h \in L'$ s.t. $f_k \leq h$ a.e. on E $\forall k$ then $\int \limsup f_k \geq \limsup \int f_k$

Dominated Conv Thm.

- $\{f_k\}$ is seq. of nonneg meas fns on E s.t $f_k \to f$ a.e in E. If $\exists$ meas. fn g s.t.
  $f_k \leq g$ a.e $\forall k$, $\int_E g < \infty \to \int f_k \to \int f$
- $\{f_k\}$ seq. of meas fns on E s.t. $f_k \to f$ a.e in E If $\exists\, g \in L'(E)$, s.t.
  $|f_k| \leq g$ a.e in E $\forall k$, then $\int f_k \to \int f$
- $\int \liminf f_k \leq \liminf \int f_k \leq \limsup \int f_k \leq \int \limsup f_k$.

Uniform Convergence.

- $\{f_k\} \in L'(E)$. $\{f_k\} \to f$ uniformly on E, $|E| < +\infty$. Then $f \in L(E)$ and
$$\int_E f_k \to \int_E f$$

Bounded Convergence

- $\{f_k\}$ meas fns. on E s.t $f_k \to f$ a.e. in E If $|E| < +\infty$, and $\exists$ a finite constant M s.t $|f_k| \leq M$ a.e. in E, then $\int_E f_k \to \int_E f$

Convergence of fns

- $\exists \{f_n\}$ simple s.t. $f_n \nearrow f$
- $\{f_n\}$ are meas and $f_n \to f \Rightarrow f$ is measurable
  - $\hookrightarrow f_n : E \to \overline{\mathbb{R}}$

- $\{f_n\}$ meas, $f_n : E \to \mathbb{R}$, $|E| < \infty$. $f_n \to f$ ptws a.e. Then $\forall \varepsilon > 0$ $\exists$ closed $E(\varepsilon) \subset \mathbb{R}$ s.t. $f_n \to f$ unif on $E(\varepsilon)$ and $|E \setminus E(\varepsilon)| < \varepsilon$
- If $f_n \xrightarrow{m} f$ $\left(\lim_{n \to \infty} |\{|f_n - f| > \varepsilon\}| = 0 \right)$ then $\exists \{f_{n_j}\} \to f$ a.e
- $f_n, g \geq 0$ are <u>simple</u> $f_1 \geq f_2 \geq \ldots \geq \lim f_n \geq g \to \lim \int f_n \geq \int g$
- If $f_n \nearrow f$ simple, $\geq 0 \Rightarrow \int f_n \to \int f$

Additive set fn.

- signed finite measure $\longrightarrow$ $\phi(E) < \infty$ $\forall E \in \Sigma$, $\phi(\cup E_n) = \sum \phi(E_n)$
- $E_1 \subset E_2 \subset \ldots \Rightarrow \phi(\cup E_n) = \lim \phi(E_n)$. $E_1 \supset E_2 \supset \ldots \Rightarrow \phi(\cap E_n) = \lim \phi(E_n)$
- $\phi(\liminf E_n) \leq \liminf(\phi(E_n)) \leq \limsup(\phi(E_n)) \leq \phi(\limsup(E_n))$

Measure

μ is a measure if

$0 \leq \mu(E) \leq \infty$, $\mu(\bigcup E_n) = \sum \mu(E_n)$ for disjoint E_n.

Absolutely Cts

- If $E \in \Sigma$ then ϕ is a.c. on E wrt μ if $\phi(A) = 0$

  $\forall \; A \subset E$ s.t. $\mu(A) = 0$

- ϕ is AC on E w

!!! · Back

- An additive set fn is a.c. on E wrt μ iff given $\varepsilon > 0$, $\exists \; \delta > 0$

  s.t. $|\phi(A)| < \varepsilon$ $\forall$ meas. $A \subset E$ s.t. $\mu(A) < \delta$

- A fn $\underline{f}$ is a.c on $[a,b]$ if: $\forall \varepsilon > 0, \exists \; \delta > 0$ s.t. $\forall$ nonoverlapping int's

  $[a_i, b_i] \in [a,b]$ s.t. $\sum |b_i - a_i| < \delta$, $\longrightarrow$ $\sum |f(b_i) - f(a_i)| < \varepsilon$

  $\Longrightarrow$ unif. continuity

  Ex: Lipschitz fns ~~on E~~, $\sqrt{x}$ on $[0,1]$, $d(x,C)$ (dist. x to Cantor set)

Variation

- $V(f,a,b) = \sup\left(\sum |f(x_i) - f(x_{i-1})|\right)$ over all partitions

  $\hookrightarrow$ BV if $V(f,a,b) < \infty$

- $\overline{V} = \sup\limits_{A \subset E} \phi(A)$ $\quad \underline{V} = -\inf\limits_{A \subset E} \phi(A)$ $\quad V = \overline{V} + \underline{V}$ (ϕ is ASF)

  $\longrightarrow \phi = \overline{V} - \underline{V}$

If f is a.c. on $[a,b]$ ⟶ f is diff of incr abs cts fns

↳ and is of BV.

ϕ is AC on E wrt μ iff $\overline{V}$ or $\underline{V}$ are a.c. iff V is a.c.

· AC & singular on $[a,b]$ ⟹ constant

· f is a.c. on $[a,b]$ ⟺ · f' exists a.e

· $f' \in \mathcal{L}^1[a,b]$,

$f(x) = f(a) + \int_a^x f'(t)\,dt$.

· If f is BV on $[a,b]$ then $\exists$ increasing g, h on $[a,b]$ s.t. $f = g - h$

Singular

- ϕ is singular on E wrt μ if there is a set $Z \subset E$ s.t. $\mu(Z)=0$ and $\phi(A)=0$ $\forall$ meas $A \subset E \setminus Z$

  → ϕ supported on a set of μ measure zero so $E = Z \cup E \setminus Z$ w/ $|Z|=0$, $\phi('A')=0$ for each meas. $A \subset E \setminus Z$

- An ASF ϕ is singular on E wrt μ iff given $\varepsilon>0$ there is a meas. subset E_0 of E s.t. $\mu(E_0)<\varepsilon$ and $V(E \setminus E_0, \phi) < \varepsilon$

- f is singular on $[a,b]$ if $f' \equiv 0$ a.e. on $[a,b]$

Fund. Thm of Calc - esque

- A fn is a.c. on $[a,b]$ iff f' exists in (a,b), $f' \in \mathcal{L}^1$, $f(x)-f(a) = \int_a^x f'$

(Radon - Nikodym)

- $\exists f$ ϕ is an AC ASF on meas $A \subset$ meas E, μ is σ-finite

$$\phi(A) = \int_A f \, d\mu$$

Φ is singular on E wrt μ iff $\bar{V}$ or $\underline{V}$ are ~~s~~ singular iff V is sing.

a.c + singular on $[a,b]$ $\Longrightarrow$ constant

Jordan Decomposition

If ϕ is an additive set function on Σ, then

$$\phi(E) = \overline{V}(E) - \underline{V}(E)$$

(also $\int f = \int f^+ - \int f^-$)

Hahn Decomposition

Let E be a meas set and let ϕ be an ASF defined on meas $A \subset E$

Then there is a measurable $P \subset E$ s.t. $(\phi(A) \geq 0 \ \forall A \subset P), (\phi(A) \leq 0 \ \forall A \subset E \setminus P)$

equivalently $\underline{V}(P) = \overline{V}(E \setminus P) = 0$ so

$\overline{V}(E) = \overline{V}(P) = \phi(P)$

$\underline{V}(E) = \underline{V}(E \setminus P) = -\phi(E \setminus P)$

Level Decomposition

Let ϕ be a nonnegative ASF defined on meas subsets $A \subset E$, meas, let μ be a measure w/ $\mu(E) < \infty$. Then given $a > 0$, there is a decomposition $E = Z \cup \left(\bigcup_{k=1}^{\infty} E_k \right)$ of E into disjoint measurable sets s.t.

i.) $\mu(Z) = 0$

ii) $a(k-1)\mu(A) \leq \phi(A) \leq a k \mu(A)$ for meas $A \subset E_k$, $k = 1, \ldots$

Lebesgue

Let ϕ be an ASF on measurable subsets of a measurable set E, and let μ be a σ-finite measure on E. Then $\exists!$ decomposition

$$\phi(A) = \alpha(A) + \sigma(A) \quad \text{for meas } A \subset E$$

where α, σ are ASF, α is AC and σ is Sing wrt μ. For $f \in L^1$, Z s.t. $\mu(Z) = 0$

$$\alpha(A) = \int_A f \, d\mu \qquad \sigma(A) = \phi(A \cap Z)$$

Random Equations

Chebyshev's Inequality:

f non neg + meas on $E \longrightarrow \forall \alpha > 0 \quad |\{f > \alpha\}| \leq 1/\alpha \int_E f$

Generalized Holders:

for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \longrightarrow \|fg\|_r \leq \|f\|_p \|f\|_q$ ($r=1$ typically)

Convolution

$(f * g)(x) = \int_{\mathbb{R}^n} f(x-t)\, g(t)\, dt. \rightarrow \int |f*g| \leq \left(\int |f|\right)\left(\int |g|\right)$

↳ equality w/out $|\cdot|$

Marcinkiewicz

$M_\lambda(x) = \int_a^b \frac{\delta^*(y)}{|x-y|^{1+\lambda}} dy$ (δ is distance fn)

Minkowski's:

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$

$\mathcal{L}^p$ norm + ℓ^p norm:

$\mathcal{L}^p$: $\left(\int |f|^p\right)^{1/p}$ $\qquad$ ℓ^p: $\left(\sum |a_n|^p\right)^{1/p}$

$\mathcal{L}^\infty$ ess sup f $\qquad$ ℓ^∞ sup a_n

Bessel's:

$\sum |c_n|^2 \leq \|f\|^2$

Parseval's

$\|f\|^2 = \sum^n |c_n|^2 + \|f - s_n\|^2$

$f = \sum c_n \phi_n$ → Fourier Series for $f \in \mathcal{L}^2$

ϕ_n is ONS

Dini #'s:

$$\overline{D}_+ = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \qquad \overline{D}_- = \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$\underline{D}_+ = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \qquad \underline{D}_- = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

Hardy - Littlewood Max Fn

$$f^*(x) = \sup_{Q \text{ cent } x} \frac{1}{|x|} \int_Q f$$

Simple Covering (Vitali)

$$\sum_{j=1}^{N} |Q_j| \geq \beta |E|_e$$ Q_j are disjoint — can prove w/ $\beta = 5^{-n}$

Hardy - Littlewood.

$$|\{x \in \mathbb{R}^n : f^*(x) > \alpha\}| \leq \frac{c}{\alpha} \int_{\mathbb{R}^n} |f|$$

Characteristic fn

$$\int_E f = \int_{\mathbb{R}^n} f \cdot \chi_E$$; p-test $$\int_1^\infty \frac{1}{x^k} = \begin{cases} \in L^1 & k > 1 \\ \infty & k \leq 1 \end{cases}$$

Bernoulli's

$$(1+y)^r \geq (1+ry)$$

Random:

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \qquad \pi/2 - \tan^{-1}(a) = \tan^{-1}(1/a) \qquad |\sin x| \leq x \text{ for } x \geq 0$$

MAT 701 HW 3.1: LEBESGUE OUTER MEASURE

`Due Wednesday 08/29/18 by the end of the day`

Problem 1. Prove that for every set $E \subset \mathbb{R}^n$ and every $\varepsilon > 0$, the Lebesgue outer measure $|E|_e$ is equal to

$$\inf \left\{ \sum v(I_k) \colon E \subset \bigcup_{k=1}^{\infty} I_k, \text{ and } \forall k \operatorname{diam} I_k < \varepsilon \right\}$$

(This is the same infimum as in the definition of $|E|_e$ but with the additional requirement $\operatorname{diam} I_k < \varepsilon$ for all k.)

Proof. Let S_1 be the set of all sums $\sum v(I_k)$ where $\{I_k\}$ is any countable cover of E by intervals I_k. Also let S_2 be the set of all sums $\sum v(I_k)$ where $\{I_k\}$ is a countable cover of E by intervals I_k which satisfy $\operatorname{diam} I_k < \varepsilon$ for all k. By definition, $|E|_e = \inf S_1$. The goal is to show that

$$|E|_e = \inf S_2$$

This will be achieved by proving that $S_2 = S_1$.

That $S_2 \subset S_1$ is immediate from the definitions of both sets. Let us take some element $z \in S_1$. By the definition of S_1 there exists a countable collection of intervals $\{I_k\}$ such that $E \subset \bigcup_k I_k$ and $\sum_k v(I_k) = z$.

For each k, let L_k be the maximal sidelength of I_k, that is $\max_{j=1,\dots,n}(b_j - a_j)$. Let N_k be a large enough integer so that $L_k/N_k < \varepsilon/\sqrt{n}$. Dividing each edge $[a_j, b_j]$ in N_k equal 1-dimensional subintervals results in N_k^n equal n-dimensional subintervals of I_k which cover I_k. Since each sidelength was reduced by the factor of N_k, their product, i.e., the volume of each piece, is $v(I_k)/N_k^n$. This means the sum of volumes of the parts is equal to $v(I_k)$. Each part has diameter at most

$$\sqrt{\sum_{j=1}^{n}((b_j - a_j)/N_k)^2} \le \sqrt{\sum_{j=1}^{n}(L_k/N_k)^2} < \sqrt{n}(\varepsilon/\sqrt{n}) = \varepsilon$$

So, the collection of all subintervals obtained after applying the above process to each k is a countable cover of E, and the sum of their volumes is exactly z. This completes the proof that $S_2 = S_1$. □

Problem 2. Suppose that the sets $E_k \subset \mathbb{R}^n$ are such that the series $\sum_{k=1}^{\infty} |E_k|_e$ converges. Prove that the outer measure of the set

$$A = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$$

is zero. (Remark: the set A is often denoted $\limsup_{k\to\infty} E_k$.)

Proof. Since $\sum_{k=1}^{\infty} |E_k|_e$ converges, the tail sums $\sum_{k=m}^{\infty} |E_k|_e$ tend to zero as $m \to \infty$. Given $\varepsilon > 0$, pick m such that $\sum_{k=m}^{\infty} |E_k|_e < \varepsilon$. By the definition of A,

$$A \subset \bigcup_{k=m}^{\infty} E_k$$

The monotonicity and countable subadditivity of outer measure imply

$$|A|_e \le \left|\bigcup_{k=m}^{\infty} E_k\right|_e \le \sum_{k=m}^{\infty} |E_k|_e < \varepsilon$$

Since ε was arbitrary, it follows that $|A|_e \le 0$. The outer measure cannot be negative, hence $|A|_e = 0$.

(Remark: as mentioned in class, Problem 2 can be solved purely on the basis of the 3 fundamental properties of outer measure. Two of them were mentioned above. The remaining one is $|\emptyset|_e = 0$: this property implies the outer measure cannot be negative, since $\emptyset \subset A$ holds for every A.) □

MAT 701 HW 3.2A: MEASURABLE SETS

Due Friday 08/31/18 by the end of the day

Problem 1. Given an arbitrary set $A \subset \mathbb{R}$ and a number $c > 0$, let $B = \{ca\colon a \in A\}$. Prove that $|B|_e = c|A|_e$.

Proof. Given $\varepsilon > 0$, let $\{[s_k, t_k]\}$ be a countable cover of A such that

$$\sum_k (t_k - s_k) \le |A|_e + \varepsilon$$

The intervals $[cs_k, ct_k]$ cover B, since every point of B is of the form ca where $a \in A$ is covered by some interval $[s_k, t_k]$. Therefore,

$$|B|_e \le \sum_k (ct_k - cs_k) = c\sum_k (t_k - s_k) \le c|A|_e + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, it follows that $|B|_e \le c|A_e|$.

It remains to observe that $A = c^{-1}B$, which by the above implies $|A|_e \le c^{-1}|B|_e$, i.e., $|B|_e \ge c|A|_e$. Thus, $|B|_e = c|A|_e$. □

Problem 2. Suppose that a set $A \subset \mathbb{R}$ is measurable. Prove that for every $c > 0$ the set $B = \{ca\colon a \in A\}$ is also measurable.

Proof. Given $\varepsilon > 0$, let G be an open set that contains A and satisfies $|G \setminus A|_e < \varepsilon/c$. Since the function $f(x) = x/c$ is continuous, the preimage of G under this function is also open. This preimage $f^{-1}(G)$ is cG. Since $A \subset G$, it follows that $B \subset cG$. Moreover, by the previous exercise

$$|(cG) \setminus B|_e = |c(G \setminus A)|_e = c|G \setminus A|_e < \varepsilon$$

Since ε was arbitrary, this proves that B is measurable. □

?

MAT 701 HW 3.2B: MEASURABLE SETS

Due Wednesday 09/05/18 by the end of the day

Problem 1. Given a sequence of continuous functions $f_k \colon \mathbb{R} \to \mathbb{R}$, let B be the set of all points $x \in \mathbb{R}$ such that the sequence $\{f_k(x)\}$ is bounded. Prove that B is a measurable set.

Hint: try to construct B from the sets $\{x \colon |f_k(x)| \leq M\}$ by using countable unions and intersections.

Proof. For $k, m \in \mathbb{N}$ let

$$A(k, m) = \{x \in \mathbb{R} \colon |f_k(x)| \leq m\} = f_k^{-1}([-m, m])$$

Being the preimage of a closed set under a continuous function, $A(k, m)$ is closed and in particular measurable. Let

$$A = \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(k, m)$$

which is also measurable, being obtained from measurable sets by countable set operations. I claim that $A = B$.

If $x \in A$, then there exists $m \in \mathbb{N}$ such that $|f_k(x)| \leq m$ for all $k \in \mathbb{N}$, which shows the sequence $\{f_k(x)\}$ is bounded.

Conversely, if the sequence $\{f_k(x)\}$ is bounded, then there exists $m \in \mathbb{N}$ such that all elements of the sequence are at most m in absolute value. This means $|f_k(x)| \leq m$ for all k, hence $x \in A$. □

Problem 2. Given a sequence of continuous functions $f_k \colon \mathbb{R} \to \mathbb{R}$, let C be the set of all points $x \in \mathbb{R}$ such that $\lim_{k \to \infty} f_k(x) = 0$. Prove that C is a measurable set.

Proof. For $k, m \in \mathbb{N}$ let

$$A(k,m) = \{x \in \mathbb{R} : |f_k(x)| < 1/m\} = f_k^{-1}((-1/m, 1/m))$$

Being the preimage of an open set under a continuous function, $A(k,m)$ is open and in particular measurable. Let

$$A = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k,m)$$

This set is also measurable, being obtained from measurable sets by countable set operations. I claim that $A = C$.

Suppose $x \in A$. Given $\epsilon > 0$, pick $m \in \mathbb{N}$ such that $1/m \leq \epsilon$. Since $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k,m)$, there exists N such that $x \in \bigcap_{k=N}^{\infty} A(k,m)$, which means $|f_k(x)| < 1/m$ for all $k \geq N$. Thus, $|f_k(x)| < \epsilon$ for all $k \geq N$, which proves $\lim_{k\to\infty} f_k(x) = 0$.

Conversely, suppose $x \in C$. Given $m \in \mathbb{N}$, use the definition of the limit $\lim_{k\to\infty} f_k(x) = 0$ to find N such that $|f_k(x)| < 1/m$ for all $k \geq N$. The latter means $x \in \bigcap_{k=N}^{\infty} A(k,m)$. Therefore, for every $m \in \mathbb{N}$ the inclusion $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k,m)$ holds. This means $x \in A$. ☐

MAT 701 HW 3.3: PROPERTIES OF LEBESGUE MEASURE

`Due Friday 09/07/18 by the end of the day`

Problem 1. Prove that the set

$$A = \{x \in \mathbb{R} \colon \exists k \in \mathbb{N} \text{ such that } |2^x - 2^k| \leq 1\}$$

is measurable and $|A| < \infty$.

(Note that $\mathbb{N} = \{1, 2, \dots\}$, not including 0.)

Proof. For each $k \in \mathbb{N}$, the inequality $|2^x - 2^k| \leq 1$ is equivalent to $\log_2(2^k - 1) \leq x \leq \log_2(2^k + 1)$. Thus $A = \bigcup_{k=1}^{\infty} I_k$ where $I_k = [\log_2(2^k - 1), \log_2(2^k + 1)]$. Each I_k is measurable, being an interval. Hence A is measurable. By countable subadditivity of measure, $|A| \leq \sum_{k=1}^{\infty} |I_k|$. It remains to show the series $\sum_{k=1}^{\infty} |I_k|$ converges. This can be done by the comparison test, limit comparison test, or the ratio test. I'll use the Limit Comparison Test with $\sum_{k=1}^{\infty} 2^{-k}$ as a reference series:

$$\begin{aligned}
\frac{|I_k|}{2^{-k}} &= \frac{\log_2(2^k + 1) - \log_2(2^k - 1)}{2^{-k}} \\
&= \frac{k + \log_2(1 + 2^{-k}) - (k + \log_2(1 - 2^{-k}))}{2^{-k}} \\
&= \frac{\log_2(1 + 2^{-k}) - \log_2(1 - 2^{-k})}{2^{-k}} \\
&= \frac{1}{\log 2} \left\{ \frac{\log(1 + 2^{-k})}{2^{-k}} + \frac{\log(1 - 2^{-k})}{-2^{-k}} \right\} \xrightarrow[k \to \infty]{} \frac{2}{\log 2}
\end{aligned}$$

Here the last step is based on $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$. Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, so does $\sum_{k=1}^{\infty} |I_k|$. □

Problem 2. Prove that the set

$$A = \{x \in [0,1] \colon \forall q \in \mathbb{N}\ \exists p \in \mathbb{N} \text{ such that } |x - p/q| \le 1/q^2\}$$

is measurable and $|A| = 0$.

Proof. Let $A_q = \bigcup_{p=1}^{\infty} E(p,q)$ where $E(p,q) = \left[\frac{p}{q} - \frac{1}{q^2}, \frac{p}{q} + \frac{1}{q^2}\right] \cap [0,1]$. This is a countable union of measurable sets $E(p,q)$ (which are intervals, possibly empty), so it is measurable. Then the set $A = \bigcap_{q\in\mathbb{N}} A_q$ is measurable too.

We have $|E(p,q)| \le 2/q^2$ by construction of $E(p,q)$. Also, when $p > q+1$, we have $\frac{p}{q} - \frac{1}{q^2} \ge 1 + \frac{1}{q} - \frac{1}{q^2} \ge 1$, which implies $E(p,q) = \emptyset$. By subadditivity,

$$|A_q| \le \sum_{p=1}^{\infty} |E(p,q)| \le \sum_{p=1}^{q+1} \frac{2}{q^2} = \frac{2q+2}{q^2}$$

By monotonicity, $|A| \le |A_q|$ for each q. Since $|A_q| \xrightarrow[q\to\infty]{} 0$, it follows that $|A| = 0$. □

MAT 701 HW 3.4: PROPERTIES OF LEBESGUE MEASURE

`Due Monday 09/10/18 by the end of the day`

Problem 1. Suppose E and Z are sets in $\mathbb{R}^n$ such that $E \cup Z$ is measurable and $|Z| = 0$. Prove that E is measurable.

Proof. Since $Z \setminus E \subset Z$, the monotonicity of outer measure implies $|Z \setminus E|_e = 0$, hence $Z \setminus E$ is measurable. And then

$$E = (E \cup Z) \setminus (Z \setminus E)$$

is measurable, being the difference of two measurable sets.

(This could be done with Carathéodory theorem or with the "G_δ minus a null set" theorem, but it's easier without.) □

Problem 2. Given a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$, define $\mathcal{M} = \{E \subset \mathbb{R}^n : f^{-1}(E) \text{ is Borel}\}$.

(a) Prove that $\mathcal{M}$ is a σ-algebra.

(b) Prove that if E is Borel, then $f^{-1}(E)$ is Borel. *Hint: use (a).*

Proof. (a) Does not involve f being continuous; the argument works for any map f. Taking preimages commutes with any set operations: for example,

$$f^{-1}(E^c) = \{x\colon f(x) \in E^c\} = \{x\colon f(x) \notin E\} = (f^{-1}(E))^c$$

and

$$f^{-1}\left(\bigcup_i E_i\right) = \{x\colon \exists i\ f(x) \in E_i\} = \bigcup_i f^{-1}(E_i)$$

So, if $E \in \mathcal{M}$, then $f^{-1}(E^c) = f^{-1}(E)^c$ is the complement of a Borel set, hence is Borel, hence $E^c \in \mathcal{M}$. Also, if $E_k \in \mathcal{M}$ for each $k \in \mathbb{N}$, then

$$f^{-1}\left(\bigcup_k E_k\right) = \bigcup_k f^{-1}(E_k)$$

is the countable union of Borel sets, hence is Borel, hence $\bigcup_k E_k \in \mathcal{M}$.

The definition of a σ-algebra in the book also requires us to check that $\mathcal{M}$ is nonempty: to do this, it suffices to notice that $f^{-1}(\emptyset) = \emptyset$ is Borel, hence $\emptyset \in \mathcal{M}$. $\square$

(b) Since f is continuous, the preimage of any open set under f is open, hence Borel. This means $\mathcal{M}$ contains all open sets. By definition, the Borel σ-algebra $\mathcal{B}$ is the **smallest** σ-algebra that contains all open sets. Thus $\mathcal{B} \subset \mathcal{M}$, which by definition of $\mathcal{M}$ means that $f^{-1}(E)$ is Borel whenever E is Borel.

(It is tempting to approach statement (b) by "writing a Borel set E in terms of open/closed sets" and concluding that $f^{-1}(E)$ can also be written in this way. But there is no such structural formula for Borel sets: one can only get the proper subclasses like G_δ, $G_{\delta\sigma}$, $G_{\delta\sigma\delta}$, and so on. The whole story is complicated: see Borel hierarchy on Wikipedia)

$\square$

MAT 701 HW 3.5: LIPSCHITZ TRANSFORMATIONS

`Due Wednesday 09/12/18 by the end of the day`

Problem 1. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function with a continuous derivative. Prove that for every measurable set E, the set $f(E)$ is also measurable.

Hint: although f need not be Lipschitz, its restriction to any bounded interval is.

Proof. For each $j \in \mathbb{N}$, the set $E_j = E \cap [-j, j]$ is measurable, as the intersection of two measurable sets. Since $E = \bigcup_j E_j$, it follows that $f(E) = \bigcup_j f(E_j)$. So it suffices to prove $f(E_j)$ is measurable for every j.

The derivative f', being continuous, is bounded on the interval $[-j, j]$. By the mean value theorem, f is Lipschitz on $[-j, j]$: indeed, $|f(a) - f(b)| \le |a - b| \sup_{[-j,j]} |f'|$. A technical detail arises: we only proved the measurability of images for Lipschitz functions on all of $\mathbb{R}^n$. To get around this, define

$$f_j(x) = \begin{cases} f(x), & x \in [-j, j] \\ f(-j), & x < -j \\ f(j), & x > j \end{cases}$$

Such extended function f_j is Lipschitz continuous on all of $\mathbb{R}$. Indeed, in each of three closed interval $(-\infty, -j]$, $[-j, j]$, $[j, \infty)$ the Lipschitz condition holds by construction. For arbitrary $a < b$, partition the interval $[a, b]$ by the points $\{-j, j\}$ should they lie there, apply the Lipschitz continuity to each interval, and use the triangle inequality.

Conclusion: $f_j(E_j)$, which is the same as $f(E_j)$, is measurable, and the proof is complete. ∎

$[a,b] = [a,-j] \cup [j,b]$

$|f(b) - f(a)| \le |f(b) - f(-j)| + |f(-j) - f(a)| \le M_1 |b - -j| + M_2 |-j - a| \le M|b - a|$

b/c $E_j \subset [-j, j]$

Note: in fact, for every set $E \subset \mathbb{R}^n$, any Lipschitz function $f\colon E \to \mathbb{R}^n$ can be extended to a Lipschitz function $F\colon \mathbb{R}^n \to \mathbb{R}^n$. Therefore, when discussing the measurability of $f(E)$ it suffices to check that f is Lipschitz on the set E.

Sketch of proof. It suffices to extend a real-valued Lipschitz function $f\colon E \to \mathbb{R}$, because the vector-valued case follows by extending each component. Let L be the Lipschitz constant of f, and define, for every $x \in \mathbb{R}^n$,

$$F(x) = \inf_{a\in E}(f(a) + L|x-a|)$$

It is an exercise with the definition of inf to prove that F is Lipschitz with constant L, and that $F(x) = f(x)$ when $x \in A$. ☐

Remark: extending a map $f\colon E \to \mathbb{R}^n$ in the above fashion, one finds the Lipschitz constant of the extension is $\le L\sqrt{n}$ where L is the Lipschitz constant of the original map. There is a deeper extension theorem (due to Kirszbraun) according to which an extension with the same Lipschitz constant L exists.

Problem 2. Given a set $E \subset [0,\infty)$, define a function $f\colon [0,\infty) \to [0,\infty)$ by $f(x) = |E \cap [0,x]|_e$.

(a) Prove that f is Lipschitz continuous.

(b) Prove that for every number b with $0 < b < |E|_e$ there exists a set $F \subset E$ such that $|F|_e = b$.

Proof. (a) I claim that $0 \le f(b) - f(a) \le b - a$ for any $a, bin[0,\infty)$ such that $a < b$; this yields the Lipschitz continuity with constant 1. On one hand, $f(b) \ge f(a)$ by the monotonicity of outer measure: $E \cap [0,a] \subset E \cap [0,b]$. On the other, $E \cap [0,b] \subset (E \cap [0,a]) \cup [a,b]$, which implies

$$f(b) \le |(E \cap [0,a]) \cup [a,b]|_e \le |E \cap [0,a]|_e + |[a,b]| = f(a) + (b-a)$$

by the subadditivity.

(b) By Theorem 3.27 in the textbook, the outer measure is continuous under nested unions even if the sets are not measurable. Since $E = \bigcup_{k \in \mathbf{N}} (E \cap [0,k])$, it follows that

$$|E|_e = \lim_{k \to \infty} |E \cap [0,k]|_e = \lim_{k \to \infty} f(k)$$

Since $b < |E|_e$, by the definition of limit there exists k such that $f(k) > b$. Also, $f(0) = |E \cap \{0\}| = 0$. Applying the intermediate value theorem to f on the interval $[0,k]$ (which is possible since f is continuous by part (a)), we conclude that there exists $x \in (0,k)$ such that $f(x) = b$. Then the set $F = |E \cap [0,x]|$ meets the requirements. $\square$

MAT 701 HW 3.6: NONMEASURABLE SETS

Due Friday 09/14/18 by the end of the day

Problem 1. Show that there exists a nested sequence of sets $E_1 \supset E_2 \supset \cdots$ such that $|E_1|_e < \infty$ and $\bigcap_{k=1}^{\infty} E_k = \emptyset$ but $\lim_{k\to\infty} |E_k|_e > 0$. That is, outer measure is not continuous under nested intersections.

(Hint: use the translates of the Vitali set.)

Proof. Let $V \subset [0,1]$ be the Vitali set described in class: recall that $|V|_e > 0$ and that the sets $V + q$ are disjoint for all $q \in \mathbb{Q}$. Let

$$E_k = \bigcup_{j=k}^{\infty} \left(V + \frac{1}{j} \right)$$

Then $E_1 \subset V + [0,1] \subset [0,2]$, hence $|E_1|_e \le 2 < \infty$.

Suppose $x \in \bigcap_{k=1}^{\infty} E_k$. This means that for each $k \in \mathbb{N}$ there exists $j \ge k$ such that $x \in V + 1/j$. In particular, $x \in V + 1/j$ for infinitely many distinct values of j. But this is impossible as the sets $V + 1/j$ are disjoint. This contradiction proves that $\bigcap_{k=1}^{\infty} E_k$ is empty.

The sets E_k are nested by construction, hence $|E_k|_e$ is a nonincreasing sequence. It is bounded from below by $|V|_e$ because each E_k contains a translated copy of V. Thus, $\lim_{k\to\infty} |E_k|_e \ge |V|_e > 0$. □

Problem 2. Show that for the standard middle-third Cantor set $C \subset [0,1]$, the difference set $C - C$ contains a neighborhood of 0.

(Hint: C is the intersection of nested sets C_n where $C_0 = [0,1]$ and $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$. Find $C_n - C_n$ using induction.)

Remark: this shows that having $|E| > 0$ is not necessary for $E - E$ to contain a neighborhood of 0.

Proof. Recall that $A - B = \{a - b \colon a \in A, b \in B\}$. This definition implies that

$$(A_1 \cup A_2) - (B_1 \cup B_2) = \bigcup_{i,j=1}^{2} (A_i - B_j) \tag{1}$$

Furthermore, for any $t \in \mathbb{R}$ we have $(A + t) - B = (A + B) + t$, $A - (B + t) = (A - B) - t$, and $tA - tB = t(A - B)$; all these follow directly from the definition.

The equality $C_0 - C_0 = [-1, 1]$ holds because on one hand, $|x - y| \le 1$ when $x, y \in [0, 1]$ while on the other, $C_0 - C_0 \supset [0, 1] - \{0, 1\} = [0, 1] \cup [-1, 0] = [-1, 1]$.

Assume $C_n - C_n = [-1, 1]$. Use the relation $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$ and distribute the difference according to (??) and other properties stated at the beginning:

$$\begin{aligned} C_{n+1} - C_{n+1} &= \left(\frac{1}{3}C_n - \frac{1}{3}C_n\right) \cup \left(\frac{1}{3}C_n - \frac{1}{3}C_n + \frac{2}{3}\right) \cup \left(\frac{1}{3}C_n - \frac{1}{3}C_n - \frac{2}{3}\right) \\ &= [-1/3, 1/3] \cup ([-1/3, 1/3] + 2/3) \cup ([-1/3, 1/3] - 2/3) \\ &= [-1/3, 1/3] \cup [1/3, 1] \cup [-1, -1/3] = [-1, 1] \end{aligned}$$

The set $(\frac{1}{3}C_n + \frac{2}{3}) - (\frac{1}{3}C_n + \frac{2}{3})$ is not included above because it is the same as $(\frac{1}{3}C_n - \frac{1}{3}C_n)$.

By induction, $C_n - C_n = [-1, 1]$ for all n.

Since $C \subset C_n$ for every n, it follows that $C - C \subset [-1, 1]$. To prove the reverse inclusion, fix $a \in [-1, 1]$. For each n, there exist $x_n, y_n \in C_n$ such that $x_n - y_n = a$. Since all these numbers are contained in $[0, 1]$, we can pick a convergent subsequence $\{x_{n_k}\}$. So, $x_{n_k} \to x$ and since $x_{n_k} - y_{n_k} = a$, we also have $y_{n_k} \to y$ where y is such that $x - y = a$.

It remains to prove that $x, y \in C$. For each $m \in \mathbb{N}$ we have $x_{n_k}, y_{n_k} \in C_m$ for $k \ge m$ by construction. Since C_m is compact, it follows that $x, y \in C_m$. And since this holds for every $m \in \mathbb{N}$, we have $x, y \in C$. □

MAT 701 HW 4.1A: MEASURABLE FUNCTIONS 1

Due Monday 09/17/18 by the end of the day

Problem 1. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function such that $f(\mathbb{R}^n)$ is countable and $f^{-1}(t)$ is measurable for every $t \in \mathbb{R}$. Prove that f is measurable.

Proof. Let $B = f(\mathbb{R}^n)$, a countable subset of $\mathbb{R}$. For any $a \in \mathbb{R}$ we have

$$\{f > a\} = \bigcup_{b \in B,\ b > a} f^{-1}(b)$$

which is a countable union of measurable sets, hence measurable. The domain of f, which is $\mathbb{R}^n$, is also measurable. Thus f is measurable. □

Problem 2. Prove that without the assumption "$f(\mathbb{R}^n)$ is countable" the statement in Problem 1 would not be true.

Proof. The statement in Problem 1 is made for any n. To disprove it, it suffices to show it fails for some n, for example $n = 1$. Let $V \subset [0, 1]$ be a Vitali set, and define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x + 1, & x \in V; \\ -|x|, & x \notin V. \end{cases}$$

By construction $\{f > 0\} = V$, which is nonmeasurable. Thus f is nonmeasurable. On the other hand, for every $t \in \mathbb{R}$ the set $f^{-1}(t)$ is finite and therefore measurable. Indeed, if t is negative, $f(x) = t$ holds for at most two values of x; and when $t \geq 0$, there is at most one such value. □

Remark: If we wanted to construct such an example on $\mathbb{R}^n$ for every n, one way is to let

$$f(x_1, \ldots, x_n) = \begin{cases} x_1 + 1, & \forall i\ x_i \in V; \\ -|x_1|, & \text{otherwise.} \end{cases}$$

Then $\{f > 0\} = V^n$ which is nonmeasurable, because on one hand, $V^n + \mathbb{Q}^n = \mathbb{R}^n$ forces $|V^n|_e > 0$; on the other, $V^n + (\mathbb{Q} \cap [0,1])^n$ is a bounded set containing infinitely many copies of V^n, which makes it impossible to have $|V^n| > 0$.

For every $t \in \mathbb{R}$, the preimage $f^{-1}(t)$ consists at most two hyperplanes of the form $\{x\} \times \mathbb{R}^{n-1}$. So it is covered by countably many sets of the form $\{x\} \times [-j, j]^{n-1}$, $j \in \mathbb{N}$. Here $|\{x\} \times [-j, j]^{n-1}| = 0$ because this set is contained in a box of dimensions $(\epsilon, 2j, \ldots, 2j)$ whose volume can be arbitrarily small. In conclusion, $|f^{-1}(t)| = 0$ for every t. Thus f is measurable.

MAT 701 HW 4.1B: MEASURABLE FUNCTIONS 2

Due Wednesday 09/19/18 by the end of the day

Problem 1. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is measurable, and $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $g' > 0$ everywhere. Prove that $f \circ g$ is measurable.

Proof. By the Mean Value Theorem, g is strictly increasing, therefore it has an inverse $h = g^{-1}$. By the Inverse Function Theorem, the inverse function h is also continuously differentiable.

Given $a \in \mathbb{R}$, consider the set $A = \{x\colon f(g(x)) > a\}$. It can be written as $\{x\colon g(x) \in B\}$ where $B = \{f > a\}$ is measurable. That is, $A = h(B)$. By #1 in Homework 3.5, the image of a measurable set under a continuously differentiable function is measurable. Thus A is measurable. □

Problem 2. (a) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that f^2 is measurable. Prove that f is measurable.

(b) Prove that the statement in (a) is false if f is not assumed continuous.

Proof. (a) Since f is continuous, it is measurable.

(b) Let $n = 1$, let V be a Vitali set, and define $f(x) = 1$ when $x \in V$ and $f(x) = -1$ when $x \notin V$. Then $f^2 \equiv 1$ is measurable, being continuous. But $\{f > 0\} = V$ is not a measurable set, so f is not measurable. □

MAT 701 HW 4.2: SEMICONTINUOUS FUNCTIONS (+2.1 BOUNDED VARIATION)

Due Friday 09/21/18 by the end of the day

Problem 1. (a) Let $E \subset \mathbb{R}^n$ be a set. Consider a sequence of lsc functions $f_k \colon E \to \overline{\mathbb{R}}$ such that $f_1 \le f_2 \le f_3 \le \dots$. Prove that $\lim_{k\to\infty} f_k$ is also an lsc function. (Note: the limit here is understood in the sense of the extended real line $\overline{\mathbb{R}}$, so it is assured to exist by monotonicity.)

(b) Give an example that shows (a) fails with "lsc" replaced by "usc".

Proof. (a) Recall the limit comparison property: if all terms of a sequence are $\le M$, then its limit (if it exists) is also $\le M$. Apply the contrapositive of this statement to $f(x) = \lim_{k\to\infty} f_k(x)$ and conclude that if $f(x) > M$, then there exists k such that $f_k(x) > M$. Thus, $\{f > M\} \subset \bigcup_{k\in\mathbb{N}}\{f_k > M\}$. The reverse inclusion is true as well, because for each k, $\{f_k > M\} \subset \{f > M\}$ by virtue of $f_k \le f$. In conclusion, $\{f > M\} = \bigcup_{k\in\mathbb{N}}\{f_k > M\}$. Each set on the right is open in E because f_k is lsc; therefore the set on the left is also open. Since M is arbitrary, this shows f is lsc.

(b) Let $f_k = \chi_{[1/k,\infty)}$, the domain being $\mathbb{R}$. This function is usc because any set of the form $\{f_k < a\}$ is either $\mathbb{R}$, $\emptyset$, or $(-\infty, 1/n)$, and all these sets are open in $\mathbb{R}$. Also, $f_k \le f_{k+1}$ because $[1/k,\infty) \subset [1/(k+1),\infty)$. But the limit $f = \chi_{(0,\infty)}$ is not usc, since the set $\{f < 1\} = (-\infty, 0]$ is not open. ☐

Problem 2. Fix $a > 0$ and define $f\colon [0,1] \to \mathbb{R}$ so that $f(1/k) = 1/k^a$ for $k \in \mathbb{N}$, and $f(x) = 0$ for all other x. Prove that f is of bounded variation on $[0,1]$ when $a > 1$, and is not of bounded variation on $[0,1]$ when $0 < a \le 1$.

Proof. Suppose $a > 1$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$. By the triangle inequality,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{n} (|f(x_i)| + |f(x_{i-1})|) \le 2\sum_{i=1}^{n} |f(x_i)|$$

(the second inequality holds because each value $|f(x_i)|$ is repeated at most twice). Ignoring any terms with $f(x_i) = 0$, we get a sum of the form

$$2\sum_{k\in B} \frac{1}{k^a}, \quad \text{where } B = \mathbb{N} \cap \{1/x_i\colon i = 1,\dots,n\}$$

Since $a > 1$, the sum $S = \sum_{k\in\mathbb{N}} 1/k^a$ is finite. From $2\sum_{k\in B} \frac{1}{k^a} \le 2S$ it follows that $V(f;0,1) \le 2S$, hence f is BV.

Now suppose $0 < a \le 1$. For $n \in \mathbb{N}$ consider the partition

$$P_n = \left\{0, \frac{1}{n}, \frac{1}{n-1/2}, \frac{1}{n-1}, \frac{1}{n-3/2}, \dots, \frac{1}{3/2}, 1\right\}$$

which can be described as $P_n = \{0\} \cup \{1/(n-k/2)\colon k = 0,\dots,2n-2\}$. The values of f at the points of P_n are

$$0, \frac{1}{n^a}, 0, \frac{1}{(n-1)^2}, 0, \dots, 0, \frac{1}{1^a}$$

Summing the absolute values of the differences of consecutive terms here, we obtain

$$V(f;0,1) \ge 1 + 2\sum_{k=2}^{n} \frac{1}{k^a}$$

As $n \to \infty$, the right hand side tends to infinity because the series $\sum_{k\in\mathbb{N}} 1/k^a$ diverges. Thus $V(f;0,1) = \infty$. □

MAT 701 HW 4.3: EGOROV AND LUSIN

`Due Monday 09/24/18 by the end of the day`

Problem 1. Suppose that $f\colon E \to \mathbb{R}$ is a measurable function, where $E \subset \mathbb{R}^n$ is measurable.

(a) Prove that there exists a Borel set $H \subset E$ such that the restriction $f_{|H}$ is Borel measurable and $|E \setminus H| = 0$.

(b) If, in addition, E is a Borel set, prove that there exists a Borel measurable function $g\colon E \to \mathbb{R}$ such that $f = g$ a.e.

Hint: for part (a), take a countable union of closed sets obtained from Lusin's theorem.

Proof. (a) By Lusin's theorem, for every $k \in \mathbb{N}$ there exists a closed set $|E_k \subset E$ such that $|E \setminus E_k| < 1/k$ and the restriction of f to E_k is continuous. Let $H = \bigcup_{k \in \mathbb{N}} E_k$. Then H is Borel, being a countable union of closed sets. Also, $|E \setminus H| \le |E \setminus E_k| < 1/k$ for every k, which implies $|E \setminus H| = 0$.

For every $a \in \mathbb{R}$ and every $k \in \mathbb{N}$ the set $A_k = \{x \in E_k : f(x) > a\}$ is open in E_k because $f_{|E_k}$ is continuous. Thus, $A_k = E_k \cap G_k$ for some open set G_k in $\mathbb{R}^n$. Since both E_k and G_k are Borel, it follows that A_k is Borel. Then $\{x \in H : f(x) > a\} = \bigcup_{k \in \mathbb{N}} A_k$ is Borel, which proves that $f_{|H}$ is Borel measurable.

(b) Let $g(x) = f(x)$ for $x \in H$ (with H as above) and $g(x) = 0$ for $x \in E \setminus H$. Then $f = g$ a.e. because $|E \setminus H| = 0$. If $a \ge 0$, then the set $\{x \in E : f(x) > a\}$ is equal to $\{x \in H : f(x) > a\}$ which is Borel by (a). If $a < 0$, then

$$\{x \in E : f(x) > a\} = \{x \in H : f(x) > a\} \cup (E \setminus H)$$

f vs g

which is Borel as the union of two Borel sets. Thus, g is a Borel measurable function on E. □

Problem 2. Suppose $\phi\colon [0,\infty) \to [0,\infty)$ is a function such that $\phi(t) \to 0$ as $t \to \infty$. Consider a sequence of measurable functions $f_k\colon \mathbb{R}^n \to \mathbb{R}$ such that $|f_k(x)| \le \phi(|x|)$ for every k, and $f_k \to f$ a.e. Prove that the conclusion of Egorov's theorem holds in this situation: that is, for every $\epsilon > 0$ there exists a closed set $E(\epsilon) \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$ and $f_k \to f$ uniformly on $E(\epsilon)$.

Hint: Follow the proof of Egorov's theorem

Proof. The proof of Egorov's theorem consists of two parts. Part 1 does not need the assumption $|E| < \infty$ and is included here unchanged, for the sake of completeness.

Part 1: It suffices to find, for each $j \in \mathbb{N}$, a measurable set E_j such that $|E_j^c| < \epsilon/2^j$ and $\sup_{E_j} |f_k - f| \le 1/j$ for all sufficiently large k. Indeed, if we can do this then the set $F = \bigcap E_j$ satisfies $|F^c| < \sum_{j\in\mathbb{N}} \epsilon/2^j = \epsilon$. On this set, f_k converge uniformly to f since for every j, the inequality $|f_k - f| \le 1/j$ holds on F for all sufficiently large k. Since F is measurable, it contains a closed subset F' where $|F \setminus F'|$ can be as small as we wish. So we can choose F' so that $|(F')^c| < \epsilon$ as well.

Part 2: To find E_j as above, fix j and consider the sets $G_m = \{x\colon |f_k(x) - f(x)| < 1/j \ \forall k \ge m\}$. By construction, the set $\bigcap_{m\in\mathbb{N}} G_m^c$ consists of points where $f_k(x) \not\to f(x)$, and thus has measure zero. We would like to conclude that $|G_m^c| < \epsilon/2^j$ for some m, which provides the desired set $E_j = G_m$. In general this does not work because the continuity of measure for nested intersections requires finite $|G_1^c|$. But here we have help from the function ϕ.

Since $\phi(x) \to 0$ as $|x| \to \infty$, there exists R such that $\phi(x) < 1/(2j)$ on the set $A_R = \{x\colon |x| > R\}$. Hence $|f_k| < 1/(2j)$ on A_R for every k, and consequently $|f| \leq 1/(2j)$ a.e. on A_R. It follows that $|f_k - f| < 1/j$ a.e. on A_R, which implies $|G_1^c \cap A_R| = 0$. Hence $G_1^c \leq |\{x\colon |x| \leq R\}| < \infty$. This allows us to apply the continuity of measure for nested intersections, and conclude that $|G_m^c| \to 0$ as $m \to \infty$; in particular there exists m such that $|G_m^c| < \epsilon/2^j$. □

Remark: the function ϕ plays the role of a "dominating function" for this sequence, which can be informally described as a function that mitigates the effects of "escaping to infinity". We will see more of this idea in Chapter 5.

MAT 701 HW 5.1: INTEGRAL OF NONNEGATIVE FUNCTIONS

Due Friday 09/28/18 by the end of the day

Problem 1. Suppose that $f\colon E \to [0,\infty)$ is a measurable function, where $E \subset \mathbb{R}^n$. Prove that $\int_E f$ is finite if and only if the series

$$\sum_{j=-\infty}^{\infty} 2^j |\{x \in E\colon f(x) > 2^j\}|$$

converges.

Note: the convergence of a doubly-infinite series $\sum_{j=-\infty}^{\infty} c_j$ means that both $\sum_{j=0}^{\infty} c_j$ and $\sum_{j=1}^{\infty} c_{-j}$ converge. In case of nonnegative terms the convergence is equivalent to partial sums $\sum_{j=-N}^{N} c_j$ being bounded.

Hint: consider the sets $E_k = \{2^k < f \le 2^{k+1}\}$ and the function $g(x) = \sum 2^k \chi_{E_k}$. Compare $\int_E g$ to the sum of series, and also to $\int_E f$.

Proof. Let E_k and g be as above. By construction, $f = 0$ iff $g = 0$, and $g < f \le 2g$ on each set E_k, which together cover $\{f > 0\}$. Thus, $g \le f \le 2g$. Since g has countable range, its integral is computed (using countable additivity over the domain) as $\int_E g = \sum_{k=-\infty}^{\infty} 2^k |E_k|$. For the same reason, $\int_E 2g = \sum_{k=-\infty}^{\infty} 2^{k+1}|E_k|$. (Note that although it's true that $\int 2g = 2\int g$ for general measurable g, we don't need this fact from 5.2 here.) These facts together with inequalities $g \le f \le 2g$ yield that $\int_E f < \infty$ if and only if $\sum_{k=-\infty}^{\infty} 2^k |E_k| < \infty$.

Let $F_k = \{x \in E\colon f(x) > 2^k\}$. It remains to prove that

$$\sum_{k=-\infty}^{\infty} 2^k |F_k| < \infty \iff \sum_{k=-\infty}^{\infty} 2^k |E_k| < \infty \tag{1}$$

To this end, note that $F_k = \bigcup_{j=k}^{\infty} E_j$, which is a disjoint union. Hence

$$\begin{aligned}\sum_{k=-\infty}^{\infty} 2^k|F_k| &= \sum_{k=-\infty}^{\infty}\sum_{j=k}^{\infty} 2^k|E_j| \\ &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{j} 2^k|E_j| \\ &= \sum_{j=-\infty}^{\infty} 2^{j+1}|E_j|\end{aligned}$$

which proves (1). □

Problem 2. Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$.

(a) Prove that $\int_B |x|^{-p}\,dx$ is finite when $0 < p < n$ and infinite when $p \geq n$.

(b) Prove that $\int_{B^c} |x|^{-p}\,dx$ is finite when $p > n$ and infinite when $0 < p \leq n$.

Note: these integrals are Lebesgue integrals, and we don't yet have anything like the Fundamental Theorem of Calculus for such integrals. Use #1. You can also use the fact that the measure of a ball of radius R is C_nR^n for some constant C_n that depends on n.

Proof. Let $f = |x|^{-p}$ and $F_k = \{x \in \mathbb{R}^n : f(x) > 2^k\}$. Note that $F_k = \{x : |x| < 2^{-k/p}\}$, so $|F_k| = C_n 2^{-kn/p}$.

(a) By #1, $\int_B f < \infty \iff \sum_{k=-\infty}^{\infty} 2^k|B \cap F_k| < \infty$. When $k \leq 0$, we have $B \subset F_k$, hence $|B \cap F_k| = |B|$. The geometric series $\sum_{k\leq 0} 2^k|B| = 2|B|$ converges regardless of p. For $k \geq 1$, $F_k \subset B$, so $|B \cap F_k| = |F_k|$, hence

$$\sum_{k=1}^{\infty} 2^k|B \cap F_k| = C_n \sum_{k=1}^{\infty} 2^{k(1-n/p)}$$

which is a geometric series that converges if and only if $1 - n/p < 0$. This proves (a).

(b) By #1, $\int_{B^c} f < \infty \iff \sum_{k=-\infty}^{\infty} 2^k |B^c \cap F_k| < \infty$. As noted above, $F_k \subset B$ when $k \geq 1$, which implies $B^c \cap F_k = \emptyset$. When $k \leq 0$, we have $B \subset F_k$, hence $|B^c \cap F_k| = |F_k| - |B|$. Since the sum $\sum_{k\leq 0} 2^k |B| = 2|B|$ converges regardless of p, it remains to consider the convergence of $\sum_{k\leq 0} 2^k |F_k|$. Writing $j = -k$, we arrive at

$$\sum_{k\leq 0} 2^k |F_k| = \sum_{j=0}^{\infty} 2^{-j} C_n 2^{jn/p} = C_n \sum_{j=0}^{\infty} 2^{j(n/p-1)}$$

which is a geometric series that converges if and only if $n/p - 1 < 0$. This proves (b). $\square$

MAT 701 HW 5.2: PROPERTIES OF THE INTEGRAL OF NONNEGATIVE FUNCTIONS 1

Due Monday 10/01/18 by the end of the day

Problem 1. (a) Suppose that $f_k\colon E \to [0,\infty]$ (where $E \subset \mathbb{R}^n$) are measurable functions such that $\int_E f_k \to 0$ as $k \to \infty$. Prove that $f_k \xrightarrow{m} 0$.

(b) Give an example where $f_k \xrightarrow{m} 0$ but $\int_E f_k \not\to 0$.

→ (5.12)

Proof. (a) For any $\epsilon > 0$, Chebyshev's inequality yields

$$|\{f_k > \epsilon\}| \le \frac{1}{\epsilon}\int_E f_k \to 0$$

which means $f_k \xrightarrow{m} 0$.

(b) Either of $f_k = k\chi_{(0,1/k)}$ or $g_k = k^{-1}\chi_{(0,k)}$ works. (Or even $h_k \equiv 1/k$). Indeed, $\{f_k \ne 0\} \to 0$, and g_k, h_k converge to zero uniformly (which is stronger than convergence in measure). Yet $\int f_k = 1$, $\int g_k = 1$, and $\int h_k = \infty$. □

Problem 2. For $k \in \mathbb{N}$ define $f_k\colon [0,1] \to [0,\infty]$ by

$$f_k(x) = \sum_{j=1}^{k} \chi_{I(j,k)}, \quad \text{where } I(j,k) = \left[\frac{j}{k} - \frac{1}{k^3}, \frac{j}{k} + \frac{1}{k^3}\right]$$

Let $f = \sum_{k=1}^{\infty} f_k$. Prove that $\int_{[0,1]} f < \infty$.

Proof. Note that $\int_E \chi_F = |E \cap F|$ by the formula for the integral of a simple function. Applying this (and the additivity of the integral) to f_k yields

$$\int_{[0,1]} f_k = \sum_{j=1}^{k} |I(j,k) \cap [0,1]| \le \sum_{j=1}^{k} |I(j,k)| = \sum_{j=1}^{k} \frac{2}{k^3} = \frac{2}{k^2}$$

By the countable additivity over nonnegative functions (Theorem 5.16),

$$\int_{[0,1]} f = \sum_{k=1}^{\infty} \int_{[0,1]} f_k \leq \sum_{k=1}^{\infty} \frac{2}{k^2} < \infty$$

(One can also say that partial sums converge to f in an increasing way, but this argument was already made in the proof of Theorem 5.16). □

MAT 701 HW 5.2B: PROPERTIES OF THE INTEGRAL OF NONNEGATIVE FUNCTIONS 2

`Due Wednesday 10/03/18 by the end of the day`

Problem 1. Suppose that $f\colon \mathbb{R}^n \to [0,\infty)$ is a measurable function such that $\int_{\mathbb{R}^n} f < \infty$. Also suppose $\{E_k\}$ is a sequence of measurable sets $E_k \subset \mathbb{R}^n$. Let $A = \limsup_{k\to\infty} E_k$ and $B = \liminf_{k\to\infty} E_k$. Prove that

$$\int_A f \geq \limsup_{k\to\infty} \int_{E_k} f$$

and

$$\int_B f \leq \liminf_{k\to\infty} \int_{E_k} f$$

Hint: $\int_E f = \int_{\mathbb{R}^n} \chi_E f$.

Proof. The second inequality follows from Fatou's lemma, using the fact (discussed in class) that $\chi_B = \liminf_{k\to\infty} \chi_{E_k}$:

$$\int_B f = \int_{\mathbb{R}^n} \chi_B f = \int_{\mathbb{R}^n} \liminf_{k\to\infty} \chi_{E_k} f$$
$$\leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} \chi_{E_k} f = \liminf_{k\to\infty} \int_{E_k} f$$

To prove the inequality for $\int_A f$, note that $f - \chi_{E_k} f \geq 0$, and apply Fatou's lemma to this sequence:

$$\int_{\mathbb{R}^n} \liminf_{k\to\infty} (f - \chi_{E_k} f) \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} (f - \chi_{E_k} f)$$

Expand both sides, recalling that $\liminf(-a_k) = -\limsup a_k$, and using the assumption that $\int_{\mathbb{R}^n} f$ is finite.

$$\int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n} \limsup_{k\to\infty} \chi_{E_k} f \leq \int_{\mathbb{R}^n} f - \limsup_{k\to\infty} \int_{\mathbb{R}^n} \chi_{E_k} f$$

Canceling $\int_{\mathbb{R}^n} f$, we get

$$\int_{\mathbb{R}^n} \limsup_{k\to\infty} \chi_{E_k} f \geq \limsup_{k\to\infty} \int_{\mathbb{R}^n} \chi_{E_k} f$$

which is precisely $\int_A f \geq \limsup_{k\to\infty} \int_{E_k} f$. □

Problem 2. Suppose that $f\colon \mathbb{R}^n \to [0,\infty)$ is a measurable function such that $\int_{\mathbb{R}^n} f < \infty$. Prove that $\int_{\mathbb{R}^n} e^{-k|x|} f(x) \to 0$ as $k \to \infty$.

Proof. For all $x \neq 0$ we have $e^{-k|x|} f(x) \to 0$ as $k \to \infty$; thus, the functions converge a.e. to 0. Also, f is a dominating function here, since its integral is finite and $e^{-k|x|} f(x) \leq f(x)$ for all x and all k. By the Dominated Convergence Theorem,

$$\int_{\mathbb{R}^n} e^{-k|x|} f(x) \to \int_{\mathbb{R}^n} 0 = 0$$ □

Dominated Convergence

MAT 701 HW 5.3A: INTEGRAL OF MEASURABLE FUNCTIONS 1

`Due Friday 10/05/18 by the end of the day`

Problem 1. Prove that under the assumptions of the Lebesgue Dominated Convergence Theorem we have $\int_E |f_k - f| \to 0$ as $k \to \infty$.

Proof. By assumption, there is an integrable dominating function for $\{f_k\}$, call it φ. By passing to the limit, $|f| \le \varphi$ (a.e.), which implies $|f_k - f| \le |f_k| + |f| \le 2\varphi$ a.e. Note that $\int_E 2\varphi = 2\int_E \varphi < \infty$. Since $f_k \to f$ a.e., it follows that $|f_k - f| \to 0$ a.e. By the DCT,

$$\int_E |f_k - f| \to \int_E 0 = 0 \qquad \square$$

Problem 2. Let $f \in L^1(E)$, where $E \subset \mathbb{R}^n$ is a measurable set. Prove that

$$\lim_{k\to\infty} k \int_E \sin\left(\frac{f}{k}\right) = \int_E f$$

Proof. By the Mean Value Theorem, $\sin t = \sin t - \sin 0 = (\cos\xi)t$ for some ξ between 0 and t. This implies two things:

(a) $|\sin t| \le |t|$ because $|\cos\xi| \le 1$;

(b) as $t \to 0$, we have $(\sin t)/t \to 1$ because $\cos\xi \to 1$.

Applying (a) to $\sin\left(\frac{f}{k}\right)$, we find that

$$k\left|\sin\left(\frac{f}{k}\right)\right| \le k\frac{|f|}{k} = |f|$$

which means $|f|$ is a dominating function for the sequence $f_k = k\sin(f/k)$.

Applying (b), we find that

$$f_k = \frac{\sin(f/k)}{f/k} f \to f, \quad k \to \infty$$

By the DCT, $\int_E f_k \to \int_E f$. $\square$

MAT 701 HW 5.3B: INTEGRAL OF MEASURABLE FUNCTIONS 2

`Due Monday 10/08/18 by the end of the day`

Problem 1. Let $f: E \to \mathbb{R}$ be a measurable function. Suppose that $|E| < \infty$ and there exists a number $p > 1$ such that

$$\limsup_{\alpha \to \infty} \alpha^p |\{x \in E : |f(x)| > \alpha\}| < \infty$$

Prove that $f \in L^1(E)$.

Proof. Let $M = \limsup_{\alpha \to \infty} \alpha^p |\{x \in E : |f(x)| > \alpha\}|$. By definition, this means

$$M = \lim_{\beta \to \infty} \sup_{\alpha \geq \beta} \alpha^p |\{x \in E : |f(x)| > \alpha\}|$$

Thus, there exists β such that $\sup_{\alpha \geq \beta} \alpha^p |\{x \in E : |f(x)| > \alpha\}| \leq M+1$. Choose an integer m such that $2^m \geq \beta$. Then for $j \geq m$ we have

$$2^{jp}|\{x \in E : |f(x)| > 2^j\}| \leq M + 1$$

hence

$$\sum_{j=m}^{\infty} 2^j \{x \in E : |f(x)| > 2^j\}| \leq \sum_{j=m}^{\infty} 2^j \frac{M+1}{2^{jp}} = (M+1) \sum_{j=m}^{\infty} 2^{(1-p)j}$$

which converges because $2^{1-p} < 1$.

Also,

$$\sum_{j=-\infty}^{m-1} 2^j \{x \in E : |f(x)| > 2^j\}| \leq \sum_{j=-\infty}^{m-1} 2^j |E| = 2^m |E| < \infty$$

by summing a geometric series. Thus $\sum_{j=-\infty}^{\infty} 2^j \{x \in E : |f(x)| > 2^j\}$ which by Homework 5.1 #1 implies $|f| \in L^1(E)$, hence $f \in L^1(E)$. ☐

Hint: use an exercise from Homework 5.1.

Problem 2. Give an example of a sequence of integrable functions $f_k\colon [0,1] \to \mathbb{R}$ such that $f_k \to f$ a.e., $\lim_{k\to\infty} \int_{[0,1]} f_k$ exists and is finite, but f is not integrable on $[0,1]$.

Hint: approximate $1/x$ *by functions with integral* 0.

Proof. We know that $\int_{[0,1]} \frac{1}{x} = \infty$ from Homework 5.1 #2. Let $C_k = \int_{(1/k,1]} \frac{1}{x}$ which is finite because the function is bounded by k on this finite interval. Define

$$f_k = -kC_k\chi_{[0,1/k)} + \frac{1}{x}\chi_{(1/k,1]}$$

Then f_k is integrable (sum of two integrable functions) and $\int_{[0,1]} f_k = -kC_k|[0,1/k)| + C_k = 0$. On the other hand, for every $x > 0$ we have $f_k(x) = 1/x$ for all k such that $k > 1/x$; thus, $f_k \to 1/x$ a.e. □

(→ look @ my example.

MAT 701 HW 5.4-5: LEBESGUE, RIEMANN, RIEMANN-STIELTJES

`Due Monday 10/15/18 by the end of the day`

Problem 1. Determine the Lebesgue-Stieltjes (That was a typo; I meant Riemann-Stieltjes) integral $\int \alpha \, d(-\omega_f(\alpha))$ corresponding to $\int_E f$ where $E = (0,3)$ and $f(x) = x + \lfloor x \rfloor$. You do not need to evaluate the integral. Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Proof. Note that $f(x) = x$ on $(0,1)$, $f(x) = x+1$ on $[1,2)$ and $f(x) = x+2$ on $[2,3)$. Hence, for any $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) > \alpha\}$ is equal to

$$\{x \in (0,1) : x > \alpha\} \cup \{x \in [1,2) : x+1 > \alpha\} \cup \{x \in [2,3) : x+2 > \alpha\}$$

The set $(a,b) \cap (c,\infty)$ can be expressed as $\max(a,c) < x < b$, so its measure is $(b - \max(a,c))^+$. This makes it possible to write $\omega_f(\alpha)$ as

$$\omega_f(\alpha) = (1 - \max(\alpha,0))^+ + (2 - \max(\alpha,1))^+ + (3 - \max(\alpha,2))^+$$

Since the range of f is $(0,5)$, the desired Riemann-Stieltjes integral is $\int_0^5 \alpha \, d(-\omega_f(\alpha))$ with ω_f given by the above formula. □

Note: one can rewrite ω_f in other ways, for example

$$\omega_f(\alpha) = \begin{cases} 3-\alpha, & 0 \le \alpha \le 1 \\ 2, & 1 \le \alpha \le 2 \\ 4-\alpha, & 2 \le \alpha \le 3 \\ 1, & 3 \le \alpha \le 4 \\ 5-\alpha, & 4 \le \alpha \le 5 \end{cases}$$

Problem 2. Suppose $E \subset [0,1]$. Prove that χ_E is Riemann integrable on $[0,1]$ if and only if $|\partial E| = 0$.

Proof. More generally, I claim that for any set $E \subset X$ in a metric space (X, d) the boundary ∂E coincides with the set of discontinuities of the characteristic function χ_E. Indeed, for $a \in X$ to be a point of continuity for χ_E we must have, for every $\epsilon > 0$, some $\delta > 0$ such that $d(x,a) < \delta \implies |\chi_E(x) - \chi_E(a)| < \epsilon$. By using this with $\epsilon = 1$ and recalling that χ_E takes only the values $0, 1$, we conclude that a is a point of continuity for χ_E if and only if χ_E is constant in some neighborhood of a. The latter means exactly one of two things: $\chi_E = 0$ in a neighborhood of a (so, a is an interior point of E^c), or $\chi_E = 1$ in a neighborhood of a (so, a is an interior point of E). It remains to recall that ∂E is the set of all points that are neither interior for E nor for E^c.

Applying the above with $X = \mathbb{R}$, we conclude that the set of discontinuities of χ_E on $\mathbb{R}$ is ∂E. When χ_E is restricted to $[0,1]$, the discontinuities at 0 and 1 may disappear (e.g., the restriction on $\chi_{[0,1/2]}$ to $[0,1]$ is continuous at 0), but the two-point set has measure zero anyway. In conclusion, $|\partial E| = 0$ if and only if the restriction of χ_E to $[0,1]$ is continuous a.e.. Since χ_E is bounded, the latter property is equivalent to Riemann integrability by Theorem 5.54.

□

MAT 701 HW 6.1-2: FUBINI AND TONELLI

`Due Friday 10/19/18 by the end of the day`

Problem 1. Prove that for any $a > 0$ the function $f(x, y) = e^{-xy} \sin x$ is in $L^1(E)$ where $E = \{(x, y) \in \mathbb{R}^2, \ x > 0, \ y > a\}$.

A remark on the relation of Riemann and Lebesgue integrals. We proved in 5.5 that if a Riemann integral $\int_a^b h(x)\,dx$ exists (with a, b finite), then it is equal to the Lebesgue integral. This can be extended to improper Riemann integrals in two ways.

First, if $h \geq 0$ and $\int_a^b h(x)\,dx$ exists as an improper Riemann integral, then it's still equal to the Lebesgue integral, by the MCT (replace h with $\min(h, k)\chi_{[-k,k]}$ and let $k \to \infty$.)

Second, if $h \in L^1((a, b))$ and the improper Riemann integral $\int_a^b h(x)\,dx$ exists, then the two integrals are equal. Indeed, $\int_c^d h(x)\,dx$ is equal to the Lebesgue integral for any $a < c < d < b$, by the above result from 5.5. As $c \to a$ or $d \to b$, we can pass to the limit in the Lebesgue integral by the DCT ($|h|$ is dominating), and in the Riemann integral, by the definition of an improper Riemann integral.

Proof. The function f is continuous and therefore measurable on E. Since $|\sin x| \leq x$ for $x \geq 0$, we have $|f| \leq g$ where $g(x, y) = xe^{-xy}$ is also continuous, hence measurable. It suffices to prove $g \in L^1(E)$, which can be done using Tonelli's theorem:

$$\int_E g = \int_0^\infty \left(\int_a^\infty xe^{-xy}\,dy \right) dx = \int_0^\infty e^{-ax}\,dx = \frac{1}{a} < \infty \qquad \square$$

Problem 2. Apply Fubini's theorem to the function f in #1 to prove that

$$\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \tan^{-1}(1/a)$$

Hint: integrate f in two different ways. You don't have to do the antiderivative $\int e^{-xy}\sin x\,dx$ by hand; just look it up.

Food for thought (not a part of the homework): how to let $a \to 0$?

Proof. By #1, Fubini's theorem applies to f. On one hand,

$$\int_E f = \int_0^\infty \left(\int_a^\infty e^{-xy}\sin x\,dy\right)dx = \int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$$

On the other,

$$\begin{aligned}\int_E f &= \int_a^\infty \left(\int_0^\infty e^{-xy}\sin x\,dx\right)dy\\ &= \int_a^\infty \left(\int_0^\infty e^{-xy}\sin x\,dx\right)dy\\ &= \int_a^\infty \left(-e^{-xy}\frac{y\sin x+\cos x}{y^2+1}\Big|_{x=0}^{x=\infty}\right)dy\\ &= \int_a^\infty \frac{1}{y^2+1}\,dy = \frac{\pi}{2} - \tan^{-1} a = \tan^{-1}(1/a)\end{aligned}$$

using the facts that $e^{-xy} \to 0$ as $x \to \infty$ (with $y > 0$), and that $\tan^{-1} x \to \pi/2$ as $x \to \infty$. □

MAT 701 HW 6.1: FUBINI'S THEOREM

Due Wednesday 10/17/18 by the end of the day

Problem 1. (a) Suppose $E \subset \mathbb{R}^2$ is a Borel set. For $x \in \mathbb{R}$, let $E_x = \{y \in \mathbb{R} \colon (x, y) \in E\}$. Prove that E_x is a Borel set in $\mathbb{R}$.

Hint: for a fixed x, prove that $\{A \subset \mathbb{R}^2 \colon A_x$ is Borel in $\mathbb{R}\}$ is a σ-algebra that contains all open subsets of $\mathbb{R}^2$.

(b) Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a Borel measurable function. Prove that for every $x \in \mathbb{R}$, the function $g(y) = f(x, y)$ is Borel measurable on $\mathbb{R}$.

(Note: in contrast with Fubini's theorem, this is no "a.e." here.)

Proof. (a) Let $M = \{A \subset \mathbb{R}^2 \colon A_x$ is Borel in $\mathbb{R}\}$. Note that $\emptyset, \mathbb{R}^2 \in M$, since their slices are $\emptyset$ and $\mathbb{R}$, respectively. For an arbitrary $A \in M$ we have $(A^c)_x = (A_x)^c$, and since the complement of a Borel set is Borel, $A^c \in M$. Also, for any countable family $A_k \in M$, $(\bigcup_k A_k)_x = \bigcup_k (A_k)_x$ is Borel, which means $A \in M$. Thus, M is a σ-algebra.

For any open set $A \subset \mathbb{R}$ the intersection of A with any set B is open as a subset of B (MAT 601; one can also see this as the definition of subspace topology in MAT 661). Therefore, A_x is open in $\mathbb{R}$ for every open set $A \subset \mathbb{R}^2$. This implies that M contains all open sets; and being a σ-algebra, it contains all Borel sets. In other words, E_x is Borel in $\mathbb{R}$ whenever E is Borel in $\mathbb{R}^2$.

??

(b) For any $a \in \mathbb{R}$ the set $\{y \in \mathbb{R} \colon g(y) > a\}$ is the x-slice of the set $\{(u, v) \in \mathbb{R}^2 \colon f(u, v) > a\}$. The latter set is Borel, hence the former is also Borel by part (a). This shows g is Borel measurable. □

A shorter proof of both (a) and (b) is to observe that, for a fixed x, the function $h(y) = (x, y)$ is a continuous map from $\mathbb{R}$ into $\mathbb{R}^2$, and therefore is Borel measurable (in the sense that the preimage of any Borel set is Borel). In class we proved that the composition of Borel measurable functions is Borel measurable. Therefore, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable, then the composition $f \circ h$ is Borel measurable; this composition is exactly g. This proves (b). Part (a) follows by applying (b) to $f = \chi_E$ and noting that $g = \chi_{E_x}$.

Problem 2. Suppose $f\colon [0,1] \to \mathbb{R}$ is a measurable function such that the function $g(x,y) = f(x) - f(y)$ is in $L^1([0,1]^2)$. Prove that $f \in L^1([0,1])$.

Proof. By Fubini's theorem, for almost every $x \in [0,1]$ the slice-function $y \mapsto g(x,y)$ is integrable. Fix such an x. Then $f(x)$ is a finite constant, hence integrable on $[0,1]$ as well (with respect to y). By linearity, $f(y) = f(x) - g(x,y)$ is integrable on $[0,1]$. $\square$

MAT 701 HW 6.3A: APPLICATIONS OF FUBINI AND TONELLI 1

Due Monday 10/22/18 by the end of the day

Problem 1. Suppose $f \in L^1([0,1])$. Let $g(x) = \int_{[x,1]} \frac{f(t)}{t}\,dt$ for $x \in (0,1]$. Prove that $g \in L^1((0,1])$ and $\int_{(0,1]} g = \int_{[0,1]} f$.

Proof. Let $h(x,t) = f(t)/t$ if $0 \le x \le t \le 1$ and $h(x,t) = 0$ otherwise. This is a measurable function on $[0,1] \times [0,1]$, because:

- The function $(x,t) \mapsto f(t)$ is measurable, as discussed in class: level sets are products of the level sets of f with $[0,1]$;
- $1/t$ is continuous a.e., hence measurable.
- The characteristic function of the closed set $\{(x,t)\colon 0 \le x \le t \le 1\}$ is measurable.

Thus, Tonelli's theorem applies to $|h|$. It yields

$$\int_{[0,1]\times[0,1]} |h| = \int_{[0,1]} \int_{[0,t]} \frac{|f(t)|}{t}\,dx\,dt = \int_{[0,1]} |f(t)|\,dt < \infty \tag{1}$$

since $f \in L^1([0,1])$. Thus $h \in L^1([0,1]\times[0,1])$, which means Fubini's theorem applies to h. Similar to (??), we get

$$\int_{[0,1]\times[0,1]} h = \int_{[0,1]} \int_{[0,t]} \frac{f(t)}{t}\,dx\,dt = \int_{[0,1]} f(t)\,dt < \infty \tag{2}$$

but the same integral is also equal to

$$\int_{[0,1]\times[0,1]} h = \int_{[0,1]} \int_{[x,1]} \frac{f(t)}{t}\,dt\,dx = \int_{[0,1]} g(x)\,dx \tag{3}$$

From (??) and (??) the result follows. □

Problem 2. Prove that convolution is associative: that is, for $f, g, h \in L^1(\mathbb{R}^n)$ we have $(f * g) * h = f * (g * h)$.

Note: we don't yet have the full change of variables formula, but we do have $\int_{\mathbb{R}^n} f(x-y)\,dx = \int_{\mathbb{R}^n} f(x)\,dx$ *as a consequence of the invariance of measure under translation.*

Proof. Since $f, g, h \in L^1$, the convolutions are in L^1 as well. Using the commutativity of convolution, $(f * g) * h = (g * f) * h$ which can be written as (with all integrals over $\mathbb{R}^n$)

$$\int (g * f)(x-t)h(t)\,dt = \int \left(\int f(s)g(x-t-s)\,ds \right) h(t)\,dt \tag{4}$$

The convolution $f * (g * h)$ can be written as $(g * h) * f$, which is

$$\int (g * h)(x-t)f(t)\,dt = \int \left(\int g(x-t-s)h(s)\,ds \right) f(t)\,dt \tag{5}$$

In (??), relabel t as s and s as t — this is not using any theorem, just changing the labels. Then the desired equality of (??) and (??) becomes

(6)
$$\int \left(\int f(s)g(x-t-s)h(t)\,ds \right) dt \stackrel{?}{=} \int \left(\int f(s)g(x-t-s)h(t)\,dt \right) ds$$

The measurability of each of the functions $(s,t) \mapsto f(s)$, $(s,t) \mapsto g(x-t-s)$, $(s,t) \mapsto h(t)$, follows as in the proof of the commutativity of convolution (the composition of a measurable function with a linear transformation is measurable).

Since the convolution $(|f| * |g|) * |h|$ is in L^1, it is finite a.e. Let x be such that $(|f| * |g|) * |h|$ is finite at x. This means that

$$\int \left(\int |f(s)||g(x-t-s)||h(t)|\,ds \right) dt < \infty$$

By Tonelli's theorem, the function $(s,t) \mapsto f(s)g(x-t-s)h(t)$ is in $L^1(\mathbb{R}^{2n})$. Hence, Fubini's theorem can be applied to the integrals in (??), meaning they are equal. Thus, $(f * g) * h = f * (g * h)$ **a.e.** in $\mathbb{R}^n$. □

Note: It is not clear to me whether $(f * g) * h = f * (g * h)$ holds in the stricter sense of both convolutions having the same domain and being identically equal on that domain. This is true when $f, g, h \geq 0$, since then we can apply Tonelli's theorem directly to (??),

MAT 701 HW 6.3B: APPLICATIONS OF FUBINI AND TONELLI 2

`Due Wednesday 10/24/18 by the end of the day`

Problem 1. Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Let $Z = \{x\colon g(x) = 0\}$ and suppose that $\mathbb{R} \setminus Z$ is bounded. Let $f(x) = 1/x^2$.

Prove that the convolution $f * g$ is finite a.e. on Z.

Proof. Since f is continuous, Z is closed. Since $\mathbb{R} \setminus Z$ is bounded, there exists $b > 0$ such that $\mathbb{R} \setminus Z \subset [-b, b]$. Let $B = (-b-1, b+1)$, $K = Z \cap B$, and $\delta(y) = \operatorname{dist}(y, K)$. The Marcinkiewitz integral

$$M_1(x) = \int_B \frac{\delta(y)}{(x-y)^2}\, dy$$

is finite a.e. on K. Since $g = 0$ on K, the Lipschitz property implies $|g(y)| \le L\delta(y)$. Also, since $g = 0$ on B^c, we have

$$\int_{\mathbb{R}} \frac{|g(y)|}{(x-y)^2}\, dy = \int_B \frac{|g(y)|}{(x-y)^2}\, dy \le LM_1(x) < \infty$$

for a.e. $x \in K$. By the integral triangle inequality, $f * g$ is finite a.e. on K.

It remains to consider $x \in Z \setminus B$. We have $|x - y| \ge 1$ for every $y \in \mathbb{R} \setminus Z$ since $y \in [-b, b]$ and $x \notin (-b-1, b+1)$. Thus,

$$\int_{\mathbb{R}} \frac{|g(y)|}{(x-y)^2}\, dy \le \int_{[-b,b]} |g(y)|\, dy < \infty$$

is finite, as an integral of a continuous function over a bounded set. □

Problem 2. Let $C \subset [0, 1]$ be the standard "middle third" Cantor set. Let $\delta(x) = \operatorname{dist}(x, C)$. For which positive numbers p is the function δ^{-p} in $L^1([0, 1])$?

Note: although Tonelli could be applied here, it's easier to use the countable additivity of integral over the set of integration.

Proof. The set $[0,1]\setminus C$ is the union of intervals $I_{k,j}$ where for each $k \in \mathbb{N}$ we have 2^{k-1} intervals of length $1/3^k$. Let's say $I_{k,j} = (a,b)$; then for $x \in (a,b)$ we have $\delta(x) = \min(x-a, b-x)$. By symmetry and substitution,

$$\int_a^b \delta^{-p} = 2\int_0^{(b-a)/2} t^{-p}\,dt$$

This immediately rules out $p \geq 1$, when the above integral diverges. For $0 < p < 1$ it evaluates to

$$\frac{2}{1-p}\left(\frac{b-a}{2}\right)^{1-p} = \frac{2^p}{1-p}|I_{k,j}|^{1-p}$$

Sum this over k, j, recalling that $|I_{k,j}| = 1/3^k$ and that there are 2^{k-1} such intervals:

$$\sum_{k,j}\int_{I_{k,j}} \delta^{-p} = \frac{2^p}{1-p}\sum_{k\in\mathbb{N}} \frac{2^{k-1}}{3^{(1-p)k}}$$

This is a geometric series with ratio $2/3^{1-p}$, so it converges if and only if $2 < 3^{1-p}$, which is equivalent to $p < 1 - \log 2/\log 3$. □

MAT 701 HW 8.1: L^p CLASSES

`Due Friday 10/26/18 by the end of the day`

Problem 1. Prove that for any $q \in (0, \infty]$, there exist:

a) A function $f\colon [2,\infty) \to \mathbb{R}$ such that $f \in L^p([2,\infty)) \iff p > q$;

b) A function $f\colon [2,\infty) \to \mathbb{R}$ such that $f \in L^p([2,\infty)) \iff p \geq q$.

Hint: use a suitable power of x, with a logarithmic factor if necessary. Recall that for nonnegative functions, improper Riemann integral agrees with the Lebesgue integral (Theorem 5.53). → fns in L^p for no values

Proof. (a) If $q = \infty$, we need f such that $f \in L^p([2,\infty)) \iff p > \infty$.

? Since $p > \infty$ is false for all p, this means $f \notin L^p([2,\infty))$ for all p. This is achieved by choosing $f(x) = x$, since this function tends to infinity as $x \to \infty$, and so do all of its positive powers.

If $0 < q < \infty$, let $f(x) = x^{-1/q}$. Then

$$\int_2^\infty |f|^p = \int_2^\infty x^{-p/q}$$

which converges iff $p/q > 1$; that is, iff $p > q$. Here and below, we use the fact (Theorem 5.53) that the convergence of an improper Riemann integral of a nonnegative function implies its Lebesgue integrability.

(b) If $q = \infty$, let $f(x) \equiv 1$. Then $f \in L^\infty$ but $f^p \equiv 1$ is never integrable, so $f \in L^p$ when $p < \infty$.

If $0 < q < \infty$, let $f(x) = (x \log^2 x)^{-1/q}$. When $p > q$,

$$\int_2^\infty |f|^p = \int_2^\infty x^{-p/q} \log^{-2p/q} x \leq \log^{-2p/q} 2 \int_2^\infty x^{-p/q} < \infty$$

When $p = q$, the antiderivative of $x^{-1} \log^{-2} x$ is $C - 1/\log x$ (check by differentiation). Since the antiderivative has a finite limit as $x \to \infty$, the integral converges.

When $p < q$, we have

$$\int_2^M |f|^p = \int_2^M x^{-p/q} \log^{-2p/q} x \geq \log^{-2p/q} M \int_2^M x^{-p/q}$$
$$= \log^{-2p/q} M \frac{M^{1-p/q} - 2^{1-p/q}}{1 - p/q}$$

L'Hospital's rule implies $\lim_{x\to\infty} \dfrac{x^\varepsilon}{\log x} = \infty$ for every $\varepsilon > 0$. Therefore, $(\log M)^{-2p/q} M^{1-p/q} \to \infty$ as $M \to \infty$, and the integral diverges. ☐

One can avoid logarithms in all these examples by using piecewise constant functions such as

$$f(x) = \sum_{j\in\mathbb{N}} j^{-2/q} 2^{-j/q} \chi_{[2^j, 2^{j+1})}$$

Indeed, $\int_2^\infty |f|^p$ is

$$f(x) = \sum_{j\in\mathbb{N}} j^{-2p/q} 2^{-jp/q} 2^j = \sum_{j\in\mathbb{N}} j^{-2p/q} 2^{(1-p/q)j}$$

which quickly shows convergence for $p \geq q$ and divergence for $p \leq q$. When $p \neq q$, the ratio test yields this result; when $p = q$, the sum is $\sum j^{-2} < \infty$.

A similar example works for #2, using intervals $(2^{-j-1}, 2^{-j}]$ instead.

Problem 2. Prove that for any $q \in (0, \infty]$, there exist:

a) A function $f\colon (0,1) \to \mathbb{R}$ such that $f \in L^p((0,1)) \iff p < q$;

b) A function $f\colon (0,1) \to \mathbb{R}$ such that $f \in L^p((0,1)) \iff p \leq q$.

Using this and #1, show that for any interval $J \subset (0, \infty]$ there exists a function f on some set $E \subset \mathbb{R}$ such that $f \in L^p(E) \iff p \in J$.

Proof. (a) If $q = \infty$, let $f(x) = \log x$. Then $f \notin L^\infty((0,1))$ but for every $p \in (0,\infty)$ we have $\lim_{x\to 0} (\log x)/x^{p/2} = 0$ by L'Hospital, hence $(\log x)/x^{p/2}$ is bounded on $(0,1)$. This means $|\log x|^p \leq C/x^{1/2}$ for some constant C, and since $\int_0^1 C/x^{1/2} < \infty$, we have $\log x \in L^p((0,1))$.

If $0 < q < \infty$, let $f(x) = x^{-1/q}$. Then

$$\int_0^1 |f|^p = \int_0^1 x^{-p/q}$$

which converges iff $p/q < 1$; that is, iff $p < q$.

(b) If $q = \infty$, let $f(x) = 1$, which is in L^p for all $p \in (0, \infty]$.

If $0 < q < \infty$, let $f(x) = (x \log^2 x)^{-1/q} \chi_{(0,1/2)}$ where the cut-off function $\chi_{(0,1/2)}$ is needed to avoid a problem with $\log x \to 0$ as $x \to 1$. When $p < q$,

$$\int_0^1 |f|^p = \int_0^{1/2} x^{-p/q} |\log x|^{-2p/q} \le \log^{-2p/q} 2 \int_0^{1/2} x^{-p/q} < \infty$$

When $p = q$, the antiderivative of $x^{-1} \log^{-2} x$ is $C - 1/\log x$ (check by differentiation). Since the antiderivative has a finite limit as $x \to 0$, the integral converges.

When $p > q$, we have

$$\int_\delta^{1/2} |f|^p = \int_\delta^{1/2} x^{-p/q} |\log x|^{-2p/q} \ge \log^{-2p/q}(1/\delta) \int_\delta^{1/2} x^{-p/q}$$

$$= \log^{-2p/q}(1/\delta) \frac{(1/2)^{1-p/q} - \delta^{1-p/q}}{1 - p/q}$$

L'Hospital's rule implies $\lim_{x\to\infty} \dfrac{x^\epsilon}{\log x} = \infty$ for every $\varepsilon > 0$. Therefore, $(1/\delta)^{p/q-1} \log^{-2p/q}(1/\delta) \to \infty$ as $\delta \to 0$, and the integral diverges. □

MAT 701 HW 8.2: HÖLDER AND MINKOWSKI

`Due Monday 10/29/18 by the end of the day`

Problem 1. Fix $r \in (0,1)$.

(a) Suppose $f \in L^p([2,\infty))$ where $1 \le p < 1/(1-r)$. Prove that

$$\int_2^\infty \frac{|f(x)|}{x^r}\,dx < \infty$$

(b) Show that the statement in (a) fails with $p = 1/(1-r)$. *Hint:* $\int_2^\infty \frac{1}{x \log x}\,dx = \infty$.

Proof. (a) Let $g(x) = 1/x^r$ and $p' = p/(p-1)$. By Hölder's inequality $\int_2^\infty |fg| \le \|f\|_p \|g\|_{p'}$ so it remains to show $\|g\|_{p'} < \infty$. If $p = 1$, then $p' = \infty$ and $\|g\|_\infty = 1 < \infty$. Otherwise, $1 < p < 1/(1-r)$ implies $1 - r < 1/p < 1$, hence $1 < 1/p' < r$. Then $\int_2^\infty |g|^{p'} = \int_2^\infty x^{-rp'} < \infty$ because $rp' > 1$.

(b) Let $f(x) = \dfrac{1}{x^{1-r}\log x}$. Then

$$\int_2^\infty \frac{|f(x)|}{x^r}\,dx = \int_2^\infty \frac{1}{x \log x}\,dx = \infty$$

(Using again the fact that improper Riemann integral of a nonnegative function is equal to its Lebesgue integral.) On the other hand, $f \in L^p$ with $p = 1/(1-r)$:

$$\int_2^\infty |f(x)|^p\,dx = \int_2^\infty \frac{1}{x \log^p x}\,dx < \infty$$

because $p > 1$. □

Problem 2. Given any sequence $\{x_1, x_2, \dots\}$ of real numbers, define

$$f(x) = \sum_{k=1}^\infty \frac{1}{k^2\sqrt{|x - x_k|}}$$

Prove that $f \in L^p([0,1])$ for $0 < p < 2$.

Proof. Lemma: for $0 < p < 2$ there exists a number $C_p \in (0, \infty)$ such that

$$\int_0^1 |x-a|^{-p/2}\,dx \le C_p$$

for all $a \in \mathbb{R}$. Assume the lemma for now; its proof appears below.

When $1 \le p < 2$, Minkowski's inequality for infinite series yields

$$\|f\|_p \le \sum_{k=1}^{\infty} \left(\int_0^1 \frac{1}{k^{2p}|x-x_k|^{p/2}} \right)^{1/p} \le C_p^{1/p} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

proving that $f \in L^p([0,1])$. For $0 < p < 1$ use the relation between L^p spaces on a set of finite measure: $f \in L^1([0,1]) \implies f \in L^p([0,1])$ for any $p \in (0,1)$. □

Proof of Lemma, version 1. If $a \in [0,1]$, then by translation

$$\int_0^1 |x-a|^{-p/2}\,dx = \int_{-a}^{1-a} |x|^{-p/2}\,dx \le \int_{-1}^{1} |x|^{-p/2}\,dx \tag{1}$$

so we can use $C_p = \int_{-1}^1 |x|^{-p/2}\,dx$ which is finite because $p/2 < 1$. If $a < 0$, then $|x-a| = x-a > x$ for all $x \in [0,1]$, hence $\int_0^1 |x-a|^{-p/2}\,dx \le \int_0^1 x^{-p/2}\,dx \le C_p$ by (??). If $a > 1$, then $|x-a| = a-x > 1-x$ for all $x \in [0,1]$, hence $\int_0^1 |x-a|^{-p/2}\,dx \le \int_0^1 |x-1|^{-p/2}\,dx \le C_p$ by (??). □

Proof of Lemma, version 2. Let $I = [-a, 1-a]$ and $J = [-1/2, 1/2]$. It suffices to prove that

$$\int_I |x|^{-p/2}\,dx \le \int_J |x|^{-p/2}\,dx$$

because then $C_p = \int_J |x|^{-p/2}\,dx$ works. Note that

$$|x|^{-p/2} \ge 2^{p/2} \quad \text{on } J, \text{ and } \quad |x|^{-p/2} \le 2^{p/2} \quad \text{on } J^c \tag{2}$$

By canceling out the integral over $I \cap J$ (which may be empty) and using (??) we get

$$\begin{aligned} \int_J |x|^{-p/2}\,dx - \int_I |x|^{-p/2}\,dx &= \int_{J\setminus I} |x|^{-p/2}\,dx - \int_{I\setminus J} |x|^{-p/2}\,dx \\ &\ge \int_{J\setminus I} 2^{p/2}\,dx - \int_{I\setminus J} 2^{p/2}\,dx \\ &= 2^{p/2}(|J\setminus I| - |I\setminus J|) = 0 \end{aligned}$$

where the last step follows from $|I| = |J|$. $\square$

The second proof is longer, but it gives the best possible bound C_p, and this idea generalizes to other sets and functions.

MAT 701 HW 8.3: SEQUENCE CLASSES ℓ^p

Due Wednesday 10/31/18 by the end of the day

Problem 1. Suppose $1 \le p \le \infty$ and $f \in L^p([1,\infty))$. Define a sequence a by $a_k = \int_k^{k+1} f$, $k \in \mathbb{N}$. Prove that $a \in \ell^p$.

Proof. *Case* $p = \infty$. By the definition of L^∞, there exists $M \in \mathbb{R}$ such that $|f| \le M$ a.e. on $[1,\infty)$. Hence

$$|a_k| \le \int_k^{k+1} |f| \le \int_k^{k+1} M = M$$

for every k, which yields $\|a\|_\infty \le M < \infty$.

Case $1 \le p < \infty$. Note that $\|\chi_{[k,k+1)}\|_{p'} = 1$ for any k and any p': for $p' < \infty$ this is because $\int_k^{k+1} 1 = 1$, and for $p' = \infty$ this is clear from the definition of the norm. By Hölder's inequality,

$$\int |f\chi_{[k,k+1)}| \le \left(\int_k^{k+1} |f|^p\right)^{1/p} \|\chi_{[k,k+1)}\|_{p'} = \left(\int_k^{k+1} |f|^p\right)^{1/p}$$

Therefore,

$$\sum_{k=1}^{\infty} |a_k|^p \le \sum_{k=1}^{\infty} \int_k^{k+1} |f|^p = \int_1^\infty |f|^p < \infty$$

using the countable additivity of integral over the domain of integration. ☐

Problem 2. Give an example of a continuous function $f\colon [1,\infty) \to \mathbb{R}$ such that the sequence a defined in #1 is in ℓ^1, but $f \notin L^1(\mathbb{R})$.

Hint: f should attain both positive and negative values so that there is some cancellation in $\int_k^{k+1} f$.

Proof. Let $f(x) = \sin(2\pi x)$. Then

$$a_k = \int_k^{k+1} \sin(2\pi x)\,dx = \frac{-1}{2\pi}\sin(2\pi x)\Big|_k^{k+1} = 0$$

for every k, so $a \in \ell^1$.

On the other hand, using symmetry properties of the sine function ($\sin(t+\pi) = -\sin t$), we get

$$\int_k^{k+1} |\sin(2\pi x)|\, dx = 2\int_k^{k+1/2} \sin(2\pi x)\, dx = \frac{-1}{\pi}\sin(2\pi x)\Big|_k^{k+1/2}$$
$$= \frac{-1}{\pi}(-1-1) = \frac{2}{\pi}$$

Therefore,

$$\int_1^\infty |f| = \sum_{k=1}^\infty \int_k^{k+1} |f| = \sum_{k=1}^\infty \frac{2}{\pi} = \infty$$

using the countable additivity of integral over the domain of integration. This shows $f \notin L^1([1,\infty))$. □

MAT 701 HW 8.4: BANACH SPACE PROPERTIES OF L^p AND ℓ^p

Due Friday 11/02/18 by the end of the day

Problem 1. Suppose that $p, p' \in [1, \infty]$ are conjugate exponents, $f_k \to f$ in $L^p(E)$, and $g_k \to g$ in $L^{p'}(E)$, where E is some measurable set. Prove that $f_k g_k \to fg$ in $L^1(E)$.

Proof. In any metric space, a convergent sequence is bounded; hence, $\{\|f_k\|_p\}$, which is $\{d(f_k, 0)\}$ in terms of the metric d on ℓ^p, is a bounded sequence. (One can also say that $\|f_k\|_p \leq \|f\|_p + \|f - f_k\|_p$ where the second term tends to zero, hence is bounded. But I wanted to emphasize that we can bring concepts from metric space theory, such as bounded sequences, into the study of ℓ^p and L^p spaces.) Choose M such that $\|f_k\|_p \leq M$ for all k. By the triangle inequality and Hölder's inequality,

$$\begin{aligned}
\|f_k g_k - fg\|_1 &= \|f_k g_k - f_k g + f_k g - fg\|_1 \\
&\leq \|f_k g_k - f_k g\|_1 + \|f_k g - fg\|_1 \\
&\leq \|f_k\|_p \|g_k - g\|_{p'} + \|f_k - f\|_p \|g\|_{p'} \\
&\leq M\|g_k - g\|_{p'} + \|f_k - f\|_p \|g\|_{p'} \to 0
\end{aligned}$$

where the convergence to 0 follows from $\|g_k - g\|_{p'} \to 0$ and $\|f_k - f\|_p \to 0$. □

Problem 2. Fix $p \in [1, \infty]$. Let $D = \{a \in \ell^p \colon \forall k \in \mathbb{N}\ 0 \leq a_{k+1} \leq a_k\}$ be the set of all nonnegative nonincreasing sequences in ℓ^p. Prove that D is a closed subset of ℓ^p.

Proof. Suppose that $a^{(j)}$ is a sequence of elements of D such that $a^{(j)} \to a$ in ℓ^p. Our goal is to prove that $a \in D$.

The definition of ℓ^p norm (either a sum or a sup) implies $|b_k| \le \|b\|_p$ for every index k and any sequence $b \in \ell^p$. In our case, this yields $|a_k^{(j)} - a_k| \le \|a^{(j)} - a\|_p \to 0$, which means $a_k = \lim_{j\to\infty} a_k^{(j)}$. It then follows that:

- $a_k \ge 0$, by letting $j \to 0$ in the inequality $a_k^{(j)} \ge 0$.
- $a_k \ge a_{k+1}$, by letting $j \to 0$ in the inequality $a_k^{(j)} \ge a_{k+1}^{(j)}$.

(This is using the comparison property of limits: if both sides of a non-strict inequality have limits, the inequality holds for the limits as well.) Thus, $a \in D$. □

Remark: there is no need to prove that $a \in \ell^p$, this is a part of the assumption "$a^{(j)} \to a$ in ℓ^p."

MAT 701 HW 8.5-6-7: HILBERT SPACE PROPERTIES OF L^2

`Due Monday 11/05/18 by the end of the day`

Problem 1. For $k \in \mathbb{N}$ let $\phi_k(t) = \sqrt{1/\pi}\sin(kt)$.

(a) Prove that $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal system in $L^2([0, 2\pi])$. *(Hint: product-of-sines formula.)*

(b) Prove that the linear span of $\{\phi_k\}$ is not dense in $L^2([0, 2\pi])$. *Hint: compute $\langle f, \phi_k \rangle$ for the constant function $f \equiv 1$.*

Proof. (a) By the product of sines formula,

$$\phi_k(t)\phi_j(t) = \frac{1}{2\pi}(\cos((k-j)t) - \cos((k+j)(t))$$

The integral $\int_0^{2\pi} \cos mt\, dt$, with $m \in \mathbb{Z}$, is equal to 2π when $m = 0$ and is 0 otherwise, because the antiderivative $m^{-1}\sin mt$ is 2π-periodic. Hence,

$$\langle \phi_k, \phi_j \rangle = \int_0^{2\pi} \phi_k(t)\phi_j(t) = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases}$$

which means $\{\phi_k\}$ is an orthonormal system.

(b) For every $k \in \mathbb{N}$ the integral $\int_0^{2\pi} 1\phi_k\, dt$ is 0 because the antiderivative of ϕ_k is $-\sqrt{1/k}\cos(kt)$ which is 2π-periodic. Thus, all Fourier coefficients $c_k = \langle 1, \phi_k \rangle$ are zeros. Recall from class that the following are equivalent for an orthonormal system in L^2 (and in Hilbert spaces in general):

(1) The closure of its linear span contains f.
(2) $\sum c_k\phi_k = f$ (the Fourier series converges to f in L^2)
(3) $\sum |c_k|^2 = \|f\|_2^2$ (Parseval's identity holds)

The conclusion follows by observing that (2) fails here (or, that (3) fails). □

Problem 2. (a) Prove that for every $f \in L^2([0, 2\pi])$

$$\lim_{k\to\infty} \int_{[0,2\pi]} f(t) \sin kt \, dt = 0$$

(b) Prove that (a) holds for every $f \in L^1([0, 2\pi])$; this is known as the Riemann-Lebesgue Lemma. *(Hint: apply (a) to a simple function g such that $\|f - g\|_1$ is small.)*

Proof. (a) We know from #1 that the functions $\phi_k(t) = \sqrt{1/\pi}\sin(kt)$ form an orthonormal system in $L^2([0, 2\pi])$. The integral $\int_{[0,2\pi]} f(t) \sin kt \, dt$ is $\sqrt{\pi} c_k$ where $c_k = \langle f, \phi_k \rangle$. Bessel's inequality $\sum |c_k|^2 \le \|f\|^2$ implies $c_k \to 0$, which proves (a).

(b) Given $\epsilon > 0$, pick a simple function g such that $\|f - g\|_1 < \epsilon/2$ (such g exists by the density of simple functions in L^p for $1 \le p < \infty$, section 8.4). Since g is a bounded function on a bounded interval, it belongs to all L^p spaces, in particular to L^2. By part (a) there exists N such that

$$\left| \int_{[0,2\pi]} g(t) \sin kt \, dt \right| < \frac{\epsilon}{2} \quad \forall k \ge N$$

By Hölder's inequality (which is just a comparison of integrals in this case),

$$\left| \int_{[0,2\pi]} (f(t) - g(t)) \sin kt \, dt \right| \le \|f - g\|_1 \|\sin kt\|_\infty < \frac{\epsilon}{2}$$

for all k. Thus,

$$\left| \int_{[0,2\pi]} f(t) \sin kt \, dt \right| < \epsilon \quad \forall k \ge N$$

which means $\int_{[0,2\pi]} f(t) \sin kt \, dt \to 0$ by definition. □

MAT 701 HW 10.1: ADDITIVE SET FUNCTIONS AND MEASURES

Due Thursday 11/08/18 by the end of the day

Problem 1. Let (X, Σ, μ) be a measure space. For $A, B \in \Sigma$ let $d(A, B) = \mu(A \triangle B)$ where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B. Prove that d satisfies the triangle inequality: $d(A, B) \leq d(A, C) + d(B, C)$ for $A, B, C \in \Sigma$.

Proof. Claim:

$$A \triangle B \subset (A \triangle C) \cup (B \triangle C) \tag{1}$$

To prove (??), let $x \in A \triangle B$. Then either $x \in A \setminus B$ or $x \in B \setminus A$; we may assume $x \in A \setminus B$, because the other case is handled by relabeling A and B. Consider two cases. Case 1: $x \in C$, then we have $x \in C \setminus B$, hence $x \in B \triangle C$. Case 2: $x \notin C$, then $x \in A \setminus C$, hence $x \in A \triangle C$. In either case (??) holds, completing the proof of the claim.

Since μ is a measure, it is monotone with respect to inclusion and subadditive (p.243 of the textbook). Therefore, (??) implies

$$\mu(A \triangle B) \leq \mu((A \triangle C) \cup (B \triangle C)) \leq \mu(A \triangle C) + \mu(B \triangle C)$$

which was to be proved. □

Problem 2. Fix a function $w \in L^1(\mathbb{R}^n)$ and define the additive set function ϕ on the Lebesgue measurable subsets of $\mathbb{R}^n$ by $\phi(E) = \int_E w$. Prove that the variations of ϕ are given by $\overline{V}(E) = \int_E w^+$, $\underline{V}(E) = \int_E w^-$, and $V(E) = \int_E |w|$.

Proof. Fix a measurable set E. For any arbitrary measurable set $A \subset E$ we have

$$\int_A w = \int_A w^+ - \int_A w^- \leq \int_A w^+ \leq \int_E w^+$$

using the fact that $w^+, w^- \geq 0$. Thus, $\overline{V}(E) \leq \int_E w^+$. To prove the reverse inequality, observe that the set $P = \{x \in \mathbb{E} \colon w(x) \geq 0\}$ satisfies $\int_P w^+ = \int_E w^+$ (because $w^+ \equiv 0$ on $E \setminus P$) and $\int_P w^- = 0$ (because $w^- \equiv 0$ on P). Thus,

$$\int_P w = \int_P w^+ - \int_P w^- = \int_E w^+$$

completing the proof of $\overline{V}(E) = \int_E w^+$.

Applying the previous result to $-w$, we obtain

$$\sup_{A \subset E} \int_A (-w) = \int_E (-w)^+$$

Since $(-w)^+ = w^-$, this yields

$$\underline{V}(E) = -\inf_{A \subset E} \int_A w = \sup_{A \subset E} \int_A (-w) = \int_E w^-$$

Finally, $V(E) = \overline{V}(E) + \underline{V}(E) = \int_E (w^+ + w^-) = \int_E |w|$. □

MAT 701 HW 10.2: MEASURABLE FUNCTIONS AND INTEGRATION

Due Monday 11/12/18 by the end of the day

Problem 1. Let (X, Σ, μ) be a measure space, and let $f\colon X \to \mathbb{R}$ be a measurable function. For each Borel set $E \subset \mathbb{R}$ define $\nu(E) = \mu(f^{-1}(E))$. Prove that ν is a measure on the Borel σ-algebra of $\mathbb{R}$.

(This measure called the pushforward of μ under f.)

Proof. Recall that a real-valued function is measurable if and only if the preimages of all Borel sets are measurable. Thus, ν is well-defined. Also, $\nu(E) \geq 0$ for every Borel E because $\mu \geq 0$.

Given disjoint Borel sets $E_k \subset \mathbb{R}$, observe that $f^{-1}(E_k)$ are also disjoint, since taking preimages commutes with all set operations. Hence

$$\begin{aligned}
\nu\left(\bigcup_k E_k\right) &= \mu\left(f^{-1}\left(\bigcup_k E_k\right)\right) \\
&= \mu\left(\bigcup_k f^{-1}(E_k)\right) \\
&= \sum_k \mu\left(f^{-1}(E_k)\right) \\
&= \sum_k \nu\left(E_k\right)
\end{aligned}$$

which proves the countable additivity of ν. □

Problem 2. With the notation of #1, prove that for every nonnegative Borel function $g\colon \mathbb{R} \to [0, \infty)$ the function $g \circ f$ is measurable on X and

$$\int_X (g \circ f)\, d\mu = \int_{\mathbb{R}} g\, d\nu$$

(Hint: begin with $g = \chi_E$ and proceed toward more general g.)

Uses a lot

Proof. For every Borel set $E \subset \mathbb{R}$ we have $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$ where $g^{-1}(E)$ is Borel by assumption and therefore $f^{-1}(g^{-1}(E))$ is measurable. This shows that $g \circ f$ is measurable.

If $g = \chi_E$ for some Borel set $E \subset \mathbb{R}$, then $g \circ f = \chi_{f^{-1}(E)}$, hence

$$\int_X (g \circ f)\, d\mu = \mu(f^{-1}(E)) = \nu(E) = \int_{\mathbb{R}} g\, d\nu$$

By linearity of integrals, the equality $\int_X (g \circ f)\, d\mu = \int_{\mathbb{R}} g\, d\nu$ extends from characteristic functions to all simple functions.

Given a general Borel function $g\colon \mathbb{R} \to [0, \infty)$, let $g_k \nearrow g$ be an approximating sequence of simple Borel functions, for example $g_k = \min(k, 2^{-k}\lfloor 2^k g \rfloor)$. By definition of measurability, g_k is measurable in whatever σ-algebra g is measurable, in this case Borel. Therefore, $\int_X (g_k \circ f)\, d\mu = \int_{\mathbb{R}} g_k\, d\nu$ holds by the preceding case. Note that $g_k \circ f \nearrow g \circ f$. Letting $k \to \infty$ and using the Monotone Convergence Theorem, we obtain $\int_X (g \circ f)\, d\mu = \int_{\mathbb{R}} g\, d\nu$. $\square$

MAT 701 HW 10.3A: ABSOLUTE CONTINUOUS AND SINGULAR ASF

`Due Wednesday 11/14/18 by the end of the day`

Problem 1. Let (X, Σ, μ) be a measure space, and let $f : X \to \overline{\mathbb{R}}$ be an integrable function (that is, $f \in L^1(X, \mu)$). Suppose that $\int_A f = 0$ for every $A \in \Sigma$. Prove that $f = 0$ μ-a.e. (that is, $f = 0$ on $X \setminus Z$ where $\mu(Z) = 0$).

Proof. For $k \in \mathbb{N}$ let $E_k = \{x \in X : f(x) \geq 1/k\}$. Then

$$0 = \int_{E_k} f \, d\mu \geq \int_{E_k} \frac{1}{k} \, d\mu = \frac{1}{k}\mu(E_k)$$

which shows $\mu(E_k) = 0$. Taking the union over k, we obtain $\mu(\{f > 0\}) = 0$. By applying this argument to $-f$ we get $\mu(\{f < 0\}) = 0$. Thus $f = 0$ μ-a.e. $\square$

Problem 2. Let (X, Σ, μ) be a measure space. Suppose that $\phi_k : \Sigma \to \mathbb{R}$ is a singular ASF with respect to μ, for each $k \in \mathbb{N}$. Suppose further that $\phi : \Sigma \to \mathbb{R}$ is an ASF such that $\phi_k(A) \to \phi(A)$ for each $A \in \Sigma$.

Prove that ϕ is singular with respect to μ.

Proof. By the definition of a singular ASF, for each k there exists a set $Z_k \subset X$ such that $\mu(Z_k) = 0$ and $\phi_k(A) = 0$ for all $A \subset Z_k^c$. Let $Z = \bigcup_k Z_k$. Then $\mu(Z) = 0$ by countable additivity. Also, for any set $A \subset Z^c$ we have $A \subset Z_k^c$ for all k, hence $\phi_k(A) = 0$ for all k, hence $\phi(A) = 0$. This shows ϕ is singular with respect to μ. $\square$

MAT 701 HW 10.3B: ABSOLUTE CONTINUOUS AND SINGULAR ASF 2

`Due Friday 11/16/18 by the end of the day`

Problem 1. For $k \in \mathbb{N}$ define $b_k\colon [0,1) \to \mathbb{R}$ by $b_k(x) = 1$ if $\lfloor 2^k x \rfloor$ is odd, and $b_k(x) = 0$ otherwise. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{2b_k(x)}{3^k}$$

Prove that: (a) f is a measurable function on $[0,1)$ with respect to the Lebesgue measure;

(b) $f([0,1]) \subset C$ where C is the standard middle-third Cantor set.

Hint: You can use the following characterization of C,

$$C = \{x \in [0,1] \colon \operatorname{dist}(3^m x, \mathbb{Z}) \le 1/3 \text{ for } m = 0, 1, 2, \dots\}$$

Proof. (a) Each term of the sum, namely $\dfrac{2b_k(x)}{3^k}$, has countably many discontinuities (specifically, the points x such that $2^k x$ is an integer). By the converse part of Lusin's theorem, it is measurable. Every partial sum of the series is measurable as the sum of measurable function. Finally, the series converges for every x, as its kth term is $\le 2/3^k$. The limit of a sequence if measurable functions is measurable.

(b) Given $m \in \{0, 1, 2, \dots\}$, observe that

$$3^m f(x) = \sum_{k=1}^{m} 2b_k(x) 3^{m-k} + \sum_{k=m+1}^{\infty} \frac{2b_k(x)}{3^{k-m}}$$

where the first sum is an integer, call it q. The second (tail) sum is nonnegative and does not exceed $\displaystyle\sum_{k=m+1}^{\infty} \frac{2}{3^{k-m}} = \frac{2/3}{1-1/3} = 1$. Thus, $q \le 3^m f(x) \le q + 1$.

To refine this further, consider two cases. If $b_{m+1}(x) = 0$, then the tail sum is at most

$$\sum_{k=m+2}^{\infty} \frac{2}{3^{k-m}} = \frac{2/9}{1-1/3} = \frac{1}{3}$$

hence $q \leq 3^m f(x) \leq q + 1/3$, proving that $|3^m f(x) - q| \leq 1/3$.

If $b_{m+1}(x) = 1$, then the tail sum is at least $\dfrac{2b_{m+1}(x)}{3^{m+1-m}} = \dfrac{2}{3}$, hence $q + 2/3 \leq 3^m f(x) \leq q + 1$. This implies $|3^m f(x) - (q+1)| \leq 1/3$.

In either case, $\operatorname{dist}(3^m f(x), \mathbb{Z}) \leq 1/3$. This proves $f(x) \in C$. □

Problem 2. Let σ be the pushforward of the Lebesgue measure on $[0, 1)$ under f from #1. That is, $\sigma(A) = |f^{-1}(A)|$ for Borel sets $A \subset \mathbb{R}$. Prove that: (a) σ is singular with respect to the Lebesgue measure on the Borel σ-algebra;

(b) $\sigma(\{p\}) = 0$ for every $p \in \mathbb{R}$. *Hint: show that for distinct $x, y \in [0, 1)$ there exists k such that $b_k(x) \neq b_k(y)$. Deduce that $f(x) \neq f(y)$.*

Proof. We know $|C| = 0$ from earlier in the semester. Also, $\sigma(\mathbb{R} \setminus C) = |f^{-1}(\mathbb{R} \setminus C)| = |\emptyset| = 0$ by #1b. Thus σ is singular.

(b) Suppose x, y are distinct points in $[0, 1)$. Without loss of generality $x < y$. For sufficiently large integers k we have $2^k(y - x) \geq 1$, hence $\lfloor 2^k y \rfloor > \lfloor 2^k x \rfloor$ (adding 1 to a number increases its integer part by 1). Let m be the smallest integer such that $\lfloor 2^m y \rfloor \neq \lfloor 2^m x \rfloor$. Then $\lfloor 2^{m-1} x \rfloor = \lfloor 2^{m-1} y \rfloor$; call this number q. Since $2^{m-1}x, 2^{m-1}j \in [q, q+1)$, it follows that $2^m x, 2^m y \in [2q, 2q + 2)$. Therefore, $\lfloor 2^m x \rfloor, \lfloor 2^m y \rfloor \in \{2q, 2q + 1\}$. Since these are distinct and $y > x$, we conclude that $\lfloor 2^m y \rfloor = 2q + 1$ and $\lfloor 2^m x \rfloor = 2q$. Thus $b_m(y) = 1$ and $b_m(x) = 0$. Also note that $b_k(x) = b_k(y)$ for all $k < m$ by the minimality of m.

The difference $f(y) - f(x)$ can be estimated from below as follows:

$$\begin{aligned} f(y) - f(x) &= \sum_{k=1}^{m-1} \frac{2(b_k(y) - b_k(x))}{3^k} + \frac{2}{3^m} + \sum_{k=m+1}^{\infty} \frac{2(b_k(y) - b_k(x))}{3^k} \\ &= 0 + \frac{2}{3^m} + \sum_{k=m+1}^{\infty} \frac{2(b_k(y) - b_k(x))}{3^k} \\ &\geq \frac{2}{3^m} + \sum_{k=m+1}^{\infty} \frac{2(0-1)}{3^k} \\ &= \frac{2}{3^m} - \frac{2/3^{m+1}}{1 - 1/3} \\ &= \frac{1}{3^m} > 0 \end{aligned}$$

This shows that f is strictly increasing. In particular it is injective, which implies that $\sigma(\{p\}) = |f^{-1}(p)| = 0$ for all p, where the set $f^{-1}(p)$ has either 0 or 1 elements. □

MAT 701 HW 7.2B: LEBESGUE DIFFERENTIATION THEOREM

Due Friday 11/30/18 by the end of the day

Problem 1. Let $E \subset \mathbb{R}^n$. Suppose there exists $c > 0$ such that every cube $Q \subset \mathbb{R}^n$ contains a cube Q' such that $Q' \cap E = \emptyset$ and $|Q'| \geq c|Q|$. Prove that $|E| = 0$.

Proof. If E is measurable and $|E| > 0$, then we know (a corollary of Lebesgue Differentiation Theorem) that $|Q \cap E|/|Q| \to 1$ as $Q \searrow x$, for a.e. $x \in E$. However, $|Q \cap E| = |Q| - |Q \setminus E| \leq |Q| - |Q'| \leq (1-c)|Q|$, which implies $|Q \cap E|/|Q| \leq 1 - c$, a contradiction. This proves $|E| = 0$ in this case.

In general, consider the closure $\overline{E}$ which is measurable, being a closed set. If Q' is a cube disjoint from E, then the interior of Q' is an open cube Q'' disjoint from $\overline{E}$. Since $|Q''| = |Q'|$, the argument from the first paragraph still applies, and shows $|\overline{E}| = 0$. Since $E \subset \overline{E}$, the claim follows. □

Alternative proof, without LDT. It suffices to show that $|E \cap Q|_e = 0$ for every cube Q, since $\mathbb{R}^n$ is a countable union of cubes. Choose an integer m such that $(m/2)^n > c^{-1}$. Divide Q into m^n equal subcubes, by partitioning each edge of Q into m equal subintervals. Each subcube has volume $m^{-n}|Q| < 2^{-n}c|Q|$. Let Q' be as in the problem statement. Let $\widehat{Q}$ be one of our subcubes that contains the center of Q'. Since $|\widehat{Q}| < 2^{-n}c|Q| \leq 2^{-n}|Q'|$, it follows that the edgelength of $\widehat{Q}$ is less than half of the edgelength of Q'. This and the fact that $\widehat{Q}$ contains the center of Q' imply $\widehat{Q} \subset Q'$. In conclusion: $E \cap Q$ is covered by

$m^n - 1$ subcubes of volume $m^{-n}|Q|$, because we do not need $\widehat{Q}$ in this cover.

Repeat the above for each of the $m^n - 1$ subcubes, getting $(m^n - 1)^2$ subsubcubes of volume $m^{-2n}|Q|$, and so on. In this way, for every $k \in \mathbb{N}$ the set $E \cap Q$ can be covered by $(m^n - 1)^k$ cubes of volume $m^{-kn}|Q|$. This implies

$$|E \cap Q|_e \le (m^n - 1)^k m^{-kn}|Q| = \left(\frac{m^n - 1}{m^n}\right)^k |Q| \xrightarrow[k\to\infty]{} 0$$

proving the claim.

Problem 2. Let $f \in L^1(\mathbb{R}^n)$. For $k \in \mathbb{N}$ let Q_k be the cube $[-1/k, 1/k]^n$ and define $h_k = |Q_k|^{-1}\chi_{Q_k}$.

(a) Prove that $f * h_k \to f$ a.e.

(b) Prove that $f * h_k \to f$ in $L^1(\mathbb{R}^n)$.

Proof. (a) Let $Q_k^x = Q_k + x$. Recall that (by the commutativity of convolution)

$$(f * h_k)(x) = \int_{\mathbb{R}^n} f(t)h_k(x-t)\,dt = \frac{1}{|Q_k^x|}\int_{Q_k^x} f(t)\,dt$$

where the second step uses the definition of h_k: the value of $h_k(x-t)$ is 0 unless $x - t \in Q_k$, equivalently $t \in Q_k^x$.

By the Lebesgue differentiation theorem, the quantity on the right converges to $f(x)$ for a.e. x, proving the claim.

(b) Given $\epsilon > 0$, pick $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_1 < \epsilon$. By the triangle inequality,

$$\|f * h_k - f\|_1 \le \|f * h_k - g * h_k\|_1 + \|g * h_k - g\|_1 + \|f - g\|_1$$

The first term on the right is estimated by the convolution inequality: $\|(f - g) * h_k\|_1 \le \|f - g\|_1\|h_k\|_1 < \epsilon$, since $\|h_k\|_1 = 1$. The last term on the right is $< \epsilon$ by the choice of g. It remains to show that $\|g * h_k - g\|_1 \to 0$ as $k \to \infty$.

Let N be large enough so that the support of g is contained in $[-N,N]^n$. If the quantity $g * h_k(x) = \int_{\mathbb{R}^n} g(t)h_k(x-t)\,dt$ is nonzero, then there must be some t such that $g(t)h_k(x-t)$ is nonzero, which requires $t \in [-N,N]^n$ and $x - t \in [-1/k, 1/k]^n$. Hence $x \in [-N-1, N+1]^n$. We have shown that the support of $g * h_k$ is contained in $[-N-1, N+1]^n$.

Since $g \in C_c(\mathbb{R}^n)$, there exists M such that $|g| \leq M$ everywhere. Then for all x

$$|g * h_k(x)| \leq \int_{\mathbb{R}^n} |g(x-t)h_k(t)|\,dt \leq M \int_{\mathbb{R}^n} h_k(t)\,dt = M$$

Thus, the sequence $g * h_k - g$ is dominated by the function $2M\chi_{[-N-1,N+1]^n}$. And since $g * h_k - g \to 0$ a.e. by part (a), the Dominated Convergence Theorem yields $\|g * h_k - g\|_1 \to 0$. □

MAT 701 HW 7.4A: DIFFERENTIABILITY OF MONOTONE FUNCTIONS

Due Monday 12/03/18 by the end of the day

Problem 1. Given an increasing function $f\colon \mathbb{R} \to \mathbb{R}$ and a number $\delta > 0$, let

$$f_\delta(x) = \sup_{0<h<\delta} \frac{f(x+h) - f(x)}{h}$$

Prove that f_δ is a measurable function on $\mathbb{R}$.

Proof. Since f is increasing, it is measurable as the sets $\{f > a\}$ are intervals for all $a \in \mathbb{R}$. Therefore, $\dfrac{f(x+h) - f(x)}{h}$ is measurable for every h, being a multiple of a difference of measurable functions. Let

$$g_\delta(x) = \sup_{0<h<\delta,\ h\in\mathbb{Q}} \frac{f(x+h) - f(x)}{h}$$

which is also measurable, being the supremum of a **countable** family of measurable functions.

We have $g_\delta \le f_\delta$ because the supremum on the left is over a smaller set. Therefore, $\{g_\delta > a\} \subset \{f_\delta > a\}$ for every $a \in \mathbb{R}$. To prove the converse inclusion, suppose $f_\delta(x) > a$. Then there exists $h \in (0, \delta)$ such that $f(x+h) - f(x) > ha$. By density of rationals there exists $h' \in (h, \delta) \cap \mathbb{Q}$, and by picking h' sufficiently close to h we can achieve $f(x+h) - f(x) > h'a$. Since f is increasing,

$$f(x+h') - f(x) \ge f(x+h) - f(x) > h'a$$

which implies $g_\delta(x) > a$. This proves $\{f_\delta > a\} = \{g_\delta > a\}$. Since the latter set is measurable, so is the former one. □

Problem 2. Given an increasing function $f\colon \mathbb{R} \to \mathbb{R}$, consider its Dini number

$$\overline{D}_{+}f(x) = \limsup_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

Prove that $\overline{D}_{+}f$ is a measurable function on $\mathbb{R}$.

Proof. Since the pointwise limit of measurable functions is measurable, it suffices to show that

$$\overline{D}_{+}f(x) = \lim_{k \to \infty} f_{1/k}(x)$$

where the functions $f_{1/k}$ are from #1. But $\overline{D}_{+}f(x) = \lim_{\delta \searrow 0} f_{\delta}(x)$ by the/a definition of lim sup, where the limit on the right exists by virtue of monotonicity in δ. Since $1/k \to 0$, the sequential limit $\lim_{k\to\infty} f_{1/k}(x)$ has the same value. ☐

Missing 7.4B

MAT 701 HW 7.5: ABSOLUTELY CONTINUOUS AND SINGULAR FUNCTIONS

Due Friday 12/07/18 by the end of the day

Problem 1. Let $C \subset [0,1]$ be the standard middle-third Cantor set. Show that the function $f(x) = \operatorname{dist}(x, C)^p$ is not absolutely continuous on $[0,1]$ when $0 < p \le \log 2/\log 3$.

Bonus (not for grade): is f absolutely continuous when $p > \log 2/\log 3$?

Proof. For each $m \in \mathbb{N}$, the set $[0,1] \setminus C$ contains 2^{m-1} disjoint open intervals of length 3^{-m} whose endpoints are in C. Let $\{(a_{mk}, b_{mk}) : k = 1, \dots 2^{m-1}\}$ be these intervals. Let $c_{mk} = (a_{mk} + b_{mk})/2$ be their midpoints. By construction, $f(a_{mk}) = 0$ and $f(c_{mk}) = (3^{-m}/2)^p$.

Note that

$$\sum_{k=1}^{2^{m-1}} (c_{mk} - a_{mk}) = 2^{m-1}3^{-m}/2 = \frac{1}{4}(2/3)^m \xrightarrow[m\to\infty]{} 0$$

Thus, for any $\delta > 0$ there is $m \in \mathbb{N}$ such that the total length of the intervals $[a_{mk}, c_{mk}]$ is less than δ. On the other hand,

$$\sum_{k=1}^{2^{m-1}} (f(c_{mk}) - f(a_{mk})) = 2^{m-1}(3^{-m}/2)^p = \frac{1}{2^{p+1}}\left(\frac{2}{3^p}\right)^m$$

where $2/3^p \ge 1$ by the choice of p. This quantity is bounded away from 0, proving that f is not absolutely continuous. □

Problem 2. Use #1 to prove that the composition of absolutely continuous functions need not be absolutely continuous.

Proof. Recall that Lipschitz functions are absolutely continuous; in particular $d(x, C)$ is absolutely continuous on $[0,1]$. Also, it was shown in class that $\sqrt{x}$ is absolutely continuous on $[0,1]$. But the composition

$f(x) = \sqrt{d(x,C)}$ is not absolutely continuous on $[0,1]$, by virtue of #1 (note that $1/2 < \log 2/\log 3$ because $2\log 2 = \log 4 > \log 3$). $\square$

Bonus content: $d(x,C)^p$ is AC on $[0,1]$ when $p > \log 2/\log 3$. The proof relies on the following useful **lemma**: Suppose

- f_k is AC on $[a,b]$ for all $k \in \mathbb{N}$;
- the sequence $\{f_k(a)\}$ has a finite limit;
- the sequence $\{f_k'\}$ converges in $L^1([a,b])$.

Then the limit $\lim f_k$ exists and is AC on $[a,b]$.

Proof of the lemma. By assumption, $f_k' \to g$ for some $g \in L^1$, the convergence being in the L^1 norm. Define

$$f(x) = \lim_{k\to a} f_k(a) + \int_a^x g'(t)\,dt$$

which is absolutely continuous because $g' \in L^1$. Applying FTC to f_k, we find

$$f_k(x) = f_k(a) + \int_a^x f_k'(t)\,dt$$

Letting $k \to \infty$ on the right yields $f_k(x) \to f(x)$. $\square$

The lemma provides another proof that the function $g(t) = t^p$ is AC on $[0,1]$ for any $p > 0$. Indeed, the function $g_k(t) = (t^p - 1/k)^+$ is Lipschitz continuous, being zero on $[0,1/k]$ and having a bounded derivative on $(1/k, 1]$. Also, $g_k \to g$ pointwise. Finally,

$$g_k'(t) = \begin{cases} 0, & 0 \le t < 1/k \\ pt^{p-1}, & 1/k < t \le 1 \end{cases}$$

which converges to pt^{p-1} in L^1. By the lemma, g is AC.

The function $d(x,C)^p$ is the pointwise sum of the series $\sum \phi_m$ where $\phi_m = 0$ outside of the intervals (a_{mk}, b_{mk}) from #1, $\phi_m(x) = (x - a_{mk})^p$ for $x \in (a_{mk}, c_{mk}]$, and $\phi_m(x) = (b_{mk} - x)^p$ for $x \in [c_{mk}, b_{mk}]$. Since ϕ_m consists of finitely many copies of $t \mapsto t^p$, it is absolutely continuous.

Also,

$$\int_0^1 |\phi'_m| = 2 \cdot \sum_{k=1}^{2^{m-1}} (c_{mk} - a_{mk})^p = 2^{m-p}/3^m$$

Since $p > \log 2/\log 3$, the series $\sum_m \int_0^1 |\phi'_m|$ converges (it is a geometric series). Hence $\sum_m \phi'_m$ converges in L^1. By the lemma, $d(x, C)^p$ is AC. □

MATH 701 MIDTERM EXAM SOLUTION

1. Suppose that $A \subset \mathbb{R}^n$ is a closed set and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Prove that $F(A)$ is a Borel set.

Proof. Every closed set is a countable union of compact sets: for example,

$$A = \bigcup_{k=1}^{\infty} (A \cap [-k,k]^n)$$

where $A \cap [-k,k]^n$ is closed and bounded, hence compact. The image of a compact set under a continuous map is compact (MAT 601). Hence,

$$F(A) = \bigcup_{k=1}^{\infty} F(A \cap [-k,k]^n)$$

is a countable union of compact sets. Each compact set is closed; so the union is a Borel set (more specifically, a F_σ-set). □

Remark: this argument was used in the proof of the main theorem of 3.5, about the measurability of Lipschitz images.

2. Suppose E_k, $k \in \mathbb{N}$, are measurable subsets of $\mathbb{R}^n$. Let E be the set of all points x such that $x \in E_k$ for more than one value of k. Prove that E is measurable.

Proof. Having $x \in E_k$ for two values of k means $x \in E_i \cap E_j$ where $i, j \in \mathbb{N}$ and $i \neq j$. So,

$$E = \bigcup_{i \in \mathbb{N}} \bigcup_{j \neq i} (E_i \cap E_j)$$

The intersection $E_i \cap E_j$ of measurable sets is measurable, and E is a countable union of these, so it is measurable. □

Remark: a shorter proof is to introduce $f = \sum_{k\in\mathbb{N}} \chi_{E_k}$, which is a measurable function, being the sum of a series of nonnegative measurable functions. Then note that $E = \{x \in \mathbb{R}^n \colon f(x) > 1\}$ is measurable.

3. Suppose $f_k, f\colon [0,1] \to [1,\infty)$ are measurable functions such that $f_k \xrightarrow{m} f$. (This means convergence in measure.) Prove that $\sqrt{f_k} \xrightarrow{m} \sqrt{f}$.

Proof. Rewrite the difference of square roots as follows:

$$|\sqrt{f_k} - \sqrt{f}| = \frac{|f_k - f|}{\sqrt{f_k} + \sqrt{f}} \leqslant \frac{|f_k - f|}{2}$$

using $f_k, f \geqslant 1$. For any $\epsilon > 0$, the above implies $\{|\sqrt{f_k} - \sqrt{f}| > \epsilon\} \subset \{|f_k - f| > 2\epsilon\}$. Hence

$$|\{|\sqrt{f_k} - \sqrt{f}| > \epsilon\}| \leqslant |\{|f_k - f| > 2\epsilon\}| \xrightarrow[k\to\infty]{} 0$$

as required. □

4. Let $f\colon E \to \mathbb{R}$ be a bounded measurable function, where $E \subset \mathbb{R}^n$ is a measurable set. Suppose there exists a number $p \in (0,1)$ such that

$$\limsup_{\alpha\to 0+} \alpha^p |\{x \in E\colon |f(x)| > \alpha\}| < \infty$$

Prove that $f \in L^1(E)$.

Proof. For $j \in \mathbb{Z}$ let $E_j = \{x \in E\colon |f(x)| > 2^j\}$. Recall from HW 5.1 that $f \in L^1(E)$ if and only if $\sum_{j\in\mathbb{Z}} 2^j |E_j| < \infty$. It remains to show this series converges.

The finiteness of lim sup implies there exist $\alpha_0 > 0$ and M such that $\alpha^p|\{x \in E\colon |f(x)| > \alpha\}| \leqslant M$ for $0 < \alpha < \alpha_0$. Let $J \in \mathbb{Z}$ be such that $2^J < \alpha_0$. Then

$$\sum_{j=-\infty}^{J} 2^j|E_j| \leqslant \sum_{j=-\infty}^{J} 2^j \frac{M}{2^{jp}} = M \sum_{j=-\infty}^{J} 2^{(1-p)j}$$

It is easier to think of this series after substitution $j = -i$, so it becomes $\sum_{i=-J}^{\infty} 2^{(p-1)i}$ which is a convergent geometric series since $2^{p-1} < 1$.

For $j > J$ we have $|E_j| \leqslant |E_J|$ since $E_j \subset E_J$. Also, there is an index K such that E_K is empty (because f is bounded). Thus,

$$\sum_{j=J+1}^{\infty} 2^j|E_j| = \sum_{j=J+1}^{K-1} 2^j|E_j| \leqslant |E_J| \sum_{j=J+1}^{K-1} 2^j$$

which is a finite sum of finite quantities, hence finite. □

Aug 2018

Complex Part

1. Show that the function $f(z) = 1/z$ has no a holomorphic anti-derivative on $\{1 < |z| < 2\}$.

2. Suppose that f is an entire function and f^2 is a holomorphic polynomial. Show that f is also a holomorphic polynomial.

3. Suppose that a function f is meromorphic on the unit disk $\mathbb{D}$ and continuous in a neighborhood of its boundary $\partial\mathbb{D}$. Show that for any number A such that $|A| > \sup_{z\in\partial\mathbb{D}} |f(z)|$ the number of zeros of the function $f - A$ is equal to the number of poles of f in $\mathbb{D}$.

4. Suppose that f and g are entire functions such that $f \circ g(x) = x$ when $x \in \mathbb{R}$. Show that f and g are linear functions.

'A 2018

1. Consider a function $f(z) = 1/z$. Proceeding by contradiction, suppose f does indeed have a holomorphic antiderivative on $\{1 < |z| < 2\}$. Let F be such a function, that is F is the primative of $f(z) = 1/z$. Consider $F(z) = \text{Log } z$. Note $F(z)$ is analytic and $F'(z) = f(z)$. Then around any closed curve in $\{1 < |z| < 2\}$, say $\{|z| = 3/2\}$ $\int_\gamma f = 0$. However

$$\int_{|z|=3/2} f = \text{Log } z \Big|_{-3/2-0i}^{-3/2+0i} = \text{Log}(-3/2+0i) - \text{Log}(-3/2-0i) = \log|3/2| + \pi i - \log|3/2| + \pi i = 2\pi i$$

This is a contradiction. Thus f has no holomorphic antiderivative on $\{1 < |z| < 2\}$

2. Suppose f is an entire function and f^2 is a holomorphic polynomial.

Since f^2 is a polynomial it can be expressed as the following product.

$$f^2 = \lambda \prod_{i=1}^{n} (z - z_i)^{m_i} \qquad n \in \mathbb{N} \cup \{0\}, \quad \lambda \in \mathbb{C}, \; z_0, \ldots, z_n \text{ zeros of order } m_i$$

Since z_i is a zero of f^2, z_i is also a zero of f. Let k_i be the order of z_i as a zero of f. (so $m_i = 2k_i$) Thus.

$$f^2(z) = \lambda \prod_{i=1}^{n} (z - z_i)^{2k_i} = \lambda \left(\prod (z - z_i)^{k_i}\right)^2$$

Let

$$g(z) = f(z) \Big/ \left(\prod (z - z_i)^{k_i}\right) \longrightarrow g^2 = \frac{f^2(z)}{\left(\prod (z - z_i)^{k_i}\right)^2} = \lambda$$

Hence $g = \pm\sqrt{\lambda}$. Since g is continuous on $\mathbb{C}$, it is either $\sqrt{\lambda}$ or $-\sqrt{\lambda}$, (because the square root has two branches) WLOG, let $g = \sqrt{\lambda}$. Thus.

$f(z) = \sqrt{\lambda} \prod (z - z_i)^{k_i}$, so f is a holomorphic polynomial ☑

3. Suppose a fn f is meromorphic on $\mathbb{D}$ and is ~~continuous~~ analytic in a nbhd of $\partial\mathbb{D}$.

Show that for any $A \in \mathbb{C}$ s.t. $|A| > \sup_{z\in\partial D} |f(z)|$ the number of zeros of the function $f - A$ is equal to the number of poles of f in D.

Suppose f is a meromorphic fn on $\mathbb{D}$ and is ~~cts~~ analytic in a nbhd of $\partial\mathbb{D}$. Consider $A \in \mathbb{C}$ s.t. $|A| \geq \sup_{z\in\partial D} |f(z)|$.

Case 1: $A \in (0, \infty)$

Let $g(z) := f(z) - A = (u(z) - A) + iv(z)$. For $z_0 \in \partial D$ $A > |f(z_0)| \geq |u(z_0)|$

So $-A < u(z_0) < A \longrightarrow u(z_0) - A < 0$. So $\text{Re } g(z) < 0 \quad \forall z \in \partial\mathbb{D}$

Hence g is a meromorphic fn on $\mathbb{D}$ that extends to be analytic on $\partial\mathbb{D}$, $g(z) \neq 0$ on $\partial\mathbb{D}$. So by thm on 224 $\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f'(z)}{f(z)-A}\,dz = N_0 - N_\infty$ (# of zeros, # of poles)

Claim $\frac{f'(z)}{f(z)-A}$ is analytic on $\mathbb{D}$, extends smoothly to $\partial\mathbb{D}$

Then by Cauchy's thm. $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)-A}\,dz = 0 \longrightarrow N_0 = N_\infty$ ☐

Not going to spend time generalizing.

4. Suppose f and g are entire functions such that $f \circ g(x) = x$ when $x \in \mathbb{R}$.

Since $\mathbb{R}$ is certainly a set w/ a non-isolated point, by the uniqueness principle

$$f \circ g(z) = z \quad \forall z \in \mathbb{C}$$

Hence g is univalent (one-to-one, analytic) on $\mathbb{C}$ $\longrightarrow$ (?).

So $g'(z) \neq 0$ for $z \in \mathbb{C}$, and is thus (thm p59) is conformal from $\mathbb{C}$ to
$\hookrightarrow$ if $g' = 0 \rightarrow (f \circ g)' = 0$

We claim $g(\mathbb{C}) = \mathbb{C}$. Suppose not. Since g is 1-1, open mapping, cts onto its image, g is a homeomorphism onto its image.

Since $\mathbb{C}$ is simply connected, $g(\mathbb{C})$ is simply connected (b/c g is homeo)

Since $g(\mathbb{C}) \subsetneq \mathbb{C}$ (by assumption) the Riemann mapping thm says $\exists$ a conformal map $h: g(\mathbb{C}) \longrightarrow \mathbb{D}$

Then $h \circ g : \mathbb{C} \rightarrow \mathbb{D}$ is conformal map onto $\mathbb{D}$.

?) But, $h \circ g$ is constant by Liouville, so $h \circ g$ is not onto $\mathbb{D}$. $\not\leq$.

Thus $g(\mathbb{C}) = \mathbb{C}$.

Expanding g as a pwr series $g(z) = \sum b_n z^n$

Let order of pole of g @ ∞ be $N \geq 1$. Then $g(z) = \sum_0^N b_n z^n$, $z \in \mathbb{C}$

So g is a polynomial. Let $g = \lambda \prod_{k=1}^{n} (z - z_i)^{m_i}$, $\{z_n\}$ distinct. Since g is 1-1 it has @ most 1 zero $\longrightarrow$ $g = \lambda (z - z_1)^{m_1}$. If $m_1 > 1 \rightarrow g' = m_1 \lambda (z - z_1)^{m_1}$ which $\Rightarrow g'(z_1) = 0$ $\not\leq$. Thus $m_1 = 1$ $g = \lambda(z - z_1) \rightarrow$ linear

$f = g^{-1} \longrightarrow f$ is also linear. ☑

REAL ANALYSIS AND MEASURE THEORY
QUALIFYING EXAM, SPRING 2018

Notation: L^p spaces are with respect to the Lebesgue measure m.

1. Let (X, Σ, μ) be a measure space (so, X is a set, Σ is a σ-algebra of subsets of X, and μ is a measure). Suppose $A_k \subset X$ for $k = 1, 2, \ldots, n$. Define the function $f\colon X \to \mathbb{R}$ as follows: $f(x)$ is the number of indices k such that $x \in A_k$.

(a) Prove that if each A_k is a measurable set, then f is a measurable function.

(b) If f is a measurable function, does it follow that each A_k is a measurable set? Prove or disprove.

2. Let $f(x) = 1/x$ for $x \in (0, 1)$. Show that there exists a sequence of Lebesgue integrable functions $f_k\colon (0,1) \to \mathbb{R}$ such that $f_k \to f$ in measure and $\int_{(0,1)} f_k = 0$ for all k.

3. Suppose that $f \in L^2((1, \infty))$. For every number $t > 0$ define

$$E_t = \{x \in (1, \infty) \colon |f(x)| > t\sqrt{x}\}$$

Prove that there exists a constant C such that $m(E_t) \leq C/t$ for all $t > 0$.

4. Consider the sequence of functions

$$f_k(x) = \frac{k}{1 + k^9 x^3}$$

on the set $[0, 1]$ equipped with the Lebesgue measure. Prove that the L^p norm of f_k tends to 0 if $1 \leq p < 3$ but not if $p \geq 3$.

A18

1a Let (X, Σ, μ) be a measure space. Suppose $A_k \subset X$ are meas. for $k = 1, 2, \ldots, n$

Define $f: X \to \mathbb{R}$ such that $f(x)$ is the number of indices k such that $x \in A_k$. To show f is measurable, I will show that $\{x \mid f(x) > \alpha\}$ is measurable, when A_k are measurable.

For $\alpha < 0$ $\{f(x) > \alpha\} = \{f(x) > \alpha\} = X$, which is measurable.

$x \in X = \left(\bigcup A_k\right) \cup X \setminus \left(\bigcup A_k\right)^c = \{x \mid x \in A_k \text{ for any \# of } k \text{ or no } k\}$

$= \{x \mid f(x) > -\varepsilon\} = \{x \mid f(x) > \alpha \text{ for } \alpha < 0\}$

$\varepsilon > 0$, arb small

For $0 \leq \alpha < 1$ $\{f(x) > \alpha\} = \bigcup_{k=1}^{n} A_k$ which is measurable (countable union of meas. sets)

$\{x \mid f(x) > \alpha\} = \{x \mid f(x) > 1 - \varepsilon\} = \{x \mid f(x) > 0\} = \{x \mid f(x) \geq 1\}$ ← b/c f is discrete

$= \{x \mid x \in A_k \text{ for at least one } k\} = \bigcup A_k$

For $j \leq \alpha < j+1$ $\{f(x) > \alpha\} = \bigcup_{B \in \mathcal{B}} \left(\bigcup_{k \in B} A_k\right)$ which is meas. again, $B \subset \{1, \ldots, n\}$ such that B contains $j+1$ elements. $\mathcal{B}$ is the collection of such sets, that is it is the collection of all $\binom{n}{j+1}$ sets which contain $j+1$ elements.

$\{x \mid f(x) > \alpha\} = \{x \mid f(x) \geq j+1\} = \{x \mid x \in A_k \text{ for } j+1 \text{ different } k\}$

$= \{x \mid x \in \bigcup_{k \in B} A_k \text{ for some } B \in \mathcal{B}\} = \bigcup_{B \in \mathcal{B}} \bigcup_{k \in B} A_k$

For $\alpha = n$ $\{f(x) > n\} = \emptyset$ which is measurable by def.

$\{x \mid f(x) > n\} = \emptyset$ b/c there are only n A_k's.

Thus, If A_k is measurable, f is measurable. ☑

1b. Suppose A_k and f are defined as above.

I posit that the claim is false. That is, the measurability of f defined does not force each A_n to be measurable.

Consider $A_1 = V \cap [0,1]$ and $A_2 = \mathbb{Q} \cap [0,1]$, $X = [0,1]$

Then $X = A_1 \cup A_2$, $X \setminus (A_1 \cup A_2) = \emptyset$ so $\forall x \in X$

So $f(x) = 1$ $\forall x \in X$, X is measurable. So

$\{f(x) > 1\} = \emptyset$

$\{f(x) > \alpha\} = X$ $\quad 0 \leq \alpha < 1$

$\{f(x) \leq 0\} = \emptyset$

which are all measurable, so f is measurable by definition,

However A_1 is not, by assumption.

* Can be generalized for disjoint A_n s.t. A_k is not measurable for at least one k, $\bigcup A_n = X$.

2. Let $f(x) = 1/x$ for $x \in (0,1)$. Consider

Let $$f_k = f \cdot \chi_{(1/k,1)} - k\left(\int_{1/k}^{1} f\right) \cdot \chi_{[0,1/k]}$$ → scales fns

Then $$\int_0^1 f_k = \int_0^1 f \cdot \chi_{(1/k,1)} - \int_0^1 \left(k\left(\int_{1/k}^{1} f\right) \cdot \chi_{[0,1/k]}\right)$$

$$= \int_{1/k}^{1} f - \int_{1/k}^{1} f \cdot \left(k \cdot (1/k - 0)\right) = 0 \quad \forall k.$$

However.
$$\lim_{k\to\infty} \left| \left\{ x \;\middle|\; |f_k(x) - f(x)| > \varepsilon \right\} \right| = 0$$

Since $f_k \longrightarrow f$ pointwise on $(0,1)$:

$$\lim_{k\to\infty} f_k(x) = f \cdot \chi_{(0,1)} - 0 = f(x)$$

3 Suppose that $f \in \mathcal{L}^2((1,\infty))$. For every number $t > 0$ define

$$E_t = \{ x \in (1,\infty) \mid |f(x)| > t\sqrt{x} \}.$$

Rewriting E_t: $E_t = \{x \in (1,\infty) \mid -t < \frac{f(x)}{\sqrt{x}} < t\} = \{x \in (1,\infty) \mid \frac{|f(x)|}{\sqrt{x}} > t\}$

By Chebyshev's inequality:

$$|E_t| = \left| \left\{ \frac{|f(x)|}{\sqrt{x}} > t \right\} \right| \leq \frac{1}{t} \cdot \int_1^\infty \frac{|f(x)|}{\sqrt{x}}$$

Applying holder's inequality, using that $f \in \mathcal{L}^2(1,\infty)$

$$\int_1^\infty \frac{|f(x)|}{\sqrt{x}} \leq \left(\int_1^\infty |f(x)|^2 \right)^{1/2} \left(\int_1^\infty \left(\frac{1}{\sqrt{x}}\right)^2 \right)^{1/2} < \infty$$ as each piece is $< \infty$

Thus $\int_1^\infty \frac{|f(x)|}{\sqrt{x}} \leq C < \infty$.

Therefore

$$|E_t| \leq \frac{1}{t} C$$

4 Consider $f_k = \frac{k}{1+k^9x^3}$ on $[0,1]$ equipped with Lebesgue measure

Beginning by examining the denominator, $1+k^9x^3$ on $[0,1]$. Since $1, k^9x^3 \geq 0$

$$\max\{1, k^9x^3\} \leq 1+k^9x^3 \leq 2\max\{1, k^9x^3\}$$

To examine $\|f\|_p$, we consider the case where $1 < k^9x^3$ and the case where $1 > k^9x^3$. That is for $x \in [0, 1/k^3)$ and $x \in (1/k^3, 1]$.

$\Big($For $x \in [0, 1/k^3)$ $k^9x^3 \leq k^9 \cdot \frac{1}{k^9} = 1$, $x \in (1/k^3, 1]$ $k^9x^3 > k^9 \, 1/k^9 = 1\Big)$

So for $[0, 1/k^3)$: $1 \leq 1+k^9x^3 \implies f_k = \frac{k}{1+k^9x^3} < \frac{k}{1} = k$

$\implies f_k^p < k^p \quad \forall p$

Thus

$$\int_{[0,1/k^3)} |f_k|^p = \int_0^{1/k^3} f_k^p \leq \int_0^{1/k^3} k^p = k^p \cdot \frac{1}{k^3} = k^{p-3} \xrightarrow{k\to\infty} 0 \text{ iff } p-3<0 \text{ iff } p<3$$

for $(1/k^3, 1]$: $k^9x^3 < 1+k^9x^3 \longrightarrow f_k = \frac{k}{1+k^9x^3} < \frac{k}{k^9x^3} = \frac{1}{k^8x^3}$

$\implies f_k^p \leq \left(\frac{1}{k^8x^3}\right)^p$

b/c $p \geq 1$

$$\int_{(1/k^3,1]} f_k^p \leq \int_{1/k^3}^1 \frac{1}{k^{8p}x^{3p}} = k^{-8p}\left[\frac{x^{1-3p}}{1-3p}\right]_{x=1/k^3}^1 = \frac{k^{-8p} - k^{-8p}\left(\frac{1}{k^3}\right)^{1-3p}}{1-3p} = \frac{k^{-8p} - k^{p-3}}{1-3p}$$

$\longrightarrow \|f_p\|^p = \int_{[0,1/k^3)} f_k^p + \int_{[1/k^3,1]} f_k^p \longrightarrow 0$ iff $1 \leq p < 3$

Since $1+k^9x^3 < 2$ on $[0, 1/k^3)$ $f_k^p \geq \left(\frac{k}{2}\right)^p \longrightarrow \|f_k\|_p^p = \int f_k^p \geq \int \left(\frac{k}{2}\right)^p = \frac{1}{2^p} k^{p-3} \not\to 0$ $p \geq 3$

Since $\int$ over $[1/k^3, 1] \geq 0$, need to see $\|f_k\|_p \not\to 0$ when $p \geq 3$. It doesn't. ☑

Qualifying Exam, Complex Analysis, January 12, 2018

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$.

1. Find a conformal map from the strip $\{0 < \operatorname{Im} z < \pi\}$ onto Δ.

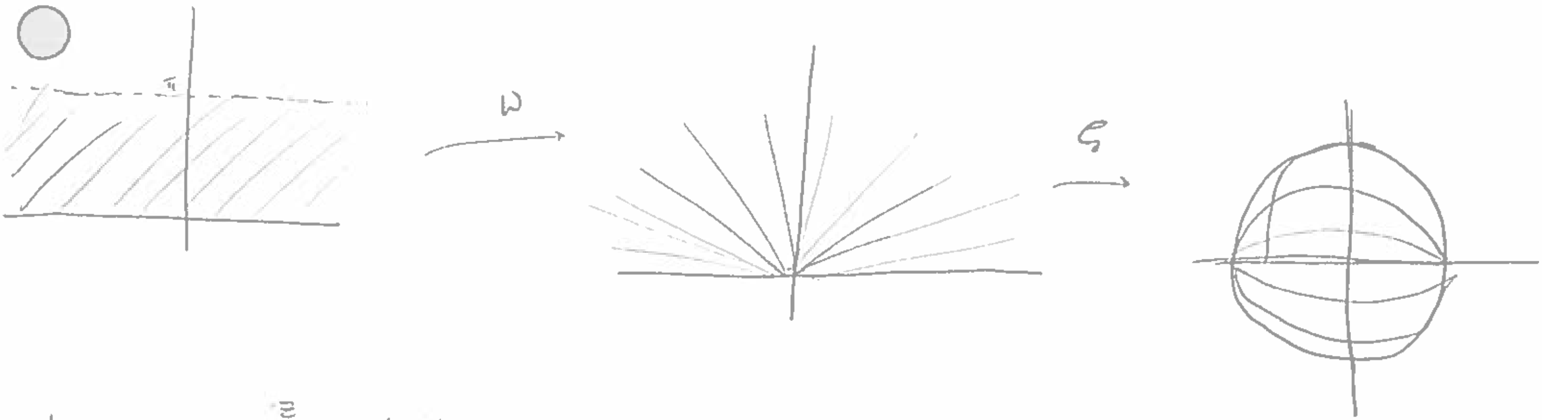
2. Find $\displaystyle\int_{|z|=7} \frac{\sin z}{4z^2 - \pi^2}\, dz$.

3. Let f be a non-constant entire function. Show that the function e^f has an isolated essential singularity at infinity.

6.2.12 HW 7

4. If $f(z) = \dfrac{1+z^2}{1-z^2}$, find $f(\Delta)$.

1. Find a conformal map from $\{0 < \operatorname{Im} z < \pi\}$ onto Δ.



Let $w = e^{z} = e^{x} e^{iy}$

$w : \{0 < \operatorname{Im} z < \pi\} \longrightarrow \mathbb{H}^+$

Let $\varsigma = \dfrac{w - i}{w + i} : \mathbb{H}^+ \longrightarrow \mathbb{D}$

Consider $\varsigma(w(z)) = \dfrac{e^{z} - i}{e^{z} + i}$

2. Find $\int_{|z|=7} \frac{\sin z}{4z^2-\pi^2}$

Let $f(z) = \frac{\sin z}{4z^2 - \pi^2}$

f has poles at

$$4z^2 - \pi^2 = (2z-\pi)(2z+\pi) = 0$$

$$z_1 = \pi/2 \ , \ z_2 = -\pi/2$$

Since $|z_1| = |z_2| = \pi/2 < 7$, $z_1, z_2 \in |z| \leq 7$.

Thus.

$$\int_{|z|=7} \frac{\sin z}{4z^2-\pi^2} = \left(\text{Res}[f, z_1] + \text{Res}[f, z_2]\right) 2\pi i$$

$$\text{Res}[f, z_1] = \lim_{z \to \pi/2} \frac{\sin z}{2z+\pi} = \frac{\left(e^{i\pi/2} - e^{-i\pi/2}\right)/2i}{2\pi} = \frac{(i - -i)}{4i\pi} = \frac{2i}{4i\pi} = \frac{1}{2\pi}$$

$$\text{Res}[f, z_2] = \lim_{z \to -\pi/2} \frac{\sin z}{2z-\pi} = \frac{\left(e^{-i\pi/2} - e^{i\pi/2}\right)/2i}{-2\pi} = \frac{-2i}{-4i\pi} = \frac{1}{2\pi}$$

$$\int_{|z|=7} \frac{\sin z}{4z^2-\pi^2} = \frac{1}{2\pi} + \frac{1}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}.$$

3. Let f be a non constant entire function. By the converse of Liouville's theorem f is unbounded. Consider e^f. Since f is entire, e is entire, e^f is entire. Thus $\forall R > 0$ $f(z)$ is analytic for $|z| > R$.

Thus, e^f has an isolated singularity at infinity. By definition a function $g = \sum_{-\infty}^{\infty} b_k z^k$ has an essential singularity at ∞ if $b_k \neq 0$ for infinitly many b_k $k > 0$. ~~Recall $e^z = \sum_{k=0}^{\infty} z^k/k!$.~~ Since f is entire it can be written as a power series $f(z) = \sum a_k z^k$.

Suppose the isolated singularity @ ∞ is removable. Then

$$\lim_{z \to \infty} e^{f(z)} < \infty \overset{*}{\Longrightarrow} \lim_{z \to \infty} f(z) < \infty \overset{**}{\Longrightarrow} f(z) = C \text{ by Liouvilles } \lightning.$$

(* e^z is unbounded, so if $e^f < \infty$, f must be bounded)

(** by Liouvilles, f is a bdd entire fn & is thus constant)

Suppose the isolated singularity @ ∞ is a pole, so $\lim_{z \to \infty} e^{f(z)} = \infty$

~~Consider $g(z) = 1/e^{f(1/z)}$. Then $\lim_{z \to 0} g(z) = 0$ so it has a removable singularity at 0. That is $e^{f(1/z)}$ has a pole at $z = 0$.~~

~~Since f is entire, $\exists$ $z_n \to 0$ s.t. $f(z_n) \to 0 \to e^{f(z_n)} \to 1$~~, not infinity

~~So $e^{f(z)}$ does not have a pole.~~

L, ↳ ~~By Piccard's Theorem~~

If e^f has a pole $|e^f| \underset{z \to \infty}{\longrightarrow} \infty \Rightarrow e^{-f} \underset{z \to \infty}{\longrightarrow} 0$

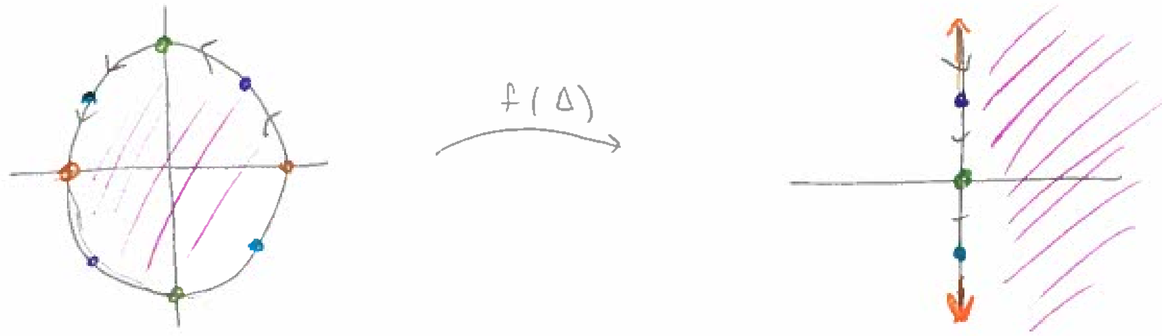
$\to$ e^{-f} is bdd, entire $\to$ constant

$e^{-f} = \lambda \to -f'e^{-f} = 0 \to -f' = 0$

$\to f$ is constant $\lightning$.

Thus e^f has an essential singularity @ ∞.

4. Let $f(z) = \frac{1+z^2}{1-z^2}$.



$f(1) = \infty = f(-1)$

$f(i) = 0 = f(-i)$

$f(e^{i\pi/4}) = 1+i/1-i \cdot \frac{1+i}{1+i} = \frac{1+2i-1}{1+1} = i = f(e^{i5\pi/4})$

$f(e^{i3\pi/4}) = 1-i/1+i \cdot \frac{1-i}{1-i} = \frac{1-2i-1}{1+1} = -i = f(e^{i7\pi/4})$

$f(\Delta)$ = the right half plane

Qualifying Exam, Real Analysis and Measure Theory
January, 2018

1. (a) Give an example that shows that the image of a measurable set under a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ may not be measurable.

   (b) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function; that is, there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \qquad \text{for every } x, y \in \mathbb{R}^n .$$

   Prove that f maps measurable sets into measurable sets.

2. Suppose that $E \subset \mathbb{R}^n$ has a finite measure and f is a measurable function on E. Prove that $f \in L^1(E)$ if and only if $\sum_{j=0}^{\infty} 2^j |\{x \in E\colon |f(x)| \geq 2^j\}| < \infty$.

3. Let $1 < p < \infty$ and $p' = p/(p-1)$ Suppose that $f, g\colon E \to [0, \infty]$ and not identically 0 (i.e., neither function equals 0 a.e.) such that $f \in L^p(E)$ and $g \in L^{p'}(E)$. Prove that the equality

$$\left|\int_E fg\right| = \left(\int_E f^p\right)^{\frac{1}{p}} \left(\int_E g^{p'}\right)^{\frac{1}{p'}}$$

   holds if and only if f^p is a multiple of $g^{p'}$ a.e.

4. (a) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support. Fix $K \in \{1, 2, \dots\}$, and divide $\mathbb{R}^n$ into cubes $\{Q_\alpha\}_{\alpha=1,2,\dots}$, $\mathbb{R}^n = \bigcup_{\alpha=1}^{\infty} Q_\alpha$, each of volume $|Q_\alpha| = 1/K$. Define

$$f_K(x) = \frac{1}{|Q_\alpha|} \int_{Q_\alpha} f(y)\, dy \qquad \text{for } x \in Q_\alpha .$$

   Show that $f_K \to f$ in $L^1(\mathbb{R}^n)$ as $K \to \infty$.

   (b) Does the statement in (a) hold for $f \in L^1(\mathbb{R}^n)$? Justify your answer.

1.a Consider $\varphi(x) = x + \mathcal{C}(x)$ where $\mathcal{C}(x)$ is the Cantor-Lebesgue function which is a continuous and strictly increasing fn. such that $|\varphi(C)| > 0$ for cantor set C. Since $|\varphi(C)| > 0$ we know $\exists$ some nonmeas set $E \subset \varphi(C)$. Since φ is increasing and continuous, we know φ^{-1} exists and is continuous. Since $\varphi^{-1}(E) \subset C$, by monotonicity $|\varphi^{-1}(E)|_e = 0$ and thus $\varphi^{-1}(E)$ is measurable. However, $\varphi(\varphi^{-1}(E)) = E$ is not measurable by construction ☑

b. Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a Lipschitz continuous function. So $\exists\ M > 0$ s.t. $|f(x) - f(y)| \leq M|x - y| \quad \forall\, x, y \in \mathbb{R}^n$.

We first show f maps F_σ sets to F_σ sets.:

For any continuous f, f maps compact sets to compact sets. We know every closed set can be written as the countable union of compact sets. So for some F_σ set H

$$H = \left(\bigcup_{j=1}^{\infty} F_j\right) = \bigcup_{j=1}^{\infty}\left(\bigcup_{k=1}^{\infty} K_{j,i}\right) \quad \text{Where } F_j \text{ are closed and } \left(F_j = \bigcup_{i=1}^{\infty} K_{j,i}\right)$$

Then $f(H) = f\left(\bigcup_{j=1}^{\infty} F_j\right) = \bigcup_{j=1}^{\infty} f(F_j) = \bigcup_{j=1}^{\infty} f\left(\bigcup K_{j,i}\right) = \bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} \overset{\text{comp}}{\widetilde{f(\ldots}}$

which is an F_σ set.

Next, we consider some Z s.t. $|Z| = 0$. Since $|Z|_e = 0$ $Z \subset \{I_k\}$ s.t. $\overset{\forall \varepsilon > 0}{\sum} |I_k| < \varepsilon$.

By monotonicity, continuity of f $|f(Z)| \leq |f(\bigcup I_k)| \leq |\bigcup f(I_k)| \overset{\text{disjoint}}{\leq} \sum |f(I_k)| < \sum M| \ldots < M\ldots$

So $|f(Z)| = 0$

Since E is meas $E = H \cup Z$, $f(E) = f(H) \cup f(Z)$ which is the union of 2 measurable sets and is thus measurable. ☑

2. Suppose $E \subset \mathbb{R}^n$ has a finite measure and f is a measurable function.

Consider the sets $E_k = \{2^k \le |f| < 2^{k+1}\}$ and $g = \sum_{k=0}^{\infty} 2^k \chi_{E_k}$

Let $\mathcal{E} = \bigcup_{k=0}^{\infty} E_k$. Then $E \setminus \mathcal{E} = \{0 \le |f| < 1\}$.

Observe $g \le f \le 2g$ on each E_k. It follows that

$$\int g \le \int f \le \int 2g \implies \int \sum_{k=0}^{\infty} 2^k \chi_{E_k} \le \int f \le \int \sum 2^{k+1} \chi_{E_k}$$

$$\sum_{k=0}^{\infty} 2^k |E_k| \le \int f \le \sum_{k=0}^{\infty} 2^{k+1} |E_k|$$

Let $F_j = \{x \in E \mid |f(x)| \ge 2^j\} = \bigcup_{k=j}^{\infty} E_k$, a disjoint union.

$$|F_j| = \sum_{k=j}^{\infty} |E_k|$$

Note $\sum_{j=0}^{\infty} 2^j \left(\sum |E_k|\right) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} 2^j |E_k| = \sum |E_k| \sum_{j=0}^{k} 2^j = \sum_{k=0}^{\infty} (2^{k+1} - 1) |E_k|$

$\Longleftarrow$ Suppose $\sum_{j=0}^{\infty} 2^j |F_j| < \infty$. Since $E_k \subset F_k \ \forall k$ we know $|E_k| \le |F_k|$

Since $\mathcal{E} = \bigcup E_k$ is a disjoint union,

$$\int_{\mathcal{E}} |f| = \sum_{k=0}^{\infty} \int_{E_k} |f| \le \sum_{k=0}^{\infty} 2^{k+1} |E_k| \le 2 \sum_{k=0}^{\infty} 2^k |F_k| < \infty$$

$$\int_{E \setminus \mathcal{E}} f \le |E \setminus \mathcal{E}| \cdot 1 \le |E| < \infty$$

So $\int_E f < \infty \rightarrow f \in \mathcal{L}(E)$

$\Longrightarrow$ Suppose $\int_E f < \infty$. As asserted above $\sum 2^j |F_j| = \sum (2^{k+1} - 1)|E_k| \le \sum 2^{k+1} |E_k|$ (−1 omitted)

$$= 2 \sum_{k=0}^{\infty} 2^k |E_k| \le 2 \sum_{k=0}^{\infty} \int_{E_k} |f| = 2 \int_{\mathcal{E}} |f| \le 2 \int_E |f| < \infty$$

$2^k \chi_{E_k} < f$ on E_k $\mathcal{E} \subset E$

3 Let $1 < p < \infty$, $p' := p/p-1$ (or equiv $\frac{1}{p} + \frac{1}{p'} = 1$). Suppose $f, g : E \to [0, \infty]$ not $\equiv$ s.t. $f \in \mathcal{L}^p$, $g \in \mathcal{L}^{p'}$.

Suppose $$\left| \int_E fg \right| = \left(\int_E f^p \right)^{1/p} \left(\int_E g^{p'} \right)^{1/p'}$$

$\Longrightarrow$ Let $F = \frac{f}{\|f\|_p}$ and $G = \frac{g}{\|g\|_{p'}}$. Note $\|F\|_p = \|G\|_{p'} = 1$

Because $$\|F\|_p = \left(\int \left| \frac{f}{\|f\|_p} \right|^p \right)^{1/p} = \left(\frac{1}{\|f\|_p^p} \int |f|^p \right)^{1/p} = \frac{1}{\|f\|_p} \|f\|_p = 1$$

Clearly $F, G \geq 0$ since $f, g \geq 0$, F, G not identically 0

Since equality holds by assumption

$$\int_E FG = \|F\|_p \|G\|_{p'} = 1 = \frac{1}{p} + \frac{1}{p'} = \frac{\|F\|_p^p}{p} + \frac{\|G\|_{p'}^{p'}}{p'}$$

$$\int_E FG = \frac{1}{p} \|F\|_p^p + \frac{1}{p'} \|G\|_{p'}^{p'} = \left(\frac{1}{p} \int |F|^p + \frac{1}{p'} \int |G|^{p'} \right) \star$$

By Young's inequality $FG \leq \frac{F^p}{p} + \frac{G^{p'}}{p'}$

Integrating we get: $\int FG \leq \frac{1}{p} \int F^p + \frac{1}{p'} \int G^{p'} < \infty$ b/c $\|F\|_p, \|G\|_{p'} < \infty$

B/c $\int FG < \infty$ we can subtract it from both sides of $\star$.

iff $$0 = \frac{1}{p} \int F^p + \frac{1}{p'} \int G^{p'} - \int FG = \int_E \left(\frac{F^p}{p} - \frac{G^{p'}}{p'} - FG \right)$$

iff a.e. E $0 = \frac{F^p}{p} - \frac{G^{p'}}{p'} - FG \longrightarrow FG = \frac{F^p}{p} - \frac{G^{p'}}{p'}$ (Youngs equality).

We know equality holds iff $F^p = G^{p'}$ a.e., i.e.

$$\frac{f^p}{\|f\|_p^p} = \frac{g^{p'}}{\|g\|_{p'}^{p'}} \longrightarrow f^p = k g^{p'} \quad k = \|f\|_p^p / \|g\|_{p'}^{p'} \quad \text{a.e.}$$

$\Leftarrow$ Suppose $f^p = k g^{p'}$

$$\left(\int f^p\right)^{1/p}\left(\int g^{p'}\right)^{1/p'} = \left(\int f^p\right)^{1/p}\cdot\left(\int \tfrac{1}{k} f^p\right)^{1-1/p} = \left(\tfrac{1}{k}\right)^{1/p'}\left(\int f^p\right)^{1-1/p+1/p}$$

$$= \left(\tfrac{1}{k}\right)^{1/p'}\cdot\int f^p = \left(\tfrac{1}{k}\right)^{1/p'}\cdot\int f\cdot\frac{f^p}{f} = \left(\tfrac{1}{k}\right)^{1/p'}\cdot\int f\cdot\left(f^{p-1}\right)$$

$$= \left(\tfrac{1}{k}\right)^{1/p'}\int f\cdot\left(f^p\right)^{1-1/p} = \left(\tfrac{1}{k}\right)^{1/p'}\int f\cdot\left(f^p\right)^{1/p'} = \int f\cdot\left(\tfrac{1}{k} f^p\right)^{1/p'} = \int f\left(g^{p'}\right)^{1/p'}$$

$$= \int fg. \checkmark$$

4. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support

Call the support of f A. Note $f(\mathbb{R}^n \setminus A) = 0$, $f(A) \neq 0$, A is compact.

Fix $k \in \{1, 2, \dots\}$ and divide $\mathbb{R}^n$ into cubes (which are closed, per Wheeden & Zygmund's construction) $\{Q_\alpha\}_{\alpha=1}^{\infty}$, $\mathbb{R}^n = \bigcup Q_\alpha$ and $|Q| = 1/k$

Define $f_k = \frac{1}{|Q_k|} \int_{Q_k} f(y)\,dy$. Note since $C_c(X)$ is dense in $\mathcal{L}^1(X)$, $f \in \mathcal{L}^1(\mathbb{R}^n)$

Consider $\|f_k - f\|_1 = \int_{\mathbb{R}^n} \left| \left(\frac{1}{|Q_\alpha|} \int_{Q_\alpha} f(y)\,dy \right) - f(x) \right| dx$

$$= \int_{\mathbb{R}^n} \left| \frac{1}{|Q_\alpha|} \int_{Q_\alpha} f(y)\chi_{S_\alpha}\,dy - \frac{1}{|Q_\alpha|} |Q_\alpha| f(x) \cdot \chi_A \right| dx$$

$$= \int_{\mathbb{R}^n} \left| \frac{1}{|Q_\alpha|} \int_{Q_\alpha} (f(y) - f(x))\chi_{S_\alpha}\,dy \right| dx$$

($\cdot \chi_{S_\alpha}$)

Could use work if time → problem w/ last bit, bit

Let $S_\alpha = Q_\alpha \cap A$, note $f(Q_\alpha \setminus S_\alpha) = 0$, $f(y) - f(x)\big|_{Q_\alpha \setminus S_\alpha} = 0$. Note S_α is the intersection of 2 sets and is thus compact. Recall $f(\mathbb{R}^n \setminus A) = 0$, A compact

$$\|f_k - f\|_1 = \int_A \left| \frac{1}{|Q_\alpha|} \int_{S_\alpha} f(y) - f(x)\,dy \right| dx = \int_A \left| k \int_{S_\alpha} f(x) - f(y)\,dy \right| dx$$

If $S_\alpha = \emptyset$, then the middle integral is $0 < \infty$. Suppose $S_\alpha \neq \emptyset$. Since f is cts and S_α is compact f is uniformly continuous on S_α (and Q_α for that matter) Further note $S_\alpha \subset Q_\alpha$ so $|S_\alpha| \leq |Q_\alpha|$ by monotonicity of measure $\exists N$ s.t. for $k > N$,
$|Q_\alpha| < 1/k \Rightarrow |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ (unif. cts).

$$\|f_k - f\|_1 = \int_A \left| k \int_{S_\alpha} f(x) - f(y)\,dy \right| dx \leq \int_A k \int_{S_\alpha} |f(x) - f(y)|\,dy\,dx < \int_A k \int_{S_\alpha} \varepsilon\,dy\,dx$$

$$= \int_A \frac{1}{|Q_\alpha|} \cdot \varepsilon |S_\alpha| < \int_A \varepsilon = \varepsilon |A| < \infty \quad \square$$

A is cmpct $\to$ closed $\to$ meas.

4b. No. Without compact support we don't know that $|A| < \infty$

This fact comes from A being closed & bdd $\to |A| < \infty$. But

if $f = \sum_{j=0}^{\infty} 2^j \chi_{[j-1, j]}$ which is cts a.e.

↳ This is heuristic justification

↳ ready to move on, will come back if time.

Qualifying Exam, Complex Analysis, August 2017

Notation: Throughout the exam $\mathbb{D}$ denotes the open unit disc.

1. Find a conformal map from the sector $D = \{z \in \mathbb{C} : 0 < \operatorname{Arg} z < \frac{\pi}{4}\}$ onto $\mathbb{D}$.

2. How many zeros (counted with multiplicity) does the function

$$f(z) = e^z - 4z^2 + 3z + 1$$

have in the disc $\{z \in \mathbb{C} : |z| < 2\}$?

Rouche's → pg 230. → bd pwr series

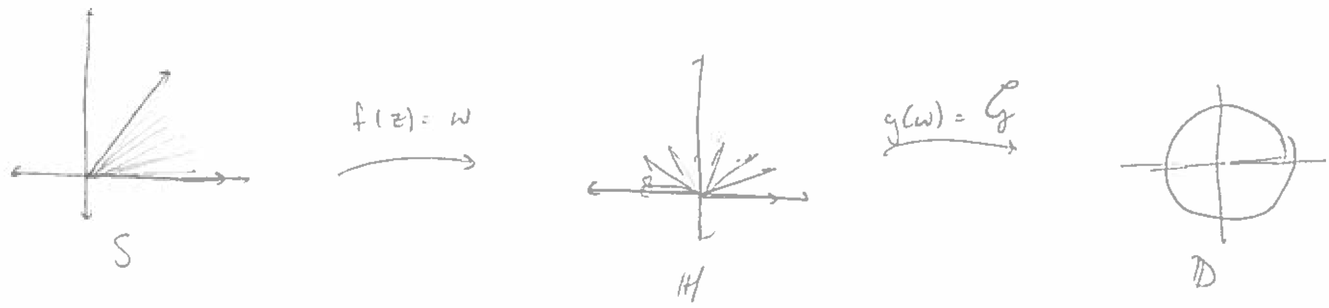
3. Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in D$, and $f : D \longrightarrow D$ be a holomorphic function such that $f(z_0) = z_0$. Show that $|f'(z_0)| \leq 1$.

4. Let $f_n : \overline{\mathbb{D}} \to \mathbb{C}$, $n \geq 1$, be continuous functions which are holomorphic on $\mathbb{D}$. Assume that $f_n(0) = 0$ and that the real part functions $u_n = \mathbf{Re}\, f_n$ converge uniformly on the unit circle $\partial\mathbb{D}$ as $n \to \infty$ to a function u. Show that the sequence $\{f_n\}$ converges normally on $\mathbb{D}$ to the function

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, u(e^{i\theta})\, d\theta\,.$$

formula (274)

1. Find a conformal map from sector $D = \{ z \in \mathbb{C} : 0 < \operatorname{Arg} z < \pi/4 \}$ onto $\mathbb{D}$



$$f(z) = z^{\pi/\alpha} = z^{\pi/\pi/4} = z^4 = w$$

$$g(w) = \frac{w-i}{w+i} \implies g(z) = \frac{z^4 - i}{z^4 - i} \ ; \ g : S \longrightarrow \mathbb{D}$$

2. How many zeros (counted w/ multiplicity) does the function

$$f(z) = e^z - 4z^2 + 3z + 1$$

have in the disc $\{z \in \mathbb{C} : |z| < 2\}$

Let $g(z) = e^z + 3z + 1$ and $h(z) = -4z^2$

Note, on $|z| = 2$:

$$|g(z)| = |e^z + 3z + 1| \le |e^z| + 3|z| + 1 \le e^{|z|} + 3|z| + 1 = e^2 + 3 \cdot 2 + 1 = e^2 + 7 < 3^2 + 7$$ (since ↓)

$$|h(z)| = |-4z^2| = 4|z|^2 = 4(2)^2 = 16$$

Thus $|g(z)| < |h(z)|$ on $|z| = 2$

Therefore, by Rouche's theorem, in $\{z \in \mathbb{C} : |z| < 2\}$, $f = g + h$ has the same number of zeros as h. That is, f has one zero of multiplicity 2.

3. Let $D \subset \mathbb{C}$ be a bdd domain, $z_0 \in D$, and $f: D \to D$ be a holomorphic fn. s.t. $f(z_0) = z_0$.

Show that $|f'(z_0)| \leq 1$.

pf. Suppose not, that is, suppose $|f'(z_0)| > 1$. Consider the iterative sequence of functions:

$$f_n(z) := \overbrace{f \circ f \circ \ldots \circ f}^{n \text{ times}}(z)$$

Observe, since $f(z_0) = z_0$, $f_n(z_0) = f(f(f(\ldots f(z_0)))) = z_0 \; \forall n$. and $f_n: D \to D$. Since D is a bounded domain $D \subset D(0, M)$ for some $M > 0$. Then $|f_n(z)| < M$.

Taking the derivative using the chain rule.

$$f_n'(z_0) = f'(z_0) \cdot (\overbrace{f \circ f \circ f \ldots \circ f}^{n-1}(z_0))' = \overbrace{f'(z_0) \cdot f'(z_0) \cdots f'(z_0)}^{n} = (f'(z_0))^n$$

Since $|f'(z_0)| > 1$ by assumption

$$\lim_{n \to \infty} f_n'(z_0) = \lim_{n \to \infty} (f'(z_0))^n = \infty.$$

Choose $\delta > 0$ s.t. $B(z_0, \delta) \subset D$, $\overline{B(z_0, \delta/2)} \subset D$.

Since f_n is analytic for $|z - z_0| \leq \delta/2$. Since $|f_n| \leq M$ for $|z - z_0| = \delta/2$, then, by the Cauchy Estimate

$$|f_n^{(1)}(z_0)| \leq \frac{1!}{\delta/2} M = \frac{2M}{\delta} < \infty \qquad \boxed{\Sigma.}$$ contradiction

Thus $|f'(z_0)| \leq 1$

4 Let $f_n : \overline{\mathbb{D}} \to \mathbb{C}$, $n \geq 1$, be continuous functions which are holomorphic on $\mathbb{D}$. Assume that $f_n(0) = 0$ and that the real part functions $u_n = \operatorname{Re} f_n$ converge uniformly on the unit circle $\partial\mathbb{D}$ as $n \to \infty$ to a function u. Show that the sequence $\{f_n\}$ converge normally on $\mathbb{D}$ to the fn

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(e^{i\theta})\, d\theta .$$

HW 10.1.4

Pf. Let $f_n : \overline{\mathbb{D}} \to \mathbb{C}$ $n \geq 1$ be cts fns which are holomorphic on $\mathbb{D}$. Assume that $f_n(0) = 0$ and that the real part functions $u_n \xrightarrow{\text{unif}} u$ on $\partial\mathbb{D}$. Let $\{f_n = u_n + iv_n\}$. Since $f_n(0) = 0 \to u_n(0) = 0$ and $v_n(0) = 0$ (equate real & imaginary). ($v_n \to v = 0$)

Since $u_n \to u$ uniformly on $\partial\mathbb{D}$, $\{u_n\}$ is uniformly Cauchy on $\partial\mathbb{D}$ (By the ✱ max principle $\{u_n\}$ is also uniformly Cauchy on $\overline{\mathbb{D}}$.) Since $\overline{\mathbb{D}}$ is compact / closed we know $\{u_n\}$ is uniformly convergent on $\overline{\mathbb{D}}$. Let $\{u_n\} \to w$ on $\overline{\mathbb{D}}$, $w\ \overline{\mathbb{D}} \to \mathbb{R}$.

Note $w|_{\partial\mathbb{D}} = u$ + w is harmonic on $\mathbb{D}$. By uniqueness of a harmonic extensions of a cts fn. on $\partial\mathbb{D}$ to $\mathbb{D}$, $w = \tilde{u}$ on $\mathbb{D}$.

Let $D \subset \mathbb{D}$ be any closed disk, D' be closed disk centered @ origin s.t. $D \subset D' \subset \mathbb{D}'$. Let ρ be radius of D'. $D' = \{|z| = \rho\}$, $0<\rho \leq 1$. Write $z = re^{i\varphi}$. Then for $0 \leq \theta \leq 2\pi$

$$\left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| = \left|\frac{e^{i\theta}-re^{i\varphi}}{e^{i\theta}-re^{i\varphi}}\right| = \left|\frac{1+re^{i(\varphi-\theta)}}{1-re^{i(\varphi-\theta)}}\right| \leq \frac{1+r}{1-r} \leq \frac{1+\rho}{1-\rho}$$

Since $\{u_n\}$ ~~conv unif~~ is Cu - unif. Cauchy on $\partial\mathbb{D}$ by def $\exists N_1$ s.t. $\forall n, m \geq N_1$.

$$|u_n(e^{i\theta}) - u_m(e^{i\theta})| < \varepsilon \left(\frac{1-\rho}{1+\rho}\right)$$

Since $\{u_n(0) = 0 \to v_n - v_m = 0\ \forall n, m$.

For $n, m \geq N$, $\forall z \in D$ it follows from Schwarz formula

$$|f_n(z) - f_m(z)| \leq \left|\int_0^{2\pi} (u_n(e^{i\theta}) - u_m(e^{i\theta}))\cdot \frac{e^{i\theta}+z}{e^{i\theta}-z}\frac{d\theta}{2\pi}\right| + 0$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |u_n(e^{i\theta}) - u_m(e^{i\theta})| \cdot \left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \varepsilon \cdot \frac{1-\rho}{1+\rho} \cdot \frac{1+\rho}{1-\rho}\, d\theta = \frac{1}{2\pi}\varepsilon$$

$\{f_n\} \xrightarrow{\text{unif}} f_0$ on D. For

NTS $f = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta}) dz$

Since $f = u(z) + iv(z)$ is analytic for $|z|<1$ → u extends to be cts on the closed disk.

$\{|z| \le 1\}$, $\tilde{u}$:

$$\tilde{u} = \operatorname{Re} \int_{-\pi}^{\pi} u(e^{i\theta}) \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \frac{d\theta}{2\pi} = \operatorname{Re} \int_0^{2\pi} u(e^{i\theta}) \frac{e^{i\theta}+z}{e^{i\theta}-z} \frac{d\theta}{2\pi}$$

expresses the Poisson integral $\tilde{u}(z)$ as the real part of an analytic fn. So

$$\tilde{f}(z) := \int_0^{2\pi} u(e^{i\theta}) \frac{e^{i\theta}+z}{e^{i\theta}-z} \frac{d\theta}{2\pi}$$

Since $\operatorname{Re}\tilde{f} = \operatorname{Re} f = u$, $f - \tilde{f}$ is a real valued fn, and must be constant.

~~By Mean value prop of harmonic fns~~ By assumption

$$f(0) = u(0) = 0$$

$$0 = \int_0^{2\pi} u(e^{i\theta}) \frac{e^{i\theta}+0}{e^{i\theta}-0} \cdot \frac{d\varphi}{2\pi} = u(0)$$

Since $f(0) - \tilde{f}(0) = i \operatorname{Im} f(0) = v(0)$:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}) \cdot \frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta$$

Used hw to get through this. Study Poisson integral / Schwartz formula more

Couldn't do this on Exam.

Aug 2017

QUALIFYING EXAM, Real Analyis and Measure Theory

Problem 1. Let $E \subset \mathbb{R}$ with positive Lebesgue measure, $|E| > 0$. Show that the set $\{x - y \mid x, y \in E\}$ contains an interval centered at the origin.

Problem 2. Let $E \subset \mathbb{R}^n$ have positive finite Lebesgue measure, $0 < |E| < \infty$, and let f be measurable on E, show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. Show by example that this is not true if $|E| = \infty$.

Problem 3. Suppose that $f(x)$ is Lebesgue measurable and finite a.e. on $(0,1) \subset \mathbb{R}$. If the function $g(x,y) = f(x) - f(y)$ is Lebesgue integrable over $(0,1) \times (0,1) \subset \mathbb{R}^2$, show that $f \in L^1(0,1)$.

see prev. exam.

Problem 4. Consider $\mathbb{R}^n$ with Lebesgue measure, $n \geq 3$, $1 < p < n$, and $p^* = \frac{np}{n-p}$. For $u \in L^{p^*}(\mathbb{R}^n)$ show that

$$\lim_{R\to\infty} R^{-p} \int_{R<|x|<2R} |u|^p dx = 0$$

↳ holder's w/ 1

1. Let $E \subset \mathbb{R}$ w/ positive Lebesgue measure, $|E| > 0$. Show the set $\{x-y \mid x, y \in E\}$ contains an interval centered at origin.

pf. Let $\varepsilon > 0$ be given. Since $|E| > 0$, $\exists$ open set $G \subset \mathbb{R}$ s.t. $E \subset G$ and $|G| \le (1+\varepsilon)|E|$. Since G is open, we can write G as a countable union of nonoverlapping intervals $\{I_k\}$; $G = \cup I_k$.

Let $E_k = E \cap I_k$, $\forall k$. Since any two I_k's have at most one pt in common, any 2 E_k's have at most one point in common. Hence

$$\sum_k |I_k| = |G| \le (1+\varepsilon)|E| = (1+\varepsilon)\sum |E_k| = \sum (1+\varepsilon)|E_k|$$

Thus $\exists$ k_0 s.t. $|I_{k_0}| \le (1+\varepsilon)|E_{k_0}|$. Let $I_{k_0} = I$ and $E_{k_0} = \mathcal{E}$

Then $|I| \le (1+\varepsilon)|\mathcal{E}|$. Let $\varepsilon = 1/3$. Then $|I| \le \frac{4}{3}|\mathcal{E}|$, $\frac{3}{4}|I| \le |\mathcal{E}|$

We claim that for any real # d, $|d| < \frac{1}{2}|I| \rightarrow d \in (-\frac{1}{2}|I|, \frac{1}{2}|I|)$

Let $\mathcal{E}_d = \mathcal{E} + d$ (translation by d). WTS $\mathcal{E}_d \cap \mathcal{E} \neq \phi$

Proc. by contradiction, suppose $\mathcal{E}_d \cap \mathcal{E} = \phi$. Consider $\mathcal{E}_d \cup \mathcal{E}$.

$\mathcal{E}_d \cup \mathcal{E} \subset I_d \cup I$

$$2|\mathcal{E}| = |\mathcal{E}_d| + |\mathcal{E}| \overset{\text{disjoint}}{=} |\mathcal{E}_d \cup \mathcal{E}| \le |I_d \cup I| = |I| + |d| \overset{b/c\ |d| < \frac{1}{2}|I|}{<} \frac{3}{2}|I| < 2|\mathcal{E}|$$

So $2|\mathcal{E}| < 2|\mathcal{E}| \rightarrow |\mathcal{E}| < |\mathcal{E}|$ Σ

Thus, $\mathcal{E}_d \cap \mathcal{E} \neq \phi \rightarrow x \in \mathcal{E}_d \cap \mathcal{E} \rightarrow x = y + d$ for $x, y \in \mathcal{E} \subset E$
$\rightarrow x - y = d$ for $x, y \in E \rightarrow d \in \{x-y \mid x, y \in E\}$

Thus, $(-\frac{1}{2}|I|, \frac{1}{2}|I|) \subset \{x-y \mid x, y \in E\}$ ☑

2. Let $E \subset \mathbb{R}^n$ have positive finite Lebesgue measure, $0 < |E| < \infty$ and let f be measurable on E. Show that $\lim_{p\to\infty} \|f_A\|_p = \|f\|_\infty$. Show by example that this is not true if $|E| = \infty$.

(8.1 WZ)

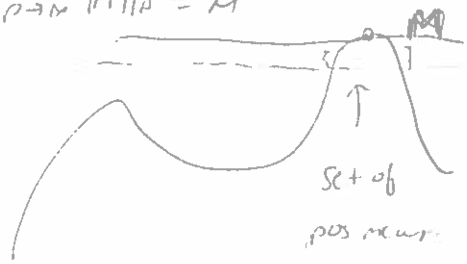
pf. Let $E \subset \mathbb{R}^{n}$ w/ pos. Lebesgue measure s.t. $0 < |E| < \infty$, and let f be meas. on E.

Let $M = \text{ess sup } f = \|f\|_\infty$. If $M' < M$, then the set $A = \{x \in E : |f(x)| > M'\}$ has positive measure. Moreover $\|f\|_p \geq \left(\int_A |f|^p\right)^{1/p} \geq M' |A|^{1/p}$.

Since $\lim_{p\to\infty} |A|^{1/p} = 1$, $\liminf_{p\to\infty} \|f\|_p \geq M'$, so $\liminf_{p\to\infty} \|f\|_p \geq M$. ⊗

However, $\|f\|_p \leq \left(\int_E M^p\right)^{1/p} = M |E|^{1/p}$. Therefore, $\limsup_{p\to\infty} \|f\|_p \leq M$

Therefore $\lim_{p\to\infty} \|f\|_p = M = \|f\|_\infty$



⊗ B/c $M' < M$, $M' = M - \varepsilon \rightarrow \liminf \geq M - \varepsilon \quad \forall \varepsilon > 0$

If $a \geq M - \varepsilon \quad \forall \varepsilon > 0 \implies a \geq M$ $\rightarrow \liminf \geq M$

CE. $|E| = +\infty$,

e.g. $f(x) = c$, $c \neq 0$, $E = (0, \infty)$

$\|f\|_\infty = c$, $\|f\|_p = \infty \rightarrow \neq$

3. See Jan 2017 #3

4 Consider $\mathbb{R}^n$ w/ Lebesgue measure, $n \geq 3$, $1 \leq p < n$ and $p^* = \frac{np}{n-p}$. For $u \in L^{p^*}(\mathbb{R}^n)$, show $\lim_{R \to \infty} \frac{1}{R^p} \int_{R<|x|<2R} |u|^p \, dx = 0$

pf. ~~Since~~ $u \in L^{p^*}(\mathbb{R}^n) \longrightarrow u^p \in L^{n/n-p}$

Let $p' = \frac{n}{n-p}$ Then $n-p > 0$, $1 < \frac{n}{n-p} = p'$.

Let $q' > 1$ satisfy $\frac{1}{p'} + \frac{1}{q'} = 1$ Then $1/q' = 1 - 1/p' = \frac{p'-1}{p'} = \frac{p'-1}{p'} = \frac{\left[\frac{n}{n-p}\right] - 1}{\left(\frac{n}{n-p}\right)} =$

By Hölder's inequality, $p^* = p \cdot \frac{n}{n-p}$

$$\int_{R<|x|<2R} |u|^p \leq \left(\int |u|^{p^*} \right)^{\frac{n-p}{n}} \left(\int 1^{n/p} \right)^{p/n}$$

$$\leq \left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{\frac{n-p}{n}} R^{p/n}$$

Let $\|u\|_{p^*} = \left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{1/p^*}$. Then $\left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{n-p/n} = \|u\|_{p^*}^{p/p^*} < \infty$

And

$$\frac{1}{R^p} \int_{R<|x|<2R} |u|^p \leq \|u\|_{p^*}^{p/p^*} R^{p/n - p} \xrightarrow[R \to \infty]{} 0 \quad \text{since } p/n < 1 < p$$

Qualifying Exam, Complex Analysis, January 2017

Directions: Attempt as many as you can of the following problems. Write neatly on one-sided sheets; explain; show work; justify your claims **(if you are using a theorem from class and/or the textbook, you must quote the theorem by its name, but you are not required to supply the theorem's proof). Write page numbers and remember to print your name.**

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$ with center at the origin, that is: $\Delta = \{z \, , \, |z| < 1\}$.

1. Let C denote the positively-oriented boundary of the domain

$$D = \left\{ z \in \mathbb{C}, \; -2 < \operatorname{Re} z < \frac{1}{2}, \quad |\operatorname{Im} z| < 2 \right\}.$$

Find

$$I = \int_C \frac{z^n}{z^4 - 1} \, dz$$

→ Aug 2016 #1.

where $n \geq 0$ is an integer. Write your answer in algebraic form "$I = a + ib$".

2. Find the domain of convergence and the sum of the following two power series. Explain.

(a.) $\sum_{k=1}^{\infty} k \cdot z^k$; (b.) $\sum_{k=1}^{\infty} k^2 \cdot z^k$

→ test radius? Yes
bdry? trivial
$|a_k| \to 0$

3. Evaluate the following integral. Explain and justify all your claims.

$$I = \int_{-\infty}^{+\infty} \frac{x^3 \sin x}{(x^2+1)^2} \, dx$$

4. Prove that there are *no*, non-constant polynomials of the form

(0.1) $$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

→ Aug 2016 #4

that satisfy

(0.2) $$|p(z)| < 1 \quad \text{when} \quad |z| = 1 \, .$$

B 1. See August 2016 # 1.

Contour integral, residue theorem.

2. Find the domain of convergence and the sum of the following two power series.

a. $\sum k z^k$: $\lim_{k\to\infty} \left|\frac{k+1}{k}\right| = 1 \longrightarrow |z|<1$ is domain.

$$\sum k z^k = z \sum_{k=1}^{\infty} k z^{k-1} = z \cdot \sum_{k=1}^{\infty} \frac{d}{dz} z^k = z\left(\frac{d}{dz} \sum z^k\right)$$

$$= z \frac{d}{dz}\left(\frac{1}{1-z}\right) = z \cdot \frac{-1}{(1-z)^2} = \frac{-z}{(1-z)^2}$$

b. $\sum k^2 z^k$: $\lim_{k\to\infty} \left|\frac{(k+1)^2}{k^2}\right| = 1 \longrightarrow |z|<1$

$$\sum k^2 z^k = z \cdot \sum k \cdot (k z^{k-1}) = z \sum k \left(\frac{d}{dz} z^k\right)$$

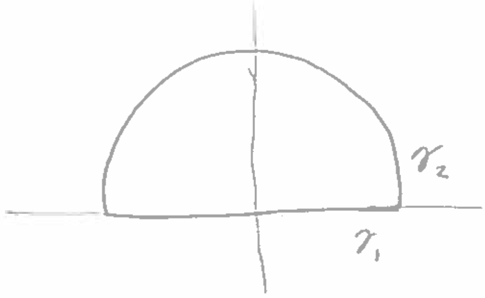
$$= z \cdot \frac{d}{dz}\left(\sum k z\right) = z \cdot \frac{d}{dz}\left(\frac{-z}{(1-z)^2}\right)$$

$$= z \cdot \frac{-(1-z)^2 - z\,2(1-z)}{(1-z)^4} = \frac{z(z-1) - 2z}{(1-z)^3}$$

$$= \frac{z^2 - 3z}{(1-z)^3}$$

3. Evaluate the integrals

$$I = \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}\, dx$$



Consider

$$\int_\Gamma f(z)\, dz \qquad f(z) = \frac{z^3 e^{iz}}{(z^2+1)^2} = \frac{z^3 \cos z}{(z^2+1)^2} + i\,\frac{z^3 \sin z}{(z^2+1)^2}$$

poles @ $z = i$, $-i$

$$\lim_{R\to\infty} \int_{\gamma_1} f(z)dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+1)^2} + i \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}$$

$$= \int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+1)^2}\, dx + i\, I$$

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\, dz \le \lim_{R\to\infty} \int_{\gamma_2} \left| \frac{z^3}{(z^2-1)^2} \right| |e^{iz}| \le \lim_{R\to\infty} \frac{R^3}{(R^2-1)^2} \int |e^{iz}||dz|$$

$$\sim \lim_{R\to\infty} \frac{1}{R}\, \pi = 0$$

$$\text{Res}[f, i] = \lim_{z\to i} \frac{d}{dz} \left(\frac{z^3 e^{iz}}{(z+i)^2} \right) = \lim_{z\to i} \frac{(z+i)^2(3z^2 e^{iz} + iz^3 e^{iz}) - z^3 e^{iz}(2(z+}{(z+i)^4}$$

$$= \frac{(2i)^2(3(i)^2 e^{-1} + e^{-1}) + i e^{-1}(4i)}{(2i)^4} = \frac{-4(-2e^{-1}) - 4e^{-1}}{16}$$

$$= \frac{4e^{-1}}{16} = \frac{1}{4e}$$

$$\int_\Gamma f(z)dz = 2\pi i \cdot \frac{1}{4e} = \frac{\pi i}{2e} = \lim_{R\to\infty} \int_\Gamma f(z)\, dz$$

Equate Real & Imaginary:

$$\boxed{I = \pi/2e}$$

4. See Aug 2016 #4

2017 January

Real Part

1. Let (X, Σ, μ) be a measure space. Show that a simple function $f = \sum_{j=1}^{n} v_j \chi_{E_j}$ is measurable if and only if all sets $E_j \in \Sigma$.

2. Compute the following limit and justify the calculation:

$$\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n}\, dx,$$

where x is the Lebesgue measure with respect to x.

3. Let f be a Lebesgue measurable function on the interval $[0, 1]$ and the measure of the set of $\{x : |f(x)| = \pm\infty\}$ is 0. If the function $g(x, y) = f(x) - f(y)$ is integrable on the unit square in $\mathbf{R}^2$ show that f is integrable on $[0, 1]$.

4. Let (X, Σ, μ) be a measure space and let $\{f_k\}$ be a sequence in $L^p(X, \mu)$, $1 \le p < \infty$, converging to f in $L^p(X, \mu)$, let $\{g_k\}$ be a sequence in $L^\infty(X, \mu)$, $\|g_k\|_\infty \le M$ for all k, converging to g in $L^\infty(X, \mu)$. Show that $f_k g_k \to fg$ in $L^p(X, \mu)$.

1. Let (X, Σ, μ) be a measure space. Show that a simple function

$f = \sum_{j=1}^{?} v_j \chi_{E_j}$ is measurable iff all sets $E_j \in \Sigma$.

$E_j \in \Sigma \Rightarrow E_j$ is meas.

pf.

Let (X, Σ, μ) be a measure space.

$\Rightarrow$ Proceeding by contra position. Suppose $E_j \in \Sigma$ are not measurable. Then, χ_{E_j} is unmeasureable and consequently f is not measurable. true?? Yes.

$\to \{x \mid f(x) = v_j\} = E_j$ is unmeas

Rick's Way:

$\Rightarrow$ f is measurable. Then this implies that, for each j, χ_{E_j} is meas. Si

$\{x \mid f = v_j\} = E_j$, this means that E_j is measurable $\forall j$.

Then $E_j \in \Sigma$.

$\Leftarrow$ Suppose all sets $E_j \in \Sigma$. Then E_j is meas then χ_{E_j} is meas

Then $f = \sum v_j \chi_{E_j}$ is measurable ☑

2. Compute the following limit and justify the calculation:

$$\lim_{n\to\infty} \int \frac{1+nx^2}{(1+x^2)^{-n}}\, dx$$

where x is the Lebesgue measure wrt x.

Consider the sequence of functions $f_n(x) = \dfrac{1+nx^2}{(1+x^2)}\, dx$

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} \overset{H}{=} \lim_{n\to\infty} \frac{2nx}{n(1+x^2)^{n-1} \cdot 2x} = \lim_{n\to\infty} \frac{+1}{(1+x^2)^{n-1}} = 0$$

Recall Bernoulli's inequality $(1+y)^r \geq 1+ry$. Let $r = n \geq 0, y = x^2 \geq 0$'s.

$$(1+x^2)^n \geq 1+nx^2 \longrightarrow 1 \geq \frac{1+nx^2}{(1+x^2)^n} = \left|\frac{1+nx^2}{(1+x^2)^n}\right|$$

Since $g(x) = 1$ dominates f_n $\forall n$, $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int \frac{1+nx^2}{(1+x^2)^n}\, dx = \int \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n}\, dx = \int 0\, dx = 0 \quad \boxtimes$$

3. Let f be a Lebesgue meas. function on the interval $[0,1]$ and the measure of the set of $\{x : |f(x)| = \pm\infty\}$ is 0. If the fn $g(x,y) = f(x) - f(y)$ is integrable on the unit square in $\mathbb{R}^2$ show that f is integrable on $[0,1]$.

Suppose, f is Lebesgue meas on $[0,1]$ + $m\{x : |f(x)| = \pm\infty\} = 0$

Consider $g(x,y) = f(x) - f(y)$ s.t. $\int_{[0,1]\times[0,1]} g(x,y)\, d\mu < \infty$

$g = f(x) - f(y)$ is diff of 2 fns incl

Since f is Lebesgue meas. and finite a.e, g is also Lebesgue meas + finite a.e.

Thus, by Fubini's Theorem

$$\int_{[0,1]\times[0,1]} g(x,y)\, d\mu = \int_0^1 \int_0^1 (f(x) - f(y))\, dx\, dy = \int_0^1 \int_0^1 (f(x) - f(y))\, dy\, dx$$

Since $\int g(x,y) = \int\left(\int (f(x) - f(y))\, dx\right) dy < \infty$, $\int_0^1 f(x) - f(y)\, dx < \infty$ a.e.

$$\int_0^1 f(x) - f(y)\, dx = \int_0^1 f(x)\, dx - \int_0^1 f(y)\, dx = \left(\int_0^1 f(x)\, dx\right) - f(y) < \infty$$

Since $f(y) < \infty$ a.e.

$$\infty > \int_0^1 f(x) - f(y)\, dx + f(y) = \int_0^1 f(x) - f(y)\, dx + \int_0^1 f(y)\, dx$$

$$= \int_0^1 f(x) - f(y) + f(y)\, dx$$

$$= \int_0^1 f(x)\, dx.$$

So $f \in \mathcal{L}^1([0,1])$

4. Let (X, Σ, μ) be a measure space and let $\{f_k\}$ be a sequence in $L^p(X,\mu)$

$1 \le p < \infty$, converging to f in $L^p(X,\mu)$, let $\{g_k\}$ be a sequence in $L^\infty(X,\mu)$,

$\|g_k\|_\infty \le M$ $\forall k$, converging to g in $L^\infty(X,\mu)$. Show that $f_k g_k \longrightarrow fg$ in

$L^p(X,\mu)$.

pf. Let (X, Σ, μ) be a measurable space, and let $\{f_k\}$ be a sequence in $L^p(X,\mu)$

$1 \le p < \infty$, converging to f in $L^p(X,\mu)$. That is, $\|f_k - f\|_p < \varepsilon/M$, $\|f\|_p = R < \infty$. Let $\{g_k\}$ be a sequence in $L^\infty(X,\mu)$, $\|g\|_\infty \le M$

$\forall k$, converging to g in $L^\infty(X,\mu)$. (That is $\|g_k - g\|_\infty = \text{ess sup} |g_k - g| < \varepsilon/R$).

Note $|g_k| \le \text{ess sup}_X |g|$ a.e. in $X \longrightarrow |g_k| \le M$. Likewise $|g_k - g| < \text{ess sup} < \varepsilon/R$

$$\|f_k g_k - fg\|_p = \|f_k g_k - f g_k + f g_k - fg\|_p$$

$$\le \|f_k g_k - f g_k\|_p + \|f g_k - fg\|_p$$

$$= \|g_k (f_k - f)\|_p + \|f(g_k - g)\|_p$$

$$= \left(\int |g_k|^p |f_k - f|^p\right)^{1/p} + \left(\int |f|^p |g_k - g|^p\right)^{1/p}$$

$$\le M \left(\int |f_k - f|^p\right)^{1/p} + \varepsilon/R \left(\int |f|^p\right)^{1/p}$$

$$\le M \, \varepsilon/M + \varepsilon/R \; R$$

$$= 2\varepsilon \quad \boxtimes$$

No Corresponding Real Component

Qualifying Exam, Complex Analysis, August 2016

Directions: Attempt as many as you can of the following problems. Write neatly on one-sided sheets; explain; show work; justify your claims **(if you are using a theorem from class and/or the textbook, you must quote the theorem by its name, but you are not required to supply the theorem's proof).** Write page numbers and remember to **print your name.**

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$ with center at the origin, that is: $\Delta = \{z\ ,\ |z| < 1\}$.

1. Let C denote the positively-oriented boundary of the domain

$$D = \left\{ z \in \mathbb{C},\ -2 < \operatorname{Re} z < \frac{1}{2},\quad |\operatorname{Im} z| < 2 \right\}.$$

Find

$$I = \int_C \frac{z^n}{z^4 - 1}\, dz$$

→ Residues, Residue Thm

where $n \geq 0$ is an integer. Write your answer in algebraic form "$I = a + ib$".

2. Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle $\{|z| = 1\}$. Suppose that there is an entire function g such that $f(z) = g(z)$ for $|z| = 1$. Prove that $f = g$ (and hence f is entire).

3. Let S be a square with center at the origin. Suppose that $F : \Delta \to S$ is analytic, one-to-one and onto and furthermore, that $F(0) = 0$. Show that

$$F(iz) = iF(z) \quad \text{for all } z \in \Delta.$$

4. Prove that there are *no*, non-constant polynomials of the form

(0.1) $$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

that satisfy

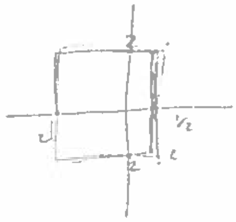
(0.2) $$|\,p(z)\,| < 1 \quad \text{when} \quad |z| = 1\,.$$

August 2016 Complex

1. Let C denote the positively-oriented boundary of the domain

$$D = \{ z \in \mathbb{C},\ -2 < \operatorname{Re} z < 1/2,\ |\operatorname{Im}| < 2 \}.$$

Find $$I = \int_C \frac{z^n}{z^4 - 1}\, dz$$



$$f(z) = \frac{z^n}{z^4-1} = \frac{z^n}{(z+1)(z-1)(z+i)(z-i)}$$

Poles in D: $i,\ -1,\ -i$

$$\operatorname{Res}[f, i] = \lim_{z \to i} \frac{z^n}{(z^2-1)(z+i)} = \frac{i^n}{-2(2i)} = \frac{-i^{n-1}}{4} = \frac{i^2\, i^{n-1}}{4} = \frac{i^{n+1}}{4}$$

$$\operatorname{Res}[f, -1] = \lim_{z \to -1} \frac{z^n}{(z-1)(z^2+1)} = \frac{(-1)^n}{-2(2)} = \frac{(-1)^{n-1}}{4} = \frac{(i^2)^{n-1}}{4} = \frac{i^{2n-2}}{4}$$

$$\operatorname{Res}[f, -i] = \lim_{z \to -i} \frac{z^n}{(z^2-1)(z-i)} = \frac{(-i)^n}{-2(-2i)} = \frac{(i^3)^n}{4i} = \frac{i^{3n-1}}{4}$$

i	-3
-1	-2
$-i$	-1
1	0
i	1
-1	2
$-i$	3
1	4

By Residue Thm $$I = 2\pi i \sum_{k=1}^{3} \operatorname{Res}[f, z_k]$$

$$I = 2\pi i \left[\frac{1}{4} \left(i^{n+1} + i^{2n-2} + i^{3n-1} \right) \right] = \frac{\pi}{2} \left[i^{n+2} + i^{2n-1} + i^{3n} \right]$$

$n = 0 \bmod 4$ $$I = \frac{\pi}{2}\left[i^2 + i^{-1} + i^0 \right] = \frac{\pi}{2}\left[-1 + -i + 1 \right] = \frac{-\pi i}{2}$$

$= 1 \bmod 4$ $$I = \frac{\pi}{2}\left[i^3 + i^1 + i^3 \right] = \frac{\pi}{2}\left[-i + i + -i \right] = \frac{-\pi i}{2}$$

$= 2 \bmod 4$ $$I = \frac{\pi}{2}\left[i^4 + i^3 + i^6 \right] = \frac{\pi}{2}\left[1 + -i - 1 \right] = \frac{-\pi i}{2}$$

$= 3 \bmod 4$ $$I = \frac{\pi}{2}\left[i^5 + i^5 + i^9 \right] = \frac{\pi}{2}\left[i + i + i \right] = \frac{3\pi i}{2}$$

$$I = \begin{cases} \frac{3\pi i}{2} & n = 3 \bmod 4 \\ \frac{-\pi i}{2} & \text{else} \end{cases}$$

2. Let f be continuous on $\mathbb{C}$ and analytic except possibly on $\{|z|=1\}$. Suppose that there is an entire fn g such that $f(z)=g(z)$ for $|z|=1$. Prove that $f=g$ on $\mathbb{C}$.

Consider $D=\{|z|<1\}$, note $\partial D=\{|z|=1\}$. Let $h(z)=f(z)-g(z)$. Since f is cts on $\mathbb{C}$ and g is entire, and $f(z)$ is analytic on D, $h(z)$ is analytic on D and extends continuously to ∂D. Since $f=g$ on $\{|z|=1\}$, $|h(z)|\le 0$ $\forall z\in\partial D$. Then by maximum principal $|h(z)|\le 0$ $\forall z\in D$. Thus $f=g$ on $\{|z|\le 1\}$.

To examine $\{|z|>1\}$ Consider the function $p(z)=h(1/z)=f(1/z)-g(1/z)$. Note that $h(z)$ is holomorphic when $|z|>1$ because both f and g are. Thus $|p(z)|$ is holomorphic for $0<|z|<1$. Since f is cts on $\mathbb{C}$, we can extend h (and consequently p) to $\{|z|=1\}$ continuously. To examine ~~the~~ Laurent expansion of $p(z)$ on $0<r<1$. consider r, $0<r<1$. We know $p(z)=\sum_{n=-\infty}^{\infty} a_n z^n$ $\quad a_n=\frac{1}{2\pi i}\int_{|z|=r}\frac{p(z)}{(z-0)^{n+1}}\,dz=\frac{1}{2\pi i}\int_{|z|=r}\frac{p(z)}{z^{n+1}}\,dz$

Since $z=re^{i\theta}$, $dz=ire^{i\theta}\,d\theta$.

$$|a_n|=\left|\frac{1}{2\pi i}\int_0^{2\pi}\frac{p(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}\,ire^{i\theta}\,d\theta\right|$$

$$=\frac{1}{2\pi i}\left|\int_0^{2\pi}\frac{p(re^{i\theta})}{r^n e^{in\theta}}\,d\theta\right|\le\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(re^{i\theta})|}{r^n}\,d\theta$$

As $r\nearrow 1$ $\quad \frac{|p(re^{i\theta})|}{r^n}\longrightarrow|p(e^{i\theta})|=0$ where convergence is uniform on $\{e^{i\theta}: 0\le\theta\le 2\pi\}$

Hence $\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(re^{i\theta})|}{r^n}\,d\theta\xrightarrow{r\to 1}0$ Thus $a_n=0$ $\forall n$. So $p(z)=0$ for $0<|z|<1$. Hence $f(1/z)=g(1/z)$ for $0<|z|<1$. So $f(z)=g(z)$ for $|z|>1$

Therefore $f(z)=g(z)$ on $\mathbb{C}$, and f is entire. $\square$

3 Let S be a square with center at the origin. Suppose that $F: \Delta \to S$ is analytic, one-to-one, and onto and furthermore, $F(0)=0$. Show that

$F(iz) = iF(z)$ for all $z \in \Delta$

pf Let S be a square with center at the origin. Suppose $F: \Delta \to S$ is analytic, one-to-one, and onto. Since F is bijective, F^{-1} exists.

Since F is one-to-one (more specifically bijective) we know $\forall z \neq w$ $F(z) \neq F(w)$, and thus $F'(z) \neq 0$ $\forall z \in \mathbb{C}$. (How much proof is needed)

Thus, by inverse mapping theorem $F^{-1}(z)$ is analytic.

Consider $G = iz$, a rotation by $\pi/2$, $G: S \to S$, which is conformal

Note $F^{-1}(G(F(z))): \Delta \to \Delta$ is a conformal self map

By the conformal mapping theorem we know that $F^{-1}(G(F))$ has the form $e^{i\varphi}(z)$ for some φ. (B/c $F(0)=0$).

As can be seen above, $\varphi = \pi/2$. thus

$$F^{-1}(G(F(z))) = iz$$

Therefore
$$G(F(z)) = F(iz)$$
$$iF(z) = F(iz)$$ ☑

4. Prove there are no nonconstant polynomials of the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

that satisfy $|p(z)| < 1$ when $|z| = 1$.

Method 1:

Seeking to apply Rouche's theorem, consider $f(z) = -z^n$. Suppose p is nonconstant

Then on $|z| = 1$ $|f(z)| = 1$, $|p(z)| < 1$

So $p(z) + f(z) = a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ has the same number of roots as $f(z) = -z^n$, that is n roots in $|z| \leq 1$

However $p + f$ has at most $n-1$ roots. Σ.

Method 2:

Suppose p is nonconstant. Then $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + 1$

$|p^{(n)}(z)| = n! > 1$ ~~for $n > 1$~~

However, by Cauchy's Estimate, since $|p(z)| < 1$, $|p(z)| \leq M < 1$ for some M

$$|p^{(n)}(z)| \leq \frac{n!}{1^n} M = n! M < n! \, 1 = n! = |p^{(n)}(z)| \quad \Sigma$$

Thus, p is constant

Qualifying Exam, Complex Analysis, August 2015

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$.

1. Find the image of the half-disc $D = \{z \in \mathbb{C} : |z| < 1, \mathrm{Im}\, z > 0\}$ by the Möbius map $f(z) = \dfrac{1+z}{1-z}$.

(graph, affine map 2.7?)

2. Let f be a holomorphic function on $\Delta \setminus \{0\}$ such that $|f(z)| > 1$ for all $z \in \Delta \setminus \{0\}$. Show that 0 is an isolated singularity of f which is either removable or a pole.

(Casorati - Weierstrass $\leq$)

3. Let $D \subsetneq \mathbb{C}$ be a simply connected domain, $z_0 \in D$, and $f : D \longrightarrow \Delta$ be a conformal map such that $f(z_0) = 0$. If $g : D \longrightarrow \Delta$ is a holomorphic map such that $g(z_0) = 0$, show that $|g'(z_0)| \leq |f'(z_0)|$, and the equality holds if and only if g is a conformal map.

?

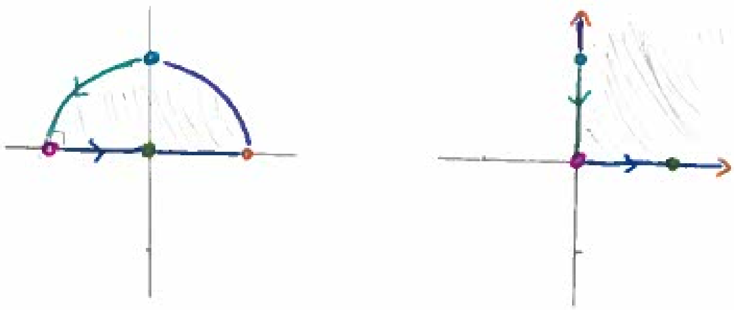
(Pick's Lemma, def of deriv)

4. Compute $F(w) = \displaystyle\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x-w)^2}\, dx$, where $w \in \mathbb{C} \setminus \mathbb{R}$. (Hint: consider the cases $\mathrm{Im}\, w > 0$, and $\mathrm{Im}\, w < 0$, separately.)

Aug. 2015 Complex

1. Find the image of $D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$ by Möbius map: $f(z) = \frac{1+z}{1-z}$

Using the method outlined in 27, we see:



$f(i) = \frac{1+i}{1-i} \cdot \frac{1-i}{1-i} = \frac{1-i^2}{1-2i+i^2} = \frac{2}{-2i} = i$

$f(-1) = \frac{1-1}{1+1} = \frac{0}{2} = 0$

$f(0) = \frac{1+0}{1-0} = \frac{1}{1} = 1$

$f(1) = \frac{1+1}{1-1} = \frac{1}{0} = \infty$

$f(D) = \{z \mid \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.

2. Let f be a holomorphic function on $\Delta \setminus \{0\}$ such that $|f(z)| > 1$ $\forall$ $z \in \Delta \setminus \{0\}$.

Show that 0 is an isolated singularity of f which is either removable or a pole

pf. Let f be a holomorphic function on $\Delta \setminus \{0\}$ such that $|f(z)| > 1$ $\forall$ $z \in \Delta \setminus \{0\}$.

Suppose to the contrary that 0 is neither removable, nor a pole, that is, 0 is an essential singularity. By the Casorati - Weierstrass Thm, $\forall$ w_0 $\exists$ $z_n \longrightarrow 0$ such that $f(z_n) \longrightarrow w_0$. Consider $w_0 = 1/2$. Since $|f(z)| > 1$ $\forall$ $z \in \mathbb{C} \setminus \{0\}$, for all $z_n \longrightarrow 0$ $|f(z_n)| \geq 1$ $\forall n$, and thus $f(z_n) \not\longrightarrow 1/2$

This contradicts the Casorati - Weierstrass Thm, Thus 0 is not an essential singularity.. Therefore 0 is either removable or a pole.

3. Let $D \subsetneq \mathbb{C}$ be a simply connected domain, $z_0 \in D$, and $f: D \longrightarrow \Delta$ be a conformal map such that $f(z_0) = 0$. If $g: D \longrightarrow \Delta$ is a holomorphic map s.t. $g(z_0) = 0$, show that $|g'(z_0)| \le |f'(z_0)|$, and equality holds iff g is conformal

(R)

pf Let $D \subsetneq \mathbb{C}$ be a simply connected domain, $z_0 \in D$, $f: D \longrightarrow \Delta$ is conformal s.t. $f(z_0) = 0$.

Let $g: D \to \Delta$, holomorphic, s.t. $g(z_0) = 0$.

pt 2

First observe that $g \circ f^{-1}: \Delta \longrightarrow \Delta$ and $(g \circ f^{-1})(0) = 0$.

By conformal mapping theorem, $(g \circ f^{-1})(z) = e^{i\varphi} z$ for some $0 \le \varphi \le 2\pi$.

Thus $|(g \circ f^{-1})'(0)| = |e^{i\varphi}| = 1$.

Since $(g \circ f^{-1})' = g'(f^{-1}(0)) \cdot (f^{-1})'(0) = g'(z_0) \cdot \dfrac{1}{f'(z_0)}$

$$|(g \circ f^{-1})'(0)| = |g'(z_0)| \cdot \frac{1}{|f'(z_0)|} = 1$$

$$|g'(z_0)| = |f'(z_0)| = \frac{1}{|(f^{-1})'(0)|}$$

Then equality holds in Pick's lemma for $g \circ f^{-1}: \Delta \longrightarrow \Delta$

So $(g \circ f^{-1})$ is a conformal self-map of Δ ———— didn't use case ~~

Thus $g = (g \circ f^{-1}) \circ f : D \longrightarrow \Delta$ is conformal $\square$

pt 1

Since f is conformal on D, deriv does not vanish on D. Hence, f admits a conformal, analytic inverse $f^{-1}: \Delta \longrightarrow D$

It follows that $f \circ f^{-1}(z) = z$ on Δ and $1 = (f \circ f^{-1})'(0) = f'(z_0)\,(f^{-1})'(0)$, so

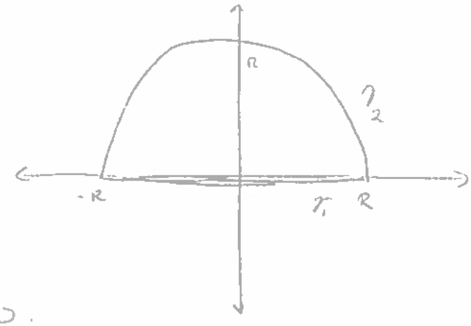
$f'(z_0) = \dfrac{1}{(f^{-1})'(0)}$; $|f'(z_0)| = \dfrac{1}{|(f^{-1})'(0)|}$ Since $g \circ f^{-1}: \Delta \to \Delta$ is analytic

Pick's lemma gives:

$$|(g \circ f^{-1})'(0)| = |g'(z_0)|\,|(f^{-1})'(0)| \le \frac{1 - |(g \circ f^{-1})(0)|^2}{1} = 1$$ ✓

So $|g'(z_0)| \le 1 / |(f^{-1})'(0)| = |f'(z_0)|$

4. Compute $F(\omega) = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2}\,dx$ where $\omega \in \mathbb{C}\setminus\mathbb{R}$.



$\Gamma = \gamma_1 \cup \gamma_2$

Suppose $\operatorname{Im} \omega > 0$.

Consider $f(z) = \frac{e^{iz}}{(z-\omega)^2}$

Rule 2. $\operatorname{Res}[f, \omega] = \lim_{z \to \omega} \frac{d}{dz} e^{iz} = ie^{i\omega}$

By Residue Thm:

$$\int_\Gamma f(z)\,dz = 2\pi i\,(ie^{i\omega})$$

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_\Gamma f(z)\,dz \quad + \lim_{R\to\infty}\int_{\gamma_1} f\,dz + \lim_{R\to\infty}\int_{\gamma_2} f\,dz = \lim_{R\to\infty}\int_\Gamma f\,dz = -2\pi e^{i\omega}$$

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{(x-\omega)^2}\,dx = I$$

Jordan

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = \lim_{R\to\infty}\int \frac{e^{iz}}{(z-\omega)^2}\,dz \le \lim_{R\to\infty}\frac{1}{R^2}\int |e^{iz}|\,|dz| \le \lim_{R\to\infty}\frac{\pi}{R^2} = 0$$

$$\therefore \int = I + 0 = I$$

AUGUST 2015 QUALIFYING EXAM IN REAL ANALYSIS

Notation: m stands for the Lebesgue measure on the real line. The spaces $L^p([0,1])$ are understood with respect to m. You may use without proof any standard results from MAT 701, MAT 601, and MAT 602.

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1. Let $(X, \mathcal{M})$ be a measurable space, and suppose $A_n \in \mathcal{M}$ for $n = 1, 2, \ldots$. Let

$$A = \{x \in X \colon x \in A_n \text{ for infinitely many } n, \text{ and } x \notin A_n \text{ for infinitely many } n\}$$

Prove that $A \in \mathcal{M}$. → liminf , limsup

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2. Suppose $f \colon [0,1) \to [0,\infty)$ is a measurable function such that $\int_0^1 \sqrt{1-x}\, f(x)\, dx < \infty$. Let $F(x) = \int_0^x f(t)\, dt$ for $x \in [0,1)$. absolute cont. $\int_0^1 \frac{1}{x^p} < \infty \; 0<p<1$

(a) Prove that F is continuous on $[0,1)$.

(b) Does F have to be bounded on $[0,1)$? Prove or disprove.

(c) Prove that $\int_0^1 F(x)\, dx < \infty$. (Tonelli's.) jk, but need to know this

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3. Give an example of a sequence of functions $f_n \colon [0,1] \to [0,1]$ such that the total variation of f_n on $[0,1]$ is at most 2, and the function $f(x) = \sup_n f_n(x)$ is not in $BV([0,1])$.

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4. Suppose that $\{f_n : n = 1, 2, \ldots\}$ is a sequence of functions on $[0,1]$ such that $\|f_n\|_{L^4([0,1])} \le 1$ for all n.

Which of the statements (a)–(c) follow from the above? Prove or give a counterexample to each.

(a) There is a constant C such that $\|f_n\|_{L^2([0,1])} \le C$ for all n.

(b) There is a constant C such that $\|f_n\|_{L^6([0,1])} \le C$ for all n.

(c) There exists a subsequence $\{f_{n_k}\}$ which converges almost everywhere on $[0,1]$.

1. Let (X, M) be a measurable space, and suppose $A_n \in M$ for $n = 1, 2, \ldots$ Let

$A = \{x \in X : x \in A_n$ for ∞ly many n, and $x \notin A_n$ for ∞ly many $n\}$

Prove that $A \in M$.

pf. Let (X, M) be a measurable space, and suppose $A_n \in M$ for $n = 1, 2, \ldots$

$A = \{x \in X : x \in A_n$ for ∞ly many n and $x \notin A_n$ for ∞ly many $n\}$

Let $B = \{x \in X \mid x \in A_n$ for ∞ly many $n\}$

$\Rightarrow B = \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

Let $C = \{x \in X \mid x \notin A_n$ for ∞ly many $n\}$

then $X \setminus C = \{x \in X \mid x \notin A_n$ for finitely many $n\} = \{x \in X \mid x \in A_n$ for all but finitely many$\}$.

So $X \setminus C = \liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$

Thus $C = X \setminus \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right)$

Since $A = B \cap C = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) \cap \left(X \setminus \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right)$

Since M is a σ-algebra $A \in M$ ☑

2. Suppose $f: [0,1) \to [0,\infty)$ is a measurable fn s.t. $\int_0^1 \sqrt{1-x}\, f(x)\,dx < \infty$

(R) Let $F(x) = \int_0^x f(t)\,dt$ for $x \in [0,1)$.

a. Prove that F is cts on $[0,1)$.

Fix $x_0 \in [0,1)$. Let $\varepsilon > 0$ Let $H: \mathcal{M} \to [0,\infty)$ be defined by $H(E) = \int_E \sqrt{1-t}\, f(t)\,dt$, for $E \in \mathcal{M}$.

where $\mathcal{M}$ is the set of Lebesgue meas subsets of $[0,1)$.

(*) Since $\int_{[0,1)} \sqrt{1-t}\, f(t)\,dt < \infty$, H is absolutely cts on $\mathcal{M}$

Thus $\exists\ \delta > 0$ s.t. $|E| < \delta \implies |H(E)| < \varepsilon$

Choose δ' st. $[x_0 - \delta', x_0 + \delta'] \subseteq [0,1)$

Let $\delta_0 = \min\{\delta, \delta'\}$

Then for $t \in (x_0 - \delta_0, x_0 + \delta_0)$, we have $t < x_0 + \delta_0 \implies 1-t > 1-(x_0+\delta_0)$

$$\implies \sqrt{1-t} > \sqrt{1-(x_0+\delta_0)} \implies \frac{1}{\sqrt{1-t}} < \frac{1}{\sqrt{1-(x_0-\delta_0)}} = \alpha$$

Let $y \in [0,1)$ s.t. $|x_0 - y| < \delta_0$, WLOG let $x_0 \le y$ so:

$$|F(x_0) - F(y)| = \int_{x_0}^y f(t)\,dt = \int_{x_0}^y \sqrt{1-t}\, f(t) \left(\frac{1}{\sqrt{1-t}}\right) dt$$

$$\le \alpha \int_{x_0}^y \sqrt{1-t}\, f(t)\,dt = \alpha\, H([x_0, y]) < \alpha\,\varepsilon. \ \square$$

2b. Does F have to be bdd on [0,1)? Prove or disprove

No, Consider the case where $f(x) = \frac{1}{1-x}$

Then $\int_0^1 \sqrt{1-x} \cdot \frac{1}{1-x}\,dx = \int_0^1 \frac{1}{\sqrt{1-x}} < \infty$

However, $F(x) = \int_0^x \frac{1}{1-t}\,dt \implies \lim_{x\to 1} F(x) = \int_0^1 \frac{1}{1-t}\,dt = \infty$

2c. Prove that $\int_0^1 F(x)\,dx < \infty$.

Note $\int_0^1 F(x)\,dx = \int_0^1 \int_0^x f(t)\,dt\,dx$

$= \int_0^1 \int_0^x \left(\frac{f(t)\sqrt{1-t}}{\sqrt{1-t}}\,dt \right) dx$

$\sqrt{1-x}$ is decr.

$\frac{1}{\sqrt{1-x}}$ is incr.

$\leq \int_0^1 \frac{1}{\sqrt{1-x}} \left(\int_0^x \sqrt{1-t}\, f(t)\,dt \right) dx$

$< \cdot \left(\int_0^1 \sqrt{1-t}\, f(t)\,dt \right) dx$

$\leq \int_0^1 \frac{1}{\sqrt{1-x}} \cdot \alpha \; dx$

$= \alpha \int_0^1 \frac{1}{\sqrt{1-x}} < \infty$

3. Give an example of a sequence of functions $f_n : [0,1] \to [0,1]$ s.t. the total variation of f_n on $[0,1]$ is at most 2, and the fn. $f(x) = \sup_n f_n(x)$ is not in $BV([0,1])$.

Let x_n be the enumeration of $\mathbb{Q}$.

Let $f_n = \chi_{x_n}$.

Then the total variation of f_n for the partition $\{0, x_n, 1\}$ is:

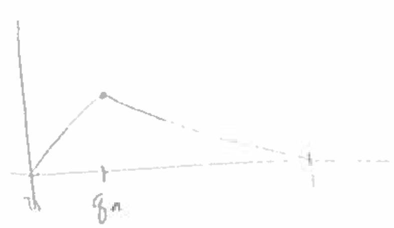
$$|f(x_n) - f(0)| + |f(1) - f(x_n)| = 1 + 1 = 2$$

However $f(x) = \sup_n f(x) = \chi_{\mathbb{Q} \cap [0,1]}$

The total variation of f for $\{0, x_1, y_1, \ldots, y_n, x_{n+1}, 1\}$ using $y_i \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $x_i < y_i < x_{i+1}$

$$|f(x_i) - f(0)| = 1 = |f(y_i) - f(x_i)| = |f(x_{i+1}) - f(y_i)|$$

So then $\lim_{n \to \infty} \sum_{i=1}^{n} \left(|f(y_i) - f(x_i)| + |f(x_{i+1}) - f(y_i)| \right) = \sum_{i=1}^{\infty} 2 = \infty$



4. Suppose that $\{f_n : n = 1, 2, \_\_\}$ is a sequence of fn on $[0,1]$ s.t.

$$\|f_n\|_{L^4([0,1])} \le 1 \quad \forall n.$$

$$\left(\int |f_n|^4\right)^{1/4} \le 1 \longrightarrow \int |f_n|^4 \le 1$$

a. There is a constant C s.t. $\|f_n\|_{L^2([0,1])} \le C \quad \forall n$

$$\int_0^1 |f_n|^2 \le \left(\int_0^1 |f_n|^4\right)^{1/2} \cdot \left(\int_0^1 1^2\right)^{1/2} = 1$$

So $\|f_n\|_{L^2} \le 1 = C$ for all n

b. No Consider $f_n(x) = \dfrac{1}{3^{1/4} x^{1/6}} \quad \forall n.$

$$\|f_n\|_{L^4} = \left(\int_0^1 \left(\frac{1}{3^{1/4} x^{1/6}}\right)^4 dx\right)^{1/4} = \left(\int_0^1 \frac{1}{3x^{2/3}}\right)^{1/4} = \left(\frac{x^{5/3}}{35}\Big|_0^1\right)^{1/4} = \left(\frac{1}{5}\right)^{1/4}$$

$$\|f_n\|_{L^6} = \left(\int_0^1 \left(\frac{1}{3^{1/4} x^{1/6}}\right)^6\right)^{1/6} = \left(\int_0^1 \frac{1}{3^{3/2}} \cdot \frac{1}{x}\right)^{1/6} = \infty$$

c. $\exists$ a subsequence $\{f_{n_k}\}$ which converges everywhere on $[0,1]$

" Consider the Banach space $L^4([0,1]) = \{f : [0,1] \to \overline{\mathbb{R}} : \int_0^1 |f|^4 < \infty\}$ "

Then $\overline{B([0,1])} = \{f \in L^4([0,1]) : \|f\|_{L^4} \le 1\}$ is a compact subset of L^4.

Since $\{f_n\} \subseteq \overline{B(0,1)}$ is compact $\exists$ $\{f_{n_k}\} \subset \{f_n\}$ s.t. $\|f_{n_k} - f\|_{L^4} \longrightarrow 0$

By Riesz - Fischer Theorem, $\exists$ a subsequence of $\{f_{n_k}\}$, & thus of $\{f_n\}$ that converges ptws to f a.e. on $[0,1]$ ☑

Complex Analysis **Qualifying Exam** **January 2015**

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from class.

Problem 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$ converges absolutely for $|z| < 1$. Also show that there are infinitely many z with $|z| = 1$ for which the series diverges.

Problem 2. Let $f(z)$ be holomorphic on $\mathbb{C}$ except for poles. At ∞ assume that f has a removable singularity or a pole.
(a) Show that f has finitely many poles on $\mathbb{C} \cup \{\infty\}$.
(b) Let $p_j(z)$ be the principal part of f at the jth pole, $1 \leq j \leq N$, show that

$$f(z) - \sum_{j=1}^{N} p_j(z)$$

is constant.

Problem 3. Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle, $|z| = 1$. Assume there is an entire function g such that $f(z) = g(z)$ for $|z| = 1$. Prove that $f = g$, and hence f is entire.

Problem 4. Let f_n be analytic in the unit disc, D, and have positive real part: $\mathcal{R}(f(z)) > 0$ on D. Assume that the f_n converge pointwise on D to a function f having $\mathcal{R}(f(z)) \leq 0$ on D. Prove that f is constant on D.

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$ converges absolutely for $|z|<1$. Also show that there are infinitely many z w/ $|z|=1$ for which the series diverges.

· Consider $\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$ Note

$$\sum \left|\frac{1}{n} z^{n!}\right| = \sum \left|\frac{1}{n}\right| |z|^{n!} \leq \sum |z|^n < \infty \quad \text{for} \quad |z|<1$$

Therefore, the series converges absolutely for $|z|<1$.

· Consider $z = e^{2\pi i/k}$. Then.

$$\sum \frac{1}{n} z^{n!} = \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi i n!/k}$$

Then $\forall$ k, for $n>k$ $e^{2\pi i n!/k} = e^{2\pi i (n)(n-1)\cdots(k+1)(k-1)} = e^{2\pi i} = 1$

So

$$\sum_{n=k+1}^{\infty} \frac{1}{n} e^{2\pi i n!/k} = \sum \frac{1}{n} \longrightarrow \infty$$

Therefore, there are infinitely values, specifically of the form $e^{2\pi i/k}$ for which the series diverges ☑

2. Let $f(z)$ be holomorphic on $\mathbb{C}$ except for poles. At ∞ assume that f has a removable singularity or a pole.

(R)

a. Show that f has finitely many poles on $\mathbb{C} \cup \{\infty\}$

Pf. Let Suppose $f(z)$ is holomorphic on $\mathbb{C}$ except for poles. At ∞ assume that f has a removable singularity or a pole. Since f has an isolated singularity at ∞ $\exists$ $R > 0$ s.t. f is analytic on $\{|z|>R\}$ [Further, since f has an isolated singularity, $f(1/z)$ has an isolated singularity at 0. Thus, $\exists$ $r>0$ s.t. $f(1/z)$ is holomorphic for $0 < |z| < r$, and thus $f(z)$ is holomorphic for $1/r < |z| < \infty$]

Suppose, for the sake of contradiction, that $f(z)$ has infinitely many poles on $\mathbb{C} \cup \{\infty\}$. By the previous statement, this is the equivalent to $f(z)$ having infinitely many poles in $\{|z| \leq R\}$ Denote these poles $\{z_n\}_{n=1}^{\infty}$. By Bolzano-Weierstrass

Since D is compact, $\exists$ a subsequence $\{z_{k_j}\} \subset \{z_k\}$ such that $z_{k_j} \to z_0 \in D$

If f is analytic at z_0, then f is analytic in an open neighborhood of z_0, which contradicts the convergence of our sequence of poles.

If f has a singularity at z_0, then it must be isolated, which again contradicts the nature of our sequence of poles. Therefore f has finitely many poles on $\mathbb{C}^*$ ☑

b. Let $p_j(z)$ be the principal part of f at the jth pole, $1 \leq j \leq N$. Show that $f(z) - \sum_{j=1}^{N} p_j(z)$ is constant

Assume: $f(z)$ has a pole @ ∞. Then let $P_N(z)$ be the principal part of f @ $z_j \in \mathbb{C}$, $j = 1, \dots, N-1$

Let $h(z) = f(z) - \sum p_j(z)$ which has a removable singularity @ each pole in $\mathbb{C}$, and can be extended analytically to each of these poles. Thus the extension is entire. We have $\lim_{z\to\infty}(f(z) - P_N(z)) = 0$. And since the principal parts are of form $\frac{1}{z - z_j}$, $\lim_{z\to\infty} P_j(z) = 0$.

So $\lim_{z\to\infty} h(z) = 0$ Thus h is bdd on $\mathbb{C}$ & thus constant by Liouville

If f has a removable singularity then $f(z) = b_n z^{-n}$ for $|z| > R$. Hence $\lim_{z\to\infty} f = 0$

As above, h is bdd and therefore constant. ☑

3 Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle $|z|=1$. Assume there is an entire function g s.t. $f=g$ for $|z|=1$. Prove that $f=g$, and hence f is entire.

- $h = f-g \longrightarrow$ max princ. $h \equiv 0$ on $\{|z| \leq 1\}$
- Laurent expansion $h = \sum a_n z^n$ $\quad a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{h(z)}{z^{n+1}} \, dz = 0$

$\hookrightarrow$ $h = 0$ o/sc

~~Problems on~~ this still

See Aug2016 for soln.

Use $p(z) = h(1/z) \longrightarrow p(z) = \sum a_n z^n$

$$a_n = 2\pi i \int \frac{p(z)}{|z|^{n+1}} \, dz \qquad \text{let } z = re^{i\theta}, \quad dz = ire^{i\theta} d\theta$$

Swap let $\left| \frac{p(z)}{r^n e^{i\theta(n+1)}} \right| = \frac{|p(z)|}{|r^n|} \longrightarrow$ $|p(z)|$ as $r \to 1$ ($|p(z)| \to 0$, $r^n \to 1$)

4. Let f_n be analytic in the unit disc, $\mathbb{D}$, and have positive real part: $\mathcal{R}(f(z)) > 0$ on $\mathbb{D}$. Assume that the f_n converge ptws on $\mathbb{D}$ to a function f having $\mathcal{R}(f(z)) \leq 0$ on $\mathbb{D}$. Prove f is constant on $\mathbb{D}$.

pf. Let f_n be analytic in the unit disc $\mathbb{D}$ & $\mathcal{R}(f_n(z)) > 0$ on $\mathbb{D}$. Assume f_n converges ptwise on $\mathbb{D}$ to a function f s.t. $Re(f(z)) \leq 0$

Let $h_n = e^{-f_n(x)}$ then $|h_n| = |e^{-f_n(x)}| \leq |e^0| = 1$ $(b/c\ (-f_n) < 0)$

Consider $g(z) = \frac{z}{1-|z|^2}$ $\mathbb{D} \longrightarrow \mathbb{C}$, analytic & bijection. Then $g^{-1}: \mathbb{C} \longrightarrow \mathbb{D}$

Then $h_n(g^{-1}(z)) : \mathbb{C} \longrightarrow \mathbb{C}$ Further h_n is entire and bdd & thus constant by Liouville's. Therefore

$h_n(g^{-1}(z)) = k_1 \longrightarrow h_n(\omega) = k \longrightarrow e^{-f_n(z)} = k \ \forall z$

$\longrightarrow f_n(z) = k \longrightarrow f = k$ ☑

Real Analysis **Qualifying Exam** **January 2015**

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from your courses.

Problem 1 Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that there are disjoint sets $E_1, E_2, \ldots$ satisfying the strict inequality

$$\mu^*(\bigcup_k E_k) < \sum_k \mu^*(E_k)$$

Problem 2. Construct a function in $L^1(\mathbb{R})$ that is not in $L^2((a,b))$ for any non-empty interval $(a,b) \subset \mathbb{R}$.

Problem 3. Let S be a measurable space and $\mathcal{F}$ a sigma algebra of subsets of S. Let ν be a positive finite measure on $\mathcal{F}$ and μ a finitely additive real-valued set function on $\mathcal{F}$. Finally, assume that both $\nu + \mu$ and $\nu - \mu$ are non-negative, finite, and countably additive on $\mathcal{F}$. Prove that μ is a signed measure on $\mathcal{F}$ whose total variation is absolutely continuous with respect to ν.

Problem 4. Let the f_n be Lebesgue integrable on $\mathbb{R}$ such that $|f_n(x)| \searrow 0$ a.e. Also assume that the series $\sum_{n=1}^{\infty} f_n(x)$ is an alternating series for almost every x. Prove that

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)\, dx.$$

1. Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that there are disjoint sets $E_1, E_2, \ldots$ satisfying the strict inequality.

$$\mu^*\left(\bigcup_k E_k\right) < \sum \mu^*(E_k).$$

Pf

We know from the subadditivity of the outer measure that

$$\mu^*\left(\bigcup_k E_k\right) \leq \sum \mu^*(E_k).$$

for ~~(disjoint?)~~ any sets E_k.

Proceeding by contradiction, suppose $\mu^*\left(\bigcup_k E_k\right) = \sum \mu^*(E_k)$ $\forall$ disjoint E_k.

More specifically suppose that for any 2 disjoint sets E_1, E_2

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$$

Let $E \subset \mathbb{R}$ be an arbitrary set. Let $A \subset \mathbb{R}$ be an arbitrary set

Then $A \cap E$ and $A \setminus E$ are disjoint, and so, by assumption (more specifically)

$$\mu^*(A) = \mu^*((A \cap E) \cup A \setminus E) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

Since A was arbitrary, it follows from Caratheodory Thm that E is measurable. But E is arbitrary, so it follows that every subset of $\mathbb{R}$ is measurable. But this contradicts vitali's thm, that $\exists$ non measurable subsets of $\mathbb{R}$
(more general)

Hence $\exists$ disjoint $E_1, E_2 \subset \mathbb{R}$ satisfying $\mu^*(E_1 \cup E_2) < \mu^*(E_1) + \mu^*(E_2)$

By induction suppose $\exists$ $E_1 \ldots E_{n-1}$ s.t. $\mu^*\left(\bigcup_{k=1}^{n-1} E_k\right) < \sum_{k=1}^{n-1} \mu^*(E_k)$

$\mu_*\left(\bigcup_{i=1}^{n} E_n\right) = \mu_*\left(\bigcup_{k=1}^{n-1} E_k \cup E_n\right) < \sum_{k=1}^{n-1} \mu E_k + \mu E_n = \sum E_n$ problem w/ $n \to \infty$

or

$\mu^*(\bigcup_k E_k) = \mu^*(E_1 \cup E_2) < \mu^*(E_1) + \mu^*(E_2) = \mu^*(E_1) + \mu^*(E_2) + \sum \mu^*(E_k)$
$= \sum$

2 Construct a function in $L^1(\mathbb{R})$ that is not in $L^2(a,b)$ for any nonempty interval $(a,b) \subset \mathbb{R}$.

(R)

Let $\{x_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q}$

For each k, let $f_k(x) = \frac{1}{\sqrt{x-x_k}} \cdot 2^{-k} \chi_{[x_k, x_k+1]}$ — not $\chi_{(k+1)}$

Then for each k:

$$\int_{\mathbb{R}} f_k = 2^{-k} \int_{x_k}^{x_k+1} (x - x_k)^{-1/2} dx = 2^{-k} \cdot 2 \cdot (x-x_k)^{1/2} \Big|_{x_k}^{x_k+1}$$

$$= 2^{-k+1}\left((x_k+1-x_k)^{1/2} - 0 \right) = 2^{-k+1}$$

this detail ⊗

Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$.

Since f_k is nonnegative on $\mathbb{R}$ for each k

$$\int_{\mathbb{R}} f = \int \sum_k f_k = \sum \int f_k = \sum_{k=1}^{\infty} 2^{-k+1} = 2\left(\sum_{k=1}^{\infty} 2^{-k}\right) = 2 \cdot 1 = 2.$$

Hence $f \in \mathcal{L}^1(\mathbb{R})$

Let $(a,b) \subset \mathbb{R}$ be a nonempty interval. We claim that $f \notin L^2((a,b))$

Since f_k is non-negative, for each k,

$$f = \sum_k f_k \geq f_j \geq 0 \text{ for any } j \in \mathbb{N}$$

Choose $j \in \mathbb{N}$ s.t. $x_j \in (a,b)$. Then $f \geq f_j \geq 0 \rightarrow f^2 \geq f_j^2 \geq 0$

So $\|f\|_{L^2(a,b)} = \left(\int_a^b f^2 \right)^{1/2} \geq \left(\int_a^b f_j^2 \right)^{1/2}$

and

$$\int_a^b f_j^2 \geq \int_{x_j}^b f_j^2 \geq \int_{x_j}^{x_j'} f_j^2 \qquad x_j' = \min\{b, x_j+1\}$$

But $\int_{x_j}^{x_j'} f_j^2 = 2^{-k} \int_{x_j}^{x_j'} \frac{1}{x-x_j} = 2^{-k} \ln(x-x_j) \Big|_{x_j}^{x_j'} = \infty$

Hence $\left(\int_a^b f_j^2 \right)^{1/2} = \infty$ and $f \notin L^2((a,b))$ $\square$

3. Let S be a measurable space and $\mathcal{F}$ a sigma algebra of subsets of S. Let ν be a positive finite measure on $\mathcal{F}$ and μ a finitely additive real-valued set function on $\mathcal{F}$.

Finally, assume that both $\nu+\mu$ and $\nu-\mu$ are nonnegative, finite, and countably additive on $\mathcal{F}$. Prove that μ is a signed measure on $\mathcal{F}$ whose total variation is absolutely cts wrt ν.

Pf.

To show μ is a signed measure on $\mathcal{F}$, we must show that:

i.) $\mu(\emptyset)=0$ and (ii) μ is countably additive on $\mathcal{F}$

Choose $E\in\mathcal{F}$. Since μ is finitely additive and real-valued,

$$\mu(E)=\mu(E\cup\emptyset)=\mu(E)+\mu(\emptyset), \text{ and thus } \mu(\emptyset)=0.$$

Since $(\nu+\mu)$ & $(\nu-\mu)$ are finite and countably additive on $\mathcal{F}$,

$$\mu = \tfrac{1}{2}\left((\nu+\mu)-(\nu-\mu)\right) \text{ is countably additive on } \mathcal{F}$$

Thus μ is a signed measure on $\mathcal{F}$. → use thm in WZ to get this connection

To show that the total variation of μ is abs continuous wrt ν, let $E\in\mathcal{F}$ s.t. $\nu(E)=0$. Suppose $\mu(E)>0$. Then

$$(\nu-\mu)(E)=\nu(E)-\mu(E)=-\mu(E)<0 \quad ↯ \text{ (b/c } \nu-\mu \text{ is non negative)}$$

Suppose $\mu(E)<0$. Then

$$(\nu+\mu)(E)=\nu(E)+\mu(E)=\mu(E)<0 \quad ↯ \text{ again}$$

Thus $\mu(E)=0$ & thus μ is abs cts wrt ν

$\mu(E)=0$ $\forall$ E s.t. $\nu(E)=0$

4. Let f_n be Lebesgue integrable on $\mathbb{R}$ s.t. $|f_n(x)| \searrow 0$ a.e. Also, assume that the series $\sum_{n=1}^{\infty} f_n$ is an alternating series for almost every x. Prove that

$$\int_{-\infty}^{\infty} \sum f_n \, dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

pf Let f_n be Lebesgue int. on $\mathbb{R}$ s.t. $|f_n| \searrow 0$ a.e. Assume $\sum f_n$ is an alternating series a.e.

Fix $x \in \mathbb{R}$ s.t. $|f_n(x)| \searrow 0$, and $\sum_{n=1}^{\infty} f_n(x)$ is alternating

By def $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} (-1)^{n-1} |f_n(x)|$ For $N \in \mathbb{N}$ let

$F_N(x) = \sum_{k=1}^{N} (-1)^{n-1} |f_n(x)|$ We claim, for each N, $0 \le F_N(x) \le f_1(x)$

<u>$0 \le F_N(x)$</u> (N even) $F_N(x) = (|f_1(x)| - |f_2(x)|) + (|f_3(x)| - |f_4(x)|) + \cdots + (|f_{N-1}(x)| - |f_N(x)|)$.

Since $|f_n(x)| \searrow 0$, $(|f_n(x)| - |f_{n+1}(x)|) \ge 0$ $n = 1, 2, \ldots, N-1$

Hence $F_N(x) \ge 0$. Holds for odd N b/c $+|f_{N+1}(x)|$ still > 0

<u>$F_N \le f_1(x)$</u>. Base case is trivial

Assume $F_k(x) \le f_1(x)$ for $k = 1, \ldots, N$.

N even $F_{N+1}(x) = F_N(x) + |f_{N+1}(x)| = F_{N-1}(x) - |f_N(x)| + |f_{N+1}(x)|$

$\le F_{N-1}(x) - |f_N(x)| + |f_N(x)| = F_{N-1} \le f_1$ (assumption)

N odd $F_{N+1}(x) = F_{N-1}(x) + |f_N(x)| - |f_{N+1}(x)| \le F_{N-1}(x) + |f_N(x)|$

$= F_N(x) \le f_1(x)$

Hence $F_N(x) \le f_1(x)$ $\forall N$. It follows $|F_N(x)| \le f_1(x)$ a.e. on $\mathbb{R}$

Thus by Dominated Convergence thm

$\lim_{N\to\infty} \int_{\mathbb{R}} F_N = \int_{\mathbb{R}} \lim_{N\to\infty} F_N = \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n$

$\lim_{N\to\infty} \int_{\mathbb{R}} F_N = \lim_{N\to\infty} \sum \int_{\mathbb{R}} f_n = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n$

Qualifying Exam, Complex Analysis, August 2014

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$.

1. Find a conformal map from the half-disc $D = \{z \in \mathbb{C} : |z| < 1, \mathrm{Re}\, z > 0\}$ onto Δ.

2. Let D be a domain in $\mathbb{C}$ containing 0 and $f : D \longrightarrow \mathbb{R}$ be a continuous function such that $f(0) = 0$ and

$$\int_{\partial R} f(z)\, dz = 0$$

for every closed rectangle $R \subset D$ with sides parallel to the coordinate axes. Prove that $f(z) = 0$ for every $z \in D$.

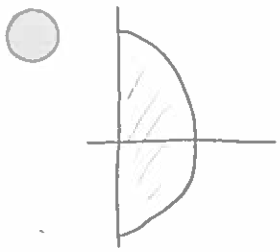
↪ f(z) analytic on D, f(0) = 0 ⟶ Since f Real valued ⟶ $v_y = 0$ → $u_x = 0$
Morera's

3. Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in D$, and $f : D \longrightarrow D$ be a holomorphic function such that $f(z_0) = z_0$. Show that $|f'(z_0)| \leq 1$.

4. Let $f_n : \Delta \longrightarrow \Delta$, $n \geq 1$, be a sequence of holomorphic functions such that f_n has a zero of order m_n at 0, where $\lim_{n\to\infty} m_n = \infty$. Show that $\{f_n\}$ converges locally uniformly to zero on Δ.

$f_n : E \to \mathbb{C}$ converges locally uniformly
if $\forall\, z \in E$ $\exists$ an open disk
$B(z, R)$ s.t. $f_n \xrightarrow[\text{unif}]{} f$ on $B(z,R) \cap E$

1. Find a conformal map from $D = \{z \in \mathbb{C} : |z| < 1\}$

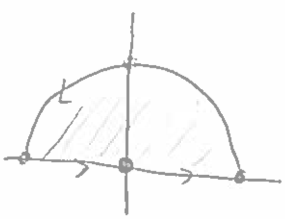


$\downarrow \varphi_1$

$\varphi_1 : D \to \mathbb{D}^+$

$\varphi_1 = e^{i\pi/2} z = iz$ (rotation by $\pi/2$)

$= e^x \cdot e^{i(y+\pi/2)}$



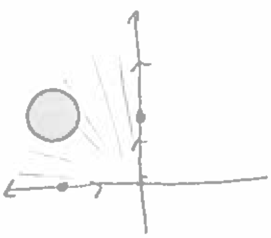
$\downarrow \varphi_2$

$\varphi_2 : \mathbb{D}^+ \longrightarrow \{z \mid \operatorname{Re} z < 0 \ \operatorname{Im} z > 0\}$

$\varphi_2(z) = i \dfrac{1+z}{1-z}$

$\varphi_2(1) = \infty \quad \varphi_2(-1) = 0 \quad \varphi(0) = i$

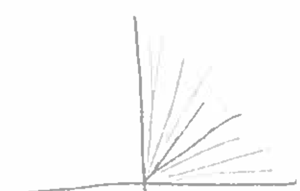
$\varphi_2(i) = i\left(\frac{1+i}{1-i} \cdot \frac{1+i}{1+i}\right) = \frac{2i}{2} i = -1$



$\downarrow \varphi_3$

$\varphi_3 : \{z \mid \operatorname{Re} z < 0, \operatorname{Im} z > 0\} \to \{z \mid \operatorname{Re} z, \operatorname{Im} z > 0\}$.

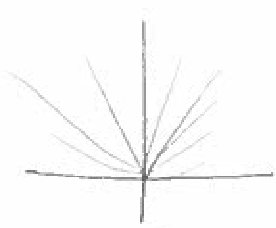
$\varphi_3 = e^{-i\pi/2} z = -iz$ (rotation by $-\pi/2$)



$\downarrow \varphi_4$

$\varphi_4 : \{z \mid \operatorname{Im} z > 0, \operatorname{Re} z > 0\} \to \mathbb{H}^+$

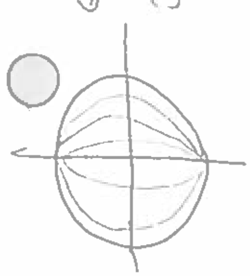
$\varphi_3 = z^{\pi/\pi/2} = z^2$ (open sector)



$\downarrow \varphi_5$

$\varphi_5 : \mathbb{H}^+ \longrightarrow \mathbb{D}$

$\varphi_5 = \dfrac{z-i}{z+i}$



$$\varphi = \varphi_5(\varphi_4(\varphi_3(\varphi_2(\varphi_1(z))))) = \frac{\left(\frac{1+iz}{1-iz}\right)^2 - 1}{\left(\frac{1+iz}{1-iz}\right)^2 + 1}$$

2. Let D be a domain in $\mathbb{C}$ containing 0 and $f: D \to \mathbb{R}$ be a continuous function such that $f(0)=0$ and $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle R with sides parallel to the coordinate axes. By Morera's Theorem f is analytic on D. Since f is real-valued and analytic, f must therefore be constant on D. Since $f(0)=0$ and $f(z)$ is constant on D, where $0 \in D$, $f(z)=0$ $\forall z \in D$. ☑

3. Let $D \subset \mathbb{C}$ be a bounded domain $z_0 \in D$. $f: D \to D$ be a holomorphic function s.t. $f(z_0) = z_0$. Consider the iterative sequence of functions

$$f_n(z) = \overbrace{f(f(\cdots(f(z)))}^{n \text{ times}})$$

Note, $f_n(z_0) = f(f(\ldots f(z_0))) = z_0$. Suppose $|f'(z_0)| > 1$.

Consider some $\overline{D_\rho(z_0)} \subset D$, a closed disk of sufficiently small radius $\rho > 0$ centered at z_0.

Since f is holomorphic on D, and therefore cts on D, f_n certainly continuous on $\overline{B_\rho(z_0)}$. Since $\overline{B_\rho(z_0)}$ is a closed, bdd subset of $\mathbb{C}$, it is compact. Since f_n is continuous on a compact set it attains a maximum value M on that set. That is, $|f_n(z)| < M \quad \forall\, |z - z_0| \leq \rho$.

Using Cauchy Estimates we see $\forall m \; |f_n^{(m)}(z_0)| \leq \frac{m!}{\rho^m} M < \infty$

However, examining the derivative of $f_n(z)$, we see

$$(f_n(z))' = f'(z)\,(f_{n-1}(z))' = \ldots = (f'(z))^n$$

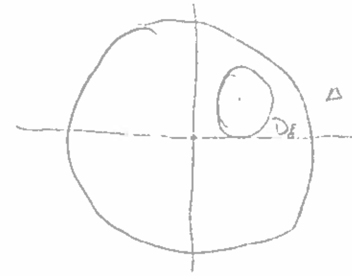
So $f_n'(z_0) = (f'(z_0))^n$. Since $|f'(z_0)| > 1$, $\lim_{n\to\infty} |f_n'(z_0)| = \infty$

But by Cauchy Estimate $|f_n'(z_0)| \leq \frac{M}{\rho} < \infty$. ↯

Thus by contradiction $|f'(z_0)| \leq 1$ ☑

4. Let $f_n : \Delta \to \Delta$ $n \geq 1$ be a sequence of holomorphic functions s.t. f_n has a zero of order m_n at 0, where $\lim_{n\to\infty} m_n = \infty$. By definition of zero

Observe that:



$$f_n(z) = z^{m_n} g_n(z), \quad g_n(0) \neq 0$$

$$f_n(0) = f_n'(0) = \dots = f_n^{(m_n-1)}(0) = 0 \quad \text{but } f_n^{(m_n)}(0) \neq 0.$$

We want to show that $\{f_n\}$ converges locally uniformly to zero on Δ

That is, for $z \in \mathbb{D}_\delta$, $|f_n(z)| < \varepsilon$, where $\delta, \varepsilon > 0$, $\mathbb{D}_\delta = \mathbb{D}_\delta(z_0) \subset \mathbb{D} = \Delta$

(Since Δ is open, we can find such a disk for every $z_0 \in \mathbb{D}$.)

Since $|f_n| < 1$ ($f_n : \Delta \to \Delta$), $f_n(0) = 0$, for fixed n, by Schwartz Lemma

$$|f_n(z)| \leq |z| \longrightarrow |z^{m_n} g_n(z)| = |z| \cdot |z^{m_n - 1} g_n(z)| \leq |z|$$

So for nonzero 0 $|z^{m_n-1} g_n(z)| \leq 1$. Again $(0)^{m_n-1} g(0) = 0$, so we can apply Schwartz Lemma again to get $|z^{m_n-1} g_n(z)| = |z| \, |z^{m_n-2} g_n(z)| \leq |z|$

doing this for m_n total iterations, we see $|g_n(z)| < 1$ for fixed n

Thus, for fixed n $|f_n(z)| = |z|^{m_n} |g(z)| \leq |z|^{m_n}$ $\forall$ $z \in D_\delta(z_0)$

Creating our $\mathbb{D}_\delta$ more precisely: let $z_0 \in \Delta$, Then $|z_0| < \rho < 1$ for some ρ.

So $z_0 \in \mathbb{D}(0, \rho) \subseteq \Delta$. By def of open (and since $\mathbb{D}(0,\rho)$ is open $\exists$ $\mathbb{D}_\delta(z_0) \subset \mathbb{D}(0,\rho)$ for some $\delta > 0$. Then for $z \in \mathbb{D}_\delta(z_0)$,

$$|f_n(z)| = |z|^{m_n} |g_n(z)| \leq |z|^{m_n} < \rho^{m_n} \xrightarrow[n\to\infty]{} 0 \quad \text{b/c } |\rho| < 1, \; m_n \to \infty$$

Thus $\{f_n\} \rightrightarrows 0$ on $\mathbb{D}_\delta(z_0)$. So $\{f_n\}$ converges locally unif to 0 on Δ ☑

Qualifying Exam   Real Analysis   August, 2014

Throughout m is Lebesgue measure.

1. Assume that E is a closed subset of $\mathbb{R}$. Prove or give a counterexample;

   (a) If E^c is dense then $m(E) = 0$.

   (b) If $m(E) = 0$ then E^c is dense.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and f a measurable function. If $f > 0$ on E a.e. and $\int_E f\,dm < \infty$, prove that

$$\lim_{n\to\infty} \int_E f^{1/n} dm = m(E).$$

3. Let f be absolutely continuous on $[0,1]$ with $f(0) = 0$ and $f' \in L^3([0,1])$. For which values of α does

$$\lim_{x\to 0+} x^{-\alpha} f(x) = 0$$

(-∞,0] (0,2/3) [2/3,1)
C.E. [1,∞)

for all such f?

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a measurable function.

   (a) Show that $E = \{(x,t) : |f(x)| > t\}$ is measurable in the product space $(X \times [0,\infty), \mathcal{A} \times \mathcal{L}, \mu \times m)$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $[0,\infty)$.

   (b) For $p > 0$ prove

$$\int_X |f|^p d\mu = \int_0^\infty p t^{p-1} \mu(x : |f(x)| > t) dt.$$

   (c) Prove that if $f \in L^p$ then

$$\lim_{t\to\infty} t^p \mu(x : |f(x)| > t) = \lim_{t\to 0+} t^p \mu(x : |f(x)| > t) = 0.$$

1. Assume E is a closed subset of $\mathbb{R}$.

a. If E^c is dense then $m(E)=0$

False. Consider $E=F$, the fat Cantor set (middle $1/4^{th}$s)

F is closed, nowhere dense so F^c is dense. However $|F|>0$.

b. If $m(E)=0$ then E^c is dense.

Consider a closed set E s.t. $m(E)=0$ Let $x_0 \in \mathbb{R}$, $\delta>0$.

Consider $I_\delta = (x_0-\delta, x_0+\delta)$.

By Caratheodory, $\forall I_\delta \subset \mathbb{R}$ (since E is meas, I_δ is meas switch $|\;|_e$ to $|\;|$)

$$2\delta = |I_\delta| = |I_\delta \cap E| + |I_\delta \setminus E|$$

$$= |I_\delta \cap E| + |I_\delta \cap E^c|$$

Monotonicity

$0 \le |I_\delta \cap E| \le |E| = 0$

$$= |E| + |I_\delta \cap E^c|$$

$$= |I_\delta \cap E^c|$$

So $|I_\delta \cap E^c| = 2\delta > 0 \rightarrow I_\delta \cap E^c \neq \emptyset \longrightarrow E^c$ is dense in $\mathbb{R}$ ☑

2 Let E be a Lebesgue measurable subset of $\mathbb{R}$ and f a measurable function

Suppose $f > 0$ on E a.e., $\int f\,dm < \infty$. Given that $X^{1/n}$ takes on different behaviors for $|X| \leq 1$ vs. $|X| > 1$, Consider two sets

$$E_1 = \{x \mid f(x) > 1\} \quad \text{and} \quad E_2 = \{x \mid f(x) \leq 1\}$$

Note $E_1 \cap E_2 = \emptyset$ So

$$\int_E f^{1/n}\,dm = \int_{E_1} f^{1/n}\,dm + \int_{E_2} f^{1/n}\,dm$$

For E_1 we will apply the dominated convergence using f as our dominating f. Justifying our choice of function, first note that $\int_{E_1} f \leq \int_E f < \infty$ by assumption Further, since $f > 1$ on E_1 $f^{1/n+1} < f^{1/n} < \dots < f^1 = f$

So f is an appropriate choice for our dominating function

By DCT

note $f < \infty$ a.e b/c $f \in \mathcal{L}^1$

$$\lim_{n\to\infty} \int_{E_1} f^{1/n}\,dm = \int_{E_1} \lim_{n\to\infty} f^{1/n}\,dm = \int_{E_1} f^0\,dm = \int_{E_1} 1\,dm = m(E_1)$$

For E_2 we will use the monotone convergence thm. Note $f > 0$ a.e. so $f^{1/n} \geq 0$ a.e further, since $f \leq 1$ $f^1 \leq f^{1/2} \leq f^{1/3} \leq \dots \leq f^{1/n}$. Therefore

$$\lim_{n\to\infty} \int_{E_2} f^{1/n}\,dm = \int_{E_2} \lim_{n\to\infty} f^{1/n}\,m = \int_{E_2} 1\,dm = m(E_2)$$

Therefore.

b/c $E_1 \cap E_2 = \emptyset$

$$\int_E f^{1/n}\,dm = \int_{E_1} f^{1/n} + \int_{E_2} f^{1/n} = m(E_1) + m(E_2) = m(E_1 \cup E_2) = m(E).$$ ☒

3. (Following from old solution) Let f be a.c. on $[0,1]$ w/ $f(0)=0$

and $f' \in \mathcal{L}^3([0,1])$. Since f is a.c. we know $f' \in \mathcal{L}^1$,

$f(x) = f(a) + \int_a^x f(t)\,dt$. Since $f' \in \mathcal{L}^3[0,1]$, $\left(\int_{[0,1]} |f'|^3\right)^{1/3} < \infty \longrightarrow \left(\int_{[0,1]} |f|^3\right) < \infty$

Note

$$|x^{-\alpha} f(x)| = \left| x^{-\alpha} \int_0^x f'(t)\,dt \right|$$

$$= \left| x^{-\alpha} \int_{\mathbb{R}} f'(t)\,\chi_{[0,x]}\,dt \right|$$

$$\leq x^{-\alpha} \int_{\mathbb{R}} |f'(t)| [\chi_{[0,x]}]\,dt$$

$$\leq x^{-\alpha} \left(\int_0^x |f'(t)|^3 dt\right)^{1/3} \left(\int \chi_{[0,x]}^{3/2}\,dt\right)^{2/3}$$

$$= x^{-\alpha} \left(\int_0^x |f'(t)|^3 dt\right)^{1/3} (x)^{2/3}$$

$$= x^{2/3-\alpha} \underbrace{\left(\int_0^1 |f'(t)|^3 dt\right)^{1/3}}$$

$\longrightarrow < \infty$ b/c $f' \in \mathcal{L}^3[0,1]$

If $\alpha = 2/3$ $\quad \lim_{x\to 0^+} x^0 \left(\int_0^x |f'(t)|^3 dt\right)^{1/3} \longrightarrow 0$

If $\alpha < 2/3$ $\quad \lim_{x\to 0^+} x^{2/3-\alpha} \left(\int_0^x |f'(t)|^3 dt\right)^{1/3} = 0 \cdot M = 0$

If $\alpha > 2/3$ $\quad \exists\, \beta$ s.t. $2/3 < \beta < \alpha$

$f = x^\beta \longrightarrow f' = \beta x^{\beta-1} \longrightarrow \int_0^1 \beta x^{\beta-1} \in \mathcal{L}^1[0,1] \to f$ a.c.

$\int |f'|^3 = \int |\beta x^{\beta-1}|^3 = \int \beta^3 x^{3\beta-3} < \infty \longrightarrow f = x^\beta$ is considered in our suppositions.

But $\rightarrow$ $x^{-\alpha} f(x) = x^{\beta-\alpha} \not\to 0$ since $\beta - \alpha < 0$

☑

$\boxed{\alpha \leq 2/3}$

4. Let (X, A, μ) be a measure space and $f: X \longrightarrow \mathbb{R}$ be a meas fn.

a. Consider $E_t = \{(x,t) : |f(x)| > t\}$. To show this is measurable, consider two functions: $F(x,t) = |f(x)|$, $G(x,t) = t$. Note $F(x,t)$ is measurable because

meas b/c f is meas, $|x|$ is cts

$$\{(x,t) \mid F(x,t) > \alpha\} = \{x \in X \mid |f(x)| > \alpha\} \times [0, \infty]$$

which are both measurable as the cartesian product of meas sets

Further G is measurable because

meas b/c t is cts

$$\{(x,t) \mid G(x,t) > \alpha\} = X \times \{t \in (0,\infty) \mid t > \alpha\}$$

which is again the cartesian product of two measurable sets.

Thus. $F - G$ is measurable, so $\{(x,t) \mid F - G > \alpha\}$ is measurable $\forall\ \alpha$

Specifically for $\alpha = 0$:

$$\{(x,t) \mid F - G > 0\} = \{(x,t) \mid |f(x)| - t > 0\} = \{(x,t) \mid |f(x)| > t\}$$ so

is measurable in $X \times [0, \infty]$

4.5 • For $p > 0$ consider

$$\int_0^\infty p t^{p-1} \mu(x : |f(x)| > t)\, dt = \int_0^\infty p t^{p-1} \mu(E_t)\, dt$$

$$= \int_0^\infty \int_{E_t} p t^{p-1}\, d\mu\, dt$$

$$= \int_0^\infty \int_X p t^{p-1} \chi_{E_t}\, d\mu\, dt$$

$$= \int_X \int_0^\infty p t^{p-1} \chi_{E_t}\, dt\, d\mu$$

b/c $t < |f(x)|$ on E_t

$$= \int_X \int_0^{|f(x)|} p t^{p-1}\, dt\, d\mu$$

$$= \int_X \left(t^p \Big|_0^{|f(x)|} \right) d\mu$$

$$= \int_X |f(x)|^p\, d\mu$$ ✓

c. Consider $f \in \mathcal{L}^p$.

$$\lim_{t \to \infty} t^p \mu(\{x \mid |f(x)| > t\}) = \lim_{t \to \infty} t^p \mu(E_t)$$

$$\leq \lim_{t \to \infty} t^p \cdot \frac{1}{t^p} \int_{E_t} |f|^p$$

$$= \lim_{t \to \infty} \int_{E_t} |f|^p$$

$$= \lim_{t \to \infty} \int_X |f|^p \chi_{E_t}$$

$$= 0 \quad (\text{since } f \in \mathcal{L}^p(X))$$

$$\lim_{t \to 0^+} t^p \mu(x . |f(x)| > t) = \lim_{t \to 0^+} t^p \mu(E_t)$$

$$= \lim_{t \to 0^+} \int_0^t p s^{p-1} ds \ \mu(E_t)$$

$$= \lim_{t \to 0^+} \int_0^t p s^{p-1} \cdot \mu(E_s) \, ds \quad \text{since } s < t$$

$$☆ \leq \lim_{t \to 0} \int_0^t M$$

$$= 0$$

☆. $p s^{p-1} \mu(E_s) \in \mathcal{L}^1[0, t_0]$, $t_0 > 0$. by absolute continuity.

$\hookrightarrow \forall \varepsilon \ \exists \delta$ s.t. $E \subset [0, t_0)$, $|E| < \delta \longrightarrow \left| \int_E p s^{p-1} \mu\{|f(x)| > s\} \, ds \right| < \varepsilon$

Qualifying Exam, Complex Analysis, January 11, 2013

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$.

1. Find a conformal map from the strip $\{0 < \mathbf{Re}\, z < 1\}$ onto Δ.

2. Let C denote the positively oriented boundary of the domain

$$D = \{z \in \mathbb{C} : -1/2 < \mathbf{Re}\, z < 2,\ |\mathbf{Im}\, z| < 2\}.$$

Find $\displaystyle\int_C \frac{z^n}{z^4 - 1}\, dz$, where $n \geq 0$ is an integer. Write your answer in algebraic form, $a + bi$.

3. Is there an entire function $f(z)$ such that $e^{f(z)}$ has a pole at ∞ ?

4. Suppose that f, g are holomorphic functions in Δ so that $f(0) = g(0) = 1$ and

$$(f'g - fg')(1/n) = 0$$

for all integers $n \geq 2$. Show that $f = g$ on Δ.

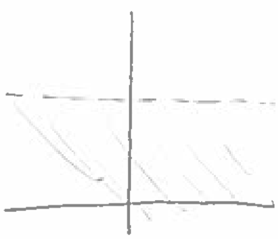
· J 2013.

1. $\{0 < \mathrm{Re} < 1\} \longrightarrow \Delta$

$\varphi_1 : D \to \{0 < \mathrm{Im} < 1\}$

$\varphi_1 = iz$

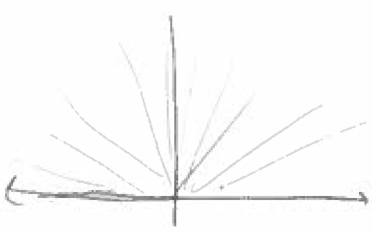
$\downarrow \; \varphi_1$



$\varphi_2 . \{0 < \mathrm{Im}\, z < 1\} \longrightarrow \mathbb{H}^+$

$\varphi_2 = e^{\pi i z} = e^{\pi i x} \cdot e^{\pi i y} = r \cdot e^{\pi i y}$

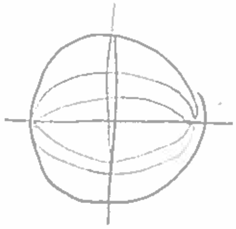
$\downarrow \; \varphi_2$



$\downarrow \; \varphi_3$

$\varphi_3 : \mathbb{H}^+ \longrightarrow \mathbb{D}$

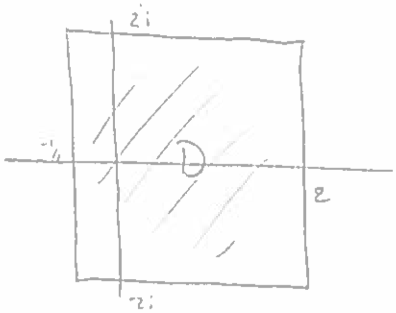
$\varphi_3 = \dfrac{z - i}{z + i}$



$$\varphi = \varphi_3(\varphi_2(\varphi_1(z))) = \boxed{\frac{e^{-\pi z} - 1}{e^{-\pi z} + 1}}$$

2. Let C denote the positively oriented boundary of the domain

$$D: \{ z \in \mathbb{C} : -1/2 < \text{Re } z < 2, \ |\text{Im } z| < 2$$



Want to find

$$\int_C \frac{z^n}{z^4 - 1}$$

$-1 < -1/2, \ i \notin D.$

Use residue thm. Let $f = \frac{z^n}{z^4-1}$, f has poles at $1, i, -1, -i$ in D

Calculating the residue at those poles using Gamelin's Rule 1 for simple poles

$$\text{Res}[f, 1] = \lim_{z \to 1} \frac{z^n}{(z-i)(z+i)(z+1)} = \frac{1^n}{2(1-i)(1+i)} = \frac{1^n}{2 \cdot 2} = \frac{1}{4}$$

$$\text{Res}[f, i] = \lim_{z \to i} \frac{z^n}{(z-1)(z+i)(z+1)} = \frac{i^n}{(i-1)(i+1)(2i)} = \frac{i^{n-1}}{2i+2} = \frac{i^{n-1}}{4i^3} = \frac{i^{n-4}}{4} = \frac{i^n}{4}$$

$$\text{Res}[f, -i] = \lim_{z \to -i} \frac{z^n}{(z-1)(z-i)(z+1)} = \frac{(i^3)^n}{(-i-1)(-i+1)(-2i)} = \frac{i^{3n}}{2i^3(-2i)^2} = \frac{i^{3n-5}}{4}$$

$$\int_C \frac{z^n}{z^4-1} = 2\pi i \sum_{i=1}^{3} \text{Res}[f, x_i] = 2\pi i \left(\frac{1}{4} + \frac{i^n}{4} + \frac{i^{3n-5}}{4} \right) = \frac{\pi i}{2}(1 + i^n + i^{3n-5})$$

$$= \pi/2 (i + i^{n+1} + i^{3n-5})$$

$$= \begin{cases} i-2 & n = 1 \bmod 4 \\ i & n = 2 \bmod 4 \\ i+2 & n = 3 \bmod 4 \\ i & n = 0 \bmod 4 \end{cases}$$

3. Is there an entire function $f(z)$ s.t. $e^{f(z)}$ has a pole at ∞?

No. Suppose $f(z)$ is constant, and thus entire. (Or by Liouville's: equivalently, any bounded entire function) Then $f(z) = k \ \forall z \in \mathbb{C}$, so $e^{f(z)} = e^k \ \forall z \in \mathbb{C}$

Thus $\lim_{|z| \to \infty} e^{f(z)} = \lim_{|z| \to \infty} e^k = e^k < \infty$.

So e^f does not have a pole at ∞.

Suppose $f(z)$ is nonconstant (and consequently unbounded, by the ~~inverse~~ converse of Liouville's)

Then, suppose e^f does have a pole @ ∞. Then

$$\lim_{|z| \to \infty} e^{f(z)} = \infty \longrightarrow \lim_{|z| \to \infty} 1/e^{f(z)} = 0.$$

That is, $\lim_{|z| \to \infty} e^{-f(z)} = 0$. Thus $\exists$ r s.t. $|z| > r$ $|e^{-f(z)}| < M_1$. (and $|e^{-f(z)}| < M_2$ on $|z| \leq r$. Thus $e^{-f(z)}$ is bdd. entire $\Rightarrow$ constant by Liouville's

Taking derivs $-f(z) e^{-f(z)} = 0$,

So $-f'(z) = 0$ or $e^{-f(z)} = 0$, Since the latter is never 0,

we conclude $f'(z) = 0$. This forces $f(z)$ to be constant, breaking our initial assertion Σ.

Therefore $e^{f(z)}$ cannot have a pole @ ∞. ☑

4. Suppose f, g are holomorphic functions in Δ s.t. $f(0) = g(0) = 1$ and

$$(f'g - fg')(1/n) = 0$$

$\forall\, n \geq 2$. Since f, g are holomorphic fns, f', g' are continuous and consequently $f'g - g'f$ is continuous. So: as

$$\lim_{n\to\infty} (f'g - g'f)(1/n) = (f'g - g'f)(0) = 0$$

~~Thus~~ Note $\left(\frac{f}{g}\right)'(0) = \dfrac{f'(0)g(0) - g'(0)f(0)}{(g(0))^2} = f'(0)(g(0)) - g'(0)f(0) = 0.$

Further. Since $f(1/n) \to f(0) = 1$, $g(1/n) \to g(0) = 1$

$\exists\, N \in \mathbb{N}$ s.t. for $n > N$ $f(1/n) > 0$, $g(1/n) > 0$. Therefore, for $n \geq N$

$$\frac{f'(1/n)g(1/n) - g'(1/n)f(1/n)}{(g(1/n))^2} = (f/g)'(1/n) = 0.$$

Let $E = \{\{1/n\}_{n\in\mathbb{N}} \cup \{0\}\}$, Note that 0 is a non isolated point.

So on E $(f/g)' = 0 \longrightarrow f/g = c$, c a constant

Since $f/g(0) = 1 \longrightarrow c = 1$.

Therefore $f = g$ on E Since E contains a non isolated point, $E \subset \Delta$, where Δ is the domain of f, g, by uniqueness principle

$f = g$ on Δ.

JANUARY 2013 QUALIFYING EXAM IN REAL ANALYSIS

Notation: m stands for the Lebesgue measure on the real line. The spaces $L^p([0,1])$ are understood with respect to m.

1. Suppose that $f\colon [-1,1] \to \mathbb{R}$ is a function of bounded variation. Prove that the function $g(x) = f(\sin x)$ belongs to $BV([a,b])$ for all $-\infty < a < b < \infty$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space such that for every set $A \in \mathcal{M}$ the measure $\mu(A)$ is a nonnegative integer. Suppose that $\{f_n\}_{n\geq 1}$ are measurable real-valued functions on X such that $\int_X |f_n|\, d\mu \to 0$ as $n \to \infty$. Prove that $f_n \to 0$ a.e.

3. Suppose that $f \in L^2([0,1])$. Prove that the function $g(x) = |f(x)|^{x+1}$ is in $L^1([0,1])$.

4. Suppose that $\{f_n\}$ is a sequence of nonnegative Borel measurable functions on $[0,1]$ such that $\int_0^1 f_n(x)\, dm(x) = 1$ for all n.

Which of the statements (a)–(d) follow from the above? Prove or give a counterexample to each.

(a) The set $A = \{x\colon f_n(x) \leq 2 \text{ for all } n\}$ is Borel

(b) The set $B = \{x\colon f_n(x) \leq 2 \text{ for infinitely many values of } n\}$ is Borel

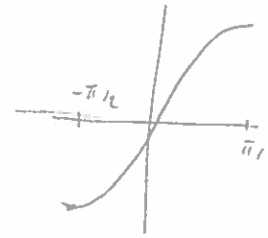
(c) $A \neq \varnothing$

(d) $B \neq \varnothing$

1. Suppose $f : [-1,1] \longrightarrow \mathbb{R}$ is BV. Consider a fn $g = f(\sin x)$. on $[a,b]$

$-\infty < a < b < \infty$. To show g is BV set $a = -\pi/2$, $b = \pi/2$. Note

$\sin x$ is monotone increasing on $[-\pi/2, \pi/2]$



Let $\Gamma_n = \{x_k\}_{k=0}^{n}$ be a partition of $[-\pi/2, \pi/2]$

Then $\Gamma_n' = \{y_k\}_{k=0}^{n}$ where $y_k = \sin(x_k)$ is

a partition of $[-1,1]$. Observe

$$\sum |g(x_i) - g(x_{i-1})| = \sum | f(\sin(x_i)) - f(\sin(x_{i-1}))|$$

$$= \sum | f(y_i) - f(y_{i-1}) |$$

$$\leq \sup_{\Gamma'} \sum_{i=1}^{n} |f(y_i) - f(y_{i-1})|$$

$$= V(f; -1, 1) < \infty$$

Hence $V[g, -\pi/2, \pi/2] \leq V[f; -1, 1] < +\infty$

Expanding this to our arbitrary $[a,b]$ consider $I_j := [-\pi/2 + \pi j, \pi/2 + \pi j]$ $j \in \mathbb{Z}$

For even j, any partition of I_j gives a partition of $[-1,1]$:

$$\Gamma_n = \{x_n\}_{k=0}^{n} \xrightarrow[y_k = \sin x_k]{\sin(x)} \Gamma_n' = \{y_k\}_{k=0}^{n}$$

By same arguement $V[g, I_j] < +\infty$ for even j.

For odd j, (e.g. $I = [\pi/2, 3\pi/2]$). $x_0 = -\pi/2 + \pi j < x_1 < \ldots < x_n = \pi/2 + \pi j$

$$\underset{y_0}{\sin(x_0)} > \underset{y_1}{\sin(x_1)} > \ldots > \underset{y_n}{\sin(x_n)}$$

Set $y_k = \tilde{y}_{n-k}$, yields Partition $\tilde{\Gamma} = \{\tilde{y}_k\}_{k=0}^{n}$ of $[-1,1]$

Again $\sum |g(x_i) - g(x_{i-1})| = \sum_{i=1}^{n} |f(y_i) - f(y_{i-1})|$

Can reorder finite sum $\rightarrow$ $= \sum_{i=n}^{1} |f(y_{i-1}) - f(y_i)|$

$= \sum_{i=1}^{n} |f(\tilde{y}_i) - f(\tilde{y}_{i-1})|$

$\leq \sup_{\Gamma'} \sum |f(y_i) - f(y_{i-1})|$

$< +\infty$

Hence $V[g, I_j] < \infty$ for j-odd.

So for any interval $I_n = [-\pi/2 + \pi(-n), \pi/2 + \pi n] = \bigcup_{j=-n}^{n} [-\pi/2 + \pi j, \pi/2 + \pi j]$

$V[g, I_n] < \infty$. b/c $V[g, I_n] = \sum_{j=-n}^{n} V[g, I_j] < \infty$

Since $[a,b]$ is bounded $\exists$ N s.t. [illegible]

- $[a,b] \subset \bigcup_{n=1}^{N} I_n$, so $V[g; a,b] \leq V[g; I_n] < +\infty$. ☑

2. Let (X, M, μ) be a measure space s.t. $\forall A \in M$ the measure $\mu(A)$ is a non negative integer. Suppose $\{f_n\}_{n \geq 1}$ are meas. real-valued functions on X s.t. $\int_X |f_n| \, d\mu \to 0$ as $n \to \infty$.

For $k, n \in \mathbb{N}$, let $E_{n,k} = \{x \in X : f_n(x) > 1/k\}$.

By Chebyshev's for each fixed k.

$$|E_{n,k}| \leq \frac{1}{1/k} \int_X |f_n(x)| \, d\mu \xrightarrow[n \to \infty]{} 0 \quad (\text{b/c } \int |f_n| \to 0, \forall \text{ fixed})$$

So $\exists N_k \in \mathbb{N}$ s.t. $n \geq N_k \Longrightarrow \mu(E_{n,k}) < 1$. Since μ is an integer $\mu(E_{n,k}) = 0$.

Let $E = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} E_{n,k} \to \mu(E) \leq \sum_{k=1}^{\infty} \left(\sum_{n=N_k}^{\infty} |E_{n,k}| \right) = 0$

Then $\mu(E) = 0$ (Since $|E_{n,k}| = 0$. We Claim $f_n \to 0$ on $X \setminus E$

Pick $x \in X \setminus E$. Let $\varepsilon > 0$. $\exists k_0 \in \mathbb{N}$ s.t. $1/k_0 < \varepsilon$. And for $n \geq N_{k_0}$, we have $|f_n(x)| \leq 1/k_0 < \varepsilon$,

Since $x \notin E_{n,k_0}$ for $n \geq N_{k_0}$.

Hence $f_n \longrightarrow 0$ p.w. on $X \setminus E \Longrightarrow f \to 0$ a.e. ☑

3. Suppose $f \in \mathcal{L}^2([0,1])$. Consider the ~~fu~~ function $g(x) = |f(x)|^{x+1}$

$$\int_0^1 g(x)\,dx = \int_0^1 |f(x)|^{x+1} dx = \int_0^1 |f(x)|^x |f(x)|\,dx$$

$$\text{(Holder's)} \quad \le \left(\int_0^1 (|f(x)|^x)^2 dx\right)^{1/2} \Bigg(\underbrace{\int_0^1 |f(x)|^2}_{<\infty,\ f\in\mathcal{L}^2 \checkmark}\Bigg)^{1/2}$$

$$= \left(\int_0^1 |f(x)|^{2x} dx\right)^{1/2} \left(\int_0^1 |f(x)|^2\right)^{1/2}$$

$E_1 \cap E_2 = \phi$

$E_1 = \{ f(x) > 1 \} \cap [0,1]$

$E_2 = \{ f(x) \le 1 \} \cap [0,1]$

$$= \left(\int_{E_1} f(x)|^{2x} dx + \int_{E_2} |f(x)|^{2x} dx\right)^{1/2} \left(\int_0^1 |f(x)|^2\right)^{1/2}$$

$$\le \left(\left(\int_0^1 |f(x)|^{2x} \chi_{E_1} dx\right)^{1/2} + \left(\int_0^1 |f(x)|^{2x} \chi_{E_2}\right)^{1/2}\right)\left(\int_0^1 |f(x)|^2\right)^{1/2}$$

$$\le \Bigg(\Bigg(\underbrace{\int_0^1 |f(x)|^2 dx}_{\hat{\infty}}\Bigg)^{1/2} + \left(\int_0^1 |f(x)|^0 dx\right)\Bigg)\left(\int_0^1 |f(x)|^2\right.$$

$\le \infty$ because all components are finite

$f \in \mathcal{L}^2([0,1]) \to \left(\int |f|^2\right)^{1/2} < \infty$, $\exists M \in \mathbb{R}$ s.t.

$$\left(\int |f|^2\right)^{1/2} = M < \infty$$

4. Suppose $\{f_n\}$ is a sequence of nonnegative Borel measurable fns on $[0,1]$ s.t.

$\int_0^1 f_n(x)\, dm(x) = 1 \quad \forall n$

a. True

a. Consider the set $A = \{x : f_n(x) \leq 2 \ \forall n\}$

$A = \bigcap_{n=1}^{\infty} \{f_n(x) \leq 2\}$, $\{f_n(x) \leq 2\}$ are Borel b/c f is Borel

A is countable intersection of Borel & is thus Borel

b. True

$B = \{x : f_n(x) \leq 2$ for infinitely many values of $n\}$

$= \limsup_{n \to \infty} \{f_n(x) \leq 2\} = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \{f_n(x) \leq 2$

Again, Borel By same argument.

c. False. A is not necessarily nonempty.

Consider E_n = the piano key sequence.

$E_1 = [0,1]$
$E_2 \ [0, 1/2]$
$E_3 = [1/2, 1]$
$\vdots$

Consider $f_n(x) = \frac{1}{|E_n|} \chi_{E_n}$. → This is Borel b/c E_n are borel,

Then $\int_x f_n = \frac{1}{|E_n|} \cdot |E_n| = 1$ ✓ However $\nexists\, x$ s.t. $f_n(x) \leq 2 \ \forall n$.

(In fact after $n = 3$, $n \geq 4$ $f_n(x) > 2$ or $f_n(x) = 0$) → would parse this better. on test to add justification

d. True. Proceeding by contradiction

Assume $B = \emptyset$. Let $E_n = \{ x \in [0,1] : f(x) > 2 \}$

Let $H_n = \bigcap_{k=n}^{\infty} E_k$. $H_n \nearrow [0,1] \Longrightarrow |H_n| \nearrow 1$

So $\exists$ N s.t. $|H_n| = 3/4$. Then

$$\frac{3}{2} = \int_{H_n} 2 \leq \int_{H_n} f_n = 1$$

b/c $f(x) > 2$; $= 1$ by assumption

But $1 < 3/2$ $\lightning$.

Thus $B \neq \emptyset$.

No solutions, just scratch work for #1.

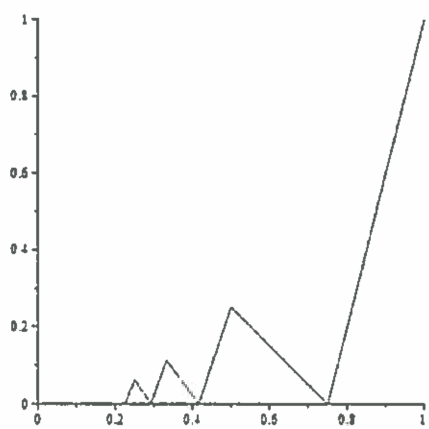
QUALIFYING EXAM, Measure Theory, August 2012

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let m be Lebesgue measure on the Borel sets of $\mathbb{R}$. For the following statement, prove OR provide a counterexample (with the details showing it is indeed a counterexample): For all Borel sets $E \subset \mathbb{R}$, if $m(E) = 0$ then $m(f(E)) = 0$.

Problem 2. A sequence of (Lebesgue) measurable functions f_n on $\mathbb{R}$ is said to converge *almost uniformly* to the measurable function f on $\mathbb{R}$ if and only if for each $\epsilon > 0$ there is a measurable set $E \subset \mathbb{R}$ such that $m(E) < \epsilon$ and $f_n \to f$ uniformly on $\mathbb{R} \setminus E$.

Give an example of $f_n \to f$ pointwise almost everywhere but NOT $f_n \to f$ almost uniformly. Show that your example works.

Problem 3. On $[0,1] \subset \mathbb{R}$ set $g(x) = \sqrt{x}$. Define f on $[0,1]$ by $f(\frac{1}{n}) = \frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$, $f\left(\frac{\frac{1}{n} + \frac{1}{n+1}}{2}\right) = 0$ for $n = 1, 2, 3, \ldots$, and otherwise f is linear. See the figure where the first few linear pieces of f are graphed.



(i) is g absolutely continuous? Why or why not.
(ii) is f absolutely continuous? Why or why not.
(iii) is $g \circ f$ absolutely continuous? Why or why not.

Problem 4. (i) For a space X with measure μ and $\mu(X) < \infty$, prove that $L^q \subset L^p$ for $0 < p < q < \infty$. (ii) Suppose that X contains disjoint sets E_k for $k = 1, 2, \ldots$ with $0 < \mu(E_k) < 2^{-k}$. Show that L^p is not contained in L^q.

Jeremy's.

$\varepsilon > 0$, $f(I_i)$

$E \subset [0, \infty)$ $E \subset \bigcup_i I_i \longrightarrow \sum |I_i| < \varepsilon$.

$f(E) \subset f(\bigcup I_i) \subset \bigcup f(I_i)$

this

WTS $|f(I_i)| \le M |I_i|$

$|E \cap [k, k+1]| = |E| = 0$ by monotonicity

$f(x)$ is Lipshitz on $E \cap [k, k+1]$

$\hookrightarrow f(E \cap [k, k+1]) = 0$

$|f(E)| \subset f(\bigcup_k E \cap [k, k+1])$
$= \sum |f(E \cap k_i)| = 0$

$M = \{ E \subset \mathbb{R} \mid |E| = 0 \Rightarrow |f(E)| = 0 \}$

Let $E_i \in M$ for $i \in \mathbb{N}$.

$\phi \in M$; $\mathbb{R} \notin M$

WTS $\bigcup E_i \in M$. If $|\bigcup E_i| > 0 \Rightarrow \checkmark$ b/c quality only applies to $|\bigcup E_i| = 0$

$|\bigcup E_i| = 0 \Rightarrow \sum |E_i| = 0 \rightarrow |E_i| = 0$ by monotonicity

$\hookrightarrow |f(E_i)| = 0$

$\hookrightarrow |f(\bigcup E_i)| \rightarrow |\bigcup f(E_i)| \le 0$

E^c in there vacuously.

If $E \in M$ | $|E| = 0 \rightarrow |E^c| = |\mathbb{R} \setminus E|$ $|\mathbb{R}| - |E| = |\mathbb{R}| > 0$

$E^c \in M$ $|E| > 0 \longrightarrow |E^c| > 0 \rightarrow \checkmark$

If $|E^c| = 0$ $|E^c| = 0 \rightarrow |f(E^c)| = |f(E)^c| = |\mathbb{R} \setminus f(E)|$
$= |f(\mathbb{R})| = 0$.

$|E| = \mathbb{R} \rightarrow |f(E)| = |\mathbb{R}|$ $|\mathbb{R}| = |\mathbb{R} \cap E| + |\mathbb{R} \setminus E|$

$|\mathbb{R}| = |\mathbb{R} \cap E| + |E^c|$

1. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let m be Lebesgue meas on Borel sets of $\mathbb{R}$. Prove that:

$\forall$ Borel sets $E \subset \mathbb{R}$, if $m(E) = 0$ then $m(f(E)) = 0$

· If E open (or G_δ) $m(E) > 0$ b/c intervals.

↳ E closed or F_σ

Let $E \subset \mathbb{R}$ s.t. $m(E) = 0$. Suppose $m(f(E)) \neq 0$.

Then $| \{ y \in f(E) \mid y = x^2 \text{ for } x \in E \} |$

Consider a Borel set E. $\longrightarrow$ $f(E) = x^2$ ~~E~~

Since f is cts f is meas, more specifically Borel measurable

↳ $f^{-1}(\mathcal{B}) = \mathcal{B}'$ $\longrightarrow$ show $f^{-1}(A)$ is a σ alg. ⇒ $\subset$ all open

$\longrightarrow$ Borel are in there $\longrightarrow$ $f^{-1}(\mathcal{B}) = \mathcal{B}$

So f ?(E)

A.C. gives $m(E) = 0 \Longrightarrow m(f(E)) = 0$

Assume f is abs cts. Let $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $\sum |b_j - a_j| < \delta \Rightarrow \sum |f(b_j) - f(a_j)| < \varepsilon$

$\longrightarrow \exists\ U$ s.t. $U = \bigcup_1^\infty (a_j, b_j)$ + $m(U) = \sum |b_j - a_j| < \delta$, $E \subset U$

f attains max and min @ x_i, y_i respectively. $\rightarrow \sum |x_j - y_j| < \sum |b_j - a_j| < \delta$

$f(E) \subset \bigcup_i^\infty |f(x_i) - f(y_i)|$ since $E \subset U$

$\Rightarrow m(f(E)) \leq \sum_i^\infty |f(x_j) - f(y_j)| < \lim$ —— $< \lim \varepsilon = \varepsilon^{d N}$ ...

Problem 1. Let f be a measurable function satisfying

$$|f(x)| \leq \frac{x^2}{1+x^4}, \quad -\infty < x < \infty.$$

a. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(nx)\,dx = 0.$$

b. Is it necessarily true that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f\left(\frac{x}{n}\right)\,dx = 0?$$

Problem 2. Let f be an integrable function satisfying $\int_0^1 f(x)\,dx = 0$. Prove that there are intervals I of arbitrarily small positive length such that

$$\int_I f(x)\,dx = 0.$$

Problem 3. Formulate and prove a version of Hölder's inequality for products of three functions. It is sufficient to obtain an upper bound on $\int_0^1 f(x)g(x)h(x)\,dx$ for non-negative measurable functions f, g, and h in terms of suitable L^p norms of the individual functions. It is permissable to use the usual (two function) Hölder inequality without proof.

Problem 4. Let C be a closed set of positive Lebesgue measure and $f(x) = d(x, C)$, the distance from the point x to the set C. Prove that there exist points x at which the derivative of f vanishes. Give an example of a closed set of measure zero for which there is no such point x.

Generalized Holder's for (3) Aug 2013 #3.

$$\int_0^1 f(x)g(x)h(x) \le \|f\|_p \|g\|_q \|h\|_r \quad \text{where } \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$$

$$\int_0^1 f(x)\, g(x)\, h(x) \le \left(\int_0^1 f(x)\, g(x)^{pq/p+q}\right)^{\frac{q+p}{pq}} \left(\int_0^1 h(x)^r\right)^{1/r}$$

$$= \left(\int_0^1 f(x)^{\frac{pq}{p+q}}\, g(x)^{pq/p+q}\right)^{\frac{q+p}{pq}} \|h(x)\|_r$$

$$\le \left(\left(\int_0^1 f(x)^{\frac{pq}{p+q}\cdot\frac{p+q}{q}}\right)^{q/p+q} \left(\int g^{\frac{pq}{p+q}}\right)^{-}\right) \|h\|_r.$$

$$= \|f\|_p \, \|g\|_q \, \|h\|_r.$$

Topics for Qualifying Exam in Complex Analysis

I Complex Plane and Elementary Function.

a) Complex Numbers
b) Polar Representation
c) Stereographic Projection
d) The Square and Square Root Functions
e) The Exponential Function
f) The Logarithm Function
g) Power Functions and Phase Factors
h) Trigonometric and Hyperbolic Functions

II Analytic Functions

a) Review of Basic Analysis
b) Analytic Functions
c) The Cauchy-Riemann Equations
d) Inverse Mappings and the Jacobian
e) Harmonic Functions
f) Conformal Mappings
g) Fractional Linear Transformations

III Line Integrals and Harmonic Functions

a) Line Integrals and Green's Theorem
b) Independence of Path
c) Harmonic Conjugates
d) The Mean Value Property
e) The Maximum Principle

IV Complex Integration and Analyticity

a) Complex Line Integrals
b) Fundamental Theorem of Calculus for Analytic Functions
c) Cauchy's Theorem
d) The Cauchy Integral Formula
e) Liouville's Theorem
f) Morera's Theorem
g) Goursat's Theorem
h) Complex Notation and Pompeiu's Formula

V Power Series

a) Infinite Series
b) Sequences and Series of Functions
c) Power Series
d) Power Series Expansion of an Analytic Function
e) Power Series Expansion at Infinity
f) Manipulation of Power Series
g) The Zeros of an Analytic Function

h) Analytic Continuation

VI Laurent Series and Isolated Singularities
- a) The Laurent Decomposition
- b) Isolated Singularities of an Analytic Function
- c) Isolated Singularity at Infinity
- d) Partial Fractions Decomposition

VII The Residue Calculus
- a) The Residue Theorem
- b) Integrals Featuring Rational Functions
- c) Integrals of Trigonometric Functions
- d) Integrands with Branch Points
- e) Fractional Residues
- f) Principal Values
- g) Jordan's Lemma
- h) Exterior Domains

VIII The Logarithmic Integral
- a) The Argument Principle
- b) Rouche's Theorem
- c) Hurwitz's Theorem
- d) Open Mapping and Inverse Function Theorems

IX The Schwarz Lemma and Hyperbolic Geometry
- a) The Schwarz Lemma
- b) Conformal Self-Maps of the Unit Disk

X Harmonic Functions and the Reflection Principle
- a) The Poisson Integral Formula
- b) Characterization of Harmonic Functions
- c) The Schwarz Reflection Principle

XI Conformal Mapping
- a) Mappings to the Unit Disk and Upper Half-Plane
- b) The Riemann Mapping Theorem
- c) Compactness of Families of Functions
- d) Proof of the Riemann Mapping Theorem

<u>References: Complex Analysis by T.W. Gamelin</u>

2012
Jan 2015

Complex Part

1. Suppose that $f(z) = u(x, y) + iv(x, y)$ is a function on a domain D and $z_0 \in D$. Show that if: a) u and v are differentiable at z_0; b) the limit

$$\lim_{\Delta z \to 0} \left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right|$$

exists, then either $f(z)$ or $\bar{f}(z)$ are complex differentiable at z_0.

2. Suppose that f is an analytic function on a disk $\{|z| < 2r\}$ given by a series $\sum_{n=0}^{\infty} c_n z^n$. Show that the series

$$F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

converges on $\mathbb{C}$ and $|F(z)| \leq M e^{|z|/r}$, where

$$M = \max_{|z|=r} |f(z)|.$$

3. Let $\mathcal{F}$ be a family of analytic functions on the open unit disk $\mathbb{D}$ such that $\Re f(z) \geq 0$ for each $f \in \mathcal{F}$ and $z \in \mathbb{D}$. Show that every sequence of functions in $\mathcal{F}$ contains a subsequence converging normally to a function in $\mathcal{F}$ or ∞.

4. Let f be a nonconstant analytic function on the unit disk $\mathbb{D}$ and let $U = f(\mathbb{D})$. Show that if ϕ is a function on U (not necessarily even continuous) and $\phi \circ f$ is analytic on $\mathbb{D}$, then ϕ is analytic on U.

Aug. 2011

1. Under what conditions on complex numbers a and b the linear function $ax + by$ is analytic as a function of $z = x_iy$?

2. Find the formula for entire analytic functions which have a simple 0 at 0. What entire analytic functions have simple zero at ∞?

3. Let f be a conformal mapping of a disk. Show that f' is never equal to 0.

4. Let $D \subset \mathbb{C}$ is a domain and $\{f_j\}$ is a sequence of analytic functions on D such that the functions

$$g_n(z) = \sum_{j=1}^{n} |f_j(z)|$$

converge normally on D. Show that the functions

$$h_n(z) = \sum_{j=1}^{n} |f_j'(z)|$$

also converge normally on D.

Qualifying Exam, Complex Analysis, August 2010

1. Let $n > 0$ be an integer. How many solutions does the equation $3z^n = e^z$ have in the open unit disk? Justify your answer in full detail.

2. Let $f(z) = \sum_{n\geq 0} a_n z^n$ be holomorphic in the unit disk U such that

$$|f'(z)| \leq \frac{1}{1-|z|}\ , \ \forall\, z \in U.$$

Prove that $|a_n| \leq e$ for all $n \geq 1$.

3. Are there any entire functions f which satisfy $|f(z)| \geq \sqrt{|z|}$ for all $z \in \mathbb{C}$? Justify your answer in full detail.

4. Show that the function $I(z) = \displaystyle\int_{-\infty}^{+\infty} e^{-(t-z)^2} dt$, $z \in \mathbb{C}$, is constant.

Jan 2010

Complex Part

1. Show that the function $f(z) = 1/z$ has no a holomorphic anti-derivative on $\{< 1 < |z| < 2\}$.

2. Suppose that f is an entire function and f^2 is a holomorphic polynomial. Show that f is also a holomorphic polynomial.

3. Suppose that a function f is meromorphic on the unit disk $\mathbb{D}$ and continuous in a neighborhood of its boundary $\partial\mathbb{D}$. Show that for any number A such that $|A| > \sup_{z\in\partial\mathbb{D}} |f(z)$ the number of zeros of the function $f - A$ is equal to the number of poles of f in $\mathbb{D}$.

4. Suppose that f and g are entire functions such that $f \circ g(x) = x$ when $x \in \mathbb{R}$. Show that f and g are linear functions.

QUALIFYING EXAM COMPLEX ANALYSIS

Thursday, January 8, 2009

Show **ALL** your work. Write all your solutions in clear, logical steps. **Good luck!**

Your Name:

Problem	Score	Max
1		20
2		20
3		30
4		30
Total		100

Problem 1. Let $f = f(z)$ be analytic in the unit disk, $f(0) = 0$. Show that the infinite series

$$\sum_{n=1}^{\infty} f(z^n)$$

is converging and represents an analytic function in the unit disk.

Problem 2.

Consider an analytic function defined in the unit disk by the following power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \,, \quad \text{where the coefficients are real numbers such that} \quad n^{-2009} \leqslant a_n \leqslant n$$

Show that f does not extend analytically near the point $z = 1$.

Problem 3. (Cauchy Formula)
Let $\mathbb{F}$ be a countable compact subset of a domain $\Omega \subset \mathbb{C}$. Suppose we are given a bounded holomorphic function

$$f : \Omega \setminus \mathbb{F} \to \mathbb{C}$$

Show that f extends holomorphically to the entire domain Ω.

a) First try a simple case when $\mathbb{F}$ is finite
b) Try the case when $\mathbb{F}$ has finite number of accumulation points
c) Try the general case.
d) The problem still remains valid if $\mathbb{F}$ is a compact set of zero length (1-dimensional Hausdorff measure), try to extend your proof to this general case. Recall that $\mathbb{F}$ has zero length if it can be covered by a finite number of disks whose diameters sum up to a number as small as we wish.

Problem 4.
Compute the following integral

$$\int_0^\infty \frac{\cos x}{(1+x^2)^2}\, dx$$

Hint. Consider the following complex function in the upper half plane

$$f(z) = \frac{e^{iz}}{(1+z^2)^2}$$

Qualifying Exam, Complex Analysis, August 2008

1. Let f be an entire function, $a \in \mathbb{C}$ and $r > |a|$. Show that

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{f(1/z)}{z-a}\, dz = f(0) .$$

2. Find the image of the first quadrant $\{x > 0, y > 0\}$ under the Möbius map $w = \dfrac{z-i}{z+i}$.

3. Find all the continuous functions $v : \mathbb{C} \to \mathbb{R}$ which have the property that for every rectangle $R \subset \mathbb{C}$ with sides parallel to the coordinate axes

$$\int_{\partial R} v\, dx = -area\ R \ , \quad \int_{\partial R} v\, dy = 0 ,$$

where ∂R is traversed counterclockwise. (Hint: Consider the function $f(z) = x + iv(x, y)$, where $z = x + iy$.)

4. Suppose that

$$f(z) = 1 + c_1 z + c_2 z^2 + \dots$$

is a holomorphic function on the closed unit disc $\overline{\Delta}$ such that $|f(z)| \leq M$ for $|z| = 1$. If $z_0 \in \Delta$ is a zero of f show that

$$|z_0| \geq \frac{1}{M+1} .$$

Qualifying Exam, Complex Analysis, January 11, 2008

Notation: Throughout the exam U denotes the open unit disc in $\mathbb{C}$.

1. Show that a complex valued function $h(z)$ on U is harmonic if and only if

$$h(z) = f(z) + \overline{g(z)},$$

where $f(z)$ and $g(z)$ are analytic on U.

2. Find $\displaystyle\int_{|z|=1} z^n \cos z \, dz$, where $n \in \mathbb{Z}$.

3. Find all the possible Laurent expansions centered at 0 of the function

$$f(z) = \frac{4z^2}{(z+1)(z-3)}.$$

Specify the annulus of convergence for each such expansion.

4. (i) Show that the Möbius transformation $h(z) = \dfrac{z-a}{1-\overline{a}z}$, where $a \in U$, is a conformal self-map of U.

(ii) Let $f : U \to U$ be a holomorphic function and assume that $a_1, \dots, a_n \in U$ are zeros of f. Prove that $|f(0)| \leq |a_1 \dots a_n|$.

Qualifying Exam, Complex Analysis, August 22, 2006

1. Find a conformal map from the strip $\{0 < \operatorname{Im} z < 1\}$ onto the unit disk.

2. Find $\displaystyle\int_{|z|=2} \frac{\sin(\pi z)}{z^2(1-z)}\, dz$.

3. Let f be a holomorphic function on the closed disk $\Delta_R = \{z \in \mathbb{C} : |z| \le R\}$. Show that

$$|f'(0)| \le \frac{3}{2\pi R^3} \iint_{\Delta_R} |f(z)|\, dx\, dy \, .$$

4. Suppose that f_n are holomorphic functions on a domain D and $\displaystyle\sum_{n=1}^{\infty} |f_n|$ converges locally uniformly on D. Show that $\displaystyle\sum_{n=1}^{\infty} |f_n'|$ converges locally uniformly on D.

Real analysis qualifying exam Aug. 22, 2006

1. Let $E \subset \mathbf{R}$ denote a countable set.

(a) Compute the Lebesgue measure of E.

(b) Construct an E that is a G_δ set (countable intersection of open sets).

(c) Construct an E that is not a G_δ set.

2. Give an example of a sequence $\{f_n\}$ for each of the requirements below or show that no such sequence exists. L^1 denotes the Lebesgue integrable functions on $\mathbf{R}$.

(a) $0 \le f_n \to 0$ in L^1, but $\{f_n\}$ does not converge pointwise a.e. to zero.

(b) $0 \le f_n \to 0$ a.e., but $\{f_n\}$ does not converge in L^1 to zero.

(c) $0 \le f_n \to f$ a.e. and $\int f_n \le 1$, but $f \notin L^1$.

3. Given a $p \ge 1$ let $f \in L^p([0,1])$ with respect to Lebesgue measure m, and let $E \subset [0,1]$ be measurable. Put $\nu(E) = \int_E f\,dm$.

(a) Show that ν is a complex measure absolutely continuous with respect to m.

(b) Let $g(x) = \nu([0,x])$ for each $x \in [0,1]$. Prove

$$\|g\|_p \le \left(\frac{1}{p}\right)^{\frac{1}{p}} \|f\|_p$$

4. For some $1 \le p \le \infty$ let $T : L^p(\mathbf{R}) \to L^p(\mathbf{R})$ be a continuous linear operator. Suppose $\|f\|_p \le \|Tf\|_p$ for all $f \in L^p(\mathbf{R})$.

(a) Show there exists a real constant C independent of f so that

$$\|Tf\|_p \le C\|f\|_p$$

for all f.

(b) Show T is $1:1$.

(c) Show T has closed range, i.e. whenever $Tf_j \to g$ in L^p there exists $f \in L^p$ such that $Tf = g$.

Qualifying Exam, Complex Analysis, January 28, 2006

1. Find a conformal map from the half-disk $\{z : |z-1| < 1, \ \operatorname{Im} z > 0\}$ onto the upper half-plane $\{\operatorname{Im} w > 0\}$.

2. Find $\displaystyle\int_{|z|=1} z^n e^{1/z} dz$, where n is an integer.

3. Let f be a holomorphic function on $U \setminus \{0\}$, where U is the open unit disk, such that $f(1/2) = 2$ and the function

$$g(z) = \bar{z}\, |f(z)|^2$$

is holomorphic on $U \setminus \{0\}$. Find f.

4. Let f be a holomorphic function in $U \setminus \{0\}$, where U is the open unit disk, which satisfies

$$|f(z)| \le -\log|z|, \ \forall\, z \in U \setminus \{0\}.$$

Prove that $f = 0$.

FALL 2005

Measure Theory Part

1. Let $\{r_n\}_{n=1}^{\infty}$ be the rationals, $f(x) = x^{-1/2}$ for $0 < x < 1$ and 0 otherwise, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Is $f(x)$ measurable? Why? Is $g(x)$ measurable? Why? What is the set of points of discontinuity of g? Is g integrable? Why? Show that g is not in L^2 on any interval.

2. Let μ be Lebesgue measure on the borel sets of the real line, and define $\nu(E)$ to be 1 if $0 \in E$ and 0 if $0 \notin E$ for all borel sets E. Is ν a measure? σ finite? Compute $\frac{d\nu}{d\mu}$.

3. Define L^p (Lebesgue measure). Is $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$? Why? Is $L^2(0,1) \subset L^1(0,1)$? Why?

4. Let $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise and $\|g_k\|_\infty \leq M$ for all k. Prove that $f_k g_k \to fg$ in L^p.

Complex Part

1. Let f be an analytic function on the unit disk and $f(z)$ is real when z is real. Show that $\bar{f}(\bar{z}) = f(z)$.

2. Let $\{f_n\}$ be a sequence of continuous functions on the closed unit disk that are analytic in the open unit disk. Suppose $\{f_n\}$ converges uniformly on the unit **circle**. Show that $\{f_n\}$ converges uniformly on the closed unit disk.

3. Suppose that f is an analytic function on an open set containing the closed unit disk, $|f(z)| = 1$ when $|z| = 1$ and f is not a constant. Prove that the image of f contains the closed unit disk.

4. Let $\mathcal{F}$ be a family of analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

on the open unit disk such that $|a_n| \leq n$ for each n. Show that $\mathcal{F}$ is normal, i.e. every sequence of functions in $\mathcal{F}$ contains a subsequence converging normally to a function in $\mathcal{F}$.

Complex Analysis
Fall 2004 & Spring 2005

1. Find all points where the polynomial $p(z,\bar{z}) = 1 + 2z + \bar{z} + z\bar{z}^2 + z^2\bar{z} + i\bar{z}^2$ is complex differentiable.

2. Find the maximal radius of the disks centered at 0, where the function $f(z) = \frac{z}{\sin z}$ can be represented by a Taylor series.

3. Suppose that a function f is holomorphic in a neighborhood of the origin and $f(z) = f(2z)$ whenever z and $2z$ are in this neighborhood. Show that f is constant.

4. Show that the function $f(z) = \bar{z}$ cannot be uniformly approximated on the unit circle by polynomials of z.

5. Show that an entire function $f(z)$ such that $|f(z)| \geq |z|^N$ for sufficiently large N is a polynomial.

6. If function f_j, $j = 1, 2, \ldots$, are holomorphic and uniformly bounded in the unit disk are not equal to 0 there and $f_j(0) \to 0$ as $j \to \infty$, then $f_j \to 0$ uniformly on compacta in the unit disk.

7. If f is holomorphic and bounded in $\{\operatorname{Im} z \geq 0\}$, real on the real axis, then f is constant.

Topics for Qualifying Exam in Analysis
MAT 701

1. σ-algebras
2. Measures, outer measures, Borel measures
3. Measurable functions
4. Lebesgue integration in abstract measure spaces and in **R**, Lebesgue measure
5. L^p spaces, Holder's and Minkowski's inequalities, approximation by continuous functions, duality of L^p and L^q.
6. Radon-Nikodym theorem, Lebesgue points, absolutely continuous functions, functions of bounded variation, fundamental theorem of calculus
7. Product measures, Fubini's theorem

References:

- Real Analysis, 2nd ed., Gerald Folland
- Real and Complex Analysis, 3rd ed., Walter Rudin
- Measure and Integral, Richard Wheeden and Antoni Zygmund
- Real Analysis, 3rd ed., H.L. Royden

MAT 701 **Qualifying Exam** **January 2012**

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from MAT 701, MAT 601, and MAT 602.

Problem 1. Let f_n be *non-decreasing* functions on $(-\infty, 0]$ such that $f_n \to 0$ in (Lebesgue) measure as $n \to \infty$. Proof or counterexample: Necessarily $f_n \to 0$ almost everywhere on $(-\infty, 0]$ with respect to Lebesgue measure.

Problem 2. Prove that any function $f \in L^p([0,1]^2), 1 \leq p < \infty$, can be approximated by a finite linear combination of functions of the form $h(x)g(y)$ with h and g continuous on [0,1]. More precisely, given $\epsilon > 0$ there is a function

$$u(x,y) = \sum_{j=0}^{n} h_j(x)g_j(y)$$

with h_j and g_j continuous on [0,1] for $j = 1, 2, \ldots, n$, such that $\|f - u\|_p < \epsilon$.

Problem 3. Let f be a continuous real-valued function on the real line that is differentiable almost everywhere with respect to Lebesgue measure and satisfies $f(0) = 0$ and

$$f'(x) = 2f(x)$$

almost everywhere. Prove that there exist infinitely many such functions, but that only one of them is absolutely continuous.

Problem 4. Let μ and ν be measures on the same measurable space. Assume that μ is finite, and define a set function μ_0 by

$$\mu_0(A) = \sup\{\mu(A \cap B) : \text{ B is measurable and } \nu(B) < \infty\}$$

for measurable sets A. Also define a set function λ on measurable sets A by $\lambda(A) = \mu(A) - \mu_0(A)$. Prove that both μ_0 and λ are measures, and that λ has the property that $\lambda(A) > 0$ implies $\nu(A) = \infty$ for measurable sets A.

Qualifying Exam Summer 2011 Analysis

(1) In Euclidean space $\mathbb{R}^n$ with Lebesgue measure m, for $k \in \mathbb{N}$ and some $1 < p < \infty$ let $f, f_k \in L^p$ with $f_k \to f$ pointwise a.e. as $k \to \infty$. Assume that $\|f_k\|_p \leq M < \infty$ for all $k \in \mathbb{N}$. Also, let $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Prove or provide a counterexample to the statement: $\|f\|_p \leq M$.

(b) True or False, explain your answer. For all $R > 0$, for all $\delta > 0$ there is $F \subset \{x \in \mathbb{R}^n \mid |x| < R\} = B(0, R)$ with $m(F) < \delta$ and $f_k \to f$ uniformly on $B(0, R) \setminus F$.

(c) Prove or provide a counterexample to the statement: For all $\epsilon > 0$ there is a $R_0 > 0$ so that $\left(\int_{|x| \geq R} |g|^q \, dm\right)^{1/q} < \epsilon$ whenever $R > R_0$.

(d) True or False, explain your answer. For all $\epsilon > 0$ there is a $\delta > 0$ so that for all $E \subset \mathbb{R}^n$ if $m(E) < \delta$ then $\int_E |g|^q \, dm < \epsilon$.

(e) Prove $\lim_{k \to \infty} \int f_k g \, dm = \int fg \, dm$

(2) Let $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure as $n \to \infty$. Prove $f_n \to f$ in L^1 as $n \to \infty$.

(3) (a) Give an example of continuous $f : \mathbb{R} \to \mathbb{R}$ and $E \subset \mathbb{R}$ with $m(E) = 0$ so that $m(f(E)) \neq 0$, m is Lebesgue measure on $\mathbb{R}$.

(b) Let f be an absolutely continuous function on the interval $[a, b]$. Show that $m(f(E)) = 0$ for all $E \subset [a, b]$ with $m(E) = 0$.

(4) For f a positive measurable function on the interval $[0, 1]$, which is larger(assume all the integrals make sense)?

$$\int_0^1 f \, dm \int_0^1 \log f \, dm \text{ OR } \int_0^1 f \log f \, dm$$

Prove your answer.

Analysis Qualifying Exam
August 2010

You must justify your answers in full detail, and explicitly check all the assumptions of any theorem you use.

1. Assume that $f, f_1, f_2, \dots \in L^1(\mathbb{R})$ (Lebesgue measure), and that as $n \to \infty$ (i) $f_n \to f$ pointwise on $\mathbb{R}$ and (ii) $\|f_n\|_1 \to \|f\|_1$. Prove that for any any measurable set $E \subset \mathbb{R}$, $\lim_{n\to\infty} \int_E f_n = \int_E f$.

2. Let $f \in L^2[1,\infty)$ (Lebesgue measure). For each of the following statements, if the statement is true, prove it, while if false give a counterexample.

   (a) If f is continuous then $f(x) \to 0$ as $x \to \infty$. (Do not assume continuity for parts (b),(c) and (d).)

   (b) $\int_{[n,n+1]} |f| \to 0$ as $n \to 0$.

   (c) $\sqrt{n} \int_{[n,n+1]} |f| \to 0$ as $n \to 0$.

   (d) $\liminf_{n\to\infty} \sqrt{n} \int_{[n,n+1]} |f| = 0$

3. Let $f \in L^2(0,\infty)$ (Lebesgue measure). Prove the following:

   (a) $\left| \int_0^x f(t)dt \right| \le x^{1/2} \|f\|_2$ for $x > 0$.

   (b) $\lim_{x\to\infty} x^{-1/2} \int_0^x f(t)dt = 0$.

4. Define

$$f(x,y) = \begin{cases} x^{-4/3} \sin(\frac{1}{xy}) & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Prove or disprove: $\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \int_0^1 f(x,y)dydx.$

Real analysis qualifying exam Jan. 13, 2010

1. (a) Let f be a *continuous* map of a metric space X into a metric space Y.

True or False. If false either give a counterexample, or make the statement true by either adding a hypothesis or modifying the conclusion. Do not prove if true.
(i) If X is compact, then so is $f(X)$.
(ii) If X is connected, then so is $f(X)$.
(iii) If f is one-to-one, then $f^{-1}: f(X) \to X$ is continuous.

(b) The Cantor set $C \subset [0,1] \subset \mathbb{R}$ consists of all sums $x = \sum_{j=1}^{\infty} \frac{n_j}{3^j}$ where the n_j are allowed to form any sequence of 0's and 2's. Let $f: C \to [0,1]$ be the canonical map defined by $f(x) = \frac{1}{2}\sum_{j=1}^{\infty} \frac{n_j}{2^j}$.

Prove or Disprove.
(i) f is onto.
(ii) f is continuous.
(iii) f is one-to-one.

2. Let $\{f_j\}$ be a sequence of Lebesgue measurable functions that converges pointwise a.e. to a function f on the interval $I = [0,1]$. Let $F \in L^p(I)$ and $g \in L^{p'}(I)$ where p and p' are dual exponents, $1 \le p \le \infty$.

(a) If $p > 1$, $\|f_j\|_p \le 1$ $(j = 1, 2, \ldots)$ and $\int_I f_j g \to \int_I Fg$, prove that $\int_I fg = \int_I Fg$.

(b) Show by example that the conclusion of part (a) is false when $p = 1$.

3. Let f be a real valued function on the interval $I = [a,b]$.

(a) Give the definition of *absolute continuity* for f on I.

(b) Suppose f is absolutely continuous on I.

True or False. If false either give a counterexample or modify the statement so that it is true. Do not prove if true.
(i) f is uniformly continuous on I.
(ii) f is differentiable at every x in the interior of I.
(iii) $f' \in L^1(I)$ and $f(x) - f(a) = \int_a^x f'(t)dt$, $a \le x \le b$.

(c) Suppose f is absolutely continuous on I. Prove that the set of values $\{y = f(x) : f'(x) \text{ is not defined}\}$ has measure *zero*.

(d) Suppose f is absolutely continuous on I. Prove that the set of values $\{y = f(x) : f'(x) = 0\}$ has measure *zero*.

4. Let Borel functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ be given so that $f(x-y)g(y)$ is a Borel function on $\mathbb{R}^2$. Prove that $\int_{-\infty}^{\infty} |f(x-y)g(y)|dy < \infty$ for a.e. x.

Qualifying Exam Measure Theory

8 January 2009

Show **ALL** your work. Write all your solutions in clear, logical steps.

Each problem has the same weight

Good luck!

Problem 1. Given $0 < p_0 < p_1 < \infty$ construct a Lebesgue measurable function f on $\mathbb{R}$ so that $f \in L^p(\mathbb{R}, m)$ if and only if $p \in [p_0, p_1]$. (m denotes Lebesgue measure)

Problem 2. Let μ be a measure on X with $\mu(X) < \infty$. For f measurable on X show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$

Problem 3. Let $Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dm(y)$ be the Hardy-Littlewood Maximal function of a function $f \in L^1(\mathbb{R}^k, m)$ **(a)** Show that there are finite positive constants c and R (that depend on f) so that $Mf(x) \geq \frac{c}{|x|^k}$ for all x with $|x| > R$. **(b)** Use part (a) to show that if $Mf(x) \in L^1(\mathbb{R}^k, m)$ then $f = 0$ a.e.

Problem 4. Suppose f_n are measurable functions on (X, μ) and that $|f_n| \leq g \in L^1(\mu)$. Show that if $f_n \to f$ in measure then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$

Measure Theory Qualifying Exam Fall 2008

Problem 1. Let $E \subset \mathbb{R}$ with $m(E) > 0$ (i.e. E has positive lebesgue measure). Show that the set $E - E = \{x - y \mid x, y \in E\}$ contains an interval centered at 0.

Problem 2. Let μ be a positive measure on X and f measurable on X. For $0 < r < p < s < \infty$ show that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$.

Problem 3. Prove that a positive measure μ on X is σ-finite if and only if there is an $f \in L^1(d\mu)$ with $f(x) > 0$ for all $x \in X$.

Problem 4. Let $1 < p < \infty$ and suppose that $f_k \to f$ in $L^p(\mathbb{R}, m)$ as $k \to \infty$ (m is Lebesgue measure on $\mathbb{R}$). In addition assume that $g_k(x) = \begin{cases} 0 & , x < k \\ 1 & , x \geq k \end{cases}$ for $k = 1, 2, \ldots$. What does the sequence $f_k g_k$ converge to in L^p? Prove it.

Analysis Qualifying Exam

You should justify nontrivial steps, referring to theorems when appropriate.

1. Fix $p \in (0, \infty)$. Give an example of a function $f \notin L^p(0,1)$ such that $f \in L^r(0,1)$ for all $r < p$.

2. Let f be a nonnegative measurable function on [0,1]. Prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$, including the case $+\infty = +\infty$.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ−finite measure spaces and let $K(x, y)$ be measurable with respect to the product σ−algebra $\mathcal{M} \times \mathcal{N}$. Assume there is a finite constant $A > 0$ such that

$$\int_Y |K(x,y)|\, d\nu(y) \leq A \text{ for all } x \in X$$

and

$$\int_X |K(x,y)|\, d\mu(x) \leq A \text{ for all } y \in Y\,.$$

Fix $p \in (1, \infty)$ and $f \in L^p(X, \mathcal{M}, \mu)$ and define

$$(Tf)(y) = \int_X f(x) K(x,y) d\mu(x)$$

Prove that $\|Tf\|_{L^p(\nu)} \leq A\|f\|_{L^p(\mu)}$

4. Let $\phi : [-\pi, \pi] \to [-1, 1]$ be measurable. Let $0 < r < 1$ and prove that

$$\lim_{r \uparrow 1} \int_{-\pi}^{\pi} \frac{dt}{1 - r\phi(t)} = \int_{-\pi}^{\pi} \frac{dt}{1 - \phi(t)}$$

Evaluate the right-hand side above for $\phi(t) = \cos t$.

Measure theory exam Jan. 28, 2006

1. Let $\mathcal{P}$ denote the σ-algebra of all subsets of $\mathbb{R}$ and define a measure ρ by $\rho(E) = 1$ if $0 \in E$ and $\rho(E) = 0$ if $0 \notin E$. Let m denote Lebesgue measure and $\mathcal{M}$ the Lebesgue measurable sets. Let f denote a real valued function on $\mathbb{R}$.

(a) Show $(\rho, \mathcal{P})$ is a σ-finite measure space.

(b) Which is true and which is false and why?
(i) If f is Lebesgue measurable, then f is ρ-measurable.
(ii) If f is ρ-measurable, then f is Lebesgue measurable.

(c)Show that if $f \in L^1(\rho)$, then there is $a.e.[\rho]$ a unique Lebesgue measurable function g such that

$$\int_E g d\rho = \int_E f d\rho$$

for all $E \in \mathcal{M}$.

(d) Show by example that g is not $a.e.[m]$ unique.

2. Let μ be a signed (or complex) Borel measure on $\mathbb{R}$ such that $|\mu|(\mathbb{R}) < \infty$. Let $E \subset \mathbb{R}$ be a measurable subset with $\mu(E) \neq 0$. Suppose for all $x \in \mathbb{R}$ and all Borel subsets $A \subset E$

$$\mu(A + x) = \mu(A)$$

Prove that $\mu = 0$.

3. Let L^1 denote the Lebesgue integrable functions on the interval $[0, 1]$ with respect to Lebesgue measure and let $||f||$ denote the L^1 norm.

(a)Construct a sequence $\{f_n\} \subset L^1$ such that $||f_n|| \to 0$, but $\{f_n\}$ converges at no point.

(b)Construct a sequence $\{f_n\} \subset L^1$ such that $f_n \to 0$ at every point, but $||f_n|| \to \infty$.

(c) Suppose $f \in L^1$, $f_n \to f$ a.e., and $||f_n|| \to ||f||$. Prove that $f_n \to f$ in L^1.

4. Let $1 < p < \infty$ and let f and g be Lebesgue measurable functions on the half-line $[0, \infty)$.

(a) Show how to use the Fubini theorem (Fubini-Tonelli) and the identity

$$\int_0^\infty \frac{f(y)}{x+y} dy = \int_0^\infty \frac{f(xy)}{1+y} dy \quad (x > 0)$$

to prove

$$\int_0^\infty \int_0^\infty \frac{f(y)}{x+y} dy \; g(x) dx \leq C_p ||f||_p ||g||_{p'}$$

where p' is the dual exponent to p.

(b) Can the Fubini theorem be used to get the same type of result when $x + y$ is replaced by $x - y$ in part (a)? Why or why not?

FALL 2005

Measure Theory Part

1. Let $\{r_n\}_{n=1}^{\infty}$ be the rationals, $f(x) = x^{-1/2}$ for $0 < x < 1$ and 0 otherwise, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Is $f(x)$ measurable? Why? Is $g(x)$ measurable? Why? What is the set of points of discontinuity of g? Is g integrable? Why? Show that g is not in L^2 on any interval.

2. Let μ be Lebesgue measure on the borel sets of the real line, and define $\nu(E)$ to be 1 if $0 \in E$ and 0 if $0 \notin E$ for all borel sets E. Is ν a measure? σ finite? Compute $\frac{d\nu}{d\mu}$.

3. Define L^p (Lebesgue measure). Is $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$? Why? Is $L^2(0,1) \subset L^1(0,1)$? Why?

4. Let $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise and $\|g_k\|_\infty \leq M$ for all k. Prove that $f_k g_k \to fg$ in L^p.

Complex Part

1. Let f be an analytic function on the unit disk and $f(z)$ is real when z is real. Show that $\bar{f}(\bar{z}) = f(z)$.

2. Let $\{f_n\}$ be a sequence of continuous functions on the closed unit disk that are analytic in the open unit disk. Suppose $\{f_n\}$ converges uniformly on the unit **circle.** Show that $\{f_n\}$ converges uniformly on the closed unit disk.

3. Suppose that f is an analytic function on an open set containing the closed unit disk, $|f(z)| = 1$ when $|z| = 1$ and f is not a constant. Prove that the image of f contains the closed unit disk.

4. Let $\mathcal{F}$ be a family of analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

on the open unit disk such that $|a_n| \leq n$ for each n. Show that $\mathcal{F}$ is normal, i.e. every sequence of functions in $\mathcal{F}$ contains a subsequence converging normally to a function in $\mathcal{F}$.

Analysis Exam 29 January 2005

Measure Theory Part

1. Let $f(x)$ be the standard Cantor function. Define $g(x) = f(x)+x$. Show that g is continuous, increasing, and 1-1 from $[0,1]$ onto $[0,2]$. Use g to show that the image of a Lebesgue measurable set under a continuous map may not be measurable.

2. Consider the real line with Lebesgue measure. A sequence of measurable real valued functions f_n converges in measure to the measurable function f. In addition $|f_n| \leq g$ for all n where g is an integrable function. Show that

$$\lim_{n\to\infty} \int |f_n - f| = 0$$

3. Suppose that $1 < p < q < r < \infty$ and that $f \in L^p \cap L^r$. Estimate the L^q norm of f in terms of a product involving the L^p and L^r norms. Something like $\|f\|_q \leq \|f\|_r^\alpha \|f\|_p^{1-\alpha}$ where $0 < \alpha < 1$.

4. Let f be measurable on the interval $[0,1]$ (Lebesgue measure on the real line). If the function $g(x,y) = x(f^2(x) - f^4(y))$ is integrable on the unit square in $\mathbb{R}^2$ show that f is integrable on $[0,1]$.

24 October 2004

Measure Theory Part

1. Define Lebesgue Outer Measure $|\cdot|_e$ on $\mathbb{R}$. Show that there exist disjoint $E_k \subset \mathbb{R}$ for $k = 1, 2, \ldots$ so that

$$|\cup_{k=1}^{\infty} E_k|_e < \sum_{k=1}^{\infty} |E_k|_e$$

2. Define convergence in measure. Construct a sequence of functions on $[0,1] \subset \mathbb{R}$ that converges in measure (Lebesgue measure) but does not converge point-wise for any point of $[0,1]$.

3. Define what it means for a set function to be absolutely continuous with respect to a measure. Let $f \in L(\mathbb{R}, dx)$ where dx is Lebesgue measure and set

$$\phi(E) = \int_E f\, dx$$

Prove that ϕ is absolutely continuous with respect to dx.

4. Let $f_k \to f$ point-wise a.e. with $|f_k| \leq g_k \in L^1$ and $g_k \to g$ in L^1 show that $f_k \to f$ in L^1.