701 Midterm Definitions

σ-algebra. A collection of subsets of X st.

(1) $X \in \Sigma$

(2) $E \in \Sigma \Rightarrow E^c \in \Sigma$

(3) $E_1, E_2, \ldots \in \Sigma \Rightarrow \bigcup E_i \in \Sigma$

Borel σ-algebra. the minimal σ-algebra containing all open sets.

properties of σ-algebras. (1) $\phi \in \Sigma$

(2) $E_1, E_2, \ldots \in \Sigma \Rightarrow \bigcap E_i \in \Sigma$

(3) $E_1, E_2, \ldots \in \Sigma \Rightarrow \overline{\lim}\, E_k \in \Sigma$

$\underline{\lim}\, E_k \in \Sigma$

(4) $E_1, E_2 \in \Sigma \Rightarrow E_1 - E_2 \in \Sigma$

$\overline{\lim}\, E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$, $\underline{\lim} = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} E_k$

additive set function. $\varphi : \Sigma \to \mathbb{R}$ st $|\varphi(E)| < \infty \ \forall E \in \Sigma$ and $\varphi(\bigcup E_k) = \sum \varphi(E_k)$ if E_k pairwise disjoint.

measure. $\mu : \Sigma \to [0, \infty]$ st $\mu(\bigcup E_k) = \sum \mu(E_k)$ if E_k pairwise disjoint.

upper variation. $\overline{V}(E) = \overline{V}(E, \varphi) = \sup \varphi(A)$, $A \subset E$, $A \in \Sigma$

lower variation. $\underline{V}(E) = \underline{V}(E, \varphi) = -\inf \varphi(A)$, $A \subset E$, $A \in \Sigma$.

total variation. $V(E) = \overline{V}(E) + \underline{V}(E)$

properties of variations. ≥ 0; $E_1 \subset E_2 \Rightarrow \underline{V}(E_1) \leq \underline{V}(E_2)$

properties of measures. (1) $\mu(\bigcup E_k) \le \sum \mu(E_k)$

(2) $E_k \nearrow E \implies \lim \mu(E_k) = \mu(E)$

(3) $E_k \searrow E$, $\mu(E_{k_0}) < \infty$ for some k_0

$\implies \lim \mu(E_k) = \mu(E)$

(4) $\mu(\underline{\lim}\, E_k) \le \underline{\lim}\, \mu(E_k)$

(5) $\mu(\bigcup_{k=k_0}^{\infty} E_k) < \infty$ for some k_0

$\implies \overline{\lim}\, \mu(E_k) \le \mu(\overline{\lim}\, E_k)$

measurable function. $f: E \to [-\infty, \infty]$ st $f^{-1}((a,\infty]) \in \Sigma$ $\forall a$.

almost everywhere. A property holds on E a.e. wrt μ if $\exists A \subset E$, $A \in \Sigma$, $\mu(A) = 0$, and the property holds everywhere on $E - A$.

integral. f – measurable, nonnegative. $E \in \Sigma$.

$\int_E f \, \partial\mu = \sup \left[\sum_{j=1}^{k} \inf_{x \in E_j} f(x) \cdot \mu(E_j) \right]$ – sup over all partitions of E.

f – measurable, not nonnegative

$f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$

$\int_E f \, \partial\mu = \int_E f^+ \partial\mu - \int_E f^- \partial\mu$

integrable. f is integrable if $\int_E f \, \partial\mu$ is finite. ($f \in L(E, \mu)$)

properties of integral. (1) $\mu(E) = 0 \implies \int_E f \, \partial\mu = 0$

(2) $f \ge g \implies \int f \partial\mu \ge \int g \partial\mu$.

outer measure. $\Gamma : X \to [0,\infty]$ st (1) $\Gamma(\emptyset)=0$, $\Gamma(A)\geq 0$

(2) $A_1 \subset A_2 \Rightarrow \Gamma(A_1) \leq \Gamma(A_2)$

(3) $\Gamma(\bigcup A_k) \leq \sum \Gamma(A_k)$

translated Vitali sets show $|\bigcup V_k|_e < \sum |V_k|_e$ even though V_k's disjoint

Γ-measurable. $E \subset X$ is Γ-measurable if

$$\Gamma(A) = \Gamma(A\cap E) + \Gamma(A \cap E^c)$$

for any $A \subset X$.

metric outer measure. $d(A,B) > 0 \Rightarrow \Gamma(A\cup B) = \Gamma(A) + \Gamma(B)$

semicontinuous functions. upper — $\overline{\lim_{x\to y}} f(x) \leq f(y)$

lower — $\underline{\lim_{x\to y}} f(x) \geq f(y)$

note. f upper semi continuous functions are Borel measurable.

regular outer measure. Γ is regular if $\forall A \subset X$ $\exists$ a Γ-measurable set E st $A \subset E$ and $\Gamma(A) = \Gamma(E)$.

Lebesgue outer measure. $f: \mathbb{R} \to \mathbb{R}$ — finite, increasing

$\lambda((a,b]) = f(b) - f(a) \geq 0$.

$A \subset \mathbb{R}$, $A \subset \bigcup (a_k, b_k]$

$\Lambda^*(A) = \inf \sum \lambda((a_k, b_k])$

Lebesgue-Stieltjes measure. Λ — restriction of Λ^* to Λ^*-measurable sets.

LS integral. g — Borel measurable function on $\mathbb{R}$

$$\int g \, d\Lambda_f$$

algebra $\mathcal{A}$ a collection of sets in X st:

(1) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$

(2) $E_1, \ldots, E_n \in \mathcal{A} \Rightarrow \bigcup_{j=1}^{n} E_j \in \mathcal{A}$

need not contain X

measure on algebra. Set function λ defined on $\mathcal{A}$ st

(1) $\lambda \geq 0$, $\lambda(\emptyset) = 0$ — nonnegative

(2) $A_1 \subseteq A_2$, $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{A}$ — monotonic

$\Rightarrow \lambda(A_1) \leq \lambda(A_2)$

(3) $A_1, A_2, \ldots \in \mathcal{A}$, $\bigcup A_j \in \mathcal{A}$ – disjoint — additive

$\Rightarrow \lambda(\bigcup^{\infty} A_j) = \sum \lambda(A_j)$

outer measure on algebra. $\mathcal{A}$ – algebra. λ – measure on $\mathcal{A}$.

$\lambda^*(E) = \inf \sum \lambda(A_j)$

$E \subset X$, $E \subset \bigcup A_j$, $A_j \in \mathcal{A}$.

(asf) absolutely continuous. $\varphi \ll \mu$ if $\mu(A) = 0 \Rightarrow \varphi(A) = 0$

(asf) singular. $\varphi \perp \mu$ if $\exists\, Z \in \Sigma$: $\mu(Z) = 0$ and $\varphi(A) = 0$ for any $A \subset X - Z$.

signed measure. A set function on Σ that can take either ∞ or $-\infty$ but not both.

(meas) absolutely continuous. $\nu \ll \mu$ on measurable E if $A \subset E$, $\mu(A) = 0 \Rightarrow \nu(A) = 0$

mutually singular. ν and μ mutually singular on E if $\exists$ disjoint E_1, E_2 with $E = E_1 \cup E_2$ and $\mu(E_2) = \nu(E_1) = 0$.

product σ-algebra. (X_j, Σ_j, μ_j) — measure spaces
$\otimes \Sigma_j$ — σ-algebra on $\prod X_j$ generated by $\prod E_j$ where $E_i \in \Sigma_i$.

product measure. (X, Σ, μ), (Y, Θ, ν) — measure spaces
For $G \subset X \times Y$, $G = \bigcup_{j=1}^{n} E_j \times F_j$ — disjoint,
$$(\mu \times \nu)(G) = \sum_{j=1}^{n} \mu(E_j)\nu(F_j)$$

For sets $B_1 \times B_2 \in \Sigma \times \Theta$,
$$(\mu \times \nu)(B_1 \times B_2) = \mu(B_1)\nu(B_2)$$

monotone class. Collection $\mathcal{C}$ of subsets of X that is closed wrt increasing unions and decreasing intersections.

slices. Let $E \subset X \times Y$. Fix $x \in X$, $E_x = \{y : (x, y) \in E\}$
Fix $y \in Y$, $E_y = \{x : (x, y) \in E\}$.

$f(x, y)$ defined on $X \times Y$.
$(f_x)^{-1}(E) = f^{-1}(E_x)$
$(f_y)^{-1}(E) = f^{-1}(E_y)$

G_δ-set. A countable intersection of open sets.
F_σ-set. A countable union of closed sets.

701 Final thms.

Ch. 3. Lebesgue Measure

thm. $E_1 \subset E_2 \Rightarrow |E_1|_e \le |E_2|_e$ $\quad |\cdot|_e$ = inf volume of covering

thm. $E = \cup E_k \Rightarrow |E|_e \le \Sigma |E_k|_e$

thm. $E \subset \mathbb{R}^n$. Given $\varepsilon > 0$ $\exists$ open G st

$$E \subset G$$

$$|G|_e \le |E|_e + \varepsilon.$$

This implies $|E|_e = \inf |G|_e$.

thm. $E = \cup E_k$. E_k measurable $\Rightarrow$ E measurable

$|E| \le \Sigma |E_k|$ ← should be $|\cdot|_e$?

lemma. $d(E_1, E_2) > 0 \Rightarrow |E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$

thm. E_1, E_2 measurable $\Rightarrow$ $E_1 - E_2$ measurable

$E_1 - E_2 = E_1 \cap E_2^c$

lemma. $E \subset \mathbb{R}^n$ measurable $\Leftrightarrow$ given $\varepsilon > 0$, $\exists$ closed $F \subset E$ st $|E - F| < \varepsilon$.

thm $\{E_k\}$ disjoint, measurable $\Rightarrow |\cup E_k| = \Sigma |E_k|$

corollary. E_1, E_2 measurable, $E_2 \subset E_1$, $|E_2| < \infty$.

$\Rightarrow |E_1 - E_2| = |E_1| - |E_2|$

thm. $\{E_k\}$ sequence of measurable sets.

(i) $E_k \nearrow E \Rightarrow \lim |E_k| = |E|$

(ii) $E_k \searrow E$ and $|E_{k_0}| < \infty$ some k_0 $\Rightarrow \lim |E_k| = |E|$ $\quad |E_{k_0}| < \infty$ needed ex in book

$E_k = [k, \infty) \searrow \emptyset$ but $|E_k| = \infty$ $\forall k$

thm. (ii) E measurable $\Leftrightarrow$ $E = H - Z$, H — G_δ and $|Z| = 0$.

(iii) E measurable $\Leftrightarrow$ $E = H \cup Z$, H — F_σ and $|Z| = 0$.

thm. (Caratheodory). E measurable $\Leftrightarrow$ $\forall$ A

$$|A|_e = |A \cap E|_e + |A - E|_e.$$

corollary. E measurable, $E \subseteq A$, then $|A|_e = |E| + |A - E|_e$

If $|E| < \infty$, $|A - E|_e = |A|_e - |E|$.

thm. Lipschitz transformations map measurable sets into measurable sets.

thm T — linear transformation of $\mathbb{R}^n$.

E — measurable.

$\Rightarrow |TE| = \delta |E|$ where $\delta = |\det T|$

det theorem in diff geom

Ch. 4. Lebesgue Measurable Functions.

thm. f measurable $\Leftrightarrow$ $\forall$ open $G \subset \mathbb{R}$,

$f^{-1}(G) \subset \mathbb{R}^n$ is measurable.

thm. A — dense in $\mathbb{R}$.

f is measurable if $\{f > a\}$ is measurable for all $a \in A$.

thm. φ continuous on $\mathbb{R}$, f finite ae in E. $\Rightarrow f^+, f^-$ meas

$\Rightarrow \varphi(f)$ is measurable if f is.

thm. If f and g are measurable, then $\{f \geq g\}$ is measurable. (and $f+g$, fg, f/g)

thm. $\{f_k\}$ measurable.
$\Longrightarrow \sup f_k$, $\inf f_k$ measurable.

thm. $\{f_k\}$ measurable
$\Longrightarrow \limsup f_k$, $\liminf f_k$ measurable
and $\lim f_k$ measurable if it exists.

thm. (i) Every function f can be written as a limit of $\{f_k\}$ — simple.

(ii). If $f \geq 0$, sequence can be chosen to increase to f.

$f_k(x) = \begin{cases} \frac{j-1}{2^k} & \frac{j-1}{2^k} \leq f(x) \leq \frac{j}{2^k} \\ k & f \geq k \end{cases}$

(iii). If f is measurable, f_k can be chosen to be measurable

thm. (i). f is usc relative to $E \Longleftrightarrow \{x : f(x) \geq a\}$ is relatively closed $\forall$ finite a.

(ii) f is lsc relative to $E \Longleftrightarrow \{x : f(x) \leq a\}$ is relatively closed $\forall$ finite a

corollary. A finite function is continuous iff it is usc and lsc.

corollary. E - measurable.
f — usc (lsc, cont)
$\Longrightarrow f$ is measurable.

thm. (Egorov). $\{f_n\}$ — measurable

$f_n \to f$ ae on E, $\mu(E) < \infty$, $f < \infty$.

$\Rightarrow$ given $\varepsilon > 0$ $\exists$ closed $F \subset E$

st $|E - F| < \varepsilon$ and $f_n \to f$ unif. on F

lemma. $\{f_n\}$ — measurable.

$f_n \to f$ ae on E, $\mu(E) < \infty$, $f < \infty$

$\Rightarrow$ given $\varepsilon, \eta > 0$ $\exists$ closed

$F \subset E$ and $K \in \mathbb{N}$ st

$|E - F| < \eta$; $|f(x) - f_n(x)| < \varepsilon$

for $x \in F$, $n > K$.

property $\mathcal{C}$

E - measurable

f defined on E has prop $\mathcal{C}$ if given $\varepsilon > 0$, $\exists$ closed

$F \subset E$ st (i) $|E - F| < \varepsilon$

(ii) f cont relative to F.

lemma. A simple meas function has property $\mathcal{C}$.

thm. (Lusin's). f defined ae and finite on msbl E.

$\Rightarrow$ f measurable $\Leftrightarrow$ f has property $\mathcal{C}$ on E.

thm. f, f_n measurable and finite ae in E.

If $f_n \to f$ ae on E, $|E| < \infty$,

then $f_n \xrightarrow{m} f$ on E.

ptwise $\Rightarrow$ in measure

thm. If $f_n \xrightarrow{m} f$ on E, $\exists$ subsequence f_{n_j} st

$f_{n_j} \to f$ ae in E.

in measure $\Rightarrow$ sub ptwise

$\mu(\{x : |f(x) - f_n(x)| > \varepsilon\}) \to 0$ as $n \to \infty$, $\forall \varepsilon > 0$.

thm. $\{f_n\}$ converges in measure on E

$\Leftrightarrow \forall \varepsilon > 0 \ \lim_{n,l\to\infty} |\{x \in E : \underbrace{|f_n(x) - f_l(x)|}_{\text{like Cauchy}} > \varepsilon\}| = 0$

Ch. 5. Lebesgue Integral.

$0 \le f \le +\infty$, E - measurable

$R(f,E) = \{(x,y) \in \mathbb{R}^{n+1} : x \in E,\ 0 \le y \le f(x)$ if $f(x) < \infty$

and $0 \le y < \infty$ if $f(x) = \infty\}$

$\int_E f(x)dx = |R(f,E)|$

$\sup \sum \inf f(x) \cdot |E_k|$

thm. $f \ge 0$, on measurable E.

$\int_E f$ exists $\Leftrightarrow$ f is measurable.

corollary. $f \ge 0$ measurable

$f = a_i$ on E_i disjoint

$E = \bigcup E_i$

$\Rightarrow \int_E f = \sum^{N} a_j |E_j|$

thm. (i) f, g - measurable

$0 \le g \le f$ on E

$\Rightarrow \int g \le \int f$ (so $\int \inf f \le \int f$)

(ii) $f \ge 0$ - measurable on E

$\int_E f < \infty$

$\Rightarrow f < \infty$ a.e. in E.

(iii). E_1, E_2 measurable, $E_1 \subset E_2$

$f \ge 0$ measurable

$\Rightarrow \int_{E_1} f \le \int_{E_2} f$

thm. $f \geq 0$ – measurable on E.

$$\Rightarrow \int_E f = \sup \sum_j \left[\inf_{x \in E_j} f(x) \right] |E_j|$$

thm. $f \geq 0$ on E.

$$|E| = 0 \Rightarrow \int f = 0$$

thm. $f, g \geq 0$ – measurable

$f \geq g$ ae $\Rightarrow \int g \leq \int f$

$f = g$ ae $\Rightarrow \int f = \int g$

thm $f \geq 0$ – measurable

$\int f = 0 \Leftrightarrow f = 0$ ae in E.

corollary. $f \geq 0$, measurable.

$$\alpha > 0 \Rightarrow |\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha} \int_E f$$

corollary. f, φ – measurable

$0 \leq f \leq \varphi$

$\int f < \infty$.

$$\Rightarrow \int_E (\varphi - f) = \int_E \varphi - \int_E f$$

thm $\{f_k\}$ – nonnegative measurable   DCT

$f_k \to f$ ae

$f_k \leq \varphi$ ae $\forall k$, φ – measurable

$\int \varphi$ finite

$$\Rightarrow \int f_k \to \int f$$

f measurable but not nonnegative

$f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$

$f = f^+ - f^-$

thm. f measurable on E.
$f \in L(E) \iff |f| \in L(E)$

thm. $f \in L(E) \implies f$ is finite a.e. in E.

thm. (i) $\int f, \int g$ exist

just repeating for non-nonnegative functions

$f \le g$ a.e. in E $\implies \int f \le \int g$

$f = g$ a.e. in E $\implies \int f = \int g$

(ii) $E_1 \subset E_2$ measurable, $\int_{E_2} f$ exists
$\implies \int_{E_1} f$ exists.

thm. $|E| = 0$ or $f = 0$ a.e. in E $\implies \int_E f = 0$.

thm. $f \in L(E)$, g – measurable, $|g| \le M$ a.e.
$\implies fg \in L(E)$.

corollary. $f \in L(E)$, $f \ge 0$ a.e.
$\alpha \le g \le \beta$ a.e.
$\implies \alpha \int f \le \int fg \le \beta \int f$

thm. (MCT). $\{f_k\}$ – measurable
(i) $f_k \nearrow f$ a.e. on E, $\varphi \in L(E)$
$f_k \ge \varphi$ a.e. $\forall k$
$\implies \int f_k \to \int f$

(ii) $f_k \searrow f$ a.e. on E, $\varphi \in L(E)$
$f_k \le \varphi$ a.e. $\forall k$
$\implies \int f_k \to \int f$

thm. $f_k \in L(E)$, $f_k \to f$ unif, $|E| < \infty$.
$\implies f \in L(E)$ and $\int f_k \to \int f$.

thm. (Fatou). $\{f_k\}$ — measurable

$\varphi \in L(E)$ st $f_k \geq \varphi$ ae $\forall k$

$\Rightarrow \int \liminf f_k \leq \liminf \int f_k$

corollary. $\varphi \in L(E)$, $f_k \leq \varphi$ ae $\forall k$

$\Rightarrow \int \limsup f_k \geq \limsup \int f_k$

thm. (DCT). $\{f_k\}$ — measurable

$f_k \to f$ ae in E

$\varphi \in L(E)$ st $|f_k| \leq \varphi$ ae $\forall k$

$\Rightarrow \int f_k \to \int f$

φ could be constant M

thm f bounded, Riemann integrable on $[a,b]$

$\Rightarrow f \in L([a,b])$ and

$\int_a^b f = (R) \int_a^b f$

thm bounded f is R-integrable on $[a,b]$

$\Leftrightarrow f$ is continuous ae in $[a,b]$.

Ch. 6. Repeated Integration.

thm (Fubini). $f(x,y) \in L(I_1 \times I_2)$.

Then (i) for almost every $x \in I_1$

$f(x,y)$ is measurable and integrable as a function of y.

finite lin combs of $\to$ still fine

$f_k \nearrow f$, $f_k \searrow f$, $f_k \to$

$f \in L(dx\, dy)$

then $f \to$

(ii). as a function of x $\int_{I_2} f(x,y)\,dy$ is msble, integrable on I_1.

$$\iint_I f(x,y)\,dx\,dy = \int_{I_1}\left[\int_{I_2} f(x,y)\,dy\right]dx$$

lemma. $E \subset \mathbb{R}^{n+m}$.

E measurable, $|E| < \infty$

$\Rightarrow \chi_E$ works with Fubini – Tonelli

$$\iint_E \chi_E(x,y)\, dx \times dy = \int_{E_x}\int_{E_y} \chi_E(x,y)\, dx\, dy$$

thm. $f(x,y)$ – measurable on $\mathbb{R}^{n+m}$

Then for almost every $x \in \mathbb{R}^n$, $f(x,y)$ is a measurable function of $y \in \mathbb{R}^m$

If E is measurable, then $E_x = \{y : (x,y) \in E\}$ is measurable in $\mathbb{R}^m$ for almost every $x \in \mathbb{R}^n$

thm. (Fubini). $f(x,y)$ – measurable on measurable $E \subset \mathbb{R}^{n+m}$

$E_x = \{y : (x,y) \in E\}$

(i) For almost every $x \in \mathbb{R}^n$, $f(x,y)$ msbl funct of y on E_x.

finiteness of multiple integral $\Rightarrow$ finiteness of corresponding iterated

(ii) If $f(x,y) \in L(E)$,

then for almost every $x \in \mathbb{R}^n$

$f(x,y)$ is integrable on E_x wrt y

Moreover, $\int_{E_x} f(x,y)\, dy$ integrable wrt x

and

$$\iint_E f(x,y)\, dx \times dy = \int_{\mathbb{R}^n} \left(\int_{E_x} f(x,y)\, dy \right) dx$$

thm. (Tonelli). $f(x,y) \geq 0$, measurable on $I = I_1 \times I_2 \subset \mathbb{R}^{n+m}$

Then for almost every $x \in I_1$,

$f(x,y)$ measurable funct of y on I_2

$\int_{I_2} f(x,y)\, dy$ measurable on I_1

$$\iint_I f(x,y)\, dx \times dy = \int_{I_1}\int_{I_2} f(x,y)\, dy\, dx$$

thm $f \geq 0$, measurable $E \subset \mathbb{R}^n$

If $R(f, E)$ is measurable in $\mathbb{R}^{n+1}$, then f is measurable.

thm. $f \in L(\mathbb{R}^n)$, $g \in L(\mathbb{R}^n)$

$\Rightarrow (f*g)(x)$ exists for almost every $x \in \mathbb{R}^n$

and $f*g \in L(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f*g|\,dx \leq \left(\int_{\mathbb{R}^n} |f|\,dx\right)\left(\int_{\mathbb{R}^n} |g|\,dx\right)$$

lemma. $f(x)$ measurable in $\mathbb{R}^n$

Then $F(x,t) = f(x-t)$ is measurable in $\mathbb{R}^{2n}$.

Ch 7. Differentiation.

$F(E) = \int_E f$ — indef integral of $f \in L(E)$

— finite and countably additive

thm. $f \in L(A) \Rightarrow F$ is absolutely continuous

$\frac{F(Q)}{|Q|} = \frac{1}{|Q|} \int_Q f\,dy$, If $\lim_{Q \searrow x} \frac{F(Q)}{|Q|} = f(x)$,

then F is differentiable at x with derivative $f(x)$.

thm. (LDT) $f \in L(\mathbb{R}^n)$.

$\Rightarrow F$ is differentiable with derivative $f(x)$ at almost every $x \in \mathbb{R}^n$.

lemma. $f \in L(\mathbb{R}^n) \Rightarrow \exists \{C_k\}$ — cont functs with compact support st

$$\int_{\mathbb{R}^n} |f - C_k|\,dx \to 0 \text{ as } k \to \infty$$

lemma. (Simple Vitali). $E \subseteq \mathbb{R}^n$ with $|E|_e < +\infty$
$\mathcal{K}$ collection of cubes Q covering E
$\Rightarrow \exists\, \beta > 0$ depending only on n and finitely many disjoint cubes $Q_1, \dots, Q_N$ in $\mathcal{K}$ st
$$\sum^N |Q_j| \geq \beta |E|_e$$

$f^*(x) = \sup \frac{1}{|Q|} \int_Q |f(y)|\, dy$ — Hardy Littlewood max funct
$$\chi_E^*(x) = \sup \left\{ \frac{|E \cap Q|}{|Q|} : Q \text{ centered at } x \right\}$$

$f^*(x) \geq \frac{c}{|x|^n}$ for $|x| \geq 1$

lemma (H-L). $f \in L(\mathbb{R}^n)$
$\Rightarrow f^*$ st $|\{x : |f^*(x)| > \alpha\}| \leq \frac{c}{\alpha}$ $\alpha > 0$
Moreover, $\exists\, c$ st
$|\{x : f^*(x) > \alpha\}| \leq \frac{c}{\alpha} \int |f|$, $\alpha > 0$

thm. E measurable.
$\Rightarrow$ almost every point of E is a point of density of E.

thm. f locally integrable.
$\Rightarrow$ Almost every pt of $\mathbb{R}^n$ is a Lebesgue pt of f
ie $\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|\, dy = 0$
for almost all x

thm. f locally integrable.
Then at ae point x, $\frac{1}{|S|} \int |f(y) - f(x)|\, dy \longrightarrow 0$
where $\{S\}$ shrinks regularly to x.
Thus $\frac{1}{|S|} \int_S f(y)\, dy \longrightarrow f(x)$ ae.

thm. (Vitali). E covered in Vitali sense by $\{K\}$ and $0 < |E|_e < +\infty$

Then given $\varepsilon > 0$, there is $\{Q_j\}$ disjoint cubes in K st $|E - \bigcup Q_j| = 0$
$\sum |Q_j| < (1+\varepsilon)|E|_e$.

$\{K\}$ covers E in Vitali sense if $\forall\, x \in E$, $\eta > 0$ there is cube in K containing x whose diameter is less η.

corollary. If $f \in BV([a,b])$, then f' exists ae, and $f' \in L([a,b])$.

thm. $f \in BV([a,b])$
$V(x)$ variation of f on $[a,x]$
$\Rightarrow V'(x) = |f'(x)|$ for ae $x \in [a,b]$.

thm. f abs cont on $[a,b] \Rightarrow f \in BV([a,b])$

thm. f abs cont, singular $\Rightarrow f$ constant

thm. f abs cont on $[a,b] \Leftrightarrow$ f' exists ae in $[a,b]$
f' integrable on $[a,b]$
$f(x) - f(a) = \int_a^x f'$ $(a \le x \le b)$

thm. $f \in BV([a,b]) \Rightarrow f = g + h$ where g abs cont
h sing

g, h unique up to additive constants.

thm. f abs cont, $V(x)$ — total variations
$\Rightarrow V$ abs cont $[a,b]$ and
$V(x) = \int_a^x |f'|$

thm. (i) If g cont on $[a, b]$, f abs cont on $[a, b]$,

then $\int g \, \partial f = \int g f' \partial x$.

(ii). f, g abs cont on $[a, b]$.

then $\int g f' \, dx = g(b)f(b) - g(a)f(a) - \int_a^b g' f \, dx$

thm. φ' exists, monotone inc $\Rightarrow \varphi$ convex (in (a,b))

φ'' exists ≥ 0 in (a,b) then φ convex.

thm. φ convex $\Rightarrow \varphi$ continuous.

thm. (Jensen's) f, p — measurable functs finite a.e. on $A \subset \mathbb{R}^n$

fp and p — integrable on A

$p \geq 0$, $\int_A p > 0$

If φ is convex in an interval containing the range of f, then $\varphi\left(\frac{\int fp}{\int p}\right) \leq \frac{\int \varphi(f) p}{\int p}$

Ch. 8. L^p classes.

thm. $|E| < \infty \Rightarrow \|f\|_\infty = \lim_{p \to \infty} \|f\|_p$

thm. $0 < p_1 < p_2 \leq \infty$, $|E| < \infty \Rightarrow L^{p_2} \subset L^{p_1}$

thm. $f, g \in L^p(E)$, $p > 0 \Rightarrow f + g \in L^p(E)$, $cf \in L^p(E)$

thm. $y = \varphi(x)$ cont, real-valued, strictly inc for $x \geq 0$, $\varphi(0) = 0$

If $x = \psi(y)$ is the inverse of φ, then for $a, b > 0$

$$ab \leq \int_0^a \varphi(x)\,dx + \int_0^b \psi(y)\,dy$$

$=$ iff $\varphi(a) = b$

corollary. (Young's inequality). $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$

$a, b \ge 0$, $1 < p < \infty$

$\frac{1}{p} + \frac{1}{p'} = 1$

thm. (Hölder) $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$

$\Rightarrow \|fg\|_1 \le \|f\|_p \|g\|_{p'}$

i.e.

$\int |fg| \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^{p'}\right)^{1/p'}$

$\int |fg| \le \text{ess sup} |f| \left(\int |g|\right)$

corollary. $\|fg\|_1 \le \|f\|_2 \|g\|_2$

hold for ℓ^p too

thm. f real-valued and measurable on E.

$1 \le p \le \infty$

Then $\|f\|_p = \sup \int_E fg$ — sup over real-valued g st $\|g\|_{p'} \le 1$, $\int fg$ exists

thm. (Minkowski). $1 \le p \le \infty$. $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

thm. $1 \le p \le \infty$. $L^p(E)$ is a Banach space with norm

$\|f\| = \|f\|_{p,E}$

thm. $1 \le p < \infty$. $L^p(E)$ separable.

thm. $0 < p < 1$. $L^p(E)$ — complete separable metric space

with $\partial(f,g) = \|f-g\|_{p,E}^p$

thm. $f \in L^p(\mathbb{R}^n)$. $1 \le p < \infty$.

$\Rightarrow \lim_{|h| \to 0} \|f(x+h) - f(x)\|_p = 0$

701 Midterm theorems.

thm. If $E_n \nearrow E$ $(E = \bigcup E_n)$ or $E_n \searrow E$ $(E = \bigcap E_n)$,
then $\varphi(E) = \lim \varphi(E_n)$

thm. (Fatou). φ — nonnegative asf.
Then $\varphi(\underline{\lim}\, E_n) \le \underline{\lim}\, \varphi(E_n) \le \overline{\lim}\, \varphi(E_n) \le \varphi(\overline{\lim}\, E_n)$.

lemma. Variations are countably subadditive.
$\overline{V}(\bigcup E_n) \le \sum \overline{V}(E_n)$, $\underline{V}(\bigcup E_n) \le \sum \underline{V}(E_n)$.

prop. Variations are finite.

thm. Variations are additive.

thm. Variations are finite measures on Σ.

Jordan decomposition. $\varphi(E) = \overline{V}(E) - \underline{V}(E)$.

thm. f is Σ-measurable $\iff f^{-1}(G) \in \Sigma$ for all G in the Borel σ-algebra

lemma. f, g — Σ-measurable $\Rightarrow \{f > g\} \in \Sigma$

thm. f, g — measurable $\Rightarrow f+g$, fg, cf, $\varphi(f)$ where φ — cont, f^+, f^-, $|f|^p$ where $p > 0$, and $1/f$ if $f \ne 0$ are measurable.

thm. $\{f_n\}$ — measurable $\Rightarrow \sup f_n$, $\inf f_n$, $\overline{\lim}\, f_n$, $\underline{\lim}\, f_n$, and $\lim f_n$ are measurable

~~prop~~

thm. f — simple on $E = \bigcup E_j$ disjoint.
f measurable $\iff$ Each $E_j \in \Sigma$.

thm. If $f \geq 0$ is measurable, then $\exists$ nonnegative, simple, measurable $f_k \nearrow f$.

In particular, $f_k(x) = \begin{cases} \frac{j-1}{2^k} & \frac{j-1}{2^k} \leq f(x) \leq j/2^k, \ 1 \leq j \leq 2^k \\ k & f(x) \geq k. \end{cases}$

thm. (Egorov). If $\mu(E) < \infty$ and $\{f_k\}$ measurable,
$f_k \to f$ ae pointwise,
then $\forall \varepsilon > 0 \ \exists A \subset E, A \in \Sigma$ st.
$f_k|_A \to f|_A$ uniformly and
$\mu(E - A) < \varepsilon$.

thm. If f is simple ($f = \sum_{j=1}^{n} v_j \chi_{E_j}$), then $\int_E f \, d\mu = \sum_{j=1}^{n} v_j \mu(E_j)$

thm. (1) if $0 \leq f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$.
(2) If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$.

lemma. If f, g simple and $c \geq 0$, then:
(1) $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$
(2) $\int cf \, d\mu = c \int f \, d\mu$.

lemma. If f simple and $E = E_1 \cup E_2$, $E_i \in \Sigma$ disjoint,
then $\int_E f \, d\mu = \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu$

thm. (Monotonic Convergence).

If $0 \le f_k \nearrow f \le g$, then $\lim \int f_k \, d\mu = \int f \, d\mu \le \int g \, d\mu$

thm. $f, g \ge 0 \implies \int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$, $\int cf \, d\mu = c \int f \, d\mu$

thm. If $E = \bigcup_{j}^{\infty} E_j$, E_j - disjoint, $\int_E f \, d\mu = \sum \int_{E_j} f \, d\mu$

thm. (1) $|\int f \, d\mu| \le \int |f| \, d\mu$.

(2) If $|f| \le |g|$ ae and $g \in L(E, \mu)$, then $f \in L(E, \mu)$.

thm. If $f = g$ ae and $\int f \, d\mu$ exists, then $\int g \, d\mu$ exists and $\int f \, d\mu = \int g \, d\mu$.

thm. If $f_k \ge 0$, then $\sum \int f_k \, d\mu = \int \sum f_k \, d\mu$

thm. (MCT II). If $f_k \nearrow f$ ae and $f_k \ge \varphi \in L(E, \mu)$ ae, then $\int f_k \, d\mu \nearrow \int f \, d\mu$.

thm. (Uniform Convergence).

If $f_k \to f$ uniformly ae, $f_k \in L(E, \mu)$, $\mu(E) < \infty$, then $\int f_k \, d\mu \to \int f \, d\mu$

lemma. (Fatou's). If $f_k \ge \varphi \in L(E, \mu)$, then $\int \underline{\lim} f_k \le \underline{\lim} \int f_k$.

corollary. If $f_k \ge 0$, $f_k \to f$ ae, $\int f_k \, d\mu \le M$, then $\int f \, d\mu \le M$.

thm. (Lebesgue dominated convergence.)

If $f_k \to f$ ae and $|f_k| \le \varphi \in L(E, \mu)$, ← ae.

then (1) $\int \underline{\lim} f_k \le \underline{\lim} \int f_k \le \overline{\lim} \int f_k \le \int \overline{\lim} f_k$

(2) $\int f_k \, d\mu \to \int f \, d\mu$.

corollary. (Bounded convergence).

Let f_n, f measurable, $f_n \to f$ ae, $\mu(E) < \infty$, and $\exists M$ st $|f_n| \le M$ ae

Then $\int f_n \, \partial\mu \to \int f \, \partial\mu$

thm. (1) If $\varphi \ll \mu$ and $\varphi \perp \mu$ on E, then $\varphi(A) = 0$ for every measurable $A \subset E$.

(2) If $\psi \ll \mu$ and $\varphi \ll \mu$, then $\psi + \varphi$, $c\varphi \ll \mu$ (or $\perp$).

(3) $\varphi \ll \mu$ ($\perp$) iff $\overline{V}, \underline{V}$ are iff V is.

(4) $\{\varphi_n\}$, $\varphi_n \ll \mu$ ($\perp$) on E and if $\varphi(A) = \lim \varphi_n(A)$ exists $\forall$ measurable $A \subset E$, then $\varphi \ll \mu$ ($\perp$).

thm. $\varphi \ll \mu$ on E $\iff$ given $\varepsilon > 0$ $\exists \delta > 0$ st $A \subset E$, $A \in \Sigma$, $\mu(A) < \delta \Rightarrow |\varphi(A)| < \varepsilon$.

thm. $\varphi \perp \mu$ on E $\iff$ given $\varepsilon > 0$ $\exists$ measurable $E_0 \subset E$ st $\mu(E_0) < \varepsilon$ and $V(E - E_0, \varphi) < \varepsilon$.

prop. $f \in L(E, \mu)$. $\varphi(A) = \int_A f \, \partial\mu$.

$\overline{V}(E) = \int_E f^+ \, \partial\mu$, $\underline{V} = \int_E f^- \, \partial\mu$

Hahn decomposition. E-measurable, φ defined on measurable subsets $A \subset E$.

Then $\exists$ measurable $P \subset E$ st $\varphi(A) \ge 0$ for $A \subset P$ and $\varphi(A) \le 0$ for $A \subset E - P$,

ie. $\overline{V}(E-P) = \underline{V}(P) = 0$

Hence, $\overline{V}(E) = \overline{V}(P) = \varphi(P)$

$\underline{V}(E) = \underline{V}(E-P) = -\varphi(E-P)$

Lebesgue decomposition. φ — asf on measurable subsets of $E \in \Sigma$.
μ — σ-finite measure on E.

Then $\exists (!)$ decomposition:

$$\varphi(A) = \alpha(A) + \sigma(A) \quad - A \text{ measurable in } E$$

where $\alpha \ll \mu$ and $\sigma \perp \mu$.

$$\alpha(A) = \int_A f \, d\mu \text{ and } \sigma(A) = \varphi(A \cap Z), \ \mu(Z) = 0.$$

Moreover, if $\varphi \geq 0$, $f \geq 0$.

thm. (Radon Nikodym). φ, μ as above.
If $\varphi \ll \mu$, $\exists (!)$ $f \in L(E, \mu)$ st
$$\varphi(A) = \int_A f \, d\mu \quad \forall A \subset E \text{ meas.}$$

thm. ν, μ — σ-finite measures on E
Then $\exists (!)$ nonnegative measble f on E and a $(!)$ measure σ st σ, μ mutually singular on E and $\nu(A) = \int_A f \, \partial\mu + \sigma(A)$, $A \subset E$ measble.

Moreover $\int_A g \, \partial\nu = \int_A g f \, \partial\mu + \int_A g \, d\sigma$, whenever $\int_A g \, d\nu$ exists.

(note: $\sigma(A) = \nu(A \cap Z)$.)

corollary. ν, μ — σ-finite.

(1) $\nu \ll \mu$ on $E \iff \exists f \geq 0$, measble, such that
$\nu(A) = \int_A f \, d\mu$, $A \subset E$ measble.

In this case, $\int_A g \, \partial\nu = \int_A g f \, \partial\mu$, any g measble

(2). $g \in L(E, \partial\nu)$. $\int g \, \partial\nu = \int g f \, \partial\mu$ for some $f \geq 0$
$\iff \int g \, \partial\nu \ll \mu$.

prop. Γ - outer measure.

(1) E is Γ - measble $\iff \Gamma(A_1 \cup A_2) = \Gamma(A_1) + \Gamma(A_2)$ for $A_1 \subset E$, $A_2 \subset E^c$.

(2) $\Gamma(Z) = 0 \implies Z$ is Γ - measble.

thm. (1) The family of Γ - measurable sets form a σ-algebra Σ.

(2) The restriction of Γ to Σ is a measure

$\implies$ If $\{E_k\} \subset \Sigma$ disjoint, $\Gamma(\bigcup E_k) = \sum \Gamma(E_k)$

and $\Gamma(A) = \sum [\Gamma(A \cap E_k)] + \Gamma(A \cap (\bigcup E_k)^c)$

$\Gamma(A \cap \bigcup E_k) = \sum \Gamma(A \cap E_k)$.

thm. If Γ is metric, every Borel set is Γ - measble

lemma. Γ - metric. $A \subset G \in \mathcal{B}$.

$A_k = \{x \in A : d(x, \partial G) \geq 1/k\}$

Then $\lim \Gamma(A_k) = \Gamma(A)$.

thm. $\lambda((a,b]) = f(b) - f(a)$

$\Lambda^*(A) = \inf \sum \lambda((a_k, b_k])$, $A \subset \bigcup (a_k, b_k]$.

All borel sets are Λ^* - measble.

thm. If Λ^* is the LS outer measure and $A \subset \mathbb{R}$, then $\exists$ a Borel set E st $A \subset E$ and $\Lambda^*(A) = \lambda(E)$

prop. μ - finite Borel measure on $\mathbb{R}$.

Then $F_\mu(x) = \mu((-\infty, x])$ is nonnegative, finite, increasing, and right continuous.

thm. If f is right continuous, increasing on $[a,b]$ and g is a bounded Borel measurable function st the RS-integral $\int_a^b g\,\partial f$ exists,

then $\int_a^b g\,\partial f = \int_{[a,b]} g\,\partial\lambda_f$.

thm. λ^* an outer measure generated by measure λ on algebra $\mathcal{A}$.

If $A \in \mathcal{A}$, then $\lambda^*(A) = \lambda(A)$ and $\mathcal{A}$ is λ^*-measble

thm. (Caratheodory - Hahn).

(1) The restriction of λ^* to $\mathcal{A}$ is λ. (~~$\mathcal{A}$ - {λ^*-measble sets}~~)

(2) If λ is σ-finite on X and Σ is any σ-algebra st $\mathcal{A} \subset \Sigma \subset \mathcal{A}^*$, then λ^* is the only measure on Σ equal to λ on $\mathcal{A}$.

corollary. Let μ and ν be Borel measures on $\mathbb{R}$ that are finite and equal ~~to~~ on any left open interval.

Then $\mu = \nu$ on $\mathcal{B}$.

corollary. The set of finite Borel measures on $\mathbb{R}$

= LS measures wrt bounded pointwise, increasing, rc. functions.

thm. A Borel measure μ on $\mathbb{R}$ which is finite on bounded sets is regular.

Lemma. If $E^y \in \Sigma$ $\forall y \in Y$, $E^x \in \Theta$ $\forall x \in X$, and if f is $\Sigma \times \Theta$ measurable, then f^y is Θ-measurable $\forall x$ and f_x is Σ-measurable $\forall y$.

thm. (X, Σ, μ), (Y, Θ, ν) — σ-finite measure spaces
If $E \in \Sigma \otimes \Theta$, then $x \longmapsto \nu(E_x)$
$y \longmapsto \mu(E^y)$ are measurable
wrt Σ, Θ respectively.
and $\mu \times \nu(E) = \int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y)$.

thm. (Fubini–Tonelli).
(X, Σ, μ), (Y, Θ, ν) — σ-finite measure spaces
(1) (Tonelli) Let $f : X \times Y \to [0, \infty]$.
Then $\int f_x(y) \, d\nu(y)$ and $\int f^y(x) \, d\mu(x)$ are measurable and $\int f \, d\mu \times \nu = \int (\int f \, d\nu) \, d\mu = g(x)$ (*)
$= \int (\int f \, d\mu) \, d\nu = h(y)$

(2) (Fubini) If $f \in L(X \times Y, \mu \times \nu)$, then $f_x \in L(Y, \nu)$ μ ae
$f_y \in L(X, \mu)$ ν ae
$g(x) = L(X, \mu)$
$h(y) = L(Y, \nu)$
and (*) holds

Zygmund and Wheeden problems.

4.2. Let f be a simple function, taking its distinct values on disjoint sets $E_1, \dots, E_N$. Show that f is measurable $\Leftrightarrow$ $E_1, \dots, E_N$ measurable.

($\Rightarrow$) f measurable, $f = \sum v_j \chi_{E_j}$

$f^{-1}(\{v_j\}) \in \mathcal{M}$ since $\{v_j\}$ measurable.

$f^{-1}(\{v_j\}) = E_j$.

($\Leftarrow$) $E_1, \dots, E_N$ measurable.

$f^{-1}((a,\infty)) = \bigcup_{v_j > a} f^{-1}(\{v_j\}) = \bigcup E_j \in \mathcal{M}$.

4.3 $f, g : \mathbb{R}^n \to \mathbb{R}$

$F(x) = (f(x), g(x)) : \mathbb{R}^n \to \mathbb{R}^2$

Prove that $F(x)$ is measurable $\Leftrightarrow$ f and g are.

$(\Rightarrow)$ $F^{-1}(G) \in \mathcal{M}$ for every open $G \subset \mathbb{R}^2$

NTS $f^{-1}((a,\infty))$, $g^{-1}((b,\infty))$ meas.

Take $G = (a,\infty) \times \mathbb{R}$

$F^{-1}(G) = f^{-1}((a,\infty)) \cap \underbrace{g^{-1}(\mathbb{R})}_{\mathbb{R}^n}$

$\Rightarrow f^{-1}((a,\infty))$ measurable for any a.

Do the same for g.

$(\Leftarrow)$ Write G as at most countable union of products of open intervals and proceed as above.

4.15 Let $\{f_k\}$ be a sequence of measurable functions defined on measurable E with $|E| < +\infty$.
If $|f_k(x)| \le M_x < \infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and a finite M st $|E - F| < \varepsilon$ and $|f_k(x)| < M$ $\forall$ k and all $x \in F$.

By Luzin's theorem, $\exists$ closed F_k with $|E - F_k| < \varepsilon 2^{-k}$
st f_k is continuous on F_k for each k.
Let $F = \cap F_k$
$|E - F| \le \sum_{k=1}^{\infty} |E - F_k| < \sum_{k=1}^{\infty} \varepsilon 2^{-k} = \varepsilon$

F closed with finite measure $\Rightarrow$ compact ?
Let $M = \max_{x \in F} M_x$ (or $\max_{x \in F} |f_k(x)| < ?$)

4.16. Prove that $f_k \xrightarrow{m} f$ on E iff given $\varepsilon > 0$ $\exists\ K$ st $|\{|f - f_k| > \varepsilon\}| < \varepsilon$ if $k > K$.
Give an analogous Cauchy criterion.

($\Longrightarrow$) Given $\varepsilon > 0$, $f^k \xrightarrow{m} f$:

$\lim_{n \to \infty} |\{x : |f(x) - f_n(x)| > \varepsilon\}| = 0$

$\forall \varepsilon' > 0$

$\Longrightarrow \exists\ K$ st $|\{|f - f_n| > \varepsilon\}| < \varepsilon'$ $n \geq K$.

Take $\tilde{\varepsilon} = \min\{\varepsilon, \varepsilon'\}$

($\Longleftarrow$) $\exists\ K$ st $|\{|f - f_k| > \varepsilon\}| < \varepsilon$ $k > K$.

Take $\varepsilon 2^{-j}$. $\exists\ K_j$ st $k > K_j \Rightarrow$

$|\{|f - f_k| > \cancel{\varepsilon} 2^{-j}\}| < \cancel{\varepsilon} 2^{-j}$

want to fix

If $\varepsilon' < \varepsilon$, $\{|f - f_k| > \varepsilon\} \subset \{|f - f_k| > \varepsilon'\}$

So $|\{|f - f_k| > \varepsilon\}| < 2^{-j} \to 0$

$|\{|f_k - f_j| > \varepsilon\}| < \varepsilon$ $k, j > K$

4.20 If f is measurable on $[a,b]$ show that given $\varepsilon > 0$ there is a continuous g on $[a,b]$ st $|\{x : f(x) \neq g(x)\}| < \varepsilon$

If f is finite, use Luzin's theorem.
If f is infinite? :
Define $f(x) = +\infty \ \forall x \in [a,b]$.
Then $f^{-1}((a, \infty]) = [a,b] \in \mathcal{M}$.
g would have to take infinite values also.
$f^{-1}(\{\infty\})$ must be measurable, but could still be uncountable etc.

3.9 If $\{E_k\}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ has measure zero.

$\limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$

Given $\varepsilon > 0$

$\sum_{k=1}^{\infty} |E_k|_e < +\infty \implies \sum_{k=K}^{\infty} |E_k|_e < \varepsilon$ some K depending on ε.

$\implies \left| \bigcup_{k=m}^{\infty} E_k \right| < \varepsilon \quad m \geq K$

Since $\limsup E_k \subset \bigcup_{k=m}^{\infty} E_k$, $|\limsup E_k|_e \leq \varepsilon$.

Since we can find such a K for any ε,
Thus the outer measure of $\limsup E_k$ must be 0.

$|\cdot|_e = 0 \implies |\cdot| = 0$.

3.17 Give an example which shows that the image of a measurable set under a continuous transf. may not be measurable.

Let f be the Cantor function.

f is continuous, increasing, and takes on every value in $[0,1]$.

~~∀x/ ∀x, β ∀y/ some/ subinterval~~

need to say a little more about the interval— have to skip constant values

Any interval in the image has positive measure (assuming it's nonempty) and so contains an unmeasurable set A.

But $f^{-1}(A) \subset C$ which has measure zero.

$\implies f^{-1}(A)$ measurable with measure zero.

3.21 Show that there exist sets $E_1, E_2, \ldots$ such that $E_k \searrow E$, $|E_k|_e < +\infty$ and $\lim_{k\to\infty} |E_k|_e > |E|_e$ with strict equality.

$E_1, E_2, \ldots$ must not be measurable

Let V be ~~[illegible]~~ a Vitali set in $[0,1]$.

Let $E_k = V \cap [0, 1/k]$.

I think! Then $E_k \searrow \{0\}$ ~~[illegible]~~, $|E_k|_e < +\infty$ $\forall k$,

but $\lim_{k\to\infty} |E_k|_e > 0$.

2.4 Let $\{f_k\}$ be a sequence of functions in $BV([a,b])$. If $V[f_k; a,b] \leq M < +\infty$ $\forall k$ and if $f_k \to f$ pointwise on $[a,b]$, show that f is of bounded variation and that $V[f; a,b] \leq M$. Give an example of a convergent sequence of bounded variation whose limit is not of bounded variation.

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{m} \left[|f(x_i) - f_k(x_i)| + |f_k(x_{i-1}) - f(x_{i-1})| \right.$$
$$\left. + |f_k(x_i) - f_k(x_{i-1})| \right]$$

Take k sufficiently large so that
$$\sum_{i=1}^{m} \left[|f(x_i) - f_k(x_i)| + |f_k(x_{i-1}) - f(x_{i-1})| \right] < \varepsilon$$

ok since finite sum can take max k for x_i, x_{i-1} etc.

$$\Rightarrow \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \leq \varepsilon + \sum_{k=1}^{m} |f_k(x_i) - f_k(x_{i-1})|$$
$$\leq M + \varepsilon$$

True for all ε + take sup to get
$$V[f; a,b] \leq M.$$

$f_k(x) = \frac{1}{x + 1/k} \longrightarrow f(x) = \frac{1}{x}$ on $[0,1]$

Maybe? Yes.

$f_k(x)$ monotone decreasing on $[0,1]$

$\Rightarrow V[f_k; 0,1] = |f(1) - f(0)| = k - \frac{k}{k+1} < +\infty$

2.5 Suppose f is finite on $[a,b]$ and of bounded variation on every interval $[a+\varepsilon, b]$, $\varepsilon > 0$, with $V[f; a+\varepsilon, b] \le M < \infty$. Show that $V[f; a,b] < +\infty$. Is $V[f; a,b] \le M$? If not, what additional assumption will make it so?

$V[f; a,b] = V[f; a, a+\varepsilon] + V[f; a+\varepsilon, b]$

needs more detail obv

Any partition of $[a, a+\varepsilon]$ can be partially absorbed in the second term, leaving

$$|f(a) - f(a+\varepsilon)| + \underbrace{\sum |f(x_i) - f(x_{i-1})|}_{\le M}$$

Since f is finite ↑ finite and $\le M$

So $V[f; a,b] < \infty$.

If f is continuous, $V[f; a,b] \le M$. $(\varepsilon \searrow 0)$

5.2 Show that the conclusions of (5.32) are not true without the assumption that $\varphi \in L(E)$.

(5.32) = MCT

Hint for $f_k \searrow f$: $f_k = \chi_{(k,+\infty)} \leq 1$

$f_k \searrow 0$ ae

$\int_{\mathbb{R}} f_k \, d\mu = \mu((k,+\infty)) = \infty \quad \forall k$

For $f_k \nearrow f$ use $-f_k$

5.3 Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \leq f$ ae on E show that $\int_E f_k \to \int_E f$.

Let $A = \{x : f_k(x) > f(x)\}$ and $B = E \setminus A$

On B, ~~[illegible]~~ not needed

$0 \leq f_k \nearrow f \implies \int_B f_k \to \int_B f$

by MCT for nonnegative functions

$\mu(A) = 0 \implies \int_A f_k = 0$ and $\int_A f = 0$

$\therefore \int_E f_k = \int_A f_k + \int_B f_k \to \int_A f + \int_B f = \int_E f$

5.4 If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,dx \to 0$.

x^k continuous on $(0,1) \Rightarrow x^k$ measurable $(k > 0)$

$|x^k| \leq 1$ on $(0,1)$ and $f \in L(0,1)$

$\Rightarrow x^k f(x) \in L(0,1)$. (thm 5.30)

5.7 Give an example of an f which is not integrable but whose improper Riemann integral exists and is finite.

$f(x) = \frac{\sin x}{x}$

Not in $L(\mathbb{R})$: $\int_{\mathbb{R}} \left| \frac{\sin x}{x} \right| = \infty$

But $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$.

5.9 If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).

Let $A_k = \{x \in E : |f - f_k| \geq \varepsilon\}$ $\varepsilon > 0$

$B_k = \{x \in E : |f - f_k| < \varepsilon\}$

$\int_E |f - f_k|^p = \underbrace{\int_A |f - f_k|^p}_{> \varepsilon^p} + \underbrace{\int_B |f - f_k|^p}_{\geq 0}$ This seems off.

$\geq \varepsilon^p \mu(A_k)$

$\lim_{k \to \infty} \int_E |f - f_k|^p = 0 \Rightarrow \lim_{k \to \infty} \mu(A_k) = 0$

5.10 If $p > 0$, $\int_E |f - f_k|^p \to 0$ and $\int_E |f_k|^p \le M$ for all k, show that $\int_E |f|^p \le M$.

$\int_E |f - f_k|^p \to 0 \implies f_k \xrightarrow{m} f$ on E by 5.9

$\implies \exists$ subsequence $f_{k_j} \to f$ ae.

$\int_E |f_k|^p \le M \quad \forall k \implies \int_E \liminf |f_k|^p \le M$ (Fatou)

$\int_E \liminf |f_{k_j}|^p = \int_E |f|^p$

5.11 For which $p > 0$ does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

$1/x \in L^p(0,1)$: $\int_0^1 1/x^p < \infty$ — need $p \le 0$

$1/x \in L^p(1,\infty)$: $\int_1^\infty 1/x^p < \infty$ — $p > 0$

$1/x \in L^p(0,\infty)$: impossible

5.13 (a) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum f_k$ converges absolutely ae in E if $\sum \int_E |f_k| < +\infty$.

$|f_k|$ non negative, measurable : oops

$\sum \int_E |f_k| = \int_E \sum |f_k| < +\infty$.

$\implies \sum |f_k|$ is finite ae.

$\implies \sum f_k$ converges absolutely ae.

5.13 (b) If $\{r_k\}$ denotes the rational numbers in $[0,1]$ and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k |x - r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$

$f_k(x) = a_k |x - r_k|^{-1/2}$

$\int_0^1 |f_k| = \int_0^1 \underbrace{|a_k|}_{\text{no } x} |x - r_k|^{-1/2}\, dx = |a_k| \int_0^1 \underbrace{|x - r_k|^{-1/2}}\, dx$

$\leq |a_k| \cdot$ finite

$\sum \int |f_k| \leq \sum |a_k| \cdot {}_{\wedge} < \infty$

times something maybe

By (a), $\sum f_k$ converges absolutely a.e.

5.21 If $\int_A f = 0$ for every measurable subset ~~A,~~ A of a measurable set E, show that $f = 0$ a.e. in E.

E is a measurable subset of itself so...

$\int_E f = 0 \iff f = 0$ ae in E

need f nonnegative

f measurable $\Rightarrow \{f = 0\}$ measurable

$\int_E f = \underbrace{\int_{\{f=0\}} f}_{=0} + \int_{\{f\neq 0\}} f = 0$

$\Rightarrow \int_{\{f \neq 0\}} f = 0$ ~ no

$|\{x \in E : |f(x)| > \alpha\}| \leq \frac{1}{\alpha} \int_E |f|$ hm.

6.1 (a) Let E be a measurable set in $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y : (x,y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}$ $\{x : (x,y) \in E\}$ has measure zero.

$E_x = \{y : (x,y) \in E\}$, $E_y = \{x : (x,y) \in E\}$
$\chi_E(x,y)$ is measurable since E is, and further note that $\chi_E(x,y) = \chi_{E_y}(x) = \chi_{E_x}(y)$

$$|E| = \iint_{\mathbb{R}^2} \chi_E(x,y) = \int_{\mathbb{R}^1}\int_{\mathbb{R}^1} \chi_E(x,y)\,dy\,dx \quad \leftarrow \text{Tonelli}$$

$$= \int_{\mathbb{R}} |E_x|\,dx = 0 \quad \text{since } |E_x| = 0 \text{ a.e.}$$

By Fubini-Tonelli,

$$\iint \chi_E(x,y)\,dy\,dx = \iint \chi_E(x,y)\,dx\,dy$$
$$= \int |E_y|\,dy = 0$$

$|E_y| \geq 0 \Rightarrow |E_y| = 0$ a.e.

(b) Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for ~~every~~ almost every $x \in \mathbb{R}$ $f(x,y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$, $f(x,y)$ is finite for almost every x.

Let $E = \{f = \infty\} \cup \{f = -\infty\}$.
Since f is measurable, E is as well.
By assumption, for almost every x, $|E_x| = 0$
Apply (a) to see that $|E_y| = 0$ for almost every y.

6.2 If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their Cartesian product $E_1 \times E_2$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$ and $|E_1 \times E_2| = |E_1|\,|E_2|$.

$h^{-1}((a,\infty)) = \{(x,y) : f(x)g(y) > \cancel{\infty}\, a\}$

[illegible]

$h(x,t) = f(x)\, g(x-t)$

$\uparrow$ measurable in $\mathbb{R}^n \times \mathbb{R}^n$

$F(x,t) = f(x)$ also measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

$\Rightarrow$ h is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

Let $h(x,y) = \chi_{E_1}(x)\chi_{E_2}(y)$ — measurable and nonnegative.

By Tonelli, $\iint_E h(x,y) = \int_{E_1}\int_{E_2} \chi_{E_1}(x)\chi_{E_2}(y)\,dy\,dx$

$\|$

$|E|$

$$= \left(\int_{E_1} \chi_{E_1}(x)\,dx\right)\left(\int_{E_2} \chi_{E_2}(y)\,dy\right)$$

$$= |E_1| \times |E_2|$$

6.3 Let f be measurable on $(0,1)$. If $f(x)-f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L(0,1)$.

By Fubini, $\iint_{[0,1]\times[0,1]} f(x)-f(y) = \int_0^1 \int_0^1 f(x)-f(y)\, dy\, dx$

$$= \int_0^1 \left(f(x) - \int_0^1 f(y)\, dy \right) dx$$

$$= \int_0^1 f(x)\, dx - \int_0^1 f(y)\, dy = 0$$

Also by Fubini, $f(x) - \underbrace{\int_0^1 f(y)\, dy}_{\text{constant}}$ is integrable wrt x

$\hookrightarrow$ must be finite.

6.5 (a) If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|$, $y > 0$, use Tonelli's theorem to prove that $\int_E f = \int_0^\infty \omega(y)\, dy$.

$\int_E f = |R(f,E)| = \iint_{R(f,E)} dx\, dy$

$R(f,E)_y = \{x : (x,y) \in R(f,E)\} = \{x \in E : f(x) \ge y\}$

By Tonelli, $\iint_{\{f(x) \ge y\}} dx\, dy$

Folland problems.

2.1.4. If $f: X \to \overline{\mathbb{R}}$ and $f^{-1}((r,\infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

For any $a \in \mathbb{R} \setminus \mathbb{Q}$, $f^{-1}((a,\infty]) = \bigcap_{r<a} f^{-1}((r,\infty])$.

Since $\mathcal{M}$ is closed under countable intersection, $f^{-1}((a,\infty]) \in \mathcal{M}$.

Because f maps to the extended real line, NTS $f^{-1}(\{\infty\})$, $f^{-1}(\{-\infty\}) \in \mathcal{M}$.

$f^{-1}(\{\infty\}) = \bigcap_{r \in \mathbb{Q}} f^{-1}((r,\infty])$.

As before, $\Rightarrow f^{-1}(\{\infty\}) \in \mathcal{M}$.

$f^{-1}(\{-\infty\}) = \bigcap_{r \in \mathbb{Q}} f^{-1}([-\infty, r]) = \bigcup_{r \in \mathbb{Q}} f^{-1}((r,\infty])^c$

$\mathcal{M}$ closed under countable union & complements, so $f^{-1}(\{-\infty\}) \in \mathcal{M}$.

$\therefore$ f is measurable.

2.1.5. If $X = A \cup B$, $A, B \in \mathcal{M}$. Show f is measurable on $X \iff f$ is measurable on A and B.

($\Rightarrow$) f measurable on X: $f^{-1}(G)$ measurable for every measurable G in codomain

$\Rightarrow f^{-1}(G) \cap A \in \mathcal{M}$ since $A \in \mathcal{M}$

$\overset{\shortparallel}{f|_A^{-1}(G)}$

Similarly, $f^{-1}(G) \cap B = f|_B^{-1}(G) \in \mathcal{M}$.

So f is measurable on A and B.

($\Leftarrow$) f measurable on A and B

$(f^{-1}(G) \cap A) \cup (f^{-1}(G) \cap B) \in \mathcal{M}$

$= f^{-1}(G) \cap (A \cup B) = f^{-1}(G) \cap X \in \mathcal{M}$

So f is measurable on X.

This is actually true for a countable union i.e. $X = \cup A_i$

2.1.8. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Assume WLOG that $f \nearrow$.

monotone $\Rightarrow$ f is discont at at most countably many points.

Choose (a,b) st x_0 is the only discont in $f^{-1}((a,b))$.

$$f^{-1}((a,b)) = \underbrace{f^{-1}((a, f(x_0))}_{\in \mathcal{B}} \cup f^{-1}(f(x_0)) \cup \underbrace{f^{-1}((f(x_0), b))}_{\in \mathcal{B}}$$

since f continuous on those pieces

$f^{-1}(f(x_0)) \in \mathcal{B}$ since it's a point or an interval.

$\Rightarrow f^{-1}((a,b)) \in \mathcal{B}$

So f is Borel measurable.

2.2.16. If $f \in L^+$ and $\int f < \infty$. Show $\forall\, \epsilon > 0$ $\exists$ $E \in \mathcal{M}$ st $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$,

701 Midterm

1 Let μ and ν be measures on a σ-algebra Σ in a space X.

(1) Show that $\lambda = \mu + \nu$ is a measure on Σ and $\int_X f \, d\lambda = \int_X f \, d\mu + \int_X f \, d\nu$.

for any Σ-measurable nonnegative function f on X.

NTS $\lambda \geq 0$, ctbly additive

$\lambda(A) = \mu(A) + \nu(A) \geq 0 \quad A \in \Sigma$

$\lambda(\phi) = \mu(\phi) + \nu(\phi) = 0$

$\lambda(\cup A_n) = \mu(\cup A_n) + \nu(\cup A_n)$

$= \sum \mu(A_n) + \sum \nu(A_n)$

If $\mu(\cup A_n) = \infty$ or $\nu(\cup A_n) = \infty$,

$\sum \mu(A_n) + \sum \nu(A_n) = \infty = \sum(\mu(A_n) + \nu(A_n))$

$= \sum \lambda(A_n)$

If both sums finite,

$\sum \mu(A_n) + \sum \nu(A_n)$

$= \sum(\mu(A_n) + \nu(A_n)) = \sum \lambda(A_n)$

$\Rightarrow$ λ is a measure on Σ.

If f is simple, $f = \sum_{i=1}^{n} a_i \chi_{E_i}$

$\int_X f \, d\lambda = \sum a_i \lambda(E_i) = \sum a_i \mu(E_i) + \sum a_i \nu(E_i)$

$= \int_X f \, d\mu + \int_X f \, d\nu$

If $f \geq 0$, take simple $0 \leq f_n \nearrow f$.

Since $\int f_n \, d\lambda = \int f_n \, d\mu + \int f_n \, d\nu \quad \forall n$

$\int f \, d\lambda = \lim_{n\to\infty} \int f_n \, d\lambda = \lim_{n\to\infty} \int f_n \, d\mu + \int f_n \, d\nu$

$= \int f \, d\mu + \int f \, d\nu$

Finally, if f just measurable, possibly nonnegative, take $f = f^+ - f^-$
Since $f^+, f^- \geq 0$, result follows.

2 Let $\{f_k\}$ be a sequence of nonnegative functions in $L(X,\mu)$. If $f_k \to f$ and $f_k \le f$ ae, then $\int_X f_k\,d\mu \to \int_X f\,d\mu$.

$f - f_k \ge 0$ ae and $f - f_k \to 0$ ae

By MCT $\int f - f_k\,d\mu \to 0$

$\Rightarrow$ $\int f - f_k\,d\mu \to 0$

$\Rightarrow$ $\int f\,d\mu - \int f_k\,d\mu$

or

$0 \le f_k \le f$ use DCT? — don't know $f \in L(E,\mu)$.

3 μ - measure on Σ in X, φ - asf on Σ.
Show that $\varphi \ll \mu \Leftrightarrow V(E,\varphi) \ll \mu$.

Assume $\varphi \ll \mu$.

$\mu(A) = 0 \Rightarrow \varphi(A) = 0$

$\Rightarrow \overline{V}(A) \cancel{\le} 0$

$\underline{V}(A) = 0$

$\Rightarrow V(A) = \overline{V} + \underline{V} = 0$.

Assume $V \ll \mu$.

$\mu(A) \Rightarrow V(A) = 0$

$\Rightarrow \overline{V}(A) + \underline{V}(A) = 0$

But $\overline{V}(A) \ge 0$ and $\underline{V}(A) \ge 0$, so $\underline{V} = \overline{V} = 0$.

$\Rightarrow \varphi(A) = \overline{V}(A) - \underline{V}(A) = 0$.

4 Let f be a finite measurable function on $[0,1]$ and let λ be the Lebesgue measure there. Show that $F(x,y) = f(x) - f(y) \in L([0,1]\times[0,1], \lambda\times\lambda)$ iff $f \in L([0,1],\lambda)$.

NTS F is measurable.

$$F^{-1}([a,\infty)) = \{(x,y) : f(x) - f(y) \ge a\}$$
$$= \{(x,y) : f(x) \ge a + f(y)\}$$

For each y, $f(y)$ finite and

$$\{x : f(x) \ge a + f(y)\} \text{ is measurable.}$$

Similarly for each x, $f(x)$.

Something like:

$$\bigcup_n \left(\{f(x) \ge a + r_n\} \cap \{f(y) \le r_n\}\right)$$
$$\longrightarrow F \text{ measurable.}$$

Assume $F \in L([0,1]\times[0,1], \lambda\times\lambda)$.

By Fubini Tonelli, F_x and F_y are integrable

where $F_x(y) = c_x - f(y)$

and $F_y(x) = f(x) - d_y$

$\Rightarrow f$ is integrable

$$\infty > \int_0^1 f(x) - d_y \, d\lambda = \int_0^1 f \, d\lambda - d_y$$

constant

poor choice of notation

Assume $f \in L([0,1],\lambda)$

Since F measurable and $\int_0^1 \int_0^1 |f(x) - f(y)| \, dx \, dy$

$$\le \int_0^1 \left(\int_0^1 f(x)\,dx\right) - f(y) \, dy < \infty$$

$\Rightarrow F \in L([0,1]\times[0,1], \lambda\times\lambda)$.

January 2013, Real

1 Suppose that $f:[-1,1] \to \mathbb{R}$ is a function of bounded variation. Prove that the function $g(x) = f(\sin(x))$ belongs to $BV([a,b])$ for all $-\infty < a < b < \infty$.

Given any partition Γ of $[a,b]$, $\sin(\Gamma)$ will be contained in a partition of $[-1,1]$.

Then $\sum |g(x_i) - g(x_{i-1})| \le \sum |f(x_i') - f(x_{i-1}')|$

$$\le \sup_{\Gamma} \sum |f(x_i') - f(x_{i-1}')| < \infty$$

$\Rightarrow V(g) = \sup \sum |g(x_i) - g(x_{i-1})| \le V(f) < \infty$.

2 Let $(X, \mathcal{M}, \mu)$ be a measure space such that for every set $A \in \mathcal{M}$ the measure $\mu(A)$ is a nonnegative integer. Suppose that $\{f_n\}_{n \geq 1}$ are measurable, real-valued functions on X st $\int_X |f_n| d\mu \to 0$ as $n \to \infty$. Prove that $f_n \to 0$ a.e.

Suppose not.

Then $\forall \varepsilon > 0$, $\mu(\{x : |f_n(x)| > \varepsilon$ for inf many $n\})$ is strictly greater than 0.

Fix ε, let A be this set and $B = X - A$.

Then $\int_X |f_n| d\mu = \int_A |f_n| d\mu + \int_B |f_n| d\mu$

$$> \varepsilon \mu(A) + \underbrace{\int_B |f_n| d\mu}_{\geq 0}$$

$$> \varepsilon \mu(A) \quad \text{for inf many } n$$

Since $\mu(A) \in \mathbb{N}$, $\varepsilon \mu(A) \geq \varepsilon$.

But $\lim_{n \to \infty} \int_X |f_n| d\mu = 0$.

This is a contradiction.

3 Suppose that $f \in L^2([0,1])$. Prove that the function $g(x) = |f(x)|^{x+1}$ is in $L^1([0,1])$.

Let $A = \{x : |f(x)| \geq 1\}$

$B = \{x : |f(x)| < 1\}$.

On A, $g(x) \leq |f(x)|^2$

On B, $g(x) \leq |f(x)|$

Then $\int_0^1 g(x)\,dm = \int_A g\,dm + \int_B g\,dm$

$$\leq \int_A |f(x)|^2\,dm + \int_B |f(x)|\,dm$$

$$< \int_A |f(x)|^2\,dm + m(B) < \infty.$$

(g nonnegative obv. so $g = |g|$)

Therefore $g \in L^1([0,1])$

4 Suppose that $\{f_n\}$ is a sequence of nonnegative Borel measurable functions on $[0,1]$ such that $\int_0^1 f_n \, dm = 1 \quad \forall n$

Which of the statements (a) – (d) follow above? Prove or give a counterexample.

(true) (a) The set $A = \{x : f_n(x) \le 2 \ \forall n\}$ is Borel.

$$A = \bigcap_n \underbrace{\{x : f_n(x) \le 2\}}$$

f Borel $\Rightarrow$ each of these sets is Borel
$\Rightarrow$ countable intersection is Borel.

(true) (b) The set $B = \{x : f_n(x) \le 2$ for infinitely many values of $n\}$ is Borel.

$$B = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{x : f_k(x) \le 2\}$$

As before, each set Borel $\Rightarrow$ limsup Borel.

(false) (c) $A \neq \emptyset$

Define $f_1 = 3\chi_{[0,1/3)}$, $f_2 = 3\chi_{[1/3,2/3)}$, $f_3 = 3\chi_{[2/3,1]}$

Let $f_n = \begin{cases} f_1 & n = 1 \bmod 3 \\ f_2 & n = 2 \bmod 3 \\ f_3 & n = 0 \bmod 3 \end{cases}$

Then $A = \emptyset$

(d) $B \neq \phi$

Suppose not.

$B = \phi \implies \forall x \in [0,1]$ $\exists N$ st $f_n(x) > 2$

for $n \geq N$.

$\implies \int_0^1 f_n \, dm > 2$ for $n \geq N$

This is a contradiction.

Real, August 2014

1 Assume that E is a closed subset of $\mathbb{R}$.
Prove or give a counterexample:

(a) If E^c is dense then $m(E)=0$.

Define $E_k = [0,1] - \bigcup \left(\frac{a}{2^k} - \frac{1}{2^{2k+1}}, \frac{a}{2^k} + \frac{1}{2^{2k+1}} \right)$
where $a, k \in \mathbb{N}$, $a/2^k$ in lowest terms.

Let $E = \bigcap E_k$.

E is closed and nowhere dense:

closed – obvious

nowhere dense – E cannot contain an interval, since any number $\frac{a}{2^k}$ must be removed from the interval.

$\Rightarrow E^c$ is open and dense.

For each k, this removes intervals adding up to at most $\frac{1}{2^{k+1}}$

$\Rightarrow m(E) \geq \frac{1}{2} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = 1 - \left(\sum_{k=0}^{\infty} \frac{1}{2^k} - \frac{1}{2} \right)$

(b) If $m(E)=0$ then E^c is dense.

$m(E)=0 \Rightarrow E$ has empty interior:

Suppose $(a,b) \subset E$ and (a,b) nonempty.

$\Rightarrow m(E) \geq m(a,b) \neq 0$. #

Since E is closed $\Rightarrow$ E is nowhere dense

(closure has empty interior)

$\Rightarrow E^c$ is an open dense set.

Note: A nowhere dense $\Leftrightarrow$ A^c contains a dense open set.

2 Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let f be a measurable function. If $f > 0$ on E a.e. and $\int_E f\,dm < \infty$, prove that:

$$\lim_{n\to\infty} \int_E f^{1/n}\,dm = m(E)$$

Let $A = \{x \in E : f < 1\}$
$B = \{x \in E : f \geq 1\}$
$N = \{x \in E : f \leq 0\}$.

By assumption, $m(N) = 0$.
$\Rightarrow m(A \cap N) = 0$

Note also that $B \cap N = \emptyset$ so $m(B \cap N) = 0$.

Let $f_n = f^{1/n}$.

On A, $f_n \nearrow 1$ a.e. and $|f_n| \leq 1 \in L(E)$.
On B, $f_n \to 1$ a.e. and $|f_n| \geq 1 \in L(E)$.
By MCT, $\lim_{n\to\infty} \int_A f_n\,dm = \int_A dm = m(A)$
(BCT?)

Similarly, $\lim_{n\to\infty} \int_B f_n\,dm = \int_B dm = m(B)$.

Since $E = (A-N) \cup B \cup N$, and these sets are disjoint, $m(E) = m(A-N) + m(B) + m(N)$
$= m(A) + m(B)$.

Finally note $\int_E f_n\,dm = \int_A f_n\,dm + \int_B f_n\,dm$.

Therefore $\int_E f_n\,dm \to m(E)$.

3 Let f be absolutely continuous on $[0,1]$ with $f(0)=0$ and $f' \in L^3([0,1])$. For which values of α does $\lim\limits_{x\to 0^+} x^{-a} f(x) = 0$ for all such f?

f absolutely continuous on $[0,1]$, $f(0)=0$

$\iff f(x) = \int_0^x f'(t)\,dt$

$x^{-\alpha} f(x) = x^{-\alpha}\int_0^x f'(t)\,dt = x^{-\alpha}\int f'(t)\,\chi_{[0,x]}\,dt$

$$
\begin{aligned}
|x^{-\alpha} f(x)| &= x^{-\alpha}\left|\int_0^1 f'(t)\chi_{[0,x]}\,dt\right| \\
&\le x^{-\alpha}\int |f'(t)\chi_{[0,x]}|\,dt \\
&\le x^{-\alpha}\left(\int_0^1 |f'(t)|^3\,dt\right)^{1/3}\underbrace{\left(\int_0^x dt\right)^{2/3}}_{= x^{2/3}} \quad \text{— Hölder} \\
&= x^{2/3-\alpha}\underbrace{\left(\int |f'(t)|^3\,dt\right)^{1/3}}_{<\infty \text{ by assumption}}
\end{aligned}
$$

$\alpha < 2/3$: $|x^{-\alpha} f(x)| \le x^c\left(\int_0^1 |f'(t)|^3\,dt\right)^{1/3}$, $c>0$.

As $x \to 0^+$, the RHS $\to 0$ as well.

So $\lim\limits_{x\to 0^+} x^{-\alpha} f(x) = 0$.

$\alpha = 2/3$: $|x^{-2/3} f(x)| \le \left(\int_0^x |f'(t)|^3\right)^{1/3} \longrightarrow 0$ as $x \to 0$

$\alpha > 2/3$: Let $2/3 < \beta < \alpha$.

Suppose $f(x) = x^\beta$.

Then $f'(x) = \beta x^{\beta-1} \in L^3$, $f(x) = \int_0^x f'(t)\,dt$,

so f is absolutely cont.

But $x^{-\alpha} f(x) = x^{-\alpha} x^\beta = x^{\beta-\alpha} \not\to 0$

since $\beta - \alpha < 0$.

Therefore $\lim\limits_{x\to 0^+} x^{-\alpha} f(x) = 0 \iff \alpha \le 2/3$

4 Let $(X, \mathcal{A}, \mu)$ be a measure space and $f: X \to \mathbb{R}$ a measurable function.

(a) Show that $E = \{(x,t) : |f(x)| > t\}$ is measurable in $(X \times [0,\infty), \mathcal{A} \times \mathcal{L}, \mu \times m)$.

Let $F(x,t) = |f(x)|$, $G(x,t) = t$.

Claim: F and G are $\mathcal{A} \times \mathcal{L}$ measurable.

$F^{-1}([a,\infty)) = \{x : |f(x)| \geq a\} \times [0,\infty)$

Since f is measurable, $F^{-1}([a,\infty)) \in \mathcal{A} \times \mathcal{L}$ for all $a \in \mathbb{R}$.

$$G^{-1}([a,\infty)) = X \times \{t : t \geq a\}$$

$$\{t : t \geq a\} = \begin{cases} [a,\infty) & \text{if } a \geq 0 \\ \emptyset & \text{if } a < 0. \end{cases}$$

In either case, $G^{-1}([a,\infty)) \in \mathcal{A} \times \mathcal{L}$.

$\Rightarrow \{F > G\}$ is measurable.

$\{F > G\} = E$.

(b) For $p \geq 0$ prove $\int_X |f|^p \, d\mu = \int_0^\infty p t^{p-1} \mu(x : |f(x)| > t) \, dt$

$\mu(x : |f(x)| > t) = \int_{E^t} d\mu$ where $E^t = \{x : (x,t) \in E\}$

Given t, $\int_{E^t} d\mu = \int_X \chi_{[0,|f(x)|]}(t) \, d\mu$ (?)

$$\int_0^\infty p t^{p-1} \mu(x : |f(x)| > t) \, dt \qquad \text{(Tonelli)}$$
$$= \int_X \int_0^{|f(x)|} p t^{p-1} \, dt \, d\mu$$
$$= \int_X |f|^p \, d\mu$$

(c) Prove that if $f \in L^p$ then

$$\lim_{t \to \infty} t^p \mu(x : |f(x)| > t) = \lim_{t \to 0^+} t^p \mu(x : |f(x)| > t) = 0$$

Since $f \in L^p$, $\mu(x : |f(x)|^p > t) \le \frac{1}{t^p} \int_{E^t} |f|^p \, d\mu$

$$\Rightarrow \lim_{t \to \infty} t^p \mu(x : |f(x)| > t) = 0$$

by Chebyshev's inequality and f finite a.e.

Details on Zygmund pg 82-83

As $t \to 0$, isn't this obvious?

$0 \cdot \infty = 0$.

January 2015, Real

1 Let μ^* be the Lebesgue outer measure on $\mathbb{R}$.
Show that there are disjoint sets $E_1, E_2, \ldots$
satisfying the strict inequality

$$\mu^*\left(\bigcup_k E_k\right) < \sum_k \mu^*(E_k)$$

Let E be the Vitali set contained in $[0,1]$ and
let r_k be an enumeration of the rational
numbers in $[-0,1]$

Define $E_k = E + r_k$ (translation by r_k).

These sets are disjoint:

Suppose $x \in E_i$ and $x \in E_j$, $i \neq j$.

Then $x = y + r_i$ and $x = z + r_j$, some
$y, z \in E$.

$\Rightarrow$ $y - z = r_j - r_i \in \mathbb{Q}$.

Since r_i and r_j are assumed to
be distinct, $y - z \neq 0$.

$\Rightarrow$ $y = z + r$ for $r \in \mathbb{Q}$, which is (redundant)
a contradiction. (by construction of E).

$\mu^*(E) \geq 0$ and $\mu^*(E) \leq 1$ $\Rightarrow$ $0 \leq \mu^*(\bigcup E_k) \leq 2$

Suppose $\mu^*(E) = \lambda \in \mathbb{R}^+$.

$\Rightarrow$ $\mu^*(E_k) = \lambda$ $\forall k$.

$\Rightarrow$ $\sum \mu^*(E_k) = \sum \lambda = \infty$.

So $\mu^*(\bigcup E_k) < \sum \mu^*(E_k)$

2 Construct a function in $L^1(\mathbb{R})$ that is not in $L^2(a,b)$ for any nonempty $(a,b) \subset \mathbb{R}$.

Let $g(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0 < x < 1 \\ 0 & o/w \end{cases}$ $\qquad g \in L^1([0,1])$, $g \notin L^2([0,1])$

and let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q}$.

Define $f(x) = \sum_k \frac{g(x-r_k)}{2^k}$.

$\int f(x) \leq \sum \int \frac{g(x-r_k)}{2^k} = \sum \frac{1}{2^k} \int \frac{1}{\sqrt{x-r_k}} < \infty$

But for any (a,b) there is some $r_k \in (a,b)$?,

so $f^2(x) \geq \frac{g^2(x-r_k)}{2^{2k}}$.

$\Longrightarrow \int f^2(x) \geq \int \frac{g^2(x-r_k)}{2^{2k}} = \infty$

3 Let S be a measurable space and $\mathcal{F}$ a σ-algebra of subsets of S. Let ν be a positive finite measure on $\mathcal{F}$ and μ a finitely additive real-valued set function on $\mathcal{F}$. Finally assume both $\nu+\mu$ and $\nu-\mu$ are nonnegative, finite, and countably additive on $\mathcal{F}$.

Prove that μ is a signed measure on $\mathcal{F}$ whose total variation is absolutely continuous wrt μ.

NTS $\mu(\emptyset)=0$, countably additive.

For any $A \in \mathcal{F}$, $(\nu+\mu)(A) \geq 0$

$$\Rightarrow \mu(A) \geq -\nu(A) \qquad (1)$$

$$(\nu-\mu)(A) \geq 0$$

$$\Rightarrow \mu(A) \leq \nu(A) \qquad (2)$$

So $0 \leq \mu(\emptyset) \leq 0 = \nu(\emptyset)$.

Let $A_1, A_2, \ldots \in \mathcal{F}$ be disjoint.

Then $(\nu+\mu)(\bigcup A_j) = \sum(\nu(A_j) + \mu(A_j)) < \infty$

$(\nu-\mu)(\bigcup A_j) = \sum(\nu(A_j) - \mu(A_j)) < \infty$

$\Rightarrow 2\mu(\bigcup A_j) = \sum(\nu(A_j) + \mu(A_j) - \nu(A_j) + \mu(A_j))$

$= 2\sum \mu(A_j)$

So μ is countably additive. (Somehow use finite add?)

$V(E,\mu) = \sup \mu(A) - \inf \mu(A)$

If $\nu(E)=0$, (1) and (2) $\Rightarrow \mu(E)=0$

For any $A \subset E$, $A \in \mathcal{F}$, $\nu(A)=0$

$\Rightarrow \mu(A)=0$ as well.

So $V \ll \nu$.

4 Let f_n be Lebesgue integrable on $\mathbb{R}$ st $|f_n(x)| \searrow 0$ ae. Also assume that $\sum f_n(x)$ is alternating for almost every x.

Prove that $\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)\,dx$

Immediate consequences of the hypotheses:

(1) $\sum f_n(x)$ converges ae (say $f(x) = \sum f_n(x)$)

(2) $\int_{-\infty}^{\infty} |f_n(x)|\,dx \to 0$ by DCT

so $\int_{-\infty}^{\infty} f_n(x)\,dx \to 0$ also.

(3) $\int_{-\infty}^{\infty} \sum |f_n(x)|\,dx = \sum \int_{-\infty}^{\infty} |f_n(x)|\,dx$ (thm 5.16)

If able to show either integral in (3) is finite, use Fubini-Tonelli.

or

Let $B = \{x : \sum f_n(x) \text{ is alternating}\}$

Define $f(x) = \begin{cases} \sum f_n(x) & x \in B \\ 0 & o/w \end{cases}$

$|f(x)| \le |f_1(x)| \implies f \in L^1$.

Let $S_n(x) = \sum_{j=1}^{n} f_j(x)$.

Then $\int |f(x) - S_n(x)| \le \int |f_{n+1}|\,dx \to 0$.

So $S_n \to f \in L^1$ and $\int f(x)\,dx = \lim \int S_n(x)\,dx$

$= \sum \int f_j(x)\,dx$

Alternating series estimation

$$\left|\sum^{\infty} a_n - \sum^{N} a_n\right| \le |a_{N+1}|$$

for $\sum a_n$ a convergent alternating series

August 2015. Real

1 Let $(X, \mathcal{M})$ be a measurable space and suppose $A_n \in \mathcal{M}$ for $n = 1, 2, \ldots$

Let $A = \{x \in X : x \in A_n$ for infinitely many n and $x \notin A_n$ for infinitely many $n\}$

Prove that $A \in \mathcal{M}$.

$\{x \in X : x \in A_n$ for inf many $n\} = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k$

$\{x \in X : x \notin A_n$ for inf many $n\}$

$= \{x \in X : x \in A_n^c$ for inf many $n\}$

$= \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k^c$

$A_n \in \mathcal{M} \Rightarrow A_n^c \in \mathcal{M}$ and $\limsup A_n \in \mathcal{M}$

$A_n^c \in \mathcal{M} \Rightarrow \limsup A_n^c \in \mathcal{M}$

So $\limsup A_n \cap \limsup A_n^c \in \mathcal{M}$.

2 Suppose $f : [0,1) \longrightarrow [0,\infty)$ is a measurable function such that $\int_0^1 f(x)\sqrt{1-x}\,dx < \infty$.
Let $F(x) = \int_0^x f(t)\,dt$ for $x \in [0,1)$.

(a) Prove that F is continuous on $[0,1)$.

(b) Does F have to be bounded on $[0,1)$?
Prove or disprove.

(c) Prove that $\int_0^1 F(x)\,dx < \infty$.

3 Give an example of a sequence of functions $f_n : [0,1] \to [0,1]$ such that the total variation of f_n on $[0,1]$ is at most 2, and the function $f(x) = \sup_n f_n(x)$ is not in $BV([0,1])$.

Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$.

Define $f_n(x) = \begin{cases} 1 & x = r_n \\ 0 & o/w \end{cases}$

Then $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & o/w \end{cases}$

Variation of f_n at most 2:

If r_n is chosen as a partition point,

$$\sum |f_n(x_i) - f_n(x_{i-1})| = 2$$

Else, $\sum |f_n(x_i) - f_n(x_{i-1})| = 0$.

$\implies \sup \sum |f_n(x_i) - f_n(x_{i-1})| = 2$.

f is not of bounded variation:

Given some partition $0 = x_0 < \dots < x_m = 1$,

suppose $\sum |f(x_i) - f(x_{i-1})| \leq M$.

If $x_1 \in \mathbb{Q}$ choose x_1' between $0, x_1$

st $x_1' \in [0,1] \setminus \mathbb{Q}$.

This will increase the variation by 2.

Similarly if $x_1 \notin \mathbb{Q}$.

ie. the variation can always be increased by adding partition points.

$\implies$ unbounded.

4 Suppose that $\{f_n\}$ is a sequence of functions on $[0,1]$ such that $\|f_n\|_4 \le 1$ for all n.

Which of the statements (a) – (c) follow from the above? Prove or give a counterexample.

(a) There is a constant C such that $\|f_n\|_2 \le C$ for all n.

Generalization of Hölder's inequality:

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$$

$\|1\|_4$

$$\|f_n\|_2 \le \|f_n\|_4 \le 1$$

(b) There is a constant C such that $\|f_n\|_6 \le C$ for all n.

Let $f_n = \frac{1}{3}x^{-1/6}$ for every n.

Then $\int_0^1 \frac{1}{3}x^{-2/3}\, dx = x^{1/3}\Big|_0^1 = 1$

But $\int_0^1 \frac{1}{3} \cdot \frac{1}{x}\, dx = \frac{1}{3}\ln\Big|_0^1 = \infty$

(c) There exists a subsequence $\{f_{n_k}\}$ which converges ae on $[0,1]$.

Qualifying Exam Real Analysis August, 2014

Throughout m is Lebesgue measure.

1. Assume that E is a closed subset of $\mathbb{R}$. Prove or give a counterexample;

   (a) If E^c is dense then $m(E) = 0$. — False

   (b) If $m(E) = 0$ then E^c is dense. — true

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and f a measurable function. If $f > 0$ on E a.e. and $\int_E f dm < \infty$, prove that

$$\lim_{n\to\infty} \int_E f^{1/n} dm = m(E).$$

3. Let f be absolutely continuous on $[0,1]$ with $f(0) = 0$ and $f' \in L^3([0,1])$. For which values of α does

$$\lim_{x\to 0+} x^{-\alpha} f(x) = 0$$

for all such f?

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a measurable function.

   (a) Show that $E = \{(x,t) : |f(x)| > t\}$ is measurable in the product space $(X \times [0,\infty), \mathcal{A} \times \mathcal{L}, \mu \times m)$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $[0,\infty)$.

   (b) For $p > 0$ prove

$$\int_X |f|^p d\mu = \int_0^\infty p t^{p-1} \mu(x : |f(x)| > t) dt.$$

   (c) Prove that if $f \in L^p$ then

$$\lim_{t\to\infty} t^p \mu(x : |f(x)| > t) = \lim_{t\to 0+} t^p \mu(x : |f(x)| > t) = 0.$$

10

Real Analysis **Qualifying Exam** **January 2015**

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from your courses.

Problem 1 Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that there are disjoint sets $E_1, E_2, \ldots$ satisfying the strict inequality

$$\mu^*(\bigcup_k E_k) < \sum_k \mu^*(E_k)$$

Problem 2. Construct a function in $L^1(\mathbb{R})$ that is not in $L^2((a,b))$ for any non-empty interval $(a,b) \subset \mathbb{R}$.

Problem 3. Let S be a measurable space and $\mathcal{F}$ a sigma algebra of subsets of S. Let ν be a positive finite measure on $\mathcal{F}$ and μ a finitely additive real-valued set function on $\mathcal{F}$. Finally, assume that both $\nu + \mu$ and $\nu - \mu$ are non-negative, finite, and countably additive on $\mathcal{F}$. Prove that μ is a signed measure on $\mathcal{F}$ whose total variation is absolutely continuous with respect to ν.

Problem 4. Let the f_n be Lebesgue integrable on $\mathbb{R}$ such that $|f_n(x)| \searrow 0$ a.e. Also assume that the series $\sum_{n=1}^{\infty} f_n(x)$ is an alternating series for almost every x. Prove that

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

Fubini Tonelli!!

I was right

$\int_{P_n} f_n + \int_{N_n} f_n \quad S_N(x) = \sum^N f_n(x)$

$\int S_N(x) = \sum \int f_n(x)$

$\int f_n(x) \to 0 \quad |f_n(x)| \searrow 0$ ae somehow

By $\Rightarrow \int |f_n(x)| \searrow 0$

$\sum f_n(x)$ converges

$f_n(x) \geq 0$

$f_{n+1}(x) \leq 0$

$P_n = \{x \; f_n(x) \geq 0\}$

$N_n = \{x \; f_{n+1}(x) \leq 0\}$

August 2014 – Real

1. Assume that E is a closed subset of $\mathbb{R}$.
   Prove or give a counterexample.

   (a) If E^c is dense, then $m(E) = 0$.

   idea: E^c dense $\Rightarrow$ E has only isolated points

   $\Rightarrow$ E is countable

   $\Rightarrow$ $m(E) = 0$

   (b) If $m(E) = 0$, then E^c is dense.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and f a measurable function. If $f > 0$ on E ae and $\int_E f\, dm < \infty$, prove that

$$\lim_{n\to\infty} \int_E f^{1/n}\, dm = m(E)$$

Suppose $A = \{x \in E : f(x) \leq 0\}$.

By assumption, $m(A) = 0$.

Consider $E - A$, partition into $E_1, E_2, \ldots, E_k$ disjoint

$$m(E) = \sum_{j=1}^{k} m(E_j)$$

f must be finite ae (why?)

Then $f^{1/n} \to 1$ as $n \to \infty$

when can you bring lim into $\int$?

$f^{1/n} \to 1$ ae ptwise

4. Δ = open unit disc in $\mathbb{C}$

$f_n : \Delta \to \Delta$, $n \geq 1$ – holomorphic

st f_n has a zero of order m_n at 0

where $\lim_{n\to\infty} m_n = \infty$

Show that $\{f_n\}$ converges locally uniformly to zero on Δ

f_n has a zero of order m_n at 0

$\Rightarrow f_n(z) = z^{m_n} g(z)$ where $g(z)$ analytic and nonzero at 0.

$\sup_{z\in\Delta} \underbrace{|f_n(z)|}_{} = \varepsilon_n$

$|z^{m_n} g(z)| < 1$ since $|z^{m_n}| < 1$ and $|g(z)| < 1$

$\mathbb{D}_r(0)$

Consider $\{|z| < r\}$ $r < 1$

$\sup_{z\in\mathbb{D}_r(0)} |f_n(z)| = \varepsilon_n$

$\sup_{z\in\mathbb{D}_r(0)} |z^{m_n} g(z)| = r^{m_n} \sup |g(z)| < r^{m_n}$

As $n \to \infty$, $m_n \to \infty$, $r^{m_n} \to 0$

So $\{f_n\}$ converges locally uniformly to zero on Δ.

For each n, $f_n(z)$ can be written as

$$f_n(z) = z^{m_n} g(z), \text{ where } g(z) \text{ is analytic}$$

and $g(0) \neq 0$. ($g(z) : \Delta \to \Delta$)

For any $r < 1$, let $\mathbb{D}_r(0) = \{ |z| < r \}$.

Then $$\sup_{z \in \mathbb{D}_r(0)} |f_n(z)| = \sup_{z \in \mathbb{D}_r(0)} |z^{m_n} g(z)|$$

$$= r^{m_n} \sup_{z \in \mathbb{D}_r(0)} |g(z)|$$

$$< r^{m_n}$$

Since $m_n \to \infty$ as $n \to \infty$, $r^{m_n} \to 0$.

$\Rightarrow \{f_n(z)\}$ converges uniformly to 0 on $\mathbb{D}_r(0)$.

3. Let $D \subset \mathbb{C}$ be a bounded domain.
$z_0 \in D$ and $f: D \to D$ be a holomorphic function.
st $f(z_0) = z_0$.
Show that $|f'(z_0)| \leq 1$.

$$|f^{(m)}(z)| \leq \frac{m!}{r^m} M$$

D bounded & $f(D) \subseteq D \Rightarrow f$ bounded.

Say $|f(z)| \leq M$.

$|f'(z_0)| \leq \frac{M}{r}$, $\{|z - z_0| \leq r\} \subset D$

$D \subseteq \{|z - z_0| \leq M\}$ — $M = \inf$ st

Suppose $z_0 = 0$.

$f(0) = 0$, $|f'(0)| \leq \frac{M}{r}$

can just look at $g(z) = f(z) - z_0$ on $D - z_0$

want $r = M$ but ?

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z_0)^2} dw \right|$$

$$\leq \left| \frac{1}{2\pi i} \right| \left| \frac{M}{r^2} \right| |2\pi r| = \frac{M}{r} \text{ dun}$$

August 2014 - Complex

2. D - a domain in $\mathbb{C}$ containing 0

$f: D \to \mathbb{R}$ continuous

$f(0) = 0$

$\int_{\partial R} f(z)\, dz = 0 \quad \forall$ rectangles in D ...

Prove that $f(z) = 0$ for every $z \in D$.

10

By Morera's thm, f is analytic in D.

(don't know if $f \in C^1(\partial D)$
or if ∂D is piecewise smooth)

$$f(0) = \cancel{\frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, d\theta}$$

$$= \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z}\, dz \qquad z = 0 + re^{i\theta}$$

$$\frac{dz}{d\theta} = re^{i\theta} \cdot i$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} f(re^{i\theta})\, d\theta \qquad dz = ire^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, d\theta = 0$$

$f = u + iv$ - but f is real valued so $v \equiv 0$.

f analytic so, u, v satisfy CR

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

correct
(~~may have sign switched~~)

$$\Rightarrow \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0 \quad \forall (x, y)$$

$$\Rightarrow f' = u_x + iu_y = 0$$

Since D is a domain, $\Rightarrow f$ is constant.
Since $f(0) = 0$, $f(z) = 0 \;\forall z \in D$.

By Morera's theorem, f is analytic in D.

Writing $f = u + iv$, we have that $v \equiv 0$

since f is real valued.

By the analyticity of f, we know that

u and v must satisfy the CR

equations.

$\Longrightarrow$ $u_x = v_y = 0$, $u_y = -v_x = 0$ $\forall (x,y)$

$\Longrightarrow$ $f' = u_x + iu_y = 0$ at every point of D.

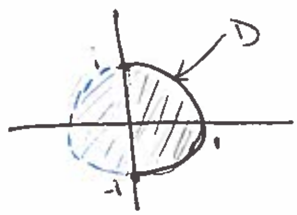
Since D is a domain, this implies that

f is constant on D.

$f(0) = 0 \Longrightarrow f(z) = 0 \quad \forall z \in D$.

1. Find a conformal map from

$D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z < 0\}$ onto Δ.



$z \longmapsto \left(-\frac{z-i}{z+i}\right)^2$ send i to 0

$-i$ to ∞

goes to upper right quadrant

$z \longmapsto \frac{z-i}{z+i}$ takes upper right to Δ.

compose :

$z \longmapsto \left(\frac{z-i}{z+i}\right)^3$