Measure Theory Problems Worth Knowing

Foland

2.1.4 If $f : X \to \bar{\mathbf{R}}$ and $f^{-1}((r,\infty]) \in M$ for each $r \in \mathbf{Q}$ then f is measurable.

2.1.5 If $X = A \cup B$, $A, B \in M$. Show f is measurable on X if and only if f is measurable on A and B.

2.1.8 If $f : \mathbf{R} \to \mathbf{R}$ is monotone then f is Borel Measurable.

2.2.15 If $\{f_n\} \subset L^+$, f_n decreases to f pointwise and $\int f_1 < \infty$ then $\int f = lim \int f_n$.

2.2.16 If $f \in L^+$ and $\int f < \infty$. Show $\forall \epsilon > 0$, $\exists E \in M$ such that $\mu(E) < \infty$and $\int_E f > (\int f) - \epsilon$.

2.4.35 Show $f_n \to f$ in measure $\Leftrightarrow$ $\forall \epsilon > 0$, $\exists N \in \mathbf{N}$such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$ $\forall n \geq N$.

3.1.2 (a) If ν is a signed measure, E is a nullset $\Leftrightarrow$ $|\nu|(E) = 0$

(b) If ν and μ are signed measures then $\nu \perp \mu \Leftrightarrow \nu^+ \perp \mu + \nu^- \perp \mu \Leftrightarrow |\nu| \perp \mu$

3.2.8 $[\nu \ll \mu] \Leftrightarrow [\nu^+ \ll \mu] + [\nu^- \ll \mu] \Leftrightarrow [|\nu| \ll \mu]$

Bass

13.3 Let (X, A) be a measure space and let μ and ν be two finite measures. We say μ is equivalent to ν if $\mu \ll \nu$ and $\nu \ll \mu$. Show μ and ν are quivalent iff there exists measurable function f that is strictly positive a.e. and integrable with respect to μsuch that $d\nu = f d\mu$

13.5 If $\mu$is a signed meaure on (X, A) and $|\mu|$ is the total variation measure. Prove there is an f (real valued, measurable wrt A) such that $|f| = 1$ a.e.(μ) and $d\mu = f d|\mu|$.

Midterm 2012

Prove $f_n \nearrow f$ and $f_1^- \in L^1(\mu)$ then $\int f_n \nearrow \int f$

$f \in L^1(m)$. Fix $a \in \mathbf{R}$. Set $F(x) = \int_a^x f(t)dt$. Prove F is continuous.

Prove or Counterexample: If $f_n \geq 0$ and $f_n \to f$ then $\int f d\mu \leq \liminf \int f_n d\mu$

Problems worth knowing

2.1.4 If $f: X \to \overline{\mathbb{R}}$ and $f^{-1}((r,\infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$ then f is measurable

Pf First consider ∞.

$$f^{-1}(\{\infty\}) = \bigcap_{r \in \mathbb{Q}} f^{-1}((r,\infty]) \subset \mathcal{M}$$

Since countable intersections are in $\mathcal{M}$.

Now consider $-\infty$

$$f^{-1}(\{-\infty\}) = \bigcap_{r \in \mathbb{Q}} f^{-1}([-\infty, r]) = \bigcap_{r \in \mathbb{Q}} f^{-1}((r,\infty])^c \in \mathcal{M}$$

For $a \in \mathbb{R}$, $f^{-1}((a,\infty)) = \bigcup_{r > a} f^{-1}((r,\infty]) \setminus f^{-1}(\{\infty\})$ is measurable on $f^{-1}(\mathbb{R})$

$\Rightarrow f$ is measurable on all of $\overline{\mathbb{R}}$.

□

need to show $\pm\infty$ msble as well $f^{-1}([a,\infty))$

2.1.5 If $X = A \cup B$, $A, B \in \mathcal{M}$ Show f is msble on X
$\Leftrightarrow$ f is measurable on A and B.

Pf $\Rightarrow$) Assume f is measurable on X.

$\Rightarrow f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{B}$.

$\Rightarrow f^{-1}(E) \cap A \in \mathcal{M}$ Since $f^{-1}(E)$ and A are,
and $f^{-1}(E) \cap B \in \mathcal{M}$ $\forall E \in \mathcal{B}$.

$\Rightarrow$ f is measurable on both A and B.

measurable on $E \Rightarrow f^{-1}(B) \cap E \in \mathcal{M}$ $\forall B \in \mathcal{B}$.

$\Leftarrow$) Assume f is msble on A and B.
Let $E \in \mathcal{B}$.

$$\begin{aligned} \Rightarrow f^{-1}(E) &= f^{-1}(E) \cap X \\ &= f^{-1}(E) \cap (A \cup B) \\ &= f^{-1}(E) \cap A \cup f^{-1}(E) \cap B \\ &\in \mathcal{M} \end{aligned}$$

$\Rightarrow$ f is msble on X.

$\square$

2.1.8 If $f:\mathbb{R}\to\mathbb{R}$ is monotone then f is Borel measurable.

Pf WLOG assume f is increasing.
f monotone $\Rightarrow$ f continuous at all but possibly countably many points.

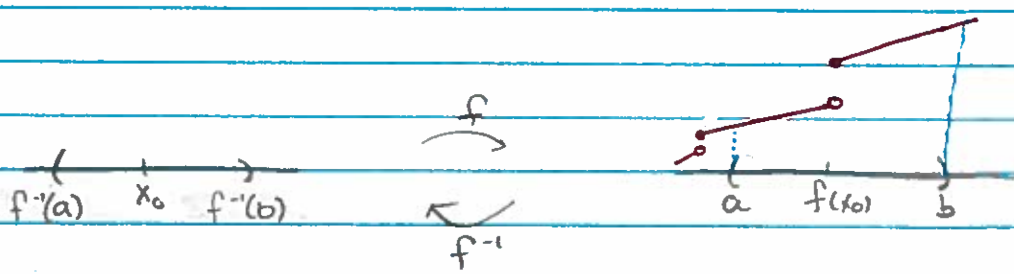
Let x_0 be a discontinuity of f.
Choose (a,b) s.t. x_0 is only discontinuity in $f^{-1}(a,b)$.

$$f^{-1}(a,b) = f^{-1}(a,f(x_0)) \cup f^{-1}(f(x_0)) \cup f^{-1}(f(x_0),b)$$

$f^{-1}(a,f(x_0)) \in \mathcal{B}$ since f cont. on $(a,f(x_0))$
$f^{-1}(f(x_0)) \in \mathcal{B}$ since either a single pt or interval
$f^{-1}(f(x_0),b) \in \mathcal{B}$ since f cont

$\Rightarrow f^{-1}(a,b) \in \mathcal{B}$
$\Rightarrow f$ is Borel measurable. $\square$

key is monotone fcns have at most countably many discontinuities.



2.2.15 If $\{f_n\} \subset L^+$, f_n decreases to f ptwise and $\int f_1 < \infty$ then $\int f = \lim \int f_n$

Pf $\int f \leq \int f_1 < \infty$ since $f \leq f_1$

$f_1 - f_n \nearrow f_1 - f$ and $f_1 - f_n \geq 0$ measurable.

So by MCT $\lim \left(\int f_1 - f_n\right) = \int \lim (f_1 - f_n)$

$= \int (f_1 - \lim f_n)$

$= \int (f_1 - f)$

Also $\lim \int (f_1 - f_n) = \lim \int f_1 - \lim \int f_n$

$= \int f_1 - \lim \int f_n.$

So $\int (f_1 - f) = \int f_1 - \lim \int f_n$

$\Rightarrow \int f_1 - \int f = \int f_1 - \lim \int f_n$

$\int f = \lim \int f_n$

$\square$

2.2.16 If $f \in L^+$ and $\int f < \infty$. Show $\forall \varepsilon > 0$, $\exists E \in M$ s.t.
$\mu(E) < \infty$ and $\int_E f > (\int f) - \varepsilon$

Pf Consider $E_n = \{x \mid f(x) > 1/n\}$ then $\mu(E_n) < \infty$

Let $f_n = f \chi_{E_n} \to f$

$\Rightarrow \int f_n \to \int f$

$\Rightarrow \int_{E_n} f \to \int f$ as $n \to \infty$

So $\forall \varepsilon > 0$ $\exists n_0$ s.t $|\int_{E_n} f - \int f| < \varepsilon$ $n \geq n_0$

but $\varepsilon > |\int_{E_n} f - \int f| = \int f - \int_{E_n} f$

$\Rightarrow \int_{E_n} f > \int f - \varepsilon$

$\square$

or

$f \in L^+$ $\Rightarrow f$ can be approximated by a simple function.

Let $f \in L^+$ s.t $\int f < \infty$

Let $\varepsilon > 0$.

By definition of $\int f$, $\exists s = \sum_1^n a_i \chi_{E_i}$ s.t. $0 \leq s \leq f$

and $\int f - \varepsilon < \int \phi$.

Let $E = \bigcup_1^n E_i \in M$ since $E_i \in M$.

Also $\int \phi \leq \int f \leq \infty$ so $\forall E_i$ $\mu(E_i) < \infty$

$\Rightarrow \mu(E) < \infty$.

$\Rightarrow \int f - \varepsilon < \int_E \phi \leq \int_E f$.

$\square$

2.4.35 Show $f_n \xrightarrow{\mu} f \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$

Pf $\Rightarrow$) Let $\varepsilon > 0$.

$\Rightarrow \mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$ as $n \to \infty$

$\Rightarrow \exists N$ s.t. $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$ $\forall n \geq N$

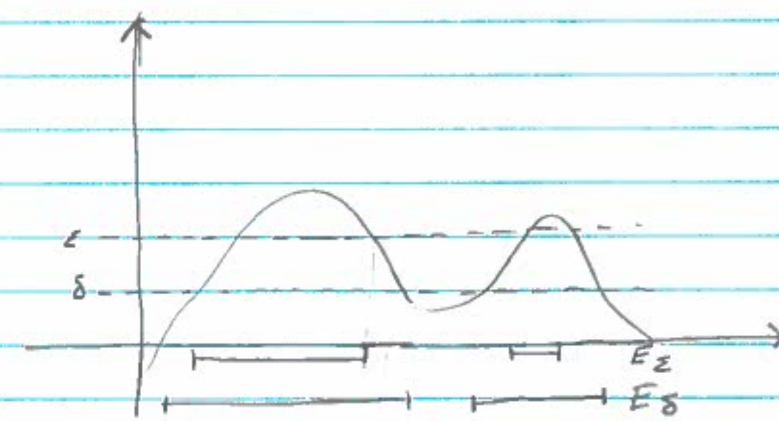
$\Leftarrow$) Let $\varepsilon > 0$.

Let $E_\varepsilon = \{x \mid |f_n(x) - f(x)| > \varepsilon\}$ $\delta < \varepsilon$

$\Rightarrow \mu(E_\varepsilon) \leq \mu(E_\delta) < \delta$ $\forall \delta < \varepsilon$.

$\Rightarrow f_n \xrightarrow{\mu} f$

$\square$



3.1.2 (a) If ν is a signed measure. E is a nullset $\Leftrightarrow |\nu|(E)=0$

(b) If ν & μ signed measure $\underset{(1)}{\nu \perp \mu} \Leftrightarrow \underset{(2)}{|\nu| \perp \mu} \Leftrightarrow \underset{(3)}{\nu^+ \perp \mu}$ and $\nu^- \perp \mu$

Pf (a) Assume E is a null set

$\Rightarrow \forall F \in \mathcal{M}$ with $F \subset E$ $\nu(F)=0$.

Let $X = P \cup N$ with $P \cap N = \phi$ Hahn decomposition

$\Rightarrow E \cap P \subset E$ and $E \cap N \subset E$

$\Rightarrow \nu^+(E) = \nu(E \cap P) = 0$

$\nu^-(E) = \nu(E \cap N) = 0$

$\Big\}$ rewrite ν^+, ν^- then they are 0.

$\Rightarrow |\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

Assume $|\nu|(E)=0$

$\Rightarrow |\nu|(E) = \nu^+(E) + \nu^-(E) = 0$

$\Rightarrow \nu^+(E) = \nu^-(E) = 0$ since $\nu^+, \nu^- \geq 0$

Let $F \subset E$, $F \in \mathcal{M}$.

$\Rightarrow 0 \leq |\nu|(F) \leq |\nu|(E) = 0$

$\Rightarrow |\nu|(F) = 0$

$\Rightarrow \nu^+(F) = \nu^-(F) = 0$

$\Rightarrow \nu(F) = 0$

(b): (1) $\Rightarrow$ (2) $\nu \perp \mu$

$\Rightarrow \exists E, F$ s.t. $E \cup F = X$ $E \cap F = \phi$ s.t. $\nu(E)=0$ $\mu(F)=0$ since $\nu \perp \mu$.

$\Rightarrow \exists N, P$ s.t. $X = N \cup P$, $N \cap P = \phi$ by Hahn Decomp

Let $T \subset E$ w/ $T \in \mathcal{M}$

$\nu^+(T) = \nu(T \cap P) = 0$ since $T \cap P \subset E$

$\nu^-(T) = \nu(T \cap N) = 0$ since $T \cap N \subset E$

$\Rightarrow E$ null for $|\nu|$.

(2) $\Rightarrow$ (3) From above

(3) $\Rightarrow$ (4)

$\nu = \nu^+ - \nu^-$ So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

$\Rightarrow \nu^+(E) = 0$ and [illegible] $\square$

328 $[\nu \ll \mu] \Leftrightarrow [|\nu| \ll \mu] \Leftrightarrow [\nu^- \ll \mu]$ & $[\nu^+ \ll \mu]$

(1) (2) (3)

Pf (1) $\Rightarrow$ (2) $\nu \ll \mu$

$\Rightarrow \mu(E) = 0 \Rightarrow \nu(E) = 0.$

Let $X = P \cup N$ be Hahn decomp of X

Let $\nu = \nu^+ - \nu^-$ be Jordan decomp of X

Then $|\nu|(E) = \nu^+(E) + \nu^-(E)$

$= \nu(E \cap P) + \nu(E \cap N)$

$= 0 - 0$

$= 0$

$\therefore |\nu| \ll \mu$

(2) $\Rightarrow$ (3) $|\nu| \ll \mu$

Let $E \in \mathcal{M}$ $\mu(E) = 0$

$\nu^+(E) < |\nu|(E) = 0$

$\nu^-(E) < |\nu|(E) = 0$

$\Rightarrow \nu^+ \ll \mu, \nu^- \ll \mu$

(3) $\Rightarrow$ (1). $\nu^- \ll \mu$ $\nu^+ \ll \mu$

Let $E \in \mathcal{M}$ $\mu(E) = 0$

$\nu(E) = (\nu^+ - \nu^-)(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0$

$\therefore \nu \ll \mu$

$\square$

6.1.9 Suppose $1 \le p < \infty$.

(a). If $\|f_n - f\|_p \to 0$ then $f_n \xrightarrow{\mu} f$

(b) If $f_n \xrightarrow{\mu} f$ and $|f_n| \le g \in \mathcal{L}^p$ $\forall n$ then $\|f_n - f\|_p \to 0$

Pf (a). Assume $\|f_n - f\|_p \to 0$.

Let $E_\varepsilon = \{x \mid |f_n(x) - f(x)| > \varepsilon^{1/p}\}$

Then $\int |f_n - f|^p \ge \int_{E_\varepsilon} |f_n - f|^p$

$= \int |f_n - f|^p \chi_{E_\varepsilon}$

$\ge \int \varepsilon \chi_{E_\varepsilon}$

$= \varepsilon \mu(\chi_{E_\varepsilon})$

$\Rightarrow \frac{1}{\varepsilon} \int |f_n - f|^p > \mu(E_\varepsilon)$

$\Rightarrow \mu(E_{E_\varepsilon}) \to 0$ as $n \to \infty$ $\forall \varepsilon$.

(b). Assume $f_n \xrightarrow{\mu} f$

Let $g_n = f_n - f$

$\Rightarrow |g_n| = |f_n - f| < |f_n| + |f| = 2g$

$\Rightarrow |g_n|^p < 2^p g^p \in \mathcal{L}^1$

Let g_{n_k} be a subsequence of g_n.

Then $f_n \xrightarrow{\mu} f \Rightarrow g_n \xrightarrow{\mu} 0$

$\Rightarrow \exists\ |g_{n_{k_j}}|^p \to 0$ a.e. and $|g_{n_{k_j}}|^p < 2^p g^p \in \mathcal{L}^1$

So by DCT $\int |g_{n_{k_j}}|^p \to \int 0 = 0$

$\Rightarrow \int |g_n|^p \to 0$ since every subseq has a convergent subseq.

$\Rightarrow \int |f_n - f|^p \to 0$.

$\square$

6.10 Suppose $1 \le p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e then $\|f_n - f\|_p \to 0 \iff \|f_n\|_p \to \|f\|_p$

Pf Assume $\|f_n - f\|_p \to 0$

$\Rightarrow \big| \|f_n\|_p - \|f\|_p \big| \le \|f_n - f\|_p \to 0$

$\Rightarrow \|f_n\|_p \to \|f\|_p$

Now assume $\|f_n\|_p \to \|f\|_p$.

$$\left.\begin{array}{l} |f_n - f|^p \le 2^p(|f_n|^p + |f|^p) = g_n \\ g_n \to 2^{p+1}|f|^p = g \end{array}\right\} \Rightarrow |f_n - f|^p \le g_n$$

$$\left.\begin{array}{rl} \lim \int g_n = & \lim \int 2^p(|f_n|^p + |f|^p) \\ = & 2^p\left[\lim \int |f_n|^p + \lim \int |f|^p\right] \\ = & 2^p(2\,\|f\|_p) \\ = & 2^{p+1}\|f\|_p = \int g. \end{array}\right\} \Rightarrow \lim \int g_n = \int g$$

Then by GDCT, $\lim \int |f_n - f|^p = \int \lim |f_n - f|^p = \int 0 = 0$

$\square$

13.3 $(X,\mathcal{A})$ m.s. μ, ν finite measures. Show μ and ν equivalent iff $\exists$ measurable $f>0$ a.e $f \in \mathcal{L}^1(\mu)$ s.t. $d\nu = f d\mu$

Pf $\Rightarrow$) Suppose μ equivalent to ν.

Then $\mu \ll \nu$ and $\nu \ll \mu$.

$\left.\begin{array}{l} \exists f \geq 0 \text{ w/ } f \in \mathcal{L}^1(\mu) \text{ s.t. } d\nu = f\, d\mu. \\ \exists g \geq 0 \text{ w/ } g \in \mathcal{L}^1(\nu) \text{ s.t. } d\mu = g\, d\nu \end{array}\right\}$ Radon Nikodym Derivatives

$\Rightarrow d\nu = f\,d\mu \Rightarrow d\nu = fg\,d\nu$

$\Rightarrow \nu(A) = \int_A fg\,d\nu$

$\Rightarrow fg = 1$ a.e. by uniqueness of RND

$\Rightarrow f \neq 0$

$\Rightarrow f > 0$ a.e.

and $d\nu = f d\nu$

$\Leftarrow$) Conversely assume $\exists f>0$ a.e, measurable, $f \in \mathcal{L}^1$ $d\nu = f d\mu$

Assume $\mu(E) = 0 \Rightarrow \int_E f d\mu = 0$

$\Rightarrow \nu(A) = 0$

$\Rightarrow \nu \ll \mu$

Assume $\nu(A) = 0$

if $\mu(A) > 0 \Rightarrow \int_A f d\mu > 0$ since $f > 0$

which contradicts since $\nu(A) = 0$

$\Rightarrow \mu(A) = 0$.

So ν and μ are equivalent

$\square$.

13.5 If μ is a signed measure on $(X, \mathcal{A})$ prove:
$\exists f$ s.t. $|f|=1$ a.e (μ) and $d\mu = f\,d|\mu|$

Pf Let $X = P \cup N$ be the Hahn decomposition of X
Let $f = \chi_P - \chi_N$

Since $P \cap N = \emptyset$ then $f(x) = \begin{cases} 1 & x \in P \\ -1 & x \in N \end{cases} \Rightarrow |f| = 1$ a.e
and f is clearly $\mathcal{A}$ measurable since its the difference of 2 characteristic functions

Now $\int_A f\,d|\mu| = \int_A (\chi_P - \chi_N)\,d|\mu|$.

$$
\begin{aligned}
&= \int_A \chi_P\,d|\mu| - \int_A \chi_N\,d|\mu| \\
&= \int \chi_{P\cap A}\,d|\mu| - \int_A \chi_{A\cap N}\,d|\mu| \\
&= |\mu|(P\cap A) - |\mu|(N\cap A) \\
&= \mu^+(P\cap A) + \mu^-(P\cap A) - \mu^+(N\cap A) - \mu^-(N\cap A) \\
&= \mu^+(A\cap P) - \mu^-(A\cap N) \\
&= \mu^+(A) - \mu^-(A) \\
&= \mu(A) \\
&= \int_A d\mu
\end{aligned}
$$

$\therefore f\,d|\mu| = d\mu$. $\square$

Midterm.

1. Prove $f_n \nearrow f$ and $f_1^- \in L^1(\mu)$ then $\int f_n \nearrow \int f$

Pf Let $g_n = f_n + f_1^-$ measurable since f_n is.
$g_n \geq 0$ since
$\int g_n \nearrow \int g \Rightarrow \int f_n + f_1^- \nearrow \int f + f_1^-$
$\Rightarrow \int f_n + \int f_1^- \nearrow \int f + \int f_1^-$
$\Rightarrow \int f_n \nearrow \int f$ since $\int f_1^-$ is constant

$\square$

2. $f \in L^1(m)$. Fix $a \in \mathbb{R}$. Set $F(x) = \int_a^x f(t)dt$. Prove F is cont.

Pf Let $x_n \to x$ in $\mathbb{R}$.
Set $f_n = f\chi_{(a,x_n]}$
Then $|f_n| \leq |f| \in L^1$ for all n.
$f_n \to f\chi_{(a,x]}$ a.e.
$\Rightarrow \int f_n = \int f\chi_{(a,x_n)} = \int_a^{x_n} f = F(x_n) \nearrow F(x)$
$\Rightarrow F$ is cont. since every sequence going to x has function values going to $F(x)$

$\square$

3. If $f_n \geq 0$, $f_n \xrightarrow{m} f$ then $\int f d\mu \leq \underline{\lim} \int f_n d\mu$

Pf Since $f_n \xrightarrow{m} f$ $\exists n_j$ s.t. $f_{n_j} \to f$ a.e.
Then by Fatou:
$\Rightarrow \int \lim f_{n_j} d\mu \leq \underline{\lim} \int f_{n_j} d\mu$
$\Rightarrow \int f d\mu \leq \underline{\lim} \int f_{n_j} d\mu$
$\Rightarrow \int f d\mu - \underline{\lim} \int f_{n_j} d\mu \leq 0$
So $\int f d\mu \leq \underline{\lim} \int f_n d\mu$

$\square$

Things worth knowing:

How to check $f \in AC$ -

- f differentiable
- f' is integrable
- $\int_a^x f'(t)\,dt = f(x) - f(a)$

How to check $f \notin AC$

- show $f \notin BV$

Counter examples.

f cont, $m(E)=0$, $m(f(E)) \neq 0$ — C = Cantor set, f = Cantor fcn

f cont, $f \not\to 0$ as $x \to \infty$, $f \in L^2$ — $f = \begin{cases} n & x = n \\ \nearrow n & x \in [n - 1/2n^2, n] \\ \searrow 0 & x \in [n, n + 1/2n^2] \\ 0 & \text{else.} \end{cases}$

$f \in L^2$, $\sqrt{n}\int_{[n,n+1]} |f| \not\to 0$ as $n \to \infty$ — $f = \begin{cases} 2^{-n/2} & x \in [2^n, 2^{n+1}] \\ 0 & \text{else} \end{cases}$

$f \notin L^p$ but $f \in L^r$ $\forall r < p$ — $f = 1/x^{1/p}$

$f_n \xrightarrow{L_1} f$ but $f_n \overset{a.e}{\not\to} f$ — $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$ ...

$f_n \xrightarrow{a.e} f$ but $f_n \overset{L_1}{\not\to} f$ — $f_n = \frac{1}{n}\chi_{[0,n]}$

- Positive msble fcn's can be approximated by simple fcns.
- msble fcn's can be approximated by continuous fcns.
- msble integrable fcns can be approx by cont fcn w/ compact supp.

Examples.

$f_n = n^{-1}\chi_{(0,n)}$ $\quad$ $f_n \to 0$ ptwise, uniformly a.e. measure not in L^1

$f_n = \chi_{(n,n+1)}$ $\quad$ not in measure. not in L^1

$f_n = n\chi_{[0,1/n]}$ $\quad$ ptwise, uniformly, a.e. measure not in L^1

Real Analysis Qualifying Exam January 2015

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from your courses.

Problem 1 Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that there are disjoint sets $E_1, E_2, \ldots$ satisfying the strict inequality

$$\mu^*(\bigcup_k E_k) < \sum_k \mu^*(E_k)$$

Problem 2. Construct a function in $L^1(\mathbb{R})$ that is not in $L^2((a,b))$ for any non-empty interval $(a,b) \subset \mathbb{R}$.

Problem 3. Let S be a measurable space and $\mathcal{F}$ a sigma algebra of subsets of S. Let ν be a positive finite measure on $\mathcal{F}$ and μ a finitely additive real-valued set function on $\mathcal{F}$. Finally, assume that both $\nu + \mu$ and $\nu - \mu$ are non-negative, finite, and countably additive on $\mathcal{F}$. Prove that μ is a signed measure on $\mathcal{F}$ whose total variation is absolutely continuous with respect to ν.

Problem 4. Let the f_n be Lebesgue integrable on $\mathbb{R}$ such that $|f_n(x)| \searrow 0$ a.e. Also assume that the series $\sum_{n=1}^{\infty} f_n(x)$ is an alternating series for almost every x. Prove that

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)\, dx.$$

Complex Analysis **Qualifying Exam** **January 2015**

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from class.

Problem 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$ converges absolutely for $|z| < 1$. Also show that there are infinitely many z with $|z| = 1$ for which the series diverges.

Problem 2. Let $f(z)$ be holomorphic on $\mathbb{C}$ except for poles. At ∞ assume that f has a removable singularity or a pole.
(a) Show that f has finitely many poles on $\mathbb{C} \cup \{\infty\}$.
(b) Let $p_j(z)$ be the principal part of f at the jth pole, $1 \leq j \leq N$, show that

$$f(z) - \sum_{j=1}^{N} p_j(z)$$

is constant.

Problem 3. Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle, $|z| = 1$. Assume there is an entire function g such that $f(z) = g(z)$ for $|z| = 1$. Prove that $f = g$, and hence f is entire.

Problem 4. Let f_n be analytic in the unit disc, D, and have positive real part: $\mathcal{R}(f_n(z)) > 0$ on D. Assume that the f_n converge pointwise on D to a function f having $\mathcal{R}(f(z)) \leq 0$ on D. Prove that f is constant on D.

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from your courses.

Problem 1 Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that there are disjoint sets $E_1, E_2, \ldots$ satisfying the strict inequality

$$\mu^*(\bigcup_k E_k) < \sum_k \mu^*(E_k)$$

Answer: Let E be the usual nonmeasurable set (intersected with $[0,1]$), whose rational translates are disjoint. For rational numbers $q_k \in (0,1)$ consider the translates $E_k = E + q_k$. Then the disjoint union is contained in $[0,2]$ while the translates all have the same positive outer measure, therefore the left hand side is finite while the right hand side is infinite. (Note that translation invariance of outer measure is used. This can be shown readily from the definition of outer measure using coverings by open intervals.)

Problem 2. Construct a function in $L^1(\mathbb{R})$ that is not in $L^2((a,b))$ for any non-empty interval $(a,b) \subset \mathbb{R}$.

Answer: Let $g(x) = \begin{cases} 1/\sqrt{x} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ then $\int_{-\infty}^{\infty} g dx = 2$, while $\int_{-\infty}^{\infty} g^2 dx = \infty$ For a countable dense subset $\{r_1, r_2, \ldots\}$ of $\mathbb{R}$ consider the function $f(x) = \sum_k g(x - r_k)/2^k$, f is clearly nonnegative, measurable, and in L^1 (the integral of f is dominated by the sum of the integrals, which is finite). For any interval (a,b) there is an r_k in the interval, and $f^2(x) \geq g^2(x - r_k)/2^{2k}$, integrating this inequality shows that $f \notin L^2(a,b)$.

Problem 3. Let S be a measurable space and $\mathcal{F}$ a sigma algebra of subsets of S. Let ν be a positive finite measure on $\mathcal{F}$ and μ a finitely additive real-valued set function on $\mathcal{F}$. Finally, assume that both $\nu + \mu$ and $\nu - \mu$ are non-negative, finite, and countably additive on $\mathcal{F}$. Prove that μ is a signed measure on $\mathcal{F}$ whose total variation is absolutely continuous with respect to ν.

Answer: Let $A, B \in \mathcal{F}$ with $B \subseteq A$. Then

$$\mu(B) = \frac{1}{2}(\nu + \mu)(B) - \frac{1}{2}(\nu - \mu)(B),$$

hence

$$|\mu(B)| \leq \frac{1}{2}(\nu+\mu)(B) + \frac{1}{2}(\nu-\mu)(B) \leq \frac{1}{2}(\nu+\mu)(A) + \frac{1}{2}(\nu-\mu)(A) = \nu(A).$$

In particular, if $B_n \in \mathcal{F}, n = 1, 2, \ldots$ are pairwise disjoint and $A = \bigcup_{n=1}^{\infty} B_n$ then

$$\sum_{n=1}^{\infty} |\mu(B_n)| \leq \sum_{n=1}^{\infty} \nu(B_n) = \nu(A) < \infty.$$

Thus μ is a signed measure. Also, if $A \in \mathcal{F}$,

$$|\mu|(A) = \sup\{|\mu(B)| : B \subseteq A, B \in \mathcal{F}\} \leq \nu(A).$$

Thus $\nu(A) = 0 \implies |\mu|(A) = 0$.

Problem 4. Let the f_n be Lebesgue integrable on $\mathbb{R}$ such that $|f_n(x)| \searrow 0$ a.e. Also assume that the series $\sum_{n=1}^{\infty} f_n(x)$ is an alternating series for almost every x. Prove that

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)\, dx.$$

Answer: Let B the set of full measure on which the series is alternating and on which we have $|f_n(x)| \searrow 0$. Define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for $x \in B$ and $f(x) = 0$ for $x \notin B$. Then f is measurable, and by the alternating series error estimate we have $|f(x)| \leq |f_1(x)|$, so f belongs to L^1. Let $S_n(x) = \sum_{j=1}^{n} f_j(x)$. Again by the alternating series error estimate and the Dominated Convergence Theorem we have

$$\int_{-\infty}^{\infty} |f(x) - S_n(x)|\, dx \leq \int_{-\infty}^{\infty} |f_{n+1}(x)|\, dx \to 0$$

as $n \to \infty$. Thus $S_n \to f$ in L^1, and we conclude that

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} S_n(x)\, dx = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} f_j(x)\, dx.$$

Throughout m is Lebesgue measure.

1. Assume that E is a closed subset of $\mathbb{R}$. Prove or give a counterexample;

   maybe (a) If E^c is dense then $m(E) = 0$.

   (b) If $m(E) = 0$ then E^c is dense.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and f a measurable function. If $f > 0$ on E a.e. and $\int_E f\,dm < \infty$, prove that

$$\lim_{n\to\infty} \int_E f^{1/n} dm = m(E).$$

3. Let f be absolutely continuous on $[0, 1]$ with $f(0) = 0$ and $f' \in L^3([0, 1])$. For which values of α does

$$\lim_{x\to 0+} x^{-\alpha} f(x) = 0$$

   for all such f?

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a measurable function.

   1/2 (a) Show that $E = \{(x, t) : |f(x)| > t\}$ is measurable in the product space $(X \times [0, \infty), \mathcal{A} \times \mathcal{L}, \mu \times m)$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $[0, \infty)$.

   (b) For $p > 0$ prove

$$\int_X |f|^p d\mu = \int_0^\infty p t^{p-1} \mu(x : |f(x)| > t) dt.$$

   1/2 (c) Prove that if $f \in L^p$ then

$$\lim_{t\to\infty} t^p \mu(x : |f(x)| > t) = \lim_{t\to 0+} t^p \mu(x : |f(x)| > t) = 0.$$

Qualifying Exam, Complex Analysis, August 2014

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$.

1. Find a conformal map from the half-disc $D = \{z \in \mathbb{C} : |z| < 1, \mathbf{Re}\, z > 0\}$ onto Δ.

2. Let D be a domain in $\mathbb{C}$ containing 0 and $f : D \longrightarrow \mathbb{R}$ be a continuous function such that $f(0) = 0$ and

$$\int_{\partial R} f(z)\, dz = 0$$

for every closed rectangle $R \subset D$ with sides parallel to the coordinate axes. Prove that $f(z) = 0$ for every $z \in D$.

3. Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in D$, and $f : D \longrightarrow D$ be a holomorphic function such that $f(z_0) = z_0$. Show that $|f'(z_0)| \leq 1$.

Mayh

4. Let $f_n : \Delta \longrightarrow \Delta$, $n \geq 1$, be a sequence of holomorphic functions such that f_n has a zero of order m_n at 0, where $\lim_{n\to\infty} m_n = \infty$. Show that $\{f_n\}$ converges $\underbrace{\text{locally uniformly}}_{\text{normally.}}$ to zero on Δ.

1. Assume E is closed in $\mathbb{R}$. Prove or counter

a) E^c is dense $\Rightarrow m(E)=0$

b) $m(E)=0 \Rightarrow E^c$ dense.

Pf a) Let $\varepsilon>0$.

Let $q_1, q_2, \ldots$ be enumeration of the rationals

Let $I_i = (q_i - \varepsilon/2^{i+1}, q_i + \varepsilon/2^{i+1})$.

Let $E = \mathbb{R} \setminus \bigcup_1^\infty I_i$, which is closed

$\Rightarrow \mathbb{Q} \subset E^c = \bigcup_1^\infty I_i$

$\Rightarrow E^c$ is dense.

$m(E^c) \leq \sum m(I_i) = \sum 2\varepsilon/2^{i+1} = \varepsilon$

$\Rightarrow m(E) = m(\mathbb{R} \setminus E^c)$

$\Rightarrow m(E) = m(\mathbb{R}) - m(E^c) > 0$

b). Assume E^c not dense but $m(E)=0$

$\Rightarrow \exists (a,b) \subset \mathbb{R}$ s.t. $(a,b) \not\subset E^c$

$\Rightarrow (a,b) \subset E$

$\Rightarrow m(a,b) < m(E)$

$\Rightarrow 0 < b-a < m(E)$

$\Rightarrow m(E) > 0$ $\lightning$

$\therefore m(E)=0 \Rightarrow E^c$ is dense.

$\square$

2 Let E be Lebesgue msble subset of $\mathbb{R}$ and f msble function If $f>0$ on E a.e and $\int_E f\,dm < \infty$ prove $\lim_{n\to\infty} \int_E f^{1/n} dm = m(E)$

Pf Let $A = \{x \in E : |f(x)| < 1\}$
$B = \{x \in E : |f(x)| \geq 1\}$

on B:
Let $f_n = f^{1/n} \chi_E \leq f\chi_E \in L^1$
$f_n \to \chi_E$ as $n \to \infty$
$\Rightarrow \int f_n = \int f^{1/n}\chi_E \to \int \chi_E = m(E \cap B)$

on A:
Let $f_n = f^{1/n}\chi_E$
$\Rightarrow 0 \leq f_n \leq f_{n+1}$ are measurable increasing.
$\Rightarrow \int_A f_n \to \int_A \chi_E$
$\Rightarrow \int_E f^{1/n} \to m(E \cap A)$

$\therefore \lim \int_E f^{1/n} = \lim \int_A f^{1/n} + \int_B f^{1/n}$
$= \lim m(E\cap A) + m(E \cap B)$
$= \lim (m(E))$ since disjoint
$= m(E)$

$\square$

3. Let f be absolutely continuous on $[0,1]$ and $f' \in L^3([0,1])$
For which α does $\lim_{x\to 0^+} x^{-\alpha} f(x) = 0 \quad \forall f$?

Pf f absolutely continuous on $[0,1]$

$\Rightarrow f(x) - f(0) = \int_0^x f'(t)\,dt \quad \forall x \in [0,1]$ by FTC for LI

$\Rightarrow f(x) = \int_0^x f'(t)\,dt$

$\Rightarrow |x^{-\alpha} f(x)| = |x^{-\alpha} \int_0^x f'(t)\,dt|$

$= |x^{-\alpha} \int f'(t) \chi_{[0,x]}\,dt|$

$\le x^{-\alpha} \int |f'(t)| \chi_{[0,x]}\,dt$

$\le x^{-\alpha} \left(\int_0^x |f'(t)|^3\right)^{1/3} \left(\int_0^x 1\right)^{2/3}$

$= x^{2/3-\alpha} \underbrace{\left(\int_0^x |f'(t)|^3\right)^{1/3}}_{<\infty \text{ since } f' \in L^3}$

$\alpha = 2/3 \Rightarrow x^{2/3-\alpha}\left(\int_0^x |f'(t)|^3\right)^{1/3} = \left(\int_0^x |f'(t)|^3\right)^{1/3} \to 0$ as $x \to 0$

$\alpha < 2/3 \Rightarrow x^{2/3-\alpha}\left(\int_0^x |f'(t)|^3\right)^{1/3} \le x^{2/3-\alpha} \|f'\|_3 \to 0$ since $2/3 - \alpha > 0$

Now if $\alpha > 2/3$, let β be s.t. $2/3 < \beta < \alpha$.

Let $f = x^\beta$

$\Rightarrow f' = \beta x^{\beta-1} \in L^1$ and f diff, and $f(x) = \int_0^x f'(t)$

$\Rightarrow f \in AC$ by FTC LI

$\int |f'|^3 = \int |\beta x^{\beta-1}|^3 = \int \beta^3 x^{3\beta-3} < \infty$ since $3\beta - 3 \ne -1$

Finally $x^{-\alpha} f(x) = x^{\beta-\alpha} \not\to 0$ since $\beta - \alpha < 0$

$\therefore \lim_{x\to 0^+} x^{-\alpha} f(x) = 0 \iff \alpha \le 2/3$

$\square$

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f: X \to \mathbb{R}$ be msble

a) Show $E = \{(x,t) : |f(x)| > t\}$ is msble

b) For $p > 0$ show $\int_X |f|^p \, d\mu = \int_0^\infty p t^{p-1} \mu\{x : |f(x)| > t\} \, dt$

c) Prove $f \in L^p \Rightarrow \lim_{t \to \infty} t^p \mu(x : |f(x)| > t) = \lim t^p \mu(x : |f(x)| > t) = 0$

Pf a). Let $F(x,t) = |f(x)|$, $G(x,t) = t$

$\Rightarrow F - G$ is msble

$\Rightarrow F - G > 0$ is msble

$\Rightarrow F > G$ is msble

$\Rightarrow \{(x,t) \mid |f(x)| > t\}$ is msble.

b) $\int_0^\infty p t^{p-1} \mu\{x : |f(x)| > t\} = \int_0^\infty \int_{E_t} p t^{p-1} \, d\mu \, dt$ where $E_t = \{x \mid |f(x)| >$

$= \int_X \int_0^{|f(x)|} p t^{p-1} \, dt \, d\mu$

$= \int_X |f(x)|^p \, d\mu$

c) $\lim_{t \to \infty} t^p \mu\{x : |f(x)| > t\} \le \lim_{t \to \infty} t^p \frac{1}{t^p} \int_{|f| > t} |f|^p$

$= \lim_{t \to \infty} \int_{|f| > t} |f|^p$

$= 0$ since $f \in L^p$

$\lim_{t \to 0} t^p \mu\{x : |f(x)| > t\} = \lim_{t \to 0} \int_0^t p s^{p-1} \, ds \, \mu\{x : |f(x)| > t\}$

$\le \lim_{t \to 0} \int_0^t \underbrace{p s^{p-1} \mu\{x : |f(x)| > s\} ds}_{\le \int_X |f(x)|^p}$ since $s < t$

$= 0$ since

$\square$

MAT 701 **Real Analysis Qualifying Exam** **August 2013**

Problem 1. Let f be a measurable function satisfying

$$|f(x)| \leq \frac{x^2}{1+x^4}, \; -\infty < x < \infty.$$

a. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(nx)\,dx = 0.$$

b. Is it necessarily true that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f\left(\frac{x}{n}\right)\,dx = 0?$$

Problem 2. Let f be an integrable function satisfying $\int_0^1 f(x)\,dx = 0$. Prove that there are intervals I of arbitrarily small positive length such that

$$\int_I f(x)\,dx = 0.$$

Problem 3. Formulate and prove a version of Hölder's inequality for products of three functions. It is sufficient to obtain an upper bound on $\int_0^1 f(x)g(x)h(x)\,dx$ for non-negative measurable functions f, g, and h in terms of suitable L^p norms of the individual functions. It is permissable to use the usual (two function) Hölder inequality without proof.

Problem 4. Let C be a closed set of positive Lebesgue measure and $f(x) = d(x, C)$, the distance from the point x to the set C. Prove that there exist points x at which the derivative of f vanishes. Give an example of a closed set of measure zero for which there is no such point x.

August 2013

1. Let f be measurable. $|f(x)| \le \frac{x^2}{1+x^4}$

(a) Prove $\lim \int_{-\infty}^{\infty} f(nx)\,dx = 0$

(b) Is it necessarily true that $\lim \int_{-\infty}^{\infty} f(\frac{x}{n})\,dx = 0$?

Pf (a) $\int_{|x|>1/n} f(nx) \le \int_{|x|>1/n} \frac{n^2x^2}{1+n^4x^4}\,dx$

$$= 2\int_{1/n}^{\infty} \frac{n^2x^2}{1+n^4x^4}\,dx \qquad u = nx \quad du = n\,dx$$

$$= \frac{2}{n}\int_{1/n}^{\infty} \frac{u^2}{1+u^4}\,du$$

$$\le \frac{2}{n}\int_1^{\infty} \frac{1}{u^2}\,du$$

$$= \frac{2}{n} \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow 0 \le \int_{-\infty}^{\infty} f(nx)\,dx \le 2/n \to 0 \quad \text{as } n \to \infty$$

(b) It is not true for $f(x) = \frac{x^2}{1+x^4}$

$$\int_{-\infty}^{\infty} f(\tfrac{x}{n}) = \int \frac{(\frac{x}{n})^2}{1+(\frac{x}{n})^4}\,dx \qquad u = \frac{x}{n} \quad n\,du = dx$$

$$= n\int_{-\infty}^{\infty} \frac{u^2}{1+u^4}$$

$$= 2n\int_{1/n}^{\infty} \frac{u^2}{1+u^4}$$

$$\ge 2n\int_1^{\infty} \frac{u^2}{1+u^4} \qquad \square$$

$$\ge 2n\int_1^{\infty} \frac{u^2}{2u^4}$$

$$= n\int_1^{\infty} \frac{1}{u^2}$$

$$= n \to \infty \quad \text{as } n \to \infty$$

when looking for counter try using upper bound.

$\lim_{n\to\infty} \int_{-\infty}^{\infty} = \lim \int_{|x|>1/n}$

2 Let f be integrable function satisfying $\int_0^1 f(x)dx = 0$
Prove that there are intervals I of arbitrarily small length s.t. $\int_I f(x)dx = 0$

Pf Let $\varepsilon > 0$.
Subdivide $[0,1]$ into k intervals of equal length $< \varepsilon/2$
If none of these k intervals satisfies $\int_K f(x)dx = 0$
then there must be an adjacent pair (K, L)
s.t. $\int_K f$, $\int_L f$ have opposite sign

{ if all had same sign it would contradict
$\int_0^1 f(x)dx = 0$ by summing on intervals

WLOG assume L is to right of K an
$\int_L f(x)dx > -\int_K f(x)dx > 0$

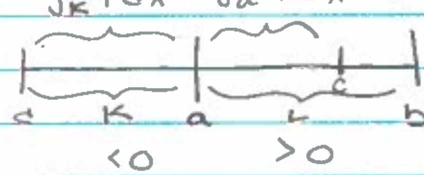
other cases are handled similarly

Let $L = [a,b]$ and define $g(x) = \int_a^x f(y)dy$
then g is continuous by standard consequence of DCT

Now $g(a) = 0$ and $g(b) > -\int_K f(x)dx > 0$
So by IVT $\exists$ $a < c < b$ s.t. $\int_a^c f(x)dx = -\int_K f(x)dx$
Let $I = K \cup [a,c]$
Then I has length $< \varepsilon$ and $\int_I f(x)dx = 0$

This completes proof since $\varepsilon > 0$ was arbitrary. $\square$

$\int_K f dx + \int_a^c f dx = 0$



$b - a < \varepsilon/2$, $a - d < \varepsilon/2$
$\Rightarrow c - d < \varepsilon$

[redacted]

Then $\int_0^1 f(x)g(x)h(x)\,dx \le \|f\|_p \|g\|_q \|h\|_r$

Pf Let $\lambda = \frac{pq}{p+q} > 0$ so $\frac{1}{\lambda} + \frac{1}{r} = 1$

$\Rightarrow \int_0^1 [f(x)g(x)]h(x) \le \left(\int_0^1 |f(x)g(x)|^\lambda\right)^{1/\lambda} \|h\|_r$

Now $\frac{\lambda}{p} + \frac{\lambda}{q} = \frac{q}{p+q} + \frac{p}{p+q} = 1$

$\Rightarrow \int_0^1 f(x)^\lambda g(x)^\lambda \le \left(\int_0^1 (f(x)^\lambda)^{p/\lambda}\right)^{\lambda/p} \left(\int_0^1 (g(x)^\lambda)^{q/\lambda}\right)^{\lambda/q}$

$= \left(\int_0^1 |f(x)|^p\right)^{\lambda/p} \left(\int_0^1 |g(x)|^q\right)^{\lambda/q}$

$\Rightarrow \left(\int_0^1 |f(x)g(x)|^\lambda\right)^{1/\lambda} \le \|f\|_p \|g\|_q$

$\Rightarrow \int_0^1 fgh \le \|f\|_p \|g\|_q \|h\|_r.$

□

use r and $\frac{pq}{p+q}$

$\frac{1}{p} + \frac{1}{r} + \frac{1}{q}$

4. Let C be closed set of positive measure and $f(x) = d(x, C)$. Prove $\exists$ points x s.t. $f'(x) = 0$. Give an example of measure 0 for which $\nexists\ x$.

Lipschitz $\Rightarrow$ BV $\Rightarrow$ derivative exists.

Pf By triangle inequality $|f(x) - f(y)| = |d(x,C) - d(y,C)| \leq |x-y|$

$\Rightarrow f$ is Lipshitz

$\Rightarrow f$ is of bounded variation.

$\Rightarrow f'(x)$ exists $\forall x$ in some set D w/ $m(D^c) = 0$

The set of isolated pts of C is at most countable ($\Rightarrow$ it has measure 0)

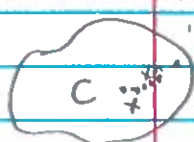
C can't be just isolated points. ?

b/c each singularity in included in a interval w/ rational endpoints that excludes all others

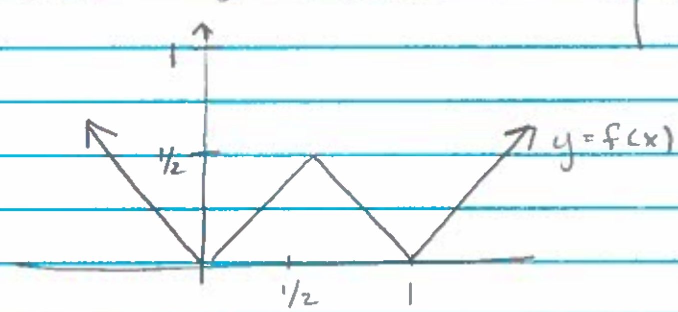
$\Rightarrow \exists\, x \in D \cap C$ that is an accumulation of points in C

$\Rightarrow \exists\, x_n \in C$ $x_n \neq x$ w/ $x_n \to x$

$\Rightarrow f'(x) = \lim_{n \to \infty} \dfrac{f(x_n) - f(x)}{x_n - x} = 0$ Since $f(x_n) = f(x) = 0$ since $x_n, x \in C$



Let $C = \{0, 1\}$ and $f(x) = \begin{cases} -x & x < 0 \\ x & 0 \leq x < 1/2 \\ 1-x & 1/2 < x < 1 \\ x-1 & 1 \leq x \end{cases}$



Here $f'(x)$ fails to exist for $x = 0, 1/2, 1$

For all other x $|f'(x)| = 1$ $\square$

MAT 701 Notes

Definition: Let X be a set. $\mathcal{A} \subseteq \mathcal{P}(X)$ is an **algebra** on X if

(1) $X \in \mathcal{A}$ whole set
(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ complements
(3) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ unions (finite)

$\mathcal{A} \subseteq \mathcal{P}(X)$ is a **σ-algebra** if (1), (2), and $A_i \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ add in countable unions

If $\mathcal{A}$ is a σ-algebra of X, then $(X, \mathcal{A})$ is called a **measurable space**.

Lemma 2.7: *If $\mathcal{A}_\alpha$, $\alpha \in I$ is a family of σ-algebras, then $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra.* intersection of σ-algebras is a σ-algebra

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Definition: Let $\mathcal{C} \subseteq \mathcal{P}(X)$. Then

$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq \mathcal{A}} \mathcal{A} \qquad (\mathcal{A} \text{ a } \sigma\text{-algebra})$$

is the **σ-algebra generated by $\mathcal{C}$**. It is a σ-algebra by the previous lemma, and is the smallest σ-algebra containing $\mathcal{C}$, i.e. if $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra, then $\sigma(\mathcal{C}) \subseteq \mathcal{A}$.

If X is a topological space, let $\mathcal{G}$ be the collection of open sets of X. Then $\sigma(\mathcal{G})$ is called the **Borel σ-algebra**, denoted $\mathcal{B}$ or $\mathcal{B}(X)$.

Proposition 2.8: *Let $X = \mathbb{R}$. Then $\mathcal{B}$ is generated by each of the following collection of sets:*

$\mathcal{C}_1 = \{(a, b) \mid -\infty < a < b < \infty\}$
$\mathcal{C}_2 = \{[a, b] \mid -\infty < a < b < \infty\}$
$\mathcal{C}_3 = \{(a, b] \mid -\infty < a < b < \infty\}$
$\mathcal{C}_4 = \{(a, \infty) \mid -\infty < a < \infty\}$

generators of Borel σ-algebra (all segments/intervals and rays)

Definition: $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **monotone class** if

(1) $A_i \in \mathcal{M},\ A_i \uparrow A \Rightarrow A \in \mathcal{M}$
(2) $A_i \in \mathcal{M},\ A_i \downarrow A \Rightarrow A \in \mathcal{M}$

increasing/decreasing nested sets converge to a set in $\mathcal{M}$.

[Note: $A_i \uparrow A$ means $A_i \subseteq A_{i+1}$ and $A = \cup_{i=1}^{\infty} A_i$. Similarly for $A_i \downarrow A$.]

Theorem: (*The Monotone Class Theorem*) *Suppose $\mathcal{A}_0$ is an algebra, $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{A}_0$, and $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}_0$. Then $\mathcal{M} = \mathcal{A}$.*

Smallest σ-algebra and smallest monotone class containing A_0 are equivalent.

Definition: Let $(X, \mathcal{A})$ be a measurable space. A **measure** on $(X, \mathcal{A})$ is a set function $\mu : \mathcal{A} \to [0, \infty]$ such that

(1) $\mu(\emptyset) = 0$
(2) $A_i \in \mathcal{A}$, A_i disjoint $\Rightarrow \mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ Countable additivity

and $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Proposition 3.5:

(0) $A_i \in \mathcal{A}$ disjoint $\Rightarrow \mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$

(i) $A, B \in \mathcal{A}$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ inclusion

(ii) $A_i \in \mathcal{A}$, $A = \cup_{i=1}^{\infty} A_i \Rightarrow \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ equality only occurs if disjoint.

(iii) $A_i \in \mathcal{A}$, $A_i \uparrow A \Rightarrow \mu(A_i) \nearrow \mu(A)$

(iv) $A_i \in \mathcal{A}$, $A_i \downarrow A$, $\mu(A_i) < \infty$ for some $i \Rightarrow \mu(A_i) \searrow \mu(A)$ need one to be finite for decreasing case.

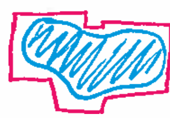
Definition: $(X, \mathcal{A}, \mu)$ is a **finite measure space** if $\mu(X) < \infty$. It is a **σ-finite measure space** if there exist sets $E_i \in \mathcal{A}$, $\mu(E_i) < \infty$ and $\cup_{i=1}^{\infty} E_i = X$.

measurable
is in A
e measu

Definition: $A \subseteq X$ is a **null set** if there exists $B \in \mathcal{A}$, $\mu(B) = 0$, and $A \supseteq B$. The measure space $(X, \mathcal{A}, \mu)$ is **complete** if $\mathcal{A}$ contains all of the null sets of X. a null set need not be in $\mathcal{A}$.

Definition: A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is called an **outer measure** if

(1) $\mu^*(\emptyset) = 0$
(2) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
(3) $\mu^*\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$



outer measure.

A set $N \subseteq X$ is a **null set** for μ^* if $\mu^*(N) = 0$.

Proposition 4.2: *Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be such that $\emptyset \in \mathcal{C}$ and $\bigcup_{C \in \mathcal{C}} C = X$. Assume*

$$\ell : \mathcal{C} \to [0, \infty]$$

satisfies $\ell(\emptyset) = 0$. For $E \subseteq X$, define

$$\boxed{\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(C_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} C_i,\ C_i \in \mathcal{C} \right\}}$$

smallest differences in lengths of sets surrounding E.

Then μ^* is an outer measure.

Definition: Let μ^* be an outer measure on $\mathcal{P}(X)$. A set A is **μ^*-measurable** if

$$\boxed{\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)}$$

for all $E \subseteq X$.

Theorem 4.6: *(Caratheodory) Let $\mathcal{A}$ be the collection of μ^*-measurable sets. Then $\mathcal{A}$ is a σ-algebra and $\mu := \mu^* |_{\mathcal{A}}$ is a measure. Moreover, $(X, \mathcal{A}, \mu)$ is a complete measure space.*

Definition: Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a right continuous function and $\mathcal{C} = \{(a, b] \mid -\infty < a \leq b < \infty\}$. Define $\ell : \mathcal{C} \to [0, \infty]$ such that $\ell((a, b]) = \alpha(b) - \alpha(a)$, and define $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that

$$m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(C_i) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} C_i,\ C_i \in \mathcal{C} \right\}$$

Surround E by smallest intervals possible the sum up their lengths

Then $m_\alpha := m^* |_{\mathcal{M}_\alpha}$ is called **Lebesgue-Stieltjes measure** where $\mathcal{M}_\alpha$ is the collection of all m^*-measurable sets.

Proposition 4.7: *$\mathcal{B} \subseteq \mathcal{M}_\alpha$ for any α.* Borel σ-algebra is contained in collection of all α measurable sets.

Lemma 4.8: *Assume $[c, d] \subseteq \cup_{k=1}^{n} J_k$, where $J_k = (a_k, b_k)$. Then*

$$\alpha(d) - \alpha(c) \leq \sum_{k=1}^{n} \alpha(b_k) - \alpha(a_k)$$

Proposition 4.9: *If $I = (a, b]$, where $-\infty < a \leq b < \infty$, then*

$$m^*(I) = \alpha(b) - \alpha(a) = \ell(I)$$

Definition: If $\alpha = \text{id}_{\mathbb{R}}$, we write $m := m_\alpha$ and $\mathcal{M} := \mathcal{M}_\alpha$ and call m **Lebesgue measure**.

Lemma: *For $a \leq b$,*

$m_\alpha((a, b]) = \alpha(b) - \alpha(a)$

$m_\alpha((a, b)) = \alpha(b-) - \alpha(a)$

$m_\alpha([a, b]) = \alpha(b) - \alpha(a-)$

$m_\alpha([a, b)) = \alpha(b-) - \alpha(a-)$

normal is left open, right closed, For others Just take the limits.

Proposition 4.14: *Let $A \in \mathcal{M}$ and $\epsilon > 0$.*

There exists $G \subseteq \mathbb{R}$ open such that $A \subseteq G$ and $m(G \setminus A) < \epsilon$ ✓ can surround arbitrarily close by an open set

There exists $F \subseteq \mathbb{R}$ closed such that $F \subseteq A$ and $m(A \setminus F) < \epsilon$ ← can put a closed set arbitrarily close to same size inside

There exists $H \in G_\delta$ such that $A \subseteq H$ and $m(H \setminus A) = 0$

There exists $K \in F_\sigma$ such that $K \subseteq A$ and $m(A \setminus K) = 0$

where G_δ is the set of countable intersections of open sets and F_σ is the set of countable unions of closed sets.

In other words, if $A \in \mathcal{M}$, there exist $H, K \in \mathcal{B}$ such that $K \subseteq A \subseteq H$ and $m(H \setminus K) = 0$.

besgue le sets are $N \cup B$, N is and B Borel

Vitali set

Theorem 4.15: *For $x, y \in [0,1]$, define $x \sim y \Longleftrightarrow x - y \in \mathbb{Q}$. Then $\sim$ is an equivalence relation. Let A be a set consisting of exactly one element from each equivalence class of $\sim$. Then A is not measurable, so m^** **is not a measure on** $\mathcal{P}(\mathbb{R})$. This is the Vitali set. Best exampe of non Lebesgue measurable set!

Definition: Let $\mathcal{A}_0$ be an algebra on X. A function $\ell : \mathcal{A}_0 \to [0, \infty]$ is a **measure on $\mathcal{A}_0$** if

(1) $\ell(\emptyset) = 0$

(2) $A_i \in \mathcal{A}_0$, $\cup_{i=1}^\infty A_i \in \mathcal{A}_0$, A_i disjoint $\Rightarrow$ $\ell(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \ell(A_i)$

Theorem: *(Caratheodeory) Assume $\mathcal{A}_0$ is an algebra and ℓ is a measure on $\mathcal{A}_0$. For $E \subseteq X$, set*

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^\infty \ell(A_i) \;\middle|\; A_i \in \mathcal{A}_0, E \subseteq \bigcup_{i=1}^\infty A_i \right\}$$

Then

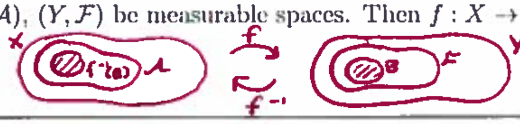
(1) μ^* is an outer measure

(2) $\mu^*(A) = \ell(A)$ for all $A \in \mathcal{A}_0$

(3) Every set in $\mathcal{A}_0$ is μ^*-measurable

(4) If ℓ is σ-finite, then ℓ has a unique extension to $\sigma(\mathcal{A}_0)$

Definition: Let $(X, \mathcal{A})$, $(Y, \mathcal{F})$ be measurable spaces. Then $f : X \to Y$ is **$(\mathcal{A}/\mathcal{F})$-measurable** if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{F}$.



Proposition: *Let $(X, \mathcal{A})$ and $(Y, \mathcal{F})$ be measurable spaces and $f : X \to Y$. Let $\mathcal{C} \subseteq \mathcal{F}$ such that $\sigma(\mathcal{C}) = \mathcal{F}$. Then f is $\mathcal{A}/\mathcal{F}$ measurable $\Longleftrightarrow f^{-1}(C) \in \mathcal{A}$, $\forall C \in \mathcal{C}$.*

Proposition 5.6: *Assume X and Y are topological spaces and $\mathcal{B}(X)$, $\mathcal{B}(Y)$ are their respective Borel sets. If $f : X \to Y$ is continuous then f is $\mathcal{B}(X)/\mathcal{B}(Y)$-measurable, i.e. f is a* **Borel function.**

continuous functions are Borel measurable

Definition: $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$. The open sets of $\overline{\mathbb{R}}$ are generated by a basis consisting of sets of the form $(a, \infty]$, $[-\infty, a)$, and (a, b), where $a, b \in \mathbb{R}$ such that $a < b$. The Borel sets of $\overline{\mathbb{R}}$ are denoted $\overline{\mathcal{B}} := \mathcal{B}(\overline{\mathbb{R}})$.

Proposition: *Let $\mathcal{C} = \{(a, \infty] \mid a \in \mathbb{R}\}$ (or $\{[a, \infty] \mid a \in \mathbb{R}\}$ or $\{[-\infty, a) \mid a \in \mathbb{R}\}$ or $\{[-\infty, a] \mid a \in \mathbb{R}\}$). Then $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$.*

Definition: Let $(X, \mathcal{A})$, $(Y, \mathcal{F})$ be measurable spaces and $f : X \to Y$ measurable.

(1) If $(Y, \mathcal{F}) = (\overline{\mathbb{R}}, \overline{\mathcal{B}})$, then f is **$\mathcal{A}$-measurable**.
(2) If further $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{M})$, then f is **(Lebesgue) measurable**.
(3) If $\mathcal{A} = \mathcal{B}(X)$, $\mathcal{F} = \mathcal{B}(Y)$, then f is **Borel-measurable** and is called a **Borel function**.

Proposition 5.5': *The following are equivalent for $f : X \to \overline{\mathbb{R}}$:*

(1) f is measurable
(2) $\{x \mid f(x) > a\} \in \mathcal{A}$, $\forall a \in \mathbb{R}$
(3) $\{x \mid f(x) \leq a\} \in \mathcal{A}$, $\forall a \in \mathbb{R}$
(4) $\{x \mid f(x) < a\} \in \mathcal{A}$, $\forall a \in \mathbb{R}$
(5) $\{x \mid f(x) \geq a\} \in \mathcal{A}$, $\forall a \in \mathbb{R}$

sets of this form go where they need to go.

Proposition 5.7: *Let $f, g : X \to \overline{\mathbb{R}}$ be measurable and $c \in \mathbb{R}$. Then $f + g$, cf, fg, $f \vee g$, $f \wedge g$ are measurable.*

Proposition 5.8: *If $f_n : X \to \overline{\mathbb{R}}$ are measurable, then*

$$\sup_n f_n, \ \inf_n f_n, \ \overline{\lim_n} \ f_n, \ \underline{\lim_n} \ f_n$$

are measurable, where

$$(\sup_n f_n)(x) := \sup_n f_n(x), \quad (\inf_n f_n)(x) := \inf_n f_n(x)$$

$$\overline{\lim_n} f_n := \inf_n \sup_{m \geq n} f_m, \qquad \underline{\lim_n} f_n := \sup_n \inf_{m \geq n} f_m$$

Proposition 5.10: *If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is a Borel function.*

Monotone functions are Borel functions

Definition: Let $(X, \mathcal{A})$ be a measurable space. If $E \in \mathcal{A}$, the function χ_E defined by

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

is called the **characteristic function of** E. A **simple function** s is a function of the form

$$s = \sum_{i=1}^{n} a_i \chi_{E_i}$$

finite

where $a_i \in \mathbb{R}$ and $E_i \in \mathcal{A}$.

Proposition 5.14: *If $f : X \to \overline{\mathbb{R}}$ is measurable and $f \geq 0$, then there exists $s_n \geq 0$ simple such that $s_n \nearrow f$ pointwise for all x.*

f positive and measurable ⇒ f can be approximated by simple functions.

Theorem 5.15: *(Lusin's Theorem) Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable and fix $\epsilon > 0$. Then there exists a closed set $F \subseteq [0,1]$ such that $m([0,1] \setminus F) < \epsilon$ and $f|_F$ is continuous.*

[Alternate version:] There exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $m(\{f \neq g\}) < \epsilon$. Further, $||g||_{\sup} \leq ||f||_{\sup}$.

measurable functions can be approximated by continuous ones.

Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and let s be a simple function

$$s = \sum_{i=1}^{n} a_i \chi_{E_i} \qquad (E_i \in \mathcal{A},\ a_i \in \mathbb{R})$$

Then s takes finitely many values, say $b_1, ..., b_m$. Let

$$B_j = s^{-1}(\{b_j\}) \qquad (\in \mathcal{A})$$

Then

$$s = \sum_{j=1}^{m} b_j \chi_{B_j}$$

with b_j all distinct, B_j disjoint, and $X = \cup_{j=1}^{m} B_j$. This is called the **canonical decomposition of** s. [It is unique].

Define the **Lebesgue integral of** s to be

$$\int s \, d\mu = \sum_{j=1}^{m} b_j \mu(B_j) \qquad \text{(where } 0 \cdot \infty := 0)$$

how to integrate simple functions

Lemma 6.A *If*

$$s = \sum_{i=1}^{n} a_i \chi_{E_i} = \sum_{j=1}^{m} c_j \chi_{D_j}$$

where E_i are disjoint, D_j are disjoint, and $X = \cup_{i=1}^{n} E_i = \cup_{j=1}^{m} D_j$, then

$$\sum_{i=1}^{n} a_i \mu(E_i) = \sum_{j=1}^{m} c_j \mu(D_j)$$

Definition: If $f \geq 0$ is measurable, set

$$\int f d\mu = \sup \left\{ \int s d\mu \,\middle|\, 0 \leq s \leq f, \ s \text{ simple} \right\}$$

If f is measurable, define $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$. These are nonnegative measurable functions. If at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite, set

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

If $A \in \mathcal{A}$, then set

$$\int_A f d\mu := \int f \chi_A d\mu$$

Proposition 6.3: *If $0 \leq f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$.*

Consequently, for $A, B \in \mathcal{A}$, $A \subseteq B$, $\int_A f \, d\mu \leq \int_A g \, d\mu$ and $\int_A f \, d\mu \leq \int_B f \, d\mu$.

basic properties of integrals

If $f \geq 0$ and $0 < c < \infty$, then $\int cf \, d\mu = c \int f \, d\mu$.

If $f \geq 0$ and $\mu(A) = 0$, then $\int_A f \, d\mu = 0$ even if $f = \infty$ on A.

If $f \equiv 0$ on A, then $\int_A f \, d\mu = 0$ even if $\mu(A) = \infty$.

Definition: If $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$, then we say f is **integrable** and we write $f \in L^1(\mu)$.

Proposition 6.4: *If $f \in L^1(\mu)$, then*

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

Proposition: *If s and t are simple functions, then*

$$\int (s+t)\, d\mu = \int s\, d\mu + \int t\, d\mu$$

linearity of integrals of simple functions

Theorem 7.1: *(Monotone Convergence Theorem) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_n : X \to \overline{\mathbb{R}}$ measurable such that $0 \le f_1 \le f_2 \le \ldots$, and set*

$$f(x) := \lim_{n\to\infty} f_n(x)$$

need positive and increasing.

Then

$$\int f_n\, d\mu \nearrow \int f\, d\mu \tag{1}$$

Also, if f_n measurable, $0 \le f_1 \le f_2 \le \ldots$ almost everywhere (and $f_n \to f$), then (1) still holds.

Theorem 7.4: *(Linearity of the integral) If f, g are nonnegative and measurable, or if they are integrable, then*

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g d\mu$$

Remark: $\int |f|\, d\mu = \int f^+\, d\mu + \int f^-\, d\mu$

Proposition 7.5: *Assume $f_n \ge 0$ are measurable. Then*

$$\int \sum_{n=1}^{\infty} f_n\, d\mu = \sum_{n=1}^{\infty} \int f_n\, d\mu$$

can pull summation out if f_n positive measurable

Theorem 7.6: *(Fatou's Lemma) If $f_n \ge 0$ are measurable, then*

need positive msble

$$\int \underline{\lim}\, f_n\, d\mu \le \underline{\lim} \int f_n\, d\mu \tag{2}$$

Also, if f_n measurable, $f_n \ge 0$ almost everywhere, then (2) still holds.

Theorem 7.7': *(Dominated Convergence Theorem) Assume f_n measurable, $f_n \to f$ pointwise, and $\sup_n |f_n| \leq g$, where $g \in L^1(\mu)$. Then $f \in L^1(\mu)$ and*

$$\int |f_n - f| \, d\mu \to 0 \tag{3}$$

need { msble; $f_n \xrightarrow{a.e.} f$; bdd by fcn.

Also, if f_n are measurable, $f_n \to f$ almost everywhere and $\sup_n |f_n| \leq g$ almost everywhere, where $g \in L^1(\mu)$, then (3) still holds.

Proposition 8.1: *If $f : X \to \overline{\mathbb{R}}$ measurable and $\int_A f \, d\mu = 0$, $\forall A \in \mathcal{A}$, then $f = 0$ almost everywhere.*

either need positive a.e. or msble.

Proposition 8.2: *If $f \geq 0$ almost everywhere and $\int f \, d\mu = 0$, then $f = 0$ almost everywhere.*

Corollary 8.3: *Let $\mathbb{R} \to \overline{\mathbb{R}}$ such that $f \in L^1(m)$ and fix $a \in \mathbb{R}$. If $\forall x \in \mathbb{R}$*

$$\int_a^x f(y) \, dy = 0$$

Then $f = 0$ almost everywhere (with respect to m).

Theorem 8.4: *Assume f is Lebesgue measurable and integrable, so $f \in L^1(m)$. Given $\epsilon > 0$ there exists $g \in C_C(\mathbb{R})$ such that*

measurable integrable functions can be approximated by continuous ones w/ compact support. i.e. bdd

$$\int |f - g| \, dm < \epsilon$$

where $C_C(\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} \mid g \text{ continuous with compact support}\}$. → set of points where $f \neq 0$

Definition: Let $-\infty \leq a < b \leq \infty$. Then $\varphi : (a, b) \to \mathbb{R}$ is convex on (a, b) if

$$\varphi(\lambda\alpha + (1-\lambda)\beta) \leq \lambda\varphi(\alpha) + (1-\lambda)\varphi(\beta)$$

$\forall \alpha, \beta \in (a, b)$, $\forall \lambda \in [0, 1]$.

For $a < z < y < x < b$,

$$\frac{\varphi(y) - \varphi(z)}{y - z} \leq \frac{\varphi(x) - \varphi(z)}{x - z}, \quad \frac{\varphi(x) - \varphi(z)}{x - z} \leq \frac{\varphi(x) - \varphi(y)}{x - y}$$

Theorem 8.A: *Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$. Let $f : X \to (a, b)$ such that $f \in L^1(\mu)$ and $\varphi : (a, b) \to \mathbb{R}$ convex. Then*

nsen's inequality

$$\varphi\left(\int f \, d\mu\right) \leq \int (\varphi \circ f) \, d\mu$$

if concave then its the opposite inequality

Definition: A function f is **continuous a.e.** if $m(\{x \mid f \text{ is not continuous at } x\}) = 0$.

Theorem 9.1: *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then*

$$f \in RI \iff f \text{ is continuous a.e. } (m)$$

classification of Reimann Integrable Fcns.

In this case, $R(f) = \int f \, dm$.

Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_n, f : X \to \overline{\mathbb{R}}$ be measurable.

The sequence (f_n) converges to f **almost everywhere**, written $f_n \to f$ a.e. (μ), if

$$\mu(\{x \mid f_n \not\to f\}) = 0$$

Types of convergence.

The sequence (f_n) converges to f **in measure**, written $f_n \xrightarrow{\mu} f$, if

$$\mu(\{x \mid |f_n - f| > \epsilon\}) \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0$$

For $1 \leq p < \infty$, the sequence (f_n) converges to f **in L^p**, written $f_n \xrightarrow{L^p} f$, if

$$\int |f_n - f|^p \, d\mu \to 0$$

Proposition 10.2:

(1) If $f_n \to f$ a.e., then $f_n \xrightarrow{\mu} f$ if μ is finite.

(2) If $f_n \xrightarrow{\mu} f$, then there exists a subsequence n_j such that $f_{n_j} \to f$ a.e.

$$ae \to \mu \leftarrow Lp$$

Lemma 10.4: *If $0 < p < \infty$, then for any $a > 0$,* $\mu(|f| \geq a) \leq \dfrac{\int |f|^p \, d\mu}{a^p}$.

Chebyshev's

Proposition 10.5: *If $f_n \to f$ in L^p, then $f_n \xrightarrow{\mu} f$.*

Theorem 10.8: *Egorov's Theorem. Assume μ is finite, $\epsilon > 0$, and $f_n \to f$ a.e. Then there exists a set $B \in \mathcal{A}$ such that $\mu(B) < \epsilon$ and $f_n \to f$ uniformly on B^c.*

Definition: Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be measure spaces and set

$$\mathcal{C}_0 = \left\{ \bigcup_{i=1}^{n} (A_i \times B_i) \;\middle|\; A_i \in \mathcal{A},\ B_i \in \mathcal{B},\ \ i \neq j \Rightarrow (A_i \times B_i) \cap (A_j \times B_j) = \emptyset,\ \ n \geq 1 \right\}$$

Then $\mathcal{C}_0$ is an algebra, and define $\mathcal{A} \times \mathcal{B} := \sigma(\mathcal{C}_0)$ the **product σ-algebra**.

If $E \in \mathcal{A} \times \mathcal{B}$, define

$$s_x(E) = \{y \in Y \mid (x, y) \in E\}, \qquad t_y(E) = \{x \in X \mid (x, y) \in E\}$$

the **x-section** of E and **y-section** of E, respectively.

If $f : X \times Y \to \overline{\mathbb{R}}$ is $\mathcal{A} \times \mathcal{B}$-measurable, set

$$S_x f(y) = f(x, y), \qquad T_y f(x) = f(x, y)$$

Lemma 11.A:

(i) $s_x(E^c) = (s_x(E))^c$ *for all* $E \subseteq X \times Y$

(ii) $s_x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} s_x(E_i)$ *for all* $E \subseteq X \times Y$

(iii) $s_x\left(\bigcap_{i=1}^{\infty} E_i\right) = \bigcap_{i=1}^{\infty} s_x(E_i)$ *for all* $E \subseteq X \times Y$

Lemma 11.1: *For all $x \in X$, $y \in Y$,*

(i) *If $E \in \mathcal{A} \times \mathcal{B}$, then $s_x(E) \in \mathcal{B}$ and $t_y(E) \in \mathcal{A}$.*

(ii) *If f is $\mathcal{A} \times \mathcal{B}$-measurable, then $S_x f$ is $\mathcal{B}$-measurable and $T_y f$ is $\mathcal{A}$-measurable.*

Definition: For $E \in \mathcal{A} \times \mathcal{B}$, define $h : X \to \overline{\mathbb{R}}$, $k : Y \to \overline{\mathbb{R}}$ by

$$h(x) = \nu(s_x(E)), \qquad k(y) = \mu(t_y(E))$$

Proposition 11.2: *Assume μ and ν are σ-finite. Then for all $E \in \mathcal{A} \times \mathcal{B}$,*

(i) *h is $\mathcal{A}$-measurable and k is $\mathcal{B}$-measurable.*

(ii) $\displaystyle\int h(x)\ \mu(dx) = \int k(y)\ \nu(dy)$

Remark:

$$h(x) = \nu(s_x(E)) = \int \chi_{s_x(E)}(y)\ \nu(dy) = \int S_x\chi_E(y)\ \nu(dy)$$

so (ii) can be rewritten as

$$\int_X \int_Y S_x\chi_E(y)\ \nu(dy)\ \mu(dx) = \int_Y \int_X T_y\chi_E(x)\ \mu(dx)\ \nu(dy)$$

usually written as

$$\int_X \int_Y \chi_E(x,y)\ \nu(dy)\ \mu(dx) = \int_Y \int_X \chi_E(x,y)\ \mu(dx)\ \nu(dy)$$

Definition: For $E \in \mathcal{A} \times \mathcal{B}$, set

$$\begin{aligned}(\mu \times \nu)(E) &= \int h(x)\ \mu(dx) = \int \nu(s_x(E))\ \mu(dx) \\ &= \int k(y)\ \nu(dy) = \int \mu(t_y(E))\ \nu(dy)\end{aligned}$$

Then $\mu \times \nu$ is called **product measure.**

Theorem 11.3: *(Fubini's Theorem) Assume μ, ν are σ-finite. Let $f : X \times Y \to \overline{\mathbb{R}}$ be $\mathcal{A} \times \mathcal{B}$-measurable such that*

(a) $f \geq 0$, or

(b) $\displaystyle\int_{X\times Y} |f|\ d(\mu \times \nu) < \infty$ (i.e. $f \in L^1(\mu \times \nu)$)

positive or in L^1!

Then

(1) $S_x f$ is $\mathcal{B}$-measurable, $\forall x$.

(2) $T_y f$ is $\mathcal{A}$-measurable, $\forall y$.

(3) $\displaystyle g(x) = \int S_x f\ d\nu = \int f(x,y)\ \nu(dy)$ is $\mathcal{A}$-measurable.

(4) $\displaystyle h(y) = \int T_y f\ d\mu = \int f(x,y)\ \mu(dx)$ is $\mathcal{B}$-measurable.

(5) $\displaystyle\int_{X\times Y} f\ d(\mu \times \nu) = \int_X \int_Y f(x,y)\ \nu(dy)\ \mu(dx) = \int_Y \int_X f(x,y)\ \mu(dx)\ \nu(dy)$

Proposition: $\mathcal{B}^2 = \mathcal{B}^1 \times \mathcal{B}^1$

Theorem 11.B: *Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous, with LS measures $d\alpha, d\beta$. Then*

$$\int_{(a,b]} \alpha(x)\beta(dx) = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_{(a,b]} \beta(x-)\alpha(dy)$$

integration by parts

Definition: Consider $(\mathbb{R}^2, \overline{\mathcal{M} \times \mathcal{M}}, \overline{m \times m})$. Then $\mathcal{M}_2 = \overline{\mathcal{M} \times \mathcal{M}}$ are the 2-dimensional Lebesgue measurable sets and $m_2 = \overline{m \times m}$ is 2-dimensional Lebsegue measure.

Theorem 11.C: *Assume $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are complete, σ-finite measure spaces and $f : X \times Y \to \mathbb{R}$ is $\overline{\mathcal{A} \times \mathcal{B}}$-measurable such that*

(a) $f \geq 0$ a.e. $(\overline{\mu \times \nu})$

or

(b) $\int_{X \times Y} |f| \; d(\overline{\mu \times \nu}) < \infty$

Then

(1) $S_x f$ is $\mathcal{B}$-measurable, for a.e. x (μ).

(2) $T_y f$ is $\mathcal{A}$-measurable, for a.e. y (ν).

(3) $g(x) = \int S_x f \; d\nu = \int f(x,y) \; \nu(dy)$ is $\mathcal{A}$-measurable (set $= 0$ when not defined).

(4) $h(y) = \int T_y f \; d\mu = \int f(x,y) \; \mu(dx)$ is $\mathcal{B}$-measurable (set $= 0$ when not defined).

(5) $\int_{X \times Y} f \; d(\overline{\mu \times \nu}) = \int_X \int_Y f(x,y) \; \nu(dy) \; \mu(dx) = \int_Y \int_X f(x,y) \; \mu(dx) \; \nu(dy)$

Theorem 11.D: *Let $f : \mathbb{R} \to [0, \infty]$ be Lebesgue measurable and let $E = \{(x,y) \mid 0 < y < f(x)\}$. Then*

(a) $E \in \mathcal{M}_2$ $\;(\in \mathcal{M} \times \mathcal{M})$

(b) $\int f \; dm = m_2(E)$

Definition: Let $(X, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \to (-\infty, \infty]$ is a **signed measure** if

can be positive or negative

i) $\mu(\emptyset) = 0$

ii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, $A_i \in \mathcal{A}$ disjoint

Definition: Let μ be a signed measure on $(X, \mathcal{A})$. A set $A \in \mathcal{A}$ is a **positive set** if $\mu(B) \geq 0$ for all $B \subseteq A$ such that $B \in \mathcal{A}$.

Similarly, $A \in \mathcal{A}$ is a **negative set** if $\mu(B) \leq 0$ for all $B \subseteq A$ such that $B \in \mathcal{A}$. A set $A \in \mathcal{A}$ is a **null set** if $\mu(B) = 0$ for all $B \subseteq A$ such that $B \in \mathcal{A}$.

Proposition 12.4: Let μ be a signed measure on $(X, \mathcal{A})$. If $E \in \mathcal{A}$ and $\mu(E) < 0$, then there exists a negative set $F \subseteq E$, $F \in \mathcal{A}$ with $\mu(F) < 0$.

Theorem 12.5: (Hahn decomposition) Let $(X, \mathcal{A}, \mu)$ be a signed measure space.

1) There exist $E, F \in \mathcal{A}$ disjoint with $E \cup F = X$, E a negative set and F a positive set.

2) If $E', F' \in \mathcal{A}$ is another such decomposition, then $E \triangle E' = F \triangle F'$ is a null set.

3) If μ is not a positive measure, then $\mu(E) < 0$; and if $-\mu$ is not a positive measure, then $\mu(F) > 0$.

Definition: Two measures μ and ν on $(X, \mathcal{A})$ are **(mutually) singular**, written $\mu \perp \nu$, if there exists a set $E \in \mathcal{A}$ with $\mu(E) = \nu(E^c) = 0$.

Theorem 12.8: (Jordan decomposition) Assume μ is a signed measure on $(X, \mathcal{A})$. Then there exist μ^+, μ^- positive measures, $\mu^+ \perp \mu^-$, and $\mu = \mu^+ - \mu^-$. Further, this decomposition is unique.

Definition: $|\mu| = \mu^+ + \mu^-$ is called the **total variation measure** of μ.

Definition: If μ, ν are measures of $(X, \mathcal{A})$, then ν is **absolutely continuous** with respect to μ, written $\nu \ll \mu$, if $\mu(A) = 0 \Rightarrow \nu(A) = 0$, $\forall A \in \mathcal{A}$.

Example: Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \geq 0$ measurable, and define $\nu : \mathcal{A} \to [0, \infty]$ by $\nu(A) = \int_A f \, d\mu$. Then ν is a measure and $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Proposition 13.2: Assume ν is finite. Then $\nu \ll \mu$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{A}$, $|\mu(A)| < \delta \Rightarrow \nu(A) < \epsilon$. alternate form of $\nu \ll \mu$

Lemma 13.3: Let μ and ν be finite measures on $(X, \mathcal{A})$. Then either

1) $\mu \perp \nu$

or

2) there exists $\epsilon > 0$ and $G \in \mathcal{A}$ with $\mu(G) > 0$ and G a positive set for $\nu - \epsilon\mu$.

Theorem 13.4: Radon-Nikodym Let μ, ν be σ-finite measures on $(X, \mathcal{A})$ such that $\nu \ll \mu$. Then there exists a measurable function f with

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$. The function is unique up to a set of μ-measure 0.

The function f is called the **Radon-Nikodym derivative** of ν with respect to μ, written $f = \frac{d\nu}{d\mu}$. It satisfies $f \geq 0$ a.e. (μ). And if ν is finite, then $f \in L^1(\mu)$.

Definition: Let ν be a signed measure and μ a measure. Then $\nu \ll \nu$ if $|\nu| \ll \mu$ and $\nu \perp \mu$ if $|\nu| \perp \mu$.

Lemma 13.A: Let ν be a signed measure and μ a measure.

1) $\nu \ll \mu \iff [\mu(A) = 0 \Rightarrow \nu(A) = 0, \ \forall A \in \mathcal{A}]$
2) $\nu \perp \mu \iff \exists A$ a null set for ν with $\mu(A^c) = 0$.

Theorem 13.4': Let μ be a σ-finite measure and ν a σ-finite signed measure with $\nu \ll \mu$. Then there exists a unique function f such that

$$\nu(A) = \int_A f \, d\mu \qquad \forall A \in \mathcal{A}$$

If ν is finite, then $f \in L^1(\mu)$.

Lemma 13.B: Let ν be a signed measure and μ a measure. If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu \equiv 0$.

absolutely continuous + mutually singular = 0.

Theorem 13.5: Lebesgue decomposition Let μ be a σ-finite measure and let ν a finite measure. Then there exists a unique pair of finite measures ν_s and ν_a with $\nu_a \ll \mu$, $\nu_s \perp \mu$ such that $\nu = \nu_a + \nu_s$.

Proposition 14.1: Assume $E \subseteq \mathbb{R}^n$ and $E = \bigcup_{\alpha \in I} B_\alpha$, where B_α are open balls of bounded diameter. Then there exists a sequence $B_1, B_2, \ldots$ of disjoint elements of $\{B_\alpha \mid \alpha \in I\}$ such that

$$m(E) \leq 5^n \sum_{i=1}^{\infty} m(B_k)$$

Definition: A function $f \in L^1_{\text{loc}}(m)$ if for any $x_0 \in \mathbb{R}^n$ there exists $r > 0$ such that $\int_{B_r(x_0)} |f|\, dm < \infty$. In other words, f is **locally integrable**. This is equivalent to $\int_K |f|\, dm < \infty$ for all compact $K \subset \mathbb{R}^n$.

For $f \in L^1_{\text{loc}}(m)$, set

$$Mf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|\, dm$$

Then Mf is called the **Hardy-Littlewood** maximal function.

Properties of Mf:

(1) $0 \leq Mf \leq \infty$

(2) $M(f+g) \leq Mf + Mg$

(3) $M(cf) = |c| Mf$

(4) Mf is measurable

Theorem 14.2: If $f \in L^1(m)$, then for every $\beta > 0$,

$$m(Mf > \beta) \leq \frac{5^n}{\beta} \int |f|\, dm$$

Theorem 14.3: Let

$$f_r(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f\, dm$$

If $f \in L^1_{\text{loc}}(m)$, then $f_r \to f$ a.e. as $r \downarrow 0$.

Theorem 14.4: If $f \in L^1_{\text{loc}}$, then

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy \to 0 \text{ a.e.}$$

Definition: If $\dfrac{m(E \cap B_r(x))}{m(B_r(x))} \to 1$, then x is called a **point of density** of E.

Theorem 14.5: Let $f : \mathbb{R} \to \mathbb{R}$ and $f \in L^1_{\text{loc}}$. Fix $a \in \mathbb{R}$ and set

$$F(x) = \int_a^x f(t)dt$$

Then F is differentiable and $F' = f$ a.e.

Lemma 14.6: Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a right continuous, increasing function and ν the associated Lebesgue-Stieltjes measure. If $\nu \perp m$, then

$$\lim_{r \downarrow 0} \frac{\nu(B_r(x))}{m(B_r(x))} = 0$$

a.e. x (m).

Proposition 14.7': Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous with L-S measure ν. Then F' exists a.e., $F' \in L^1_{\text{loc}}(m)$, and $F' = \frac{d\nu_a}{dm}$ a.e. m. As a consequence,

$$\int_a^b F' \, dm \leq F(b) - F(a)$$

Theorem 14.8: Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F' exists a.e., $F' \in L^1_{\text{loc}}$, and for $a < b$,

$$\int_a^b F'(x)dx \leq F(b-) - F(a+) \leq F(b) - F(a)$$

Lebesgue Differentiation Thm:

$\frac{1}{|Q|} \int_Q |f| \to f(x)$ as $Q \to \{x\}$

In particular $\lim_{I \to \{x\}} \frac{|E \cap I|}{|I|} = \chi_E(x)$

$(Q = I \quad \& \quad f = \chi_E)$

$|f+g|^p \leq 2^p (|f|^p + |g|^p)$

Definition: Let $f : [a,b] \to \mathbb{R}$ and for $a \leq y \leq b$, set

$$Vf(y) = \sup\left\{ \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \;\middle|\; a = x_0 < \cdots < x_k = y, k \geq 2 \right\}$$

Then $Vf(y)$ is the **total variation** of f over $[a,y]$. Say $f \in BV$ if $Vf(b) < \infty$.

Set

$$P(y) = \sup\left\{ \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+ \;\middle|\; a = x_0 < \cdots < x_k = y, k \geq 2 \right\}$$

$$N(y) = \sup\left\{ \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^- \;\middle|\; a = x_0 < \cdots < x_k = y, k \geq 2 \right\}$$

The functions P and N are called the **positive** and **negative variations** of f over $[a,y]$.

Note: $Vf(a) = P(a) = N(a) = 0$.

Lemma 14.10: If $f \in BV$, then $f(y) - f(a) = P(y) - N(y)$ and $Vf(y) = P(y) + N(y)$.

Definition: A function $f : [a,b] \to \mathbb{R}$ is **absolutely continuous**, written $f \in AC$, if $\forall \epsilon > 0 \exists \delta > 0$ such that

$$\sum_{i=1}^{k} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{k} |f(b_i) - f(a_i)| < \epsilon$$

whenever $(a_i, b_i) \subset [a,b]$ are disjoint.

Lemma 14.12: $f \in AC \Rightarrow f \in BV$.

Lemma 14.13: Let $f \in AC$, so in particular $f \in BV$. Then $Vf \in AC$ and hence so are P and N.

Theorem 14.14: If $F \in AC$, then F' exists a.e., $F' \in L^1([a,b])$, and

$$\int_a^x F'(t)dt = F(x) - F(a) \qquad (*)$$

for all $a \leq x \leq b$. Conversely, if F' exists, $F' \in L^1([a,b])$, and $(*)$ holds, then $F \in AC$.

undamental Thm of Calculus for besgue Integrals

To check AC check
1. derivative exists
2. derivative is integrable
3. $F(x) - F(a) = \int_a^x F'(t)dt$.

Assume $f \in BV$ on $[a,b]$ and f is right continuous. Write $f(x) - f(a) = P(x) - N(x)$. Let $\mu_P \sim P$, $\mu_N \sim N$. Then $Vf(x) = P(x) + N(x)$. Let $\nu \sim Vf$, so $\nu = \mu_P + \mu_N$. Let $\mu = \mu_P - \mu_N$.

Theorem 14.C: $\nu = |\mu|$, or equivalently, $\mu_P = \mu^+$, $\mu_N = \mu^-$.

Definition: A function $F : [a,b] \to \mathbb{R}$ is **singular** if $F' = 0$ a.e.

Theorem 14.D: Let $F : [a,b] \to \mathbb{R}$ be right continuous and $F \in BV$. Then

$$F(x) - F(a) = F_a(x) - F_s(x)$$

where $F_a \in AC$ and F_s is singular. This decomposition is unique if $F_a(a) = F_s(a) = 0$.

Note: If $\overline{\alpha}(x) = \int_c^x a(t)dt$, $\overline{\beta}(x) = \int_c^x b(t)dt$ for $a, b \geq 0$, $a, b \in L^1([c,d])$, then

$$\int_c^d \overline{\alpha}(x)b(x)dx = \overline{\alpha}(d)\overline{\beta}(d) - \int_c^d \overline{\beta}(x)a(x)dx$$

If $\alpha, \beta \in AC$, write $\alpha(x) - \alpha(c) = \int_c^x \alpha'(t)dt$. Then

$$\int_c^d \alpha(x)\beta'(x)dx = \alpha(d)\beta(d) - \alpha(c)\beta(c) - \int_c^d \beta(x)\alpha'(x)dx$$

Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$. For $1 \leq p < \infty$, define

$$||f||_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$$

For $p = \infty$, set

$$||f||_\infty = \inf\{M \mid \mu(|f| > M) = 0\} = \text{ess sup } |f|$$

For $1 \leq p \leq \infty$, set

$$L^p = \{f \mid ||f||_p < \infty\}$$

For $1 < p < \infty$, let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then q is called the **conjugate exponent** of p. If $p = 1$, let $q = \infty$; if $p = \infty$, let $q = 1$.

Proposition 15.1: (Holder's Inequality) If p, q are conjugate exponents ($1 \le p, q \le \infty$), then

$$\int |fg| \, d\mu \le ||f||_p ||g||_q$$

Lemma 15.2: If $a, b \ge 0$ and $1 \le p < \infty$, then $(a+b)^p \le 2^{p-1}(a^p + b^p)$

Proposition 15.3: (Minkowski's Inequality) If $1 \le p \le \infty$, then $||f+g||_p \le ||f||_p + ||g||_p$

Define an equivalence relation $\sim$ on L^p such that $f \sim g \Longleftrightarrow ||f-g||_p = 0$, i.e. $f = g$ a.e. Let $\mathbb{L}^p = L^p/\sim$. For $F \in \mathbb{L}^p$, define $|||F|||_p = ||f||_p$ for every $f \in F$. Then $||| \cdot |||_p$ is a norm on $\mathbb{L}^p$.

Theorem 15.4: L^p, $1 \le p \le \infty$, is complete.

Proposition 15.5': If $1 \le p < \infty$, then $C_C(\mathbb{R})$ is dense in $L^p(m)$.

Definition: Let $1 \le p \le \infty$. A function $H : L^p \to \mathbb{R}$ is a **linear functional** if

(i) $H(f+g) = H(f) + H(g)$

(ii) $H(cf) = cH(f)$, $c \in \mathbb{R}$

If in addition $||H|| := \sup\{|Hf| \mid ||f||_p < 1\} < \infty$, then H is called a **bounded linear functional.**

The set

$$(L^p)^* = \{H \mid H \text{ is a bounded linear functional on } L^p\}$$

is called the **dual space** of L^p. It is a vector space with addition and scalar multiplication of functions and $|| \cdot ||$ is a norm.

Observation: If $f \in L^p$ and $||f||_p \neq 0$, then $|Hf| \le ||H|| \, ||f||_p$.

Lemma 15.A: If H is a linear functional on L^p, the following are equivalent:

(1) H is bounded

(2) H is continuous

(3) H is continuous at one point

Theorem 15.8: For $1 \leq p \leq \infty$,

$$||f||_p = \sup\left\{\int fg\ d\mu \,\middle|\, ||g||_q \leq 1\right\}$$

Proposition 15.10: $(1 \leq p \leq \infty)$ Let $g \in L^q$ and define $H \in (L^p)^*$ by $Hf = \int fg\ d\mu$. Then

$$||H|| = ||g||_q$$

Proposition 15.B: Simple functions are dense in L^p, $1 \leq p \leq \infty$. In fact, for any $f \in L^p$, $\forall \epsilon > 0$ there exists r simple such that $|r| \leq |f|$ and $||f - r||_p < \epsilon$.

Corollary 15.9': If $1 \leq p \leq \infty$, then

$$||f||_p = \sup\left\{\int fs\ d\mu \,\middle|\, ||s||_q \leq 1,\ s \text{ simple}\right\}$$

Theorem 15.11': Let $1 \leq p < \infty$ and fix $H \in (L^p)^*$. Then there exists $g \in L^q$ such that $H(f) = \int fg\ d\mu$. Then by Proposition 15.10, $||H|| = ||g||_q$.

Definition: Let U, V be normed linear spaces, $\Lambda : U \to V$ a linear transformation. Then Λ is an **isometric isomorphism** of U onto V if Λ is bijective and norm-preserving, i.e. $||\Lambda u|| = ||u||$.

Thus for $1 \leq p < \infty$, $\Lambda : L^q \to (L^p)^*$ by $g \mapsto \Lambda_g$, where $\Lambda_g(f) = \int fg\ d\mu$, is an isometric isomorphism.

Proposition: Given $(X, \mathcal{A}, \mu)$ a σ-finite measure space. Let $M = \{\nu \mid \nu \text{ is a finite signed measure and } \nu \ll \mu\}$. Then M is a vector space with norm the total variation norm $||\nu|| = |\mu|(X)$.

In addition, $L^1(\mu) \cong M$ (an isometric isomorphism).

JANUARY 2013 QUALIFYING EXAM IN REAL ANALYSIS

Notation: m stands for the Lebesgue measure on the real line. The spaces $L^p([0,1])$ are understood with respect to m.

1. Suppose that $f\colon [-1,1] \to \mathbb{R}$ is a function of bounded variation. Prove that the function $g(x) = f(\sin x)$ belongs to $BV([a,b])$ for all $-\infty < a < b < \infty$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space such that for every set $A \in \mathcal{M}$ the measure $\mu(A)$ is a nonnegative integer. Suppose that $\{f_n\}_{n\geq 1}$ are measurable real-valued functions on X such that $\int_X |f_n|\, d\mu \to 0$ as $n \to \infty$. Prove that $f_n \to 0$ a.e.

3. Suppose that $f \in L^2([0,1])$. Prove that the function $g(x) = |f(x)|^{x+1}$ is in $L^1([0,1])$.

4. Suppose that $\{f_n\}$ is a sequence of nonnegative Borel measurable functions on $[0,1]$ such that $\int_0^1 f_n(x)\, dm(x) = 1$ for all n.

Which of the statements (a)–(d) follow from the above? Prove or give a counterexample to each.

(a) The set $A = \{x\colon f_n(x) \leq 2 \text{ for all } n\}$ is Borel

(b) The set $B = \{x\colon f_n(x) \leq 2 \text{ for infinitely many values of } n\}$ is Borel

(c) $A \neq \varnothing$

(d) $B \neq \varnothing$

J13

1. Suppose $f:[-1,1]\to\mathbb{R}$ is a function of Bounded Variation. Prove $g(x)=f(\sin(x))\in BV([a,b])$ $\forall\ -\infty<a<b<\infty$.

Pf $f\in BV \Rightarrow f\in BV([a,b])$

$$\Rightarrow \sup\{\textstyle\sum_1^k |f(x_i)-f(x_{i-1})| : -1=x_0<x_1<\cdots<x_n=1\}<\infty$$

We know $\sin x\in BV([a,b])$ $\forall -\infty<a<b<\infty$

$h=\sin x \Rightarrow h'(x)<1$ by mean value thm.

We can partition $[a,b]$ into a finite # of sub intervals $I_0, I_1, \ldots, I_n$ s.t. $\sin(x)$ is monotone on each I_i.

$V(g)_{\text{on } I_i}=\sup_{\text{part of } I_i}\{\sum_1^k |f(\sin(x_i))-f(\sin(x_{i-1}))|\}$

$< \sup\{\sum|f(x_i)-f(x_{i-1})|\}$ by the monotonicity

$<\infty$ since $f\in BV$.

$V(g)=\sum_{I_i} V(g)$
$<\infty$ since there is a finite # of I_i's.

$\square$

$f\in BV \Rightarrow \sup\{\sum_1^k |f(x_i)-f(x_{i-1})| : x_0<x_1<\cdots<x_n\}<\infty$

- bdd derivative $\Rightarrow f\in BV$
- making partition smaller makes sum smaller

7. Let $(X, \mathcal{M}, \mu)$ measure space $\forall A \in \mathcal{M}$ $\mu(A) \in \mathbb{N}$
Suppose $\{f_n\}_{n\geq 1}$ msble real valued fcns on X
s.t. $\int_X |f_n| d\mu \to 0$ as $n \to \infty$. Prove $f_n \to 0$ a.e.

Pf Assume not. and let $E_n = \{x \mid |f_n - 0| > c\} = \{x \mid |f_n| > c\}$
$\Rightarrow m(E_n) > 0 \quad \forall n$
E_n is a msble set since f_n are msble
So $\mu(E_n) \in \mathbb{N} \Rightarrow m(E_n) \geq 1$

$$\begin{aligned}
\int_X |f_n| d\mu &= \int_{X \setminus E_n} |f_n| + \int_{E_n} |f_n| \\
&= \int |f_n| \chi_{X \setminus E_n} + \int |f_n| \chi_{E_n} \\
&\geq 0 + c\,\mu(E_n) \\
&> 0 + c \cdot 1 \\
&> c \quad \forall n
\end{aligned}$$

which contradicts since $\int_X |f_n| d\mu \to 0$.

□

f msble $\Rightarrow \{f > a\}$ msble
$E_n = \{x \mid |f_n| > c\}$

3 Suppose $f \in L^2[0,1]$ Prove $g(x) = |f(x)|^{x+1} \in L^1([0,1])$

Pf $f \in L^2[0,1] \Rightarrow \left(\int_0^1 |f|^2\right)^{1/2} < \infty$ or $\|f\|_2 < \infty$

Let $A = \{x \in [0,1] \mid |f(x)| > 1\}$
$B = \{x \in [0,1] \mid |f(x)| < 1\}$

$$\begin{aligned}\int g &= \int_A g + \int_B g \\ &= \int_A |f(x)|^{x+1} + \int_B |f(x)|^{x+1} \\ &\leq \int_A |f(x)|^2 + \int_B |f(x)| \quad \text{since } x+1 \leq 2 \text{ on } A \text{ and } x+1 > 1 \text{ on } B \\ &\leq \|f\|_2^2 + \|f\|_2 \left(\int_B 1^2\right)^{1/2} \quad \text{by Hölders} \\ &\leq \|f\|_2^2 + \|f\|_2 \\ &< \infty \quad \text{since } \|f\|_2 < \infty\end{aligned}$$

$\therefore g \in L^1$

Hölders: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \int |fg| \leq \|f\|_p \|g\|_q$ $\square$

consider f pos/neg differently

4) $\{f_n\} \geq 0$ Borel measurable. s.t. $\int_0^1 f_n(x)\,dm(x) = 1 \quad \forall n$.
Prove or counterexample
(a) $A = \{x : f_n(x) \leq 2 \quad \forall n\}$ is Borel
(b) $B = \{x : f_n(x) \leq 2$ for a.e $n\}$ is Borel
(c) $A \neq \emptyset$
(d) $B \neq \emptyset$.

Pf (a). $A = \{x : f_n(x) \leq 2 \quad \forall n\}$
$= \bigcap_1^\infty f_n^{-1}[0,2]$
which is Borel since $f_n^{-1}[0,2]$ is
and intersects are

(b) $B = \{x ; f_n(x) \leq 2$ a.e $n\}$
$= \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty f_n^{-1}[0,2]$
is still Borel.

(c). Consider $f_1 = 3\chi_{[0,1/3)}$ $f_2 = 3\chi_{[1/3,2/3)}$ $f_3 = 3\chi_{[2/3,1]}$
let $f_n = \begin{cases} f_1 & n = 1 \bmod 3 \\ f_2 & n = 2 \bmod 3 \\ f_3 & n = 0 \bmod 3 \end{cases}$
then $A = \emptyset$ for this set.

(d) Assume BWOC $B = \emptyset$
then $\exists N$ s.t. $\forall n > N$ $f_n(x) \geq 2 \quad \forall x$.
$\Rightarrow \int_0^1 f_N(x) \geq \int_0^1 2\,dx = 2$
which contradicts since $\int f_n(x) = 1$

countable unions
and intersections
preserve measurability

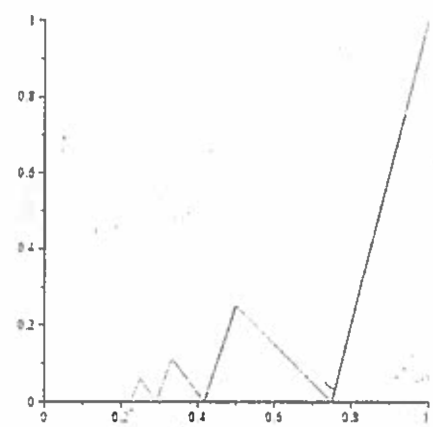
QUALIFYING EXAM, Measure Theory, August 2012

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let m be Lebesgue measure on the Borel sets of $\mathbb{R}$. For the following statement, prove OR provide a counterexample (with the details showing it is indeed a counterexample): For all Borel sets $E \subset \mathbb{R}$, if $m(E) = 0$ then $m(f(E)) = 0$.

Problem 2. A sequence of (Lebesgue) measurable functions f_n on $\mathbb{R}$ is said to converge *almost uniformly* to the measurable function f on $\mathbb{R}$ if and only if for each $\epsilon > 0$ there is a measurable set $E \subset \mathbb{R}$ such that $m(E) < \epsilon$ and $f_n \to f$ uniformly on $\mathbb{R} \setminus E$.

Give an example of $f_n \to f$ pointwise almost everywhere but NOT $f_n \to f$ almost uniformly. Show that your example works.

Problem 3. On $[0,1] \subset \mathbb{R}$ set $g(x) = \sqrt{x}$. Define f on $[0,1]$ by $f(\frac{1}{n}) = \frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$, $f\left(\frac{\frac{1}{n} + \frac{1}{n+1}}{2}\right) = 0$ for $n = 1, 2, 3, \ldots$, and otherwise f is linear. See the figure where the first few linear pieces of f are graphed.



(i) is g absolutely continuous? Why or why not.
(ii) is f absolutely continuous? Why or why not.
(iii) is $g \circ f$ absolutely continuous? Why or why not.

Problem 4. (i) For a space X with measure μ and $\mu(X) < \infty$, prove that $L^q \subset L^p$ for $0 < p < q < \infty$. (ii) Suppose that X contains disjoint sets E_k for $k = 1, 2, \ldots$ with $0 < \mu(E_k) < 2^{-k}$. Show that L^p is not contained in L^q.

A12

1. Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^2$. Let m be Lebesgue measure on $\mathcal{B}$. Prove or Counter
$\forall$ Borel sets $E\subset\mathbb{R}$ if $m(E)=0$ then $m(f(E))=0$.

Pf • First WTS f absolutely continuous gives us
$m(E)=0 \Rightarrow m(f(E))=0$.

Assume f absolutely continuous. Let $\varepsilon>0$.
$\Rightarrow \exists\delta>0$ s.t. $\sum|b_j-a_j|<\delta \Rightarrow \sum|f(b_j)-f(a_j)|<\varepsilon$
$\Rightarrow \exists U$ s.t. $U=\bigcup_1^\infty(a_j,b_j)$ and $m(U)=\sum_1^\infty|b_j-a_j|<\delta$ $E\subset U$
$\Rightarrow$ On each $[a_j,b_j]$ f attains max & min at x_j, y_j respecti
$\Rightarrow \sum|x_j-y_j|<\sum|b_j-a_j|<\delta$
$f(E)\subset\bigcup_1^\infty(f(x_j)-f(y_j))$ since $E\subset U$
$\Rightarrow m(f(E))\le\sum_1^\infty|f(x_j)-f(y_j)|$
$\le\lim\sum_1^N|f(x_j)-f(y_j)|$
$\le\lim\varepsilon$
$=\varepsilon$ $\forall N$ since f is abs continuous

• Now $f=x^2$ is absolutely continuous on any Compact set even though it is not even uniformly continuous on $\mathbb{R}$.
$\Rightarrow \forall$ Compact set containing E $m(E)=0 \Rightarrow m(f(E))=0$
$\Rightarrow$ we can do this infinitely many times to cover E if necessary.
$\Rightarrow m(E)=0 \Rightarrow m(f(E))=0$

$\square$

AC: $\forall\varepsilon>0$ $\exists\delta>0$ s.t. $\sum|b_j-a_j|<\delta$ $\Rightarrow \sum|f(b_j)-f(a_j)|<\varepsilon$

AC $\Rightarrow |E|=0 \Rightarrow |f(E)|=0$
use FTC to prove $x^2\in AC$

2. A sequence of measurable functions f_n on $\mathbb{R}$ converge almost uniformly to msble $f \iff \forall \varepsilon > 0$ $\exists$ msble $E \subset \mathbb{R}$ s.t. $m(E) < \varepsilon$ and $f_n \to f$ uniformly on $\mathbb{R} \setminus E$.
Give an example of $f_n \to f$ pointwise a.e. but not $f_n \to f$ almost uniformly.

Pf Consider $f_n = \chi_{[n,n+1]}$.

$f_n \to 0$ a.e. clearly.

WTS $f_n \not\to 0$ almost uniformly.
Let $\varepsilon = 1/2$.
Let $E \subset \mathbb{R}$ s.t. $m(E) < 1/2$
WTS $\chi_{[n,n+1]} \overset{u}{\not\to} 0$ on $\mathbb{R}/E$.

There will always be a place and N s.t. $|f_N(x) - f(x)| = 1$ so $\chi_{[n,n+1]} \overset{u}{\not\to} 0$.

This is also example for ptwise but not uniform convergence.

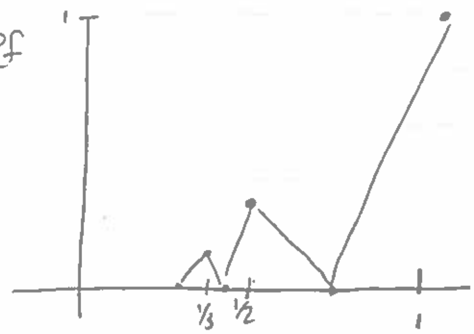
3. On $[0,1] \subset \mathbb{R}$, $g(x) = \sqrt{x}$. Define f on $[0,1]$ by $f(1/n) = 1/n^2$ $f\left(\frac{1/n + 1/n+1}{2}\right) = 0$ otherwise f linear.

(i) is g absolutely continuous?
(ii) is f absolutely continuous?
(iii) is $g \circ f$ absolutely continuous?

FTC for LI
- f' exists
- $f' \in L[a,b]$
- $f(x) - f(a) = \int_a^x f'(t)\,dt$

$\Big\} \Leftrightarrow$ AC

Pf



(i). Let $\varepsilon > 0$. $\delta = \varepsilon^2$

Let $(a_j, b_j) \subset [0,1]$ s.t. $\sum |b_j - a_j| < \delta$

$$\Rightarrow \sum |f(b_j) - f(a_j)| = \sum |\sqrt{b_j} - \sqrt{a_j}|$$
$$\le \sum |\sqrt{b_j - a_j}|$$
$$\le \sqrt{\sum |b_j - a_j|}$$
$$\le \sqrt{\varepsilon^2} = \varepsilon$$

$\Rightarrow$ g is absolutely continuous on $[0,1]$

or use FTC for Lebesgue Integrals.

(ii) Let $\varepsilon > 0$.

$\forall (a_j, b_j) \subset [0,1]$, $\sum_1^N |f(b_j) - f(a_j)| \le \sum_1^\infty 1/n^2 = M$.

Since it is finite we can choose our (a_j, b_j) intervals small enough to make our differences small.

(iii) $g \circ f$ is not absolutely continuous since $g \circ f$ is not of bounded variation

$AC \Rightarrow BV$

$$Vf(y) = \sup\{\textstyle\sum_1^k |f(x_i) - f(x_{i-1})| \mid 0 = x_0 < x_1 < \dots < x_k = 1\}$$
$$= \sup\{\textstyle\sum_1^k |1/n| \mid n \in \mathbb{N}\}$$
$$= \infty$$

4 (i) For a space X w/ measure μ and $\mu(X) < \infty$. Prove $L^q \subset L^p$ for $0 < p < q < \infty$

(ii) Suppose X contains disjoint E_k for $k = 1, 2, \dots$ w/ $0 < \mu(E_k) < 2^{-k}$. Show $L^p \not\subset L^q$

Hölder's $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \int |fg| \leq \|f\|_p \|g\|_q$

Pf: Let $f \in L^q$ then $\|f\|_p < \infty$.

$$\|f\|_p^p = \int |f|^p \cdot 1$$
$$\leq \| |f|^p \|_{q/p} \|1\|_{q/q-p} \qquad \text{since } p/q + \tfrac{q-p}{q} = 1$$
$$= \left(\int (|f|^p)^{q/p}\right)^{p/q} \|1\|_{\frac{q}{q-p}}$$
$$= (\|f\|_q)^p \, \mu(X)^{\frac{q-p}{q}}$$
$$< \infty \quad \text{since } \|f\|_q \text{ and } \mu(X) \text{ are.}$$

$\therefore f \in L^p$

(ii) Let $f = \sum_0^\infty \mu(E_n)^{-1/q} \chi_{E_n}$ wts $f \in L^p$ but $f \notin L^q$

note f is a simple function since E_n's are disjoint.

$$\int |f|^q = \int \textstyle\sum_0^\infty \mu(E_n)^{-1} \chi_{E_n}$$
$$= \textstyle\sum_0^\infty \mu(E_n)^{-1} \mu(E_n)$$
$$= \textstyle\sum_0^\infty 1$$
$$= \infty \qquad \therefore f \notin L^q$$

$$\int |f|^p = \int \textstyle\sum_0^\infty \mu(E_n)^{-p/q} \chi_{E_n}$$
$$= \textstyle\sum_0^\infty \mu(E_n)^{-p/q} \mu(E_n)$$
$$= \textstyle\sum_0^\infty \mu(E_n)^{1-p/q}$$
$$\leq \textstyle\sum_0^\infty \left(\frac{1}{2^k}\right)^{1-p/q}$$
$$= \textstyle\sum_0^\infty \left(\frac{1}{2^{1-p/q}}\right)^k$$
$$< \infty \quad \text{since } \frac{1}{2^{1-p/q}} < 1 \text{ for any } p \,\&\, q \text{ where } p < q. \;\square$$

MAT 701 Qualifying Exam January 2012

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from MAT 701, MAT 601, and MAT 602.

Problem 1. Let f_n be *non-decreasing* functions on $(-\infty, 0]$ such that $f_n \to 0$ in (Lebesgue) measure as $n \to \infty$. Proof or counterexample: Necessarily $f_n \to 0$ almost everywhere on $(-\infty, 0]$ with respect to Lebesgue measure.

Problem 2. Prove that any function $f \in L^p([0,1]^2), 1 \leq p < \infty$, can be approximated by a finite linear combination of functions of the form $h(x)g(y)$ with h and g continuous on $[0,1]$. More precisely, given $\epsilon > 0$ there is a function

$$u(x,y) = \sum_{j=0}^{n} h_j(x)g_j(y)$$

with h_j and g_j continuous on $[0,1]$ for $j = 1, 2, \ldots, n$, such that $\|f - u\|_p < \epsilon$.

Problem 3. Let f be a continuous real-valued function on the real line that is differentiable almost everywhere with respect to Lebesgue measure and satisfies $f(0) = 0$ and

$$f'(x) = 2f(x)$$

almost everywhere. Prove that there exist infinitely many such functions, but that only one of them is absolutely continuous.

Problem 4. Let μ and ν be measures on the same measurable space. Assume that μ is finite, and define a set function μ_0 by

$$\mu_0(A) = \sup\{\mu(A \cap B) : \text{ B is measurable and } \nu(B) < \infty\}$$

for measurable sets A. Also define a set function λ on measurable sets A by $\lambda(A) = \mu(A) - \mu_0(A)$. Prove that both μ_0 and λ are measures, and that λ has the property that $\lambda(A) > 0$ implies $\nu(A) = \infty$ for measurable sets A.

J 2012

1. Let f_n be nondecreasing on $(-\infty, 0]$ s.t. $f_n \to 0$ in Lebesgue measure as $n \to \infty$. Proof or counter example $f_n \to 0$ a.e. on $(-\infty, 0]$ w.r.t. Lebesgue measure

Pf $f_n \xrightarrow{m} 0$

$\Rightarrow m(\{x \mid |f_n| > \varepsilon\} < \varepsilon$ as $n \to \infty$ $\forall \varepsilon > 0$

$\Rightarrow \exists N$ s.t. $0 < |f_n(-\varepsilon)| \le \varepsilon$ for all $n \ge N$

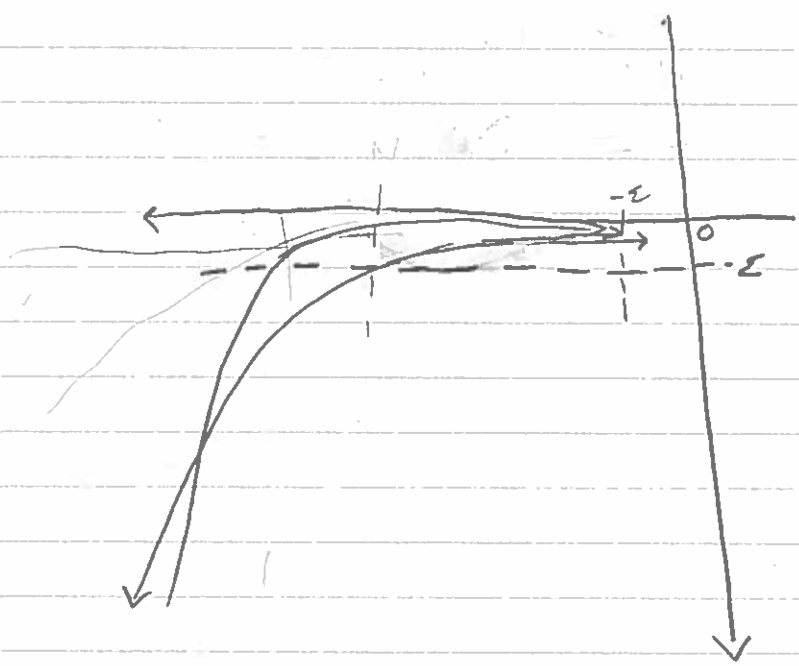
$\Rightarrow$ $0 \le |f_n(x)| \le \varepsilon$ $\forall x < -\varepsilon$ since f_n non decreasing

$\Rightarrow$ $f_n \to 0$ on $(-\infty, -\varepsilon)$

$\Rightarrow$ $f_n \to 0$ on $(-\infty, 0)$ as $\varepsilon \to 0$

$\Rightarrow$ $f_n \to 0$ a.e on $(-\infty, 0]$.

□



7. Prove any $f \in L^p([0,1]^2)$ $1 \le p < \infty$ can be approximated by a finite linear combination of functions of form $h(x)g(y)$ with h, g continuous on $[0,1]$. More precisely given $\varepsilon > 0$ $\exists$ $u(x,y) = \sum h_j(x) g_j(y)$ w/ h_j and g_j continuous on $[0,1]$ s.t. $\|f-u\|_p < \varepsilon$

Pf Let $f = \chi_E$ where $E \subset [0,1]\times[0,1]$

$\Rightarrow \exists B = \cup_1^\infty [a_j, b_j] \times [c_j, d_j]$ s.t. $\mu(B \triangle E) < \varepsilon$.

Consider $[a_j, b_j] \subset [0,1]$.

$$\Rightarrow \exists h_j(x) = \begin{cases} 1 & x \in [a_j, b_j] \\ 0 & x \le a_j - \frac{\varepsilon}{2^{j/2p}}, x > b_j + \frac{\varepsilon}{2^{j/2p}} \\ \text{interpolates linearly otherwise} \end{cases}$$

Similarly define $g_j(y)$ for $[c_j, d_j] \in [0,1]$.

$\Rightarrow u(x,y) = \sum h_j(x) g_j(y)$

$\Rightarrow \|f-u\|_p = \sum_1^\infty \left(\frac{\varepsilon}{2^{j/2}}\right)^p \left(\frac{\varepsilon}{2^{j/2}}\right)^p = \sum_1^\infty \frac{\varepsilon^{2p}}{2^j} = \varepsilon^{2p} < \varepsilon$

Now let $f = \sum_i^n a_i \chi_{A_i}$ be a simple function. We know from above each χ_{A_i} can be approximated as desired. We can extend this to f by linearity

Let $f \in L^+$ the $\exists$ simple $s_n \nearrow f$. So by MCT it follows

$f \in L^p$ follows by a decomp.

$\square$

3. Let f be a continuous real valued on real line thats differentiable a.e. wrt m and $f(0)=0$ and $f'(x)=2f(x)$ a.e. Prove $\exists$ infinitely many such functions but only one is absolutely continuous

Pf $f'(x)=2f(x) \Rightarrow f(x)=e^{2x}g(x)$. where $g(0)=0$
$\Rightarrow f'(x)=2e^{2x}g(x)+e^{2x}g'(x)$
$\Rightarrow g'(x)=0 \quad \forall x$.
$\Rightarrow g(x)=K\cdot\alpha(x)$ where K is constant and $\alpha(x)$ is extended cantor fn

So there are infinitely many such functions.

If $f(x)$ is absolutely continuous
$\Rightarrow g(x)=e^{-2x}f(x)$
$\Rightarrow g(x)$ is absolutely continuous. Since e^{2x} and f is.

$$\sum|e^{-2a_i}f(a_i)-e^{-2b_i}f(b_i)|$$
$$\le\sum\underbrace{|e^{-2a_i}|}_{\text{bad}}\underbrace{|f(a_i)-f(b_i)|}_{\text{small}}+\underbrace{|f(b_i)|}_{\text{bad}}\underbrace{|e^{-2a_i}-e^{-2b_i}|}_{\text{small}}=\text{small}$$

So $g(x)$ is.

$\Rightarrow g$ absolutely continuous and $g'(x)=0$
$\Rightarrow g\equiv 0$
$\Rightarrow f(x)=e^{2x}\,0=0$ is only absolutely continuous one

$\square$

- $g,f\in AC \Rightarrow fg\in AC$
- Cantor fcn has 0 derivative.
-

4. Let μ and ν be measures on X. Assume μ finite and $\mu_0(A) = \sup\{\mu(A\cap B) : B \text{ measurable } \nu(B)<\infty\}$ for msble A. $\lambda(A) = \mu(A) - \mu_0(A)$.
Prove μ_0 and λ are measures and $\lambda(A)>0 \Rightarrow \nu(A)=\infty$

Measure: $\mu(\phi)=0$
$\mu(\bigcup E_i)=\sum \mu(E_i)$

Pf $\mu_0(\phi) = \sup\{\mu(\phi\cap B), B \text{ msble } \nu(B)<\infty\}$

$$= \sup\{\mu(\phi)\}$$

$$= \sup\{0\} \quad \text{since } \mu \text{ is a measure.}$$

$$= 0$$

$$\lambda(\phi) = \mu(\phi) - \mu_0(\phi) = 0 - 0 = 0$$

Let A_i be disjoint in X.

$$\mu_0(\cup_1^\infty A_i) = \sup\{\mu((\cup_1^\infty A_i)\cap B)\}$$

$$= \sup\{\mu(\cup_1^\infty (A_i\cap B))\} \quad A_i\cap B \text{ disjoint}$$

$$= \sup\{\textstyle\sum_1^\infty \mu(A_i\cap B)\}$$

$$= \textstyle\sum_1^\infty \sup \mu(A_i\cap B)\}$$

$$= \textstyle\sum_1^\infty \mu(A_i)$$

$$\lambda(\cup_1^\infty A_i) = \mu(\cup_1^\infty A_i) - \mu_0(\cup_1^\infty A_i)$$

$$= \textstyle\sum_1^\infty \mu(A_i) - \sum_1^\infty \mu_0(A_i)$$

$$= \textstyle\sum_1^\infty \mu(A_i) - \mu_0(A_i)$$

$$= \textstyle\sum_1^\infty \lambda(A_i)$$

So μ_0 and λ are measures

Assume $\nu(A)>0 \Rightarrow \mu(A) > \mu_0(A)$

$$\Rightarrow \mu(A) > \sup\{\mu(A\cap B) : B \text{ msble } \nu(B)<\infty\}$$

$$\Rightarrow \mu(A) > \mu(A) \} \text{ if } \nu(A)<\infty$$

$$\Rightarrow \nu(A) = \infty$$

Qualifying Exam Summer 2011 Analysis

(1) In Euclidean space $\mathbb{R}^n$ with Lebesgue measure m, for $k \in \mathbb{N}$ and some $1 < p < \infty$ let $f, f_k \in L^p$ with $f_k \to f$ pointwise a.e. as $k \to \infty$. Assume that $\|f_k\|_p \leq M < \infty$ for all $k \in \mathbb{N}$. Also, let $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Prove or provide a counterexample to the statement: $\|f\|_p \leq M$.

(b) True or False, explain your answer. For all $R > 0$, for all $\delta > 0$ there is $F \subset \{x \in \mathbb{R}^n \mid |x| < R\} = B(0, R)$ with $m(F) < \delta$ and $f_k \to f$ uniformly on $B(0, R) \setminus F$.

(c) Prove or provide a counterexample to the statement: For all $\epsilon > 0$ there is a $R_0 > 0$ so that $\left(\int_{|x| \geq R} |g|^q \, dm \right)^{1/q} < \epsilon$ whenever $R > R_0$.

(d) True or False, explain your answer. For all $\epsilon > 0$ there is a $\delta > 0$ so that for all $E \subset \mathbb{R}^n$ if $m(E) < \delta$ then $\int_E |g|^q \, dm < \epsilon$.

(e) Prove $\lim_{k \to \infty} \int f_k g \, dm = \int fg \, dm$

(2) Let $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure as $n \to \infty$. Prove $f_n \to f$ in L^1 as $n \to \infty$.

(3) (a) Give an example of continuous $f : \mathbb{R} \to \mathbb{R}$ and $E \subset \mathbb{R}$ with $m(E) = 0$ so that $m(f(E)) \neq 0$, m is Lebesgue measure on $\mathbb{R}$.

(b) Let f be an absolutely continuous function on the interval $[a, b]$. Show that $m(f(E)) = 0$ for all $E \subset [a, b]$ with $m(E) = 0$.

(4) For f a positive measurable function on the interval $[0, 1]$, which is larger(assume all the integrals make sense)?

$$\int_0^1 f \, dm \int_0^1 \log f \, dm \text{ OR } \int_0^1 f \log f \, dm$$

Prove your answer.

Summer 2011

Definitions and concepts

Lebesgue measure

measure of an interval is simply it's length.

$f \in \mathcal{L}^p$

$\|f\|_p = (\int |f|^p)^{1/p} < \infty$

Egorov's thm

μ finite, $\varepsilon > 0$, $f_n \to f$ a.e.

$\Rightarrow \exists B \in \mathcal{A}$ s.t. $\mu(B) < \varepsilon$ and $f_n \to f$ uniformly on B^c

Dominated Convergence

Assume f_n measurable $f_n \to f$ pointwise $\sup |f_n| \leq g \in \mathcal{L}$

$\Rightarrow f \in \mathcal{L}'(\mu)$ and $\int |f_n - f| d\mu \to 0$

absolutely continuous

$\mu(E) = 0 \Rightarrow \nu(E) = 0$.

$f_n \xrightarrow{\mu} f$

$\mu(\{x \,|\, |f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$

$x_n \to x$

every subsequence of $\{x_n\}$ has a convergent sub-subsequence.

absolutely continuous function.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\sum b_j - a_j < \delta \Rightarrow \sum |f(b_j) - f(a_j)|$

Convex $\Rightarrow F(\int f) \leq \int F(f)$

concave $\Rightarrow F(\int f) \geq \int F(f)$

A11

(1) In $\mathbb{R}^n$ w/ L.M. m. $k \in \mathbb{N}$ and $1 < p < \infty$. Let $f, f_k \in L^p$ w/ $f_k \to f$ ptwise a.e. as $k \to \infty$. Assume $\|f_k\|_p \leq M < \infty$ $\forall k \in \mathbb{N}$. Let $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$

(a) PoC: $\|f\|_p \leq M$

(b) ToF: $\forall R > 0$, $\forall \delta > 0$ $\exists F \subset \{x \in \mathbb{R}^n \mid |x| < R\} = B(0,R)$ w/ $m(F) < \delta$ and $f_k \to f$ uniformly on $B(0,R) \setminus F$

(c) PoC: $\forall \varepsilon > 0$, $\exists R_0 > 0$ s.t. $\left(\int_{|x| \geq R} |g|^q dm\right)^{1/q} < \varepsilon$ $R > R_0$

(d) ToF: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall E \subset \mathbb{R}^n$ if $m(E) < \delta$ then $\int_E |g|^q dm < \varepsilon$

(e) Prove $\lim_{k \to \infty} \int f_k g \, dm = \int f g \, dm$

Fatou $\int \underline{\lim} f_k \leq \underline{\lim} \int f_k$

Pf (a) True

$$\|f\|_p^p = \int |f|^p = \int \lim_{k \to \infty} |f_k|^p \overset{\text{fatou}}{\leq} \underline{\lim}_{k \to \infty} \int |f_k|^p \leq \underline{\lim} M = M$$

Egorov's Thm
Assume $\mu < \infty$ $\varepsilon > 0$
Then $f_n \to f$ a.e. $\exists B \in \mathcal{A}$ s.t. $\mu(B) < \varepsilon$ and $f_n \overset{u}{\to} f$ on B^c

(b) True

Consider finite measure space $B_R(0)$
By Egorovs thm $\exists F \in B_R(0)$ s.t. $m(F) < \delta$ and $f_n \to f$ uniformly on $B_R(0) \setminus F$.

To use Egorovs we have to restrict to a finite space

use Egorov's when looking for a set where sequence converges uniformly

(c) True

Consider $g_n = g \chi_{B_{r_n}(0)} \to g$ a.e. where $r_n \nearrow \infty$

Then $|g_n| \leq |g| \in L^q$

DCT
h integ L^1
$f_n \to f$ a.e.
$\Rightarrow \lim \int f_n = \int f$

By DCT $\int |g_n|^q \to \int |g|^q$

equivalently $\forall \varepsilon > 0$, $\exists n_0$ s.t. $\int_{B_{n_0}(0)} |g|^q \geq \|g\|_q^q - \varepsilon$

$\Rightarrow \varepsilon > \|g\|_q^q - \int_{B_{n_0}(0)} |g|^q$

$\Rightarrow \varepsilon > \int_{B_n(0)^c} |g|^q$

$\int_{B_n(0)} |g|^q \nearrow \|g\|_q^q$,
So, subtracting "ε" will be smaller given n is large enough,

$\Rightarrow \varepsilon^{1/q} > \left(\int_{|x| \geq R} |g|^q\right)^{1/q}$

$\Rightarrow \varepsilon > \left(\int_{|x| \geq R} |g|^q\right)^{1/q}$

$\nu(E) = \int_E f$ is abs cont wrt m

(d) True

$\nu(E) = \int_E |g|^q$ defines a measure thats absolutely continuous wrt m

(e) Note $\int |(f_n - f)g| = \int_{B_R(0)} |f_n - f||g| + \int_{B_R(0)^c} |f_n - f||g|$

Holders: $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \int |fg| \le \|f\|_p \|g\|_q$

On $B_R(0)^c$:

$$\int_{|x|\ge R} |f_n - f||g| \overset{\text{Holder}}{\le} \underbrace{\left(\int_{|x|\ge R} |f_n - f|^p\right)^{1/p}}_{\le 2M} \left(\int_{|x|\ge R} |g|^q\right)^{1/q}$$

$$< 2M\varepsilon'$$

$$< \varepsilon$$

($R > R_1$ as in c) w/ ε)

On $B_R(0)$

Break up into where g negligible where $f_n \to f$ uniformly rest has measure 0

$$\int_{B_R(0)} |f_n - f||g| \le \hat{\varepsilon} \int_{B_R(0)\setminus F} |g| + \int_F |f_n - f||g|$$

$$\le \hat{\varepsilon}\, m(B_R(0))^{1/p} \left(\int_{B_R(0)\setminus F} |g|^q\right)^{1/q} + \left(\int_F |f_n - f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

$$\le \varepsilon + 2M\varepsilon'$$

$$< 2\varepsilon$$

Then $\int |(f_n - f)g| \le 3\varepsilon$.

So $\lim_{n\to\infty} \int f_n g\, dm = \int fg\, dm$

$\square$

2. Let $|f_n| \le g \in L^1$ and $f_n \to f$ in measure as $n \to \infty$.
Prove $f_n \to f$ in L^1 as $n \to \infty$

Pf Consider a subsequence of $\int f_n$
$\int f_{n_1}, \int f_{n_2}, \dots \int f_{n_k}$

$\{f_{n_k}\} \to f$ in μ so $\exists$ subsequence $f_{n_{k_\ell}} \to f$ a.e.

$\left.\begin{array}{l} \cdot\ f_{n_{k_\ell}} \text{ measurable} \\ \cdot\ |f_{n_{k_\ell}}| \le |g| \\ \cdot\ f_{n_{k_\ell}} \text{ pointwise} \end{array}\right\}$ so by DCT $\int f_{n_{k_\ell}} \to \int f$

So every subsequence has a convergent subseq.

$\Rightarrow \int f_n \to \int f$

$\Rightarrow f_n \to f$ in L_1

$\square$

$f_n \xrightarrow{\mu} f$ $\exists$ subsequence $f_{n_k} \to f$ a.e.

DCT - f_n msble, $f_n \to f$ a.e. $|f_n| \le g \in L^1$
$\Rightarrow \lim \int f_n = \int f$.

subsubsequence argument
$x_n \to x$ if every subsequence of $\{x_n\}$ has a convergent sub-subsequence.

J11

3 (a) Give an example of continuous $f: \mathbb{R} \to \mathbb{R}$ and $E \subset \mathbb{R}$ w/ $m(E)=0$ so $m(f(E)) \neq 0$

(b) Let f be abs cont. function on $[a,b]$. Show $m(f(E))=0$ $\forall E \subseteq [a,b]$ w/ $m(E)=0$

Pf (a) Let f be the extended cantor function

Let C be the Cantor set.

$m(f(C)) = m([0,1]) = 1$

$m(C)=0$

(b) Let $\varepsilon>0$ then since f is absolutely continuous $\exists \delta>0$ s.t. $\sum_{j=1} b_j - a_j < \delta \Rightarrow \sum |f(b_j)-f(a_j)| < \varepsilon$

Now $\exists\ \mathcal{U} = \bigcup_{k=1}^{\infty} (a_n, b_n)$ s.t. $m(U) < \delta$ and $E \subset U$ where $m(\mathcal{U}) = \sum_{k=1}^{\infty} b_k - a_k$.

On each $[a_k, b_k]$ f attains a max $f(y_j)$ and $\min f(x_j)$

$\Rightarrow \sum_{j=1}^{N} |x_j - y_j| < \sum_{j=1}^{N} |b_j - a_j| < \delta \quad \forall N$

Also $f(E) \subset \bigcup_{j=1}^{\infty} (f(x_j), f(y_j))$

$\Rightarrow m(f(E)) \leq m(\bigcup_{j=1}^{\infty} (f(x_j), f(y_j)))$

$\leq \sum_{j=1}^{\infty} |f(y_j) - f(x_j)|$

$= \lim_{N\to\infty} \sum_{j=1}^{N} |f(y_j) - f(x_j)|$

$\leq \varepsilon \quad \forall N$

$AC \Rightarrow \forall \varepsilon>0, \exists \delta>0$ s.t. $\sum |b_j - a_j| \Rightarrow \sum |f(b_j) - f(a_j)|$

$m(E)=0 \Leftrightarrow \exists U$ s.t. $E \subset U = \bigcup_j (a_j, b_j)$ w/ $\sum |b_j - a_j| < \delta$

4 For f a positive measurable function on the interval which is larger? $\int_0^1 f\,dm \int_0^1 \log f\,dm$ or $\int_0^1 f \log f\,dm$

Pf Let $F(x) = x\log x$ then $F'(x) = \log x + 1$
and $F''(x) = \frac{1}{x} > 0$ on $[0,1]$

Let $G(x) = \log x$ $\quad G'(x) = \frac{1}{x}$, $G''(x) = -\frac{1}{x^2} < 0$ on $[0,1]$

Note $F(x)$ is convex and $G(x)$ is concave

So $F\left(\int_0^1 f\,dm\right) \le \int_0^1 F(f) = \int_0^1 f \log f$ $\quad (*)$
and $G\left(\int_0^1 f\,dm\right) \ge \int_0^1 G(f) = \int_0^1 \log f$ $\quad (**)$

Equivalently $F\left(\int_0^1 f\,dm\right) = \left(\int_0^1 f\,dm\right)\left(\log \int_0^1 f\,dm\right)$

$\Rightarrow \left(\int_0^1 f\,dm\right)\left(\log \int_0^1 f\,dm\right) \le \int_0^1 f \log f\,dm$ by $(*)$
$\Rightarrow \left(\int_0^1 f\,dm\right)\left(\int_0^1 \log f\,dm\right) \le \int_0^1 f \log f\,dm$ by $(**)$

$\square$

Jensen's Inequality
$F'' > 0 \Rightarrow F$ convex $\Rightarrow F(\int f) \le \int (F \circ f)$
$F'' < 0 \Rightarrow G$ concave $\Rightarrow G(\int f) \ge \int (G \circ f)$

Analysis Qualifying Exam
August 2010

You **must** justify your answers in **full detail**, and
explicitly check all the assumptions of any theorem you use.

1. Assume that $f, f_1, f_2, \dots \in L^1(\mathbb{R})$ (Lebesgue measure), and that as $n \to \infty$ (i) $f_n \to f$ pointwise on $\mathbb{R}$ and (ii) $\|f_n\|_1 \to \|f\|_1$. Prove that for any any measurable set $E \subset \mathbb{R}$, $\lim_{n\to\infty} \int_E f_n = \int_E f$. GDCT

2. Let $f \in L^2[1, \infty)$ (Lebesgue measure). For each of the following statements, if the statement is true, prove it, while if false give a counterexample.

   (a) If f is continuous then $f(x) \to 0$ as $x \to \infty$. (Do not assume continuity for parts (b),(c) and (d).)

   (b) $\int_{[n,n+1]} |f| \to 0$ as $n \to \infty$

   (c) $\sqrt{n} \int_{[n,n+1]} |f| \to 0$ as $n \to \infty$

   (d) $\liminf_{n\to\infty} \sqrt{n} \int_{[n,n+1]} |f| = 0$

3. Let $f \in L^2(0, \infty)$ (Lebesgue measure). Prove the following:

   (a) $\left| \int_0^x f(t)dt \right| \le x^{1/2} \|f\|_2$ for $x > 0$.

   (b) $\lim_{x\to\infty} x^{-1/2} \int_0^x f(t)dt = 0$.

4. Define

$$f(x,y) = \begin{cases} x^{-4/3} \sin(\frac{1}{xy}) & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Prove or disprove: $\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \int_0^1 f(x,y)dydx.$

August 2010

Definitions and Concepts

Generalized Dominated Convergence Thm

Let $\{f_n\}$ be msble and $f_n \to f$ pointwise.

If $|f_n| \leq g_n \in L^1$ and $\lim \int g_n \, d\mu = \int g \, d\mu$

$\Rightarrow \lim \int f_n \, d\mu = \int f \, d\mu$

$(\int |f| \, dx)^2 \leq \int |f|^2 dx$. by Hölders.

Hölders Inequality

If $p+q=1$ $\int |fg| \, d\mu \leq \|f\|_p \|g\|_q$.

Fubini

Assume μ, ν σ-finite. Let $f: X \times Y \to \overline{\mathbb{R}}$ be $\mathcal{A} \times \mathcal{B}$ msble.

s.t. $f \geq 0$ or $f \in L^1(\mu \times \nu)$

$\Rightarrow$ Can interchange order of integration.

A10

1. Assume $f, f_1, f_2, \ldots \in L^1(\mathbb{R})$ and $f_n \to f$ ptwise on $\mathbb{R}$ $\|f_n\|_1 \to \|f\|_1$ Prove $\forall$ measurable $E \subset \mathbb{R}$ $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Pf Note $f_n \chi_E \le |f_n| \in L^1$

$\|f_n\|_1 \to \|f\|_1 \Rightarrow \int |f_n| \to \int |f| \Rightarrow |f_n| \xrightarrow{L_1} |f|$

So $\int_E f_n = \int f_n \chi_E \to \int f \chi_E = \int_E f$

by generalized DCT.

$\square$

Generalized Dominated Convergence Thm:

Let $\{f_n\}$ be sequence of measurable fcns s.t. $f_n \to f$ ptwise.

If $|f_n| \le g_n \in L^1$ and $\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu$

then $\int f \, d\mu = \lim \int f_n \, d\mu$.

i.e. f_n bounded by L_1-convergent sequence g_n converging to integrable fcn.

Pf Since $|f_n| \le g_n$, $-g_n \le f_n \le g_n$ w/ $g_n \ge 0$

$\Rightarrow g_n - f_n \ge 0$ and $g_n + f_n \ge 0$

$$\Rightarrow \int g + \int f = \int \underline{\lim}\, g_n + \underline{\lim}\, f_n = \int \underline{\lim}\, g_n + f_n \le \underline{\lim} \int g_n + f_n = \int g + \underline{\lim} \int f_n$$

$$\Rightarrow \int g - \int f = \int \underline{\lim}\, g_n - \underline{\lim}\, f_n \le \underline{\lim} \int g_n - \int f_n = \int g + \underline{\lim}\left(-\int f_n\right) = \int g - \overline{\lim} \int f_n$$

$\Rightarrow \int f \le \underline{\lim} \int f_n$

$\Rightarrow \int f \ge \overline{\lim} \int f_n$

$\Rightarrow \overline{\lim} \int f \le \int f \le \underline{\lim} \int f_n$

$\Rightarrow \lim \int f_n = \int f$

$\square$

2 Let $f \in L^2[1,\infty)$ T or F

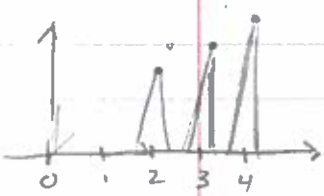
(a) If f is cont then $f(x) \to 0$ as $x \to \infty$

(b) $\int_{[n,n+1]} |f| \to 0$ as $n \to \infty$

(c) $\sqrt{n} \int_{[n,n+1]} |f| \to 0$ as $n \to \infty$

(d) $\liminf_{n\to\infty} \sqrt{n} \int_{[n,n+1]} |f| = 0$.

Pf (a) False consider



$$f(x) = \begin{cases} 0 & \text{otherwise} \\ n & x = n \in \mathbb{N} \setminus \{1\} \\ \text{line increasing to } n & x \in [n - \frac{1}{n^4}, n] \\ \text{line decreasing to } 0 & x \in [n, n + \frac{1}{n^4}] \end{cases}$$

triangles w/ area $1/2n$

$$\int_1^\infty |f|^2 dm = \sum_{n=2}^\infty \frac{1}{2} n^2 \frac{1}{2n^4} = \sum_2^\infty \frac{1}{4} \frac{1}{n^2} < \infty$$

So $f \in L^2[1,\infty)$ and f is continuous but $f(x) \not\to 0$

(b). True

break up integral

$\int_1^\infty |f|^2 < \infty \Rightarrow \sum_1^\infty \int_n^{n+1} |f|^2 < \infty$ since $f \in L^2$

$\Rightarrow \int_{[n,n+1]} |f|^2 \to 0$

$\Rightarrow (\int_{[n,n+1]} |f|)^2 \to 0 \Rightarrow \int_{[n,n+1]} |f| \to 0$

(c) false

pick ~~n~~ to be specific values.

Consider $f(x) = \begin{cases} 2^{-n/2} & x \in [2^n, 2^n+1] \\ 0 & \text{else} \end{cases} = \sum 2^{-n/2} \chi_{[2^n, 2^n+1]}$

$\int |f|^2 = \sum_1^\infty 2^{-n} = 1 < \infty$ so $f \in L^2$

If $n = 2^k \Rightarrow \sqrt{2^k} \int_{[2^k, 2^{k+1}]} |f| = \sqrt{2^k}\, 2^{-k/2} = 1 \neq 0$

2(d) Assume false i.e. $\lim_{n} \sqrt{n} \int_{[n,n+1]} |f| \not\to 0$.

$\Rightarrow \sqrt{n} \int_{[n,n+1]} |f| > c > 0$ for all but finitely many n

$\Rightarrow \int_{[n,n+1]} |f| > c/\sqrt{n}$

$$\begin{aligned}
\Rightarrow \int_1^\infty |f|^2 dx &= \sum_1^\infty \int_n^{n+1} |f|^2 dx \\
&\geq \sum_1^\infty \left(\int_n^{n+1} |f| dx\right)^2 \quad \text{by } * \\
&\geq \sum_1^\infty \left(\frac{c}{\sqrt{n}}\right)^2 \\
&= \sum_1^\infty \frac{c^2}{n} \\
&= \infty
\end{aligned}$$

which contradicts since $f \in L^2$

$$\left.\begin{aligned}
\int |f| dx &\leq \|f\|_2 \|1\|_2 \quad \text{by Hölders} \\
&= \left(\int_n^{n+1} |f|^2\right)^{1/2} \left(\int_n^{n+1} 1 dx\right)^{1/2} \\
&= \left(\int_n^{n+1} |f|^2\right)^{1/2} \\
\Rightarrow \left(\int |f| dx\right)^2 &\leq \int |f|^2 dx.
\end{aligned}\right\} *$$

□

$\left(\int |f| dx\right)^2 \leq \int |f|^2 dx$

3. Let $f \in L^2(0,\infty)$

(a) Prove $\left|\int_0^x f(t)dt\right| \le x^{1/2}\|f\|_2 \quad \forall x>0$

(b) $\lim_{x\to\infty} x^{-1/2}\int_0^x f(t)dt = 0$

Pf (a) $\left|\int_0^x f(t)dt\right| \le \int_0^x |f(t)|dt$

$= \int_0^x 1\cdot|f(t)|dt$

$\le \left(\int_0^x 1^2\right)^{1/2}\|f\|_2$ by Hölders.

$= \sqrt{x}\,\|f\|_2.$ ✓

Hölders $1/p + 1/q = 1 \Rightarrow \int|fg| \le \|f\|_p\|g\|_q$

(b) $f\in L^2$ so $\exists N_\varepsilon$ s.t. $\int_{N_\varepsilon}^\infty |f|^2 < \varepsilon \Rightarrow \int_{N_\varepsilon}^x |f|^2 < \varepsilon \quad \forall x \ge N_\varepsilon$

$\left|x^{-1/2}\int_0^x f(t)dt\right| \le x^{-1/2}\int_0^x |f(t)|dt$

$= x^{-1/2}\left(\int_0^{N_\varepsilon}|f(t)| + \int_{N_\varepsilon}^x |f|\right)$

$= \frac{1}{\sqrt{x}}\int_0^{N_\varepsilon}|f| + \frac{1}{\sqrt{x}}\int_{N_\varepsilon}^x 1\cdot|f|$

$\le \frac{M}{\sqrt{x}} + \frac{1}{\sqrt{x}}\left(\int_{N_\varepsilon}^x 1\right)^{1/2}\left(\int_{N_\varepsilon}^x |f|^2\right)^{1/2}$

or break into $[0,\sqrt{N}], [\sqrt{N}, N]$

finite since $L^2((0,N_\varepsilon)) \subset L^1((0,N_\varepsilon))$

$\le \frac{M}{\sqrt{x}} + \frac{1}{\sqrt{x}}\sqrt{x}\sqrt{\varepsilon} = \frac{M}{\sqrt{x}} + \sqrt{\varepsilon}.$

$\longrightarrow \sqrt{\varepsilon}$ as $x \to \infty$

□

$f\in L^2 \Rightarrow f \in L^1$ on finite measure space

$f \in L^2$ means has negligible tail.

4 Define $f(x,y) = \begin{cases} x^{-4/3}\sin\left(\frac{1}{xy}\right) & \text{if } 0<y<x<1 \\ 0 & \text{otherwise.} \end{cases}$

Prove or disprove $\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1\int_0^1 f(x,y)\,dy\,dx$

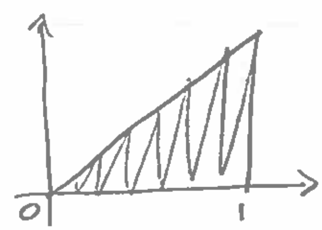
Pf true.

$$\int_0^1\int_0^1 |f(x,y)|\,dy\,dx \le \int_0^1\int_0^x x^{-4/3}\,dy\,dx = \int_0^1 x^{-4/3}\,y\Big|_0^x\,dx = \int_0^1 x^{-1/3}\,dx = 3/2 < \infty$$

So by Tonelli $\int_{[0,1]^2} |f|\,d(x\times y) = \int_0^1\int_0^1 |f(x,y)|\,dy\,dx = \iint |f|\,dx\,dy$

So by Fubini the claim holds.

$\square$



$0<y<x<1$

Real analysis qualifying exam Jan. 13, 2010

1. (a) Let f be a *continuous* map of a metric space X into a metric space Y.

True or False. If false either give a counterexample, or make the statement true by either adding a hypothesis or modifying the conclusion. Do not prove if true.

(i) If X is compact, then so is $f(X)$.
(ii) If X is connected, then so is $f(X)$.
(iii) If f is one-to-one, then $f^{-1} : f(X) \to X$ is continuous.

(b) The Cantor set $C \subset [0,1] \subset \mathbb{R}$ consists of all sums $x = \sum_{j=1}^{\infty} \frac{n_j}{3^j}$ where the n_j are allowed to form any sequence of 0's and 2's. Let $f : C \to [0,1]$ be the canonical map defined by $f(x) = \frac{1}{2}\sum_{j=1}^{\infty} \frac{n_j}{2^j}$.

Prove or Disprove.
(i) f is onto.
(ii) f is continuous.
(iii) f is one-to-one.

2. Let $\{f_j\}$ be a sequence of Lebesgue measurable functions that converges pointwise a.e. to a function f on the interval $I = [0,1]$. Let $F \in L^p(I)$ and $g \in L^{p'}(I)$ where p and p' are dual exponents, $1 \leq p \leq \infty$.

(a) If $p > 1$, $\|f_j\|_p \leq 1$ $(j = 1, 2, \ldots)$ and $\int_I f_j g \to \int_I Fg$, prove that $\int_I fg = \int_I Fg$.

(b) Show by example that the conclusion of part (a) is false when $p = 1$.

3. Let f be a real valued function on the interval $I = [a, b]$.

(a) Give the definition of *absolute continuity* for f on I.

(b) Suppose f is absolutely continuous on I.

True or False. If false either give a counterexample or modify the statement so that it is true. Do not prove if true.

(i) f is uniformly continuous on I.
(ii) f is differentiable at every x in the interior of I.
(iii) $f' \in L^1(I)$ and $f(x) - f(a) = \int_a^x f'(t)dt$, $a \leq x \leq b$.

(c) Suppose f is absolutely continuous on I. Prove that the set of values $\{y = f(x) : f'(x) \text{ is not defined}\}$ has measure *zero*.

(d) Suppose f is absolutely continuous on I. Prove that the set of values $\{y = f(x) : f'(x) = 0\}$ has measure *zero*.

4. Let Borel functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ be given so that $f(x-y)g(y)$ is a Borel function on $\mathbb{R}^2$. Prove that $\int_{-\infty}^{\infty} |f(x-y)g(y)|dy < \infty$ for a.e. x.

January 2010
Definitions and Concepts.

Compact
Every open cover has a finite subcover

Connected
$\nexists X = E \cup F$, $E \cap F = \emptyset$ E & F open.

f^{-1} 1-1 $\Rightarrow f^{-1}$ Cont $\Leftrightarrow f'(x) \neq 0$ $\forall x$.

Fatou's lemma
$f_n \geq 0$ measurable $\Rightarrow \int \underline{\lim} f_n d\mu \leq \underline{\lim} \int f_n d\mu$.

Lousin's Thm
Suppose $f: [0,1] \to \mathbb{R}$ is Borel msble, m lebesgue measure for $\varepsilon > 0$, $\exists F \subset [0,1]$ s.t. $m([0,1] \setminus F) < \varepsilon$ and $f|_F$ is continuous on F.

absolutely continuous
$\forall \varepsilon > 0$ if $\sum b_j - a_j < \delta$ then $\sum |f(b_j) - f(a_j)| < \varepsilon$ $\forall n \in \mathbb{N}$
→ uniformly continuous
and for $f \in L'$ $f(x) - f(a) = \int_a^x f'(t)dt$.
→ Sends null sets to null sets.

Fubini
μ, ν are σ-finite. Let $f: X \times Y \to \mathbb{R}$ be $A \times B$ measurable s.t. if $f \geq 0$ or $f \in L'(\mu \times \nu) \Rightarrow$ can interchange order of integrati

J10.

1(a) $f: X \to Y$ continuous. True or false

(i) X compact $\Rightarrow f(x)$ is too

(ii) X connected $\Rightarrow f(x)$ connected

(iii) f^{-1} 1-1 $\Rightarrow f^{-1}: f(x) \to X$ is cont

Pf (i). Let V_α be an open cover of $f(x)$.

then f cont $\Rightarrow f^{-1}(V_\alpha)$ open and covers X

$\Rightarrow \exists$ finite subcover $f^{-1}(V_i)$ $i=1 \ldots n$

$\Rightarrow f(f^{-1}(V_i))$ is finite subcover of $f(X)$

$\Rightarrow f(X)$ compact

continuous preserves compact and connected

f cont, f^{-1} 1-1, $f' \neq 0$ $\Rightarrow f^{-1}$ cont.

(ii) True

(iii) only true if $f'(x) \neq 0$ $\forall x$.

(b) Prove or disprove.

$C = \{x \in [0,1] \mid x = \sum_1^\infty \frac{n_j}{3^j}\}$ $f: C \to [0,1]$ $f(x) = \frac{1}{2} \sum_1^\infty \frac{n_j}{2^j}$

PoD (i) f onto, (ii) f cont, (iii) f 1-1

Pf (i) Let $x \in [0,1] \Rightarrow x = 1/2 \sum_1^\infty \frac{n_j}{2^j}$ $n_j \in \{0,1\}$

$\hat{x} = 2 \sum \frac{n_j}{3^j} \in C$ and $f(\hat{x}) = x$

Cantor set is cont and bijective

(ii) Fix $\varepsilon > 0$

$|x-y| = |\sum_1^\infty \frac{n_j}{3^j} - \sum_1^\infty \frac{\hat{n}_j}{3^j}| = |\sum \frac{n_j - \hat{n}_j}{3^j}| < \frac{1}{3n}$ wLOG $x > y$

only if $n_j = \hat{n}_j$ $\forall j \leq n$

$\Rightarrow \sum_1^\infty \frac{n_j - \hat{n}_j}{3^j} = \sum_{n+1}^\infty \frac{n_j - \hat{n}_j}{3^j}$

or onto and increasing means continuous

$|f(x) - f(y)| = f(x) - f(y) = \frac{1}{2} \sum \frac{n_j}{2^j} - \frac{\hat{n}_j}{2^j} \leq \sum_{n+1}^\infty \frac{1}{2^{j-1}} \to 0$

(iii) $f(x) = f(y) \Rightarrow 1/2 \sum \frac{n_j}{2^j} = 1/2 \sum_1^\infty \frac{\hat{n}_j}{2^j} \Rightarrow x = y$

So 1-1

1(b) $C \subset [0,1]$ consists of all sums $x = \sum_{j=1}^{\infty} \frac{n_j}{3^j}$

Let $f : C \to [0,1]$ s.t. $f(x) = \frac{1}{2} \sum_{1}^{\infty} \frac{n_j}{2^j}$

PoD

i) f is onto

ii) f is cont

iii) f is 1-1

2. $\{f_j\}$ sequence of Lebesgue msble functions that converge ptwise a.e. to f on $I = [0,1]$.
Let $F \in L^p(I)$ $g \in L^{p'}(I)$ where p and p' are dual exponents $1 \le p \le \infty$

(a) If $p > 1$ $\|f_j\|_p \le 1$ and $\int_I f_j g \to \int_I Fg$ prove $\int_I fg = \int Fg$

(b) Show by examples (a) fails if $p = 1$.

Pf $\|f_j\|_p \le 1 \Rightarrow \int |f|^p \le \underline{\lim} \int |f_j|^p \le 1 \Rightarrow f \in L^p(I)$

$\Rightarrow \int_I fg$ & $\int_I Fg$ make sense by Hölder

Fatou $\int f \le \underline{\lim} \int f_n$

$\forall \delta > 0$, $\exists E$ w/ $m(I \setminus E) < \delta$ s.t. $f_j \to f$ uniformly on E

Egerov's, $\exists E$ w/ $\forall \varepsilon > 0$, $m(I \setminus E) < \varepsilon$ s.t. $f_j \to f$ uniformly on E

$$\int_I |(f_j - f) g| = \int_E |f_j - f||g| + \int_{E^c} |f_j - f||g|$$
$$\le \left(\int_E |f_j - f|^p\right)\left(\int_E |g|^q\right)^{1/q} + \left(\int_{E^c} |f_j - f|^p\right)^{1/p}\left(\int_{E^c} |g|^q\right)^{1/q}$$
$$< \varepsilon\, m(E)\|g\|_q + (\|f_j\|_p + \|f_j\|_p)\, \varepsilon^{1/q}$$
$$\le \varepsilon \|g\|_q + 2\|f_j\|_p \varepsilon^{1/q}$$
$$< \varepsilon \|g\|_q + 2\varepsilon^{1/q} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Note: • $\left(\int_{E^c} |f_j - f|^p\right)^{1/p} \le \|f_j - f\|_p \le \|f_j\|_p + \|f_j\|_p$ by Minkowski

• $m(E) < 1$, $\|f_j\|_p < 1$

and • if $\nu(E) = \int_E |g|^q\, dm$ then $\nu \ll m$

So $m(E^c) < \delta \Rightarrow \nu(E) < \varepsilon \Rightarrow \left(\int_E |g|^q\right)^{1/q} < \varepsilon$

Thus $\int f_j g \to \int fg$ and $\int f_j g \to \int Fg$ $\therefore \int fg = \int Fg$ by uniqueness

(b) Let $g \equiv 1$ and $f_n = n^{-1/p} \chi_{[0,n]} = n^{-1} \chi_{[0,n]}$ when $p = 1$.

Then $f_n \to 0$ as $n \to \infty$

$\frac{1}{n}\chi_{[0,n]}$ is good example for not convergence in L^1.

$$\int f_n g = \int_0^n n^{-1} \cdot 1 = 1 \to 1$$

but $\int fg = \int 0 \cdot 1 = 0$

So $\int f_n g \not\to \int fg$ when $p = 1$.

3. f real valued on $I = [a,b]$.

(a) Give definition of absolute continuity

(b) Suppose f absolutely continuous of I

true or false (i) f uniformly cont on I

(ii) f diff at every $x \in I$

(iii) $f' \in L^1(I)$, $f(x) - f(a) = \int_a^x f'(t)\,dt$.

(c) Suppose f abs cont on I. Prove $m(y = f(x) : f'(x)$ not defined$) = 0$

(d) Prove $m\{y = f(x) : f'(x) = 0\} = 0$.

Pf (a) f abs cont $\Leftrightarrow$ $\forall \varepsilon > 0$ if $\sum_1^n b_j - a_j < \delta$ $(a_j, b_j) \subset (a,$
then $\sum |f(b_j) - f(a_j)| < \varepsilon$ $\forall n \in \mathbb{N}$

FTC for LI

f AC $\Leftrightarrow$ f' exists a.e $f' \in L^1[a,b]$ $f(x) - f(a) = \int_a^x f'(t)\,dt$

(b). (i) true ($n=1$ in def)

(ii) almost every x.

(iii) true.

abs. cont functions map null sets to null sets

(c) $\exists f'(x)$ for a.e x

$\Rightarrow m(\{x \mid f'(x)$ not defined$\}) = 0$

$\Rightarrow m(\{f(x) \mid f'(x)$ not defined$\}) = 0$

Since absolutely continuous functios map null sets to null sets.

not rigourous and possibly incorrect

(d) $\{x : f'(x) = 0\} = \cup I_n \cup \bigcup \{x_j\}$.

where I_n are intervals and x_j are extrema so $f'(x) \neq 0$ in nbhd of x_j

We need to show that absolute continuity $\Rightarrow \exists$ at most countably many points and intervals f is uniformly continuous so this holds.

We can surrond each pt & interval by non overlapping open sets Each contains a rational so there can be at most countably many

$\Rightarrow \{f(x) : f'(x) = 0\}$ is at most countable

$\Rightarrow m\{f(x) : f'(x) = 0\} = 0$

$\square$

4 Let Borel functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ be s.t. $f(x-y)g(y)$ is Borel on $\mathbb{R}^2$, Prove $\int_{-\infty}^{\infty} |f(x-y)g(y)|dy < \infty$ for a.e x

Pf $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)g(y)|dxdy = \int_{-\infty}^{\infty} |g(y)| \int_{-\infty}^{\infty} |f(x-y)|dxdy$

$= \|f\|_1 \int_{-\infty}^{\infty} |g(y)|dy$

$= \|f\| \, \|g\|$

$< \infty$

$\Rightarrow \int\int |f(x-y)g(y)| < \infty$

So by Torelli $\int\int |f(x-y)g(y)|dydx < \infty$

$\Rightarrow \int_{-\infty}^{\infty} |f(x-y)g(y)|dy < \infty$

$\square$

3d alternative since previous might be wrong.

Pf Let $g = \chi_A$ where $A = \{y = f(x) : f'(x) = 0\}$

Consider $\int g(f(x))f'(x)dx = \int g(u)du$ (can do since $f \in AC$)

$\Rightarrow \int \chi_A(f(x))f'(x)dx = \int \chi_A \, du$

$\Rightarrow \qquad 0 \qquad = \mu(A)$

since has mass only when $f'(x)=0$ so always 0

$\therefore \mu(\{y = f(x) : f'(x) = 0\})$

Qualifying Exam Measure Theory

8 January 2009

Show **ALL** your work. Write all your solutions in clear, logical steps.

Each problem has the same weight

Good luck!

1. **Problem 1.** Given $0 < p_0 < p_1 < \infty$ construct a Lebesgue measurable function f on $\mathbb{R}$ so that $f \in L^p(\mathbb{R}, m)$ if and only if $p \in [p_0, p_1]$. (m denotes Lebesgue measure) In Foland pg 187

Problem 2. Let μ be a measure on X with $\mu(X) < \infty$. For f measurable on X show that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$

Problem 3. Let $Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dm(y)$ be the Hardy-Littlewood Maximal function of a function $f \in L^1(\mathbb{R}^k, m)$ (a) Show that there are finite positive constants c and R (that depend on f) so that $Mf(x) \geq \frac{c}{|x|^k}$ for all x with $|x| > R$. (b) Use part (a) to show that if $Mf(x) \in L^1(\mathbb{R}^k, m)$ then $f = 0$ a.e.

Problem 4. Suppose f_n are measurable functions on (X, μ) and that $|f_n| \leq g \in L^1(\mu)$. Show that if $f_n \to f$ in measure then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$

January 2009
Definitions and Concepts.

$f \in L^p(\mathbb{R}, m)$

$\|f\|_p = (\int |f|^p dm)^{1/p} < \infty$

$\|f\|_\infty$

$\inf \{M \mid \mu(|f| > M) = 0\}$.

Dominated Convergence Thm

f_n msble, $f_n \to f$ ptwise, $\sup |f_n| \le g \in L^1(\mu)$

$\Rightarrow f \in L^1(\mu)$ and $\int |f_n - f| d\mu \to 0$

$f_n \to f$

every subsequence has a convergent Subsequence

J09

1. Given $0 < p_0 < p_1 < \infty$ construct a Lebesgue measurable function f on $\mathbb{R}$ so that $f \in L^p(\mathbb{R}, m)$ iff $p \in [p_0, p_1]$

Pf

Let $f(x) = \frac{1}{x^{1/p_1}} \ln(x)^{-2/p_1} \chi_{[0,1/e]} + \frac{1}{x^{1/p_0}} \ln(x)^{-2/p_0} \chi_{[e,\infty)}$

$$\int |f|^{p_0} = \int_0^{1/e} \frac{1}{x^{p_0/p_1}} |\ln(x)|^{\frac{-2p_0}{p_1}} dx + \int_e^{\infty} \frac{1}{x} (\ln x)^{-2} dx$$

$$\leq \int_0^{1/e} 1/x^{p_0/p_1} dx + \int_1^{\infty} u^{-2} du$$

$$= \frac{x^{1-p_0/p_1}}{1-p_0/p_1} \Big|_0^{1/e} + (-u^{-1}) \Big|_1^{\infty}$$

$$= \frac{1}{(1-p_0/p_1) e^{1-p_0/p_1}} + 1 < \infty \quad \text{so } f \in L^{p_0}$$

Similarly $f \in L^{p_1}$

check endpoints then use Littlewood's inequality to show it holds inside.

$\Rightarrow$ if $p \in [p_0, p_1]$, $\|f\|_p \leq \|f\|_{p_0}^{\lambda} \|f\|_{p_1}^{1-\lambda} < \infty$

$\Rightarrow f \in L^p \ \forall p \in [p_0, p_1]$

Now if $p < p_0$ then since $|\ln(x)| \leq |x|^{\varepsilon}$ on $(1,\infty)$

$\Rightarrow \int |f|^p > \int_e^{\infty} \left| \frac{1}{x^{1/p_0}} \ln x^{-2/p_0} \right|^p$

$\geq \int_e^{\infty} \frac{1}{x^{p/p_0 + \varepsilon}} = \infty$ for ε s.t. $p/p_0 + \varepsilon < 1$

$\Rightarrow f \notin L^p$ for $p < p_0$

If $p > p_1$ then $|\ln x| \leq |1/x|^{\varepsilon}$ on $(0, 1/e)$

$\Rightarrow \int |f|^p > \int_0^{1/e} \frac{|x|^{\varepsilon}}{x^{p/p_0}} = \int_0^1 x^{-p/p_1 + \varepsilon} = \frac{x^{1-p/p_1+\varepsilon}}{1-p/p_1+\varepsilon} \Big|_0$

so for ε small enough the integral is ∞.

$\Rightarrow f \notin L^p$ for $p > p_1$.

$\square$

2. Let μ be a measure on X w/ $\mu(X) < \infty$
For f measurable on X show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$

Pf Case 1 f bounded
Then $f \in L^\infty \Rightarrow f \in L^p \ \forall \ 1 \le p \le \infty$

$\|f\|_p \le \|f\|_1^\lambda \|f\|_\infty^{1-\lambda}$ where $\lambda = \frac{1}{p}$

$\Rightarrow \overline{\lim_{p\to\infty}} \|f\|_p \le \|f\|_\infty$

Now let $E_\varepsilon = \{x \mid f(x) > \|f\|_\infty - \varepsilon\}$

$\|f\|_q \ge \|f\chi_{E_\varepsilon}\|_q = \left(\int_{\{f > \|f\|_\infty - \varepsilon\}} |f|^q d\mu\right)^{1/q} \ge (\|f\|_\infty - \varepsilon)^{q/q} \underbrace{\mu(E_\varepsilon)^{1/q}}_{\to 1 \text{ as } q \to \infty}$

$\Rightarrow \underline{\lim} \|f\|_q \ge \|f\|_\infty - \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \underline{\lim} \|f\|_q \ge \|f\|_\infty$

$\Rightarrow \lim \|f\|_q = \|f\|_\infty$

Case 2 f unbounded
Let $E_M = \{x \mid f(x) > M\}$ and $\|f\|_\infty = \infty$

$\Rightarrow \|f\|_p \ge \left(\int_{E_M} |f|^p\right)^{1/p} > M\mu(E_M)^{1/p} \to M \quad \forall M$
as $p \to \infty$

So $\lim \|f\|_p = \|f\|_\infty$

$\square$

$\|f\|_p \le \|f\|_1^\lambda \|f\|_\infty^{1-\lambda}$
$\lambda = \frac{1}{p}$

3. Let $Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\,dm(y)$, $f \in L^1(\mathbb{R}^k, m)$

(a) Show there are finite positive constants c & R (depending on f) so that $Mf(x) \geq \frac{c}{|x|^k}$ $\forall |x| > R$.

(b) Show if $Mf(x) \in L^1(\mathbb{R}^k, m)$ then $f = 0$ a.e.

Pf Note: $Mf(x) \geq \frac{1}{m(B(x,r))} \int_{B(x,|x|)} |f(y)|\,dm$.

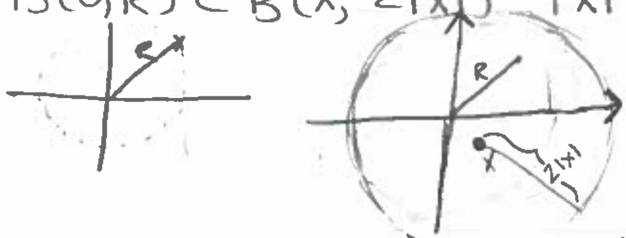
$\exists R$ s.t. $\int_{B(0,R)^c} |f| < \varepsilon$ i.e $\int_{B(0,R)} |f| \geq \|f\|_1 - \varepsilon$

Since $f \in L^1$

Now $m(B(x,r)) = r^n m(B(x,1))$
$= r^n m(B(0,1))$ translation invariant

$\Rightarrow Mf(x) \geq \frac{1}{|x|^n m(B(0,2))} \int_{B(x,2|x|)} |f(y)|\,dm$

$B(0,R) \subset B(x, 2|x|)$ $|x| > R$



move ball to some constant

bdd on some ball

$\geq \frac{1}{|x|^n m(B(0,2))} \int_{B(0,R)} |f(y)|\,dm$

$\geq \frac{\|f\|_1 - \varepsilon}{m(B(0,2))} \frac{1}{|x|^n}$

$= \frac{c}{|x|^n}$ □

(b) BWOC assume $f \neq 0$

then $c\sigma(S^k) \int_R^\infty \frac{r^{k-1}}{r^k} dr = c\sigma(S^{m-1}) \log r \Big|_R^\infty \geq \infty$ ↯

$\Rightarrow f = 0$ a.e.

$\frac{1}{|x|^k}$ is not integrable at ∞ $\Rightarrow c = 0$ $\Rightarrow f = 0$ Corollary 2.5, Folland

4. Suppose f_n are measurable functions on (X, μ) and $|f_n| \le g \in L^1(\mu)$. Show if $f_n \to f$ in measure then $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$

Pf Let $\int f_{n_k}$ be a subseq of $\int f_n$

$\Rightarrow \quad f_{n_k} \xrightarrow{\mu} f$

$\Rightarrow \quad \exists f_{n_{k_\ell}} \to f$ a.e.

$|f_{n_{k_\ell}}| \le g \overset{DCT}{\Rightarrow} \int f\,d\mu = \lim \int f_{n_{k_\ell}}\,d\mu$

So every subsequence of $\int f_n\,d\mu$ has a subsequence converging to $\int f\,d\mu$

$\Rightarrow \int f_n\,d\mu \longrightarrow \int f\,d\mu$

$\Rightarrow \int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$

$\square$

Measure Theory Qualifying Exam Fall 2008

Problem 1. Let $E \subset \mathbb{R}$ with $m(E) > 0$ (i.e. E has positive lebesgue measure). Show that the set $E - E = \{x - y \mid x, y \in E\}$ contains an interval centered at 0.

Problem 2. Let μ be a positive measure on X and f measurable on X. For $0 < r < p < s < \infty$ show that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$.

Problem 3. Prove that a positive measure μ on X is σ-finite if and only if there is an $f \in L^1(d\mu)$ with $f(x) > 0$ for all $x \in X$.

Problem 4. Let $1 < p < \infty$ and suppose that $f_k \to f$ in $L^p(\mathbb{R}, m)$ as $k \to \infty$ (m is Lebesgue measure on $\mathbb{R}$). In addition assume that $g_k(x) = \begin{cases} 0 & , x < k \\ 1 & , x \geq k \end{cases}$ for $k = 1, 2, \ldots$. What does the sequence $f_k g_k$ converge to in L^p? Prove it.

Fall 2008

Definitions and Concepts.

$\|f\|_p \le \|f\|_r^{\lambda} \|f\|_s^{1-\lambda}$

σ-finite measure

$X = \bigcup_1^\infty E_n$ w/ $\mu(E_n) < \infty$

$f_n \xrightarrow{L^p} 0$

$\int |f_n - f|^p \to 0$

Minkowski's Inequality

$\|f+g\|_p \le \|f\|_p + \|g\|_p$

Sets of positive measure are almost intervals.

If $E \in \mathcal{M}$, $m(E) > 0$ then $\forall \alpha < 1$ $\exists$ open interval I s.t. $m(E \cap I) > m(I)$ (Foland page 40)

F08

1. Let $E \subset \mathbb{R}$ w/ $m(E)>0$ Show that the set $E-E=\{x-y \mid x,y \in E\}$ contains an interval centered at 0

Pf $m(E)>0 \Rightarrow \exists$ Interval I s.t. $m(E \cap I) > \alpha m(I)$ where $3/4 < \alpha < 1$ (Foland page 40) $*$

Let $\hat{E} = E \cap I$ then $\hat{E} \subset I$ & $\hat{E} \subset E$.

WTS $\exists r>0$ s.t. $(t+\hat{E}) \cap \hat{E} \neq \emptyset$ $\forall |t|<r$

$\Rightarrow (-r,r) \subset \hat{E}-\hat{E}$

$\Rightarrow (-r,r) \subset E-E$

{will show we're moving it along interval $(-r,r)$ and its non empty everywhere along the interval.}

Suppose $(t+\hat{E}) \cap \hat{E} = \emptyset$ for some $|t| < \frac{m(I)}{4}$

$3/2\, m(I) = m(I) + \frac{m(I)}{2} \geq m(I) + 2|t|$

$(t+\hat{E}) \cup (\hat{E}) \subset (t+I) \cup (I) \subset (a-|t|, b+|t|)$ if $I=(a,b)$

$\Rightarrow m((t+\hat{E}) \cup (\hat{E})) \leq m(I) + 2|t|$

then $m(I) + 2|t| \geq m((t+\hat{E}) \cup \hat{E})$

$= m(t+\hat{E}) + m(\hat{E})$ since we're assuming $(t+\hat{E}) \cap \hat{E} = \emptyset$

$= 2m(\hat{E})$ since m is translation invarient

$\geq 2\alpha m(I)$ by $(*)$

$> 3/2\, m(I)$

$\Rightarrow 3/2\, m(I) > 3/2\, m(I)$ which contradicts

$\Rightarrow (t+\hat{E}) \cap \hat{E} \neq \emptyset$ $\forall |t| < \frac{m(I)}{4}$

$\Rightarrow (-\frac{1}{4} m(I), \frac{1}{4} m(I)) \subset E-E$

$\square$

Sets of positive measure are almost intervals

2 Let μ be positive measure on X and f msble on X. For $0 < r < p < s < \infty$ show $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$

Pf $\|f\|_p \leq \|f\|_r^{\lambda} \|f\|_s^{1-\lambda}$ by Littlewoods inequality.

$\leq \max\{\|f\|_r, \|f\|_s\}^{\lambda} \max\{\|f\|_r, \|f\|_s\}^{1-\lambda}$

$\leq \max\{\|f\|_r, \|f\|_s\}$ since $\lambda + 1 - \lambda = 1$

$\square$

3. Prove that a positive measure μ on X is σ-finite if and only if $\exists f \in L'(d\mu)$ w/ $f(x) > 0 \;\; \forall x \in X$

Pf $\Rightarrow$) Assume positive measure μ is σ-finite

$\Rightarrow X = \bigcup_{n=1}^{\infty} E_n$ w/ $\mu(E_n) < \infty \;\; \forall n$

Let $f = \sum_{n=1}^{\infty} \frac{1}{n^2 \mu(E_n)} \chi_{E_n}$

Then $f > 0$ and $\int f \leq \sum_1^{\infty} \frac{1}{n^2 \mu(E_n)} \mu(E_n) = \sum_1^{\infty} \frac{1}{n^2} < \infty$

so $f \in L'(d\mu)$.

$\Leftarrow$) Assume $\exists f \in L'(d\mu)$ w/ $f(x) > 0 \;\; \forall x \in X$,

Let $E_n = f^{-1}([n, n+1))$ $n \in \mathbb{N}$.

Then $\bigcup E_n = X$. since $\sum [n, n+1) = [0, \infty) \Rightarrow \bigcup f^{-1}([n, n+1)) = X$

WTS $\mu(E_n) < \infty \;\; \forall n$

Let $f = \sum_1^{\infty} f \chi_{E_n} \Rightarrow \|f\|_1 = \int f \, d\mu \geq \int \underbrace{f \chi_{E_n}}_{\geq n} d\mu \geq \int n \chi_{E_n} d\mu = n\mu(E$

$\|f\|_1$ is finite $\Rightarrow n\mu(E_n) < \infty$

$\Rightarrow \mu(E_n) < \infty$

$\square$

4. Let $1 < p < \infty$ and suppose that $f_k \to f$ in $L^p(\mathbb{R}, m)$ as $k \to \infty$. In addition assume that $g_k(x) = \begin{cases} 0 & x < k \\ 1 & x \geq k \end{cases}$ $k \in \mathbb{N}$.
What does the sequence $f_k g_k$ to in L^p? Prove it.

Pf I claim $f_n g_n \xrightarrow{L^p} 0$ as $n \to \infty$.
Since $f_n \to f$, $\exists N$ s.t. $\|f_N - f\|_p < \varepsilon$.

$$\|f_k g_k\|_p = \|f_k g_k - f g_k + f g_k\|_p$$
$$= \|(f_k - f) g_k + f g_k\|_p$$
$$\leq \|(f_k - f) g_k\|_p + \|f g_k\|_p \quad \text{by Minkowski}$$

Let $\varepsilon > 0$.
Notice $\int (|f_N - f||g_N|)^p < \int |f_N - f|^p < \varepsilon/2$ since $g_N \leq 1$
and since $f \in L^p$ $\exists R > 0$ s.t. $\int_{B(R,0)^c} |f|^p < \varepsilon/2$
i.e. $\int_{|x| \geq R} |f|^p < \varepsilon/2$
As such $\int |f g_R|^p = \int_{x \geq R} |f|^p dm < \varepsilon/2$ since $g_R \equiv 0$ for $x < R$

$$\Rightarrow \|f_k g_k\|_p \leq \|(f_k - f) g_k\|_p + \|f g_k\|_p$$
$$\leq \int (|f_N - f| g_N)^p + \int_{x \geq R} |f|^p dm$$
$$< \varepsilon/2 + \varepsilon/2$$
$$< \varepsilon \quad \text{for } R \text{ and } N \text{ large enough.}$$
$\Rightarrow \|f_k g_k\|_p \to 0$ as $k \to \infty$

$\square$

January 2008

Analysis Qualifying Exam

You should justify nontrivial steps, referring to theorems when appropriate.

1. Fix $p \in (0, \infty)$. Give an example of a function $f \notin L^p(0,1)$ such that $f \in L^r(0,1)$ for all $r < p$.

2. Let f be a nonnegative measurable function on $[0,1]$. Prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$, including the case $+\infty = +\infty$.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K(x,y)$ be measurable with respect to the product σ-algebra $\mathcal{M} \times \mathcal{N}$. Assume there is a finite constant $A > 0$ such that

$$\int_Y |K(x,y)|\, d\nu(y) \leq A \text{ for all } x \in X$$

and

$$\int_X |K(x,y)|\, d\mu(x) \leq A \text{ for all } y \in Y .$$

Fix $p \in (1, \infty)$ and $f \in L^p(X, \mathcal{M}, \mu)$ and define

$$(Tf)(y) = \int_X f(x) K(x,y) d\mu(x)$$

Prove that $\|Tf\|_{L^p(\nu)} \leq A \|f\|_{L^p(\mu)}$

4. Let $\phi : [-\pi, \pi] \to [-1, 1]$ be measurable. Let $0 < r < 1$ and prove that

$$\lim_{r \uparrow 1} \int_{-\pi}^{\pi} \frac{dt}{1 - r\phi(t)} = \int_{-\pi}^{\pi} \frac{dt}{1 - \phi(t)}$$

Evaluate the right-hand side above for $\phi(t) = \cos t$.

January 2008
Definitions & Concepts

Littlewoods inequalty
$\|f\|_p \le \|f\|_1^{\lambda} \|f\|_\infty^{1-\lambda}$

Hölders Inequality
$p+q=1 \Rightarrow \int |fg|\, d\mu \le \|f\|_p \|g\|_q$

$f \in L^p$
$\int |f|^p < \infty$

J08

1. Fix $p \in (0,\infty)$. Give example of a function $f \notin L^p(0,1)$ s.t. $f \in L^r(0,1)$ $\forall r < p$

Pf $f \notin L^p \Rightarrow \int_0^1 |f|^p = \infty$

$f \in L^r \Rightarrow \int_0^1 |f|^r < \infty$

Let $f = 1/x^{1/p}$

$$\int_0^1 |1/x^{1/p}|^p dx = \int_0^1 \frac{1}{x} dx = \ln(1) - \ln(0) = \infty$$

$$\int_0^1 |1/x^{1/p}|^r dx = \int_0^1 1/x^{r/p} dx$$ where $r < p \Rightarrow r/p < 1$

$$= \frac{x^{1-r/p}}{1-r/p}\Big|_0^1 = \frac{1}{1-r/p} < \infty$$

So $f \notin L^p$ but $f \in L^r$ $\forall r < p$.

$\square$

2. Let f be a nonnegative measurable function on $[0,1]$.
Prove $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$

Pf <u>Case 1</u> f bounded

f bdd $\Rightarrow f \in L^\infty \cap L^p \ \forall p$.

$\|f\|_p \leq \|f\|_1^{\lambda} \|f\|_\infty^{1-\lambda}$ by Littlewoods inequality

Now $\lambda = \frac{p^{-1}-r^{-1}}{q^{-1}-r^{-1}} = \frac{1}{p}$ so we have

$\|f\|_p \leq \|f\|_1^{1/p} \|f\|_\infty^{1-\frac{1}{p}} \longrightarrow \|f\|_\infty$ as $p \to \infty$

$\Rightarrow \lim \|f\|_p \leq \|f\|_\infty$

Let $E_\varepsilon = \{x \mid |f(x)| \geq (\|f\|_\infty - \varepsilon)\}$

Now $\|f\|_q \geq \|f\chi_\varepsilon\|_q$

$= \left(\int |f\chi_{E_\varepsilon}|^q\right)^{1/q}$

$\geq \left((\|f\|_\infty - \varepsilon)^q \int |\chi_{E_\varepsilon}|^q\right)^{1/q}$

$= (\|f\|_\infty - \varepsilon)\, \mu(E_\varepsilon)^{1/q}$

$\to \|f\|_\infty - \varepsilon$ as $q \to \infty$ since $\mu(E_\varepsilon) > 0$

$\Rightarrow \lim \|f\|_q \geq \|f\|_\infty$

$\Rightarrow \lim \|f\|_p = \|f\|_\infty$

<u>Case 2</u> f unbounded

Let $E_M = \{x \mid f(x) > M\}$

$m(E_M) > 0 \ \forall M$ since f is unbounded

$$\|f\|_q \geq \|f\chi_{E_M}\|_q = \left(\int_{\{f(x)>M\}\cap[0,1]} |f|^q dm\right)^{1/q} \geq M\, m(E_M)^{1/q} \longrightarrow M$$

$\Rightarrow \varliminf_{q\to\infty} \|f\|_q \geq M \ \forall M.$

$\Rightarrow \|f\|_q \to \|f\|_\infty$ as $q \to \infty$

$\square$

3 $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ σ-finite m.s. $K(x,y)$ msble $(\mathcal{M}\times\mathcal{N})$
$0 < A < \infty$ $\int_Y |K(x,y)| d\nu(y) \leq A$ $\forall x \in X$
and $\int_X |K(x,y)| d\mu(x) \leq A$ $\forall y \in Y$. Fix $p \in (1,\infty)$, $f \in L^p(X, \mathcal{M}, \mu)$
$(Tf)(y) = \int_X f(x) K(x,y) d\mu(x)$. Prove $\|Tf\|_{L^p(\nu)} \leq A \|f\|_{L^p(\mu)}$

Pf First consider $\int_X |f(x) K(x,y)| dx$

$$= \int_X |f(x)| |K(x,y)|^{1/p} |K(x,y)|^{1/q} dx \quad \text{if } \frac{1}{p} + \frac{1}{q} = 1$$

$$\leq \left(\int_X |f(x)|^p K(x,y) dx\right)^{1/p} \left(\int K(x,y) dx\right)^{1/q} \quad \text{by Hölder}$$

$$\leq A^{1/q} \left(\int_X |f(x)|^p |K(x,y)| dx\right)^{1/p}$$

Now

$$\|Tf\|_p^p \leq \left(\int_Y A^{p/q} \left(\int_X |f(x)|^p |K(x,y)| dx\right) dy\right)$$

$$= A^{p/q} \int_X |f(x)|^p \int_Y |K(x,y)| dy dx \quad \text{by Fubini!}$$

$$\leq A^{p/q+1} \int_X |f(x)|^p dx$$

$$\leq A^p \int_X |f(x)|^p dx \quad \text{since } \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow 1 + \frac{p}{q} = p$$

$$\Rightarrow \|Tf\|_p \leq A \left(\int_X |f(x)|^p dx\right)^{1/p}$$

$$= A \|f\|_p$$

$\square$

4. Let $\phi : [-\pi,\pi] \to [-1,1]$ be msble. Let $0<r<1$ and prove
$\lim_{r \nearrow 1} \int_{-\pi}^{\pi} \frac{dt}{1-r\varphi(t)} = \int_{-\pi}^{\pi} \frac{dt}{1-\varphi(t)}$. Evaluate rhs for $\varphi(t) = \cos t$

Pf Let $A = \{x \in [-\pi,\pi] \mid \varphi(x) \in [0,1]\}$
$B = \{x \in [-\pi,\pi] \mid \varphi(x) \in [-1,0)\}$

$$\int_{-\pi}^{\pi} \frac{dt}{1-r\varphi(t)} = \int_A \frac{dt}{1-r\varphi(t)} + \int_B \frac{dt}{1-r\varphi(t)}$$

On A $1-r\varphi(t)$ can approach 0 since $r \to 1$ and $\varphi(t) \leq 1$
Let $\{r_n\} \nearrow 1$ then $0 < \frac{1}{1-r_n\varphi(t)} \nearrow \frac{1}{1-\varphi t}$ are msble
So by MCT $\int_A \frac{1}{1-r_n\varphi(t)} \nearrow \int_A \frac{1}{1-\varphi(t)}$

On B, $\varphi(t) \in [-1,0)$. $r_n \nearrow 1$
$\frac{1}{1-r_n\varphi(t)} < \frac{1}{1-r_n(0)} = \frac{1}{1+r_n(0)} < 1 \quad \in L^1(\mu)$
$\frac{1}{1-r_n\varphi(t)} \to \frac{1}{1-\varphi(t)}$ pointwise
So by DCT $\int_B \frac{1}{1-r_n\varphi(t)} \to \int_B \frac{1}{1-\varphi(t)}$

Now $\int_{-\pi}^{\pi} \frac{dt}{1-r\varphi(t)} = \int_A \frac{dt}{1-r\varphi(t)} + \int_B \frac{dt}{1-r\varphi(t)}$
$\to \int_A \frac{dt}{1-\varphi(t)} + \int_B \frac{dt}{1-\varphi(t)}$
$= \int_{-\pi}^{\pi} \frac{dt}{1-\varphi(t)}$

$\square$

Real analysis qualifying exam Aug. 22, 2006

1. Let $E \subset \mathbb{R}$ denote a countable set.

(a) Compute the Lebesgue measure of E.

(b) Construct an E that is a G_δ set (countable intersection of open sets).

(c) Construct an E that is not a G_δ set.

2. Give an example of a sequence $\{f_n\}$ for each of the requirements below or show that no such sequence exists. L^1 denotes the Lebesgue integrable functions on $\mathbb{R}$.

(a) $0 \leq f_n \to 0$ in L^1, but $\{f_n\}$ does not converge pointwise a.e. to zero.

(b) $0 \leq f_n \to 0$ a.e., but $\{f_n\}$ does not converge in L^1 to zero.

(c) $0 \leq f_n \to f$ a.e. and $\int f_n \leq 1$, but $f \notin L^1$.

3. Given a $p \geq 1$ let $f \in L^p([0,1])$ with respect to Lebesgue m[illegible]et $E \subset [0,1]$ be measurable. Put $\nu(E) = \int_E f\,dm$.

(a) Show that ν is a complex measure absolutely continuo[illegible]t to m.

(b) Let $g(x) = \nu([0,x])$ for each $x \in [0,1]$. Prove

$$\|g\|_p \leq \left(\frac{1}{p}\right)^{\frac{1}{p}} \|f\|_p$$

4. For some $1 \leq p \leq \infty$ let $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be a continuous linear operator. Suppose $\|f\|_p \leq \|Tf\|_p$ for all $f \in L^p(\mathbb{R})$.

(a) Show there exists a real constant C independent of f so that

7.

$$\|Tf\|_p \leq C\|f\|_p$$

for all f.

(b) Show T is $1:1$.

(c) Show T has closed range, i.e. whenever $Tf_j \to g$ in L^p there exists $f \in L^p$ such that $Tf = g$.

August 2006

Definitions and Concepts.

Lebesgue measure

$m(E) = \inf \{ \sum_1^\infty b_k - a_k \mid E \in \bigcup_1^\infty (a_k, b_k) \}$

G_δ set

a countable intersection of open sets

$f_n \to f$ a.e (μ)

$\mu(\{x \mid f_n \not\to f\}) = 0$

$f_n \xrightarrow{\mu} f$

$\mu(\{x \mid |f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$

$f_n \xrightarrow{L^p} f$

$\int |f_n - f|^p d\mu \to 0$

Fatou's lemma

If $f_n \geq 0$ are msble then $\int \underline{\lim} f_n \, d\mu = \underline{\lim} \int f_n \, d\mu$

$f \in L^p([0,1])$

$\|f\|_p = (\int |f|^p)^{1/p} < \infty$

ν absolutely continuous wrt m

$\nu \ll m$, $m(E) = 0 \Rightarrow \nu(E) = 0$

Continuous linear operator

$\Rightarrow$ bounded.

A06

1. Let $E \subset \mathbb{R}$ be countable.

(a) Compute $m(E)$

(b) Construct E s.t. E is a G_δ set

(c) Construct E that is not a G_δ set

Pf (a). $E = \bigcup_{n=1}^{\infty} \{a_n\}$.

$$m(\{a_n\}) = \inf\{ \sum_{k=1}^{\infty} b_k - c_k \mid a_n \in \bigcup_{k=1}^{\infty} (c_k, b_k) \}$$
$$\leq |(a_n - \tfrac{1}{k}, a_n + \tfrac{1}{k})| = \tfrac{2}{k} \quad \forall k.$$

$\Rightarrow m(\{a_n\}) = 0 \quad \forall n$

$\Rightarrow m(E) = 0$

(b). $E = 0 = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$

(c) $E = \mathbb{Q}$, $\mathbb{Q}$ is countable but every open set contains infinitely many of them so cannot intersect down to one.

Bae category.

2. Give example of $\{f_n\}$ for each below or show not possible

a) $0 \le f_n \to 0$ in L_1 but $\{f_n\} \not\to 0$ ptwise a.e.

b). $0 \le f_n \to 0$ a.e but $\{f_n\} \overset{L_1}{\not\to} 0$

c) $0 \le f_n \to f$ a.e and $\int f_n \le 1$ but $f \notin L'$

Pf a) $f_1 = \chi_{[0,1]}$ $f_2 = \chi_{[0,1/2]}$ $f_3 = \chi_{[1/2,1]}$ $f_4 = \chi_{[0,1/4]}$ $f_5 = \chi_{[1/4,\ldots]}$ ...

then $\int f_n < \int \frac{1}{2^k} \to 0$ for some k $k \to \infty$ as $n \to \infty$.

However $f_n \not\to 0$ for any point in $(0,1)$ since it cycles through each pt ∞ times.

b) $f_n = \frac{1}{n} \chi_{[0,n]}$

$f_n \to 0$ a.e

$\int f_n = \int \frac{1}{n} \chi_{[0,n]} dm = \frac{1}{n} m(\chi_{[0,n]}) = 1$

So $\int f_n \not\to 0$

c) Assume $f_n \to f$ a.e. and $\int f_n \le 1$

By Fatou's Lemma

$\int f = \int \underline{\lim} f_n \le \underline{\lim} \int f_n \le \underline{\lim} 1 = 1$

$\Rightarrow f \in L'$

3. Given a $p \geq 1$ let $f \in L^p([0,1])$ wrt m. let $E \subset [0,1]$ msble. Put $\nu(E) = \int_E f\,dm$

(a) Show ν is complex measure absolutely cont wrt m

(b). Let $g(x) = \nu([0,x])$ $\forall x \in [0,1]$ Prove $\|g\|_p \leq (\frac{1}{p})^{1/p} \|f\|_p$

Pf (a) $|\nu(E)| = |\int_E f\,dm|$ ← Let E s.t. $m(E) = 0$

$\leq \int_E |f|\,dm$

$= \int |f| \chi_E\,dm$

$\leq \|f\|_p (\int_E dm)^{\frac{1}{p}q}$ Holders

$= 0$ finite, 0

So $\nu(E) = 0$.

(b). $\|g\|_p^p = \int_{[0,1]} |g|^p\,dm$

$= \int_{[0,1]} \left| \int_{[0,x]} f(y)\,dy \right|^p dx$

$\leq \int_{[0,1]} \left(\int_{[0,x]} |f|\,dy \right)^p dx$ triangle inequality

$\leq \int_0^1 \left(\|f\|_p \left(\int_0^x dy \right)^{1/q} \right)^p$ where $\frac{1}{p} + \frac{1}{q} = 1$

$= \|f\|_p^p \int_0^1 x^{p/q}\,dx$

$= (\frac{1}{p})^{1/p} \|f\|_p^p$

4. $1 \leq p \leq \infty$ let $T: L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be cont. lin. op
$\|f\|_p \leq \|Tf\|_p$ $f \in L^p(\mathbb{R})$
(a) Show $\exists$ real C so $\|Tf\|_p \leq C\|f\|_p$.
(b) Show T is 1-1
(c) Show T has closed range

Pf (a) I claim linear + continuous - bounded

Note: T is continuous at 0 $\Rightarrow$ $\exists \delta$ s.t. $\{|f|<\delta\} \subset T^{-1}(\{|f|<1\})$
$\Rightarrow \|f\|_p < \delta \Rightarrow \|Tf\|_p < 1$

Let $\varepsilon > 0$.
$f \in L^p \Rightarrow \left\| \frac{\delta f}{\|f\|_p + \varepsilon} \right\|_p < \delta$
$\Rightarrow 1 > \left\| T\left(\frac{\delta f}{\|f\| + \varepsilon} \right) \right\|_p = \frac{\delta}{\|f\|_p + \varepsilon} \|Tf\|_p$

Now $\frac{\|f\|_p + \varepsilon}{\delta} > \|Tf\|_p$
let $\varepsilon \to 0$ then $\frac{1}{\delta}\|f\|_p \geq \|Tf\|_p$ ✓

(b). Let f, g be s.t. $Tf = Tg$ $\Rightarrow T(f-g) = 0$
So $\|f-g\|_p \leq \|T(f-g)\|_p = 0$
$\Rightarrow$ $\|f-g\|_p \leq 0$
$\Rightarrow$ $f = g$ a.e.

(c) We know Tf_j converges so
$\exists N$ s.t. $m, n \geq N$ $\Rightarrow$ $|Tf_m - Tf_n| < \varepsilon$
$\Rightarrow |f_m - f_n| < \varepsilon$
So f_n converges $\Rightarrow f_n \to f$
We can show $Tf_j \to Tf$ but $Tf_j \to g$
Thus $Tf = g$ $\square$

Measure theory exam Jan. 28, 2006

1. Let $\mathcal{P}$ denote the σ-algebra of all subsets of $\mathbb{R}$ and define a measure ρ by $\rho(E) = 1$ if $0 \in E$ and $\rho(E) = 0$ if $0 \notin E$. Let m denote Lebesgue measure and $\mathcal{M}$ the Lebesgue measurable sets. Let f denote a real valued function on $\mathbb{R}$.

(a) Show $(\rho, \mathcal{P})$ is a σ-finite measure space.

(b) Which is true and which is false and why?
(i) If f is Lebesgue measurable, then f is ρ-measurable.
(ii) If f is ρ-measurable, then f is Lebesgue measurable.

(c)Show that if $f \in L^1(\rho)$, then there is $a.e.[\rho]$ a unique Lebesgue measurable function g such that

$$\int_E g d\rho = \int_E f d\rho$$

for all $E \in \mathcal{M}$.

(d) Show by example that g is not $a.e.[m]$ unique.

2. Let μ be a signed (or complex) Borel measure on $\mathbb{R}$ such that $|\mu|(\mathbb{R}) < \infty$. Let $E \subset \mathbb{R}$ be a measurable subset with $\mu(E) \neq 0$. Suppose for all $x \in \mathbb{R}$ and all Borel subsets $A \subset E$

$$\mu(A + x) = \mu(A)$$

Prove that $\mu = 0$.

3. Let L^1 denote the Lebesgue integrable functions on the interval $[0, 1]$ with respect to Lebesgue measure and let $||f||$ denote the L^1 norm.

(a)Construct a sequence $\{f_n\} \subset L^1$ such that $||f_n|| \to 0$, but $\{f_n\}$ converges at no point.

(b)Construct a sequence $\{f_n\} \subset L^1$ such that $f_n \to 0$ at every point, but $||f_n|| \to \infty$.

(c) Suppose $f \in L^1$, $f_n \to f$ $a.e.$, and $||f_n|| \to ||f||$. Prove that $f_n \to f$ in L^1.

4. Let $1 < p < \infty$ and let f and g be Lebesgue measurable functions on the half-line $[0, \infty)$.

(a) Show how to use the Fubini theorem (Fubini-Tonelli) and the identity

$$\int_0^\infty \frac{f(y)}{x+y} dy = \int_0^\infty \frac{f(xy)}{1+y} dy \quad (x > 0)$$

to prove

$$\int_0^\infty \int_0^\infty \frac{f(y)}{x+y} dy \; g(x) dx \leq C_p ||f||_p ||g||_{p'}$$

where p' is the dual exponent to p.

(b) Can the Fubini theorem be used to get the same type of result when $x + y$ is replaced by $x - y$ in part (a)? Why or why not?

January 2006
Definitions

σ-finite measure space.
$\exists E_i \in \mathcal{A}$ s.t. $\mu(E_i) < \infty$ and $\bigcup_i^\infty E_i = X$.

f $\mathcal{L}$-measurable.
$f^{-1}(B) \in \mathcal{L}$ $\forall B \in \mathcal{B}$ $(\mathcal{L} \subset \mathcal{P}(\mathbb{R}))$

Borel measure
a measure on $\mathbb{R}$ w/ domain $\mathcal{B}_{\mathbb{R}}$

$|\mu|$
$\mu^+ + \mu^-$

Jordan Decomposition
$\mu = \mu^+ - \mu^-$ (both positive measures)

For positive measures can choose set F s.t.
$\mu(E \cap F) < \varepsilon$.

Fubini Thm
μ, ν σ finite. Let $f: X \times Y \to \overline{\mathbb{R}}$ be $\mathcal{A} \times \mathcal{B}$-msble.
w/ $f \geq 0$ or $\int_{X \times Y} |f| \, d(\mu \times \nu) < \infty$.
$\Rightarrow$ Can switch order of integration.

J06

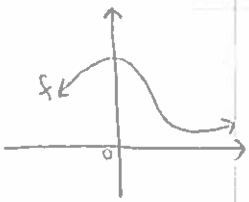
1. $\mathcal{P}$ denotes σ algebra of all subsets of $\mathbb{R}$.
$\rho(E)=1$ if $0\in E$, $\rho(E)=0$ if $0\notin E$.
(a) Show $(\rho,\mathcal{P})$ is a σ finite measure space
(b). Which is true
(i) If f is Lebesgue measurable then f is ρ-measurable
(ii) If f is ρ-measurable then f is Lebesgue msble
(c) Show if $f\in L^1(\rho)$ then there is a.e $[\rho]$ a(!) L.m. function g s.t. $\int_E g\,d\rho=\int_E f\,d\rho$ $\forall E\in M$
(d) Show by example g is not a.e. $[m]$ unique

pf (a) $\mathbb{R}\in\mathbb{R}$ and $\mathbb{R}=\mathbb{R}$ w/ $\rho(\mathbb{R})=1<\infty$ since $0\in\mathbb{R}$
so $(\rho,\mathcal{P})$ is a σ-finite measure space
(b) $f: X\to\mathbb{R}$.
f $\mathcal{L}$-measurable $\Rightarrow f^{-1}(B)\in\mathcal{L}$ $\forall B\in\mathcal{B}$.
$\mathcal{L}\subset\mathcal{P}(\mathbb{R})$ so $f^{-1}(B)\in\mathcal{P}(\mathbb{R})$ $\forall B\in\mathcal{B}$
so f is ρ-measurable
$\Rightarrow$ (i) is true.

If f is ρ-measurable then $f^{-1}(B)\in\mathcal{P}(\mathbb{R})$ $\forall B\in\mathcal{B}$
as counterexample let A be a $\mathcal{L}$ non-measurable set
Let $f=\chi_A$ then $f^{-1}(\{1\})=A\notin\mathcal{L}$ but
$A\in\mathcal{P}(\mathbb{R})$ so (ii) is false.

(b) Vitali sets not lebesgue msble but works for ρ

(c) Note: $\int_E f\,d\rho = \begin{cases} 0 & 0 \notin E \\ f(0) & 0 \in E \end{cases}$



$\int_E f\,d\rho$
$= \int_{\{0\}} f\,d\rho + \int_{E \setminus \{0\}} f\,d\rho$
$= f(0)\rho(\{0\}) + 0$
$= f(0)$

Let $g = f(0)$. g is constant and thus $\mathcal{L}$ measurable

$\int_E g\,d\rho = \int_E f(0)\,d\rho = f(0)\rho(E) = \begin{cases} 0 & 0 \notin E \\ f(0) & 0 \in E \end{cases}$

Assume $\exists \hat{g}$ since $\int_E \hat{g} = \int_E f = \begin{cases} 0 & 0 \notin E \\ f(0) & 0 \in E \end{cases}$

$f(0) = g(0) = \hat{g}(0)$

$\Rightarrow 0 \notin \{x \mid g(x) \neq \hat{g}(x)\}$

$\Rightarrow \rho\{x \mid g(x) \neq \hat{g}(x)\} = 0$

$\Rightarrow g$ is ρ unique.

(d) $g_1 = f(0)\chi_{[-1,1]}$ shows g is not (m) unique

$m\{x \mid g(x) \neq g_1(x)\} = m\{x \mid x < -1 \text{ and } x > 1\}$
$\neq 0$ clearly

$\Rightarrow g_1 \neq g$ a.e.

$\Rightarrow g$ is not m unique

$\square$

2. Let μ be a signed (or complex) Borel measure on $\mathbb{R}$ s.t. $|\mu|(\mathbb{R})<\infty$. Let $E\subset\mathbb{R}$ be a measurable subset w/ $\mu(E)\neq 0$. Suppose for all $x\in\mathbb{R}$ and all Borel subsets $A\subset E$ $\mu(A+x)=\mu(A)$. Prove $\mu=0$.

Pf $\mu(E)\neq 0 \Rightarrow \mu(E)=\mu^+(E)-\mu^-(E)\neq 0$ (Jordan decomposition)
WLOG $\mu^+(E)>\mu^-(E)$. choosing nonzero one.
$\exists P$ s.t. $\mu^+(E)=\mu(P\cap E)>0$. Let $E\cap P=E^+$

Now $E^+=\bigcup_1^\infty (E^+\cap[-n,n])$ and $\exists N$ s.t. $\mu(E^+\cap[-N,N])>\varepsilon$ for some ε.

Now. $\infty>|\mu|(\mathbb{R})=(\mu^++\mu^-)(\mathbb{R}) \geq \mu^+(\mathbb{R})$
$\geq \mu^+\left(\bigcup_{k=1}^\infty (E^+\cap[N,N]+2Nk)\right)$
$=\sum_{k=1}^\infty \mu(E^+\cap[N,N]+2NK)$
$=\sum \mu(E^+\cap[-N,N])$ since $\mu(A+x)=\mu(A)$
$\geq\sum_1^\infty \varepsilon$
$=\infty$ since $\varepsilon>0$

which contradicts.
So $\mu=0$.

Note: $\bigcup_{k=1}^\infty (E^+\cap[-N,N]+2NK)$ covers $\mathbb{R}$
and each $E^+\cap[-N,N]+2NK$ is disjoint.

$\square$

1. Assume $\exists E$ w/ $\mu(E)>0$
2. Break up μ
3. Consider only biggest part.

3. Let L^1 denote the Lebesgue integrable functions on the interval $[0,1]$ wrt Lebesgue measure and let $\|f\|$ denote the L^1 norm.

(a) Construct a sequence $\{f_n\} \subset L^1$ s.t. $\|f_n\| \to 0$ but $\{f_n\}$ converges at no point

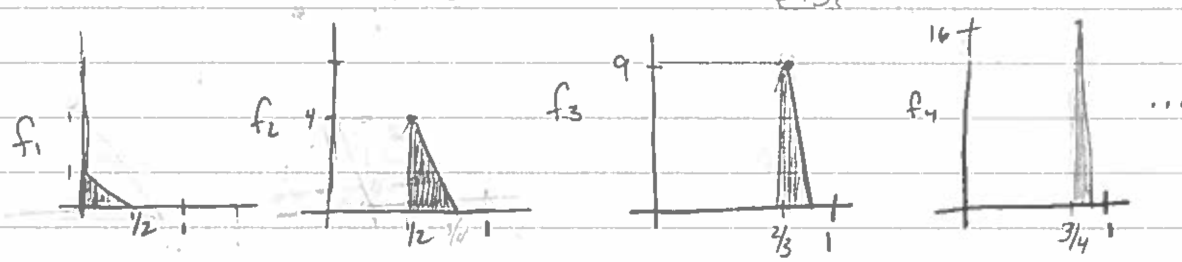
(b) Construct a sequence $\{f_n\} \subset L^1$ s.t. $f_n \to 0$ at every point but $\|f_n\| \to \infty$.

(c) Suppose $f \in L^1$, $f_n \to f$ a.e. and $\|f_n\| \to \|f\|$. Prove that $f_n \to f$ in L^1

Pf (a) Let $f_n = \chi_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]}$ $\quad n = 2^k + j$, $0 \le j \le 2^k$

$\|f_n\|_1 = \frac{1}{2^k} \to 0$ as $n \to \infty$

(b) Let $f_n(x) = \begin{cases} n^2 & x = 1 - \frac{1}{n} \\ \text{decreasing to } 0 & x \in \left[1 - \frac{1}{n}, 1 - \frac{1}{n} + \frac{1}{2n}\right] \\ 0 & \text{else} \end{cases}$



$\|f_n\|_1 = \underbrace{\frac{1}{2n}}_{\text{base}} \underbrace{(n^2)}_{\text{height}} \frac{1}{2} = \frac{n}{4} \to \infty \qquad f_n \xrightarrow{a.e.} 0$

(c) Let $g_n = f_n - f$. Then $g_n \to 0$ a.e.

$|g_n| \le |f_n| + |f| \to 2|f|$ so $|f_n| + |f|$ is our dominated sequence. So

$\int g_n \to \int 0 = 0$

$\Rightarrow \int (f_n - f) \to 0$ so $f_n \xrightarrow{L^1} f$

4. Let $1 < p < \infty$ and let $f \,\&\, g$ be Lebesgue measurable functions on the half-line $[0, \infty)$

(a) Show how to use Fubini and $\int_0^\infty \frac{f(y)}{x+y}\,dy = \int_0^\infty \frac{f(xy)}{1+y}\,dy$ x2 to $\int_0^\infty \int_0^\infty \frac{f(y)}{x+y}\,dy\, g(x)\,dx \le C_p \|f\|_p \|g\|_{p'}$ where p' is the dual exponent to p

(b). Can the Fubini thm be used to get the same type of result when $x+y$ is replaced by $x-y$ in part (a)? why or why not.

Pf (a) $\frac{f(y)g(x)}{x+y} \ge 0$ so can use Fubini

$$\int_0^\infty \int_0^\infty \frac{f(y)}{x+y}\,dy\, g(x)\,dx = \int_0^\infty \int_0^\infty \frac{f(y)}{x+y} g(x)\,dx\,dy \quad \text{by Fubini}$$

$$= \int_0^\infty \int_0^\infty \frac{f(xy)}{1+y} g(x)\,dx\,dy$$

$$= \int_0^\infty \frac{1}{1+y} \int_0^\infty f(xy) g(x)\,dx\,dy$$

$$\le \int_0^\infty \frac{1}{1+y} \int_0^\infty |f(xy)|^p \left(\int_0^\infty |g(x)|^q dx\,dy\right)^{1/q} \quad u = xy$$

$$\le \int_0^\infty \frac{1}{1+y} \left(\int_0^\infty y^{-1} |f(u)|^p\right)^{1/p} \|g\|_q\,dy \quad du = y\,dx$$

$$= \int_0^\infty y^{-1/p} \frac{1}{1+y} \left(\int_0^\infty |f(u)|^p\right)^{1/p} \|g\|_q\,dy$$

$$= \|f\|_p \|g\|_q \int_0^\infty y^{-1/p} \frac{1}{1+y}\,dy$$

$$\le \|f\|_p \|g\|_q \int_0^\infty \frac{1}{y^{1+1/p}}\,dy$$

$$\le \|f\|_p \|g\|_q C_p \quad \text{where } C_p = \int_0^\infty \frac{1}{y^{1+1/p}} < \infty$$

since $1 + 1/p > 1$

(b) Fubini can no longer be used since $\frac{f(y)g(x)}{x-y}$ may not be positive and it also might not be in L^1 since $x-y$ may put a 0 in denominator so area area might shoot off to ∞.

$\square$

FALL 2005

Measure Theory Part

1. Let $\{r_n\}_{n=1}^{\infty}$ be the rationals, $f(x) = x^{-1/2}$ for $0 < x < 1$ and 0 otherwise, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Is $f(x)$ measurable? Why? Is $g(x)$ measurable? Why? What is the set of points of discontinuity of g? Is g integrable? Why? Show that g is not in L^2 on any interval.

2. Let μ be Lebesgue measure on the borel sets of the real line, and define $\nu(E)$ to be 1 if $0 \in E$ and 0 if $0 \notin E$ for all borel sets E. Is ν a measure? σ finite? Compute $\frac{d\nu}{d\mu}$.

3. Define L^p (Lebesgue measure). Is $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$? Why? Is $L^2(0,1) \subset L^1(0,1)$? Why?

4. Let $f_k \to f$ in L^p, $1 \le p < \infty$, $g_k \to g$ pointwise and $\|g_k\|_\infty \le M$ for all k. Prove that $f_k g_k \to fg$ in L^p.

Complex Part

1. Let f be an analytic function on the unit disk and $f(z)$ is real when z is real. Show that $\bar{f}(\bar{z}) = f(z)$.

2. Let $\{f_n\}$ be a sequence of continuous functions on the closed unit disk that are analytic in the open unit disk. Suppose $\{f_n\}$ converges uniformly on the unit **circle.** Show that $\{f_n\}$ converges uniformly on the closed unit disk.

3. Suppose that f is an analytic function on an open set containing the closed unit disk, $|f(z)| = 1$ when $|z| = 1$ and f is not a constant. Prove that the image of f contains the closed unit disk.

4. Let $\mathcal{F}$ be a family of analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

on the open unit disk such that $|a_n| \le n$ for each n. Show that $\mathcal{F}$ is normal, i.e. every sequence of functions in $\mathcal{F}$ contains a subsequence converging normally to a function in $\mathcal{F}$.

Fall 2005

Definitions & concepts.

measurable fcn

$f:(X,\mathcal{A})\to(Y,\mathcal{B})$ msble if $f^{-1}(B)\in\mathcal{A}$ $\forall B\in\mathcal{B}$.

continuous a.e $\Rightarrow f$ measurable.

sums, scalars, limits, sups, infs preserve measurability

Monotone Convergence Thm

f_n msble, $0\le f_1\le f_2\le\cdots$ $f_n:X\to\overline{\mathbb{R}}$ $f(x)=\lim f_n(x)$

$\Rightarrow \int f_n\,d\mu \nearrow \int f\,d\mu$

measure

$\mu(\phi)=0$ A_i disjoint $\Rightarrow \mu(\cup A_i)=\Sigma(\mu(A_i))$

Compute $\frac{d\nu}{d\mu}$

$\nu\perp\mu \Rightarrow \exists A$ s.t. $\nu(A)=0$ $\mu(A^c)=0$

$\Rightarrow \frac{d\nu}{d\mu}=0$

? $L^2(\mathbb{R})\not\subset L^1(\mathbb{R})$ counter $f=\frac{1}{x}\chi_{[1,\infty)}$

$L^2([0,1])\subset L^1([0,1])$ use Holders.

$f_n\xrightarrow{L^p} f$

$\int|f_n-f|^p\,d\mu\to 0$ $(\Rightarrow f_n\xrightarrow{\mu} f)$

$g_n\to g$ pt wise

$\mu(\{x\mid g_n\not\to g\})=0$

$\Rightarrow \|g\|_\infty$ bdd

F05

1. Let $\{r_n\}_{n=1}^{\infty} = \mathbb{Q}$, $f(x) = \begin{cases} x^{-1/2} & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ $\quad g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x-r_n)$

Is $f(x)$ msble? Is $g(x)$? discontinuities of g?
g integrable? Show $g \notin L^2$ on any interval.

Pf (i). f msble since continuous a.e

(ii) $g(x) = \lim_{K\to\infty} \underbrace{\sum_{n=1}^{K} 2^{-n} f(x-r_k)}$ msble.

since sum's, scalar multiplication and msble limits preserve measurability

(iii) g is discontinuous everywhere :.
Let $M>0$.
Let (a,b) be an interval. Since $\mathbb{Q}$ is dense
$\exists j$ s.t. $r_j \in (a,b)$.
$\Rightarrow 2^{j} \frac{1}{\sqrt{x-r_j}} \geq M$ for some $x \in (a,b)$
$\Rightarrow g(x) \geq 2^{j} \frac{1}{\sqrt{x-r_j}} \geq M$

$\Rightarrow g$ is unbounded on any interval (*)

(iv). $g \in L^1$.
$g = \sum_{n=1}^{\infty} f_n$ where $f_n = 2^{-n} f(x-r_n)$

$$\int f_n = \int_{-\infty}^{\infty} 2^{-n} \frac{1}{\sqrt{x-r_n}} \chi_{(0,1)}$$
$$= \int_{r_n}^{r_n+1} 2^{-n} (x-r_n)^{-1/2} \quad \text{its 0 everywhere els}$$
$$= \int_0^1 2^{-n} x^{-1/2} \quad \text{shift-change variable}$$
$$= 2^{-n}\, 2x^{1/2}\Big|_0^1 = 2^{-n+1}$$

by MCT $\int g = \sum \int f_n = \sum_{n=1}^{\infty} 2^{-n+1} = \sum_{n=0}^{\infty} 2^{-n} = 2$

(iv) $g \notin L^2(a,b)$

$$\int_a^b |g|^2 = \int_a^b g^2 = \int_a^b \left(\sum_1^\infty 2^{-n} f(x-r_n)\right)^2$$
$$\geq \int_a^b \sum_1^\infty 2^{-n} f^2(x-r_n)$$
$$= \int_{a-r_n}^{b-r_n} 2^{-n} \frac{1}{x} \chi_{(0,1)}(x)$$
$$= \int_0^{\min\{b-r_n, 1\}} 2^{-n} \frac{1}{x} dx = \infty$$

(x) Let $x_0 \in \mathbb{R}$.

Case 1 $g(x_0) = \infty$

$\forall \delta > 0$, $\exists x \in B_\delta(x_0)$ s.t. $g(x)$ is finite.

Since $g \in C'$.

$\Rightarrow$ g not cont at x_0.

Case 2 $g(x_0) = C < \infty$.

Since g is unbdd on any $B_\delta(x_0)$ we can find x s.t. $g(x) > C+1$

$\Rightarrow |g(x) - g(x_0)| > 1$

So not cont at x_0

F05

2. Let μ be $\mathcal{L}m$ on $\mathcal{B}$. $\nu(E) = \begin{cases} 1 & 0 \in E \\ 0 & 0 \notin E \end{cases}$

Is ν a measure? Is ν σ finite? Compute $\frac{d\nu}{d\mu}$.

Pf (a) $\nu(\phi) = 0$ since $0 \notin \phi$

Let $A_i \in \mathcal{B}$ disjoint.

$\mu(\bigcup_{i=1}^{\infty} A_i) = \begin{cases} 0 & 0 \notin A_i \ \forall A_i \\ 1 & 0 \in A_i \text{ for some } i \end{cases}$

$\Sigma(\mu(A_i)) = 0$ or 1 since disjoint 0 can be in at most 1 A_i $\Rightarrow \nu$ a measure

(b) $\nu(\mathbb{R}) = 0$ $\Rightarrow \sigma$ finite

(c). $\nu \perp \mu$. $\mathbb{R} = \{0\} \cup (\mathbb{R} \setminus \{0\})$ (mutually singular)

$\nu(\{0\}) = 1$ $\quad$ $\nu(\mathbb{R} \setminus \{0\}) = 0$

$\mu(\{0\}) = 0$ $\quad$ $\mu(\mathbb{R} \setminus \{0\}) \neq 0$

$\Rightarrow \frac{d\nu}{d\mu} = 0$ since $\nu \perp \mu$

$\square$

James's way

$\lim_{E_n \to \{x\}} \frac{\nu(E_n)}{\mu(E_n)} = f(x)$ $\quad$ pg 99 3.22

$\quad = 0$ a.e.

So $f' = 0$ a.e.

3. Define L^p. Is $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$ Is $L^2(0,1) \subset L^1(0,1)$?

Pf (i) $L^p(X) = \{f : X \to \mathbb{R} \mid \int_X |f|^p < \infty\}$

(ii) No!

Let $f(x) = \frac{1}{x} \chi_{[1,\infty)}$

$$\int f = \int \frac{1}{x}\chi_{[1,\infty)} = \int_1^\infty \frac{1}{x} = \log|x|\Big|_1^\infty = \infty$$

$$\int |f(x)|^2 = \int_1^\infty \frac{1}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1$$

So $f \in L^2(\mathbb{R})$ but $f \notin L^1(\mathbb{R})$.

(iii) Yes!

Let $f \in L^2(0,1)$

Then $\int f\,dm = \int f \cdot 1\,dm \le (\int f^2)^{1/2} (\int 1^2)^{1/2} = \|f\|_2 < \infty$

$\Rightarrow f \in L^1$.

4. Let $f_k \to f$ in L^p, $1 \le p < \infty$. $g_k \to g$ ptwise $\|g_k\|_\infty \le M$ $\forall k$.
Prove $f_k g_k \to fg$ in L^p

Pf $g_k \to g$ a.e $\Rightarrow \|g\|_\infty \le M$

So $|f_k g_k| \le |f_k| M$

Consider f_{n_k} a subseq. of f_n.
$\exists$ Subseq $f_{n_{k_\ell}} \to f$ a.e — since $f_{n_k} \xrightarrow{L^p} f \Rightarrow f_{n_k} \xrightarrow{\mu} f$
$\Rightarrow f_{n_{k_\ell}}^p \to f^p$ a.e
gives a.e convergence $\Rightarrow |f_{n_{k_\ell}} g_{n_{k_\ell}}|^p \to f^p g^p$ a.e — since $g_{n_{k_\ell}}^p \to g^p$
gives dominated $\Rightarrow |f_{n_{k_\ell}} g_{n_{k_\ell}}|^p \le M |f_{n_{k_\ell}}|^p \in L^1$ — since $\|g\|_\infty \le M \Rightarrow g^p \le M$ $\forall p$
$\Rightarrow \int |f_{n_{k_\ell}} g_{n_{k_\ell}}|^p \to \int |fg|^p$ — by GDCT where $M|f_{n_{k_\ell}}|^p$ is dominating converging sequence.

So every subsequence of $f_k g_k$ has a convergent subsequence thus $f_k g_k \xrightarrow{L^p} fg$

□

Analysis Exam 29 January 2005

Measure Theory Part

1. Let $f(x)$ be the standard Cantor function. Define $g(x) = f(x)+x$. Show that g is continuous, increasing, and 1-1 from $[0,1]$ onto $[0,2]$. Use g to show that the image of a Lebesgue measurable set under a continuous map may not be measurable.

Talk through this one

2. Consider the real line with Lebesgue measure. A sequence of measurable real valued functions f_n converges in measure to the measurable function f. In addition $|f_n| \leq g$ for all n where g is an integrable function. Show that

$$\lim_{n\to\infty} \int |f_n - f| = 0$$

3. Suppose that $1 < p < q < r < \infty$ and that $f \in L^p \cap L^r$. Estimate the L^q norm of f in terms of a product involving the L^p and L^r norms. Something like $\|f\|_q \leq \|f\|_r^\alpha \|f\|_p^{1-\alpha}$ where $0 < \alpha < 1$.

4. Let f be measurable on the interval $[0,1]$ (Lebesgue measure on the real line). If the function $g(x,y) = x(f^2(x) - f^4(y))$ is integrable on the unit square in $\mathbb{R}^2$ show that f is integrable on $[0,1]$.

January 2005
Definition & Concepts.

Cantor Function.

$f_n \xrightarrow{\mu} f$
$\mu(\{x \mid |f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$

Dominated convergence Thm.
f_n msble, $f_n \to f$ pointwise, $\sup |f_n| \le g$ $g \in L^1(\mu)$
$\Rightarrow f \in L^1(\mu)$ and $\int |f_n - f| \, d\mu$.

L^q norm
$f \in L^q \Rightarrow \|f\|_q = (\int |f|^q)^{1/q} < \infty$

Fubini.
μ, ν σ-finite $f : X \times Y \to \overline{\mathbb{R}}$ $\mathcal{A} \times \mathcal{B}$ msble.
s.t. $f \geq 0$ or $\int_{X \times Y} |f| \, d(\mu \times \nu) < \infty$
$\Rightarrow$ can interchange order of integration

$f^n \in L^1 \Rightarrow f \in L^n$?

a set of positive Lebesgue measure always contains a non lebesgue measurable set.

J05

1. $f(x)=$ Cantor Function. $g(x)=f(x)+x$
Show g is cont, increasing and 1-1 from $[0,1]\to[0,2]$
Use g to show image of Lebesgue msble set under cont. map may not be msble.

Pf $\cdot f$ cont, x cont $\Rightarrow f+x=g$ cont
- f strictly increasing $\Rightarrow g$ strictly increasing.
- g strictly increasing $\Rightarrow g$ 1-1
- $g(0)=f(0)+0=0$ $g(1)=f(1)+1=2$ $\Rightarrow g$ onto

Note g^{-1} is also continuous. since g is both bijective and continuous.

Let C be the Cantor set
Then $g(C)=1$
$\Rightarrow \exists\, A\notin\mathcal{L}$ s.t. $A\subset g(C)$ since a set of posit measure always contains a non Lebesgue measurable set
Let $B=g^{-1}(A)$
I claim $B\in\mathcal{L}$ but $B\notin\mathcal{B}$ $\Rightarrow \mathcal{L}\not\subset\mathcal{B}$
$g^{-1}(A)\subset g^{-1}(g(C))$ and $m(C)=0$.
$\Rightarrow g^{-1}(A)$ msble b/c m is a complete measure.

If $B\in\mathcal{B}$ then $(g^{-1})^{-1}(B)\in\mathcal{B}$ since g^{-1} is Continuous

$\mathcal{B}\ni(g^{-1})^{-1}(B)=g(B)=g(g^{-1}(A))=A\notin\mathcal{L}$
but $\mathcal{B}\subset\mathcal{L}$ ↯

2 Consider the real line w/ Lebesgue measure $f_n \xrightarrow{\mu} f$, $|f_n| \leq g$ $\forall n$. g integrable. Show $\lim_{n\to\infty} \int |f_n - f| = 0$

Pf Note: $\int |f_n - f| \to 0 \iff f_n \xrightarrow{L^1} f$

Let $f_{n_k} - f$ be a subsequence of $f_n - f$

$f_{n_k} - f$ has subsequence $f_{n_{k_\ell}} - f \to 0$

and $|f_{n_{k_\ell}} - f| \leq 2g \in L^1$

by DCT $\int |f_{n_{k_\ell}} - f| \to \int 0 = 0$

$\Rightarrow \int |f_n - f| \to 0$

$\square$

3. Suppose $1 < p < q < r < \infty$ and $f \in L^p \cap L^r$. Estimate L^q norm of f in terms of a product involving L^p and L^r norms.

Pf | claim $\|f\|_q \le \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$ $\quad \lambda = \dfrac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r}}$

or choose λ s.t. $\frac{p}{\lambda q} + \frac{r}{(1-\lambda)q} = 1$

$$\|f\|_q^q = \int |f|^q$$
$$= \int |f|^{\lambda q} |f|^{(1-\lambda)q}$$
$$\le \||f|^{\lambda q}\|_{\frac{p}{\lambda q}} \, \||f|^{(1-\lambda)q}\|_{\frac{r}{(1-\lambda)q}} \quad \text{by Holders } \tfrac{p}{\lambda q} + \tfrac{r}{(1-\lambda)q}$$
$$= \left[\int |f|^p\right]^{\frac{\lambda q}{p}} \left[\int |f|^r\right]^{\frac{(1-\lambda)q}{r}}$$

$\Rightarrow \|f\|_q \le \|f\|_p^{\lambda} \|f\|_r^{(1-\lambda)}$ as desired.

Note $\dfrac{\lambda q}{p} + \dfrac{(1-\lambda)q}{r} = \dfrac{\lambda q r + pq - pq\lambda}{pr}$

$$= \frac{\lambda(qr - pq) + pq}{pr}$$
$$= \frac{\frac{r-q}{rq} \cdot \frac{pr}{r-p}(qr - pq) + pq}{pr}$$
$$= \frac{\frac{rp - qp}{qr - pq}(qr - pq) + pq}{pr}$$
$$= \frac{rp - qp + pq}{rp}$$
$$= 1$$

✓

□

4 Let f be msble on $[0,1]$. If $g(x,y) = x(f^2(x) - f^4(y))$ in integrable on $[0,1]\times[0,1]$ show f is integrable on $[0,1]$

Pf $\int_{[0,1]^2} g(x,y) = \iint x(f^2(x) - f^4(y))$ by Fubini $g \in L^1$

$= \iint x(f(x) - f^2(y))(f(x) + f^2(y))$

$\iint g(x,y)\,dy\,dx < \infty \Rightarrow \iint x(f^2(x) - f^4(y))\,dx\,dy < \infty$ a.e x

$\Rightarrow \int_{[0,1]} |f^2(x) - f^4(y)|\,dy < \infty$

$\Rightarrow \int_{[0,1]} f^4(y)\,dy < \infty$

$\Rightarrow f \in L^4([0,1])$

$\Rightarrow f \in L^1([0,1])$ since $m([0,1]) < \infty$

$\square$

24 October 2004

Measure Theory Part

1. Define Lebesgue Outer Measure $|\cdot|_e$ on $\mathbb{R}$. Show that there exist disjoint $E_k \subset \mathbb{R}$ for $k = 1, 2, \ldots$ so that

$$|\cup_{k=1}^{\infty} E_k|_e < \sum_{k=1}^{\infty} |E_k|_e$$

2. Define convergence in measure. Construct a sequence of functions on $[0,1] \subset \mathbb{R}$ that converges in measure (Lebesgue measure) but does not converge point-wise for any point of $[0,1]$.

3. Define what it means for a set function to be absolutely continuous with respect to a measure. Let $f \in L(\mathbb{R}, dx)$ where dx is Lebesgue measure and set

$$\phi(E) = \int_E f\,dx$$

Prove that ϕ is absolutely continuous with respect to dx.

4. Let $f_k \to f$ point-wise a.e. with $|f_k| \le g_k \in L^1$ and $g_k \to g$ in L^1 show that $f_k \to f$ in L^1.

October 2004

Defintions and Concepts.

Lebesgue Outer Measure

$\inf \{ \sum_i^\infty \ell(\mathcal{C}_i) \mid E \subset \bigcup_i^\infty \mathcal{C}_i \quad \mathcal{C}_i \in \mathcal{C} \}$

$f_n \xrightarrow{\mu} f$

$\mu(\{x \mid |f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$

$f_n \xrightarrow{a.e.} f$

$\mu(\{x \mid f_n \not\to f\}) = 0$

absolutely continuous

$\nu \ll \mu \Rightarrow \mu(E) = 0 \Rightarrow \nu(E) = 0.$

Building steps:

- $f = \chi_E$
- f a simple fcn
- $f \in L^+$ $\exists$ simple fcns $\nearrow$ to f.
- $f \in L \Rightarrow f = f_+ - f_-$ for $f_+, f_- \in L^+$

Fatou

$f_n \geq 0$ msble

$\Rightarrow \int \underline{\lim} f_n \, d\mu \leq \underline{\lim} \int f_n \, d\mu$

Dominated Convergence Thm

f_n msble, $f_n \to f$ ptwise. $\sup |f_n| \leq g$ $g \in L^1(\mu)$

$\Rightarrow \int |f_n - f| \, d\mu \to 0$

Oct 04

1. Define Lebesgue Outer Measure $|\cdot|_e$ on $\mathbb{R}$.
Show $\exists$ disjoint $E_k \subset \mathbb{R}$ for $k \in \mathbb{N}$ s.t.
$$|\bigcup_{k=1}^{\infty} E_k|_e < \sum_{k=1}^{\infty} |E_k|_e$$

Pf Consider the equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
Let $N = \{$exactly one representative from each class$\}$
$N_r = \{x + r \mid x \in N \cap [0, 1-r)\} \cup \{x - 1 + r \mid x \in N \cap [1-r, 1)\}$
for $r \in [0,1] \cap \mathbb{Q}$

if leave $(0,1)$ shift back.

Claim N_r disjoint
$\forall x \in [0,1)$, $\exists r$ s.t. $x \in N_r$.
$\Rightarrow m_e(\bigcup N_r) \leq \sum m_e(N_r) = \sum m_e(N) = \infty$
and $m_e(\bigcup N_r) = m([0,1)) = 1$

has to be bigger than 0 since N not lebesgue msble.

So $m_e(N_r) \leq m_e(N)$

□

2. Define convergence in measure. Construct a sequence of fcns on $[0,1] \subset \mathbb{R}$ that converge in Lebesgue measure but does not converge pointwise $\forall$ pt of $[0,1]$

Pf

Def $f_n \xrightarrow{\mu} f \Leftrightarrow \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t. $\mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$
$\forall n \geq n_0$.

$f_{11} = \chi_{[0,1]}$

$f_{21} = \chi_{[0,\frac{1}{2})}$ $\quad f_{22} = \chi_{[1/2, 1]}$

$f_{31} = \chi_{[0,\frac{1}{3}]}$ $\quad f_{32} = \chi_{[1/3, 2/3]}$ $\quad f_{33} = \chi_{[2/3, 1]}$

$f_{nk} = \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$ for $k = 1, 2, \dots, n$

$\mu \{|f_{nk}(x) - f(x)| > \varepsilon\} = \frac{1}{n} \longrightarrow 0.$

but $f_{nk}(x) = 1$ for infinitely many combinations of n, k.

3. Define: a set function absolutely continuous wrt μ.
Let $f \in L(\mathbb{R}, dx)$ (dx Lebesgue measure) and set $\phi(E) = \int_E f dx$
Prove ϕ is abs. cont. wrt dx

Pf $\nu \ll m \Leftrightarrow \forall E$ in a σ-algebra $m(E)=0 \Rightarrow \nu(E)=0$

Let E be a Lebesgue null set.
- If $f = \chi_F$ for some F
  then $\phi_E = \int_E \chi_F dm = \int \chi_{F \cap E} = m(E \cap F) \leq m(E) = 0$
- If f is a simple function
  then holds by linearity.
- If $f \in L^+$ then $\exists f_n$ simple $\nearrow f$
  $\Rightarrow \int_E f dm = \int_E \lim f_n dm \overset{MCT}{=} \lim \int_E f_n dm = 0$
- If $f \in L \Rightarrow f = f_+ - f_-$ for $f_+, f_- \in L^+$
  $\Rightarrow \int_E f dm = \int_E f_+ - \int_E f_- = 0 - 0 = 0.$

So ϕ is absolutely continuous wrt dx or m

or

Let A_n be sets s.t. $m(A_n) \to 0$
Then let $f_n = f\chi_{A_n}$.
then $f_n \to 0$ a.e. and $|f_n| \leq |f| \in L'$

So by DCT $\lim \int f_n = \int \lim f_n$
$\Rightarrow \lim \int_{A_n} f = \int 0$
$\Rightarrow \lim \int_{A_n} f = 0$

Just asking for proof of GDCT

4. Let $f_k \to f$ pointwise a.e. w/ $|f_k| \le g_k \in L^1$ and $g_k \to g$ in L^1. Show $f_k \to f$ in L^1

Pf Note $|f_k| \le g_k \Rightarrow -g_k \le f_k \le g_k$ and $g_k \ge 0$
So $g_k + f_k \ge 0$ and $g_k - f_k \ge 0$ a.e.

Then
$$\int g + \int f = \int(\underline{\lim}\, g_n + \underline{\lim}\, f_n)$$
$$= \int \underline{\lim}(g_n + f_n)$$
$$\le \underline{\lim}\left(\int g_n + \int f_n\right) \text{ by Fatou + linearity}$$
$$= \int g + \underline{\lim} \int f_n$$

and
$$\int g - \int f = \int(\underline{\lim}\, g_n - \underline{\lim}\, f_n)$$
$$= \underline{\lim}\left(\int g_n - \int f_n\right)$$
$$= \int g + \underline{\lim} - \int f_n$$
$$= \int g - \overline{\lim} \int f_n$$

$\Rightarrow \overline{\lim} \int f_n \le \int f \le \underline{\lim} \int f_n$
$\Rightarrow f_n \to f$ in L^1

Note. If $g_n \to g$ in L^1
Consider subsequence f_{n_k}, $g_{n_k} \to g$ in L^1
$\exists$ subseq $g_{n_{k_l}} \to g$ a.e.

Now use above argument.
If $a_n \to a$, b_n seq then $\underline{\lim}\, a_n + b_n = a + \underline{\lim}\, b_n$

□