## * Some word problems and strategies for them.

Jan 2015/P3.

$f, g : \mathbb{R} \to \mathbb{R}$.
$f$ is differentiable
$\forall x, h \in \mathbb{R}$, $f(x+h) - f(x-h) = 2h\, g(x)$.

} Prove that $f$ is a polynomial of degree at most 2.

* To prove that $f$ is a polynomial of degree at most 2, we want to prove that $f'' =$ constant.

* See sth about $(x-y)$ or $\underline{f(x) - f(y)}$ may not state → think about MVT ex

$p x^p (x-y) < x^p - y^p < p y^p (x-y)$ for $0 < x < y < \dots$, $p > 1$, $p \in \mathbb{R}$

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* every uncountable set of real number has a limit point (Fall 2001 > P1, Aug 1991, P3),

Some useful results that are used in this problems:

- every infinite + bounded subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.
- If $A \subseteq B$, $B$ has no limit point } $\Rightarrow$ $A$ has no limit point.
- If we let $A_n = A \cap [-n, n]$ $\Rightarrow$ $A_n$ is bounded.

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* Aug 2005, P4 > Stirling's formula:

$\ln(n!) = n \log n - n$.

---

* Some relations between $f$ continuous vs $f'$ vs one-to-one property.

- Aug 2008 > P2 > : $f$ is continuous on $E$, $f$ is one-to-one } if $f$ is differentiable on $E$ then $f' > 0$ on $E$ or $f' < 0$ on $E$ } $f$ is "strictly" monotonic on $E$.

- Aug 2015, P6.

Suppose $f$ is continuous on $E$
$f'(x_0) > 0$ for some $x_0 \in E$
$f'(x_1) < 0$ for some $x_2 \in E$
} $\Rightarrow$ $f$ is not one-to-one on $E$ ( $\wedge$

MA 6021 HW 4.3, Aug 2009 >

* $\mathbb{N}$ is closed in $\mathbb{R}$

because $\mathbb{R} \setminus \mathbb{N} = (-\infty, 0) \cup (0,1) \cup (1,2) \dots \cup \dots$ is open in $\mathbb{R}$ (any union of open is open).

$\mathbb{N} = \{n,\ n = 1, 2, 3, \dots\}$ is closed in $\mathbb{R}$.
$H = \{n + \frac{1}{n},\ n = 1, 2, 3, \dots\}$ is closed in $\mathbb{R}$
} $\mathrm{dist}(\mathbb{N}, H) = 0$ but $\mathbb{N} \cap F = \phi$

Aug 2016 7/4 7.

* Finite intersection of many dense subsets may not be dense.

finitely ~~many~~ intersection of many open + dense subset is dense.

countable intersection of many open + dense subsets is nonempty

# Chapter 1

* **Def**: A relation $\alpha$ is an order in some set $X$ if
$\forall x, y \in X$, only one of the following hold $x \alpha y$ or $y \alpha x$ or $y = x$.

(A way to define a relation:
create an injective function : $f : X \longrightarrow \mathbb{R}$ define $x \alpha y$ if $f(x) < f(y)$)

Example $f : \mathbb{Z} \longrightarrow \mathbb{R}$

$$\begin{cases} f(n) = \frac{1}{n} \\ f(0) = 0 \end{cases}$$

and so relation
$x \alpha y$ iff $\frac{1}{x} < \frac{1}{y}$

* **Def**:

- $s$ is sup $E \iff \begin{cases} \forall x \in E, x \leq s \\ \text{If } t < s, t \text{ is not an upper bound} \end{cases}$
- $\phi$ has a lot of upper bounds $\phi$ has no greatest upper bound.
  has no lower bound.

* $E \neq \phi$,

**1.10 Def** An ordered set $S$ is said to have the least-upper-bound property if

$$\left.\begin{matrix} \forall E \neq \phi, E \subseteq S \\ E \text{ is bounded above} \end{matrix}\right\} \text{ then } \begin{cases} \exists \sup E \\ \sup E \in S \end{cases}$$

- $(\mathbb{R}, <)$ is an ordered set with least upper bound property.
- $(\mathbb{Q}, <)$ is ~~not an ord.~~ an ordered field set that does not have least upper bound property.
  $A = \{q^2, q < 2\}$ $\exists \sup A$ but $\sup A \notin \mathbb{Q}$.

**1.11** Every ordered set has least upper bound property also have great lower bound property

(Suppose $S$ is an ordered set, has least upper bound property.

$$\left.\begin{matrix} \text{Let } B \neq \phi, B \subseteq S \\ B \text{ is bounded below} \end{matrix}\right\} \begin{matrix} \text{Then let } L = \text{all of lower bound of } B \\ \text{then } \exists \alpha = \sup L, \alpha \in S \end{matrix}$$

and $\alpha = \inf B$, $\alpha \in S$ $\Rightarrow$ $S$ has greatest lower bound p

- $(\mathbb{R}, <)$ is an ordered field set with $\begin{cases} \text{greatest } ~~\text{lower}~~ \text{ bound property} \\ \text{least upper bound property.} \end{cases}$

1.20

Achimède If $\left.\begin{array}{l} x \in \mathbb{R},\ x > 0 \\ y \in \mathbb{R} \end{array}\right\} \Rightarrow \exists\, n, \begin{cases} n \in \mathbb{N} \\ n > 0 \end{cases}$ such that $nx > y$

b) If $\left.\begin{array}{l} x \in \mathbb{R} \\ y \in \mathbb{R} \\ x < y \end{array}\right\}$ then $\exists\, q \in \mathbb{Q},\ x < q < y$

1.21: For every $\left.\begin{array}{l} \text{real } x > 0 \\ \text{integer } n > 0 \end{array}\right\} \Rightarrow \exists!\, y \in \mathbb{R},\ y^n = x$ (write $y = \sqrt[n]{x}$) or $x^{1/n}$).

- If $y^n = z^n = x$, then $y = z$.

* Complex number :

$(a,b)$ $\quad z = \underset{\substack{\downarrow \\ \text{Re}\,z}}{a} + \underset{\substack{\downarrow \\ \text{Im}(z)=b}}{b}i$ $\quad$ or $\quad a+bi \Leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix}$

$\Leftrightarrow (a+bi) + (c+di) = (a+c) + (b+d)i$

- $z = a+bi$

$|z| = \sqrt{a^2+b^2} = \det\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

- $\bar{z} = a - bi$ : the conjugate of $z$

- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{pmatrix}$

$\Leftrightarrow (a+bi)\cdot(c+di) = (ac-bd) + (ad+bc)i$

(multiplication is commutative)

* If $z$ and $w$ are complex, then

$\overline{z+w} = \bar{z} + \bar{w}$ $\quad | \quad z + \bar{z} = 2\,\text{Re}(z)$

$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ $\quad | \quad z - \bar{z} = 2\,\textcircled{i}\,\text{Im}\,z$

$z\bar{z} = |z|^2$ $\qquad |z| = \sqrt{z \cdot \bar{z}}$

* 1.33 Let $z$ and $w$ are complex number, then

- $|z| \geq 0,\ \forall z$ $\qquad |z| = 0 \Leftrightarrow z = 0$
- $|\bar{z}| = |z| = |-z|$
- $|zw| = |z| \cdot |w|$ $\qquad$ • $|z+w| \leq |z| + |w|$ $\qquad$ • $|z-w| \geq \big||z| - |w|\big|$

*1.35 Cauchy Suatch inequality.

If $\left.\begin{matrix} z_1, \dots, z_n \\ w_1, \dots, w_n \end{matrix}\right\}$ are complex number $\quad$ Then $\left|\sum_{i=1}^{n} z_i w_i\right| \leq \sqrt{\sum_{i=1}^{n} z_i^2 \ \sum_{i=1}^{n} w_i^2}$

$\langle z, w \rangle \leq \|z\| \, \|w\|$

1.11 Theorem

Prove that every ~~set~~ ordered set $S$ has least upper bound property also has great lower bound property.

(Let $S$ is an ordered set has least upper bound property.
Let $B \neq \phi$, $B \subseteq S$
$B$ is bounded below } Prove that $\exists \inf B$ and $\inf B \in S$.

* Let $\perp = \{$set of all lower bounds of $B\}$, then we have

- $L \neq \phi$ (because $B$ is bounded below and $B \neq \phi$)
- $L \subseteq S$
- By assumption, $S$ has least upper bound property.
- $L$ is bounded above because $\forall x \in \perp$, $x \leq y$, $\forall y \in B$

} $\Rightarrow \exists \sup \perp$ and $\sup \perp \in S$.

Put $\alpha = \sup \perp$, this means we have $\begin{cases} \forall x \in \perp, x \leq \alpha. \\ \forall \varepsilon > 0, \exists x_0 \in \perp \text{ such that } \alpha - \varepsilon < x_0. \end{cases}$ (1)

* Now we will prove that $\alpha = \inf B$ (we already know $\alpha \in S$ from above).

$\Leftrightarrow$ We NTP $\begin{cases} \forall y \in B, y \geq \alpha \\ \forall \varepsilon > 0, \exists y_0 \in B, \alpha + \varepsilon > y_0. \end{cases}$

- Now we prove $\forall y \in B$, $y \geq \alpha$.

Assume that $\exists y_0 \in B$, $y_0 < \alpha \Rightarrow \alpha - y_0 > 0$, Put $\varepsilon = \alpha - y_0$

Then by (1), $\exists x_0 \in \perp$ s.t $\alpha - \varepsilon < x_0 \Leftrightarrow \alpha - (\alpha - y_0) < x_0$
$\Leftrightarrow y_0 < x_0$ (contradict with $\perp$ is a set of lower bound of $B$)

$\Rightarrow \forall y \in B$, $y \geq \alpha$ $\square$

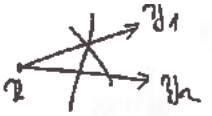
- Now we will prove that $\forall \varepsilon > 0$, $\exists y_0 \in B$, $\alpha + \varepsilon > y_0$.

Assume $\exists \varepsilon > 0$, $\forall y \in B$, $\alpha + \varepsilon < y_0$,

This mean $(\alpha + \varepsilon)$ is an lower bound of $B$

this means $(\alpha + \varepsilon) \in \perp$ and so $\alpha + \underset{>0}{\underbrace{\varepsilon}} \leq \alpha$ (contradiction) $\square$.

# Finite, countable, and uncountable sets.



* A and B are given

$f: A \longrightarrow B$ is a "well defined" function $\overset{\text{def}}{\Leftrightarrow}$ $\forall x \in A$, $\exists$ at much $y \in B$ $y = f(x)$.

A: domain of f    B: codomain

- $f(A) \subseteq B$.

f(A): range of f

- $f(A) = B$ f maps A onto B

* $f: A \longrightarrow B$

$E \subset A$    f(E): image of E under f

* $A \sim B$ : A and B have the same cardinality (equivalent)
→ equivalent relation

iff (def) $\exists f: A \longrightarrow B$ bijective

f: bijective ⇔ { f injective, f onto

$\Leftrightarrow \begin{cases} f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \\ f(A) = B \end{cases}$

- $A \sim A$
- $A \sim B$ ~~then~~ $B \sim A$
- $A \sim B$, $B \sim C \Rightarrow A \sim C$

(this is because $f: A \longrightarrow B$ injective/onto/bijective then $g \circ f$ injective/onto/bijective
also have $g: B \longrightarrow C$

24 Def:

countable ⇒ infinite

infinite, ⊂ countable ⇒ countable

a) A is finite $\overset{\text{def}}{\Leftrightarrow}$ $\left[\begin{array}{l} A = \phi \\ A \sim \{1, \dots, n\} \text{ for some } n \end{array}\right.$

b) A is infinite $\overset{\text{def}}{\Leftrightarrow}$ A is not finite $\Leftrightarrow \begin{cases} A \neq \phi \\ \nexists f: A \longrightarrow \{1, \dots, n\} \text{ bijective} \end{cases}$

→ then A can be countable ($\mathbb{Z} \sim \mathbb{R}$ (subset of Z)).

c) A is countable $\overset{\text{def}}{\Leftrightarrow}$ $A \sim \mathbb{N}$

d) A is uncountable $\Leftrightarrow \begin{cases} A \text{ is infinite} \\ A \text{ is not countable} \end{cases} \Leftrightarrow \begin{cases} A \neq \phi \\ A \nsim \{1, \dots, n\} \\ A \nsim \mathbb{N} \end{cases}$

e) A is at most countable $\Leftrightarrow \left[\begin{array}{l} A \text{ is finite} \\ A \text{ is countable} \end{array}\right.$

$\mathbb{Q}$ is countable

* $\mathbb{Z}$ is countable.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{N}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
| $\mathbb{Z}$ | 0 | 1 | -1 | 2 | -2 | 3 | -3 | 4 | -4 | |

$\mathbb{N}$ is countable $f(n) = n$

* A finite, B finite, $A \subsetneq B$ } $A \nsim B$.

* A finite, B finite, ~~c A ∼ B~~ card B = card A, $f: A \to B$ } f is injective ⇔ f is onto ⇔ f is bijective.

* EX: A "distinct" sequence $x_1, x_2, \dots, x_n$ is countable.

$*$ $A \subset B$
$B$ is countable $\Big\} \Rightarrow$ $A$ is finite or countable (every subset of a countable set is either finite or countable).

$A \subset B$
$A$ is infinite
$B$ is countable $\Big\} \Rightarrow$ $A$ is countable.

$*$ 2.7 Def: By a sequence: a function: $f : \mathbb{N} \longrightarrow A$, $n \longmapsto x_n$

$f = \{x_n\} = \{x_n, n \in \mathbb{N}\}$
$\downarrow$
terms of the sequence

If $x_n \in A$, $\forall n \in \mathbb{N}$
we say: $\{x_n\}$: a sequence in $A$
or a sequence of element in $A$

- Note that terms $x_1, \ldots, x_n$ of a sequence need not be distinct

$*$ If $A$ countable, $A \sim \mathbb{N}$, then $A$ can be regarded as a range of a sequence of "distinct" terms.

$A = \{a_i, i = 1, 2, \ldots, a_i \neq a_j \text{ if } i \neq j\}$ (A can be arranged in a sequence

$*$ 2.11: Let $\{E_n\}$, $n = 1, 2, \ldots$ be a sequence of countable set $\Rightarrow \bigcup_{n=1}^{\infty} E_n$ is countable.

---

$*$ Fall 2001/1: every uncountable set of real line has a limit point.

---

$*$ A set that can be arranged in a sequence is countable

$*$ $\mathbb{N}$ is infinite.

$*$ $\mathbb{Q}$ is infinite
$\mathbb{Q}$ is countable

$*$ countable $\Rightarrow$ minimum infinite
infinite $\not\Rightarrow$ countable
infinite $\longrightarrow$ countable
$\searrow$ uncountable.

## * Metric space

+ Example:

Let $X = \mathbb{R}^2$

$\vec{x} = (x_1, x_2)$

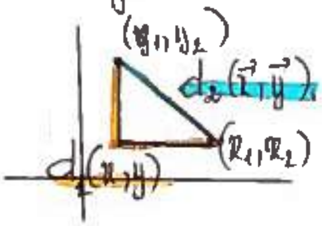
$\vec{y} = (y_1, y_2)$

$d_1(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$ (Taxicab metric)

$d_2(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$d_\infty(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

* Neighborhood : $N_r(x) = \{y, d(x, y) < r\}$.



*2.18 Def: $(X, d)$: metric space $\quad E \subseteq X$

$p \in X$ ($p$ is a point of $X$). $\quad$ a set.

a) Neighborhood

- A ⓡ neighborhood of $p$ : $\quad N_r(p) = \{x \in X,\ d(x,p) < \overset{\text{radius}}{r}\}$
- $M$ is a neighborhood of $p \Leftrightarrow \begin{cases} p \in M \\ \exists\, B(p, \varepsilon) \subseteq M \end{cases}$

b) Limit points:

$p \in X$ is a limit point of $E \Leftrightarrow$ every neighborhood of $p$ contain a point $x \in E$, $x \neq p$

$\forall N_r(p),\ (N_r(p) \setminus \{p\}) \cap E \neq \emptyset$.

$\Leftrightarrow \forall r > 0,\ \exists\, x \in E,\ x \neq p,\ d(x,p) < r$

- sup E, inf E are limits point of E

* Isolated point

p is an isolated point of $E \Leftrightarrow \begin{cases} p \in E \\ p \text{ is } \underline{\text{not}} \text{ a limit point of } E \end{cases}$.

$\Leftrightarrow \nexists$ a neighborhood of p that does not contain $\underbrace{\text{any point of } E}_{(\text{different } p)}$

$\exists N_r(p),\ N_r(p) \cap E = \{p\}$.

$\Leftrightarrow \exists N_r(p),\ (N_r(p) \setminus \{p\}) \subseteq (X \setminus E)$

- $x$ is an isolated point of $E \Rightarrow \exists\, (x_n) \subseteq E,\ x_n = x,\ \forall n,\ x_n \longrightarrow x$

* Interior point.

- p is an interior point of $E \Leftrightarrow \exists N_r(p),\ N_r(p) \subseteq E$.
- $E$ is a neighborhood of $p \Leftrightarrow \exists N_r(p) \subseteq E \Leftrightarrow$ p is an interior point of E.

- p is interior point of E $\Longrightarrow$ p is a limit point of E.
- If p is a limit point of $E^c \Rightarrow$ p is not an interior point of E.

- p is not an interior point $\Leftrightarrow \forall N_r(p),\ N_r(p) \nsubseteq E$

$\Leftrightarrow \forall N_r(p),\ N_r(p) \cap (X \setminus E) \neq \emptyset$

* $\text{dist}(A, B) = \inf\{d(a,b),\ a \in A,\ b \in B\}$.

$E \text{ is closed} \Leftrightarrow E = \bar{E}$

$E \text{ is open} \Leftrightarrow E = E^{o}$

$E \text{ is perfect} \Leftrightarrow E = E'$

*** Open set**

$E \subseteq X$ is open $\Leftrightarrow \forall p \in E$, $p$ is an interior point of $E$.

$\Leftrightarrow \forall p \in E, \exists r > 0, N_r(p) \subseteq E$.

$\Leftrightarrow \forall p \in E, \exists r > 0, N_r(p) \cap (X \setminus E) = \phi$

*** Closed set**

$E \subseteq X$ is closed $\Leftrightarrow \forall p$ is a limit point of $E$, then $p \in E$

$\Leftrightarrow \forall p$, s.t $\forall r > 0$, $N_r(p) \cap E \neq \phi$, then $p \in E$

- $E$ is closed $\Leftrightarrow E = \bar{E} \Leftrightarrow \forall (x_n)$ in $E$ $x_n \to x$, then $x \in E$
- $E$ is not closed $\Leftrightarrow \exists (x_n)$ in $E$, $x_n \to x$ but $x \notin E$.

*** Perfect set**  [ ] perfect  ( ) not perfect  [ ] $\cup \{x\}$ not perfect

$E$ is perfect $\Leftrightarrow$ { $E$ is closed
every point of $E$ is a limit point of $E$. (contains no isolated point)

$\Leftrightarrow$ { $E$ is closed
$E$ contains no isolated point.

$\Leftrightarrow E = E'$

$\Leftrightarrow E \neq \phi$ | $\forall x \in E$, $x$ is a limit point of $E$.

- Union of 2 perfect sets is perfect
- Intersection of 2 perfect sets may not perfect
$[0,1] \cap [1,2] = \{1\}$

*** Bounded set**

$E \subseteq X$ is bounded $\Leftrightarrow \exists r > 0$, $E \subseteq N(r, x)$ for some $x \in X$

*** Discrete set** : $\Leftrightarrow$ A set is made up by only isolated point.

*** Dense subset** (let $E \subseteq X$, we want to $E$ dense in $X$, then prove $\forall x \in X$, $x \in \bar{E}$) (Rd 4.4)

$E \subseteq X$ is dense $\Leftrightarrow \bar{E} = X$ (every point of $X$ is a limit point of $E$)

$\Leftrightarrow$ $E$ intersects with every neighborhood $N_r(x)$ of every point $x \in X$

$\Leftrightarrow \forall x \in X, \forall r > 0, N_r(x) \cap E \neq \phi$.

$\Leftrightarrow \forall U \subseteq X$, $U$ is open in $X$, then $U \cap E \neq \phi$.

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* A set having (no) limit point $\Rightarrow$ closed
A set containing (no) point $\rightarrow$ open

$\Rightarrow$ • $\phi$ is open / closed in $(\mathbb{R}, d_2)$

• $\mathbb{Q}$ is not open not closed in $\mathbb{R}$

• $\{x\}$ (discrete points) is not open (contains 1 interior point) / is closed (contains no limit point) in $\mathbb{R}$

• $\mathbb{N}$ is closed in $\mathbb{R}$ ($\cup$ of discrete pts)

$\{n + \frac{1}{n}\}$ is closed

$\mathbb{R}$ is closed / open in $\mathbb{R}$

• $[0, +\infty)$: closed in $\mathbb{R}$

2.26 + Exercise

Jan 2003 .

$(X,d)$ metric space

$E \subseteq X$ is a subset

- $E' = \{$all of limit point of $E\} = \{p \in X, \forall r > 0, N_r(p) \cap E \neq \phi\}$ .

$E^\circ = \{$all of interior point of $E\}$ .

$\partial E = \overline{E} \cap \overline{E^c} = \overline{E} \setminus E^\circ$

$p \in \partial E \Leftrightarrow \forall N_r(p)$ , $N_r(p)$ contains $\begin{cases} \text{at least one point in } E \\ \text{at least one point in } E \end{cases}$

$\overline{E} = E \cup E' = E^\circ \cup \partial E$

---

* $\overline{E}$ is closed

- $E = \overline{E} \Leftrightarrow E$ is closed (means $E' \subseteq E$ iff $E$ closed)
- $\overline{E}$ is the smallest closed subset of $X$ containing $E$

$\Leftrightarrow \left.\begin{array}{l} \forall F \boxed{\text{closed}} \text{ in } X \\ E \subset F \end{array}\right\} \Rightarrow \overline{E} \subseteq F$

- ~~E and $\overline{E}$ have the same limit point~~ .

$A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$

$\Rightarrow$ If $\underline{B}$ has no limit point $\Rightarrow A$ has no limit po

$x \in E \Leftrightarrow \forall r > 0, N_r(x) \cap E \neq \phi$

$\overline{\bigcup_{i=1}^{\infty} A_i} \supseteq \bigcup_{i=1}^{n} \overline{A_i}$

$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

- $E$ has $\underline{\text{no}}$ isolated point $\Rightarrow \overline{E}$ has $\underline{\text{no}}$ isolated p

---

* $E'$ is closed

- $p$ is a limit point of $E' \Longrightarrow p$ is a limit point of $E$

$\not\Leftarrow$

- $(A \cup B)' \subseteq A' \cup B'$

---

* $E^\circ$ is open

- $E = E^\circ \Leftrightarrow E$ is open
- $E^\circ$ is the biggest open subset of $E$ .

$\left.\begin{array}{l} \forall F \text{ open} \\ F \subseteq E \end{array}\right\} \Rightarrow F \subseteq E^\circ$

- $X \setminus (E^\circ) = \overline{(X \setminus E)}$

2.19:

- Every neighborhood is an open set

2.20: Theorem: If p is a limit point of E $\Leftrightarrow$ every neighborhood of p contains infinitely many points of E

Cor: A set containing finite points has no limit point

( A set containing infinite many points may contains limit point.

2.23 $(X, d)$ metric space (complement of open set $\to$ closed)

$E \subseteq X$ is open $\Leftrightarrow$ $X \setminus E$ is closed

$E \subseteq X$ is closed $\Leftrightarrow$ $X \setminus E$ is open.

Complement of dense may $\left[\begin{array}{l}\text{dense}\\ \text{not dense}\end{array}\right.$.

$\mathbb{Q}$ $\quad R/\mathbb{Q}$ : dense in $\mathbb{R}$

$\mathbb{N}$ not dense $\quad R/\mathbb{N}$ dense in $\mathbb{R}$.

2.24:

- For any collection $\{G_\alpha\}$ of open sets, then $\bigcup_\alpha G_\alpha$ is open

$$\bigcap_{i=1}^{n} G_{\alpha_i} \text{ is open}$$

$$\underset{\text{finite}}{\uparrow}$$

- For any collection $\{G_\alpha\}$ of closed sets, $\bigcap_{\alpha \in I} G_\alpha$ is closed, $\bigcup_{i=1}^{n} G_{\alpha_i}$ is closed

$$\underset{\text{finite}}{\uparrow}$$

2.28

$\left.\begin{array}{l} E \subseteq \mathbb{R},\ E \neq 0 \\ E \text{ is is bounded above} \end{array}\right\}$ then $\sup E \in \bar{E}$

~~Let~~

E is closed $\Rightarrow$ $\sup E \in E$

$\inf E \in E$

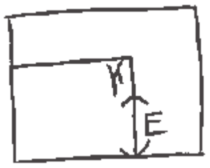
2.29

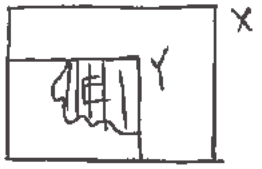
- E is open (relative to Y) $\Leftrightarrow$ $\forall x \in E$, $\exists N_r(x)$, $N_r(x) \subseteq Y$, $N_r(x) \subseteq E$.

neighborhood of x in Y.

- We know if $E \subseteq Y \subseteq X$, then E may be open in Y

$\lfloor$ not open in X





2.30: $E \subseteq Y \subseteq X$

E is open relative to Y $\Longleftrightarrow$ $E = Y \cap G$ for some G open relative to X

(closed) $\qquad$ (closed)

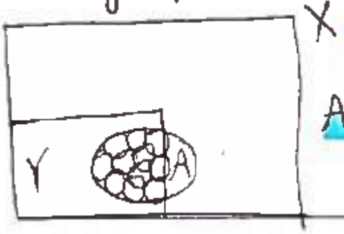
* E has no isolated point } ⇒ $E \cap G$ has no isolated point .
G is open

* Set of single points is not open in $\mathbb{R}$.

* A is an infinite bounded subset of the real line $\mathbb{R}$ } A contains some limit point $a_0$ (Aug 2011 P1)
⇒ ∃ a set $D \subseteq A$ which is neither open nor closed in $\mathbb{R}$ } ⇒ $\exists \{a_n\} \subset A$ , $a_n \to a_0$
Then $(a_n)$ is neither open nor closed in $\mathbb{R}$.

## * Compact sets.

The idea of compact set is that we can say a set is compact (without the space)



Let $A \subseteq Y \subseteq X$

A is open in Y $\Leftrightarrow$ $A = B \cap Y$, B open in X

$A \subseteq Y \subset X$

A is compact in Y $\Longleftrightarrow$ A is compact in X.

* Def. Let $(X, d)$: metric space

$K \subseteq X$ is compact $\Leftrightarrow$ every open cover of K in X, there is a finite subcover

$\Leftrightarrow$ if $K \subseteq \bigcup_{\alpha \in I} G_\alpha$, $G_\alpha$: open in X $(\forall \bigcup_\alpha G_\alpha)$, then $\exists\, \alpha_1, \dots, \alpha_n \in I$ s.t. $K \subset \bigcup_{i=1}^{n} G_{\alpha_i}$

* Every finite set is compact $\Rightarrow$ every finite set is closed

compact is the next best thing to being finite

Compactness does not depend on the space.

* Suppose $K \subseteq Y \subset X$

K is compact relative to Y $\Leftrightarrow$ K is compact relative to X

* Heine-Borel theorem

K is compact $\xRightarrow{\text{always}}$ $\left\{ \begin{array}{l} K \text{ is closed} \\ K \text{ is bounded} \end{array} \right.$

$\xLeftarrow{\ }$ in $\mathbb{R}^k(d)$

* Closed subset of a compact set $\Rightarrow$ compact

* $\left. \begin{array}{l} F \text{ is closed} \\ K \text{ is compact} \end{array} \right\} \Rightarrow F \cap K$ is compact (the intersection of a closed and compact set is compac

* every intersection (finite/infinite) of compact set is compact

$\{K_\alpha\}_{\alpha \in I}$ compact $\Rightarrow \bigcap_{\alpha \in I} K_\alpha$ is compact

(because closed + subset of compact → compact.)

* Finite union of compact sets is compact

* K is compact $\Leftrightarrow$ every infinite subset E of K has a limit point in K

$\Rightarrow$ (K compact, $\Rightarrow$ every sequence in K has a convergent subsequence)

* Weirestrass theorem:

Every bounded, infinite subset in $\mathbb{R}^n$, has a limit point in $\mathbb{R}^n$

→ (every bounded sequence in $\mathbb{R}$ has a convergent subsequence)

* X compact $\Longleftrightarrow$ $\left\{ \begin{array}{l} X \text{ complete} \\ X \text{ is bounded} \end{array} \right.$

$\left. \begin{array}{l} X \text{ compact} \\ \{x_n\} \text{ Cauchy sequence in } X \end{array} \right\} \Rightarrow \{x_n\}$ converges → $x \in X$

* K is compact $\Leftrightarrow$ $\exists$ finitely $q_1, q_2, \dots, q_n \in K$, s.t. $\begin{cases} K \subseteq W_{q_1} \cup W_{q_2} \cup \dots \cup W_{q_n} \\ W_{q_i}, i = \overline{1,n} \end{cases}$

' Example of subset of compact set is not compact
$K = (0) \cup \{\frac{1}{n}, n \in \mathbb{N}\}$ is compact
$K \setminus \{0\} \subseteq K$ is not compact .

* Example of closed/bounded but not compact
$[0, +\infty)$ closed in $\mathbb{R}$ but not compact (unbounded)
$(0,1)$ bounded but not compact (because not closed)

* In $\mathbb{R}^k$, E is closed + bounded
$\Leftrightarrow$ E is compact
$\Leftrightarrow$ every infinite subset of E has a limit point in E .

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* Note that when $x_n \longrightarrow x$, then the set $A = \{x_n\} \cup \{x_0\}$ is a compact set .
* So we have $f : X \longrightarrow Y$
and assume that $f(x_n) \longrightarrow y$ then we have $\left(\{f(x_n)\} \cup \{y\}\right)$ is a compact set . (this used in May 2006, 247

## * Connected set

* Def Let $(X,d)$ : metric space.
$A, B \subseteq (X,d)$

(Note that if $A \subseteq (X,d)$
then $\bar{A}$ : closure of $A$ in $(X,d)$ ($\boxed{\text{not}}$ in $(A \cup B)$)

A and B are seperated $\Leftrightarrow \begin{cases} \bar{A} \cap B = \phi \\ A \cap \bar{B} = \phi \end{cases}$ (no point of B are limit point of A)

$\Leftrightarrow \begin{cases} A \cap B = \phi \\ \text{both } A \text{ and } B \text{ are open} \end{cases}$ (in $A \cup B$). $\Leftrightarrow \begin{cases} A \cap B = \phi \\ \text{both } A \text{ and } B \text{ are closed in } A \cup B \end{cases}$

* A and B are disjoint $\Leftrightarrow A \cap B \neq \phi$

seperated $\Leftrightarrow$ disjoint + both open
$\Leftrightarrow$ disjoint + both closed.

* $E \subseteq X$ is said to be connected $\Leftrightarrow$ E is not a union of two nonempty seperated set

$\Leftrightarrow$ E is not a union of 2 nonempty + disjoint + both open in E
$\Leftrightarrow$ E is not " " " " nonemphy + disjoint + both closed in
$\Leftrightarrow$ the only 2 clopen subsets of E are $\phi$ and E.
(closed + open in E)

E is connected
$E = A \cup B$
where A, B seperates $\Big\} \rightarrow \Big[ \begin{array}{l} A = \phi \\ B = \phi \end{array}$
(used in Aug 2007)

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### + Conneted set of $\mathbb{R}$

$E \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ E is an interval or a point.

$\Leftrightarrow$ (E is an interval of R $\Leftrightarrow \left.\begin{array}{l} \forall x, y \in E \\ \text{if } z \in \mathbb{R},\ x < z < y \end{array}\right\} \Rightarrow z \in E$)

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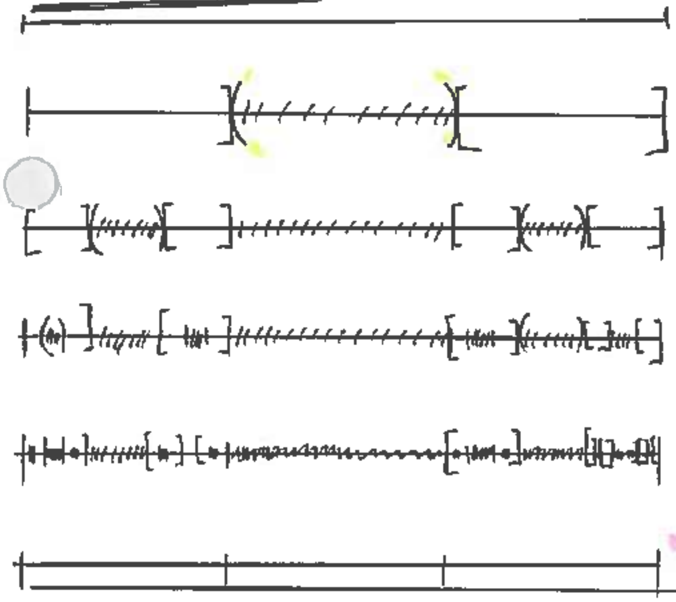
+ Compact + vs + connected
( Countable... union of nested nonempty compact + conneted sets is connected

Let $K_1 \supset K_2 \supset \cdots \supset \cdots$ nonempty + compact + conneted sets in $(X,d) \Longrightarrow \bigcap_{n=1}^{\infty} K_n$ connecte

*** Cantor set**



$K_0 = [0, 1]$

$K_1 = [0, \frac{1}{3}] \cup [\frac{1}{3}, 1]$.

$K_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

| # of interval removed | length of eac interval remo |
|---|---|
| 0 | |
| 1 | 1/3 |
| $2^1$ | $1/3^2$ |
| $2^2$ | $1/3^3$ |

$K_n$ { has $2^n$ interval
closed
compact (because closed + subset of compact).

Cantor set $C := \bigcap_{n=1}^{\infty} K_n$

*** The cantor set has length 0 (the total length removed is 1).**

- Now we prove that the total length removed is 1:

Total length removed $= \sum_{n=1}^{\infty} 2^{n-1} \frac{1}{3^n} = \frac{1}{2} \sum_{n=\textcircled{1}}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{2} \frac{2/3}{1-\frac{2}{3}} = \frac{1}{2} \frac{\frac{2}{3}}{\frac{1}{3}} = 1$

*** The Cantor set contains no interval.**

It contains no interval because it has length 0.

$\boxed{* \; C \neq 0}$ because $0 \in C$ (or because C is a intersection of sequence of compact set which ~~has~~ every finite subcollection has nonempty intersectio

$\boxed{* \; C \text{ is compact}}$ because C is a intersection of closed ⇒ closed
subset of [0,1] compact } ⇒ compact.

*** C is a perfect set ($C = C'$)**

* Every point of the cantor set C is a limit point of $[0,1] \setminus C$

* Jan 2009, P1, Let $A = R/C$, then $A' = \mathbb{R}$.

2.23 * $E \subseteq (X,d)$

$E$ is open $\iff (X \setminus E)$ is closed.

($\Longrightarrow$): Let $E$ is open | Prove $(X\setminus E)$ is closed.

$\iff \forall x \in E$, $x$ is an interior point of $E$ | Let $p$ is a limit point of $(X\setminus E)$. Prove $p \in (X\setminus E)$.

* Assume a contradiction that $p \notin (X/E)$. We NTP $p$ is not a limit point of $(X\setminus E)$

Because $p \notin (X/E) \iff p \in E$, we have $E$ is open $\Big\}$ $\Rightarrow$ $p$ is a interior point of $E$.

$\Rightarrow \exists N_r(p),\ N_r(p) \subseteq E$

$\Rightarrow \exists N_r(p),\ N_r(p) \cap (X\setminus E) \neq \phi$

$\Rightarrow$ $p$ is not a limit point of $(X\setminus E]$ □.

($\Longleftarrow$): $(X\setminus E)$ is closed. Prove that $E$ is open.

Prove that $\forall x \in E$, then $\exists N_r(x) \subseteq E$

Prove by contradiction, assume that $x \in E$, but $\forall r>0,\ N_r(x) \nsubseteq E$

this means $x \in E$, $\underbrace{\forall r>0,\ N_r(x) \cap (X\setminus E) \neq \phi}$,

$x$ is a limit point of $(X\setminus E)$, we have $(X\setminus E)$ is closed $\Big\}$ $\Rightarrow x \in X\setminus E$

contradiction

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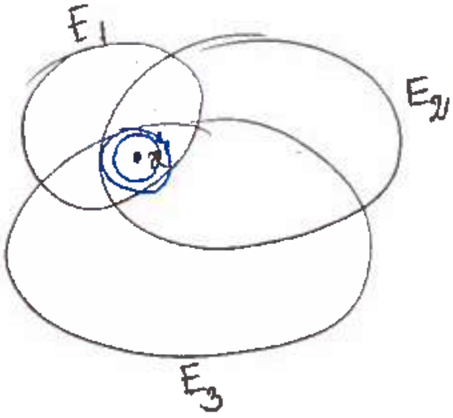
2.24 * Prove that { any union of open sets is open; finite ~~union~~ intersection of open set is open | For closed, use $E$ open $\iff (X\setminus E)$ is closed

* Let $E_1, \dots, E_n$ : open sets in $(X, d)$

Prove that $\bigcap_{i=1}^{n}(E_i)$ is open in $(X,d)$ $\iff$ NTP $\forall x \in \bigcap_{i=1}^{n} E_i$, $x$ is a interior point of $\bigcap_{i=1}^{n} E_i$

$\iff$ NTP, $\forall x \in \bigcap_{i=1}^{n} E_i$, $\exists r>0,\ N_r(x) \subseteq \bigcap_{i=1}^{n} E_i$

Let $x \in \bigcap_{i=1}^{n} E_i$, then $\forall i = \overline{1,n}$, $x \in E_i$, we have $E_i$ is open $\Big\}$ $\Rightarrow \exists r_i>0,\ N_{r_i}(x) \subseteq E_i,\ \forall i = \overline{1,n}$



Then because of the finitity of $\{\overline{1,n}\}$

Choose $r = \min\{r_1, \dots, r_n\}$,

We have $N_r(x) \subseteq E_i,\ \forall i = \overline{1,n}$

$N_r(x) \subseteq \bigcap_{i=1}^{n} E_i \Rightarrow$ □

Note if infinite intersect $\Rightarrow$ min may not exist.

2.20. Theorem:

$(X,d)$ metric space, $E \subseteq X$

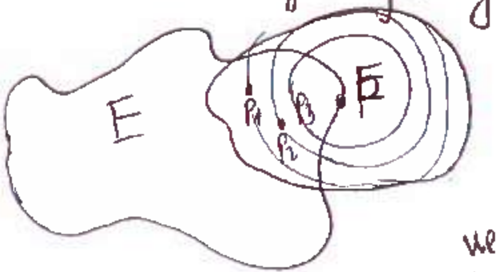
p is limit point of E $\Rightarrow$ every neighborhood of p contain infinitely many point of E.

Corollary: A set that contains finitely many points $\Rightarrow$ has $\boxed{\text{no}}$ limit point

We prove this by contradiction:

We will prove that if $\exists$ a neighborhood N of p such that N only contains finitely many point of E then p is $\boxed{\text{not}}$ a limit point of E.

NTP, If $\exists N$, neighborhood of p, N contains finitely many points of E $\Big\}$ Then $\exists r > 0$, $(N_r(p) \setminus \{p\}) \cap E = \phi$.



Assume N only contains $p_1, p_2, \dots, p_n$

$p_i \in E$, $\forall i = \overline{1,n}$

and $p_i \neq p$.

we have $d_i = d(p, p_i) > 0$

then choose $r = \min\{d(p,p_i),\ i = \overline{1,n}\} - \varepsilon$

then we have $N_r(p) \setminus \{p\} \cap E = \phi \Rightarrow \square$

2.27 $(X,d)$: metric space, $E\subseteq X$
$E'=\{$all limit point of $E\}$
$\bar E = E\cup E'$

a) Prove that $\bar E$ is closed
b) Prove that $E=\bar E \Longleftrightarrow E$ is closed
c) $\bar E$ is the smallest closed subset of $X$ that contains $E$
d) $E$ and $\bar E$ have the same limit point.
e) $E$ has no isolated point $\Rightarrow$ $\bar E$ has no isolated point.
#: $A\subseteq B$, then $\bar A\subseteq \bar B$

a) Prove that $\bar E$ is closed. NTP, $\forall x$ is a limit point of $\bar E$, then $x\in\bar E$
NTP, $\forall x$ s.t $\forall r, (N_r(x)\setminus\{x\})\cap\bar E\neq\phi$, then $\left[\begin{array}{l} x\in E\\ x\in E'\end{array}\right.$.

We will prove that if $x\notin E$, then $x\in E'$

NTP, if $x$ is a point s.t $\forall r, (N_r(x)\setminus\{x\})\cap\bar E\neq\phi$, then ~~$\forall d, (N_d(x)\setminus\{x\})\cap E\neq\phi$~~
$(N_r(x)\setminus\{x\})\cap E\neq\phi$

- We have $N_r(x)\setminus\{x\}\cap\bar E\neq\phi$

then $\exists p, \left|\begin{array}{l} p\in N_r(x)\setminus\{x\} \quad (1)\\ p\in\bar E \Leftrightarrow \left[\begin{array}{l} p\in E \quad (2)\\ p\in E' \quad (3)\end{array}\right.\end{array}\right.$

① If $p\in E \Rightarrow$ because $(1)+(2) \Rightarrow p\in(N_r(x)\setminus\{x\})\cap E \Rightarrow$

(Note that $N_r(x)$ is always open)

② If $p\in N_r(x)\setminus\{x\}$
$p\notin E$, $p\in E'$, then $\forall N_\ell(p)$, $N_\ell(p)\setminus\{p\}\cap E\neq\phi$
Choose $\ell$ such that $N_\ell(p)\subseteq N_r(x)$
this means we have
$\Rightarrow N_r(x)\setminus\{x\}\cap E\neq\phi \Rightarrow \square$

b) $E=\bar E \Longleftrightarrow E$ is closed

$(\Rightarrow)$ $E=\bar E$
from a) $\bar E$ is closed $\Big\}\Rightarrow E$ is closed

$(\Leftarrow)$: $E$ is closed. Prove that $E=\bar E$

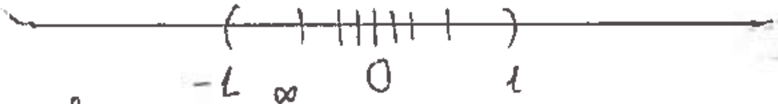
Let $E$ is closed. We need to prove $\bar E\subseteq E$
Let $x\in\bar E$, we NTP $x\in E$.

Because $x\in\bar E = E\cup E' \Rightarrow \left[\begin{array}{l} x\in E \Rightarrow \text{done}\\ x\in E' \Leftrightarrow x \text{ is a limit point}\end{array}\right.$
we have $E$ is closed $\Big\}\Rightarrow x\in E$ done.

* Every finite set is compact $\Rightarrow$ every finite set is closed

* Example of ~~infinite~~ countable set but is compact:

Let $K = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N}\}$ we have $K$ is compact. | $K \setminus \{0\}$ is not compact because



We have $K \subseteq \bigcup_{\alpha=1}^{\infty} G_\alpha$ where $G_1 = (-1, 1)$ then ~~K~~ $K \subseteq G_1$ & $K$ is compact.

$K \subseteq Y \subseteq X$

* $K$ is compact relative to $Y \Longleftrightarrow K$ is compact relative to $X$

(Idea: We use the property that every $G$ open in $Y$, then $G = B \cap Y$ for some $B$ open in $X$)

$\Rightarrow$ $K$ is compact relative to $Y \overset{\text{def}}{\Longleftrightarrow} \forall K \subseteq \bigcup_{\alpha \in I} G_\alpha$, $G_\alpha$ is open in $Y$ then $\exists \alpha_1, \dots, \alpha_n$, $K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

we have because $G_\alpha (\alpha \in I)$ open in $Y \Longleftrightarrow G_\alpha = B_\alpha \cap Y$ for some $B_\alpha$ open in $X$

then ~~∀K~~ $K \subseteq \bigcup_{\in I} B_\alpha$, $B_\alpha$, $\alpha \in I$, $B_\alpha$ open in $X$, then $\exists$ finite open cover in $X$ $K \subseteq \bigcup_{i=1}^{n} B_{\alpha_i}$ $\Rightarrow$ $K$ is compact in $X$.

$\Leftarrow$: Similarly, we use the property that if $\left.\begin{array}{l} G_\alpha = B_\alpha \cap Y \\ B_\alpha \text{ open in } X \end{array}\right\} \Rightarrow G_\alpha$ open in $Y$.

## Compact subset of a metric space is closed + bounded

- Let $(X,d)$: metric space

K is compact ~~metr~~ subset of $(X,d)$

**Prove that K is closed**

We want to prove $(X \setminus K)$ is open

NTP $\forall x \in (X\setminus K), \exists N_r(x) \subset (X\setminus K)$.

The idea of this prove is that:
K is compact $\Rightarrow \exists q_1, q_2, \dots, q_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} W_{q_i}$
$W_{q_i}$ is neighborhood of $q_i$

Then $N_r(x)$ is a neighborhood that does not intersect with those $W_{q_i}$

- Because K is compact
$\Rightarrow \exists q_1, q_2, \dots, q_n$ such that $K \subseteq \bigcup_{i=1}^{n} W_{q_i}$
$\boxed{q_i \in K}$
$W_{q_i}$ is a neighborhood of $q_i$

- Then consider $x \in X\setminus K$
We have $q_i \in K$, $x \in X\setminus K$, then $\exists V_i$ such that $x \in V_i$ and $V_i \cap W_i = \phi$

- Now consider $V = V_{q_1} \cap V_{q_2} \cap \dots \cap V_{q_m}$, we have $x \in V_i$

because $K \subseteq \bigcup_{i=1}^{n} W_{q_i}$
then $V = \bigcap V_{q_i}$

$$\left.\begin{array}{l} \text{because } K \subseteq \bigcup_{i=1}^{n} W_{q_i} \\ \text{then } V = \bigcap V_{q_i} \end{array}\right\} \Rightarrow V \cap K \subseteq V \cap \left(\bigcup_{i=1}^{n} W_{q_i}\right) = \bigcap_{i=1}^{n} \underbrace{(V \cap W_{q_i})}_{=\phi} = \phi \quad \Rightarrow V \subseteq X\setminus K$$

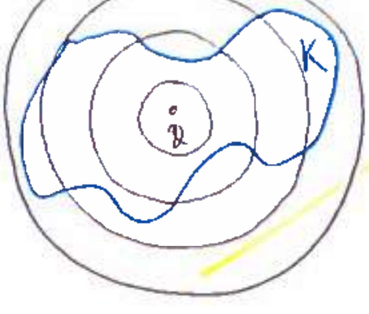
$\Rightarrow$ V is a open neighborhood of x that in $X\setminus K$
$\Rightarrow X\setminus K$ is open
$\Rightarrow$ K is closed.

- **K is a compact set of a metric space $(X,d)$. Prove that K is bounded.**

K is compact $\Leftrightarrow$ every open cover of K contains a finite subcover

~~$\Leftrightarrow \left.\begin{array}{l}\forall \{G_\alpha\}_{\alpha\in I} \text{ open in } X \\ K \subseteq \bigcup_{\alpha \in I} G_\alpha \end{array}\right\} \Rightarrow \text{then } \exists \alpha_1, \dots, \alpha_n, K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$~~

Then consider $x_0 \in K$
we create open cover for K from neighborhood of x (with increasing radius)



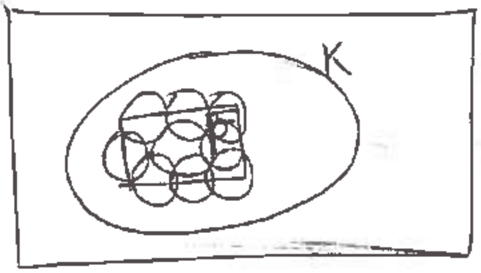
$K \subseteq \bigcup_{r>0} N_r(x)$

because K is compact $\Rightarrow$ every subcover contains finite subcover
$\Rightarrow \exists r_1, \dots, r_n$, $K \subseteq \bigcup_{i=1}^{n} N_{r_i}(x)$

Then choose $r = \max\{r_1, \dots, r_n\} \Rightarrow K \subseteq N_r(x) \Rightarrow$ K is bounded

* Prove that closed + subset of a compact set ⇒ compact

Let $(X, d)$ metric space
$K$ is compact
$F \subset K$, $F$ is closed
} Prove that $F$ is compact.



We have $F$ is closed $\Rightarrow (X \setminus F)$ is open.

We want to prove $F$ is compact

$\Leftrightarrow$ NTP $\forall \{F_{\alpha_i}\}_{\alpha_i \in I}$ open in $X$, $F \subseteq \bigcup_{\alpha_i \in I} F_{\alpha_i}$ } $\Rightarrow$ then $\exists$ finite subcover.

- Let $\{F_i\}_{i \in I}$ open cover of $F$

$$F \subseteq \bigcup_{i \in I} F_i$$

then we have

$$K \subseteq \underbrace{\bigcup_{\alpha_i \in I} \underbrace{F_i}_{\text{open}} \cup \underbrace{(X \setminus F)}_{\text{open}}}_{\text{this is a open cover of } K}$$

$K$ is compact

$\Rightarrow \exists \alpha_1, \alpha_2, \ldots, \alpha_n,$

$$K \subseteq \bigcup_{i=1}^{n} F_{\alpha_i} \cup \{X \setminus F\}$$

$\Rightarrow F \subseteq \bigcup_{i=1}^{n} F_{\alpha_i}$ (this means $F_{\alpha_i}, i = \overline{1,n}$ is finite open cover)

$\Rightarrow F$ is compact.

* Example of the convergence depends not only on $X$, but also on $d$.

| | Convergent sequence | divergent sequence |
|---|---|---|
| $X=\mathbb{R},\ d(x,y)=\lvert x-y\rvert$ | $x_n=\frac{1}{n}$ | $x_n=n$ |
| $X=\mathbb{R},\ d(x,y)=\begin{cases}1 & x\neq y\\ 0 & x=y\end{cases}$ | $x_n=a,\ \forall n$<br>a is a const | $x_n=\frac{1}{n}$ |
| $X=\mathbb{R},\ d(x,y)=\begin{cases}\lvert x\rvert+\lvert y\rvert & x\neq y\\ 0 & x=y\end{cases}$ | $x_n=\frac{1}{n}$<br>or $x_n=\frac{(-1)^n}{n}$ | $x_n=1+\frac{1}{n}$ |
| $X=\mathbb{R},\ d(x,y)=\sqrt{x-y}$ | $x_n=\frac{1}{\sqrt{n}}$ | $x_n=(-1)^n$ |

have the same convergent sequences

* Important result

$(X,d)$: metric space

$E\subseteq X$ a) $x\in\overline{E}\Leftrightarrow\forall N_r(x),\ N_r(x)\cap E\neq\phi$.

then b) $x\in\overline{E}\Longleftrightarrow\exists(x_n)\subseteq E,\ x_n\longrightarrow x$

$(x_n\in E,\ \forall n,$

* Prove a: $x\in\overline{E}\Leftrightarrow\forall N_r(x),\ N_r(x)\cap E\neq\phi$.

$(\Longrightarrow)$: $x\in\overline{E}\Rightarrow\left[\begin{array}{l}x\in E\ \Rightarrow\ \forall N_r(x),\ x\in N_r(x)\cap E\ \Rightarrow\ N_r(x)\cap E\neq\phi\\ x\in E'\overset{\Leftrightarrow}{\text{def}}\forall N_r(x),(N_r(x)\setminus\{x\})\cap E\neq\phi\end{array}\right.$

we have $(N_r(x)\setminus\{x\})\cap E\subseteq(N_r(x)\cap E)$ $\Big\}\Rightarrow N_r(x)\cap E\neq\phi$

$(\Longleftarrow)$: Assume $\forall r,\ N_r(x)\cap E\neq\phi$. Prove that $x\in\overline{E}$

Let $x\notin E$, we prove that $x\in E'$

$\left.\begin{array}{l}\text{because } x\notin E\\ \text{we have } \forall r, N_r(x)\cap E\neq\phi\end{array}\right\}\Rightarrow\forall r,(N_r(x)\setminus\{x\})\cap E\neq\phi\overset{\text{def}}{\Rightarrow}x\in\overline{E}.$

Prove b:

$(\Longrightarrow)$ We have from a, $x\in\overline{E}\Longrightarrow\forall N_r(x),\ N_r(x)\cap E\neq\phi$.

Then chose $r=\frac{1}{n}$, then this means, $\forall n,\ N_{\frac{1}{n}}(x)\cap E\neq\phi$.

this means $\exists x_n,\begin{cases}d(x_n,x)<\frac{1}{n} & \forall n.\\ x_n\in E.\end{cases}$

$\Rightarrow\exists(x_n)\subseteq E,\ x_n\rightarrow x$

$(\Longleftarrow)$: Assume $\exists(x_n)\subseteq E,\ x_n\rightarrow x$. Prove that $x\in\overline{E}$

We have $x_n\subseteq E$

$x_n\rightarrow x$ then every neighborhood of $x$ contain all but finitely many point of $x_n$

$\Longleftrightarrow\forall N_r(x),\ N_r(x)\cap E\neq\phi\Rightarrow x\in\overline{E}$

* Prove that the two definitions of seperated are equivalent.

$(X,d)$ metric space, $A, B \subseteq X$

| | |
|---|---|
| $A, B$ are seperated in $X$ $\Leftrightarrow \begin{cases} \overline{A}\cap B = \phi \\ A \cap \overline{B} = \phi \end{cases}$ | $A, B$ are seperated in $X$ $\Leftrightarrow \begin{cases} A\cap B = \phi \\ A \text{ and } B \text{ are both open in } (A\cup B) \end{cases}$. |

$(\Rightarrow)$: We have $\left.\begin{matrix} A\cap B \subseteq \overline{A}\cap B \\ \overline{A}\cap B = \phi \end{matrix}\right\} \Rightarrow A\cap B = \phi$.

- Now we need to prove $\left.\begin{matrix} \overline{A}\cap B = \phi \\ A\cap \overline{B} = \phi \end{matrix}\right\} \Rightarrow A$ and $B$ are both open in $(A\cup B)$.

$A = \underbrace{(A\cup B)\setminus \underbrace{\overline{B}}_{\text{closed}}}_{\text{open}}$ $\Rightarrow A$ is open

$B = \underbrace{(A\cup B)\setminus \underbrace{\overline{A}}_{\text{closed}}}_{\text{open}}$ $\Rightarrow B$ is open.

note that we have
$(X,d)$: metric space
$A \subseteq (X,d)$
$\rightarrow \overline{A}$ : closure of $A$ in $(X,d)$
(not in $A\cup B$)

$(\Leftarrow)$: $\begin{cases} A\cap B = \phi \\ A \text{ and } B \text{ are both open in } (A\cup B) \end{cases}$ Prove $\begin{cases} \overline{A}\cap B = \phi \\ A\cap\overline{B} = \phi \end{cases}$.

we have $\left.\begin{matrix} A \text{ is open in } A\cup B \\ A\cap B = \phi \end{matrix}\right\} \Rightarrow B = \underbrace{(A\cup B)\setminus \underbrace{A}_{\text{open}}}_{\text{closed}}$ closed in $(A\cup B)$

$\Rightarrow B = (A\cup B)\cap E$ for some $E$ closed in $X$

$\Rightarrow \overline{B} \subseteq E$

$\Rightarrow A\cap\overline{B} \subseteq A\cap E \subseteq (A\cup B)\cap E = B$

$\Rightarrow A\cap\overline{B} \subseteq \underline{(A\cap B) = \phi}, \Rightarrow A\cap\overline{B} = \phi$

* $E$ is $(X,d)$: metric space
$E \subset (X,d)$.

* Prove that the two definitions of connected set are equivalent.

| | |
|---|---|
| E is connected<br>$\Rightarrow$ E is not a union of 2 nonempty + disjoint + open subsets of E. | E is connected<br>$\Leftrightarrow$ The only clopen subsets of E are $\phi$ and E. |

$\Rightarrow$) : Prove by contradiction

Assume $\exists$ a nonempty open + closed subset U of E such that $U \neq \phi$ and $U \neq E$

Then we put $V = (E \setminus U)$, we have $U \cap V = \phi$

$U, V \neq \phi$ (because assumption $U \neq 0$, $V = E \setminus U \neq E \setminus E \neq \phi$.

V is open because U is closed.

$\Leftarrow$) Assume $E = V \cup W$ where $V, W \neq \phi$

$V \cap W = \phi$

V, W are both open in E

$\Rightarrow$ Prove that $\exists$ a subset of E that $\neq \phi$, $\neq E$ that is both open and closed in E

We have $E = V \cup W$

V is open in E

$V \cap W = \phi$

$\Rightarrow$ $W = E \setminus U$ is closed in E and $W \neq E$ because $U \neq \phi$

$\Rightarrow$ W is the set that $\neq 0$, $\neq E$ and both open / + closed in E.

# MAT 601 REMARKS ON 2.4-5: PERFECT SETS AND CONNECTED SETS

**Bonus Theorem 1 from 9/26.** I overstated the result, claiming it's true for an arbitrary set $E \subset \mathbb{R}$ (it can't be for many reasons). The correct statement is: every closed set $E \subset \mathbb{R}$ is the union of a perfect set and an at most countable set.

*Proof.* Let $\mathcal{J}$ be the set of all intervals $I$ with rational endpoints such that $E \cap I$ is at most countable. Let $C = \bigcup_{I\in\mathcal{J}}(E \cap I)$; this is an at most countable set. It is also open in $E$.

If $x \in E \setminus C$, then $E \cap N_r(x)$ is uncountable for every $r > 0$, for otherwise $x$ would be contained in some interval $I \in \mathcal{J}$. Therefore, $(E \setminus C) \cap N_r(x)$ is also uncountable. This shows that $x$ is a limit point of $E \setminus C$. Finally, $E \setminus C$ is closed in $E$ and since $E$ is closed in $\mathbb{R}$, it follows that $E \setminus C$ is closed in $\mathbb{R}$.

Summary: $E \setminus C$ is perfect and $C$ is at most countable. ☐

(Note that the assumption that $E$ is closed is used only to show that $E \setminus C$ is closed.)

**Bonus Theorem 2 from 9/26.** Suppose that $K_1 \supset K_2 \supset \cdots$ are nonempty compact connected sets in a metric space $X$. Then the set $K = \bigcap_{n=1}^{\infty} K_n$ is also connected.

*Proof.* Suppose to the contrary that $K = A \cup B$ where $A$ and $B$ are nonempty, disjoint and open in $K$. We have $A = U \cap K$ where $U$ is open in $X$. Let $V = X \setminus \overline{U}$; this set is also open in $X$. We have $V \cap K = B$ because on one hand, $V$ is disjoint from $A$, and on the other, $\overline{U}$ is disjoint from $B$.

The sets $E_n = K_n \backslash (U \cup V)$ are compact and nested. Since $K \subset U \cup V$, the intersection of $E_n$ is empty. Hence, there exists $n$ such that $E_n = \emptyset$, meaning that $K_n \subset U \cup V$. But the sets $U \cap K_n$ and $V \cap K_n$ are nonempty, disjoint, and open in $K_n$, so $K_n$ being covered by them contradicts the assumption that $K_n$ is connected. ☐

**Hint for homework problem 2.** The key step is to prove that after $K_1, \dots, K_n$ have been constructed, the set $I_{n+1} \setminus (K_1 \cup \cdots \cup K_n)$ is nonempty. Here's a hint for this step.

$K_{n+1}$

Pick any $x \in I_{n+1}$. If it's none in $K_1, \dots, K_n$, done. Otherwise it's in exactly one of them, say $K_j$. Then there is a neighborhood $N_r(x)$ that is contained in $I_{n+1}$ and is disjoint from $K_i$ for $i \in \{1, 2, \dots, n\} \setminus \{j\}$. (Why?) Once you have this $N_r(x)$, the conclusion follows since $K_i$ does not contan any interval.

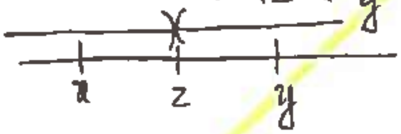
* Prove that

If $E \subseteq \mathbb{R}$, $E$ is connected $\Big\}$ $\Leftrightarrow$ $E$ is an interval $[a,b]$.
(If $x, y \in E$, Let $z \in \mathbb{R}$ s.t. $x < z < y$ $\Big\}$ then $z \in E$

Not done

($\Rightarrow$) Prove by contradiction.

Assume $E$ is not an interval (which means if $x, y \in E$, $x < z < y$ but $z \notin E$) $\Big\}$ Prove that $E$ is not connected.



Let $A = E \cap (-\infty, z)$
$B = E \cap (z, +\infty)$.

Then we have

- $E = E \cap \mathbb{R} \underset{z \notin E}{\overset{\uparrow}{=}} E \cap \big((-\infty, z) \cup (z, +\infty)\big) = \big(E \cap (-\infty, z)\big) \cup \big(E \cap (z, +\infty)\big) = A \cup B$
- $\overline{A} \cap B \in \phi$
  $A \cap \overline{B} = \phi$

$\rightarrow$ $E$ is not conneted.

($\Leftarrow$) We prove if $E = [a,b]$. $E$ is not connected $\Big\}$ $\Rightarrow$ could not happen

This is one way to prove a problem
Want to prove $A \Rightarrow B$
we prove $A$ and ($\rceil B$)
is impossible

Assume $E = [a,b]$
$E = A \cup B$
for $\overline{A} \cap B = \phi$
$A \cap \overline{B} = \phi$

(Because $A \cap B = \phi$ $\Rightarrow$ we care about points that are in the boundary of $A$)

Let consider $\Big|$ $z \in \partial A \Leftrightarrow \overline{A} \cap \overline{(E \setminus A)}$
$z \in E$

$\Leftarrow$): (Rudin's book) A $\longrightarrow$ B

Let $E \subseteq \mathbb{R}$ such that (In this case we understand that E is a set of real numbers). Prove that E is connected.

$\forall x, y \in E$

if $z \in \mathbb{R}$ is a point st $x<z<y$ } then $z \in E$

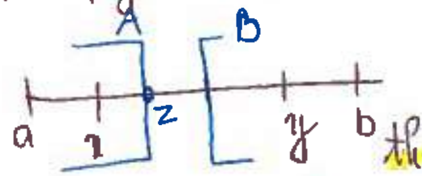
---

Assume E is not connected $\Leftrightarrow E = A \cup B$ for $A, B \neq \phi$, $\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$

Let A, B open in $\mathbb{R}$

We prove that $\exists x, y \in E$, $\exists z$ st $x<z<y$ } and $z \notin E$

Pick a, b $\in$ E such that $a \in A$ and $b \in B$. Prove that $\exists z$



Let $z = \sup(A \cap [a,b])$ We will

then $z \in \bar{A} \Leftrightarrow \begin{cases} z \in A \\ ~~z \notin A~~ (z \in A') \ (z \notin A) \end{cases}$

- We have $z \in \bar{A}$, we have $E = A \cup B$, $\bar{A} \cap B = \phi$ } $\Rightarrow z \notin B$

if $z \notin A$

- if $z \in A$

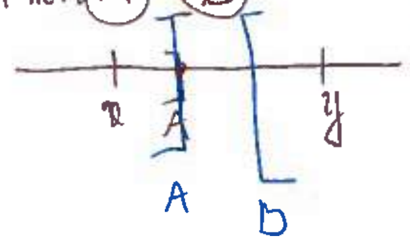
**Kovalev**

A, B are both open in E

+ let $E = A \cup B$, where A, B, seperated nonempty. Pick $x \in A$ $y \in B$.

Pick $z = \sup(A \cap [x,y])$

Then A B



Then $z = \bar{A}$, then $\begin{cases} z \in A \\ z \in A' \end{cases}$

- If $z \in A$, then because $A \cap B = \phi$ } $\Rightarrow z \notin B \Rightarrow$

but B is open $\Rightarrow z + \frac{1}{n} \in B, \forall n \Rightarrow z \in \bar{B}$

- If $\begin{cases} z \in A' \Rightarrow z \in B \\ z \notin A \end{cases}$

2.36:

Suppose $\{K_\alpha\}$ compact subsets of a metric space
every finite subcollection of it has nonempty intersection $\Big\} \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \phi$.

2.37 * Corollary:

Let $\{K_n\}$ is a sequence of nonempty, nested, compact set of a metric space
Then $\bigcap_{n=1}^{\infty} K \neq \phi$

3.10 * Compact nested set theorem

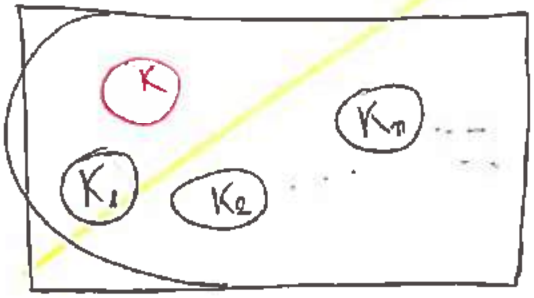
Let $\{K_n\}$ be a sequence of nonempty, nested, compact sets
$\lim_{n\to\infty} \text{diam}\, K_n = 0$ $\Big\} \Rightarrow \bigcap_{n=1}^{\infty} K_n$ contains one point

---

* Prove 2.36: $\{K_\alpha\}$ compact subsets
every finite subcollection of it has nonempty intersection $\Big\} \Rightarrow \bigcap_\alpha K_\alpha \neq \phi$

We prove this by contradiction.

Assume that $\bigcap K_\alpha = \phi$. Then we NTP that $\exists$ a $\bigcap_{i=1}^{\ell} K_i = \phi$



Let $U_\alpha = X \setminus K_\alpha$, then we have $U_i$ is open.

~~Assum~~ Let $K$ is one of $K_\alpha$. ~~$\Rightarrow$~~

Assume $\bigcap_\alpha K_\alpha = \phi$, ~~then~~ $K \cap (\bigcap_{K_\alpha \neq K} K_\alpha) = \phi$.

then we have $\{U_\alpha\}$ is open cover of $K$ $(K \subset \bigcup_\alpha U_\alpha)$

Then because $K$ is compact, then $\exists$ a finite subcover $K \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3} \dots \cup U_{\alpha_n}$

but then $K \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \phi$ (contradiction) $\Rightarrow$ □ 2.36.

* Prove 2.37: easy because nested $\Leftrightarrow$ every finite subcollection has nonempty intersection.

* Prove 3.10

We have $\{K_n\}$: sequence of nonempty, nested, compact set

Then by corollary 2.37 $\rightarrow \bigcap_{n=1}^{\infty} K_n \neq \phi \Rightarrow$ there is at least 1 point in $\bigcap_{n=1}^{\infty} K_n$

$\rightarrow$ It suffices to prove that $\bigcap_{n=1}^{\infty} K_n$ can't contains more than 1 point.

- Assume $\bigcap_{n=1}^{\infty} K_n$ contains more than 1 point, then put $K = \bigcap_{n=1}^{\infty} K_n$

$\Rightarrow \text{diam}\, K > 0$
but $\text{diam}\, K_n \xrightarrow{n\to\infty} 0$ $\Big\} \Rightarrow$ then $\exists n_0 \in \mathbb{N}, \forall n \geqslant n_0$
$\text{diam}\, K > \text{diam}\, K_n$
(impossible because $K = \bigcap K_n$
$\Rightarrow K \subseteq K_n$.

# §3: Numerical Sequences and Series

Focus on $d(x,y) = |x-y|$

* A sequence $\{p_n\}$ in $X$ is a function $f : \mathbb{N} \longrightarrow X$, $n \longmapsto p_n$

* Def: A sequence $\{p_n\}$ is said to "converge in $X$" with metric $d$

$\overset{def}{\Longleftrightarrow}$ $\exists p \in X$, st $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $d(p_n, p) < \varepsilon$

means $\forall n \geq n_0$, $p_n \in N_\varepsilon(p)$

$\overset{write}{\Longleftrightarrow}$ $p_n \longrightarrow p$

$\overset{write}{\Longleftrightarrow}$ $\lim_{n \to \infty} p_n = p$

- Note: the convergence depends on $\begin{cases} \text{metric } d \\ \text{space } X \end{cases}$
- $p_n$ converges in $X$ $\Leftrightarrow$ $p_n \to p$ with $p \in X$

* $\{p_n\}$ diverges $\Leftrightarrow$ $\{p_n\}$ does not converges $\Leftrightarrow$ $\exists \varepsilon > 0$, $\forall n_0 \in \mathbb{N}$, $\exists n \geq n_0$, $d(p_n, p) \geq \varepsilon$.

---

* Definition (About range)

Let $\{p_n\}$ sequence, then the range of $\{p_n\}$ is $A = \{p_n, n = 1, 2, 3, \dots\}$

- The range of a sequence may be a finite or an infinite set
- $\{p_n\}$ is said to be bounded iff its range $A$ is bounded $\Leftrightarrow$ $\exists M$, $|p_n| < M$, $\forall n$

* Theorem: Let $\{p_n\}$ be a sequence in a metric space $(X, d)$. (capare 2.20)

a) $\{p_n\} \longrightarrow p \in X \Longleftrightarrow$ every neighborhood of $p$ contains $p_n$ for all but finitely many $n$

b) If $\left.\begin{array}{l} p_n \longrightarrow p \in X \\ p_n \longrightarrow p' \in X \end{array}\right\}$ then $p = p'$ (Limit of a sequence is unique)

c) $\{p_n\}$ converges $\underset{\not\Longleftarrow}{\Longrightarrow}$ $\{p_n\}$ bounded

d) $E \subset X$

$p$ is a limit point of $E$ $\Longleftrightarrow$ $\exists \{p_n\} \subset E$, $p_n \longrightarrow p$

($p \in \overline{E}$) (note that the sequence in $E$).

$x \in \overline{E} \Leftrightarrow \forall r > 0, N_r(x) \cap E \neq \emptyset$

$\not\Longleftarrow$ If we have $\exists \{p_n\}$, $p_n \to p$

does not enough to deduce $p$ is a limit point of any set.

* A way to create a sequence (from the limit definition)

$p$ is a limit point of $E$

$\Leftrightarrow \forall \varepsilon > 0, \exists p$

---

- $\{s_n\} \longrightarrow s$ $\underset{\not\Longleftarrow}{\Longrightarrow}$ $\{|s_n|\} \longrightarrow |s|$

With series:

- Weierstrass theorem:

Every bounded + infinite set in $\mathbb{R}^n$ has a limit in $\mathbb{R}^n$

3.37 <u>Theorem</u>:

Suppose $\left\{\begin{array}{l}\{s_n\}, \{t_n\} \text{ are complex sequences}\\ \lim\limits_{n\to\infty} s_n = s\\ \lim\limits_{n\to\infty} t_n = t\end{array}\right.$

Then a) $\lim(s_n+t_n) = s,$

b) $\lim c s_n = c \lim s_n = cs$

$\lim(c_n+s_n) = c+s$

c) $s_n t_n \longrightarrow st$

d) $\dfrac{1}{s_n} \longrightarrow \dfrac{1}{s}$

$(\text{if } s_n \neq 0, \forall n\ ;\ s \neq 0)$

Note that with series

$$\left.\begin{array}{l}\sum a_n \text{ converges}\\ \sum b_n \text{ converges}\end{array}\right\} \underset{\uparrow}{\not\Rightarrow} \sum a_n b_n \text{ converges}$$

happens when $a_n \geq 0, \forall n$

$b_n \geq 0, \forall n$

## Supsequence :

* Def : Give a sequence $\{p_n\}$. $\{n_k\}$ : sequence of positive integers, such that $n_1 < n_2 < n_3 < \cdots$

Then $\{p_{n_k}\}$ : sub sequence of $\{p_n\}$.

( • $f : \mathbb{N} \longrightarrow X$, $n \longmapsto p_n$ (sequence). $g : \mathbb{N} \longrightarrow \mathbb{N}$, $k \longmapsto n_k$

$$\begin{tikzcd} \mathbb{N} \arrow[r, "g"] & \mathbb{N} \arrow[r, "f"] & X \\ k \arrow[r, mapsto] \arrow[rr, bend right, "f \circ g"'] & n_k \arrow[r, mapsto] & p_{n_k} \end{tikzcd}$$

If $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$

note : $n_k \geq k$

$n_k \geq n$

- every sequence has a normal subsequence is itself.
- If $\{p_{n_k}\}$ converges, its limit is called a subsequence limit of $\{p_n\}$.

* Theorem : $p_n \longrightarrow p \iff$ every subsequence $\{p_{n_k}\}$ converges to $p$.

---

* $X$ compact $\Rightarrow$ every sequence in $X$ has a convergent subsequence (converges to a point in $X$).

* Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Note : Consider a sequence $(x_n)$ in $(a, b)$ $\Rightarrow$ $(x_n)$ converges to a point in $[a, b]$

(Veatiros theorem : every infinite bounded subset in $\mathbb{R}$ has a $\boxed{\text{limit point}}$

---

37* Theorem : The subsequence limits of a sequence $\{p_n\}$ in a metric space $X$ forms a closed set

(Let $\{p_n\}$ is a sequence in metric space $X$

$S = \{x \in X \mid \exists \{p_{n_k}\}, \{p_{n_k} \to x\}\} \setminus \{\pm\infty\}$ then $S$ is closed in $X$ $\boxed{S = \overline{S}}$

↖ note that in this we only consider limit that $\neq \pm\infty$.

$(S = \overline{S} \iff \forall x \in S, \exists (x_n)^{\in S}, x_n \longrightarrow x \}$.

- Let $\{x_n\}$ is a sequence that goes through $\mathbb{Q}$, then $S = \mathbb{R}$.

* One important property of $x \notin S$ :

$x \notin S \iff \forall N_\lambda(x)$, $N_\lambda(x)$ contains finitely many terms of $\{p_n\}$. (on the pg

# Cauchy sequence.

3.8 Def: $(X,d)$: metric space.
$\{p_n\}$ Cauchy $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, d(p_m, p_n) < \varepsilon$

---

3.11
* $\{p_n\}$ Cauchy $\Longrightarrow \{p_n\}$ bounded (in $\mathbb{R}$ (sequence in $\mathbb{R}$)

( def of bounded sequence : range $K$ is bounded
$\to \exists N_1(x)$ $K \subseteq N_1(x)$)
if in $\mathbb{R}$: $\{p_n\}$ is bounded $\Leftrightarrow \exists M, |p_n| \leq M, \forall n$

(If $\{f_n\}$ Cauchy in $(\mathscr{C}(X)$ (see chapter 7, 7.14) $(\mathscr{C}(X), d(f,g) = \|f-g\|$
$\Rightarrow \{f_n\}$ bounded (in $\mathscr{C}(X)$) $\Leftrightarrow \exists M, \|f_n(x)\| \leq M, \forall n$
$\Rightarrow \sup_{x \in X} |f(x)| \leq M$.)

---

* Converge sequence $\Rightarrow$ Cauchy.

---

3.11
* If a Cauchy sequence has a convergent subsequence $\Rightarrow$ it converges

let $\{p_n\}$ is a Cauchy sequence
$\exists \{p_{n_k}\}$ convergent subsequence (to $p$)
$\Big\} \Rightarrow \{p_n\}$ converges (to $p$)

---

* $X$ is a Compact metric space
$\{p_n\}$ is a Cauchy sequence in $X$
$\Big\} \Rightarrow \{p_n\}$ converges to some point of $X$

* In $\mathbb{R}^n$, Cauchy $\Longleftrightarrow$ converges     In Compact, Cauchy $\Leftrightarrow$ converges

## * Complete metric space

3.12* Def $X$ is a complete metric space $\Leftrightarrow$ every Cauchy sequence in $X$ is converges in $X$

EX: $\mathbb{R}$, $\mathbb{R}^d$: complete spaces.

$\mathbb{R}\setminus\{0\}$, $\mathbb{Q}$ are not complete

* $X$ compact $\Rightarrow$ $X$ is complete $\Rightarrow$ $X$ is closed

$X$ compact $\Leftrightarrow$ $\begin{cases} X \text{ complete} \\ \text{bounded} \end{cases}$

* $E$ is closed, subset of a complete space $\Rightarrow$ $E$ is complete (similar to compact).

$E$ is closed, $K$ is complete $\Rightarrow$ $E \cap K$ is complete.

If $\left.\begin{array}{l} X \text{ complete metric space} \\ E \subseteq X \text{ is complete} \end{array}\right\} \Rightarrow E$ is closed

If $E \subseteq X$ is complete $\Rightarrow$ $E$ is closed in $X$

(this is interesting because when we talk about closed it depends on what metric space is.

## * 3.9, 3.10 Diameter and nested closed set theorem.

* 3.9 def:

Let $E \subseteq (X,d)$, $E \neq \phi$.

then diameter of $E$ = $\operatorname{diam} E = \sup\limits_{x,y\in E}\{d(x,y), x,y \in E\}$

• Actually $A \subseteq B \Rightarrow \operatorname{diam} A \leq \operatorname{diam} B$

• $\operatorname{diam} E < +\infty \overset{\text{def}}{\Longleftrightarrow} E$ is bounded.

• $\operatorname{diam} E = \operatorname{diam} \overline{E}$

* $\{P_n\}$ is Cauchy, then $\operatorname{diam}\{P_n, n \geq m\} \xrightarrow{m\to\infty} 0$

*3.10 (Nested closed sets theorem) $(X,d)$ is complete

Let $\begin{cases} E_1 \supset E_2 \supset E_3 \supset \cdots \text{ are nonempty, nested, closed, bounded subsets.} \\ \operatorname{diam} E_n \xrightarrow{n\to\infty} 0 \end{cases}$

Then $\bigcap\limits_{n=1}^{\infty} E_n$ constains exactly one point

(Base on theorem)

* Nested compact sets theorem

$\begin{cases} K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots \text{ nonempty, nested, compact} \\ \operatorname{diam} K_n \xrightarrow{n\to\infty} 0 \end{cases}$

then $\bigcap\limits_{n=1}^{\infty} K_n$ contains exactly one point.

# Convergence vs (monotonic + bounded) (in $\mathbb{R}^n$)

Idea : We have converges $\Rightarrow$ bounded

From here: bounded + monotone $\Rightarrow$ converge

* **3.13 Def**:

A sequence $\{s_n\}$ of real numbers is said to be

a) monotonically increasing $\overset{def}{\iff} s_n \le s_{n+1}, \forall n$.

b) monotonically decreasing $\iff s_n \geq s_{n+1}, \forall n$

c) monotonical $\iff \{s_n\} \left[\begin{array}{l}\text{monotonically increasing}\\ \text{monotonically decreasing}\end{array}\right.$.

**3.14 Theorem**.

$\{s_n\}$ converges $\Rightarrow$ $\{s_n\}$ bounded

$\{s_n\}$ bounded + monotonic $\Rightarrow$ $\{s_n\}$ converges. to $\left[\begin{array}{l}\sup\{s_n\}\\ \inf\{s_n\}\end{array}\right.$

* Important:

A monotone sequence has a bounded subsequence $\Rightarrow$ it converges

* From this we have the relation between sequence vs series.

- $\sum a_k$ is a series with partial sum $s_n = \sum_{k=1}^{n} a_k$

If $a_n \geq 0, \forall n \Rightarrow s_n$ is monotonically increasing.

Then $\begin{cases}\sum a_k \text{ conver} \\ a_n \geq 0, \forall n\end{cases} \iff \{s_n\}$ is bounded (above)

or $\begin{cases}\sum a_k \\ a_n \leq 0, \forall n\end{cases}$, $\sum a_n$ converges $\iff \{s_n\}$ bounded (below)

# limsup and liminf

**3.15** Let $\{s_n\}$: sequence of real number, $s_n \to +\infty \overset{\text{def}}{\iff} \forall M(>0), \exists n_0 \in \mathbb{N}, \forall n \geq n_0, s_n \geq M$

$s_n \to -\infty \overset{\text{def}}{\iff} \forall M\,(<0), \exists n_0 \in \mathbb{N}, \forall n \geq n_0, s_n \leq M$

**3.16 Def**:

$\{s_n\}$: sequence of real numbers $\quad E = \{x, \exists s_{n_k}, s_{n_k} \to x\} = \{\text{subsequential limit}\} \cup \{\pm$

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} \sup_{N \geq n}\{s_N\} = \inf_n \left\{ \sup_{N\geq n}\{s_N\} \right\} = \boxed{\sup E}$$

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} \inf_{N\geq n}\{s_N\} = \sup_n \left\{ \inf_{N\geq n}\{s_N\} \right\} = \boxed{\inf E}$$

---

**3.17**

- $\exists\{s_{n_k}\}, s_{n_k} \to \limsup s_n$ $\quad$ ($\limsup \in E$)

(Give $a = \limsup s_n$, then we can chose a subsequence of positive integers s.t $a = \lim_{k\to\infty} s_{n_k}$

- If $a > \limsup s_n \Rightarrow \exists n_0 \in \mathbb{N}, \forall n \geq n_0, a > s_n$

- $\exists\{s_{n_k}\}, s_{n_k} \to \liminf s_n$
- If $a < \liminf s_n \Rightarrow \exists n_0 \in \mathbb{N}, \forall n \geq n_0, s_n > a$

* If $\limsup s_n = \liminf s_n = L < +\infty$, then $\lim s_n \Rightarrow L$

(means ~~lim~~ $s_n \to s \iff \limsup s_n = \liminf s_n = s < +\infty = \lim s_n$.)

* For any two sequence $\{a_n\}$ and $\{b_n\}$

$$\limsup\{a_n + b_n\} \leq \limsup a_n + \limsup b_n.$$

* $\liminf a_n \leq \lim \boxed{a_{n_k}} \leq \limsup a_n$ for any convergentint subsequence $\{a_{n_k}\}$

---

* If $a_n \leq b_n, \forall n \Longrightarrow \limsup a_n \leq \limsup b_n$

* If $\exists n_0 \in \mathbb{N}, \forall n \geq n_0, x_n \leq M \Longrightarrow \limsup x_n \leq M$.

* $a > \limsup s_n = \lim_{N\to\infty} \sup_{n \geq N} s_n \longrightarrow \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \sup s_n \to \alpha < a$.

$\Rightarrow \exists n_0 \in \mathbb{N}, \forall n \geq n_0, s_n \leq \alpha < a$.

$a < \limsup s_n \Rightarrow s_n > a$ for infinitly many $n$.

* If $\left.\begin{array}{l} s_{2n} \to K \\ s_{2n+1} \to L \end{array}\right\} \to \limsup \geq \max\{K, L\}$.

---

* If $a = \limsup_{n\to\infty} s_n$, then $\exists$ infinitely many $s_n$ such that $s_n > a - \varepsilon$. (proof T3.K

* If $\limsup_{n\to\infty} s_n \leq \beta$ true for all $\beta \geq \alpha$, then we have $\limsup_{n\to\infty} s_n \leq \alpha$.

* Assume $\limsup a_n = \alpha > 1$.
  then $\exists\, a_{n_k}$, $a_{n_k} \longrightarrow \alpha > 1$.
  this means $\exists\, k_0 \in \mathbb{N}$, $\forall k \geq k_0$, $a_{n_k} > 1$.

* $\limsup s_n = \alpha$ then for $\beta > \alpha$, $\exists N \in \mathbb{N}, \forall n \geq N, s_n < \beta$

* We want to prove $\limsup s_n = \alpha \Leftrightarrow$ NTP $\exists N \in \mathbb{N}, \forall n \geq N, s_n < \alpha$ (see proof T 3.34)

* We have $\limsup s_n \leq \beta$ is true for all $\beta \geq \alpha$ (see proof T 3.34).
  then we have $\limsup s_n \leq \alpha$

---

* $\limsup\limits_{n\to\infty} a_n = A \Leftrightarrow A$ is the smallest number s.t $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, a_n < A + \boxed{\varepsilon}$

* Aug 2005, 12, Let $\{a_n\}$: sequence of positive integers, $\sum a_n$ converges $\Big\} \Rightarrow \liminf\limits_{n\to\infty} na_n = 0$

  ( $\not\Rightarrow \lim\limits_{n\to\infty} na_n = 0$ because $\lim\sup b_n \xRightarrow{na_n} 0$ when

## 3.8.7 Remainder, estimation, Irrationality.

(Motivation: simple $a_n$ produce complicated $\sum_{n=1}^{\infty} a_n$)

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ | $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational (Apéry, 1978) | $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ | $\sum \frac{1}{n^5}$ is still unknown if rational or irrational.

* ?? How to prove sth ~~is~~ irrational

- Way 1: Directly: EX $\sqrt{2}$
- Way 2: Rational root theorem:

If $x$ satisfies $x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$, $a_{n-1}, \cdots, a_0 \in \mathbb{Z}$ $\Longrightarrow$ $x$ is either integer or irrational

EX: $x^2 - 2 = 0$

- Way 3: Rational approximation:

If $x$ can be well approximated by rational number, $\Rightarrow$ it's irrational

$$\left.\begin{array}{l} \text{Let } x \in \mathbb{R}, \text{ assume } \exists \left\{\frac{p_n}{q_n}\right\} \subset \mathbb{Q} \text{ such that } \boxed{q_n} \left|x - \frac{p_n}{q_n}\right| \longrightarrow 0 \\ \frac{p_n}{q_n} \neq x \end{array}\right\} \Rightarrow x \notin \mathbb{Q}$$

- Way 4: For consider if $\sum \boxed{\frac{p_n}{q_n}}$ rational / irrational

Consider the series $\sum_{n=1}^{\infty} \frac{p_n}{q_n}$, has $\begin{cases} \text{partial sum } s_n = \sum_{k=1}^{n} \frac{p_k}{q_k} & \text{with denominator } \operatorname{lcm}(q_1, \cdots, q_n) \\ \text{remainder } r_n = \sum_{k=n+1}^{\infty} \frac{p_k}{q_k} \end{cases}$

If $r_n \operatorname{lcm}(q_1, \cdots, q_n) \longrightarrow 0$, then $\sum \frac{p_n}{q_n}$ is $\boxed{\text{irrational}}$

---

* Remainder estimate

Suppose $\exists \boxed{b < 1}$ s.t $\left|\frac{a_{k+1}}{a_k}\right| < \underset{\text{depending on } n}{b}$, $\forall \boxed{k \geq n}$ (or $n+1$)....)

then $r_n \leq \left|\sum_{k=n+1}^{\infty} a_k\right| \leq \boxed{\frac{|a_{n+1}|}{1-b}}$

## * Some special sequence

* If $0 \leq x_n \leq s_n$ for $n \geq N$
uher N is some fixed number

If $s_n \to 0$ then $x_n \to 0$

- If $p > 0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$
- If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$

∀ polynomial p, then $\lim\limits_{n\to\infty} |p|^{1/n} = 1$.

- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ $\qquad \lim\limits_{n\to\infty} \sqrt[c]{n} = +\infty$
for all $c > 0$

* Binomial theorem
$(1+nx) \leq (1+x)^n$, $\forall n \in \mathbb{N}$
$x > -1$

d) If $p > 0$ $\lim\limits_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
$\alpha \in \mathbb{R}$

Koralev: $\lim\limits_{n\to\infty} \frac{n^d}{a^n} = 0$, $\forall d \in \mathbb{R}$, $a > 1$

e) If $|x| < 1$, then $\lim\limits_{x\to\infty} x^n = 0$.

- $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1 \Leftrightarrow \lim\limits_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = 1$ (we in Aug2003 ℓ2)

---

* Example of Cauchy sequence but not converge
We have in $\mathbb{R}$, Cauchy ⟺ converges, ⇒ we need to find a space E (Ex $\mathbb{R}\setminus\{0\}$)
such that a sequence $\{x_n\}$ is Cauchy but not converge in E

EX: $x_n = \frac{1}{n}$, $\forall n$ in $E = \mathbb{R}\setminus\{0\}$. $\{x_n\}$ Cauchy but does not converge in $E = \mathbb{R}\setminus\{0\}$

* Unbounded sequence containing a Cauchy subsequence:

$$x_n = \begin{cases} n & , n \text{ even} \\ \frac{1}{n} & , n \text{ odd} \end{cases}$$

when we need to find a sequence with property a
that has a subsequence with property b
we need to divide the sequence in to parts.
Then we have subsequence with property a
subsequence with property b.

* A Cauchy sequence but not monotone.

$x_n = \frac{1}{n}, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \cdots$

we have $\frac{1}{n} > 0$ but $\frac{1}{n} \to 0$
then we create a sequence with $\frac{1}{n}$, and 0

* Monotone but not Cauchy
$x_n = n$

* Bounded but not Cauchy
$(x_n) = (-1)^n$.

Chapter 3, [illegible] part.

*: Some common functions that we can apply $\lim_{n\to\infty} f(x_n) \overset{}{\underset{f\ \mathrm{cont}}{=\!=\!=}} f(\lim_{n\to\infty} x_n)$ : $e^x$, $\ln x$, $x^{\ell}$

(use to find $\lim \ln(f(n))$  $\lim e^{f(n)}$  $\lim [f(n)]^{\ell}$ ).

EX: find $\lim(\ln\frac{2n+1}{3n+4}) = \ln(\lim_{n\to\infty}\frac{2n+1}{3n+4}) = \ln\frac{2}{3}$ $\square$.

* ~~Find~~ $\lim_{n\to\infty} \ln(5^n) - \ln(n!)$ converg or not? We can apply $x = e^{\ln x}$.

$\ln(5^n) - \ln(n!) = \ln_{n\to\infty}(\frac{5^n}{n!})$ , we have $\lim_{n\to\infty}\frac{5^n}{n!} = 0$ so $\ln(\frac{5^n}{n!})$ diverges when $n \to \infty$.

* If we need to find $\lim f(x)^{g(x)}$ , or $\lim_{n\to\infty} f(n)^{g(n)}$

we can solve by way $a^b = e^{\ln a^b} = e^{b\ln a}$

so $\lim a^b = e^{\lim(b\ln a)}$

- For example: $\lim_{x\to 0} x^x = 1$ because ~~$x^x = e^{\ln x^x} = e^{x\ln x}$~~ ?

  $\lim_{n\to\infty} \sqrt[n]{n} = 1$

* Weird problems

- Aug2003 + Aug2015 P 2 P2 ⇒ Prove that $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln n = \gamma$ for some $\gamma \in (\frac{1}{2}, 1)$

Note that even this problem looks complicated, we just put $a_n = \sum_{k=1}^{n}\frac{1}{k} - \ln n$ and then prove that $\begin{cases} a_n - a_{n+1} \geq 0 \Rightarrow \text{decreasing} \\ \text{bounded below} \end{cases}$ by noticing $\ln n = \int_1^n \frac{1}{x}\,dx$

---

* With problem $x_n = (-\frac{1}{2})^n + \sin(\frac{n\pi}{2})$ converges or diverges? (See Aug1997, P2)

> can't use another way ⇒ write down to see any clue.

* $x_n = \frac{n^n + (-n)^n}{\text{limit2}} + (1 + \frac{1}{2n})^n$ converge or diverge?

Evaluate

* Aug2013, P8 ⇒ $x_n = \prod_{k=1}^{n}\left(1 - \frac{1}{2^k}\right) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^n}\right)$ and we want to find the limit of this $(x_n)$ to prove that $\lim_{n\to\infty} x_n \neq 0$.

Note that with this problem we need to find limit of a $\prod$

⇒ we want to solve by using $\lim x_n = \lim e^{\ln x_n} = e^{\lim(\ln x_n)} = e^{\lim \sum_{n=1}^{\infty} \ln\left(1 - \frac{1}{2^n}\right)}$

- Also note that $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$ ⇒ $\lim_{n\to\infty}\frac{\ln\left(1 - \frac{1}{2^n}\right)}{-\frac{1}{2^n}} = 1$. ⇒ $\sum \ln\left(1 - \frac{1}{2^n}\right)$ and $\sum\left(-\frac{1}{2^n}\right)$ both converge or diverge.

*

* Some ways to prove that a sequence is convergent (Aug 1997 2)

- If $(x_n)$ $\overset{\text{decreasing}}{\text{increasing}}$, prove that $x_n$ is bounded. (monotone + bounded ⇒ converges).

  EX: $x_n = \sum_{k=1}^{n} a_k$ where $a_k \geq 0$ EX: $x_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}$.

  Trick: $\boxed{\frac{1}{n^2+k} \leq \frac{1}{n^2+1}, \forall k \geq 1}$

- Consider if the terms are sum/−/·/: of other terms of convergent ~~sub~~sequences.

- If all the term $\geq 0$, $\leq 0$ ⟹ prove monotonic + bounded.

- $\left.\begin{array}{l} a_n \leq x_n \leq b_n \\ a_n, b_n \longrightarrow \text{to the same limit } L \end{array}\right\} \Rightarrow x_n \to L$.

- def
- every subsequences converge to the same limit.

- Cauchy (in $\mathbb{R}$)

- sequence with both positive terms, negative terms } ⇒ prove every subsequences converges to 0.

- $\left.\begin{array}{l} \lim a_n = L < +\infty \\ |a_n - b_n| \longrightarrow 0 \\ (\text{or which means } \exists n_0, \forall n \geq n_0, |a_n - b_n| < \epsilon \end{array}\right\}$ then $\lim_{n\to\infty} b_n = 0$ $\overset{\text{apl}}{(\text{Jan 2009/22})}$.

---

* Ways to compute value of limit:

① $f$ is continuous, then $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$

② L'Hospital rule $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$

* Prove that 3.7 Theorem.

The set S of subsequence limits is a closed set.

(X, d) : metric space

(p_n) : sequence in X

$S = \{x \in X, \exists p_{n_k}, p_{n_k} \to x\}$

$\Rightarrow$ Then S is closed in X.

---

* **Way 1**: Let s is a limit point of S.

$\Leftrightarrow \forall N_\varepsilon(s), (N_\varepsilon(s)\setminus\{s\}) \cap S \neq \phi$

$\Leftrightarrow \forall \varepsilon_1 > 0, \exists x \in S, 0 < d(x, s) < \varepsilon_1$

We prove that $s \in S$

NTP $\exists (p_{n_k}) \longrightarrow s$.

We have when s is a limit point of S $\Rightarrow \forall \varepsilon_1 > 0, \exists x \in S, 0 < d(x, s) < \varepsilon_1$.

because $x \in S$, then $\exists p_{n_k} \to x \Leftrightarrow \forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, d(p_{n_k}, x) < \varepsilon_2$ $\Big\} \Rightarrow$

$\Rightarrow \forall k \geq k_0, \quad d(p_{n_k}, s) \leq d(x, s) + d(x, p_{n_k}) < \varepsilon_1 + \varepsilon_2 \Rightarrow p_{n_k} \to s \Rightarrow s \in S \Rightarrow \square$

---

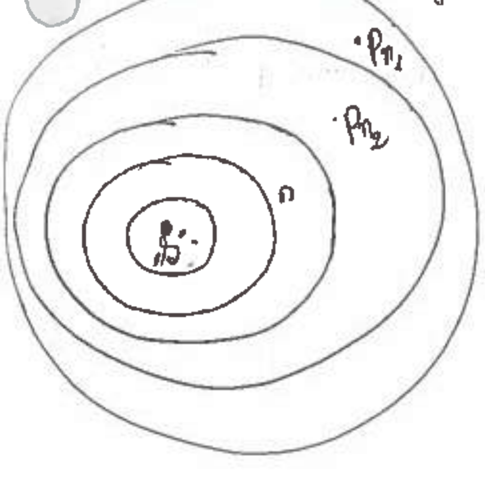
* **Way 2** (Kovaler's) We will prove that $(X \setminus S)$ is open

$\Leftrightarrow$ NTP, $\forall s \in (X \setminus S)$, then $\exists N_\lambda(s) \subset X \setminus S$.

- **Step 1**: Let $s \in X \setminus S$. We will prove that $\exists N_\lambda(s)$ that contains finitely many terms of $p_n$

Assume claim is false: $\forall \lambda > 0, N_\lambda(s)$ contains infinitely many term of $\{p_n\}$

$\Leftrightarrow \forall n > 0, N_{1/n}(s)$



Choose $n_1$ s.t $p_{n_1} \in N_{1/n_1}(x)$.

$n_2 > n_1$ s.t $p_{n_2} \in N_{1/n_2}(x)$

Choose $n_3 > n_2$ s.t $p_{n_3} \in N_{1/n_3}(x)$

$\vdots$

$n_k > n_{k+1}$ s.t $p_{n_k} \in N_{1/k}(x)$

$\Rightarrow p_{n_k} \to s$ because $d(p_{n_k}, s) < k$ (contradiction with $s \in X \setminus S$

? s is a point s.t $\forall \lambda > 0, N_\lambda(s)$ contains infinitely many point of $\{p_n\}$ $\Longrightarrow$ s is a limit of some subsequence $p_{n_k}$ $(p_{n_k} \to s)$

- **Step 2**: Now we prove $s \in (X \setminus S)$ then $\exists N_\lambda(s) \subseteq (X \setminus S)$

$\exists N_\lambda(s) \cap S = \phi$.

Assume a contradiction that $\exists y \in S, y \in \underbrace{N_\lambda(s)}_{\text{open}}$

$y \in S \Rightarrow \exists p_{n_k} \longrightarrow y \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n_k \geq n_0,$ ~~d~~ $p_{n_k} \in N_\varepsilon(y)$

Choose ε so that $N_\varepsilon(y) \subseteq N_\lambda(s)$

Then $N_\lambda(s)$ contains all $p_{n_k}$, where $n_k \geq n_0$ (contradiction with step 1

* Prove that (3.6T)

a) $X$ is compact, then every sequence $\{p_n\}$ (in $X$) has a convergent subsequence

b) Every ~~sequence~~ (bounded) sequence in $\mathbb{R}^n$ has a convergent subsequence

Prove a (Way 1) (Kovalev's) (Use property that $S$ is closed in $X$)

(Assume every subsequence of $\{p_n\}$ does not converge (which means $S = \phi$)
$X$ is compact } → (we want to have some contradiction)

Assume $S = \phi$, then $\forall x \in X$, $x \notin S$

this means $\forall N_\lambda(x)$, $N_\lambda(x)$ contains finitely many points of $\{p_n\}$ (1)

(See the proof for this claim on back)

and we also have $X \subseteq \bigcup_{x \in X} N_\lambda(x)$

we have $X$ is compact } ⇒ ∃ finite subcover $X \subseteq \bigcup_{i=1}^{n} N_\lambda(x)$ (2)

⇒ $X$ contains finitely many points of $\{p_n\}$, this contradicts with the fact that $\{p_n\}$ have infinitely many terms.

---

* Way 2 (Rudin's book) (In fact these ways are similar, because the Rudin's way prove directly, doesn't use property of $x \notin S$)

(Prove directly)

Remind: Weierstrass's theorem (is a corollary of below theorem)

every (bounded) (infinite) subset of ($\mathbb{R}$) has a limit point.

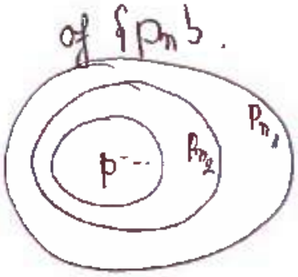
(More theorem with compact.. $X$ (compact)
$K$ is a (infinitely) subset of $X$ } ⇒ $K$ has a limit point in $X$

* We have $\{p_n\}$ is a sequence in $X$ compact (⇒ closed + bounded) ⟹ $\{p_n\}$ is a bounded sequence

let $K = \{$range of $p_n\}$, then we have $K$ is bounded.

- In case $K$ has finitely value: then there are some value ~~happens~~ appear many times
  ⇒ take the subsequence by $p \in \{p_n\}$
  ↑ appears many times

- In case $K$ has infinitely value in $X$
  then by theorem that every infinitely subset $K$ of a compact space $X$ has a limit in $X$
  we have ∃ $p$ is a limit point of $K$
- Because $p$ is a limit of $K$, then every neighborhood of $p$ contains infinitely many terms of $\{p_n\}$.



Choose $n_1$ s.t $p_{n_1} \in N_1(p)$

$n_2 > n_1$ s.t $p_{n_2} \in N_{1/2}(p)$.

⋮

$n_k$ s.t $p_{n_k} \in N_{1/k}(p)$

⇒ we have $(p_{n_k}) \longrightarrow p$ because $d(p, p_{n_k}) < \frac{1}{k}$.

then ⇒ $\square$.

* Prove a Cauchy sequence has a convergent subsequence (assume to p) ⇒ converges to p. (* Important result)

Let $\{p_n\}$ is a Cauchy sequence

$\{p_n\}$ has a subsequence $\{p_{n_\ell}\}$ (converges) $\iff$ $\{p_n\}$ converges.

($\Longleftarrow$) obvious.

($\Longrightarrow$): $\{p_n\}$ Cauchy $\iff \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0 \; d(p_m, p_n) < \varepsilon$ (1)

$\exists \{p_{n_\ell}\}$ converges (assume to p). $\iff \forall \varepsilon > 0, \exists \ell_0 \in \mathbb{N}, \forall \ell \geq \ell_0, d(p_{n_\ell}, p) < \varepsilon$ (2).

Choose $N = \max\{n_0, \ell_0\} + L$

Then $\forall m, ~~\ell~~ \geq N$, ~~d(p_n~~ (because $n_\ell \geq n$), $\left.\begin{array}{l} d(p_m, p_{n_\ell}) < \varepsilon \\ d(p_{n_\ell}, p) < \varepsilon \end{array}\right\} \longrightarrow$

$\Rightarrow d(p_m, p) \leq d(p_m, p_{n_\ell}) + d(p_{n_\ell}, p) < 2\varepsilon \Rightarrow \{p_n\} \to p$. □

* Prove that in a Compact metric space every Cauchy sequence is convergent

Prove that Compact metric space ⇒ Complete metric space

~~We have from theorem 3.7 that~~

$\left.\begin{array}{l}\text{Let } (X, d)\text{: compact metric space} \\ \{p_n\} \text{ is a Cauchy sequence in } (X, d)\end{array}\right\}$ Prove that $\{p_n\}$ converges

We have because $\{p_n\}$ is a sequence in a compact metric space

by theorem 3.7 (every sequence in a compact metric space has a convergent subsequence)

$\Rightarrow \exists p_{n_\ell}$ converges.

$\{p_n\}$ Cauchy

(then by above result) a Cauchy sequence that has a convergent subsequence ⇒ converges $\Big\} \to p_n$ Conv

□

* X is a complete metric space ⇒ Prove that X is closed. | easy by def of complete space.

* $\left.\begin{array}{l} X \text{ is a complete metric space.} \\ E \text{ is closed in } X\end{array}\right\} \Rightarrow$ Prove that E is a complete metric space.

We NTP If $(p_n)$ is a Cauchy sequence in E, then $p_n \to p$, $p \in E$

We have because $\left.\begin{array}{l}(p_n) \text{ is Cauchy in } E \\ E \subset X\end{array}\right\} \Rightarrow \left.\begin{array}{l}(p_n) \text{ Cauchy in } X \\ \text{we have } X \text{ complete}\end{array}\right\} \Rightarrow (p_n) \longrightarrow y, y \in X$

$\left.\begin{array}{l}\text{Because } (p_n) \subseteq E,\; p_n \to y \in X \Rightarrow y \text{ is a limit point of } E \\ E \text{ is closed}\end{array}\right\} \Rightarrow y \in E$

(or we can understand: because $(p_n) \to y$, then every neighborhood of y contains all but finitely many pts of $p_n$) $\iff$ ~~$d(y, p_n)$~~ $N_r(y) \cap E \neq \phi \Rightarrow y \in \bar{E} \Rightarrow y \in E$

Need to redo
for better understd

* Prove that (Theorem 3.17).

$$\boxed{\text{If } \alpha = \limsup s_n \quad \text{then } \exists\, s_{n_k},\ s_{n_k} \longrightarrow \alpha}$$

* If $\alpha = \infty$, then we have this equivalent with $x_n$ has no upper bound.

$\Rightarrow x_{n_k} \longrightarrow \infty$.

* In case $\alpha \in \mathbb{R}$, we have $\exists$ infinitely many $x_n$, s.t. $x_n > a - \varepsilon$
(otherwise $x_n \leq a - \varepsilon$, $\forall n \geq N$, we can't have subsequence limit $> a - \varepsilon$, $\Rightarrow$ contrad)

Pick $x_{n_1} > a - 1$

$x_{n_2} > a - \frac{1}{2}$ $\quad n_2 > n_1$

so we get a subsequence (this subsequence increasing, bounded) $\Rightarrow x_{n_k} \rightarrow L$.

- We have $x_{n_k} > a - \frac{1}{k} \Rightarrow L \geq a$
$a = \sup S \Rightarrow L \leq a$

$$\left.\begin{array}{l} x_{n_k} > a - \frac{1}{k} \Rightarrow L \geq a \\ a = \sup S \Rightarrow L \leq a \end{array}\right\} \Rightarrow L = a \quad \square$$

* Prove that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.

- The above series has partial sum $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ with lcm$(1, 2!, 3!, \dots, n!) = n!$

- We have

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}\right| = \left|\frac{1}{k+1}\right| \boxed{\le \overset{b}{\overset{\shortparallel}{\frac{1}{n+2}}}} \text{ for } \boxed{k \geq n+1}$$

So we have (by remainder estimate):

$$r_n = \left|\sum_{k=n+1}^{\infty} a_k\right| \le \frac{a_{n+1}}{1-b} = \frac{1}{(n+1)!\left(1-\frac{1}{n+2}\right)}$$

So we have

$$r_n \cdot \text{lcm}(q_1, \dots, q_n) \le \frac{n!}{(n+1)!\left(1-\frac{1}{n+2}\right)} = \frac{1}{(n+1)\frac{(n+1)}{(n+2)}} = \frac{(n+2)}{(n+1)^2} \xrightarrow{n\to\infty} 0$$

So we have e is irrational.

* ? Consider ~~the rational~~ if this series is rational / irrational $\sum_{n=1}^{\infty}\frac{1}{2^n}$ (we know this series $= \frac{1}{1-\frac{1}{2}} = 2$)

we want to use the test for this series

- The partial sum $s_n = \sum_{k=1}^{n}\frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n}$ has lcm(denominator) $= 2^n$

- $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{1}{2^{k+1}}}{\frac{1}{2^k}}\right| = \left|\frac{1}{2}\right| < \left(b = \frac{3}{4}\right)$

Then the remainder $|r_n| < \frac{a_{n+1}}{1-b} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{4}} = \frac{1}{4\cdot 2^{(n+1)}} = \frac{1}{2^{n+3}}$

Then by the test $r_n \cdot \text{lrm}(\text{denominator}) = \frac{2^n}{2^{n+3}} = \frac{1}{2^3} \not\to 0$ ⇒ do not tell that $\sum$ is irrational.

* Way ‡:

$$r_n = \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}}\left(1+\frac{1}{2}+\cdots\right) = \frac{1}{2^{n+1}}\frac{1}{1-\frac{1}{2}} = \frac{1}{2^n}$$

Then lcm(denominator) $\cdot r_n = 1 \not\to 0$ ⇒ no conclusion (that the series is irrational)

+ Example: Consider if the series is rational or irrational?

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

- The partial sum $s_n = \sum_{k=1}^{n} \frac{p_k}{q_k}$ has $\text{lcm}(q_1, \dots, q_n) = \text{lcm}(1, 2^3, 3^3, 4^3, \dots, n^3) =$
$= [\text{lcm}(1, \dots, n)]^3$

- Now consider,

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{(k+1)^3}}{\frac{1}{(k)^3}} = \frac{k^3}{(k+1)^3} = 1$$ (does not $< b$ for some $b < 1$) $\Rightarrow$ can use ratiotest.

- Now we use condensation:

Remainder $$r_n = \sum_{k=n+1}^{\infty} \frac{1}{k^3} = \underbrace{\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \cdots + \frac{1}{(2n)^3}}_{\le n \frac{1}{n^3}} + \underbrace{\frac{1}{(2n+1)^3} + \cdots + \frac{1}{(4n)^3}}_{\le 2n \frac{1}{(2n)^3}} + \underbrace{\frac{1}{(4n+1)^3} + \cdots + \frac{1}{(8n)^3}}_{\le \frac{1}{4n} \frac{1}{(4n)^3}}$$

$$\le \frac{1}{n^2} + \frac{1}{(2n)^2} + \frac{1}{(4n)^2} + \cdots$$

$$= \frac{1}{n^2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) = \frac{1}{n^2}\left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{3}{4n^2}$$

Then we have $r_n \cdot \text{lcm(denominator)} = \frac{3}{4n^2} \cdot [\text{lcm}(1, \dots, n)]^3 \not\longrightarrow 0$

$\Rightarrow$ no conclusion $\square$.

* Series:

* 3.21: Def: Given a sequence $\{a_n\}$.

- $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=10}^{\infty} a_n$ is an infinite series / or series.
- $s_n = \sum_{\ell=1}^{n} a_\ell$ : partial sum of the series.

- If $s_n \to s \iff \sum a_n \to s$, and we write $\sum_{n=1}^{\infty} a_n = s$.
- If $s_n$ diverges $\Leftrightarrow \sum a_n$ diverges.

* Sequence can be stated in term of series $\begin{cases} a_1 = s_1 \\ a_n = s_{n+1} - s_n, \end{cases} \forall n > 1$ $\Bigg|$ $\sum_{\ell=n}^{m} a_\ell = s_m - s_n$

* 3.22: (Cauchy criterion)

$\sum a_n$ converges $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, \left|\sum_{\ell=n}^{m} a_\ell\right| < \varepsilon$

$\left|\sum_{\ell=n}^{n+k} a_\ell\right| < \varepsilon, \forall k > 0$

* 3.23: $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges.

* 3.24: Given $a_n \in R$, $\boxed{a_n \geq 0}$, $\forall n$

$\sum a_n$ converges $\Leftrightarrow \{s_n\}$ form a bounded sequence

(monotonically increasing)

* 3.25 Comparison test:

- If $\exists n_0 \in \mathbb{N}$, $\boxed{\forall n \geq n_0}$, $0 \leq |a_n| \leq c_n$ ; $\sum c_n$ converges $\Big\} \Rightarrow \sum |a_n|$ converges $\Rightarrow \sum a_n$ converges.
- If $\exists n_0 \in \mathbb{N}, \forall n \geq n_0$, $a_n \geq d_n \geq 0$ ; $\sum d_n$ diverges $\Big\} \Rightarrow \sum a_n$ diverges

---

* If we want to prove 2 series of $\boxed{\text{nonnegative}}$ terms both converges or diverge

→ prove 2 partial sum are both ~~diverges~~ bounded / unbounded $\begin{pmatrix} t_k \leq N s_n \\ s_n \geq M t_m \end{pmatrix}$

## * Series of nonnegative terms

Note: $\sum_{n=1}^{\infty} a_n$, a series with $a_n \geq 0$ $\Rightarrow \{s_n\}$: increasing sequence

$\sum a_n$ converges $\Leftrightarrow$ $\{s_n\}$ bounded sequence

---

* 3.26 (Geometric theory)

$$\sum_{n=0}^{\infty} x^n \begin{cases} = \dfrac{1}{1-x} & \text{if } 0 \leq x < 1 \\ \text{diverges} & \text{if } x \geq 1 \end{cases}$$

~~Note: this series begins with n=0~~

$$* \sum_{n=0}^{\infty} (-1)^n x^n = 1 + x + x^2 - x^3 + x^4 - x^5 + \cdots = \begin{cases} \dfrac{1}{1+x} & \text{if } x < 1 \\ \text{diverges} & x \geq 1 \end{cases}$$

---

* 3.27 Cauchy condensation test. Fall 1993

Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ (Note: $a_n \geq 0$, $\forall n$ and decreasing)

then $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$ converges.

---

*

$$\sum_{n=1} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

(note: $\sum \frac{(-1)^n}{n}$ converges non absolutely)

* 3.28:

$$\sum_{n=2} \frac{1}{n(\log n)^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

* $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log(\log n))}$ diverges.

$\sum_{n=3}^{\infty} \frac{1}{n \log n [\log(\log n)]^2}$ converges.

---

* $\left.\begin{array}{l} \sum a_n \text{ converges} \\ a_n \geq 0 \end{array}\right\} \Rightarrow a_n \longrightarrow 0$ $\Rightarrow a_n < 1$ for n large enough

$\Rightarrow a_n^k \leq a_n$ for n large

$\Rightarrow \sum_{n=1}^{\infty} a_n^2, \sum_{n=1}^{\infty} a_n^k$ converges, for $k \geq 2$

* $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges

$\left.\begin{array}{l} \sum a_n \text{ converges} \\ a_n \geq 0 \end{array}\right\} \Rightarrow \sum |a_n|$ converges.

---

* There is no 'smallest divergent' or 'biggest convergent' series (positive terms)

- $\left.\begin{array}{l} \sum a_n \text{ diverges} \\ a_n > 0 \end{array}\right\} \Rightarrow \exists\, b_n > 0$, $\frac{b_n}{a_n} \to 0$, $\sum b_n$ diverges
- $\left.\begin{array}{l} \sum a_n \text{ converges} \\ a_n > 0 \end{array}\right\} \Rightarrow \exists\, b_n > 0$, $\frac{b_n}{a_n} \longrightarrow \infty$, $\sum b_n$ converges.

+ Prove that with series have (positive terms), there is $\boxed{\text{no}}$ ... divergent series
there is $\boxed{\text{no}}$ .... convergent series

---

+ Prove that there is no smallest divergent series (series with positive terms).

$$\left.\begin{array}{l}\text{Let } a_n \ \sum a_n \text{ diverges} \\ \quad a_n > 0\end{array}\right\} \text{ Then } \exists \{b_n\} \quad \begin{array}{l} b_n > 0 \\ \lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = 0 \\ \sum b_n \text{ diverges}\end{array}$$

# * The root and the ratio tests.

3.33 **Root test**:

Given $\sum a_n$. Put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \in [0, +\infty]$

a) $\alpha < 1 \Rightarrow \sum |a_n|$ converges

b) $\alpha > 1 \Rightarrow \sum |a_n|$ diverges

c) $\alpha = 1 \Rightarrow$ the test gives no information → need to check

We can also use this test in case $a_{2n}$ and $a_{2n+1}$ have ≠ induction formula
we only need to compute
$\limsup \sqrt[2n]{a_{2n}}$ and $\limsup \sqrt[2n+1]{a_{2n+1}}$
and compute max $< 1$
or min $> 1$.

3.34 * **Ratio test**: $\alpha = \limsup \left|\frac{a_{n+1}}{a_n}\right|$

a) $\alpha < 1 \Rightarrow \sum |a_n|$ converges

b) $\alpha > 1 \Rightarrow \sum |a_n|$ diverges

c) $\alpha = 1 \Rightarrow$ gives no information

- if $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the ratio test does not apply

3.35 Theorem: For $\{c_n\}$ is a positive sequence

$$\liminf \frac{c_{n+1}}{c_n} \le \liminf \sqrt[n]{c_n} \le \limsup \sqrt[n]{c_n} \le \limsup \frac{c_{n+1}}{c_n}.$$

* In both root test and ratio test, we note that if lim exist then
$\limsup = \liminf = \lim \Rightarrow$ the root/ratio test can be applied with lim.

* In case $\sum a_n$ converges $\rightarrow \left[\begin{array}{l}\alpha < 1\\ \alpha = 1\end{array}\right.$

- If $\lim \frac{a_{n+1}}{a_n} < 1$ then $\lim \frac{a_n}{a_{n+1}} > 1$ (Aug 1999)

- If we can prove $\limsup \frac{b_n}{a_k} \le n$ for all $k$

# * The number e

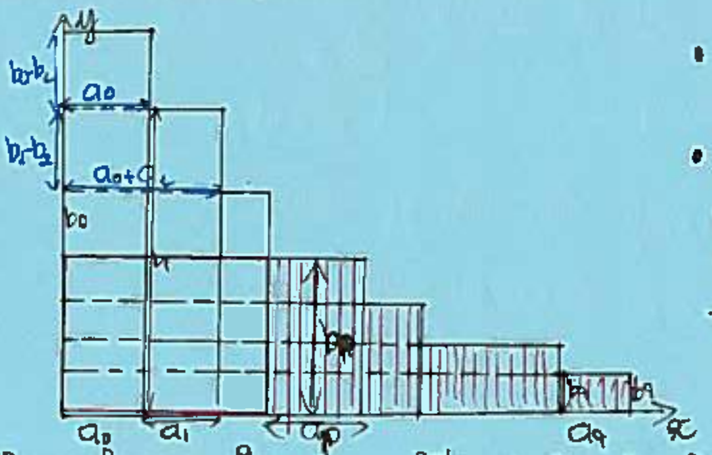
$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

* Remine: Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

## * Summation by parts:

There are 2 ways to sum the area of below rectangles: width $a_n$, heigh $b_n$ $\boxed{b_n \text{ decreasing}}$



- Total area $= \sum_{n=0}^{q} a_n b_n$

- Another way:
  width: $b_n - b_{n-1}$
  height: $A_n = a_0 + a_1 + \cdots + a_n$.
  Total area $= \sum_{n=0}^{q-1} A_n(b_n - b_{n-1}) + A_q b_q$

Then we have: $\sum_{n=0}^{q} a_n b_n = \sum_{n=0}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q$

More generally $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$ , $0 \le p \le q$

## * Theorem Dirichlet test:

Suppose $\sum a_n$ has bounded partial sum
$b_n \geq 0, \forall n$, decreasing, $\lim_{n\to\infty} b_n = 0$ $\Big\} \Rightarrow \sum a_n b_n$ converges.

## * Exercise 3.8.

$\sum a_n$ converges.
$\{b_n\}$ monotonic + bounded $\Big\} \Rightarrow \sum a_n b_n$ converges

$\bullet \sum a_n \longrightarrow a$
$\sum b_n \longrightarrow b$
$a_n \geq 0, \forall n$ $b_n \geq 0, \forall n$ $\Big\} \Rightarrow \sum a_n b_n$ converges to ab

---

## * Theorem: Alternative series test.

a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$ (or we only need $|c_n| \downarrow$ when n is large enough)
$c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $m = 1, 2, 3, \ldots$
$\lim_{n\to\infty} c_n = 0$
Then $\sum c_n$ converges.

---

## * Theorem: Give a series $\sum c_n z^n$

Suppose a) The convergent radius of this series is 1
b) $c_0 \geq c_1 \geq c_2 \geq \cdots$
c) $\lim c_n = 0$

$\Rightarrow$ Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except at $z = 1$ 'pointly' (we need to check at $z = 1$, the series may converges or diverge at this point)

---

* In Dirichlet theorem, we need $\{b_n\}$ decreasing

EX: Let $\{a_n\} = (-1)^n$ $b_n = \frac{1+(-1)^n}{n}$, $b_n \geq 0$, $b_n \longrightarrow 0$ but $b_n$ is not decreasing

then $\sum a_n b_n = \sum \frac{(-1)^n + 1}{n} = \frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum \frac{1}{n} \to$ diverges. 'harmonic series

# Absolute convergence.

* **Def**: $\Sigma a_n$ is said to converge absolutely $\overset{def}{\Longleftrightarrow}$ $\Sigma|a_n|$ converges.

$\Sigma a_n$ converges nonabsolutely $\overset{def}{\Longleftrightarrow} \begin{cases} \Sigma a_n \text{ converges} \\ \Sigma|a_n| \text{ diverges} \end{cases}$

* Theorem: $\Sigma a_n$ converges absolutely $\underset{\not\Leftarrow}{\Rightarrow}$ $\Sigma a_n$ converges

* $\Sigma a_n$ converges unconditionally $\Leftrightarrow$ $\Sigma a_n^k$ converges for any order of terms.

$\Sigma a_n$ converges conditionally $\Leftrightarrow$ $\Sigma a_n^k$ converges for this order, but not all

* For $a_n \in \mathbb{R}$, or $a_n \in \mathbb{C}$, converge conditionally $\Leftrightarrow$ non-absolute convergence

* Remark. $a_n \geq 0, \forall n$ then converges $\Leftrightarrow$ converges absolutely.

* Ratio test, root test $\Rightarrow$ test for absolutely convergence

comparison test $\Rightarrow$ test for absolute convergence

Dirichlet test $\Rightarrow$ yeild non-absolute convergence.

[redacted]

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* $\Sigma a_n$ converges absolutely ($\Sigma|a_n|$ converges) $\Rightarrow$ $\Sigma a_n^k$ converges, $\forall k \in \mathbb{N}$.

---

* One important property of absolute convergent sequence: (prelim 1993 e3).

If $\Sigma a_n$ converges absolutely $\Rightarrow$ any rearrangement has the same sum

**We have**: $\left.\begin{matrix} \Sigma a_n \text{ converges} \\ a_n \geq 0 \end{matrix}\right\} \Rightarrow$ any $\Sigma a_n'$ converges (rearrangement)

# * Addition and multiplication of series.

## * Addition:

* **Theorem**: If $\sum a_n = A$, $\sum b_n = B$ $\Big\}$ — this means we only need $\sum a_n, \sum b_n$ converges.

Then $\sum (a_n + b_n) = A + B$

$\sum (c a_n) = c A$ . for only fixed c

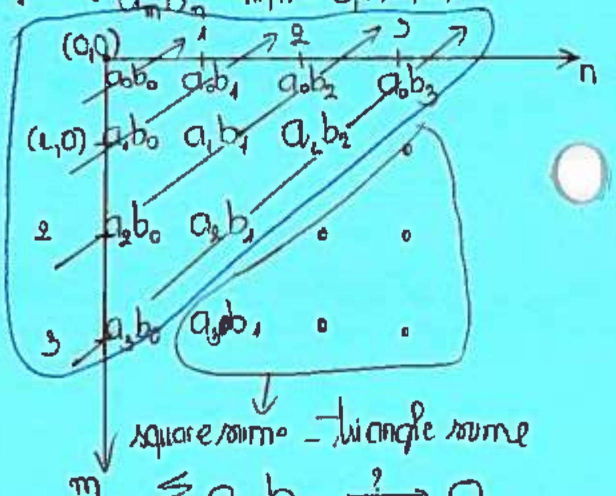
* If $\sum a_n, \sum b_n$ converges absolutely $\Rightarrow$ $\sum (a_n + b_n)$ converges absolutely, then $\sum |a_n + b_n| = \sum |a_n| + \sum |b_n|$.

( If $a_n, b_n \geqslant 0$ , converges $\Leftrightarrow$ converges absolutely ;

In general case : $\sum_{n=p}^{q} |a_n + b_n| \leq \sum_{n=p}^{q} |a_n| + \sum_{n=p}^{q} |b_n|$

meas $\sum_{n=1}^{\infty} |a_n + b_n| \leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n|$

* Multiplication of series : Want to compute $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = ?$ ( we need to arrange all $a_m b_n$ $m, n = 0, 1, 2, 3, \dots$



square sum - triangle sume

$$\sum_{\substack{m+n > N \\ m, n \leq N}} a_m b_n \xrightarrow{?} 0$$

+ Given $\sum a_n$ and $\sum b_n$

Cauchy product $\underbrace{c_n = \sum_{k=0}^{n} a_k b_{n-k}}_{\text{triangle sum}}$ $(n = 0, 1, 2, \dots)$

- Square sum $\left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{n} b_k\right) \rightarrow$ square sum .

of these require the methods of this section, while others are drawn from the preceding sections (just to keep you thinking about the big picture). For the sake of convenience, we summarize our convergence tests in the table that follows.

| Test | When to use | Conclusions | Section |
|---|---|---|---|
| **Geometric Series** | $\sum_{k=0}^{\infty} ar^k$ | Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$;<br>diverges if $\lvert r\rvert \geq 1$. | 8.2 |
| **kth Term Test** | All series | If $\lim_{k\to\infty} a_k \neq 0$, the series diverges. | 8.2 |
| **Integral Test** | $\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and<br>$f$ is continuous, decreasing and $f(x) \geq 0$ | $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$<br>both converge or both diverge. | 8.3 |
| **p-series** | $\sum_{k=1}^{\infty} \frac{1}{k^p}$ | Converges for $p > 1$; diverges for $p \leq 1$. | 8.3 |
| **Comparison Test** | $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$ | If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.<br>If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges. | 8.3 |
| **Limit Comparison Test** | $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $\boxed{L \geq 0}$<br>$a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L < \cancel{0} + \infty$ | $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$<br>both converge or both diverge. | 8.3 |
| **Alternating Series Test** | $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all $k$ | If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all $k$, then the series converges. | 8.4 |
| **Absolute Convergence** | Series with some positive and some negative terms (including alternating series) | If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then<br>$\sum_{k=1}^{\infty} a_k$ converges (absolutely). | 8.5 |
| **Ratio Test** | Any series (especially those involving exponentials and/or factorials) | For $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = L$,<br>if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely<br>if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges,<br>if $L = 1$, no conclusion. | 8.5 |
| **Root Test** | Any series (especially those involving exponentials) | For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$,<br>if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely<br>if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges,<br>if $L = 1$, no conclusion. | 8.5 |

prove in Aug 1999 P2

If $a_n \geq 0$, $b_n \geq 0$, $\forall n$.
$\limsup \frac{a_n}{b_n} \leq M < +\infty$
$\sum b_n$ converges
$\Rightarrow \sum a_n$ converges. ( Aug 1999 P 2).

don't need $\boxed{M > 0}$

+ Some important results

- $\left.\begin{array}{l}\sum a_n \text{ converges} \\ a_n \geq 0\end{array}\right\} \Rightarrow \begin{array}{l}\sum a_n^k \text{ converges } k \geq 1 \\ \sum \frac{\sqrt{a_n}}{n} \text{ converges. (}\end{array}$

$\left.\begin{array}{l}\sum a_n, \sum b_n \text{ which } a_n, b_n > 0, \forall n \\ \sum b_n \text{ converges.} \\ \frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}, \forall n \geq n_0.\end{array}\right\} \Rightarrow \sum a_n \text{ conv.}$

Aug 2007.

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* Important : $\left.\begin{array}{l}\sum a_n^2 \text{ converges} \\ \sum b_n^2 \text{ converges}\end{array}\right\} \Rightarrow \sum a_n b_n$ converges (comparison test

- Want to prove $\sum \frac{a_n}{b_n}$ converges $\Rightarrow$ we NTP $\left\{\begin{array}{l}\sum a_n^2 \text{ converges} \\ \sum \frac{1}{b_n^2} \text{ converges}\end{array}\right.$.

---

$\left\{\begin{array}{l}\text{or if } a_n \leq 1, \forall n. \\ \Rightarrow \frac{a_n}{b_n} \leq \frac{1}{b_n} \\ \sum \frac{1}{b_n} \text{ converges}\end{array}\right.$ $\Rightarrow \sum \frac{a_n}{b_n}$ converges

---

+ Important inequality :

- $(1+nx) \leq (1+x)^n, \forall n \in \mathbb{N}, \boxed{x > -1}$
- $(a+b)^n = \sum C_n^k a^k b^{n-k} \geq \underbrace{C_n^i a^i b^{n-i}}_{\text{depends to the exercise to chose } i}, \forall i = \overline{1, n}$
- $a^n - 1 \geq \frac{1}{2} a^n$ $\boxed{a > 1}$ for $n$ big enough.
- $\left\{\begin{array}{l}x_1 \\ x_{n+1} = x_n + d\end{array}\right.$ then $S = x_1 + \cdots + x_n = \frac{n(x_1 + x_n)}{2} = \frac{n(x_1 + (n-1)d)}{2}$
- $x_n = x_1 x^{n-1}$ then $S = x_1 + \cdots + x_n = \frac{x_1 (1-x)^n}{(1-x)}$
- $\ln n \leq n$

For ex $\sum \frac{1}{\ln n}$ we have $\left.\begin{array}{l}\frac{1}{\ln n} > \frac{1}{n} \\ \sum \frac{1}{n} \text{ diverges}\end{array}\right\} \Rightarrow \sum \frac{1}{\ln n}$ diverges.

- $\left.\begin{array}{l}\lim\limits_{n \to \infty} \frac{a^n}{n!} = 0 \\ \lim \frac{n^d}{a^n} = 0, \forall d \in \mathbb{R} \quad \boxed{a > 1}\end{array}\right\} \Rightarrow n^d \ll a^n \ll n!$

* Important results relating to chapter 3 / Series.

- Sample A/1:
  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges. $\sum (a_n)^2 = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

$$\left.\begin{array}{l}\sum a_n \text{ converges} \\ \sum b_n \text{ converges}\end{array}\right\} \not\Rightarrow \sum a_n b_n \text{ converges}$$

- $\left.\begin{array}{l}\sum a_n \text{ converges}; \boxed{a_n \geq 0}, \forall n \\ \sum b_n \text{ converges}, \boxed{b_n \geq 0} \forall n\end{array}\right\} \Rightarrow \sum a_n b_n$ converges.
- ⇒ $\sum a_n$ converges, $\boxed{a_n \geq 0, \forall n} \Rightarrow \sum a_n^k$ converges, $\forall k \geq 1$
- use ratio test / root test Jan 2008.
- prove that $\left.\begin{array}{l}\sum a_n^2 \\ \sum b_n^2\end{array}\right\}$ converges

* Aug 1993, ②

If $a_n \geq 0$, $\sum a_n$ converges

Prove that $\exists \{b_n\}$, $\lim_{n\to\infty} b_n = +\infty$ and $\sum a_n b_n$ converges

Also another problems (Kov)
think about $r_n = \sum_{k=n}^{\infty} a_k$
(in this problem $b_n = \frac{1}{\sqrt{r_n}}$

* Aug 1992, P1 Also MAT601 HW3.5, P1,

$a_n \longrightarrow a$

Then $\frac{1}{n} \sum_{k=1}^{n} a_k \longrightarrow a$.

Note that with problem requiring us to prove that
$\lim_{n\to\infty} k_n = a$ ⇒ we may consider $|a_n - a|$.

---

* Kind: Given $a_n \longrightarrow L$. Prove that $A(p) = \sum_{k=1}^{\infty} f_k(p) a_k$ converges.

- Aug 2002,.

$\{a_n\}$ sequence of real number, $a_\ell \xrightarrow{\ell\to\infty} L$

For $0 < p < 1$, prove that $\sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} a_\ell \xrightarrow{\quad p\to\infty \quad} L$

We note that $|a_\ell - L| < \varepsilon$ when $\ell$ large engh.
and $\sum_{\ell=0}^{\infty} p(1-p)^{\ell} = 1$.

$$\Rightarrow L = L \sum_{\ell=1}^{\infty} (p)(1-p)^{\ell-1}$$

$$\Rightarrow \left|\sum_{\ell=1}^{n} (p(1-p)^{\ell-1} a_\ell - L\right| = \left|\sum p(1-p)^{\ell-1} a_\ell - \sum p(1-p)^{\ell-1}\right.$$

- Jan 2009

$\left.\begin{array}{l}\{a_n\}: \text{sequence of real number}, a_\ell \xrightarrow{\ell\to\infty} L \\ b_n = \frac{1}{n^2} \sum_{k=1}^{n} k a_k.\end{array}\right\}$ Prove that $b_n \longrightarrow \frac{L}{2}$

* Template $\sum_{n=1} a_n \le \sum_{n=1} b_n$ | Aug2013 prove $p_\ell \geq 0$, $\sum p_\ell = 1$ $\Big\}$ $\to$ prove $\left(\sum \ell p_\ell\right)^2 \le \sum_{\ell=1}^{\infty} \ell^2 p_\ell$

Just need to prove $\sum_{n=1}^{N} a_n \le \sum_{n=1}^{N} b_n \sum_{n=1}^{N} c_n$

$$\left.\begin{array}{l} \text{take } N \longrightarrow \infty \\ \sum_{n=1}^{\infty} c_n = 1 \end{array}\right\} \Rightarrow \text{done} \quad \square .$$

+ One good trick can be used in $\sum a_n$ is $\sum a_n = \sum_{n=1}^{N} a_n + \sum_{n=N}^{\infty} a_n$

Sea Aug 1993

* Prove that there is $\left\{\begin{array}{l}\text{no biggest (positive) convergent series.}\\ \text{no smallest (positive) divergent series.}\end{array}\right.$

Which means.

a) Give $\left.\begin{array}{l}\sum a_n \text{ convergent}\\ a_n \geqslant 0\end{array}\right\}$ Then $\exists \{b_n\}$, $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ convergent.

b) Give $\left.\begin{array}{l}\sum a_n \text{ divergent}\\ a_n \geqslant 0\end{array}\right\}$ Then $\exists \{b_n\}$ $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ divergent.

---

* Prove a) there is no biggest convergent (positive) series.

Give $\left.\begin{array}{l}\sum a_n \text{ convergent}\\ a_n > 0, \forall n\end{array}\right\}$ Then $\exists \{b_n\}$. $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ convergent.

Let $r_n$ be the remainder of the series. $r_n = \sum_{k=(n+1)}^{\infty} a_k$

Then put $b_n = \sqrt{r_{n-1}} - \sqrt{r_n}$, we have.

- $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\sqrt{r_{n-1}} - \sqrt{r_n}}{r_{n-1} - r_n} = \lim_{n\to\infty} \frac{1}{\sqrt{r_{n-1}} + \sqrt{r_n}} = \infty$

(note that $r_n$ is the remainder of a convergent series, then $(r_n \downarrow)$ and $(r_n \downarrow 0)$

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\sqrt{r_{n-1}} - \sqrt{r_n})$ convergent because $\left\{\begin{array}{l}\sum \sqrt{r_{n-1}}\\ \sum \sqrt{r_n}\end{array}\right.$ convergent. □a)

---

b) Prove that there is no smallest divergent series with positive terms.

Give $\left.\begin{array}{l}\sum a_n \text{ divergent}\\ a_n \geqslant 0, \forall n.\end{array}\right\}$ Prove that $\exists \{b_n\}$, ~~$b_n \geqslant 0$~~

$\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ $\sum_{n=1}^{\infty} b_n$ divergent.

Put $s_n = \sum_{k=1}^{n} a_k$ then we have $\{s_n\}$ increasing and $s_n \longrightarrow \infty$ because $\sum a_n$ divergent

$\{s_n\}\uparrow$, $s_n \longrightarrow \infty$

Put $b_n = \sqrt{s_{n+1}} - \sqrt{s_n}$.

- $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\sqrt{s_{n+1}} - \sqrt{s_n}}{s_{n+1} - s_n} = \lim \frac{1}{\sqrt{s_{n+1}} + \sqrt{s_n}} = 0$ (because $s_n \uparrow \infty$).

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\sqrt{s_{n+1}} - \sqrt{s_n})$ divergent because $\sqrt{\text{partial sum}}: \sqrt{s_n} \longrightarrow \infty$

* Prove Theorem 3.34

For $\{a_n\}$: sequence of positive integ numbers. Prove that

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty}\sqrt[n]{a_n} \leq \limsup_{n\to\infty}\sqrt[n]{a_n} \leq \limsup_{n\to\infty}\frac{a_{n+1}}{a_{n+1}}$$

* Prove that $\limsup_{n\to\infty}\sqrt[n]{a_n} \leq \limsup_{n\to\infty}\frac{a_{n+1}}{a_n}$

- Note that when $\alpha = \limsup_{n\to\infty}\frac{a_{n+1}}{a_n} = +\infty$ then the inequality is hold $\Rightarrow$ only need to care when $\alpha < +$

- Put $\alpha = \limsup_{n\to\infty}\frac{a_{n+1}}{a_n}$ $<+\infty$ ~~$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, \left|\sup_{k\geqslant n}\frac{a_{k+1}}{a_k} - \alpha\right| < \varepsilon$~~

Let $\beta \geqslant \alpha \Rightarrow$ $, \exists N \in \mathbb{N}, \forall n \geqslant N, \left|\frac{a_{n+1}}{a_n}\right| < \beta$ (Just property of lim sup)

So we have $|a_{N+1}| \leq \beta |a_N|$

$\vdots$

$a_{N+\ell} \leq \beta^\ell |a_N|$

$\Rightarrow \forall n \geqslant N,\ a_n \leq \beta^{n-N}|a_N$

$\Rightarrow \forall n \geqslant N,\ \sqrt[n]{a_n} \leq \beta^{1-\frac{N}{n}}\sqrt[n]{a_N} = \beta\sqrt[n]{\frac{a_N}{\beta^n}}$

We note that $\left.\begin{matrix} a_N > 0 \\ \beta^n > 0 \end{matrix}\right\} \Rightarrow \frac{a_N}{\beta^n} > 0 \Rightarrow \lim_{n\to\infty}\sqrt[n]{\frac{a_N}{\beta^n}} = 1$

So we have $\forall n \geqslant N, \sqrt[n]{a_n} \leq \beta$.

$\Rightarrow \limsup_{n\to\infty}\sqrt[n]{a_n} \leq \beta,\ \forall \beta > \alpha$ $\Rightarrow \limsup_{n\to\infty}\sqrt[n]{a_n} \leq \alpha = \limsup_{n\to\infty}\frac{a_{n+1}}{a_n}$ □

Note that in this proof and also in the proof of root test and ratio test:

+ with root test + ratio test : we use definition of lim sup liminf

+ with this problem : comparing about $\frac{a_{n+1}}{a_n}$ and $\sqrt[n]{a}$ we begin with letting $\alpha = \limsup\frac{a_{n+1}}{a_n}$

* Prove that $\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty}\sqrt[n]{a_n}$ (Really similar).

- Put $\alpha = \liminf_{n\to\infty}\frac{a_{n+1}}{a_n}$ and $\beta < \alpha$, we have $\exists N \in \mathbb{N}, \forall n \geqslant N, \left|\frac{a_{n+1}}{a_n}\right| > \beta > 0$

- When $\alpha = -\infty$

$\Rightarrow$ always true

$\Rightarrow$ only need to care when $\alpha \in \mathbb{R}$.

So we have $a_{N+1} > \beta a_N$

$a_{N+2} > \beta^2 a_N$

$\vdots$

$a_{N+\ell} > \beta^\ell a_N$

$\Rightarrow a_n > \beta^{n-N} a_N,\ \forall n \geqslant N$

So we have $\sqrt[n]{a_n} > \beta^{1-\frac{N}{n}}\sqrt[n]{a_N} = \beta\underbrace{\sqrt[n]{\frac{a_N}{\beta^N}}}_{\to 1}$.

So we have $\sqrt[n]{a_n} > \beta,\ \forall n$

$\Rightarrow \liminf\sqrt[n]{a_n} \geqslant \beta,\ \forall \beta < \alpha$

$\Rightarrow \liminf\sqrt[n]{a_n} \geqslant \alpha$ □

* Learn from this problem

- We need to prove $\limsup a_n \leq \alpha$

$\Leftrightarrow$ NTP, $\exists N \in \mathbb{N}, \forall n \geqslant N, a_n \leq \alpha$.

- If $\limsup a_n \leq \beta,\ \forall \beta > \alpha$ then $\limsup a_n \leq \alpha$.

* Prove the root test for convergence test for series :

Let $\sum a_n$ be a series with $a_n \in \mathbb{C}, \forall n$.
$\alpha = \limsup\limits_{n\to\infty} |a_n|^{1/n} \quad (\alpha \in [0, +\infty))$
} Prove that: If $\alpha < 1$, $\sum |a_n|$ convergent
$\alpha > 1$, $\sum a_n$ divergent
$\alpha = 1$, no conclusion .

* Prove in case $\alpha < 1$ : (the idea is compare $\sum |a_n|$ with geometric series)

We have $\alpha = \limsup\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{N\to\infty} \sup\limits_{n \geq N} \{|a_n|^{1/n}\} \Leftrightarrow \exists N \in \mathbb{N}, \forall n \geq N, \sup\limits_{n\geq N}\{|a_n|^{1/n}\} = \alpha$

this means $\forall n \geq N$, $|a_n|^{1/n} < \alpha$ .

$|a_n| \leq \alpha^{n}$ for $\alpha < 1$
we have $\sum\limits_{n=1}^{\infty} \alpha^n$ convergent
$\Big\} \Rightarrow \sum\limits_{n=1}^{\infty} |a_n|$ converges

* Prove in case $\alpha > 1$

We have $\alpha = \limsup\limits_{n\to\infty} |a_n|^{1/n}$, then $\exists \{a_{n_k}\}$, $|a_{n_k}|^{1/n_k} \longrightarrow \alpha > 1$

this means $\exists K_0 \in \mathbb{N}, \forall k \geq K_0$, $|a_{n_k}|^{1/n_k} > 1$

$\Rightarrow |a_{n_k}| > 1, \forall k \geq K_0 \Rightarrow a_n \not\to 0$

$\Rightarrow \sum a_n$ diverges .

- Another way (by using contradiction)

Assume $\sum a_n$ converges, then $a_n \xrightarrow{n\to\infty} 0$ , this means $\exists N$, $|a_n| < 1$ , $\forall n \geq N$

$\Rightarrow |a_n|^{1/n} \leq 1$ , $\forall n \geq N$

$\Rightarrow \limsup\{|a_n|^{1/n}\} \leq 1, \forall n$

(contradicts with $\limsup\{|a_n|^{1/n}) = \alpha >$

## Prove the Ratio test

Given $\sum a_n$, $\boxed{a_n \neq 0}$

Let $\alpha = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

Prove that:

If $\alpha < 1$, the series $\sum a_n$ converges.

$\alpha > 1$, the series $\sum a_n$ diverges.

~~$\alpha = 1$, no conclusion~~.

+ $\sum a_n$, $a_n \neq 0$, $\alpha = \lim \frac{|a_{n+1}|}{|a_n|}$, $\alpha < 1$. Prove that $\sum a_n$ converges.

We have $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \alpha < 1 \Leftrightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}, \forall n \geq N, \left|\frac{a_{n+1}}{a_n} - \alpha\right| < \varepsilon$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < \underbrace{\alpha + \varepsilon}$ put $r := \alpha + \varepsilon$

So we have $|a_{n+1}| < r|a_n|$, $\forall n \geq N$

$\Rightarrow a_{N+1} < r|a_N|$

$a_{N+\ell} < r^{\ell}|a_N|$

$\Rightarrow \sum_{i=N+1}^{N+\ell} a_i < |a_N| \sum_{i=N+1}^{N+\ell} r^i \quad (1)$

So we have $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{\infty} a_i$

we devide the sum into 2 parts and care about the tail.

From (1) $\sum_{i=N+1}^{\infty} a_i$ converges since $|a_N| \underbrace{\sum_{i=N+1}^{\infty} r^i}$ converges (geometry series.)

$\Rightarrow \sum_{i=1}^{\infty} a_i$ converges by comparison test.

* In case $\alpha > 1$

We have $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \alpha > 1 \Leftrightarrow \forall n \geq N, |a_{n+1}| \geq \underset{>1}{\alpha}|a_n| > |a_n|$

$\Rightarrow \lim_{n\to\infty} |a_n| \neq 0 \Rightarrow \sum a_n$ diverges.

* In case $\alpha = 1$: there is no conclusion.

EX $\sum \frac{1}{n}$ diverges ~~whete~~ $\lim \frac{|a_{n+1}|}{|a_n|} = 1$

$\sum \frac{(-1)^n}{n}$ converges. while $\lim \frac{|a_{n+1}|}{|a_n|} = 1$.

conditionally conv

$\sum \frac{1}{n^2}$ converges.

## Limits of functions

X, Y : metric spaces

$E \subset X$, $f : E \to Y$ E: domain of function

+ point continuous.

Let p is a limit point of E, (every neighborhood of $p \setminus \{p\} \cap E \neq \emptyset$).

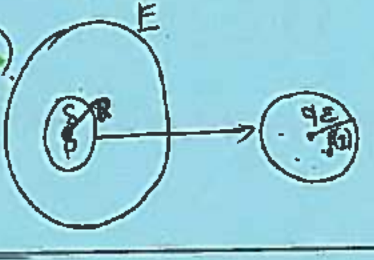
we don't need f(p) here

$$\lim_{x \to p} f(x) = q \iff \forall \varepsilon > 0, \exists \delta > 0, \forall x \in E \; 0 < d_X(x,p) < \delta, \; d_Y(f(x), q) < \varepsilon$$

$$\iff \forall \varepsilon > 0, \exists \delta > 0, \; f((N_\delta(p) \setminus \{p\}) \cap E) \subset N_\varepsilon(q)$$

$$\iff \lim_{n \to \infty} f(x_n) = q, \; \forall \{x_n\} \subset E, \; x_n \to p$$

(but we first need to prove the $x_n \neq p$ $\lim f(x_n)$ exist (by property cauchy for example))



* Corollary

If f has a limit at p, this limit is unique

(If ∃ 2 values of limit at p, then $\nexists \lim_{x \to p} f(x)$)

4.4* $E \subset X$, $f : X \to \mathbb{C}$, X,Y metric spaces, p limit point of E, $\lim_{x \to p} f(x) = a$ $\lim_{x \to b} g(x) = b$

- $\lim_{x \to p} (f \pm g)(x) = a \pm b$

  $\lim_{x \to p} (f \cdot g)(x) = a \cdot b$

  $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{a}{b}$ (if $b \neq 0$)

*

Value of f at p does not affect $\lim_{x \to p} f(x)$

EX: $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $f(0) = 0$ $\lim_{x \to 0} f(x) = 1 > 0$

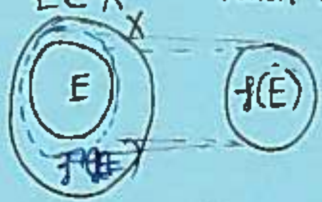
- Note that $\lim_{x \to p} f(x) = q$ does not ~~affect~~ related f(p)

EX: $\lim_{x \to 0} f(x) > 0 \not\Rightarrow f(x) > 0$ $\forall x \in (-\delta, \delta)$

$f(x) = \begin{cases} 1, & x \neq 0 \\ -1, & x = 0 \end{cases}$

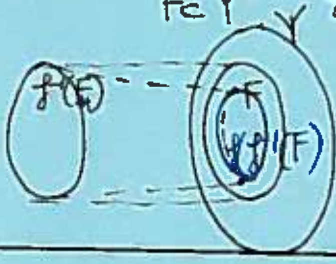
- $f : X \to Y$, $E \subset X$ then $f^{-1}(f(E)) \supseteq E$

proper subset when f is not an injection



- $f : X \to Y$, $F \subset Y$ $f(f^{-1}(F)) \subset F$

proper subset in case f is not a surjection



If $f : X \to Y$

$A, B \subset X \Rightarrow f(A \cup B) = f(A) \cup f(B)$

$f(A \cap B) \subseteq f(A) \cap f(B)$

$E, F \in Y \Rightarrow f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$

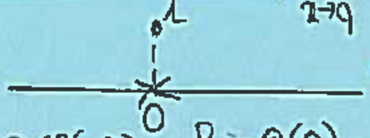
$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$

$$\lim_{x \to p} f(x) = q \Rightarrow \bigcap_{n=1}^{\infty} \overline{f(E \cap N_{\frac{1}{n}}(p) \setminus \{p\})} = \{q\}$$

⟸ the converse happens when (Y compact)

- $\lim_{x \to p} f(x) = q$, $\lim_{x \to q} g(x) = r$ Does not mean $\lim_{x \to p} g(f(x)) = r$ | only true when f & g continuous

$g(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$



$\lim_{x \to 0} g(x) = 0$

$f(x) = 0, \forall x$ $\lim_{x \to 0} g(f(x)) = \lim_{x \to 0} g(0) = 1$

Note that value of a function at a point does NOT affect $\lim_{x \to x_0} f(x)$

* We want to prove that $\lim_{x\to p} f(x) = L \Leftrightarrow$ WNTP $\forall (x_n) \to p,\ x_n \neq p$ then $\lim_{x_n \to p} f(x_n) = L$ (Jan 2006 3).

but we just need to prove that $f(x_n)$ converges

If $x_n \to p$ and $f$ has bounded derivative $f: R \to R$ $\Big\}$ then $f(x_n)$ converges (because $x_n$ converges $\Rightarrow$ $x_n$ cauchy $|x_m - x_n| < \varepsilon$

$|f(x_n) - f(x_n)| = f'(\xi)\ |x_m - x_n| < M\varepsilon$.

$\Rightarrow$ Cauchy $\Rightarrow$ $f(x_n)$ converges.

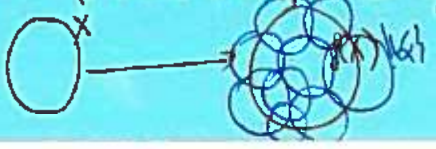
* Prove $\lim_{x\to a} f(x) = L$ (See example about some special functs C5)

$$\left.\begin{array}{l} \text{NTP } h(x) \leq f(x) \leq g(x) \\ \lim_{x\to a} g(x) = \lim_{x\to b} h(x) = L \end{array}\right\} \Rightarrow \lim_{x\to a} f(x) = L$$

* $\lim_{x\to +\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \boxed{\exists N > 0}, \forall x,\ x > N$, then $|f(x) - L| < \varepsilon$.

---

* $f$ is continuous
If $\{V_\alpha\}$ is a open cover of $\underline{f(X)}$ $\Big\}$ then $\{f^{-1}(V_\alpha)\}$ is a open cover of $X$



note that this only true for open cover of $f(X)$
(not true if not $f(X)$ in case $f$ is not surjection.

# Continuity

* X, Y metric space, $E \subset X$

p is a point of E (p does not need to be a limit point of E as in limit of f)

f is continuous at p

$\Leftrightarrow \lim_{x \to p} f(x) = f(p) \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in E, d(x,p) < \delta$, then $d(f(x), f(p)) < \varepsilon$

does not need to bigger than 0 (like in first ca) ($x$ can be $\equiv p$)

$f(p)$ not a q

in case p is a limit point of E

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, f(N_\delta(p) \cap E) \subset N_\varepsilon(f(p))$

$\Leftrightarrow \lim_{n \to \infty} f(x_n) = f(p)$ for all $x_n \in E, x_n \to p$

$= \cancel{\lim} f(\lim_{n \to \infty}(x_n))$ if $\{x_n\}$ converse

- f has to be well-defined at p in order to be continuous at p

---

- If p is an isolated point of E, then f is continuous at p.
(because $N_\delta(p) = \{p\}$ $f(N_\delta(p)) = \{f(p)\} \subset N_\varepsilon(f(p)), \forall \varepsilon > 0$.
- A function $f: X \to Y$ then f is continuous
discrete

---

4.9

* In case $Y = \mathbb{R}^k, \mathbb{C}^k$ : $f: X \longrightarrow \mathbb{R}^k/\mathbb{C}^k$

Let f, g complex functions on a metric space X

Then $f+g$, $f \cdot g$, $f/g$ are continuous
$g \neq 0$

4.10

* For vector values functions

Let $f, g: X \longrightarrow \mathbb{R}^k$ s.t. $f = (f_1, \dots, f_k)$
$g = (g_1, \dots, g_k)$

a) f is continuous $\Leftrightarrow$ each $f_i$ continuous

b) f, g continuous $\longrightarrow$ $\underbrace{f+g}_{\in \mathbb{R}^k}$, $\underbrace{f \cdot g}_{\in \mathbb{R}}$, $f/g$ continuous
$g \neq 0$

---

4.8

* $f: X \longrightarrow Y$ continuous at p $\Leftrightarrow$ whenever $f(p)$ is an interior point of $B \subset Y$.
then p is an interior point of $f^{-1}(B)$

- $f: X \to Y$ continuous $\Leftrightarrow$ $\forall$ V open in Y then $f^{-1}(V)$ open in X $\quad f^{-1}(K^c) \subset \cancel{f}K^c)$

$\Leftrightarrow \forall$ U ~~open~~ closed in Y, then $f^{-1}(U)$ open in X.

---

* $f: X \to Y$ continuous, $g: Y \to Z$ continuous $\Big\} \Rightarrow g \circ (f): X \longrightarrow Z$ continuous.

this means: if $\left.\begin{array}{l} \lim_{x \to p} f(x) = q \\ \lim_{x \to q} g(x) = r \end{array}\right\}$ then $\lim_{x \to p} g(f(x)) = r$ (this is not true when f and g are not continuous)

Important example (Kobsley) (Prog Aug 2014 Q7).

* Let $X$ metric space, $\forall a \in X$

$x$

$\bullet a$

$y$

Put $f(x) = d(x,a)$ $\quad f: X \longrightarrow \mathbb{R}$

$x \longmapsto f(x) = d(x,a)$

Then $f$ is a continuous function

(means, $y_n \longrightarrow x$, then $d(y_n,a) \longrightarrow d(x,a)$

or $f(x) = \inf_{x \in E} \{d(x,a)\}$ !

also continuous.

---

4.2/98

* $f: X \longrightarrow Y$ continuous

a) $E \subset X$, then $f(\overline{E}) \subseteq \overline{f(E)}$

b) $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$

Jan 2011/Q1

$f: X \longrightarrow Y$ continuous $\Leftrightarrow \forall V \subset Y$, $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$

* 4.4/98: $X, Y$ metric spaces

$f, g: X \longrightarrow Y$ continuous in $X$

a) if $E$ is dense in $X$ then $f(E)$ is dense in $f(X)$ ( $\overline{E} = X$, then $\overline{f(E)} = f(X)$ ).

b) If $f(x) = g(x)$, $\forall x \in E$, then $f(x) = g(x)$, $\forall x \in X$

* $f$ is continuous at all $f \in \mathbb{R} \setminus \{0\}$ does not mean $f$ is continuous at $0$

~~+ $f$ is not continuous at $\mathbb{R} \setminus \{0\}$ does not mean $f$ is not continuous at $0$~~

~~imp (0 is an isolated point.~~

# * Continuity and Compactness

* 4.13 Def:

A mapping : $f: X \longrightarrow \mathbb{R}^\ell$ is said to be a bounded mapping $\Leftrightarrow$ $\exists M \in \mathbb{R}, |f(x)| \leq M, \forall x \in X$.

* 4.14 Theorem + 4.15.

$f: X \longrightarrow Y$ continuous
$K \subseteq X$, K is compact
$\Big\} \Rightarrow \not\Leftarrow$ $f(K)$ is compact in Y
(closed + bounded) ( f(K) is bounded subset of Y )

* 4.16:

$f: X \longrightarrow \mathbb{R}^k$ continuous
X is compact
$\Big\} \Rightarrow$ f achives maximum and minimum in $\mathbb{R}^\ell$.
$\Leftrightarrow$ means $\exists a, b \in X$ s.t. $f(a) \leq f(x) \leq f(b)$, $\forall x \in X$

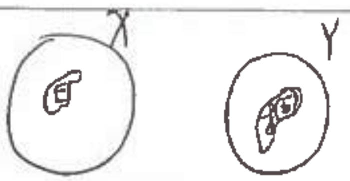
* 4.17 Theorem.

$f: X \longrightarrow Y$ bijection continuous
X compact
$\Big\} \Rightarrow \exists f^{-1}$, and $f^{-1}: Y \longrightarrow X$ is continuous function

# * Continuity and connectedness

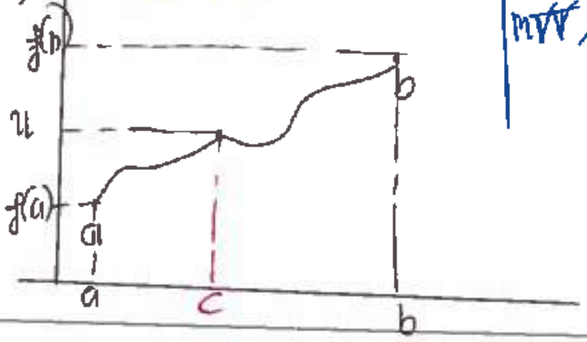
4.22:

$f: X \longrightarrow Y$ continuous
$E \subseteq X$, E is connected
$\Big\} \Rightarrow$ f(E) connected



4.23: Intermediate value theorem :

If : $f: [a,b] \longrightarrow \mathbb{R}$ continuous
$f(a) < f(b)$
$u \in [a,b]$ s.t $f(a) < u < f(b)$
$\Big\} \Rightarrow \exists c \in [a,b]$, $f(c) = u$



Note that IMV theorem only requires f cont
MVT requires f differenti

* Special case

$f: X$ (continuous connected) $\longrightarrow \mathbb{R}$
$f(a) < u < f(b)$
$\Big\} \Rightarrow \exists c \in X, f(c) = u$

* Some Remark:

- We don't have $f: X \longrightarrow Y$ continuous then $B$ compact in $Y$ $f^{-1}(B)$ compact in $X$.

  EX: Let $f(x) = 0, \forall x$, then $\{0\}$ compact in $\mathbb{R}$

  $f^{-1}(0) = X$, is not compact (if we let $X$ is not compact)

* Prove that for all $a < \alpha < b$, then $\alpha$ can be obtain as a value of some $f(x)$ (Jan 2016) ④

→ We just need to prove $f$ continuous on $E$

$a < f(x) < b$

* Jan 2014:

$f: X \longrightarrow Y$ cont

$X$ compact

$y_0 \in Y$ is a point st $\exists! \, x_0 \in X$ st $y_0 = f(x_0)$ $\Bigg\}$ $\forall$ $U$ open neighborhood of $x_0$ in $X$, $\exists$ $V$ open neighborhood of $y_0$ in $Y$ s.t. $f^{-1}(V) \subseteq U$

(fails if $X$ is not compact.

* Fall 1992

$(X, \rho)$ compact $(Y, d)$ metric space

a) $f: X \longrightarrow Y$ continuous + onto $\Longrightarrow$ $X$ is compact $\Longrightarrow$ $Y$ is complete,

b) If $f$ is bijective + above $\longrightarrow$ $g^{-1}$ is continuous.

easy to prove just some lines

# Uniform continuity and compactness

* $f: X \to Y$, $f$ is uniformly continuous on $X$ → in whole ( in normal continuous defined at a point )

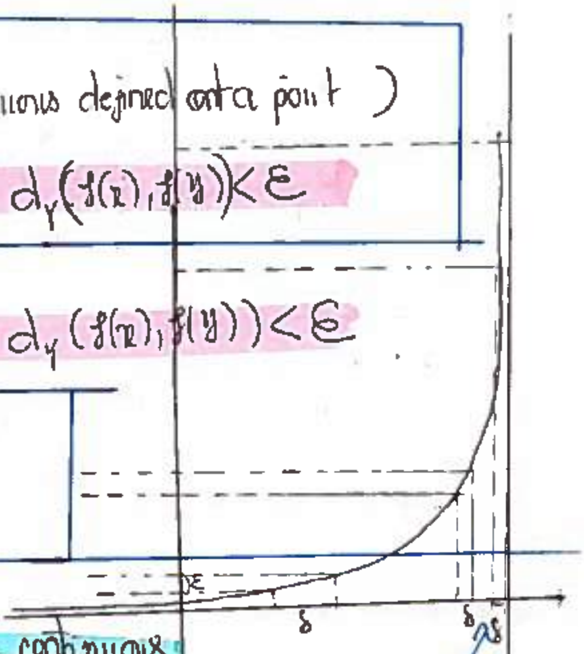
$\Leftrightarrow \forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x, y \in X,\ d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \epsilon$

- ( continuous function ( defined at a point $x$ )
  $\forall \epsilon > 0, \exists \delta_{\epsilon,x}, \forall y \in X,\ d_X(x,y) < \delta_{\epsilon,x}$ then $d_Y(f(x), f(y)) < \epsilon$

4.19

* $f$ is uniformly continuous $\Rightarrow$ $f$ continuous

* $f$ is continuous on $X$, $X$ compact $\Rightarrow$ $f$ is uniformly continuous



one way to prove $f$ is uniformly cont

* If $|f(x) - f(y)| \leq k\,|x - y|$ $\Rightarrow$ then $f$ is uniformly continuous
Lipschitz inequality
Lipschitz constant

A function is not uniformly continuous if ~~δ small~~ but ε big.

* $f: X \longrightarrow Y$ uniformly continuous, $\{x_n\}$ Cauchy in $X$ $\Big\} \Rightarrow \{f(x_n)\}$ Cauchy in $Y$
(Jan 2016, P2)

---

* A function $f: X \longrightarrow \mathbb{R}$ is NOT uniformly continuous on $X$.
iff $\exists\, \epsilon > 0$, $\exists \{x_n\}_{n\in\mathbb{N}} \subset X$, $\exists \{y_n\}$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$ $\forall n$.
$|x_n - y_n| \xrightarrow{n \to \infty} 0$
(Aug 2002)

* $f: \mathbb{R} \longrightarrow \mathbb{R}$ differentiable, $f'$ is bounded $\Big\} \Rightarrow f$ is uniformly continuous.

---

* Want to prove that $f$ is uniformly continuous on $E$
$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in E,\ |x - y| < \delta$ then $\underbrace{|f(x) - f(y)|}_{\text{to estimate this}} < \epsilon$
Choose $\delta$ dependent on $\epsilon$
consider $x, y \in E$ to find out the property of $x, y$ → to estimate this.

* Want to prove that $f$ is not uniformly continuous on $E$.
$\Leftrightarrow \exists \epsilon > 0, \forall \delta > 0, \exists x, y \in E,\ |x - y| < \delta$ but $|f(x) - f(y)| > \epsilon$
choose $x, y \in E$, $x, y$ depend on $\delta$

* $f: \mathbb{R} \longrightarrow \mathbb{R}$ uniformly continuous
$E$ (bounded) $\subseteq \mathbb{R}$.
$\Big\} \to f$ is bounded on $E$.

The conclusion is false if boundedness of $E$ is omitted from the hypothesis.

uniformly continuous $\not\Rightarrow$ bounded

* Let $f = kx + b$ then $f$ is uniformly continuous on $\mathbb{R}$.
(?$k<1$)
but $f$ is not bounded in $\mathbb{R}$.

* $f$ is uniformly continuous $\Leftrightarrow \forall$ ~~$E \subset X$, diam $E < \delta$~~
$\forall \varepsilon > 0, \exists \delta > 0, \forall E \subset X$, diam $E < \delta$ then diam $f(E) < \varepsilon$.

Jan 2015, P2.
* One more example when $f$ is uniformly continuous in $[0, +\infty)$ (example of uniformly cont $\not\Rightarrow$ bounded] Jan 2015;
not bound

$f: [0, +\infty) \longrightarrow \mathbb{R}$ continuous
$\lim_{x \to \infty} f(x) = x$
Then $f$ is uniformly continuous in $[0, +\infty)$

Example $f(x) = x$.

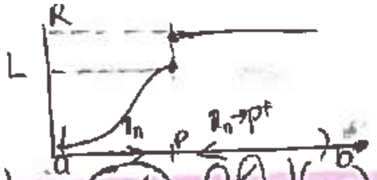
We can prove $f$ uniformly cont on $[0, +\infty)$ by proving that $f$ is uniformly continuous on $[0, N]$ and $[N + \infty)$,

---

* Jan 2012 P2 : $f: \mathbb{R} \to \mathbb{R}$ uniformly continuous
$\exists A, B$ positive constant s.t $|f(x)| \le A|x| + B, \forall x \in \mathbb{R}$.

* One important strategy to prove in case $f$ is uniformly continuous $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in E$, $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$
$\forall x$, we consider $|f(x) - f(x_0)| \le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$
(where we choose $n = \left(\frac{|x|}{\delta} + 1\right)$)

## 4.6 Monotone function



4.25 Def: one-sided limits:

$f(p-) = \lim_{\substack{x\to p \\ x<p}} f(x) = L$ means $\exists L$ s.t if $\boxed{x<p}$, $0 \boxed{<} d(x,p) < \delta$, then $d(f(x),L)) < \varepsilon$

$\Leftrightarrow \forall x_n \in (a,p)$, $x_n \to p$ then $f(x_n) \to L$

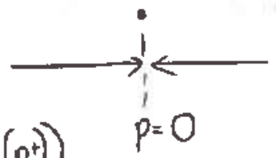
$f(p+) = \lim_{\substack{x\to p \\ x>p}} f(x) = R$ $\Leftrightarrow \exists R$, $\forall \boxed{x>p}$ $0 \boxed{<} d(x,p) < \delta$, then $d(f(x),R) < \varepsilon$

$\Leftrightarrow \forall x_n \in (p,b)$, $x_n \to p$ then $f(x_n) \to R$

---

* It is clear that $\forall p \in (a,b)$,

- $\lim_{x\to p} f(x)$ $\boxed{\text{exists}}$ iff $\underbrace{\lim_{x\to p^+} f(x)}_{f(p^+)} = \underbrace{\lim_{x\to p^-} f(x)}_{f(p^-)} = \cancel{\lim_{x\to p} f(x)} \boxed{< +\infty}$



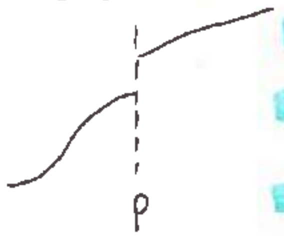
- $f$ is $\boxed{\text{continuous}}$ at p if $\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x) \boxed{= f(x)}$ $(f(p^-) = f(p) = f(p^+))$

---

4.29 * **Theorem:** If $\left.\begin{array}{l} f:[a,b] \longrightarrow \mathbb{R} \\ f \text{ is } \boxed{\text{monotone}} \end{array}\right\}$ then $f(p^-)$ and $f(p^+)$ exist for all $p \in (a,b)$

Apply this to prove that if $f:[0,1]\to[0,1]$ monoton ! onto ⇒ continuous prelim 199

* If $f$ is $\boxed{\text{increasing}}$ then



$f(p^-) = \sup\{f(x), x<p\}$

$f(p^+) = \inf\{f(x), x>p\}$

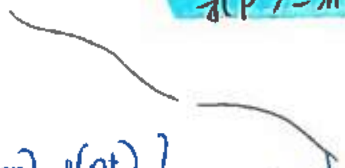
exist and

$f(p^-) \boxed{\leq} f(p) \boxed{\leq} f(p^+)$ ⇒ $\left.\begin{array}{l} \text{if } f(p^-) = f(p^+) \\ f \text{ is monotone} \end{array}\right\}$ ⇒ $f$ is continuous (prelim 1991)

(because $f(p^-) \leq f(x)$, $x>p$
$f(p^-)$ is a lower bound ⇒ $f(p^-) \leq \inf\{f(x), x>p\} = f(p^+)$)

$f$ decreasing

$f(p^-) = \inf\{f(x), x<p\}$.



4.30. **Theorem:** If $f:(a,b) \longrightarrow \mathbb{R}$ is monotone
the set of discontinuity $D$ of $f$ is $\boxed{\text{at most countable}}$ ($\phi$, finite, countable)

# * Discontinuity :

+ Def 4.26

Let $f : (a,b) \longrightarrow \mathbb{R}$

We say $\left[\begin{array}{l} f \text{ has discontinuity of the first kind} \\ f \text{ has simple discontinuity} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \exists f(x^+) \\ \exists f(x^-) \\ \left[\begin{array}{l} f(x^+) \neq f(x^-) \\ f(x^+) = f(x^-) \text{ but } \neq f(x) \end{array}\right. \end{array}\right.$ .

We say f has discontinuity of the second kind $\Leftrightarrow \left[\begin{array}{l} \nexists f(x^+) \\ \nexists f(x^-) \end{array}\right.$ .

- We have with monotonic function $, \exists f(x^+), \exists f(x^-), \forall x \Rightarrow f$ (monotonic) does not have discontinuity of $2^{st}$ kind .

---

* Dirichlet function

$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not continuous at every $x$
(it has discontinuity of the second kind $\forall x$ )

- $f(x) = \begin{cases} x \text{ ~~$\neq 0$~~} & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is continuous at 0
not continuous ( $2^{nd}$ kind ) at every other point ) also $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 2x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

- $f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has discontinuity of $2^{nd}$ kind at 0 .
(not continuous at 0 ) and continuous at every other point .

- $f(x) = \begin{cases} x , & x \in \mathbb{Q} \\ 1-x , & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ continuous at 1/2 and discontinuous at other points .

- $f(x) = \begin{cases} 1/n & x \in \mathbb{Q} , x = \frac{m}{n} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ then f continuous at all irrational numbers
discontinuous at all rational number

* Problems relating to uniformly continuous functions + expanding functions

* Jan 2010, P2,

$f:(0,1]\longrightarrow\mathbb{R}$ is a bounded + continuous function
$\forall t\in\mathbb{R}$, the set $\{x\in(0,1], f(x)=t\}$ is finite $\Big\}\Rightarrow f$ is uniformly continuous on $(0,1]$.

uniformly continuous in (...

* <u>Strategy to prove that $f$ is uniformly continuous on $(a,b]$</u> (for ex: above problem).

- We prove that $f(a+)$ exist

---

* <u>Sample C, P1</u>: Let $f:D\longrightarrow\mathbb{R}$
D is dense in $[0,1]$.
$f$ is uniformly continuous on D $\Big\}$ Show that $f$ can be extended to a uniformly continuous function on $[0,1]$.

- <u>Solve by</u>: let $(x_n$ for $x\in[0,1]$, $\exists(x_n)\subseteq D$, $x_n\to x$.
and we prove that $(x_n)$ Cauchy
$f$ uniformly cont on D $\Big\}\Rightarrow\{f(x_n)\}$ Cauchy $\Rightarrow$ converges to

\+ <u>Math stack</u>: $f$ has a continuous extension to $[a,b]\iff f$ is uniformly continuous on $(a,b)$.

---

* Fall 2001, P3. Template: consider if a function $f(x)$ (for ex in this problem $f(x)=x^{3/2}\log x$) is uniformly continuous on $(0,1)$ (a open set)

$\Rightarrow$ <u>Solve</u>: in this case, we have $f$ is already continuous on $(0,1)$.
and we want to find $\lim_{x\to0^+}f(x)$ and $\lim_{x\to1^-}f(x)$

and put $g(x)=\begin{cases}\lim_{x\to0^+}f(x)\\ x^{3/2}\log x\\ \lim_{x\to1^-}f(x)\end{cases}$ then $g$ is continuous on $[0,1]$. $\Rightarrow$ uniformly continuous on $[0,1]$

$\Rightarrow f$ is uniformly on $[0,1]$

* Aug ~~1999, P3~~ 2002, P1.

$f$ is bounded on $(a,b)$, $f:(a,b)\longrightarrow\mathbb{R}$
$f$ is continuous + increasing $\Big\}$ Prove that $f$ is uniformly continuous on $(a,b)$.

* Aug 2006, P3.

$f:[0,1)\longrightarrow\mathbb{R}$ differentiable with bounded derivative. Prove that $f$ can be extended to a continuous function on $[0,1]$.

* Fall 2001/P3

Prove or dispute $f(x)=x^{3/2}\log x$ is uniformly continuous on $(0,1)$.

(with this problem, just compute $\lim_{x\to0}f(x)$ (L'Hopital) and extend to g
$\lim_{x\to1}f(x)$

* Aug2008, P2, Aug2011 P5.

$$\left.\begin{array}{l} f \text{ is continuous} \\ f \text{ is on-to-one} \end{array}\right\} \Rightarrow f \text{ strictly monotonic in } \mathbb{R}.$$

* Jan 2016, P2

$$\left.\begin{array}{l} f: X \longrightarrow Y \\ f \text{ is uniformly continuous in } X \\ \{x_n\} \text{ Cauchy} \end{array}\right\} \Rightarrow \{f(x_n)\} \text{ Cauchy in } Y.$$

* Aug 2010, P3.

$$\left.\begin{array}{l} X, Y: \text{metric spaces} \\ f: X \longrightarrow Y \text{ has the property:} \\ \qquad \forall g: Y \longrightarrow \boxed{\mathbb{R}} \text{ continuous } f, \text{ then } g \circ f \text{ is continuous.} \end{array}\right\} \text{ Prove that } f \text{ is continuous.}$$

* Jan 2009, P3.

A periodic + continuous function $\Longrightarrow$ attains its min/max.

* Prove that the two definitions of limit of function are equivalent : Let $f:(X,d_x) \to (Y,d_Y)$

$\lim_{x\to p} f(x) = q$ (I)

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall (x \in X, \boxed{0<d}(x,p) < \delta$ then $d_Y(f(x),q) < \varepsilon$

$\lim f(x) = q$ (II)

$\Leftrightarrow \forall (x_n) \subset X,\ x_n \to p,\ x_n \neq p$ then $f(x_n) \longrightarrow q$.

($\Rightarrow$) Given (I)

Let $(x_n) \subset X,\ x_n \to p,\ x_n \neq p$

We want to prove that $f(x_n) \longrightarrow q$.

- We have $(x_n) \subset X,\ x_n \to p,\ x_n \neq p$ means $\forall \delta > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0,\ 0 < d_x(x_n,p) < \delta$

by (I) we have $d_Y(f(x_n),q) < \varepsilon$.

$\Rightarrow f(x_n) \longrightarrow q$

($\Leftarrow$): Given (II): $\forall (x_n) \subset X,\ x_n \to p,\ x_n \neq p$ then $f(x_n) \to q$

Harder

Want to prove $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X,\ 0 < d_x(x,p) < \delta$ then $d_Y(f(x),q) < \varepsilon$.

Prove by contradiction, assume a contradiction that

$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in X,\ 0 < d_x(x,p) < \delta$ but $d_Y(f(x),q)) \geq \varepsilon$.

taking $\delta = \frac{1}{n}$, then $\underbrace{\forall n, \exists x_n,\ 0 < d_x(x_n,p) < \frac{1}{n}}_{}$ but $d_Y(f(x_n),q) \geq \varepsilon$.

This means we already have a sequen $(x_n) \subset X,\ x_n \to p,\ x_n \neq p$ but $f(x_n) \not\longrightarrow q$

(contradicts with I).

---

* Note that (I) $\Leftrightarrow \lim_{n\to\infty} f(x_n) = q$

then if $\lim_{n\to\infty} f(x_n) \neq q \Leftrightarrow$ two case [ the limit does not exist / the limit exist but different $\neq q$.

* Prove the claim:

X, Y metric spaces $\quad \lim_{x\to p} f(x) = q \Rightarrow \bigcap_{n=1}^{\infty} \overline{f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E)} = \{q\}$.

$f : E \subset X \longrightarrow Y \quad \xleftarrow{\text{when } Y \text{ is compact}}$

* **Step 1**: Prove that $\{q\} \subseteq \bigcap_{n=1}^{\infty} \overline{f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E)}$

We NTP $q \in \overline{f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E)}$, $\forall n$

NTP $\forall \lambda > 0$, $N_\lambda(q) \cap f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E) \neq \phi$

We have $\lim_{x\to p} f(x) = q \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in E, 0 < d_X(x,p) < \delta$, $d_Y(f(x),q) < \varepsilon$

$\forall x \in E$

Choose $\boxed{\varepsilon = \lambda}$, $\boxed{\delta = \frac{1}{n}}$, this means $\forall \lambda > 0$, $\forall x \in N_{\frac{1}{n}}(p)\setminus\{p\}$, $f(x) \in N_\lambda(q)$

$\Leftrightarrow \forall \lambda > 0, N_\lambda(q) \cap f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E) \neq \phi$ □.

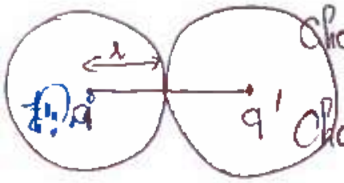
* **Step 2**: Want to we assume $\exists q'$ such that $q' \neq q$, then $q' \notin \bigcap_{n=1}^{\infty} \overline{f(N_{\frac{1}{n}}(p)\setminus\{p\})\cap E}$

We want to prove that $\bigcap_{n=1}^{\infty} A_n = \{q\}$ then we need to prove "q" is the "unique point" in $A_n$

$\Leftrightarrow$ NTP $\begin{cases} q \in A_n, \forall n \\ \text{If } q' \neq q, \text{ then } \exists n, q' \notin A_n \end{cases}$

We want to prove ~~$\exists n$ such that~~ $q' \notin \overline{f(N_{\frac{1}{n}}(p)\setminus\{p\})\cap E}$

We want to prove $\exists \lambda$ such that $N_\lambda(q') \cap f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E) = \phi$.



Choose $\boxed{\lambda = \frac{1}{2} d_Y(q,q')}$

Choose $\varepsilon = \lambda$ and $\delta = \frac{1}{n}$,

we have from above, $f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E) \subseteq N_\varepsilon(q)$
$N_\lambda(q) \cap N_\lambda(q') = \phi$ $\Big\}$ $\Rightarrow$

$\Rightarrow f(N_{\frac{1}{n}}(p)\setminus\{p\}\cap E) \cap N_\lambda(q') = \phi$

This picture also give us idea about $x_n \to p$ then $f(x_n) \longrightarrow q$

$x_n \neq p \qquad f(x_n) \subseteq N_\varepsilon(q)$

Theorem 4.8

Prove the connection between continuity and $f^{-1}(V)$ open.

Let $f: X \longrightarrow Y$. Prove that.

a) $f$ is continuous $\Longleftrightarrow$ ~~whenever $f(p)$ is an interior point of $B \subseteq Y$~~
~~$p$ is an interior point of $f^{-1}(B)$~~.

b) $f$ is continuous $\Leftrightarrow$ $\forall V$ open in $Y$, $f^{-1}(V)$ open in $X$

c) $f$ is continuous $\Leftrightarrow$ $\forall V$ closed in $Y$, $f^{-1}(Y)$ closed in $X$.

Prove a) $f$ is continuous $\Leftrightarrow$ whenever $f(P)$ is an interior point of $B \subset Y$, $p$ is an interior point of $f^{-1}(B)$

$(\Longrightarrow)$: Let $f: X \to Y$ continuous.
$f(p)$ is an interior point of $B \subseteq Y$ } Prove $p$ is an interior point of $f^{-1}(B)$.

- $f(P)$ is an interior point of $B \subseteq Y$. | NTP $p$ is an interior point of $f^{-1}(B)$

$\Leftrightarrow \exists \varepsilon > 0$, ~~st~~ $N_\varepsilon(f(P)) \subseteq B$ (1) | $\Leftrightarrow$ NTP, $\exists \delta > 0$, $N_\delta(p) \subseteq f^{-1}(B)$

Then because $f$ is continuous $\Rightarrow$ continuous at every $p \in X$ $\Leftrightarrow$ NTP $\forall \delta > 0$, $f(N_\delta(p)) \subseteq B$

$\forall \varepsilon > 0, \exists \delta > 0$, $f(N_\delta(p)) \subseteq N_\varepsilon(f(P)) \underset{\text{bec}(1)}{\subseteq} B \Rightarrow \square \Rightarrow$.

$(\Longleftarrow)$: Given $\varepsilon$, let $B = N_\varepsilon(f(P))$, ~~because~~ $\Rightarrow$ $f(P)$ is an interior point of $B$

then $p$ is an interior point of $f^{-1}(B)$

$\Leftrightarrow \exists \delta > 0$, $N_\delta(p) \subseteq f^{-1}(N_\varepsilon(f(P)))$

$\Rightarrow$ $f(N_\delta(p)) \subseteq N_\varepsilon(f(P))$ $\Rightarrow$ $f$ is continuous.

b) Prove that $f$ is continuous $\Leftrightarrow$ $\forall V$ open in $Y$, $f^{-1}(V)$ open in $X$.

$(\Longrightarrow)$: $f$ is continuous.
$V$ open in $Y$ } $\Rightarrow$ NTP $f^{-1}(V)$ open in $X$ $\Leftrightarrow$ NTP, $\forall p \in f^{-1}(V)$, $p$ is an interior point of $f^{-1}(V)$.

We have $p \in f^{-1}(V) \Rightarrow f(P) \in V$
we have $V$ is open } $\Rightarrow$ $f(p)$ is an interior point of $V$
from a) $p$ is an interior point of $f^{-1}(V) \Rightarrow \square$

$(\Longleftarrow)$: $V$ open in $Y$, then $f^{-1}(V)$ open in $X$. NTP $f$ is continuous.

$V$ open in $Y$, then $f^{-1}(V)$ open in $X$

$\Rightarrow$ Put $V = N_\varepsilon(f(p))$. Then we have $f(P)$ is an interior point of $V$
from a) $p$ is an interior point of $f^{-1}(V)$
$\Leftrightarrow \exists \delta > 0$, $N_\delta(p) \subseteq f^{-1}(V)$ $\Rightarrow$ $f(N_\delta(p)) \subseteq V = N_\varepsilon(f(P))$
$\Rightarrow f$ is continuous

c) Prove that $f$ is continuous $\Leftrightarrow$ $\forall B$ closed in $Y$, $f^{-1}(B)$ closed in $X$.

This is because $[f^{-1}(B^c)] \boxed{=} [f^{-1}(B)]^c$

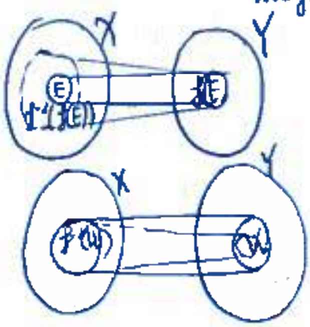
**Theorem 4.14, 4.15**

$\left.\begin{array}{l} f: X \longrightarrow Y \text{ is continuous} \\ E \subseteq X \text{ is compact} \end{array}\right\} \Rightarrow f(E) \text{ is compact in } Y$

We want to prove that $f(E)$ is compact in $Y$.

$\Leftrightarrow$ NTP. Let $\{W_\alpha\}_{\alpha \in I}$ is open cover of $f(E)$ (which means $f(E) \subseteq \bigcup_{\alpha \in I} W_\alpha$, $W_\alpha$ : open in $Y$) | We NTP it contains a finite subcover $f(E) \subseteq \bigcup_{i=1}^{n} W_{\alpha_i}$

Remind: Some useful properties:



$E \subseteq f^{-1}(f(E))$

$f(f^{-1}(W)) \subseteq W$.

$A \subseteq B \Rightarrow f(A) \subseteq f($

* $\left.\begin{array}{l} \text{We have } W_\alpha : \text{is open in } Y, \forall \alpha \in I \\ \text{we have } f: X \longrightarrow Y \text{ continuous} \end{array}\right\} \Leftrightarrow f^{-1}(W_\alpha) \text{ is open in } X, \forall \alpha \in I \quad (1)$

* We have because $\{W_\alpha\}_{\alpha \in I}$ covers $f(E)$ $\Rightarrow \{f^{-1}(W_\alpha)\}_{\alpha \in I}$ covers $E$ (2)

(because $\forall x \in E, f(x) \in f(E) \subseteq \bigcup W_\alpha \Rightarrow \exists \alpha \in I, f(x) \in W_\alpha$

$\Rightarrow \exists \alpha \in I, x \in f^{-1}(W_\alpha) \subseteq \bigcup_{\alpha \in I} f^{-1}$

$\left.\begin{array}{l} (1)+(2) \Rightarrow \{f^{-1}(W_\alpha)\}_{\alpha \in I} \text{ is an open cover of } E \\ \text{we have } E \text{ is compact in } X \end{array}\right\} \Rightarrow \exists \text{ a finite subcover.}$

$E \subseteq \bigcup_{i=1}^{n} f^{-1}(W_{\alpha_i})$

So we have $f(E) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(W_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} W_{\alpha_i} \Rightarrow \square$.

\+ **Theorem 4.17**

$\left.\begin{array}{l} f: X \longrightarrow Y \text{ bijective, continuous} \\ X \text{ is compact} \end{array}\right\} \text{Then } f^{-1}: Y \longrightarrow X \text{ is continuous}$

Put $g := f^{-1}$. We NTP: $g: Y \longrightarrow X$ is continuous.

NTP, $\forall$ $E$ closed in $X$, then $g^{-1}(E) = f(E)$ is closed in $Y$

$\left.\begin{array}{l} \left.\begin{array}{l} \text{We have } E \text{ is closed in } X \\ X \text{ compact} \end{array}\right\} \Rightarrow E \text{ is compact} \\ \text{we have } f \text{ is continuous} \end{array}\right\} \xrightarrow[\text{4.14}]{\text{Theorem}} f(E) \text{ compact in } Y$

$\Rightarrow f(E)$ is closed in $Y$

$\Rightarrow$ done $\square$ :)

Proof Theorem 4.19

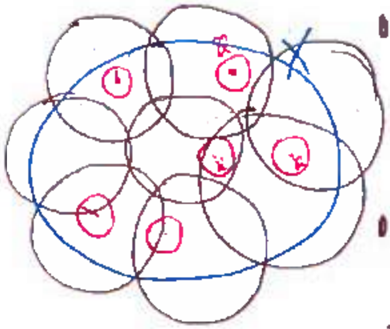
$f: X \to Y$ a) $f$ is uniformly continuous $\Rightarrow$ $f$ is continuous (done).

b) $\left.\begin{array}{l} f \text{ is continuous} \\ X \text{ is compact} \end{array}\right\} \Rightarrow f$ is uniformly continuous in $X$

* We first prove Lebesgue number lemma

If $U_\alpha$ is an open cover of $X$ $(X \subseteq \bigcup_{\alpha \in I} U_\alpha)$, ($X$ compact)

Then $\exists \lambda > 0$ such that $\forall x \in X$, $N_\lambda(x) \subseteq U_\alpha$ in some $U_\alpha$.



- For all $x \in X \subseteq \bigcup_{\alpha \in I} U_\alpha$

then $\exists \alpha \in I$, $x \in U_\alpha$

Then because $U_\alpha$ is open, then $\exists \lambda_x$, $N_{\lambda_x}(x) \subseteq U_\alpha$ (1)

- Then we have $X \subseteq \bigcup_{x \in X} N_{\frac{\lambda_x}{2}}(x)$ ($\{N_{\frac{\lambda x}{2}}(x)\}_{x \in X}$ is an open cover of $X$)

$\Rightarrow$ $\exists$ a finite subcover $X \subseteq \bigcup_{i=1}^{n} N_{\left(\frac{\lambda_x}{2}\right)}(x_i)$ (2)

- Choose $\lambda = \min_{i=1,n} \left\{ \lambda_{x_i/2} \right\}$ (3), then we have from (1)(2)(3)

for each $x \in X$, $\exists x_k$, $x \in N_\lambda(x_k) \subseteq N_{\lambda(x_k)}(x_k) \subseteq U_\alpha$ $\rightarrow \square$

* Now we prove the theorem b)

- We have $f$ continuous on $X$ $\Leftrightarrow$ continuous at all $x \in X$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_{(x)} > 0, \forall y \in X, d_X(x,y) < \delta_{(x)}$ then $d_Y(f(x), f(y)) < \varepsilon$ (3)

We want to prove $\forall$

$\forall \varepsilon > 0, \exists \lambda > 0, \forall y, y' \in X$, $d(y, y') < \lambda$ then $d(f(y), f(y')) < \varepsilon$

- We have $\left.\begin{array}{l} \{N_{\delta(x)}(x)\}_{x \in X} \text{ is an open cover of } X \\ \text{we also have } X \text{ is compact} \end{array}\right\} \Rightarrow$ Then by Lebesgue number lemma

$\exists \lambda$ such that $\forall y \in X$, $N_\lambda(y) \subseteq N_{\delta(x)}(x)$ for some $x$.

Then $\forall y' \in X$ such that $d_X(y, y') < \lambda$, we have $y' \in N_\lambda(y) \subseteq N_{\delta(x)}(x)$

$\Rightarrow \begin{cases} d(y,x) < \delta(x) \\ d(y',x) < \delta(x) \end{cases}$

$\overset{(3)}{\Rightarrow} \begin{cases} d_Y(f(y), f(x)) < \varepsilon \\ d_Y(f(y'), f(x)) < \varepsilon \end{cases}$

Then by (3), we have

$\forall \varepsilon > 0 \ \exists \lambda \ \forall y, y' \in X, d(y, y') < \lambda, \ d(f(y), f(y'))$

$\leq d(f(y), f(x)) + d(f(y'), f(x)) \leq 2\varepsilon$

$\Rightarrow \square$

* If $\{x_n\}$ Cauchy in $X$ .
$f: X \longrightarrow Y$ uniformly continuous. $\Big\} \Rightarrow \{f(x_n)\}$ Cauchy in $Y$.

The proof is simple (by just using definition) (1)

- $f$ is uniformly continuous $\overset{\text{def}}{\Longleftrightarrow} \forall \varepsilon > 0, \exists \delta > 0, \forall (x, y \in X, d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) <$
- $f$: $\{x_n\}$ Cauchy in $X$

$\Leftrightarrow \forall \delta > 0, \exists n_0 \in \mathbb{N}, \forall \overset{m}{n} \geq n_0, d_X(x_m, x_n) < \delta$ (2)

We need to prove
$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall \overset{m}{n} \geq N, d_Y(f(x_n), f$
$< \varepsilon$

We have from (1)+(2)

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, d_X(x_m, x_n) < \delta$ then $d_Y(f(x_m), f(x_n)) < \varepsilon$

$\Rightarrow \{f(x_n)\}$ Cauchy in $Y$.

* Quiz questions:

a> $X$ compact
$f: X \longrightarrow \mathbb{R}$ continuous.
$f(x) > 0, \forall x \in X$
$\Big\} \overset{?}{\Rightarrow} \exists \varepsilon > 0, f(x) \geq \varepsilon, \forall x$.

b> $X$ compact
$f: X \longrightarrow X$ continuous
$f(x) \neq x, \forall x$
$\Big\} \xrightarrow{?} \exists \varepsilon > 0, d(f(x), x) \geq \varepsilon, \forall x$

a> is True because $f: X \to \mathbb{R}$ continuous, $X$ compact $\Big\} \Rightarrow f$ is uniformly continuous in $X \Rightarrow$ True.

b> Put $g(x) = d(f(x), x)$
Then we have $g: X \longrightarrow \mathbb{R}$ continuous, $g(x) > 0, \forall x \in X$ $\Big\} \xrightarrow{\text{apply a}} \Rightarrow \exists \varepsilon > 0, g(x) \geq \varepsilon, \forall x$

$\Rightarrow d(f(x), x) \geq \varepsilon, \forall x \Rightarrow \overset{\text{True}}{\square}$

* About continuity and connection.

* Prove theorem 4.22 —

$$\left.\begin{array}{l} f: X \longrightarrow Y \text{ continuous} \\ E \text{ connected in } X \end{array}\right\} \Rightarrow \text{Prove that } f(E) \text{ is connected in } Y.$$

* Assume that $f(E)$ is not connected in $Y$,
this means $f(E) = A \cup B$, where $A \neq \phi$, $B \neq \phi$ and $\begin{cases} \bar{A} \cap B = \phi \\ A \cap \bar{B} = \phi \end{cases}$

- Now let $E_1 = E \cap f^{-1}(A)$
  $E_2 = E \cap f^{-1}(B)$

then we have $E_1 \cup E_2 = (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) = E \cap (f^{-1}(A) \cup f^{-1}(B)) =$
$= E \cap f^{-1}(A \cup B) = E \cap f^{-1}(f(E)) = E \cap E = E$. (1)

* Now we already have $E = E_1 \cup E_2$, we now prove that $E_1$ and $E_2$ seperated.

- Put $\left.\begin{array}{l} K_1 = f^{-1}(\bar{A}) \\ K_2 = f^{-1}(\bar{B}) \end{array}\right.$, $\left.\begin{array}{l} \text{where } \bar{A}, \bar{B} \text{ closed} \\ f \text{ is continuous} \end{array}\right\} \Rightarrow K_1$ and $K_2$ are closed in $X$.

- We have

$$\left.\begin{array}{l} \left.\begin{array}{l} E_1 = E \cap f^{-1}(A) \subseteq f^{-1}(A) \subseteq f^{-1}(\bar{A}) = K_1 \\ K_1 \text{ is closed} \end{array}\right\} \Rightarrow \bar{E_1} \subseteq K_1 \\ E_2 \cap K_1 = [E \cap f^{-1}(B)] \cap f^{-1}(\bar{A}) = E \cap [f^{-1}(B) \cap f^{-1}(\bar{A})] = \\ = E \cap [f^{-1}(\underbrace{B \cap \bar{A}}_{=\phi})] = \phi \end{array}\right\} \Rightarrow \bar{E_1} \cap E_2 = \phi. \quad (2)$$

- Similarly, $E_1 \cap \bar{E_2} = \phi$ (3)

$(1)+(2)+(3) \Rightarrow E_1 \neq \phi$, $E_2 \neq \phi$ ⇒ $E$ is not connected (contradiction) ⇒ $f(E)$ has to be connected ⇒ □.

* Prove 4.23: Intermediate value theorem.

$$\left.\begin{array}{l} \text{Let } f: [a,b] \longrightarrow \mathbb{R} \text{ continuous} \\ f(a) < \lambda < f(b) \end{array}\right\} \text{Then } \exists c \in [a,b],\ f(c) = \lambda$$

Prose

# Chapter 5 Differentiation

(confine our attention to real function $f:[a,b]\to\mathbb{R}$
$f:(a,b)\to\mathbb{R}$).

## 5.1 * Definition

Let $f:[a,b]\longrightarrow\mathbb{R}$

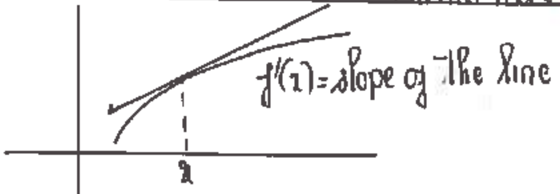
Function $f$ is differentiable at $x\in[a,b]$ if there exist the limit.

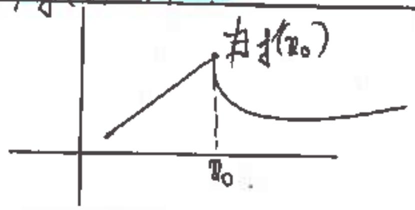
$$\underbrace{f'(x)}_{\uparrow\ \text{function of } x} = \lim_{\substack{t\to x\\ \boxed{t\neq x}}} \underbrace{\frac{f(t)-f(x)}{\boxed{t-x}}}_{\uparrow\ \text{function of } t} = \lim_{\substack{h\to 0\\ \boxed{h\neq 0}}} \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{function of } h}$$

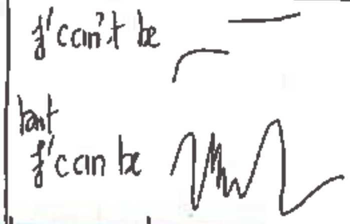
If $x=a$ or $x=b$, then $f(a^+)$ $f(b^-)$: one-side limit.

* Claim $f'(x)$ exists $\Longleftrightarrow \exists A\in\mathbb{R}\ \exists\lambda(t)\to 0$, s.t $f(t)=f(x)+A(t-x)+\boxed{\lambda(t)}(t-x)$

Furthermore, $f'(t)=A$ where $\lambda(t)\xrightarrow[t\to x]{}0$







## 5.2 Theorem

$f$ is differentiable at $x\in[a,b]$ $\underset{\not\Longleftarrow}{\Longrightarrow}$ $f$ is continuous at $x$.

## 5.3 Derivative rule

Let $f,g$ defined on $[a,b]$
$f,g$ are differentiable at $x\in[a,b]$.

- Then $f\pm g$, $f\cdot g$, $f/g$ (if $g(x)\neq 0$) are differentiable at $x$, and
- $(f\pm g)'(x) = f'(x)+g'(x)$

$(fg)'(x) = f'(x)g(x)+f(x)g'(x)$ … * Every polynomial is differentiable.

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$$

## 5.5 Theorem Chain rule (Prelim Aug 2006 #1)

Suppose $f:[a,b]\longrightarrow\mathbb{R}$ continuous

$\exists f'(x)$ exists

$g$ is differentiable at $f(x)$. $(\exists g'(f(x)))$

$[a,b]\xrightarrow{f} f[a,b]\xrightarrow{g} g(f[a,b]))$

If $h(t)=g(f(t))$, $\forall t\in[a,b]$.

Then $h$ is differentiable at $x$, and $h'(x)=g'(f(x))\ f'(x)$

## lean value theorems

7: Def:

et $f: X \longrightarrow R$

We say $f$ has local maximum at $p \in X \overset{\text{def}}{\Longleftrightarrow} \exists \delta > 0, \forall x \in X, d(x,p) < \delta$ then $f(x) < f(p)$

$f$ has local minimum at $p \in X \overset{\text{df}}{\Longleftrightarrow} \exists \delta > 0, \forall x \in X, d(x,p) < \delta$, then $f(x) > f(p)$.

### .8 Theorem:

et $f: [a,b] \longrightarrow R$

If $f$ has a local maximum at $x \in (a,b)$ (/ local minimum) $\longrightarrow \begin{cases} \nexists f'(x) \\ \exists f'(x), \text{ then } f'(x) = 0. \end{cases}$

### 5.9 Theorem (Generalized mean value theorem) Jan2004 7 P37

.et $f, g: [a,b] \longrightarrow R$ are continuous

$f, g$ differentiable in $(a,b)$

hen $\exists c \in (a,b)$ such that $[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$.

---

### Rolle's theorem:

$\left.\begin{array}{l} f: [a,b] \longrightarrow R \text{ continuous on } [a,b] \\ f \text{ is differentiable in } (a,b) \\ f(a) = f(b) \text{ (does not need equals 0)} \end{array}\right\} \Rightarrow \exists c \in (a,b), \quad f'(c) = 0$

### 5.10 Mean value theorem:

$\left.\begin{array}{l} f: [a,b] \longrightarrow R \text{ continuous on } [a,b] \\ f \text{ is differentiable in } (a,b) \end{array}\right\} \Rightarrow \exists c \in (a,b), \ f(b) - f(a) = f'(c)(b-a)$

$$\Leftrightarrow \frac{f(b) - f(a)}{b-a} = f'(c)$$

---

### 5.11 Theorem

7 $f'(x) \geq 0, \forall x \in (a,b) \Longrightarrow f$ monotonically increasing

7 $f'(x) \leq 0$ decreasing.

7 $f'(x) = 0 \Longrightarrow f$ is a constant.

---

$\left.\begin{array}{l} \text{If } f: R \longrightarrow R \text{ differentiable} \\ f'(x) \text{ is bounded}, \forall x \in \overset{\overset{(a,b)}{R}}{[a,b]} \end{array}\right\} \Rightarrow f$ is uniformly continuous on $\underset{\underset{R}{(a,b)}}{[a,b]}$.

(note that $f$ is uniformly continuous on $[a,b] \not\Rightarrow f'(x)$ is bounded on $[a,b]$.

EX: $f(x) = \sqrt{x}$ uniformly cont on $[0,1]$.

$f'(x) = \frac{1}{2\sqrt{x}}$ on $(0,1)$

## * Derivative and limit (The continuity of derivative $f'$)

* We know $f'$ is not always continuous.

The existence of $f'(p)$ does not $\overset{\not\Rightarrow}{}$ mean existence of $\lim_{x\to p} f'(x)$

$\Longleftarrow$ (The converse is true.

### * Theorem (From Kor)

$$\left.\begin{array}{l} f \text{ is continuous on an interval } I \\ \lim_{x\to p} f'(x) \text{ exists for some } p\in I \end{array}\right\} \Rightarrow \exists f'(p) \text{ exists and } f'(p) = \lim_{x\to p} f'(x)$$

(this means if $f'(x)$ exists near p

$\exists \lim_{x\to p} f'(x) \Longrightarrow \exists f'(p)$

$\not\Leftarrow$

$f(x) = \begin{cases} x^2\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

### * Theorem 5.12 Intermidiate value theorem (Jan 2015 P47

$$\left.\begin{array}{l} f \text{ is a differentiable function on } [a,b] \longrightarrow \mathbb{R} \\ f'(a) < \lambda < f'(b) \end{array}\right\} \Rightarrow \exists c\in (a,b),\ f'(c) = \lambda$$

$f: [a,b] \to \mathbb{R}$ continuous

$f(a) < \lambda < f(b)$

$\Rightarrow \exists c \in [a,b],\ f(x$

* Cor If $f$ is differentiable on $[a,b]$ $\Rightarrow$ $f'$ can $\boxed{\text{not}}$ have discontinuity of kind 1

may have discontinuity of kind 2.

### 5.3 * L'Hospital theorem

$f, g: (a,b) \longrightarrow \mathbb{R}$ or in a neighborhood of a.

$f, g$ differentiable in $(a,b)$, $-\infty \leq a < b \leq +\infty$

$g'(x) \neq 0,\ \forall x\in (a,b)$

$\frac{0}{0}$ or $\frac{}{\mp\infty}$

Suppose $\frac{f'(x)}{g'(x)} \xrightarrow{x\to a} A \in \overline{\mathbb{R}}$

If $\begin{cases} f(x) \xrightarrow{x\to a} 0 \\ g(x) \xrightarrow{x\to a} 0 \end{cases}$ or $g(x) \xrightarrow{x\to a} \pm\infty$

Then

$\frac{f(x)}{g(x)} \xrightarrow{x\to a} A$

which means

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Mat601 HW5.34

Aug 2013, Q 1.

### * Stolze Cesaro theorem:

$\frac{}{\pm\infty}$ form

If $\exists \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$.

$\{b_n\}$ strictly increasing, $\lim_{n\to\infty} b_n = +\infty$ $\Big\}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

$\boxed{\frac{0}{0} \text{ form.}}$

If $\exists \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, $\Big\}$ $\{b_n\}$ strictly mon

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

# Higher order derivative and Taylor's theorem

In order to have $f^{(n)}(x)$ exists at a point $x$, $\begin{cases} f^{(n-1)} \text{ has to be defined in a neighborhood of } x \\ f^{(n-1)} \text{ has to be differentiable at } x. \end{cases}$

## 15 Taylor's theorem (approximate a function by a polynomial, using its derivative)

egree d Taylor polynomial $\boxed{\text{at } \alpha}$ is (we need $f^{(d)}(\alpha)$ exists

$$(x) = \sum_{l=0}^{d} \frac{f^{(l)}(\alpha)}{l!}(x-\alpha)^l = f(\alpha) + \frac{f'(\alpha)}{1}(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^{(d)}(\alpha)}{d!}(x-\alpha)^d$$

eano form  not only can be applied for $x$, but also specific point.

$f$ is defined on some interval containing $\alpha$

- Assum $f^{(d)}(\alpha)$ exists, then  $f^{(d)}$ only need to exists at $\alpha$

$$f(x) = \sum_{l=0}^{\boxed{d}} \frac{f^{(l)}(\alpha)}{l!}(x-\alpha)^l + \lambda(x)\,(x-\alpha)^{\boxed{d}}, \quad \text{where } \lambda(t) \xrightarrow{t\to\alpha} 0$$

proof use

$f(1) = f(x) + f'(x)(1-x) + \lambda(t)(1-x$

## Lagrange form

- Assume $\begin{cases} f^{(d)}(x) \text{ exists and } \boxed{\text{continuous}} \text{ on } [\alpha, \beta] \\ \boxed{\text{differenthable}} \text{ on } (\alpha, \beta) \end{cases}$ (means $\exists\, f^{(d+1)}$ exists, on $(\alpha, \beta)$)

$$f(x) = \sum_{l=0}^{d} \frac{f^{(l)}(\alpha)}{l!}(x-\alpha)^l + \frac{f^{(d+1)}(\xi)}{(d+1)!}(x-\alpha)^{d+1}, \quad \text{for some } \xi \in (\alpha, x)$$

( (Proof use mean v the.)

---

Basic Taylor series.

$$^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{l=0}^{\infty} \frac{x^l}{l!}$$

$$x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} = \sum (-1)^l \frac{x^{2l+1}}{(2l+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum (-1)^{l} \frac{x^{2l}}{(2l)!}$$

$$x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$ (for $\boxed{p \in \mathbb{R}}$)

*Note that we can have the Taylor of $g(x)$ if we have Taylor expansion of $f(x)$

EX. $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

then $g(x) = [f(x)]' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2$

we can also find

$f(x) = \frac{2x}{1-5x^3} = 2x + 10x^4 + 50x^7 + 250x^{10} + \cdots$

* Problems that can be solve by using Taylor's theorem.

(we notice that there are some special Taylor series that need t remember sin, cos, e and t to use Taylor ser in case it's han to use another na

- Jan 2012 (P3)

$f : \mathbb{R} \to \mathbb{R}$ be a function differentiable at 0, $\boxed{f(0)=0}$

Show that the following limit exists and find : $\lim_{x\to 0} \frac{f(x)-\sin f(x)}{x^3}$

- Practice Taylor : $\lim_{x\to 0} \frac{1-\cos(2x)}{3x^2}$ $\lim_{x\to 0} \frac{\sin x^2}{1-\cos(2x)}$

* We can also find Taylor series of $f(x)$ through finding Taylor series of $F(x)$.

We have $F'(x) = f(x)$. ( ⇒ when define $F(x) = \int_0^x f(t)\,dt$ )

Then $\{\text{Taylor series of } F'(x)\} \equiv \{\text{Taylor series of } f(x)\}$.

* We can also use Taylor theorem to investigate the convergence of a series.

- 3.6/78. $\sum_{n=1}^{\infty} \frac{1}{n^4 e^n}$ converges or diverges? $e^n = 1 + \frac{n}{1} + \frac{n^2}{2} + \frac{n^3}{3!} + \frac{n^4}{4!} \geq \frac{n^4}{4!}$

then $e^n - n^4 \geq \frac{n^4}{4!} - n^4 = \frac{(1-4!)n^4}{4}$.

$0 \leq \frac{1}{e^n - n^4} \leq \underbrace{\frac{4!}{(1-4!)n^4}}_{\sum \text{converges}}$ $\Big\}$ ⇒ $\sum \frac{1}{e^n - n^4}$ converges ⇒ the series conv

* We can also use Taylor theorem to prove $\exists x$ satisfies some inequality associate with value of $f^{(k)}(x)$ at some point $x$

HW 5.5, 5.6?

- P1? $f$ has third derivative at any point in $\mathbb{R}$, $f'(p) = f^{(2)}(p) = f^{(3)}(p) = 0$ and $f^{(4)}(p) > 0$ $\Big\}$ then $p$ is a point of strictly local minimum of $f$

- P2? $f : [0,2] \to \mathbb{R}$ is continuous. $|f''(x)| < 1$, $\forall x \in (0,2)$. $\Big\}$ Prove that $|f(0) - 2f(1) + f(2)| \leq 1$.

- Rudin 5.17

$f$ is real, three time differentiable function on $[-1, 1]$ such that.

$f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, $f'(0) = 0$.

Prove that $f^{(3)}(x) \geq 3$, for some $x \in (-1, 1)$.

* Note that apply Taylor series at $(x+h)$ and $x$ is a really interesting trick.

$f(x+h) = f(x) + \frac{f'(x)}{1!}(h)$

$f(x+2h) = \dots\dots$

with problem requiring proving some inequality with $f(x)$ (for $x$ not specific)
⇒ need to do this with $x+h$ and $x$.

EX: Rudin 5.15.

Suppose $a \in \mathbb{R}^1$, $f$ is twice differentiable real function on $(a, +\infty)$

$M_0 = \sup_{x \in (a,+\infty)} |f(x)|$

$M_1 = \sup_{x \in (a,+\infty)} |f'(x)|$

$M_2 = \sup_{x \in (a,+\infty)} |f''(x)|$

then $M_1^2 \leq 4 M_0 \cdot M_2$.

We can also use Taylor theorem to find $\lim_{n\to\infty}\int_0^1 g(n)\, f(x,n)\,dx$, note that with this kind $g(x)$ can be $e^x, \sin x, \cos x$ $\longrightarrow$ use Taylor series to approximate these function.

m2003: Prove $\lim n^2\int_0^1 e^{x^2} x^n(1-x)\,dx = e$.

We use $e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!}$ $\quad e^{2x} = \sum_{l=0}^{\infty}\frac{x^{2l}}{l!}$ and $\boxed{\lim n^2\int_0^1 x^{n+2}(1-x) = 1}$

## * Differentiation of vector-value functions.

5.16 Remarks:

Let $f: \cancel{\mathbb{R}} \longrightarrow \mathbb{C}$, $t \longmapsto f(t)$ | $f(t)$ can be defined through the real and imaginary parts of $f$; that is,

If $f(t) = f_1(t) + i \cdot f_2(t)$, $a \leq t \leq b$ | where $f_1: \mathbb{R} \longrightarrow \mathbb{R}^1$, $f_2: \mathbb{R} \longrightarrow \mathbb{R}^1$.

Then we clearly have

$$f'(x) = f_1'(x) + i f_2'(x)$$

, $f$ is differentiable at $x$ iff both $f_1$ and $f_2$ are differentiable at $x$.

* Now consider $f: \mathbb{R}^1 \longrightarrow \mathbb{R}^k$

$$x \longmapsto f(x) = \left(f_1(x), f_2(x), \dots, f_k(x)\right)$$

then $f$ is differentiable at $x$ iff $f_i$, $i = \overline{1,k}$ is differentiable at $x$.

The definition of derivative

$f$ is differentiable $\Rightarrow$ $f$ is continuous.

$$(f \pm g)'(x) = f'(x) + g'(x)$$

dot product

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}, g(x) \neq 0$$

* However, the mean value theorem, L's Hospital rule { does not work when $f: R^1 \longrightarrow R^k$. (only works in case $f: R^1 \longrightarrow R^1$)

* Theorem 5.10 (mean value theory)

$$f(b) - f(a) = f'(x)(b-a) \Rightarrow |f(b) - f(a)| \leq \sup_{a \leq x \leq b} |f'(x)| (b-a)$$

* Theorem 5.19 (a result weaker ~~the result~~ than mean-value theorem but works in case $f: R \longrightarrow \mathbb{R}^k$)

$f: [a,b] \longrightarrow \mathbb{R}^k$ continuous

$f$ is differentiable in $(a,b)$

Then there exists $x \in (a,b)$ such that $|f(a) - f(b)| \leq |f'(x)| \, |b-a|$

* Rudin5.1//114

Let f is defined for all real x
Suppose $|f(x)-f(y)| \leq (x-y)^2, \forall x, y$ } Then f is a constant
+ f is continuous + differentiable on $\mathbb{R}$.

* Rudin5.8.

If f' is continuous on $[a,b]$. Then f is uniformly continuous on $[a,b]$ which means.

$$\forall \varepsilon > 0, \exists \delta > 0, 0<|t-x|<\delta, \text{ then } \Big|\underbrace{\frac{f(t)-f(x)}{t-x}}_{f'(\xi)} - f'(x)\Big| < \varepsilon$$

$x, t \in [a,b]$.

* Rudin5.5.

f is defined and differentiable for $x>0$
$\lim_{x\to+\infty} f'(x) = 0$ } Then $\lim_{x\to+\infty} f(x+1) - f(x) = 0$.

- Aug2013.

I3, If f be a real valued function on $\mathbb{R}$ that satisfies $A=\{x, |f(x)|>\varepsilon\}$ is compact.
Then $\lim_{|x|\to+\infty} f(x) = 0$

* Note that (From Aug2013 P1, HW601, 5.3,4 P1. (See more explain in this problem).

If we have $f''(x)$ exists at some point $x$, we have:

$f'(x)$ exists in the neighborhood of $x$.

we <u>don't</u> have $f''(x)$ exist in a neighborhood of $x$.

- <u>Assume we compu</u>

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*See in Aug1994 >

$\lim_{x\to+\infty}\frac{f(x)}{x}=0 \Leftrightarrow \forall\varepsilon>0, \exists N$ such that $\forall x>N$, then $\boxed{\frac{1}{2}}\frac{f(x)}{x}\Big| < \varepsilon$

---

* <u>Group of problems relating to $\lim_{x\to+\infty}\frac{f(x)}{x}=0$ or compute $\lim_{x\to+\infty}\frac{f(x)}{x}$</u>

- Aug1994 > P2

$f$ is a differentiable function on $(0,+\infty)$

$\lim_{x\to\infty}\frac{f(x)}{x}=0$ $\lim_{x\to\infty}f'(x)=a$

} Prove that $a=0$.

- Jan2004 > P2

$f:(0,+\infty)\to\mathbb{R}$ be differentiable

$\lim_{x\to+\infty}\frac{f(x)}{x}=0$

} Prove that $\exists(x_n)$, $x_n\uparrow+\infty$, $\lim_{n\to\infty}f'(x_n)=0$

- <u>Aug2007 > P2</u>

$f$ is defined on $[0,+\infty)$.

bounded on any $[0,a]$, $a<+\infty$)

$\lim_{x\to+\infty}[f(x+1)-f(x)]$ exists.

} Show that $\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}[f(x+1)-f(x)]$. (apply Cesaro theorem).

A very good trick used in this problem is that

$\lim_{x\to+\infty}\frac{f(x)}{x}=0 \Leftrightarrow \forall\varepsilon>0, \exists N, \forall x>N, \boxed{\frac{1}{2}}\Big|\frac{f(x)}{x}\Big|<\varepsilon$.

then $\Big|\frac{f(x)}{2}\Big|<\varepsilon \Leftrightarrow \Big|\frac{f(x)}{2x}-\frac{f(2x)}{2x}+\frac{f(2x)}{2x}\Big|<\varepsilon$

$\Leftrightarrow \Big|\frac{f(x)-f(2x)}{2x}+\frac{f(2x)}{2x}\Big|<\varepsilon$

$\Leftrightarrow \Big|\frac{f'(\xi)}{2}\Big|-\Big|\frac{f(2x)}{2x}\Big|\leq \;<\varepsilon$

$\Rightarrow \Big|\frac{f'(\xi)}{2}\Big| < \underbrace{\Big|\frac{f(2x)}{2x}\Big|+\varepsilon}_{\to 0}$ $\Rightarrow f'(\xi) \to 0$

* MAT601. HW5.3.4.

$f:\mathbb{R}\to\mathbb{R}$ is a function s.t $\|f'(x)\|\leq 1$, $\forall x\in\mathbb{R}$; Prove that $\lim_{x\to\infty}\frac{f(x)}{x^p}=0$, $\boxed{\forall p>2}$

Note that L Hospital is extremely useful when we want to find $\boxed{\lim_{n\to\infty}\frac{f(n)}{g(n)}}$ or ex $\lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}$ (see Sample B, P4, a)

* Prove the claim

$$\boxed{f'(x) \text{ exists} \iff \exists A \in \mathbb{R}, \exists \lambda(t) \text{ s.t. } f(t) = f(x) + A(t-x) + \lambda(t)(t-x) \quad \text{where } \lambda(t) \xrightarrow[t\to x]{} 0}$$

($\Longrightarrow$) Because $f'(x)$ exist. Put $A = f'(x)$,

we have $\lim\limits_{t\to x} \dfrac{f(t)-f(x)}{t-x} = A \iff \lim\limits_{t\to x}\left(\dfrac{f(t)-f(x)}{t-x} - A\right) = 0$

Put $\lambda(t) = \dfrac{f(t)-f(x)}{t-x} - A$, then we have $\lim\limits_{t\to x} \lambda(t) = 0$ and $f(t) = f(x) + A(t-x) + \lambda(t)$

($\Longleftarrow$): We have $f(t) =$ [illegible] $A(t-x) + \lambda(t)(t-x)$ where $t \neq x$.

then $\dfrac{f(t)-f(x)}{t-x} = A + \lambda(t)$

$\Rightarrow \lim\limits_{t\to x} \dfrac{f(t)-f(x)}{t-x} = \lim\limits_{t\to x}(A+\lambda(t)) = A$ then $\exists f'(x)$ and $f'(x) = A$.

* Theorem 5.2 : $\boxed{f \text{ is differentiable at } x \Rightarrow f \text{ is continuous at } x}$

We have from the above claim,

$f'(x)$ exist $\Rightarrow \exists A \in \mathbb{R}, \exists \lambda(t) \xrightarrow{t\to x} 0$ s.t $f(t) = f(x) + A(t-x) + \lambda(t)(t-x)$

$\Rightarrow \lim\limits_{t\to x} f(t) = f(x) \Rightarrow f$ is continuous at $x$ □

Prove Theorem 5.3 Derivative rule.

et $f$ and $g$ defined on $[a,b]$.
$f$ and $g$ are differentiable at $x$

Then $(f\pm g)$, $(f\,g)$, $(f/g)$ (if $g(x)\neq 0$) differentiable at $x$ and

$$(f+g)'(x) = f'(x)+g'(x)$$

$$(f\,g)(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$(f/g)(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}.$$

We have $f$ and $g$ differentiable at $x$ $\Leftrightarrow$ $\exists\, f'(x) = \lim\limits_{\substack{t\to x\\ t\neq x}} \frac{f(t)-f(x)}{t-x}$

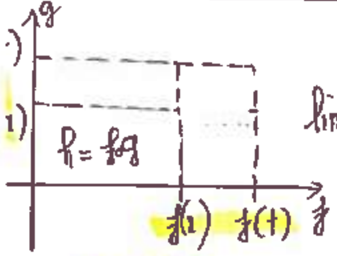
$\exists\, g'(x) = \lim\limits_{\substack{t\to x\\ t\neq x}} \frac{g(t)-g(x)}{t-x}.$

Now, prove $(f+g)$ differentiable and $(f+g)'(x) = f'(x)+g'(x)$

we have $\dfrac{(f+g)(t)-(f+g)(x)}{t-x} = \dfrac{f(t)-f(x)}{t-x} + \dfrac{g(t)-g(x)}{t-x}$

then find $(f+g)'(x) = \lim\limits_{\substack{t\to x\\ t\neq x}} \dfrac{(f+g)(t)-(f+g)(x)}{t-x} = \lim\limits_{\substack{t\to x\\ t\neq x}} \dfrac{f(t)-f(x)}{t-x} + \lim\limits_{\substack{t\to x\\ t\neq x}} \dfrac{g(t)-g(x)}{t-x} = f'(x)+g'(x)$

Now prove that $(fg)(x$ differentiable at $x$ and $(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$



We want to prove that

$$\lim \frac{(f\cdot g)(t)-(f\cdot g)(x)}{t-x} = f'(x)g(x) + g'(x)\,\cancel{f(x)} = \frac{f(t)-f(x)}{t-x}g(x) + \frac{g(t)-g(x)}{t-x}$$

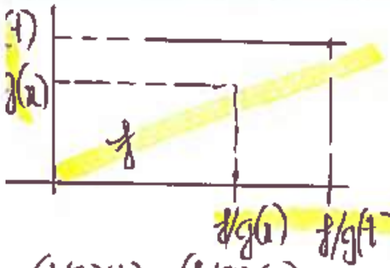
Put $h(= f\cdot g$ ), we have $(f\cdot g)(t)-(f\cdot g)(x) = h(t)-h(x) = [f(t)-f(x)]\cdot g(t) + [g(t)-g(x)]f(x$

$$\Rightarrow \frac{(f\cdot g)(t)-(f\,g)(x)}{t-x} = \frac{f(t)-f(x)}{t-x}\,g(t) + \frac{g(t)-g(x)}{t-x}\,f(x)$$

$$\Rightarrow \lim_{t\to x} \frac{(f\,g)(t)-(fg)(x)}{t-x} = \lim \frac{f(t)-f(x)}{t-x}\,\underset{g(x)}{\underset{\downarrow}{g(t)}}\ \lim \frac{g(t)-g(x)}{t-x}\,f(x)$$

$$= f'(x)\,g(x) + g'(x)\,f(x) \quad \square.$$

Prove $(f/g)$ differentiable at $x$. And $(f/g)'(x) = \ldots$



$$f(t) - f(x) = [g(t)-g(x)]\,\frac{f}{g}(x) + \left[\frac{f}{g}(t) - \frac{f}{g}(x)\right]g(t)$$

$$\Rightarrow \frac{f}{g}(t) - \frac{f}{g}(x) = \frac{1}{g(t)}\left\{ f(t) - f(x) - [g(t)-g(x)]\,\frac{f}{g}(x) \right\}$$

$$= \frac{1}{g(x)g(t)}\Big[\, [f(t)-f(x)]\,g(x) - [g(t)-g(x)]\,f(x) \,\Big].$$

$$\Rightarrow \frac{(f/g)(t)-(f/g)(x)}{t-x} = \frac{1}{g(x)\,g(t)}\left[\frac{[f(t)-f(x)]\,g(x) - [g(t)-g(x)]\,f(x)}{t-x}\right] \xrightarrow{t\to x} \frac{1}{g^2(x)}\,[f'(x)\,g(x) + f(x)\,g'(x)$$

55 Theorem <u>Chain rule (Rudin Aug 2006, p17</u>

$f:[a,b] \longrightarrow \mathbb{R}$ is continuous

At $x \in [a,b]$, $\exists f'(x)$

$\exists g'(f(x))$

Let $h(t) = g(f(t))$, $a \leq t \leq b$

Then $h$ is differentiable at $x$, and

$h'(x) = g'(f(x))\, f'(x)$.

* We have $\exists f'(x) \Leftrightarrow f(t) = f(x) + f'(x)(t-x) + \lambda(t)(t-x)$, where $\lambda(t) \xrightarrow{t\to x} 0$ (1

* Put $y = f(x)$ $z = f(t)$ $x \xmapsto{f} y$

$t \xmapsto{f} z$

We have $\exists g'(f(x)) \Leftrightarrow \exists g'(y)$

$\Leftrightarrow g(z) = g(y) + g'(y)(z-y) + \upsilon(t)(z-y)$, where $\upsilon(t) \xrightarrow{t\to x} 0$ (

* We want to prove that $h(t) = h(x) + B(t-x) + R(t)(t-x)$

where $\begin{cases} B = g'(f(x))\, f'(x) = g'(y)\, f'(x) \\ R(t) \xrightarrow{t\to 0} 0 \end{cases}$

By def of $h$, we have

$$h(t) = g(f(t)) = g(z) \underset{\text{by (2)}}{=\!=} g(y) + g'(y)(z-y) + \upsilon(t)(z-y)$$

$$= g(f(x)) + g'(y)[f(t) - f(x)] + \upsilon(t)(z-y)$$

$$\underset{\text{by (1)}}{=} h(x) + g'(y)[f'(x)(t-x) + \lambda(t)(t-x)] + \upsilon$$

$$+ \upsilon(t)[f'(x)(t-x) + \lambda(t)(t-x)]$$

$$= h(x) + g'(y) f'(x)(t-x) + \underbrace{[g'(y)\lambda(t) + \upsilon(t) f'(x) + \upsilon(t)\lambda(t)]}_{R(t) \xrightarrow[t\to x]{} 0}$$

Prove theorem 5.9 (Generalized mean value theorem).

$$\left.\begin{array}{l} f, g \text{ continuous on } [a,b] \longrightarrow \mathbb{R} \\ f, g \text{ differentiable in } (a,b) \end{array}\right\} \Rightarrow \exists c \in (a,b) \text{ st } [g(b)-g(a)]\, f'(c) = [f(b)-f(a)]\, g'(c)$$

Proof: this is an interesting proof from Kovalev (more interesting than in Rudin's book)

Put $h(x) = \begin{vmatrix} f(x) & g(x) \\ f(b)-f(a) & g(b)-g(a) \end{vmatrix}$

Then we have

$h(b)-h(a) = \begin{vmatrix} f(b)-f(a) & g(b)-g(a) \\ f(b)-f(a) & g(b)-g(a) \end{vmatrix} = 0 \quad \to h(b) = h(a)$

$\Rightarrow$ By Rolle's theorem.

$\exists c \in (a,b) \quad h'(c) = 0$

$\Rightarrow f'(c)[g(b)-g(a)] = g'(c)\,[f(b)-f(a)]$

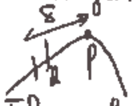
* Prove theorem 5.8

Let $f : [a,b] \longrightarrow \mathbb{R}$
$f$ has a local maximum at $p \in (a,b)$
$\exists f'(p)$
$\Big\} \Rightarrow f'(p) = 0.$

We have $f$ has local maximum at $p \in [a,b]$
$\Leftrightarrow \exists \delta > 0, \forall x \in [a,b], d(x,p) < \delta$ then $f(x) < f(p)$

We NTP $f'(p) = 0$
$\Leftrightarrow$ NTP $\lim_{x \to p} \frac{f(x)-f(p)}{x-p}$

- Then for $x \in X$, $p-\delta < x < p$, we have $\frac{\overbrace{f(x)-f(p)}^{<0}}{\underbrace{x-p}_{<0}} > 0 \Rightarrow f'(p) \geq 0$



- Then for $x \in X$, $p < x < p+\delta$, we have $\frac{f(x)-f(p)}{\underbrace{x-p}_{>0}} < 0 \Rightarrow f'(p) \leq 0$

$\Big\} \Rightarrow f'(p) = ($

* Prove Rolle's theorem

Let $f : [a,b] \longrightarrow \mathbb{R}$, $f$ continuous on $[a,b]$.
$f$ is differentiable ~~at~~ in $(a,b)$
$f(a) = f(b)$
$\Big\} \Rightarrow$ Prove that $\exists c \in (a,b)$, $f'(c) = 0$.

We have $f : [a,b] \longrightarrow \mathbb{R}$ continuous on $[a,b] \Rightarrow$ by extreme value theorem, $f$ attains $\underbrace{\text{global maximum}}_{\text{Put} =: x_n}$ and $\underbrace{\text{global minimum}}_{:=}$ in $[a,b]$.

Then in case $x_n$ or $x_m \in (a,b)$, then it il is the point c, $f'(c) = 0$.

- In case $x_n$ and $x_m$ are endpoints,
because $f(a) = f(b)$ $\Big\} \Rightarrow f(x_n) = f(x_m) \Rightarrow f$ is contant in $[a,b]$
$\Rightarrow f'(x) = 0, \forall x \in [a,b]$ ,

* Prove mean value theorem ( Theorem 5.9 (Generalized mean value theorem ← Past pa

$f : [a,b] \longrightarrow \mathbb{R}$ continuous
$f$ is differenthable on $(a,b)$
$\Big\} \rightarrow$ Then $\exists c \in (a,b)$ $f(b) - f(a) = f'(c)(b-a)$.

Put $g(x) = f(x) - \frac{f(b)-f(a)}{(b-a)} x.$

Then we have $g(b) = f(b) - \frac{f(b)-f(a)}{b-a} b = \frac{f(b)(b-a) - f(b) \cdot b + f(a) b}{b-a} = \frac{-a f(b) + bf}{b-a}$

$g(a) = f(a) - \frac{f(b)-f(a)}{b-a} a = \frac{f(a)(b-a) - f(b) \cdot a + f(a) a}{b-a} = \frac{b(f(a) - a f}{b-a}$

$\Rightarrow g(b) = g(a)$

From Rolle's theorem, $\exists c, g'(c) = 0 \Leftrightarrow \exists c \in (a,b), 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$ $\Rightarrow$

If $f: \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable
$f'(x)$ is bounded ~~$\forall x \in [a,b]$~~ $\Big\} \Rightarrow f$ is uniformly continuous ~~on $[a,b]$~~

: Prove
x) is bounded ~~$\forall x \in [a,b]$~~, by mean value theorem

$\exists M > 0, \forall x \in$ ~~$[a,b]$~~, $\left|\frac{f(t)-f(x)}{t-x}\right| \leq M$

$\Leftrightarrow \quad |f(t)-f(x)| \leq M|t-x|$

We need to prove $f$ is uniformly continuous

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall t, x \in \mathbb{R},$
$|t-x| < \delta$, then $|f(t)-f(x)| < \varepsilon$

hen $\forall \varepsilon > 0$, Choose $\delta$ such that $M\delta < \varepsilon$, then $\forall t, x \in \mathbb{R}$, $|t-x| < \delta$, then

$$|f(t)-f(x)| \leq M|t-x| < M\delta < \varepsilon \Rightarrow \square$$

Note that $f$ is uniformly continuous $\not\Rightarrow f'(x)$ is bounded

EX: $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$.
$f'(x) = \frac{1}{2\sqrt{x}}$ on $(0,1)$

~~An weird example that haven't not knew how to find solution by myself~~:

If $f(1) = 2$ and $f(2) = 0$
$f$ is differentiable on $[1,2]$. $\Big\}$ Prove that $\exists c$ such that $c^2 f'(c) = -4$.
$\in (1,2)$.

Put $g(t) = f\left(\frac{1}{t}\right)$

$g(1) = f(1) = 2 \qquad g\left(\frac{1}{2}\right) = f(2) = 0$

Then by means value theorem, $\exists d \in \left(\frac{1}{2}, 1\right), (2-0) = g'(d)\left(\frac{1}{2}\right)$.

$\Leftrightarrow \quad 4 = g'(d) = f'\left(\frac{1}{d}\right) = -\frac{1}{d^2} f'\left(\frac{1}{d}\right)$

Then let $c = \frac{1}{d} \Rightarrow c \in (1,2)$

$\Leftrightarrow \exists c \in (1,2), 4 = -c^2 f'(c)$

Note that: in this example $\frac{1}{1} = 1$
trick here: put $g(t) = f\left(\frac{1}{t}\right) \qquad \left(\frac{1}{t}\right)' = -\frac{1}{t^2}$

* Prove theorem from Kor

$f$ is continuous on an interval $I$
$\lim\limits_{x\to p} f'(x)$ exists for $p \in I$
$\Big\} \to \exists f(p)$ and $f'(p) = \lim\limits_{x\to p} f'(x)$

(this means, $f'(x)$ exists when $x$ near $p$

* We have $\exists f'(x) = \dfrac{f(t) - f(P)}{t - p}$ for $x$ between $t$ & $p$ (Note that the mean value theorem is extremely important

Then when $t \to p$, $x \to p$

$$\Rightarrow \lim_{t\to p} \frac{f(t) - f(P)}{t - p} = \lim_{x\to p} f'(x)$$

This means $\exists f'(p)$ and $f'(P) = \lim\limits_{x\to p} f'(x)$ □

* Prove theorem 5.12 (Intermidiatte value theorem) for derivative Jan 2015 47

Let $f$ ~~be~~ a differentiable $[a,b] \longrightarrow \mathbb{R}$
……, $f'(a) < \lambda < f'(b)$
$\Big\} \to$ Then $\exists c \in (a,b)$, $f'(c) = \lambda$.

Put $g(x) = f(x) - \lambda x$ Note that we want $g'(x) = f'(x) - \lambda \Rightarrow$ put $g(x) = f(x) - \lambda x$

Then we have because $f$ differentiable on $[a,b]$. $\Rightarrow$ $g$ is differentiable in $(a,b)$.

$g'(b) = f'(b) - \lambda > 0$

$g'(a) = f'(a) - \lambda < 0$

* We need to prove that $\exists c \in (a,b)$ such that $g'(c) = 0$.

$\Rightarrow$ Neither of $a$ or $b$ are a point of min of $g$.

$\Rightarrow$ So its min is at some $c \in (a,b)$

$\Rightarrow g'(c) = 0 \Rightarrow f'(c) = \lambda \Rightarrow$ □.

## Prove Taylor theorem (Peano form)

ppose $f$ is defined on some interval containing $\alpha$

$f^{(d)}(d)$ exists

en $f(x) = \sum_{k=0}^{d} \frac{f^{(k)}(d)}{k!}(x-d)^k + \lambda(x)(x-\alpha)^d$ , where $\lambda(x) \xrightarrow{x\to\alpha} 0$

et $P(x) := \sum_{k=0}^{d} \frac{f^{(k)}(d)}{k!}(x-d)^k = f(d) + \frac{f'(d)}{1!}(x-d)^1 + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^{(d)}(d)}{d!}(x-\alpha)^d$

we want to prove that $\lambda(x) = 0$

$f^{(i)}(d) - P^{(i)}(d) = 0, \forall i = \overline{1,d}$

nt $\lambda(x) = \frac{f(x) - P(x)}{(x-d)^d}$

We first have that: $f^{(i)}(x) - P^{(i)}(x) \xrightarrow{x\to d} 0$ (at $x = d$), for all $i = 1, d$

Thus, with $i = 1$, $f'(x) - P'(x) = f'(x) - \left[\frac{f'(d)}{1!} + \frac{f''(d)}{2} 2(x-\alpha) + \cdots + \frac{f^{(d)}(d)}{d!} d(x-\alpha)^{d-1}\right] \xrightarrow{x\to\alpha} 0$

with $i = 2$, $f''(x) - P'(x) = f''(x) - \left[\frac{f''(d)}{2} 2 + \cdots + \frac{f^{(d)}(d)}{d!} d(d-1)(x-d)^{d-2}\right] \xrightarrow{x\to\alpha} 0$

then by induction, we can prove the above claim.

apply L'Hospital $(d-1)$ time, it suffice to prove that

$$\lim_{x\to\alpha} \lambda(x) = \lim_{x\to\alpha} \frac{f^{(d-1)}(x) - P^{(d-1)}(x)}{x-\alpha} = 0 \quad (*)$$

Note that we have $f^{(d)}(\alpha)$ exists means $f^{(d-1)}(x)$ exists in a neighborhood of $x$.
→ only use L'Hospital $(d-1)$ time.

e have $P^{(d-1)}(x) = f^{(d-1)}(d) + f^{(d)}(d)(x-\alpha)$

$$\text{So } (*) \Leftrightarrow \lim_{x\to\alpha} \frac{f^{(d-1)}(x) - f^{(d-1)}(d) - f^{(d)}(\alpha)(x-\alpha)}{x-\alpha} = 0$$

$$\Leftrightarrow \underbrace{\lim_{x\to\alpha} \frac{f^{(d-1)}(x) - f^{(d-1)}(\alpha)}{x-\alpha}}_{\longrightarrow f^{(d)}(\alpha)} - f^{(d)}(\alpha) = 0.$$

thus this is true □.

The idea of this prove:

e have $f^{(d)}(\alpha)$ exists and we want to prove that

$$f(x) = \underbrace{\sum_{k=1}^{d} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k}_{P(x)} + \lambda(x)(x-\alpha)^d.$$

we only need to put $\lambda(x) = \frac{f(x) - P(x)}{(x-\alpha)^d}$.

d we want to prove that $\lim_{x\to\alpha} \lambda(x) = 0$ by using the fact that $f^{(k)}(\alpha) - P^{(k)}(\alpha) = 0$

then use

* Prove Taylor theorem ( Lagrange form)

Prove that if f is a function such that $f^{(d+1)}(x)$ exists for $x \in (\alpha, \beta)$.

Then we have

$$f(\beta) = \sum_{\ell=1}^{d} \frac{f^{(\ell)}(\alpha)}{\ell!}(\beta-\alpha)^{\ell} + \frac{f^{(d+1)}(\xi)}{(d+1)!}(\beta-\alpha)^{d+1}.$$

* We let K is the number satisfies

$$f(\beta) = \sum_{\ell=1}^{d} \frac{f^{(\ell)}(\alpha)}{\ell!}(\beta-\alpha)^{\ell} + K(\beta-\alpha)^{d+1}.$$

(This means we want to prove that $\exists$ some $p$ between $(\alpha,\beta)$ such that $K = \frac{f^{(d+1)}(\xi)}{(d+1)!}$

* Now consider $f(x) = \underbrace{\sum_{\ell=1}^{d} \frac{f^{(\ell)}(\alpha)}{\ell!}(x-\alpha)^{\ell}}_{P_d(x)} + K(x-\alpha)^{d+1}$.

Let $F(x) = f(x) - P_d(x) - K(x-\alpha)^{d+1}$

Then we have $F(\alpha)=0,\ F(\beta)=0 \Rightarrow \exists c_1$ s.t $F'(c_1)=0$.

$F'(\alpha)=0,\ F'(c_1)=0 \Rightarrow \exists c_2$ between 0 and $c_1$, $F'(c_2)=0$.

$F''(\alpha)=0,\ F''(c_2)=0 \Rightarrow \exists c_3$ between 0 and $c_2$, $F'(c_3)=0$

⋮

$F^{(d)}(\alpha)=0,\ F^{(d)}(c_d)=0 \Rightarrow \exists c_{d+1}$ between 0 and $c_d$, $\underbrace{F^{(d+1)}(c_{d+1})=0}$

this explains why we need $f^{(d+1)}(x)$ exists for all $x$ in $(\alpha, \beta)$ (because we don't know where $c_{d+1}$ can be

We also have $F^{(d+1)}(x) = f^{(d+1)}(x) - K(d+1)!$.

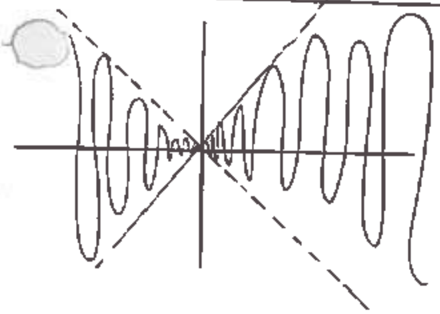
So we have $K = \frac{f^{(d+1)}(\xi)}{(d+1)!}$ □.

* Example about some special functions.

See Jan 2015 ④

See HW5 3-4 601 for another $f(x)=x|x|$

- $f(x)=\begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$, this function has derivative at $x\neq 0$ does not have derivative at $x=0$.



- At $x\neq 0$

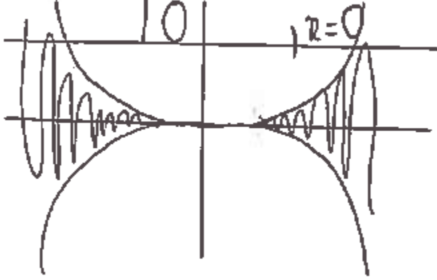
$$f'(x)=\sin\frac{1}{x}+x\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}.$$

- At $x=0$,

If $t\neq 0$, $\frac{f(t)-f(0)}{t-0}=\frac{t\sin\frac{1}{t}-0}{t}=\sin\frac{1}{t}\ \cancel{\longrightarrow}$ (does not converges)

$\Rightarrow \nexists f'(0)$

* $f(x)=\begin{cases} x^2\sin\frac{1}{x} & , x\neq 0 \\ 0 & , x=0 \end{cases}$ Then $f$ is differentiable at $\forall x\in\mathbb{R}$ but $f'(x)$ is ~~not continuous at $x=0$~~ & $\boxed{\nexists}\lim_{x\to0}f'(x)$



- At $x\neq0$

$$f'(x)=2x\sin\frac{1}{x}+x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

- At $x=0$

If $t\neq0$, $\left|\frac{f(t)-f(0)}{t-0}\right|=\left|\frac{t^2\sin\frac{1}{t}-0}{t}\right|=\left|t\sin\frac{1}{t}\right|<|t|$

then $\lim_{t\to0}\frac{f(t)-f(0)}{t-0}=0$

* Away to prove that $\lim_{t\to0}\frac{f(t)-f(0)}{t-0}=0$

Note $\begin{cases} 0\leq\left|\frac{f(t)-f(0)}{t-0}\right|\leq g(t) \\ \lim_{t\to0}g(t)=0 \end{cases}$

$\Rightarrow \lim_{t\to0}\left|\frac{f(t)-f(0)}{t-0}\right|=0$

$\Rightarrow \lim_{t\to0}\frac{f(t)-f(0)}{t-0}=0$

Then we have

$$f'(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x} & , x\neq0 \\ 0 & x=0 \end{cases}$$

- $\lim_{x\to0}f'(x)=\lim_{\substack{x\to0\\x\neq0}}2x\sin\frac{1}{x}-\cos\frac{1}{x}$ does not exist.

* $f(x)=\begin{cases} x^3\sin\frac{1}{x} & , x\neq0 \\ 0 & , x=0 \end{cases}$, for

- $f'(x)=\begin{cases} 3x^2\sin\frac{1}{x}-x\cos\frac{1}{x} & , x\neq0 \\ 0 & , x=0 \end{cases}$

- At $x\neq0$ $f''(x)=6x\sin\frac{1}{x}-3\cos\frac{1}{x}-\cos\frac{1}{x}-\frac{1}{x}\sin\frac{1}{x}$.

At $x=0$ when $t\neq0$ $\lim_{t\to0}\frac{f(t)-f'(0)}{t-0}=\lim_{t\to0}\frac{3t^2\sin\frac{1}{t}-t\cos\frac{1}{t}}{t}=\lim 3t\sin\frac{1}{t}-\cos\frac{1}{t}$ does not exist.

* $\lim_{x\to0}\left|x\sin\frac{1}{x}\right|=0$ $\lim\left(\sin\frac{1}{x}\right)$ does not exist.

$(x) = |x|$ , we have $\begin{cases} f \text{ continuous in } \mathbb{R}. \\ f' \text{ does not exist at } 0, \end{cases}$ $f'$ exists for all $x \neq 0$.

$x) = x|x| = \begin{cases} x^2 & ; x>0 \\ 0 & ; x=0 \\ -x^2 & ; x<0 \end{cases}$ $f'(x) = \begin{cases} 2x & , x>0 \\ 0 & x=0 \\ -2x & , x<0 \end{cases}$ $f''(x) \begin{cases} 2 & , x>0 \\ -2 & , x<0 \\ \nexists \text{ at } x=0 \end{cases}$

have: $\left.\begin{array}{l} \text{for } x>0,\ f'(x) = 2x \\ \text{for } x<0,\ f'(x) = -2x \end{array}\right\} \Rightarrow \lim_{x\to 0} f'(x) = 0 \to f'(0) = 0 \Rightarrow f'(x) = \begin{cases} 2x & , x \geq 0 \\ 0 & , x=0 \\ -2x & , x<0 \end{cases}$

or $x>0,\ f''(x) = 2$
$x<0,\ f''(x) = -2$ $\Rightarrow \nexists \lim_{x\to 0} f''(x)$ $\nexists f''(0)$.

$(x) = |x|^3$ , $f'(x)$ , $f''(x)$ exists $\forall x \in \mathbb{R}$
$f'''(x)$ exists for all $x \neq 0$ , $\nexists f'''(0)$

$x) = |x|^3 = \begin{cases} x^3 & , x>0 \\ -x^3 & , x<0 \end{cases}$

$'(x) = \begin{cases} 3x^2 & , x>0 \\ -3x^2 & , x<0 \end{cases}$ $\lim_{x\to 0} f'(x) = 0 \to f'(0) = 0$

$'(x) = \begin{cases} 6x & , x>0 \\ -6x & , x<0 \\ 0 & , x=0 \end{cases}$

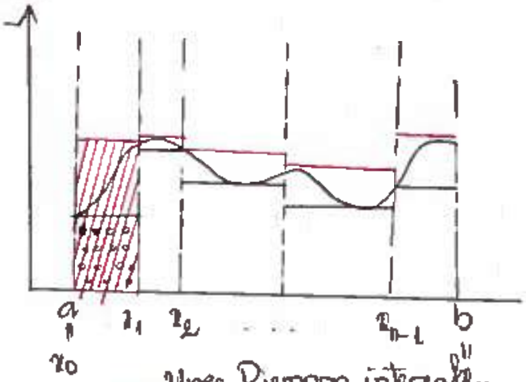
$'(x) = \begin{cases} 6 & , x>0 \\ -6 & , x=0 \end{cases}$ $\nexists f'''(0)$.

# §6: The Riemann - Stieltjes integral

We consider in $[a,b]$ bounded interval

## * 6.1 Riemann integral on $[a,b]$

Let $[a,b]$ be a given interval



- Define partition $P$ on $[a,b]$

$P = \{(x_0, \dots, x_n), a = x_0 \le x_1 \le x_2 \le \dots \le x_n = b\}$

$\uparrow$ finite set

- $\Delta x_i = x_i - x_{i-1}$ $\quad \boxed{i = \overline{1,n}}$

* Suppose $f$ is a $\boxed{\text{bounded}}$ real function on $[a,b]$

$M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$ | upper Riemann sum $U(P,f) = \sum\limits_{i=1}^{n} M_i \Delta x_i$

$m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$ | lower Riemann sum $L(P,f) = \sum\limits_{i=1}^{n} m_i \Delta x_i$

Uper Riemann integral:

$$\overline{\int_a^b} f dx = \inf_P U(P,f)$$

Lower Riemann integral

$$\underline{\int_a^b} f dx = \sup_P L(P,f)$$

We say $f$ is Riemann integrable $\overset{\text{def}}{\Longleftrightarrow}$ $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx \xrightarrow{\text{denote}} \int_a^b f dx$ or $\int_a^b f(x) dx$

---

* Now we show that the upper Riemann integral $\overline{\int} f dx$ and $\underline{\int} f dx$ are defined for $f$ is $\boxed{\text{bounded}}$

- We have $f$ is bounded $\Rightarrow \exists m, M$, $\forall x \in [a,b]$, $m \le f(x) \le M$

$$\Rightarrow \forall P \quad m(b-a) \le \underset{\sum m_i \Delta x_i}{\underset{\parallel}{L(P,f)}} \le \underset{\sum M_i \Delta x_i}{\underset{\parallel}{U(P,f)}} \le M(b-a)$$

$\Rightarrow$ The numbers $L(P,f)$, $U(P,f)$ form a bounded set $\Rightarrow \exists \inf\limits_P U(P,f) = \overline{\int} f dx$

$\exists \sup\limits_P L(P,f) = \underline{\int} f dx$

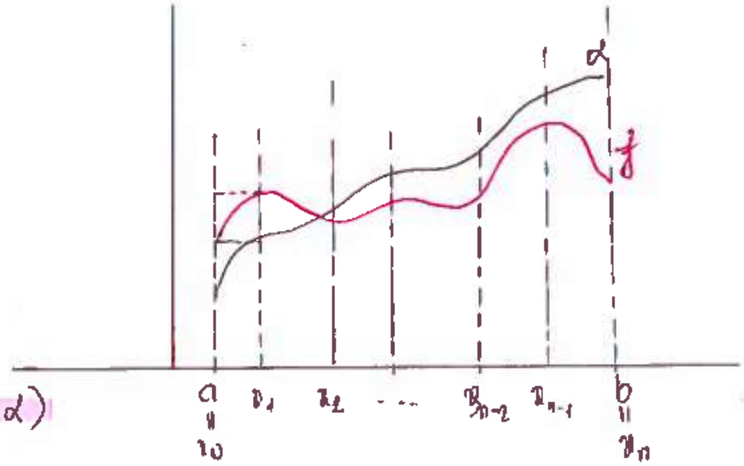
* Now we need to investigate the equality

---

* $f$ is Riemann integrable $\underset{\not\Leftarrow}{\Longrightarrow}$ $f$ is bounded

$$f = \begin{cases} \pi, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

# iemann- Stieltjes integral



t α : monotonically (increasing) on [a, b]

α is bounded on [a, b] )

efine $\Delta\alpha_i = \underbrace{\alpha(x_i) - \alpha(x_{i-1})}_{\geq 0}$, (for $i = \overline{1, n}$)

et f : real, (bounded) function on [a, b]

fine

$P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ | $\overline{\int} f d\alpha = \inf_{P} U(P, f, \alpha)$

$P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$ | $\underline{\int_a^b} f d\alpha = \sup_{P} L(P, f, \alpha)$

say f is R-S integrable $\Longleftrightarrow$ $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha \overset{\text{denote}}{=} \int f d\alpha$ R-S integrable of f with α over [a, b]

$\mathscr{R}(\alpha) = \{ f : [a, b] \longrightarrow \mathbb{R},$ f is R-S integrable wrt α $\}$

Remark:

α may not be continuous !

If α(x) = x , the R integral is a special case of R-S integral

$L(P, f, \alpha) \leq \underline{\int} f d\alpha \leq \overline{\int} f d\alpha \leq U(P, f, \alpha)$ , $\forall$ ⓟ

---

Definition

Partition P* is the refinement of P $\overset{(\text{def})}{\text{iff}}$ $P^* \supset P$

Given 2 partitions $P_1$ and $P_2$,

* is their common refinement iff $P^* = P_1 \cup P_2$

---

$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) \geq U(P^*, f, \alpha) - L(P^*, f, \alpha)$

Theorem:

If P* is a refinement of P, then $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$

---

.5 Theorem: $L(P, f, \alpha) \leq \underline{\int} f d\alpha \leq \overline{\int} f d\alpha \leq U(P, f, \alpha)$ , $\forall P$

---

. 6 Theorem:

$\in \mathscr{R}(\alpha)$ on $[a, b]$ $\Longleftrightarrow \begin{cases} f \text{ is bounded} \\ \forall \epsilon > 0, \exists \text{ partition } P, \ U(P, f, \alpha) - L(P, f, \alpha)| < \epsilon \end{cases}$

## 6.7 Corollary :

If $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ hold for partition $P = \{x_0, \dots, x_n\}$, then

a) $U(P_1^*, f, \alpha) - L(P_1^*, f, \alpha) < \varepsilon$ holds for any $P_1^*$: refinement of $P$

b) If $s_i, t_i$ are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} | f(s_i) - f(t_i) | \Delta\alpha_i < \varepsilon$$

c) If $t_i$ are arbitrary points in $[x_{i-1}, x_i]$, then

$$\left| \int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \right| < \varepsilon$$

## *6.8 Theorem

$f$ continuous on $[a,b]$ $\Longrightarrow$ $f \in \mathscr{R}(\alpha)$ on $[a,b]$

## *6.9 Theorem

$$\left.\begin{array}{l} f \text{ is monotonic on } [a,b] \\ \alpha : \text{monotonic + continuous} \end{array}\right\} \Rightarrow f \in \mathscr{R}(\alpha) \text{ on } [a,b]$$

Jan 2010, #4

## * 6.10 Theorem

in case $f$ and $\alpha$ have the same points of discontinuity $\Rightarrow f \notin \mathscr{R}(\alpha)$

$$\left.\begin{array}{l} f \text{ is bounded + has finitely many point of discontinuity on } [a,b] \\ \alpha \text{ is continuous at every point at which } f \text{ is discontinuous} \end{array}\right\} \Rightarrow f \in \mathscr{R}(\alpha) \text{ on } [a,b]$$

## *6.11 Theorem

$$\left.\begin{array}{l} f \in \mathscr{R}(\alpha) \text{ on } [a,b] \quad m \le f(x) \le M \\ \phi \text{ is continuous on } [m,M] \\ h = \phi(f(x)) \text{ on } [a,b] \end{array}\right\} \Rightarrow h \in \mathscr{R}(\alpha) \text{ on } [a,b]$$

For example

$f \in \mathscr{R}(\alpha)$ on $[a,b]$

$\Rightarrow f^2, f^2, -f \in \mathscr{R}(\alpha$

on [

## Properties of integral

2 Theorem:

If $f_1 \in \mathcal{R}(\alpha)$ on $[a,b]$, $f_2 \in \mathcal{R}(\alpha)$. Then i) $f_1 \pm f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ $\quad \int (f_1 \pm f_2)\,d\alpha = \int f_1\,d\alpha \pm \int f_2\,d\alpha$

ii) $cf \in \mathcal{R}(\alpha)$ on $[a,b]$ $\quad \int cf_1\,d\alpha = c\int f_1\,d\alpha$.

If $f_1(x) \le f_2(x)$ on $[a,b]$, then $\int f_1\,d\alpha \le \int f_2\,d\alpha$ | means $f \geq 0$ on $[a,b] \Rightarrow \int_a^b f\,d\alpha \geq 0$

If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $c$ is a number s.t. $a<c<b$. Then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and $[c,b]$

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $|f(x)| \le M$ on $[a,b]$ $\Rightarrow \int_a^b f\,d\alpha \le M\,[\alpha(b)-\alpha(a)]$.

---

If $f \in \mathcal{R}(\alpha_1)$ on $[a,b]$, $f \in \mathcal{R}(\alpha_2)$ on $[a,b]$. Then $f \in \mathcal{R}(\alpha_1+\alpha_2)$

$$\int f\,d(\alpha_1+\alpha_2) = \int f\,d\alpha_1 + \int f\,d\alpha_2$$

$f \in \mathcal{R}(\alpha)$, $c$ : positive constant. Then $f \in \mathcal{R}(c\alpha)$

$$\int f\,d(c\alpha) = c\int f\,d\alpha.$$

---

13 If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$. Then $fg \in \mathcal{R}(\alpha)$.

7 If $f \in \mathcal{R}(\alpha)$, then $|f| \in \mathcal{R}(\alpha)$, $\left|\int f\,d\alpha\right| \le \int |f|\,d\alpha$

Jan 2001, P 4

### Mean value theorem (Intermediate Value Theorem for integrals) (used in Jan 2009 [3])

Let $f \in [a,b] \longrightarrow \mathbb{R}$, $f$ continuous on $[a,b]$, $\alpha$: monotonic function on $[a,b]$ $\Rightarrow$ $f \in \mathcal{R}(\alpha)$ on $[a,b]$ (theorem 6.8)

$$\exists\, \xi \in [a,b] \quad \int_a^b f\,d\alpha = f(\xi)\,[\alpha(b)-\alpha(a)].$$

for some $\xi \in [a,b]$

* Way to compute integral with $\alpha$ is a step function

6.14 Def

The characteristic function $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$

The unit step function $I(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$

$= \chi_{(0,+\infty)}(x)$

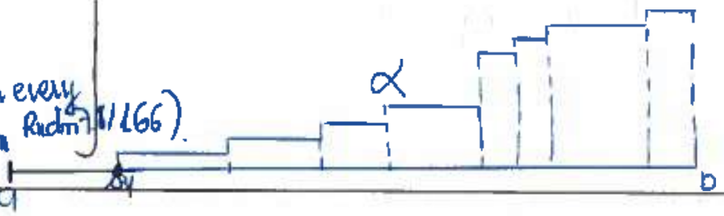
, then $I(x-s) = \begin{cases} 0, & x \leq s \\ 1, & x > s \end{cases}$

6.15 : Let $a < s < b$
$f$ continuous at $s$
$\alpha(x) = I(x-s)$
$\Rightarrow \int_a^b f d\alpha = f(s)$

* According to exercise Rudin 6.3.
It's ok to have $f \in \mathcal{R}(\alpha)$
when $\begin{cases} f \text{ is right cont at } x_0 \\ \alpha \text{ is left continuous at } x_0 \end{cases}$.

6.16 : $\{s_n\}$ : sequence of distinct point in $(a,b)$.
$f$ continuous on $[a,b]$.
$c_n \geq 0$, $\forall n$, $\sum c_n$ converges
$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ (α cont for every $x \neq s_n$ Rudin 1/166).

Then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$



6.17 Theorem

(If $\alpha$ has derivative, the R-S integral reduces to ordinary Riemann-integral integrable

$\alpha$ monotonically increasing, $\alpha' \in \mathcal{R}$ on $[a,b]$
$f$ bounded in $[a,b]$

Then $\begin{cases} f \in \mathcal{R}(\alpha) \Longleftrightarrow (f\alpha') \in \mathcal{R} \\ \int_a^b f d\alpha = \int_a^b f(x)\, \alpha'(x)\, dx \end{cases}$

6.19 (Change of variable)

$\varphi$

# Integration and differentiation (are inverse operation)

20 Theorem

Let $f \in \mathcal{R}$ on $[a,b]$
for $a \le x \le b$, $F(x) = \int_a^x f(t)\,dt$ $\Big\} \Rightarrow$ F is continuous $\boxed{\text{on } [a,b]}$ (uniformly continuous

* If $f$ is periodic $\Rightarrow F(x)$ is also periodic
(with the same frequency (Jan 2009, Q37.

is continuous at $\boxed{x_0 \in [a,b]}$ $\Rightarrow$ F is differentiable at $x_0$, $F'(x_0) = f(x_0)$.

$$\left(\int_a^{x_0} f(t)\,dt\right)'_{x_0} = f(x_0)$$

## 21 The FT o Calculus

$f \in \mathcal{R}$ on $[a,b]$
$\exists\, F$, st $F' = f$ $\Big\}$ Then $\int_a^b f(x)\,dx = F(b) - F(a)$

means $\int_a^b F'(x)\,dx = F(b) - F(a)$

## 22 Integration by parts

suppose F

$$\int F(x)G'(x)\,dx = F(x)G(x)\Big|_a^b - \int F'(x)G(x)\,dx \quad \left(\int F\,dG = FG\Big|_a^b - \int G\,dF\right)$$

Let $F(x), G(x)$ $\begin{cases} \text{continuously differentiable function defined on } [a,+\infty) \\ \lim\limits_{b\to+\infty} F(b)G(b) \text{ exist} \\ \int_a F(x)G'(x)\,dx \text{ converges} \end{cases}$

Then $\begin{cases} \int_a^b F'(x)G(x)\,dx \text{ converges} \\ \underset{\Big\downarrow \text{ converges}}{\int_a^\infty F'(x)G(x)\,dx} = \underbrace{\lim\limits_{b\to\infty} F(b)G(b)}_{\text{exists}} - F(a)G(a) - \underbrace{\int_a^\infty F(x)G'(x)\,dx}_{\text{converges}}. \end{cases}$

* Improper integral (Def)

Suppose $f$ is a real function on $(0,1]$

$f \in \mathcal{R}$ on $[c,1]$ for every $c>0$

Def: $\int_0^1 f(x)dx = \lim_{c \to 0} \int_c^1 f(x)dx$ (if the limit exists and is finite).

(If $f \in \mathcal{R}$ on $[0,1]$, the above definition agrees with the old one).

* Let $a$ is fixed, $f \in \mathcal{R}$ on $[a,b]$, $\forall b>0$.

Define $\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$ (if the limit exists and is finite).

- If the limit $\lim_{b\to\infty} \int_a^b |f(x)|dx$, we say $\int f(x)dx$ converges absolutely.

* Integral test for convergence of series.

Assume $f$ (eventually) $\geq 0$

$f$ (eventually) decreasing on $[1,+\infty)$ $(f'<0, \forall x \in [1,+\infty))$

Then $\int_1^\infty f(x)dx$ and $\sum_{n=1}^{\infty} f(n)$ both converge or diverge.

Chapter 6 Template + strateg

* Some problems of the form find $\lim_{n\to\infty}(n+1)\int_0^1 x^n f(x)\,dx = f(a)$.

Jan 2015 > P5.
$f: \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function. Show that $\lim (n+1)\int_0^1 x^n f(x)\,dx = f(1)$.

and ~~lim(n~~ Important in this $\lim_{n\to\infty}(n+1)\int_0^1 x^n P_k(x)\,dx = P_k(1)$, $\forall k$

and we use $P_k(x) \rightrightarrows f(x)$ on $[0,1]$.

Jan 2013 > P3 > $f$ be a continuous function s.t. $\lim_{x\to+\infty} f(x) = c$. Prove that $\forall\ \alpha > 0$
$$\lim_{N\to\infty}\frac{\alpha+1}{N^{\alpha+1}}\int_0^N x^\alpha f(x)\,dx = c$$

With this problem we use ~~c =~~ $c = \frac{1}{N}\int_0^N c\,dx$. and note that $\lim_{x\to+\infty} f(x) = c$ → divide the $\int$ to $\int_0^K + \int_K^N$
and let $N \to \infty$

Notee that $1 = \frac{\alpha+1}{N^{\alpha+1}}\int_0^N x^\alpha\,dx$.

---

* In this kind of question:
Way 1: Notice $1 = \int_a^b \cancel{(b-a)}\,\frac{1}{(b-a)}\,dx$.

Way 2: $f$ is continuous then $\exists\, P_k(x) \rightrightarrows f(x)$.

xam: Give ~~$\int_0^1 g(x,n) f(x)dx = 0$ Prove that $f(x) = 0$~~

0 Rudin + ~~Fall 1991~~, P4.

xam: Investigate the convergence/divergence of improper integral $\int_0^{\infty} \frac{\sin x}{x} dx$, $\int_0^{\infty} \cos(x^2)dx$

we use integral by part

and notice that we can treat this $\int_1^{\infty}$ as a $\sum_1^{\infty} f(n)$ this means can use comparison and $\int_0^{\infty} \frac{1}{x^2} dx$ converges.

xam. prove that there is a number a s.t $\int_a^x f(t)\,dt \geq 0$.

ian2009, P47

ppose $f(x+1)=f(x)$ for all real x; f is real; Riemann integrable on every compact set, $\int_0^1 f(x)dx = 0$.

Prove that $\exists x_0$ s.t $\int_{x_0}^x f(t)\,dt \geq 0$, for all x.

put $G(x)=\int_0^x f(t)\,dt$ then prove that G attain min/~~max~~ in R at $x_0$ $\to$ $G(x) \geq G(x_0)$

$\to G(x)-G(x_0) = \int_{x_0}^x f(t)\,dt \geq 0$.

lay 2016, P47

t $f:[0,1] \longrightarrow \mathbb{R}$ be a integrable function

Prove that there exist $a \in (0,1)$ s.t $\int_0^a |f(x)|dx \leq \int_a^1 |f(x)|dx$. $\Leftrightarrow \int_a^1 |f(x)|dx \to \int_0^a |f(x)|dx \geq 0$

t $F(y) = \int_0^y f(x)dx$. then this is a nonnegative, increasing, continuous, then $\exists a$ $F(1)-2F(a) \geq 0$ $\Leftrightarrow \int_0^1 |f(x)|dx - 2\int_0^a |f(x)|dx \geq 0$. $\Rightarrow \square$.

## * Form: Using Rieman integral to find limit

Form and strateg

Idea: Consider $\int_0^1 f(x)\,dx$

Assume we decide $[0,1]$ in to $\substack{3n\\2n\\n}$ part $\{x_0 = 0 < x_1 < \cdots < x_n = 1\}$ such that $\Delta x_i = \frac{1}{n}$

$\Delta x_i = \frac{1}{2n}$

$\Delta x_i = \frac{1}{3n}$

Then we have $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k \xlongequal[\substack{x_k = \frac{k}{n}\\ \Delta x_k = \frac{1}{n}}]{} \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$ ⟹ we

or $x \longleftrightarrow \left(\frac{k-1}{n}\right)$

$\left(x \longleftrightarrow \left(\frac{k}{n}\right)\right)$

⟹ We can compute $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{3n} f\left(\frac{k}{n}\right)$ through $\int_0^1 f(x)\,dx$.

where $\sum_{k=1}^{2n}$ ⟹ Partition includes 2n points $\Delta x_i = \frac{1}{2n}$, $x_k$

Example: $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right) \Longrightarrow \int_0^1 f(x)dx$ where $f(x) = x$ $= \int_0^1 x\,dx = \frac{1}{2}$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{(i-1)^2+n^2}$ $\qquad \lim \sum_{n=1}^{n} \frac{(2k+1)}{n^2+k^2}$

$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\sum_{k=1}^{n}(n+k)}$ $\qquad$ (more advance Jan 2009: $\left.\begin{matrix}\{a_n\} \text{ be a sequence, } a_n \to L\\ b_n = \frac{1}{n^2}\sum_{k=1}^{n} k a_k\end{matrix}\right\} b_n$

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## * Form: investigate the convergence/divergence of imformal integral (Aug 2015), Jan 2012.

- Aug 2015: Investigate the convergence/divergence of $\int_1^\infty \frac{\sin x}{x}\,dx$.

Jan 2012 ———————— $\int_0^\infty \cos(x^2)dx$, $\int_0^\infty \sin(x^2)dx$, $\int_1^\infty \frac{\sin x}{x}dx$, $\int \frac{x}{1+x}$

⟹ With this kind of question, just use some integration by part and then use + ~~[illegible]~~ $\sum_{1}^{+\infty} f(n)$ and $\int_1^\infty f(x)\,dx$ is both converge or diverge if f is increasing function.

+ comparision.

In the proof of problem: $\left.\begin{array}{l} f \text{ cont on } [a,b] \\ F(x)=\int_a^x f(t)\,dt, \ x\in[a,b] \end{array}\right\}$ Prove that $F'=f$

We can't prove directly that $\lim\limits_{\substack{t\to x \\ s\to x}} \frac{F(t)-F(s)}{t-s} = f(x)$ or $\lim\limits_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$

we have to prove that $\left|\frac{F(t)-F(s)}{t-s} - f(x)\right| \longrightarrow 0$

This is a good trick that be use a lot in this chapter when we use $\boxed{C = c\int_a^b \frac{1}{b-a}\,dx}$ □ .

⇒ Want to use $\lim \int g(x) = f(x)$. ⇒ NTP $\int_a^b g(x) - f(x)\frac{1}{b-a}\int_a^b 1\,dx \Rightarrow 0$

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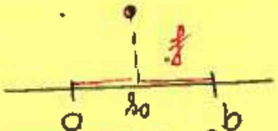
If problem requires us to prove $\exists \xi$ s.t

x: Jan 2011 P4 $f:\mathbb{R}\to\mathbb{R}$ continuous. Show that $\exists \xi$, $\cancel{f}\int_0^1 f(x)\,x^2\,dx = \frac{1}{3} f(\xi)$

hink about using Intermediate value for integral.

Results from Rudin

$$\left.\begin{array}{l} \underline{6.1}: \alpha \text{ increasing on } [a,b], \text{ continuous at } x_0,\ a \le x_0 \le b \\ f(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0 \end{cases} \end{array}\right\} \text{Then } f \in \mathscr{R} \text{ and } \int f(x)dx = 0$$



$$\left.\begin{array}{l} \underline{6.2}.\ f \geq 0,\ f \text{ is } \boxed{\text{continuous}} \text{ on } [a,b] \\ \int_a^b f(x)dx = 0 \end{array}\right\} \text{Then } f(x) = 0,\ \text{for all } x \in [a,b].$$

Results from Prelim.

* Note that when we create a partition, we don't create a partition with specific point, Result
we have to create a partition with $x_0-\epsilon$, $x_0$, $x_0+\epsilon$ to refine the partition (Aug1997, 5
or depend on $n, \delta$ (

+ Theorem 16:

~~Want to compute $\int f d\alpha$~~

$\alpha$ monotony on [a,b]

Have $\left\{\begin{array}{l} f_n \in R(\alpha) \\ f_n \rightrightarrows f \end{array}\right\}$ NTP $f \in R(\alpha)$ and $\int f d\alpha = \lim \int f_n d\alpha$

We want to prove that $\int f_n d\alpha - \epsilon < \underline{\int} f d\alpha < \overline{\int} f d\alpha < \int f_n d\alpha + \epsilon$ for $n \geq n_0$.

* In the problem requiring computing $\int_a^b f d\alpha - M$, we use $M = \frac{1}{b-a}\int_a^b 1 \, d\alpha$. $\Rightarrow \int_a^b f d\alpha - M = \int_a^b \left(f - \frac{1}{b-a}\right) d\alpha$

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* Jan2013/3. $f$ is continuous.

NTP $\lim_{N\to\infty} \frac{\alpha+1}{N^{\alpha+1}} \int_0^N x^\alpha f(x) dx = C$

Way 1: $\int_0^N x^\alpha dx = \frac{1}{\alpha+1}\int_0^N (\alpha+1) x^\alpha dx = \frac{1}{\alpha+1}\int_0^N d(x^{\alpha+1}) = \frac{1}{\alpha+1} N$

$\Rightarrow \frac{\alpha+1}{N^{\alpha+1}} \int_0^N x^\alpha = 1$

Way 2: Notice that $F(N) = \int_0^N x^\alpha f(x) dx$ is diff. $F'(x) = x^\alpha f(x)$

Jan2006/P4

* If $\sum \frac{a_k}{k+1} = 0$ Prove that the polynomial $\sum_{k=0}^{n} a_k x^k$ has at least one root in the interval $(0, 1)$.

(even we have $\frac{a_k}{k+1} = \int_0^1 a_k x^k$, we need to put ~~$F(x) = \int_0^x a_k t^k dt$~~ $F(x) = \frac{a_k}{k+1} x^{k+1}$

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* Some results relating to $F(x)$.

(Jan 2009, P3).

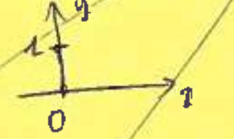
- $F(x)$ is continuous
- If $\left\{\begin{array}{l} \boxed{\int_0^a f(x) dx = 0} \\ f(x) \text{ is periodic with frequency } (a) \end{array}\right\} \Rightarrow$ then $F(x)$ is periodic with the same freque
- $F(x)$ is continuous + periodic $\Rightarrow$ attain min + maximum in $\mathbb{R}$ (Jan 2009, P3).

Some important results (need) to remember:

(Rudin p138) $\alpha$: increases on $[a,b]$, $\alpha$ continuous at $x_0$

$a \leq x_0 \leq b$

$f(x_0) = 1$, $f(x) = 0$ if $x \neq x_0$



Prove that $f \in R(\alpha)$ and $\int f d\alpha = 0$

Suppose $f \geq 0$
$f$ continuous on $[a,b]$
$\int_a^b f(x) dx = 0$
$\Bigg\}$ $\Rightarrow$ Prove that $f(x) = 0$, $\forall x \in [a,b]$ 2/ Rudin )

f $f, g : [a,b] \longrightarrow \mathbb{R}$ are both Riemann integrable

en $\phi : [a,b] \longrightarrow \mathbb{R}$

$x \longmapsto \phi(x) = \max\{f(x), g(x)\}$ is also Riemann integrable.

# Chapter 6 V/s chapter 7.

* If we have $\left.\begin{array}{l} f_n \in \mathscr{R}(\alpha) \\ f_n \rightrightarrows f \end{array}\right|$ We want to prove that $f \in \mathscr{R}(\alpha)$. (see theorem 7.16).

We can prove $f \in \mathscr{R}(\alpha)$ by proving :

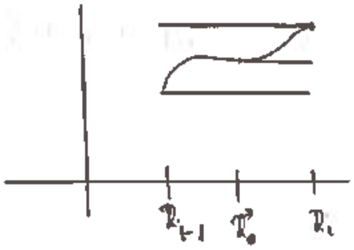
$$\int (f_n - \epsilon) d\alpha \leq \underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f(d\alpha) \leq \int f_n + \epsilon \, d\alpha .$$

6.4 Theorem:

Let $P^*$ is a refinement of $P$, then we have $L(P,f,\alpha) \underset{(1)}{\leq} L(P^*,f,\alpha) \underset{(2)}{\leq} U(P^*,f,\alpha) \underset{③}{\leq} U(P,f,$

Obviously, we have (2), now we will prove (1), the case(3) is similar with case(1)

We want to prove that $L(P^*,f,\alpha) - L(P,f,\alpha)$ for $P^*$ is a refinement of $P$



Assume $P^*$ is a refinement of $P$ by adding a point $x^*$ in between $(x_{i-1}, x_i)$

We have $m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \leq \inf_{x\in[x_{i-1},x^*]} f(x) =: w_1$

and $m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \leq \inf_{x\in[x^*,x_i]} f(x) = w_2$

Then $L(P^*,f,\alpha) - L(P,f,\alpha) = w_1(\alpha(x^*)-\alpha(x_{i-1})) + w_2(\alpha(x_i)-\alpha(x_i^*)) - m_i(\alpha(x_i)-\alpha(x$

$$= \underbrace{(w_1 - m_i)}_{\geq 0 \quad \geq 0} \underbrace{(\alpha(x^*)-\alpha(x_{i-1}))}_{\geq 0} + \underbrace{(w_2-m_i)}_{\geq 0}\underbrace{(\alpha(x_i)-\alpha(x^*))}_{\geq 0}$$

$\alpha$ increasing

*6.5 Theorem

$$\boxed{\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha.}$$

We have $\underline{\int_a^b} f\,d\alpha = \sup_{P_1} L(P_1,f,\alpha)$, ~~assume~~ $\underline{\int_a^b} f\,d\alpha = L(P_1,f,\alpha)$

$\overline{\int_a^b} f\,d\alpha = \inf_{P_2} U(P_2,f,\alpha)$, assume $\overline{\int_a^b} f\,d\alpha = U(P_2,f,\alpha)$

Let $P^* = P_1 \cup P_2$, then we have

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha)$$

So we have $\sup_{P_1} L(P_1,f,\alpha) \leq \inf_{P_2} U(P_2,f,\alpha) \Rightarrow \square$.

'rove theorem 6.6

rove that if $f$ is bounded

hen $f \in \mathcal{R}(\alpha) \Longleftrightarrow \forall \varepsilon > 0, \exists$ partition $P$, $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

=>): Let $f$ is bounded, $f \in \mathcal{R}(\alpha)$ $\}$ Prove that $\forall \varepsilon > 0, \exists$ partition $P$, $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$U(P,f,\alpha) < L(P,f,\alpha) + \varepsilon.$

e have $f \in \mathcal{R}(\alpha) \Leftrightarrow$

$$\int f\,d\alpha = \underset{\substack{\parallel \\ \sup L(P_1,f,\alpha)}}{\underline{\int} f\,d\alpha} = \underset{\substack{\parallel \\ \inf U(P_2,f,\alpha)}}{\overline{\int} f\,d\alpha}$$

Then $\exists P_1$, $\int f\,d\alpha - \varepsilon \leq L(P_1,f,\alpha)$ (1)

$\exists P_2$, $\int f\,d\alpha + \varepsilon \geq U(P_2,f,\alpha)$. (2)

.et $P = P_1 \cup P_2$, we have

$$U(P,f,\alpha) \leq U(P_2,f,\alpha) \underset{(2)}{\leq} \int f\,d\alpha + \varepsilon \underset{(1)}{\leq} \int L(P_1,f,\alpha) + \varepsilon + \varepsilon \leq L(P,f,\alpha) + 2\varepsilon$$

$\Rightarrow \square (\Rightarrow)$.

=): We have

$$L(P,f,\alpha) \leq \underline{\int} f\,d\alpha \leq \overline{\int} f\,d\alpha \leq U(P,f,\alpha)$$

Because $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$\} \Rightarrow \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha \leq \varepsilon, \forall \varepsilon$

then $\overline{\int} f\,d\alpha = \underline{\int} f\,d\alpha \Rightarrow f \in \mathcal{R}(\alpha)$

$\square \Leftarrow$

## 6.8 Theorem (Sample A)E6).

Let $f$ continuous on $[a,b]$. Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$.
$\alpha$ monotonically increasing.

$f$ is continuous on $[a,b]$, then $\forall x \in [a,b]$

$\Leftrightarrow \forall \xi > 0, \exists \delta > 0, \forall y \in [a,b], |y-x| < \delta$, then $|f(y) - f(x)| < \xi$ (1)

NTP $f \in \mathcal{R}(\alpha)$ on $[a,b]$

$\Rightarrow$ NTP, $\forall \varepsilon > 0, \exists$ partition $P$

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$\Leftrightarrow$ NTP, $\forall \varepsilon > 0, \exists$ partition $P$

$\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i < \varepsilon$

* For all $\varepsilon > 0$, choose $\xi$ such that

$$[\alpha(b) - \alpha(a)]\,\xi < \varepsilon$$

Then because of (1), choose partition $P$ such that $\Delta x_i < \delta$, then we have

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} \underbrace{(M_i - m_i)}\Delta\alpha_i \leq \xi \sum_{i=1}^{n} \Delta\alpha_i = \xi[\alpha(b) - \alpha(a)] < \varepsilon.$$

Then we have $f \in \mathcal{R}(\alpha)$ on $[a,b]$ $\square$.

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## 6.9 Theorem:

$f$ is monotonically on $[a,b]$.
$\alpha$ is monotonically increasing + $\boxed{\text{continuous}}$ on $[a,b]$
} Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

* $\alpha$ is continuous on $[a,b]$ $\Leftrightarrow \forall x \in [a,b]$

$\forall \xi > 0, \exists \delta > 0, \forall y \in [a,b], |y-x| < \delta$, then $|\alpha(y) - \alpha(x)| < \xi$ (1)

NTP $f \in \mathcal{R}(\alpha)$ on $[a,b]$

NTP $\exists$ partition $P$,

$|U(P,f,\alpha) - L(P,f,\alpha)| < \varepsilon$

NTP $\left|\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i\right| < \varepsilon$

* We choose $\xi$ such that $|f(b) - f(a)|\,\xi < \varepsilon$.

Because $\alpha$ is continuous, then because of (1), $\exists \delta > 0, |y-x| < \delta, |\alpha(y) - \alpha(x)| < \xi$

Then choose a partition $P$ such that $\Delta x_i < \delta$ (This means $\frac{b-a}{n} < \delta$)

Then

$$|U(P,f,\alpha) - L(P,f,\alpha)| = \left|\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i\right| \leq \sum_{i=1}^{n} |M_i - m_i|\underbrace{|\Delta\alpha_i|}_{<\xi} \leq \xi \sum_{i=1}^{n} |M_i - m$$

$$= \xi\,|f(b) - f(a)| < \varepsilon$$

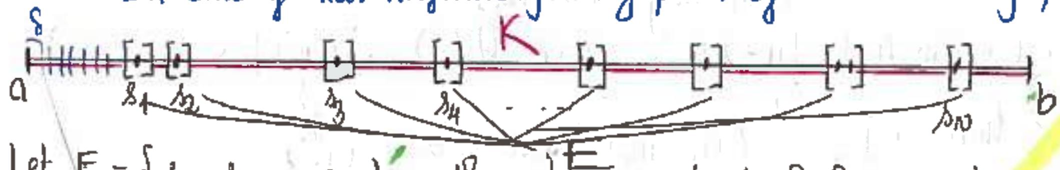
↑
notice that
$f$ is monotonic.

Proe Theorem 6.10 See Jan 2010, P4

Suppose $f$ is bounded on $[a,b]$
$f$ has finitely many points of discontinuity on $[a,b]$.
$\alpha$ is continuous at every point at which $f$ is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

Note: If $f$ and $\alpha$ have a common point of discontinuity, then $f$ need not be in $\mathcal{R}(\alpha)$. See exercise 3/ (See Jan 2010, P4).

- In case $f$ has infinitely many point of discontinuity, $f \notin \mathcal{R}$ (see ex 4)



- Let $E = \{s_1, s_2, \dots, s_{10}\}$ = the set of point at which $f$ is discontinuous (Note that E f

- $\alpha$ is continuous at those $s_i$, $i = \overline{1,N}$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_{\varepsilon,i} > 0, \forall s \in [a,b], |s - s_i| < \delta$, then $|\alpha(s) - \alpha(s_i)| < \frac{\varepsilon}{2N}$

Note that E is finite, Card E = N

$\Rightarrow$ We cover E by finitely disjoint interval $[u_i, v_i]$ such that $\sum_{i=1}^{N} [\alpha(v_i) - \alpha(u_i)]$

* Then let $K = [a,b] \setminus \underbrace{\bigcup_{i=1}^{n} \underbrace{(u_i, v_i)}_{\text{open}}}_{\text{finite union of open} \Rightarrow \text{open}}$ K closed + bounded in $\mathbb{R}$ $\Rightarrow$ compact

closed

$f$ continuous in K compact $\Rightarrow$ uniformly continuous in K.

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_\varepsilon, \forall (x, y \in [a,b], |x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$ (2

* Now we create partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ as follow:
Each $u_i, v_i \in P$, no point of segment $[u_i, v_i]$ in P
If $x_{i-1}$ is not one of $u_j$, then $\Delta x_i < \delta_\varepsilon$

Then we have

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{x_{i-1} \in K} \underbrace{(M_i - m_i)}_{< \varepsilon \text{ (because of (2))}} \Delta\alpha(x_i) + \sum_{u_i = 1}^{N} \underbrace{(M_i - m_i)}_{< 2M} [\Delta\alpha(v_i) - \alpha(u_i)]$$

$$= < \varepsilon \sum_{x_{i-1} \subset K} \Delta\alpha(x + 2M \underbrace{\sum_{i=1}^{n} (\alpha(v_i) - \alpha(u_i)]}_{< \varepsilon \text{ because of } (1)}$$

$$\leq \varepsilon [\alpha(b) - \alpha(a)] + 2M \cdot \varepsilon$$

$\rightarrow f \in \mathcal{R}(\alpha)$.

// Theorem

$\in \mathscr{R}(\alpha)$ on $[a, b]$ , $m \le f(x) \le M$
continuous on $[m, M]$
$(x) = \phi(f(x))$
} Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$

We have $f \in \mathscr{R}(\alpha)$ on $[a, b]$

$\Rightarrow \forall \varepsilon > 0$, $\exists$ partition $P = \{x_0 = a, \dots, x_i, \dots, x_n = b\}$, $\sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i < \varepsilon \quad (1)$

We have $\phi$ is continuous on $[m, M]$ $\Rightarrow$ uniformly continuous.

$\forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in [m, M]$, $|u - v| < \delta$ then $|\phi(u) - \phi(v)| < \varepsilon \quad (2)$

In (1), we choose $\boxed{\varepsilon = \delta^2}$, then $\exists P$, $\sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i < \overset{f(x)}{\delta^2} \quad \overset{f(y)}{(I)}$

Let $M_i^*$, $m_i^*$ are analogous point of $M_i$ and $m_i$ for $h$.

We divide $i$ into 2 groups, $\begin{cases} i \in A, \text{ if } |M_i - m_i| < \delta \\ i \in B \text{ if } (M_i - m_i) \ge \delta \end{cases}$ (in (2)) $\Rightarrow M_i^* - m_i^* < \varepsilon$ . so we have

$$\delta \sum_{i=B}^{n} \Delta_i \le \sum \underbrace{(M_i - m_i)}_{\delta <} \Delta\alpha_i < \delta^2 \quad (\text{by } I)$$

$$\Rightarrow \underset{i \in B}{\Delta\alpha_i} < \delta \qquad (*)$$

$$\Rightarrow \sum_{i \in B} \underbrace{(M_i^* - m_i^*)}_{\le 2K} \Delta\alpha_i \text{ , where } K = \sup|\phi(u)| \; u \in [m, M].$$

$\Bigg\}$ $\sum_{i=B} (M_i^* - m_i^*) \Delta\alpha_i < 2K\delta$

So Sum up, we have

$$\sum_{i=1}^{n} (M_i^* - m_i^*) \Delta\alpha_i = \sum_{i \in A} \underbrace{(M_i^* - m_i^*)}_{< \varepsilon} \Delta\alpha_i + \underbrace{\sum_{i \in B} (M_i^* - m_i^*) \Delta\alpha_i}_{\boxed{\le 2K\delta}}$$

$$\le \varepsilon \underbrace{\sum_{i \in A} \Delta\alpha_i}_{\substack{\le \sum_{i=1}^{n} \Delta\alpha_i \\ = \alpha(b) - \alpha(b)}}$$

$$= \varepsilon [\alpha(b) - \alpha(a)] + 2K\delta .$$

$$< [2K + \alpha(b) - \alpha(a)] \varepsilon \Rightarrow h \in \mathscr{R}(\alpha) \text{ on } [a, b]$$

6.12) Theorem: Properties of integral

a) $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$ $\}$ Then $\begin{cases} (f+g) \in \mathcal{R}(\alpha) \\ \int (f+g)\,d\alpha = \int f\,d\alpha + \int g\,d\alpha \end{cases}$

\+ Let $f, g \in \mathcal{R}(\alpha)$. Prove that $(f+g) \in \mathcal{R}(\alpha)$

- $f \in \mathcal{R}(\alpha)$

$\Leftrightarrow \forall \varepsilon > 0, \exists P_1, U(P_1, f, \alpha) - L(P_2, f, \alpha) < \varepsilon$

- $g \in \mathcal{R}(\alpha)$

$\Leftrightarrow \forall \varepsilon > 0, \exists P_2, U(P_2, g, \alpha) - L(P_2, g, \alpha) < \varepsilon$

NTP $(f+g) \in \mathcal{R}(\alpha)$

$\Leftrightarrow$ NTP, $\forall \varepsilon > 0, \exists P^*$,
$U(P^*, f+g, \alpha) - L(P^*, f+g, \alpha) < \varepsilon$

\+ Let $P = P_1 \cup P_2$, then we have

- $U(P^*, f+g, \alpha) \underset{\uparrow}{\leq} U(P^*, f, \alpha) + U(P^*, g, \alpha) \leq U(P_1, f, \alpha) + U(P_2, g, \alpha)$. (1

because $\sup\limits_{x \in [x_{i-1}, x_i]} (f(x) + g(x)) \leq \sup\limits_{x \in [x_{i-1}, x_i]} (f(x)) + \sup\limits_{x \in [x_{i-1}, x_i]} (g(x))$

- $L(P^*, f+g, \alpha) \underset{\uparrow}{\geq} L(P^*, f, \alpha) + L(P^*, g, \alpha) \geq L(P_1, f, \alpha) + L(P_2, g, \alpha)$ (2)

because $\inf\limits_{x} (f+g) \geq \inf\limits_{x} f + \inf\limits_{x} g$

(1)+(2) $\Rightarrow$
$U(P^*, f+g, \alpha) - L(P^*, f+g, \alpha) \leq U(P_1, f, \alpha) - L(P_1, f, \alpha) + U(P_2, g, \alpha) - L(P_2, g\, \alpha$

* We want to prove $\int (f+g)\,d\alpha = \int f\,d\alpha + \int g\,d\alpha$ $\Rightarrow$

$\Leftrightarrow$ We want to prove that $\forall \varepsilon > 0$, $\int f\,d\alpha + \int g\,d\alpha - \varepsilon \leq \int (f+g)\,d\alpha \leq \int f\,d\alpha + \int g\,d\alpha + \varepsilon$

- We have $f \in \mathcal{R}(\alpha) \Rightarrow \exists P_1, \int f\,d\alpha + \varepsilon > U(P_1, f, \alpha)$
  $g \in \mathcal{R}(\alpha) \Rightarrow \exists P_2, \int g\,d\alpha + \varepsilon > U(P_2, g, \alpha)$

Then let $P^* = P_1 \cup P_2$,

we have $\int (f+g)\,d\alpha = \inf U(P, f+g, \alpha) \leq U(P^*, f+g, \alpha) \leq U(P_1, f, \alpha) + U(P_2, g, \alpha) \leq \int f\,d\alpha + \int g\,d\alpha$. (1

- Similarly, we have

$\int (f+g)\,d\alpha = \sup L(P, f+g, \alpha) \geq L(P^{*\prime}, f+g, \alpha) \geq L(P_1', f, \alpha) + L(P_2', g, \alpha) \geq \int f\,d\alpha + \int gc$ (

(1)+(2) $\Rightarrow$ $\int f\,d\alpha + \int g\,d\alpha - 2\varepsilon \leq \int (f+g)\,d\alpha \leq \int f\,d\alpha + \int g\,d\alpha + 2\varepsilon$

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Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ <br>
c is a constant $\Big\}$ Prove that $\begin{cases} (cf) \in \mathcal{R}(\alpha) \\ \int cf\, d\alpha = c \int f\, d\alpha \end{cases}$

e-that when $c \geq 0$, $c \sup\{f(x)\} = \sup\{c f(x)\}$. $c \inf\{f(x)\} = \inf\{c f(x)\}$.

$c < 0$, $c \sup\{f(x)\} = \inf\{c f(x)\}$ $c \inf\{f(x)\} = \sup\{c f(x)\}$.

⇒ we need to divide the problem into two cases, when $c \geq 0$ and when $c < 0$.

We have $f \in \mathcal{R}(\alpha) \Leftrightarrow \forall \varepsilon > 0, \exists$ a partition $P$ such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \Leftrightarrow \left| \sum_{i=1}^{n} \Big( \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \Big) \Delta\alpha_i \right| < \varepsilon. \quad (*)$$

Put $I_i = [x_{i-1}, x_i]$

When $c > 0$, we have

$$) \Rightarrow \sum_{i=1}^{n} \Big[ \sup_{x \in I_i}(c f(x)) - \inf_{x \in I_i}(c f(x)) \Big] \Delta\alpha_i < c\varepsilon \Rightarrow (cf) \in \mathcal{R}(\alpha).$$

$$\overline{\int} cf\, d\alpha = \inf_{P} U(P, cf, \alpha) = \inf_{P} \sum_{i=1}^{n} \sup_{x \in I_i}(c f(x)) \underbrace{\Delta\alpha_i}_{> 0} = c \inf_{P} \sum_{i=1}^{n} \sup_{x \in I_i} f(x)\, \Delta\alpha_i = c \overline{\int} f\, d\alpha.$$

$$\underline{\int} c f\, d\alpha = \quad = c \underline{\int} f\, d\alpha$$

Because $f \in \mathcal{R}(\alpha) \Rightarrow c \overline{\int} f\, d\alpha = c \underline{\int} f\, d\alpha = c \int f\, d\alpha \Rightarrow \overline{\int} cf\, d\alpha = \underline{\int} cf\, d\alpha = c \int f\, d\alpha$

$$\Rightarrow \int cf\, d\alpha = c \int f\, d\alpha.$$

When $c < 0$

$$) \Rightarrow \sum_{i=1}^{n} \Big( c \sup_{x \in I_i} f(x) - c \inf_{x \in I_i} f(x) \Big) \Delta\alpha_i > c\varepsilon.$$

$$\Leftrightarrow \sum_{i=1}^{n} \Big( \inf_{x \in I_i}(c f(x)) - \sup_{x \in I_i}(c f(x)) \Big) \Delta\alpha_i > c\,\varepsilon$$

$$\sum_{i=1}^{n} \Big( \sup_{x \in I_i}(c f(x)) - \inf_{x \in I_i}(c f(x)) \Big) \Delta\alpha_i < c\varepsilon \Rightarrow (cf) \in \mathcal{R}(\alpha).$$

$$\overline{\int} cf\, d\alpha = \inf_{P} U(P, cf, \alpha) = \inf_{P} \sum_{i=1}^{n} \sup_{x \in I_i}(c f(x)) \Delta\alpha_i = \inf_{P} c \sum_{i=1}^{n} \inf(f(x)) \Delta\alpha_i =$$

$$= c \sup_{P} \sum_{i=1}^{n} \inf(f(x)) \Delta\alpha_i = c \cdot \sup L(P, f, \alpha) = c \underline{\int} f\, d\alpha = c \int f\, d\alpha$$

Similarly $\underline{\int} cf\, d\alpha = c \overline{\int} f\, d\alpha = c \int f\, d\alpha$

Hence $\int cf\, d\alpha = c \int f\, d\alpha$.

6.12c :

$$\text{If } f \in \mathcal{R}(\alpha) \text{ on } [a,b],\ a<c<b \Rightarrow \begin{cases} f \in \mathcal{R}(\alpha) \text{ on } [a,c] \text{ and } [c,b] \\ \int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha . \end{cases}$$

* We have $f \in \mathcal{R}(\alpha)$ on $[a,b] \iff \exists$ Partition $P = \{x_0 = a \le x_1 \le \cdots \le x_n = b\}$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ .

Then let $P^* = P \cup \{c\}$ .

$\Rightarrow U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$

Then consider $P_a$ = take from partition $P^*$ ~~until~~ on $[a,c] = P^* \cap [a,c]$ .

$P_b = P^* \cap [c,b]$ .

Then we have $$\underbrace{\left(U(P_a,f,\alpha) - L(P_a,f,\alpha)\right)}_{\ge 0} + \underbrace{\left(U(P_b,f,\alpha) - L(P_b,f,\alpha)\right)}_{\ge 0} = U(P^*,f,\alpha) - L(P^*,f$$

$$\Rightarrow \begin{cases} U(P_a,f,\alpha) - L(P_a,f,\alpha) \le \epsilon \\ U(P_b,f,\alpha) - L(P_b,f,\alpha) \le \epsilon \end{cases} \iff \begin{cases} f \in \mathcal{R}(\alpha) \text{ on } [a,c] \\ f \in \mathcal{R}(\alpha) \text{ on } [c,b] . \end{cases}$$

* ~~But~~ Now we prove $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

not checked

Let $$f_1 = \begin{cases} f(x) & , x \in [a,c) \\ 0 & , x \in [c,b] \end{cases}$$

$$f_2 = \begin{cases} 0 & , x \in [a,c] \\ f(x) & , x \in (c,b] \end{cases}$$

Then we have

$f(x) = f_1(x) + f_2(x)$

$$\int_a^b f(x)\,d\alpha = \int_a^c f_1(x)\,d\alpha + \int_c^b f_2(x)\,d\alpha = \int_a^c f(x)\,d\alpha + \int_c^b f(x)\,d\alpha .$$

* Way 2 :

Bec~~ause f∈R(α)~~ $\Rightarrow \overline{\int_a^b} f\,d\alpha \le U(P^*,f,\alpha) = U(P_a,f,\alpha) + U(P_b,f,\alpha)$

$\le \overline{\int_a^c} f\,d\alpha + \epsilon + \overline{\int_c^b} f\,d\alpha + \epsilon = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ (1

$\underline{\int_a^b} f\,d\alpha \ge L(P^*,f,\alpha) = L(P_a,f,\alpha) + L(P_b,f,\alpha)$

$\ge \underline{\int_a^c} f\,d\alpha - \epsilon + \underline{\int_c^b} f\,d\alpha - \epsilon = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha - 2\epsilon$ (2)

$(1)+(2) \Longrightarrow \int_a^b f\,d\alpha = \int_a^c + \int_c^b \quad \to 0$

* 6.13 Theorem

a) If $\left.\begin{array}{l} f \in \mathcal{R}(\alpha) \\ g \in \mathcal{R}(\alpha) \end{array}\right\}$ Then $\left\{ fg \in \mathcal{R}(\alpha) \right.$

b) If $f \in \mathcal{R}(\alpha)$, then $\left\{\begin{array}{l} |f| \in \mathcal{R}(\alpha) \\ \left|\int f d\alpha\right| \leq \int |f| d\alpha \end{array}\right.$

a) We have $\left.\begin{array}{l} f \in \mathcal{R}(\alpha) \\ g \in \mathcal{R}(\alpha) \end{array}\right\} \Rightarrow f \pm g \in \mathcal{R}(\alpha)$ $\left.\right\} \Rightarrow (f \pm g)^2 \in \mathcal{R}(\alpha)$

because $x^2$ is a continuous function

we have $fg = \dfrac{(f+g)^2 - (f-g)^2}{4} \Rightarrow fg \in \mathcal{R}(\alpha)$ □a).

b) If $f \in \mathcal{R}(\alpha)$. Prove that $|f| \in \mathcal{R}(\alpha)$

- We have $\left.\begin{array}{l} f \in \mathcal{R}(\alpha) \\ |x| \text{ is a continuous function} \end{array}\right\} \Longrightarrow$ by theorem 6.11 $|f| \in \mathcal{R}(\alpha)$

- ~~Choose $c = \pm 1$ so that $c\int f d\alpha \geq 0$~~

$|\int f d\alpha| = c\int f d\alpha = \int c f d\alpha \leq \int |f| d\alpha$ ~~since $cf \leq |f|$~~

---

* Prove the mean value theorem for integral (Intermediate Value Theorem for integrals

Let $f : [a,b] \longrightarrow \mathbb{R}$, $f$ is continuous on $[a,b]$. $\alpha$: monotonically $\overset{\text{incre}}{}$ on $[a,b]$ $\Bigg\}$ Then (theorem 6.8) $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\exists c \in [a,b]$ st $\displaystyle\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha$ (*)

---

* Because $f$ continuous on $[a,b]$ compact in $\mathbb{R}$ $\Rightarrow$ $f([a,b])$ attains min, max value in $\mathbb{R}$.

Let $m = \min\limits_{x \in [a,b]} f(x)$ $\quad M = \max\limits_{x \in [a,b]} f(x)$

Then we have $m \leq f(x) \leq M$, $\forall x \in [a,b]$ $\xRightarrow[\text{Theorem 6.12d}]{}$ $m[\alpha(b)-\alpha(a)] \leq \displaystyle\int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$

- In case $\alpha(b) = \alpha(a)$ we have $\alpha$ monotonically increasing $\Big\} \Rightarrow \alpha$ is constant in $[a,b]$ $\Rightarrow \displaystyle\int_a^b f d\alpha = 0$

and $\alpha(b) - \alpha(a) = 0$

Then (*) satisfy for all $c \in [a,b]$

- In case $\alpha(b) > \alpha(a)$

then we have $m \leq \dfrac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)} \leq M$

The idea of this proof is we want to prove that $\exists \xi$ st $f(c) = \dfrac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)}$ So we want $m \leq \dfrac{\int_a^b f d\alpha}{\alpha(b)-\alpha(a)} \leq$

Then by Intermediate Value theorem, $\exists c \in [a,b]$, $f(c) = \dfrac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)}$

$\Leftrightarrow$ ~~$f(c)$~~ $\displaystyle\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha(a)]$

Theorem 6.15, 6.16 Computing integral when $\alpha$ is a step function

<u>6.15</u>: $a < s < b$

$$\left.\begin{array}{l} \alpha(x) = I(x-s) = \begin{cases} 0, & x \leq s \\ 1, & x > s \end{cases} \\ f \text{ continuous at } s \end{array}\right\} \text{Then } \int_a^b f\, d\alpha = f(s)$$

<u>6.16</u>

$$\left.\begin{array}{l} \{s_n\} \text{ sequence of distinct points in } (a,b) \\ c_n \geq 0, \forall n \text{ , } \sum c_n \text{ converges} \\ \alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n) \\ f \text{ continuous on } [a,b] \end{array}\right\} \int f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

6.17 Theorem Fall 1993

(If $\alpha$ has an integrable derivative, the integral reduces to an ordinary Riemann integra

$$\left.\begin{array}{l} \alpha : \text{monotonically increasing} \\ \alpha' \in \mathscr{R} \text{ on } [a,b] \\ f \text{ is bounded real function on } [a,b] \end{array}\right\} \text{Then } \left\{\begin{array}{l} f \in \mathscr{R}(\alpha) \iff (f\alpha') \in \mathscr{R} \text{ on } [a,b] \\ \int f\, d\alpha = \int f(x)\, \alpha'(x)\, dx\,. \end{array}\right.$$

$(\Longrightarrow)$:

6.20 Theorem (Important) ~~Aug 2003, Jan 2002~~

Let $f \in \mathcal{R}$ on $[a,b]$
For $a \leq x \leq b$, put $F(x) = \int_a^x f(t)\,dt$
Then $F$ is continuous on $[a,b]$.

b) Furthermore, if $f$ is continuous at a point $x_0 \in [a,b]$
Then $F$ is differentiable at $x_0$ and
$F'(x_0) = f(x_0)$.

* Prove that $F$ is continuous on $[a,b]$, NTP for each $x \in [a,b]$.

~~NTP~~ $\forall \varepsilon > 0, \exists \delta > 0$, st. $\forall y \in [a,b]$, $|y-x| < \delta$, then $|F(y) - F(x)| < \varepsilon$

Now consider $|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right| \leq M|y-x|$

(because $f \in \mathcal{R}$ in $[a,b]$ $\Rightarrow$ $f$ is bounded in $[a,b]$ $\Rightarrow |f(t)| \leq M, \forall t \in [a,b]$.

then $\forall \varepsilon > 0$, chose $\delta$ such that $M\delta \leq \varepsilon$,
then $\forall |y-x| < \delta$, we have $M|y-x| \leq \varepsilon$, which means $|F(y) - F(x)| < \varepsilon$
$\rightarrow F$ continuous.

* Prove that if $f$ is continuous at a point $x_0 \in [a,b]$. Then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$ (note that we can't prove directly that $\lim \frac{F(x+h) - F(x)}{h} = f(x)$ have to prove

- $f$ is continuous at a point $x_0 \in [a,b]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall y \in [a,b]$ st $|y - x| < \delta$
then $|f(y) - f(x_0)| < \varepsilon$.

We want to prove that $\lim F(h) -$ $= 0$
$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| < \varepsilon$$

Let $t, s \in [a,b]$ st $t, s \in (x_0 - \delta, x_0 + \delta)$ and $x_0 - \delta < s \leq x_0 \leq t \leq x_0 + \delta$

This is just because wlog, we want $t > s$ so that we can have $\int_a^t - \int_a^s = \int_s^t$

and take $t$, $x$ in this.

Then we have

$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{(t-s)} \int_a^t f(x)\,dx - \int_a^s f(x)\,dx \overset{(*)}{-} \frac{1}{(t-s)} \int_s^t f(x_0)\,dx \right|$$

because $f(x_0) = ca$
$$f(x_0) = \frac{1}{(t-s)} \int_s^t f(x_0)$$

$$= \left| \frac{1}{(t-s)} \int_s^t f(x)\,dx - \frac{1}{(t-s)} \int_s^t f(x_0)\,dx \right|$$

$$\leq \frac{1}{(t-s)} \int_s^t \underbrace{|f(x) - f(x_0)|}_{<\varepsilon \text{ because } x \in (s,t) \subset (x_0-\delta, x_0+\delta) \text{ and } f \text{ is continuous}} dx$$

$$= \frac{1}{(t-s)} \varepsilon (t-s).$$

$= \varepsilon$. ■

* Or we can use $\left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right|$ in the same way.

Idea of the proof

Instead of proving $\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon$

we use $s, t \in (x_0 - \delta, x_0 + \delta)$
and $s \leq x_0 \leq t$.

This is a good trick to remember!

$M = \frac{1}{b-a} \int_a^b M\,dx$.

1 Fundamental of Calculus.

1 $f \in \mathscr{R}$ on $[a,b]$.
$\exists$ a differentiable function on $[a,b]$, $F' = f$ $\Bigg\}$ $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

Note that in here we don't have $F$ is monotonically increasing
$\Rightarrow$ we can't apply theorem 6.17 with $F = \alpha$ $\Rightarrow$ just use def.

$\in \mathscr{R}$ on $[a,b]$
$\forall \varepsilon > 0$, $\exists$ a partition $P = \{x_0 = a \le x_1 \le \cdots \le x_n = b\}$ such that $U(P,f) - L(P,f) < \varepsilon$ (*)

n consider

$$F(x_i) - F(x_{i-1}) = F'(t_i)[x_i - x_{i-1}] \quad \text{for some } t_i \in [x_{i-1}, x_i]$$
$$= f(t_i)[x_i - x_{i-1}].$$

Then $F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}]$. (1)

We also have ~~ball~~ partition $P$ satisfies (*), $\left|\int_a^b f\,dx - \sum f(t_i)[x_i - x_{i-1}]\right| < \varepsilon$ (2)

$+(2) \Rightarrow \left|\int_a^b f(x)\,dx - (F(b) - F(a))\right| < \varepsilon$

$\Rightarrow \int_a^b f\,dx = F(b) - F(a)$ □.

# Chapter 7. Sequences and series of function

Focus on real functions

* Discussion of main problem (why do we need "uniformly convergence"?
(what's the limit of 'pointwise convergence'?")

* Definition of pointwise convergence

- For sequence:

Let $f_n : E \longrightarrow \mathbb{R}$.

Suppose $(f_n)$ converges "pointwise" for every $x \in E$

We can define $f(x)$ by $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in E$

We say
- $\{f_n\}$ converges on E
- $f$: the "pointwise" limit of $\{f_n\}$

or $\{f_n\}$ converges 'pointwise'

* For series

Similarly, if $\sum_{n=1}^{\infty} f_n(x)$ converges (pointwise) for every $x \in E$

we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for each $x \in E$.

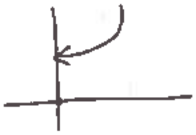
The function $f$ is called the sum of the series $\sum f_n$

* With pointwise convergence, we have some problems:

- A convergent series of continuous functions may have a discontinuous sum.

$f_n(x) = \dfrac{x^2}{(1+x^2)^n}$ ($x$ real, $n = 0, 1, 2 \dots$) continuous

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 0 & x = 0 \\ 1+x^2 & x \neq 0 \end{cases}$$



not continuous at 0.

* Important: we don't need $f_n(x)$ to be continuous to ensure that $f_n(x) \rightrightarrows$.

because ex $f_n(x) = f(x) + \frac{1}{n} \Longrightarrow f(x)$

but $f(x)$ does not need to be continuous.

## Uniform convergence

Note that with uniformly convergence,
~~we need to say uniformly convergent~~ in where

$f_n$ converges pointwise (at $x$) to $f$ $\qquad$ $f_n \rightrightarrows f$ then $f_n \to f$.

$\forall \varepsilon > 0, \exists N_{\varepsilon,x} \in \mathbb{N}, \forall n \geq N_{\varepsilon,x}, |f_n(x) - f(x)| < \varepsilon$

If $E$ is infinite (for ex, $E = [a,b]$, there are infinite many $N_{\varepsilon,x}$, we can't find max $N_{\varepsilon,x}$).

$f_n$ converges uniformly on $E$

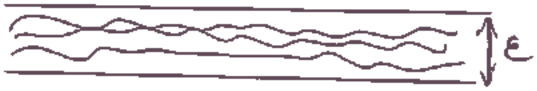
$\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$

$\sum f_n(x)$ converges uniformly on $E$

$\Rightarrow$ the partial sum $s_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly on $E$ | $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon,$ $|s_n(x) - s(x)| < \varepsilon$

## Uniformly convergence criteria:

8 Theorem Cauchy criterion



The sequence $\{f_n\}: E \to \mathbb{R}$ converges uniformly on $E$

$\Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall m, n \geq N_\varepsilon, \forall x \in E, |f_n(x) - f_m(x)| \leq \varepsilon$

$M_n = \sup_x |f_n(x) - f_m(x)|$, $M_n \not\to 0$ then $f_n \not\rightrightarrows$

The series $\sum f_n(x)$ converges uniformly on $E$

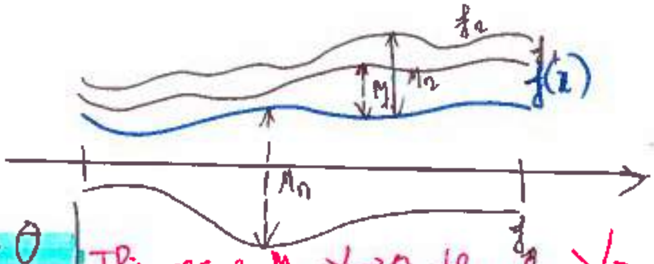
$\Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall m, n \geq N_\varepsilon, \forall x \in E, \left|\sum_{k=n}^{m} f_k(x)\right| < \varepsilon$

This means, put $M_n = \sup_x |f_n$

For comparison Theorem 7.10.

Suppose $\begin{cases} f_n(x) \xrightarrow{\text{pointwise}} f(x), \forall x \in E \\ M_n = \sup_{x \in E} |f_n(x) - f(x)| \end{cases}$

Then $f_n(x) \rightrightarrows f(x)$ iff $M_n \xrightarrow[\text{pointwise}]{n \to \infty} 0$



This means $M_n \not\to 0$ then $f_n \not\rightrightarrows$

Suppose $\{f_n(x)\}$ is a sequence of function defined on $E$

Suppose $|f_n(x)| \leq M_n$, $\forall x \in E$, $n = 1, 2, 3$

$(\sup_{x \in E} |f_n(x)| \leq M_n)$

$\sum M_n$ converges $\Longrightarrow$ $\sum f_n(x)$ converges uniformly

Dirichlet test for uniformly convergence of a series.

Consider $\sum_{n=1}^{\infty} f_n(x) g_n(x)$

$\sum f_n(x)$ has uniformly bounded partial sum

$g_1(x) \geq g_2(x) \geq \ldots$

$g_n(x) \rightrightarrows 0$ (uniformly)

$\Rightarrow \sum f_n(x) g_n(x) \rightrightarrows$

$\left(\text{EX } \sum (-1)^n \frac{x^2 + n}{n^2} \quad \text{EX 7.6}\right)$

## * Uniform convergence and boundedness

RB 1 $\left.\begin{array}{l}\{f_n\} \text{ bounded} \\ f_n \rightrightarrows f\end{array}\right\} \Rightarrow \begin{array}{l}\{f_n\} \text{ uniformly bounded} \\ f \text{ bounded.}\end{array}$

- $\{f_n\}$ sequence of bounded functions $\Leftrightarrow |f_n(x)| \le M_n$

* $\left.\begin{array}{l}f_n \rightrightarrows f \\ f \text{ bounded}\end{array}\right\} \Rightarrow \{f_n\}$ uniformly bounded $\exists M, |f_n(x)| \le M, \forall n, x$

* $\left.\begin{array}{l}f_n \rightrightarrows f \\ g_n \rightrightarrows g\end{array}\right\} \Rightarrow (f_n + g_n) \rightrightarrows f+g)$ (RE7.2)

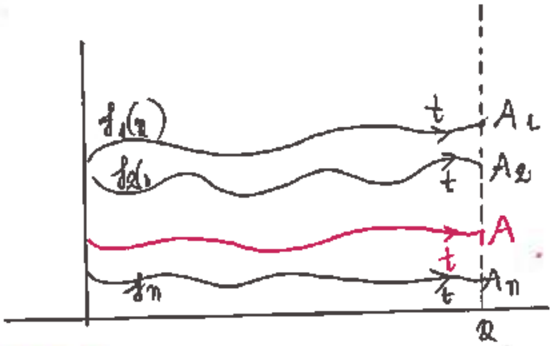
* $\left.\begin{array}{l}\left.\begin{array}{l}f_n \rightrightarrows f \\ g_n \rightrightarrows g\end{array}\right\} \\ \{f_n\}\{g_n\} \text{ sequences of bounded functions}\end{array}\right\} \rightarrow (f_n g_n) \rightrightarrows fg$. (RE7.2

## * Uniformly convergence and continuity

*** 7.11 Theorem**

$\left\{\begin{array}{l}\text{Suppose } f_n \rightrightarrows f \text{ uniformly on } E \text{ (a metric space)} \\ x \text{ is a limit point of } E. \\ \lim\limits_{t \to x} f_n(t) = A_n\end{array}\right.$

Then: $A_n$ converges in E; $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n$



$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

* 7.12 Theorem (uniformly convergent + continuous → continuous)

- $\left.\begin{array}{l}\{f_n\} \text{ sequence of continuous functions on } E \\ f_n \rightrightarrows f \text{ on } E\end{array}\right\} \Rightarrow f$ continuous on E

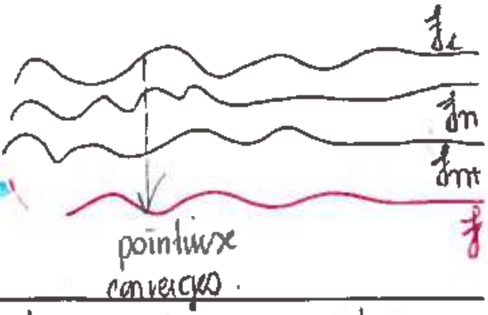
- $\left.\begin{array}{l}\{f_k\} \text{ sequence of continuous function on } E \\ S_n(x) = \sum\limits_{k=1}^{n} f_k(x) \rightrightarrows f = \sum\limits_{k=1}^{\infty} f_k(x)\end{array}\right\} \Rightarrow f = \sum\limits_{k=1}^{\infty} f_k(x)$ continuous on E

One more uniform convergence criteria (Prelim Aug 2005, P1, 7)

* 7.13 Theorem: (continuous, decreasing, pointwise converges in a compact set ⇒ uniformly converges)

(can apply for $f_n(x) = h_n(x)\, g_n(x)$)

$\left.\begin{array}{l}K \text{ compact} \\ \{f_n\} \text{ continuous } (\{f_n\}\text{: sequence of continuous function}) \\ f \text{ continuous.} \\ f_n \xrightarrow{\text{pointwise}} f \\ f_n \text{ decreasing (pointwise)}: f_n(x) \geq f_{n+1}(x), \forall x \in E\end{array}\right\} \Rightarrow f_n \rightrightarrows f.$



* A sequence of continuous function can converges pointwise to a continuous function (not uniformly)

7.9/166 Rudin

$$\left.\begin{array}{l}\{f_n\}: \text{sequence of } \boxed{\text{continuous}} \text{ functions} \\ f_n \rightrightarrows f\end{array}\right\} \Rightarrow \lim_{n\to\infty} f_n(x_n) = f(x) \quad \text{for every sequence of } \{x_n\} \in E,\ x_n \to x$$

# Metric space of continuous function on $X$

* 7.14. Def

$X$: metric space

define $\|\cdot\|$ by sup $\Rightarrow$ need $f$ to be bounded

$\mathcal{C}(X) = \{ f : X \longrightarrow \mathbb{C},\ f$ continuous, bounded on $X \}$

(If $X$ is compact $\Rightarrow$ the boundedness is redundant)).

- bounded
  pointwise bounded
  uniformly bounded

$\cdot \forall f \in \mathcal{C}(X) \quad \|f\| := \sup_{x \in X} |f(x)|$

$$d(f,g) := \|f-g\| = \sup_{x\in X} |f(x) - g(x)|$$

$(\mathcal{C}(X),\ d(f,g) = \|f-g\|)$ is a metric space.

---

* Theorem 7.9 ( Note that wea can only apply this if $\{f_n\}$ is a sequence of continuous + bor

Suppose $\begin{cases} f_n(x) \longrightarrow f(x) \\ M_n = \sup_{x\in E} |f_n(x) - f(x)| = d(f_n, f) \end{cases}$

Then $f_n(x) \rightrightarrows f(x)$ iff $M_n \xrightarrow{\text{pointwise}} 0$

$f_n \rightrightarrows f$ in $X$ $\Longleftrightarrow$ $f_n \longrightarrow f$ in $\mathcal{E}(X)$ )

$(d(f_n, f) \xrightarrow{\text{in } \mathcal{E}(X)} 0$

$\|f_n - f\| \xrightarrow{n\to\infty} 0$

---

7.15 * $(\underset{\text{continuous, bounded}}{\mathcal{C}(X)},\ d(f,g))$ is a complete metric space

$(\{f_n\}$ Cauchy sequence in $\mathcal{E}(X)$, then $\exists f \in \mathcal{E}(X)$ (continuous bounded), $f_n \to f$ in $\mathcal{C}(X)$

means $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, |d(f_n(x), f_m(x)) < \varepsilon$, then $f_n \longrightarrow f$ in $\mathcal{C}$

$\| f_n(x) - f_m(x) | < \varepsilon \qquad f_n \rightrightarrows f$ in

* Note: $\{f_n\}$ Cauchy in $(\mathcal{C}(X), d(f,g)) \rightarrow f_n$ bounded in $\mathcal{C}(X)$

$\Leftrightarrow \exists M, \|f_n\| \leq M, \forall n$

means $\sup_{x} |f_n(x)| \leq M, \forall n$.

---

$$\left.\begin{array}{l} * \ f_n \rightrightarrows f \\ f_n \text{ bounded (with norm } \|\cdot\|) \end{array}\right\} \Rightarrow f \text{ bounded (with norm } \|\cdot\|)$$

means $\|f\| \leq M \Leftrightarrow \sup_{x\in E} |f(x)| \leq M$.

# Uniformly convergence and integration

7.16 Theorem

Let $\alpha$ be a monotonically (increasing) on $[a,b]$.

$$\left.\begin{array}{l} f_n \in \mathscr{R}(\alpha) \text{ on } [a,b], \text{ for } n=1,2,3. \\ f_n \rightrightarrows f \end{array}\right\} \text{Then } \left\{\begin{array}{l} f \in \mathscr{R}(\alpha) \\ \int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha \end{array}\right.$$

* Corollary:

$$\left.\begin{array}{l} \alpha: \text{monotonically (increasing) on } [a,b] \\ \sum f_n(x) \rightrightarrows f(x) \text{ on } [a,b] \end{array}\right\} \text{Then } \left\{\begin{array}{l} f \in \mathscr{R}(\alpha) \\ \int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha \end{array}\right.$$

$$\left( \int_a^b \left(\sum_{n=1}^{\infty} f_n\right) d\alpha = \sum_{n=1}^{\infty} \left(\int_a^b f_n\,d\alpha\right) \right)$$

(when $\sum f_n$ converges uniformly $\Rightarrow$ we can swap $\int_a^b$ and $\sum_{n=1}^{\infty}$ )

---

# * Uniformly convergence and differentiation

7.17 Theorem

$$\text{Suppose} \left\{\begin{array}{l} \{f_n\}: [a,b] \longrightarrow \mathbb{C} \text{ sequence of (differentiable) functions on } [a,b] \to \mathbb{C} \\ \exists\, x_0 \in [a,b], \text{ s.t } \{f_n(x_0)\} \text{ converges pointwise} \\ \{f_n'\} \rightrightarrows \end{array}\right\}$$

$$\text{Then } \exists\, f \text{ (differentiable), such that} \left\{\begin{array}{l} f_n \rightrightarrows f \\ f'(x) = \lim_{n\to\infty} f_n'(x), \text{ for all } x \in [a,b] \end{array}\right.$$

* Theorem 7.17 (for series)

$$\left\{\begin{array}{l} \sum f_n(x) \text{ is a series where} \\ \quad \{f_n\}: \text{sequence of differentiable function } (\exists\, f_n', \forall n) \text{ on } [a,b] \\ \sum f_n(x_0) \text{ converges (pointwise) for some } x_0 \in [a,b] \\ \sum f_n' \text{ converges} \rightrightarrows \text{ on } [a,b] \end{array}\right.$$

$$\text{Then, } \exists\, f \text{ differentiable s.t} \left\{\begin{array}{l} \sum f_n \rightrightarrows f \\ f'(x) = \sum_{n=1}^{\infty} f_n'(x) \end{array}\right.$$

# Equicontinuous families of functions

Theorem 3.6
$\{x_n\}$ : sequence of bounded sequence $\to$ $\exists$ a convergent subsequ
($\exists$ $x_{n_k}$, $x_{n_k}$ converge

**7.19 Theorem:**

Let $\{f_n\} : E \longrightarrow \mathbb{C}$ be a sequence of functions $\left(M_x : \underset{x \mapsto M_x}{X \longrightarrow \mathbb{R}}\right)$

We say
- $\{f_n\}$ is pointwise bounded on $E \Leftrightarrow \forall x \in E, \exists M_x, |f_n(x)| \leq M_x, \forall n \in \mathbb{N}$
- $\{f_n\}$ is uniformly bounded on $E \Leftrightarrow \exists M, |f_n(x)| \leq M, \forall x \in E, \forall n \in \mathbb{N}$

(Note that when $E$ is compact, $f_n$ continuous $\sup|f_n(x)| \leq M$

- $\{f_n\}$ sequence of bounded functions $\Leftrightarrow$ each $f_n$ is bounded, $\forall n$
  $\Leftrightarrow |f_n(x)| \leq M_n, \forall n \in \mathbb{N}$

**7.22: Def:**

$\mathcal{F} = \{f : X \longrightarrow \mathbb{C}\}$ (a family of functions defined on $X$)
is said to be equicontinuous on $X \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ $\forall x, y \in X, d_X(x,y) < \delta$, then $|f(x) - f(y)| <$ for all $f \in \mathcal{F}$

$\Rightarrow$ Every member of a equicontinuous family is uniformly continuous

Theorem 7.23 Aug 2015 > P1

Let $X$ : countable
$\{f_n\} : X \longrightarrow \mathbb{C}$ is a pointwise bounded sequence of functions

$\Rightarrow$ $\{f_n\}$ contains a pointwise convergent subsequence.
($\exists f_{n_k}$, $f_{n_k}(x)$ converges, $\forall x \in E$

* 7.24 Theorem

K compact metric space
$f_n \in C(K) = \{$set of bounded, continuous functions on K$\}$
$f_n \rightrightarrows$ on K . don't use in the proof

$\Rightarrow$ $\{f_n\}$ : equicontinuous on K.

* 7.25 Theorem (Arzela-Ascoli theorem).

K is compact
$f_n \in C(K)$
$\{f_n\}$ pointwise bounded, equicontinuous on K.

$\Rightarrow$ $\{f_n\}$ uniformly bounded on K
$\{f_n\}$ contains a uniformly convergente subsequence

Aug 2015 ⑤

K compact
$\{f_n\}$ equicontinuous on K
$f_n \xrightarrow{\text{pointwise}}$ on K

$\Rightarrow$ $f_n \overset{\text{uniformly}}{\rightrightarrows}$ on K (Review the proof)

Any element of a equicontinuous family $F$ is uniformly continuous.

Every finite family of uniformly continuous functions $\Rightarrow$ is an equicontinuous family.

$X$: compact  
$f: X \longrightarrow \mathbb{R}$ continuous $\Big\} \Rightarrow f$ is uniformly continuous.

---

A very useful result (prove and apply in Jan2009, P5).

$\{f_n\}$: sequence of differentiable on $[a,b]$  
; $\{f_n'\}$ uniformly bounded (which means, $\exists M>0, |f_n'(x)| \leq M, \forall x, \forall n$ $\Big\} \Rightarrow \{f_n\}$ equicontinuous.

Also Aug2015)

$\{f_n\}$ equicontinuous.  
$f_n \longrightarrow f$ on $K$  
$K$ compact $\Big\} \Rightarrow f_n \rightrightarrows f$ on $K$

$\{f_n\}$ sequence of differential function in $\mathbb{R}$  
$\{f_n'\}$ uniformly bounded  
$f_n \rightrightarrows 0$ $\Big\}$ $\{f_n\}$ equicontinuous

(used and proved in Jan2002, P6)

we have .....

$K$ compact  
$f_n \longrightarrow f$ pointwise  
$|f_n'(x)| \leq M$ (uniformly bounded) $\Big\}$ then $\Big\{$ $\{f_n\}$ equicontinuous  
$f_n \rightrightarrows f$.

$f_n$, $f$ defined on $[a,b] \longrightarrow \mathbb{R}$  
$f_n \rightrightarrows f$ | $h$ is uniformly continuous on $\mathbb{R}$ $\Big\}$ $h(f_n) \rightrightarrows h(f)$ on $[a,b]$. (apply + prove in Jan 2009)

---

Aug2015/1)

$\{$sequence of functions $f_n: \mathbb{R} \longrightarrow [0,1]$.

ne that $\exists$ subsequence $n_k$ along which $f_{n_k}(q)$ converges $\forall q \in \mathbb{Q}$.

in2014 :, P5.

: real value, cont on $[0,1]$.

$\{f^n\}_{n=1}^{\infty}$, (sequence of power of $f$) is equicontinuous $\Leftrightarrow \overset{\text{supnorm.}}{\|f\|} < 1$

idTerm 602, P3

: family of continuous function defined on $[0,1]$.  
i) $f(0)=0, \forall f \in F$  
ii) $F$ is equicontinuous $\Bigg\}$ Prove that  
a) $F^2 = \{f^2, f \in F\}$ is equicontinuous.  
b) $f(0)=0, \forall f \in F$ is necessary  
(Prove this by an example).

Jan 2016) P5.

$F$ = equicontinuous family of nonnegative functions on $(M,d)$

$S$ dense in $M$

Suppose that for each $x \in S$, we have $f(x)=0$ for some $f \in F$.

Prove that for any $y \in M$, we have $\inf\{f(y), f \in F\} = 0$

**Stone Weierstrass** (Even a "badly" continuous function is a uniform limit of polynomial)

7.26 Weierstrass approximation theorem:

$f: [a,b] \xrightarrow{\text{seperal}} \mathbb{C}$ (continuous)

$\Rightarrow \begin{cases} \exists \{p_n\} \text{ sequence of polynomial} \\ p_n \rightrightarrows f \text{ on } [a,b] \end{cases}$

Furthermore, if $f$ is real value, we can find real value

* Corollary: The metric space $\mathcal{C}([a,b], \mathbb{C})$ contains a countable dense subset.

* Corollary: Let $[-a,a]$: interval

Then $\exists \{p_n\}$ sequence of real polynomial, $\begin{cases} p_n(x) \rightrightarrows |x| \text{ on } [-a,a] \\ p_n(0) = 0, \forall n \end{cases}$

---

7.28 * Definition: A family $\mathcal{A} = \{f : X \longrightarrow \mathbb{C}$ (complex value function$\}$ is said to be an algebra if

$\forall f, g \in \mathcal{A}$
$\forall c \in \mathbb{C}$
$\begin{cases} i) \ f + g \in \mathcal{A} \\ ii) \ fg \in \mathcal{A} \\ iii) \ cf \in \mathcal{A} \end{cases}$

- $\mathcal{A}$ is uniformly closed if $\forall \{f_n\} \subset \mathcal{A}$, $f_n \rightrightarrows f$, then $f \in \mathcal{A}$
- $\mathcal{B}$: the uniform closure of $\mathcal{A}$: iff $\mathcal{B} = \{f, \exists \{f_n\}, f_n \rightrightarrows f\}$

(The set $\mathcal{P}$ of all polynomials is an algebra in $C(X, \mathbb{C})$
the uniform closure of $\mathcal{B}$ is $\mathcal{B} = C(X, \mathbb{C})$)

* Theorem 7.29

$\left.\begin{array}{l} \mathcal{A} \text{ is an algebra of (bounded) function on a set } X \\ \mathcal{B} \text{ is its uniform ~~bounded~~ closure.} \end{array}\right\} \rightarrow \mathcal{B}$ is an uniformly bounded algebra

* Definition: $\mathcal{A}$: a family of complex value function on $X$

$\mathcal{A}$ separates points iff $\forall x, y \in X$, $x \neq y$, then $\exists f \in \mathcal{A}$, $f(x) \neq f(y)$

$\mathcal{A}$ vanishes at no point iff $\forall x \in X$, $\exists f \in \mathcal{A}$, $f(x) \neq 0$

---

* If $\mathcal{A}$ is an algebra generated (for ex $e^{-\frac{kx}{1-x}}$), then we can prove that $\mathcal{A}$ seperated point by consider

$h(x) = e^{\frac{-kx}{1-x}}$ and prove that $h'(x) > 0$ or $h'(x) < 0$, $\forall x \in X$ (Aug 2003 / 5 heorem 7.31:

: an algebra of functions on $X$

seperates points, vanishes at no point

et $x, y \in X$, $x \neq y$ and $c, d \in \mathbb{C}$

$\Rightarrow \exists f \in \mathcal{A}$ st $\begin{cases} f(x) = c \\ f(y) = d \end{cases}$

If $\mathcal{A}$ is a real algebra, the result holds for $c, d$ are real

heorem 7.32 Stone Weierstrass, real version

(compact) metric space

: an algebra of real-value (continuous) function on $X$

separates points, vanishes at no point on $X$

$\Rightarrow$ The closure of $\mathcal{A}$ is all $C(X, \mathbb{R})$

uniformly

In general:

Assume $f : X \longrightarrow \mathbb{C}$ continuous , $X = [a, b]$ (compact)

then $\exists f_n(x)$ , $f_n(x) \rightrightarrows f$ .

we can prove that $\{f_n(x)\}$ is an ~~ge~~ algebra generated by sth $\left\{\begin{array}{l} \text{vanish at no point} \\ \text{seperates point} \end{array}\right\}$, for example .

$f_n(x) = \sum_{k=1}^{n} c_k e^{kx}$, $f_n(x) = \sum_{k=0}^{n} c_k e^{kx}$

$f_n(x) = \sum_{k=0}^{n} c_k x^k$ (if $k$ goes from 1, then it ~~does no~~ vanishes at $x = 0$ )

$f_n(x) = \sum_{k=0}^{n} c_k x^{2017k}$ (because 2017 is odd $\Rightarrow$ seperates point )

Any (real) polynomial can be approximated by polynomials with (rational) coefficient .

For complex

* Some problems and strategies (these problems were given many times) using Stone Weierstrass theorem      Prove that $f \equiv$ constant on $[a,b]$.

- Aug2003, P5. Let $f$ be a continuous function on $[0,1]$ s.t. $\int_0^1 e^{\frac{-nx}{1-x}} f(x)\,dx = 0$, $\forall n \geq 0$ } Prove that $f \equiv 0$ on $[0,1]$.
  $P_n(x) = \sum_{k=1}^{n} c_k e^{-\frac{kx}{1-x}}$

- 7.20 Rudin/169. Let $f$ is continuous on $[0,1]$, $\int_0^1 x^n f(x)\,dx = 0$, $\forall n = 0,1,2,\dots$ } Prove that $f(x) = 0$ on $[0,1]$.
  $P_n(x) = \sum_{k=1}^{n} c_k x^k$

- Aug 2007 P4 (A bit different, but done) Let $f$ continuous on $[0,1]$, $\int_0^1 x^n f(x)\,dx = \frac{1}{n+1}$, $\forall n = 0,1,2,\dots$ } What can we say about the function $f(x)$.
  (we have $f(x) = 1$ on $[0,1]$, just by using $g(x) = f(x$
  $P_n(x) = \sum_{k=1}^{n} c_k x^k$

- Aug1998 Let $f$ real continuous on $[0,1]$, $\int_0^1 e^{-\lambda x^2} f(x)\,dx = 0$, $\boxed{\forall \lambda \geq 0}$ } Prove that $f = 0$ on $[0,1]$.
  $P_n(x) = \sum_{k=1}^{n} c_k e^{-kx^2}$

* Note: we need to use $P_n(x) \rightrightarrows f$
then $P_n(x)\cdot f \Longrightarrow f^2$ (because $P_n(x)$ bounded (uniformly bounded converges uniformly).
and $f$ is bounded

- and $\left.\begin{array}{l} g(x) \geq 0 \\ \int_0^1 g(x) = 0 \end{array}\right\}$ then $g(x) = 0$.

* Two more advance problems relating to using Stone-Weierstrass theorem.

- Aug2013: Let $f : [1,+\infty) \to \mathbb{R}$ continuous function.
  $\lim_{x\to\infty} f(x) = 0$
  Prove that $\forall \varepsilon > 0$, $\exists n$ and $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that $\left| f(x) - \sum c_k e^{-kx} \right| < \varepsilon$

We can use variable changing to prove.

* May 2017
$f : \mathbb{R} \to \mathbb{R}$ strictly increasing continuous, $f(0) = 0$
$g : [0,1] \to \mathbb{R}$ be a continuous function such that
$\int_0^1 f^n(x)\, g(x)\,dx = 0$, $n = 0,1,2,\dots$ } Prove that $g \equiv 0$ on $[0,1]$  Use integration by part

* There are ~~7~~ 2 ways to consider if $\sum F_n(x)$ converge uniformly or not :

**Way 1** : $F_n(x) \le M_n$ $\quad (\sup_{x\in E}|F_n(x)| \le M_n)$
$\sum M_n$ converges . $\Big\} \Rightarrow \sum_{n=1}^{\infty} F_n(x) \rightrightarrows$ .

**Way 2** : $\sum F_n(x) = \sum f_n(x)\, g_n(x)$ .

Dirichlet test where $\sum_{n=1}^{\infty} f_n(x)$ has uniformly bounded partial sum
$g_n(x) \ge g_{n+1}(x)$ , $g_n(x) \rightrightarrows 0$ $\Big\} \Rightarrow \sum f_n(x)\, g_n(x)$

* Dirichlet test (way 2) is extremely usually usefull when we consider $\sum_{n=1}^{\infty}(-1)^n g_n(x)$

forexample $\sum(-1)^n \frac{x^2+n}{n^2}$ (E7.6) $\qquad \sum(-1)^n x^n$ (Midterm 602, 22)

* Problem about proving that a sequence $\{f_n\}$ contains a uniformly convergent subsequence

: Jan 2011, 25.
$f_1 : [0,1] \to \mathbb{R}$ continuous .
nsider sequence $\{f_n\}$, $f_{n+1} = \cos f_n(x)$
ve that $\{f_n\}$ contains a uniformly convergent subsequence

⇒ The key point is proving that $\{f_n\}$ equicontinuous.
(with this problem, use
$|\cos a - \cos b| = (\sin c)\, |b-a|$
$\Rightarrow |\cos(f_n(x)) - \cos f_n(y)| \le f_n(x) - f_n(y) \cdots \le$
$\le |f_1(x) - f_1(y)| < \varepsilon$

* More results relating to uniformly convergence in a compact set. Results

* * Aug2003, P4.

for each $n$, let $f_n : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ nondecreasing function
$f_n \longrightarrow f$ point wise,
$f$ is continuous.

} Prove that $f_n \rightrightarrows f$ on compact sets

* Aug1999 7.

$\{f_n\}$ : sequence of uniformly bounded, Riemann integrable function on $[0,1]$
$F_n(s) = \int_0^s f_n(t)\,dt$ for $0 \le s \le 1$.

} Prove that $\exists$ a subsequence $\{F_n\}$ converges uniformly on

---

* A very useful trick to prove $\{f_n\}$ equicontinuous when we have $f_n'(x) < g(x)$ is by using

$$f_n(x) - f_n(y) = \int_x^y f_n'(t)\,dt$$

to prove that $|f_n(x) - f_n(y)| < \varepsilon$
$\forall x, y, |x-y| < \delta, \forall n$.

See Aug2012 7

$f_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n = 1, 2, \dots$ is a $C^1$ function
$\forall n, |f_n'(x)| \le \frac{1}{\sqrt{x}}, 0 < x \le 1$
$\int_0^1 f_n(x)\,dx = 0$

} Prove that the sequence has a subsequence that converges uniformly on $[0,1]$.

* Rudin 7.7: trong TH bài toán có $x$ in both numerator and denominator | * Check Rudin 7.9. Shake

$\Rightarrow$ try to eliminate $x$ by considering 2 cases $\begin{cases} x \neq 0 \\ x = 0 \end{cases}$

EX $f_n(x) = \dfrac{x}{1+nx^2}$ (Rudin 7.7)

* Rudin 7.4: Prove that the series $\sum (-1)^n \dfrac{x+n}{n^2}$ converges uniformly in a bounded interval

- We note that if $\left.\begin{matrix} f_n \rightrightarrows f \\ g_n \rightrightarrows g \end{matrix}\right\} \Rightarrow f_n + g_n \rightrightarrows f + g$
- We also note that if $\left.\begin{matrix} g_n \to g \\ g_n \text{ does } \underline{\text{not}} \text{ depends on } x \end{matrix}\right\} \Rightarrow g_n \rightrightarrows g$

---

*Result $f : [0,1] \longrightarrow \mathbb{R}$ continuous. Aug2013, Jan 2015.

Then $\displaystyle\lim_{n\to\infty} \int_0^1 f\, n x^{n-1}\, dx = f(1)$ (Aug2013) $\displaystyle\int_0^1 P_l^n(x)\, x^n\, dx \longrightarrow P_l(1)$.

$\displaystyle\lim_{n\to\infty} (n+1) \int_0^1 f x^n\, dx = f(1)$ (Jan 2015)

*

* In case the series has $f_n \cdot f_m \equiv 0, \forall m \neq n$ $\Rightarrow$ convienence in investigating $\sum_{n=1}^{\infty} f_n(x)$. (See Sample B, 2Q).
because for $n > N_0$, $\exists$ only one $n$ s.t. $f_n \neq 0$.

## Pointwise convergence /

$_n(x) \xrightarrow[\text{pointwise}]{n\to\infty} f(x) \Leftrightarrow \forall \varepsilon > 0, \exists n_{\varepsilon,x} \in \mathbb{N}, \forall n \geq n_{\varepsilon,x}, |f_n(x) - f(x)| < \varepsilon$

$\Leftrightarrow$ NTP $, \forall \varepsilon > 0, \exists n_x, \forall n \geq n_x, |f_n(x) - f(x)| < \varepsilon$ .

$_n(x) \underset{\text{pointwise}}{\not\longrightarrow} f(x)$ in $E$ $\Leftrightarrow$ NTP $\exists x_0 \in E, f_n(x_0) \overset{n\to\infty}{\not\longrightarrow} f(x_0)$

## Uniformly convergence

TP $f_n(x) \rightrightarrows f(x) \Leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n > n_\varepsilon, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$ .

$\boxed{\text{on } E} \Leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon, \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$

TP $f_n(x) \not\rightrightarrows f(x)$, Can prove $f_n(x) \not\longrightarrow f(x)$ on $E$ , $(\exists x_0 \in E, f_n(x_0) \not\longrightarrow f(x_0)$ .

$\boxed{\text{on } E}$, Can prove $\exists \varepsilon > 0, \forall n_\varepsilon \in \mathbb{N}, \exists n \geq n_\varepsilon, \exists x_n, |f_n(x_n) - f(x_n)| \not< \varepsilon$

$\exists \varepsilon > 0$, (no matter how large n is), $\exists x_n, |f_n(x_n) - f(x_n)| > \varepsilon$

Example of function $f$ continuous but not uniformly continuous.

Exa

- $f(x) = \frac{1}{x}$ is continuous on $(0,2)$ but not uniformly continuous on $(0,2)$.
- $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbb{R}$. (see Aug 2006

* $f$

* Example of uniformly continuous in $\mathbb{R}$.

- $f(x) = \sin(x)$ is uniformly continuous in $\mathbb{R}$.
-

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* One important example of $f_n(x) \rightrightarrows f(x)$ is by putting $f_n(x) = f(x) + \frac{1}{n}$ (see Aug 2006.

- For example Let $f(x) = x$ Put $f_n(x) = x + \frac{1}{n}$, then $f_n(x) \rightrightarrows f(x)$.

$$f(x) = \begin{cases} \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \quad f_n(x) = f(x) + \frac{1}{n}, \text{ then } f_n(x) \rightrightarrows f(x)$$

* Aug 1996 > 26

$f$ is not continuous on $[0,1]$.

a> Example when $f_n \rightrightarrows f$ on $[0,1]$ } But $\phi \circ f_n \not\rightrightarrows \phi \circ f$.
$\phi$ continuous on $\mathbb{R}$

$f_n(x) = f(x) + \frac{1}{n}$, $f(x) = 1/x$
and $\phi(x) = x^2$ (continuous but n

b> $f_n \rightrightarrows f$ on $[0,1]$.
($f$ may cont or not continuous)
$\phi$ is uniformly continuous on R } then $\phi \circ f_n \rightrightarrows \phi \circ f$.

c> $f_n \rightrightarrows f$ on $[0,1]$.
$f$ continuous on $[0,1]$
$\phi$ is ~~uniform~~ continuous on $\mathbb{R}$ } then $\phi \circ f_n \rightrightarrows \phi \circ f$.

~~sequence of~~ uniformly convergent → uniformly bounded.

Rudin 7.1 : $\{f_n\}$ , $f_n \rightrightarrows$ then $\{f_n\}$ is uniformly bounded

means $\exists M, |f_n(x)| < M, \forall n, \forall x$.

* Example when $\left.\begin{array}{l} f_n(x) \text{ continuous in } E \\ f_n(x) \longrightarrow f(x) \text{ pointwise} \end{array}\right\} \not\Rightarrow f(x)$ continuous in E.

Let $E = [0,1]$ $f_n(x) = x^n$ continuous in $[0,1]$.

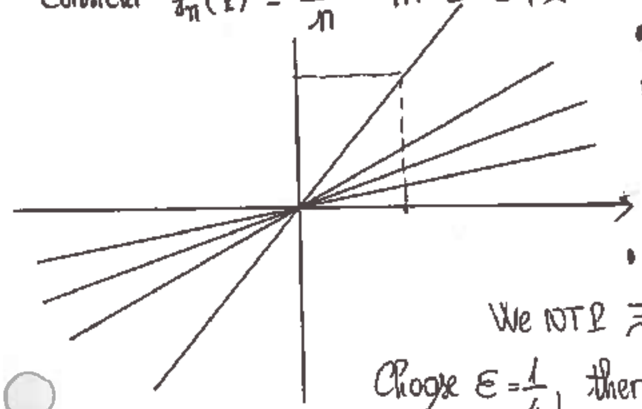
$\left.\begin{array}{l} \bullet \text{ When } x \in [0,1) ,\ f_n(x) \longrightarrow 0 \\ \bullet \text{ when } x = 1,\ f_n(x) = 1 \longrightarrow 1 \end{array}\right)$ this means $f_n(x) \xrightarrow[\uparrow \text{ pointwise}]{} f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$

$f(E)$ is <u>not</u> continuous in $[0,1]$

* Example of $\begin{cases} f_n(x) \longrightarrow f(x) \text{ in } E \\ f_n(x) \not\rightrightarrows f(x) \end{cases}$

Consider $f_n(x) = \dfrac{x}{n}$ in $E = [0,1]$

Let $f(x) = 0$, $\forall x$



- We have $f_n(x) \longrightarrow f(x)$ pointwise:

For each $x \in [0,1]$

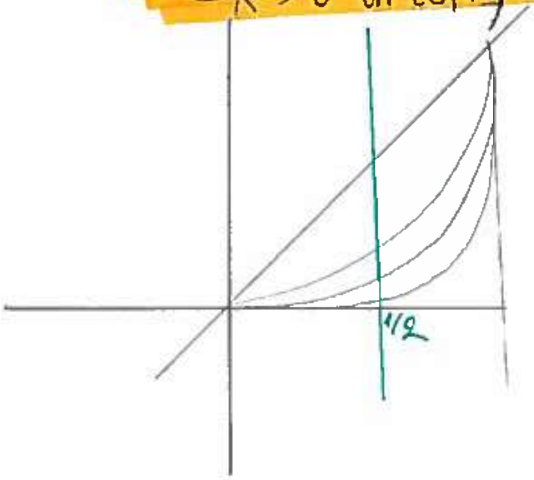
$$\lim_{n \to \infty} \frac{x}{n} = 0$$

- We prove $f_n(x) \not\rightrightarrows f(x)$.

We NTP $\exists \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \exists x_n, |f(x_n) - f(x)| > \varepsilon$

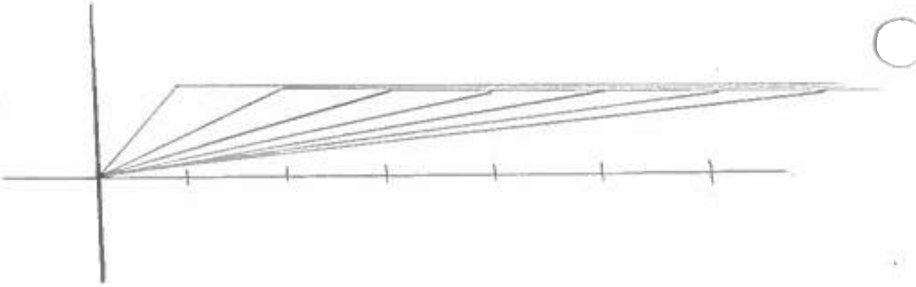
Choose $\varepsilon = \frac{1}{4}$, then no matter how large n is, $\exists x = n, |f(x_n) - f(x)| = |1-$

* $\left.\begin{array}{l} f_n(x) = x^n \longrightarrow 0 \text{ on } [0,1] \\ \qquad\qquad \not\rightrightarrows 0 \text{ on } [0,1] \end{array}\right\}$ | $f_n(x) \Longrightarrow 0$ on $[0, \frac{1}{2}]$.



$D = \mathbb{R}^+$, $f_n : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined by

$$x \longmapsto f_x(x) = \begin{cases} \frac{x}{n} & , 0 \leq x \leq n \\ 1 & , x > n \end{cases}$$



## Prove 7.8 Cauchy criterion of uniformly convergence

$f_n(x) \rightrightarrows f(x)$ in E

$\overset{def}{\Leftrightarrow} \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$ | $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall m, n \geq n_\varepsilon, \underset{\forall x \in E}{,} |f_m(x) - f_n($

($\Rightarrow$): Because $f_n \rightrightarrows f$ in E.

Then for $m, n \geq n_\varepsilon$, $\left.\begin{array}{l} |f_m(x) - f(x)| < \varepsilon/2 \\ |f_n(x) - f(x)| < \varepsilon/2 \end{array}\right\} \Rightarrow |f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x)$

$< \varepsilon \Rightarrow \square.$

($\Leftarrow$): From Cauchy criterion

$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall m, n \geq n_\varepsilon, \overset{\forall x \in E}{,} |f_m(x) - f_n(x)| < \varepsilon \quad (1)$

$\Rightarrow$ ~~$\forall n$~~ $x \in E$, $f_n(x)$ Cauchy sequence

then $f_n(x) \xrightarrow{\text{pointwise}} f(x)$.

Then for fix $n$, let $m \xrightarrow{\infty} (1)$, we have $|f(x) - f_n(x)| < \varepsilon$, $\forall n \geq n_\varepsilon$, $\forall x \in E \Rightarrow \square$

## * Prove 7.10: Uniformly convergent criteria for series

Suppose $\{f_n\}$: sequence of functions defined on E

$|f_n(x)| \leq M_n, \overset{\forall x \in E}{\forall n} = 1, 2, 3, \dots$

$\sum M_n$ converges.

Prove that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly

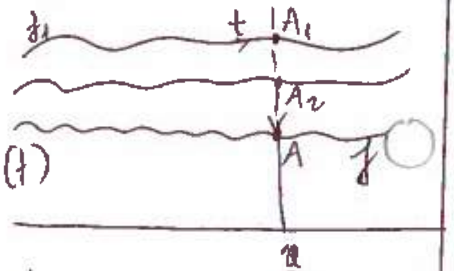
We have $\sum M_n$ converges $\Rightarrow \{M_n\}_1^\infty$ Cauchy sequence | We need to prove

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, \left|\sum_{\ell=n}^{m} M_\ell\right| < \varepsilon$ | $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall m, n \geq n_\varepsilon, \forall x \in E,$ $\left|\sum_{\ell=n}^{m} f_\ell(x)\right| < \varepsilon$

because $|f_n(x)| \leq M_n, \forall x \in E, \forall n$

$$\Rightarrow \left|\sum_{\ell=n}^{m} f_\ell(x)\right| \leq \sum_{\ell=n}^{m} |f_\ell(x)| \leq \sum_{\ell=n}^{m} M_\ell < \varepsilon \Rightarrow \square.$$

roof Theorem 7.11 and 7.12 Uniformly convergence and continuity



ppose $f_n \rightrightarrows f$ in $E$
: is a limit point of $E$
$\lim\limits_{t \to x} f_n(t) = A_n$

} Then $A_n$ converges and $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n$

(this means $\lim\limits_{t \to x} \lim\limits_{n \to \infty} f_n(t) = \lim\limits_{n \to \infty} \lim\limits_{t \to x} f_n(t)$)

heorem 7.12

ppose $f_n \rightrightarrows f$ in $E$
$\{f_n\}$ sequence of continuous function on $E$
} ⇒ $f$ is continuous on $E$ | $f_n(x) \rightrightarrows f$, $f_n$ continuous } ⇒ $f$ cont

roof Theorem 7.11:

First, we prove that $A_n$ converges ⇔ NTP $\{A_n\}$ Cauchy sequence
(⇔ NTP $\forall \varepsilon > 0\ \exists N \in \mathbb{N}, \forall m, n \geq N, |A_m - A_n| < \varepsilon$

have $f_n \rightrightarrows f$ in $E \underset{\text{uniform Cauchy}}{\Longleftrightarrow} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, \forall t \in E, |f_m(t) - f_n(t)| < \varepsilon$ (1)

have $f_n(t) \xrightarrow[t \to x]{} A_n \Leftrightarrow \forall \varepsilon > 0, \exists \delta_n, \forall t \in E, |t - x| < \delta_n$, then $|f_n(t) - A_n| < \varepsilon$ (2)

$f_m(t) \xrightarrow[t \to x]{} A_m \Leftrightarrow \forall \varepsilon > 0, \exists \delta_m, \forall t \in E, |t - x| < \delta_m$, then $|f_m(t) - A_m| < \varepsilon$ (3)

ien Choose $\delta = \min\{\delta_n, \delta_m\}$, choose $N = n_0$, we have
$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N,$

$$|A_m - A_n| < |A_m - f_m(t)| + |f_m(t) - f_n(t)| + |f_n(t) - A_n| < 3\varepsilon \Rightarrow \blacksquare.$$

$\Rightarrow A_n$ Cauchy $\Rightarrow \{A_n\}$ converges.

Now we will prove that $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n$

Let $A = \lim\limits_{n \to \infty} A_n$, we want to prove $\lim\limits_{t \to x} f(t) = A$

⇔ NTP $\forall \varepsilon > 0, \exists \delta > 0, \forall t \in E, |t - x| < \delta$, then $|f(t) - A| < \varepsilon$

We have $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall t \in E, |f_n(t) - f(t)| < \varepsilon/3$

We have $f_n(t) \xrightarrow{t \to x} A_n \Leftrightarrow \forall \varepsilon > 0, \exists \delta_n, \forall t \in E, |t - x| < \delta_n$, then $|f_n(t) - A_n| < \varepsilon/3$

We have $A_n \longrightarrow A \Leftrightarrow \forall \varepsilon > 0, \exists n_1 \in \mathbb{N}, \forall n \geq n_1, |A_n - A| < \varepsilon/3$

hen $\forall \varepsilon > 0$, ~~$\exists \delta$~~ $\underset{n = \max\{n_0, n_1\}}{\underline{\text{for } n \text{ large enough}}}$, $\exists \delta = \delta_n$,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| < \varepsilon \Rightarrow \square.$$

Proof theorem 7.12 (directly next page)
Let $A_n = f(x)$, because $f_n$ continuous, $\lim\limits_{t \to x} f_n(t) = f_n(x)$ | From T7.11, $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n = \lim\limits_{n \to \infty} f_n(x) = f$

□

* Prove theorem 7.12 directly

Let $\{f_n\}$ : sequence of continuous function in E, $f_n \rightrightarrows f$ $\Big\} \Rightarrow$ $f$ is continuous in $E$.

- We have $f_n$ continuous in $E$ $(\forall n)$, then $\forall x \in E$
  $\forall \varepsilon > 0, \exists \delta_{n,\varepsilon}, \forall t \in E, |t - x| < \delta_{n,\varepsilon}, |f_n(t) - f_n(x)| < \varepsilon$. (1)
- $f_n \rightrightarrows f$ in $E$
  $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall t \in E, |f_n(t) - f(t)| < \varepsilon$. (2)

* We want to prove that $f$ continuous $\forall x \in E \Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists \delta > 0, \forall t \in E, |t - x| < \delta$ then $|f(t) - f(x)|$

From (1) and (2), choose $n = n_0$,

then we have $\forall \varepsilon > 0, \exists \delta = \delta_{n_0,\varepsilon}, \forall t \in E, |t - x| < \delta_{n_0,\varepsilon}$ $|f_{n_0}(t) - f_{n_0}(x)| < \varepsilon$

and $\forall t \in E, |f_{n_0}(t) - f_{n_0}(t)| < \varepsilon$

Then $\forall \varepsilon > 0, \exists \delta = \delta_{n_0,\varepsilon}, \forall t \in E, |t - x| < \delta_{n_0}$,

$$
\begin{aligned}
|f(t) - f(x)| &\leq |f(t) - f_{n_0}(t) + f_{n_0}(t) - f_{n_0}(x) + f_{n_0}(x) - f(x)| \\
&\leq |f(t) - f_{n_0}(t)| + |f_{n_0}(t) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)| \\
&\leq 3\varepsilon \quad \square
\end{aligned}
$$

---

* Theorem 7.15 $(\mathscr{C}(X), d(f,g))$ is a complete metric space

$\mathscr{C}(X) = \{f : X \longrightarrow \mathbb{R},\ f$ is continuous, bounded$\}$
$\|f\| := \sup\limits_{x \in X} |f(x)|$
$d(f,g) = \|f - g\| = \sup\limits_{x \in X} |f(x) - g(x)|$

$\mathscr{C}(X)$ is a complete metric space.

We need to prove that $\mathscr{C}(X)$ is a complete metric space $\Leftrightarrow$
NTP $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$, then $f_n \xrightarrow{\text{in } \mathscr{C}(X)} f$, with $f \in \mathscr{C}(X)$
NTP $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$, then $f_n \rightrightarrows f$, $f \in \mathscr{C}(X)$

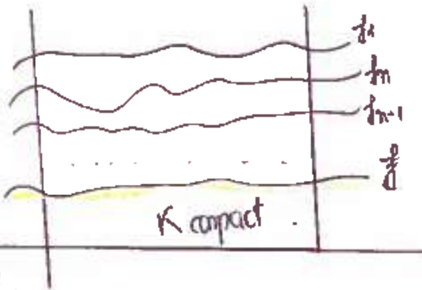
- We have $\{f_n\}$ Cauchy in $\mathscr{C}(X)$, $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geq n_0, d(f_m, f_n) < \varepsilon$
  $\Leftrightarrow \sup\limits_{x \in X} |f_m(x) - f_n(x)| < \varepsilon$
  $\Rightarrow \forall x \in X, |f_m(x) - f_n(x)| < \varepsilon$.

  this means $\{f_n\}$ uniformly Cauchy in $\mathbb{R}$
  $\Leftrightarrow f_n \rightrightarrows f$ in $\mathbb{R}$. $\Leftrightarrow f_n \xrightarrow{\text{in } \mathscr{C}(X)} f$
- Now we need to prove $f$ is $\underbrace{\text{continuous}}_{\text{theorem 7.12}} + \underbrace{\text{bounded}}_{\text{Rudin 7.1}} \Rightarrow \square$.

## 7.13 Theorem: Aug 2003.

K compact
$\{f_n\}$: sequence of continuous function on K
$f_n \xrightarrow[\text{pointwise}]{} f$ , f is continuous on K.
$f_n \geqslant f_{n+1}(x)$, $\forall x \in K$, $n = 1,2,3$

$\Rightarrow$ Then $f_n \rightrightarrows f$ on K.



Put $g_n = f_n - f$, then from the assumption, we have.

$g_n \geqslant 0$
$\{g_n\}$: sequence of continuous functions
$g_n \xrightarrow[\text{pointwise}]{} 0$
$g_n \geqslant g_{n+1}$ , $\forall x \in K$ compact , $n = 1,2,3,\dots$

We want to prove that $g_n \rightrightarrows 0$
NTP $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, \forall x \in K,$
$\underline{|g_n(x) - 0| < \varepsilon}$
$\Leftrightarrow 0 \leq g_n(x) < \varepsilon$

Put $K_n = \{x \in K, |g_n(x)| \geqslant \varepsilon\}$, We NTP $(\exists n_0 \in \mathbb{N})(\forall n \geqslant n_0)(K_n = \phi)$. This is what we need to do.

*Now consider $\{K_n\}$, we have

- $K_n = \underbrace{g_n^{-1}\underbrace{[\varepsilon, +\infty)}_{\text{closed in } \mathbb{R}}}_{\text{closed in K (because } g_n \text{ cont )}}$

We have $K_n$ closed in K
K compact
$\Big\} \Rightarrow K_n$ compact. (*)

- $g_n \geqslant g_{n+1} \Rightarrow K_n \supseteq K_{n+1}$ (**)
- Fix $x \in K$, we have because $g_n(x) \longrightarrow 0$ pointwise

$\Leftrightarrow \forall \varepsilon > 0, \exists n_1 \in \mathbb{N}, \forall n \geqslant n_1, |g_n(x)| < \varepsilon \Rightarrow x \notin K_n$ for $n \geqslant n_1$
$\Rightarrow x \notin \bigcap\limits_n K_n$

for any $x$ fixed in K, $x \notin \bigcap\limits_n K_n \Rightarrow \bigcap\limits_n K_n = \phi$ (***)

* Now we consider the result from (*), (**) and (***):

$\{K_n\}$ family of compact subsets
$K_n \supseteq K_{n+1}$
$\bigcap\limits_n K_n = \phi$

We recall the corollary of theorem 2.36:
(An) family of nonempty compact subsets
$A_n \supseteq A_{n+1}$
then $\bigcap\limits_{n=1}^{\infty} A_n \neq \phi$

So we have $K_{n_0} = \phi$ for some $n_0$
and because $K_n \supset K_{n+1}$
$\Big\} \Rightarrow K_n = \phi$, $\forall n \geqslant n_0$, this is what we need to do.

*Note that the compactness is really needed here
EX $f(x) = \dfrac{1}{nx+1}$ $0 < x < 1$ , $n = 1,2,3,\dots$
then $f_n \longrightarrow 0$ monotonically in (0,1)
but $f_n \not\rightrightarrows 0$ in (0,1).

Idea of the proof:
Put $K_n = \{x \in K, g_n(x) \geqslant \varepsilon\}$, NTP $\phi$ $\forall n \geqslant n_0$
NTP $\{K_n\}$ is a family of $\left\{\begin{array}{l}\text{nested, compact}\\ \bigcap K_n = \phi\end{array}\right\}$

to show $K_n = \phi, \forall n \geqslant n_0$.

* Prove theorem 7.16 : Uniform convergence and integration

Let $\{f_n\}$ : sequence of function that $\in \mathcal{R}(\alpha)$ on $[a,b]$
($\alpha$: monotonically increasing on $[a,b]$.
$f_n \rightrightarrows f$ on $[a,b]$.

Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = \lim \int_a^b f_n\,d\alpha$

* Note that

$$f \in \mathcal{R}(\alpha) \Leftrightarrow \begin{cases} f \text{ is bounded} \\ \forall\varepsilon, \exists \text{ a partition } P_\varepsilon, |U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha)| < \varepsilon \end{cases}$$

* Now we will prove that $f$ is bounded.

We have $\{f_n\} \in \mathcal{R}(\alpha), \forall n \Rightarrow \{f_n\}$ : sequence of bounded function, $|f_n(x)| \le M_n$, $\forall x \in [a,b$

Because $f_n \rightrightarrows f \overset{\text{Uniformly Cauchy}}{\Leftrightarrow} \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \ge N_0, \forall x \in [a,b], |f_n(x) - f(x)| < \varepsilon$.

This means $\forall n \ge N_0$, $\forall x \in [a,b]$, $|f(x)| \le |f_{N_0}(x)| + \varepsilon \le M_{N_0} + \varepsilon$

Then choose $M = \max\{M_1, M_2, \dots, M_{N_0}, M_{N_0+\varepsilon}\}$, we have

$|f(x)| \le M \Rightarrow f$ is bounded.

* Now we will prove that $\forall \varepsilon > 0, \exists P_\varepsilon, |U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha)| < \varepsilon$

* We have $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon_1 > 0, \exists n_0 \in \mathbb{N}, \forall n \ge n_0, \forall x \in [a,b], |f_n(x) - f(x)| < \varepsilon_1$ (*).

• We also have $f_{n_0+1} \in \mathcal{R}(\alpha) \Leftrightarrow \forall \varepsilon_2 > 0$, $\exists$ partition $P^{f_{n_0+1}}$, $|U(P^{f_{n_0+1}}_{\varepsilon_2}, f_{n_0+1}, \alpha) - L(P^{f_{n_0+1}}_{\varepsilon_2}, f_{n_0+1}, \alpha)|$

$\Leftrightarrow \sum_{i=1}^{k} (M_{i, f_{n_0+1}} - m_{i, f_{n_0+1}}) \Delta\alpha_i < \varepsilon_2$. (1)

Then because of (*), choose $n = n_0 + 1$, we have $|f(x)| \le |f_{n_0+1}(x)| + \varepsilon_1$. (2)

Then choose $P_\varepsilon = P_\varepsilon^{f_{n_0+1}}$, we have

$$m_{i, f_{n_0+1}} - \varepsilon_1 \le m_{i,f} \le M_{i,f} \le M_{i, f_{n_0+1}} + \varepsilon_2$$

$$\Rightarrow |U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha)| = \sum_{i=1}^{k} (M_{i,f} - m_{i,f}) \Delta\alpha_i = \sum_{i=1}^{k} (M_{i, f_{n_0+1}} - m_{i, f_{n_0+1}} + 2\varepsilon_2) \Delta\alpha_i$$

$$= \sum_{i=1}^{k} (M_{i, f_{n_0+1}} - m_{i, f_{n_0+1}}) \Delta\alpha_i + \sum_{i=1}^{k} 2\varepsilon_1 \Delta\alpha_i$$

$$\underset{\text{by (1)}}{\le} \varepsilon_2 + 2\varepsilon_1 \underbrace{(\alpha(b) - \alpha(a))}_{\text{bounded}}.$$

Then $\forall \varepsilon > 0, \exists P_\varepsilon$, $|U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha)| < \varepsilon$ + $f$ is bounded $\Rightarrow f \in \mathcal{R}(\alpha)$

* Now we will prove $\int_a^b f\,d\alpha = \lim \int_a^b f_n\,d\alpha \Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \ge N, \left|\int_a^b f\,d\alpha - \int_a^b f_n\,d\right.$

because of (*), and because $f, f_n \in \mathcal{R}(\alpha)$, then $\exists n_0 \in \mathbb{N}, \forall n \ge n_0$,

$$\left|\int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha\right| = \left|\int_a^b (f - f_n)\,d\alpha\right| \le \int_a^b |f - f_n|\,d\alpha \underset{(*)}{\le} \int_a^b \varepsilon\,d\alpha = \varepsilon(\alpha(b) - \alpha(a)) \Rightarrow$$

* About Equicontinuous family (Contains Some propositions and Ascoli theorem)

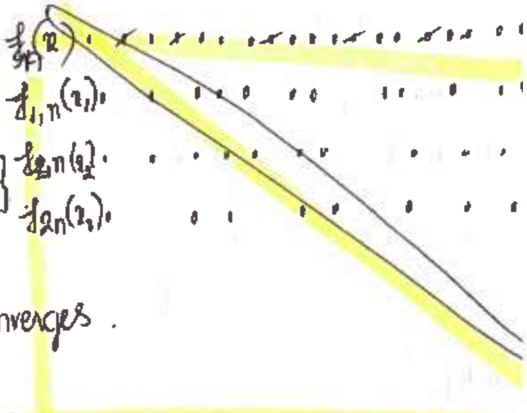
\+ Theorem 7.23.

X : countable

$f_n : X \longrightarrow \mathbb{C}$ give a $\boxed{\text{pointwise bounded}}$ sequence of functions $\Bigg\} \Rightarrow \{f_n\}$ has a subsequence that converges pointwise.

* Let $X = \{x_i\}_{i=1}^{n}$

Note that in here $x_1$ is fixed, we can consider $s_n = f_n(x_1,$



• Now consider $x_1$ : We have because $f_n(x_1)$ is bounded

$\Rightarrow \exists$ subsequence $f_{1,n}(x_1)$ converges.

• Now the sequence $\{f_{1,n}(x_2)\}$ (is a subsequence of $\{f_n(x_2)\}$) is a bounded sequence $\Rightarrow$

$\Rightarrow \exists$ a subsequence (of $f_{1,n}(x_2)$) such that $f_{2,n}(x_2)$ converges.

• Similarly, $\{f_{2,n}(x_3)\}$ is a bounded sequence

$\rightarrow \exists$ a subsequence (of $\{f_{2,n}(x_3)\}$) such that $f_{3,n}(x_3)$ converges...

... This means we have a sequence $\{f_{1,n}, f_{2,n}, f_{3,n}, \dots, f_{k,n}, \dots\}_k$ such that $\{f_{k,n}(x_i)\}$ converges for all $x_i = \overline{1, k}$

Repeating this, ~~we have~~ let $\{f_{k+1,n}\}_n$ be a subsequence of $f_{k,n}$ such that $\{f_{k+1,n}(x_k$ converges

$\rightarrow$ ~~Pick~~ In general, we have ~~spe~~ created

$\{f_{k+1,n}\}$ is a subsequence of $\{f_{k,n}\}$.

$\{f_{k,n}(x_i)\}$ converges, $\forall\, i \le k$.

Choose $S = \{f_{11}, f_{22}, f_{33}, \dots, f_{nn} \dots\}$ this is a subsequence of $f_n$

and $f_{n,n}(x_i)$ converges when $n \longrightarrow \infty$ for any $x_i$, $\overline{i = 1, \infty}$ □

eorem 7.24 See Aug 2015

compact (The role of X compact here is f cont in X compact ⇒ uniformly cont.

$_n \in C(X) = \{ f_n : X \to C,\ f_n$ is bounded, continuous in $X$

( this means, $\forall n$, $|f_n(x)| \le M_{(n)}$, $\forall x \in X$ )
$f_n(x)$ continuous function

$_n \rightrightarrows$ in $X$

$\Bigg\}$ $\{f_n\}$ equicontinuous on $X$.

We need to prove $\{f_n\}$ equicontinuous on $X$ $\boxed{\forall n \in \mathbb{N}}$

⇒ NTP $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X, |x-y| < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon$

We have $f_n \rightrightarrows$ in $X$

$\Longrightarrow$ uniformly Cauchy $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \ge N$, ~~$\forall x \in X$~~, $|f_n(x) - f_N(x)| < \varepsilon$ (1)

This is a very good trick to use when $f_n \rightrightarrows$.

We have $\{f_n\}$ is continuous in $X$ compact ⇒ $f_n$ is uniformly continuous, $\boxed{\forall n}$

⇒ $\forall \varepsilon > 0, \boxed{\exists \delta_{\varepsilon,n}}\ \forall x, y \in X, |x-y| < \delta_{\varepsilon,n}$, then $|f_n(x) - f_n(y)| < \varepsilon$ (2)

for each $n \in \mathbb{N}$ $\exists \delta_{\varepsilon,n}$
⇒ we can't choose $\delta = \min_{n \in \mathbb{N}} \{\delta_{\varepsilon,n}\}$
⇒ consider case when $n < N$
$n \ge N$

So we have $\boxed{\text{for } n < N}$

because of (2), Choose $\delta_1 = \min\{\delta_{\varepsilon,1}, \delta_{\varepsilon,2}, \dots, \delta_{\varepsilon,N-1}\}$

then $\forall \varepsilon > 0, \exists \delta_1, \forall x, y \in X, |x-y| < \delta_1$, ~~$\forall$~~ $\underline{n < N}$, $|f_n(x) - f_n(y)| < \varepsilon$ (I)

For $\boxed{n \ge N:}$

We have: $\forall \varepsilon > 0, \exists \delta_{\varepsilon,N}, \forall x, y \in X, |x-y| < \delta_{\varepsilon,N}, \forall n \ge N, |f_n(x) - f_n(y)| < \varepsilon$ (II)

$$|f_n(x) - f_n(y)| \le \underbrace{|f_n(x) - f_N(x)|}_{\substack{<\varepsilon \\ \text{because of (1), } n \ge N}} + \underbrace{|f_N(x) - f_N(y)|}_{\substack{<\varepsilon \\ \text{because of (2)} \\ \text{when } |x-y| < \delta_{\varepsilon,N}}} + \underbrace{|f_N(y) - f_n(y)|}_{\substack{<\varepsilon \\ \text{because of (1) } n \ge N}}$$

Choose $\delta = \min\{\delta_1, \delta_{\varepsilon,N}\}$

⇒ $\forall \varepsilon > 0, \exists \delta, \forall x, y \in X, |x-y| < \delta, \underline{\forall n \in \mathbb{N}}, |f_n(x) - f_n(y)| < \varepsilon$ ⇒ ☐

Note in here we don't use $f_n(x) \le M_n$ ( $f_n(x)$ is bounded for each n)

In fact this is because we need $f_n(x)$ continuous in $X$, $X$ compact $\Big\}$ ⇒ we have $f_n$ is bounded

## Theorem 7.25. (Arzela - Ascoli Theorem)

K is compact
$f_n \in C(K)$
$\{f_n\}$ pointwise bounded, equicontinuous on K

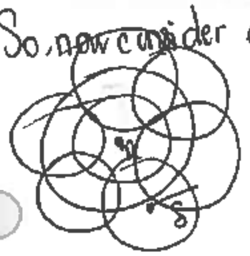
Prove that
a) $\{f_n\}$ uniformly bounded.
b) $\{f_n\}$ contains a uniformly convergent subsequence.

a) • $f_n \in C(K) = \{$bounded, continuous functions on K$)$
$f_n$ bounded $\Leftrightarrow$ for each $n$, $\exists M_n$, $|f_n(x)| \le M_n$, $\forall x \in K$. (1)
• $\{f_n\}$ pointwise bounded
$\Leftrightarrow$ for each $x \in K$, $|f_n(x)| \le \phi_x$, $\forall n \in \mathbb{N}$. (2)
• $\{f_n\}$ equicontinuous.
$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in K, |x-y| < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon, \forall n$ (3)
• K compact $\Leftrightarrow$ every open cover contains a finite subcover
then consider $\bigcup_{x \in K} B(x_0, \delta)$, $\exists x_i, i = \overline{1,\ell}$, $K \subseteq \bigcup_{i=1}^{\ell} B(x_i, \delta)$

NTP $\{f_n\}$ uniformly bou
$\Rightarrow$ NTP $\exists M$,
$|f_n(x)| \le M, \forall n, \forall x$

This is a very good trick and be used a lo when we have $\{f_n\}$ equicontinuous on K compact

*So, now consider every $x \in K$, we have $\exists B(x_i, \delta)$ for some $i \in \overline{1,\ell}$, $x \in B(x_i, \delta)$



* Also choose $M^* = \max\{M_1, M_2, \dots, M_\ell\}$
Then we have $|f_n(x_i)| \le M^*$, $\forall x_i$, $\forall$

So we have $|f_n(x) - f_n(x_i)| \le \varepsilon \Rightarrow |f_n(x)| \le \underbrace{|f_n(x_i)| + \varepsilon}_{\le M^* + \varepsilon}$ for some $i \in \overline{1,\ell}$

Let $M = M^* + \varepsilon$, we have $|f_n(x)| \le M$, $\forall n, \forall x$ □a

b) NTP $\{f_n\}$ contains a uniformly convergent subsequence.

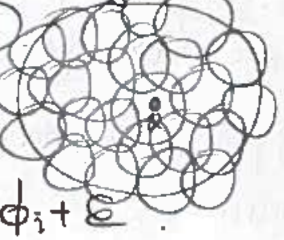
7.25 Theorem :

If $K$ is compact
$f_n \in \mathscr{E}(K)$ for $n = 1, 2, 3, \dots$ $\mathscr{E}(K)$ = set of continuous, bounded on $K$.
$\{f_n\}$ is pointwise bounded and equicontinuous on $K$, then
a) $\{f_n\}$ is uniformly bounded on $K$
b) $\{f_n\}$ contains a uniformly convergence subsequence.

What we have: • $\{f_n\}$ pointwise bounded
$\Leftrightarrow$ For fixed $p_i$, $\exists \phi(p_i)$ such that $|f_n(p_i)| \leq \phi(p_i)$, $\forall n$. ?
• $\{f_n\}$ are ~~pointwise continuous on K~~ equicontinuous on $K$
$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in K$, $d(x,y) < \delta$ then $\|f_n(x) - f_n(y)\| < \varepsilon$ $\forall n$

a) We need to prove. $\exists M > 0$, such that $|f_n(x)| \leq M$, $\forall x \in K, \forall n \in \mathbb{N}$.

Because $K$ is compact.
$\Leftrightarrow \exists q_1, q_2, \dots, q_N$ are finitely many points in $K$ such that
$K \subset W_{q_1} \cup W_{q_2} \cup \dots \cup W_{q_N}$ where $W_{q_i}$ is a neighborhood of $q_i$ with radius less than $\delta$



Then $\forall x \in K, \exists W_{q_i}$ such that
$d(x, p_i) < \delta$ and $f_n(p_i) < \phi_i$
$\Rightarrow |f_n(x) - f_n(p_i)| < \varepsilon \Rightarrow |f_n(x)| < \phi_i + \varepsilon$.

$\Rightarrow$ Choose $M = \max\{\phi_1, \phi_2, \dots, \phi_N\} + 1$.
then $\forall x \in K, \forall n$, $|f_n(x)| < M$, ∎

$f_n(p_1) < M_1$ $n = 1, 2, \dots$
$f_n(p_2) < M_2$
⋮
$f_n(p_N) < M_N$.

~~b) Need to prove $\{f_n\}$ contains a uniformly convergent subsequence.~~

From Exercise 2.25 45:
Every compact metric space $K$ has a countable base.
A base of a metric space $K$ is a collection $\{V_\alpha\}$ of open subsets of $K$ has the following properties : $\forall x \in K$
$\forall G$ open $\subset K$, $x \in G$ then $\exists V_\alpha$ such that $x \in V_\alpha \subset G$.
in other words : every open set in $X$ is the union of a subcollection of $\{V_\alpha\}$.

## Theorem 7.25b :

If K compact
$\mathscr{C}(K) = \{$set of continuous, bounded functions on K
$f_n \in \mathscr{C}(K)$.
$\{f_n\}$ is pointwise bounded and equicontinuous on K,

$\Rightarrow$ $\{f_n\}$ contains a uniformly convergent subsequence.

read about the purpose of this part again

Idea of this proof:

K compact ~~⇒~~ ⇒ let E is a contable dense subset of K.

$f_n$ is pointwise bounded on E
$\Rightarrow \exists f_{n_i}$ pointwise convergent

finitely
$\exists W x_1, x_2, \dots, x_m$ ~~and~~ E such that.
$K \subset W(x_1, \delta) \cup W(x_2, \delta) \cup \dots \cup W(x_m, \delta)$

$f_n$ equicontinuous on K ⇒ equicontinuous on

~~We have~~ Need to prove $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall i, j \geq N, \forall x \in K, |g_i(x) - g_j(x)| < \varepsilon$.

Let $\varepsilon > 0$, ~~pick $\delta > 0$~~ at the begining of this proof what is E review?

- Let E is a contable dense subset of K (exercise 2.25 shows the existence of this set).
  then $\exists$ finitely many point $x_1, x_2, \dots, x_m \in E$ such that
  $K \subset W(x_1, \delta) \cup W(x_2, \delta) \cup \dots \cup W(x_m, \delta)$ (1) where $W(x_i, \delta)$ is a neighborhood with radius $\delta$ of $x_i$.

- $\{f_n\}$ is pointwise bounded on contable set E
  $\Rightarrow$ by theorem 7.25 $\Rightarrow \exists\, g_i = f_{n_i}$ is a ~~subse~~ pointwise convergent subsequence of $\{f_n\}$ on E.

$\Rightarrow g_i$ pointwise convergent at $x_1, \dots, x_m$
$\Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall i, j \geq N, \forall s, 1 \leq s \leq m, |g_i(x_s) - g_j(x_s)| < \varepsilon$. (2)

- Decause of (1), $\forall x \in K, \exists x_s \in \{x_1, \dots, x_m\}$ such that, choose and fix $x \in W(x_s, \delta)$.
  and because $\{f_n\}$ is equicontinuous on K $\Leftrightarrow \forall \varepsilon > 0, \forall x, y \in K, d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$
  this property also be true for $\forall g_i$ so

  $\{f_n\}$ equicontinuous
  $\Rightarrow \{f_{n_i}\}$ equicontinuous.

$$|g_i(x) - g_i(x_s)| < \varepsilon \quad \forall i. \quad (3).$$

equicontinuous

max$\{N_1, \dots, N_m\}$

$(2) + (3) \Rightarrow \forall i, j \geq N, \forall x \in K,$ ~~$\forall x_s$~~

$$|g_i(x) - g_j(x)| \leq \underbrace{|g_i(x) - g_i(x_s)|}_{} + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$$

$\leq 3\varepsilon$ ◻.

$N_1$: so that this satisfy for $x_1$

$N_s$: ———— $x_s$

Theorem 7.29

Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions. Then $\mathcal{B}$ is a uniformly closed algebra.

We have:

- $\mathcal{B} = \{ f \mid \exists f_n \in \mathcal{A}, f_n \rightrightarrows f \}$.
- $\mathcal{A}$: algebra of bounded function
  $\underbrace{|f_n(x)| \leq M_n}$, $\forall f_n \in \mathcal{A}$

note that from E7.1/165 Rudin
because $f_n \rightrightarrows$ | then $\{f_n\}$ is uniformly bounded $\exists M, |f_n(x)| \leq M, \forall x, \forall n$.

We need to prove
$\begin{cases} \mathcal{B} \text{ is an algebra} \\ \mathcal{B} \text{ is uniformly closed} \end{cases}$.

$\Rightarrow$ NTP
$$\begin{cases} \forall f, g \in \mathcal{B} \quad f+g \in \mathcal{B} \\ \qquad\qquad\quad fg \in \mathcal{B} \\ \qquad\qquad\quad \alpha f \in \mathcal{B} \quad , \forall \alpha \in \mathbb{C} \\ \forall \{h_n\} \in \mathcal{B}, h_n \rightrightarrows h, \text{ then } h \in \mathcal{B} \end{cases}$$ (*)

* We now prove (*): $\mathcal{B}$ is an algebra

Let $f, g \in \mathcal{B}$, then because $\mathcal{B}$ is the uniform closure of $\mathcal{A}$, we have

- $\exists (f_n) \in \mathcal{A}$, $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$
  and $\{f_n\}$ sequence of bounded function $\Rightarrow |f_n(x)| \leq M_n$
- $\exists (g_n) \in \mathcal{A}$, $g_n \rightrightarrows g \underset{\substack{\uparrow \\ \text{because}}}{\Leftrightarrow} \forall \varepsilon > 0, \exists n_1 \in \mathbb{N}, \forall n \geq n_1, \forall x \in E, |g_n(x) - g(x)| < \varepsilon$

Then for $n \geq \max\{n_0, n_1\}$, we have

$$|(f_n(x) + g_n(x)) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < 2\varepsilon .$$

then we have $\exists (f_n + g_n) \in \mathcal{A}$, $f_n + g_n \rightrightarrows f + g \rightarrow f + g \in \mathcal{B}$

+ Consider $fg$: we have because of (1) and (2)

$$|f_n g_n(x) - fg(x)| = |f_n(x) g_n(x) - fg$$
$$= \underbrace{|f_n(x)|}_{\leq M} \underbrace{|g_n(x) - g(x)|}_{\leq \varepsilon} + \underbrace{|g(x)|}_{\leq N} \underbrace{|f_n(x) - g(x)|}_{\leq \varepsilon}$$

We need the boundness assumption here

Also use 7.1 Rudin result

+ Consider $\alpha f$, We have

$$|\alpha f_n(x) - \alpha f(x)| = |\alpha (f_n(x) - f(x)| = \alpha \varepsilon \quad \Rightarrow \quad \alpha f \in \mathcal{B} .$$

Hence, $\mathcal{B}$ is an algebra.

---

Result of E7.1 Rudin:
Every uniformly convergent sequence of bounded function is uniformly bounded.

Let $\{f_n\}$, with $|f_n(x)| \leq M_n$, $f_n \rightrightarrows f$ $\Big\}$ then $\exists M, |f_n(x)| \leq M, \forall n, \forall x$ | Also $f$ is bound ow we prove (**), $\mathcal{B}$ is ~~an~~ uniformly closed

$\left.\begin{array}{l}\text{Let } (h_n) \in \mathcal{B} \\ h_n \rightrightarrows h\end{array}\right)$ NTP $h \in \mathcal{B}$

te: We have ~~$\forall f \in \mathcal{B}, \exists (f_n) \in \mathcal{A}$ such that $f_n \rightrightarrows f$~~

~~then bec~~ if $\{h_n\}$ is a sequence in $\mathcal{B}$ converging uniformly to $h$ in $\mathcal{B}$.

n we find a $g_n \in \mathcal{A}$ for every n such that for all $x \in \mathcal{E}$, we have

$$|g_n(x) - h_n(x)| < \frac{1}{n}.$$

Is this still true in case the convergence is pointwise ?

$$\begin{array}{cccccccc} g_{11} & g_{12} & g_{13} & \cdots & g_{1n} & \cdots & h_1 \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2n} & \cdots & h_2 \\ & & \ddots & & & & \\ g_{m1} & g_{m2} & g_{m3} & \cdots & g_{mn} & \cdots & h_m \end{array}$$

e have $\{h_n\}$ is a sequence in $\mathcal{B}$, $h_n \rightrightarrows h$

$$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in E, |h_n(x) - h(x)| < \varepsilon \quad (1)$$

$h_n\}$ is a sequence in $\mathcal{B}$

) is the closure of $\mathcal{A}$, then we can find $g_n \in \mathcal{A}$ such that $\forall x \in E$,

$$|g_n(x) - h_n(x)| < \frac{1}{n}. \quad (2)$$

n from (1) and (2) choose $N \geq n_0$ and $N$ large enough such that $\frac{1}{N} < \varepsilon$.

en, $\forall n \geq N$, ~~we have~~, $\forall x \in E$, we

$$|g_n(x) - h(x)| < \underbrace{|g_n(x) - h_n(x)|}_{< \frac{1}{n} < \frac{1}{N} < \varepsilon} + \underbrace{|h_n(x) - h(x)|}_{< \varepsilon} \leq 2\varepsilon$$

this means $\exists (g_n)$ in $\mathcal{A}$, $g_n \rightrightarrows h$, this means $h \in \mathcal{B}$ □.

y 2: In Rudin's book:

e have the set of bounded function on E is a metric space, $\mathcal{C}(X) = \left\{\begin{array}{l}\text{the set of continuous} \\ \text{bounded in } X\end{array}\right\}$

$(\mathcal{C}(X), d(f,g) = \|f - g\|)$ is a metric space.

en we have $\mathcal{B} = \bar{\mathcal{A}} \Rightarrow \mathcal{B}$ is closed

$\Rightarrow \mathcal{B}$ is uniformly closed.

**Proposition (Theorems 9.2 and 9.3 in Rudin):** Suppose that $X$ is a vector space.

(i) If $X$ is spanned by $d$ vectors, then $\dim X \leq d$.
(ii) $\dim X = d$ if and only if $X$ has a basis of $d$ vectors (and so every basis has $d$ vectors).
(iii) In particular, $\dim \mathbb{R}^n = n$.
(iv) If $Y \subset X$ is a vector space and $\dim X = d$, then $\dim Y \leq d$.
(v) If $\dim X = d$ and a set $T$ of $d$ vectors spans $X$, then $T$ is linearly independent.
(vi) If $\dim X = d$ and a set $T$ of $m$ vectors is linearly independent, then there is a set $S$ of $d - m$ vectors such that $T \cup S$ is a basis of $X$.

*Proof.* Let us start with (i). Suppose that $S = \{x_1, \ldots, x_d\}$ span $X$. Now suppose that $T = \{y_1, \ldots, y_m\}$ is a set of linearly independent vectors of $X$. We wish to show that $m \leq d$. Write

$$y_1 = \sum_{k=1}^{d} \alpha_1^k x_k,$$

which we can do as $S$ spans $X$. One of the $\alpha_1^k$ is nonzero (otherwise $y_1$ would be zero), so suppose without loss of generality that this is $\alpha_1^1$. Then we can solve

$$x_1 = \frac{1}{\alpha_1^1} y_1 - \sum_{k=2}^{d} \frac{\alpha_1^k}{\alpha_1^1} x_k.$$

In particular $\{y_1, x_2, \ldots, x_d\}$ span $X$, since $x_1$ can be obtained from $\{y_1, x_2, \ldots, x_d\}$. Next,

$$y_2 = \alpha_2^1 y_1 + \sum_{k=2}^{d} \alpha_2^k x_k,$$

As $T$ is linearly independent, we must have that one of the $\alpha_2^k$ for $k \geq 2$ must be nonzero. Without loss of generality suppose that this is $\alpha_2^2$. Proceed to solve for

$$x_2 = \frac{1}{\alpha_2^2} y_2 - \frac{\alpha_2^1}{\alpha_2^2} y_1 - \sum_{k=3}^{d} \frac{\alpha_2^k}{\alpha_2^2} x_k.$$

In particular $\{y_1, y_2, x_3, \ldots, x_d\}$ spans $X$. The astute reader will think back to linear algebra and notice that we are row-reducing a matrix.

We continue this procedure. Either $m < d$ and we are done. So suppose that $m \geq d$. After $d$ steps we obtain that $\{y_1, y_2, \ldots, y_d\}$ spans $X$. So any other vector $v$ in $X$ is a linear combination of $\{y_1, y_2, \ldots, y_d\}$, and hence cannot be in $T$ as $T$ is linearly independent. So $m = d$.

Let us look at (ii). First notice that if we have a set $T$ of $k$ linearly independent vectors that do not span $X$, then we can always choose a vector $v \in X \setminus \operatorname{span}(T)$. The set $T \cup \{v\}$ is linearly independent (exercise). If $\dim X = d$, then there must exist some linearly independent set of $d$ vectors $T$, and it must span $X$, otherwise we could choose a larger set of linearly independent vectors. So we have a basis of $d$ vectors. On the other hand if we have a basis of $d$ vectors, it is linearly independent and spans $X$. By (i) we know there is no set of $d + 1$ linearly independent vectors, so dimension must be $d$.

For (iii) notice that $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

To see (iv), suppose that $Y$ is a vector space and $Y \subset X$, where $\dim X = d$. As $X$ cannot contain $d + 1$ linearly independent vectors, neither can $Y$.

For (v) suppose that $T$ is a set of $m$ vectors that is linearly dependent and spans $X$. Then one of the vectors is a linear combination of the others. Therefore if we remove it from $T$ we obtain a set of $m - 1$ vectors that still span $X$ and hence $\dim X \leq m - 1$.

For (vi) suppose that $T = \{x_1, \ldots, x_m\}$ is a linearly independent set. We follow the procedure above in the proof of (ii) to keep adding vectors while keeping the set linearly independent. As the dimension is $d$ we can add a vector exactly $d - m$ times. ☐

**Definition:** A mapping $A\colon X \to Y$ of vector spaces $X$ and $Y$ is said to be *linear* (or a *linear transformation*) if for every $a \in \mathbb{R}$ and $x, y \in X$ we have

$$A(ax) = aA(x) \qquad A(x+y) = A(x) + A(y).$$

We will usually just write $Ax$ instead of $A(x)$ if $A$ is linear.

If $A$ is one-to-one an onto then we say $A$ is *invertible* and we define $A^{-1}$ as the inverse.

If $A\colon X \to X$ is linear then we say $A$ is a *linear operator* on $X$.

We will write $L(X,Y)$ for the set of all linear transformations from $X$ to $Y$, and just $L(X)$ for the set of linear operators on $X$. If $a, b \in \mathbb{R}$ and $A, B \in L(X,Y)$ then define the transformation $aA + bB$

$$(aA+bB)(x) = aAx + bBx.$$

It is not hard to see that $aA + bB$ is linear. $(aA+bB \in L(X,Y)$

If $A \in L(Y,Z)$ and $B \in L(X,Y)$, then define the transformation $AB$ as $X \xrightarrow{B} Y \xrightarrow{A} Z$

$$ABx = A(Bx).$$

It is trivial to see that $AB \in L(X,Z)$.

Finally denote by $I \in L(X)$ the *identity*, that is the linear operator such that $Ix = x$ for all $x$.

Note that it is obvious that $A0 = 0$.

**Proposition:** If $A\colon X \to Y$ is invertible, then $A^{-1}$ is linear.

*Proof.* Let $a \in \mathbb{R}$ and $y \in Y$. As $A$ is onto, then there is an $x$ such that $y = Ax$, and further as it is also one-to-one $A^{-1}(Az) = z$ for all $z \in X$. So

$$A^{-1}(ay) = A^{-1}(aAx) = A^{-1}(A(ax)) = ax = aA^{-1}(y).$$

Similarly let $y_1, y_2 \in Y$, and $x_1, x_2 \in X$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$, then

$$A^{-1}(y_1 + y_2) = A^{-1}(Ax_1 + Ax_2) = A^{-1}(A(x_1 + x_2)) = x_1 + x_2 = A^{-1}(y_1) + A^{-1}(y_2).$$

$\square$

**Proposition:** If $A\colon X \to Y$ is linear then it is completely determined by its values on a basis of $X$. Furthermore, if $B$ is a basis, then any function $\tilde{A}\colon B \to Y$ extends to a linear function on $X$.

*Proof.* For infinite dimensional spaces, the proof is essentially the same, but a little trickier to write, so let's stick with finitely many dimensions. Let $\{x_1, \ldots, x_n\}$ be a basis and suppose that $A(x_j) = y_j$. Then every $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} b^j x_j$$

for some numbers $b^1, \ldots, b^n$. Then by linearity

$$Ax = A\sum_{j=1}^{n} b^j x_j = \sum_{j=1}^{n} b^j Ax_j = \sum_{j=1}^{n} b^j y_j.$$

The "furthermore" follows by defining the extension $Ax = \sum_{j=1}^{n} b^j y_j$, and noting that this is well defined by uniqueness of the representation of $x$. $\square$

**Theorem 9.5:** If $X$ is a finite dimensional vector space and $A\colon X \to X$ is linear, then $A$ is one-to-one if and only if it is onto.

## + Distance. norms.

+ **Def**: X: vector space,

$$\|\cdot\| \text{ is a norm } \iff \begin{cases} \|x\| \geq 0, \forall x \in X & \|x\|=0 \iff x=0 \\ \|cx\| = |c|\,\|x\|, \forall c\in\mathbb{R},\ x\in X \\ \|x+y\| \le \|x\|+\|y\|, \forall x,y\in X \end{cases}$$

+ **Def**: Euclidean norm:

Let $(x_1,\dots,x_n)\in\mathbb{R}^n$. Then

Euclidean norm $\|x\| = \sqrt{(x_1)^2+(x_2)^2+\dots+(x_n)^2}$

Standard metric on $\mathbb{R}^n$: $d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}$, $(X,d)$: metric space

+ **Def**: Norm of operator. (note $L(X,Y)$: vector space).

Let $A\in L(X,Y)$, the operator norm:

$$\|A\| = \sup_{x\in X}\{\|Ax\|, x\in X, \|x\|=1\} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Ax\|}{\|x\|}$$

- $\underset{\substack{\uparrow\\ \text{norm in } Y}}{\|Ax\|} \le \underset{\substack{\uparrow\\ \text{operator norm}}}{\|A\|}\ \underset{\nwarrow \text{ norm in } X}{\|x\|}$

$\|Ax\| \le \lambda\|x\|, \forall x\in\mathbb{R}^n$

$\implies \|A\| \le \lambda$

- $\|A\| = 0 \iff A = 0_{L(X,Y)} \iff Ax = 0, \forall x\in X$
- ⊕ For $\dim X < +\infty \Rightarrow \|A\| < +\infty$
  - ⊕ For $\dim X \nleq +\infty \Rightarrow \|A\| \not< +\infty$, $\forall A\in L(X,Y)$

<u>Example</u>. Let $X = C([0,1]) = \{f: [0,1]\to\mathbb{R}, f \text{ continuous}\}$.

Let $f(x) = \sin(nx)$ then $\|f\| = \sup_{x\in X}\{\sin(nx), x\in X, \|x\|=1\} = 1$.

$f'(x) = n\sin(nx)$ (also a linear operator)

$\|f'\| = \sup\{n\sin(nx), x\in X, \|x\|=1\} = +\infty$

+ <u>Proof $\|Ax\| \le \|A\|\,\|x\|^{(*)}, \forall x\in\mathbb{R}^n$</u>

- when $x=0 \Rightarrow (*)$ is trivial
- when $\|x\|\neq 0$, we need to prove $\frac{\|Ax\|}{\|x\|} \le \|A\|$

⊕ Put $\boxed{u = \frac{x}{\|x\|}}$, then we have $\|Au\| = \left\|A\frac{x}{\|x\|}\right\| = \left\|\frac{1}{\|x\|}A(x)\right\| = \frac{\|Ax\|}{\|x\|}$

⊕ We have $u = \frac{x}{\|x\|} \Rightarrow \|u\| \le 1$

then $\frac{\|Ax\|}{\|x\|} = \|Au\| \textcircled{$\le$} \sup_{\|u\|\le 1}\|Au\| = \|A\|$ □

Theorem 9.7: In $L(\mathbb{R}^n,\mathbb{R}^m)$,

If $A \in L(\mathbb{R}^n,\mathbb{R}^m)$, then $\begin{cases} \|A\| < +\infty \\ A \text{ is uniformly continuous (Lipchitz with constant } \|A\|) \end{cases}$

$L(\mathbb{R}^n,\mathbb{R}^m)$ is a metric space with $d(X,Y) = \|X - Y\|$

If $A, B \in L(\mathbb{R}^n,\mathbb{R}^m)$, then $\begin{cases} \|A+B\| \leq \|A\| + \|B\| \\ \|cA\| = |c|\,\|A\| \end{cases}$

$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{B} \mathbb{R}^k$

$\left.\begin{matrix} A \in L(\mathbb{R}^n,\mathbb{R}^m) \\ B \in L(\mathbb{R}^m,\mathbb{R}^k) \end{matrix}\right\} \Rightarrow \|BA\| \leq \|B\|\,\|A\|$

**Need to note when consider linear transformation on $\mathbb{R}^n \to \mathbb{R}^m$**

$x \in \mathbb{R}^n$

A is defined by its action on a basis

$\Rightarrow$ write $x = \sum_{i=1}^{n} c_i e_i$

* Note $\|x\| = 1 \Rightarrow |c_i| \leq 1, \forall i$

Prove $\|A\| < +\infty$, we want to prove $\sup_{\|x\|=1} \|Ax\| < +\infty$

$\in L(\mathbb{R}^n,\mathbb{R}^m)$ defined by its value on a basis, we consider standard basis

$= \sum_{i=1}^{n} c_i e_i$, then $\|Ax\| = \|A(\sum_{i=1}^{n} c_i e_i)\| = \|\sum_{i=1}^{n} c_i A(e_i)\| \leq |c_i| \sum_{i=1}^{n} \|A(e_i)\|$

te $\|x\| = 1 \Rightarrow c_i \leq 1, \forall i = \overline{1,n}$

$$\Rightarrow \|Ax\| \leq \underbrace{\sum_{i=1}^{n} \|A(e_i)\|}_{\text{does not depends on } x} < +\infty \Rightarrow \square$$

(i7)

rove that A is uniformly continuous (Lipchitz with constant $\|A\|$)

e want to prove $\|A(x-y)\| \leq \|A\|\,\|x-y\|$

e have ~~$\|A(x-y)\|$~~ $\|Ax\| \leq \|A\|\,\|x\|$

$$\Rightarrow \|A(x-y)\| \leq \underbrace{\|A\|}_{\text{from above } \|A\| < +\infty} \|x-y\| \Rightarrow \square$$

Prove $\|A+B\| \leq \|A\| + \|B\|$ We use property that if $\|A+B(x)\| \leq \overset{\|A\|+\|B\|}{\overset{\|}{\lambda}} \|x\|, \forall x$, then $\|A+B\| \leq \lambda$

e have $\|(A+B)(x)\|_{\mathbb{R}^m} = \|A(x) + B(x)\|_{\mathbb{R}^m} \leq \|Ax\|_{\mathbb{R}^m} + \|Bx\|_{\mathbb{R}^m} \leq \|A\|\,\|x\| + \|B\|\,\|x\| = (\|A\| + \|B\|)\|x\|$

$\uparrow$ we have this for any linear transformation.

$$\Rightarrow \|A+B\| \leq \|A\| + \|B\|$$

ove $\|cA\| \leq |c|\,\|A\|$

$\left.\begin{matrix} \|(cA)x\| = \|c(Ax)\| = |c|\,\|Ax\| \leq |c|\,\|A\|\,\|x\|, \forall x \Rightarrow \|cA\| \leq |c|\,\|A\| \\ |c|\,\|A(x)\| = \|c(A(x))\| = \|cA(x)\| \leq \|cA\|\,\|x\| \Rightarrow |c|\,\|A\| \leq \|cA\| \end{matrix}\right\} \Rightarrow \square$

Prove $\|BA\| \leq \|B\|\,\|A\|$, we want to prove $\|BA(x)\| \leq \|B\|\,\|A\|\,\|x\|, \forall x \in X$

We have $\forall x \in X$, $\|BA(x)\| \leq \|B\|\,\|Ax\| \leq \|B\|\,\|A\|\,\|x\| \Rightarrow \square$.

---

$(L(\mathbb{R}^n,\mathbb{R}^m), d)$ with $d(A,B) = \|A - B\|$ is a metric space

→ We can talk about open/closed/continuity/convergence ... in $L(\mathbb{R}^n,\mathbb{R}^m), d)$ (see Th 9.8

* **Theorem 9.8** (Utilizer the concept of open set in $L(\mathbb{R}^n)$ and continuity)

Consider in $L(\mathbb{R}^n)$

Consider $\Omega \subset L(\mathbb{R}^n)$, $\Omega$ = the set of invertible linear operators $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

$= \{A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $A$ is linear operators, $A$ is invertible$\}$ $(\det A \neq 0)$

a) $\left.\begin{array}{l} A \in \Omega \\ B \in L(\mathbb{R}^n) \\ \|A-B\| < \frac{1}{\|A^{-1}\|} \end{array}\right\} \rightarrow B \in \Omega$ (means $B$ is invertible)

b) $\Omega$ is an open subset of $L(\mathbb{R}^n)$

$\Leftrightarrow \forall A \in \Omega$, $\exists \left(\lambda = \frac{1}{\|A^{-1}\|}\right)$, $N_\lambda(A) \subseteq \Omega$

$N_\lambda(A) = \{B \in L(\mathbb{R}^n), \|B-A\| \leq \frac{1}{\|A^{-1}\|}\} \subseteq \Omega$

b) ii) The map $f : \Omega \longrightarrow \Omega$, is continuous on $\Omega$

$A \longmapsto f(A) = A^{-1}$

a) We want to prove $B \in \Omega \Leftrightarrow$ NTP $B$ is surjection

because $B \in L(\mathbb{R}^n)$, it suffices to prove $B$ is a injection.

NTP $Bx \neq 0$ if $x \neq 0$

NTP $\|Bx\| \geq k\|x\|$, $\forall x \in X$ (for $k > 0$)

- Put $\beta = \|A-B\|$ $\lambda = \frac{1}{\|A^{-1}\|}$, we have $\beta < \lambda$ (1).
- Then $\forall x \in \mathbb{R}^n$, we have

$$\lambda\|x\| \overset{\circ}{=} \lambda\|A^{-1}Ax\| \leq \lambda\|A^{-1}\|\,\|Ax\| = \|Ax\| \overset{\circ}{=} \|(A-B)x + Bx\|$$

$$\overset{\circ}{\leq} \|(A-B)x\| + \|Bx\|$$

$$\leq \|(A-B)\|\|x\| + \|Bx\|$$

$$\Rightarrow \|Bx\| \geq \underbrace{(\lambda-\beta)}_{>0\text{ by }(1)}\|x\| \quad (2) \Rightarrow B \text{ is a injection} \quad \square\text{a)}.$$

b) from a $\Rightarrow$ $\Omega$ is an open subset of $L(\mathbb{R}^n)$.

b) ii) Now prove $f : \Omega \longrightarrow \Omega$ is continuous function on $\Omega$.

$A \longmapsto f(A) = A^{-1}$

We want to prove if $\|B-A\| \longleftrightarrow 0$ then $\|B^{-1} - A^{-1}\| \longrightarrow 0$

(when $\|B-A\| \longrightarrow 0$ means $\beta \longrightarrow 0$ | we want to prove $\|B^{-1} - A^{-1}\| \xrightarrow{\beta \to 0} 0$

- Replacing $x$ by $B^{-1}(y)$ in (2), we have

$$(\lambda-\beta)\|B^{-1}(y)\| \leq \|B(B^{-1}(y))\| = \|y\|, \forall y \in \mathbb{R}^n$$

$$\Rightarrow \frac{\|B^{-1}(y)\|}{\|y\|} \leq \frac{1}{(\lambda-\beta)}, \forall y \in \mathbb{R}^n$$

- Put $u = \frac{y}{\|y\|}$, then $\|B^{-1}(u)\| \leq \frac{1}{(\lambda-\beta)}$, $\forall \|u\| \leq 1$ $\Rightarrow$ $\|B^{-1}\| \leq \frac{1}{(\lambda-\beta)}$

* We have $(B^{-1} - A^{-1}) = A^{-1}(A-B)B^{-1}$ $\quad (A^{-1}(A-B)B^{-1} = A^{-1}(AB^{-1} - I) = B^{-1} - A^{-1})$

$$\Rightarrow \|B^{-1} - A^{-1}\| = \|A^{-1}(A-B)B^{-1}\| \leq \underbrace{\|B^{-1}\|}_{<\frac{1}{\lambda-\beta}}\underbrace{\|(A-B)\|}_{\beta}\underbrace{\|A^{-1}\|}_{\frac{1}{\lambda}} = \frac{\beta}{\lambda(\lambda-\beta)} \xrightarrow{\beta \to 0} 0 \quad \square$$

## Matices (Because it's convenient way to represent finite dimensional operators.)

* Consider $A \in \mathcal{L}(X, Y)$

$X \subseteq \mathbb{R}^n$, has basis $\{x_1, \dots, x_n\}$

$Y \subseteq \mathbb{R}^m$, has basis $\{y_1, \dots, y_m\}$.

Then linear operator $A$ is defined on its values on basis

$$\underset{\uparrow}{Ax_j} = \sum_{i=1}^{m} a_{ij} y_i \quad (1)$$

j^th column vector of [A]

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

$\boxed{Ax_j}$

$$Ax_j = [A] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j =$$

---

* If $x = \sum_{j=1}^{n} c_j x_j$ (2)

then $Ax = A\left(\sum_{j=1}^{n} c_j x_j\right) = \sum_{j=1}^{n} c_j A(x_j) = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} a_{ij} y_i\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_j a_{ij}\right) y_i$

⇒ give rise to the familiar rule for matrix multiplication

⇒ $[A]$ is a matrix of operator $A$ (associate with basis $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$.

---

* Then we have if

$$\begin{tikzcd} X \arrow[r, "A"] & Y \arrow[r, "B"] & Z \\ \text{basis } \{x_1, \dots, x_n\} \arrow[u] & \text{basis } \{y_1, \dots, y_m\} \arrow[u] & \text{basis } \{z_1, \dots, z_p\} \arrow[u] \end{tikzcd}$$

$Ax_j = \sum_{i=1}^{m} a_{ij} y_i$, $j = \overline{1, n}$ ([A] matrix for

$By_i = \sum_{k=1}^{p} b_{ki} z_k$, $i = \overline{1, m}$ ([B] matrix for B)

if $\left.\begin{matrix} A \in \mathcal{L}(X, Y) \\ B \in \mathcal{L}(Y, Z) \end{matrix}\right\} \Rightarrow BA \in \mathcal{L}(X, Z)$ with $(BA)(x_j) = \sum_{k=1}^{p} c_{kj} z_k$, $j = \overline{1, n}$ $\Big($ $[C] = [B][A$

with $c_{kj} = \sum_{i=1}^{n} b_{ki} a_{ij}$

---

* Remark: (If we consider $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ standard basis of $\mathbb{R}^m$ and $\mathbb{R}^m$

- $\|Ax\|^2 \underset{(2)}{=} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_j a_{ij}\right)^2 \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_j\right)^2 \left(\sum_{j=1}^{n} (a_{ij})^2\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (a_{ij})^2\right) \|x\|^2$

then $\|A\| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij})^2}$

this means if $[B] - [A] \longrightarrow 0$ then $\|B\| - \|A\| \longrightarrow 0$

## he derivative

onsider the case $f: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$U$ open in $\mathbb{R}^1$, $U = (a,b)$ $\quad f: U \longrightarrow \mathbb{R}^1$ is differentiable at $x \in U$ iff (def)

$$\exists \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = a \qquad a = f'(x)$$

$$\exists\, a = f'(x) \text{ such that } \lim \frac{|f(x+h) - f(x) - ah|}{|h|} = \lim \left| \frac{f(x+h) - f(x)}{h} - a \right| = 0$$

Note $a = f'(x) \in L(\mathbb{R}^1, \mathbb{R}^1)$

**Def** (9.11)

et $U$ open, $U \subseteq \mathbb{R}^n$

$: U \text{ open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

en $f$ is differentiable at $x \in U \Leftrightarrow A \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t $\displaystyle \lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{\| f(x+h) - f(x) - Ah \|}{\| h \|} = 0$

norm in $\mathbb{R}^m$

norm in $\mathbb{R}^n$

nd $A = Df(x) = f'(x)$ is the derivative of $f$ at $x$

$f$ is differentiable on $U \overset{\text{def}}{\Leftrightarrow} f$ is differentiable at all $x \in U$

ote: • $h \in \mathbb{R}^n$

• If $h$ is small enough, then $(x+h) \in U$ (because $U$ is open).

$\Rightarrow f(x+h)$ is well defined

**12**: Uniqueness of derivative.

t $U \subseteq \mathbb{R}^n$, $f: U \longrightarrow \mathbb{R}^m$

Suppose that $x \in U$, and $\exists\, A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - Ah \|}{\| h \|} = 0$$

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - Bh \|}{\| h \|} = 0$$

then $A = B$

(this means derivative is unique

**Example**: $f(x) = Ax$ for a linear mapping $A$ then $f'(x) = A$ because

$$\frac{\| f(x+h) - f(x) - Ah \|}{\| h \|} = \frac{\| A(x+h) - A(x) - Ah \|}{\| h \|} = \frac{0}{\| h \|} = 0$$

Proposition: $f: U \text{ open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$f$ is differentiable at $x_0 \in U$ $\Big\} \Longrightarrow f$ is continuous at $x_0$

Prove because $f$ is differentiable at $x_0$

$\lambda(h) = f(x_0+h) - f(x_0) - f'(x_0)h$ $\quad$ where $\dfrac{\| \lambda(h) \|}{\| h \|} \xrightarrow{h \to 0} 0$ means $(\lambda(h) \xrightarrow{h \to 0} 0$

$\longrightarrow f'(x_0)h$ continuous (because of linear $\mathbb{R}^n \to \mathbb{R}^m$) Then $f(x_0+h) \xrightarrow{h \to 0} f(x_0) \Rightarrow f$ continuous

9.15) Theorem (Chain rule)

$U \subseteq \mathbb{R}^n$ open, $f: U \longrightarrow \mathbb{R}^m$ differentiable at $x_0 \in U$

$V \subseteq \mathbb{R}^m$ open, $f(U) \subseteq V$, $g: V \longrightarrow \mathbb{R}^\ell$, $g$ is differentiable at $f(x_0)$

Then $F(x) = g(f(x))$ is differentiable at $x_0$,

and $F'(x_0) = \underbrace{g'(f(x_0))}\,\underbrace{f'(x_0)}$.

$$\begin{tikzcd} U \subset \mathbb{R}^n \arrow[r, "f"'] & V \subset \mathbb{R}^m \arrow[r, "g"'] & \subset \mathbb{R}^\ell \end{tikzcd}$$

* Proof: Put $D = g'(f(x_0))$ $\quad A = f(x_0)$

We want to prove $F'(x_0) = BA \iff$ NTP $\displaystyle\lim_{h \to 0} \frac{\|F(x_0+h) - F(x_0) - BAh\|}{\|h\|} = 0$

$$\frac{\|F(x_0+h) - F(x_0) - BAh\|}{\|h\|} = \frac{\|g(f(x_0+h)) - g(f(x_0)) - BAh\|}{\|h\|}$$

Put $y_0 = f(x_0)$ $\quad k = f(x_0+h) - f(x_0)$

then $r(h) = f(x_0+h) - f(x_0) - f'(x_0)\cdot h = k - A\cdot h$.

- Then

$$\frac{\|F(x_0+h) - F(x_0) - BAh\|}{\|h\|} = \frac{\|g(y_0+k) - g(y_0) - B(k - r(h))\|}{\|h\|}$$

$$\leq \frac{\|g(y_0+k) - g(y_0) - Dk\|}{\|h\|} + \|D\| \frac{\|r(h)\|}{\|h\|}$$

$$= \underbrace{\frac{\|g(y_0+k) - g(y_0) - Dk\|}{\|k\|}}_{\text{this term } \xrightarrow{k \to 0} 0 \text{ because } g \text{ differentiable at } y_0} \frac{\|f(x_0+h) - f(x_0)\|}{\|h\|} + \underbrace{\|B\|}_{\text{constant}} \underbrace{\frac{\|r(h)\|}{\|h\|}}_{\to 0 \text{ because } f \text{ differentiable at}}$$

$f$ differentiable at $x_0 \Rightarrow k \xrightarrow{h \to 0} 0$

Besides,

$$\frac{\|f(x_0+h) - f(x_0)\|}{\|h\|} \leq \frac{\|f(x_0+h) - f(x_0) - Ah\|}{\|h\|} + \frac{\|Ah\|}{\|h\|} \leq \frac{\|f(x_0+h) - f(x_0) - Ah\|}{\|h\|} + \|A\| < +$$

Then $\displaystyle\lim \frac{\|F(x_0+h) - F(x_0) - BAh\|}{\|h\|} = 0 \Rightarrow \square$.

## Partial derivative / (Total) derivative

$\{e_1, \dots, e_n\}$ standard basis of $\mathbb{R}^n$
$\{u_1, \dots, u_m\}$ standard basis of $\mathbb{R}^m$

$x = (x_1, \dots, x_n)^t = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$e_j = (0, \dots, \underset{\overset{\uparrow}{j}}{1}, \dots 0)^T$

$\subseteq \mathbb{R}^n$ a open set, $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

; the following limit exists, we write

$$_j f = \frac{\partial f}{\partial x_j}(x) = \lim_{h \to 0} \frac{f(x_1, \dots, x_j + h, x_{j+1}, \dots, x_n) - f(x)}{h} = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

$_j f = \frac{\partial f}{\partial x_j}(x)$ : the partial derivative of $f$ wrt $x_j$

---

When $\underset{\overset{\uparrow}{\text{open}}}{U} \subseteq \mathbb{R}^n$, $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

$(x_1, \dots, x_n) \longmapsto (f_1(x), f_2(x), \dots, f_m(x))$

hen $D_j f_i(x) = \frac{\partial f_i}{\partial x_j}(x)$ derivative of $f_i$ wrt to $x_j$ $= \lim_{h \to 0} \frac{f_i(x + he_j) - f_i(x)}{h}$

$f_i(x) = f(x)\, u_i$

$u = \{u_1, \dots, u_m\}$: standard basis of $\mathbb{R}$

$f(x) = \sum_{i=1}^{m} f_i(x)\, u_i$

---

17 Theorem (Compute total derivative from partial derivative)

et $U$ open $\subseteq \mathbb{R}^n$, $f : U \longrightarrow \mathbb{R}^m$, $f$ is differentiable at $x_0 \in U$

en: All partial derivative exist at $x_0$, and

$$f'(x_0)]^{e}_{B} = \begin{bmatrix} D_1 f_1(x_0) & D_2 f_1(x_0) & \dots & D_n f_1(x_0) \\ D_1 f_2(x_0) & D_2 f_2(x) & & D_n f_2(x_0) \\ & & & \\ D_1 f_m(x_0) & D_2 f_m(x_0) & \dots & D_n f_m(x_0) \end{bmatrix}_{m \times n} \; f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \frac{\partial f_m}{\partial x_3} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n} = \begin{bmatrix} \nabla f_1 \\ - \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix}$$

$$'(x) e_j = \sum_{i=1}^{m} (D_j f_i)(x)\, u_i, \quad 1 \le j \le n$$

the $j^{th}$ column of $[f'(x_0)]^{e}_{B}$

If $h$ is any vector in $\mathbb{R}^n$

$$h = \sum_{j=1}^{n} h_j e_j \qquad f'(x) h = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} (D_j f_i)(x)\, h_j \right) u_i$$

---

Corollary:

Let $f : U$ open $\subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f$ is differentiable at $x$. Then

$f' : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function $\Longleftrightarrow$ all $\frac{\partial f_i}{\partial x_j}$ are continuous fnll

$x \longmapsto f'(x)$.

---

Def Gradient

$f : U$ open $\subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$,

$(x_1, \dots, x_n) \longrightarrow f(x_1, \dots, x_n)$

Then Gradient of $f$ (is total derivative of $f$)

$$\nabla f_x = \sum \frac{\partial f}{\partial x_j} e_j = \left[ \frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right]$$

## 9.18 Example (Warm up for direction derivative)

Let $\begin{cases} \gamma : (a,b) \subseteq \mathbb{R}^1 \longrightarrow U \text{ open} \subseteq \mathbb{R}^n \\ \gamma \text{ is differentiable} \end{cases}$

$\begin{cases} f : U \text{ open} \subseteq \mathbb{R}^n \longrightarrow \boxed{\mathbb{R}^1} \\ f \text{ is differentiable} \end{cases}$

$$(a,b) \subseteq \mathbb{R}^1 \xrightarrow{\ \gamma\ } U \subseteq \mathbb{R}^n \xrightarrow{\ f\ } \mathbb{R}^1$$

* Define

$g : (a,b) \longrightarrow \mathbb{R}^1$

$t \longmapsto g(t) = f(\gamma(t))$

Then by Chain rule $g'(t) = f'(\gamma(t))\ \gamma'(t) \quad (*) = \nabla f(\gamma(t))\ \gamma'(t)$

$\left.\begin{array}{l} * \gamma'(t) \in \perp(\mathbb{R}^1, \mathbb{R}^n) \\ f'(\gamma(t)) \in \perp(\mathbb{R}^n, \mathbb{R}^1) \end{array}\right\} \Rightarrow g'(t) \in \perp(\mathbb{R}^1, \mathbb{R}^1)$ ($g'(t)$ is a linear operator on $\mathbb{R}^1$)

* However, $g'(t)$ can also be regarded as a real number:

Compute $g'(t)$ through $(*)$:

- $\gamma : (a,b) \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^n$

  $t \longmapsto (\gamma_1(t)\ \ \gamma_2(t)\ \dots\ \gamma_n(t))$

  then for $t \in (a,b)$ $\quad \gamma'(t) = \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix} \quad (1)$

- $f : U \text{ open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$

  $(\gamma_1(t)\ \gamma_2(t) \dots \gamma_n(t)) \longrightarrow f(\gamma_1(t)\ \ \gamma_2(t)\ \dots\ \gamma_n(t))$

  Put $y_1 = \gamma_1(t)\ \dots\ y_n = \gamma_n(t)$ $\quad$ Then $f : \mathbb{R}^n \longrightarrow \mathbb{R}^1$

  $y = (y_1\ \ y_2\ \dots\ y_n)$ $\qquad (y_1, \dots, y_n) \longmapsto f(y_1, \dots, y_n)$

  Then $f'(y) = \nabla f(y) = \left[\frac{\partial f}{\partial y_1}(y)\ \ \frac{\partial f}{\partial y_2}(y)\ \cdots\ \frac{\partial f}{\partial y_n}(y)\right] = [D_1 f(y)\ \ D_2 f(y) \dots D_n f($

  (2

- $(*) + (1) + (2)$

$$\Rightarrow g'(t) = f'(\gamma(t))\ \gamma'(t) = [D_1 f(y)\ D_2 f(y)\ \dots\ D_n f(y)] \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$$

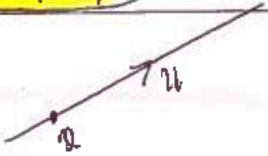
$$= \nabla f(\gamma(t))\ \gamma'(t)\ \square$$

irection derivative. Aug 2005, P 57

$(a,b) \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^n$

$t \longmapsto \gamma_u(t) = \vec{x}_0 + t\vec{u}$

$\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$



where $\vec{x} = (x_1, \dots, x_n)^t \subseteq$ Open in $\mathbb{R}^n$.

$\begin{cases} \vec{u} = (u_1, \dots, u_n)^t \text{ such that} \\ \|u\|_{\mathbb{R}^n} = 1 \end{cases}$

g is a function with variable t.

fine $g \; \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$t \longmapsto g(t) = f(\gamma(t)) = f(x_0 + tu)$

$D_u f(x) = \nabla f(x) \bullet \vec{u}$

say f has direction derivative at $x_0$ in the direction of vector u if $\exists\, g'(0)$

note the direction derivative at $x_0$ with direction u is $D_u f(x)$, we have $D_u f(x) = g'(0)$ (1)

has derivative in E $\Longrightarrow$ f has direction derivative of any vector u at every $x \in E$

$$ {}_u f(x_0) \doteq \nabla f(x_0) \bullet u = \left( \frac{\partial f}{\partial x_1}(x_0) \; \frac{\partial f}{\partial x_2}(x_0) \cdots \frac{\partial f_n}{\partial x_n}(x_0) \right) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

note that x is a fixed point in E

$$= \lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t} \quad (2)$$

$$= \lim_{t \to 0} \frac{f(x_1 + tu_1, x_2 + tu_2, \dots, x_n + tu_n) - f(x_1, \dots, x_n)}{t}$$

Proof: (1)

et $\gamma(t) = x + tu$

then $\gamma'(t) = u$ and $g(t) = f(\gamma(t))$

$\Rightarrow g'(t) = f'(\gamma(t)) \cdot \gamma'(t) = [(\nabla f)(x + tu)] \cdot [u]$

Then $D_u f(\vec{x}) := \nabla f(x) \bullet \vec{u} = g'(0)$

Proof (2):

ve have $g(t) = f(x + tu)$

$g(0) = f(x)$

$$\Rightarrow g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

## 9.19 Theorem (Mean value theorem (inequality) for vector value function)

i) If $\varphi : [a,b] \longrightarrow \mathbb{R}^n$
$\varphi$ is differentiable on $(a,b)$
$\varphi$ is continuous on $[a,b]$

$\Rightarrow \exists t$

$$\|\varphi(b)-\varphi(a)\| \leq \|\varphi'(t)\| (b-a)$$

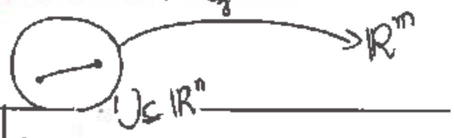
ii) Def: A set $U$ is convex $\overset{def}{\Longleftrightarrow}$ $\forall x,y \in U$, $\overset{x+t(y-x) \subseteq X}{(1-t)x+ty} \subseteq U$, $\forall t \in [0,1]$
(the line ~~of~~ segment from $x$ to $y$ lies in $U$)

- In $\mathbb{R}$, every connected interval is convex
- In $\mathbb{R}^2$, $B(x,r)$ is always convex (by triangle inequality)

ii) Theorem:
$U \subseteq \mathbb{R}^n$ convex, open set
$f : U \longrightarrow \mathbb{R}^m$ differentiable function
$\exists M$ such that $\|f'(x)\| \leq M$, $\forall x \in U$

$\Rightarrow$ $f$ is Lipchitz with constant $M$:

$$\|f(x)-f(y)\| \leq M \|x-y\|$$



* Corollary (Jan 2001)
$U \subseteq \mathbb{R}^n$ is connected, open
$f : U \longrightarrow \mathbb{R}^m$ is differentiable
$f'(x) = 0$, $\forall x \in U$

$\Rightarrow$ $f$ is constant

Way to use this theorem is apply theorem for
$N_\lambda(x)$ because $N_\lambda(x)$ convex, open
$\lambda <$

Jan 2001 proof the corollary

9.20 Continuous differentiable $\mathcal{E}'(U, \mathbb{R}^n)$

* **Def**: $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous differentiable $\overset{def}{\iff}$ $\begin{cases} f \text{ is differentiable} \\ f' \text{ is continuous } (f' : U \longrightarrow L( \end{cases}$

$(f \in C'(U, \mathbb{R}^n))$

9.21 Theorem

$f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Then, $f \in C'(\mathbb{R}, \mathbb{R}^n) \iff$ All the partial derivative $D_j f_i$ $\begin{cases} \text{exist} \\ \text{continuous on } E \end{cases}$

* Remark

All $D_j f_i$ need to exist and be continuous

$D_i f_i$ can ~~be~~ exist but $f$ even is not continuous.

* Application of contractive map : (

* Proposition: (The derivative test for contractive mapping)

Let $I$ : closed + bounded in $\mathbb{R}$

$f : I \longrightarrow I$ is a $C^1$ function

$|f'(x)| < 1$ , $\forall x \in I$

$\Big\}$ $\Rightarrow$ $f$ is a contraction

## * Contraction mapping principle

* Def:

Let $(X, d)$ and $(X_1, d_1)$ are metric spaces

Then $f : (X, d) \longrightarrow (X_1, d_1)$ is a contraction $\overset{\text{def}}{\Longleftrightarrow} \exists 0 \leq k \boxed{<} 1,$ (f is Lipchitz with constant $k <$

(or a contractive map) $d_1(f(x), f(y)) \boxed{\leq} k \, d(x, y), \forall x$

* $f : X \longrightarrow X$, $x \in X$ is a fixed point $\Longleftrightarrow f(x) = x$ | $f$ contraction $\Longrightarrow f$ continuo

* Theorem: Contraction mapping principle in $\mathbb{R}^n$

$$\left.\begin{array}{l} E \subseteq \boxed{\mathbb{R}^n} \text{ be a } \boxed{\text{closed}} \text{ subset} \\ f : E \longrightarrow E \text{ be a contraction mapping} \end{array}\right\} \Rightarrow \begin{array}{l} f \text{ has a unique fixed point.} \\ (\exists!\, x \in E,\ f(x) = x) \end{array}$$

* Theorem (Contraction mapping principle / Fixed point theorem in nonempty complete subs

$$\left.\begin{array}{l} (X, d) : \boxed{\text{nonempty}} \ \boxed{\text{complete}} \text{ metric space} \\ f : X \longrightarrow X \text{ be a } \boxed{\text{contraction}} \text{ mapping} \end{array}\right\} \Rightarrow \begin{array}{l} \exists!\, x \in X,\ f(x) = x \\ (f \text{ has a unique fixed point.} \end{array}$$

* Proof for contraction mapping principle in $\mathbb{R}^n$.

$$\left.\begin{array}{l} (E \subseteq \mathbb{R}^n) \text{ be a closed subset} \\ f : E \rightarrow E \text{ be a contraction mapping} \end{array}\right\} \Rightarrow \exists!\, x \in E,\ f(x) = x.$$

- Recall $\vec{x}_k \in \mathbb{R}^n$, $\sum_{k=1}^{\infty} \|\vec{x}_k\|$ converges $\longrightarrow \sum_{k=1}^{\infty} \vec{x}_k$ converges

Note: the uniqueness hold

if $f$ is a contraction

(we don't need $X$ to be co

$$\left.\begin{array}{l} \text{if } f \text{ has a fixed point} \\ f \text{ is a contraction} \end{array}\right\} \Rightarrow$$

Proof contraction mapping principle

$(X, d)$ non zero complete metric space $\Big\}$ Prove that $\exists!\, x \in X,\ f(x) = x$
$f: X \to X$ contraction mapping

Pick any $x_0 \in X$

Define a sequence $\{x_n\} \subseteq X$ recursively by $\vec{x}_{n+1} = f(\vec{x}_n)$

al: we want to prove that $\{x_n\}$ convergent (because
because $f$ is contraction $\Rightarrow$ $f$ is continuous, then if $x_n \to x$, then
$x = \lim\limits_{n\to\infty} x_{n+1} = \lim f(x_n) = f(\lim\limits_{n\to\infty} x_n) = f(x)$. $\Big\}$ $\Rightarrow$ $x$ is a fixed point.
and besides, because $X$ complete $\Rightarrow$ closed, then if $x_n \to x$ then $x \in X$.)

e want to prove $(x_n)$ convergent
we have $X$ complete metric space $\Big\}$ $\Rightarrow$ NTP $(x_n)$ $\boxed{\text{Cauchy}}$ sequence.
NTP $\forall\, m > n,\ d(x_m, x_n) < \frac{k^n}{1-k} d(x_1, x_0)$

We have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq k\, d(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0) \qquad (*)$$

Then for $m \geq n$

$$\begin{aligned} d(x_m, x_n) &\boxed{\leq} d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\underset{\text{by}(*)}{\leq} (k^{m-1} + k^{m-2} + \cdots + k^n)\, d(x_1, x_0) \\ &= k^n \underbrace{[k^{m-1-n} + \cdots + 1]}\, d(x_1, x_0) \\ &\boxed{\leq} k^n \underbrace{\Big[\sum_{i=1}^{\infty} k^i\Big]}_{\text{Geometric series } (k<1)} d(x_1, x_0) \\ &\leq k^n \underbrace{\frac{1}{1-k} d(x_1, x_0)}_{\text{constant}} \xrightarrow[\text{(note } k<1)]{n\to\infty} 0 \end{aligned}$$

$(x_n)$ Cauchy
$X$ complete $\Big\}$ $\Rightarrow \exists!\, x \in X,\ x_n \to x$ | From the explanation above $\Rightarrow$ $x$ is a fixed point

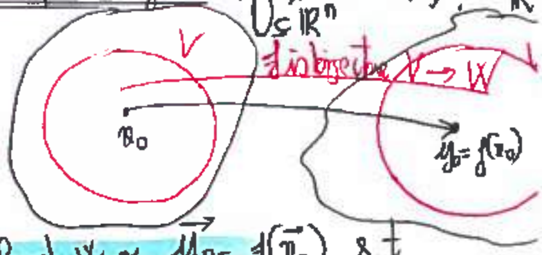
Prove the uniqueness of $x$ (by the contractivity of $f$).

Assume $\exists\, x, y$ such that $f(x) = x$
$f(y) = y$

because $f$ is a contraction, $d(\underset{x}{\underset{\parallel}{f(x)}}, \underset{y}{\underset{\parallel}{f(y)}}) \leq k\, d(x, y)$, $\boxed{k < 1}$

$\Rightarrow$ $d(x, y) < k\, d(x, y)$ for $k < 1$ $\Rightarrow$ $d(x, y) = 0$ $\Rightarrow$ $x = y$

## 9.24: Inverse function theorem / (A special case of Implicit function theorem)



Let $f: U \text{ open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a $C^1$ function.

Suppose $\begin{cases} \vec{x}_0 \in U \\ f'(x_0) \text{ is invertible} \end{cases}$. (means $Df(x_0) \neq 0$)

Then:

i) $\exists$ a open neighborhood $V$ of $x_0$ and $\exists$ open neighborhood $W$ of $\vec{y}_0 = f(\vec{x}_0)$ s.t

$f: V \longrightarrow W$ is bijective

ii) $\exists$ a $C^1$, bijective inverse function of $f$: $g: W \longrightarrow V$

$y \longmapsto g(y) = f\big|_V^{-1}(y)$

means $g(\vec{y}) = f^{-1}(\vec{y})$, $\forall y \in W$

$g(f(\vec{x})) = \vec{x}$, $\forall x \in V$

and $g'(\vec{y}) = [f'(\vec{x})]^{-1}$, $\forall x \in V$, $y \in W$

[note that this formula is not true fo
$(x_0, y_0)$ but true for $(x, f(x))$ when
$x$, $y = f(x)$ in $V$ and $W$

①) Note that even if a function $f$ does not satisfy the condition of IFT, we can also find the inverse function directly (Jan 2012, P5)

* **Corollary:**

$\left.\begin{array}{l} f: U \text{ open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ f \text{ is a } C^1 \text{ function in } U \\ f'(x) \text{ is invertible}, \forall \vec{x} \in U \end{array}\right\} \to f$ is a open mapping

$\forall$ given $V$ open in $U$, $f(V)$ is open.

## + Example of inverse function theorem:

E1 Given $z, w > 0$. Can you find $x, y \in \mathbb{R}$ such that $\begin{cases} z = x+y \\ w = xy \end{cases}$

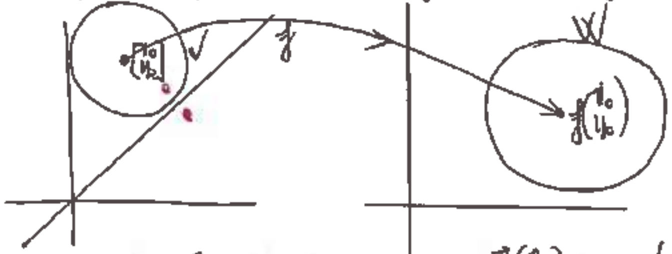
(note that $\begin{cases} x+y = y+x \\ xy = yx \end{cases}$, so unless $x=y$, if there are solution, then there are at least tw

* Solve $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Put $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x+y \\ xy) \end{bmatrix}$ f is $C^1$ function.

- We have $f'(x_0, y_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ y_0 & x_0 \end{bmatrix}$ then $f'(x_0, y_0)$ is invertible when $x_0 \neq y_0$

- Suppose $x_0 \neq y_0$ then $f'(x_0, y_0)$ is nonsingular,

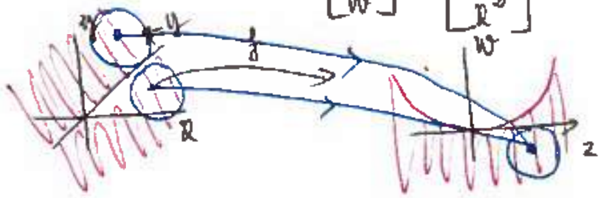


If we start at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

(the neighborhood V can't contain both red points)

- What $\begin{bmatrix} z \\ w \end{bmatrix}$ can be written as $f\begin{pmatrix} x \\ y \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \subseteq \mathbb{R}^2$

(In case $x = y$, then $\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 2x \\ x^2 \end{bmatrix}$ then $w = \left(\frac{z}{2}\right)^2$ then.



(Note $(x+y)^2 \geqslant 2xy + x^2 + y^2 \geqslant 2xy + 2xy =$

$\Rightarrow \quad z^2 \geqslant 4w$

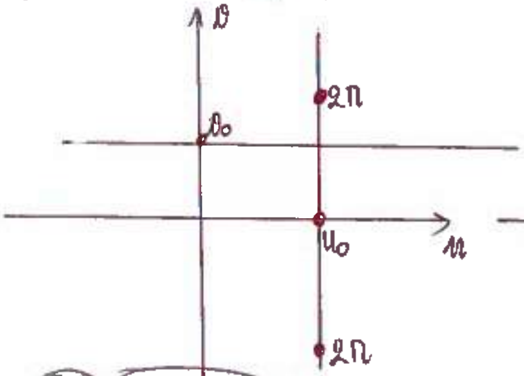
$\left(\frac{z}{2}\right)^2 \geqslant w$

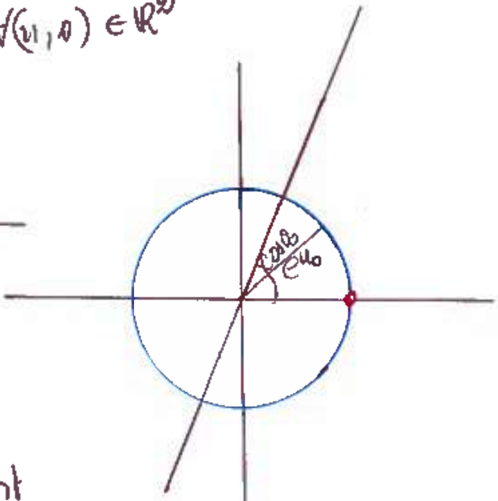
Then for $\begin{bmatrix} z \\ w \end{bmatrix} \subseteq \mathbb{R}^2$ $w < \left(\frac{z}{2}\right)^2$ then

? Let $\vec{f}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} e^u \cos v \\ e^u \sin v \end{bmatrix}$, then $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $f$ is $C^1$ (in fact $C^\infty$)

$$Df\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{bmatrix} = e^{2u} > 0$$

Because $\det Df\begin{pmatrix} u \\ v \end{pmatrix} > 0$, $\forall (u, v) \in \mathbb{R}^2$





) $\boxed{\text{local}}$ $\boxed{C^1 \text{ inverse}}$ near everypoint

ut each value (other than 0 is taken infinitely many point) (<u>not</u> $\boxed{\text{global}}$ $\boxed{C^1 \text{ inverse}}$)

## * The Implicit function theorem

* Idea: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f$ is $C^1$

Assume $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ is a solution of $f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

Then in a neighborhood of $(x_0, y_0)$, we can solve $f(x,y)=0$ $y = g(x)$ if $\frac{\partial f}{\partial y} \neq 0$

we can solve $f(x,y)=0$ $x = g(y)$ if $\frac{\partial f}{\partial x} \neq 0$.

---

*9.26: Notation

Let $\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathbb{R}^{n+m}$ $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$

$\vec{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$

$(\vec{x}, \vec{y}) = (x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}$

* Then consider $\vec{f}: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$

$(\vec{x}, \vec{y}) \longmapsto \vec{f}(\vec{x}, \vec{y})$

$$D\vec{f} = [\underbrace{Df_x}_{n \times n} \,\vdots\, \underbrace{Df_y}_{n \times m}] \updownarrow n+m$$

$\longleftrightarrow n$

$Df(\vec{y}) =$

Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ splits into 2 linear tr

$A_x h = A(h, 0)$, $\boxed{h \in \mathbb{R}^n}$

$A_y k = A(0, k)$ $k \in \mathbb{R}^m$

Then $A_x \in L(\mathbb{R}^n)$ and $A(h,k) = A_x h + A_y$

$A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$ $A = [\underbrace{A_x}_{n \times n} \,\vdots\, \underbrace{A_y}_{n \times m}$

9.27:

If $D\vec{f} = [Df_x \,\vdots\, Df_y]$

$Df_x = \frac{\partial f}{\partial x}$ is nonsingular

then $\exists g$

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$

$A_x$ is invertible $\Bigg\}$ $\forall k \in \mathbb{R}^m$, $\exists !\, h \in \mathbb{R}$

such that $A(h,k) =$

and $h$ computed by $k$ through

$h = -(A_x)^{-1}(A_y) k$ $\left(\begin{matrix} h \text{ is a linear} \\ \text{function of } k \end{matrix}\right)$

---

9.28 * **Implicit function theorem**:

| $x \in \mathbb{R}^n$ $y \in \mathbb{R}^m$

$f: U \overset{\text{open}}{\subseteq} \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$, $f$ is $C^1$ mapping

$(\vec{x}_0, \vec{y}_0)$ is a solution of $f(\vec{x}, \vec{y}) = \vec{0}_{\mathbb{R}^n}$ ( $f(\vec{x}_0, \vec{y}_0) = 0$ for $(\vec{x}_0, \vec{y}_0) \in U$ )

→ note: one more formula next page.

Put $A = Df$ $A_{x (n \times n)} = \frac{\partial f}{\partial x}(x_0, y_0)$ invertible (at $(x_0$, $y_0)$)

We explain $f \in C^1$ by computing $Df$ and explaining that all $\frac{\partial_i f}{\partial x_j}$ exists + conti

Then: i) $\exists$ open neighborhood $V \subseteq \mathbb{R}^{n+m}$ of $(\vec{x}_0, \vec{y}_0)$ such that

open neighborhood $W \subseteq \mathbb{R}^m$ of $\vec{y}_0$

$\forall y \in W$, $\exists !\, x$ such that $\begin{cases} (x,y) \in V \\ f(x,y) = 0 \end{cases}$

ii) If $x = g(y)$, then $g: W_{\text{open}} \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a $C^1$ mapping

$y \longmapsto g(y) = x$.

$f(g(y), y) = 0$, $\forall y \in W$

$g'(y_0) = -[A_{x_0}]^{-1} A_{y_0}$ (at $(x_0, y_0)$)

of:

We have $f: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$

$$(\vec{x},\vec{y}) \longmapsto \left(f^1(\vec{x},\vec{y}),\cdots,f^n(\vec{x},\vec{y})\right)$$

en consider $F: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$

$$(\vec{x},\vec{y}) \longmapsto F(\vec{x},\vec{y}) = \left(f(\vec{x},\vec{y}),\vec{y}\right) = \left(f^1, f^2, \cdots, f^n, y_1, \cdots, y_m\right)$$

e have $F$ is $C^1$ near $(x_0, y_0)$

ve have

$$F'(x_0,y_0) = \left[\begin{array}{ccc|c} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \\ & & & \\ \hline & & & \\ & & & \end{array}\right]_{(n+m)\times(n+m)} = \begin{array}{ccc} & \begin{array}{cc} n & m \end{array} & \\ \begin{array}{c} n \\ m \end{array} & \left[\begin{array}{c|c} Df_x & Df_y \\ \hline O & I \end{array}\right] & \begin{array}{c} n \\ m \end{array} \end{array}$$

$\Rightarrow \det F'(x_0,y_0) = \det A_x \neq 0$

$\Rightarrow$ For every $\underset{\in W}{y}$ near $y_0$, $\exists!$ $x$ near $x_0$ so that $f(x,y)=0$

Denote $x$ at $x = g(y)$

Denote $G$ is $F^{-1}$, we have $G(F(x,y)) = (x,y)$

$\Leftrightarrow G(f(x,y),y) = (x,y)$ | By inverse function theorem $G$ is $C^1$

~~with $x=g(y)$, we have $f(x,y)=0$~~ )

$F(x,y) := (f(x,y),y) := (X,Y)$

then $G(X,Y) = G(f(x,y),y) = (x,y)$

where $x = g(y)$, we have $f(x,y)=0$

Then $G(0,y) = (g(y),y)$

because $G(0,y)$ is $C^1$ function $\Rightarrow g$ is $C^1$ function. □

Example Rudin/227 <u>Implicit function theorem</u>. with $n=2,\ m=3$.

Consider $f:\mathbb{R}^{\textcircled{2+3}} \longrightarrow \mathbb{R}^{\textcircled{2}}$

$$f=(f_1,f_2) \text{ with } \begin{cases} f_1(x_1,x_2,y_1,y_2,y_3) = 2e^{x_1}+x_2y_1-4y_2+3. \\ f_2(x_1,x_2,y_1,y_2,y_3) = x_2\cos x_1-6x_1+2y_1-y_3 \end{cases}$$

With $a=(x_1^\circ,x_2^\circ)=(0,1)$
$b=(y_1^\circ,y_2^\circ,y_3^\circ)=(3,2,7)$. Then $f(a,b)=0$.

- Wrt to the standard basis, the matrix of transformation

$$A=f'(a,b)=\left[\begin{array}{cc|ccc} f^1_{x_1} & f^1_{x_2} & f^1_{y_1} & f^1_{y_2} & f^1_{y_3} \\ f^2_{x_1} & f^2_{x_2} & f^2_{y_1} & f^2_{y_2} & f^2_{y_3}\end{array}\right]_{\text{at } a,b} = \begin{bmatrix} 2e^{x_1} & y_1 & x_2 & -4 & 0 \\ -x_2\sin x_1 - 6 & \cos x_1 & 2 & 0 & -1\end{bmatrix}_{\text{at } a,b}.$$

$$=\underbrace{\left[\begin{array}{cc|ccc} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1\end{array}\right]}_{A_x \qquad A_y}.$$

- We have $\boxed{F(x,y)=(f(x,y),y)} = (f^1,f^2,y^1,y^2,y^3)$

Then
$$F'(x,y)=\left[\begin{array}{cc|ccc} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \leftarrow f^1 \\ \leftarrow f^2 \\ \leftarrow y^1 \\ \leftarrow y^2 \\ \leftarrow y^3 \end{array}$$

$\det F'(x,y) = 20 \neq 0$

$$A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$$

that $A^{-1}=\frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$

- $\boxed{(x,y)=G(f(x,y),y)} \neq G\circ F = Id$

$$\Rightarrow G'\cdot F' = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{G'=[F']^{-1}} \neq \left[\begin{array}{c|c} A_x & A_y \\ \hline 0 & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} A_x^{-1} & -A_x^{-1} \\ \hline 0 & I\end{array}\right]$$

$$= \left[\begin{array}{c|c} \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2\end{bmatrix} & -\frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2\end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 0\end{bmatrix} \\ \hline 0 & I \end{array}\right]$$

$g'(3,2,7)$

Then
$$g'(3,2,7) = -(A_x)^{-1}A_y$$
$$= -\frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2\end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1\end{bmatrix} = \begin{bmatrix} 1/4 & 1/5 & -3/20 \\ -1/2 & 6/5 & 1/10\end{bmatrix}$$

$$\underset{m\times n}{g'} = -\underset{n\times n}{[f_x]^{-1}}\ \underset{n\times m}{[f_y]}$$

Some things need to know about Implicit function theorem :.

roblem: Aug 2009, #6 :

c is a parameter, prove that $x^7+x+c$ has a unique real root and that this root is a differentiable function of c.

- $F: \mathbb{R}^2 \to \mathbb{R}$
  $(x,c) \longmapsto F(x,c) = x^7+x+C$

$DF = [7x^6+1 \;\vdots\; 1]$ then ... ∃ real root and it is a differential fnt of c.

and $A_x \underline{>0}\ \forall x$ → the root is unique.

## 938 Jacobian

$f: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$f$ is differentiable at $x \in E$

Then the Jacobian of $f$ at $x$ $J_f(x) = \det [f'(x)] = \det \begin{vmatrix} f_1^1(x) & f_2^1(x) & \cdots & f_n^1(x) \\ f_1^2(x) & f_2^2(x) & \cdots & f_n^2(x) \\ f_1^n(x) & f_1^n(x) & \cdots & f_n^n(x) \end{vmatrix}$

If $(y_1, \dots, y_n) = f(x_1, x_2, \dots, x_n)$, use the notation $J_f(x) = \dfrac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}$

- If the implicit function theorem stated

- When $n = 1$, $J_f(x) = f'(x)$.
- From chain rule, we have $J_{f \circ g}(x) = J_f(g(x))\, J_g(x)$
- Restate the inverse function theorem using the Jacobian:

  $f: U \longrightarrow \mathbb{R}^n$ is locally invertible near $x$ if $\boxed{J_f(x) \neq 0}$
  (continuous differentiable)

  - If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible at a point $x$ (means $J_f(x) \neq 0$)

    then $f$ is an invertible function near $x$ (that is, an inverse function $f^{-1}$ exists in a
    neighborhood of $f(x)$)

    Moreover, $f^{-1}$ is also continuous differentiable

    and $J_{f^{-1}}(f(x)) = [J_f(x)]^{-1}$

(wiki - inv
function th

# * The Rank Theorem

*9.30 Definition.

$X, Y$ : vector spaces. $A \in L(X,Y)$ .

- The null space of $A$, $\mathcal{N}(A) = \{x \in X, Ax = 0_Y\}$ . is a vector space in $X$ .
- Range of $A$, $\mathcal{R}(A) = A(X)$, is a vector space in $Y$ .
- Rank of $A$ = $\dim(\mathcal{R}(A))$ = $\left[\begin{array}{l}\#\text{ of independent columns of } A \\ \#\text{ of independent rows of } A\end{array}\right.$ .

$\text{Rank } A = \text{Rank } A^T$

+ $\text{Rank } A = \dim Y \Rightarrow A$ is onto .

+ Dimensional theorem $\dim(\text{Ker} A) + \dim(\text{Rang } A) = \dim X$ . for $A : X \longrightarrow Y$ .

$\dim(\text{Nul} A) + \dim(R(A)) = \dim X$

* 9.31 Def + Prop about Projection .

- Def : $X$ : be a vector space

  $P \in L(X)$ is said to be a projection in $X$ if $P^2 = P$ .

ug 1997, P8 (Need to review).

s F ~~a glob~~ has a global inverse ? $\Longleftrightarrow$ ? is F a global $\boxed{\text{bijective}} \left< \begin{array}{l} \text{injective} \\ \text{surjective} \end{array} \right.$

periodic $\Rightarrow$ not ~~bijec~~ injective.

The first step is considering using Implicit function theorem $(f : \mathbb{R}^{n+m} \longrightarrow R^n)$

Inverse function theorem $(f : R^n \longrightarrow \mathbb{R}^n)$.

$$A^{-1} = \frac{1}{\det A}\left[\operatorname{adj} A\right]^{\textcircled{t}} ?$$

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all 2001, P4.

nd the range of a function $f: R^2 \longrightarrow \mathbb{R}^2$

$(x,y) \mapsto (u,v) = f(x,y)$ $\quad u = x^2 - y^2$, $v = 2xy$

so the range is $\mathbb{R}^2$.

We note that each $(x,y)$ associates with $z = x + iy$.

then ~~$u = z^2$~~ ~~$v =$~~ $z^2 = (x+iy)(x+iy)$

$= (x^2 - y^2 + i2xy)$

---

all 1992, P5.

t $\rho : R^2 \longrightarrow R$

$(x,y) \longmapsto \rho(x,y) = (xy-1)^2 + x^2$ in $\mathbb{R}^2$

Find the inf $\{\rho(x,y), x,y \in \mathbb{R}\}$

te that $\rho(x,y)$ attains local min/max if $\rho_x = \rho_y = 0$

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Aug 1998, P3, b.

~~Prove~~ IF $f(x,y,u,v) = (F_1, F_2)$

Prove that there is no open set in the plan on which the resulting equation

efine $x, y$ as a function $g(u,v)$.

We want to prove that for each pair $(u,v) \in \mathbb{R}^2$ $\left\{ \begin{array}{l} \text{there is no } (x,y) \quad (x,y) = g(x,y) \\ \text{there are more than 2 values } (x,y) = g(u,v) \end{array} \right.$

* Template Let $\boxed{f:\mathbb{R}^2} \to \boxed{\mathbb{R}}$ Justify the difference of $f$ at $(0,0)$.

$(x,y) \mapsto f(x,y)$

Aug 2008/17 $f(x,y)=\begin{cases} \frac{x^2y}{x^4+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0)\end{cases}$ + also Aug Jan 2004.

Jan 2013 6 $f(x,y) = (x^3+y^3)^{1/3}$.

**Step 1**: find $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$

$f_y(0,0) = \lim \frac{f(0,h)-f(0,0)}{h}$

**Step 2**: Find $\lim \frac{f(h_1,h_2) - f(0,0) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}}$ $\underline{=0}$ differentiable or $\begin{cases}\neq 0 \\ \text{not exist}\end{cases}$ not differentiable

Find ~~the~~ $\phi$ by: $\lim \frac{\phi(h_1,h_2)-\phi(0,}{\|(h_1,h_2)\|}$ (See Jan 2006

* 2 ways to prove that $f$ is an open map.

- Way 1: $\left.\begin{array}{l} f: ~~Vopen\mathbb{s}~~ \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ Df \neq 0, \forall (\vec{x} \in \mathbb{R}^n \end{array}\right\} \Rightarrow f$ is an open map □

- Way 2: If we can find $g = f^{-1}$ and $g$ is continuous $\mathbb{R}^n \to \mathbb{R}^n$ $\Big\}$ then $\forall V$ open in $\mathbb{R}^n$, $g^{-1}(V)$ is open in $\mathbb{R}^n$ $\|$ which means $\forall V$ open in $\mathbb{R}^n$, $f(V)$ is open in $\mathbb{R}^n$ □. (See Aug 20

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* Show that $f$ is not one to one in a neighborhood of $(0)$

- Way 1: We can consider $(-x,-y)$ and some have $f(-x,-y)=f(x,y)$ or similar ways to prove

- In case $f: R \to R$ (Aug 2015, P6) $f: \begin{cases} x + 2x^2 \sin\frac{1}{x}, & x\neq 0 \\ 0 & , x=0\end{cases}$

and we want to prove that $f$ is not 1-1 in a neighborhood of $(0)$.

we want to find $x_n = \frac{1}{2\pi n}$ $f'(x_n) = -1 < 0$ $\Rightarrow$ not one to one.

$x_n' = \frac{1}{(2n+1)\pi}$ $f'(x_n) = 3 > 0$

it's ok even in this case $x, x'$ in the same side of $0$.

~~* Another problem about approximate f by a sequence of polynomial where f vanishes at some points, except 1 point~~

$$\left(f_n(x) - f(x)\right) = \left(\int_a^x f_n'(t)\,dt + f_n(a) - \int_a^x f'(t)\,dt - f(a)\right)$$

$$\leq \int_a^b f_n'(t) - f'(t)\,dt + \underbrace{f_n(a) - f(a)}_{=0}$$

then $\sup\limits_{x\in[a,b]} \left|f_n(x) - f(x)\right| \leq |x-a| \sup\limits_{x} \|f_n' - f'\|$

$$\leq (b-a) \underbrace{\sup \|P_n + g_n - f'\|}_{<\varepsilon \text{ by } (3)}$$

~~then~~ so we have $f_n(x) \rightrightarrows f(x)$. (II)

(I)+(II)+(2) $\Rightarrow$ all of things that we need to prove.

The words *complete* and *contraction* are necessary. For example, $f\colon (0,1) \to (0,1)$ defined by $f(x) = kx$ for any $0 < k < 1$ is a contraction with no fixed point. Also $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ is not a contraction ($k = 1$) and has no fixed point.

Existence

*$\star$ Proof.* Pick any $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_{n+1} := f(x_n)$. ← Also know how to find the fix point

$$d(x_{n+1}, x_n) = d\big(f(x_n), f(x_{n-1})\big) \le kd(x_n, x_{n-1}) \le \cdots \le k^n d(x_1, x_0). \qquad (1)$$

Suppose $m \ge n$, then

$x_0 \quad x_1 \quad \cdots \quad x_n$

$$\begin{aligned} d(x_m, x_n) &\overset{\text{triangle inequality}}{\le} \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\overset{\text{by (1)}}{\le} \sum_{i=n}^{m-1} k^i d(x_1, x_0) \\ &= k^n d(x_1, x_0) \sum_{i=0}^{m-n-1} k^i \\ &\le k^n d(x_1, x_0) \sum_{i=0}^{\infty} k^i = k^n d(x_1, x_0) \frac{1}{1-k}. \xrightarrow{n \to \infty} 0 \end{aligned}$$

In particular the sequence is Cauchy (why?). Since $X$ is complete we let $x := \lim x_n$ and we claim that $x$ is our unique fixed point.

Fixed point? Note that $f$ is continuous because it is a contraction. Hence

$$f(x) = \lim f(x_n) = \lim x_{n+1} = x.$$

$\star$ Unique? Let $y$ be a fixed point. (Assume there are 2 fixed point $x$ and $y$) :

$f(x) = x$
$f(y) = y$

$$d(x, y) = d\big(f(x), f(y)\big) \le kd(x, y). \quad k < 1$$

As $k < 1$ this means that $d(x, y) = 0$ and hence $x = y$. The theorem is proved. ☐

Note that the proof is constructive. Not only do we know that a unique fixed point exists. We also know how to find it.

We've used the theorem to prove Picard's theorem last semester. This semester, we will prove the inverse and implicit function theorems.

Do ~~also note the proof of uniqueness holds even if $X$ is not complete.~~ If $f$ is a contraction, then if it has a fixed point, that point is unique.

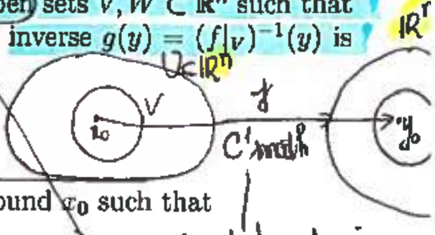
## Inverse function theorem

The idea of a derivative is that if a function is differentiable, then it locally "behaves like" the derivative (which is a linear function). So for example, if a function is differentiable and the derivative is invertible, the function is (locally) invertible.

$\mathbb{R}^n$ — same dimensional space $\mathbb{R}^n$

**Theorem 9.24:** Let $U \subset \mathbb{R}^n$ be a set and let $f\colon U \to \mathbb{R}^n$ be a continuously differentiable function. Also suppose that $x_0 \in U$, $f(x_0) = y_0$, and $f'(x_0)$ is invertible. Then there exist open sets $V, W \subset \mathbb{R}^n$ such that $x_0 \in V \subset U$, $f(V) = W$ and $f|_V$ is one-to-one and onto. Furthermore, the inverse $g(y) = (f|_V)^{-1}(y)$ is continuously differentiable and

$$g'(y) = \underbrace{(f'(x))^{-1}}_{\in \mathbb{R}^n \to \mathbb{R}^n}, \qquad \text{for all } x \in V,\ y = f(x).$$



*Proof.* Write $A = f'(x_0)$. As $f'$ is continuous, there exists an open ball $V$ around $x_0$ such that

Choose $\lambda = \frac{1}{2\|A^{-1}\|}$ $\quad \|f'(x_0) - f'(x)\|$

$$\|A - f'(x)\| < \frac{1}{2\,\|A^{-1}\|} \qquad \text{for all } x \in V. \quad (x \approx x_0)$$

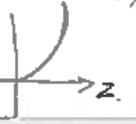
(first derivative is continuous.

Note that $f'(x)$ is invertible for all $x \in V$. $\Rightarrow \exists \|A^{-1}\|$ ?

$f(x) = f(x_0) + f'(x_0)(x - x_0) + 0$

$\star$ Example $f(z) = z^2$ $\quad f'(z) = 2z$

then for $z > 0$ $f$ is bijective



* Prove that $f|_V$ is one-to-one .

* → $x_1 + A^{-1}(y - f(x_1)) = x_2 + A^{-1}(y - f(x_2))$ implies $x_1 = x_2$. In particular, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Given $y \in \mathbb{R}^n$ fixed we define $\varphi_y : C \to \mathbb{R}^n$ , consider $y \in \mathbb{R}^n$ fixed . (*)

$$\varphi_y(x) = x + A^{-1}(y - f(x)). \quad , x \in U$$

As $A^{-1}$ is one-to-one, we notice that $\varphi_y(x) = x$ ($x$ is a fixed point) if only if $y - f(x) = 0$, or in other words $f(x) = y$. Using chain rule we obtain.

$$\varphi_y'(x) = I - \overbrace{A^{-1} f'(x)}^{?} = A^{-1}(A - f'(x)).$$

so for $x \in V$ we have

$$\|\varphi_y'(x)\| \leq \|A^{-1}\| \, \|A - f'(x)\| \leq \|A\| \cdot \frac{1}{2\|A^{-1}\|} = \frac{1}{2}$$

As $V$ is a ball it is convex, and hence

$$\varphi(x_1) - \varphi(x_2) = \int_0^1 \frac{d}{dt}\left(\varphi(tx_1 + (1-t)x_2)\right)dt = \int_0^1 \varphi'(tx_1+(1-t)x_2)(x_1-x_2)dt = \left[\int_0^1 \varphi'(tx_1+(1-t)x_2)dt\right](x_1-x_2)$$

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in V. \Rightarrow$$

φ has at most one fixed point in V

In other words $\varphi_y$ is a contraction defined on $V$, though we so far do not know what is the range of $\varphi_y$. We cannot apply the fixed point theorem, but we can say that $\varphi_y$ has at most one fixed point (note proof of uniqueness in the contraction mapping principle). That is, there exists at most one $x \in V$ such that $f(x) = y$, and so $f|_V$ is one-to-one. , □ $f|_V$ one-to-one

* Let $W = f(V)$. We need to show that $W$ is open. Take a $y_1 \in W$, then there is a unique $x_1 \in V$ such that $f(x_1) = y_1$. Let $r > 0$ be small enough such that the closed ball $C(x_1, r) \subset V$ (such $r > 0$ exists as $V$ is open).

Suppose $y$ is such that

$\left\{ \varphi(x_1) = x_1 \text{ and } \varphi(x_2) = x_2 \Rightarrow |x_1 - x_2| \leq \frac{1}{2}|x_1 - x_2| \right.$ / Nothing to do with the (Banach fixed point theorem)

$$\|y - y_1\| < \frac{r}{2\|A^{-1}\|}.$$

If we can show that $y \in W$, then we have shown that $W$ is open. Define $\varphi_y(x) = x + A^{-1}(y - f(x))$ as before. If $x \in C(x_1, r)$, then

$$\begin{aligned}
\|\varphi_y(x) - x_1\| &\leq \|\varphi_y(x) - \varphi_y(x_1)\| + \|\varphi_y(x_1) - x_1\| \\
&\leq \frac{1}{2}\|x - x_1\| + \|A^{-1}(y - y_1)\| \\
&\leq \frac{1}{2}r + \|A^{-1}\| \, \|y - y_1\| \\
&< \frac{1}{2}r + \|A^{-1}\| \frac{r}{2\|A^{-1}\|} = r.
\end{aligned}$$

$\varphi_y$ takes the ball into itself

So $\varphi_y$ takes $C(x_1, r)$ into $B(x_1, r) \subset C(x_1, r)$. It is a contraction on $C(x_1, r)$ and $C(x_1, r)$ is complete (closed subset of $\mathbb{R}^n$ is complete). Apply the contraction mapping principle to obtain a fixed point $x$, i.e. $\varphi_y(x) = x$. That is $f(x) = y$. So $y \in f(C(x_1, r)) \subset f(V) = W$. Therefore $W$ is open. □ W is open.

* Next we need to show that $g$ is continuously differentiable and compute its derivative. First let us show that it is differentiable. Let $y \in W$ and $k \in \mathbb{R}^n$, $k \neq 0$, such that $y + k \in W$. Then there are unique $x \in V$ and $h \in \mathbb{R}^n$, $h \neq 0$ and $x + h \in V$, such that $f(x) = y$ and $f(x + h) = y + k$ as $f|_V$ is a one-to-one and onto mapping of $V$ onto $W$. In other words, $g(y) = x$ and $g(y + k) = x + h$. We can still squeeze some information from the fact that $\varphi_y$ is a contraction.

$$\varphi_y(x + h) - \varphi_y(x) = h + A^{-1}(f(x) - f(x + h)) = h - A^{-1}k.$$

So

$$\|h - A^{-1}k\| = \|\varphi_y(x + h) - \varphi_y(x)\| \leq \frac{1}{2}\|x + h - x\| = \frac{\|h\|}{2}.$$

By the inverse triangle inequality $\|h\| - \|A^{-1}k\| \leq \frac{1}{2}\|h\|$ so

$$\|h\| \leq 2\|A^{-1}k\| \leq 2\|A^{-1}\| \, \|k\|.$$

In particular as $k$ goes to 0, so does $h$.

As $x \in V$, then $f'(x)$ is invertible. Let $B = \big(f'(x)\big)^{-1}$, which is what we think the derivative of $g$ at $y$ is. Then

$$\begin{aligned}
\frac{\|g(y+k) - g(y) - Bk\|}{\|k\|} &= \frac{\|h - Bk\|}{\|k\|} \\
&= \frac{\big\|h - B\big(f(x+h) - f(x)\big)\big\|}{\|k\|} \\
&= \frac{\big\|B\big(f(x+h) - f(x) - f'(x)h\big)\big\|}{\|k\|} \\
&\leq \|B\| \frac{\|h\|}{\|k\|} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|} \\
&\leq 2\,\|B\|\,\|A^{-1}\| \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|}.
\end{aligned}$$

As $k$ goes to 0, so does $h$. So the right hand side goes to 0 as $f$ is differentiable, and hence the left hand side also goes to 0. And $B$ is precisely what we wanted $g'(y)$ to be.

We have that $g$ is differentiable, let us show it is $C^1(W)$. Now, $g\colon W \to V$ is continuous (it's differentiable), $f'$ is continuous function from $V$ to $L(\mathbb{R}^n)$, and $X \to X^{-1}$ is a continuous function. $g'(y) = \big(f'(g(y))\big)^{-1}$ is the composition of these three continuous functions and hence is continuous. □

**Corollary:** Suppose $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^n$ is a continuously differentiable mapping such that $f'(x)$ is invertible for all $x \in U$. Then given any open set $V \subset U$, $f(V)$ is open. ($f$ is an open mapping).

*Proof.* WLOG suppose $U = V$. For each point $y \in f(V)$, we pick $x \in f^{-1}(y)$ (there could be more than one such point), then by the inverse function theorem there is a neighbourhood of $x$ in $V$ that maps onto an neighbourhood of $y$. Hence $f(V)$ is open. □

The theorem, and the corollary, is not true if $f'(x)$ is not invertible for some $x$. For example, the map $f(x,y) = (x, xy)$, maps $\mathbb{R}^2$ onto the set $\mathbb{R}^2 \setminus \{(0,y) : y \neq 0\}$, which is neither open nor closed. In fact $f^{-1}(0,0) = \{(0,y) : y \in \mathbb{R}\}$. Note that this bad behaviour only occurs on the $y$-axis, everywhere else the function is locally invertible. In fact if we avoid the $y$-axis it is even one to one.

Also note that just because $f'(x)$ is invertible everywhere doesn't mean that $f$ is one-to-one globally. It is definitely "locally" one-to-one. For an example, just take the map $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = z^2$. Here we treat the map as if it went from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}^2$. For any nonzero complex number, there are always two square roots, so the map is actually 2-to-1. It is left to student to show that $f$ is differentiable and the derivative is invertible (Hint: let $z = x + iy$ and write down what the real an imaginary part of $f$ is in terms if $x$ and $y$).

Also note that the invertibility of the derivative is not a necessary condition, just sufficient for having a continuous inverse and being an open mapping. For example the function $f(x) = x^3$ is an open mapping from $\mathbb{R}$ to $\mathbb{R}$ and is globally one-to-one with a continuous inverse.

**Implicit function theorem:**

The inverse function theorem is really a special case of the implicit function theorem which we prove next. Although somewhat ironically we will prove the implicit function theorem using the inverse function theorem. Really what we were showing in the inverse function theorem was that the equation $x - f(y) = 0$ was solvable for $y$ in terms of $x$ if the derivative in terms of $y$ was invertible, that is if $f'(y)$ was invertible. That is there was locally a function $g$ such that $x - f\big(g(x)\big) = 0$.

OK, so how about we look at the equation $f(x,y) = 0$. Obviously this is not solvable for $y$ in terms of $x$ in every case. For example, when $f(x,y)$ does not actually depend on $y$. For a slightly more complicated example, notice that $x^2 + y^2 - 1 = 0$ defines the unit circle, and we can locally solve for $y$ in terms of $x$ when 1) we are near a point which lies on the unit circle and 2) when we are not at a point where the circle has a vertical tangency, or in other words where $\frac{\partial f}{\partial y} = 0$.

$f = (f_1, \dots, f_n)$

$x = (x_1, \dots, x_n)$ $y = (y_1, \dots, y_m)$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$x \longmapsto g(x) = y$

$f(x,y) = 0$
$\Rightarrow f(x, g(x)) = 0$

To make things simple we fix some notation. We let $(x, y) \in \mathbb{R}^{n+m}$ denote the coordinates $(x^1, \dots, x^n, y^1, \dots, y^m)$. A linear transformation $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^m)$ can then always be written as $A = [A_x \ A_y]$ so that $A(x, y) = A_x x + A_y y$, where $A_x \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $A_y \in L(\mathbb{R}^m)$.

Note that Rudin does things "in reverse" from what the statement is usually. I'll do it in the usual order as that's what I am used to, where we are taking the derivatives of $y$, not $x$ (but it doesn't matter really in the end). First a linear version of the implicit function theorem.

**Proposition (Theorem 9.27):** Let $A = [A_x \ A_y] \in L(\mathbb{R}^{n+m}, \mathbb{R}^m)$ and suppose that $A_y$ is invertible, then let $B = -(A_y)^{-1} A_x$ and note that

$$0 = A(x, Bx) = A_x x + A_y Bx.$$

The proof is obvious. We simply solve and obtain $y = Bx$. Let us therefore show that the same can be done for $C^1$ functions.

**Theorem 9.28 (Implicit function theorem):** Let $U \subset \mathbb{R}^{n+m}$ be an open set and let $f : U \to \mathbb{R}^m$ be a $C^1(U)$ mapping. Let $(x_0, y_0) \in U$ be a point such that $f(x_0, y_0) = 0$. Write $A = [A_x \ A_y] = f'(x_0, y_0)$ and suppose that $A_y$ is invertible. Then there exists an open set $W \subset \mathbb{R}^n$ with $x_0 \in W$ and a $C^1(W)$ mapping $g : W \to \mathbb{R}^m$, with $g(x_0) = y_0$, and for all $x \in W$, we have $(x, g(x)) \in U$ and

$$f(x, g(x)) = 0.$$

Furthermore,

$$g'(x_0) = -(A_y)^{-1} A_x.$$

*Proof.* Define $F : U \to \mathbb{R}^{n+m}$ by $F(x, y) = \big(x, f(x, y)\big)$. It is clear that $F$ is $C^1$, and we want to show that the derivative at $(x_0, y_0)$ is invertible.

Let's compute the derivative. We know that

$$\frac{\|f(x_0 + h, y_0 + k) - f(x_0, y_0) - A_x h - A_y k\|}{\|(h, k)\|}$$

goes to zero as $\|(h, k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ goes to zero. But then so does

$$\frac{\left\|\big(h, f(x_0 + h, y_0 + k) - f(x_0, y_0)\big) - (h, A_x h + A_y k)\right\|}{\|(h, k)\|} = \frac{\|f(x_0 + h, y_0 + k) - f(x_0, y_0) - A_x h - A_y k\|}{\|(h, k)\|}.$$

So the derivative of $F$ ate $(x_0, y_0)$ takes $(h, k)$ to $(h, A_x h + A_y k)$. If $(h, A_x h + A_y k) = (0, 0)$, then $h = 0$, and so $A_y k = 0$. As $A_y$ is one-to-one, then $k = 0$. Therefore $F'(x_0, y_0)$ is one-to-one or in other words invertible and we can apply the inverse function theorem.

That is, there exists some open set $V \subset \mathbb{R}^{n+m}$ with $(x_0, 0) \in V$, and an inverse mapping $G : V \to \mathbb{R}^{n+m}$, that is $F(G(x, s)) = (x, s)$ for all $(x, s) \in V$ (where $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$). Write $G = (G_1, G_2)$ (the first $n$ and the second $m$ components of $G$). Then

$$F\big(G_1(x, s), G_2(x, s)\big) = \big(G_1(x, s), f(G_1(x, s), G_2(x, s))\big) = (x, s).$$

So $x = G_1(x, s)$ and $f\big(G_1(x, s), G_2(x, s)\big) = f\big(x, G_2(x, s)\big) = s$. Plugging in $s = 0$ we obtain

$$f\big(x, G_2(x, 0)\big) = 0.$$

Let $W = \{x \in \mathbb{R}^n : (x, 0) \in V\}$ and define $g : W \to \mathbb{R}^m$ by $g(x) = G_2(x, 0)$. We obtain the $g$ in the theorem.

Next differentiate

$$x \mapsto f\big(x, g(x)\big),$$

at $x_0$, which should be the zero map. The derivative is done in the same way as above. We get that for all $h \in \mathbb{R}^n$

$$0 = A\big(h, g'(x_0)h\big) = A_x h + A_y g'(x_0) h,$$

and we obtain the desired derivative for $g$ as well. □

In other words, in the context of the theorem we have $m$ equations in $n+m$ unknowns.

$$f^1(x_1,\ldots,x_n,y_1,\ldots,y_m)=0$$
$$\vdots$$
$$f^m(x_1,\ldots,x_n,y_1,\ldots,y_m)=0$$

And the condition guaranteeing a solution is that this is a $C^1$ mapping (that all the components are $C^1$, or in other words all the partial derivatives exist and are continuous), and the matrix

$$\begin{bmatrix} \frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial y^1} & \cdots & \frac{\partial f^m}{\partial y^m} \end{bmatrix}$$

is invertible at $(x_0,y_0)$.

**Example:** Consider the set $x^2+y^2-(z+1)^3=-1$, $e^x+e^y+e^z=3$ near the point $(0,0,0)$. The function we are looking at is

$$f(x,y,z)=(x^2+y^2-(z+1)^3+1,e^x+e^y+e^z-3).$$

We find that

$$Df=\begin{bmatrix} 2x & 2y & -3(z+1)^2 \\ e^x & e^y & e^z \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2(0) & -3(0+1)^2 \\ e^0 & e^0 \end{bmatrix}=\begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

is invertible. Hence near $(0,0,0)$ we can find $y$ and $z$ as $C^1$ functions of $x$ such that for $x$ near 0 we have

$$x^2+y(x)^2-(z(x)+1)^3=-1, \qquad e^x+e^{y(x)}+e^{z(x)}=3.$$

The theorem doesn't tell us how to find $y(x)$ and $z(x)$ explicitly, it just tells us they exist. In other words, near the origin the set of solutions is a smooth curve that goes through the origin.

Note that there are versions of the theorem for arbitrarily many derivatives. If $f$ has $k$ continuous derivatives, then the solution also has $k$ derivatives.

So it would be good to have an easy test for when is a matrix invertible. This is where determinants come in. Suppose that $\sigma=(\sigma_1,\ldots,\sigma_n)$ is a permutation of the integers $(1,\ldots,n)$. It is not hard to see that any permutation can be obtained by a sequence of transpositions (switchings of two elements). Call a permutation even (resp. odd) if it takes an even (resp. odd) number of transpositions to get from $\sigma$ to $(1,\ldots,n)$. It can be shown that this is well defined, in fact it is not hard to show that

$$\operatorname{sgn}(\sigma)=\operatorname{sgn}(\sigma_1,\ldots,\sigma_n)=\prod_{p<q}\operatorname{sgn}(\sigma_q-\sigma_p)$$

is $-1$ if $\sigma$ is odd and 1 if $\sigma$ is even. The symbol $\operatorname{sgn}(x)$ for a number is defined by

$$\operatorname{sgn}(x)=\begin{cases} -1 & \text{if } x<0, \\ 0 & \text{if } x=0, \\ 1 & \text{if } x>0. \end{cases}$$

This can be proved by noting that applying a transposition changes the sign, which is not hard to prove by induction on $n$. Then note that the sign of $(1,2,\ldots,n)$ is 1.

Let $S_n$ be the set of all permutations on $n$ elements (the *symmetric group*). Let $A=[a^i_j]$ be a matrix. Define the *determinant* of $A$

$$\det(A)=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}a^i_{\sigma_i}.$$

**Proposition (Theorem 9.34 and other observations):**

(i) $\det(I) = 1$.
(ii) $\det([x_1 x_2 \dots x_n])$ where $x_j$ are column vectors is linear in each variable $x_j$ separately.
(iii) If two columns of a matrix are interchanged determinant changes sign.
(iv) If two columns of $A$ are equal, then $\det(A) = 0$.
(v) If a column is zero, then $\det(A) = 0$.
(vi) $A \mapsto \det(A)$ is a continuous function.
(vii) $\det\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc$ and $\det[a] = a$.

In fact, the determinant is the unique function that satisfies (i), (ii), and (iii). But we digress.

*Proof.* We go through the proof quickly, as you have likely seen this before.

(i) is trivial. For (ii) Notice that each term in the definition of the determinant contains exactly one factor from each column.

Part (iii) follows by noting that switching two columns is like switching the two corresponding numbers in every element in $S_n$. Hence all the signs are changed. Part (iv) follows because if two columns are equal and we switch them we get the same matrix back and so part (iii) says the determinant must have been 0.

Part (v) follows because the product in each term in the definition includes one element from the zero column. Part (vi) follows as det is a polynomial in the entries of the matrix and hence continuous. We have seen that a function defined on matrices is continuous in the operator norm if it is continuous in the entries. Finally, part (vii) is a direct computation. □

**Theorem 9.35+9.36:** If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$. In particular, $A$ is invertible if and only if $\det(A) \neq 0$ and in this case, $\det(A^{-1}) = \frac{1}{\det(A)}$.

*Proof.* Let $b_1, \dots, b_n$ be the columns of $B$. Then

$$AB = [Ab_1\ Ab_2\ \cdots\ Ab_n].$$

That is, the columns of $AB$ are $Ab_1, \dots, Ab_n$.

Let $b_j^i$ denote the elements of $B$ and $a_j$ the columns of $A$. Note that $Ae_j = a_j$. By linearity of the determinant as proved above we have

$$\begin{aligned}\det(AB) = \det([Ab_1\ Ab_2\ \cdots\ Ab_n]) &= \det\left(\left[\sum_{j=1}^{n} b_1^j a_j\ Ab_2\ \cdots\ Ab_n\right]\right)\\ &= \sum_{j=1}^{n} b_1^j \det([a_j\ Ab_2\ \cdots\ Ab_n])\\ &= \sum_{1 \le j_1, \dots, j_n \le n} b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n} \det([a_{j_1}\ a_{j_2}\ \cdots\ a_{j_n}])\\ &= \left(\sum_{(j_1,\dots,j_n)\in S_n} b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n} \operatorname{sgn}(j_1, \dots, j_n)\right) \det([a_1\ a_2\ \cdots\ a_n]).\end{aligned}$$

In the above, we note that we could go from all integers, to just elements of $S_n$ by noting that the determinant of the resulting matrix is just zero.

The conclusion follows by recognizing the determinant of $B$. Actually the rows and columns are swapped, but a moment's reflection will reveal that it does not matter. We could also just plug in $A = I$.

For the second part of the theorem note that if $A$ is invertible, then $A^{-1}A = I$ and so $\det(A^{-1})\det(A) = 1$. If $A$ is not invertible, then the columns are linearly dependent. That is suppose that

$$\sum_{j=1}^{n} c^j a_j = 0.$$

Without loss of generality suppose that $c^1 \neq 1$. Then take

$$B = \begin{bmatrix} c^1 & 0 & 0 & \cdots & 0 \\ c^2 & 1 & 0 & \cdots & 0 \\ c^3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c^n & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

It is not hard to see from the definition that $\det(B) = c^1 \neq 0$. Then $\det(AB) = \det(A)\det(B) = c^1 \det(A)$. Note that the first column of $AB$ is zero, and hence $\det(AB) = 0$. Thus $\det(A) = 0$. ☐

**Proposition:** Determinant is independent of the basis. In other words, if $B$ is invertible then,

$$\det(A) = \det(B^{-1}AB).$$

The proof is immediate. If in one basis $A$ is the matrix representing a linear operator, then for another basis we can find a matrix $B$ such that the matrix $B^{-1}AB$ takes us to the first basis, apply $A$ in the first basis, and take us back to the basis we started with. Therefore, the determinant can be defined as a function on the space $L(\mathbb{R}^n)$, not just on matrices. No matter what basis we choose, the function is the same. It follows from the two propositions that

$$\det \colon L(\mathbb{R}^n) \to \mathbb{R}$$

is a well defined and continuous function.

We can now test whether a matrix is invertible

**Definition:** Let $U \subset \mathbb{R}^n$ and $f \colon U \to \mathbb{R}^n$ be a differentiable mapping. Then define the *Jacobian* of $f$ at $x$ as

$$J_f(x) = \det\big(f'(x)\big).$$

Sometimes this is written as

$$\frac{\partial(f^1, \ldots, f^n)}{\partial(x^1, \ldots, x^n)}.$$

To the uninitiated this can be a somewhat confusing notation, but it is useful when you need to specify the exact variables and function components used.

When $f$ is $C^1$, then $J_f(x)$ is a continuous function.

The Jacobian is a real valued function, and when $n = 1$ it is simply the derivative. Also note that from the chain rule it follows that:

$$J_{f \circ g}(x) = J_f\big(g(x)\big) J_g(x).$$

We can restate the inverse function theorem using the Jacobian. That is, $f \colon U \to \mathbb{R}^n$ is locally invertible near $x$ if $J_f(x) \neq 0$.

For the implicit function theorem the condition is normally stated as

$$\frac{\partial(f^1, \ldots, f^n)}{\partial(y^1, \ldots, y^n)}(x_0, y_0) \neq 0.$$

It can be computed directly that the determinant tells us what happens to area/volume. Suppose that we are in $\mathbb{R}^2$. Then if $A$ is a linear transformation, it follows by direct computation that the direct image of the unit square $A([0,1]^2)$ has area $|\det(A)|$. Note that the sign of the determinant determines "orientation". If the determinant is negative, then the two sides of the unit square will be flipped in the image. We claim without proof that this follows for arbitrary figures, not just the square.

Similarly, the Jacobian measures how much a differentiable mapping stretches things locally, and if it flips orientation. We should see more of this geometry next semester.

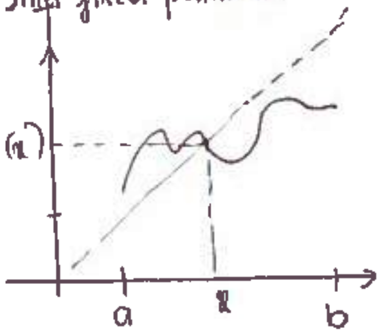
Example
Rudin 9.6. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

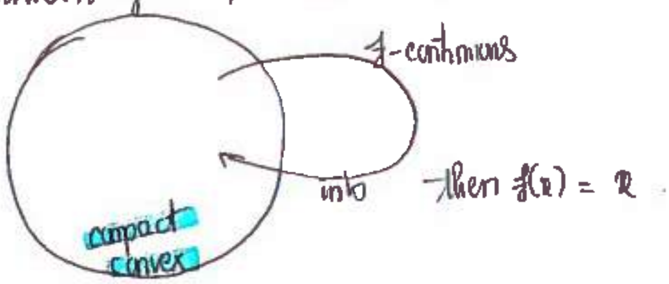
$D_1 f(x,y) =$ exist

$D_2 f(x,y) =$ exists

But $f$ is not continuous at $(0,0)$, so not differentiable

Other fixed point theorem : Brouwer's fixed point theorem :





- $f$ continuous $0 \longrightarrow 1$, a convex function
Then $f$ can be approximated by $a_\lambda$ ~~function~~ polynomial. 2015 Jan $4 \rightarrow 3$ ?

Problem 2: Consider a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the rule:

$$F(x,y) = \left(x^{2017} + y \, , \, y^{2017} - x\right)$$

Show that F is local homomorphism .

NTP F is local homomorphism $\Leftarrow$ NTP $\det J_F(x,y) \neq 0$ .

we have

$$J_F(x,y) = \begin{bmatrix} f^1_x & f^1_y \\ f^2_x & f^2_y \end{bmatrix} = \begin{bmatrix} 2017\,x^{2016} & 1 \\ -1 & 2017\,y^{2016} \end{bmatrix} = \cancel{20}$$

$$\Rightarrow \det J_F(x,y) = (2017)^2 x^{2016} y^{2016} + 2 > 0$$

$\Rightarrow$ (See Restate inverse function theorem by Jacobian) $\Rightarrow$ F is an invertible function near $(x,y)$ then $\Rightarrow$ F is local homomorphism .

b) Prove that F is a proper map, that is, the set $F^{-1}(K) \overset{def}{=} \{(x,y) \in \mathbb{R}^2, F(x,y) \in K\}$ is compact whenever $K \subset \mathbb{R}^2$ is compact

- Consider a transformation $T : C[0,1] \longrightarrow C[0,1]$

$$f \longmapsto T(f(x)) = \frac{1}{\lambda(x)} \mathrm{Exp}\left(\int_0^1 f(y)\,\lambda(y)\,dy\right)$$

where $\lambda$ is a positive continuous function defined on $[0,1]$.

Does T have a fixed point?

Suppose that we have T has a fixed point, this means

$$\exists f,\ f(x) = Tf(x) = \underbrace{\frac{1}{\lambda(x)}}_{\text{positive}} \underbrace{\mathrm{Exp}\left(\int_0^1 f(y)\,\lambda(y)\,dy\right)}_{\text{positive}}.$$

Put $u(x) := f(x)\cdot\lambda(x)$.

Then we have $u(x) = \exp\left(\int_0^1 f(y)\lambda(y)\,dy\right) = \exp\left(\int_0^1 u(y)\,dy\right)$

Thus we have u is a positive constant c and $\underbrace{c = e^c = \sum_{l=0}^{\infty} \frac{c^l}{l!} = 1 + \frac{c}{1} + \frac{c^2}{2!} + \frac{c^3}{3!} \cdots}_{\text{impossible}}$ (~~impossible~~)

there is no c satisfies this

⇒ In conclusion, T has no fixes point.

**Problem 5**: Investigate a famous John Ball's example of elastic deformation ~~of~~ in a mathematical model of Nonlinear Elasticity.

$f(z) = z + \frac{z}{|z|}$ Notice $z = x+iy$ $f(z) = (x+iy) + \frac{x+iy}{\sqrt{x^2+y^2}}$

a) Show that $\det f(z) \neq 0 \Rightarrow$ so $f$ is locally 1-1

b) Find the range of $f : \mathbb{C}\setminus\{0\} \xrightarrow{\text{into}} \mathbb{C}$

c) Write the formula for the inverse of $f$ (from its range to $\mathbb{C}\setminus\{0\}$

Need to learn by heart: Give $f(z)$ where $z \in \mathbb{C}$, then $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$

$|z| = (z \cdot \bar z)^{1/2}$

$\overset{\uparrow}{\text{consider } \bar z \text{ as a constant}}$ $\overset{\uparrow}{\text{consider } z \text{ as a const}}$

(For ex: $g(z) = z^5 \bar z^7$

then $g_z = 5z^4\bar z^7$ $g_{\bar z} = 7z^5\bar z^6$

then $J_g(z) = (5z^4\bar z^7)^2 - (7z^5\bar z^6)^2$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

a) Show that $\det f'(z) \neq 0 \Rightarrow f$ is locally 1-1

- We have $f(z) = z + \frac{z}{|z|} = z + \frac{z}{(z\bar z)^{1/2}} = z + \frac{z^{1/2}}{(\bar z)^{1/2}}$

From this, we have

- $f_z = 1 + \frac{\frac{1}{2}z^{-1/2}}{(\bar z)^{1/2}} = 1 + \frac{1}{2}\frac{1}{(z\bar z)^{1/2}} = 1 + \frac{1}{2|z|}$
- $f_{\bar z} = z^{1/2}\left(-\frac{1}{2}\right)\bar z^{-3/2} = -\frac{1}{2}\frac{z^{1/2}}{\bar z^{3/2}} = -\frac{1}{2}\frac{z^{1/2}z^{1/2}}{\bar z^{1/2}z^{1/2}\bar z} = -\frac{1}{2}\frac{z}{|z|\bar z}$

So we have

$$J_f(z) = |f_z|^2 - |f_{\bar z}|^2 = \left(1 + \frac{1}{2|z|}\right)^2 - \frac{1}{4}\frac{1}{|z|^2} = 1 + \frac{1}{2|z|} > 0$$

So we have $J_f(z) = \det f'(z) \neq 0 \Rightarrow f$ is locally 1-1 □a).

b) Find the range of $f : \mathbb{C}\setminus\{0\} \xrightarrow{\text{into}} \mathbb{C}$

We want to find ~~f(z) for~~ values for all $f(z)$, where $|z| > 0$ (because $f$ is defined in $\mathbb{C}$

Put $w = f(z) = z + \frac{z}{|z|} = z\left(1 + \frac{1}{|z|}\right)$

$\Rightarrow |w| = |z|\left(1 + \frac{1}{|z|}\right) = |z| + 1$

We have $|z| > 0$

$\Rightarrow |w| = |z| + 1 > 1$.

$\Rightarrow$ Range of $f : \mathbb{C}\setminus\{0\}$ are all complex number $w$ with $|w| > 1$.

nother way to find range of $f:\mathbb{C}\setminus\{0\}\longrightarrow\mathbb{C}$ . (in fact, the same written in + way)

We know that $\mathbb{C}\setminus\{0\} = \bigcup_{r>0}\left(B(0,r)\setminus\{0\}\right)$

So we want to find the image of $f$ on the circle

onsider $z = x+iy$ where $|z|=r$ where $r>0$.

hen we have $f(z) = z+\frac{z}{|z|} = z\left(1+\frac{1}{r}\right)$

$$\Rightarrow |f(z)| = |z|\left(1+\frac{1}{r}\right) = |z|\left(1+\frac{1}{|z|}\right) = |z|+1.$$

this ~~means, through $f$, $z \longrightarrow f(z)$ r~~

this means, the range of $f:\mathbb{C}\setminus\{0\}\longrightarrow\mathbb{C}$ ~~is~~ contains all the complex number with $|w|>1$.

$\boxed{> \text{Find the inverse of } f \text{ (from it range back to } \mathbb{C}\setminus\{0\})}$

om b, we have $w = f(z) = z+\frac{1}{|z|}$

$$\Rightarrow z = \frac{w}{1+\frac{1}{|z|}} = \frac{w}{1+\frac{1}{|w|-1}} = \frac{w(|w|-1)}{|w|} = \left(1-\frac{1}{|w|}\right)w = w-\frac{w}{|w|} \text{ where } |w|>1.$$

$\Rightarrow$ The inverse of $f$ $\quad z = f^{-1}(w) = w-\frac{w}{|w|}$ where $|w|>1$.

P4. Using complex notation $z = x + iy \in \mathbb{C} \equiv \mathbb{R}^2$,
consider $f(z) = z + \frac{z^2}{|z|^2}$ defined for $z \neq 0$.

a) Identify the subset of $\mathbb{C}$ in which the Jacobian determinant of $f$ vanishes, that is, the subset of $\mathbb{C}$ in which the range rank of the matrix $f'(z)$ is less than 2 (equal to 1 or 0). Observe that outside this so-called "singular" set, the map $f$ is a local homeomorphism.

b) What is the image of the singular set

Singular set is the set such that Jacobiant determinant of $f$ is vanishes

$\boxed{a)}$ $f(z) = z + \frac{z^2}{|z|^2} = z + \frac{z^2}{z \cdot \bar{z}} = z + \frac{z}{\bar{z}}$

then $f_z = 1 + \frac{1}{\bar{z}}$

$f_{\bar{z}} = (-1)\frac{z}{(\bar{z})^2}$

The Jacobian determinant of $f$ vanishes when

$J_f(z) = 0$

$\Leftrightarrow |f_z|^2 - |f_{\bar{z}}|^2 = 0$

$\Leftrightarrow \left|1 + \frac{1}{\bar{z}}\right|^2 - \left|\frac{z}{(\bar{z})^2}\right|^2 = 0$

$\Leftrightarrow \left|\frac{\bar{z}+1}{\bar{z}}\right|^2 - \frac{1}{|\bar{z}|^2} = \frac{|\bar{z}+1|^2 - 1}{|\bar{z}|^2} = 0$

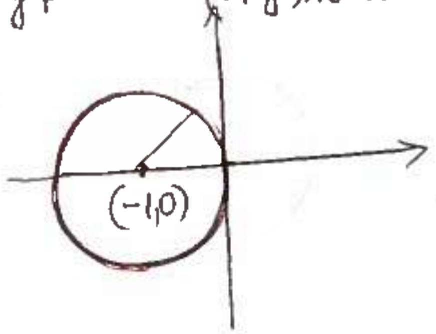
when $z = x + iy$, then $\bar{z} = x - iy \Rightarrow (\bar{z} + 1) = (x+1) - iy$

$\Rightarrow |\bar{z} + 1| = \sqrt{(x+1)^2 + y^2}$.

$\Rightarrow |\bar{z}+1|^2 - 1 = 0 \Leftrightarrow |\bar{z}+1|^2 = 1$

$\Leftrightarrow (x+1)^2 + y^2 = 1$.

f only when $(x, y)$ is on the circle with center $(-1,0)$ and radius 1 (*)



+ Outside of the so called "singular set", the set of z where Jacobian determinant of f vanishes, we have Jacobian determinant $J_f(z) > 0 \Rightarrow f$ is a local homeomorp

## * Using Rieman sum to find limit

Idea: Assume we have that we can compute $\int_0^1 f(x)\,dx$ . ( $f \in \mathcal{R}$ in $[0,1]$ )

Let divide $[0,1]$ into $n$ parts, with the length $\frac{1}{n}$ $P = \{x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \dots, x_1 = 1$

we have $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$

So we have we can compute $\lim \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$ $\left(x \Longleftrightarrow \frac{k}{n}\right)$ $\frac{k-1}{n}$

$\frac{k}{n} \Longleftrightarrow x$ $\Leftrightarrow$ $f(x) = x$ .

*** Compute $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)$**

We consider $\int_0^1 x\,dx$ , we have $f(x) = x$ continuous on $[0,1]$ $\Rightarrow$ Rieman integrable on $[0,1]$

and $\int_0^1 x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k$ we divide $[0,1]$ into $n$ equal partes. $P = \{x_0 = 0, x_1 = \frac{1}{n}, \dots, x_n = \frac{n}{n} = 1\}$ .

$= \lim_{n\to\infty} \sum_{k=1}^{n}\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)$

So we have $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$ □ .

*** Compute $\lim_{n\to\infty}\left(\frac{n}{n^2} + \frac{n}{1+n^2} + \frac{n}{2+n^2} + \dots + \frac{n}{(n-1)^2+n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(i-1)^2+n^2}$ =:(*)**

(note that in here we have $\sum_{i=1}^{n}$ ⇒ ok to use $\int$, and we need the form $\frac{i}{n}$ or $\frac{i-1}{n}$

Consider $\int_0^1 \frac{1}{x^2+1}\,dx$ , we have $f(x) = \frac{1}{x^2+1}$ continuous on $[0,1]$ $\Rightarrow$ $f \in \mathcal{R}$ on $[0,1]$ .

Consider partition $P = \{x_0 = 0, x_1 = \frac{1}{n}, \dots, x_n = \frac{n}{n} = 1\}$ including $n$ points with $x_0 = 0$, $x_n = 1$ ; $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$

we have $\int_0^1 \frac{1}{x^2+1}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta x_i \underset{t_i = x_{i-1}}{\overset{\text{choose}}{=\!=}} \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\frac{1}{n}$

( for $t_i$ is an arbitrary point in $x_{i-1}, x_i$)

$\overset{\uparrow}{=\!=} \lim \frac{1/n}{\left(\frac{i-1}{n}\right)^2 + 1}$

note $x_{i-1} = \frac{i-1}{n}$

Note that $(*) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(i-1)^2+n^2} \underset{\text{nume deno for } n^2}{\overset{\text{divide}}{=\!=}} \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{1}{n}}{\left(\frac{i-1}{n}\right)^2 + 1} \overset{\text{from above}}{=\!=\!=} \int_0^1 \frac{1}{x^2+1}\,dx$

$= \arctan x\Big|_0^1 = \arctan 1$ . □ .

* Find $\lim\limits_{n\to\infty}\sum_{k=1}^{(2n)}\frac{1}{n+k}$ $\left(\text{For } \lim\limits_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{n}f\left(\frac{k}{n}\right)\right)$

---

Let $(*) := \lim\limits_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{n+k} \xrightarrow[\text{ \& de for } 2n]{\text{divide num}} \lim\limits_{n\to\infty}\sum_{k=1}^{2n}\frac{\frac{1}{2n}}{\frac{1}{2}+\frac{k}{2n}}$

Now consider $\int_0^1 f(x)dx$ where $f(x)=\frac{1}{\frac{1}{2}+x}$

Consider partition $P=\left\{x_0=0, x_1=\frac{1}{2n}, x_2=\frac{2}{2n}, \cdots, x_k=\frac{k}{2n}, \cdots, x_{2n}=\frac{2n}{2n}=1\right\}$

(including $2n$ points with distant between each pair is $\Delta x_i=\frac{1}{2n}$)

We have $\int_0^1 f(x)dx = \lim\limits_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i = \lim \sum_{i=1}^{n}\frac{1}{\frac{1}{2}+\frac{i}{2n}}\cdot\frac{1}{2n}$

So we have ~~lim~~ $(*) = \int_0^1 \frac{1}{x+\frac{1}{2}}dx = \ln\left(x+\frac{1}{2}\right)\Big|_0^1 = \ln\left(\frac{3}{2}\right)-\ln\left(\frac{1}{2}\right) = \ln 3$

---

* Find $\lim \frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(n+n)} = \cancel{\lim\frac{1}{n}\sqrt[n]{\sum_{k=1}^{n}(n+k)}} \leftarrow$ don't do this way

---

$(*) = \lim\limits_{n\to\infty}\frac{1}{n}\sqrt[n]{\prod(n+k)} = \lim\sqrt[n]{\frac{\prod_{k=1}^{n}(n+k)}{n^n}} = \lim\limits_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\frac{n+k}{n}}$

$= \lim\limits_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)} = \lim\limits_{n\to\infty}(**)$

Notice that $\ln a^b = b\ln a$ and $a=e^{\ln a}$, we consider

~~ln(**)~~ $= \ln(**) = \ln\left[\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right]^{1/n} = \frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$

So $(**) = e^{\ln(**)} = e^{\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)}$.

$(*) = \lim\limits_{n\to\infty}(**) = \lim\limits_{n\to\infty} e^{\ln(**)} = e^{\lim\limits_{n\to\infty}(\ln(**))}$ (1)

- We now want to compute $\lim\limits_{n\to\infty}(\ln(**)) = \lim\limits_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\right]$

Consider $f(x)=\ln(1+x)$, consider $\int_0^1 f(x)dx = \lim\sum_{i=1}^{n} f(x_i)\Delta x_i = \lim\sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\cdot\frac{1}{n}$

so $\lim\limits_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\right] = \int_0^1 \ln(1+x)dx = (1+x)\ln(1+x)\Big|_0^1 - (1+x)\Big|_0^1 = 2\ln 2 - 1$ [✓ Checked]

$\Rightarrow (*) = e^{2\ln 2 - 1}$ ☐ checked

MAT602 Midterm exam, P2:

a) Show that the infinite series, defined for $0 \leq x < 1$, by the rule $f(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$ converges absolutely on $[0,1)$ to a continuous function on $[0,1)$.

b) Compute (explicitly) the function $f(x) = \dots$ to see that $f$ extends continuously to $[0,1]$.

c) Does the infinite series converge uniformly on the interval $[0,1)$?

a) Put $f_n(x) = \cancel{x^n} (-1)^n x^n$

We have $f(x) = \sum_{n=0}^{\infty} f_n(x)$

Now consider $\sum_{n=0}^{\infty} |f_n(x)| = \sum_{n=0}^{\infty} x^n$ converges for $x \in [0,1)$ to $\frac{1}{1-x}$, a continuous function.

Thus, the infinite series converges (absolutely) to $\frac{1}{1-x}$, a continuous function on $[0,1)$.

b) Compute explicity the function $f(x)$ to see that $f$ extend continuously to $[0,1]$.

We have $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$,

Thus, this is a continuous function on $[0,1]$.

c) Does the infinite series converge uniformly on the interval $[0,1)$.

Way 1: Use Dirichlet test for the uniform convergence of series.

Dirichlet test $\sum_{n=1}^{\infty} f_n(x) g_n(x)$
$\left.\begin{array}{l} \{f_n\} \text{ has uniformly bounded partial sum} \\ g_n(x) \geq g_{n+1}(x), \forall x \in E; \ g_n \rightrightarrows 0 \text{ in } E \end{array}\right\} \Rightarrow \sum f_n(x) g_n(x) \rightrightarrows \text{ on } E$.

We consider $\sum_{n=1}^{\infty} (-1)^n x^n = \sum_{n=1}^{\infty} h_n(x) g_n(x)$ where $h_n(x) = (-1)^n$, $g_n(x) = x^n$ $x \in [0,1)$.

we have $\left\{\begin{array}{l} \{h_n\} \text{ has uniformly bounded partial sum } \left|\sum_{n=1}^{k} h_n(x)\right| \leq 1, \forall k \\ g_n(x) = x^n \geq x^{n+1}, \forall x \in [0,1) \\ g_n(x) = x^n \cancel{\rightrightarrows} 0 \text{ on } [0,1) \end{array}\right.$

wrong we can't use this because $g_n(x) = x^n \not\rightrightarrows 0$ on $[0,1)$

Then by Dirichlet test, $\sum (-1)^n x^n$ converges uniformly on $[0,1)$.

Way 2 (Next page)

c) Does the series $\sum_{n=1}^{\infty} (-1)^n x^n$ converges uniformly on $[0,1)$ ?

We want to use comparison test: $\left.\begin{array}{l} \sum f_n(x) \\ \sup_{x\in E} |f_n(x)| \le M_n \\ \sum M_n \text{ converges} \end{array}\right\}$ Then $\sum f_n(x) \rightrightarrows$ in $E$

Put $s_\ell(x) = \sum_{n=1}^{\ell} f_n(x) = \sum_{n=1}^{\ell} (-1)^n x^n = \frac{1-(-x)^{\ell+1}}{1+x}$

We have from (a), $\sum_{n=1}^{\infty} f_n(x) \xrightarrow{\text{absolutely}} \frac{1}{1+x}$ on $[0,1)$ $\Rightarrow$ converges pointwise.

We want to consider if $s_\ell(x) \rightrightarrows \frac{1}{1+x}$ on $[0,1)$.

So we want to ~~prove~~ that $\left|s_\ell(x) - \frac{1}{1+x}\right| \rightrightarrows 0$ on $[0,1)$ or not.

- Now we consider

$$\left|s_\ell(x) - \frac{1}{1+x}\right| = \left|\frac{1-(-x)^{\ell+1}}{1+x} - \frac{1}{1+x}\right| = \left|\frac{(-x)^{\ell+1}}{1+x}\right|$$

Now we want $\exists \varepsilon > 0$, $\forall n$ large, $\exists$ $x$ on $[0,1)$, $\left|\frac{(-x)^{\ell+1}}{1+x}\right| > \varepsilon$.

Let $\ell = 2n-1$, then $\left|\frac{(-x)^{\ell+1}}{1+x}\right| = \left|\frac{(-x)^{2n}}{1+x}\right|$

Set $x = 1 - \frac{1}{2n}$ (notice that $x^n \not\rightrightarrows 0$ at $[0,1)$ with "special point near 1)

then $$\left|\frac{(-x)^{2n}}{1+x}\right| = \frac{\overbrace{\left(1-\frac{1}{2n}\right)^{2n}}^{\to e^{-1}}}{\underbrace{1+1-\frac{1}{2n}}_{\to 2}} \longrightarrow \frac{2}{e} \neq 0$$

So $\left|s_\ell(x) - \frac{1}{1+x}\right| \not\longrightarrow 0$ thus, $\left|s_\ell(x) - \frac{1}{1+x}\right| \not\rightrightarrows$ on $[0,1)$ $\square$

Mat602 Midterm P3.

Let $F$: family of continuous functions defined on $[0,1]$ s.t

i) $f(0)=0$, $\forall f\in F$.

ii) $F$ is equicontinuous.

a) Prove that $F^2=\{ f^2 \mid f\in F\}$ is equicontinuous.

b) Give an example showing that the condition (i) is neccesary.

* More advance problem (taken from the advance problem from 602 midterm exam).

Consider $f(x) = \ln x$ , $0 < x \leq 1$.

Show that $\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$

## * Fiding limit by using Riemann sum.

* Example: Find $\lim \sum_{k=1}^{n} \frac{2k+1}{n^2+k^2}$

- We have
$$\sum_{k=1}^{n} \frac{2k+1}{n^2+k^2} = \sum_{k=1}^{n} \frac{2k}{n^2+k^2} + \sum_{k=1}^{n} \frac{1}{n^2+k^2}$$

- Consider $\sum_{k=1}^{n} \frac{2k}{n^2+k^2} = \sum_{k=1}^{n} \frac{\frac{2k}{n}}{1+\left(\frac{k}{n}\right)^2} \cdot \frac{1}{n}$

  put $f(x) = \frac{2x}{1+x^2}$

  Then we have
$$\sum_{k=1}^{n} \frac{2k}{n^2+k^2} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{1}{n} \xrightarrow[n\to\infty]{} \int_0^1 f(x)\,dx = \int_0^1 \frac{2x}{1+x^2}\,dx =$$
$$\xrightarrow[n\to\infty]{} \ln(1+x^2)\Big|_0^1 = \ln 2$$

- Consider $\sum_{k=1}^{n} \frac{1}{n^2+k^2} = \sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} \frac{1}{n^2}$

  put $g(x) := \frac{1}{1+x^2}$

  Then $\sum_{k=1}^{n} \frac{1}{n^2+k^2} = \sum_{k=1}^{n} g\left(\frac{k}{n}\right) \frac{1}{n^2} = \int$

Evaluating a limit of series using Riemann integral (more advance)

Find $\lim_{n\to\infty} n \sum_{j=1}^{n} \frac{\cos\left(\frac{n}{j}\right) f\left(\frac{n}{j}\right)}{j^2}$ where $f$ is $C^\infty$ and monotonically decreasing $\lim_{x\to\infty} f(x) = 0$.

+ Define $g(x) = \frac{\cos\left(\frac{1}{x}\right) f\left(\frac{1}{x}\right)}{x^2}$

then $n \sum_{j=1}^{n} \frac{\cos\left(\frac{n}{j}\right) f\left(\frac{n}{j}\right)}{j^2} \textcircled{=} \sum_{j=1}^{n} \frac{1}{n}\, g\left(\frac{j}{n}\right) \xrightarrow[n\to\infty]{} \int_0^1 g(x)\,dx$ . =

- $\int_0^1 g(x)\,dx = \int_0^1$

* Example: $\lim_{n \to \infty} \left[ \sin\left(\frac{n}{n^2+1}\right) + \sin\left(\frac{n}{n^2+2^2}\right) + \cdots + \sin\left(\frac{n}{n^2+n^2}\right) \right]$

# MAT 602 Fundamentals of Analysis

## Practice Exam 1

### March 7. 2017

**Choose 4 out of the following 5 problems.**

1. Find the limit $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right]$. Justify your answer.

2. Let $f_k : [0,1] \hookrightarrow \mathbb{R}$,
$$f_k(x) = x^k(1-x).$$
   (a) Prove that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[0,a]$ for $0 < a < 1$.
   (b) Does the series $\sum_{k=1}^{\infty} f_k$ converge uniformly on $[0,1]$? Justify your answer.

3. For $n = 1, 2, 3, \ldots$, let
$$f_n(x) = \begin{cases} 1 & \text{if } \ x \in \{1, \frac{1}{2}, \ldots, \frac{1}{n}\} \\ 0 & \text{otherwise.} \end{cases}$$
Assume that $\alpha : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function, show that
$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\, \mathrm{d}\underbrace{\alpha(x)} = \int_{-1}^{1} \underbrace{\lim_{n\to\infty} f_n(x)}\, \mathrm{d}\alpha(x). \quad = \underbrace{\int_{\varepsilon}^{1} f\, d\alpha}_{0} + \underbrace{\int_{0}^{\varepsilon}}_{\text{[illegible]}}$$

4. Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, \ldots$ with $|f_n'(x)| \leqslant 1$ for all $n$ and $x$. Assume
$$\lim_{n\to\infty} f_n(x) = g(x) \qquad \text{for all } x \in \mathbb{R}.$$
Prove that $g\colon \mathbb{R} \to \mathbb{R}$ is continuous.

5. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $f' \in \mathcal{R}$ (Riemann integrable) on $[0,1]$, show that
$$\lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\, \mathrm{d}x = 0.$$
   (b) Is the claim in part (a) true if $f$ is only continuous on $[0,1]$? Justify your answer.

1) Find the limit $\lim \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right]$.
Justify your answer

$$\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\;+\left(\frac{n}{n}\right)^2\right]=\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 \underset{\text{Put } f(x)=x^2}{=\!=\!=} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \cancel{\int_0^1 x^2\,dx} =$$

$$\xrightarrow{n\to\infty}\int_0^1 f(x)\,dx=\int_0^1 x^2\,dx=\left.\frac{x^3}{3}\right|_0^1=\frac{1}{3}.$$

2) Let $f_k:[0,1]\longrightarrow R$
$x \longmapsto f_k(x)=x^k(1-x)$

a) Prove that the series converge uniformly on $[0,a]$ for $0<a<1$.

* We have for $x\in[0,a]$, for $0<a<1$:

$$\left.\begin{array}{l}|x^k(1-x)|<|x^k|<a^k\\ \sum_{k=1}^{\infty}a^k \text{ converges}\end{array}\right\}\text{ then by theorem 7.10 } \sum_{k=1}^{\infty}f_k(x)\text{ converges.}$$

b) Does the series $\sum_{k=1}^{\infty}f_k$ converges uniformly on $[0,1]$. Justify your answer.

*Way 1: Use the property that $\left.\begin{array}{l}s_n \text{ continuous}\\ s_n\rightrightarrows s\end{array}\right\}$ then $s$ continuous: $(*)$

- Step 1: Find the 'pointwise' limit of the series.
  + We have

$$\sum_{k=1}^{\infty}f_k(x)=\sum_{k=1}^{\infty}x^k(1-x)=\begin{cases}(1-x)\sum_{k=1}^{\infty}x^k\xrightarrow{k\to\infty}(1-x)\frac{1}{1-x}=1 & \text{when } 0<x<1\\ 0 & \text{when } x=0,1.\end{cases}$$

- This means $f(x)=\begin{cases}1 & 0<x<1\\ 0 & x\in\{0,1\}\end{cases}$
- Step 2: Use the $(*)$ property to conclude that $\sum_{k=1}^{\infty}f_k$ does not converge uniformly

Way 2: To prove that the series $\sum_{l=1}^{\infty} x^l(1-x)$ does not converge uniformly

Put $s_N(x) = \sum_{l=1}^{N-1} f_l(x) = \sum_{l=1}^{N-1} x^l(1-x) = (1-x)\sum_{l=1}^{N-1} x^l \textcircled{=} 1 - x^N$

We have the series $\sum_{l=1}^{\infty} f_l(x)$ converges uniformly iff $s_N(x)$ converges uniformly

- We have for $0<x<1$, $s_N(x) \xrightarrow[N\to\infty]{} 0$

  and also $s_N(x)$ is continuous $[0,1]$ then $s(x)$ has to be continuous on $[0,1]$

  If $s_N(x) \rightrightarrows s(x)$ ) this means $s(0) = s(1) = 0$

- Now we prove that $s_N(x) \not\rightrightarrows 0$ by proving that .

  $\exists \epsilon > 0$, $\forall n \in \mathbb{N}$, $\exists x_N \in [0,1]$ such that $|s_N(x) - s(x)| > \epsilon$

Choose $x_N = ~~1~~ 1 - \frac{1}{N}$ n

Then we have $s_N(x_N) = 1 - \left(1 - \frac{1}{N}\right)^N \xrightarrow[N\to\infty]{} 1 - \frac{1}{e} \neq 0$ ∎

Note that $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$

37 For $n = 1, 2, 3, \dots$

Let $f_n(x) = \begin{cases} 1 & \text{if } x \in \{1, \frac{1}{2}, \dots, \frac{1}{n}\} \\ 0 & \text{otherwise} \end{cases}$

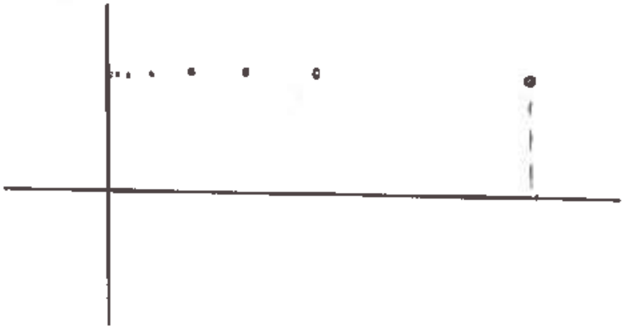
Assume that $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing continuous function. Show that

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\, d\alpha(x) = \int_{-1}^{1} \lim_{n\to\infty} f_n(x)\, d\alpha(x) \quad (*)$$

The idea of this exercise is proving that in this case $f_n \not\rightrightarrows$ ut we still have (*)

* First, I prove one result that we can understand the ideal of exercise :

Prove that $f_n(x) \not\rightrightarrows$ in $[-1, 1]$.



* Put $f(x) = \begin{cases} 1 & x = \frac{1}{k}, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

Then we have $f_n(x) \xrightarrow[\text{pointwise}]{n\to\infty} f(x)$

Thus: • for $x \neq \frac{1}{k}, k \in \mathbb{N}$, $\begin{cases} f_n(x) = 0 \\ f(x) = 0 \end{cases}$

• For $x = \frac{1}{k}$ for some $k \in \mathbb{N}$, then $\forall n \geq k$, $\frac{1}{n} \leq \frac{1}{k}$

Then $\forall m \geq n$, we have $|f_m(x) - f(x)| = |1 - 1| = 0 < \varepsilon$, $\forall \varepsilon >$

* But we have $f_n(x) \not\rightrightarrows f(x)$.

We need to prove $\exists \varepsilon > 0, \forall n \in \mathbb{N}, \exists x_0, |f_n(x_0) - f(x_0)| > \varepsilon$

$f_n(x) = \begin{cases} 1, & x \in \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\} \\ 0, & \text{otherwise} \end{cases}$

$f(x) = \begin{cases} 1, & x \in \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots, \frac{1}{n+l}, \dots\} \\ 0, & \text{otherwise} \end{cases}$

Then $\forall n$, $\exists x_0 = \frac{1}{n+1}$, at $x_0$, $f_n(x_0) = 0$

$f(x_0) = 1$

So $|f_n(x_0) - f(x_0)| > \cancel{1} > \varepsilon \cdot \frac{1}{2}$

Thus $f_n \not\rightrightarrows f(x)$.

* Now we prove the problem:

$\alpha : R \longrightarrow \mathbb{R}$ increasing function, $\alpha$ is continuous

Prove that $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\, d\alpha(x) = \int_{-1}^{1} \lim_{n\to\infty} f_n(x)\, d\alpha(x)$ (*)

where $f_n(x) = \begin{cases} 1 & x = \{1, \frac{1}{2}, \dots, \frac{1}{n}\} \\ 0 & \text{otherwise} \end{cases}$

* First, evalute LHS (*)

• $\int_{-1}^{1} f_n(x)dx = \underbrace{\int_{-1}^{0} f_n(x)\, d\alpha(x)}_{\substack{=0 \\ \text{because } f_n(x) = 0}} + \underbrace{\int_{0}^{1} f_n(x)\, d\alpha(x)}$

We note that $f_n(x)$ has finitely many points that it is discontinuous
$\alpha$ is continuous at those points

$\Rightarrow$ we have $f_n(x) \in \mathscr{R}(\alpha)$ in $[0,1]$

* Now because $\alpha$ is continuous, we can choose partion $P$ such that $\Delta\alpha_i < \varepsilon$

Review Rudin E 6.1

Then we have $\left| \int f_n(x)\, d\alpha(x) - \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \right| < \varepsilon$ for $t_i$ is an arbitrary point in $[x_{i-1}, x_i]$

$\rightarrow \left| \int_{0}^{1} f_n(x)\, d\alpha(x) \right| \leq \sum f(t_i)\Delta\alpha_i + \varepsilon \leq 1 \cdot \varepsilon + \varepsilon = 2\varepsilon$

$\Rightarrow \int_{0}^{1} f_n(x)\, d\alpha(x) = 0$

$\Rightarrow \int_{-1}^{1} f_n(x)dx = 0, \forall n$

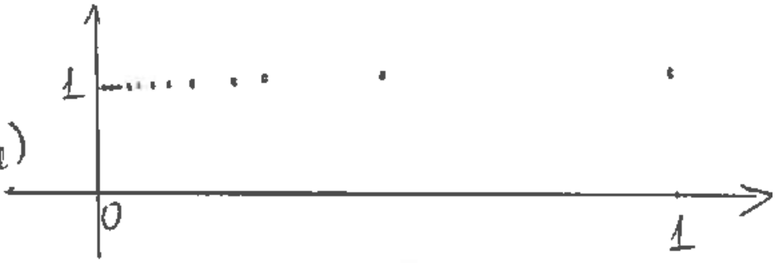
$\rightarrow \lim \int_{-1}^{1} f_n(x)dx = 0, \forall n$

+ Second, we now evalute $\int_{-1}^{1} \lim_{n\to\infty} f_n(x)\, d\alpha(x) = \int_{-1}^{1} f(x)\, d\alpha(x)$

where $f(x) = \begin{cases} 1 & , x = \frac{1}{k} \ , k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$.

+ we have

$$\int_{-1}^{1} f(x)\, d\alpha(x) = \underbrace{\int_{-1}^{0} f(x)\, d\alpha(x)}_{=0} + \int_{0}^{1} f(x)\, d\alpha(x)$$



Because $\alpha$ is continuous, we could choose partition $P = \{x_0, x_1, \dots, x_n\}$ such that $\Delta\alpha_i < \varepsilon$.

then we have Let $\varepsilon > 0$, $\exists n$ st $\frac{1}{n} < \delta$, such that $\alpha(\frac{1}{n}) - \alpha(0) < \varepsilon$

5a) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be differentiable

$f'\in\mathcal{R}$ (Riemann integrable) on $[0,1]$.

Show that $\lim_{n\to\infty}\int_0^1 f(x)\,dx\,\sin(nx)dx = 0$

b) Is the claim in part (a) true if $f$ is only continuous on $[0,1]$?

---

a) In case $f'\in\mathcal{R}$

we have $(\sin nx)$ is a differentiable function

$\Big\}\Rightarrow$ we can use integration by part to solve this problem.

- Now consider $\int_0^1 f(x)\sin(nx)\,dx$.

Put $g(x) = -\cos nx$

$g'(x) = n\sin nx$.

Then $\int_0^1 f(x)\sin(nx)dx = \frac{1}{n}\int_0^1 f(x)\,n\sin nx\,dx = \frac{1}{n}\int_0^1 f(x)\,g'(x)\,dx = \frac{1}{n}\left[fg\Big|_0^1 - \int_0^1 f'(x)g(x)dx\right.$

$$= \underbrace{\frac{1}{n}\left[-f(1)\underbrace{\cos(n)}_{\le 1}+f(0)\underbrace{\cos 0}_{=1}\right]}_{:=(I)} + \underbrace{\frac{1}{n}\int_0^1 f'(x)\cos nx\,dx}_{:=(II)}$$

* Now consider (I)

we have $f$ is continuous on $[0,1]$ $\Rightarrow$ bounded in $[0,1]$.

$$\Rightarrow \cancel{\tfrac{1}{n}}\ I \le \frac{1}{n}M \xleftrightarrow{n\to\infty} 0$$

* Consider (II)

$$\left|\int_0^1 f'(x)\cos nx\,dx\right| \le \int_0^1 |f'(x)\cos nx|\,dx \le \int_0^1 |f'(x)|\,dx.$$

$f'$ is Riemann integrable $\Rightarrow$ bounded

$$\le \int_0^1 M_2\,dx = M_2$$

$$\rightarrow \quad (II) \le \frac{1}{n}M \xrightarrow{n\to\infty} 0$$

Then from evaluating (I) and (II), we have $\lim_{n\to\infty}\int_0^1 f(x)\sin(nx)\,dx = 0$

$$\boxed{\text{B7 Prove that we still have } \lim_{n\to\infty}\int_0^1 f(x)\sin nx = 0 \text{ when we only have the assumption that } f \text{ is continuous on } [0,1]}$$

* Because $f:[0,1] \longrightarrow \mathbb{R}$ is continuous, by Weierstrass approximation theorem, there exists a sequence $\{P_k\}$ of polynomials such that

$$P_k(x) \rightrightarrows f(x) \text{ on } [0,1].$$

. This means $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, \forall x \in [0,1], \ |f(x) - P_k(x)| < \varepsilon$

Take $k$ large enough, we have $|f(x) - P_k(x)| < \varepsilon$

- Now consider $\left|\int_0^1 f(x)\sin nx \, dx\right| = \left|\int_0^1 \{[f(x) - P_k(x)] + P_k(x)\}\sin(nx)\, dx\right|$

$$\leq \underbrace{\int_0^1 \underbrace{|f(x) P_k(x)|}_{<\varepsilon}\underbrace{|\sin nx|}_{<1} dx}_{<\varepsilon} + \underbrace{\int_0^1 |P_k(x)||\sin nx| \, dx}_{=(*)}.$$

* We can prove $(*) \xrightarrow{n\to\infty} 0$ by applying 5a or we can prove directly

$$\int_0^1 P_k(x)\sin nx\, dx = \sum_{i=1}^{k}\int_0^1 a_k x^i \sin nx \, dx = \sum_{i=1}^{k} a_i \int_0^1 x^i \sin nx\, dx.$$

$$\int_0^1 x^i \sin nx \, dx = \underbrace{\frac{1}{n}\left(1^i \cos n + 0\right)}_{\xrightarrow{n\to\infty} 0} + \underbrace{\frac{1}{n}\int_0^1 i\, x^{i-1}\cos nx \, dx}_{\xrightarrow{n\to\infty} 0}$$

Rudin 8 1/196

Tran Le MAT 6

HW 9

Define $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ Prove that $f$ has derivatives of all order at $x = C$ and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots$

10/10

**+ Compute $f'(x)$**

- $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{x\to 0} \frac{\frac{1}{x}}{e^{-1/x^2}} = \lim_{x\to 0} \frac{x \frac{1}{x^2}}{e^{1/x^2}}$

Put $y = \frac{1}{x^2}$, we have $y \xrightarrow[x\to 0]{} +\infty$, then we have $\lim_{x\to 0} \frac{\frac{1}{x^2}}{e^{1/x^2}} = \lim_{y\to+\infty} \frac{y}{e^{y}} = 0$ (theorem 8.6

then $f'(0) = 0$

- when $x \neq 0$, $f'(x) = (e^{-1/x^2})' = 2 \frac{1}{x^3} e^{-1/x^2}$ $(*)$

+ Claim that for $x \neq 0$, $f^{(n)}(x) = \frac{p(x)}{q(x)} e^{-1/x^2}$ for some $p(x)$ and $q(x)$ are polynomials and $q(x)$ is only have the form $x^k$ for some $k$.

Prove claim by induction:

- By $(*)$, claim is true when $n = 1$.
- Assume $f^{(n)}(x) = \frac{p(x)}{q(x)} e^{-1/x^2}$ for $p(x)$ and $q(x)$ are polynomial $q(x) = x^k$ for some $k \geq 1$

then we have

$$f^{(n+1)}(x) = \left(\frac{p(x)}{q(x)}\right)' e^{-\frac{1}{x^2}} + \frac{p(x)}{q(x)} \frac{2}{x^3} e^{-\frac{1}{x^2}}$$

$$= \left[\frac{p'(x)q(x) + p(x)q'(x)}{q^2(x)} + \frac{p(x)}{q(x)} \frac{2}{x^3}\right] e^{-\frac{1}{x^2}}$$

$$= \frac{p'(x)q(x)x^3 + p(x)q'(x)x^3 + 2p(x)q(x)}{q^2(x)x^3} e^{-\frac{1}{x^2}}$$

$$= \frac{P(x)}{Q(x)} e^{-\frac{1}{x^2}}$$ for $P(x), Q(x) = q^2(x)x^3 = x^{2k+3}$ are polynomials and $Q(x)$ has form $x^k$ for some $k \geq$ □ claim

+ Now we prove that $f^{(n)}(0) = 0$ also by induction,

- From above $f'(0) = 0$
- Assume $f^{(n)}(0) = 0$

Then we have $f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{p(x)}{q(x)\, x} e^{-1/x^2} =$

$$= \lim_{x\to 0} \frac{P(x)}{Q(x)} e^{-1/x^2}$$ (where $P(x) = p(x)$, $Q(x) = q(x) \cdot x$.)

by dividing $P(x)$ for $Q(x) = x^k$ $\longrightarrow$ $= \lim_{x\to 0} \left(\sum_{i=1}^{n} a_i x^i\right) e^{-1/x^2} = 0$

theorem 8.6 f ($\lim y^n e^{-y} = 0$)

HW
Tran L

20/20

Rudn 8.6/197

Suppose $f(x) f(y) = f(x+y)$ for all real $x$ and $y$

a) Assume that $f$ is differentiable and not zero, prove that $f(x) = e^{cx}$ where $c$ is a constant.

b) Prove the same thing, assuming only that $f$ is continuous.

$\boxed{a}$

* We first notice that:

$$\left.\begin{array}{l} f(x) f(y) = f(x+y) \xrightarrow{\text{let } y=0} f(x) f(0) = f(x) \\ \text{by assumption, } f(x) \text{ is not zero} \end{array}\right\} \Rightarrow f(0) = 1 \qquad (*)$$

* We have by assumption, $f$ is differentiable $\Rightarrow \exists f'(0)$. put $c := f'(0)$. $\quad (**)$

Then for $x \neq 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)[f(h) - 1]}{h} =$$

$$= f(x) f'(0) = c \cdot f(x).$$

So we have $f'(x) = c \cdot f(x) \qquad (***)$

* Now we put $g(x) := e^{-cx} f(x)$

$$\Rightarrow g'(x) = -c e^{-cx} f(x) + e^{-cx} f'(x) \underset{\text{by}(***)}{=} -c\, e^{-cx} f(x) + c f(x) e^{-cx} =$$

$\forall$

This means $g(x)$ is a constant function $g(x) = g(0) = f(0) \underset{\uparrow \text{ by}(*)}{=} 1 \quad \forall x$.

$$\Rightarrow e^{-cx} f(x) = 1, \ \forall x$$

$$\Rightarrow f(x) = e^{cx}$$

$$\left.\begin{array}{l} f(x+y)=f(x)f(y),\ \forall x,y\in\mathbb{R} \\ f \text{ is continuous},\ \text{~~not zero~~} \end{array}\right\} \Rightarrow \text{Prove that } f(x)=e^{cx} \text{ for some constant } c$$

Note that from $f(x+y)=f(x)f(y)$, $\forall x,y\in\mathbb{R}$ we have $f(x)>0, \forall x$ (1)

( This is because . $f(x)=f\left(\frac{x}{2}+\frac{x}{2}\right)=\left[f\left(\frac{x}{2}\right)\right]^2 \geqslant 0 \Rightarrow f(x)\geqslant 0, \forall x$.

. Then assume $\exists x_0$ such that $f(x_0)=0$,

then $f(x)=f(x_0+(x-x_0))=\underbrace{f(x_0)}_{=0} f(x-x_0)=0, \forall x$ (nothing to do with this) )

Then we have

$$\underbrace{f(x)}_{>0,\forall x}=f(x+0)=\underbrace{f(x)}_{\substack{>0,\forall x\\ \text{by }(1)}} f(0) \Rightarrow f(0)=1. \quad (2).$$

Put $g(x)=\log(f(x))$. Then it suffices to prove that $g(x)=cx$ for some constant c (*

~~From~~ We have $\left\{\begin{array}{l} g(x) \text{ is a continuous function (property of log)} \quad (3) \\ g(x+y)=\log(f(x+y))=\log(f(x)\cdot f(y))=\log(f(x))+\log(f(y))=g(x)+g(y) \text{ (} \\ \forall x,y \end{array}\right.$

Now we prove that (*) is true for $n\in\mathbb{Z}$

⊕ For $n\in\mathbb{N}$

$$\left.\begin{array}{l}\left.\begin{array}{l} g(n)=g(1+1+\cdots+1) \underset{\text{by }(4)}{=\!=} n\,g(1) \\ \text{Put } c:=g(1) \end{array}\right\} \Rightarrow g(n)=cn,\ \forall n\in\mathbb{N} \\ \ominus\ \begin{array}{l} g(-n+n) \underset{\text{by }(4)}{=\!=} g(-n)+g(n) \\ \| \\ g(0)=\log(f(0))=0 \end{array} \Rightarrow g(-n)=0-g(n)=-cn \end{array}\right\} \Rightarrow (*) \text{ is true for } n\in\mathbb{Z}$$

Now we prove (*) is true for all $\frac{p}{q}$, $\begin{array}{l} p\in\mathbb{Z} \\ q\in\mathbb{N}, q\neq 0 \end{array}$.

- We have

$$\begin{array}{l} g(p)=g\left(q\cdot\frac{p}{q}\right) \underset{\substack{\text{by }(4)\text{ and}\\ \text{because } q\in\mathbb{N}}}{=\!=\!=} q\,g\left(\frac{p}{q}\right) \\ \| \ p\in\mathbb{Z} \text{ from above} \\ cp \end{array} \Rightarrow g\left(\frac{p}{q}\right)=c\cdot\frac{p}{q} \Rightarrow \checkmark$$

(*) is true for all $x\in\mathbb{R}$ / because rational numbers are dense in $\mathbb{R}$

$$\left.\begin{array}{l} \text{then } \forall x \text{ rational}, \exists (x_n) \text{ irrational } x_n\longrightarrow x \\ \text{From above } g(x_n)=c\,x_n, \forall x_n\in\mathbb{Q} \\ g \text{ is continuous (by 3) and } c\,x_n \text{ continuous} \end{array}\right\} \Rightarrow \begin{array}{l} g(x)= \\ \forall x\in\mathbb{R} \end{array}$$

∎

* Question: (Relating to Picard's existence and Uniqueness theorem).
Let $\phi : [a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ be continuous
$|\phi(t,x) - \phi(t,y)| \le L|x-y|$ ($\phi$ is Lipchitz with 2nd variable).
Consider IVP: $\begin{cases} f'(x) = \phi(t, f(t)) & \text{for } a \le t < b \\ f(t_0) = t_0 \end{cases}$ (*)
Prove that this IVP has unique solution near $t_0$

* Notice that under the assumptions of existence, we have the IVP(*) is equivalent to

$$f(t) = f(t_0) + \int_{t_0}^{t_x} \phi(s, f(s))\,ds.$$

* So now we define an operator $T : f(\ ) \longmapsto T(f)(\ )$ with

$$T(f)(t) = f(t_0) + \int_{t_0}^{t} \phi(s, f(s))\,ds \qquad (**)$$

Then if we can prove that (**) has a unique solution near to (which means T(f) has a fixed point), then it means we can prove that (*) has a unique solution near $t_0$

* We will prove (**) has a unique solution by proving that T is a contraction (from a complete space to itself):

- We have

$$T(f_1)(t) - T(f_2)(t) = \int_{t_0}^{t} \underbrace{\phi(s, f_1(s)) - \phi(s, f_2(s))}_{\le L|f_1(s) - f_2(s)|}\,ds.$$

(because of assumption that $\phi$ is Lipchitz with 2nd variabl

$$\le \int_{t_0}^{t} L\,|f_1(s) - f_2(s)|\,ds.$$

$$\le \int_{t_0}^{t} L\,\|f_1 - f_2\|\,ds.$$

$$|T(f_1)(t) - T(f_2)(t)| \le L\,\|f_1 - f_2\|\,|t - t_0|.$$

Then when we choose t near to $t_0$, we have

$$|T(f_1)(t) - T(f_2)(t)| \le \underbrace{k}_{<1}\|f_1 - f_2\|$$ ⇒ T is a contraction ⇒ done. □

uestion :

: any set

$\varphi : X \longrightarrow X$

here is $k$ such that the $k^{th}$ iteration $\underbrace{\varphi\circ\varphi\circ\varphi\cdots\circ\varphi}_{k\text{ times}} : X \longrightarrow X$ has exactly one fixed point

rove that : $\varphi$ has exactly one fixed point .

e have $\varphi^{k} : X \longrightarrow X$ has exactly one fixed point $\Longleftrightarrow \exists! x \in X,\ \varphi^{k}(x) = x$

$\Rightarrow \quad \varphi(\varphi^{k}(x)) = \varphi(x)$ .

$\Rightarrow \quad \varphi^{k}[\varphi(x)] = \varphi(x)$ . ~~for any~~

$\Rightarrow \varphi(x)$ is also a fixed point of $\varphi^{k}$
by the uniqueness of fixed point $\Big\} \Rightarrow \varphi(x) = x$

$\Rightarrow$ ~~$\varphi$ is~~ $x$ is a fixed point of $\varphi$
~~bec~~ from above $x$ is unique $\Big\} \Rightarrow$ done $\square$

* Rudin 9.9/239

If $f$ is a differentiable mapping of a connected open set $E \subseteq \mathbb{R}^n$ } Prove that $f$ is
$f'(x) = 0$ for every $x \in E$ . } constant in $E$

* Now we prove that $f$ is locally constant:

Because $E$ is open, $\forall x \in E$, $\exists N_\delta(x) \subset E$ .

then $\forall y \in N_\delta(x)$, $|f(y) - f(x)| \leq M|y - x|$ where $M = \sup_{x \in E} \|f'(x)| \underset{\uparrow}{=} 0$

because $f'(x) = 0, \forall x \in$

$\Rightarrow |f(y) - f(x)| \leq 0|y - x| \Rightarrow f(y) = f(x)$, $\forall y \in N_\delta(x)$

which means, $f$ is locally constant .

* Now consider $x_0 \in E$, Let $A := \{x \in E,\ f(x) = f(x_0)\}$ .

then because $f$ is locally constant, $A$ is open in $E$

* We also have $A$ is a closed subset of $E$ (intersection of $E$ and a closed set in $\mathbb{R}^n$)

$\Rightarrow$ We have $A \neq \emptyset$, closed and open in $E$ . } $\Rightarrow A = E$, which means $f$ is
we also have assumption that $E$ is connected } constant in $E$ $\square$ .

Aug 2002, 1 One of Prof Kovalev's review questions

Let $f:(0,1) \longrightarrow \mathbb{R}$ be continuous, bounded, and decreasing
Prove $f$ is uniformly continuous on $(0,1)$.

o understand more about this pu
we consider a similar problem
on back of this page

* Some things needed to notice in this question:

- Theorem: every bounded sequence in $\mathbb{R}$ has a convergent subsequence
  then because $(0,1)$ bounded in $\mathbb{R}$, a sequence $(x_n)$ in $(0,1)$ has a convergent subsec
  in $\mathbb{R}$.

  and note that $(x_{n_k}) \longrightarrow x \in [0,1]$

\+ If $f$ ~~cont~~ $:[0,1] \to \mathbb{R}$, $f$ continuous in $[0,1] \Rightarrow f$ is uniformly continuous in $[0,1]$.
In this question: $f:(0,1) \to \mathbb{R}$. $f$ need to be bounded + monotone

\+ Assume $f$ is not uniformly continuous on $(0,1)$
$\Leftrightarrow \exists \varepsilon_0 > 0$, $\exists (x_n)(y_n)$ on $(0,1)$ $\boxed{\forall n}$ $|x_n - y_n| < \frac{1}{n}$, and $|f(x_n) - f(y_n)| > \varepsilon_0$ $\boxed{\forall n}$ (I

\+ We consider $(x_n)$ in $(0,1)$. $\Rightarrow (x_n)$ bounded in $\mathbb{R}$.
$\Rightarrow \exists (x_{n_k})$, $x_{n_k}$ converges to a point $x$ in $[0,1]$

- In case $x \in (0,1)$, we have $f$ continuous on $(0,1)$ $\Rightarrow$ $f(x_{n_k}) \to f(x)$. (1)

We also have because $|x_n - y_n| < \frac{1}{n}$, $\forall n$ $\Rightarrow y_{n_k} \longrightarrow x$ (see back of this page for more detail)
and so because $f$ is continuous on $(0,1)$
$f(y_{n_k}) \longrightarrow f(x)$ (2)

(1) + (2) $\Rightarrow$ for $k$ big enough, $|f(x_{n_k}) - f(y_{n_k})| < \varepsilon$, $\forall \varepsilon$ this contradicts with (

- In case $x = 0$ (we need the assumption $f$ bounded + decreasing here).

* Note: If $E \neq \phi$, $E \subset \mathbb{R}$.
If $\exists \sup E \Rightarrow \exists (x_n)$ in $E$, $x_n \to \sup E$. (Jan 2016, E1)

Prove that: If $f$ is not uniformly continuous in $X$ ~~then~~ iff

$\exists \varepsilon_0 > 0$, $\exists (x_n)(y_n) \in X$ s.t $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geqslant \varepsilon_0$, $\forall n \in \mathbb{N}$.

Prove that $\left.\begin{array}{l} f \text{ continuous in } X \\ X \text{ compact} \end{array}\right\} \Rightarrow f$ uniformly continuous in $X$.

---

Def of uniformly continuous: $f$ is uniformly continuous in $X$

$\iff \forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x, y \in X, |x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$

$\Leftarrow$): $f$ is not uniformly continuous in $X$

$\exists \varepsilon > 0, \forall \delta_\varepsilon > 0, \exists x_\delta, y_\delta \in X, |x_\delta - y_\delta| < \delta$

but $|f(x_\delta) - f(y_\delta)| \not< \varepsilon$ (1)

Prove $\exists \varepsilon_0 > 0, \exists (x_n)(y_n)$ s.t. $\lim\limits_{n\to\infty} |x_n - y_n| = 0$

and $|f(x_n) - f(y_n)| \geqslant \varepsilon_0$ $\forall n \in \mathbb{N}$ (2)

For $\varepsilon > 0$, Choose $\delta_n = \frac{1}{n}$, then by (1), $\exists x_n, y_n$ st $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \not< \varepsilon$.

because this is true for all $n$, let $\varepsilon_0 = \varepsilon$ and $n \to \infty$, we have

$\exists \varepsilon_0, \exists (x_n), (y_n)$ in $X$, $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$.

$\Rightarrow$): From (2), because $\lim\limits_{n\to\infty} |x_n - y_n| = 0$, we have

$\forall \delta > 0, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, |x_n - y_n| < \delta$

This means, $\exists \varepsilon_0 > 0, \forall \delta > 0, \exists x_n, y_n \in X, |x_n - y_n| < \delta$ but $|f(x_n) - f(y_n)| \geqslant \varepsilon_0$ $\forall n \in \mathbb{N}$ □

---

$\left.\begin{array}{l} f \text{ continuous in } X \\ X \text{ compact} \end{array}\right\}$ Prove that $f$ is uniformly continuous

have $f$ is continuous in $X$

$\forall (x_n)$ in $X$, $x_n \to x$ then $f(x_n) \to f(x)$.

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, |f(x_n) - f(x)| < \varepsilon$

Need to prove.

$f$ is uniformly continuous.

$X$ is compact

Assume $f$ is not uniformly continuous $\Leftrightarrow \exists \varepsilon_0 > 0, \exists (x_n), (y_n)$ in $X$, $\lim\limits_{n\to\infty} |x_n - y_n| = 0$, (1)

$|f(x_n) - f(y_n)| > \varepsilon_0, \forall n$ (2)

because $(x_n)$ in $X$, which is compact

$\Rightarrow \exists (x_{n_\ell}), x_{n_\ell} \longrightarrow x$ in $X$. (3)

When $|x_n - y_n| \longrightarrow 0$ ? we have
$(x_n) \longrightarrow x$ $y_n \longrightarrow x$.

then because of (1), we have

$$\lim_{\ell\to\infty} y_{n_\ell} = \lim_{\ell\to\infty} \left[ (y_{n_\ell} - x_{n_\ell}) + x_{n_\ell} \right] = \lim_{\ell\to\infty} (y_{n_\ell} - x_{n_\ell}) + \lim_{k\to\infty} x_{n_\ell} = x. \quad (4)$$

we have from (3), (4) $x_{n_\ell} \to x$ $\xrightarrow{}$ $f(x_{n_\ell}) \to f(x) \Leftrightarrow \forall \varepsilon > 0, \exists \ell_0 \in \mathbb{N}, \forall \ell \geqslant \ell_0, |f(x_{n_\ell}) - f(x)| < \varepsilon$

$y_{n_\ell} \to x$. $f$ cont $f(y_{n_\ell}) \to f(x) \Leftrightarrow \forall \varepsilon > 0, \exists \ell_1 \in \mathbb{N}, \forall \ell \geqslant \ell_1, |f(y_{n_\ell}) - f(x)| < \varepsilon$

$\Rightarrow$ Choose $K = \max\{\ell_0, \ell_1\}$, we have $|f(x_{n_\ell}) - f(y_{n_\ell})| \leq |f(x_{n_\ell}) - f(x)| + |f(y_{n_\ell}) - f(x)| < 2\varepsilon$

this contradicts with (2)

* $f : K \longrightarrow \mathbb{R}$ continuous.

$K \subset \mathbb{R}$ compact $\Rightarrow f(K)$ compact.

18 <u>6.3/138.</u>

Remind (Theorem 6.10) If $f$ has finitely many points at which $f$ is not continuous

$\alpha$ continuous at those points (where $f$ is discontinuous) $\}$

Then $f \in R(\alpha)$ on $[a,b]$.

(Note that we keep the assumption that $f$ is always bounded in $[a,b]$.

$\alpha$ is monotonic (assume increasing in $[a,b]$),)

Then this exercise is for a statement:

If $f$ and $\alpha$ have a common point of discontinuity, then $f$ need not be in $R(\alpha)$

# Chapter 8 : Some special functions :

* **Power series:** $f(x)=\sum_{n=0}^{\infty} c_n x^n$ (1) or more generally $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ (2).

analytic functions.

→ restrict to real values of $x$.

→ restrict from circles of convergence to intervals of convergence

↓

If $f(z)=\sum c_n z^n$
then put $\alpha=\limsup \sqrt[n]{|c_n|}$ $R=\frac{1}{\alpha}$ then $f(x)$ converges if $|z|<R$
diverges if $|z|>R$.

If (1) converges for all $x$ in $(-R,R)$, for some $R>0$ ($R$ may be $+\infty$)
we say : $f$ is expanded in a power series about the point $x=0$

If (2) converges for $|x-a|<R$,
we say : $f$ is expanded in a power series about the point $x=a$.

⇒ We shall often take $a=0$, without lostany any lost of generality

**8.1 Theorem:** Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ (3) converges for $|x|<R$ (pointwise convergence)

Define $f(x)=\sum_{n=0}^{\infty} c_n x^n$ $(|x|<R)$ $|x|<(R-\varepsilon)$

then i) (3) converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ for all $\forall \varepsilon>0$ is chosen (

ii) The function $f$ is continuous and differentiable in $(-R,R)$ and

$$f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1} \qquad (|x|<R)$$

i) Prove that $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly ~~on~~ for $|x|<R-\varepsilon$
$\forall x\in[-R+\varepsilon, R-\varepsilon]$

we have $|f_n(x)|=|c_n x^n| \leqslant |c_n(R-\varepsilon)^n|$, but we have $\sum c_n(R-\varepsilon)^n$ converges

then by theorem: ( if $|f_n(x)|\leq M_n$ $(\forall x\in E,\ n=1,2,3,\ldots)$
then $\sum f_n(x)$ converges uniformly on $E$ if $\sum M_n$ converges ).

$\Rightarrow \sum_{n=0}^{\infty} c_n x^n$ converges uniformly.

b) Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$.
define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ $|x| < R$.

Then a) $\sum c_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ (means $|x| < R-\varepsilon$)

b) $f$ is continuous & differentiable in $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad (|x| < R.) \quad (2)$$

- The idea of this prove is that : ~~proving~~

put $f_n'(x) = n c_n x^{n-1}$, we ~~prove that~~ use the theorem 7.17.

( assume : $\sum f_n(x)$ is a ~~sequence~~ series of differentiable function on $[a,b]$
$\sum\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$
$\sum\{f_n'(x)\}$ converges uniformly on $[a,b]$
then $\sum f_n \rightrightarrows f$ on $[a,b]$, where $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$. )

then we prove $\sum f_n'(x)$ converges uniformly.

- because $\exists f'(x)$ then $f$ is continuous.
- put $f_n'(x) = ~~\lim~~ n c_n x^{n-1}$ $f_n(x) = c_n x^n$

because $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$ $\limsup_{n\to\infty} \sqrt[n]{n c_n} = \limsup_{n\to\infty} \sqrt[n]{c_n}$ (pointwise)

$\Rightarrow \sum f_n'(x)$ and $\sum f_n(x)$ have the same interval of convergence.

- According to (a) $\Rightarrow \sum f_n'(x)$ converges uniformly on ~~$[-R, R]$~~ $[-R+\varepsilon, R-\varepsilon]$.

then by theorem 7.17, $\overset{(2)}{f'(x)} = \sum f_n'(x) = \sum n c_n x^{n-1}$ for $x \in [-R+\varepsilon, R-\varepsilon]$

- But given any $x$ such that $|x| < R$, we can find $\varepsilon$ such that $|x| < R-\varepsilon$
$\Rightarrow$ (2) also holds for $|x| < R$.

( what is hold for $|x| < R-\varepsilon$ also hold for $|x| < R$, vice versa.)

- Continuity of $f$ is deduced from the existence of $f'$.

* Corollary : Under the hypotheses of theorem 8.1, $f$ has derivative of all orders in $(-R, R)$, which are given by :

$$f^{(k)}(x) = \sum_{n=k}^{\infty} (n)(n-1)\cdots(n-k+1) c_n x^{n-k}$$

In particular $f^{(k)}(0) = k!\, c_k$ $(k = 0, 1, 2, \ldots)$

We have $f^{(k)}(x) = \underbrace{k(k-1)\cdots 1 \cdot c_k \overset{1}{x^0}}_{= k!c_k} + \underbrace{(k+1)(k)\cdots(2) c_{k+1} x^{(k+1)-k} + \cdots}_{= 0}$

at $x = 0$ then

Prove theorem 8.2 (Abel's theorem) (Rudin' book)

$\sum c_n$ converges.

$\sum c_n x^n$ converges pointwise in $(-1, 1)$. Put $f(x) = \sum_{n=0}^{\infty} c_n x^n$

Then we have $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$

(And also in this proof, we don't use $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly in $[-1+\delta^*,$

Put $s = \sum_{n=0}^{\infty} c_n$

$s_\ell = \sum_{n=0}^{\ell} c_n$

We have $s_\ell \to s$

$\Rightarrow \forall \varepsilon_1 > 0, \exists k_{\varepsilon_1}, \forall \ell \geq k_{\varepsilon_1}, |s_\ell - s| < \varepsilon_1$

We want to prove that $\lim_{x \to 1} f(x) = s$

$\Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists \delta_{\varepsilon,1}, \forall x \in [-1, 1],$

$|x - 1| < \delta_{\varepsilon,1}, |f(x) - s| < \varepsilon$

We want to estimate $|f(x) - s|$, we have $|s_\ell - s|$

$\Rightarrow$ we want to compute $f(x)$ according to $s$

* We have $f(x) = \sum_{n=0}^{\infty} c_n x^n$

Now we compute

$$\sum_{n=0}^{\ell} c_n x^n \textcircled{=} \sum_{n=0}^{k} (s_n - s_{n-1}) x^n = \sum_{n=0}^{k} s_n x^n - \sum_{n=0}^{k} s_{n-1} x^n$$

$$= \sum_{n=0}^{\ell} s_n x^n - \sum_{n=-1}^{\ell-1} s_n x^{n+1} \qquad (s_1 = 0)$$

$$= \sum_{n=0}^{\ell-1} s_n x^n - \sum_{n=0}^{\ell-1} s_n x^{n+1} + s_\ell x^\ell$$

$$= \sum_{n=0}^{\ell-1} s_n (x^n - x^{n+1}) + s_\ell x^\ell$$

$$\textcircled{=} (1-x) \sum_{n=0}^{\ell-1} s_n x^n + \underbrace{s_\ell x^\ell}_{\downarrow \, 0}$$

For $|x| < 1$, let $\ell \to \infty$, we have

$$f(x) \textcircled{=} (1 - x) \sum_{n=0}^{\infty} s_n x^n$$

Note : $(1-x) \sum_{n=0}^{\infty} x^n = (1-x) \frac{1}{1-x}$ (for |

*
$$|f(x) - s| = \left|(1-x) \sum_{n=0}^{\infty} s_n x^n - s\right| \textcircled{=} \left|(1-x) \sum_{n=0}^{\infty} s_n x^n - (1-x) \sum_{n=0}^{\infty} x^n s\right|$$

$$= \left|(1-x) \sum_{n=0}^{k_{\varepsilon_1}} x^n (s_n - s)\right| + \left|(1-x) \sum_{n=k_{\varepsilon_1}}^{\infty} x^n (s_n - s)\right|$$

8.2 Abel's theorem Suppose $\sum C_n$ converges

Put $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (-1 < x < 1)$

Then $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} C_n$

→ Corolary

$\sum C_n R^n$ converges

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $-R < x < R$

(means $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ ) (pointwise)

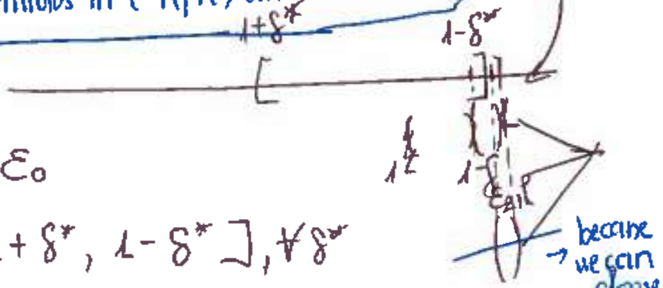
note, that we can choose $\delta^*$

Then from 8.1, $\sum c_n x^n$ converges uniformly in $[-R+\delta^*, R]$

$f(x)$ continuous in $(-R, R)$ and at $x = R$

+ Put $s_k = \sum_{n=0}^{k} C_n \qquad f_k(x) = \sum_{n=0}^{k} c_n x^n$

$s = \sum_{n=0}^{\infty} C_n$



because → we can choose $\delta^*$

We have

- $s_k \xrightarrow{k \to \infty} s \Leftrightarrow \forall \varepsilon_0 > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, |s_k - s| < \varepsilon_0$
- By theorem 8.1: $f_k(x) \overset{\leftarrow}{\rightrightarrows} f(x)$ ~~for~~ in $[-1+\delta^*, 1-\delta^*], \forall \delta^*$

  $\Leftrightarrow \forall \varepsilon_1 > 0, \exists k_{\varepsilon_1, \delta^*}, \forall k \geq k_{\varepsilon_1, \delta^*}, \forall x \in [1+\delta^*, 1-\delta^*], |f_k(x) - f(x)| < \varepsilon_1$.
- $f_k(x) \xrightarrow{x \to 1} s_k \Leftrightarrow \forall \varepsilon_2 > 0, \exists \delta_{\varepsilon_2, k}, \forall x, |x-1| < \delta_{\varepsilon_2, k}, |f_k(x) - s_k| < \varepsilon_2$.

  $1 - \delta_{\varepsilon_2, k} \leq x < 1 + \delta_{\varepsilon_2, k}$

NTP: $\forall \epsilon > 0, \exists \delta_\varepsilon > 0, \forall x, |x-1| < \delta_\varepsilon$ then $|f(x) - s| < \varepsilon$

can't use this way, too complicated.

from (1) and by the theorem 8.2 $\left( \begin{array}{l} \Sigma c_n \text{ converges.} \\ \text{that, } f(x) = \sum_{n=0}^{\infty} c_n x^n \quad -1<x<1 \\ \text{Then } \lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n \end{array} \right)$

$\Rightarrow \lim_{x \to 1} c(x) = \lim_{x \to 1} a(x) \times \lim_{x \to 1} b(x) \Rightarrow C = A \times B.$

**8.3 Theorem:**

Given a double sequence $\{a_{ij}\}$ $i = 1, 2, 3, \dots$
$j = 1, 2, 3, \dots$

Suppose that $\begin{cases} \sum_{j=1}^{\infty} |a_{ij}| = b_i & i = 1, 2, 3, \dots \quad (12) \\ \sum_{i=1}^{\infty} b_i \text{ converges.} & (13) \end{cases}$

Then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$

We can establish this result by a direct procedure similar to the one we in theorem 3.55

$\left( \begin{array}{l} \Sigma |a_n| \text{ converges.} \\ \Sigma a_n = A \end{array} \right\} \Rightarrow \Sigma a_n' = A$ where $\Sigma a_n'$ is a rearrangement of $\Sigma a_n$

Let E is a countable set, consisting $x_0, x_1, \dots$ and $x_n \xrightarrow{n \to \infty} x_0$

Define $f_i(x_0) = \sum_{j=1}^{\infty} a_{ij} \quad i = 1, 2, 3, \dots \quad (14)$

$f_i(x_n) = \sum_{j=1}^{n} a_{ij}. \quad i = 1, 2, \dots \quad (15)$

$g(x) = \sum_{i=1}^{n} f_i(x), \ \forall x \in E \quad (16)$

(This is well defined because $\Sigma a_{ij}$ converges.

- $(12)+(14)+(15) \Rightarrow$ ~~$f$~~ each $f_i$ is continuous at $x_0$ $\left( \lim_{n \to \infty} f_i(x_n) = f_i(\lim_{n \to \infty} x_n) = f_i(x_0) \right)$ (I)
- $(14)+(15) \Rightarrow |f_i(x)| \le b_i, \ \forall x \in E$
  (13) $\Sigma b_i$ converges. $\Big\} \Rightarrow \Sigma f_i(x)$ converges uniformly, $\forall x \in E$ (II)

$(I)+(II) \Rightarrow g$ is continuous at $x_0$

$\Rightarrow$ We have: $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n \to \infty} g(x_n)$

$= \lim_{n \to \infty} \left( \sum_{i=1}^{\infty} f_i(x_n) \right) = \lim_{n \to \infty} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} \right)$

$\textcircled{=} \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$

8.47 Theorem

Suppose $f(x)=\sum_{n=0}^{\infty} c_n x^n$, the series converging in $|x|<R$

If $|a|<R$, then $f$ can be expanded in a power series about the point $x=a$

$$f(x)=\sum_{m=0}^{\infty} b_m (x-a)^m$$ which converges in $|x-a|<R-a$,

and $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \qquad (|x-a|<R-|a|)$

$$f(x)=\sum_{n=0}^{\infty} c_n\left[(x-a)+a\right]^n=\sum_{n=0}^{\infty} c_n\left(\sum_{m=0}^{n}\binom{n}{m}(x-a)^m a^{(n-m)}\right)$$

$$=\sum$$

**Example**: Let $f(x) = f(x,y,z) = x^2 z + y^3 z^2 - xyz$ in the direction of $\vec{v} = \langle -1, 0, 3 \rangle$.

$\frac{\partial f}{\partial x} = 2xz - yz \quad \frac{\partial f}{\partial y} = 3y^2 z^2 - xz \quad \frac{\partial f}{\partial z} = x^2 + 2y^3 z - xy$.

$\Rightarrow D_{\vec{v}} f(x,y,z) = -1(2xz - yz) + 0 + 3(x^2 + 2y^3 z - xy)$

we have $\|\vec{v}\| = \sqrt{10} \neq 1$.

$\Rightarrow$ convert $\vec{v}$ (not a unit vector) to $\vec{u}$) $|\vec{u}| = 1$.

$u = \langle -\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \rangle$.

$\Rightarrow D_{\vec{u}} f(x,y,z) = -\frac{1}{\sqrt{10}}(2xz - yz) + 0 + \frac{3}{\sqrt{10}}(x^2 + 2y^3 z - xy)$.

b) Find $D_{\vec{u}} f(\vec{x})$ for $f(\vec{x}) = f(x,y,z) = \sin(yz) + \ln x^2$ at $(1,1,\pi)$ in the direction of $\vec{v} = \langle 1, 1, -1 \rangle$.

- $\nabla f(\vec{x}) = \left\langle \frac{2}{x}, z\cos(yz), y\cos(yz) \right\rangle$

  $\nabla f(\vec{x})$ at $\vec{x} = (1,1,\pi) = \langle 2, -\pi, -1 \rangle$

- $\|\vec{v}\| = \sqrt{3} \Rightarrow u = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

Finally, the directional derivative at $(1,1,\pi)$ in the direction of $\vec{v}$ is:

$\langle 2, -\pi, -1 \rangle \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle = \frac{1}{\sqrt{3}}(2 - \pi + 1)$

* **Theorem**: Fix $f$ and $x$. Let $u$ varies | $(D_u f)(x)$ attains its maximum $(\|\nabla f(x)\|)$ when $u$ is a positive scalar multiple of $(\nabla f)(x)$. (means when $u$ is pointing in the same direction as the gradient $(\nabla f)(x)$.

$$D_u f(x) = (\nabla f)(x) u = \underset{\text{fixed}}{\|\nabla f(x)\|}\ \underset{=1}{\|\vec{u}\|} \cos\theta \qquad \theta = \widehat{\langle \nabla f(x), \vec{u} \rangle}$$

$\Rightarrow D_u f(x)$ attains its maximum when $\cos\theta = 1 \Rightarrow \theta = 0$ means $\vec{u} \uparrow\uparrow \nabla f(x)$.

## 9.19 Theorem :

Let $E$ convex, open set $\subset \mathbb{R}^n$
$f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$f$ is differentiable in $E$
$\exists M$ real, $\|f'(x)\| \le M$, $\forall x \in E$
$\Rightarrow |f(b)-f(a)| \le M|b-a|$, $\forall a \in E$, $b \in E$

- With $a, b$ fixed $\in E$. $\boxed{t \in [0,1]}$
  Put $\boxed{\gamma(t) = (1-t)a + tb}$
  $E$ convex
  $\Rightarrow \gamma(t) \in E$ | $\gamma'(t) = b-a$.
- Put $g(t) = f(\gamma(t))$
  $|g'(t)| = |f'(\gamma(t))|\,|\gamma'(t)| \le M\,|b-a|$.
- By Theorem 5.19 : Suppose $g$ continuous mapping of $[a,b] \longrightarrow \mathbb{R}^k$.
  $g$ is differentiable on $(a,b)$.
  $\Rightarrow \exists x \in (a,b)$ $|g(b)-g(a)| \le |g'(x)|(b-a)$.

$\Rightarrow |g(1)-g(0)| \le g'(t)(1-0) = M|b-a|$

$\Rightarrow |f(\gamma(1)) - f(\gamma(0))| \le M|b-a|$.

$\Rightarrow |f(b)-f(a)| \le M|b-a|$, $\forall a, b \in E$.

### Corollary :

If, in addition, $f'(x) = 0$, $\forall x \in E$.
then $f$ is constant.

## 9.20 Definition :

Consider $f : E\,\text{open} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$f$ is differentiable.
} $f$ is said to be continuously differentiable in $E$
if $f'$ is a continuous mapping of $E$ into $L(\mathbb{R}^n, \mathbb{R}^m)$

$\Leftrightarrow \forall x \in E, \forall \varepsilon > 0, \exists \delta > 0$, $\|f'(y) - f'(x)\| < \varepsilon$, if $y \in E$, $|x-y| < \delta$

$\Rightarrow f \in \mathscr{C}^1$ mapping, or $f \in C^1(E)$.

HW3.5, Mat601, / Also Fall 1992, P1

P1, Suppose that $\{x_n\}$ is a sequence of real numbers, $x_n \longrightarrow a$.

Put $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

Prove that $y_n \longrightarrow a$.

Note that if the problem requires us to compute $\lim\limits_{n\to\infty} x_n = a$ constant

$\Rightarrow$ we consider $|x_n - a|$.

← Key point
only need to kno
this then don

- We have $x_n \to a$

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |x_n - a| < \varepsilon$ (*)

We want to prove that $y_n \longrightarrow a$

$\Leftrightarrow$ NTProve : $\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0, |y_n - a| <$

This means we need to consider, for $n \geq n_0$

$$|y_n - a| = \left|\frac{1}{n}\sum_{i=1}^{n} x_i - a\right| = \left|\frac{\sum_{i=1}^{n} x_i}{n} - n\frac{a}{n}\right| = \left|\frac{x_1 + \cdots + x_{n_0} + x_{n_0+1} + \cdots + x_n}{n} - \frac{a + \cdots +}{n}\right.$$

$$\leq \underbrace{\frac{1}{n}\underbrace{\sum_{i=1}^{n_0-1}|x_i - a|}_{\text{bounded}}}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\sum_{i=n_0}^{n}\frac{1}{n}|x_i - a|}$$

$$= \frac{1}{n}\underbrace{\sum_{i=n_0}^{n}|x_i - a|}_{<\varepsilon \text{ because of } (*)}$$

$$= \frac{1}{n}(n - n_0)\varepsilon \leq \frac{n}{n}\varepsilon = \varepsilon$$

So we have $y_n \xrightarrow{n\to\infty} a$ $\square$.

Note that we can explain a li
bit more careful that:

$x_n$ converges $\Rightarrow$ $x_n$ bounde

$\Rightarrow |x_n| \leq n, \forall n$

$\Rightarrow |x_n - a| \leq |x_n| + |a| \leq M+$
$\forall M$ $M$

P2, Suppose $\{x_n\}$ in a sequence of real number, $x_n \to a$.

$$y_n = \frac{1}{n^2}(x_1 + 2x_2 + 3x_3 + \cdots + nx_n) \longrightarrow \frac{a}{2}$$

* We have $x_n \to a \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |x_n - a| < \varepsilon$
* We want $y_n \to a \Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0, |y_n - \frac{a}{2}| < \varepsilon$.

The same idea with above problem, we want to prove $y_n \to \frac{a}{2}$ $\Rightarrow$ consider $|y_n - \frac{a}{2}|$

$$\left|y_n - \frac{a}{2}\right| = \left|\frac{1}{n^2}(x_1 + 2x_2 + 3x_3 + \cdots + nx_n) - \frac{a}{2}\right| \; \text{\textcircled{=}} \; \left|\frac{2[x_1 + 2x_2 + \cdots + nx_n] - n^2 a}{2n^2}\right|$$

$$= \left|\frac{\cancel{2}\{(x_1 - a) + [2x_2 - 2a] + \cdots + [(n_0 - 1)x_{n_0-1} - (n_0-1)a] + \cdots + [n(x_n - na] - n^2a +}{\cancel{2}n^2}\right.$$

this problem just needed to write down carefully ~~and~~ $|y_n - a|$

$$\leq \underbrace{\boxed{\sum_{k=1}^{n_0-1}}\frac{k}{n^2}\underbrace{|x_k - a|}_{<M' \text{ (as above problem)}}}_{< M'\frac{1}{n^2}\sum_{k=1}^{n_0-1} \xrightarrow[n\to\infty]{} 0} + \underbrace{\boxed{\sum_{k=n_0}^{n}}\frac{k}{n^2}\underbrace{|x_k - a|}_{<\varepsilon}}$$

$$< \frac{\varepsilon}{n}\sum_{k=n_0}^{n} k \leq \frac{\varepsilon}{n^2}(n - n_0)\cancel{n} < \frac{n}{n}\varepsilon = \varepsilon$$

→ done.

# MAT601 HW 5.5-6 Higher derivative and The Taylor theorem.

Importa

**P1** ⇒ $f:\mathbb{R}\to\mathbb{R}$ has the third derivative at any point of $\mathbb{R}$.

$\exists p\in\mathbb{R}$ s.t $f'(p)=f''(p)=f'''(p)=0$ and $f^{(4)}(p)>0$.

Prove that $p$ is a point of "strict" local minimum for $f$

(that is $\exists\delta>0,\ \forall x\ 0<|x-p|<\delta$ then $f(x)>f(p)$).

need to prove there exists this $\delta$.

(This associates with what we learned that: $\left.\begin{matrix} f'(p)=0 \\ f''(p)>0 \end{matrix}\right\} \Rightarrow f$ attain local minimum

+ Taylor series (Peano form):

condition: $f^{(d)}(p)$ exist, then $f(x)=P_d(x)+\lambda(x)(x-p)^d$ ; $\lambda(x)\to 0$.

+ Lagrange form.

condition $f^{(d)}$ exist for every $[x,p)$, $f^{(d+1)}$ exists at on $(x,p)$. , then $f(x)=P_d(x)+\dfrac{f^{(d+1)}(\xi)}{(d+1)!}(x-p)^{d+1}$

* Way 1: Solve the problem by Taylor series (Peano form).

Note that we have $f$ has the third derivative at any point and $\exists f^{(4)}(p)$, then we conside

Taylor series of $f$ at $p$:

$$f(x)=P_4(x)+\lambda(x)(x-p)^4=f(p)+\underbrace{\frac{f'(p)}{1!}(x-p)^1}_{0}+\underbrace{\frac{f''(p)}{2!}(x-p)^2}_{0}+\underbrace{\frac{f'''(p)}{3!}(x-p)^3}_{0}+\frac{f^{(4)}(p)}{4!}(x-p)^4 \quad \lambda(x)(x-p)^4$$

$$=f(p)+\frac{f^{(4)}(p)}{4!}(x-p)^4+\lambda(x)(x-p)^4 \qquad \text{where } \lambda(x)\longrightarrow 0$$

and because $\lambda(x)\xrightarrow[x\to p]{}0 \Leftrightarrow \forall\varepsilon>0,\ \exists\delta_\varepsilon,\ \forall x,\ |0<|x-p|<\delta$, then $|\lambda(x)|<\varepsilon$

Choose $\varepsilon=\dfrac{f^{(4)}(p)}{4}$, then $\exists\delta>0,\ \forall x,\ 0<|x-p|<\delta,\ |\lambda(x)|<$

So we have

$$f(x)=f(p)+\underbrace{\left[\frac{f^{(4)}(p)}{4!}+\lambda(x)\right]}_{>0}\underbrace{(x-p)^4}_{>0} > f(p) \qquad \square \text{ way } 1.$$

* Way 2: Solve the problem using Taylor series (Lagrange's form).

Note that by assumption that $f'''$ exist at any point of $\mathbb{R}$, and we only have that $f^{(4)}$ exist at (only

⇒ we can only use Taylor series (Lagrange's form) up to $d=\cancel{3}\ 2$

We have Taylor series of $f$ at $p$:

$$f(x)=f(p)+\frac{f'(p)}{1!}(x-1)^1+\frac{f''(p)}{2!}(x-p)^2+\frac{f'''(\xi)}{3!}(x-p)^3+\cancel{\frac{f^{(4)}(\xi)}{4!}(x-p)^4}.$$

$$=f(p)+\frac{f'''(\xi)}{3!}(x-p)^3 \quad (*) \qquad \text{for some } \xi \text{ between } x \text{ and } p.$$

Note that $f^{(4)}(p)>0 \Leftrightarrow \lim\limits_{x\to p}\dfrac{f'''(x)-f'''(p)}{x-p}>0 \Leftrightarrow \lim\limits_{x\to p}\dfrac{f'''(x)}{x-p}>0$

This means $\exists\delta>0,\ \forall\ |x-p|<\delta$, $\begin{cases} x<p, & \text{then } f'''(x)<0 \\ x>p, & \text{then } f'''(x)>0. \end{cases}$

Then apply this to (*) we have $f(x)>f(p)$.

601, HW 5.5-6, P2 — See 5.16, 5.17, Rudin, 5.15.

ppose $f : [0,2] \longrightarrow \mathbb{R}$ is continuous. $|f''(x)| \leq 1$, $\forall x \in (0,2)$ $\Bigg\}$ Prove that $|f(0) - 2f(1) + f(2)| \leq 1$

ote that we have $f''(x)$ exist for all $x \in (0,2)$ $\Rightarrow$ we can apply Taylor series with $\alpha = 1$.

Apply Taylor series for $f(x)$ (Lagrange form), we have.

$$(0) = f(1) + \frac{f'(1)}{1!}(0-1) + \frac{f''(\xi)}{2!}(0-1). \quad , \text{ for some } \xi \text{ in } (0,1)$$

$$(2) = f(1) + \frac{f'(1)}{1!}(2-1) + \frac{f''(\beta)}{2!}(2-1) \quad , \text{ for some } \beta \text{ in } (1,2).$$

$$f(0) + f(2) - 2f(1) = \frac{f''(\xi)}{2!}(-1) + \frac{f'(\beta)}{2!}1.$$

$$\text{en } |f(0) - 2f(1) + f(2)| = \left| \frac{f''(\beta)}{2!} - \frac{f''(\xi)}{2!} \right| \leq \frac{1}{2!}\Big[\underbrace{|f''(\beta)|}_{\leq 1} + \underbrace{|f''(\xi)|}_{\leq 1}\Big] = \frac{2}{2!} = 1.$$

---

7/1/16:

## * Some more practicing on series and Taylor series.

P17 Use ~~the forth degree Taylor~~ polynomial of $\cos(2x)$ to find the exact value of $\lim \frac{1-\cos(2x)}{3x^2}$ Similar with Jan2012 P3.

We have $\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8}x^8 + \cdots$

$$\Rightarrow \cos 2x \approx 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \frac{1}{8}(2x)^8 - \cdots$$

$$\approx 1 - \frac{1}{2!}2^2x^2 + \frac{1}{4!}2^4x^4 - \frac{1}{6!}2^6x^6 + \cdots$$

$$\frac{1-\cos 2x}{3x^2} \approx \frac{2 + \frac{1}{2!}2^2x^2 - \frac{1}{4!}2^4x^4 - \cdots}{3x^2} \approx \frac{2}{3} - \frac{2^4}{3\cdot 4!}x^2$$

$$\text{So } \lim_{x\to 0}\frac{1-\cos(2x)}{3x^2} = \lim_{x\to 0}\left(\frac{2}{3} - \frac{2^4}{3\cdot 4!}x^2 + ??x^2 + \cdots\right\} = \frac{2}{3}.$$

---

* Let $f(x) = \ln(1+x^2)$ Find the Taylor series of $f(x)$ with center $x_0 = 0$ and its radius o
convergence.

Note that we can find Taylor series of $f(x)$ through finding Taylor series of $F(x)$.

If $F'(x) = f(x)$ $\left(\overset{\text{because}}{\Rightarrow} f(x) = \int_0^x F(t)\,dt\right)$

Then $\{$Taylor series of $F'(x)\} = \{$Taylor series of $f(x)\}$

Note $f(x) = \int_0^x f'(t)\,dt$.

$\{$Taylor series of $f(x)\} = \int_0^x [$Taylor series $f'(t$

*We have $f'(x) = \frac{2x}{1+x^2} = F(x)$

Now we want to find Taylor series of $F(x) = \frac{2x}{1+x^2}$

- We have $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12}$$

$$F(x) = \frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + 2x^9 - 2x^{11} + 2x^{13}.$$

$$\text{So } f(x) = \int_0^x \frac{2x}{1+x^2}\,dx = \int_0^x F(t)\,dt = \int_0^x 2t - 2t^3 + 2t^5 - 2t^7 + 2t^9 - 2t^{11}\cdots dt$$

$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^{10}}{10} - \frac{x^{12}}{12} + \cdots\right]$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \frac{x^{12}}{6} \cdots$$

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} (-3)^n \sqrt{n+1}\, |x+1|^{2n+1}$.

~~e have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(-3)^{n+1}\sqrt{n+2}\,|x+1|^{2(n+1)+1}}{(-3)^n\sqrt{n+1}\,|x+1|^{2n+1}} \xrightarrow[n\to\infty]{} (-3)|x+1|^2$~~

~~The series converges when $(-3)|x+1|^2 < 1 \Leftrightarrow |x+1|^2 > \frac{1}{3}$~~

Find the limit $\lim_{x\to0}\frac{\sin x^2}{1-\cos 2x}$ without using L'Hospital's rule

We can find a limit of function by computing limit of its Taylor series.

(If use L'Hospital's rule

$$\lim_{x\to0}\frac{\sin x^2}{1-\cos 2x} \overset{\frac{0}{0}}{=} \lim_{x\to0}\frac{2x\cos(x^2)}{-2\sin(2x)} = \lim_{x\to0}\frac{x\cos x^2}{\sin 2x} \overset{\frac{0}{0}}{=} \lim_{x\to0}\frac{\cos x^2 + x\,2x\sin x^2}{2\cos 2x} = \frac{1}{2}$$

Without using L'Hospital's rule

- We have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

$$\Rightarrow \sin x^2 = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \frac{(x^2)^9}{9!} - \frac{(x^2)^{11}}{11!} + \cdots$$

- We have $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$

$$\Rightarrow \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \frac{(2x)^{10}}{10!}$$

$$1 - \cos 2x = \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \frac{2^8x^8}{8!} - \frac{2^{10}x^{10}}{10!}$$

o we have

$$\frac{\sin x^2}{1-\cos 2x} = \frac{\cancel{x^2}\left[1 - \frac{(x^2)^2}{3!} + \frac{(x^2)^4}{5!} - \frac{(x^2)^6}{7!} + \frac{(x^2)^8}{9!} - \frac{(x^2)^{10}}{11!} + \cdots\right]}{\cancel{x^2}\left[2 + \frac{2^4}{4!}x^2 - \frac{2^6}{6!}x^4 + \frac{2^8}{8!}x^6 + \cdots\right]}$$

Then $\lim_{x\to0}\frac{\sin x^2}{1-\cos 2x} = \frac{1}{2}$

MAT601 HW 5.3. 4 Derivative and limit

**P1**: Suppose $f:\mathbb{R}\to\mathbb{R}$ is a function s.t $f''(x)$ exists at some $x\in\mathbb{R}$.

a) Prove that $\lim\limits_{h\to 0}\dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$ $(*)$

b) Prove that (by example) the limit on the LHS may exist even $f''(x)$ does not exist

If $f''$ exists at $x$, then
$f''(x)=(*)$
but this is not the definition of $f''(x)$.

Note that in here,
we only have $f''$ exists at $x$ (we don't have $f''(x)$ exists in a neighborhood of $x$.
? $\nexists$ $f''(x+\varepsilon)$ $f''(x+h)$? → don't know $\exists$?

but we have $\begin{cases} f' \text{ exists in a neighborhood of } x & (1)\\ f' \text{ differentiable at } x \Rightarrow f' \text{ continuous at } x & (2).\end{cases}$

- We have LHS $=\lim\limits_{h\to 0}\dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}\xrightarrow[\text{wrt } h]{\frac{0}{0}\text{ L'Hospital}}\lim\limits_{h\to 0}\dfrac{f'(x+h)-f'(x-h)}{2h^2}$

$(h^2)'=2h\neq 0$ for $h\neq 0$

- We also have

$$f''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}=\lim_{h\to 0}\frac{f'(x)-f'(x-h)}{h}$$

So we have

$$f''(x)=\frac{1}{2}\left[\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}+\lim\frac{f'(x)-f'(x-h)}{h}\right]$$

$$=\frac{1}{2}\left[\lim\frac{f'(x+h)-f'(x-h)}{h}\right]=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

→ LE
=F

* Note that, with this problem, we can't prove it directly from LHS or RHS

- from LHS $=\lim\limits_{h\to 0}\dfrac{f'(x+h)-f'(x-h)}{2h}$ form $\frac{0}{0}$ but we can't use L'Hospital 2nd time because we don't know if $f''$ exists at $(x+h)$ or $(x-h)$ or not.

- from RHS $=f''(x)=\lim\limits_{h\to 0}\dfrac{f'(x+h)-f'(x)}{h}$

we notice that $f'(x+h)=\lim\limits_{h_1\to 0}\dfrac{f(x+h+h_1)-f(x+h)}{h_1}$

notice that $h\neq h_1\neq h$

and $f'(x)=\lim\limits_{h_2\to 0}\dfrac{f(x+h_2)-f(x)}{h_2}$

Show by example that the limit $\lim\limits_{h\to 0} \dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}$ can be exist while $f''(x)$ does not exist.

te that $f''(x_0)$ does not exist when $f'$ is not differenthable at $x_0$.

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} f' \text{ continuous at } x_0 \\ f' \text{ is not differenthable at } x_0 \end{cases} \\ f' \text{ is not continuous at } x_0 . \end{array}\right.$$

et $f(x) = x|x| = \begin{cases} x^2 , & x > 0 \\ 0 , & x = 0 \\ -x^2 , & x < 0 \end{cases}$ $\quad f'(x) = \begin{cases} 2x & , x > 0 \\ -2x & , x < 0 \\ 0 & , x = 0 \end{cases}$ $\quad f''(x) = \begin{cases} 2 & , x > 0 \\ -2 & , x < 0 \\ \nexists & \text{at } x = 0 . \end{cases}$

cause we have $f'(x)$ exist for all $x \in \mathbb{R}$

$\Rightarrow f(x)$ differenthable at all $x \in \mathbb{R}$ $\Rightarrow$ continuous $\forall x \in \mathbb{R}$.

$\Rightarrow \lim \cdots = 0$.

be more specific,

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \underset{f \text{ is differentiable}}{\overset{\text{form } \frac{0}{0} \text{ (note that}}{=\!=}} \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = 0.$$

AT601 HW5.3.4 Derivative and limit.

P2) Suppose $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a function st $|f''(x)|\leq 1, \forall x\in\mathbb{R}$ } Prove that $\lim\limits_{x\to\infty}\frac{f(x)}{x^p}=0, \forall p>2$.

* Note that $\exists f''(x), \forall x\in\mathbb{R}$

Put $g(x)=x^p \Rightarrow \lim\limits_{x\to\infty} g(x)=\pm\infty$ (form $\frac{\quad}{\pm\infty}$)

Note that $g'(x)=p(x^{p-1})\neq 0$ for all $|x|\in(M,+\infty)$ and $p>2$

→ Apply L Hospital theorem, we have

$$\lim_{x\to\infty}\frac{f(x)}{x^p}=\lim_{x\to\infty}\frac{f'(x)}{px^{p-1}} \quad (*)$$

* Similarly, $(*)$ has form $\frac{\quad}{\pm\infty}$

$(px^{p-1})'=(p(p-1)x^{p-2}\neq 0$ (in a neighborhood of $\infty$).

$f''$ exists

$$(*)=\lim_{x\to\infty}\frac{f''(x)}{p(p-1)x^{p-1}} \quad (1)$$

* We have

$$0\leq\left|\frac{f''(x)}{p(p-1)x^{p-1}}\right|<\underbrace{\left|\frac{1}{p(p-1)x^{p-1}}\right|}\xrightarrow[x\to\infty]{}0 \Rightarrow \lim_{x\to\infty}\left|\frac{f''(x)}{p(p-1)x^{p-1}}\right|=0$$

$$\Leftrightarrow \lim\frac{f''(x)}{p(p-1)x^{p-1}}=0 \quad (2)$$

$(1)+(2)\Rightarrow \lim\limits_{x\to\infty}\frac{f(x)}{x^p}=0, \forall p>2$ □

Rudin 4.4/98

$X, Y$ metric spaces

$f : X \to Y$, $g : X \to Y$ $f$ and $g$ $\boxed{\text{continuous}}$

Let $E$ be a dense subset in $X$

a) Prove that $f(E)$ is dense in $\boxed{f(X)}$

b) Let $g(p) = f(p)$, $\forall p \in E$. Prove that $g(p) = f(p)$, $\forall p \in X$.

(a continuous mapping is determined by its value on a dense subset of its

NTP $\bar{E} = X$

we can prove by def: $\forall x \in X$, $\forall r$, $N_r(x) \cap E \neq$

or prove by: let $x \in X$, nt p, $\left[\begin{array}{l} x \in E \\ x \in E' \end{array}\right.$

a) Let $\left.\begin{array}{l} f : X \longrightarrow Y \text{ cont} \\ E \text{ is dense in } X \end{array}\right\}$ Prove that $f(E)$ dense in $f(X)$.

We have $\left.\begin{array}{l} E \subseteq X \Rightarrow f(E) \subseteq f(X) \\ \text{We want to prove } f(E) \text{ dense in } f(X) \end{array}\right\} \Rightarrow$ NTP $\forall y \in f(X)$ then $\left[\begin{array}{l} y \in f(E) \\ y \text{ is a limit po} \\ f( \end{array}\right.$

- Let $y \in f(X)$, then $\exists x \in X$, $y = f(X)$

because $x \in X = \bar{E} \Rightarrow \left[\begin{array}{l} x \in E \text{, this means } y = f(x) \in f(E) \\ x \in E', \Rightarrow \exists (x_n) \subseteq E,\ x_n \to x \xRightarrow[f \text{ cont}]{} \underset{\subset f(E)}{f(x_n)} \longrightarrow f(x) = y \end{array}\right.$

this means $y$ is a limit pon

$\Rightarrow ($

b) Let $g(p) = f(p)$, $\forall p \in E$, Prove that $g(p) = f(p)$, $\forall p \in X$.

Notice that $\bar{E} = X$, this means $\forall p \in X \Rightarrow \left[\begin{array}{l} p \in E \\ p \in E' \end{array}\right.$

+ In case $p \in E \Rightarrow f(p) = g(p) \Rightarrow \square$

+ In case $p \in E'$, then $\exists (p_n) \subseteq E$, $p_n \longrightarrow p$

$$\left.\begin{array}{l} \text{because } \left.\begin{array}{l} f \text{ cont} \\ p_n \to p \end{array}\right\} \Rightarrow f(p_n) \longrightarrow f(p) \\ \text{Similarly } \left.\begin{array}{l} g \text{ cont} \\ p_n \to p \end{array}\right\} \Rightarrow g(p_n) \longrightarrow g(p) \\ \text{~~because~~ } f(g_n) = g(p_n),\ \forall n \text{ (because } (p_n) \subseteq E \end{array}\right\} \Rightarrow f(p) = g(p) \Rightarrow \square.$$

* To be more specific, we prove $\left.\begin{array}{l} a_n \to a \\ b_n \to b \\ a_n = b_n, \forall n \end{array}\right\} \Longrightarrow a = b$.

$a_n \to a \Leftrightarrow \forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $|a_n - a| < \varepsilon$.

$b_n \to b \Leftrightarrow \forall \varepsilon > 0$, $\exists n_1 \in \mathbb{N}$, $\forall n \geq n_1$, $|a_n - b| < \varepsilon$.

Choose $n = \max\{n_0, n_1\}$

Then $\forall \varepsilon > 0$, $|a - b| \leq \underbrace{|a - a_n|}_{<\varepsilon} + \underbrace{|a_n - b_n|}_{=0} + \underbrace{|b_n - b|}_{<\varepsilon} < 2\varepsilon \Rightarrow a = b$ $\square$

Rudin 3.1/78

Prove that $\{s_n\}$ converges (to $p$), then $|s_n|$ converges to $|p|$.

$|s_n| \longrightarrow 0 \Longleftrightarrow s_n \longrightarrow 0$

↑

~~does~~ is not true when $p \neq 0$.

Rudin 3.2/78. Is $\lim\limits_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$

(we have fore $n > 0$, then $n^2+n \geq 0$ )

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} =$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{2}.$$

Rudin 3.3/78

Let $s_1 = \sqrt{2}$

$s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ $n = 1, 2, 3, \ldots$

Prove that $\{s_n\}$ converges and that $s_n < 2, \forall n = 1, 2, 3, \ldots$

- We have $s_1 = \sqrt{2} \Rightarrow s_1^2 =$

  $s_2 = \sqrt{2 + \sqrt{2}} > s_1$

  assume that $s_n = \sqrt{2 + \sqrt{s_{n-1}}} \geq s_{n-1} \geq 0$

  We want to prove that $s_{n+1} = \sqrt{2 + \sqrt{s_n}} > s_n = \sqrt{2 + \sqrt{s_{n-1}}}$

  because of introduction assumption $s_n > s_{n-1} \Rightarrow 2 + \sqrt{s_n} > 2 + \sqrt{s_{n-1}} > 0$

  $\Rightarrow \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}}$

  $\Leftrightarrow s_{n+1} > s_n$

  So we have $s_n \geq s_{n-1}, \forall n$. (1).

- Now we will prove that $s_n < 2, \forall n$ (this means $\{s_n\}$ is bounded) (2):

  we also prove this by induction.

(1)+(2) $\Rightarrow \{s_n\} \longrightarrow$ converges.

1/78 Find the upper and lower limit of the sequence $\{s_n\}$ defined by

$$\begin{cases} s_1 = 0 \\ s_{2m} = \frac{s_{2m-1}}{2} \\ s_{2m+1} = \frac{1}{2} + s_{2m} \end{cases}$$

$_1 = 0$

$_2 = \frac{0}{2} = \boxed{0}$

$_3 = \frac{1}{2} + s_2 = \frac{1}{2} + 0 = \frac{1}{2}$

$= \frac{s_3}{2} = \boxed{\frac{1}{4}}$

$= \frac{1}{2} + s_4 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$= \frac{s_5}{2} = \boxed{\frac{3}{8}}$

$= \frac{1}{2} + s_6 = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$

$= \frac{s_7}{2} = \boxed{\frac{7}{16}}$

$= \frac{1}{2} + \frac{7}{16} = \frac{15}{16}$

$= \frac{s_9}{2} = \boxed{\frac{15}{32}}$

$= \frac{1}{2} + s_{10} = \frac{1}{2} + \frac{15}{32} = \frac{31}{32}$

$= \boxed{\frac{31}{64}}$

We have $\begin{cases} \lim\limits_{n\to\infty} s_{2n+1} = 1 \\ \lim\limits_{n\to\infty} s_{2n} = \frac{1}{2} \end{cases}$

* We prove $\begin{cases} s_{2n+1} = \frac{2^n - 1}{2^n} \text{ by ir} & n = \overline{0,\infty} \\ s_{2n} = \frac{2^{n-1} - 1}{2^n} & n = \overline{0,\infty} \end{cases}$ by induction

- Now we prove $s_{2n+1} = \boxed{\frac{2^{n-1}}{2^n}}$ by induction $\forall n = \overline{0,\infty}$

\+ We have $s_1 = 0$

\+ Induction hypothesis $s_{2(n-1)+1} = s_{2n-1} = \frac{2^{n-1} - 1}{2^{n-1}}$

\+ So we have

$$s_{2n} = \frac{s_{2n-1}}{2} = \frac{2^{n-1} - 1}{2^n}$$

$$s_{2n+1} = \frac{1}{2} + s_{2n} = \frac{1}{2} + \frac{2^{n-1} - 1}{2^n} = \frac{2^{n-1} + 2^{n-1} - 1}{2^n} = \frac{2^n - 1}{2^n}.$$

- Now we prove $s_{2n} = \boxed{\frac{2^{n-1} - 1}{2^n}}$ by induction ..

similar ...

$\boxed{\text{Then } S = \{s \in \mathbb{R},\ \exists s_{n_k},\ s_{n_k} \to s\} = \{1, \tfrac{1}{2}\}}.$

$\boxed{\limsup s_n = \sup S} = 1$

$\boxed{\liminf s_n = \inf S} = \frac{1}{2}$

In case S contains only 1 element, $\limsup s_n = \liminf s_n = L\ \boxed{< \infty}$

then $s_n \longrightarrow L$.

* Rudin 3.5/78

For any two real sequence $\{a_n\}$ and $\{b_n\}$.
Prove that $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$

$\liminf_{n\to\infty}(a_n+b_n) \geqslant \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

We only prove for case lim sup
case liminf is similar

* In case $\limsup_{n\to\infty}(a_n) = \infty$ or $\limsup_{n\to\infty} b_n = \infty$, the inequality is always true.

* Let $\alpha = \limsup_{n\to\infty} a_n$, $\alpha < +\infty$

$\beta = \liminf_{n\to\infty} b_n$, $\beta < +\infty$

- $\alpha = \limsup_{n\to\infty} a_n$

$\Leftrightarrow$ ..., $\exists N_0 \in \mathbb{N}, \forall n \geqslant N_0$,
$\sup_{\ell\geqslant n}\{a_\ell\} \leqslant \alpha$

- $\beta = \limsup_{n\to\infty} b_n$

$\Leftrightarrow$ ..., $\exists N_1 \in \mathbb{N}, \forall n \geqslant N_1$
$\sup_{\ell\geqslant n}\{b_\ell\} \leqslant \beta$.

We need to prove that
$\limsup(a_n+b_n) \leq \alpha+\beta$.
$\Leftrightarrow$ NTP $\lim_{N\to\infty}\sup_{n\geqslant N}(a_n+b_n) \leq \alpha+\beta$.
$\Leftrightarrow$ NTP $\exists N_\varnothing \in \mathbb{N}, \forall n \geqslant N_\varnothing, \sup_{\ell\geqslant n}(a_\ell+b_\ell) \leq \alpha+$

So, choose $N = \max\{N_0, N_1\}$, we have $\forall n \geqslant N$, $\sup_{k\geqslant n}\{a_\ell\} + \sup_{\ell\geqslant n}\{b_\ell\} < \alpha+\beta$ (

* But we also have

$$\left.\begin{array}{l} a_i \leq \sup_{\ell\geqslant n}\{a_\ell\}, \forall i \geqslant n \\ b_i \leq \sup_{\ell\geqslant n}\{b_\ell\}, \forall i \geqslant n \end{array}\right\} \Longrightarrow \sup_{\ell\geqslant n}(a_\ell+b_\ell) \leq \sup_{\ell\geqslant n}(a_\ell) + \sup_{\ell\geqslant n}(b_\ell) \quad ($$

From (1)+(2) $\Rightarrow$
So $\exists N, \forall n \geqslant n$, $\sup_{\ell\geqslant n}(a_\ell+b_\ell) \leq \sup_{\ell\geqslant n}(a_\ell) + \sup_{\ell\geqslant n}(b_\ell) \leq \alpha+\beta$.

This means $\limsup(a_n+b_n) \leq \alpha+\beta = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ $\square$.

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* A easier way to solve this problem is simply understand that $\limsup_{n\to\infty} s_n = \lim_{n\to\infty}\sup_{\ell\geqslant n} s_\ell$

We have $\sup_{\ell\geqslant n}(a_\ell+b_\ell) \leqslant \sup_{\ell\geqslant n}(a_\ell) + \sup_{k\geqslant n}(b_\ell)$

Let $n\to\infty$, we have

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty} b_n \quad \square.$$

---

* Another way next page ⟶

Prove that $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

In case $\limsup_{n\to\infty} a_n = +\infty$ or $\limsup_{n\to\infty} b_n = +\infty$ then the inequality is always true.

By theorem about lim sup, let $s_n = a_n + b_n$, $\limsup_{n\to\infty} s_n = \limsup_{n\to\infty}(a_n+b_n)$

: have $\exists (s_{n_\ell})\ s_{n_\ell} \longrightarrow \limsup s_n$

n mean $\exists \{a_{n_\ell}\}\{b_{n_\ell}\}$, $\lim_{\ell\to\infty}(a_{n_\ell}+b_{n_\ell}) = \limsup_{n\to\infty}(a_n+b_n)$ (1)

Let $a = \limsup_{\ell\to\infty} a_{n_\ell}$

Then $\exists \{a_{n_{\ell_m}}\}$, $\lim_{m\to\infty} a_{n_{\ell_m}} = \limsup_{\ell\to\infty} a_{n_\ell} = a$ (2).

because of (1), we also have

$\lim_{m\to\infty}(a_{n_{\ell_m}} + b_{n_{\ell_m}}) = \limsup_{n\to\infty}(a_n+b_n)$ (3)

cause $(a_{n_\ell})$ bounded above.

have $b := \lim(b_{n_{\ell_m}}) = \lim(a_{n_{\ell_m}} + b_{n_{\ell_m}} - a_{n_{\ell_m}}) = \underbrace{\lim_{m\to\infty}(a_{n_{\ell_m}} + b_{n_{\ell_m}})}_{= \limsup_{n\to\infty}(a_n+b_n) \text{ by } (3)} - \underbrace{\lim_{m\to\infty}(a_{n_{\ell_m}})}_{= \limsup a_{n_\ell} = a \text{ by } (2)}$

$= \limsup_{n\to\infty}(a_n+b_n) - a.$

So we have $\limsup(a_n+b_n) = a + b = \underbrace{\limsup_{\ell\to\infty} a_{n_\ell}}_{\leq \limsup_{n\to\infty} a_n} + \underbrace{\lim b_{n_{\ell_m}}}_{\leq \limsup_{n\to\infty} b_n} \leq \limsup a_n + \limsup b$

□.

Rudin 3.6/7 8. Investigate the behavior (convergence/divergence) of $\sum a_n$ if.

$\boxed{a)\ a_n = \sqrt{n+1} - \sqrt{n}}$

We have $a_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2\sqrt{n+1}}$.
$\sum \frac{1}{2\sqrt{n+1}}$ diverges
$\Rightarrow \sum a_n$ diverges.

$\boxed{b)\ b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}}$

We have $0 < b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{(n+1)-n}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n(2\sqrt{n})} = \frac{1}{2n^{3/2}}$
$\sum \frac{1}{2n^{3/2}}$ converges.
$\Rightarrow \sum b_n$ converg

$\boxed{d)\ d_n = \frac{1}{1+z^n}, \text{ for complex value of } z.}$

* We have for $|z| < 1$

Then $|1+z^n| \leq 1 + |z^n| = 1 + |z|^n < 2$

so $\frac{1}{1+z^n} > \frac{1}{2}$
$\sum_{n=1}^{\infty} \frac{1}{2}$ diverges
$\Rightarrow \sum \frac{1}{1+z^n}$ diverges

* For $|z| = 1$

$\sum d_n = \sum \frac{1}{2}$ diverges

* For $|z| > 1$

We have $|z^n + 1| \geq \frac{1}{2}|z^n|$ when $n$ large enough and $z > 1$

So we have $\frac{1}{|1+z^n|} < \frac{2}{|z^n|}$
$\sum \frac{2}{|z|^n}$ converges when $z > 1$
$\Rightarrow \sum \frac{1}{|1+z^n|}$ converges. $\Rightarrow \sum \frac{1}{1+z^n}$ converges.

(It's easy to remember that when $z > 1$, $|z^n + 1| \geq \frac{1}{2} z^n$. We can have another way to explain this:
We have $|z^n| = |z^n + 1 - 1| \leq |z^n + 1| + |1|$
$\Rightarrow |z^n + 1| \geq |z^n| - 1 = \frac{|z^n|}{2} + \underbrace{\left(\frac{|z^n|}{2} - 1\right)}_{>0 \text{ when } z > 1} \geq \frac{|z^n|}{2}$ )

not wrong but don't really need

Investigate the behavior of $\sum a_n$ when:

$\sum_{n=4}^{\infty} \frac{1}{n-3}$

We have for $n \geq 4$, $n-3 \leq n$

$$\left.\begin{array}{l} \Rightarrow \frac{1}{n-3} > \frac{1}{n} \geq 0 \\ \sum_{n=4}^{\infty} \frac{1}{n} \text{ diverges} \end{array}\right\} \Rightarrow \sum_{n=4}^{\infty} \frac{1}{n-3} \text{ diverges.}$$

$\sum_{n=1}^{\infty} \frac{1}{n+3}$

Use comparison test

$$\left.\begin{array}{l} \sum a_n \quad \sum b_n \text{ where } a_n > 0,\ b_n > 0 \\ \lim_{n\to\infty} \frac{a_n}{b_n} = L \ \boxed{>0} \end{array}\right\} \Rightarrow \sum a_n \ \& \ \sum b_n \text{ both converge or diverge}$$

e consider $\sum \frac{1}{n+3}$ and $\sum \frac{1}{n}$

$$\left.\begin{array}{l} \text{we have } (n+3) > 0,\ n > 0 \\ \lim_{n\to\infty} \frac{n+3}{n} = 1,\ \sum \frac{1}{n} \text{ diverges} \end{array}\right\} \Rightarrow \sum \frac{1}{n+3} \text{ diverges.}$$

$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$

$$\left.\begin{array}{l} \text{Ve have } \cancel{2}\frac{e^{-n}}{n^2} \leq \frac{1}{2}\left[e^{-2n} + \frac{1}{n^4}\right] \\ \sum \left(\frac{1}{e^2}\right)^n \text{ converges},\ \sum \frac{1}{n^4} \text{ converges} \Rightarrow \sum \left(e^{-2n} + \frac{1}{n^4}\right) \text{ converges.} \end{array}\right\} \Rightarrow \sum \frac{e^{-n}}{n^2} \text{ converge}$$

$\sum \frac{1}{3^n+1}$ converges by comparison test .

$\sum \frac{1}{n^4+e^n}$

$$\left.\begin{array}{l} \frac{1}{n^4+e^n} \leq \frac{1}{n^4} \\ \sum \frac{1}{n^4} \text{ converges} \end{array}\right\} \Rightarrow \sum \frac{1}{n^4+e^n} \text{ converges.}$$

$\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$

e have $e^n = \sum_{l=0}^{\infty} \frac{n^l}{l!} \overset{(1)}{=} 1 + \frac{n}{1} + \frac{n^2}{2!} + \frac{n^3}{3!} + \frac{n^4}{4!} + \cdots \geq \frac{n^4}{4!}$

o we have $e^n - n^4 \geq \frac{n^4}{4!} - n^4 = \frac{(1-4!)\,n^4}{4!}$

$$\left.\begin{array}{l} \text{So we have } 0 \leq \frac{1}{e^n - n^4} \leq \frac{4!}{(1-4!)\,n^4} \\ \text{we have } \sum \frac{1}{n^4} \text{ converges.} \\ \cancel{\frac{1}{e^n-n^4}} \end{array}\right\} \begin{array}{l} \Rightarrow \sum \frac{1}{e^n - n^4} \text{ converges} \\ \Rightarrow \sum \frac{1}{n^4 - e^n} \text{ converges.} \end{array}$$

te that it's important
make sure we have
s

$\boxed{* \sum \frac{1}{\ln n}}$

We have $\boxed{\lim_{n\to\infty} \frac{\ln n}{n} \underset{\substack{\uparrow \\ \text{LHos}}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0}$

so we have $\ln n \leq n$ when $n$ is large enough.

$$\left.\begin{array}{l} \frac{1}{\ln n} \geqslant \frac{1}{n} \geqslant 0 \\ \sum \frac{1}{n} \text{ diverges} \end{array}\right\} \Rightarrow \sum \frac{1}{\ln n} \text{ diverges.}$$

$\boxed{* \sum \frac{2^n+1}{n2^n-1}}$ we can use compairson test.

way 2: We have

$$\left.\begin{array}{l} \frac{2^n+1}{n2^n-1} \geqslant \frac{2^n}{n2^n-1} \geqslant \frac{2^n}{n\,2^n} = \frac{1}{n} \\ \sum \frac{1}{n} \text{ diverges} \end{array}\right\} \Rightarrow \sum \frac{2^n+1}{n2^n-1} \text{ diverges.}$$

Rudin 3.11/79

Suppose $a_n > 0$, $s_n = \sum_{k=1}^{n} a_k$ and $\sum a_n$ diverges.

a) Prove that $\sum_{n\to\infty} \frac{a_n}{1+a_n}$ diverges.

b) Prove that $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$, and deduce that $\sum \frac{a_n}{s_n}$ diverges.

c) Prove that $\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$, and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

d) What can be said about $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1+n^2 a_n}$

a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

* We prove this by contradiction. Assume that $\sum \frac{a_n}{1+a_n}$ converges,

so we have $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0 \xRightarrow[\text{note } a_n > 0]{} \lim_{n\to\infty} \frac{\overset{1}{\cancel{a_n}}}{\frac{1}{a_1}+1} = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{a_n} = \infty \Rightarrow \boxed{\lim_{n\to\infty} a_n = (}$

- This means $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |a_n| < 1$. (1)

- We also have $\sum \frac{a_n}{1+a_n}$ converges $\Rightarrow$ Cauchy

$$\Leftrightarrow \forall \varepsilon > 0, \exists n_1 \in \mathbb{N}, \forall n \geq n_1 \left| \sum_{k=n_1}^{n} \frac{a_k}{1+a_k} \right| < \varepsilon$$

$$\Rightarrow \sum_{k=n_1}^{n} \frac{a_k}{1+a_k} < \varepsilon \Big( \begin{matrix} \text{ca} \\ a_n \end{matrix}$$

note that because of (1), we have $a_n < 1$

$$\Rightarrow a_n + 1 < 2$$

$$\Rightarrow \frac{a_n}{a_n+1} > \frac{a_n}{2}, \forall n \geq n_0$$

Then choose $N = \max\{n_0, n_1\}$, $\forall n \geq N$,

$$\sum_{k=N}^{n} \frac{a_k}{2} < \sum_{k=N}^{n} \frac{a_k}{1+a_k} < \varepsilon \Rightarrow \sum_{n=1}^{\infty} a_n \text{ Cauchy}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges (contradiction)}$$

The idea of this proof is by contradiction:

assume $\sum \frac{a_n}{1+a_n}$ converges, we want to get a contradiction by prove that $\sum a_n$ conver

$\longrightarrow$ Cauchy

want to prove this by proving $\sum a$ Cauchy

$\uparrow$

we do this by compare

$\frac{a_n}{2} < \frac{a_n}{1+a_n}$ ....

Suppose $a_n > 0$, $s_n = \sum_{k=1}^{n} a_k$, $\sum a_n$ diverges.

ae that $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} > 1 - \frac{s_N}{s_{N+k}}$, and deduce that $\sum \frac{a_n}{s_n}$ diverges

Prove that $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} > 1 - \frac{s_N}{s_{N+k}}$ :

e have LHS $= 1 - \frac{s_N}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = \frac{\sum_{i=N+1}^{i=N+k} a_i}{s_{N+k}} = \frac{a_{N+1} + a_{N+2} + \cdots + a_{N+k}}{s_{N+k}}$

$$= \frac{a_{N+1}}{s_{N+k}} + \frac{a_{N+2}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}}$$

note that $s_{N+k} = \sum_{i=1}^{N+k} a_i > s_{N+l}$ where $l < k$.

$$< \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+k}}{s_{N+k}}.$$

Deduce that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges.

e want to prove that $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, \left|\sum_{i=N+1}^{N+k} a_i\right| > \varepsilon$.

$$\left.\begin{aligned} &\text{e have } \left|\sum_{i=N+1}^{N+k} a_i\right| \underset{\text{above}}{>} 1 - \frac{s_N}{s_{N+k}}. \\ &\text{note that } s_{N+k} > s_{N+1} \\ &\quad \rightarrow \frac{s_N}{s_{N+k}} < \frac{s_N}{s_{N+1}} \\ &\quad \rightarrow 1 - \frac{s_N}{s_{N+k}} > 1 - \frac{s_N}{s_{N+1}} \end{aligned}\right\} \rightarrow \left|\sum_{i=N+1}^{N+k} a_i\right| > 1 - \frac{s_N}{s_{N+1}}$$

because $\{s_n\}$ increasing + divergent choose $N$ s.t. $\frac{s_N}{s_{N+1}} < \frac{1}{2}$

$$\rightarrow \left|\sum_{i=N+1}^{N+k}\right| > \frac{1}{2} \Rightarrow \text{not Cauchy} \Rightarrow \text{divergent} \ \square$$

$c > a_n > 0$, $s_n = \sum_{k=1}^{n} a_k$, $\sum a_n$ diverges.

Prove that $\frac{a_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

* Prove that $\frac{a_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$

We have RHS $= \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1} s_n} = \frac{a_n}{s_{n-1} s_n}$

note that $a_n > 0 \Rightarrow \{s_n\}$ increasing $\Big\}$ RHS $< \frac{a_n}{s_n^2}$ .

* Prove that $\sum \frac{a_n}{s_n^2}$ converges.

Rudin5.1/114

Let $f$ be defined for all real $x$
Suppose that $|f(x)-f(y)| \leq (x-y)^2$ for all real $x$ and $y$ } Prove that $f$ is constant.

We want to prove that $\exists f'(x), \forall x \in \mathbb{R}$ and $f'(x)=0$

Now consider $0 \leq \left|\frac{f(y)-f(x)}{y-x}\right| < |x-y|$
we have $\lim_{x \to y} |x-y| = 0$ } $\Rightarrow \exists \lim_{y \to x} \left|\frac{f(y)-f(x)}{y-x}\right|$ and $\lim_{y \to x} \frac{f(y)-f(x)}{y-x} =$

So we have $\exists f'(x), \forall x \in \mathbb{R}$ and $f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x} = 0 \rightarrow f$ is a constant.

* note that from this we have
$|f(x)-f(y)| \leq (x-y)^2 \Rightarrow f$ cont on $\mathbb{R}$.

Rudin5.2/114

Suppose $f'(x) > 0$ in $(a,b)$
a> Prove that $f$ is strictly increasing in $(a,b)$
b> Let $g$ be a inverse function.
Prove that $g$ is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$ $a<x<b$.

a> Let $y > x$ and $x,y \in (a,b)$
We have $f(y)-f(x) = \underbrace{f'(\xi)}_{>0 \text{ in} (a,b)} \underbrace{(y-x)}_{>0 \text{ cause } (y>x)}$
So when $y > x$, $f(y) > f(x)$ in $(a,b) \Leftrightarrow f$ is strictly increasing in $(a,b)$

b> We have $f'(x) > 0$ in $(a,b) \Rightarrow f$ is one-to-one in $(a,b)$ + the fact $f$ ~~monot~~ strict mon
$\Rightarrow f$ is bijective from $(a,b) \longrightarrow (f(a), f(b))$
$\Rightarrow g$ is well defined.

We have $g \circ f(x) = x \Rightarrow (g \circ f)'(x) = 1$
$\Leftrightarrow g'(f(x)) \underbrace{f'(x)}_{>0} = 1$
$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$.

din 5.3/114

ppose $g$ is a real function on $\mathbb{R}^1$, $|g'| \le M$

x $\varepsilon > 0$, define $f(x) = x + \varepsilon g(x)$.

ve that $f$ is one-to-one if $\varepsilon$ is small enough.

e want if $\varepsilon$ is small then $f$ is one-to-one $\Leftrightarrow$ $f(x) - f(y) \neq 0$ when $x \neq y$

e have $f(x) - f(y) = x + \varepsilon g(x) - y - \varepsilon g(y) = (x-y) + \varepsilon(g(x) - g(y)) =$

$= (x-y) + \varepsilon\, g'(\xi)(x-y)$ when $\xi$ between $x$ and $y$.

$= (x-y)\,[1 + \varepsilon g'(\xi)]$.

o we need $1 + \varepsilon g'(\xi) \neq 0$

n case $1 + \varepsilon g'(\xi) > 0 \Rightarrow \varepsilon g'(\xi) > -1 \Rightarrow \varepsilon < \left|\frac{1}{g'(\xi)}\right| \le \frac{1}{M}$.

~~consider when $g'(\xi) < 0$ and~~

n case ~~$1 + \varepsilon g'(\xi) < 0$~~

consider when

we have $-M \le g'(\xi) \le M$

$1 - \varepsilon M \le 1 + \varepsilon g'(\xi) \le 1 + \varepsilon M$ $\Big\}$ so $f(x) \neq f(y)$ when $\left[\begin{array}{l} 1 - \varepsilon M \geqslant 0 \\ 1 + \varepsilon M \leqslant 0 \text{ (not happ)} \end{array}\right.$

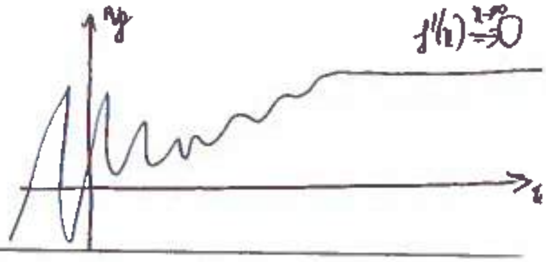
$1 - \varepsilon M \geqslant 0$ when $1 \geqslant \varepsilon M \Rightarrow \varepsilon < \frac{1}{M}$ □.

idin 5.5/114

ppose $f$ is defined and differentiable for every $x > 0$

$f'(x) \xrightarrow[x \to \infty]{} 0$

t $g(x) = f(x+1) - f(x)$ Prove that $g(x) \xrightarrow{x \to \infty} 0$



e have $g(x) = f(x+1) - f(x) = f'(\xi)\,(1)$

Then $g(x) \xrightarrow{x \to \infty} 0$ becomes $f'(\xi) \xrightarrow{\substack{x \to \infty \\ \xi \to \infty}} 0$

Rudin 5.8/114

Suppose $f'$ is continuous on $[a,b]$

a> Prove that $\forall \varepsilon>0, \exists \delta>0$, $\left|\dfrac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$, $\forall t$ $0<|t-x|<\delta$, $a \le x,t \le b$

(this could be expressed by saying that $f$ is uniformly differentiable on $[a,b]$.

b> Does it hold for vector-valued functions too?

a> Way 1:

- We have $f'$ is continuous on $[a,b]$ $\Rightarrow$ continuous at $x$, $\forall x \in [a,b]$.

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0, \forall t \in [a,b]$, $|t-x|<\delta$ then $|f'(t)-f'(x)|<\varepsilon$.

- We also have that $f'(t) = \lim\limits_{u\to t} \dfrac{f(u)-f(t)}{u-t}$ $\Rightarrow$ can't use this way.

- Way 2

① ~~$f'$ is cont exists on $[a,b]$~~ $\Rightarrow$ $f'$ is continuous on $[a,b]$ $\Rightarrow$ $f'$ uniformly continuous on $[a,b]$

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0, \forall u, x \in [a,b]$, $|u-x|<\delta$, then $|f'(u)-f'(x)|<\varepsilon$ (1)

② $f'$ exists on $[a,b]$, then by MVT

$\dfrac{(f(t)-f(x))}{(t-x)} = f'(u)$ for some $u \in (\min(x,t), \max(x,t))$ (2)

and because $0<|t-x|<\delta \Rightarrow |u-x|<\delta$

From (1)+(2) $\Rightarrow$ $\left|\dfrac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$.

- One example when $f'$ exist but not continuous on $[a,b]$ (See Jan 2010, P3)

then $\exists \varepsilon>0, \forall \delta>0, \exists t$, $|t-x|<\delta$ but $\left|\dfrac{f(t)-f(x)}{t-x} - f'(x)\right| > \varepsilon$

when $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x=0 \end{cases}$

we have $f'$ exists for all $x \in \mathbb{R}$

$f'(0)=0$

and $\exists \varepsilon>0, \forall \delta>0, \exists t$, ~~$|t|<\delta$~~ but $\left|\dfrac{f(t)-f(s)}{t-s}\right| > \varepsilon$

$-\delta<s<t<\delta$

ex 5.7/114

Suppose $f'(x), g'(x)$ exist

$g'(x) \neq 0$ , $f(x) = g(x) = 0$

Prove that $\lim\limits_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

Look similar with L'Hospital's Theorem the difference in here is the hypothesis $f(x) = g(x) = 0$ so that we can prove this ex by using def

We have $\lim\limits_{t\to x} \frac{f(t)}{g(t)} \underset{f(x)=g(x)=0}{\overset{\text{because}}{=}} \lim\limits_{t\to x} \frac{f(t)-f(x)}{g(t)-g(x)} = \lim\limits_{t\to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} =$

$$= \frac{\lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}}{\lim\limits_{t\to x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)} .$$

ex 5.13/114

Suppose a and c are real number, $c > 0$

f is defined on $[-1,1]$ by $f(x) = \begin{cases} x^a \sin(|x|^{-c}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

a) Prove that f is continuous iff $a > 0$

Note that in this ex $x \in [-1,1] \to x$ may have positive & negative value

In case $a > 0$ Proving $f(x) \xrightarrow[x\to 0]{} 0$: $0 \leq |f(x)| \leq |x|^a$ (because $|\sin(g(x))| \leq 1, \forall g(x)$).

~~Chose $\varepsilon$ s.t $|x|^a < \varepsilon$~~

For $\varepsilon$ is given, find $\delta$ s.t $||x|^a - f(0)| < \varepsilon$ where $|x-a| < \delta$.

In case $a \leq 0$ We prove that f is not continuous when $a \leq 0$ by finding $\{x_n\}$ such that $f(x_n) \not\longrightarrow f(0)$ when $x_n \longrightarrow 0$

Def f is continuous iff $f(x_n) \xrightarrow[x_n \to 0]{} f(0)$

( )

- Choose $x_n = \left(2\pi n + \frac{\pi}{4}\right)^{\boxed{-\frac{1}{c}}} = t_n^{-\frac{1}{c}} =$ where $t_n = 2\pi n + \frac{\pi}{2}$

we use $2\pi n$ here to have $2\pi n \xrightarrow[n\to\infty]{} \infty$ and $\sin(y + 2\pi n) = \sin y, \forall y$

we need $\left(-\frac{1}{c}\right)$ here because we want it will deleted with $(-c)$ $\left(-\frac{1}{c}\right)(c) = 1$.

then we have $x_n \xrightarrow[n\to\infty]{} 0$

⊕ $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1, \forall p > 0$

$\lim\limits_{n\to\infty} \sqrt[c]{n} = \infty, \forall c \geq 0$

- $f(x_n) = t_n^{\frac{-a}{c}} \sin(t_n) = \frac{\sqrt{2}}{2} t_n^{\frac{-a}{c}}$
- when $a = 0$, $f(x_n) = \frac{\sqrt{2}}{2} \neq f(0)$ ⇒ does not continuous.
- $a < 0$ , $f(x_n) = \frac{\sqrt{2}}{2} t_n^{\frac{-a}{c}} \xrightarrow{n\to\infty} \infty$ ⇒ does not contin.

$a, c \in \mathbb{R}$, $c > 0$

f defined on $[-1,1]$ by $f(x) = \begin{cases} x^a \sin(|x|^{-c}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

$\Leftarrow$ ): Let $a > 0$, prove that f is continuous

We have $\forall a \in \mathbb{R}$, $f(x)$ continuous at ~~$x \neq 0$~~ $x \in [-1,1] \setminus \{0\}$.

At $x = 0$, we have

$$-|x^a| < x^a \sin(|x|^c) < |x^a|$$

$$x^a \xrightarrow[a>0]{x\to 0} 0$$

Since $|x^a| \xrightarrow[a>0]{x\to 0} 0$

$\Rightarrow \lim\limits_{x\to 0} f(x) = 0 = f(0) \rightarrow f$ is continuous at $x = 0$.

? $f'(0)$ exist iff $a > 1$.

($\Leftarrow$): $a > 1$, prove that $f'(0)$ exist.

$f'(0)$ exists, if below function ($\phi$) has limit when $x \to 0$.

$$\phi(x) = \frac{f(x) - f(0)}{x - 0} = \frac{x^a \sin(|x|^{-c}) - 0}{x - 0} = x^{a-1} \sin(|x|^{-c})$$

~~Cause $a > 0$~~ $-|x^{a-1}| < |\phi(x)| < |x^{a-1}|$

When $a > 1$ $\quad |x^{a-1}| \xrightarrow{x\to 0} 0$

then $\phi(x) \xrightarrow{x\to 0} 0 \Rightarrow f'(0)$ exists.

$\Rightarrow$) $f'(0)$ exist, prove that $a > 1$. ($\Leftrightarrow$ we prove that if $a \leq 1$, then $f'(0)$ does no exist.

$$f'(0) = \lim_{\substack{x\to 0 \\ x\neq 0}} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{x\to 0 \\ x\neq 0}} \frac{x^a \sin(|x|^{-c}) - 0}{x - 0} = \lim x^{a-1} \sin(|x|^{-c}).$$

Chose a sequence $(x_n)$, where $x_n = \left(\frac{\pi}{4} + 2\pi n\right)^{-\frac{1}{c}}$ means $x_n = (t_n)^{-\frac{1}{c}}$ where $t_n = \frac{\pi}{4} + 2\pi n$.

Hence we have $x_n = \frac{1}{\sqrt[c]{\left(\frac{\pi}{4} + 2\pi n\right)}} \xrightarrow{n\to\infty} 0$

but $f(x_n) = t_n^{\frac{1-a}{c}} \sin(|t_n|) = \frac{\sqrt{2}}{2} t_n^{\frac{1-a}{c}}$ $\qquad f(0) = 0$.

when $a < 1$ then $\quad f(x_n) \xrightarrow{n\to\infty} +\infty \neq f(0)$

$a = 1$ then $\quad f(x_n) = \frac{\sqrt{2}}{2} \neq f(0)$

5.15 / 115. Rudin. Same section : 5.16, 5.17, 5.18 Rudin + HW 5.5 - 5.6

Suppose $a \in \mathbb{R}^1$, $f$ is twice differentiable real function on $(a, +\infty)$

$M_0 = \sup_{x \in (a,+\infty)} |f(x)|$

$M_1 = \sup_{x \in (a,+\infty)} |f'(x)|$ Prove that $M_1^2 \leq 4 M_0 M_2$.

$M_2 = \sup_{x \in (a,+\infty)} |f''(x)|$

Note that $f$ is twice differentiable real function → we can apply Taylor series (of Lagrange form) with $d = 1$, where $P_d(x)$ : ~~Lagrange~~ Taylor polynomial :

$$f(\beta) = f(\alpha) + \frac{f'(\alpha)}{1!}(\beta-\alpha)^1 + \frac{f''(\xi)}{2!}(\beta-\alpha)^2 \quad , \text{ for } \xi \text{ between } (\alpha, \beta).$$

- Apply above formula with $\alpha = x$, $\beta = x+h$ ) we have

This is a really good trick that needed to remember : when be requiring to pr some inequalit with $f(x)$ use Taylor theorem Lagrang

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2!}h^2 \quad , \text{ for some } \xi \in (x, x+h)$$

$$\text{Then } \left|f'(x)\cdot \cancel{h}\right| = \left|\frac{f(x+h) - f(x) + \frac{f''(\xi)}{2!}h^2}{h}\right| \leq \frac{1}{h}|f(x+h)| + \frac{1}{h}|f(x)| + \frac{1}{2}h\,|f''(\xi)|$$

$$\Rightarrow M_1 \leq \frac{2}{h}M_0 + \frac{h}{2}M_2 \leq 2\sqrt{\frac{2}{h}M_0 \cdot \frac{h}{2}M_2} = 2\sqrt{M_0 M_2}. \Rightarrow M_1^2 \leq 4 M_0 M_2$$

with $\beta = (x+h)$ $\alpha = (x)$

---

* Note that we can also apply Taylor series with $\beta = x + 2h$, $\alpha = x$.

$$f(x+2h) = f(x) + f'(x)\,2h + \frac{f''(\xi)}{2!}(2h)^2$$

$$\Rightarrow \left|f'(x)\cdot 2h\right| = \left|f(x+2h) - f(x) - 2f''(\xi)h^2\right|$$

$$\Rightarrow 2h\,M_1 \leq M_0 + M_0 + 2M_2 h^2.$$

$$\Rightarrow M_1 \leq \frac{M_0}{h} + M_2 \cdot h \leq 2\sqrt{\frac{M_0}{h} M_2 h} = 2\sqrt{M_0 M_2}$$

$$\Rightarrow M_1 \leq 4 M_0 M_2 \Rightarrow \text{done.}$$

din5.16/116

ppose $f$ is twice differentiable on $(0,a)$
$f''$ is bounded on $(0,+\infty)$
$f(x) \xrightarrow[x\to+\infty]{} 0$

Prove that $f'(x) \xrightarrow[x\to+\infty]{} 0$.

pply the result of exercise 5.15 $\Rightarrow M_1^2 \le 4M_0M_2$.

Let $a \to +\infty$, then $M_0 \longrightarrow 0$
$M_2$ bounded

$\Rightarrow M_1^2 \longrightarrow 0 \Rightarrow M_1 \longrightarrow 0$,

this means $\sup_{x\in(a,+\infty)}|f'(x)| \xrightarrow[a\to+\infty]{} 0$, this mean $f'(x) \xrightarrow[x\to+\infty]{} 0$.

din5.17/116:

ppose $f$ is real, three time differentiable function on $[-1,1]$ such that
$f(-1)=0$ $f(0)=0$ $f(1)=1$ $f'(0)=0$.
Prove that $f^{(3)}(x) \geqslant 3$ for some $x\in(-1,1)$.

e have $f$ is three time differentiable on $[-1,1]$, then we can apply Taylor theorem with gree of Taylor polynomial $d=2$, so we have Taylor expand of $f$ is

$$f(\beta) = f(\alpha) + \frac{f'(\alpha)}{1}(\beta-\alpha)^1 + \frac{f''(\alpha)}{2!}(\beta-\alpha)^2 + \frac{f'''(\xi)}{3!}(\beta-\alpha)^3$$ for some $\xi$ between $(\alpha,\beta)$

pply with $\begin{cases}\beta=1\\ \alpha=0\end{cases}$ and $\begin{cases}\beta=-1\\ \alpha=0\end{cases}$, we have

$$. = f(1) = \cancel{f(0)} + \cancel{\frac{f'(0)}{1}(1-0)^1} + \frac{f''(0)}{2!}(1-0)^2 + \frac{f'''(\xi_1)}{3!}1^3, \quad \xi_1\in(0,1)$$

$$) = f(-1) = \cancel{f(0)} + \cancel{\frac{f'(0)}{1}(-1-0)^1} + \frac{f''(0)}{2!}(-1-0)^2 + \frac{f'''(\xi_2)}{3!}(-1)^3, \quad \xi_2\in(-1,0)$$

ien $1 = f(1) - f(-1) = \frac{f'''(\xi_1)}{3!} + \frac{f'''(\xi_2)}{3!}$

So we have $f'''(\xi_1) + f'''(\xi_2) = 6$

If $f'''(\xi_1) < 3$, then $f'''(\xi_2) = 6 - f'''(\xi_1) \geqslant 3$ and vice versa.

$\Rightarrow$ ~~At least~~ one of $\xi_1$ or $\xi_2$ satisfies $f(\xi) \geqslant 3$. □

$$\Rightarrow \frac{f'''(\zeta)}{6} + \frac{f'''(\xi)}{6} = 1 \Leftrightarrow f'''(\zeta) + f'''(\xi) = 6 \quad \text{for } \xi \in (-1,0),\ \zeta \in (0,1).$$

5.18/116: Suppose $f$ is a real function on $[a,b]$
$n$: positive integer
$f^{(n-1)}$ exists for every $t \in [a,b]$
Let $\alpha, \beta, P$ be as in Taylor's theorem
Define $Q(t) = \dfrac{f(t) - f(\beta)}{t-\beta}$ for $t \in [a,b]$, $t \neq \beta$

a) differentiate $f(t) - f(\beta) = (t-\beta)\,Q(t)$ $(n-1)$ times at $t = \alpha$.
b) and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

$*$ $f(t) - f(\beta) = (t-\beta)\ Q(t)$
take derivative of this equation (with variable $t$), we have

$$f'(t) = Q(t) + (t-\beta)\,Q'(t)$$

$$f''(t) = Q'(t) + Q'(t) + (t-\beta)\,Q''(t) = 2Q'(t) + (t-\beta)\,Q''(t)$$

$$\Rightarrow f^{(k)}(t) = k\,Q^{(k-1)}(t) + (t-\beta)\,Q^{(k)}(t) \quad . \text{ for } k = \overline{1, n-1}$$

$$\Rightarrow \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k = \frac{\cancel{k}\,Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^k + \frac{(\beta-\alpha)^{k+1}\,Q^{(k)}(t)}{k!}$$

$$\Rightarrow \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k = Q(\alpha)(\beta-\alpha) - \frac{(\beta-\alpha)^n\,Q^{(n-1)}(t)}{(n-1)!}$$

$$= f(\beta)$$

5.19/116 :

Suppose $f$ is defined in $(-1,+1)$, $f'(0)$ exists.
Suppose $-1 < \alpha_n < \beta_n < 1$,
$\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$
Define the difference quotients $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$
Prove the following statements

a) If $\boxed{\alpha_n < 0} < \beta_n$ then $\lim_{n\to\infty} D_n = f'(0)$.

muốn cm $\lim D_n = f'(0)$
ta cm $|D_n - f'(0)| < \varepsilon$
by choosing $\varepsilon$ bigger than

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} + \frac{f(0) - f(\alpha_n)}{\beta_n - \alpha_n}$$

$$= \frac{\beta_n}{\beta_n - \alpha_n}\,\frac{f(\beta_n) - f(0)}{\beta_n} + \frac{\alpha_n}{\beta_n - \alpha_n}\,\frac{f(0) - f(\alpha_n)}{\alpha_n}$$

Cause $f'(0)$ exist, according to exercise 5.8/114, we have
$\forall \varepsilon > 0, \exists \delta$ such that $\left|\dfrac{f(x) - f(0)}{x} - f'(0)\right| < \varepsilon$ whenever $0 < |x| < \delta$

Cause $\beta_n \to 0$
$\alpha_n \to 0$ $\Rightarrow$ choose $N$ such that $0 < |\beta_n| < \delta$ for all $n \geq N$
$0 < |\alpha_n| < \delta$

$$\Rightarrow D_n - f'(0) = \left(\frac{\beta_n}{\beta_n - \alpha_n}\,\frac{f(\beta_n) - f(0)}{\beta_n} - \frac{\beta_n}{\beta_n - \alpha_n} f'(0)\right) + \left(\frac{\alpha_n}{\beta_n - \alpha_n}\,\frac{f(0) - f(\alpha_n)}{\alpha_n} + \frac{\alpha_n}{\beta_n - \alpha_n} f\right)$$

$$|D_n - f'(0)| \leq \left|\frac{\beta_n}{\beta_n - \alpha_n}\left(\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right)\right| + \left|\frac{(-\alpha_n)}{\beta_n - \alpha_n}\left(\frac{-(f(0) - f(\alpha_n))}{\alpha_n} - f'(0\right)\right|$$

$$\Rightarrow |D_n - f'(0)| \leq \frac{\beta_n}{\beta_n - \alpha_n}\varepsilon + \frac{\alpha_n}{\beta_n - \alpha_n}\varepsilon = \varepsilon \quad \square$$

b) $\boxed{\begin{array}{l}\text{If } (0 < \alpha_n) < \beta_n \\ \dfrac{\beta_n}{\beta_n - \alpha_n} \text{ is bounded}\end{array}\Bigg\}\text{ then } \lim_{n\to\infty} D_n = f'(0).}$

- $\dfrac{\beta_n}{\beta_n - \alpha_n}$ is bounded $\Leftrightarrow \exists M$, $\left|\dfrac{\beta_n}{\beta_n - \alpha_n}\right| \leq M$

$\Rightarrow$ ~~$|\beta_n| \leq M|\beta_n - \alpha_n| \leq M|\beta_n| + M|\alpha_n|$~~

~~$\Rightarrow |\alpha_n|$~~

$$\Rightarrow -M \leq \frac{\beta_n}{\beta_n - \alpha_n} \leq M$$

$\Rightarrow \quad \alpha_n \le \dfrac{\beta_n + M\beta_n}{M} \Longrightarrow N$

means $\dfrac{\alpha_n}{\beta_n - \alpha_n}$ also bounded $\dfrac{\alpha_n}{\beta_n - \alpha_n} < N$

Choose $L = \max\{M, N\} \Rightarrow \dfrac{\beta_n}{\beta_n - \alpha_n} < L \quad \dfrac{\alpha_n}{\beta_n - \alpha_n} < L$.

- ~~$D_n - f'(0)$~~

Similarly to the answer in question a,

~~Choose N such that~~ $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\left| \frac{f(x) - f(0)}{x} - f'(0) \right| < \frac{\varepsilon}{2M}$$

and choose $N$ such that $0 < |\beta_n| < \delta$ for all $n \ge N$.
$0 < |\alpha_n| < \delta$

$$|D_n - f'(0)| \le \frac{\beta_n}{\beta_n - \alpha_n}\frac{\varepsilon}{2L} + \frac{\alpha_n}{\beta_n - \alpha_n}\frac{\varepsilon}{2L} \le L\frac{\varepsilon}{2L} + L\frac{\varepsilon}{2L} = \varepsilon. \quad \blacksquare$$

c) $\boxed{\text{If } f' \text{ is continuous in } (-1,1), \text{ then } \lim_{n\to\infty} D_n = f'(0)}$

- By mean value theorem : $\exists\, c_n$ between $\alpha_n$ and $\beta_n$ such that

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \frac{f'(c_n)(\beta_n - \alpha_n)}{\beta_n - \alpha_n} = f'(c_n)$$

- $\left.\begin{array}{l} \alpha_n \to 0 \\ \beta_n \to 0 \\ c_n \text{ between } \alpha_n, \beta_n \end{array}\right\} \Rightarrow c_n \to 0$

- $f'$ continuous $\Rightarrow \lim\limits_{n\to\infty} D_n = \lim\limits_{n\to\infty} f'(c_n) = f'(\lim\limits_{n\to\infty} c_n) = f'(0)$.

d) Give an example in which $f$ is differentiable in $(-1,1)$, but $f'$ is not continuous at $0$ and in which $\alpha_n, \beta_n \to 0$ in such a way that $\lim D_n$ exist but is different from $f'(0)$

Hàm sin quan trọng là vì
$\sin(2\pi n) = 1$ always, even when $n \to \infty$
hữu hạn

- Let $f(x) = \begin{cases} x\sin\frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

then $f'(x) = \begin{cases} \sin\frac{1}{x} + x\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

- Let $\beta_n = \dfrac{1}{2\pi n}$
$\alpha_n = +\dfrac{1}{2\pi n + \frac{\pi}{2}}$

then $\alpha_n, \beta_n \xrightarrow{n\to\infty} 0$

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \frac{\frac{1}{2\pi n}\overbrace{\sin(2\pi n)}^{=1} - \frac{1}{2\pi n + \frac{\pi}{2}}\overbrace{\sin\left(+2\pi n + \frac{\pi}{2}\right)}^{-1}}{\frac{1}{2\pi n} - \frac{1}{2\pi n + \frac{\pi}{2}}}$$

$1 \quad 1$

5.21/117:

Let $E$ = closed subset of $R^1$
We saw in exercise 22, chapter 4, that there is a real continuous function $f$ on $R^2$ whose zero set in $E$ .
Is it possible, for each closed set $E$, to find such an $f$ which is differentiable on $R^2$, or one which is n

1 Prove that $\int_a^b x^2 = \frac{b^3-a^3}{3}$ by definition.

Mat 632, Sprin
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Tran Le

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~~Consider~~ * We first consider case $b > a \geq 0$

Consider a partition $P$ of $[a, b]$, $P = \{x_0, x_1, \dots, x_n\}$.

Because ~~$x^2$~~ $f(x) = x^2$ is an increasing function on $[0, +\infty)$, we have $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = x_i$

$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = x_{i-1}$

+ We now estimate the upper Riemann sum.

$$\cdot\ U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{n} x_i^2 (x_i - x_{i-1})$$

$$\cdot\ x_i^3 - x_i^2 x_{i-1} = \frac{x_i^3 - x_{i-1}^3}{3} + \frac{2x_i^3}{3} + \frac{x_{i-1}^3}{3} - \frac{3x_i^2 x_{i-1}}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} + \frac{2x_i^3 - 2x_i^2 x_{i-1} + x_{i-1}^3 - x_i^2 x_{i-1}}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} + \frac{2x_i^2 (x_i - x_{i-1}) - x_{i-1}(x_i^2 - x_{i-1}^2)}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} + \frac{(x_i - x_{i-1})[2x_i^2 - x_{i-1}(x_i + x_{i-1})]}{3}$$

$$= \frac{x_{i+1}^3 - x_{i-1}^3}{3} + \frac{(2x_i + x_{i-1})(x_i - x_{i-1})^2}{3}$$

Then we have
$$U(P, f) = \sum_{i=1}^{n} \frac{x_i^3 - x_{i-1}^3}{3} + \sum_{i=1}^{n} \frac{(2x_i + x_{i-1})(x_i - x_{i-1})^2}{3}$$

$$= \frac{b^3 - a^3}{3} + \frac{1}{3} \sum_{i=1}^{n} (2x_i + x_{i-1})(x_i - x_{i-1})^2$$

+ We now estimate Lower Riemann sum

$$\cdot\ L(P, f) = \sum_{i=1}^{n} x_{i-1}^2 (x_i - x_{i-1})$$

$$\cdot\ x_i x_{i-1}^2 - x_{i-1}^3 = \frac{x_i^3 - x_{i-1}^3}{3} + \left(\frac{-x_i^3}{3} + \frac{-2x_{i-1}^3}{3} + \frac{3x_i x_{i-1}^2}{3}\right)$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} + \frac{(x_i x_{i-1}^2 - x_i^3) + (2x_i x_{i-1}^2 - 2x_{i-1}^3)}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} + \frac{x_i (x_{i-1}^2 - x_i^2) + 2x_{i-1}^2 (x_i - x_{i-1})}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} + \frac{(x_i - x_{i-1})(2x_{i-1}^2 - x_i^2 - x_i x_{i-1})}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} - \frac{(x_i - x_{i-1})[(x_i - x_{i-1})(x_i + x_{i-1}) + x_{i-1}(x_i - x_{i-1})}{3}$$

$$= \frac{x_i^3 - x_{i-1}^3}{3} - \frac{(x_i - x_{i-1})^2 (x_i + 2x_{i-1})}{3}$$

hen we have $L(P,f) = \underbrace{\sum_{i=1}^{n} \frac{x_i^3 - x_{i-1}^3}{3}}_{= \frac{b^3-a^3}{3}} - \sum_{i=1}^{n} \frac{(x_i + 2x_{i-1})(x_i^2 - x_{i-1})^2}{3}$

t remains to show that $\begin{cases} \inf\left(\frac{1}{3}\sum_{i=1}^{n}(2x_i + x_{i-1})(x_i - x_{i-1})^2\right) = 0 & (1) \\ \inf \frac{1}{3}\sum_{i=1}^{n}(x_i + 2x_{i-1})(x_i - x_{i-1})^2 = 0 & (2) \end{cases}$

nsider (1):

We have $\frac{1}{3}\left(\sum_{i=1}^{n}(2x_i - x_{i-1})(x_i - x_{i-1})^2\right) \le \frac{1}{3}(3b)\sum_{i=1}^{n}(x_i - x_{i-1})^2$

hoose $(x_i - x_{i-1}) < \frac{\varepsilon}{b(b-a)}$

then we have $b\sum_{i=1}^{n}(x_i - x_{i-1})^2 \le (b)\frac{\varepsilon}{(b)(b-a)}\sum_{i=1}^{n}(x_i - x_{i-1}) = \varepsilon$

nsider 2:

Ve have $\frac{1}{3}\left(\sum_{i=1}^{n}(x_i + 2x_{i-1})(x_i - x_{i-1})^2\right) \le \frac{1}{3}(3b)\sum_{i=1}^{n}(x_i - x_{i-1})^2$

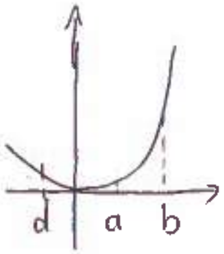
hoose $(x_i - x_{i-1}) < \frac{\varepsilon}{b(b-a)}$

Then we have $b.\sum_{i=1}^{n}(x_i - x_{i-1})^2 \le (b)\frac{\varepsilon}{(b)(b-a)}\sum_{i=1}^{n}(x_i - x_{i-1}) = \varepsilon$

~~so~~ $(x_i - x_{i-1}) \le \frac{\varepsilon}{b(b-a)}$

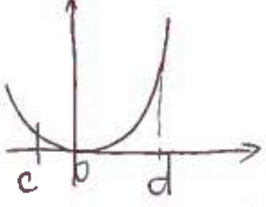
Then we have (1) and (2) are both true $\Rightarrow$ $f$ is Riemann integral and $\int f\,dx = \frac{b^3-a^3}{3}$ when $b > a \geq 0$.

In case we want to compute $\int_c^d x^2\,dx$, where $c < d \le 0$



we have $\int_c^d x^2\,dx \underset{\substack{\uparrow \\ \text{where} \\ a=-d \\ b=-c}}{=} \int_a^b x^2\,dx \underset{\text{from above}}{=} \frac{b^3-a^3}{3} = \frac{(-c)^3-(-d)^3}{3} = \frac{d^3-c^3}{3}$.

In case we want to compute $\int_c^d x^2\,dx$, where $c < 0 < d$,



then $\int_c^d x^2\,dx = \int_c^0 x^2\,dx + \int_0^d x^2\,dx \underset{\substack{\uparrow \\ \text{from above}}}{=} \frac{0-c^3}{3} + \frac{d^3-0^3}{3} = \frac{d^3-c^3}{3}$ ∎

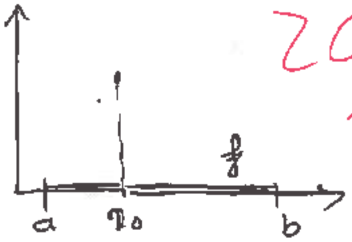
6.4/138

Suppose $\alpha$ increases on $[a,b]$, $a \le s_0 \le b$
$\alpha$ is continuous at $s_0$

$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \ne x_0 \end{cases}$$

Prove that $f \in \mathcal{R}(\alpha)$ and $\int_a^b f d\alpha = 0$

Tran Le



20/2

- We have $\alpha$ continuous at $s_0 \iff \forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in [a,b]$, $|x - s_0| \le \delta$, then $|\alpha(x) - \alpha(s_0)| < \varepsilon$
- We create a partition $P = \{\underset{a}{\overset{\shortparallel}{x_0}}, x_1, x_2, \dots, x_n = b\}$ such that $\Delta x_i < \delta$

then we have $\left[\begin{array}{l} s_0 \text{ can be one of } x_{k-1}\text{, for some } k = \overline{1,n}, \quad \text{(call this is case (1))} \\ s_0 \text{ belongs to a segment } [x_{k-1}, x_k]\text{, for some } k = \overline{1,n} \quad \text{(call this is case (2))} \end{array}\right.$

we can skip this step by using theorem 6.10

Then we have :

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=0}^{n} (M_i - m_i) \, [\alpha(x_i) - \alpha(x_{i-1})]$$

$$= \left[\begin{array}{l} 1[\alpha(s_0) - \alpha(x_{k-2})] + 1[\alpha(x_k) - \alpha(s_0)] \text{, for ca} \\ 1[\alpha(x_k) - \alpha(x_{k-1}) \text{, for case (2)} \end{array}\right.$$

$$= \left[\begin{array}{l} 2\varepsilon \text{, for case (1)} \\ \le \alpha(x_k) - \alpha(s_0) + \alpha(s_0) - \alpha(x_{k-1}) \le 2\varepsilon \text{, for case (2)} \end{array}\right.$$

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then we have $f \in \mathcal{R}(\alpha)$

* Because $f \in R(\alpha)$, we have (by theorem 6.7c):

$$\left|\int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i) \Delta\alpha_i\right| < \varepsilon \quad \text{(where } t_i \text{ is an arbitrary point in } [x_{i-1}, x_i]\text{)}.$$

$$\left|\int_a^b f d\alpha\right| < \sum_{i=1}^{n} f(s_i) \Delta\alpha_i + \varepsilon \le 1 \Delta\alpha_k + \varepsilon \le \varepsilon + \varepsilon = 2\varepsilon.$$

$$\Rightarrow \int_a^b f d\alpha = 0.$$

Good! But you can make life much easier by citing Theorem 6.10 :)

Suppose $f \geq 0$

$f$ is continuous on $[a,b]$

$\int_a^b f(x)\,dx = 0$

Prove that $f(x) = 0$ for all $x \in [a,b]$.

Prove by contradiction!

Assume $\exists\, x_0 \in [a,b]$ such that $f(x_0) > 0$.

Since $f$ is continuous on $[a,b]$ $\Rightarrow$ continuous at $x_0$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in [a,b], |x - x_0| \leq \delta$, then $|f(x) - f(x_0)| < \varepsilon$

choose $\varepsilon = \frac{f(x_0)}{2}$, we have $-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}$

$\Rightarrow 0 < \frac{f(x_0)}{2} < f(x)$

Choose $\gamma = \min\{\delta, x_0 - a, b - x_0\}$

We have $f \geq 0$ on $[a,b]$

then we have

$$\int_a^b f(x)\,dx \geq \underbrace{\int_a^{x_0-\gamma} f(x)\,dx}_{\geq 0} + \underbrace{\int_{x_0-\gamma}^{x_0+\gamma} f(x)\,dx}_{\geq \int_{x_0-\gamma}^{x_0+\gamma} \frac{f(x_0)}{2}\,dx} + \underbrace{\int_{x_0+\gamma}^{b} f(x)\,dx}_{\geq 0}$$

$$\boxed{= f(x_0)\gamma > 0}$$

so we have $\int_a^b f(x)\,dx > 0$ (contradiction)

even we have $\gamma$ in here, it OK to put $\boxed{>0}$

In conclusion, $f(x) = 0, \forall x \in [a,b]$ ∎

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6.3/138 Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$\beta_j(x)=0$ if $x<0$ $\quad \beta_j(x)=1$ if $x>0$ $\quad \beta_1(0)=0$

$\beta_2(0)=1$

$\beta_3(0)=\frac{1}{2}$

Let $f$ be a bounded function on $[-1,1]$

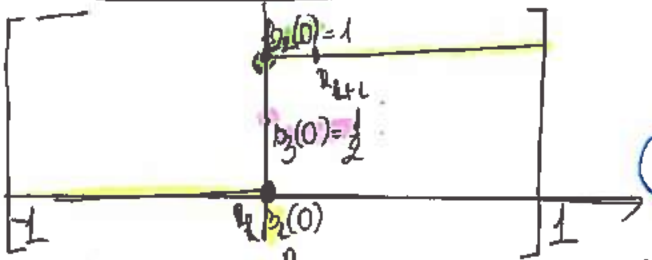
a) Prove that $f\in \mathcal{R}(\beta_1)$ iff $f(0+)=f(0)$ and $\int f\,d\beta_1=f(0)$

see theorem 6.15

b) State a similar result for $\beta_2(x)$ iff $f(0^-)=f(0$

c) Prove that $f\in\mathcal{R}(\beta_3)$ iff $f$ cont at $0$

d) Prove that if $f$ cont at $x$ then $\int f\,d\beta_1=\int f\,d\beta_2=\int f\,d\beta_3=f(0)$



Consider a partition on $[-1,1]$

$P=\{x_1<x_2<x_3<\cdots<x_n\}$ such that

$\boxed{\exists x_\ell \text{ s.t } x_\ell=0}$

$$U(P,f,\beta_j)=\sum_{i=1}^{n}M_i\,\Delta\beta_{ji}=\sum \sup_{x_\ell\le x\le x_{\ell+1}} f(x)\,\Big(\underset{1}{\beta_j(x_{\ell+1})}-\underset{0}{\beta_j(x_\ell)}\Big)=\sup_{x\in[x_\ell,x_{\ell+1}]} f(x)\,\big(\beta(x_{\ell+1})-\beta(x_\ell)\big)$$

$$L(P,f,\beta_j)=\sum_{i=1}^{n}m_i\,\Delta\beta_{ji}=\sum \inf_{x_\ell\le x\le x_{\ell+1}} f(x)\,\big(\underset{1}{\beta_j(x_{\ell+1})}-\beta_j(x_\ell)\big)$$

a) In case $\beta_1(0)=0$ then $U(P,f,\beta_j)=\sup\limits_{x_\ell\le x\le x_{\ell+1}} f(x)(1-0)=M_\ell$ $\quad x_\ell=0.$

$$L(P,f,\beta_j)=\inf_{x_\ell\le x\le x_{\ell+1}} f(x)=m_\ell$$

($\Rightarrow$):

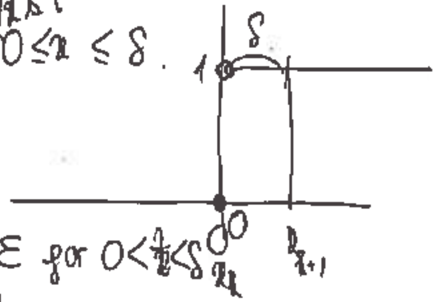
We have $f\in\mathcal{R}(\beta_2)$ iff exist a partition $P^*$ such that $U(P^*,f,\beta_j)-L(P^*,f,\beta_j)<\varepsilon$ (actually, $P^*$ is a refinement of $P$)

$M_\ell-m_\ell<\varepsilon$.

If such a partition exist, put $x_{\ell+1}=\delta$

$\Rightarrow$ Then we have $|f(x)-f(0)|<M_\ell-m_\ell<\varepsilon$ $\forall x$ s.t for $0\le x\le\delta$.

hence $\lim\limits_{x\to 0^+} f(x)=f(0)$



($\Leftarrow$): If $f(0^+)=f(0)$

then for $\forall\varepsilon>0$, let $\delta>0$ be such that $|f(t)-f(0)|<\varepsilon$ for $0<t<\delta$

then let $P$ a partition such that $x_\ell=0$, $x_{\ell+1}<\delta$

Then $|M_\ell-f(0)|<\varepsilon$

$|m_\ell-f(0)|<\varepsilon$

$$\Rightarrow |M_\ell-m_\ell|\le|(M_\ell-f(0))-(m_\ell-f_0)|\le$$
$$\le|M_\ell-f(0)|+|m_\ell-f_0|\le\varepsilon$$

$$\Rightarrow U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

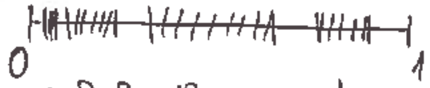
$$\Rightarrow f\in\mathcal{R}(\alpha)$$

and because $\sup\limits_{0\le x\le x_{\ell+1}} f(x)\xrightarrow{x_{\ell+1}\to 0} f(0)$, $\inf f(x)\longrightarrow f(0)$ $\Big\}\Rightarrow \int_a^b f\,d\alpha=f(0)$.

6/138 : Let $P$ be the Cantor set constructed in section 24. 20/20 Tran Le

Let $f$ be a bounded real function on $[0,1]$ which is continuous at every point outside

Prove that $f \in \mathcal{R}$ on $[0,1]$ .



Total length removed from $[0,1]$ $= \frac{1}{3} + 2 \cdot \frac{1}{3^2} + 4\frac{1}{3^3} + 2^3 \frac{1}{3^4} + \cdots$

$$= \frac{1}{3}\left[\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k\right] = \frac{1}{3}\,\frac{1}{1-\frac{2}{3}} = 1 .$$

The Cantor set has measure 0

We can cover $P$ by finitely many segments whose total length can be made as small as desired.

Then apply 6', we have $f \in \mathcal{R}$ on $[0,1]$ .

6/138:

Def: Jordan zero measure: A set $E \subset [a,b]$ is said to have Zordan ~~mee~~ zero measure if it can be covered by finitely many segments whose total length can be made as small as desired.

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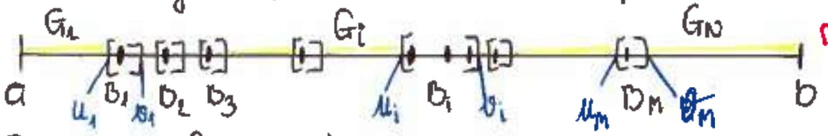
Exercise 6': Let $f$ be a bounded real function on $[a,b]$ which is continuous at every point outside E (a set of Jordan zero measure)
Prove that $f$ is Riemann-integrable on $[a,b]$.

---

- Assume $f$ is discontinuous at some point in E



nice picture :)

Because E has Zordan zero measure, we can cover E by finitely many segment $D_1, \dots, ($
such that total length can be made as small as desired

Then we have $|D_1| + |D_2| + \dots + |D_M| < \varepsilon$ (1)

- By assumption, $f$ is continuous outside E $\Rightarrow$ $f$ is continuous on $G_1, G_2, \dots,$ and $G_N$
$\Rightarrow$ $f$ is Riemann integrable on $G_1, \dots, G_N$ (I explain this on the last page) (2

$\underset{\text{theorem 6.6}}{\Longleftrightarrow}$ $\forall \varepsilon > 0, \exists$ partition $P_i$, such that $U(P_i, f) - L(P_i, f) < \varepsilon/N$, $\forall i = \overline{1}$

- Then in $[a,b]$, we have a partition $P = P_1 \cup P_2 \cup \dots \cup P_N \cup \left( \bigcup_{j=1}^{n} \{u_j, v_j\} \right)$ $j = \overline{1, M}$
where $u_j, v_j$ are 2 end points of $D_j$

Then we have

$$U(P,f) - L(P,f) = [U(P_1,f) - L(P_1,f)] + \dots + [U(P_N,f) - L(P_N,f)] +$$

$$+ \sum_{j=1}^{M} \underbrace{(M_i - m_i)}_{\leq 2K} \underbrace{[v_j - u_j]}_{= |D_j|}$$ (if $|f| \leq K$ (by assumption $f$ is bounded

$$\leq \underset{(\text{because of } (2))}{N \varepsilon/N} + 2K \underbrace{\sum_{j=1}^{M} D_j}_{< \varepsilon \text{ (because of (1))}}$$

$$= \varepsilon + 2K\varepsilon = (1+2K)\varepsilon$$

So $f \in \mathcal{R}$ on $[a,b]$ ∎

10/10

finition : A set $E \subset [a,b]$ is said to have a zero Lebesgue measure if it can be covered a countable family of segment whose total length can be made as small as desired

xercise 6'. Let $f$ be Riemann-integrable on $[a,b]$. Prove that :

) $f$ is bounded

) The set of point at which $f$ fails to be continuous has zero Lebesgue measure

, $f$ is Riemann-integrable on $[a,b]$. Prove that $f$ is bounded on $[a,b]$

e have $f$ is Riemann integrable on $[a,b]$

$\Leftrightarrow \forall \varepsilon > 0$, $\exists$ partition $P$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

hoose $\varepsilon = 1$ $\Leftrightarrow$ $\sum_{i=1}^{n} (M_i - m_i) \underbrace{\Delta x_i}_{>0} < 1$

$\rightarrow 0 \leq M_i - m_i < +\infty \rightarrow f$ is bounded on $[a,b]$

$f$ is Riemann integrable. Prove that the set of points at which $f$ ~~is~~ fails to be continuous is zero Lebesgue measure.

Let $E$ be the set of all points at which $f$ is not continuous. We want to show $E$ has zero Lebesgue measure

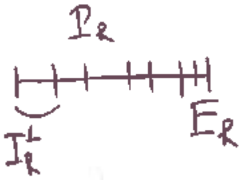
y definition of discontinuity

$x_1 \in E \Leftrightarrow \exists \delta > 0, \forall$segment $I$ containing $x_1$, $\sup_{s,t \in I} |f(t) - f(s)| > \delta$.

r each $k > 0$, Let $E_k = \left\{ \begin{matrix} x \in [a,b], \\ I \text{ is segment containg } x \end{matrix} \middle| \sup_{t,s \in I} |f(t) - f(s)| > \frac{1}{k} \right\}$

ien we have $E = E_1 \cup E_2 \cup \ldots \cup E_k \cup \ldots$

jiven $k \in \{1, 2, \ldots\}$, because $f$ is Riemann integrable in $E_k$, there is a partition $P_k$ ch that $U(P_k, f) - L(P_k, f) < \frac{\varepsilon}{k 2^k}$ (1)



Let $I_k^1, I_k^2, \ldots, I_k^{n_k}$ be intervals of $P_k$ which contain points in $E_k$.

$$(1) \Leftrightarrow \sum_{i=1}^{n_k} \underbrace{(M_i - m_i)}_{> \frac{1}{k}} (x_i - x_{i-1}) < \frac{\varepsilon}{k 2^k}$$

$$\Rightarrow \frac{1}{k} \sum_{i=1}^{n_k} (x_i - x_{i-1}) < \frac{\varepsilon}{k 2^k}$$

$$\Rightarrow \quad \frac{1}{k}\left(|I_k^1|+|I_k^2|+\cdots+|I_k^{n_k}|\right) < \frac{\varepsilon}{k2^k}$$

So $E_k$ is covered by interval $I_k^1,\dots,I_k^{n_k}$ such that

$$|I_k^1|+|I_k^2|+\cdots+|I_k^{n_k}| < \frac{\varepsilon}{2^k}.$$

- Hence $E$ is covered by a countable family $E_1, E_2, E_3, \dots$ with ~~total~~ length

$$|E| = \sum_{k=1}^{\infty} E_k < \sum_{k=1}^{\infty}\frac{\varepsilon}{2^k} = \varepsilon\sum_{k=1}^{\infty}\frac{1}{2^k} = \varepsilon \quad \blacksquare$$

This means the set of points at which $f$ fails to be continuous has zero Lebesgue measure $\blacksquare$

10/10

vith $u_i, v_i$ are two ends point of each segment $D_i$

e consider $K = \underbrace{[a,b] \setminus \underbrace{\bigcup_{i=1}^{N} (u_i, v_i)}_{\text{finite union of open} \to \text{open}}}_{\text{closed}}$

$G_1 \cup G_2 \cup \dots \cup G_N = K$ is closed + bounded in $\mathbb{R}$ $\Rightarrow$ compact.

f continuous in K compact → f ~~continuous in~~ uniformly continuous in K.

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_\varepsilon, \forall x, y \in [a,b], |x-y| < \delta_\varepsilon$, then $|f(x) - f(y)| < \varepsilon/N$

ve create partition $P_i$ in $G_i$ ~~as~~ : $P_i = \{x_i^0, x_i^1, \dots, x_i^{n_i}\}$ such that

$$x_i^0 = v_{i-1},\ x_i^{n_i} = u_i,\ \Delta x_i < \delta_\varepsilon$$

So we have in ~~d~~ each $G_i$ :

$$U(P_i, f) - L(P_i, f) = \sum_{j=0}^{n_i} (M_j - m_j) \Delta x_i < \varepsilon/N \sum \Delta x_i$$

$$= \varepsilon/N \, |D_i| \le \frac{\varepsilon}{N}$$

6.7/138

Tran Le

MAT632

⑳ 30/30

Suppose $f$ is a real function on $[0,1]$

$f \in \mathscr{R}$ on $[c,1]$ for every $c>0$.

Define $\int_0^1 f(x)dx = \lim_{c\to 0}\int_c^1 f(x)dx \; (*)$ if the limit exists and is finite.

a) If $f\in\mathscr{R}$ on $[0,1]$. Show that this definition of the integral agrees with the old

b) Construct a function $f$ such that the above limit exists, although it fails to exist with $|f|$ in place of $f$.

---

a) Because $f\in\mathscr{R}$ on $[0,1]$, we have

$$\left|\int_0^1 f(x)dx - \int_c^1 f(x)dx\right| = \left|\int_0^c f(x)dx\right| \le \int_0^c |f(x)|dx$$

because $f\in\mathscr{R}$ on $[0,1]$ $\Rightarrow$ $f$ is bounded in $[0,1]$ $\Rightarrow \exists M, |f| < M$ on $[0,1]$

$$\Rightarrow \left|\int_0^1 f(x)dx - \int_c^1 f(x)dx\right| \le M(c-0) = Mc$$

⑤

Let $c\to 0$, we have the definition $(*)$ agrees with the old one.

b) Consider $f(x) = \begin{cases} (-1)^n(n+1) & , \frac{1}{n+1} < x \le \frac{1}{n}, n = 1,2,\dots \\ 0 & \text{otherwise.} \end{cases}$

- Then $\int_c^1 f(x)dx = \int_c^1 (-1)^n(n+1)\chi_{\left[\frac{1}{n+1},\frac{1}{n}\right]}(x)dx$. $\left(\text{for } \frac{1}{N+1} \le c \le \frac{1}{N}\right)$

$$= \underbrace{(-1)^n(N+1)\frac{1}{N} - (-1)^N(N+1)c}_{\substack{N\to\infty \\ \Rightarrow c\to 0}} + \underbrace{\sum_{k=1}^{N-1}\frac{(-1)^k}{k}}_{\text{converges (alternating series)}}$$

⑤

$$\downarrow 0$$

This means $\int_c^1 f(x)dx$ converges when $c\to 0$

- $$\int_c^1 |f(x)|dx = \underbrace{(N+1)\left(\frac{1}{N}-c\right)}_{\xrightarrow[\substack{N\to\infty\\(c\to 0)}]{} 0} + \underbrace{\sum_{k=1}^{N-1}\frac{1}{k}}_{\longrightarrow \infty \text{ (when } N\to\infty)}$$

69/139. Show that the integration by part can sometimes be applied to the "improper" integral defined in exercise 7 and 9

(State appropriate ~~theorem~~ hypotheses, formulate a theorem and prove it)

For instance, show that $\int_0^{+\infty} \frac{\cos x}{1+x} dx = \int_0^{\infty} \frac{\sin x}{(1+x)^2} dx$.

* Integration by part for "improper" integral:

Suppose F and G are differentiable functions on $[a,b]$ such that

$$\begin{cases} \lim_{b\to\infty} F(b)G(b) = \text{exists} \\ \int_a^{+\infty} F(x)G'(x)\,dx \text{ converges} \end{cases}$$

Then

$$\begin{cases} \int_0^{+\infty} F'(x)G(x)\,dx \text{ converges,} \\ \text{and } \int_a^{+\infty} F'(x)G(x) = \lim_{b\to\infty} F(b)G(b) - F(a)G(a) - \int_a^b F(x)G'(x)\,dx. \end{cases}$$

* Apply this integration by part for $F(x) = \sin x$, $G(x) = \frac{1}{1+x}$, $a = 0$

- $F(b)G(b) = \frac{\sin b}{1+b}$

  because $|\sin b| < 1 < 1+b$ $(b>0)$ $\Rightarrow \lim_{b\to\infty} F(b)G(b) = \lim_{b\to+\infty} \frac{\sin b}{1+b} = ($

- $F(0)G(0) = \frac{\sin 0}{1} \Longrightarrow 0$

- $\int_0^{\infty} F(x)G'(x)\,dx = -\int_0^{+\infty} \frac{\sin x}{(1+x^2)} dx$.

  i.e. here $\left|\frac{\sin x}{1+x^2}\right| < \left|\frac{1}{1+x^2}\right|$ decrease monotonically on $[1,+\infty)$

  $\sum_{n=1}^{\infty} \frac{1}{1+x^2}$ converges.

⑤

Then by integral test, $\int_0^{+\infty} F(x)G'(x)$ converges

In conclusion $\int_0^{+\infty} \cos x \frac{1}{1+x} = 0 - 0 + \int_0^{+\infty} \frac{\sin x}{(1+x^2)} dx$ ∎

6.10/139. Let p and q dre positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Prove the following statements.

a) If $u \geq 0$ and $v \geq 0$, then $uv \leq \frac{u^p}{p}+\frac{v^q}{q}$ (*)

equality hols iff $u^p = v$

* We have when $u=0$, LHS $\leq \frac{v^q}{q}$, $\forall v \geq 0$

$v=0$, LHS $\leq \frac{u^p}{p}$, $\forall u \geq 0$

$u=v=0$, the equality holds.

* Now consider when $u>0, v>0$

For fixed $v$, define $f:(0,+\infty) \longrightarrow \mathbb{R}$

$$u \longmapsto f(u) = \frac{u^p}{p}+\frac{v^q}{q}-uv$$

- $f'(u) = u^{p-1}-v$ • $f''(u) = (p-1)u^{p-2}$
- From $\frac{1}{p}+\frac{1}{q}=1$, $p>0, q>0$ $\Big\}$ $\Rightarrow \frac{1}{p}<1$ and $\frac{1}{q}<1 \Rightarrow p>1$ and $q>1$ (1

$\Rightarrow \frac{1}{q} = 1-\frac{1}{p} = \frac{p-1}{p} \Rightarrow (p-1)q = p$ (2)

- Because of (1), we have $f''(u)>0$, $\forall u>0$, hence $f$ attains it minimu at $u_0$ such that $f'(u_0)=0 \Leftrightarrow u_0^{p-1} = v$

$$\Rightarrow \underset{\text{because of (2)}}{(u_0^{p-1})^q = u_0^p} = v^q$$

and we also have $f(u_0) = \frac{u_0^p}{p}+\frac{v^q}{q}-u_0 v = \frac{v^q}{p}+\frac{v^q}{q}-u_0u_0^{p-1} = v^q\left(\frac{1}{p}+\frac{1}{q}\right)-$

$= v^q - v^q = 0$

So we have $f(u) \geq f(u_0) = 0$, $\forall u$ $\Rightarrow$ (*) □.

and equality holds iff $u^p = v^q$. □✓

5

If $f \in \mathcal{R}(\alpha)$
$g \in \mathcal{R}(\alpha)$
$f \geq 0$
$g \geq 0$
$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha$

Prove that $\int_a^b fg \, d\alpha \leq 1$.

Because $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, we have $fg \in \mathcal{R}(\alpha)$

$f \geq 0$, $g \geq 0$ then apply 10a, we have $(fg) \leq \frac{f^p}{p} + \frac{g^q}{q}$

$$\to \int fg \, d\alpha \leq \int \frac{f^p}{p} d\alpha + \int \frac{g^q}{q} d\alpha$$

$$= \frac{1}{p} \underbrace{\int f^p d\alpha}_{= 1 \text{ by assumption}} + \frac{1}{q} \underbrace{\int g^q d\alpha}_{=1}$$

⑤

$$= \frac{1}{p} + \frac{1}{q} = 1$$

7 If f and g are complex function in $\mathcal{R}(\alpha)$, then
Prove $\left| \int_a^b fg \, d\alpha \right| \leq \left[ \int_a^b |f|^p d\alpha \right]^{1/p} \left[ \int_a^b |g|^q d\alpha \right]^{1/q}$

nsider $F(x) = \frac{|f(x)|}{\left[\int_a^b |f|^p d\alpha\right]^{1/p}}$ then we have $\int_a^b F \, d\alpha = \int_a^b \left| \frac{f^p}{\int_a^b |f|^p d\alpha} \right|^p d\alpha = 1$

$G(x) = \frac{|g(x)|}{\left[\int_a^b |g|^q d\alpha\right]^{1/q}}$ then $\int G^q \, d\alpha = 1$

Then we have $F \geq 0$, $G \geq 0$

Apply 10b, we have $\int_a^b F(x) G(x) \, d\alpha(x) \leq 1$

$$\Leftrightarrow \int_a^b \frac{|f| \, |g|}{\left[\int_b^b |f|^p d\alpha\right]^{1/p} \left[\int_a^b |g|^q d\alpha\right]^{1/q}} \leq 1$$

$$\Rightarrow \left| \int_a^b fg \, d\alpha \right| \leq \int_a^b |fg| \, d\alpha \leq \left( \int |f|^p d\alpha \right)^{1/p} \left( \int_a^b |g|^q d\alpha \right)^{1/q}.$$

6.7/138 Suppose $f$ is a real function on $(0, 1]$.

$f \in \mathcal{R}$ on $[c,1]$ for every $c>0$

Define $\int_0^1 f(x)\,dx = \lim_{c\to 0} \int_c^1 f(x)\,dx$ if the limit exists and finite. ?

a) If $f \in \mathcal{R}$ on $[0,1]$, show that the definition of the integral agrees with the /old one

b) Construct a function $f$ such that the above limit exists, although it fail s to exist with $|f|$ in place of $f$.

a) If $f \in \mathcal{R}$ on $[0,1]$ We have $f$ is bounded on $[0,1]$ $\Rightarrow \exists M$, $|f| < M$.

$$\begin{array}{rcl} -M & \le \int_0^1 f(x)\,dx \le & M \\ -M(1-c) & \le \int_c^1 f(x)\,dx \le & M(1-c) \\ \hline -Mc & \le \left| \int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx \right| \le & Mc \end{array}$$

When $c \to 0$, then

$$\left| \int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx \right| < \varepsilon$$

68/138

Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where $a$ is fixed.

Define $\displaystyle\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$ if the limit exists and is finite.

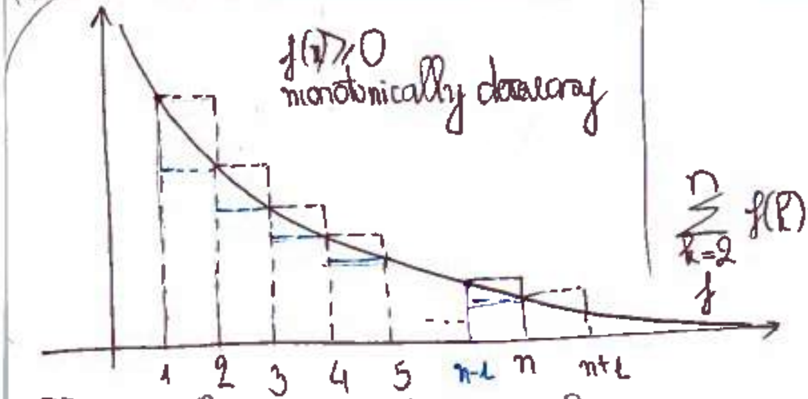
In that case, we say that the integral on the left $\boxed{\text{converges}}$.

If it also converges after $f$ has been replaced by $|f|$, it is said to converge absolutely.

- Assume $\begin{cases} f(x) \geq 0 \\ f \text{ decreases monotonically on } [1, +\infty) \end{cases}$

Prove that $\displaystyle\int_1^{\infty} f(x)\,dx$ converges $\iff \displaystyle\sum_{n=1}^{\infty} f(n)$ converges

Integral test for the convergence / divergence of the series.



Partition $[1, n+1]$ into unit length,

By the figure, we have

$$\underset{f}{\sum_{k=2}^{n}} f(k) \times 1 \leq \int_1^n f(t)\,dt \leq \sum_{k=1}^{n-1} f(k) \times 1$$

Put $S_n = \displaystyle\sum_{k=1}^{n} f(k)$, then we have

$$S_n - f(1) \underset{(1)}{\leq} \int_1^n f(t)\,dt \underset{(2)}{\leq} \cancel{\sum} S_{n-1}$$

- Because of (1), if $\displaystyle\int_1^n f(t)\,dt$ converges $\Rightarrow S_n$ converges, means $\displaystyle\sum_{k=1}^{n} f(k)$ converges.

  if $S_n$ diverges $\Rightarrow \displaystyle\int_1^n f(t)\,dt$ diverges.

- Because of (2) if $\displaystyle\int_1^n f(t)\,dt$ diverges $\Rightarrow S_{n-1}$ diverges $\Rightarrow \displaystyle\sum_{k=1}^{n} f(k)$ diverges.

  if $S_{n-1}$ converges $\Rightarrow \displaystyle\int_1^n f(t)\,dt$ converges.

~~6.9: Show that integration by parts can sometimes be applied to the "improper" integrals defined in exercises 7 & 8.~~

~~(State appropriate theorem, formulate a theorem, and prove it.~~

~~For instant, show that~~ $\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx$.

$$\int_0^M \frac{\cos x}{1+x}dx = \int_0^M [\sin x]' \frac{1}{1+x}dx = \frac{\sin x}{1+x}\Big|_0^M - \int_0^M \sin x\,(-1)\left(\frac{1}{1+x}\right)^2 dx.$$

$$= \frac{\sin x}{1+x}\Big|_0^M + \int_0^M \frac{\sin x}{(1+x)^2}dx.$$

$$= \frac{\sin M}{1+M} + \int_0^M \frac{\sin x}{(1+x)^2}dx.$$

đã làm.

Letting $M \to \infty$, $|\sin M| < 1 < 1+M \Rightarrow \frac{\sin M}{1+M} \xrightarrow{M\to\infty} 0$

$$\Rightarrow \int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx.$$

---

EG.11/140

Let $\alpha$ be a fixed increasing function on $[a,b]$

For $u \in \mathscr{R}(\alpha)$, define $\|u\|_2 = \left[\int_a^b |u|^2 d\alpha\right]^{1/2}$

Suppose $f, g, h \in \mathscr{R}(\alpha)$. Prove the triangle inequality:

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2$$

as a consequence of the ~~Swatch~~ Schwarz inequality, as in the proof of Theo 1.3

By Hölder's inequality $\left|\int u\bar{v}\,d\alpha\right| \leq \left(\int u^2 d\alpha\right)^{1/2}\left(\int v^2 d\alpha\right)^{1/2}$ (*)

We have

$$\|u+v\|^2 = \int |u+v|^2 d\alpha = \int u^2 + \int v^2 + \int \bar{u} v + \int u \bar{v}$$

$$\leq \int u^2 + \int v^2 + 2\left(\int u^2 d\alpha\right)^{1/2}\left(\int v^2 d\alpha\right)^{1/2} \quad (\text{by}(*))$$

$$= \|u\|^2 + \|v\|^2 + 2\|u\|^{1/2}\|v\|^{1/2}$$

$$= (\|u\| + \|v\|)^2$$

$$\Rightarrow \|u+v\| \leq \|u\| + \|v\|$$

put $u = f - g$   $v = g - h$

~~$\|u+v\| = \|$~~ $\|f - h\| = \|u+v\| \leq \|u\| + \|v\| = \|f-g\| + \|g-h\|$

Using integral test to test the convergence / divergence of

$\sum n e^{-n^2}$

* We need to consider the convergence / divergence of the integral

$\int_1^{+\infty} f(x)\,dx$ where $f(x) = x e^{-x^2}$ and $\begin{cases} f(x) \geqslant 0, \forall x \\ f \text{ monotonically decreases on } [1, +\infty) \end{cases}$

\+ We have $f(x) = x e^{-x^2} \geqslant 0, \forall x$.

\+ $f'(x) = \cancel{x} e^{-x^2} + x(-2x) e^{-x^2} = e^{-x^2}(1 - 2x^2) \leqslant 0$ for $x \in [1, +\infty)$

$1 - 2x^2 = 0 \Leftrightarrow x = \pm \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

* $\int_1^{+\infty} x e^{-x^2} dx$ put $u = e^{-x^2} \Rightarrow du = -2x e^{-x^2} dx$.

$x = 1 \Rightarrow u = e^{-1}$

$x = +\infty \Rightarrow u = e^{-\infty} = 0$

$= -\frac{1}{2} \int_{e^{-1}}^{0} du = -\frac{1}{2}(-e^{-1}) = \frac{1}{2e}$.

Because the ~~s~~integral converges $\Rightarrow$ The series converges.

* $\int_1^{+\infty} \frac{1}{x^2} dx = 1 \Rightarrow \sum \frac{1}{n^2}$ also convergent

$\int_1^{\infty} \frac{1}{x} dx$ divergent $\Rightarrow \sum \frac{1}{n}$ divergent

Rudin 612/140

Homewok

Tran L

Let $\alpha$ fixed increasing function on $[a,b]$, $f \in \mathscr{R}(\alpha)$, $\varepsilon > 0$.

Define $\|f\|_2 = \left[\int_a^b |f|^2 d\alpha\right]^{1/2}$

Prove that there exists a continuous function $g$ on $[a,b]$ such that $\|f-g\|_2 < \varepsilon$

* We have $f \in \mathscr{R}(\alpha)$ on $[a,b]$

$\Rightarrow \forall \varepsilon > 0, \exists$ a partition $P = \{x_0, \dots, x_n\}$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$$\Leftrightarrow \sum_{i=1}^{n} (M_i - m_i)\left(\alpha(x_i) - \alpha(x_{i-1})\right) <$$

(*

* Now we define

$$g(t) := \begin{cases} \frac{t - x_0}{\Delta x_0} f(x_1) \\ \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) & , x_{i-1} \le t \le x_i \quad , i = \overline{1,n} \end{cases} \qquad (**)$$

- We have $\forall x_i \in P$, $i = \overline{1,n}$

$$\left.\begin{array}{l} g(x_i^-) = \lim\limits_{t \to x_i^-} \left( \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right) = f(x_i) \\ g(x_i^+) = \lim\limits_{t \to x_i^+} \left( \frac{x_{i+1} - t}{\Delta x_{i+1}} f(x_i) + \frac{t - x_i}{\Delta x_{i+1}} f(x_{i+1}) \right) = f(x_i) \end{array}\right\} \text{equal}$$

then because $g(x_i^-) = g(x_i^+)$ and the definition of $g(x)$ $\Rightarrow$ g is continuous function

Now we want to prove that $\|f - g\|_2 < \varepsilon$

* By (**), we have

$$\left[\begin{array}{ll} m_i \le f(x_i) \le g(x) \le f(x_{i+1}) \le M_i & , \text{if } f(x_{i+1}) \geqslant f(x_i) \\ m_i \le f(x_{i+1}) \le g(x) \le g(x_i) \le M_i & , \text{if } f(x_{i+1}) \le f(x_i) \end{array}\right.$$

$$\Rightarrow \quad |g(x) - f(x)| \le M_i - m_i \quad , \forall x \in [x_i, x_{i+1}] \qquad (***)$$

* ~~Also~~ We also have because $f \in \mathscr{R}(\alpha) \Rightarrow$ bounded

$$\Rightarrow \exists M, \quad |f(x)| \le M \quad , \forall x \in [a,b] \qquad (**$$

* Then we have:

$$\|f - g\|_2 = \int_a^b |f - g|^2 d\alpha \le \sum_{i=1}^{n} \left(\max\{f(x_i) - g(x_i)\}\right)^2 [\alpha(x_i) - \alpha(x_{i-1})]$$

$$\underset{\text{by }(***)}{\le} \sum_{i=1}^{n} (M_i - m_i)^2 [\alpha(x_i) - \alpha(x_{i-1})] \quad \#$$

$$\Rightarrow \|f-g\|_2 \underset{\text{by }(***)}{\leq} M \underbrace{\sum_{i=1}^{n} |M_i - m_i| \left[\alpha(x_i) - \alpha(x_{i-1})\right]}_{\leq \varepsilon \ (\text{by } *)}$$

$$< M\varepsilon$$

because $\varepsilon$ is arbitrary $\Rightarrow$ done

⑩

165

7.1a) Prove that every uniformly convergent sequence of bounded functions is uniformly bounded

$\left.\begin{array}{l}\forall n, f_n \text{ bounded} \\ f_n \rightrightarrows\end{array}\right\}$ Then $\{f_n\}$ uniformly bounded.

From this, we can have
$\left.\begin{array}{l}\{f_n\}\text{: sequence of bounded functions} \\ f_n \rightrightarrows f\end{array}\right\}$ Then $f$ bounded

We have :

- $\{f_n\}$ : sequence of bounded functions.

$\Leftrightarrow \quad |f_n(x)| \leq M_n \ , \forall n \ , \forall x.$ (1)

- $f_n \rightrightarrows$ , then $\{f_n\}$ satisfies Cauchy criterion

$\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall m \geq N_\varepsilon, |f_m(x) - f_{N_\varepsilon}(x)| < \varepsilon$ (2)

$\Rightarrow \ |f_m(x)| \leq |f_{N_\varepsilon}(x)| + \varepsilon$

We need to prove

- $\{f_n\}$ uniformly bounded.

$\Leftrightarrow$ NTP $\exists M$,

$|f_n(x)| \leq M, \forall n, \forall$

Then we have , $\forall m < N_\varepsilon$ , Choose $M^* = \max\{M_1, M_2, \dots, M_{N_\varepsilon}\}$

then $|f_m(x)| \leq M^*$ (because of (1))

$\forall m \geq N_\varepsilon$ , Choose $M^{**} = |f_{N_\varepsilon}(x)| + 1$

, then $|f_m(x)| \leq M^{**}$

Choose $M = \max\{M^*, M^{**}\}$, then $|f_m(x)| \leq M$ , $\forall m \in \mathbb{N}$ ∎

b) Prove that $\left.\begin{array}{l}\{f_n\}\text{ : sequence of bounded function} \\ f_n \rightrightarrows f\end{array}\right\}$ then $f$ is bounded.

- $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon, |f_n(x) - f(x)| < \varepsilon$

$|f(x)| \leq |f_n(x)| + \varepsilon$ (3)

- Because a) $\Rightarrow |f_n(x)| \leq M, \forall n, \forall x$ (4)

(3)+(4) $\Rightarrow |f(x)| \leq M+1, \forall x$.

( Note that from this we have
If $\{f_n\}$ uniformly bounded and $|f_n(x)| \leq M, \forall n, \forall x$
and $f_n \rightrightarrows f$
then $|f(x)| \leq M + 1$
(not $M$ )

2/165: Prove that  See Fall 1996, I3 $>$

$>$ If $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ } Then $(f_n+g_n) \rightrightarrows (f+g)$.

$>$ If $f_n \rightrightarrows f$, $g_n \rightrightarrows g$, $\{f_n\}, \{g_n\}$ are sequences of bounded functions } Then $(f_n g_n)(x) \rightrightarrows (fg)(x)$

Prove that : $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ } Then $(f_n+g_n) \rightrightarrows (f+g)$

By definition. $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon > 0, \exists N_{1\varepsilon}, \forall n \geq N_{1\varepsilon}, \forall x \in E, |f_n(x) - f(x)| < \frac{\varepsilon}{2}$ (1)

$g_n \rightrightarrows g \Leftrightarrow \forall \varepsilon > 0, \exists N_{2\varepsilon}, \forall n \geq N_{2\varepsilon}, \forall x \in E, |g_n(x) - g(x)| < \frac{\varepsilon}{2}$ . (2)

Choose $N = \max\{N_{1\varepsilon}, N_{2\varepsilon}\}$ .

Then $\forall n \geq N$, $\forall x \in E$, we have $|(f_n(x)+g_n(x)) - (f(x)+g(x))| \leq |f_n(x)-f(x)| + |g_n(x)-g$

$< \varepsilon$

this means $(f_n+g_n) \rightrightarrows (f+g)$

If $f_n \rightrightarrows f$, $g_n \rightrightarrows g$, $\{f_n\}, \{g_n\}$: sequences of bounded functions } Then $(f_n \cdot g_n) \rightrightarrows (fg)$ (Note that $\{f_n\}, \{g_n\}$ not sequences of continuous + bounded → can't use $(C(X), \|\cdot\|)$)

$\{f_n\}$: sequences of bounded functions $\Leftrightarrow \exists M, |f_n(x)| \leq M, \forall n, \forall x$. (3)

and because $g_n \rightrightarrows g$, by exercise 7.1, $g$ is bounded

$\Leftrightarrow |g(x)| \leq N, \forall x$ (4)

Then also choose $n \geq N$ as above,

consider $|(f_n g_n)(x) - (fg)(x)|$, we have

$$|f_n(x) g_n(x) - f(x) g(x)| \leq |f_n(x) g_n(x) - f_n(x) g(x)| + |f_n(x) g(x) - f(x) g(x)|$$

$$= \underbrace{|f_n(x)|}_{\leq M} \underbrace{|g_n(x) - g(x)|}_{< \frac{\varepsilon}{2}} + \underbrace{|g(x)|}_{\leq N} \underbrace{|f_n(x) - f(x)|}_{< \frac{\varepsilon}{2}}$$

$$\leq \frac{M}{2}\varepsilon + \frac{N}{2}\varepsilon$$

Then we have $(f_n g_n) \rightrightarrows f g$

Tran Le

HW

Mat 602

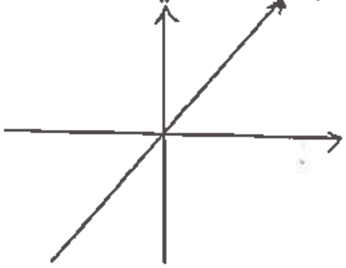
19/20

7.3/165.

Construct sequence $\{f_n\}$, $\{g_n\}$ which converges uniformly on some set $E$ but $\{f_n g_n\}$ does not converge uniformly on $E$.

(Of course, $\{f_n g_n\}$ must converge (pointwise) on $E$).

* Put $f_n(x) = x$, $\forall n$

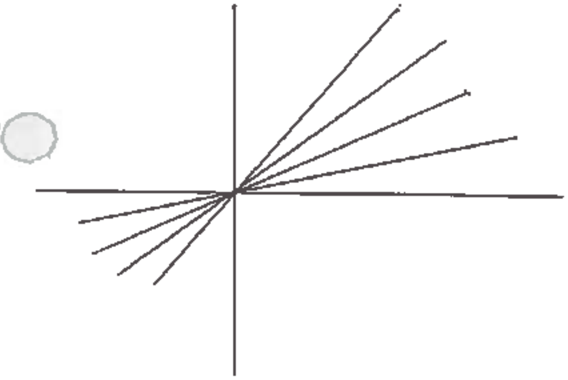


Then we have $f_n(x) \rightrightarrows g(x)$ where $g(x) = x$

* Put $g_n(x) = \frac{1}{n}$, $\forall n$.

Then we have $g_n(x) \rightrightarrows 0$

* Consider $(f_n g_n)(x) = \frac{x}{n} \not\rightrightarrows$ ✓ explain why not! we can always find $x \ni f_n g_n > 1$



Of course, we have $(f_n g_n)(x) = \frac{x}{n} \xrightarrow[n\to\infty]{\text{pointwise}} 0$

4/5

* A question supports for ex 7.4 (Rudin) : ——— Practicing exercise

Investigate the uniform convergence of the following sequence and series on the interval $[a,b]$, $a,b \in \mathbb{R}$.

a) $f_n(x) = \frac{1}{1+n^2x^2}$ b) $\sum_{n=1}^{\infty} \frac{n}{x^n}$ $(a>0)$.

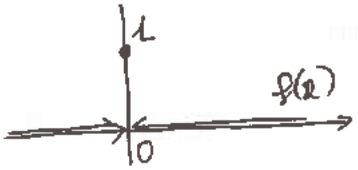
* <u>First, we find the pointwise limit of</u> $f_n(x)$

- when $x = 0$, $f_n(x) = 1$.
- when $x \neq 0$, $f_n(x) = \frac{1}{1+n^2x^2} \xrightarrow{n\to\infty} 0$

Then we have $f_n(x) \xrightarrow{\text{point wise}} f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$

* We note that $x^2$ is the coeffici of $n^2$ $\Rightarrow$ we have to consider in case $x^2 = 0$

*



* Now, we consider $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$

We will show that, the convergence is uniform on $[a,b]$ iff $0$ is not in $[a,b]$.

- <u>Now, show that $f_n(x)$ is not convergent uniformly on $[0,b]$.</u>

$f_n(x) \underset{\text{on } E}{\rightrightarrows} f(x)$, $\Leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$.

$f_n(x) \not\rightrightarrows f(x)$ on $E \Leftrightarrow \exists \varepsilon > 0, \forall n_\varepsilon \in \mathbb{N}, \exists n \geq n_\varepsilon, \exists x_0 \in E, |f_n(x_0) \neq f(x_0)| \geq \varepsilon$

$\hookrightarrow$ we want to find $x_0 = h(n)$ such that $f_n(x_0) \neq f(x_0)$

no matter how large n is taken

- Choose $\varepsilon = \frac{1}{2}$, then nomatter how large $n$ is taken, $\exists x_n = \frac{1}{n}$ )

$$|f_n(x_n) - f(x_n)| = \left|\frac{1}{1+1} - 0\right| = \frac{1}{2}$$

this means $f_n(x) \not\rightrightarrows f(x)$ on $[0,b]$

- <u>Similarly, $f_n(x) \not\rightrightarrows f(x)$ on $[-b,0]$</u>
- <u>Now, show that $f_n(x) \rightrightarrows f(x)$ on $[a,b]$ for $a>0$</u> | or $b<0$

We consider $M_n = \sup_{x\in E} |f_n(x) - f(x)| = \sup_{\substack{x\in[a,b] \\ a>0}} |f_n(x) - 0| = \sup_{\substack{x\in[a,b] \\ a>0}} \left|\frac{1}{1+n^2x^2}\right| = \frac{1}{1+n^2a^2}$

we have $M_n = \frac{1}{1+n^2a^2} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow f_n(x) \rightrightarrows f(x)$ on $[a,b]$, $a>0$.

Investigate the convergence of the following series on the interval $[a|b]$ , $a,b\in\mathbb{R}$

$$\sum_{n=1}^{\infty}\frac{n}{x^n} \quad , a>0$$

Consider

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,x^n}{x^{n+1}\;n}\right| < 1 \text{ when } \frac{1}{|x|}<1 \Leftrightarrow |x|>1 .$$

Then the series does not converge pointwise for $|x|<1$

$\Rightarrow$ does not converges uniformly

We will show that the series converges uniformly on $[a,+\infty)$ , for $\boxed{a>1}$ .

we have for $x \geq a > 1$

$$x^n \geq a^n > 1 .$$

$$\frac{n}{x^n} \leq \frac{n}{a^n}$$

- Now consider the series $\sum_{n=1}^{\infty}\frac{n}{a^n}$ , $a>1$ .

\# 7/4 / Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$

a) For what values of $x$ does the series converge absolutely?

b) On what intervals does it converge uniformly?

b₂ On what interval does it fail to converges uniformly?

c) Is $f$ continuous wherever the series converges? Is $f$ bounded.

a) <u>For what value of $x$ does the series converge absolutely?</u>

- When $x = 0$, $f(x) = \sum_{n=1}^{\infty} 1$ does not converge $\Rightarrow$ does not converge absolutely
- When $x = -\frac{1}{n^2}$, $1+n^2x = 0$ $\Rightarrow$ $f(x)$ is undefined
- Consider when $x \neq 0$ and $x \neq -\frac{1}{n^2}$, we have $x \in (-\infty,-1) \cup \left(\bigcup_{l=1}^{\infty}\left(-\frac{1}{l^2}, -\frac{1}{(l+1)^2}\right)\right) \cup ($

<u>Incase $x\in(0,+\infty)$</u> $|1+n^2x| \geq |n^2x| = |x|\, n^2$

then $0 < \left|\frac{1}{1+n^2x}\right| \leq \frac{1}{|x|\, n^2}$, $\forall n$

$\frac{1}{|x|}\sum \frac{1}{n^2}$ converges

By comparison test $\sum_{n=1}^{\infty}\left|\frac{1}{1+n^2x}\right|$ converges

$\Leftrightarrow f(1) = \sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ converges absolutely

- <u>Incase $x \in (-\infty,-1)$ and $x \in \left(-\frac{1}{l^2}, -\frac{1}{(l+1)^2}\right)$ for some $l \in \mathbb{N}$</u>

we have $|1+n^2x| \boxed{>} \left|\frac{n^x}{2}\right|^2$ when $n$ is large enough.

then we have $\left|\frac{1}{1+n^2x}\right| < \frac{4}{n^2}$, for some $n \geq n_0$

$\sum \frac{1}{n^2}$ converges

$\Rightarrow$ by comparison test $\sum\left|\frac{1}{1+n^2x}\right|$ converges.

In conclusion, the series converges when $x \in (0,+\infty)$, $(-\infty,-1)$, $\left(-\frac{1}{l^2}, -\frac{1}{(l+1)^2}\right)$ and finite union of those intervals

Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$

On what interval does it converge uniformly

We have $f$ does not converge at $x = 0$ and $x = -\frac{1}{n^2}$

in any interval that contains $-\frac{1}{n^2}$ or $0$ $f$ does not converge uniformly

We consider when $x \in [a, c]$, for $a \gneq 0$ c finite / infinite

We claim that $f$ is uniformly convergent in those intervals

we have, $|1+n^2x| \geqslant |n^2x| = |x|n^2 \geqslant an^2$, for $>0$

then $\left|\frac{1}{1+n^2x}\right| \leqslant \frac{1}{a.n^2} = M_n$

we have $\sum_{n=1}^{\infty} M_n$ converges

By theorem 7.10

We have $\sum f_n(x)$ converges uniformly

imilaly, $f$ convergent uniformly on all closed interval $[-\infty, b]$, for $b \gneq 0$ :

except at the points $x = -\frac{1}{n^2}$

When we choose $n \geqslant \sqrt{\frac{2}{b}}$, then we have

$$\left|\frac{1}{1+n^2x}\right| \leqslant \frac{1}{n^2\left(b-\frac{1}{n^2}\right)} \leqslant \frac{2}{bn^2}$$

$$\sum_{n=1}^{\infty} \frac{2}{bn^2} \text{ converges}$$

→ by theorem 7.10

→ $\sum f_n(x)$ converges uniformly

so it converges unif. on $[\epsilon, \infty)$ $\forall \epsilon > 0$

and $(-\infty, -1)$

as well as the intervals between the discontinuities.

b) Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$

On what interval does it fail to converge ~~a~~ uniformly

I claim that it fails to converge uniformly on :

(1) • any open/closed interval that contain 0 or $-\frac{1}{n^2}$, for some n.

(2) • any interval that has 0 or $-\frac{1}{x^2}$ is a limit point.

(3) •

*(1) We have in any closed/open interval that contains 0 or $-\frac{1}{n^2}$ ~~can't~~, $f(x)$ can't have convergence at these points

$\Rightarrow f(x)$ doesn't converge uniformly on those intervals

*(2) • First consider $[0,a)$ or $(0,a)$ (a can be finite / infinite)

+ We have $\forall x \in [0,a)$, $|f_n(x)| \le 1, \forall x, \forall n$ (each term of the series is bounded on $[0,a)$

then if $\sum f_n(x) \rightrightarrows f(x)$, we have $f(x)$ has to be a bounded function (by exercise 7.1)

but we have $f\left(\frac{1}{m^2}\right) = \sum_{n=1}^{\infty} \frac{1}{1+\frac{n^2}{m^2}} = m^2 \sum_{n=1}^{\infty} \frac{1}{m^2+n^2} \xrightarrow{m\to\infty} \infty$ (*)

• Another cases are similar.

10/10

Is $f$ continuous whenever the series ~~bounded~~ converge ?
Is $f$ bounded ?

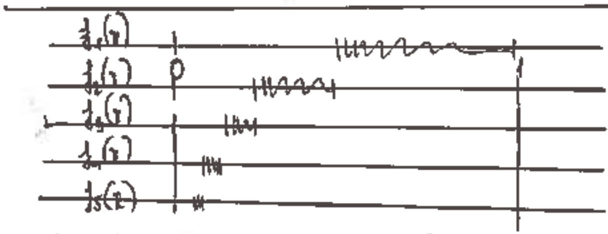
$f$ is continuous on those intervals that it converges uniformly because of (*) (~~test~~ (on part b2) $f$ is unbounded.

Rudin 7.5/111

$$f_n(x)=\begin{cases}0 & x<\frac{1}{n+1}\\ \sin^2\frac{\pi}{x} & \frac{1}{n+1}\leq x\leq\frac{1}{n}\\ 0 & \frac{1}{n}<x\end{cases}$$

a) Show that $\{f_n\}$ converges to a continuous function, but not uniformly

b) Use the serie $\sum f_n$ to show that the absolute convergence, even for all $x$ does not imply uniform convergence



Put $f(x)=0$, $\forall x\in\mathbb{R}$.

We want to show that $f_n(x)\xrightarrow{\text{pointwise}} f(x)$.

- We have $\forall x\leq 0$ or $x\geq 1$ $f_n(x)=0 \Rightarrow$ true $\Rightarrow$ done.

- Consider $x\in(0,1)$, we have $\exists n_0$ such that $0<\frac{1}{n_0}<x<1$.

So $\forall n\geq n_0$, $\frac{1}{n}\leq\frac{1}{n_0}<x \Rightarrow f_n(x)=0$, $\forall n\geq n_0$.

this means $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, $\forall n\geq n_0$, $|f_n(x)|<\varepsilon \Rightarrow$ done.

* Now we prove that $f_n(x) \not\rightrightarrows f(x)$.

We have $f_n\rightrightarrows f \overset{\text{def}}{\Leftrightarrow} \forall\varepsilon>0, \exists n_0\in\mathbb{N}, \forall n\geq n_0, \forall x\in\mathbb{R}, \|f_n(x)-0|<\varepsilon$.

$f_n \not\rightrightarrows 0 \overset{\text{def}}{\Leftrightarrow} \exists\varepsilon>0, \forall n_0\in\mathbb{N}, \exists n\geq n_0, \exists x\in\mathbb{R}, \quad |f_n(x)|>\varepsilon$

this means, how large $n$ is, $\exists x_n$ st. $|f_n(x)|>\varepsilon$

F how large $n$ is, $\exists x=\frac{1}{n+\frac{1}{4}}$ $f_n(x)=\sin^2\frac{\pi}{x}=\frac{1}{2}-\frac{1}{2}\cos\frac{2\pi}{x}=\frac{1}{2}-\frac{1}{2}\cos 2\pi\left(n+\frac{1}{4}\right)$

$$=\frac{1}{2}-\frac{1}{2}\underbrace{\cos\left(2\pi n+\frac{\pi}{2}\right)}_{=0} \geq \frac{1}{2} \Rightarrow \square.$$

b) * Prove that $\sum|f_n|$ converges for all $x\in\mathbb{R}$.

Put $g(x)=\begin{cases}0 & x\leq\frac{1}{(1)}\\ \sin^2\frac{\pi}{x} & 0<x<1\\ 0 & x\geq 1.\end{cases}$ Put $s_\ell(x)=\sum_{n=1}^{\ell}|f_n(x)|$

We NTP, $\forall x\in\mathbb{R}$, $\sum_{n=0}^{\infty}|f_n|\longrightarrow g(x)$. $(*)$

$\Leftrightarrow$ NTP, $\forall x\in\mathbb{R}, \forall\varepsilon>0, \exists k_0\in\mathbb{N}, \forall k\geq k_0, |s_\ell(x)-g(x)|<$

We have for $x\leq 0$ or $x\geq 1$ $(*)$ is true.

Now we will prove that $(*)$ is true for $0<x<1$

We have for $x\in(0,1)$, $\exists!\,\ell_0$ such that ~~$f$~~ $\frac{1}{\ell_0+1}<x<\frac{1}{\ell_0}$

then $\forall\ell\geq\ell_0$ $s_\ell(x)=\sum_{n=1}^{\ell}f_n(x)=f_{\ell_0}(x)=\sin^2\frac{\pi}{x}\,\chi_{\left(\frac{1}{\ell_0+1},\frac{1}{\ell_0}\right)}$ $\Rightarrow \square$.

* Prove that $\sum_{n=1}^{\infty}|f_n(x)|\not\rightrightarrows g(x)$ NTP $\exists\varepsilon>0$, for all $\ell$ large, $\exists x_{(\ell)}$ such that $|s_\ell(x)-g(x)|>$

We will find $x_{(n)}$ such that $\sin^2\frac{\pi}{x_n}=1$ let $\frac{\pi}{x_{(n)}}=2n\pi \Rightarrow x_{(n)}=\frac{\pi}{2n\pi}=\frac{1}{2n}$

Then $s_\ell(x_{(n)})=$

Rudin 7.6

Prove that the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$

a) converges uniformly on every bounded interval

b) does not converge absolutely for any value of $x$

vcheck.

a) <u>Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly on every bounded interval</u>

We use the property $\left.\begin{matrix} f_n \rightrightarrows f \\ g_n \rightrightarrows g \end{matrix}\right\} \Rightarrow f_n + g_n \rightrightarrows f+g$

and property $\left.\begin{matrix} g_n \to g \\ g_n \text{(does not depend on } x) \end{matrix}\right\} \Rightarrow g_n \rightrightarrows g$

* We consider

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$$

we have on every bounded interval $|x^2| \leq M$

Then $\left.\begin{matrix} \left|(-1)^n \frac{x^2}{n^2}\right| \leq \frac{M}{n^2} \\ \sum_{n=1}^{\infty} \frac{n}{n^2} \text{ converges} \end{matrix}\right\} \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$ converges uniformly

* We consider

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Put $c_n = \frac{(-1)^n}{n}$

then we have $\left\{\begin{matrix} |c_1| \geq |c_2| \geq \cdots \\ c_{2n-1} \geq 0 \quad c_{2n} \leq 0 \\ \lim c_n = 0 \end{matrix}\right.$

Then by alternative series test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, this series does not depend on $x$ $\Rightarrow$ converges uniformly

We have $\left.\begin{matrix} s_n \rightrightarrows s \\ g_n \rightrightarrows g \end{matrix}\right\} \Rightarrow s_n + g_n \rightrightarrows s+g$

$\Rightarrow$ The above series converge uniformly on every bounded interval.

b) <u>Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely for any value of $x$.</u>

We have $\forall x$, $x^2 \geq 0$

$\left.\begin{matrix} \Rightarrow \left|(-1)^n \frac{(x^2+n)}{n^2}\right| \geq \left|\frac{1}{n}\right| \\ \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \end{matrix}\right\} \Rightarrow \sum_{n=1}^{\infty} \left|(-1)^n \frac{x^2+n}{n^2}\right|$ diverges.

$\Rightarrow$ The above series does not converge absolutely, $\forall x$.

---

<u>Another way</u> Use Dirichlet test for uniformly convergence (with $f_n(x) = (-1)^n$ $g_n(x) = \frac{x^2+n}{n^2}$

For ~~fixed~~ a bounded interval $[a,b]$

i) $\sum f_n(x)$ has uniformly bounded partial sums

ii) $g_n(x) \rightrightarrows 0$ on $[a,b]$

iii) $g_n(x) \geq g_{n+1}(x)$ on $[a,b]$

$\Rightarrow \sum f_n(x) g_n(x) \rightrightarrows$

**Exercise 7.7.** Rudin

For $n = 1, 2, 3, \ldots$

$x$ real

$f_n(x) = \frac{x}{1+nx^2}$

Show that $f_n \rightrightarrows$ to $f$

And that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$

fail if $x = 0$

- When $x = 0$, $f_n(x) = 0 \longrightarrow 0$

  when $x \neq 0$, $f_n(x) \longrightarrow 0$

  Then we have $f_n(x) \xrightarrow[\text{pointwise}]{n\to\infty} f(x) = 0, \forall x$

The idea of this ex is to show that $f_n \rightrightarrows f$ does not mean $f_n' \rightrightarrows f'$ | even $f_n' \not\to$

(see theorem 7.17 uniform convergence vs differentiable

* We now want to show that $f_n \rightrightarrows f$

[ We want to show $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x, |f_n(x) - f(x)| < \varepsilon$ (1)

We want to show $M_n = \sup_x |f_n(x) - f(x)|$ $\quad M_n \longrightarrow 0$ (2)

In this proof by using (2)

- Consider $M_n = \sup_x |f_n(x) - f(x)| = \sup \left|\frac{x}{1+nx^2}\right|$

  we have $\left|\frac{x}{1+nx^2}\right| \boxed{\leq} \left|\frac{x}{2\sqrt{n}\,x^2}\right| = \frac{1}{2\sqrt{n}}$

  and we have $\frac{1}{2\sqrt{n}} \longrightarrow 0$

Then by (2), we have $f_n \rightrightarrows f$

Note: In order to have $|f_n(x) - f(x)| \leq \underbrace{M_n}_{\text{only depe on } n}$, we want to use some inequality such that we can "delete" $x$

* Prove that the relation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$

fail if $x = 0$

- $$\begin{cases} f_n'(x) = \dfrac{1+nx^2 - x(1+2nx)}{(1+nx^2)^2} = \dfrac{1+nx^2 - x - 2nx^2}{(1+nx^2)^2} = \dfrac{1-nx^2-x}{(1+nx^2)^2} \xrightarrow[x\neq 0]{n\to\infty} 0 \\ f_n'(0) = 1 \end{cases}$$

- $f'(x) = 0, \forall x$

Then we have $f_n'(x) \xrightarrow[x\neq 0]{n\to\infty} f'(x)$

$f_n'(0) \not\longrightarrow f'(0)$

/166

$I(x)=\begin{cases}0, & x\le 0\\ 1, & x>0\end{cases}$ if $\{x_n\}$ is a sequence of distinct points of $(a,b)$

$\sum |c_n|$ converges.

Prove that the series $f(x)=\sum_{n=1}^{\infty} c_n I(x-x_n)$ $a\le x\le b$ converges uniformly, and that $f$ is continuous for every $x\neq x_n$.

easy
remember the
result

ies result of this exercise is applied in theorem 616 (actually not applied but related to).

$$\left.\begin{array}{l}\{s_n\} \text{ sequence of distinct point.}\\ f \text{ continuous on } [a,b]\\ c_n\geqslant 0,\ \forall n,\quad \sum c_n \text{ converges}\\ \alpha(x)=\sum_{n=1}^{\infty} c_n I(x-s_n)\end{array}\right\}\text{Then } \int f\,d\alpha=\sum_{n=1}^{\infty} c_n f(s_n)$$

Prove that $f(x)=\sum_{n=1}^{\infty} c_n I(x-x_n)$, $a\le x\le b$ converges uniformly

Put $f(x)=\sum_{n=1}^{\infty} f_n(x)$, where $f_n(x)=c_n I(x-x_n)$

then we have $\left.\begin{array}{l}|f_n(x)|\le |c_n|\\ \sum|c_n| \text{ converges}\end{array}\right\}\Rightarrow$ By Wietrass M test, $\sum |f_n(x)|$ converges.

Prove that $f$ is continuous for every $x\neq x_n$

Put $s_k(x)=\sum_{n=1}^{k} f_n(x)$

en we have $\left.\begin{array}{l}s_k(x) \text{ continuous for every } x\neq x_n\\ \text{we have } s_k(x)\rightrightarrows f(x)\end{array}\right\}\Rightarrow f$ is continuous for every $x\neq x_n$ $\square$

7.9/166 Rudin

Let $\{f_n\}$ be a sequence of continuous functions
$f_n \rightrightarrows f$ on E

a) Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence of point $x_n \in E$ s.t $x_n \to x$, $x \in E$.

b) Q Is the conclusion still be holds if convergent pointwise ?

c) Is the inverse of a) true ?

- $f_n(x) \rightrightarrows f$ in E.

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in E,$
$$|f_n(x) - f(x)| < \varepsilon. \quad (1)$$

- $x_n \in E, x_n \longrightarrow x_0,$

$\Leftrightarrow \forall \delta > 0, \exists n_1 \in \mathbb{N}, \forall n \geq n_1, |x_n - x_0| < \delta \quad (2)$

- $\left.\begin{matrix} f_n \rightrightarrows f \\ f_n \text{ continuous} \end{matrix}\right\} \Rightarrow f$ is continuous

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$ (3)

NTP
$\lim_{n\to\infty} f_n(x_n) = f(x_0)$

$\Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N,$
$|f_n(x_n) - f(x_0)| < \varepsilon$

We have chose $N = \max\{n_0, n_1\}$, then we have because of (1), $|f_n(x_n) - f(x)| < \varepsilon$
because of (2) + (3) $\to \forall \varepsilon > 0, \exists \delta > 0, \forall n \geq N \geq n_1,$
$|x_n - x_0| < \delta$ then $|f(x_n) - f(x_0)| <$

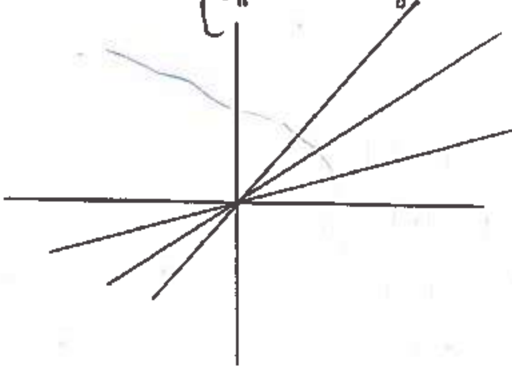
$\Rightarrow |f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < 2\varepsilon.$

b) Is the conclusion still be hold if $f_n \longrightarrow f$ pointwise ?

(Idea: If $f_n \longrightarrow f$ pointwise, $\forall \varepsilon$, $\exists n_{\varepsilon, x} \in \mathbb{N}$ $\Rightarrow$ can't find max }.
$\forall x \in E$ )

It is not true in general case, But we can find some examples in special cases :

* Example when $\begin{cases} f_n \longrightarrow f \text{ pointwise} \\ f_n(x_n) \longrightarrow f(x) \text{ when } x_n \to x \end{cases}$ But $f_n \not\rightrightarrows f$



- We have $f_n(x) = \frac{x}{n} \xrightarrow{\text{pointwise}} f(x) = 0, \forall x \in \mathbb{R}.$
- $f_n(x) \not\rightrightarrows f(x)$ on $\mathbb{R}$.
- Let $x_n = \frac{1}{n}, \forall n$ then $x_n \longrightarrow x = 0$
and $f_n(x_n) = \frac{1}{n^2} \longrightarrow 0$ when $x_n \to 0$.

Another example when $\begin{cases} f_n(x_n) \not\rightrightarrows f(x) \\ f_n(x) \longrightarrow f(x) \\ f_n(x_n) \longrightarrow f(x) \text{ when } x_n \to x. \end{cases}$

Let $f_n(x) = x^n$ on $[0,1]$

$$f(x) = \begin{cases} 0 & , x \in [0,1) \\ 1 & , x \leq 1 \end{cases}$$

?

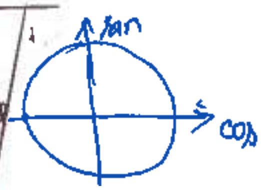
Then $f_n(x) \longrightarrow f(x)$ on $[0,1]$

$f_n(x) \not\rightrightarrows f(x)$ on $[0,1]$

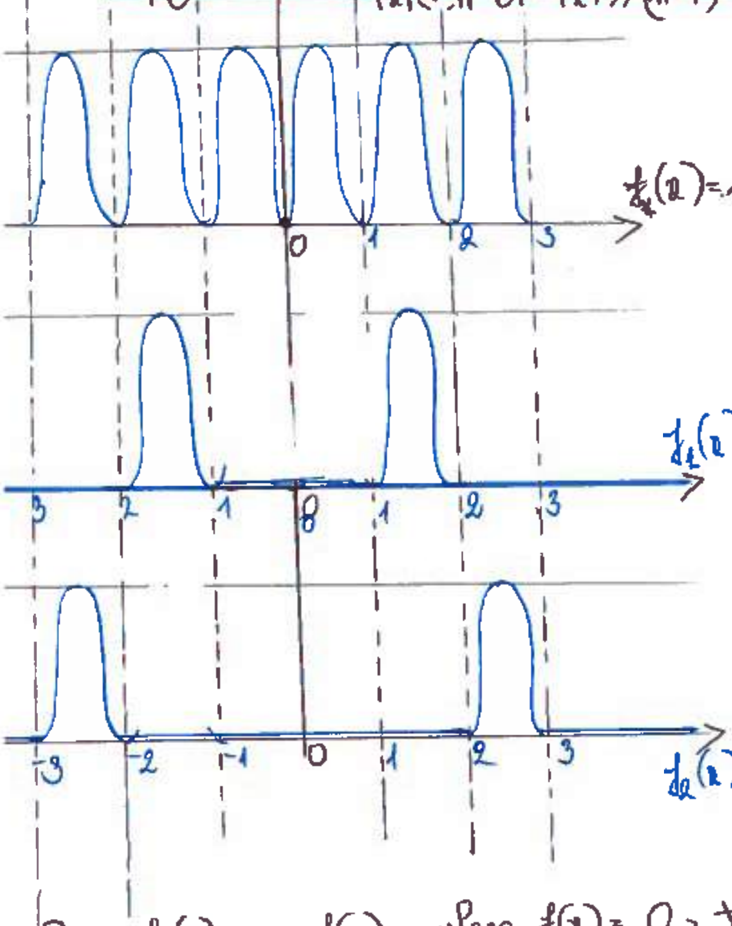
Then let
$x_n = x^n$ on $[0,1]$
Then

?

**Example c7 The inverse is not true : Example when** $\begin{cases} f_n(x) \longrightarrow f(x) \\ f_n(x_n) \longrightarrow f(x) \end{cases}, x_n \to x$
But $f_n(x) \not\rightrightarrows f(x)$.



$$f_n(x) = \begin{cases} \sin^2 \pi x & n \leq |x| \leq n+1 \\ 0 & |x| \leq n \text{ or } |x| \geq (n+1) \end{cases}.$$



$f_0(x) = \sin^2(\pi x)$.

$f_1(x) = \begin{cases} \sin^2 \pi x & , 1 \leq |x| \leq 2 \\ 0 & |x| \leq 1 \text{ or } |x| \geq 2 \end{cases}$.

$f_2(x) = \begin{cases} \sin^2 \pi x & 2 \leq |x| \leq 3 \\ 0 \end{cases}$

- Prove $f_n(x) \longrightarrow f(x)$, where $f(x) = 0$, $\forall x \in \mathbb{R}$.

NTP $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists n_{\varepsilon,x}$, $\forall n \geq n_{\varepsilon,x}$, $|f_n(x)| < \varepsilon$

We have $\forall x \in \mathbb{R}$, $\exists n_{\varepsilon,x} \in \mathbb{N}$ such that ~~$n \leq |x| \leq n+1$~~ $|x| \leq n_{\varepsilon,x}$.

then ~~Choose $n_{\varepsilon,x} =$~~ for every $n \geq n_{\varepsilon,x}$; because $|x| \leq n_{\varepsilon,x} \leq n$, $f_n(x) = 0 < \varepsilon$.

And we have $f_n(x_n) \longrightarrow 0$ when $x_n \to 0$ But $f_n(x) \not\rightrightarrows f(x)$.

7.11 Rudin . Dirichlet test for uniformly convergence

Suppose $\{f_n\}, \{g_n\}$ are defined on E

a) $\sum_{n=1} f_n$ has ~~bam~~ uniformly bounded partial sums.

b) ~~#~~ $g_n \rightrightarrows 0$ on E

c) $g_1(x) \geq g_2(x) \geq \dots \geq g_n(x) \geq \dots \; \forall x \in E$

$\Rightarrow \sum_{n=1}^{\infty} f_n g_n$ converges uniform on E

(For problems related to $\sum a_n b_n$ we need to use a trick related to partial sum as bel

- Put $F_k(x) = \sum_{n=1}^{k} f_n(x)$, we have $F_k(x)$ is uniformly bounded

$\Leftrightarrow \exists M, \; |F_k(x)| \leq M, \; \forall x \in E, \forall k \geq 1$ (a)

- We have $g_n \rightrightarrows 0$ on E

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, \; |g_m(x) - g_n(x)| < \varepsilon$ (b)

$|g_n(x)| < \varepsilon$

- $g_n(x)$ decreasing (c).

NTP $\sum_{n=1}^{\infty} f_n g_n$ converges unifor

Put $S_k(x) = \sum_{n=1}^{k} f_n g_n$, we

NTP $S_k(x) \rightrightarrows$ on E

NTP $\{S_k(x)\}$ uniformly Cauchy

$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k, \ell \geq k_0$

$|S_k(x) - S_\ell(x)| < \varepsilon$ .

* We have (don't need to compute this way, see next page)

$$S_k(x) = \sum_{n=1}^{k} f_n(x) g_n(x) = f_1(x) g_1(x) + \sum_{n=2}^{k} f_n(x) g_n(x)$$

$$= f_1(x) g_1(x) + \sum_{n=2}^{k} (F_n(x) - F_{n-1}(x)) \; g_n(x).$$

$$= f_1(x) g_1(x) + \sum_{n=2}^{k} F_n(x) g_n(x) - \sum_{n=2}^{k} F_{n-1}(x) g_n(x).$$

$$= \sum_{n=1}^{\ell} F_n(x) \; g_n(x) - \sum_{n=2}^{k} F_{n-1}(x) g_n(x)$$

Then for $\boxed{k > \ell}$ large enough

$$S_k(x) - S_\ell(x) = \sum_{n=\ell+1}^{k} F_n(x) g_n(x) - \sum_{n=\ell+1}^{k} F_{n-1}(x) \; g_n(x).$$

$$= \sum_{n=\ell+1}^{k} F_n(x) \; g_n(x) - \sum_{n=\ell}^{k-1} F_n(x) \, g_{n+1}(x)$$

Compute $S_k(x) - S_\ell(x)$ directly next page

$$= F_k(x) g_k(x) - F_\ell(x) \, g_{\ell+1}(x) + \sum_{n=\ell+1}^{k-1} F_n(x) \underbrace{\left[ g_n(x) - g_{n+1}(x) \right.}_{\geq 0}$$

The key idea is because we have $|F_\ell(x)|$ ... $|g_m(x) - g_n(x)$ ... $\forall m, n$ so we want to change $S_k(x)$- ... to sth related to these things

So

$$|S_k(x) - S_\ell(x)| < M \left[ g_k(x) + g_{\ell+1}(x) + \sum_{n=\ell+1}^{k-1} [g_n(x) - g_{n+1}(x)] \right].$$

and $g_n(x) - g_n(x) <$ ... $\forall n \geq n$ ... because of b

We choose $k_0 = n_0$, then $\forall k, \ell \geq k_0$

$$|S_k(x) - S_\ell(x)| \leq M \, [\underbrace{|g_k(x)| + |g_{\ell+1}(x)|}_{<\varepsilon \text{ by b}} + \underbrace{|g_{\ell+1}(x) + g_k(x)|}_{<\varepsilon \text{ by b}} \leq 3M \, \varepsilon \Rightarrow \square.$$

done :)

~~We have~~ * (compute $S_k(x) - S_l(x)$ directly): for $k > l$ big enough.

e have

$$S_k(x) - S_l(x) = \sum_{n=1}^{k} f_n(x)\, g_n(x) - \sum_{n=1}^{l} f_n(x)\, g_n(x)$$

$$= \sum_{n=l+1}^{k} f_n g_n = \sum_{n=l+1}^{k} \left(F_n - F_{n-1}\right) g_n$$

$$= \sum_{n=l+1}^{k} F_n g_n - \sum_{n=l+1}^{k} F_{n-1} g_n$$

$$= \sum_{n=l+1}^{k} F_n g_n - \sum_{n=l}^{k-1} F_n g_{n+1}(x).$$

$$= F_k g_k - F_l g_{l+1}(x) + \sum_{n=l+1}^{k-1} F_n(x) \left[ g_n(x) - g_{n+1}(x) \right] \dots$$

7.20/169 Important . Aug2003 Q5

If $f$ continuous on $[0,1]$

$$\int_0^1 f(x)\, x^n\, dx = 0 \quad (n = 0, 1, 2, ..)$$

10/10

Prove that $f(x) = 0$ on $[0,1]$ . Tian Le

Hint: The integral of the product of $f$ with any polynomial is zero .

Use the W theorem to show $\int_0^1 f^2(x)\,dx = 0$ .

- Because $f$ continuous on $[0,1]$

  Then by theorem 7.26 , $\exists$ sequence of polynomial $(P_n)$ , $P_n \rightrightarrows f$ Note: yellow part is important in this proof .

- Besides because $f$ is continuous on $[0,1]$, compact , we have $f$ bounded on $[0,1]$

$$\left.\begin{array}{l} \text{then } P_n \rightrightarrows f \\ f \text{ is bounded} \end{array}\right\} \Rightarrow \{P_n\} \text{ is uniformly bounded}$$

(I explain this at the end of this proof )

- Then we have

$$\left.\begin{array}{l} \text{because } \{P_n\} \text{ uniformly bounded} \\ P_n \rightrightarrows f \end{array}\right\} \text{ by applying the result of EX 7.2 ,} \quad fP_n \rightrightarrows f^2 \text{ on } [0,1]$$

* Now by using theorem 7.16 (uniform convergence and integration) we have .

$$\int_0^1 f^2\,dx = \lim_{n\to\infty} \int_0^1 f(x)\,P_n(x)\,dx$$

Assume $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ , we have

$$\int_0^1 f(x)\,P_n(x)\,dx = \sum_{i=0}^{n} a_i \underbrace{\int_0^1 f(x)\, x^i\,dx}_{=0 \text{ (by assumption)}} = 0$$

$$\left.\begin{array}{l} \text{Then we have } \int_0^1 f^2(x)\,dx = 0 \\ \text{We also have } f^2(x) \geq 0 ,\ \forall x \end{array}\right\} \Rightarrow \text{Apply the result of ex 6.2}$$

$\Rightarrow f^2(x) = 0 ,\ \forall x \in [0,1]$ .

$\Rightarrow f(x) = 0 ,\ \forall x \in [0,1]$

10/10

7.23/169

Put $\begin{cases} P_0 = 0 \\ P_{n+1}(x) = P_n(x) + \dfrac{x^2 - P_n^2(x)}{2} \end{cases}$ , $n = 0, 1, 2, \dots$

Prove that $P_n(x) \rightrightarrows |x|$ on $[-1, 1]$.

We want to prove $P_n(x) \rightrightarrows |x|$ on $[-1, 1]$

$\Leftrightarrow$ We want to prove $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in [-1, 1], |P_n(x) - |x|| < \varepsilon$

* Consider $|x| - P_{n+1}(x)$, we have

$$|x| - P_{n+1}(x) = |x| - P_n(x) - \frac{x^2 - P_n^2(x)}{2} =$$

$$= [|x| - P_n(x)] - \frac{(|x| + P_n(x))(|x| - P_n(x))}{2}$$

$$= [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2}\right] \qquad (1)$$

* We have $P_0 = 0 \leq |x|$ , for $x \in [-1, 1]$

- Assume $P_n(x) \leq |x|$ , for $x \in [-1, 1]$

then consider (1) : $|x| - P_{n+1}(x) = \underbrace{[|x| - P_n(x)]}_{\geq 0} \underbrace{\left[1 - \frac{|x| + P_n(x)}{2}\right]}$

(by induction hypothesis)

$\left( 0 \leq 1 - \frac{|x| + P_n(x)}{2} \leq 1 \right.$ (when $|x$

because $\left. \frac{|x| + P_n(x)}{2} \leq \frac{|x| + |x|}{2} = |x| \right)$

This means by induction, we have that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x| \leq 1$

And so, we have

$$|x| - P_n(x) = [|x| - P_{n-1}(x)] \left[1 - \frac{|x| + P_{n-1}(x)}{2}\right] \leq$$

$$\leq [|x| - P_{n-1}(x)] \left[1 - \frac{|x| + P_0}{2}\right]$$

$$= [|x| - P_{n-1}(x)] \left[1 - \frac{|x|}{2}\right]$$

$$\leq [|x| - P_{n-2}(x)] \left[1 - \frac{|x|}{2}\right]^2$$

$$\vdots$$

$$\leq [|x| - P_0] \left[1 - \frac{|x|}{2}\right]^n$$

$$= |x| \left[1 - \frac{|x|}{2}\right]^n$$

k Now put $g(x) := x\left(1-\frac{x}{2}\right)^n \quad x \in [0,1]$

$g'(x) = \left(1-\frac{x}{2}\right)^n + xn\left(1-\frac{x}{2}\right)^{n-1}\left(-\frac{1}{2}\right)$

$g'(x) = 0$ at $x_0 = \frac{2}{n+1}$ ✓

$g''(x) = n\left(-\frac{1}{2}\right)\left(1-\frac{x}{2}\right)^{n-1} + \left(-\frac{1}{2}\right)n\left(1-\frac{x}{2}\right)^{n-1} + \left(-\frac{1}{2}\right)nx(n-1)\left(1-\frac{x}{2}\right)^{n-2}\left(-\frac{1}{2}\right)$

$= \left(1-\frac{x}{2}\right)^{n-2}\left[\frac{n^2x}{4} - \frac{nx}{4} - n\right] < 0$ for n large enough ✓
and $x \in [0,1]$.

enwe have $x\left(1-\frac{x}{2}\right)^n \leq \frac{2}{(n+1)}\left(1-\frac{\frac{2}{n+1}}{2}\right)^n = \frac{2}{n+1}\left(1-\frac{1}{n+1}\right)^n \xrightarrow{n\to\infty} \frac{2}{n+1}$

✓

Coming back to our problem:

we have $|x|\left(1-\frac{|x|}{2}\right)^n < \frac{2}{n+1} \xrightarrow{n\to\infty} 0$ ✓

Then we have $|x| - P_n(x) \xrightarrow[\forall x\in[-1,1]]{n\to\infty} 0$ which means $P_n(x) \rightrightarrows |x|$.

10/10

HW 602

Tran Le

* Given an example of a polynomial $p(x,y)$ such that

$$\begin{cases} p(x,y) > 0 \text{ everywhere} \\ \inf_x \inf_y p(x,y) = 0 \end{cases}$$

* We want $p(x,y) > 0$ everywhere so we should have

$$p(x,y) = [\alpha(x,y)]^2 + [\beta(x,y)]^2$$ where $\alpha(x,y)$ and $\beta(x,y)$ are not both equal 0 with the same value $x, y$

- Let consider $p(x,y) = (xy-1)^2 + x^2 \not\geq 0$

( We have $p(x,y) \neq 0$ because assume $p(x,y) = 0$, then $\begin{cases} xy - 1 = 0 \Rightarrow -1 = 0 \\ x = 0 \end{cases}$ (could not happe

- $\inf \inf p(x,y) = \lim\limits_{\substack{x = \frac{1}{n} \to 0 \\ y = n \to \infty}} p(x,y) = 0$

10/10

* Another example that we can think about is the probability density function of bivariate normal distribution (let consider "standard" bivariate normal distribution) :

$$p(x,y) = \phi(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad , x,y \in \mathbb{R}^2$$

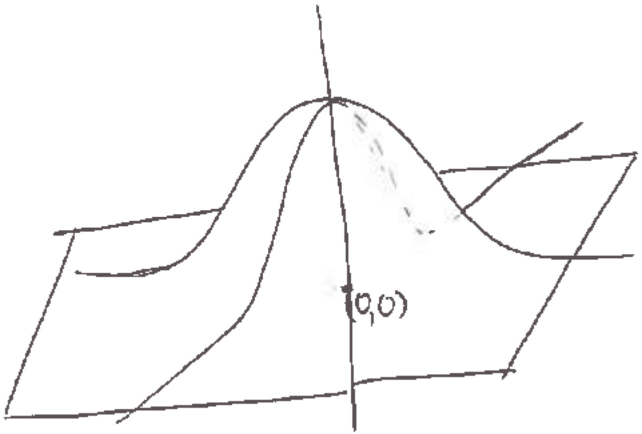
But we want a polynomial!

- We have $p(x,y) > 0$ since

Put $z = x^2 + y^2$ ; $p(x,y) = \frac{1}{2\pi} \underbrace{e^{-\frac{1}{2}z}}_{\text{exponential function} > 0}$

- and $\inf_z e^{-\frac{1}{2}z} = 0$ when $(z \to \infty)$

$(e^w \to 0$ as $w \to -\infty$, Theorem 8.6 e$)$



awesome picture! :)

$f_n(x) - g_x(x) =$

$=$ $\forall \varepsilon > 0, \nexists n, \quad |f_n(x) - g_n(x)| < \varepsilon$

$|f_n(x) - g_n(x)| < \varepsilon$

TranLe
HW
MAT632

* Question: (Relating to Picard's existence and Uniqueness theorem).
Let $\phi : [a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ be continuous

$$|\phi(t,x)-\phi(t,y)| \leq L|x-y| \quad (\phi \text{ is Lipchitz with } 2^{nd} \text{ variable})$$

Consider IVP:
$$\begin{cases} f'(t) = \phi(t, f(t)) & \text{for } a \leq t < b \\ f(t_0) = t_0 \end{cases} \quad (*)$$

Prove that this IVP has unique solution near to.

20/20

* Notice that under the assumptions of existence, we have the IVP(*) is equivalent to

$$f(t) = f(t_0) + \int_{t_0}^{t} \phi(s, f(s))\,ds.$$

* So now we define an operator $T : f(\ ) \longmapsto T(f)(\ )$ with

$$T(f)(t) = f(t_0) + \int_{t_0}^{t} \phi(s, f(s))\,ds \qquad (**)$$

Then if we can prove that (**) has a unique solution near to (which means T(f) has a fixed point), then it means we can prove that (*) has a unique solution near to

* We will prove (**) has a unique solution by proving that T is a contraction (from a complete space to itself):

- We have

$$T(f_1)(t) - T(f_2)(t) = \int_{t_0}^{t} \underbrace{\phi(s, f_1(s)) - \phi(s, f_2(s))}_{\leq L|f_1(s)-f_2(s)|}\, ds.$$

(because of assumption that $\phi$ is Lipchitz with $2^{nd}$ variable)

$$\leq \int_{t_0}^{t} L\,|f_1(s) - f_2(s)|\,ds.$$

$$\leq \int_{t_0}^{t} L\,\|f_1 - f_2\|\,ds.$$

$$|T(f_1)(t) - T(f_2)(t)| \leq L\,\|f_1 - f_2\|\,|t - t_0|.$$

Then when we chose t near to $t_0$, we have

$$|T(f_1)(t) - T(f_2)(t)| \leq \underbrace{k}_{<1}\|f_1 - f_2\| \Rightarrow T \text{ is a contraction} \Rightarrow \text{done.}\ \square$$

*Question:

$X$: any set

$\varphi : X \longrightarrow X$

There is $k$ such that the $k^{th}$ iteration $\underbrace{\varphi \circ \varphi \circ \varphi \cdots \circ \varphi}_{k \text{ times}} : X \longrightarrow X$ has exactly one fixed point

<u>Prove that</u>: $\varphi$ has exactly one fixed point.

We have $\varphi^k : X \longrightarrow X$ has exactly one fixed point $\Longleftrightarrow \exists! x \in X, \varphi^k(x) = x$.

$\Rightarrow \quad \varphi(\varphi^k(x)) = \varphi(x)$.

$\Rightarrow \quad \varphi^k[\varphi(x)] = \varphi(x)$. ~~for unique~~

$\Rightarrow \varphi(x)$ is also a fixed point of $\varphi^k$.

by the uniqueness of fixed point

$\Bigg\} \Rightarrow \varphi(x) = x$

$\Rightarrow$ ~~$\varphi$ is~~ $x$ is a fixed point of $\varphi$

~~bec~~ from above $x$ is unique

$\Bigg\} \Rightarrow$ done $\square$

Rudin 9.7/239

Suppose $E \subseteq \mathbb{R}^n$, $f$ defined on $E$, $f: E \longrightarrow \mathbb{R}$.
The partial derivative $D_1 f, \dots, D_n f$ are bounded in $E$.
Prove that $f$ is continuous in $E$

20/20

Let $x = (x_1, \dots, x_n)$ be an arbitrary point in $E$.

We choose $\delta_0 > 0$ be sufficiently small such that $N_{\delta_0}(x) \subseteq E$.

Let $M$ be an upper bound of partial derivative,

$$\text{choose } \delta = \min\left(\delta_0, \frac{\varepsilon}{(n+1)M}\right)$$

* We want to prove that $|f(y) - f(x)| < \varepsilon$, $\forall y \in N_\delta(x)$.
Using triangle inequality, we have: $y = (y_1, \dots, y_n)$

$$|f(y) - f(x)| = |f(y_1, \dots, y_n) - f(x_1, \dots, x_n)|$$
$$\leq \underbrace{|f(y_1, \dots, y_n) - f(x_1, y_2, \dots, y_n)|}_{(*)}$$
$$+ |f(x_1, y_2, \dots, y_n) - f(x_1, x_2, \dots, y_n)| + \cdots$$
$$\cdots + |f(x_1, x_2, \dots, x_{n-1}, y_n) - f(x_1, x_2, \dots, x_n)|.$$

(Note that each term in each $|\dots|$ differs in only one coordinate)

⇒ Applying the mean value theorem to that single coordinate:

we have $|\text{each term}| \leq |D_i f(\xi)|\, \delta \leq M\, \delta$

$$\Rightarrow |f(y) - f(x)| \leq n \cdot M\delta \leq n \cdot M \frac{\varepsilon}{(n+1)M} \leq \varepsilon \quad \Rightarrow \square.$$

Rudin 9.8/239:

Suppose $f$ is differentiable real function on an open set $E \subseteq \mathbb{R}^n$ | $f: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$.

$f$ has a local maximum at a point $x \in E$.

Prove $f'(x) = 0$

Let $y \in \mathbb{R}^n$

- Define $g: \mathbb{R} \longrightarrow \mathbb{R}^n \Rightarrow g$ is a differentiable function at $t$

$$t \longmapsto x + ty$$

Then consider $V = N_\delta(t)$, then consider $F: V_\delta \longrightarrow \mathbb{R}$

$$t \longmapsto F(t) = f(g(t)) = f(x+ty)$$

Then by theorem 9.15 (Chain rule) $F$ is differentiable at $t$ and

$$F'(t) = f'(g(t))\, g'(t) = y \cdot f'(x+ty)$$

Then we have $F$ has a maximum at $t = 0$

$$\Rightarrow F'(0) = 0 \quad \Rightarrow f'(x) \cdot y = 0, \ \forall y \in \mathbb{R}^n$$

$$\Rightarrow f'(x) = 0 \quad ▣$$

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Rudin 9.24/242 $f : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2$

For $(x,y) \neq (0,0)$

Define $f = (f_1, f_2)$ by $\begin{cases} f_1(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2} \\ f_2(x,y) = \dfrac{xy}{x^2 + y^2} \end{cases}$

a. compute the rank of $f'(x,y)$

b. Find the range of $f$

a) Compute the rank of $f'(x,y)$

$$f'(x,y) = \begin{bmatrix} f^1_x & f^1_y \\ f^2_y & f^2_y \end{bmatrix} = \begin{bmatrix} \dfrac{2x(x^2+y^2) - (x^2-y^2)2x}{(x^2+y^2)^2} & \dfrac{-2y(x^2+y^2) - (x^2-y^2)(2y)}{(x^2+y^2)^2} \\ \dfrac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} & \dfrac{x(x^2+y^2) - (xy)2y}{(x^2+y^2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4xy^2}{(x^2+y^2)^2} & \dfrac{-4x^2y}{(x^2+y^2)^2} \\ \dfrac{y^3 - x^2y}{(x^2+y^2)^2} & \dfrac{x^3 - xy^2}{(x^2+y^2)^2} \end{bmatrix}$$

$\det f'(x,y) = 0$, $\forall x,y$ then rank is 0 or 1 at every point. ?

* Prove that $f'(c)$ exists (at c) (see sample A, question 4).

We prove that ~~$\lim_{\xi} f'(\xi_x)$~~ $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f'(\xi_x)$

* Prove that $\lim_{x\to c^+} f(x)$ exist ⇒ we prove that for $\{x_n\} \to c$ then $\{f(x_n)\}$ converges

(see Homework 5.2).

- by proving that $\{f(x_n)\}$ Cauchy
- in case $\{f(x_n)\}$ monotone, we can prove that $f(x_n)$ bounded ⇒ converges.

in case f is monotone + bounded then one side limit exists.

* Prove that

\+ If f is a increasing function on $(a,b)$ $\lim_{x\to b^-} f(x)$ does <u>not</u> exist $\Big\}$ ⇒ $\sup_{x\in(a,b)} f(x) = +\infty$ (see Homework 5.2 question 2).

* We want to prove that $\lim_{x\to c} f(x) = 0$
we have $|f(x)| < |g(x)|$ $\Big\}$ then we can prove this by proving $-|g(x)| < f(x) < |g(x)|$ (see ex 5.13)

and then prove $-|g(x)|$ and $|g(x)| \to 0$

# GRADUATE PRELIMINARY EXAMINATION

## ANALYSIS

### Sample A

1. (a) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of real numbers. Show by example that the series $\sum_{n=1}^{\infty} a_n b_n$ need not converge.

(b) If, in addition, $a_n \geq 0$ and $b_n \geq 0$ for all $n$, prove that $\sum_{n=1}^{\infty} a_n b_n$ does converge.

2. Prove $x^{-1}\arctan(x)$ is decreasing on $[1, \infty)$.

3. (a) Prove that if $f$ is a continuous, strictly positive function on $[0,1]$, then $\int_0^1 f(x)dx > 0$. You may assume only the definition of the integral.

(b) Prove the same thing if $f$ is only assumed to be Riemann integrable and strictly positive on $[0,1]$. Here you may assume basic facts from analysis, other than what you are asked to prove. For instance, you may assume $\int_0^1 f(x)dx \geq 0$ for nonnegative Riemann integrable functions $f$.

4. (a) Suppose $f$ is a continuous, real valued function defined on $(a,b)$. Let $c \in (a,b)$, and suppose that $f$ is differentiable on $E = (a,c) \cup (c,b)$ and $f'(x) \to \lambda$ as $x \to c$ in $E$. Prove that $f'(c)$ exists and equals $\lambda$.

(b) Let $g$ be a continuous, real valued function on $[a,b]$. Suppose that $g$ is differentiable on $(a,b)$, and that $g'(x) \neq 0$ for all $x \in (a,b)$. Prove that $g^{-1}(y)$ is a finite set for all $y$ in the range of $g$.

5. Is it possible to solve

$$\begin{aligned} xy^2 + xzu + yv^2 &= 3 \\ u^3yz + 2xv - u^2v^2 &= 2 \end{aligned}$$

for $u(x,y,z)$ and $v(x,y,z)$ near $(x,y,z) = (1,1,1)$ such that $(u(1,1,1), v(1,1,1)) = (1,1)$? Why? If it is possible, compute $\frac{\partial v}{\partial y}$ at $(1,1,1)$.

6. Suppose $f$ is differentiable on the interval $[a,b]$, $f(a) = 0$, and there is a finite constant $A$ such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$. Hint: For $a \leq c \leq b$ let $M_c = \sup\{|f(x)| : a \leq x \leq c\}$, and show that $|f(x)| \leq AM_c(x-a)$ for all $x \in [a,c]$.

# GRADUATE PRELIMINARY EXAMINATION

## ANALYSIS

### Sample B

1. Let $\{a_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=0}^{\infty} a_n = 1$. The power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [-1, 1]$. If $L$ denotes the left-hand derivative of $f$ at $x = 1$, $L = \lim_{x \to 1-} \frac{f(1) - f(x)}{1 - x}$, show that

$$L = \sum_{n=1}^{\infty} n a_n ,$$

including the case $+\infty = +\infty$.

2. Let $\{f_n\}$ be a sequence of continuously differentiable functions on $\mathbf{R}$ such that $f_n(0) = 0$ for all $n$, $f'_n \cdot f'_m \equiv 0$ for all $m \neq n$, and $f'_n \to 0$ uniformly on $\mathbf{R}$ as $n \to \infty$.

(a) Prove that $\sum_1^{\infty} f'_n$ converges uniformly and absolutely on $\mathbf{R}$. Let $g = \sum_1^{\infty} f'_n$.

(b) Prove that $\sum_1^{\infty} f_n$ converges pointwise on $\mathbf{R}$. Let $f = \sum_1^{\infty} f_n$.

(c) Show that $f$ is differentiable on $\mathbf{R}$, and that $f'(x) = g(x)$ for all $x \in \mathbf{R}$.

3. Let $g, f_n, n = 1, 2, \ldots$ be real valued functions defined on $[0, \infty)$ such that: (i) each $f_n$ is Riemann integrable on every interval $[0, T]$, $T < \infty$; (ii) $|f_n(x)| \leq g(x)$ for all $n$ and $x$; (iii) $\int_0^{\infty} g(x)dx < \infty$, and (iv) there is a function $f$ such that $f_n \to f$ uniformly on every interval $[0, T]$ as $n \to \infty$. Prove, without using results from Lebesgue integration theory, that the improper Riemann integrals $\int_0^{\infty} f_n(x)dx$ and $\int_0^{\infty} f(x)dx$ exist, and

$$\lim_{n \to \infty} \int_0^{\infty} f_n(x)dx = \int_0^{\infty} f(x)dx .$$

4. Determine the convergence (absolute or conditional) or divergence of the following series:

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} n^2[\pi^{1/n} - 1]^n$

(c) $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}$

(d) $\sum_{n=1}^{\infty} n! e^{-n}$

Sample A

1a) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of real numbers.

Show by example that $\sum a_n b_n$ need not converge

b) Prove that if $\sum a_n$ converges, $a_n \geq 0, \forall n$
$\sum b_n$ converges, $b_n \geq 0, \forall n$ $\Big\} \Rightarrow \sum_{n=1}^{\infty} a_n b_n$ does converge

a) Example that satisfies :

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{(n+1)}} \qquad \sum_{n=1}^{\infty} b_n = \sum \frac{(-1)^n}{\sqrt{n+1}}$$

then $\sum a_n$ and $\sum b_n$ are convergent because

$$\begin{cases} a_{2k} > 0, a_{2k+1} < 0, \forall k = 0, \dots \\ |a_1| \geq |a_2| \geq \dots \geq |a_n| \geq \dots \\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0 \end{cases}$$

$\sum a_n b_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)}$ does not converge

b) Prove that if $\sum a_n$ converges, $a_n \geq 0, \forall n$
$\sum b_n$ converges, $b_n \geq 0, \forall n$ $\Big\} \Rightarrow \sum a_n b_n$ converges.

$\cancel{\sum_{n=1}^{\infty} a_n b_n =}$ | Cause $\sum b_n$ converges $\Rightarrow b_n \xrightarrow{n\to\infty} 0$

$\Rightarrow \exists n_0 \in \mathbb{N}, \forall (n \geq n_0), |b_n| < 1$ (1)

$0 < b_n < 1$

Then we have

$$\sum_{n=1}^{\infty} a_n b_n = \underbrace{\sum_{n=1}^{n_0} a_n b_n} + \sum_{n=n_0}^{\infty} a_n b_n$$

we have $\sum_{n=1}^{\infty} a_n b_n$ converges if $\sum_{n=n_0}^{\infty} a_n b_n$ converges.

we have because (1) $\Rightarrow$ $\cancel{\sum_{n \geq n_0}^{\infty} a_n} b_n$ $a_n b_n \leq a_n$ , $\forall n \geq n_0$ $\Big\} \Rightarrow$

$\sum a_n$ converges.

we need the condition $a_n \geq 0$, $b_n \geq 0$ here.

$\Rightarrow \sum_{n_0}^{\infty} a_n b_n$ converges.

checked

2) If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent.

we have $\sin x \leq x$ if $x$ is positive

$\Rightarrow \sum \sin(a_n)$ is also convergent

* If $\sum a_n$ diverges, $\sum b_n$ diverges $\Rightarrow$ Give an example that $\sum(a_n+b_n)$ converges.

$\sum a_n = \sum n$ diverges.

$\sum b_n = \sum(-n)$ diverges

$\sum a_n+b_n = \sum 0$ converges

See theorem 6.12

3) a) Prove that if $f$ is continuous, strictly positive function on $[0,1]$, then $\int_0^1 f(x)\,dx > 0$

You may assume only the definition of the integral

b) Prove the same thing if $f$ is only assumed to Riemann integrable ← strictly positive in $[0,1]$.

Here you may assume basic facts from analysis, other than what you are asked to prove. For instance, you may assume $\int_0 f(x)\,dx \geq 0$ for nonnegative Riemann integable function $f$.

Use the result from this ex to prove.

* Now we prove that if $f_1, f_2 \in R(\alpha)$, $f_1 \leq f_2$ on $[a,b]$ } then $\int_a^b f_1\,d\alpha \leq \int_a^b f_2\,d\alpha$. (theorem 6.12 b)

We use the fact that:

- if $f_1, f_2 \in R(\alpha)$ then $f_1 + f_2 \in R(\alpha)$ and $\int(f_2+f_1)\,d\alpha = \int f_2\,d\alpha + \int f_1\,d\alpha$

$$\Rightarrow \int(f_2-f_1)\,d\alpha = \int f_2\,d\alpha - \int f_1\,d\alpha \quad (1)$$

- and $\int f\,d\alpha \geq 0$ if $f \geq 0$ (2)

$$(1)+(2) \Rightarrow \int f_2\,d\alpha - \int f_1\,d\alpha = \int(f_2-f_1)\,d\alpha \geq 0$$

a)* First, we prove that if $\left.\begin{cases} f \text{ is continuous on } [a,b] \\ \alpha \text{ monotonically increasing} \end{cases}\right\} \Rightarrow f \in \mathscr{R}(\alpha) \text{ on } [a,b]$.

- $f$ is continuous on $[a,b] \Leftrightarrow \forall \eta > 0, \exists \delta > 0, \forall x, y \in [a,b], |x-y| < \delta$ then $|f(x)-f(y)| < \eta$ (1)
  $\Leftrightarrow f$ is uniformly continuous
- We choose $\eta$ such that $\boxed{[\alpha(b)-\alpha(a)]\eta < \varepsilon}$
  we have (1) also holds in this case
- Choose a partition $P$ such that $\Delta x_i < \delta$
  Then we have
  $U(P,f,\alpha) - L(P,f,\alpha) = \sum \underbrace{(M_i - m_i)}_{< \eta} \Delta\alpha_i \leq \eta \sum \Delta\alpha_i \leq [\alpha(b)-\alpha(a)]\eta <$
  $\Rightarrow f \in \mathscr{R}(\alpha)$ on $[a,b]$.

* Then we have

$$\int_a^b f d\alpha \geq L(P,f,\alpha) = \sum_{i=1}^n \underbrace{m_i}_{>0} \underbrace{\Delta x_i}_{>0} > 0.$$

because $f$ is strictly positive.

b) In case $f \in \mathscr{R}(\alpha)$ on $[a,b] \Rightarrow$ similarly.

4/ a) Suppose $f$ is continuous, real value function defined on $[a,b]$

Let $c \in (a,b)$, & suppose that $f$ is differentiable on $E = (a,c) \cup (c,b)$

$$f'(x) \xrightarrow{x\to c} \lambda \text{ in } E \quad (1)$$

Prove that $f'(c)$ exists and equals $\lambda$

b) ~~Let~~ Let $g$ be a continuous, real valued function on $[a,b]$

Suppose that $g$ is differentiable on $(a,b)$ $\Rightarrow$ $g$ cont $+ g'(x) \neq 0 \Longrightarrow$ monotonic

$g'(x) \neq 0$ for all $x \in (a,b)$ Aug 2008, #2, $\left.\begin{array}{l} f \text{ is monotone} \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$ strictly monotonic

Prove that $g^{-1}(y)$ is a finite set for all $y$ in the range of $g$

a) We want to prove that $\exists \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ and this lim equals $\lambda$.

We have $\dfrac{f(x)-f(c)}{x-c} = f'(\xi_x)$, where $\xi_x$ between $x$ and $c$

- when $x > c$, $f'(\xi_x)$ in $(c,b)$, by (1) $f'(\xi_x) \xrightarrow{x\to c} \lambda$
- when $x < c$, $f'(\xi_x)$ in $(a,c)$, by (1) $f'(\xi_x) \xrightarrow{x\to c} \lambda$

$\Rightarrow \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c} f'(\xi_x) = \lambda$ ☐

b) Warm up: We have if $g$ has local min/max in $[a,b]$ and $\exists\, g'(x)$ $\Big\} \Rightarrow$ then $g'(x) = 0$

local min/max

in here $g'(x) \neq 0$, $\forall x \in (a,b)$ $\begin{cases} g'(x) > 0 \\ g'(x) < 0 \end{cases}$ $\Rightarrow$ $g$ is increasing or decreasing in all $[a,b]$

then $|g^{-1}(y)| = 1 \Leftarrow$ finite.

* Assume $\exists\, x_1, x_2 \in g[a,b]$ such that $g(x_1) = g(x_2) = y$

$x_1 \neq x_2$ wlog assume $x_1 < x_2$.

then we have $\dfrac{g(x_2)-g(x_1)}{x_2-x_1} = g'(c)$ for some $c \in (x_1, x_2)$.

$\Rightarrow g'(c) = 0$ (contradiction)

then we have for $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

$g$ is a one to one function then $g^{-1}(y)$ is a finite set for all $y$ in the range of $g$

ample A, 6:

*

Suppose $f$ is differentiable on the interval $[a,b]$

$f(a) = 0$

there is a finite constant $A$ such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$.

Prove that $f(x) = 0$ for all $x \in [a,b]$.

also $f(x)$ here, not $x$

note that $f(x)$ is not $\neq 0$ $\forall x \in [a,b]$

Sample № 1

Let ~~$\sum a_n$~~ $\{a_n\}$ be a sequence of nonnegative numbers and that $\sum_{n=0}^{\infty} a_n = 1$.

The power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in [-1, 1]$.

If L denotes the left hand derivative of $f$ at $x = 1$, $L = \lim_{x \to 1^-} \frac{f(1) - f(x)}{1 - x}$

Show that $L = \sum_{n=1}^{\infty} n a_n$ including the case

Sample 47 147

Determine the convergence (absolutely, conditionally) or divergence of the following series.

a> $\sum (-1)^n \frac{\ln n}{\sqrt{n}}$ b> $\sum_{n=1}^{\infty} n^2 [\pi^{1/n} - 1]^n$ c> $\sum_{n=1}^{\infty} \frac{2\cdot 4 \cdot 6 \cdots (2n)}{3\cdot 5\cdot 7 \cdots (2n+1)}$

d> $\sum_{n=1}^{\infty} n!\, e^{-n}$

a> $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n c_n$ where $c_n = \frac{\ln n}{\sqrt{n}}$

* Determine the convergence/ divergence.

- we have $f(x) = \frac{\ln x}{\sqrt{x}}$, where $x > 0$ have $f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}} = \frac{\frac{1}{\sqrt{x}}\left[1 - \frac{\ln x}{2}\right]}{\sqrt{x}}$.

So we have when $x > e^2$, $\ln x > 2 \Rightarrow f'(x) < 0$

this means $\{c_n\}$ eventually decreasing

- $\lim_{n\to\infty} c_n = \boxed{\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \overset{\text{L'Hospital}}{\underset{\frac{\infty}{\infty}}{=}}} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$

So $c_n \downarrow 0 \Rightarrow \sum (-1)^n c_n$ converges.

* Determine absolute convergence ?

Consider $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n}{n} \right| = \sum_{n=1}^{\infty} \left| \frac{\ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$

we have $\ln n > 1$ when $n > 1$

$\Rightarrow \frac{\ln n}{n} > \frac{1}{n}$

we also have $\sum \frac{1}{n}$ diverges

$\Rightarrow \sum \frac{\ln n}{n}$ diverges

In conclusion, the series converges conditionally.

b> $\sum_{n=1}^{\infty} n^2 [\pi^{1/n} - 1]^n$

Use Root test to find $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n^2 \cdot [\pi^{1/n} - 1]^n} = \lim_{n\to\infty} n^{2/n} [\pi^{1/n} - 1]$

we have $\lim_{n\to\infty} n^{2/n} = 1$

$\lim_{n\to\infty} [\pi^{1/n} - 1] = 0$

$\Rightarrow \lim_{n\to\infty} n^{2/n} [\pi^{1/n} - 1] = 0$

So $\lim \sqrt[n]{a_n} = 0 < 1 \rightarrow$ the series converges absolutely.

c) $\sum \frac{2\cdot 4\cdot 6\cdot 8 \cdots (2n-2)(2n)}{3\cdot 5\cdot 7\cdot 9 \cdots (2n-1)(2n+1)}$.

Use comparison test,

we have $\frac{2\cdot 4\cdot 6\cdot 8 \cdots (2n-2)(2n)}{3\cdot 5\cdot 7\cdot 9 \cdots (2n-1)(2n+1)} > \frac{2}{2n+1}$ ⇒ $\sum a_n$ diverges.

$> 1$

$\sum \frac{2}{2n+1}$ diverges

$$\boxed{d)\ \sum_{n=1}^{\infty} n!\, e^{-n}}$$

We have $\frac{n!}{e^n} = \frac{1\cdot 2\cdot 3 \cdots n}{e\ e\ \underbrace{e \cdots}_{>1}\underbrace{\cdots e}_{>1}} > \frac{2n}{e^3}$ ⇒ our series diverges □.

$\frac{2}{e^3}\sum n$ diverges

We can also use Ratio test ....

Sample 07 I-2 :

Let $\{f_n\}$: sequence of continuously differentiable functions on $\mathbb{R}$

$f_n(0) = 0 \;, \forall n$

$f_n' \cdot f_m' \equiv 0$ for all $m \neq n$.

$f_n' \underset{n \to \infty}{\rightrightarrows} 0$ on $\mathbb{R}$.

a) Prove that $\sum_1^{\infty} f_n'$ converges uniformly and absolutely on $\mathbb{R}$

b) Prove that $\sum f_n$ converges pointwise on $\mathbb{R}$

c) Let $g = \sum_{n=1}^{\infty} f_n'$ $\quad f := \sum_{n=1}^{\infty} f_n$

Prove that $f$ is differentiable in $\mathbb{R}$ and $g(x) = f'(x) ; \forall x \in \mathbb{R}$

a

a) Prove that $\sum f_n'$ converges uniformly on $\mathbb{R}$.

NTP
$\sum f_n'$ converges uniformly
$\Leftrightarrow$ NTP $\sum f_n'$ uniformly Cauchy
$\Rightarrow$ NTP $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \left|\sum_{\ell=n}^{m} f_\ell'(x)\right|$

- $f_n' \rightrightarrows 0$ on $\mathbb{R} \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |f_n' - 0| < \varepsilon, \forall x \in \mathbb{R}$

- Note that because $f_n' \cdot f_m' \equiv 0, \forall m \neq n$, So

so for every $\ell \in \overline{n,m}$ , $\exists$ at most one $\ell_0$ such that $f_{\ell_0}' \not\equiv 0$.

So we have $\left|\sum_{\ell=n}^{m} f_\ell'(x)\right| = \left|f_{\ell_0}'(x)\right|$ for only one $\ell_0 \in \overline{n,m}$

it's great that we have $f_n' \cdot f_m' = 0, \forall m \neq n$

at most one of $f_n' \not\equiv 0$

This means $\forall \varepsilon > 0, \exists N = n_0, \forall m, n \geq n, \left|\sum_{\ell=n}^{m} f_\ell'(x)\right| = \left|f_{\ell_0}'(x)\right| < \varepsilon$ □

* Prove that $\sum_{n=1}^{\infty} |f_n'|$ converges pointwise on $\mathbb{R}$.

We ~~also~~ need to prove that ~~$f_n$~~ $\forall$ fixed $x$ in $\mathbb{R}$, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \left|\sum_{\ell=n}^{m} f_\ell'(x)\right| < \varepsilon$
because what we prove above is true for all $x \in \mathbb{R}$, thus this case can be proved in the same way with above problem.

b) Prove that $\sum f_n$ converges pointwise on $\mathbb{R}$.

# GRADUATE PRELIMINARY EXAMINATION

## ANALYSIS

### Sample C

Important template.

1. Let $f$ be a real valued function defined on a set $D$ which is dense in $[0,1]$. If $f$ is uniformly continuous on $D$, show that $f$ can be extended to a uniformly continuous function on $[0,1]$.

2. Fix a real number $a > 1$ and define a sequence of numbers $\{x_n\}$ inductively by $x_1 = 0$ and

$$x_{n+1} = \frac{a(1+x_n)}{a+x_n} \quad \text{for } n = 0, 1, \ldots .$$

Show that $\lim_{n\to\infty} x_n$ exists and find this limit.

3. Let $f: \mathbf{R}^3 \to \mathbf{R}$ be a given function, and consider 3 variables $x, y, z$ which are related by the equation

(A) $$f(x,y,z) = 0 .$$

In some textbooks in Thermodynamics it is claimed that (A) implies the formula

(B) $$(\partial x/\partial y)(\partial y/\partial z)(\partial z/\partial x) = -1 ,$$

where it is understood that equation (A) may be solved for each variable in terms of the other two. Thus, for example, $x$ may be expressed as function of $y$ and $z$, and it is this function which is differentiated in the symbol $\frac{\partial x}{\partial y}$. By making use of the Implicit Function Theorem, formulate and prove a precise theorem, including appropriate hypotheses on $f$, which shows that (B) is indeed a consequence of (A).

4. Let $f$ be a continuous, nonnegative function defined on $[a,b]$ with $M = \sup_{x \in [a,b]} f(x)$. Prove

$$\lim_{n\to\infty} \Big( \int_a^b f^n(x)\,dx \Big)^{1/n} = M .$$

5. Define $f: \mathbf{R}^2 \to \mathbf{R}$ by $f(0,0) = 0$, and $f(s,t) = t^3/(s^2+t^2)$ otherwise. Prove the following facts:
   - (a) $f$ is continuous on $\mathbf{R}^2$,
   - (b) $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ exist at all points of $\mathbf{R}^2$,
   - (c) $f$ is not differentiable at $(0,0)$.

# GRADUATE PRELIMINARY EXAMINATION

## ANALYSIS

Fall 1991

*Instructions:* Do all problems. Each problem is worth 10 points.

Same.

See solution in Fall 2001 P1

1. Show that every uncountable subset of the real numbers has a limit point.

2. The sequence of real numbers $\{x_n\}$ is defined recursively by $x_1 = 1$ and

$$x_{n+1} = (x_n + x_n^2)^{1/3}.$$

Prove that $x_n$ converges, and find the limit.

3. Let $\{f_n\}$ be a sequence of continuous functions defined on a compact metric space $K$, and suppose $f_n$ converges uniformly on $K$ to a function $f$ Prove that $f_n^2$ converges uniformly to $f^2$ on $K$.

4. Prove the following: if $f$ is a continuous, real valued function on $[0,1]$ such that ~~$f(0) = 0$~~ and

7.20/169 Rudin

$$\int_0^1 x^n f(x)dx = 0 \qquad \text{for } n = 1, 2, \ldots,$$

don't need this

then $f(x) = 0$ for all $x \in [0,1]$. *Hint:* Show that $\int_0^1 f^2(x)dx = 0$.

template

5. Let $F(x,y,z) = 3x + 2y + z - y\sin(xz)$.

(a) Can the equation $F(x,y,z) = 0$ be solved for $z = f(x,y)$ in a neighborhood of the point $(0,-1)$ satisfying $f(0,-1) = 2$? Justify your answer.

(b) State a precise version of what is asked for in (a). Be as complete as possible.

6. The function $f$ maps $[0,1]$ onto $[0,1]$, and is monotone. Prove $f$ is continuous on $[0,1]$.

Sample … template.

I ↳ Let $f: D \to \mathbb{R}$ or a complete space ⟷ Show that $f$ can be extended to a uniformly continuous function on $[0, 1]$. unique

$D$ is dense in $[0, 1]$

$f$ is uniformly continuous on $D$

* Stt noted before solving the problem:

Let $(x_n)$ Cauchy in $\mathbb{R}$, $f$ uniformly continuous $\}$ $\Rightarrow$ $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$ $\Rightarrow$ $\{f(x_n) \longrightarrow L$

$\mathbb{R}$ complete

• If $x_n \longrightarrow x$, $x'_n \longrightarrow x$ then if $g(x_n) \longrightarrow L$ $\Rightarrow$ we have $f(x'_n) \longrightarrow L$

Step 1. Prove that if $z \in [0, 1]$ $\Rightarrow \exists x_n \to z$ and because $f$ uniformly cont then $\{f(x_n)\}$ converges.

* We have $f$ is uniformly continuous on $D$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in D, |x - y| < \delta, |f(x) - f(y)| < \varepsilon$ (1)

* $D$ is dense in $[0, 1]$ $\Leftrightarrow \forall z \in [0, 1], \exists \{x_n\} \subseteq D,$

$x_n \to z$.

• We have because ~~$\{x_n\} \subseteq D$ and~~ $x_n \longrightarrow z$, then we have $\{x_n\}$ is a Cauchy sequence

this means $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, |x_m - x_n| < \varepsilon$

Let $\varepsilon = \delta$, we have $\forall m, n \geq n_0, |x_m - x_n| < \delta$

Because of (1), we have $|f(x_m) - f(x_n)| < \varepsilon$.

This means $\{f(x_n)\}_n$ is a Cauchy sequence in $\mathbb{R}$, we have $\mathbb{R}$ is complete $\}$ $\Rightarrow$ $f(x_n)$ converges, Assume $f(x_n) \xrightarrow[n\to\infty]{} L$

Step 2. We prove that if $\{x'_n\} \subseteq D$ is another sequence s.t. $x'_n \to z$ Then we also have $f(x'_n) \longrightarrow L$

(this means $L$ is uniquely defined)

$x_n \longrightarrow z$ $\Rightarrow$ for $\delta$, $\exists n_1 \in \mathbb{N}, \forall n \geq n_1, |x_n - z| < \delta/2$

$x'_n \longrightarrow z$ $\Rightarrow$ $\exists n_2 \in \mathbb{N}, \forall n \geq n_2, |x'_n - z| < \delta/2$.

then $|x_n - x'_n| \leq |x_n - z| + |z - x'_n| < \delta$

$\Rightarrow |f(x_n) - f(x'_n)| < \delta$.

this means $\lim\limits_{n\to\infty} f(x'_n) = \lim\limits_{n\to\infty} f(x_n) = L$.

Step 3. Because the limit $L$ is unique,

we put $\boxed{g(z) = \lim\limits_{n\to\infty} f(x_n)}$ when $\{x_n\} \subset D$, $x_n \longrightarrow z$.

This is the extention of $f$ to $[0, 1]$

* **Step 4** Now we will prove that $\hat{f}$ is uniformly continuous ~~for every~~ on [0, 1] (with the [illegible] of $\hat{f}$ :)

We want to prove that $\forall \varepsilon > 0$, with $\delta$ above, $\forall x, y \in [0, 1]$, $|x-y| < \delta/3$
$\delta_1 = \delta/3$
then $|\hat{f}(x) - \hat{f}(y)| < \varepsilon$.

. We have

$x \in [0,1] \Rightarrow \exists (x_n) \subset D,\ x_n \to x \Leftrightarrow \forall \delta > 0, \exists N_1 \in \mathbb{N}, \forall n \geq N_1, |x_n - x| < \delta/3$

$y \in [0,1] \Rightarrow \exists (y_n) \subset D,\ y_n \to y \Leftrightarrow \exists N_2 \in \mathbb{N}, \forall n \geq N_2, |y_n - y| < \delta/3$

$\forall n \geq \max\{N_1, N_2\}$
then $|x_n - y_n| \leq |x_n - x| + |x - y| + |y_n - y| < \delta$

$$\left.\begin{array}{l} \text{and because of (1)}, \ |f(x_n) - f(y_n)| < \varepsilon \\ \text{we have } f(x_n) \to \hat{f}(x) \\ \qquad f(y_n) \to \hat{f}(y) \end{array}\right\} \begin{array}{l} , \forall n \geq \max\{N_1, N_2\}. \\ \Rightarrow |\hat{f}(x) - \hat{f}(y)| < \varepsilon. \end{array}$$

this means $\hat{f}$ is uniformly continuous on [0, 1], we win □

:)

In fact we have

$$\left.\begin{array}{l} f: D \to \mathbb{R} \\ f \text{ is uniformly on } D \\ D \text{ is densen in } [0,1] \end{array}\right\} \Longleftrightarrow f \text{ can be extended to a continuous function on } [0,1].$$

$\Rightarrow$) : above problem.

$\Leftarrow$): Let f, f can be extended to a continuous function on [0, 1]. Prove that f is uniformly continous on D

We have because f can be ~~extension~~ extended to a continuous function on [0, 1].
this means $\exists\ g : [0,1] \longrightarrow \mathbb{R}$ continuous.
where $g(x) = f(x) =\ , \forall x \in D$.

$$\left.\begin{array}{l} \text{because } g \text{ is cont on } [0,1] \\ [0,1] \text{ compact} \end{array}\right\} \Rightarrow g \text{ is uniformly cont on } [0,1].$$

$\Rightarrow$ This clearly restrict to x, y in D $\Rightarrow$ f is uniform continuous on D.

$\Rightarrow$ Corollary:

Eg: f has a continuous extension on [a, b] $\Longleftrightarrow$ f is uniformly continuous on (a, b)

Sample C.

①

2) Fix $a > 1$ and define a sequence $\{x_n\}$: $\begin{cases} x_1 = 0 \\ x_{n+1} = \dfrac{a(1+x_n)}{a+x_n} \end{cases}$, $n \in \mathbb{N}$

Show that the limit exist and find the limit

~~* Put $f(x) = \dfrac{a(1+x)}{a+x}$ $f'(x) = \dfrac{a(a+x) - (a+1+x)}{(a+x)^2}$~~.

Way to do this problem:
1) write some first terms to determine $\{x_n\}$ increase / decrease
2) Compute $x_{n+1} - x_n$ to find the relation depend on $a$.
3) ~~Cup~~ ~~Compute~~ prove that $x_n$ is bounded by $g(a)$ and from this result, we also know that the sequence increase or decrease.

1)* $x_1 = 0$

$$x_2 = \frac{a(1+0)}{a+0} = \frac{a}{a} = 1.$$

$$x_3 = \frac{a(1+1)}{a+1} = \frac{2a}{a+1} \quad x_3 - x_2 = \frac{2a}{a+1} - 1 = \frac{2a-a-1}{a+1} = \frac{\overset{>0}{a-1}}{\underset{>0}{a+1}}.$$

$\Rightarrow$ the sequence may increasing

2)* $$x_{n+1} - x_n = \frac{a(1+x_n)}{a+x_n} - x_n = \frac{a + \cancel{ax_n} - \cancel{ax_n} - x_n^2}{a+x_n} = \frac{a - x_n^2}{a+x_n} \quad (1)$$

3) (Because the sequence may increase, we want $a - x_n^2 > 0 \Rightarrow \sqrt{a} > x_n$.

We now prove that $x_n < \sqrt{a}$, $\forall n$.

We have $x_1 = 0 < \sqrt{a}$
$x_2 = 1 < \sqrt{a}$

Assume $x_{n-1} < \sqrt{a}$, do we have $x_n \overset{?}{<} \sqrt{a}$ ?

$$\Leftrightarrow \frac{a(1+x_{n-1})}{a+x_{n-1}} \overset{?}{<} \sqrt{a}$$

$$\Leftrightarrow \frac{\sqrt{a}(1+x_{n-1})}{a+x_{n-1}} \overset{?}{<} 1$$

$$\Leftrightarrow \sqrt{a}(1+x_{n-1}) < a + x_{n-1}$$

$$(\sqrt{a}-1)x_{n-1} < a - \sqrt{a}$$

$$\cancel{(\sqrt{a}-1)}(x_{n-1}) < \sqrt{a}\,\cancel{(\sqrt{a}-1)}$$

$$(x_{n-1}) < \sqrt{a}$$

So we have $x_n < \sqrt{a}$.

By induction, we have $x_n < \sqrt{a}$, $\forall n$. (2)

and from (1) + (2) $\Rightarrow$ we have $(x_n)$ increasing + bounded by $\sqrt{a}$ $\Rightarrow \exists$ limit.

* Find the limit. Assume $\lim x_n = \ell$ we have $\lim x_{n+1} = \ell$

$$\Rightarrow \ell = \frac{a(1+\ell)}{a+\ell} \Rightarrow \ldots \Rightarrow \ell = \sqrt{a}$$

Fix $a > 1$ and define the sequence $\{x_n\}$ $\begin{cases} x_1 = 0 \\ x_{n+1} = \dfrac{a(1+x_n)}{a+x_n} \end{cases}$, $n \in \mathbb{N}$.

Show that the limit exist and find the limit

Another way, this way is not as good as 1st way but can learn sth from this).

Put $f(x) = \dfrac{a(1+x)}{a+x}$ $\quad f'(x) = \dfrac{a(a+x) - a(1+x)}{(a+x)^2} = \dfrac{a^2+ax-a-ax}{(a+x)^2} = \dfrac{a^2-a}{(a+x)^2} = \dfrac{a(a-1)}{(a+x)^2} > 0$

$\Rightarrow$ $f$ is increasing function.

By definition: if $x < y$ then $f(x) < f(y)$.

$\Rightarrow$ If $x_{n-1} < x_n$ then $f(x_{n-1}) < f(x_n) \Rightarrow x_n < x_{n+1}$. $\quad (1)$

we already know $x_1 = 0$

$x_2 = \dfrac{a(1+0)}{a+0} = \dfrac{a}{a} = 1$ $\quad x_1 < x_2$

Assume $x_{n-1} < x_n$ then by (1), we have $x_n < x_{n+1}$

$\Rightarrow$ this is an increasing sequence (I)

+ We now prove that the sequence is bounded

We have $a > 1 \Rightarrow a + x_n > 1 + x_n$

$\Rightarrow a(1+x_n) < a(a+x_n)$ $\Big\} \Rightarrow \cancel{\dfrac{a(a+x_n)}{a+x_n} > \dfrac{a(1+x_n)}{1+}}$

$\Rightarrow \dfrac{a(1+x_n)}{a+x_n} < \dfrac{a(a+x_n)}{a+x_n} = a$ $\quad (2)$

So we have $x_1 = 0 < a$

$x_2 = 1 < a$

If $0 < x_n < a$ then by (2), $x_{n+1} < a$ $\Big\} \rightarrow x_n < a$, $\forall n$. $\quad$ (II)

(I) + (II) $\Rightarrow \exists$ limit.

**1991** / Let $\{f_n\}$ be a sequence of continuous functions defined on a compact metric space K

$f_n \rightrightarrows f$ on K.

Prove that $f_n^2 \rightrightarrows f^2$ on K.

---

- $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in K, |f_n(x) - f(x)| < \varepsilon$

We NTP $f_n^2 \rightrightarrows f^2$ on K

NTP $\forall \varepsilon_1 > 0, \exists n_1 \in \mathbb{N}, \forall n \geq n_1, |f_n^2(x) - f^2(x)| < \varepsilon$.

* We consider $|f_n^2(x) - f^2(x)| = |f_n(x) + f(x)| \, \underbrace{|f_n(x) - f(x)|}_{< \varepsilon \text{ for } n \geq n_0 \text{ by assumption}}$.

- We have $\{f_n\}$ sequence of continuous on K compact

$\Rightarrow \{f_n\}$ sequence of bounded functions

According to 7.1/65 Rudin, every uniformly convergent sequence of bounded functions is uniformly bounded

and because $f_n \rightrightarrows f$, $f$ is also bounded.

Then we have $|f_n(x) + f(x)| \leq M + L$

This means $|f_n^2(x) - f^2(x)| \leq (M+L)\varepsilon$ for $n \geq n_0$, $\forall x \in K$, $\forall \varepsilon$

This means $f_n^2 \rightrightarrows f^2$ on K $\square$.

---

* We can prove the blue line above directly as in 7.1/65 Rudin,

or we can also have the result by using:

**Thm 7.24**
$$\left.\begin{array}{l} K \text{ compact} \\ f_n \in C(K) \\ f_n \rightrightarrows \end{array}\right\} \Rightarrow \{f_n\} \text{ equicontinuous.}$$

$$\left.\begin{array}{l} K \text{ compact} \\ f_n \text{ equicontinuous} \\ f_n \xrightarrow[\text{pointwise}]{} \end{array}\right\} \Rightarrow \begin{array}{l} \{f_n\} \text{ uniformly bounded} \\ (\text{contains a uniformly convergent subsequence.}) \end{array}$$

Template ①

5) Let $F(x,y,z) = 3x + 2y + z - y\sin(xz)$

a) Can the equation $F(x,y,z) = 0$ be solve for $z = f(x,y)$ in a neighborhood of the point $(0,-1)$ satisfying $f(0,-1) = 2$ ? Justify

b) State a precise version of what is asked for in (a). Be as complete as possible

a) • We have at $(0,-1,2)$ $F(x_0,y_0,z_0) = 3\cdot 0 + 2(-1) + 2 + 1\sin 0 = -2+2=0$

$\Rightarrow (0,-1,2)$ is a solution of $F(x,y,z)=0$ (1)

• We have $DF = \left[\frac{\partial F}{\partial x} \ \frac{\partial F}{\partial y} \ \frac{\partial F}{\partial z}\right] = \left[3 - yz\cos(xz) \quad 2 - \sin(xz) \quad 1 - yx\cos z\right]$

we have all $D_j f_i$ exist and continuous $\Rightarrow f$ is continuous differentiable (2)

• $\frac{\partial F}{\partial z}(0,-1,2) = 1 \neq 0$ (3)

(1)+(2)+(3) $\Rightarrow$ By Implicit function theorem, the equation $F(x,y,z)=0$ can be solve for $z = f(x,y)$ in a neighborhood of $(0,-1)$ satisfying $f(0,-1)=2$ ◻ a)

b) To ask if there exist an open neighborhood of $U \subseteq \mathbb{R}^3$ of $(x_0,y_0,z_0) = (0,-1,2)$ and a neighborhood $W$ of $(x_0, y_0) = (0,-1)$

such that for all z ~~∈ W, there exist! (x, y) such that $\begin{cases} F(x,y,z)=0 \\ z = f(x,y)\end{cases}$~~ ∈ U

such that for all $(x,y)$, $\exists!$ z such that $\begin{cases}(x,y,z) \in U \\ F(x,y,z)=0\end{cases}$

this means we can get $z = f(x,y)$ ◻

Need to review.

6 Fall 1991: The function $f$ maps $[0,1]$ onto $[0,1]$ and is monotone. Prove that $f$ is continuous on $[0,1]$

* $f$ is monotone then we have $f(x^-)$ and $f(x^+)$ exists for all $x\in[0,1]$
($x=0$, $f(0^+)$ exists only | $x=1$, $f(1^-)$ exist only)

\+ besides, $f$ is monotone then, wlog, assume $f$ is increasing..

$$f(x^-)\leq f(x)\leq f(x^+).$$

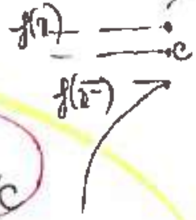
- We have if $f(x^-)=f(x^+)$ then $f$ is continuous on $[0,1]$ $\Rightarrow$ done.

- Assume $f(x^-)<f(x^+)$, ~~then there are~~ there cases that $f$ can be discontinuous

$$\left[\begin{array}{l} f(x^-)<f(x)<f(x^+) \\ f(x^-)<f(x)=f(x^+) \\ f(x^-)=f(x)<f(x^+) \end{array}\right.$$

\+ Then we consider if $f(x^-)<f(x)$, then $\exists c$ such that $f(x^-)<c<f(x)$

then $\underline{\forall y<x}$



$$f(x^-)=\sup\{f(y),\ y<x\}.$$

$$f(y)\leqslant \sup\{f(y), y<x\}=f(x^-)<c$$

$\underline{\forall y>x}$ $f(y)\geqslant f(x)>c$

$\underline{y=x}$ , $f(y)=f(x)>C$

Then $\exists c\in(f(x^-),f(x))\subset[0,1]$ such that $\nexists\, y\in[0,1]$, $c=f(y)$)

$\Rightarrow$ $f$ is not onto (contradiction)

* Some property about $\boxed{\text{monotone}}$ function $f:(a,b)\longrightarrow\mathbb{R}$.
$f$ is monotone, then $\forall p\in(a,b)$, $f(p^-)$ and $f(p^+)$ exists and $\left[\begin{array}{l} f(p^-)\leq f(p)\leq f(p^+) \\ f(p^-)\geqslant f(p)\geqslant f(p^+)\end{array}\right.$

if futhermore, if $f(p^-)=f(p^+)$ $\Rightarrow$ then $f$ is continuous.

- there are 3 cases that $f$ can be discontinuous.

4) Prove the following : If $f$ is continuous, real valued function on $[0,1]$ such that

=. 10/169 Rudin done.

$$f(0)=0$$

$$\int_0^1 x^n f(x)\,dx = 0 \quad \text{for } n = 1,2,3,\dots$$

Prove that $f(x)=0$, $\forall x \in (0,1)$.

# GRADUATE PRELIMINARY EXAMINATION

## Analysis

(Fall 1992)

See solution MAT602 #W3.5. | also Rudin.

1. Let $\{x_n\}$ be a sequence of complex numbers converging to $a$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_j = a.$$

2. a) If $f_n \in C^1(0,2)$, $n = 1, 2, \ldots$, and $f_n'$ converges uniformly to zero, while $f_n(1)$ converges to 1, prove that $f_n$ converges uniformly on $(0,2)$.

   b) Is the result true if each $f_n$ is only differentiable on $(0,2)$?

3. Let $(X, \rho)$ be a compact metric space and $(Y, d)$ be a metric space.

   a) If $f : X \to Y$ is continuous and onto show that $(Y, d)$ is complete.

   b) If $f$ is also one-to-one prove that $f^{-1} : Y \to X$ is continuous.

4. Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is $C^1$. If $f_{xy}$ exists in a neighborhood of $(0,0)$ and is continuous at $(0,0)$, prove that $f_{yx}$ exists at $(0,0)$ and $f_{yx}(0,0) = f_{xy}(0,0)$.

5. Let $p(x,y) = (xy-1)^2 + x^2$ for $(x,y) \in \mathbf{R}^2$. Find $\inf\{p(x,y) : (x,y) \in \mathbf{R}^2\}$.

6. Suppose $f$ is continuous and greater than 1 on $[0,1]$. Prove that for positive $a$

$$\lim_{a\to 0} \left( \int_0^1 \mid f(x) \mid^a dx \right)^{\frac{1}{a}} = \exp\left( \int_0^1 \ln \mid f(x) \mid dx \right)$$

   Hints: First establish the limit formally. Then attend to the intermediate results that require justification.

# Graduate Proficiency Examination
# Analysis

## Fall 1993

Instructions: Do all problems. Each problem is worth 10 points.

1. Given a $C^1$ function $F : \mathbf{R}^n \to \mathbf{R}^n$ satisfying

$$\|F(x)\| \leq \|x\|^2, \qquad x \in \mathbf{R}^n,$$

prove that there is an $\epsilon > 0$ such that the equation $F(x) = x + \alpha$ has a solution $x$ whenever the vector $\alpha$ satisfies $\|\alpha\| < \epsilon$.

② If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$ prove that there exists a sequence $b_n$ such that $\lim_{n\to\infty} b_n = +\infty$ and $\sum_{n=1}^{\infty} a_n b_n$ converges.

3. Assume that the family $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on $[0,1]$ is equicontinuous and pointwise bounded. Also assume $\int_a^b f_n(x)\,dx \to 0$ as $n \to \infty$ for every $0 \leq a < b \leq 1$. Prove that $f_n \to 0$ uniformly.

4. Let $P_E$ denote the set of real-valued polynomials which involve no odd powers of the variable, i.e., the coefficient of each odd power term is zero. Prove that $P_E$ is dense in $C([0,1])$ with the sup norm. For which closed intervals other than $[0,1]$ can the same be proved?

5. For which non-decreasing functions $\beta$ on $[0,1]$ does the Riemann-Stieltjes integral $\int_0^1 \beta\, d\beta$ exist? Prove your assertion.

6. If $f$ is continuous and $\lim_{s\to\infty} f(s) = a$, prove that $\frac{1}{\log t}\int_1^t \frac{f(s)}{s}\,ds \to a$ as $t \to \infty$.

Fall 1998

3) Let $(X,\rho)$ be a compact metric space
$(Y,d)$ be a metric space

a) If $f: X \to Y$ is continuous and onto show that $(Y,d)$ is complete.

b) If $f$ is also one-to-one, prove that $f^{-1}: Y \to X$ is continuous.

3a We have $(X,\rho)$ compact
$f: X \longrightarrow Y$ continuous
$\Big\} \Rightarrow f(X)$ compact
$f$ is onto
$\Big\} \Rightarrow Y = f(X)$ compact $\Rightarrow Y$ comp

3b: $f$ onto
$f$ one-to-one
$\Big\} \Rightarrow \exists f^{-1}: Y \to X$

We want to prove that $f^{-1}$ is continuous $\Leftrightarrow$ ~~we need to prove $(f^{-1})^{-1}$~~ =

We need to prove for all A closed in X, $(f^{-1})^{-1}(A) = f(A)$ is closed in Y.

Because A closed in X
X compact
$\Big\} \Rightarrow$ then $f(A$ is compact
$f$ continuous
$\Big\} \Rightarrow f(A)$ compact
compact set is closed
$\Big\} \Rightarrow f(A)$ clos
Y.

---

* Now we prove some theorems that we have applied in above proof.

Aug1992 → P5.

Let $\rho:\mathbb{R}^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto \rho(x,y) = (xy-1)^2 + x^2$ for | Find $\inf\{\rho(x,y), x,y \in \mathbb{R}^2\}$.

Note that $\rho$ attains local max/min iff $\rho_x = \rho_y = 0$

So we compute $\begin{cases} \rho_x = 2(xy-1)y + 2x \\ \rho_y = 2x(xy-1). \end{cases}$

- We have $\rho_y = 0$ when $2x(xy-1) = 0 \Leftrightarrow \left[\begin{array}{l} x = 0 \\ xy = 1 \Leftrightarrow x = \frac{1}{y}. \end{array}\right.$
- We have $\rho_x = 0$ when $2(xy-1)y + 2x = 0$
  - + when $x = 0$, then $\Rightarrow -2y = 0 \Rightarrow y = 0$
  - + when $x = \frac{1}{y}$, then this means $2(1-0)y + 2x = 0 \Rightarrow 2x = 0 \Rightarrow$ this case could not happen

So we have $\rho_x = \rho_y = 0 \Leftrightarrow (x,y) = (0,0)$

This means $\rho(x,y)$ attains local max/min at $(0,0)$
Besides we have $\rho(x,y) = (xy-1)^2 + x^2 \geq 0, \forall (x,y)$ $\Big\} \Rightarrow \inf = 0$

g 1998, B6 →.

ppose $f$ is continuous and greater than $1$ on $[0,1]$.
ove that for $a>0$, $\lim_{a\to 0}\left(\int_0^1 |f(x)|^a dx\right)^{1/a} = \exp\left[\int_0^1 \ln|f(x)|\,dx\right]$

Put $g(a) := \left[\int_0^1 |f(x)|^a dx\right]^{1/a} \geq \left[\int_0^1 1\,dx\right]^{1/a} > 1.$

ote that $\lim_{a\to 0} g(a) = \lim_{a\to 0} e^{\ln g(a)} = e^{\lim_{a\to 0}[\ln g(a)]}$

ow we compute $\ln[g(a)] = \left[\ln\left(\int_0^1 |f(x)|^a dx\right)^{1/a}\right] = \frac{1}{a}\ln\int_0^1 |f(x)|^a dx =$

$= \frac{1}{a}\int_0^1 \ln|f(x)|^a dx = \frac{1}{a}\int_0^1 a\ln|f(x)|\,dx = \int_0^1 \ln|f(x)|\,dx.$

$\Rightarrow \lim_{a\to 0}[\ln(g(a))] = \int_0^1 \ln|f(x)|\,dx.$ wrong here.

$\Rightarrow \lim_{a\to 0} g(a) = e^{\int_0^1 \ln|f(x)|dx}.$ □.

Fall 1993

Vier

P1 > Given a $C^1$ function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying $\|F(x)\| \le \|x\|^2 \quad x \in \mathbb{R}^n$
Prove that $\exists\, \varepsilon > 0$ s.t the equation $F(x) = x + \alpha$ has a solution $x$ whenever the vector $\alpha$ satisfies $\|\alpha\| < \varepsilon$

Now. we put

$$G(x,\alpha) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(x,\alpha) \longmapsto F(x) - x - \alpha.$$

So we have $G(0,0) = F(0) = 0$ and because $\|F(0)\|^2 \le \|0\|^2 = 0 \Rightarrow \|F(0)\| = 0 \Rightarrow F(0) = 0$

+ we have

$$DG = \begin{bmatrix} \frac{\partial F}{\partial x} - I & -1 \end{bmatrix}$$

we have

$$\frac{\partial F}{\partial x} = \lim \frac{\|F(x) - \overset{0}{F(0)} - \overset{=0}{DF(0)}(x-0)\|}{\|x\|} \longrightarrow 0 \Rightarrow \left|\frac{F(x)}{\|x\|}\right| \longrightarrow 0.$$

$$\frac{DG(x,\alpha)}{\partial x} = D_x G(x,\alpha) = DF(x) - I.$$

$$\Rightarrow D_x G(0,0) = -I \Longrightarrow \det[D_x G(0,0)] \ne 0$$

So by Implicit $\Rightarrow$ ...

Aug 1993

) If $a_n \geqslant 0$, $\sum a_n < +\infty$

Prove that $\exists \{b_n\}$, $\lim\limits_{n\to\infty} b_n = +\infty$ and $\sum a_n b_n$ converges.

See Jan 2012)

$\{c_n\}$ be a sequence so that $c_n > 0, \forall n \geqslant 1$, $\lim\limits_{n\to\infty} c_n = 0$

Prove that $\exists \{a_n\}$; $a_n > 0, \forall n \geqslant 1$; $\sum a_n$ divergent; and $\sum c_n a_n$ is convergent.

Case 1: $a_n = 0$, $\forall n$.

then we just choose $b_n = n, \forall n$, we have $\lim\limits_{n\to\infty} b_n = \infty$ and $\sum a_n b_n = \sum 0$ converges.

Case 2: $a_n > 0$, $\forall n$.

hen put $\lambda_n = \sum\limits_{k=(n+1)}^{\infty} a_k$, then because $a_n > 0$ and $a_n \to 0$ $\Big\}$ we have $\lambda_n \downarrow 0$ and $\lambda_n > 0, \forall n$.

hen put $b_n = \dfrac{1}{\sqrt{\lambda_n}}$ ← well define because $\lambda_n > 0$, we have $\lim\limits_{n\to\infty} b_n = +\infty$.

Now we want to prove that $\sum a_n b_n$ converges.

e have $0 \leqslant a_n b_n \preceq \dfrac{\lambda_n - \lambda_{n-1}}{\sqrt{\lambda_n}} \enclose{circle}{\leq} \dfrac{\lambda_n - \lambda_{n-1}}{\sqrt{\lambda_n} + \sqrt{\lambda_{n-1}}} = \sqrt{\lambda_n} - \sqrt{\lambda_{n-1}}$

$\sum\limits_{n=1}^{\infty} \left(\sqrt{\lambda_n} - \sqrt{\lambda_{n-1}}\right) = \sqrt{\lambda_1}$

$\Big\} \Rightarrow$ By comparison test $\sum a_n b_n$ converges.

Case 3: $a_n = 0$ for some $n$

Because $a_n \geqslant 0$, $\sum a_n$ converges, $\Rightarrow$ we can arange $\{a_n\}$ and still have the same sum

We let $a_1 = 0, \cdots, a_\ell = 0, a_{\ell+1}, a_{\ell+2}, \cdots$, (if there are $\ell$ elements in $\{a_n\}$ equal 0.

Then we put $b_1 = \cdots = b_\ell = 0$, and $b_n = \dfrac{1}{\sqrt{\lambda_n}}$ for $n = \ell+1, \infty$ ← well defined.

imilar to above, $\Big\{ \begin{array}{l} \lim b_n = \infty \\ \sum a_n b_n \text{ converges} \end{array}$ □

Fall 1993

57 For which non-decreasing function $\beta$ on $[0,1]$ does the R-S integral $\int_0^1 \beta \, d\beta$ exists
(increasing)

all 1993 7 P6

f is continuous, $\lim_{s\to\infty} f(s) = a$. Prove that $\lim_{t\to\infty} \frac{1}{\log t}\int_1^t \frac{f(s)}{s}\,ds = a$

ote that $\frac{1}{\log t}\int_1^t \frac{1}{s}\,ds = \frac{1}{\log t}\ln s\Big|_1^t = \frac{1}{\log t}\log t = 1$.

So $a = \frac{1}{\log t}\int_0^t \frac{a}{s}\,ds$.

le also have $\lim_{s\to\infty} f(s) = a \Leftrightarrow \forall \varepsilon > 0, \exists S$ such that for $s > S$, $|f(s)-a| < \varepsilon$.

now we have

$$\left|\frac{1}{\log t}\int_1^t \frac{f(s)}{s}\,ds - a\right| \leq \left|\frac{1}{\log t}\int_1^t \frac{1}{s}|f(s)-a|\,ds\right|$$

$$= \frac{1}{\log t}\int_1^S \frac{1}{s}\underbrace{|f(s)-a|}_{\leq M}\,ds + \underbrace{\frac{1}{\log t}\int_S^t \frac{1}{s}\underbrace{|f(s)-a|}_{<\varepsilon}\,ds}$$

$$= \underbrace{\frac{1}{\log t}\underbrace{\int_1^S \frac{1}{s}M\,ds}_{\text{bounded}}}_{\xrightarrow[t\to\infty]{} 0} + \underbrace{\leq \varepsilon \cdot \underbrace{\frac{1}{\log t}\int_1^t \frac{1}{s}\,ds}_{=1}}_{\leq \varepsilon}$$

o we have What we need to prove □.

# GRADUATE PRELIMINARY EXAMINATION

## ANALYSIS

26 August 1994

(1) For which real $x$ does the series $\sum_{n=1}^{\infty} ne^{-nx}$ converge?

(2) Suppose that $f$ is a differentiable function on $(0, \infty)$. $\lim_{x\to\infty} f(x)/x = 0$, and $\lim_{x\to\infty} f'(x) = a$. Prove that $a = 0$. Similar with Jan 2004 , See Aug 2007 I.2

bounded

(3) Find $\lim_{n\to\infty} x_n$ when $x_{n+1} = \sqrt{x_n + a}$, $a > 0$, and $x_1 = \sqrt{a}$. 3/3

(4) Prove that if a function $f(x)$ is integrable on $[a, b]$ then its absolute value $|f(x)|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

(5) Let $f$ be a complex valued function on a set $D$ and suppose that $|f(x)| < 1$ for each $x \in D$.

- (a) Show that the sequence of powers of $f$, $\{f, f^2, f^3, \dots\}$ converges pointwise.
- (b) Find necessary and sufficient conditions for the convergence to be uniform.

(6) Let $K(x, y)$ be continuous on the rectangle $[a, b] \times [c, d] \subset \mathbb{R}^2$. For integrable functions $f$ on $[c, d]$ define an operator $T$ by

looks hard but try → do.

$$(Tf)(x) = \int_c^d K(x, y) f(y)\,dy.$$

- (a) Show that $(Tf)(x)$ is a continuous function on $[a, b]$.
- (b) Show that $S = \{Tf \mid \int_c^d |f(x)|dx \leq 1\}$ is an equicontinuous family of functions on $[a, b]$.

(7) Let $U = \{(u, v) \in \mathbb{R}^2 \mid u > 0\}$ and define $F : U \to \mathbb{R}^2$ by $F(u, v) = (u \cos v, u \sin v) = (x, y)$.

- (a) Show that F is an open mapping on $U$.
- → (b) Find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$.

(8) Let $f(x, y) = x^2 + y^2 - 5$ be a function on $\mathbb{R}^2$.

- (a) Describe thoroughly the results of applying the implicit function theorem in a neighborhood of the point $(2, 1)$.
- (b) Describe thoroughly the results of applying the implicit function theorem in a neighborhood of the point $(\sqrt{5}, 0)$.

# Preliminary Examination

## Analysis

18 August 1997

1. Let $K \subset \mathbb{R}^n$ be a compact set and let $\epsilon > 0$. Set $J = \{x \in \mathbb{R}^n \mid \text{dist}(x, K) \leq \epsilon\}$, where $\text{dist}(x, K) = \inf\{\|x - y\|_2 \mid y \in K\}$ and $\|t\|_2$ is the usual norm in $\mathbb{R}^n$. Prove that $J$ is compact.

2. Determine the convergence or divergence of the following sequences $\{x_n\}_{n=1}^{\infty}$.

   (a) $x_n = \dfrac{1}{n^2+1} + \dfrac{2}{n^2+2} + \cdots + \dfrac{n}{n^2+n}$

   (b) $x_n = \left(-\dfrac{1}{2}\right)^n + \sin\left(\dfrac{n\pi}{2}\right)$

   (c) $x_n = \dfrac{n^n + (-n)^n}{2} + \left(1 + \dfrac{1}{2n}\right)^n$

3. Determine whether or not $\sum\limits_{n=1}^{\infty} u_n(x)$ converges uniformly on $I$, where $u_n(x)$ and $I$ are given in parts (a) and (b) below

   (a) $I = \mathbb{R}$ and $u_n(x) = \begin{cases} 0 & , |x| \leq n \text{ or } |x| \geq n+1 \\ n\sin(1/n^2) & , n < |x| < n+1 \end{cases}$

   (b) $I = [1, \infty)$ and $u_n(x) = \int_1^x e^{-nt^2} dt$, $x \in I$.

4. Let $D^+$ and $D^-$ denote the operation of taking derivatives of real functions from the right and left respectively, for example $D^+ f(x) = \lim\limits_{y \to x^+} \dfrac{f(y) - f(x)}{y - x}$, $D^-$ is defined similarly.

   (a) Give an example of a function for which $D^+ f(0)$, $D^- f(0)$ both exist but are not equal.

   (b) Prove or disprove: if $D^+ f(0)$, $D^- f(0)$ both exist then the function $f$ is continuous at $x = 0$.

5. Suppose that $f(x) = x$ and $g(x) = \begin{cases} 0 & , 0 \leq x < 1/2 \\ 1/2 & , x = 1/2 \\ 1 & , 1/2 < x \leq 1 \end{cases}$, evaluate:

   (a) $\int_0^1 f\, dg$

   (b) $\int_0^1 g\, df$

6. For a nonnegative integer $l$ let $P_l(x) = \sum_{k=0}^{l} a_k x^k$ for real numbers $a_k$ and $x \in [-1, 1]$. Given a positive integer $n$ set $\mathcal{F}(n) = \{P_l(x) \mid 0 \leq l \leq n \text{ and } |a_k| < 1 \text{ for } k = 0, \ldots, l\}$. So $\mathcal{F}(n)$ is

the set of polynomials of degree less than or equal $n$ whose coefficients all have absolute value less than 1. Prove or disprove, for each $n$ the set $\mathcal{F}(n)$ is equicontinuous.

~~7.~~ Let $f(x,y) = |x|^{1/2}|y|^{1/2} + xy$ be a real function on $\mathbb{R}^2$.

(a) Find the partial derivatives of $f$ at the origin.

(b) Discuss the differentiability of $f$ at the origin.

~~8.~~ Let $x = r\cos(\theta)\sin(\phi)$, $y = r\sin(\theta)\sin(\phi)$, and $z = r\cos(\phi)$ define the map $F(r,\theta,\phi) = (x,y,z)$ from $(r,\theta,\phi) \in \mathbb{R}^3$ to $(x,y,z) \in \mathbb{R}^3$.

(a) Prove or disprove, $F$ has a global inverse on $\mathbb{R}^3$.

(b) Find $\dfrac{\partial}{\partial x}\theta(0,1,0)$.

uclel.

**Preliminary Examination** **Analysis** **August 21, 1998**

**Instructions: Work all 6 questions in the bluebook. You do not need to reprove standard results in basic analysis. Everything else should be carefully justified.**

1. Construct an open set containing every rational number, but not every real number. What can be said about the closure of *any* such set ? (Use the standard topology on the set of real numbers.)

2. Prove the inequalities

$$py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y),$$

where $x$ and $y$ are real numbers satisfying $0 < y < x$, and $p$ is a real number satisfying $1 \le p < \infty$.

3. Let $F(x,y,u,v) = 3x^2 - y^2 + u^2 + 4uv + v^2$, and $G(x,y,u,v) = x^2 - y^2 + 2uv$.

a) Show that the equations

$$F(x,y,u,v) = 9,$$
$$G(x,y,u,v) = -3$$

determine $x$ and $y$ as functions of $u$ and $v$ in a neighborhood of $u = 1, v = 1$ with $x(1,1) = 2$ and $y(1,1) = 3$. Also find $\frac{\partial y}{\partial u}$ at $(u,v) = (1,1)$.

b) If the numbers 9 and -3 on the right-hand sides of the equations above are both replaced by 0, show that there is no open set in the $(u,v)$ plane on which the resulting equations define $x$ and $y$ as functions of $u$ and $v$.

4. Let $f$ be a real valued continuous function on [0,1) such that

$$\lim_{x\to 1^-} f(x) = f(0).$$

Prove that f cannot be one-to-one.

Aug 2003 P5
7.20 Rudin
Aug 2003 P 4
A

5. Suppose $f$ is real-valued continuous on [0,1] and

$$\int_0^1 f(x)e^{-\lambda x^2}\,dx = 0, \quad \text{all } \lambda \ge 0.$$

Prove that

Aug 1994 P1)

1) For which real $x$, does $\sum_{n=1}^{\infty} n e^{-nx}$ converges. ?

* When $x<0$: we have $ne^{-nx} \xrightarrow[n\to\infty]{} +\infty$ $\Rightarrow$ the series diverges.

* When $x=0$, we have $\sum n e^{-nx} = \sum_{n=1}^{\infty} n$ diverges.

* When $x>0$, using the Ratio test, we have

$$\alpha = \lim \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)\,e^{nx}}{e^{(n+1)x}\,(n)} = \frac{1}{e^x}.$$

So the series converges when $\alpha < 1 \Leftrightarrow \frac{1}{e^x} < 1 \Leftrightarrow e^x > 1 \Leftrightarrow x > 0$.

diverges when $\alpha > 1 \Leftrightarrow x < 0$

when $x = 0$ from above diverges $\square$.

ng 1994 7 ~~12~~ See Jan 2004 12 (Very similar).

Suppose that $f$ is a differentiable function on $(0,+\infty)$ $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\infty} f'(x) = a$ $\Big\}$ Prove that $a = 0$

**Way 1: Use L'Hospital theorem:**

e have $\lim_{x\to+\infty} x = +\infty$ $\quad x' = 1 \neq 0, \forall x \in (0,+\infty)$

and $\lim_{x\to+\infty} \frac{f'(x)}{x'} = \lim_{x\to+\infty} f'(x) = a$

$f$ is differentiable on $(0,+\infty)$

$\Big\}$ By L'Hospital $a = \lim_{x\to\infty} \frac{f'(x)}{x'} = \lim_{x\to\infty} \frac{f(x)}{x} = 0$ $\square$

**Way 2: Use definition:** In this way, we prove that $\exists \lim_{x\to\infty} f'(x)$ and limit equals ⓪

We have $\lim_{x\to\infty} \frac{f(x)}{x} = 0 \Leftrightarrow \forall \varepsilon > 0, \exists N$, such that $(\forall x,\ N < x$ [illegible]$)$, $\frac{1}{2}\left|\frac{f(x)}{x}\right| < \varepsilon/2$

$\cancel{\left|\frac{f(2x)}{2x}\right| < \varepsilon}$

and $\cancel{\left|\frac{f(2x)}{2x} - \frac{f(x)}{x}\right| < \varepsilon}$

We have (because $f$ is differentiable on $(0,+\infty)$):

$$\left|\frac{f(2x)}{2x} - \frac{f(x)}{x}\right| = \left|\frac{f(2x) - 2f(x)}{2x}\right| = \left|\frac{f(2x) - f(x)}{2x} - \frac{f(x)}{x}\right| =$$

$$= \left|\frac{f'(\xi)x}{2x} - \frac{f(x)}{x}\right| = \left|\frac{f'(\xi)}{2} - \frac{f(x)}{x}\right| \text{ for some } \xi \in (x, 2x).$$

So we have

$$\frac{1}{2}|f'(\xi)| - \left|\frac{f(x)}{x}\right| < \left|\frac{f'(\xi)}{2} - \frac{f'(x)}{x}\right| < \varepsilon$$

$$\Rightarrow |f'(\xi)| < 2\left[\varepsilon + \underbrace{\left|\frac{f(x)}{x}\right|}_{\xrightarrow{x\to\infty} 0}\right]$$

So when $x \to \infty$, $\lim_{x\to\infty} f'(x) = 0$ $\square$

o we have

$$\frac{1}{2}\left|\frac{f(x)}{x}\right| < \frac{\varepsilon}{2} \Leftrightarrow \left|\frac{f(x)}{2x} - \frac{f(2x)}{2x} + \frac{f(2x)}{2x}\right| < \frac{\varepsilon}{2}$$

$$\Leftrightarrow \left|\frac{f'(\xi)x}{2x} + \frac{f(2x)}{(2x)}\right| < \varepsilon/2$$

$$\Rightarrow \left|\frac{f'(\xi)}{2}\right| - \left|\frac{f(2x)}{2x}\right| < \left|\frac{f'(\xi)x}{2x} + \frac{f(2x)}{2x}\right| < \frac{\varepsilon}{2}.$$

$$0 \Rightarrow \leq \left|\frac{f'(\xi)}{2}\right| < \underbrace{\frac{f(2x)}{2x} + \frac{\varepsilon}{2}}_{\longrightarrow 0} \quad \text{so } \lim_{\xi\to\infty} f'(\xi) = 0$$

Aug 1994?

P3) Find $\lim\limits_{n\to\infty} x_n$ when $\begin{cases} a > 0 \\ x_1 = \sqrt{a} \\ x_2 = \sqrt{x_n + a} \end{cases}$

* First, we will prove that the sequence $\{x_n\}$ increasing.

- Base case $x_1 = \sqrt{a}$ $\quad$ we have $x_1^2 = a$
  $x_2 = \sqrt{x_1 + a} = \sqrt{\sqrt{a} + a}$ $\quad$ $x_2^2 = \sqrt{a} + a$ $\Big\} \Rightarrow x_2^2 > x_1^2$
  we have $x_1, x_2 > 0$ $\Big\} \Rightarrow x_2 > x_1$
- Induction hypothesis: $x_n \geq x_{n-1}$
- We want to prove that $x_{n+1} \geq x_n$
  + We have $x_n = \sqrt{x_{n-1} + a}$, by induction hypothesis, we have $\sqrt{x_{n-1} + a} \geq x_{n-1}$ (1)
  + Now consider $x_{n+1} = \sqrt{x_n + a} = \sqrt{\sqrt{x_{n-1} + a} + a}$

  We have $x_{n+1}^2 = \sqrt{x_{n-1} + a} + a$
  $x_n^2 = \sqrt{x_{n+1} + a}$
  by (1): $\sqrt{x_{n-1} + a} \geq x_{n-1}$ $\Bigg\} \Rightarrow x_{n+1}^2 \geq x_n^2$
  we also have $x_n, x_{n+1} \geq 0$ $\Big\} \Rightarrow x_{n+1} \geq x_n$

So by induction, we have $\{x_n\}$ increasing (I)

* Because we have $x_{n+1} \geq x_n, \forall n,$

$$\Leftrightarrow \sqrt{x_n + a} \geq x_n$$

$$\Leftrightarrow x_n + a \geq x_n^2 \quad \Rightarrow x_n^2 \leftarrow x_n - a \leq 0$$

$\Delta = b^2 - 4ac = 1^2 + 4a > 0$ $\quad \Rightarrow$ solution $x_n = \dfrac{1 \pm \sqrt{1 + 4a}}{2}$

so we have $\forall n,$ $\quad 0 < x_n \leq \dfrac{1 + \sqrt{1 + 4a}}{2}$ (II)

From (I) and (II), the sequence $\{x_n\}$ increasing + bounded above $\Rightarrow$ converges.

* we assume that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n-1}$. Assume that $\lim\limits_{n\to\infty} x_n = \alpha$, we have $\alpha$ is a solution of: $\quad \alpha = \sqrt{\alpha + a}$ $\Rightarrow$ similar to above, the solution is $\alpha = \dfrac{1 + \sqrt{1 + 4a}}{2}$

So $\lim\limits_{n\to\infty} x_n = \dfrac{1 + \sqrt{1+4a}}{2}$ □.

ig1994?

17 Prove that if a function $f(x)$ is integrable on $[a,b]$, then $|f(x)|$ is also integrable on $[a,b]$ and $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

We have

is integrable on $[a,b]$

$(x) = |x|$ is a continuous function $\Big\}$ By theorem 6.11 $|f|$ is integrable on $[a,b]$.

Choose $c = \pm 1$ so that $c\int_a^b f(x)\,dx \geq 0$

we have $\left|\int_a^b f(x)\,dx\right| = c\int_a^b f(x)\,dx = \int_a^b \boxed{c\, f(x)\,dx} \leq \int_a^b |f|\,dx$ because $\boxed{c\, f(x) \leq |f(x)|}$ $\square$.

(Or: we can understand that because $f(x) \leq |f(x)|$

$-f(x) \leq |f(x)|$

$\cancel{\left|\int_a^b f(x)\,dx\right| =}$

So we have $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$ ) $\square$.

Theorem 6.11:

$f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f(x) \leq M$

$\phi$ continuous on $[m, M]$

$h(x) = \phi(f(x))$ $\Big\}$ Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

Aug 1994>

P5> Let $f$ be a complex valued function on a set $D$, and $|f(x)| < 1$ for each $x \in D$.

a> Show that the sequence of powers of $f$ $\{f, f^2, f^3, \dots\}$ converges pointwise

ⓑ> Find the necessary and sufficient conditions for the convergence to be uniform.

a> Put $g_n(x) = f^n(x)$. NTP $\{g_n\}$ converges pointwise.

$\Leftrightarrow$ NTP for every $x$ in $D$, $g_n(x) \xrightarrow[\text{point wise}]{n \to \infty}$ in $D$.

We have for each fixed $x$, $|g_n(x)| \geq |g_{n+1}(x)|$ (because $|f(x)| < 1$ and $g_{n+1} = g_n \cdot f < g_n$

and $g_n(x) \geq 0$.

Thus ~~it is a~~ $\{g_n(x)\}$ decreasing and have $0$ as a greatest lower bound $\Rightarrow g_n(x) \longrightarrow 0$

So, the sequence of power of $f$ converges pointwise to $0$.

(b> <u>Find the necessary and sufficient conditions for the convergence to be uniform</u>

We have $f_n(x) \longrightarrow 0$
If $\sup|g_n(x)$ ~~$\leq M$~~ $| \leq M_n$
and $\{M_n\}$ converges

then $g_n(x) \rightrightarrows 0$.

($\Rightarrow$) Prove that $|f_n(x)| \leq M$, where $M < 1$ then $g_n \rightrightarrows 0$

We have $|f(x)| \leq M < 1$
Then $g_n(x) \leq M^n$
we have when $M < 1$, $\{M^n\}$ converges to $0$
$g_n \xrightarrow{\text{ptwise}} 0$

$\Rightarrow g_n \rightrightarrows 0$

($\Leftarrow$): We have $|f(x)| < 1$
$g_n(x) = f^n(x)$, $g_n \rightrightarrows 0$

Prove that $\sup|f(x)| = M$, then $M < 1$
(means $\sup\limits_{x \in D}|f(x)| < 1$

Assume $\sup\limits_{x \in D}|f(x)| = 1 \Leftrightarrow \forall \varepsilon > 0, \exists x \in D$, $|f(x)| > 1 - \varepsilon > 0$
$|f^n(x)| > (1-\varepsilon)^n > 0$

(Note: $a < b$ then $a^2 < b^2$

Choose $\delta = (1-\varepsilon)^n$, this means
$\exists \delta, \forall n$ large, $\exists x \in D$, $|f^n(x)| > \delta$

$\Rightarrow f^n(x) \not\rightrightarrows 0$ contradiction

In conclusion, $f^n \rightrightarrows 0$ iff $\sup\limits_{x \in D} f(x) < 1$.

* Note that we may think about a theorem:
$D$ compact
$g_n \longrightarrow g$
$g_n$ decreasing (satisfies)
$g_n$ (continuous), $g$ (continuous)

then $g_n \rightrightarrows g$

But in here, we only use the above (blue) criterion. □

g19947 26

t $K(x,y)$ be continuous on the rectangle $[a,b]\times[c,d]\subseteq\mathbb{R}^2$.

or integrable function $f$ on $[c,d]$ define an operator $T$:

$$(Tf)(x) = \int_c^d K(x,y)\, f(y)\, dy$$

a) Show that $(Tf)(x)$ is a continuous function on $[a,b]$.

b) Show that $S=\{Tf \mid \int_c^d |f(x)|dx \leq 1\}$ is an equicontinuous family of functions on $[a,b]$.

---

Note that $K$ is continuous on $[a,b]\times[c,d]$ ⇒ $K$ is uniformly continuous on $[a,b]\times[c,d]$.

$\forall \varepsilon>0, \exists \delta>0, \forall x, x' \in [a,b], |x-x'|<\delta$, then $|K(x,y)-K(x',y)|<\varepsilon$

we have

$$|Tf(x)-Tf(x')| = \left|\int_c^d K(x,y) f(y)\, dy - \int_c^d K(x',y)\, f(y)\, dy\right| = \left|\int_c^d [K(x,y)-K(x',y)] f(y)\, dy\right|$$

$$\leq \int_c^d |K(x,y)-K(x',y)|\, |f(y)|\, dy.$$

because $f$ is integrable in $[c,d]$, $\int_c^d |f(y)|\, dy < n$

Then $|Tf(x) - Tf(x')| \leqslant \varepsilon \int_c^d |f(y)|dy \leqslant n\varepsilon$

So $(Tf)$ is a continuous function on $[a,b]$.

From above $\forall \varepsilon>0, \exists \delta>0, \forall x, x' \in [a,b], |x-x'|<\delta, |Tf(x)-Tf(x')|<\varepsilon$

⇒ for all $Tf$

So we have $S=\{Tf \mid \cdots\}$ is $\cdots$ □.

Aug 1994 ✓Checked

7) $U = \{(u,v) \in \mathbb{R}^2, u > 0\}$

$F : U \longrightarrow \mathbb{R}^2$

$(u,v) \longmapsto F(u,v) = (x,y) = (u\cos v, u\sin v)$

a) Show that F is an open mapping

b) Find $du/dx$, $du/dy$, $dv/dx$, $dv/dy$

Proof: by theorem 9.25, F is an open mapping iff $\begin{cases} F \text{ is a } C^1 \text{ mapping from open } U \to \\ F'(u,v) \text{ invertible } \forall \underset{(u,v)}{x} \in U \end{cases}$

- We already know $\left| \begin{array}{l} F \text{ is } C \text{ mapping} \\ U \text{ is open} \end{array} \right.$ (1)
- Now we need to prove $F'(u,v)$ invertible $\forall x \in U$

(⇔) NTP $\det F'(u,v) \neq 0$, $\forall (u,v) \in U$. (2) F is $C^1$ because all

$$\det F'(u,v) = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix} = u\cos^2 v + u\sin^2 v = u > 0$$

(1)+(2) ⇒ F is ~~is~~ an open mapping

b) We use implicit theorem

Let g be the inverse function of F

then $$g' = (F')^{-1} = \frac{1}{u}\begin{pmatrix} u\cos v & u\sin v \\ -\sin v & \cos v \end{pmatrix} = \begin{pmatrix} u\cos v & \sin v \\ -\frac{\sin v}{u} & \frac{\cos v}{u} \end{pmatrix}$$

then $\frac{du}{dx} = u\cos v$ $\quad \frac{du}{dy} = \sin v$ $\quad \frac{dv}{dx} = -\frac{\sin v}{u}$ $\quad \frac{dv}{dy} = \frac{\cos v}{u}$

ug 1994 728.

Let $f(x,y) = x^2 + y^2 - 5$ be a function on $\mathbb{R}^2$ .

a) Describe thoroughly

Aug 1997

P1) Let $K \subseteq \mathbb{R}^n$ be a compact set, and let $\varepsilon > 0$.

Set $J = \{x \in \mathbb{R}^n, \text{dist}(x, K) < \varepsilon\}$, where $\text{dist}(x, K) = \inf\{\|x-y\|_2, y \in K\}$.
and $\|t\|_2$ is usual norm in $\mathbb{R}^2$.

Prove that J is compact.

* We have K is compact in $\mathbb{R}^n$
So we have K is closed and bounded in $\mathbb{R}^n$.

We want to prove that J is compact in $\mathbb{R}^n$.
$\Leftrightarrow$ We need to prove J is closed and bounded in $\mathbb{R}$

* We first prove that J is bounded in $\mathbb{R}^n$.

- Note that K is bounded in $\mathbb{R}^n \iff \exists a \in \mathbb{R}^n, \exists r > 0, \quad K \subseteq B(a, r)$.

Then $\forall x \in J$, $d(x,a) \leq d(x,y) + d(y,a)$, $\quad \boxed{\forall y \in K}$

$$\leq d(x,y) + r$$

Note that this inequality true for all $y \in K$, so

$$d(x,a) \leq \inf_{y\in K}\{d(x,y)\} + r.$$

$$\Rightarrow d(x,a) \leq \varepsilon + r \quad \Rightarrow \quad J \subseteq B(a, \varepsilon + r) \Rightarrow J \text{ is bounded.}$$

* Second, we will prove that J is closed in $\mathbb{R}^n$.

We need to prove that for $x \in \mathbb{R}^n$, and $\exists (x_n) \subseteq J$, $x_n \to x$, then $x \in J$.

- We have because $x_n \to x$, then $\forall \delta > 0, \exists N \in \mathbb{N}, \forall n \geq N_0, \|x_{n_0} - x\| < \delta$

we have $\forall y \in K$, $|x - y| \leq |x - x_n| + |x_n - y|$

take inf over y, $\inf_{y\in K}\|x-y\|\} \leq \delta + \inf_{y\in K}\|x_n - y\| = \delta + \varepsilon$.

then $d(x, K) \leq \delta + \varepsilon$.

Since $\delta$ is arbitrary small $d(x,K) \leq \varepsilon \Rightarrow x \in J \Rightarrow$ J closed ☒.

* In conclusion, because J closed + bounded in $\mathbb{R}^n \Rightarrow$ compact □.

* Note $a_n \leq b_n$, $\forall n$
then $\inf_n a_n \leq \inf_n b_n$

$d(x,y) \leq d(a,y)$
$\Rightarrow \inf_y d(x,y) \leq \inf_y d(a,y)$ for $x, a$ fixed.

ug1997, P3.

Determine whether or not $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $I$, where $u_n(x)$ and $I$ are given in part (a) and (b) below.

) $I=\mathbb{R}$, $\quad u_n(x)=\begin{cases} 0 & , x \le n \text{ or } |x| > n+1 . \\ n \sin\left(\frac{1}{n^2}\right) & , n < |x| < n+1 . \end{cases}$

Aug. 1997

(6) For a nonnegative integer $\ell$, Let $P_\ell(x) = \sum_{k=0}^{\ell} a_k x^k$ $a_k \in \mathbb{R}$ $x \in [-1,$

n $n \in \mathbb{N}$, Let $F(n) = \{ P_\ell(x) \mid 0 \le \ell \le n \}$ and $|a_k| < 1$ $k \in \{0,$

Prove for each $n$, $F(n)$ is equicontinuous.

proof: By defn, $F(n)$ equicontinuous if $\forall \varepsilon > 0$ $\exists \delta$ s.t.

$|f_n(x) - f_n(y)| < \varepsilon$ when $|x-y| < \delta$ $f_n \in F(n)$

So let $f \in F(n) \Rightarrow f$ a polynomial of deg $\le n$ on $[-1, 1]$, a compact set $\Rightarrow f$ u.cont.

~~so choose~~

Also note $x^k$ un. cts. on $[-1, 1]$ so choose $\delta_k$ s.t.

$|x-y| < \delta_k \Rightarrow |x^k - y^k| < \frac{\varepsilon}{n}$. so let $\delta = \min \{\delta_k\}_{k=1}^{n}$ and $|x-y|<$

$$\text{Then } |f(x) - f(y)| = |a_n x^n + \cdots + a_0 - a_n y^n - \cdots - a_0|$$

$$= |a_n (x^n - y^n) + \cdots + a_1 (x-y)|$$

$$\le |a_n||x^n - y^n| + \cdots + |a_1||x-y|$$

$$\le |x^n - y^n| + \cdots + |x-y|$$

note $|x-y| < \delta \Rightarrow |x^k - y^k| < \varepsilon/n$ $\forall$ $k \in \{1, \ldots, n\}$

so $\le \varepsilon/n + \cdots + \varepsilon/n = n(\varepsilon/n) = \varepsilon$.

Thus $F(n)$ is equicontinuous. //

Aug 1997:

2) Determine the convergence or divergence of the following sequence $\{x_n\}_{n=1}^{\infty}$

a) $x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \dots + \frac{n}{n^2+n}$

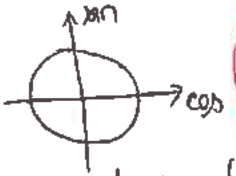
We have $x_n = \sum_{k=1}^{n} a_k$ where $a_k \geq 0, \forall k$. $\Rightarrow$ $(x_n)$ is an increasing sequence.

we want to prove that $(x_n)$ is bounded, $\forall n$.

We have $x_n = \sum_{k=1}^{n} \frac{k}{n^2+k} \leq \sum_{k=1}^{n} \frac{k}{n^2+1} \leq \sum_{k=1}^{n} \frac{n}{n^2+1} = \frac{n^2}{n^2+1} \leq 1$. $\Rightarrow x_n$ bounded.

$\Rightarrow$ $(x_n)$ converges.

b) $\boxed{x_n = \left(-\frac{1}{2}\right)^n + \sin\left(\frac{n\pi}{2}\right)}$



$\left.\begin{array}{l}\left(-\frac{1}{2}\right)^n \text{ makes } x_n \text{ alternate} \\ \sin\left(\frac{n\pi}{2}\right) \text{ bounded}\end{array}\right\} \Rightarrow$ diverges

$x_1 = \left(-\frac{1}{2}\right) + \sin\left(\frac{\pi}{2}\right) = -\frac{1}{2} + 1 = \frac{1}{2}$

$x_2 = \left(-\frac{1}{2}\right)^2 + \sin\left(\frac{2\pi}{2}\right) = \frac{1}{2} + 0 = \frac{1}{2}$

$x_3 = \left(-\frac{1}{2}\right)^3 + \sin\left(\frac{3\pi}{2}\right) = -\frac{1}{2} - 1 = -\frac{3}{2}$

$x_4 = \left(-\frac{1}{2}\right)^4 + \sin\left(\frac{4\pi}{2}\right) = \frac{1}{2} + 0 = \frac{1}{2}$

$x_5 = \left(-\frac{1}{2}\right)^5 + \sin\left(\frac{5\pi}{2}\right) = -\frac{1}{2} + \sin\left(\frac{\pi}{2}\right) = x_1$

$x_6 = \left(-\frac{1}{2}\right)^6 + \sin\left(\frac{6\pi}{2}\right) = \frac{1}{2} + \sin\left(\frac{2\pi}{2}\right) = x_2$

$x_7 = \left(-\frac{1}{2}\right)^7 + \sin\left(\frac{7\pi}{2}\right) = -\frac{1}{2} + \sin\left(2\pi + \frac{3\pi}{2}\right) = x_3 = -$

Then we have
$$x_n = \begin{cases} -\frac{3}{2} & n = 3+4k, \ k = 0,1,2,\dots \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

then we have $(x_n)$ has subsequence $x_{3+4k} \longrightarrow -\frac{3}{2}$ $\Rightarrow$ $(x_n)$ diverges.

$x_{2n} \longrightarrow \frac{1}{2}$

c) $\boxed{x_n = \frac{n^n + (-n)^n}{2} + \left(1 + \frac{1}{2n}\right)^n}$

- We have $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
  $\left(1+\frac{1}{2n}\right)^n \leq \left(1+\frac{1}{n}\right)^n \Rightarrow \left\{\left(1+\frac{1}{2n}\right)^n\right\}$ converges

- When n is even we have $\frac{n^n+(-n)^n}{2} = \frac{2n^n}{2} = n^n \longrightarrow \infty$

$\Rightarrow \frac{n^n+(-n)^n}{2}$ diverges.

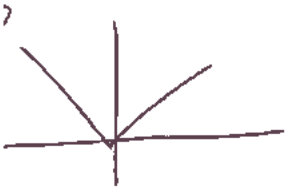
($\{a_n\}$ converges to p $\Leftrightarrow$ every subsequence converges to p
$\Rightarrow$ If $\exists$ a divergent subsequence $\Rightarrow$ diverges.

$x_n = a_n + b_n$ $\Rightarrow$ $x_n$ diverges.
$a_n$ ↓ diverges, $b_n$ ↓ converges

The sequence $\{x_n\}$ diverges since subsequence $\{x_{2n}\}$ diverges

Let $D^+$ and $D^-$ denote the operation of taking derivatives of real functions from the right and left respectively, $D^+f(x) = \lim_{y\to x^+} \frac{f(y)-f(x)}{y-x}$ and $D^-f(x) = \lim_{y\to x^-} \frac{f(y)-f(x)}{y-x}$

Give an example of a function for which $D^+f(0)$ & $D^-f(0)$ exist but are not equal



Let $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

Then $D^+f(0) = \lim_{y\to 0^+} \frac{f(y)-f(0)}{y-0} = \lim_{y\to 0} \frac{y-0}{y} = +1$.

$D^-f(0) = \lim_{y\to 0^-} \frac{f(y)-f(0)}{y-0} = \lim_{y\to 0} \frac{-y-0}{y} = -1$.

both exist but are not equal.

Prove or disprove: If $D^+f(0)$, $D^-f(0)$ both exist ~~but~~, then the function $f$ is continuous at $x=0$

If $\lim_{x\to p} f'(x)$ exists $\Rightarrow \exists f'(p)$ $\Rightarrow f$ continuous at $p$.

$\left(\lim_{x\to p^+} f'(x) = \lim_{x\to p^-} f'(x)\right)$ (and $f'(p) = \lim_{x\to p^+} f'(x) = \lim_{x\to p^-} f'(x)$)

In case $\exists D^+f(0)$, $\exists D^-f(0)$ but $D^+f(0) \neq D^-f(0)$ $\Rightarrow \nexists f'(p)$. $\boxed{\text{but } f \text{ still continuous}}$

5 Aug 1997, 5

Suppose $f(x) = x$ $g(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases}$

a) Evaluate $\int_0^1 f\,dg$

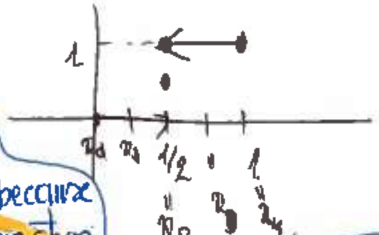
* Remind : $f$ continuous on $[a,b]$
$\alpha$ monotone on $[a,b]$
$\Rightarrow f \in \mathcal{R}(\alpha)$ .

Need to n

$\boxed{\text{Theorem } 6.15}$

~~$g'(x) = \{0$~~ We have g is a step function .
$f$ is continuous on $[0,1]$

~~Then $\int_0^1 f\,dg = f(\frac{1}{2}) = \frac{1}{2}$~~



We can not apply theorem 6.15 here because $g$ is not a step function $\searrow$ we have to use partition .

* Another way : ~~Prove~~ Similar the Theorem 6.15 .

We define a partition $P = \{x_0, x_1, x_2, \dots x_{k-1} = \frac{1}{2} - \delta, x_k = \frac{1}{2}, x_{k+1} = \frac{1}{2} + \delta, \dots, x_n = 1\}$ .

( we note that we devide $[0,1]$ to n equal part $x_0 = 0, x_1 = \delta, \dots, x_n = n\delta$

where $\delta = \frac{1}{n}$ .

Then we have

$$U(P,f,g) = \sum_{i=1}^{n} M_i (g_i - g_{i-1}) = f(\tfrac{1}{2}) \underbrace{\Big[\underbrace{g(\tfrac{1}{2})}_{\frac{1}{2}} - \underbrace{g(\tfrac{1}{2} - \delta)}_{=0}\Big]}_{\frac{1}{2}} + f(\tfrac{1}{2} + \delta) \underbrace{\Big[\underbrace{g(\tfrac{1}{2} + \delta)}_{1} - g}_{=\frac{1}{2}}$$

$$= \frac{1}{2}\left[f(\tfrac{1}{2}) + f(\tfrac{1}{2} + \delta)\right] = \frac{1}{2}(1 + \delta)$$

~~$L(P,f,g) =$~~

- We have $f$ is continuous on $[0,1]$
$g$ is an increasing function on $[0,1]$ $\Big\} \Rightarrow f \in \mathcal{R}(g)$ .

then $\int_0^1 f\,dg = \inf_P \big(U(P,f,g)\big) = \frac{1}{2}\left[\cancel{\frac{1}{2} + \frac{1}{2} + \delta}\right]$

This result = the res if $g(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x < \end{cases}$ becaus $\frac{1}{2}$ is in the mid and $g(\frac{1}{2}) = \frac{1}{2} = \frac{1+0}{2}$

- $L(P,f,g) = \sum m_i [g_i - g_{i-1}] = f(\frac{1}{2} - \delta)\left[g(\frac{1}{2}) - g(\frac{1}{2} - \varepsilon)\right] + f(\frac{1}{2})\left[g(\frac{1}{2} + \delta) - g(\frac{1}{2}\right.$

$= \frac{1}{2}\left[f(\frac{1}{2} - \delta] + f(\frac{1}{2})\right]$

$= \frac{1}{2}\left(\frac{1}{2} - \delta + \frac{1}{2}\right) = \frac{1}{2}(1 - \delta)$ .

Then $U(P,f,g) - L(P,f,g) = \delta$

$f \in R(\alpha)$ when $\delta \longrightarrow 0$ , and so $\int_0^{1} f\,dg = \frac{1}{2}$ .

$\boxed{\text{b) Evaluate } \int_0^1 g\,df}$

$f' = 1 \in \mathcal{R}$ on $[0,1]$ and we have $g \in \mathcal{R}(f) \Leftrightarrow g\alpha' \in \mathcal{R}$

$$\int_0^1 g\,df = \int_0^1 g\,f'dx = \int_0^1 g\,dx = \int_0^{1/2} g\,dx + \int_{1/2}^{1} g\,dx = \int_{1/2}^{1} 1\,dx = 1 - \frac{1}{2} = \frac{1}{2} .$$

Aug 1997 / 7

Let $f(x,y) = |x|^{1/2}|y|^{1/2} + xy$ be a real function on $\mathbb{R}^2$

ⓐ Find the partial derivative of $f$ at the origin. → In the formula of $f$ has $|\;|$ → could not compute normally

ⓑ Discuss the differentiability of $f$ at the origin. — Similar Aug 2008

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a) Find the partial derivative of $f$ at the origin.

$$D_x f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$D_y f(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

b) Discuss the differentiability of $f$ at the origin.

$$\underset{\text{at } x=(x,y)}{f \text{ is differentiable}} \overset{\text{def}}{\iff} \exists A = Df \text{ s.t. } \lim_{\substack{h\to 0 \\ h\in\mathbb{R}^2}} \frac{\| f(x+h) - f(x) - A\,D_f(x) \|}{\|h\|} = 0$$

Then consider (let $x = (0,0)$ $h = (x,y)$)

$$\lim_{(x,y)\to 0} \frac{\| f(x,y) - f(0,0) - D_x f(0,0)\, x - D_y f(0,0) \cdot y \|_{\mathbb{R}^1}}{\|(x,y)\|_{\mathbb{R}^2}}$$

$$= \lim_{(x,y)\to 0} \frac{\| |x|^{1/2}|y|^{1/2} + xy \|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to 0} \frac{\left| |x|^{1/2}|y|^{1/2} + xy \right|}{\sqrt{x^2+y^2}} =$$

Take $x = y$, then we have

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{|x| + x^2}{\sqrt{2x^2}} = \lim_{x\to 0} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x = \frac{1}{\sqrt{2}} \neq 0$$

So we have $f$ is not differentiable at the origin □

ıg 1997/8

need to remember.

note: in here $(x, y, z)$ $\downarrow \downarrow \downarrow$ are functions of $\lambda, \theta, \varphi$

$\Rightarrow$ inverse function theorem (not implicit funct T)

et $\begin{cases} x = \lambda\cos(\theta)\sin(\varphi) \\ y = \lambda\sin(\theta)\sin(\varphi) \\ z = \lambda\cos\varphi \end{cases}$ Define the map $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $(\lambda, \theta, \varphi) \longmapsto (x, y, z)$

? Prove or disprove, F has a global inverse on $\mathbb{R}^3$.

? Find $\frac{\partial\theta}{\partial x}(0, 1, 0)$. note that the formula is $g'(\vec{y}) = [f'(\vec{x})]^{-1}$

~~$\Rightarrow (0,1,0)$ is $(x,y,z)$ not $(\lambda, \theta, \varphi)$~~

Prove or disprove, F has a global inverse on $\mathbb{R}^3$.

F has a global inverse on $\mathbb{R}^3$ $\overset{def}{\Longleftrightarrow}$ F is global surjective $\left(\begin{cases} \text{injective} \\ \text{surjective} \end{cases}\right)$

In this case F is not a injection because sin and cos are periodic.

$\Rightarrow$ F is not a global homomorphism

$\Rightarrow$ F does not have a global inverse on $\mathbb{R}^3$.

? Find $\frac{\partial\theta}{\partial x}(0, 1, 0)$

~~First, we have at $(\lambda_0, \theta_0, \varphi_0) = (0, 1, 0)$, $F(x_0, y_0, z_0) = F(\lambda_0, \theta_0, \varphi_0) = (0, 0, 0)$~~.

$$DF = \begin{bmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \lambda} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\varphi & -\lambda\sin\theta\sin\varphi & \lambda\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \lambda\cos\theta\sin\varphi & \lambda\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\lambda\sin\varphi \end{bmatrix}$$

. note that at $(0, 1, 0)$ $\begin{cases} \overset{\neq 0}{\lambda}\cos\theta \overset{\neq 0}{\sin\varphi} = 0 & \Rightarrow \cos\theta = 0 \\ \lambda\sin\theta\sin\varphi = 1 & \Rightarrow \sin\varphi \neq 0 \text{ and } \lambda \neq 0 \text{ and } \sin\theta \neq 0. \\ \lambda\cos\varphi = 0 & \Rightarrow \cos\varphi = 0 \end{cases}$

not done

$$DF = \begin{bmatrix} 0 & -1 & 0 \\ 1/\lambda & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \det DF = -\frac{1}{\lambda}$$

$$\Rightarrow DF^{-1} = -\frac{1}{\lambda}\begin{bmatrix} 0 & +1/\lambda & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}^t = -\frac{1}{\lambda}\begin{bmatrix} 0 & -1 & 0 \\ 1/\lambda & 0 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}.$$

$\partial\theta$ $\partial\varphi$

Aug 1998

(P1) Construct an open set containing every rational numbers, but not every real number. What can be ~~we~~ said about the closure of any such set.

- Let $A = \mathbb{R} \setminus \sqrt{2}$

So it's obvious that $\mathbb{Q} \subseteq A$ and $\sqrt{2} \notin A$.

- What can be said about the closure of any such set.

we have $\bar{\mathbb{Q}} = \mathbb{R}$ so we have $\bar{\mathbb{Q}} \subseteq A$ and this means $\mathbb{R} \subseteq \bar{A} \Rightarrow \bar{A} = \mathbb{R}$

it's not required to consider, but we have an interesting fact that $\left( \begin{array}{l} \mathbb{Q} \not\subset A \text{ (because } \sqrt{2} \in A, \sqrt{2} \notin \mathbb{Q}) \\ A \not\subset \mathbb{R} \text{ because } (\sqrt{2} \notin A) \end{array} \right)$

Aug 1998

P2 → Prove the inequality $py^{p-1}(x-y) \leq x^p - y^p \leq px^{p-1}(x-y)$.
where $x, y \in \mathbb{R}$, $0 < y < x$; $p \in \mathbb{R}$, $1 \leq p < +\infty$ ← note this.

sees sth $(x-y)$ ⟹ think about MVT

* Consider $f(x) = x^p$ when $x > 0$ and $p \geq 1$

this is a continuous, differentiable function on $x > 0$, $+\infty > p \geq 1$

so by mean value theorem: $f(x) - f(y) = x^p - y^p = f'(\xi)(x-y)$ for some $\xi \in (x, y)$

$= p\xi^{p-1}(x-y)$

but notice that $g(x) = x^{p-1}$, is a ~~cont~~ increasing function $\Rightarrow 0 < y^{p-1} < \xi^{p-1} < x^{p-1}$
$(x > 0,\ 1 \leq p < +\infty)$

so we have $py^{p-1}(x-y) < p(\xi^{p-1}(x-y) < px^{p-1}(x-y)$ □

g1998 ——— * ——— *

7 Let $F(x,y,u,v) = 3x^2 - y^2 + u^2 + 4uv + v^2$

$G(x,y,u,v) = x^2 - y^2 + 2uv$

Show that the equation $F(x,y,u,v) = 9$ determine x and y as a functions of
$G(x,y,u,v) = -3$ u, v in a neighborhood of $u = 1, v = 1$

ith $x(1,1) = 2$ $y(1,1) = 3$

lso, find $\frac{\partial y}{\partial u}$ at $(u,v) = (1,1)$.

If the number 9 and -3 on the RHS of the equations above are both replaced by 0, ow that there is no open set in the (u, v) plan on which the resulting equation define nd y as functions of u and v.

---

nsider $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$

$$(x,y,u,v) \longmapsto f(x,y,u,v) = (\overset{f_1}{\overbrace{F(x,y,u,v) - 9}}, \overset{f_2}{G(x,y,u,v) + 3})$$

consider at $(2,3,1,1)$.

we have

$$Df = \begin{array}{c|cccc} & x & y & u & v \\ F & 6x & -2y & 2u+4v & 4u+2v \\ G & 2x & -2y & 2v & 2u \end{array}_{2\times 4}$$

we have all $D_j f_i$ exist and continuous $\Rightarrow$ f is a $C^1$ function.

$$A_x = \begin{bmatrix} 6x & -2y \\ 2x & -2y \end{bmatrix}_{(2,3,1,1)} = \begin{bmatrix} 12 & -6 \\ 4 & -6 \end{bmatrix} \qquad \det A_x = -12 \neq 0$$

$$A_y = \begin{bmatrix} 6 & 6 \\ 2 & 2 \end{bmatrix}$$

$f(x,y,u,v) = (0,0)$

hen by implicit the function theorem,
here is an open neighborhood $V \subset \mathbb{R}^4$ of $(2,3,1,1)$ and an open neighborhood W of $(1,1)$
such that $\forall (u,v) \in W$, $\exists! (x,y)$ s.t $\begin{cases} (x,y,u,v) \in V \\ f(x,y,u,v) = 0 \end{cases}$

So we can have $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ $(x,y) = g(u,v)$

where $g'(u,v)\Big|_{(u,v)=(1,1)} = -[A_x]^{-1}[A_y] = -\frac{1}{96}\begin{bmatrix} -6 & 6 \\ -4 & 12 \end{bmatrix}\begin{bmatrix} 6 & 6 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$

So we have $\frac{\partial y}{\partial u}\Big|_{(u,v)=(1,1)} = [0 \quad 0]$.

Aug1998, P3, b.

b) when 9 and 3 are replaced by 0,

we want to prove that there is no ~~open~~ open set in the $(u, v)$ plan on which $(x, y)$ can be defined as a function of $(u, v)$

$\Rightarrow$ we want to prove that for each $(u, v)$ in a open subset of $\mathbb{R}^2$,

$\left[\begin{array}{l}\text{there is no } (x, y) \\ \text{or there are more than a value of } (x, y)\end{array}\right.$ such that $\{ f(x, y, u, v) = 0$.

Now consider $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$

$(x, y, u, v) \longmapsto (f_1 = F, f_2 = G)$.

Then the equation $\begin{cases} F = 0 \iff 3x^2 - y^2 + u^2 + 4uv + v^2 = 0 \\ G = 0 \iff x^2 - y^2 + 2uv = 0 \end{cases}$

$$F - G \iff 2x^2 + (u + v)^2 = 0.$$

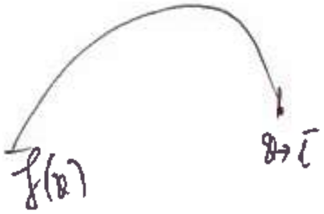
So the above system of equations can only be hold ~~when~~ at $(0,0,0,0)$;

In case $(x, y, u, v) \neq (0,0)$ $\boxed{2x^2 = -(u+v)^2}$ $\Rightarrow$ can't find $x \neq 0$ satisfies th

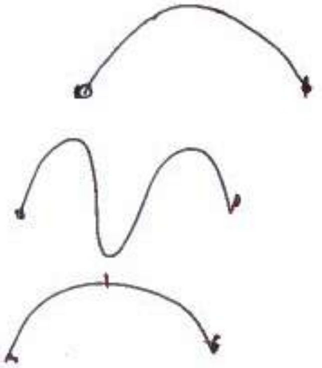
$\Rightarrow$ no open set in the $(u, v)$ plan on which $(x, y)$ can be defined as a functio of $(u, v)$ $\square$.

is real continuous
is real continuous on.

f is real continuous on [0, 1]?



case s e $x_0 \in [0,1]$



* Problem 4 > Aug 1998 >.

Let $f$ be real-valued function function on $[0,1)$ such that
$$\lim_{x \to 1^-} f(x) = f(0).$$
Prove that $f$ can not be one-to-one.

* **Case 1**: When $f(x)=f(0)$ for all $x \in [0,1)$ so we have $f$ is a constant function, thus can not be a 1-1 function.

* **Case 2**: When $\exists x_0 \in [0,1)$ such that $f(x_0) \neq f(0)$

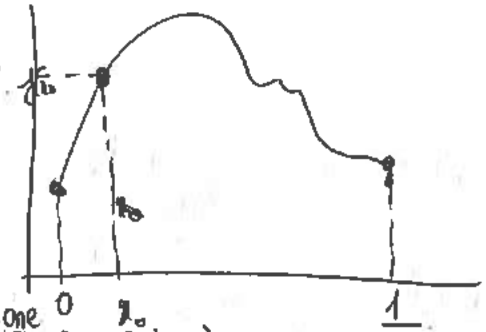
Then we consider $\frac{f(x_0)+f(0)}{2}$ (~~is~~ wlog assume $f(x_0) > f(0)$)

$f(x_0)$ then $> \frac{f(x_0)+f(0)}{2} > f(0)$

Then by IMT, we have the value $\frac{f(x_0)+f(0)}{2}$
because $f$ is continuous

is attained by at least one $x$ in $(0, x_0)$
and by at least one $y$ in $(x_0, 1)$.

$$\Rightarrow f(x) = f(y) = \frac{f(x_0)+f(0)}{2} \Rightarrow f \text{ is not one to one} \square$$



Aug 1998, P5.

Suppose that $f$ is real value, continuous on $[0,1]$. $\int_0^1 f(x) e^{-\lambda x^2} dx = 0$, for all $\lambda \geq 0$ $\Big\}$ Prove that $f \equiv 0$ on $[0,1]$.

* Now we first consider~~ed~~ $\{P_n\}$, with $\{P_n(x) = \sum_{k=0}^{n} c_k e^{-kx^2}$ ; $n \in \mathbb{N}$ ; $c_k \in \mathbb{R}, \forall k = \overline{1,n}$.

we will prove that $\{P_n\}$ is an algebra, separates points and vanishes at no point.

- Prove that $\{P_n\}$ is an algebra.

+ Consider $P_n$ and $P_m$, wlog, assume $m > n$, $P_n(x) = \sum_{k=0}^{n} c_k e^{-kx^2}$

$$P_m(x) = \sum_{\ell=0}^{m} b_\ell e^{-\ell x^2}$$

$$P_n(x) + P_m(x) = \sum_{\ell=0}^{m} (c_\ell + b_\ell) e^{-\ell x^2} \in \mathcal{A}, \text{ where } c_\ell = 0 \ \forall \ell = \overline{(m-n), m}$$

+ $P_n(x) \cdot P_m(x) = \sum_{\substack{k=\overline{0,n} \\ \ell=\overline{1,m}}} c_k b_\ell e^{-(k+\ell)x^2} \in \mathcal{A}$

+ $c P_n(x) = \sum_{k=1}^{n} (c \cdot c_k) e^{-kx^2} \in \mathcal{A}$.

- Prove that $\mathcal{A} = \{P_n\}$ vanishes at no point $\boxed{\text{on } [0,1]}$

NTP that $\forall x \in [0,1]$, $\exists P_n \in \mathcal{A}$, $P_n(x) \neq 0$
$\forall x \in [0,1]$, we choose $P_n(x) = e^{-x^2} \neq 0$, $\forall x \in [0,1]$.

- Prove that $\mathcal{A} = \{P_n\}$ distinguishes points.

$\forall x, y \in [0,1]$, $x \neq y$ need to prove $\exists P_n \in \mathcal{A}$, $P_n(x) \neq P_n(y)$

we choose $P_n(x) = e^{-x^2}$, then we have for $x \neq y$ $e^{-x^2} \neq e^{-y^2}$ → done.

So we have $\mathcal{A} = \{P_n(x)\}$ is an algebra ~~m~~ distinguishes points, vanishes at no points $\boxed{\text{of continuous functions}}$ $\Big\}$
$[0,1]$ compact

⇒ by Stone Weierstrass theorem

→ $P_n(x) \rightrightarrows f$, when $f$ is continuous on $[0,1]$.

* So now we have $P_n(x) \rightrightarrows f$ $\Big\}$ $P_n(x) \cdot f \rightrightarrows f^2$ on $[0,1]$
we also have $P_n(x)$ $\boxed{\text{~~sequence~~ sequence of bounded functions}}$
$f$ bounded

* So we have

$$\int f^2(x) dx = \lim \int P_n(x) \cdot f(x) = \lim \int \left(\sum c_k e^{-kx^2} f(x) dx\right) = \lim \sum c_k \underbrace{\int_0^1 e^{-kx^2} f(x) dx}_{=0}$$

- So $\int f^2(x) dx = 0$
we have $f^2 \geq 0$ $\Big\}$ ⇒ $f \equiv 0$ on $[0,1]$ ☐.

chec

# Preliminary Exam

## 21 August 1999

1. Use $e = 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots$ to prove that $e$ is irrational.

2. Let $a_n, b_n \geq 0$, assume that $\sum a_n$ converges and that $\limsup_{n\to\infty} \dfrac{b_n}{a_n} \leq M < \infty$ show that $\sum b_n$ converges.

See Aug2006 ℓ3.
Sample C ℓ5 (template)

3. Let $f$ be bounded on the real interval $(a,b)$, show that if in addition $f$ is both continuous and monotone then $f$ is uniformly continuous.

4. Define $f(x) = \begin{cases} 0 & , x \text{ irrational} \\ \frac{1}{n} & , x = m/n \text{ where } m \text{ and } n \text{ relatively prime} \end{cases}$. Prove that $f$ is integrable on $[0,1]$.

5. Let $\{f_n\}$ be a sequence of uniformly bounded Riemann integrable functions on $[0,1]$, set $F_n(s) = \int_0^s f_n(t)dt$ for $0 \leq s \leq 1$. Prove that a subsequence of $\{F_n\}$ converges uniformly on $[0,1]$.

Important
Same Fall2001 ℓ2

6. Let $f(x)$ be a differentiable mapping of the connected open subset $V$ of $\mathbf{R}^n$. Suppose that $f'(x) = 0$ on $V$, prove that $f$ is constant on $V$.

Same Important
Fall2001

7. Let $f(x,y) = (u,v)$ where $u = x^2 - y^2$ and $v = 2xy$ describe a map from $\mathbf{R}^2$ to $\mathbf{R}^2$. (a) What is the range of this map? (b)Show that if $(u,v) \neq (0,0)$ then $f$ has an inverse in a neighborhood of $(u,v)$. (c) Show that there is no neighborhood of $(0,0)$ in which $f$ has an inverse.

back.

# Analysis Preliminary Examination
## Fall 2001

1. Let $A$ be an uncountable set of real numbers. Prove that $A$ has an accumulation point.

Same Aug 1999 P

2. Let $f(x)$ be a differentiable mapping of the connected open subset $V$ of $\mathbf{R}^n$. Suppose that $f'(x) = 0$ on $V$, prove that $f$ is constant on $V$.

3. Prove or disprove: the function $f(x) = x^{3/2} \log x$ is uniformly continuous on the interval $(0, 1)$.

Same [illegible] 1998

4. Let $f(x, y) = (u, v)$ where $u = x^2 - y^2$ and $v = 2xy$ describe a map from $\mathbf{R}^2$ to $\mathbf{R}^2$.

   (a) What is the range of this map?

   (b) Show that if $(u, v) \neq (0, 0)$ then $f$ has an inverse in a neighborhood of $(u, v)$.

   (c) Show that there is no neighborhood of $(0, 0)$ in which $f$ has an inverse.

5. Prove that

$$\sum_{n=1}^{\infty} \frac{\sin(n^4 x)}{n^2}$$

defines a continuous function on $\mathbf{R}$.

6. (a) Find the limit

$$\lim_{\lambda \to \infty} \lambda \int_{-1}^{1} e^{-\lambda |y|}\, dy.$$

(b) Let $g : \mathbf{R} \to \mathbf{R}$ be a bounded, continuous function. For $x \in \mathbf{R}$, find the limit

$$\lim_{\lambda \to \infty} \lambda \int_{-1}^{1} g(x + y) e^{-\lambda |y|}\, dy.$$

*Hint: Try a "nice" $g$ first, formulate a guess, and then try to prove your guess is correct.*

Aug 1999, P17.

$$\boxed{\text{Use } e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{l=0}^{\infty} \frac{1}{l!} \text{ to prove that } e \text{ is irrational.}}$$

* **Way 1:** use 2 theorems :

**Theorem 1**: Put $lcm := lcm\{q_0, \dots, q_n\}$, $r_n = \sum_{l=n+1}^{\infty} \frac{p_l}{q_l}$ } ~~Ife~~ If $lcm . r_n \xrightarrow{n\to\infty} 0$ then $\sum_{n=0}^{\infty} \boxed{\frac{p_n}{q_n}}$ is irrational

**Theorem 2**: If $\left|\frac{a_{k+1}}{a_k}\right| < b$ for $k \geq n$

Then $|r_n| \leq \boxed{\frac{|a_n|}{1-b}}$

- Now consider $e = \sum_{n=0}^{\infty} \frac{p_n}{q_n} = \sum_{n=0}^{\infty} \frac{1}{n!}$ this means $p_n = 1, \forall n$
  $q_n = n!$

So $lcm := lcm(q_0, \dots, q_n) = lcm\{0!, 1!, \dots, n!\} = n!$

- Now we have $\left|\frac{a_{l+1}}{a_l}\right| = \left|\frac{(l)!}{(l+1)!}\right| = \frac{1}{l+1} \leq \frac{1}{n+2}$. for $l \geq n+1$.

Then we have $r_n \underset{\uparrow \text{ Theorem 2}}{<} \frac{a_{n+1}}{1 - \frac{1}{n+2}} = \frac{\cancel{a_{n+1}}\ \frac{1}{(n+1)!}}{1 - \frac{1}{n+2}} = \frac{\frac{1}{(n+1)!}}{\frac{(n+1)}{n+2}} = \frac{(n+2)}{(n+1)(n+1)}$

So we have $0 \leq lcm . r_n < n! \frac{(n+2)}{(n+1)(n+1)!} = \underbrace{\frac{(n+2)}{(n+1)^2}}_{\xrightarrow[n\to\infty]{} 0}$ ☐ way 1

So $lcm . r_n \xrightarrow[n\to\infty]{} 0$, $\underset{\uparrow \text{ Theorem 1}}{\Longrightarrow}$ e is irrational ☐

---

* **Way 2**.

We have $e = \sum_{l=1}^{\infty} \frac{1}{l!} = \underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}}_{\text{put } := s_n} + \underbrace{\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots}_{\text{put } := r_n .}$

we have $r_n := \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots = \frac{1}{(n+1)!}\left(1 + \frac{1}{(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots\right) \leq \cancel{\frac{1}{(n+1)!}}$

$\leq \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+1)} + \frac{1}{(n+2)^2} + \frac{1}{(n+1)^3} + \cdots\right]$

$= \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)!} \frac{1}{\frac{n}{n+1}} = \frac{1}{n! n}$

Assume that e is rational, which means

$e = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $gcd(p,q) = 1$. , of course $p, q > 0$

din

So we have $e - s_q = r_q < \frac{1}{q! q} \Rightarrow 0 < q! [e - s_q] < \frac{1}{q}$

$\Leftrightarrow \boxed{0 <} \boxed{q! e - q! s_q} < \frac{1}{q} \boxed{< 1}$ — we will prove if $e = \frac{p}{q}$ rational then is nn
is an integer

Aug1999, P2.

Let $a_n, b_n \geq 0$

$\sum a_n$ converges.

$\limsup \frac{b_n}{a_n} \leq M < +\infty$

Show that $\sum b_n$ converges.

We have $\limsup \frac{b_n}{a_n} \leq M \Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon, \forall n \geq N, \frac{b_n}{a_n} < M + \varepsilon$

Choose $\varepsilon = 1$,

then $\exists N_\varepsilon, \forall n \geq N, \frac{b_n}{a_n} \leq (M+1)$.

$\Rightarrow b_n \leq (M+1) a_n, \forall n \geq N$.

we have $(M+1)$ is a constant.

$(M+1) \sum a_n$ converges since $\sum a_n$ converges.

Then by comparison test (note that $a_n, b_n \geq 0$) $\Rightarrow \sum b_n$ converges.

* Prove the limit comparison test

Let $a_n > 0, \forall n$; $b_n \geq 0, \forall n$.

~~$\sum b_n$~~ Let $c = \lim_{n \to \infty} \frac{b_n}{a_n} < +\infty$

$c > 0$ ← important

Prove that then $\sum a_n$ and $\sum b_n$ both converge or diverge.

We have $\lim_{n\to\infty} \frac{b_n}{a_n} = c \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \left| \frac{b_n}{a_n} - c \right| < \varepsilon$

$c - \varepsilon < \frac{b_n}{a_n} < c + \varepsilon$

Note that because $c > 0$, then $\exists m, M > 0$ s.t.

$0 < m < c - \varepsilon < \frac{b_n}{a_n} < c + \varepsilon < M$

so $0 < m < \frac{b_n}{a_n} < M, \forall n \geq N$.

Then it is easy to see that $\sum a_n, \sum b_n$ both converge or diverge

(Note that it is important to have $c > 0$

because if $c = 0$, there is a case when $\sum a_n$ diverges

& $b_n = 0, \forall n$

then $c = 0$

but $\sum b_n$ converges.

ig 1999 → P3 ——— See Aug 2006 P3

Let f is bounded on the real interval $(a,b)$. $\Big\}$ f is uniformly continuous on $(a,b)$.
f is both continuous + monotone $\Big\}$

log assume that f is monotone increasing.

ve first ~~that~~ prove that $\exists f(b^-)$.

have b is a limit point of $(a,b)$, then $\exists \{x_n\} \subset (a,b)$, $x_n \xrightarrow{\text{increas}} b$.

consider $\{f(x_n)\}$, we have this is a increasing sequence (since f is increasing).

by the assumption f is bounded

→ $\{f(x_n)\}$ increasing + bounded ⇒ converges. $\exists \lim_{x_n \to b} f(x_n) = L$

Because f is ~~monotone~~ increasing ⇒ $f(b^-) = \lim_{x_n \to b^-} f(x_n) = \sup_{\substack{x \in (a,b) \\ x<b}} \{f(x), x<b\}$.

note that we need one more step to explain that $\forall \{x_n'\}$, $x_n' \xrightarrow[\text{increas}]{} b$ then $\lim f(x_n') = \lim f(x_n)$ (see Sample P1).

Similarly, $\exists f(a+)$.

Now define $F(x) = \begin{cases} f(a+) & , x = a. \\ f(x) & , x \in (a,b) \\ f(b^-) & , x = b \end{cases}$

e have F is continuous on $[a,b]$, thus uniformly continuous on $[a,b]$.

f is the restriction of F on $(a,b)$ ⇒ f is uniformly continuous on $[a,b]$ □.

Better way next page. →

Aug1999, P4 *

define $f(x) = \begin{cases} 0 & , x \text{ is } \mathbb{R}\setminus\mathbb{Q}. \\ \frac{1}{n} & , \text{if } x = \frac{m}{n} \end{cases}$, where m, n are relative prime.

Prove that f is Riemann integrable on [0, 1].

* A really good observe that we can get when partizing [0, a] to p part when p is pri

is that  we have all $x_l$ in the partition have value $x_l = \frac{l}{p}$, where l and p are relative prime

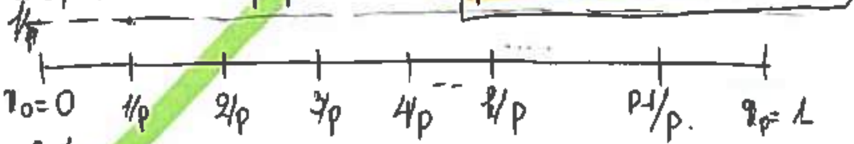
* We want to prove that f is Riemann integrable on [0, 1].

$\Leftrightarrow$ We need to prove that there is a partition $P = \{x_0 = 1 \le x_1 \le \cdots \le x_p = 1\}$ so that $U(P, f) - L(P, f) < \varepsilon$.

- Note that because $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$, $L(P, f) = 0$, $\forall$ partition P

Now it suffices to show that $\exists$ partition P such that $U(P, f) < \varepsilon$

- Assume we divide [0, 1] into p parts with p is a prime number

then we have 

So we have $f(x) = \begin{cases} \frac{1}{p} & , \text{if } x = x_i \\ 0 & , \text{otherwise} \end{cases}$

So $\sum M_i \Delta x_i = \sum_{i=1}^{p} \frac{1}{p} \cdot \frac{1}{p} = p \sum \frac{1}{p^2} = \frac{1}{p}$.

So when we chose p prime such that $\frac{1}{p} < \varepsilon \Leftrightarrow p > \frac{1}{\varepsilon}$

$\forall \varepsilon > 0$, $\exists$ partition $P = \{x_0 = 1 < \cdots \le x_p = 1\}$, $U(P, f) - L(P, f) < \varepsilon$

$\Rightarrow$ f is Rieman integrable.

Aug 1999, P5.

Let $\{f_n\}$ be a sequence of uniformly bounded Riemann integrable function on $[0,1]$, set $F_n(s)=\int_0^s f_n(t)dt$ for $0\le s\le 1$.

Prove that there is a subsequence of $\{F_n\}$ converges uniformly on $[0,1]$.

*(1), we know that $[0,1]$ is compact.

*(2), we have $F_n(s)$ is continuous on $[0,1]$ by theorem 6.12 Rudin book.

*(3), we now prove that $F_n(s)$ is a sequence of bounded functions on $[0,1]$.
(even more than that, we have $\{F_n\}$ uniformly bounded).

We have $\{f_n\}$ uniformly bounded $\iff \exists M$, $|f_n(x)|\le M$, $\forall x\in[0,1]$, $\forall n\in\mathbb{N}$.

so we have
$$|F_n(s)|=\left|\int_0^s f_n(t)dt\right| \le \int_0^s |f_n(t)|dt \le \int_0^s M\,dt = Ms \le M \quad, \forall s\in[0,1], \forall n\in\mathbb{N}.$$

*(4), we now prove that $\{F_n\}$ equicontinuous:

We want to prove that $\forall \varepsilon>0, \exists\delta>0, \forall x,y\in[0,1], |x-y|<\delta$, then
$|F_n(x)-F_n(y)|<\varepsilon, \forall n\in\mathbb{N}$

We have
$$|F_n(x)-F_n(y)| \overset{\text{wlog}}{\underset{\text{assume } x<y}{=}} \left|\int_0^x f_n(t)dt - \int_0^y f_n(t)dt\right| = \left|\int_x^y f_n(t)dt\right| \le \int_x^y |f_n(t)|dt \le$$
$$\le \int_x^y M = M|y-x|$$

Then $\forall\varepsilon$, choose $\delta>0$, st $M\delta<\varepsilon$, we have $|F_n(x)-F_n(y)|<\varepsilon$, $\forall n$.

$\Rightarrow$ Then from (1)+(2)+(3)+(4)+ applying Arzela Ascoli,
we have $\{F_n\}$ contains a ~~sub~~ convergent subsequence.

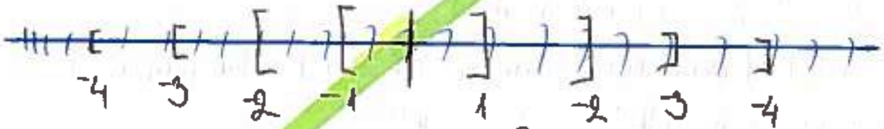
need to Review . #

Fall 2001 / 1

Let $A$ be an uncountable set of real numbers.
Prove that $A$ has an accumulation point.

Put $A_n = A \cap [-n, n]$

Then we have $A = \bigcup_{n=1}^{\infty} A_n$



* Prove this by contradiction, Assume $A$ has no limit point } ⇒ $A_n$ has no limit
we have $A_n \subseteq A$

So we have $A_n$ is a bounded set in $\mathbb{R}$ and $A_n$ has no limit point.
we also have property that every infinite, bounded subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$ } ⇒

⇒ $A_n$ has to be finite.

countable

\+ So we have $A = \bigcup_{n=1}^{\infty} A_n$ } ⇒ $A$ is countable ✓, contradicts with the assumption that
$A_n$ finite
$A$ is an uncountable set of $\mathbb{R}$.

⇒ $A$ has to have a limit point □.

PP2001 ② ~~Rudin 9.9/239~~ ~~Review ex 9.7, 9.8/239~~ (Important) Theorem 9.19 Rudin ✱ Hard

Need to review.

t $f(x)$ be a differentiable mapping of connected open subset $V$ of $\mathbb{R}^n$

$f'(x) = 0$ on $V$

ve that $f$ is a constant in $V$

rem 9.19 (Mean value theorem for vector value function)

convex open in $\mathbb{R}^n$, $f : U \longrightarrow \mathbb{R}^n$

is differentiable in $U$

$\|f'(x)\| \le M$, $\forall x$

$\Rightarrow$ Then $\|f(y) - f(x)\| \le M \|y - x\|$, $\forall x, y \in U$

f is locally constant.

We have $V$ is open, then $\forall x \in V$, $\exists N_\varepsilon(x) \subseteq V$, we prove $\forall y \in N_\varepsilon(x)$, $f(y) = f(x)$ (1)

We have $N_\varepsilon(x)$ is convex + open in $\mathbb{R}^n$

$f$ is differentiable in $E \Rightarrow$ in $N_\varepsilon(x)$

$f'(x) = 0$, $\forall x \in N_\varepsilon(x)$

$\Rightarrow \forall y \in N_\varepsilon(x)$, $|f(y) - f(x)| \le M |y - x|$

where $M = 0$

$\Rightarrow \forall y \in N_\varepsilon(x)$, $f(y) = f(x)$.

Let fix $x_0$ in ~~E~~ V

Now put $A = \{x \in$ ~~E~~ V $, f(x) = f(x_0)\}$

We need to prove $f$ is constant in $V \Longleftrightarrow$ We need to prove $A =$ ~~E~~ V

We have $V$ is op connected, then by the property that a connected set has only 2 sets that are both open and closed in $V$ is $\phi$ and $V$

$\Rightarrow$ We need to prove that $A$ is open and closed in $V$ (because $A \ne \phi$)

Now we prove that $A$ is open in $E$ $\Leftrightarrow$ NTP, $\forall x \in A$, $\exists N_\varepsilon(x) \subseteq A$.

We have $\forall x \in A$, $f(x) = f(x_0)$

From (1), $\forall x \in A \subseteq V$, $\exists N_\varepsilon(x) \subseteq V$, $\forall y \in N_\varepsilon(x)$, $f(y) = f(x) = f(x_0)$ $\Big\} \Rightarrow N_\varepsilon(x) \subseteq A$

Now we prove that $A$ is closed in $E$

($A$ is closed because $A$ is an intersection of $E$ and closed set in $\mathbb{R}^n$).

Another way to prove $A$ is closed is by proving that $(E \setminus A)$ is open

Let $x \in E \setminus A$, then $f(x) \ne f(x_0)$

From (1), $\forall x \in E \setminus A$, $\exists N_\lambda(x)$, $\forall y \in N_\lambda(x)$, $f(y) = f(x) \ne f(x_0)$

$\Rightarrow N_\lambda(x) \subseteq E \setminus A$

$\Rightarrow E \setminus A$ is open.

Note: we know another way (see solution from file.

* Rudin 9.9/239 ~~Prelim Jan 2002. See better solution in Jan 2002.~~

If $f$ is a differentiable mapping of a connected open set $E \subseteq \mathbb{R}^n$, $f'(x) = 0$ for every $x \in E$. } Prove that $f$ is constant in $E$

\+ Now we prove that $f$ is locally constant:

Because $E$ is open, $\forall x \in E$, $\exists N_\delta(x) \subset E$.

then $\forall y \in N_\delta(x)$, $|f(y) - f(x)| \leq M|y-x|$ where $M = \sup_{x \in E} \|f'(x)\| = 0$ ↑ because $f'(x) = 0, \forall x \in$

$\Rightarrow |f(y) - f(x)| \leq 0|y-x| \Rightarrow f(y) = f(x), \forall y \in N_\delta(x)$

which means, $f$ is locally constant ✓

* Now consider $x_0 \in E$, Let $A := \{x \in E, f(x) = f(x_0)\}$ ✓

then because $f$ is locally constant, $A$ is open in $E$.

* We also have $A$ is a closed subset of $E$ (intersection of $E$ and a closed set in $\mathbb{R}^n$)

$\Rightarrow$ We have $A \neq \emptyset$, closed and open in $E$.
we also have assumption that $E$ is connected } $\Rightarrow A = E$, which means $f$ is constant in $E$ ☐. ✓

Fall 2001 : (E3) See sample C I1,

Prove or disprove : the function $f(x) = x^{3/2} \log x$ is uniformly continuous on $(0,1)$

* We find $\lim_{x\to 0^+} f(x)$:

○ We have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{\log x}{x^{-3/2}}$ $\quad \lim_{x\to 0^+} \log x = -\infty$ $\quad \lim_{x\to 0^+} x^{-3/2} = +\infty$

$= \lim_{x\to 0} \frac{(\log x)'}{(x^{-3/2})'}$ (because $\log, x^{-3/2}$ are different) $\lim \frac{(\log x)'}{(\lim x^{-3/2})'} =$

$= \lim_{x\to 0} \frac{1/x}{-3/2\, x^{-5/2}}$

$= \lim_{x\to 0} -\frac{2}{3} x^{3/2} = 0$

f is uniformly continuous on D }
D is dense in [0,1] } ⇒ f ~~uniform~~ has a (uniformly) contin extension on [0,1]

f has a continuous extension on [a, b]
⟺ f is uniformly continuous on (a, b) (Sample C, I 1).

* We find $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^{3/2} \log x = 0$

* Now we define $g(x) = \begin{cases} 0 & x = 0 \\ x^{3/2}\log x & x \in (0,1) \\ 1 & x = 1 \end{cases}$

○ Then we have g continuous on [0, 1] }
[0, 1] compact } ⇒ g is uniformly continuous on [0, 1]
⇒ f is uniformly continuous on (0, 1) ◻

---

* Learn from this problem :

This problem belong to "extending function to a uniformly continuous function on a compact set".

- We already know that f is continuous on (0, 1).
  so we want to extend f to a continuous function g ~~on~~ (cont on [0, 1].
  by finding $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$
  Then put $\begin{cases} g(0) = \lim_{x\to 0^+} f(x) \\ g(x) = f(x) \text{ when } x \in (0,1). \\ g(1) = \lim_{x\to 1^-} f(x) \end{cases}$ then g is cont on [0, 1]
  ⇒ uniformly cont [0, 1]
  ⇒ f is uniformly continuous on (0, 1).

○

all 2001/4

*

t $f(x,y) = (u,v)$, where $u = x^2 - y^2$, $v = 2xy$ describe a map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

) What is the range of this map?

Show that if $(u,v) \neq (0,0)$ then $f$ has an inverse in a neighborhood of $(u,v)$

Show that there is no neighborhood of $(0,0)$ in which $f$ has an inverse.

What is the range of this map.

te that a point $(x,y) \in \mathbb{R}^2$ associates with $z = (x+iy \in \mathbb{C}$ ⇒ In this problem, we it to find [illegible]

it $z = x+iy$ then we have $z^2 = x^2 - y^2 + 2ixy = (x^2-y^2, 2xy)$

$= (x,y)$ So the map $f: \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto z^2$

So range of $f$ is $\mathbb{R}^2$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto f(x,y) = (u,v)$ $\quad u = x^2 - y^2$, $v = 2xy$

have $f$ is a $C'$ function (1).

$$x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

et$[f'] = 4x^2 + 4y^2$

hen $f'(x,y)$ is invertible $\Leftrightarrow (x,y) \neq 0$ (2)

Note that $(x,y) = \overline{0} \Longrightarrow (u,v) = \overline{0}$

$(u,v) = \overline{0} \Leftrightarrow \begin{cases} x^2 - y^2 = 0 \\ 2xy = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ $\Rightarrow$ $(x,y) = 0 \Leftrightarrow (u,v) = 0$

$(x,y) \neq 0 \Leftrightarrow (u,v) \neq 0$ (3)

Note: inverse function theorem.

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a $C'$ function

$\vec{x}_0 \in U$

$f'(\vec{x}_0)$ invertible.

⇒ ∃ a neighborhood of $\vec{x}_0$, such that $f$ is bijective

In this problem: neighborhood of $(u_0)$.

(we need to deduce neighborhood of $x_0$ ⇔ neighb of $u_0$

~~hen in any $(u,v) \neq (0,0)$~~

Then from (1)+(2)+(3) + Inverse function theorem ∃ a open neighborhood $V$ of $(x,y)$ where $(x,y) \neq 0$ nd a neighborhood $W$ of $(u,v)$ where $(u,v) \neq (0,0)$ such that $f: V \longrightarrow W$ bijective.

⇒ $f$ has an inverse in a neighborhood of $(u,v)$ $\quad (u,v) \neq (0,0)$

Show that there is no neighborhood of $(0,0)$ in which $f$ has an inverse

: prove this by consider neighborhood of $(0,0)$ then prove that in this neighborhood $f$ is not ~~bijective~~ injective

nsider interval $(-\varepsilon, \varepsilon)$,

then $\exists\, t \in (0,\varepsilon)$ st $(-t) \in (-\varepsilon, 0)$

$f(t,t) = (0, 2t^2)$

$f(-t,-t) = (0, 2t^2)$

In ~~a~~ ev neighborhood of $(0,0)$

$f$ is not injective

⇒ No neighborhood of $(0,0)$ in which $f$ has an inverse.

Fall 2001 , Q5 :

Prove that $\sum_{n=1}^{\infty} \frac{\sin(n^4 x)}{n^2}$ defines a continuous function on $\mathbb{R}$.

* Put $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^4 x)}{n^2} = \sum_{n=1}^{\infty} f_n(x)$

where $f_n(x) = \frac{\sin(n^4 x)}{n^2}$

$$\left.\begin{array}{l} \bullet \text{ We have } |f_n(x)| = \left|\frac{\sin(n^4 x)}{n^2}\right| \leq \left|\frac{1}{n^2}\right| = M_n \\ \text{we have } \sum_{n=1}^{\infty} M_n \text{ converges} \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly}$$

$$\left.\begin{array}{l} * \text{ Note that } f_k(x) = \sum_{n=1}^{k} f_n(x) \text{ is continuous.} \\ f_k(x) \rightrightarrows f \end{array}\right\} \Rightarrow f \text{ is continuous} \quad \square .$$

Fall 2001:

Find the limit $\lim_{\lambda\to+\infty}\lambda\int_{-1}^{1}e^{-\lambda|y|}dy$

Let $g:\mathbb{R}\to\mathbb{R}$ be a bounded, continuous function.

For $x\in\mathbb{R}$, find the limit $\lim_{\lambda\to\infty}\lambda\int_{-1}^{1}g(x+y)\,e^{-\lambda|y|}dy$

---

We first consider $(*)=\lambda\int_{-1}^{1}e^{-\lambda|y|}dy=\lambda\int_{-1}^{0}e^{\lambda y}dy+\lambda\int_{0}^{1}e^{-\lambda y}dy$.

$$:=\lambda\int_{-1}^{0}e^{\lambda y}dy=\int_{-1}^{0}e^{\lambda y}d(\lambda y)\underset{\substack{u=\lambda y\\ y=-1\Rightarrow u=-\lambda\\ y=0\Rightarrow u=0}}{=\!=\!=}\int_{-\lambda}^{0}e^{u}du=e^{u}\Big|_{-\lambda}^{0}=e^{0}-e^{-\lambda}$$

$$=\lambda\int_{0}^{1}e^{-\lambda y}dy=-\int_{0}^{1}e^{-\lambda y}d(-\lambda y)\underset{\substack{u=-\lambda y\\ y=0\Rightarrow u=0\\ y=1\Rightarrow u=-\lambda}}{=\!=\!=}-\int_{0}^{-\lambda}e^{u}du=-e^{u}\Big|_{0}^{-\lambda}=-e^{-\lambda}+e^{0}$$

Then we have $(*)=A+B=2e^{0}-2e^{-\lambda}\xrightarrow[\lambda\to+\infty]{}2$

so $\lim_{\lambda\to+\infty}\lambda\int_{-1}^{1}e^{-\lambda|y|}dy=2$.

We have g cont ⇒ integrable ⇒ the integral is well defined.
+ assumption that g is bounded
⇔ $\exists M,\ |g(x+y)|\le M,\ \forall y\in[-1,1],\ x\in\mathbb{R}$.

$$\left|\lambda\int_{-1}^{1}g(x+y)\,e^{-\lambda|y|}dy\right|\underset{\uparrow}{\le}\cancel{g(x+y)}\,M\,\lambda\int_{-1}^{1}e^{-\lambda|y|}dy$$

note that we consider
$\lambda\to+\infty$
so $\lambda>0$.

not done.

Checked ✓

# Analysis Preliminary Exam, August 2002

Solution on
See Aug 1999

1. Let $f : (0,1) \to \mathbb{R}$ be continuous, bounded and decreasing. Prove that $f$ is uniformly continuous on $(0,1)$.

2. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, given by $f(x) = \dfrac{\sum_{j=1}^n x_j^3}{\|x\|^2}$ if $x \neq 0$, and $f(0) = 0$, where $x = (x_1, \ldots, x_n)$ and $\|x\|$ is the Euclidean norm of $x$. Prove that $f$ is continuous on $\mathbb{R}^n$.

template

3. Prove that the system

$$xy^5 + yu^5 + zv^5 = 1,$$
$$x^5y + y^5u + z^5v = 1,$$

has a unique solution $u = f(x,y,z)$, $v = g(x,y,z)$, in a neighborhood of the point $(u,v,x,y,z) = (1,0,0,1,1)$. Find $\dfrac{\partial u}{\partial x}(0,1,1)$.

4. Let $\mathbb{Q}_0$ be the set of rationals in the interval $[0,1]$. For a bounded function $f : \mathbb{Q}_0 \to \mathbb{R}$, and $n = 1, 2, \ldots$, define

$$S_n(f) = \frac{1}{n}\sum_{k=1}^{n} f(k/n)\,.$$

If $\lim_{n\to\infty} S_n(f)$ exists, we say that $f$ is $S$-summable, and let $S(f) = \lim_{n\to\infty} S_n(f)$ denote this limit. Let $f_1, f_2, \ldots$ be bounded functions on $\mathbb{Q}_0$ which are $S$-summable, and suppose that $f_k \to f$ uniformly on $\mathbb{Q}_0$ as $k \to \infty$. Prove that $f$ is $S$-summable, and that $\lim_{k\to\infty} S(f_k) = S(f)$.

See Aug 2002, P5
Jan 2004 P 4
Jan 2009 P2

5. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $\lim_{k\to\infty} a_k = L \in \mathbb{R}$ exists. For $0 < p < 1$ define

$$A(p) = \sum_{k=1}^{\infty} p(1-p)^{k-1} a_k\,.$$

Prove that this sum converges, and that $\lim_{p\to 0} A(p) = L$.

Use $\sum_{l=1}^{\infty} p(1-p)^{l-1} = 1$
and devide $\sum_{l=1}^{\infty} = \sum_{l=1}^{K_0} + \sum_{K_0}^{\infty}$

6. Prove that

$$\lim_{n\to\infty} \frac{1}{n^{5/2}} \sum_{k=1}^{n} k^{3/2} = \frac{2}{5}\,.$$

checked ✓

# Preliminary Exam - January 2002

1. Let $A$ and $B$ be subsets of a metric space. Prove that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and give an example when $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

2. Let $f$ and $f'$ be continuous functions on $\mathbb{R}$. Prove that that the sequence of functions

$$g_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}$$

converges to $f'(x)$ uniformly on every interval $[a, b]$, $-\infty < a < b < \infty$.

3. Let $f$ be a Riemann integrable function on $[0, 1]$ and

Also Aug 2005(2)
Theorem 6.20

$$F(x) = \int_0^x f(t)\,dt.$$

a) Show that there is a constant $C$ such that $|F(x) - F(y)| \le C|x - y|$ for every $x, y \in [0, 1]$.

b) Give an example of $f$ such that $F$ is not differentiable at some point.

? 4. Show that the sequence

$$f_n(x) = \frac{\tan^{-1}(nx)}{\sqrt{n}}$$

is equicontinuous on $\mathbb{R}$ and converges uniformly to $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f_n'(x)$ does not converge uniformly to $f'(x)$.

5. Determine the values of $\alpha$ for which $f$ is differentiable at $(0, 0)$ when

$$f(x, y) = \begin{cases} (x^2 + y^2)^\alpha \sin \frac{1}{x^2+y^2}, & (x, y) \neq (0, 0); \\ 0, & (x, y) = (0, 0). \end{cases}$$

6. Show that if $\phi(y)$ is a continuously differentiable function on $(-a, a)$, $a > 0$, such that $\phi(0) = 0$ and $|\phi'(y)| \le k < 1$ on $(-a, a)$, then there is $\varepsilon > 0$ and a unique differentiable function $g$ on $(-\varepsilon, \varepsilon)$ satisfying the equation $x = g(x) + \phi(g(x))$.

Jan 2008

P17 Let A and B be subsets of a metric space.
Prove that $\overline{A\cap B} \subset \bar{A}\cap\bar{B}$. Give an example when $\overline{A\cap B} \neq \bar{A}\cap\bar{B}$.

* Prove that $\overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$

- Let $x \in \overline{A\cap B}$ then $\forall r$, $N_r(x) \cap (A\cap B) \neq \phi$

$$\Rightarrow \quad (N_r(x)\cap A) \cap (N_r(x)\cap B) \neq \phi .$$

$$\Leftrightarrow \left.\begin{cases} N_r(x)\cap A \neq \phi, \forall r \Leftrightarrow x\in\bar{A} \\ N_r(x)\cap B \neq \phi, \forall r \Leftrightarrow x\in\bar{B} \end{cases}\right\} \Rightarrow x\in\bar{A}\cap\bar{B}.$$

* Another way to prove $\overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$

Notice that $A\cap B \subseteq A \subseteq \bar{A}$
$A\cap B \subseteq B \subseteq \bar{B}$ $\Rightarrow A\cap B \subseteq \bar{A}\cap\bar{B}$
note that $\bar{A}\cap\bar{B}$ closed
$\overline{A\cap B}$ is the smallest closed set containing $A\cap B$ $\Big\} \Rightarrow \overline{A\cap B} \subseteq \bar{A}\cap$

* Give an example that $\overline{A\cap B} \not\supseteq \bar{A}\cap\bar{B}$

- let $A = \mathbb{Q}$ $B = \mathbb{R}\setminus\mathbb{Q}$ then $A\cap B = \phi$. $\Rightarrow \overline{A\cap B} = \phi$
$\bar{A} = \mathbb{R}$, $\bar{B} = \mathbb{R}$ $\Rightarrow \bar{A}\cap\bar{B} = \mathbb{R}$.

- Or let $A = (0,1)$ $B = (1,2)$
$A\cap B = \phi \Rightarrow \overline{A\cap B} = \phi$
$\bar{A} = [0,1]$, $\bar{B} = [1,2]$ $\Rightarrow \bar{A}\cap\bar{B} = 1 \neq \phi$. ☐

Jan 2007 #2.

Let $f$ and $f'$ be continuous functions on $\mathbb{R}$.
Prove that the sequence of functions $g_n(x) = \dfrac{f(x+1/n) - f(x)}{1/n} \rightrightarrows f'(x)$ on $[a,b]$

$-\infty \leq a < b < +\infty$.

---

We have

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n} =$$

$$= \lim_{n\to\infty} f'(\xi) = f'(\xi) \text{ for some } \xi \in (x, x+1/n)$$

NTP

$g_n(x) \rightrightarrows f'(x)$

$\Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists N > 0, \forall n \geq N, \forall x \in [a,b]$,

$|g_n(x) - f'(x)| < \varepsilon$.

Note that $f'$ is continuous on $R \Rightarrow$ uniformly continuous on $[a,b]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x, \xi \in [a,b], |x - \xi| < \delta$, then $|f'(\xi) - f'(x)| < \varepsilon$.

Then for $\frac{1}{n} < \delta$, $|f'(\xi) - f'(x)| < \varepsilon$

This means choose $N = \frac{1}{\delta} + 1$, $\forall n \geq N$, $\forall x \in [a,b]$, $|g_n(x) - f'(x)| < \varepsilon$ $\square$

Jan 2002/3 See Theorem 6.20, Aug 2003, Template

3 Let $f$ be a Riemann integrable function on $[a,b]$

$$F(x) = \int_a^x f(t)\,dt$$

a) Show that $\exists$ constant $c$, $|F(x) - F(y)| \leq c\,|x-y|$

c) Give an example of $f$ s.t $F$ is not differentiable at some point.

b) Prove that if $f$ is continuo
at $x_0 \in [a,b]$
Then $F$ is differentiable at $x_0$
and $F'(x_0) = f(x_0)$.

a) Show that there is a constant $c$ such that $F(x) - F(y) \leq c\,|x-y|$.

We have because $f$ is a Riemann integrable on $[a,b]$ $\Rightarrow$ bounded on $[a,b]$.

$\Leftrightarrow \exists c$ constant s.t $|f(t)| \leq c$, $\forall t \in [a,b]$.

Then wlog, assume $x > y$, we have

$$|F(x) - F(y)| = \left|\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right| = \left|\int_y^x f(t)\,dt\right| \leq \int_y^x |f(t)|\,dt \leq \int_y^x c\,dt = c\,|x-y| \;\square a$$

b) Prove that if $f$ is continuous at $x_0 \in [a,b]$, then $F$ is differentiable at $x_0$, and $F'(x_0) = f(x_0)$.

- $f$ cont at $x_0 \in [a,b]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall y \in [a,b], |y - x_0| < \delta$, then $|f(y) - f(x_0)| < \varepsilon$ (1)

NTP $F'(x_0) = f(x_0)$

NTP $\displaystyle\lim_{\substack{y < x_0 < t \\ t \to y}} \frac{F(t) - F(y)}{t-y} = f(x_0)$

Let $y$ and $t \in [a,b]$ such that $x_0 - \delta < y < x_0 \leq t < x_0 + \delta$, (2)

we want to prove that $\forall \varepsilon > 0$, $\left|\frac{F(t) - F(y)}{t-y} - f(x_0)\right| < \varepsilon$

- We have

$$\left|\frac{F(t) - F(y)}{t-y} - f(x_0)\right| = \left|\left(\frac{1}{t-y}\right)\left(\int_a^t f(x)\,dx - \int_a^y f(x)\,dx\right) - f(x_0)\right|$$

$$= \left|\frac{1}{(t-y)}\int_y^t f(x)\,dx - \frac{1}{(t-y)}\int_y^t f(x_0)\,dx\right|$$

$$= \left|\frac{1}{(t-y)}\int_y^t \big(f(x) - f(x_0)\big)\,dx\right|$$

$$\leq \left|\frac{1}{(t-y)}\right| \int_y^t \underbrace{|f(x) - f(x_0)|}_{<\varepsilon \text{ because of (1) and (2)}}\,dx$$

$$\leq \frac{1}{|t-y|}(t-y)\cdot\varepsilon = \varepsilon$$

So we have $F$ is differentiable at $x_0$, and $F'(x_0) = f(x_0)$.

Note From this we have if $f$ continuous on $(a,b) \Longrightarrow \begin{cases} F \text{ differentiable on } (a,b) \\ F'(x) = f(x), \quad x \in (a,b) \end{cases}$

27 Give an example of $f$ such that $F$ is not differentiable at some point.

* Example 1 : (when $F$ is continuous but not differentiable at $x_0$).

Let $f(x) : [0,2] \longrightarrow \mathbb{R}$

$$x \longmapsto f(x) = \begin{cases} 0 , & x \leq 1 \\ 1 , & x > 1 . \end{cases}$$

Then we have $F(x) = \displaystyle\int_0^x f(t)\,dt = \begin{cases} 0 , & x \leq 1 \\ (x-1) , & x > 1 . \end{cases}$

- We have $f$ discontinuous at finite point $\Rightarrow f \in \mathcal{R}$ in $[0,2]$.
- $F(x)$ is continuous at $x_0 = 1$ but not differentiable.

$$\left.\begin{aligned} \lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} &= \lim_{x \to 1^+} \frac{x-1}{x-1} = 1 . \\ \lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} &= \lim_{x \to 1^-} \frac{0-0}{x-1} = 0 \end{aligned}\right\} \Rightarrow \nexists\, F'(1) .$$

Jan 2008 >

P4 > Prove that the sequence $f_n(x) = \frac{\tan^{-1}(nx)}{\sqrt{n}}$ is equicontinuous on $\mathbb{R}$. and $\rightrightarrows f(x) = \lim_{n\to\infty} f_n(x)$

b > Show that $f_n'(x)$ does not $\rightrightarrows f'(x)$.

b > $f_n'(x) \not\rightrightarrows f'(x)$

because $f_n'(x) = \frac{\sqrt{n}}{1+n^2x^2} \overset{\text{at } x=0}{=} \frac{\sqrt{n}}{1+0} \cancel{\rightrightarrows}$

* Prove that $f_n(x) = \frac{\tan^{-1}(nx)}{\sqrt{n}} \rightrightarrows f(x) = \lim_{n\to\infty} f_n(x)$

we note that $|\tan^{-1}(nx)| < \frac{\pi}{2}$

so we have $\left|\frac{\tan^{-1}(nx)}{\sqrt{n}}\right| < \frac{\pi}{2\sqrt{n}}$

notice that $M_n := \frac{\pi}{2\sqrt{n}}$, then $\{M_n\} \xrightarrow{n} 0$ $\Bigg\}$ $\frac{\tan^{-1}(nx)}{\sqrt{n}} \rightrightarrows 0$ on $\mathbb{R}$.

* Prove that $\{f_n(x)\}$ equicontinuous on $\mathbb{R}$: NTP $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in \mathbb{R}, |x-y| < \delta$ then $|f_n(x) - f_n(y)| < \varepsilon, \forall n$.

• We prove a more general result

(*) $f_n \rightrightarrows \boxed{0}$ $\boxed{\text{in } \mathbb{R}}$ and $\{f_n'\}$ uniformly bounded. $\Big\}$ then $\{f_n\}$ equicontinuous. in $\mathbb{R}$.

(See in Jan 2009 > P5 $\{f_n\}$ sequence of differentiable function $\boxed{\text{in } [a,b]}$, $\{f_n'\}$ uniformly bounded $\Big\}$ then in $[a,b]$ equicontinuous)

Prove (*): $f_n \rightrightarrows 0 \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \overset{\forall x \in \mathbb{R}}{,} |f_n(x)| < \varepsilon/2$

this means $\underline{\forall n \geq N}, \forall x, y \in \mathbb{R}$, $|f_n(x) - f_n(y)| \leq |f_n(x)| + |f_n(y)| < \varepsilon$

• Now consider in case $n < N$

because $\{f_n'\}$ uniformly bounded $\Leftrightarrow \exists M > 0$, $|f_n'(x)| < M$, $\forall n, \forall x$.

then $|f_n(x) - f_n(y)| \underset{\substack{\uparrow \\ \xi \in (x,y)}}{=} |f_n'(\xi)|\ |x-y| < M\ |x-y|$.

So $\forall \varepsilon > 0$, Choose $\delta$ s.t $M\delta < \varepsilon$, we have $\forall x, y \in \mathbb{R}$, $\forall n$, $|f_n(x) - f_n(y)| < \varepsilon$ $|x-y| < \delta$

$\Rightarrow \{f_n\}$ equicontinuous.

---

Come back to our problem: (could not apply directly the above result but the idea is quite similar)

We have $\underbrace{\frac{\tan^{-1}(nx)}{\sqrt{n}}}_{f_n(x)} \rightrightarrows 0 \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |f_n(x)| < \varepsilon/2$

$\Rightarrow$ for all $n \geq N, \overset{\forall x, y \in \mathbb{R}}{,} |f_n(x) - f_n(y)| \leq \varepsilon$

• In case $n < N$

we have $|f_n(x) - f_n(y)| = \left|\frac{\tan^{-1}(nx)}{\sqrt{n}} - \frac{\tan^{-1}(ny)}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} f_n'(\xi)\ |nx - ny| \leq \sqrt{n}\,|x-y|$

for some $\xi$ between $(nx, ny)$

$\Rightarrow$ Choose $\delta < \frac{\varepsilon}{\sqrt{N}}$

$\Rightarrow \ldots$

equicontinuous

Jan 2008, K3

Determine the value of $\alpha$ for which $f$ is differentiable at $(0,0)$ when

$$f(x,y) = \begin{cases} (x^2+y^2)^\alpha \sin \frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 \end{cases}$$

We have $f$ is differentiable at $(0,0)$ where

$$\exists \lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \frac{\partial f}{\partial x}(h_1,h_2)\, h_1 - \frac{\partial f}{\partial y}(h_1,h_2)\, h_2}{\sqrt{h_1^2+h_2^2}} = 0 = \exists \lim_{(h_1,h_2)\to(0,0)} (*)$$

~~$(h_1^2+h_2^2)^\alpha \sin\left(\frac{1}{h_1^2+h_2^2}\right) - \alpha (x^2+y^2)^{\alpha-1} \cdot 2x \sin \frac{1}{x^2+y^2}\Big|_{00} - (x^2+y^2)^\alpha \underbrace{\left[\sin \frac{1}{x^2+y^2}\right]'}$~~

notice that $f_x(0,0) = \alpha \cdot 2x\, (x^2+y^2)^{\alpha-1} \sin \frac{1}{x^2+y^2} + (x^2+y^2)^\alpha \overset{=0}{\left[\sin \frac{1}{x^2+y^2}\right]'} = 0$ at $(0,0)$.

$= f_y(0,0)$.

So we need $$\lim_{(h_1,h_2)\to 0} \frac{f(h_1,h_2) - f(0,0)}{\sqrt{h_1^2+h_2^2}} = 0$$

Need $$\lim_{(h_1,h_2)\to(0,0)} \frac{(h_1^2+h_2^2)^\alpha \sin \frac{1}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} = 0$$

Need $$\exists \lim_{(h_1,h_2)\to(0,0)} (h_1^2+h_2^2)^{\alpha - 1/2} \sin \frac{1}{h_1^2+h_2^2} = 0$$

- We have that when $\boxed{\alpha > \frac{1}{2}}$, $0 \leq \left| (h_1^2+h_2^2)^{\alpha-1/2} \sin \frac{1}{h_1^2+h_2^2} \right| \leq \underbrace{(h_1^2+h_2^2)^{\alpha-1/2}}_{\longrightarrow 0}$

then $\exists \lim(*) = 0$

- When $\alpha \leq 1/2$,

we have assume choose $h_1 \neq 0$, $h_1 = 0$, $h_1 \to 0$,

we have $$\lim_{h_1\to 0} \frac{f(h_1,0) - f(0,0)}{\sqrt{h_1^2}} = \lim_{h_1\to 0} \frac{(h_1^2)^\alpha \sin \frac{1}{h_1^2}}{\sqrt{h_1^2}} = \lim_{h_1\to 0} (h_1^2)^{\alpha-1/2} \sin \frac{1}{h_1^2}$$ does not exist when $\alpha \leq 1/2$.

so $\lim_{(h_1,h_2)\to(0,0)} (*)$ does not exist when $(h_1,h_2) \to (0,0)$

Jan 2007

P67 Show that if $\begin{cases} \phi(y) \text{ is a continuously differentiable function on } (-a,a),\ a>0 \\ \phi(0) = 0 \\ |\phi'(y)| \leq k < 1 \quad \text{on } (-a,a) \end{cases}$

then $\exists\, \varepsilon > 0$, $\exists !$ $g$ differentiable function on $(-\varepsilon, \varepsilon)$ satisfying the equation

$$x = g(x) + \phi(g(x))$$

Need to review

note that we have if $f(x) = x + \phi(x)$

then $g(x) + \phi(g(x)) = f \circ g(x)$.

and we want to prove that $\exists !$ $f \circ g(x) = x$

this means $g = f^{-1}$

* Put $f(x) = x + \phi(x)$. then because $\phi$ continuously differentiable $\Rightarrow$ $f$ is continuously differentiable

- $f(0) = \phi(0) = 0$
- $f'(x) = 1 + \phi'(x) > 0$ because $|\phi'(x)| \leq k < 1$, on $(-a,a)$

So by Inverse function theorem, $\exists$ a neighborhood $V = (-\delta, \delta)$ of $0$ and a neighborhood $W = (-\varepsilon, \varepsilon)$ of $0$ s.t

$f : V \longrightarrow W$ is bijective and $\exists !$ $g : W \longrightarrow V$ a differentiable function
st $\qquad x \longmapsto g_x = f^{-1}\big|_V (x)$.

this means $f \circ g(x) = x$

$\Leftrightarrow g(x) + \phi(g(x)) = x$ $\quad \square$.

Aug 2008/3

a) Prove that the system $xy^5 + yu^5 + zv^5 = 1$

$x^5y + y^5u + z^5v = 1$

has a unique solution $u = f(x,y,z)$ in a neighborhood of the point $(u, v, x, y, z) = (1,0,0,1,1)$

$v = g(x,y,z)$

note: we need to care about the order of $(u, v, x, y, z)$

$\boxed{\text{Check} \checkmark}$

b) Find $\frac{\partial u}{\partial x}(0,1,1)$

---

a) Let $F : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$

$(u,v,x,y,z) \longmapsto \left(F_1(\cdots), F_2(\cdots)\right)$

$F_1(u,v,x,y,z) = xy^5 + yu^5 + zv^5 - 1.$

$F_2(u,v,x,y,z) = x^5y + y^5u + z^5v - 1$

(1) First, we have at $(1,0,0,1,1)$, $\begin{bmatrix} F_1(1,0,0,1,1) \\ F_2(1,0,0,1,1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \bar{O}_{\mathbb{R}^2}$

$\Rightarrow (1,0,0,1,1)$ is a solution of $F(X) = 0$

(2) $DF = \begin{bmatrix} \nabla F_1 \\ \nabla F_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ - & - & - & -- & -- \end{bmatrix} = \left[\begin{array}{cc|ccc} 5yu^4 & 5zv^4 & y^5 & 5xy^4 + u^5 & v^5 \\ y^5 & z^5 & 5x^4y & x^5 + 5y^4u & 5z^4v \end{array}\right]$

we have all $D_j F_i$ exist and continuous $\Rightarrow$ F is continuously differentiable

(3) At $(1,0,0,1,1)$

Put $\vec{a} = (1,0)$

$\vec{b} = (0,1,1)$

we have $A_a = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ $\Rightarrow \det A_a = 5 \Rightarrow A_a$ invertible.

$A_b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix}$

From (1)+(2)+(3) $\Rightarrow$ By Implicit function Theorem, the above system has a unique solution

$u = f(x,y,z)$ in a neigh ....

$v = g(x,y,z)$

b) Find $\frac{\partial u}{\partial x}(0,1,1).$

Put $(u,v) = G(x,y,z)$

then we have

$$G'(u_0,v_0) = -[A_a]^{-1}[A_b] = \frac{-1}{5}\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -1/5 & -1/5 & 0 \\ 1/5 & -24/5 & 0 \end{bmatrix}$$

then $\frac{\partial u}{\partial x}(0,1,1) = -\frac{1}{5}$

Aug2002, P4:

Let $Q_0 := \{$rational number in the interval $[0,1]\}$

For a bounded function : $f : Q_0 \longrightarrow \mathbb{R}$

for $n = 1, 2, 3, \dots$ define $S_n(f) = \frac{1}{n}\sum_{l=1}^{n} f\left(\frac{l}{n}\right)$

If $\lim_{n\to\infty} S_n(f)$ exists, we say that $f$ is S_summable.

Let $S(f) = \lim_{n\to\infty} S_n(f)$ denote the limit.

Let $f_1, f_2$ be bounded function on $Q_0$ with are Summable.

Suppose $f_l \overset{l\to\infty}{\Longrightarrow} f$ on $Q_0$.

Prove that $f$ is Summable, and that $\lim_{l\to\infty} S(f_l) = S(f)$.

---

- Note that each $f_i$ is bounded on $Q_0$ and Summable.

Define $S_n(f_i) = \frac{1}{n}\sum_{l=1}^{n} f_i\left(\frac{l}{n}\right)$ (1)

$S(f_i) = \lim_{n\to\infty} S_n(f_i)$

- We also have $f_i \overset{i\to\infty}{\Longrightarrow} f$ on $Q_0$

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall i, j \geq n_0, \overset{\forall x \in Q_0}{,} |f_i(x) - f(x)| < \varepsilon$

$|f_i(x) - f_j(x)| < \varepsilon$ (2)

Need to prove $f$ is summable.

$\Leftrightarrow$ NT prove

$S_n(f)$ converges ~~$\longrightarrow$~~ $S(f)$

and

$(S(f) = \lim_{l\to\infty} S(f_l)$.

* Now consider

$$|S_n(f) - S_n(f_i)| = \left|\frac{1}{n}\sum_{l=1}^{n} f\left(\frac{l}{n}\right) - \frac{1}{n}\sum f_i\left(\frac{l}{n}\right)\right| =$$

$$= \underbrace{\left|\frac{1}{n}\sum_{l=1}^{n_0}\underbrace{\left[f\left(\frac{l}{n}\right) - f_i\left(\frac{l}{n}\right)\right]}_{\text{bounded}}\right|}_{\text{tend to } 0} + \left|\frac{1}{n}\sum_{l=n_0}^{n}\underbrace{\left[f(l/n) - f_i(l/n)\right]}_{<\varepsilon \text{ because of } (2)}\right|$$

$$< \varepsilon \quad + \quad \frac{1}{n}(n - n_0)\varepsilon$$

$$< 2\varepsilon, \forall i \quad (I)$$

- Note that because $|S_n(f_i) - S_n(f_j)| = \frac{1}{n}\left|\sum_{l=1}^{n}|f_i\left(\frac{l}{n}\right) - f_j\left(\frac{l}{n}\right)\right| \leq \max(f_i - f_j) < \varepsilon$

$\Leftrightarrow$ ~~$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0, |S_n(f_i) - S_n(f_j)| <$ [illegible], $\forall i$ (II)~~.

$$|S_n(f_i) - S_n(f_j)| < \varepsilon \quad \overset{\leq \frac{1}{2}}{\text{for}} \; i, j \text{ large } \geq n_0.$$

* So now we consider $|S_n(f) - S_m(f)|$ for all $m, n, i \geq \max\{n_0, N_0\}$.

$$|S_n(f) - S_m(f)| \leq \underbrace{|S_n(f) - S_n(f_i)|}_{<2\varepsilon \text{ because of (I)}} + \underbrace{|S_n(f_i) - S_m(f_i)|}_{<\varepsilon \text{ because conv. then Cauchy}} + \underbrace{|S_m(f_i) - S_m(f)|}_{<2\varepsilon \text{ because of (I)}}$$

$\leq 5\varepsilon$ ~~(III)~~

~~So~~ because Cauchy $\Rightarrow \{S_n(f)\}_n$ converges.

Put $S(f) = \lim S_n(f)$ (III)

* Now we need to prove $|S(f) - S(f_i)| < \varepsilon$ for $i$ large enough. $i, n \geq \max\{n_0, N_0\}$

$$|S(f) - S(f_i)| \leq \underbrace{|S(f) - S_n(f)|}_{\varepsilon \text{ by (III)}} + \underbrace{|S_n(f) - S_n(f_i)|}_{2\varepsilon \text{ (by I)}} + \underbrace{|S_n(f_i) - S(f_i)|}_{<\varepsilon \text{ because } S(f_i) = \lim_{n\to\infty} S_n(f_i),\ \forall i}$$

$\leq$

$< 4\varepsilon$

This means $S(f) = \lim_{i \to \infty} S(f_i)$

Aug2002, #07.

Let $a_1, a_2, \cdots$ be a sequence of of real number such that $\lim_{\ell\to\infty} a_\ell = L \in \mathbb{R}$.

For $0<p<1$ define $A(p) = \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} a_\ell$.

Prove that the sum converges and that $\lim_{p\to 0} A(p) = L$

This problem not only depends on $n\to\infty$ but also depends on $p$.

See Jan 2009, #2 b) $\{a_n\}$ a sequence of real number, $a_n \to L$, $b_n = \frac{1}{n^2}\sum \ell a_\ell$ } Prove that $b_n \to L$.

With those kind of problems, we try to use $|a_\ell - L| < \varepsilon$ when $\ell$ large enough.
and try to use $\sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} = 1 \Rightarrow L = \sum_{\ell=1}^{\infty}(p(1-p)^{\ell-1})L$

Prove the sum converges:

Now consider $A(p) = \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} a_\ell = \sum_{\ell=1}^{\infty} b_\ell$ where $b_\ell := p(1-p)^{\ell-1} a_\ell$

So we have $\left|\frac{b_{\ell+1}}{b_\ell}\right| = \left|\frac{p(1-p)^{\ell} a_{\ell+1}}{p(1-p)^{\ell-1} a_\ell}\right| = |1-p| \underbrace{\left|\frac{a_{\ell+1}}{a_\ell}\right|}_{\to 1} \longrightarrow 1-p < 1$ because $0<p<1$

so the series converges.

* Prove that $\lim_{p\to 0} A(p) = L$.

Note that we divide $\sum_{\ell=1}^{\infty} A(p)$ into $\underbrace{\sum_{\ell=1}^{N}}_{\text{bounded}}$ and $\underbrace{\sum_{N}^{\infty}}_{<\varepsilon}$ since $a_n \to L$.

- We have $a_\ell \xrightarrow{\ell\to\infty} L \Leftrightarrow \forall\varepsilon>0, \exists K_0 \in \mathbb{N}, \forall \ell \geq K_0, |a_\ell - L| < \varepsilon$
- Note that $\sum_{\ell=0}^{\infty}(1-p)^{\ell} = \frac{1}{1-(1-p)} = \frac{1}{p}$

So we have $\sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} = 1$ This means $L = \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} L$

So we have

$$\left|\sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} a_\ell - L\right| = \left|\sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} a_\ell - \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} L\right|$$

$$= \left|\sum_{\ell=1}^{K_0} p\underbrace{(1-p)^{\ell-1}}_{<1}\underbrace{|a_\ell - L|}_{\leq M \text{ bounded}}\right| + \underbrace{\left|\sum_{\ell=K_0}^{\infty} p(1-p)^{\ell-1}\underbrace{|a_\ell - L|}_{<\varepsilon}\right|}_{\underbrace{\leq \varepsilon \underbrace{\sum_{\ell=K_0}^{\infty} p(1-p)^{\ell-1}}_{\leq 1}}_{\leq\varepsilon}}$$

$$\underbrace{M\sum_{\ell=1}^{K_0} p}_{= \underbrace{MK_0 p}_{\xrightarrow{p\to 0} 0}} =$$

Aug 2007

P2 Consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$x \longmapsto f(x) = \begin{cases} \dfrac{\sum_{j=1}^{n} x_j^3}{\|x\|^2} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$x = (x_1, \dots, x_n)$

$\|x\|$: Euclidean norm of $x$ ✱

Prove that $f$ is continuous on $\mathbb{R}^n$

Step 2:

We have

$$|f(x) - f(y)| = \left| \frac{\sum_{i=1}^{n} x_i^3}{\sum_{i=1}^{n} x_i^2} - \frac{\sum_{j=1}^{n} y_j^3}{\sum_{j=1}^{n} y_j^2} \right| = \left| \frac{\sum_{i,j} x_i^3 y_j^2 - \sum_{i,j} x_i^2 y_j^3}{\sum_{i,j} x_i^2 y_j^2} \right|$$

$$= \left| \frac{\sum_{i,j} [x_i^2 y_j^2 (x_i - y_j)]}{\sum x_i^2 y_j^2} \right|$$

note that $\dfrac{x_i^2 y_j^2}{\sum x_i^2 y_j^2} \leq 1$

So we have $|f(x) - f(y)| \leq n^2 \sum_{i,j} |x_i - y_i|$

note that when $x \to y$, we have $|x_i - y_i|$ is really small

Now we will prove that for $\|x - y\| < \delta \Leftrightarrow \sum (x_i - y_i)^2 < \delta$

$\Rightarrow \quad (x_i - y_i)^2 < \delta$

$\Rightarrow \quad |x_i - y_i| < \sqrt{\delta}$

So we choose $\delta$ such that $n^3 \sqrt{\delta} < \varepsilon \Leftrightarrow \sqrt{\delta} < \dfrac{\varepsilon}{n^3}$

we have $\forall \varepsilon > 0$, $\exists \delta$ s.t $\sqrt{\delta} < \dfrac{\varepsilon}{n^3}$, $\forall \|x - y\| < \delta$, then $|f(x) - f(y)| \leq n^2 n \sqrt{\delta} < \varepsilon$

$\Rightarrow f$ is uniformly continuous on $\mathbb{R}^n$ $\square$

Step 1

✱ we have

$$|f(x)| = \left| \frac{\sum x_i^3}{\sum x_i^2} \right| = \left| \frac{\sum x_i \, x_i^2}{\sum x_i^2} \right|$$

note that when $\|x\| < \delta$, then each $|x_i| < \delta$, $\forall i$

$$\Rightarrow \quad |f(x)| < \delta \left| \frac{\sum x_i^2}{\sum x_i^2} \right| = \delta$$

So for all $\varepsilon > 0$, $\exists \delta > 0$, $\delta = \varepsilon$, for all $x \in \mathbb{R}^n$, $\|x\| < \delta$ then $|f(x)| < \varepsilon$.

$\Rightarrow f$ is continuous at $0$.

Assignment 16:

Prove that $\lim_{n\to\infty} \frac{1}{n^{5/2}} \sum_{k=1}^{n} k^{3/2} = \frac{2}{5}$

$$\frac{1}{n^{5/2}} \sum_{k=1}^{n} k^{3/2} = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{3/2} \frac{1}{n} \longrightarrow \int_0^1 x^{3/2}\,dx = \frac{2}{2} x^{5/2}\Big|_0^1 = \frac{2}{5}$$

# Preliminary Examination in Analysis
## January 10, 2003

(1) Prove that a continuous function on $\mathbb{R}$ has a finite or countable number of strict local maxima.

(2) Proof or counterexample: Let $f$ be a continuous function on $[0,1]$ that is differentiable on a dense subset. Also, $f' > 0$ whereever it is defined. Then $f$ is increasing. (Hint: think about the Cantor function.)

(3) Find
$$\lim_{n\to+\infty} n^2 \int_0^1 e^{x^2} x^n (1-x)\,dx.$$
Hint: $\lim_{n\to\infty} n^2 \int_0^1 x^n(1-x)\,dx = 1$.

(4) Let $a_n, b_n \geq 0$. Assume that $\sum a_n$ converges and that $\limsup_{n\to\infty} \frac{b_n}{a_n} \leq M < \infty$. Show that $\sum b_n$ converges.

Solution in

(5) Let $f(x)$ be a differentiable mapping of the connected open subset $V$ of $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that $f'(x) = 0$ on $V$. Prove that $f$ is constant on $V$.

(6) Let $f(x,y) = (u,v)$, where $u = x^4 - y^4$ and $v = 2xy$, be a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. (a) Show that if $(u,v) \neq (0,0)$ then $f$ has an inverse in a neighborhood of $(u,v)$. (b) Show that there is no neighborhood of $(0,0)$ in which $f$ has an inverse.

Jan 2003, P2 NTK *

1) Prove that a continuous function on $\mathbb{R}$ has a finite or countable number of strict local maxima.

- We have if $x^*$ is a strict local maxima
  Then $\exists \delta_x$, $\forall x \in \mathbb{R}$, $0<|x-x^*|<\delta$ then $f(x)<f(x^*)$
- For each $n$, consider $E_n = \{ x^* \mid \forall x \; 0<|x-x^*|<\frac{1}{n}, \text{ then } f(x)<f(x^*) \}$.

  Then we have "the set of local maxima" $= \bigcup_{n=1}^{\infty} E_n$.

- Now we have

$\forall a,b \in E_n$, $\left[\begin{array}{l} \text{if } |a-b|<\frac{1}{n}, \text{ then we have } \begin{cases} f(a)<f(b) \\ f(a)>f(b) \end{cases} \Rightarrow a=b. \\ \text{if } |a-b| \geq \frac{1}{n}, \text{ ~~then because a~~} \end{array}\right.$

So we have each $E_n$ is countable or finite

$\Rightarrow$ The set of local maxima is finite or countable $\square$.

Jan 2003, P3

Find $\lim\limits_{n\to\infty} n^2\int_0^1 e^{x^2} x^n(1-x)\,dx$ Hint : $\lim\limits_{n\to\infty} n^2\int_0^1 x^n(1-x)\,dx = 1$

* We first prove a useful claim that is used in this problem.

$$\lim_{n\to\infty} n^{2}\int_0^1 x^{n+k}(1-x)\,dx = 1 \quad , \quad \forall k$$

we have $n^2\int_0^1 x^{n+\ell}(1-x)dx = n^2\int_0^1 x^{n+\ell} - x^{(n+1)+\ell}\,dx = n^2\left(\frac{1}{n+\ell+1} - \frac{1}{n+\ell+2}\right) =$

$$= n^2\frac{1}{(n+\ell+1)(n+\ell+2)} \xrightarrow{n\to\infty} 1$$

* Now we have $e^x = \sum\limits_{\ell=0}^{\infty}\frac{x^\ell}{\ell!}$ Then $e^{2x} = \sum\limits_{\ell=0}^{\infty}\frac{x^{2\ell}}{\ell!}$

(using Taylor theorem is a very important step in this problem)

so we have

$$\underbrace{n^2\int_0^1 e^{x^2}x^n(1-x)dx}_{(*)} = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}n^2\int_0^1 x^{2k+n}(1-x)\,dx.$$

$$\lim(*) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\underbrace{\lim_{n\to\infty}\left(n^2\int_0^1 x^{2\ell+n}(1-x)dx\right)}_{=1 \text{ from above.}}$$

$$= \sum_{\ell=0}^{\infty}\frac{1}{\ell!} = e \qquad \square .$$

Jan 2003, P4.

$a_n, b_n \geqslant 0$

$\sum a_n$ converges

$\limsup\limits_{n\to\infty}\frac{b_n}{a_n} \leq M < +\infty$

Show that $\sum b_n$ converges.

Jan 2003, #10.

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad u = x^4 - y^4$ be a map

$(x,y) \longmapsto f(x,y) = (u,v) \quad v = 2xy$

a) Show that if $(u,v) \neq (0,0)$ then $f$ has an inverse in a neighborhood of $(u,v)$

b) Show that there is no neighborhood of $(0,0)$ in which $f$ has an inverse.

a)

We have $D_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{bmatrix} 4x^3 & -4y^3 \\ 2y & 2x \end{bmatrix}$ $\quad \det D_f = 8x^4 + 8y^4 = 8(x^4 + y^4)$

a) If $(u,v) \neq (0,0)$,

this means $\begin{cases} x^4 - y^4 \neq 0 \\ 2xy \neq 0 \end{cases} \Rightarrow \Big\{ x \neq 0 \text{ and } y \neq 0 \quad \Rightarrow x^4 + y^4 \neq 0 \Rightarrow \det D_f \neq 0.$

Then by inverse function theorem, if $(u,v) \neq (0,0)$, $f$ has an inverse in a neighborhood of $(u,v)$.

b) But in case $(u,v) \neq (0,0)$.

$\begin{cases} x^4 - y^4 = 0 \\ 2xy = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow \det D_f = 0 \Rightarrow$ does not satisfy IFT

Now consider a neighborhood of $(0,0)$, $N_\varepsilon(0,0)$, in any neighborhood

we have $f(-x,-y) = f(x,y) \rightarrow$ this means $f$ is not injective

of $(0,0)$

$\Rightarrow$ there is no neighborhood of $(0,0)$ in which $f$ has an inverse $\square$.

Checked.

**Analysis Preliminary Exam** **August 16 2003**

1. If $f$ is continuous on $[a,b]$ and

See Theorem 6.20
Jan 2002/3.

one kind that need to pay attention.

$$F(x) = \int_a^x f(t)\,dt$$

for $x \in [a,b]$, show that $F' = f$ on $(a,b)$.

Same Aug 2015

2. Prove that

$$\left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n \to \gamma$$

for some $\gamma \in (1/2, 1)$.

3. Let $f : \mathbf{R}^1 \to \mathbf{R}^1$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational or } x = 0 \\ p \sin\frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with no common divisors.} \end{cases}$$

Where is $f$ continuous?

4. For each $n$ let $f_n : \mathbf{R}^1 \to \mathbf{R}^1$ be a non-decreasing function, and assume $f_n$ converges point-wise to a continuous function $f$. Prove that $f_n$ converges uniformly on compact sets to $f$.

Template.
Rudin 7.20

5. Let $f$ be a continuous function on $[0,1]$ such that

Template use Stone Weierstrass theorem

$$\int_0^1 e^{-\frac{nx}{1-x}} f(x)\,dx = 0$$

for all $n \geq 0$. Show that $f$ is identically zero.

form prove $f \equiv 0$.

6. Show that there is an open interval $I$ containing 0 and a unique curve $(x(t), y(t)), t \in I$ with $(x(0), y(0)) = (1,1)$ satisfying

$$(*) \qquad \begin{aligned} x + y^2 + \sin t &= 2 \\ x^2 + ty^2 &= 1. \end{aligned}$$

Find the velocity of the curve at $t = 0$. For a given $t_0 \in I$ is there a unique solution $(x, y)$ to $(*)$ with $t = t_0$?

# Preliminary Examination in Analysis, January 2004

checked.

1. Show that if $E \subseteq \mathbb{R}^k$ is not compact then there is a continuous function $f : E \to \mathbb{R}$ which is unbounded.

See Aug 2007 P2. Aug 1994

2. Let $f : (0, +\infty) \to \mathbb{R}$ be a differentiable function such that $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = 0$. Prove that there is a sequence $x_n \nearrow +\infty$ such that $f'(x_n) \to 0$.

3. Let $f : [x_1, x_2] \to \mathbb{R}$ be a differentiable function, where $0 < x_1 < x_2$. Prove that there exists $c \in (x_1, x_2)$ such that

$$\frac{1}{x_1 - x_2} \begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = f(c) - cf'(c).$$

See Jan 2009, P2

4. Let $f, \rho : [0, +\infty) \to \mathbb{R}$ be functions which are Riemann integrable on each interval $[0, A]$, $A > 0$. Assume that $\rho(x) \geq 0$ for all $x \geq 0$ and

$$\int_0^{+\infty} \rho(x)\, dx = 1, \quad \lim_{x \to +\infty} f(x) = L \in \mathbb{R}.$$

(i) Calculate $t \displaystyle\int_0^{+\infty} \rho(tx)\, dx$, where $t > 0$.

(ii) Show that $\lim\limits_{t \searrow 0} t \displaystyle\int_0^{+\infty} \rho(tx) f(x)\, dx = L$.

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n + x^{2n}}$. Find all the values $x \geq 0$ where the series is convergent. Show that the series converges uniformly on the set $[0, 1/2] \cup [2, +\infty)$. Is the series uniformly convergent on $[0, 1)$? Justify your answer.

Not do

Almost agree with Aug 2008 P1.

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that $f$ is uniformly continuous on $\{(x, y) : x^2 + y^2 \leq 1\}$. Find the first order partial derivatives of $f$ at $(0, 0)$. Is $f$ differentiable at $(0, 0)$? Justify your answer.

Aug 2005

P7 If $f$ is a continuous function on $[a,b]$.

$$F(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a,b].$$

Show that $F' = f$ on $(a,b)$.

Note that we can't prove directly that $\lim\limits_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$:

consider: $$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h} \;???$$

* We have $f$ is continuous on $[a,b]$, $[a,b]$ compact $\Big\}$ $\Rightarrow$ $f$ is uniformly continuous on $[a,b]$ $\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0, \forall |s-t| < \delta$, then $|f(t)-f(s)| < \varepsilon$ (*)

* We want to prove $F'(= f$ on $[a,b]$

Then fixed any $x_0$ in $[a,b]$, we want to prove

$$\lim_{t\to s} \frac{F(t)-F(s)}{t-s} = f(x_0).$$

(when $x_0 - \delta < x_0 < t < x_0 + \delta$

We have for any $s,t$ s.t $x_0-\delta < s < x_0 < t < x_0+\delta$, we have

$$\left|\frac{F(t)-F(s)}{t-s} - f(x_0)\right| = \left|\frac{\int_a^t f(x)dx - \int_a^s f(x)dx}{t-s} - f(x_0)\right| = \left|\frac{\int_s^t f(x)dx}{t-s} - \frac{1}{t-s}\int_s^t f(x_0)dx\right.$$

$$\leq \left|\frac{1}{t-s}\right| \int_s^t \underbrace{|f(x)-f(x_0)|}_{<\varepsilon \text{ (because of (*))}} dx = \varepsilon.$$

don't really need

This means $F' = f$ on $[a,b]$ □.

just use:

$$\left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| = \left|\frac{\int_a^t f(x)dx - \int_a^{x_0} f(x)dx}{t-x_0} - f(x_0)\right| = \left|\frac{1}{t-x_0}\int_{x_0}^t f(x)dx - f(x_0)\right|$$

$$= \left|\frac{1}{t-x_0}\int_{x_0}^t f(x)dx - \frac{1}{t-x_0}\int_{x_0}^t f(x_0)dx\right|$$

$$\leq \left|\frac{1}{t-x_0}\right| \int_{x_0}^t \underbrace{|f(x)-f(x_0)|}_{<\varepsilon} dx = \varepsilon$$ □.

Aug 200

P2 > Prove that $\left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n \longrightarrow \gamma$ for some $\gamma \in (1/2, 1)$ | Solution in Aug 2005

Aug 2003, P3 >

Need to review.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto f(x) = \begin{cases} x & , x = 0, x \in \mathbb{R}\setminus\mathbb{Q} \\ p \sin \frac{1}{q} & , \text{if } x = \frac{p}{q} \end{cases}$$ where $p, q$ are integers with no common divisors.

Where is $f$ continuous?

* First, $f$ is continuous at 0: NTP $\forall \varepsilon > 0, \exists \delta, \forall y \in N_\delta(x) \cap \mathbb{R}$, $|f(y) - f(x)| < \varepsilon$.

we have when $y \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow$ chose $\delta = \varepsilon$, $|y - x| < \delta$, then $|f(y) - f(x)| = |y - x| < \varepsilon$.

$y \in \mathbb{Q} \Rightarrow y = \frac{p}{q}$ where $\frac{p}{q} < 1 \Leftrightarrow p < q$ and $y \to x$ $\Leftrightarrow q \to \infty$ $f(y) = \left|p \sin \frac{1}{q}\right| = |p| \frac{1}{q}$ $\frac{\sin \frac{1}{q}}{\frac{1}{q}} = \frac{p}{q}$

So $f$ is continuous at 0.

* Prove that $f$ is not continuous at all $x \in \mathbb{Q}\setminus\{0\}$

- Note that we have $q$ integer, $q \geq 1 \Rightarrow \frac{1}{q} \leq 1 \Rightarrow \sin \frac{1}{q} < \frac{1}{q} \Rightarrow p \sin \frac{1}{q} < \frac{p}{q}$.

This means $\boxed{f(x) < x}$, $\forall x \in \mathbb{Q}\setminus\{0\}$.

- Then let $\{x_n\} \subseteq \mathbb{R}\setminus\mathbb{Q}$, $x_n \longrightarrow x$

if $f$ continuous at $x$, then $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \xlongequal[f(x_n) = x_n = (x_n \in \mathbb{R}\setminus\mathbb{Q})]{} \lim x_n = x$

contradicts with $\boxed{f(x) < x}$.

So $f$ is not continuous at all $x \in \mathbb{Q}\setminus\{0\}$.

* Prove that $f$ is continuous at all $x_0 \in \mathbb{R}\setminus\mathbb{Q}$.

- If $y \in \mathbb{R}\setminus\mathbb{Q}$ and $|y - x| < \delta$ (chose $\delta = \varepsilon$), then $|f(y) - f(x)| = |y - x| < \varepsilon$.

- If $y \in \mathbb{Q}$, $y = \frac{p}{q}$, chose $\delta = \varepsilon/2$

then $|f(y) - f(x)| = \left|p \sin \frac{1}{q} - x\right| = \left|p \sin \frac{1}{q} - \frac{p}{q} + \frac{p}{q} - x\right|$

$$\leq \underbrace{\left|\frac{p}{q}\left(q \sin \frac{1}{q} - 1\right)\right|}_{< \varepsilon/2} + \underbrace{\left|\frac{p}{q} - x\right|}_{< \delta}$$

note that $\frac{\sin \frac{1}{q}}{\frac{1}{q}} - 1 \xrightarrow{q \to \infty} 0 \longrightarrow < \varepsilon/2$

$< \varepsilon \Rightarrow$

Aug 2003 7.

* Sth need to learn
Need to learn

I 4) For each $n$, let $f_n: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ be a non decreasing function, assume $f_n \longrightarrow f$ pointwise, $f$ is continuous. Prove that $f_n \rightrightarrows f$ on compact sets.

* $f_n$: increasing function. (1)

* $f_n \longrightarrow f$ pointwise in $K$ compact.

$\Leftrightarrow \forall x \in K, \forall \epsilon > 0, \exists n_{\epsilon,x}, \forall n \geqslant n_{\epsilon x}, |f_n(x) - f(x)| < \epsilon$. (2)

* $f$ is continuous on $K$ compact $\Rightarrow$ uniformly continuous.

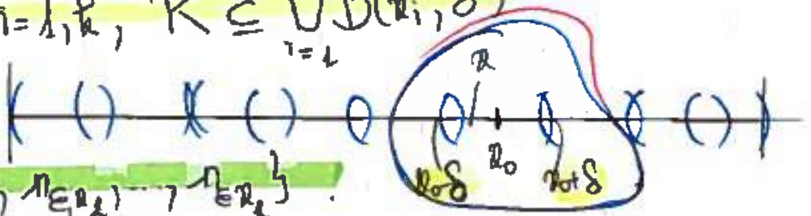
$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in K, |x - y| < \delta, |f(x) - f(y)| < \epsilon$ (3)

NTP $f_n \rightrightarrows f$ on $K$.

NTP
$\forall \epsilon > 0, \exists N > 0, \forall n \geqslant N, \forall x \in K, |f_n(x) - f(x)| < \epsilon$

* We have $K$ is compact in $\mathbb{R}$, then every open cover of $K$ contain a finite sub cover.

consider $K \subset \bigcup_{x \in K} B(x, \delta)$ then $\exists x_i, i = \overline{1,k}$, $K \subseteq \bigcup_{i=1}^{k} B(x_i, \delta)$



Now because of (2), (2) is true for all $x_i, i = \overline{1,k}$, we choose $N = \max\{n_{\epsilon,x_1}, n_{\epsilon,x_2}, \dots, n_{\epsilon x_k}\}$.

we have from this, $\forall x_i, i = \overline{1,k}, \forall \epsilon > 0, \exists N, \forall n \geqslant N, |f_n(x_i) - f(x_i)| < \epsilon$ (2')

• Consider any $x \in K$, we have because $K \subset \bigcup_{i=1}^{k} B(x_i, \delta)$, $\exists x_0$ such that $x \in B(x_0, \delta)$

So we have

$$|f_n(x) - f(x)| \leq \underbrace{|f_n(x) - f_n(x_0)|}_{\leq |f_n(x_0+\delta) - f_n(x_0)| \text{ because } f_n \text{ increasing}} + \underbrace{|f_n(x_0) - f(x_0)|}_{<\epsilon \text{ because of } (2')} + \underbrace{|f(x_0) - f(x)|}_{<\epsilon \text{ because } x \in B(x_0,\delta) \text{ and } (3)}.$$

this is an important step when we use $f_n$ is increasing in $K$ capact (in $\mathbb{R}$).

$$\leq \underbrace{|f_n(x_0+\delta) - f(x_0+\delta)|}_{<\epsilon \text{ becaus } (2')} + \underbrace{|f(x_0+\delta) - f(x_0)|}_{<\epsilon \text{ beca } f \text{ cont}} + \underbrace{|f(x_0 - f_n(x_0)|}_{<\epsilon \text{ because } (2')}$$

So $|f_n(x) - f(x)| \leq 5\epsilon$, $\forall n \geqslant N, \forall x \in K$.

this is what we need to prove $\square$

Aug 2003, 267

*

Show that there is an open interval $I$ containing $0$ and a unique curve $(x(t), y(t))$, $t = I$ with $(x(0), y(0)) = (1, 1)$

satisfying $\begin{cases} x + y^2 + \sin t = 2 \\ x^2 + t y^2 = 1 \end{cases}$ $(*)$

a) Find the velocity of the curve at $t = 0$

b) For a given $t_0 \in I$, is there a unique solution $(x, y)$ to $(*)$ with $t = t_0$ ?

---

Put $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x,y,t) \longmapsto F(x,y,t) = \left( F_1(x,y,t) = x + y^2 + \sin t - 2 \;;\; F_2(x,y,t) = x^2 + ty^2 - 1 \right)$$

So we have

$$DF = \begin{array}{c|cc|c} & x & y & t \\ F_1 & 1 & 2y & \cos t \\ F_2 & 2x & 2ty & y^2 \\ & & A_{xy} & A_t \end{array}$$

- ~~becau~~ So we have $F(x,y,t) = 0$ (1)
- because all partial derivative exist and continuous.

F is a continuous differentiable function (2)

So we have $A_{xy} = \begin{bmatrix} 1 & 2y \\ 2x & 2ty \end{bmatrix}$ $\quad A_t = \begin{bmatrix} \cos t \\ y^2 \end{bmatrix}$

At $t = 0$, $x(0) = y(0) = 1$

$$A_{xy}\big|_{(1,1,0)} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \qquad \text{have } \det A_{xy(1,1,0)} = -4 \neq \quad (3)$$

From (1)(2)(3), by Implicit Function theorem, there is a open neighborhood $V \subset \mathbb{R}^3$ of $(1,1,0)$ and a open neighborhood $I$ ~~of~~ $\mathbb{R}$ of $0$ such that

$$\forall t \in I, \ \exists! (x,y) \in \mathbb{R}^2 \text{ such that } \begin{cases} (x,y,t) \in V \\ F(x,y,t) = 0 \end{cases}$$

this means, we can define $(x,y) = (x(t), y(t))$

a) The velocity of the curve at $t = 0$

we have
$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix}_{(1,1,0)} = -\left[A_{xy}\right]^{-1}_{(1,1,0)} \left[A_t\right]_{(1,1,0)} = -\frac{1}{-4} \begin{bmatrix} 0 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$$

So the velocity of the curve at $t = 0$ is $\begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$

b) For $t = t_0$, there are 2 points $(x, y), (x, -y)$ satisfies $(*)$ ?

there is more than

Aug 2015

* template.

5) Let $f$ be a continuous function on $[0,1]$ such that

$$\int_0^1 e^{\frac{-nx}{1-x}} f(x)\,dx = 0, \ \forall n \geq 0$$

Show that $f$ is identically zero

form $\int_0^1 e^{k(x)} f(x)$ .

+ Let $\mathcal{A} = \left\{ P_n(x) = \sum_{i=0}^{n} a_i e^{\frac{-ix}{1-x}} = a_0 + a_1 e^{\frac{-x}{1-x}} + a_2 e^{\frac{-2x}{1-x}} + \dots + a_n e^{\frac{-nx}{1-x}} \;\middle|\; a_i \in \mathbb{R}, \forall i = \overline{1,n} \right\}$

We will prove that $\mathcal{A}$ is an algebra, seperates points and vanishes at no point .

- Prove that $\mathcal{A}$ is an algebra

Let $f, g \in \mathcal{A}$, then $f = \sum_{i=1}^{n} a_i e^{\frac{-ix}{1-x}}$ and $g = \sum_{j=1}^{m} b_j e^{\frac{-jx}{1-x}}$, then we have .

⊕ WLOG, assume $n \geq m$, let $b_j = 0, \forall j \geq m$, we have $(f+g) = \sum_{i=1}^{n} (a_i + b_i) e^{\frac{-ix}{1-x}}$

⊕ $(f \cdot g) = \sum_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}} (a_i b_j) e^{\frac{-(i+j)x}{1-x}}$

⊕ $\forall c \in \mathbb{R}, (cf) = c \sum_{i=1}^{n} a_i e^{\frac{-ix}{1-x}} = \sum_{i=1}^{n} (ca_i) e^{\frac{-ix}{1-x}}$ ▪

- Prove that $\mathcal{A}$ seperates points : Let $x \neq y$, Prove that $\exists P_n(x)$ such that $P_n(x) \neq P_n(y)$

Consider $P_1(x) = e^{\frac{-x}{1-x}}$ we can't take $h(x_1) - h(x_2)$ directly, but computing $h'(x)$ is a general way .

Put $h(x) = \frac{-x}{1-x}$, we have $h'(x) = \frac{-1(1-x) + x(-1)}{(1-x)^2} = \frac{-1}{(1-x)^2} < 0, \forall x$

$n=1$ $\Rightarrow$ $h$ is a ~~mo~~ strictly monotone functions $\Rightarrow$ if $x \neq y$, we have $\frac{-x}{1-x} \neq \frac{-y}{1-y}$ . ▪

+ Besides exp is a bijective function from $[0,+\infty) \to \mathbb{R}$ $\Rightarrow$ $e^{\frac{-x}{1-x}} \neq e^{\frac{-y}{1-y}}$ if $x \neq y$

- Prove that $\mathcal{A}$ vanishes at no point, $(\forall x \in [0,1])$, Prove that $\exists P_n(x) \in \mathcal{A}$, s.t $P_n(x) \neq 0$

We also choose $P_1(x) = e^{\frac{-x}{1-x}}$

we have exp is a monotone increasing function of continuous functions, and $e^x > 0, \forall x$ ▪

$\Rightarrow$ Because $\mathcal{A}$ is an algebra, seperates points, vanishes at no point + $[0,1]$ compact .

then the uniform closure of $\mathcal{A}$ is $C([0,1], \mathbb{R})$ .

$f$ is a continuous function $[0,1] \longrightarrow \mathbb{R}$ ,

then by Wierstrass theorem , $\exists P_n$, $P_n \rightrightarrows f$ on $[0,1]$ .

* we have $P_n \rightrightarrows f$ $\{P_n\}$ sequence of bounded funcs $\Rightarrow$ $P_n f \rightrightarrows f^2$ then by the theorem about uniformly convergence and integration

$$\int_0^1 f^2 dx = \int \lim_{n\to\infty} P_n f\,dx = \lim_{n\to\infty} \int_0^1 P_n(x) f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \underbrace{\int_0^1 e^{\frac{-kx}{1-x}} f(x)}_{=0} = 0$$

By Homework 6 L Rudin

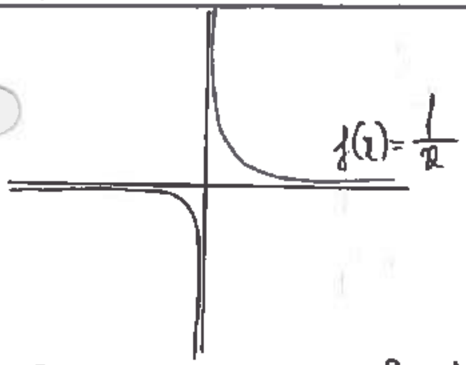
$$\left( \left.\begin{matrix} \int g(x) = 0 \\ g(x) \geq 0 \end{matrix}\right\} \Rightarrow g \equiv 0 \text{ on } [0,1] \right)$$

then we have because $\left.\begin{matrix} \int_0^1 f^2(x) = 0 \\ f^2(x) \geq 0 \end{matrix}\right\} \Rightarrow f^2(x) = 0 \quad , \forall x \in [0,1]$

$\Rightarrow f(x) = 0 \quad , \quad \forall x \in [0,1]$ . We win .

Jan 2004 $f: E \longrightarrow \mathbb{R}:$

I) Show that if $E \subseteq \mathbb{R}^l$ is not compact then $\exists$ continuous function which is unbounded



We have $E \subseteq \mathbb{R}^l$ is compact ~~iff~~ then $\begin{cases} E \text{ is closed} \\ E \text{ is bounded.} \end{cases}$

So $E$ is not compact if $\begin{cases} E \text{ is not closed} \\ E \text{ is unbounded} \end{cases}$.

* In case $E$ is not closed $\Leftrightarrow \exists$ a limit point of $E$ that is not belong to $E$

Assume the point $a$ is a point that is a limit of $E$ and is not belong to $E$.

Now define
$$f: E \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{1}{\|x-a\|}$$

- $f$ is continuous because $1$ is continuous, $x-a$ is continuous
- $f$ is unbounded beccause $\lim_{x \to a} \frac{1}{x-a} = +\infty$.

* In case $E$ is unbounded,

Define
$$f: E \subseteq \mathbb{R}^l \longrightarrow \mathbb{R}.$$
$$x \longmapsto \|x\|$$

- $f$ is continuous
- Actually $f$ is unbounded $\Rightarrow$ done $\square$

Remine : $f: \mathbb{R}^l \longrightarrow \mathbb{R}$, $x \longmapsto \|x\|$ is continuous.

Jan 2004, P2) See 1994 P2 (very similar)

P2) Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be a differentiable function

$$\lim_{x \to +\infty} \frac{f(x)}{x} = 0$$

Prove that there exist a sequence $x_n \nearrow +\infty$ s.t $f'(x_n) \longrightarrow 0$.

We first redo a problem from Prelim Aug 1994, P2) :

$f (0,+\infty)$ be a differentiable function

$\lim_{x \to +\infty} \frac{f(x)}{x} = 0$ } then $\lim_{\xi \to \infty} f'(\xi) = 0$.

We have $\lim_{x \to +\infty} \frac{f(x)}{x} = 0 \iff \forall \varepsilon > 0, \exists N > 0$ s.t $\forall x > N$ $\left(\frac{1}{2}\right) \left|\frac{f(x)}{x}\right| < \frac{\varepsilon}{2}$.

So we have

$$\frac{1}{2}\left|\frac{f(x)}{x}\right| < \frac{\varepsilon}{2} \Rightarrow \left|\frac{f(x)}{2x} - \frac{f(2x)}{2x} + \frac{f(2x)}{2x}\right| < \frac{\varepsilon}{2}$$

$$\Rightarrow \left|\frac{f'(\xi)(-x)}{2x} + \frac{f(2x)}{2x}\right| < \frac{\varepsilon}{2}$$ for some $\xi$ between $(x, 2x)$

$$\Rightarrow \left|\frac{f'(\xi)}{2}\right| - \left|\frac{f(2x)}{2x}\right| \leq \left|\frac{f'(\xi)}{2} + \frac{f(2x)}{2x}\right| < \frac{\varepsilon}{2}$$

$$\Rightarrow \left|\frac{f'(\xi)}{2}\right| < \left|\frac{f(2x)}{2x}\right| + \frac{\varepsilon}{2} < \varepsilon$$

So $\lim_{\xi \to \infty} |f'(\xi)| = 0 \Rightarrow \lim_{\xi \to \infty} f'(\xi) = 0$.

* So now we create a sequence $(x_n)$ increasing by setting $\begin{cases} x_1 = 1 \\ x_n = 2x_{n-1}, \quad n = 2, 3, \dots \end{cases}$

So we have a increasing $\{x_n\}$.

From above, we have a increasing $\{\xi_n\}$, $\xi_n \nearrow \infty$ (because $\xi_n$ between $(x_n, x_{n+1})$)

and $f'(\xi_n) \longrightarrow 0$ $\square$.

Jan 2004

I3) Let $f:[x_1, x_2] \longrightarrow \mathbb{R}$ be a differentiable function, where $0 < x_1 < x_2$.

Prove that $\exists c \in [x_1, x_2]$ $\frac{1}{x_1 - x_2}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = f(c) - c f'(c)$. (*)

* Generalized mean value theorem

Let $f, g$ continuous on $[a,b]$, $f, g$ differentiable on $(a,b)$ $\Big\}$ Then $\exists c \in [a,b]$ $[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\,f'(c)$

(Prove put $h(x) = \begin{vmatrix} f(x) & g(x) \\ f(b)-f(a) & g(b)-g(a) \end{vmatrix}$ Then $h(b) - h(a) = 0$ means $h(b) = h(a) =$ by mean value theorem $\exists c$ s.t $h'(c) = 0 \Rightarrow$ What NTP

From the generalized mean value theorem, we have (change to h and g).

$$\frac{h(b) - h(a)}{g(b) - g(a)} = \frac{h'(c)}{g'(c)}.$$

We have the RHS(*) $= f(c) - c f'(c) = -[c f'(c) - f(c)] = -c^2\left[\frac{f(c)}{c}\right]' = \dfrac{\left[\frac{f(c)}{c}\right]'}{\left[+\frac{1}{c}\right]'}$

So put $h(x) = \frac{f(x)}{x}$ $\quad g(x) = \frac{1}{x}$.

note that one important trick here is using this to have $h(x) = \frac{f(x)}{x}$ $\Rightarrow$ then we can get $g(x)$ from that

So by generalized mean value theorem, $\exists c$ s.t

$$\frac{h(x_2) - h(x_1)}{g(x_2) - g(x_1)} = \frac{h'(c)}{g'(c)} = \frac{\left[\frac{f(c)}{c}\right]'}{\left[\frac{1}{c}\right]'} = \cdots = f(c) - c f'(c). = \text{RHS}$$

$$\parallel$$

$$\frac{\frac{f(x_2)}{x_2} - \frac{f(x_1)}{x_1}}{\frac{1}{x_2} - \frac{1}{x_1}} = \frac{\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 x_2}}{\frac{x_1 - x_2}{x_1 x_2}} = \frac{1}{x_1 - x_2}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = \text{LHS}$$

$\square$.

147 Jan 2047. [illegible]

Let $f, \rho : [0,+\infty) \longrightarrow \mathbb{R}$ be functions which are Riemann integrable on each interval $[0,A]$, $A>0$

Assume that $\rho(x) \geq 0$, $\forall x \geq 0$ and $\displaystyle\int_0^{+\infty} \rho(x)\,dx = 1$ $\displaystyle\lim_{x\to+\infty} f(x) = L$.

a) Calculate $\displaystyle t\int_0^{+\infty} \rho(tx)\,dx$, where $t>0$

b) Show that $\displaystyle\lim_{t\searrow 0} t\int_0^{+\infty} \rho(tx) f(x)\,dx = L$. *

a) Calculate $\displaystyle t\int_0^{+\infty} \rho(tx)\,dx$, where $t>0$ :

Put $u = tx \Rightarrow du = t\,dx$ $\begin{cases} x = 0 \Rightarrow u = 0 \\ x = +\infty \Rightarrow u = +\infty \quad (t>0) \end{cases}$

So $\displaystyle t\int_0^{+\infty} \rho(tx)\,dx = \int_0^{\infty} \rho(u)\,du = 1$ □a)

This is a really good trick can be use in both improper integral and finding value of series by dividing to $\displaystyle\int_0^{N}$ and $\displaystyle\int_N^{+\infty}$ or $\displaystyle\sum_{k=1}^{N}$ $\displaystyle\sum_{k=N}^{\infty}$ (Jan 2009)

b) Show that $\displaystyle\lim_{t\searrow 0} t\int_0^{+\infty} \rho(tx) f(x)\,dx = L$.

- We have $\displaystyle\lim_{x\to+\infty} f(x) = L \Leftrightarrow \forall \varepsilon>0, \exists N>0$ s.t $\forall x$, $x>N$, $|f(x) - L| < \varepsilon$

Then we want to prove $\displaystyle\lim t\int_0^{+\infty} \rho(tx) f(x)\,dx = L \Leftrightarrow$ WTP $\displaystyle\left| t\int_0^{+\infty} \rho(tx) f(x)\,dx - L \right|$

We have
$$\left| t\int_0^{+\infty} \rho(tx) f(x)\,dx - L \right| = \left| t\int_0^{+\infty} \rho(tx)\, f(x)\,dx - \underbrace{t\int_0^{+\infty} \rho(tx) L\,dx}_{=L} \right|$$
$$\leq \underbrace{t\int_0^{N} |\rho(tx)|\,|f(x) - L|\,dx}_{\text{integrable} \xrightarrow[t\to 0]{} 0} + t\int_N^{+\infty} \rho(tx) \underbrace{|f(x) - L|}_{<\varepsilon}\,dx$$
$$< \varepsilon\, t\int_N^{+\infty} \rho(tx)\,dx$$
$$< \varepsilon\, \underbrace{t\int_0^{\infty} \rho(tx)\,dx}_{=1}$$
$< \varepsilon$ □.

Jan2004>P5.

Consider the series $(*) = \sum_{n=1}^{\infty} \frac{x^n}{n+x^{2n}}$

a> Find all the value $x \geq 0$ where the series is convergent.

b> Show that the series converges uniformly on $[0, 1/2] \cup [2, +\infty)$

c> Is the series uniformly convergent on $[0, 1)$

Justify your answer.

a> Find all the value $x \geq 0$ where the series is convergent

Note that when $x \geq 0$, we have $\begin{cases} n + x^{2n} \geq n \\ n + x^{2n} \geq x^{2n} \end{cases}$ $\Rightarrow \frac{x^n}{n+x^{2n}} \leq \frac{x^n}{n}$ (1)

$\Rightarrow \frac{x^n}{n+x^{2n}} \leq \frac{x^n}{x^{2n}} = \frac{1}{x^n}$ (2).

- When $x > 1$:

we have $\sum \frac{1}{x^n}$ converges, then because of (2), the series converges.

- When $x = 1$

$(*) = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

- when $x < 1$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1}}{(n+1)+x^{2(n+1)}} \frac{[(n)+x^{2n}]}{x^n} = x \frac{n+x^{2n}}{(n+1)+x^{2(n+1)}} \longrightarrow x < 1 \text{ converges. } ~~\Rightarrow \text{diverges.}~~$$

- when $x = 0 \Rightarrow$ converges

So the series converges $\forall x \geq 0, x \neq 1$.

b> Show that the series converges uniformly on $[0, 1/2] \cup [2, +\infty)$

- When $x \in [0, 1/2]$

because of (1) $\left|\frac{x^n}{n+x^{2n}}\right| \leq \left|\frac{x^n}{n}\right| \leq \left|\frac{(1/2)^n}{n}\right| = \left|\frac{1}{2^n n}\right|$ and we have $\sum_{n=1}^{\infty} \frac{1}{2^n n}$ converges $\Bigg\} \Rightarrow (*)$ converges uniformly.

- When $x \in [2, +\infty)$

because of (2), $\left|\frac{x^n}{n+x^{2n}}\right| \leq \left|\frac{1}{x^n}\right| \leq \frac{1}{2^n}$ we also have $\sum \frac{1}{2^n}$ converges $\Bigg\} \Rightarrow (*)$ converges uniformly

c> Is the series uniformly convergent on $[0, 1)$?

Note ~~that when $x < 1$, then $x = \frac{p}{q}$ where~~

Jan 2004 > I6

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$(x,y) \longmapsto \begin{cases} \frac{x^2 y}{x^2+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0). \end{cases}$$

a> Show that $f$ is uniformly continuous on $\{(x,y), x^2+y^2 \leq 1\}$.

b> Find the first order partial derivatives of $f$ at $(0,0)$.

c> Is $f$ differentiable at $(0,0)$. Justify.

a> Show that $f$ is uniformly continuous on $\{(x,y) , x^2+y^2 \leq 1\}$

* Note that with this question, we already have $f$ is continuous $\forall (x,y) \neq (0,0)$

$\Rightarrow$ we only care when $(x,y) \longrightarrow (0,0)$, and use $f$ continuous on compact set $\Rightarrow$ uniformly co

* We have

$$\left|\frac{x^2 y}{x^2+y^2}\right| \underset{\substack{\uparrow \\ \text{because} \\ x^2+y^2 \leq 1}}{\leq} |x^2 y| \underset{\substack{\text{because} \\ x^2+y^2 \leq 1}}{\leq} |y| \qquad \text{So we have } \lim_{(x,y)\to(0,0)} |f(x,y)| \Longrightarrow 0$$

So we have $f$ is continuous at $(0,0)$.

By formula of $f$, actually we have $f$ is continuous $\forall (x,y) \neq (0,0)$.

$\Bigg\} \rightarrow f$ is continuous for all $\{(x,y), x^2+y^2 \leq 1\}$

which is a compact set $\Longrightarrow$ $f$ is uniformly continuous on $\{(x,y), x^2+y^2 \leq 1\}$.

b> Find the first order partial derivative of $f$ at $(0,0)$.

~~$D_1 f = \frac{\partial f}{\partial x} = \frac{2xy(x^2+y^2) - x^2 y \, 2x}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}$~~ , when $(x,y) \neq (0,0)$.

* We don't compute $D_1 f$ by above (crossed way) we use definition.

$$D_1 f = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{0-0}{0} = 0 \qquad D_2 f = \lim \frac{f(0,h) - f(0,0)}{|h|} = \frac{0}{h} \to 0$$

c> Is $f$ differentiable at $(0,0)$.

We have $f$ is differentiable at $(0,0)$ iff

$$\lim_{(h_1,h_2)\to 0} \frac{\overbrace{f(h_1,h_2) - f(0,0) - f_x(0,0)(h_1) - f_y(0,0) h_2}^{:=*}}{\sqrt{h_1^2+h_2^2}} = 0$$

We have

$$(*) = \frac{\frac{h_1^2 h_2}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} = \frac{h_1^2 h_2}{(h_1^2+h_2^2)^{3/2}}$$

when $h_1 = h_2$

$$(*) = \frac{h_1^3}{(2h_1^2)^{3/2}} = \frac{h_1^3}{2^{3/2} h_1^3} \longrightarrow \frac{1}{2^{3/2}} \neq 0 .$$

So $f$ is not differentiable at $(0,0)$ $\square$.

checked
X

## Analysis Preliminary Exam, August 2005

NTR 1. Let $g$ be a continuous function on $[0,1]$ with $g(1)=0$, and let $h_n(x)=x^n g(x)$ for $n=1,2,\ldots$ Prove that $h_n$ converges uniformly on $[0,1]$.

NTR 2. Let $a_n$, $n=1,2,\ldots$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n$ converges.
(a) Prove that $\liminf_{n\to\infty} na_n = 0$.
(b) Show by example that $\limsup_{n\to\infty} na_n > 0$ is possible.

3. Let $F(x_1,x_2,y_1,y_2)=(x_1x_2+x_1y_1+y_2, x_1y_2+x_2y_1^2)$. Check that $F(1,1,1,1)=(3,2)$.
(a) Prove that there is a neighborhood $U$ of $(1,1,1,1)$ and a neighborhood $W$ of $(1,1)$ and a function $g : W \to \mathbb{R}^2$ such that for all $(y_1,y_2)\in W$ there is a unique $(x_1,x_2)\in\mathbb{R}^2$ given by $g(y_1,y_2)$ such that $(x_1,x_2,y_1,y_2)\in U$ and $F(x_1,x_2,y_1,y_2)=(3,2)$.
(b) Find $g'(1,1)$.
NTR (c) Find an approximate solution to the equation $F(x_1,x_2,1.001,1.003)=(3,2)$. Assume that $(1.001,1.003)\in W$.

4. Prove that

$$\lim_{n\to\infty}\frac{\ln(2)+\ln(3)+\cdots+\ln(n)}{n\ln(n)}=1.$$

5. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a continuously differentiable function such that that

$$f(tx)=t^5 f(x)\ ,\ \forall t>0\ ,\ \forall x=(x_1,\ldots,x_n)\in\mathbb{R}^n.$$

Prove that $f$ satisfies the partial differential equation

$$\sum_{j=1}^{n} x_j\frac{\partial f}{\partial x_j}(x)=5f(x),\ \forall x\in\mathbb{R}^n.$$

See page 334 6. Prove that if $\{a_n\}$ is a sequence of positive numbers, then

$$\limsup_{n\to\infty}(a_n)^{1/n}\le\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}$$

checked.

AUG 1 0 2006

**Analysis Preliminary Exam** **January 2006**

1. Prove the chain rule: if $g$ is differentiable at $a$, $g(a) = b$, and $f$ is differentiable at $b$, then $f \circ g$ is differentiable at $a$ and $(f \circ g)'(a) = f'(b)g'(a)$.

2. Let $f(0) = 0$ and $f(t) = t^2 \sin(1/t)$ for $t \neq 0$, and let $\phi(x, y) = f(x) + f(y)$.

   (a) Prove that $\frac{\partial \phi}{\partial x}$ exists everywhere in $\mathbf{R}^2$ but is not continuous at (0,0).

   (b) Prove that $\phi$ is differentiable at (0,0) and find $\phi'(0, 0)$.

Aug 2009, P3.
Sample C, P3.

3. Let $f : [0, 1) \to \mathbf{R}$ be differentiable with bounded derivative. Prove that $f$ can be extended to a continuous function on $[0, 1]$.

4. If $\sum_0^n \frac{a_k}{k+1} = 0$, prove that the polynomial $\sum_0^n a_k x^k$ has at least one root in the interval $(0, 1)$.

NTR

5. Assume $f : [0, \infty) \to \mathbf{R}$ is nonnegative, Riemann integrable on $[0, b]$ for every $b > 0$, and

$$\lim_{b \to \infty} \int_0^b f(t)\, dt < \infty.$$

Prove or give a counterexample;

   (a) $\lim_{x \to \infty} f(x) = 0$,

   (b) $f$ is continuous implies $\lim_{x \to \infty} f(x) = 0$,

   (c) $f$ is uniformly continuous implies $\lim_{x \to \infty} f(x) = 0$.

NTR.

6. Let $f, f_n : [0, 1] \to \mathbf{R}$ and $\phi : \mathbf{R} \to \mathbf{R}$. Prove or give a counterexample to each of the following statements;

   (a) If $f_n \to f$ uniformly on $[0, 1]$ and $\phi$ is continuous, then $\phi \circ f_n \to \phi \circ f$ uniformly.

   (b) If $f_n \to f$ uniformly on $[0, 1]$ and $\phi$ is uniformly continuous, then $\phi \circ f_n \to \phi \circ f$ uniformly.

   (c) If $f_n \to f$ uniformly on $[0, 1]$, and $f$ and $\phi$ are continuous, then $\phi \circ f_n \to \phi \circ f$ uniformly.

+ Aug 2007

P 1) Let $g$ be a continuous function on $[0,1]$, $g(1)=0$ add 1 more condition: $g(x)\geq 0$ $\forall x\in[0,1]$.
Let $h_n(x)=x^n g(x)$, $n=1,2,3,\dots$ } Prove that $h_n$ converges uniformly on $[0,1]$

(Review theorem 7.13 (sequence of function: continuous + pointwise converges + decreasing on compact set) → uniformly converges.
Let K

Theorem 7.13:
$\left.\begin{array}{l} K \text{ compact} \\ f_n \to f \\ f_n, f \text{ continuous} \\ f_n \geq f_{n+1} \end{array}\right\} \Rightarrow f_n \rightrightarrows f$.

(1) * We have $K=[0,1]$ compact

(2), We put $f_n(x)=x^n$, $\forall x\in[0,1]$, $n=1,2,3,\dots$

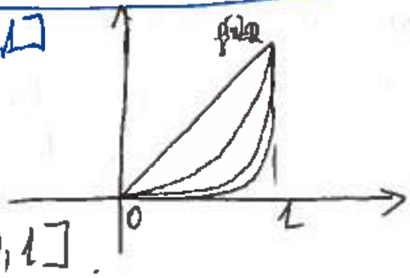
We have $\left.\begin{array}{l} f_n(x) \text{ continuous on } [0,1],\ \forall x, \forall n \\ g(x) \text{ continuous on } [0,1] \end{array}\right\} \Rightarrow h_n$ continuous on $[0,1]$.

(3) We have $f_n(x)=x^n \geq x^{n+1}=f_n(x)$, $\forall x\in[0,1]$

note that we can't use this way because we don't know if $g(x)\geq 0$, $\forall x\in[0,1]$
→ can't prove $h_n(x)\geq h_{n+1}(x)$, $\forall x\in[0,1]$

Keep doing this way in case we had $g(x)\geq 0$ $\forall x\in[0,1]$

Because $g(x)\geq 0$, $\forall x\in[0,1]$, we have
$h_n(x)=x^n g(x) \geq x^{n+1} g(x) = h_{n+1}(x)$, $\forall x$.



(4) ~~We have~~ Now we prove that $h_n(x)\longrightarrow 0$ (point wise) on $[0,1]$.

(Note that we have $f_n(x)\longrightarrow f(x)$, where $f(x)=\begin{cases} 0 & , x\in[0,1) \\ 1 & , x=1 \end{cases}$.

- For $x\in[0,1)$: $f_n(x)\longrightarrow 0$, $\Leftrightarrow \forall\varepsilon>0, \exists n_0\in\mathbb{N}, \forall n\geq n_0, |f_n(x)|<\varepsilon$ $\forall x\in[0,1)$

  $g$ continuous on $[0,1]$ $\Rightarrow$ bounded $\exists M, |g(x)|\leq M$, $\forall x\in[0,1]$

  $\Rightarrow \forall n\geq n_0$, $|f_n(x)\cdot g(x)|<\varepsilon$.

  This means $h_n(x)\longrightarrow 0$ on $[0,1)$

- At $x=1$: $\left.\begin{array}{l} f_n(x)=1, \forall n \\ g(1)=0 \end{array}\right\} \Rightarrow h_n(1)=f_n(1)g(1)\longrightarrow 0, \forall n$.

(1)+(2)+(3)+(4) $h_n(x)\rightrightarrows$ on $[0,1]$ □.

Case we don't have $g(x)\geq 0$, $\forall x\in[0,1]$ ⟶.

Aug2007

[1] Let $g$ is a continuous function on $[0,1]$.
$g(1) = 0$
Put $h_n(x) = x^n g(x)$, $n = 1,2,3,\ldots$

Prove that $h_n \rightrightarrows$ on $[0,1]$.

In case we have $g(x) \geq 0, \forall x \in [0,1]$ (last page).

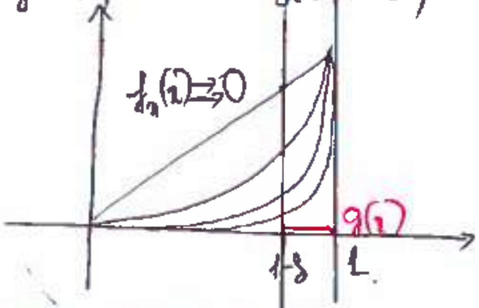
Note that we have $f_n(x) = x^n \rightrightarrows 0$ on $[0, 1-\delta]$.

$f_n(x) = x^n \not\rightrightarrows 0$ on $[0,1]$, but compensate for this, we have $g$ is really small near $1$, $\Rightarrow$ we devide $[0,1]$ into $[0, 1-\delta], [1-\delta, 1]$

* We have $g(x)$ continuous on $[0,1] \Rightarrow$ uniformly cont on $[0,1]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1]$, $|x-y| < \delta$ then $|g(x) - g(y)| < \varepsilon$

Let $y = 1$, because $g(y) = 0$, we have $\forall x$ s.t $|x - 1| < \delta$, $|g(x)| < \varepsilon$. (1).
$\Leftrightarrow x \in [1-\delta, 1]$



* Consider $x \in [0, 1-\delta]$, we have $f_n(x) := x^n \rightrightarrows 0$ on $[0, 1-\delta]$

This means $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $|x^n| < \varepsilon$

$g$ continuous $\Rightarrow$ bounded on $[0, 1-\delta]$ $\Rightarrow \forall n \geq n_0$, $|x^n g(x)| < \varepsilon$

$\Rightarrow h_n(x) \rightrightarrows 0$ on $[0, 1-\delta]$. (I)

* Consider in case $x \in [1-\delta, 1]$.

$x^n$ even not converges uniformly to $0$ but is still less than $1$

We have $|h_n(x)| = |x^n g(x)| \leq \boxed{1} \cdot |g(x)| < \varepsilon$ (by (1)).

note that $|x^n| < 1, \forall x \in [0,1]$

$\Rightarrow h_n(x) \rightrightarrows 0$ on $[1-\delta, 1]$ (II)

(I) + (II) $\Rightarrow$ done $\square$.

---

* In case we want to prove that $\{h_n\}$ uniformly Cauchy:

NTP $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall m, n \geq n_0$, ~~$h_m(x)$~~ $\forall x \in [0,1]$, $|h_m(x) - h_n(x)| < \varepsilon$

Consider $|h_m(x) - h_n(x)| = |(x^m - x^n) g(x)|$

Because we only have $x^n \rightrightarrows 0$ on $[0, 1-\delta]$, we also need to divide $[0,1] \begin{cases} [0, 1-\delta] \\ [1-\delta, 1] \end{cases}$.

Aug 2005

Need to review.

P2) Let $a_n$, $n = 1, 2, \ldots$ be a sequence of positive numbers s.t $\sum a_n$ converges.
a) Prove that $\liminf\limits_{n\to\infty}(na_n) = 0$
b) Show by example that $\limsup\limits_{n\to\infty} n a_n > 0$ is possible.

a) $a_n > 0, \forall n; \sum a_n$ converges. Prove that $\liminf\limits_{n\to\infty} na_n = 0$

* We have because $a_n > 0$, $\sum a_n$ converges $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow \liminf\limits_{n\to\infty} na_n \geq 0$

So we need to prove that the case $\liminf\limits_{n\to\infty} na_n > 0$ does not happen. Prove by contradiction

* Assume $\liminf\limits_{n\to\infty}(na_n) \geq \beta > 0$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \; na_n \geq (\beta - \varepsilon) \geq 0$

$\Rightarrow a_n \geq \dfrac{\beta - \varepsilon}{n} \geq 0, \; \forall n \geq N$

we also have $\sum\limits_{n=1}^{\infty} \dfrac{\beta - \varepsilon}{n}$ diverges.

$\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverges (contradiction) $\Rightarrow$ ∎a

b) Show by example that $\limsup\limits_{n\to\infty} na_n > 0$ is possible.

Let $a_n = \begin{cases} \frac{1}{n} & \text{, when } n = 2^k \\ 0 & \text{, when } n \neq 2^k \end{cases}$

Then we have $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{k=1}^{\infty} \dfrac{1}{2^k}$ converges $(= 1)$

~~and $\lim\limits_{n\to\infty} na_n = 1$ ∎~~ and $\limsup\, na_n = 1$.

* Another (similar) question from mathstackexchange:

If $\{a_n\}$: non increasing sequence of positive real numbers such that $\sum\limits_{n=1}^{\infty} a_n$ converges.
Prove $\lim\limits_{n\to\infty}(na_n) = 0$

* If $\{a_n\}$ : sequence of non increasing sequence of positive numbers s.t. $\sum_{n=1} a_n$ converges.
Prove that $\lim_{n\to\infty} na_n = 0$.

---

* **Way 1**: We have $a_n > 0, \forall n$ then $\lim_{n\to\infty} na_n \geq 0$, ~~$\forall n$~~.
We NTP that the case $\lim_{n\to\infty} na_n > 0$ does not happen.
We assume a contradiction that $\lim_{n\to\infty} na_n > 0$, assume $\lim_{n\to\infty} na_n = \alpha > 0$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \quad |na_1 - \alpha| < \varepsilon .$$

$$\alpha - \varepsilon < |na_n| < \alpha + \varepsilon$$

$$\alpha - \varepsilon < na_n < \alpha + \varepsilon$$

$$\left.\begin{array}{l} a_n > \frac{\alpha-\varepsilon}{n} \geq 0. \\ \text{but we always have } \sum \frac{(\alpha-\varepsilon)}{n} \text{ diverges} \end{array}\right\} \Rightarrow \sum a_n \text{ diverges (contradiction)}.$$

* **Way 2**: Use condensation test ( $\sum_{n=1}^{\infty} a_n$ and $\sum_{\ell=0}^{\infty} 2^{\ell} a_{2^{\ell}}$ both converges or diverges ).

* By condensation test $\sum_{n=1}^{\infty} a_n$ converges $\rightarrow \sum_{\ell=0}^{\infty} 2^{\ell} a_{2^{\ell}}$ converges

$$\left.\begin{array}{l} \text{we have } \forall n \in \mathbb{N}, \exists \ell \text{ st } \quad 2^{\ell} \leq n \leq 2^{\ell+1} \\ \text{note that } a_n \text{ decreasing} \Rightarrow a_{2^{\ell}} \geq a_n \geq a_{2^{\ell+1}} \end{array}\right\} \Rightarrow 2^{\ell} a_{2^{\ell+1}} \leq na_n \leq 2^{\ell+1} a_{2^{\ell}} \quad (1)$$

- Note that $\sum 2^{\ell} a_{2^{\ell}}$ converges $\Rightarrow \lim_{\ell\to\infty} 2^{\ell} a_{2^{\ell}} = 0$

$$\Rightarrow \begin{cases} \lim \frac{1}{2} \cdot 2^{\ell} a_{2^{\ell+1}} = \lim_{\ell\to\infty} 2^{\ell+1} a_{2^{\ell+1}} = 0 \rightarrow \lim 2^{\ell} a_{2^{\ell+1}} = 0 \\ \lim 2^{\ell+1} a_{2^{\ell}} = 2 \lim 2^{\ell}_{a^{2^{\ell}}} = 0. \end{cases} \quad (2)$$

$(1) + (2) \Rightarrow \lim_{n\to\infty} na_n = 0. \square$

Aug 2005, P4

Prove that $\lim_{n\to\infty} \frac{\ln(2)+\ln(3)+\cdots+\ln(n)}{n\ln(n)} = 1$.

* We have Stirling formular

$$\ln(n!) = n\ln n - n + O(\ln n).$$

* Apply Stirling formula to the problem, we have

$$\frac{\ln 2+\ln 3+\cdots+\ln(n)}{n\ln n} = \frac{\ln(n!)}{n\ln n} \cong \frac{n\ln n - n}{n\ln n} = 1-\frac{1}{\ln n} \xrightarrow[n\to\infty]{} 1 \quad \square.$$

⁍ Let $F(x_1, x_2, y_1, y_2) = (x_1 x_2 + x_1 y_1 + y_2,\ x_1 y_2 + x_2 y_1^2)$
Check that $F(1,1,1,1) = (3,2)$

⁍ Prove that there is a neighborhood $U$ of $(1,1,1,1)$ and a neighborhood $W$ of $(1,1)$ and a function $g : W \to \mathbb{R}^2$ s.t $\forall (y_1, y_2) \in W$, $\exists! (x_1, x_2) \in \mathbb{R}^2$ given by $g(y_1, y_2)$ s.t $(x_1, x_2, y_1, y_2) \in U$ and $F(x_1, x_2, y_1, y_2) \in (3,2)$

⁍ Find $g'(1,1)$

⁍ Find an approximate solution to the equation $F(x_1, x_2, 1.001, 1.003) = (3,2)$.
Assume that $(1.001, 1.003) \in W$.

---

⁍ Note that the implicit theorem applies for function $\bar{F}$ with $\bar{F}(\overset{\in\mathbb{R}^2}{X_0}, \overset{\in\mathbb{R}^2}{Y_0}) = 0_{\mathbb{R}^2}$.

Let $\bar{F}(x_1, x_2, y_1, y_2) = \begin{pmatrix} x_1 x_2 + x_1 y_1 + y_2 - 3 \\ x_1 y_2 + x_2 y_1^2 - 2 \end{pmatrix}$ transfer $F$ to $\bar{F}$ with $\bar{F}(1,1,1,1) = \vec{0}$

So we have $\bar{F}$ is a $C^1$ function.

$$\bar{F}(1,1,1,1) = (0,0).$$

$$D\bar{F} = \begin{pmatrix} x_2 + y_1 & x_1 & x_1 & 1 \\ y_2 & y_1^2 & 2x_2 y_1 & x_1 \end{pmatrix} = DF$$

We have $\overline{A_x}(1,1,1,1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1 \neq 0$

So by implicit function theorem, there is a neighborhood $U$ of $(1,1,1,1)$ and a neighborhood of $W$ of $(1,1)$ s.t

$\forall (y_1, y_2) \in W$, $\exists! (x_1, x_2)$ such that $\begin{cases} (x_1, x_2, y_1, y_2) \in U \\ \bar{F}(x_1, x_2, y_1, y_2) = 0 \end{cases} \Rightarrow F(x_1, x_2, y_1, y_2) = (3,$

- This means $\exists \bar{g_0}$ :
  $(x_1, x_2) = (\bar{g}_1(y_1, y_2), \bar{g}_2(y_1, y_2))$
- Define
  $(x_1, x_2) = (g_1(y_1, y_2), g_2(y_1, y_2)) := (\bar{g}_1(y_1, y_2) + 3, \bar{g}_2(y_1, y_2) + 2)$

Then we have $g : W \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

and $F(x_1, x_2, y_1, y_2) = F(g_1(y_1, y_2), g_2(y_1, y_2), y_1, y_2) = F(\bar{g}_1(y_1, y_2) + 3, \bar{g}_2(y_1, y_2) + 2, y_1, y_2$

$F(1,1,1,1) = (3,2)$ ☐ a7.

b7 From Implicit F theorem, we have

$$\bar{g}^{1}(1,1) = -[\bar{A}_x]^{-1}[\bar{A}_y] \text{ (at } (1,1,1,1))$$

$$= -1\begin{bmatrix}2&1\\1&1\end{bmatrix}^{-1}\begin{bmatrix}1&1\\2&1\end{bmatrix}$$

$$= -1\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix} = \begin{bmatrix}1&0\\-3&-1\end{bmatrix}$$

Note that by the way we set up F and $\bar{F}$, we have $\bar{A}_x = A_x$ and $\bar{A}_y = A_y$

So we also have $g^{-1}(1,1) = \bar{g}^{-1}(1,1) = \begin{bmatrix}1&0\\-3&-1\end{bmatrix}$ □b.

c7. Find an approximation solution to the equation $F(x_1, x_2, 1.001, 1.003) = (3,2)$

Assume that $(1.001, 1.003) \in W$

We have

$$g(1.001, 1.003) = g(1,1) + g'(\xi_1, \xi_2)\begin{pmatrix}1.001-1\\1.003-1\end{pmatrix}$$

$$\approx g(1,1) + g'(1,1)\begin{pmatrix}0.001\\0.003\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1&0\\-3&-1\end{pmatrix}\begin{pmatrix}0.001\\0.003\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0.001\\-0.006\end{pmatrix} = \begin{pmatrix}1.001\\0.994\end{pmatrix}$$ □

Note that in this case we need to find $(x_1, x_2)$ where $(x_1, x_2) = g(y_1, y_2)$.

~~So we a~~ and from above we have $g'(y)$

So we apply $g(Y_2) = g(\cancel{X}_1) + g'(\xi)(\cancel{X}_2 - Y_1)$.

where $X_2 = g(Y_2)$ $X_1 = g(Y_1)$.

57 Aug 2003

Set $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function such that

$$f(tx) = t^5 f(x) \ , \ \forall t > 0, \ \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

Prove that $f$ satisfies the partial differentiable equation .

$$\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}(x) = 5 f(x)$$

Direction derivative .

- Let $\gamma_x : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$t \longmapsto \gamma_x(t) = t\vec{x}$$

Put $g(t) = f(\gamma(t)) = f(t\vec{x}) = t^5 f(x)$.

Then by Chain rule , we have

$$g'(t) = \underbrace{f'(\gamma(t))\,\gamma'(t) = \nabla f(tx)\cdot\vec{x}}_{\text{apply chain rule for } g(t) = f(\gamma(t))} = \underbrace{5t^4 f(x) = 5t^4 f(x).}_{\text{compute } g'(t) \text{ with } g(t) = t^5 f(x)} \qquad (*)$$

We want to compute

$$\sum x_j \frac{\partial f}{\partial x_j}(x) = \nabla f(x)\cdot\vec{x}$$

So we consider $(*)$ at $t = 1$, we have $g'(1) = \nabla f(x)\cdot x = 5 f(x)$ $\Big\} \rightarrow \displaystyle\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}(x) = 5f(x)$ □.

Jan 2006 Prove the chain rule:

2L) Let $g$ is differentiable at $a$. $g(a)=b$
$f$ is differentiable at $b$
Then $(f\circ g)$ is differentiable at $a$, and $(f\circ g)'(a) = f'(b)\,g'(a)$.

- We have

$g$ is differentiable at $a$ $\Leftrightarrow \exists\, \overset{<+\infty}{g'(a)}$ and $\exists\, \lambda(t) \xrightarrow{t\to a} 0$ for $t\to a$:

$\Leftrightarrow g(t) = g(a) + g'(a)(t-a) + \lambda(t)(t-a)$.

- $f$ is differentiable at $b$, $\exists\, f'(b)$ and $\vartheta(u) \xrightarrow{u\to b} 0$

$\Leftrightarrow f(u) = f(b) + f'(b)(u-b) + \vartheta(u)(u-b)$.

We consider when $u = g(t)$, because $g$ continuous at $a$ $\Rightarrow$ $t \to a$
$g(t) \to g(a)$
$\Leftrightarrow u \to b$.

Then we have

$$\begin{aligned} f(g(t)) &= f(g(a)) + f'(b)[g(t)-g(a)] + \vartheta(u)(g(t)-g(a)) \\ &= f(g(a)) + f'(b)[g'(a)(t-a)+\lambda(t)(t-a)] + \\ &\quad + \vartheta(u)[g'(a)(t-a)+\lambda(t)(t-a)] \\ &= f(g(a)) + f'(b)\,g'(a)(t-a) + \underbrace{[f'(b)\lambda(t) + g'(a)\vartheta(u) + \vartheta(u)\lambda(t)]}_{\xrightarrow{t\to a} 0}(t-a) \end{aligned}$$

This means $(f\circ g)$ is derivative at $a$, and $(f\circ g)'(a) = f'(b)g'(a)$ where $b = g(a)$

Another ways: By using def: $g$ is derivative at $a$ $\Leftrightarrow \exists\, g'(a)$ and $\lambda(t) \xrightarrow{t\to a} 0$ s.t.

$$g(t) - g(a) = (t-a)[g'(a) + \lambda(t)] \quad (1)$$

- $f$ is differentiable at $b$ $\Leftrightarrow f(u) - f(b) = [u-b][f'(b) + \vartheta(u)]$
where $\vartheta(u) \xrightarrow{u\to b} 0$

when $u = g(t)$ we have $t \to a$
$g(t) \to g(a)$ (because $g$ is differentiable at $a$ $\Rightarrow$ cont at $a$)
$u \to b$

$$\begin{aligned} \text{so } f(g(t)) - f(g(a)) &= [g(t)-g(a)][f'(b)+\vartheta(u)] \\ &\overset{(1)}{=} [g'(a)+\lambda(t)](t-a)[f'(b)+\vartheta(u)] \\ &= f'(b)g'(a)(t-a) + \underbrace{[\lambda(t)f'(b) + \vartheta(u)g'(a) + \lambda(t)\vartheta(u)]}_{\to 0}(t-a) \end{aligned}$$

$\Rightarrow$ what we need to prove □

7. $f : [0,1] \to R$ be a ~~derivative~~ differentiable with bounded derivative
Prove that $f$ can be extended to a continuous function on $[0,1]$

**Idea**. The idea of extending a function is that of this problem is that.
we already have $f$ continuous on $[0,1)$
we now want to find a function $g$ continuous in $[0,1]$.

$$g(x) = f(x) \ , \forall x \in [0,1)$$

so we need to prove that $\exists \lim_{x\to 1^-} f(x) = L$ and then put $g(1) = L$ then $g$ is the function we need to find

---

We want to prove

$\exists \lim_{x \to 1^-} f(x) = L \Leftrightarrow$ NTP $\forall (p_n)$ in $[0,1)$, $p_n \to 1$ then $\exists \lim_{n\to\infty} f(p_n) = L$ and ~~$\lim_{n\to\infty} f(p_n) \neq L$~~

$p_n \not\Rightarrow 1$

~~we have to prove 2 steps~~

* Prove $\exists \lim_{n\to\infty} f(p_n) \Leftrightarrow$ NTP $f(p_n)$ converges.

We have $(p_n)$ converges $\Rightarrow (p_n)$ Cauchy

$$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall \overset{m >}{n \geq} n_0, |p_m - p_n| < \varepsilon$$

We have $f$ is differentiable with bounded derivative $|f'(x)| \leq M$, $\forall x$, then we have

$$|f(p_m) - f(p_n)| = |f'(\xi)| |p_m - p_n| \qquad < M\varepsilon$$

(for $\xi$ between $p_n$ and $p_m$

then $\{f(p_n)\}$ Cauchy in $\mathbb{R}$ $\Rightarrow$ converges in $\mathbb{R}$ $\Rightarrow$ $\exists \lim_{n\to\infty} f(p_n)$.

\+ ~~Prove~~ Assume $\lim_{n\to\infty} f(p_n) = L$

We have $f$ continuous on $[0,1)$
$\exists \lim_{n\to\infty} f(p_n) = L$
$p_n \longrightarrow 1^-$

$\Rightarrow$ ......
$L = \lim_{n\to\infty} f(p_n) = f(\lim_{n\to\infty} p_n) = (f(1^-)$

Then put $g(x) = \begin{cases} f(x) & , x \in [0,1) \\ L = \lim_{n\to\infty} f(p_n) \text{ where } p_n \to 1^- \end{cases}$

$g$ is the extention ...

Jan 2006

P2) Let $f(0)=0$

$f(t)=t^2\sin\frac{1}{t}$, for $t\neq 0$

Let $\phi(x,y)=f(x)+f(y)$

a) Prove that $\frac{\partial\phi}{\partial x}$ exists everywhere in $\mathbb{R}^2$ but is not continuous at $(0,0)$

b) Prove that $\phi$ is differentiable at $(0,0)$ and find $\phi'(0,0)$.

---

a) We have $\phi(x,y)=f(x)+f(y)$

$\Rightarrow \frac{\partial\phi}{\partial x}=f'(x)$ $\left(\frac{\partial\phi}{\partial x} = \lim_{h\to0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to0}\frac{[f(x+h)+f(y)]-[f(x)+f(y)]}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x).\right.$

~~We have~~ * We now consider $f(x)=x^2\sin\frac{1}{x}$.

we have • for $x\neq0$, $f'(x)=2x\sin\frac{1}{x}+x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x} = 2x\sin\frac{1}{x}-\cos\frac{1}{x}$

• at $x=0$, $f'(0)=\lim_{t\to0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to0}\frac{t^2\sin\frac{1}{t}}{t} = \lim t\sin\frac{1}{t} = 0$

because (we have $0\leq\left|t\sin\frac{1}{t}\right|\leq \underset{\to 0}{t}$ $\Rightarrow \lim_{t\to0}\left|t\sin\frac{1}{t}\right|=0$ )

So we have $f'(x)$ exist everywhere in $\mathbb{R}$

$\Rightarrow \frac{\partial\phi}{\partial x}=f'(x)$ exists everywhere in $\mathbb{R}^2$.

* We have

$\lim_{x\to0}f'(x)$ does not exist since $\nexists \lim_{x\to0}\cos\frac{1}{x}$ $\Rightarrow f'(x)$ is not continuous at 0

$\Rightarrow \frac{\partial\phi}{\partial x}$ is not continuous at $(0,0)$.

b) Prove that $\phi$ is differentiable at $(0,0)$ and find $\phi'(0,0)$.

We have $\phi$ is differentiable at $(0,0)$ $\Leftrightarrow \exists \lim_{(h_1,h_2)\to(0,0)}\frac{\|\phi((0,0)+(h_1,h_2))-\phi(0,0)\|_{\mathbb{R}}}{\|(h_1,h_2)\|_{\mathbb{R}^2}} = \lim(*)$

Now we have

$$\frac{\|\phi(h_1,h_2)-\phi(0,0)\|_{\mathbb{R}^1}}{\|(h_1,h_2)\|_{\mathbb{R}^2}} = \frac{|f(h_1)+f(h_2)-f(0)-f(0)|}{\sqrt{h_1^2+h_2^2}} = \frac{\left|h_1^2\sin\frac{1}{h_1}+h_2^2\sin\frac{1}{h_2}\right|}{\sqrt{h_1^2+h_2^2}}.$$

$$0\leq\frac{\left|h_1^2\sin\frac{1}{h_1}+h_2^2\sin\frac{1}{h_2}\right|}{\sqrt{h_1^2+h_2^2}}\leq\frac{h_1^2+h_2^2}{\sqrt{h_1^2+h_2^2}} = \underbrace{\sqrt{h_1^2+h_2^2}}\xrightarrow[(h_1,h_2)\to0]{}0$$

So we have

$\lim_{(h_1,h_2)\to(0,0)}(*)=0$ this means $\phi$ is differentiable at $(0,0)$

and $\phi'(0,0)=0$ □.

Note that we have $\phi:\mathbb{R}^2\longrightarrow\mathbb{R}$ $\Rightarrow$ $\phi\in L(\mathbb{R}^2,\mathbb{R})$ $\Rightarrow$ $\phi(0,0)\in\mathbb{R}$

Jan 2006

P47 If $\sum_{\ell=1}^{n} \frac{a_\ell}{\ell+1} = 0$. Prove that the polynomial $\sum_{\ell=0}^{n} a_\ell x^\ell$ has at least one root in the interval (0, 1).

Stratergy: We want to prove that $g(x)$ has a root in $(a,b)$.

we want to prove that $g(x) = f'(x)$ with $f(b) = f(a)$

then apply Rolle' theorem $\underbrace{f(b)-f(a)}_{=0} = f'(\xi)\underbrace{(b-a)}_{\neq 0}$

$\Rightarrow \exists \xi \in (a,b)$ such that $g(\xi) = f'(\xi) = 0$.

this means $g(x)$ has at least one root in $(a,b)$.

- Look at this problem we have $\int_0^1 a_\ell x^\ell dx = \frac{a_\ell}{\ell+1}$ so we want to put $F(x) = \int_0^x f(t)\,dt$

$f(t) = a_\ell t^\ell$.

*Put $F(x) = \sum_{\ell=1}^{n} \int_0^x a_\ell t^\ell dt = \sum_{\ell=1}^{n} \frac{1}{(\ell+1)} \int_0^x a_\ell (\ell+1) t^\ell dt = \sum_{\ell=1}^{n} \frac{a_\ell}{\ell+1} \int_0^x d(t^{\ell+1}) dt =$

$= \sum_{\ell=1}^{n} \frac{a_\ell}{\ell+1} x^{\ell+1}$

So we have $F(1) = \sum_{\ell=1}^{n} \frac{a_\ell}{\ell+1} \underset{\substack{\uparrow \\ \text{assumpt}}}{=} 0$ $\quad F(0) = \sum_{\ell=1}^{n} \int_0^0 \cdots = 0 \quad \Rightarrow F(1) = F(0)$

We also have because $f(t) a_\ell t^\ell$ is a continuous function on $[0,1]$ $\Rightarrow$ F is differentiable on $[0,1]$.

So by Rolle's theorem : $\exists \xi \in (0,1)$ such that $F'(\xi) = 0$

$\Rightarrow \exists \xi \in (0,1)$ s.t $f(t) = \sum a_\ell t^\ell = 0$ $\Rightarrow$ done $\square$.

Aug 2006

P5> Assume $f:[0,+\infty)\to\mathbb{R}$ is nonnegative, Riemann integrable on $[0,b]$ for every $b>0$.
and $\lim_{b\to\infty}\int_0^b f(t)\,dt < +\infty$

○ Prove or give a counter example.

a> $\lim_{t\to\infty} f(t) = 0$ (not right)

b> $f$ is continuous implies $\lim_{x\to\infty} f(x) = 0$ (not right).

c> $f$ is uniformly continuous implies $\lim_{x\to\infty} f(x) = 0$ .

---

a> Now we consider

$$f(x) = \begin{cases} 1, & x\in\mathbb{N} \ \cancel{x\in[0,b]} \\ 0, & \text{otherwise}. \end{cases}$$

then we have $\int_0^b f(x)\,dx = 0 \rightarrow \lim_{b\to\infty}\int_0^b f(x) = 0.$

But we have $\lim_{t\to+\infty} f(t) \neq 0$ because the sequen $f_n(x) = 1$ does not converge.

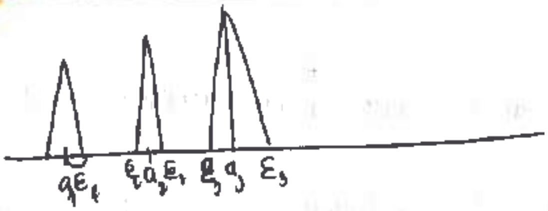
---

○ b> Give a counter example that $\lim_{b\to\infty}\int_0^b f(t)\,dt < +\infty$

$f$ is continuous on $[0,+\infty)$, non negative

$\lim_{x\to+\infty} f(x) \neq 0$

* This problem is harder than problem a.



Choose $f(x) = \begin{cases} 1 + \frac{x-a}{\varepsilon}, & \text{when } a-\varepsilon < x \leq a \\ 1 - \frac{x-a}{\varepsilon}, & \text{when } a \leq x \leq a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$

choose when $\boxed{a=n}$ and $\boxed{\varepsilon = \frac{1}{2^n}}$

This means $f(x) = \begin{cases} 1 + \frac{x-n}{1/2^n}, & \text{when } n-\frac{1}{2^n} < x \leq n \\ 1 - \frac{x-n}{1/2^n}, & \text{when } n \leq x < n+\frac{1}{2^n} \\ 0 & \text{otherwise} \end{cases}$

○ * So we have $f$ is non negative (obviously) and $f$ is co

at $x = n - \frac{1}{2^n}$, $f(x) = 1 + \frac{n - \frac{1}{2^n} - n}{1/2^n} = 0$

at $x = n + \frac{1}{2^n}$, $f(x) = 1 - \frac{x-n}{1/2^n} = 0$

* So we have

$$\int_0^b f(x)\,dx = \sum_{n=0}^{k} \int_{n-\frac{1}{2^n}}^{n+\frac{1}{2^n}} f(x)\,dx = \int_{n-\frac{1}{2^n}}^{n} 1+\frac{x-n}{\frac{1}{2^n}}\,dx + \int_{n}^{n+\frac{1}{2^n}} 1-\frac{x-n}{\frac{1}{2^n}}\,dx =$$

not sure check!

$$= \cancel{\frac{1}{2^n}} + \frac{1}{2^n}\frac{(x-n)^2}{2}\Bigg|_{n-1/2^n}^{n} - \cancel{\frac{1}{2^n}} - \frac{1}{2^n}\frac{(x-n)^2}{2}\Bigg|_{n}^{n+1/2^n}$$

$$= \frac{1}{2^n}\left[0-\frac{\frac{1}{2^n}}{2}\right] - \frac{1}{2^n}\,\frac{\frac{1}{2^n}}{2} \cancel{=0}\ 0$$

but $\lim_{x\to\infty} f(x) \neq 0.$ $\square$

---

c) Prove that $f$ is uniformly continuous, nonnegative on $[0,\infty)$, $\lim_{b\to\infty}\int_0^b f(t)\,dt < +\infty$ } then $\lim_{t\to\infty} f(t) = 0$.

$f$ is uniformly continuous
$\Rightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x, t \in [0,+\infty), |x-t| < \delta, |f(x) - f(t)| < \epsilon$

NTP that $\lim_{t\to+\infty} f(t) = 0$
NTP $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall t \geq N,$
$|f(t)| < \epsilon$

* We have because $f$ is continuous then $F(x) = \int_0^x f(t)\,dt$ is differentiable and $F' = f$ on $\mathbb{R}$.

we have $f$ nonnegative $\Rightarrow$ F is increasing

we also have $F(b) - F(0) < +\infty$ $\Rightarrow$ $F(b)$ is bounded $\forall b \in \mathbb{R}^+$

So we have $F(b) \xrightarrow{b\to\infty} a < +\infty$
* So we have because $F' = f$
} $\Rightarrow f(b) \xrightarrow{b\to\infty} 0$ $\square$.

Aug 2007 I0

Let $f, f_n : [0,1] \longrightarrow \mathbb{R}$ . Prove or give a counterexample to each of the following statement. ✱

$\phi : \mathbb{R} \longrightarrow \mathbb{R}$

a) $\left.\begin{array}{l} f_n \rightrightarrows f \text{ uniformly} \\ \phi \text{ continuous (in } \mathbb{R}) \end{array}\right\} \not\Rightarrow \phi \circ f_n \longrightarrow \phi \circ f$ uniformly (on $[0,1]$) Important

b) $\left.\begin{array}{l} f_n \rightrightarrows f \text{ on } [0,1] \\ \phi : \text{uniformly continuous} \end{array}\right\} \Rightarrow \phi \circ f_n \rightrightarrows \phi \circ f$ ?

c) $\left.\begin{array}{l} f_n \rightrightarrows f \text{ on } [0,1] \\ f \text{ and } \phi \text{ are cont} \end{array}\right\} \Rightarrow \phi \circ f_n \rightrightarrows \phi \circ f$

a) A counter example show that : $\left.\begin{array}{l} f_n \rightrightarrows f \text{ uniformly} \\ \phi \text{ continuous (in } \mathbb{R}) \end{array}\right\}$ but $\phi \circ f_n \not\rightrightarrows \phi(f)$

* Let $f_n(x) = x + \frac{1}{n}$ , for $x \in [0,1]$ , $n = 1, 2, 3, \ldots$

$f(x) = x$ , $x \in [0,1]$

Important to remember this exaple $f_n = f(x) + \frac{1}{n}$ then $f_n \rightrightarrows f$

Then we have $f_n(x) \rightrightarrows f(x)$ on $[0,1]$ because $|f_n(x) - f(x)| = \left|\frac{1}{n}\right| \leq M_n$

WRONG

where $M_n = \frac{1}{n}$ and $M_n \to 0$.

(Remind If $\left.\begin{array}{l} \sup|f_n(x) - f(x)| \leq M_n \text{ on } E \\ \text{and } M_n \longrightarrow 0 \text{ on } E \end{array}\right\} \Rightarrow f_n \rightrightarrows f$ on E )

* Let $\phi(x) = x^2$ on $\mathbb{R}$. (note that $\phi(x) = x^2$ is one impotant example of a function which is continuous but not uniformly continuous in $\mathbb{R}$ )

Then we have $\phi(f(x)) = x^2$ on $[0,1]$

$\phi(f_n(x)) = \left(x + \frac{1}{n}\right)^2$

Now we prove that $\phi(f_n(x)) \not\rightrightarrows \phi(f(x))$ on $[0,1]$

NTP $\exists \varepsilon > 0$, $\forall n$ large $\exists x \in [0,1]$, $|\phi(f_n(x)) - \phi(f(x))| > \varepsilon$

this counterexaple was wrong because $x \in [0,1]$ so can't chose $x = n$ satisfies this. In fact in this case $\phi \circ f_n \rightrightarrows \phi \circ f$

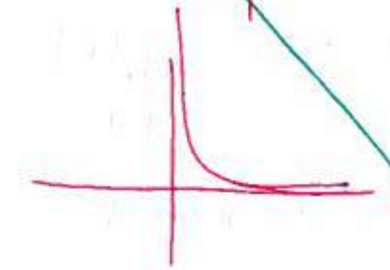
$|\phi(f_n(x)) - \phi(f(x))| = \left|\left(x + \frac{1}{n}\right)^2 - x^2\right| = \left|2x\frac{1}{n} + \frac{1}{n^2}\right|$

→ In fact $\left[f(x) + \frac{1}{n}\right]^2 - f^2(x) = 2f(x) \cdot \frac{1}{n} + \frac{1}{n^2}$ and we choose $f(x) = x$ in here and

Choose $x = n$, then $|\phi(f_n(x)) - \phi(f(x))| = \left|2 + \frac{1}{n^2}\right| \geq 2 > \varepsilon$.

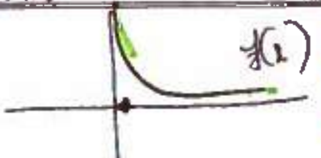
⇒ So this example only work in case $\left.\begin{array}{l} f_n(\mathbb{R} \longrightarrow \mathbb{R} \\ f_n \longrightarrow f \text{ in } \mathbb{R} \\ \phi \text{ continuous} \end{array}\right\} \Rightarrow \phi \circ f_n \not\rightrightarrows \phi \circ f$

* ⇒ Base on this fact, let try $\begin{cases} f_n(x) = f(x) + \frac{1}{n} \\ \text{where } f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \end{cases}$



* Give an counter example to show that $f_n : [0,1] \longrightarrow \mathbb{R}$, $f_n \rightrightarrows f$ on $[0,1]$ $\left.\begin{matrix} \\ \end{matrix}\right\}$ and $\phi(f_n) \not\rightrightarrows \phi(f)$;
$\phi : \mathbb{R} \longrightarrow \mathbb{R}$ continuous

- Let $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, x \in [0,1] \\ 0, & x = 0 \end{cases}$



So f, $f_n$ not continuous on [0,1]

Note that one way to have $f_n(x) \rightrightarrows f(x)$ is by setting $f_n(x) = f(x) + \frac{1}{n}$

$f_n(x) = f(x) + \frac{1}{n}$, for $x \in [0,1]$, $n = 1, 2, \ldots$

Then we have $\sup\limits_{x \in [0,1]} |f_n(x) - f(x)| = \sup\limits_{x \in [0,1]} \left\{ \left| \frac{1}{n} \right| \right\} \le M_n$, where $M_n = \frac{1}{n} \xrightarrow{n \to \infty} 0$

so we have $f_n(x) \rightrightarrows f(x)$.

* Let $\phi(x) = x^2$, then we have $\phi$ is a continuous but not uniformly continuous function on $\mathbb{R}$

Then $\phi(f_n(x)) = \begin{cases} \left(\frac{1}{x} + \frac{1}{n}\right)^2, & x \in (0,1] \\ 0, & x = 0 \end{cases}$

$\phi(f(x)) = \begin{cases} \left(\frac{1}{x}\right)^2, & x \in (0,1] \\ 0, & x = 0 \end{cases}$

$\sup |\phi(f_n(x)) - \phi(f(x))| = \begin{cases} \sup\limits_{x \in [0,1]} \left| \frac{2}{xn} + \frac{1}{n^2} \right| \\ 0, & x = 0 \end{cases}$ (when $x \neq 0$, $\sup(\cdots) = \infty$ →

$\Rightarrow \phi \circ f_n \not\rightrightarrows \phi \circ f$

Another way to explain is because we can prove
$\exists \varepsilon > 0, \forall \delta > 0, \forall n \geq N, \exists x_n \in [0,1]$ s.t
$|\phi \circ f_n - \phi \circ f| > \varepsilon$.

$|\phi \circ f_n(x) - \phi \circ f(x)| = \left| \left(\frac{1}{x} + \frac{1}{n}\right)^2 - \frac{1}{x^2} \right| = \left| \frac{2}{xn} + \frac{1}{n^2} \right|$

choose $x \in [0,1]$, $x = \frac{1}{n}$, then $|\phi \circ f_n - \phi \circ f| \geq 2$

b) Prove that $\left.\begin{matrix} f_n \rightrightarrows f \text{ on } [0,1] \\ \phi \text{ uniformly continuous on } \mathbb{R} \end{matrix}\right\}$ Then $\phi \circ f_n \rightrightarrows \phi \circ f$

- We have $f_n(x) \rightrightarrows f$ on $[0,1]$
$\Leftrightarrow \forall \delta > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in [0,1], |f_n(x) - f(x)| < \delta$ (1)
- $\phi$ is uniformly continuous in $\mathbb{R}$
$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in \mathbb{R}, |u - v| < \delta$ then $|\phi(u) - \phi(v)| < \varepsilon$ (2)

We want to prove that
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in [0,1], |\phi(f_n(x)) - \phi(f(x))| < \varepsilon$.

Let $u, v \in \mathbb{R}$, $u = f_n(x)$ then we have $\overset{\text{by } (1)}{\exists N \in \mathbb{N}}, \forall n \geq N$, $|u - v| < \delta$
$v = f(x)$ → by (2), $|\phi(u) - \phi(v)| < \varepsilon$,

this means $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \phi(f_n(x)) - \phi(f(x))| < \varepsilon$.

c7 Prove or give a counter example $f_n \rightrightarrows f$ on $[0,1]$
$f$ and $\phi$ are continuous.
($f$ cont on $[0,1]$, $\phi$ cont on $\mathbb{R}$) $\Big\}$ $\Rightarrow \phi\circ f_n \rightrightarrows \phi\circ f$.

---

* We have $f$ is continuous on $[0,1]$
$[0,1]$ is compact $\Big\}$ $\Rightarrow$ ($f$ is bounded, $\exists M, |f(x)| \le M, \forall x \in [0,1]$)

because $f_n \rightrightarrows f \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |f_n(x) - f(x)| < \varepsilon$

$\Rightarrow \forall n \geq n_0, |f_n(x)| \le M + 1$

* $\phi$ is continuous in $\mathbb{R}$ $\Rightarrow$ uniformly continuous in $[-(M+1), M+1]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in [-(M+1), M+1], |u - v| < \delta, |\phi(u) - \phi(v)| < \varepsilon$

So let $u = f_n(x)$, $v = f(x)$, then $\forall \delta > 0$, ~~$\forall m$~~, $n \geq n_0$, $|f_m(x) - f(x)| < \delta$

$\Rightarrow |u - v| < \delta$

$\Rightarrow |\phi(f_m(x)) - \phi(f(x))| < \varepsilon$

$\Rightarrow \{\phi(f_m)\} \rightrightarrows \phi\circ f$

* ~~Note that~~

~~$|\phi(f_m(x)) - \phi(f(x))| \le \phi(f_n$~~

* ~~We can~~ Instead of having the assumption that $f$ is continuous, the statement is also true when $\{f_n\}$ sequence of continuous function on $[0,1]$.
$f_n \rightrightarrows f$
$\phi$ continuous in $\mathbb{R}$ $\Big\}$ $\Rightarrow \phi(f_n) \rightrightarrows \phi(f)$

Online :

Let $\{f_n\}$: uniformly convergent of sequence of continuous real-valued functions defined on $M$

$\phi$: continuous function on $\mathbb{R}$.

Define $h_n(x) = \phi(f_n(x))$

a) Let $M = [0,1]$. Prove that $\{h_n(x)\} = \{\phi \circ f_n\}$ converges uniformly on $[0,1]$.

b) Let $M = \mathbb{R}$. Either prove that $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$
or provide a counter example.

---

a) We have $f_n \rightrightarrows f$
$f_n$ sequence of continuous functions $\Big\} \to f$ is cont on $[0,1]$
we have $[0,1]$ compact $\Big\} \Rightarrow$ $f$ is bounded

$\Leftrightarrow \Rightarrow \exists M, |f(x)| \le M, \forall x \in [0,1]$.

$\Rightarrow |f_n(x)| \le M + 1, \forall x \in [0,1], \forall n$.

- We have $\phi$ continuous in $\mathbb{R}$ $\Rightarrow$ uniformly continuous in $[-(M+1), M+1]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in \underset{[-(M+1), M+1]}{\mathbb{R}}, |u - v| < \delta$, then $|\phi(u) - \phi(v)| < \varepsilon$ (1)

- $f_n(x) \rightrightarrows f \Rightarrow$ uniformly Cauchy $\forall x \in [0,1]$.

$\Leftrightarrow \forall \delta > 0, \exists N \in \mathbb{N}, \forall \overset{m}{n} \ge N, |f_m(x) - f_n(x)| < \delta$. (2)

From (1)+(2), (apply when $u = f_m(x)$, $v = f_n(x)$)

So we have $\forall \varepsilon > 0\ \exists N \in \mathbb{N}, \forall m, n \ge N, \forall x \in [0,1], |\phi(f_m(x)) - \phi(f_n(x))| < \varepsilon$.

$\Rightarrow h_n (= \phi \circ f_n$ converges uniformly on $[0,1]$

b) Let $M = \mathbb{R}$. Give a counter example that $f_n \rightrightarrows f$ on $\mathbb{R}$, $\phi$ continuous on $\mathbb{R}$ $\Big\}$ but $h_n = \phi \circ f_n \not\rightrightarrows$ on $\mathbb{R}$.

---

Let $f(x) = x$, $x \in \mathbb{R}$

$f_n(x) = x + \frac{1}{n}$, $x \in \mathbb{R}$, $n = 1, 2, 3, \ldots$

Then $f_n(x) \rightrightarrows f(x)$

- $\phi(x) = x^2$ is a continuous (not uniformly continuous) function on $\mathbb{R}$

$$|\phi_n(f_n(x)) - \phi(f(x))| = \left|\left(x + \tfrac{1}{n}\right)^2 - x^2\right| = \left|2\frac{x}{n} + \frac{1}{n}\right|$$

So $\exists \underset{2}{\overset{\|}{\varepsilon}} > 0, \forall n$ large, $\exists x = n$, $|\phi(f_n(x)) - \phi(f(x))| \le |2 + \frac{1}{n}| \ge 2 \ge \varepsilon$

So $\phi \circ f_n \not\rightrightarrows \phi \circ f$ $\square$.

Clare

## Preliminary Exam Jan 2007

1. Let $X$ be a metric space and let $A_j$ be subsets of $X$, $j = 1, 2, \ldots$. For each of the following statements, prove it or give a counterexample (the $'$ means limit points):

(i) $(A_1 \cup A_2)' \subseteq A_1' \cup A_2'$

(ii) $\overline{\bigcup_{j=1}^{\infty} A_j} \subseteq \bigcup_{j=1}^{\infty} \overline{A_j}$

2. Prove that the series $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ is convergent and find its sum.

3. Let $f : (-1, 1) \to \mathbf{R}$ be a differentiable function such that $f(0) = 0$ and $f''(0) \in \mathbf{R}$ exists. Prove that the limit $\lim_{x \to 0} \frac{f(2x) - 2f(x)}{x^2}$ exists.

4. (a) Let $f^4 \in \mathcal{R}$ (this means $f^4$ is integrable $dx$ on some closed interval) prove or disprove, $f \in \mathcal{R}$.
(b) Let $f^5 \in \mathcal{R}$ prove or disprove, $f \in \mathcal{R}$.

5. Let $f(x, y)$ be a real continuous function on the rectangle $[0, 1] \times [0, 2]$. Given $\epsilon > 0$ show that there exists $n$ and real continuous functions $g_i(x)$ on $[0, 1]$ and $h_i(y)$ on $[0, 2]$ for $i = 1, \ldots, n$ so that

$$|f(x, y) - \sum_{i=1}^{n} g_i(x) h_i(y)| < \epsilon$$

for all $(x, y)$ in the rectangle.

6. Given the equations $x - f(u, v) = 0$ and $y - g(u, v) = 0$ (a) give conditions that assure you can solve for $(x, y)$ in terms of $(u, v)$ and (b) similarly that you can solve for $(u, v)$ in terms of $(x, y)$. (c) Assuming these conditions are satisfied prove that

$$\frac{\partial x(u, v)}{\partial u} \frac{\partial u(x, y)}{\partial x} = \frac{\partial y(u, v)}{\partial v} \frac{\partial v(x, y)}{\partial y}$$

# Analysis Exam August 2007

1. Show that any set $E$ in a connected metric space $X$ with no boundary in $X$ is either $X$ or empty. Note: if we denote the closure of $E$ by $\overline{E}$ and the complement of $E$ by $E^c$ then the boundary of $E$ is given by $\overline{E} \cap \overline{E^c}$.

NTR. See Jan 2004 p2. Aug 1994

2. Suppose that a function $f$ is defined on $[0, \infty)$, bounded on any interval $[0, a]$, $a < \infty$, and $\lim_{x\to\infty}(f(x+1) - f(x))$ exists. Show that

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} (f(x+1) - f(x)).$$

Hard + weird. Cesaro theorem.

3. Suppose that $\sum a_n$ and $\sum b_n$ are series with non-negative terms and the series $\sum b_n$ converges. Show that if

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for all $n \geq n_0$, then the series $\sum a_n$ also converges.
Derive that $\sum a_n$ converges if $a_n > 0$ and if there is a $p > 1$ so that $\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n}$ for all $n$. Hint: use $b_n = n^{-p}$.

f(x)=1
f cont on compact. $\Rightarrow \exists P_n(x) \rightrightarrows f$

4. Let $f(x)$ be continuous on $[0, 1]$ and suppose that

$$\int_0^1 f(x)\, x^n\, dx = \frac{1}{n+1}$$

for all $n = 0, 1, 2, \ldots$. What can you say about the function $f(x)$? Prove your answer.

Weird NTR.
very tricky

5. Prove that the only function $f(x)$ satisfying $f^2(x)$ is Riemann Integrable on $[0, 1]$ and

$$f(x) = \int_0^x f^2(t)\, dt \text{ for } x \in [0, 1]$$

is the function $f(x) \equiv 0$.

Nothing is special.

6. Consider the map $(u, v) = \mathrm{f}(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $u = x^2 + y^2$, $v = x^2 + y^2 - y$.
(a) Find all the points $(x, y)$ so that $\mathrm{f}(x, y) = (1, 1/2)$.
(b) Choose one of the points you found in (a) and call it $\mathbf{a} = (x_0, y_0)$. What does the inverse function theorem say about $\mathrm{f}$ near $\mathbf{a}$? State your answer carefully.
(c) Why is (a) not a contradiction to (b)?

Jan2001

Let $X$ be a metric space. For each of the following statements

$A_j$ be a subset of $X$, $j = 1, 2, \ldots$ Prove it or give counter example

i) $(A_1 \cup A_2)' = A_1' \cup A_2'$ (True)

ii) $(A_1 \cap A_2)' \overset{\not\supseteq}{\subseteq} A_1' \cap A_2'$ ;

iii) $\overline{\bigcup_{i=1} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ (True)

IV) $\overline{\bigcup_{i=1}^{\infty} A_i} \not\supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$

$\overline{\bigcup_{i=1}^{\infty} A_i} \not\subseteq \bigcup_{i=1}^{\infty} \overline{A_i}$

---

i) Prove that $(A_1 \cup A_2)' \subseteq A_1' \cup A_2'$

Let $x \in (A_1 \cup A_2)' \Leftrightarrow \forall r > 0, N_r(x) \cap (A_1 \cup A_2) \neq \phi$

$\Leftrightarrow \forall r > 0, (N_r(x) \cap A_1) \cup (N_r(x) \cap A_2) \neq \phi$

$\Leftrightarrow \forall r > 0, \left[\begin{array}{l} N_r(x) \cap A_1 \neq \phi \\ N_r(x) \cap A_2 \neq \phi \end{array}\right. \Rightarrow \begin{array}{l} x \in A_1' \\ x \in A_2' \end{array}$

ii Prove that $(A_1 \cap A_2)' \underset{\not\supseteq}{\subseteq} A_1' \cap A_2'$

- Prove that $(A_1 \cap A_2)' \leq A_1' \cap A_2'$

Let $x \in (A_1 \cap A_2)' \Leftrightarrow \forall r > 0, (N_r(x) \setminus \{x\}) \cap (A_1 \cap A_2) \neq \phi$

$\Rightarrow \forall r > 0, \begin{cases} (N_r(x) \setminus \{x\}) \cap A_1 \neq \phi \\ (N_r(x) \setminus \{x\}) \cap A_2 \neq \phi \end{cases}$

$\Leftrightarrow \begin{cases} x \in A_1' \\ x \in A_2' \end{cases} \Leftrightarrow x \in A_1' \cap A_2'$

- An example that $A_1' \cap A_2' \notin (A_1 \cap A_2)'$

Let $A_1 = (0, 1)$ $A_2 = (1, 2)$

then $(A_1 \cap A_2) = \phi \Rightarrow (A_1 \cap A_2)' = \phi$

$A_1' = [0, 1]$ $A_2' = [1, 2]$. $A_1' \cap A_2' = \{1\}$

$\Big\} \Rightarrow A_1' \cap A_2' \notin (A_1 \cap A_2)'$

* Prove that $\bigcup_{i=1}\overline{A_i} \subseteq \bigcup_{i=1}A_i$

$\bigcup_{i=1}^{n}\overline{A_i} \supseteq \overline{\bigcup_{i=1}^{n}A_i}$

Give an example that $\bigcup_{i=1}^{\infty}\overline{A_i} \not\supseteq \overline{\bigcup_{i=1}^{\infty}A_i}$

Note that in here, we use a result that if $(A_i)_{i\in I}$, $\underline{A_i \subseteq B, \forall i \in I}$
then $\bigcup_{i\in I} A_i \subset B$

* Prove that $\bigcup_{i=1}^{\infty}\overline{A_i} \subseteq \underbrace{\overline{\bigcup_{i=1}^{\infty}A_i}}$

Put $B := \overline{\bigcup_{i=1}^{\infty}A_i}$, then we have $B$ is closed

We have $B$ contains $A_i, \forall i$
$B$ is close $\Big\}\Rightarrow \overline{A_i} \subseteq B, \forall i$
from the result above $\Big\}\Rightarrow \bigcup_{i=1}^{\infty}\overline{A_i} \subseteq B = \overline{\bigcup_{i=1}^{\infty}A_i}$ □ ⇒.

* Prove that $\overline{\bigcup_{i=1}^{n}A_i} \subseteq \bigcup_{i=1}^{n}\overline{A_i}$ (This means, when $n$ is finite, $\bigcup_{i=1}^{n}\overline{A_i} = \overline{\bigcup_{i=1}^{n}A_i}$

- We have $\bigcup_{i=1}^{n}\overline{A_i}$ is a finite union of closed sets $\Rightarrow \bigcup_{i=1}^{n}\overline{A_i}$ closed

$A_i \subseteq \overline{A_i}, \forall i \Rightarrow \bigcup_{i=1}^{n}A_i \subseteq \bigcup_{i=1}^{n}\overline{A_i}$ $\Bigg\} \Rightarrow \overline{\bigcup_{i=1}^{n}A_i} \subseteq \bigcup_{i=1}^{n}\overline{A_i}$ □

theorem: $\bar{E}$ is the "smallest" closed set containing $E = \bigcup_{i=1}^{n}A_i$

* Given an example that $\bigcup_{i=1}^{\infty}A_i \not\subseteq \bigcup_{i=1}^{\infty}\overline{A_i}$

~~We need to find $\{A_i\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty}\overline{A_i}$ is open~~

- Let $\{A_i\}$ = set of single rational point

Notice that $\mathbb{Q} = \bigcup_{i=1}^{\infty}A_i$

We have $\mathbb{Q}$ is dense in $\mathbb{R} \Rightarrow \overline{\bigcup_{i=1}^{\infty}A_i} = \overline{\mathbb{Q}} = \mathbb{R}$

- Because $\mathbb{Q}$ contains no limit point $\Rightarrow \bigcup_{i=1}^{\infty}\overline{A_i} = \mathbb{Q}$ and $\mathbb{R} \not\subseteq \mathbb{Q}$.

- in this example, we notice ~~en~~ 2 important properties of $\mathbb{Q}$: $\begin{cases} \mathbb{Q} \text{ is countable} \\ \mathbb{Q} \text{ is dense in } \mathbb{R} \quad \overline{\mathbb{Q}} = \mathbb{R}. \end{cases}$

- Another thing that we need to notice is that a collection of single points has no limit point. $\{x\}$ or $\bigcup_{i=1}^{\infty} x_i$ contains no limit point.

Jan 2007 / 2

Prove that the series $\sum_{n=1}^{\infty}\frac{n^2}{n!}$ is convergent and find its sum

* Prove that the above series converge

$$\cdot \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)!}\,\frac{n!}{n^2}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\,\frac{1}{(n+1)}\right| = 0 < +1$$

by ratio test, the series convergent

* Find its sum:

Note that we only know sum of some common series, ex. $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

⇒ with problems requiring find series' sum, we try to use these results.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

* Way 1: (Simple, just write down)

$$\sum_{n=1}^{\infty}\frac{n^2}{n!} = \frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \frac{5^2}{5!} + \frac{6^2}{6!} + \frac{7^2}{7!} + \cdots$$

$$= 1 + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!} + \frac{6}{5!} + \frac{7}{6!} + \cdots$$

(because the series has $a_n \geq 0$ ⇒ we can rearrange

$$= 1 + \frac{1+1}{1!} + \frac{2+1}{2!} + \frac{3+1}{3!} + \frac{4+1}{4!} + \frac{5+1}{5!} + \frac{6+1}{6!} + \cdots$$

$$= 1 + 1 + 1 + \frac{2}{2!} + \frac{1}{2!} + \frac{3}{3!} + \frac{1}{3!} + \frac{4}{4!} + \frac{1}{4!} + \frac{5}{5!} + \frac{1}{5!} + \frac{6}{6!} + \frac{1}{6!} + \cdots$$

$$= \left(1 + 1 + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \frac{5}{5!} + \cdots\right) + \underbrace{\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots\right)}$$

$$= \underbrace{\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right)}_{=e} + \quad = e$$

$$= 2e$$

* Way 2: (Look more advange, but the idea is the same with way 1.)

$$\sum_{n=1}^{\infty}\frac{n^2}{n!} \textcircled{=} \sum_{n=1}^{\infty}\frac{n\,n}{(n-1)!\,n} \textcircled{=} \sum_{n=1}^{\infty}\frac{n}{(n-1)!} \textcircled{=} \sum_{n=0}^{\infty}\frac{(n+1)!}{n!} \underset{\uparrow}{=} \sum_{n=0}^{\infty}\frac{n}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!}$$

$a_n \geq 0, \forall n$
(we can arrange)

$$= \cancel{1+}\sum_{n=1}^{\infty}\frac{1}{(n-1)!} + \sum_{n=0}^{\infty}\frac{1}{n!}$$

$$= \cancel{1+}\,2\sum_{n=0}^{\infty}\frac{1}{n!} = 2e \quad \square$$

Jan 2007 E3

Let $f:(-1,1)\longrightarrow\mathbb{R}$ be a differentiable function
$f(0)=0$
$f''(0)\in\mathbb{R}$ exist
} Prove $\lim\limits_{x\to0}\frac{f(2x)-2f(x)}{x^2}$ exists.

* Consider $\frac{f(2x)-2f(x)}{x^2}$

Step 1

We have $\lim\limits_{x\to0} f(2x)-2f(x) = f(0)-2f(0)=0$

$\lim\limits_{x\to0} x^2 = 0$

Then we can use LHospital (note that f is a differentiable funct)

$$\lim_{x\to0}\frac{f(2x)-2f(x)}{x^2} = \lim_{x\to0}\frac{[f(2x)-2f(x)]'}{(x^2)'} = \lim_{x\to0}\frac{2f'(2x)-2f'(x)}{2x} = (*)$$

Step 2

Note that $f''(0)$ exists ⇒ $f'(0)$ exist.

$$(*) = \lim_{x\to0}\frac{f'(2x)-f'(0)-f'(x)+f'(0)}{x} = \lim_{x\to0}\left(\frac{f'(2x)-f'(0)}{x} + \frac{f'(x)-f'(0)}{x}\right)$$

Because $f''(0)$ exist ⇒ $\lim\limits_{x\to0}\frac{f'(2x)-f'(0)}{x}$ exist

$\lim\limits_{x\to0}\frac{f'(x)-f'(0)}{x}$ exist.

⇒ $\lim\limits_{x\to0}\frac{f(2x)-2f(x)}{x^2}$ exist □

---

* Sth learn I notice from the problem

- We have we can only use L Hospital rule if f is differentiable, $\frac{0}{0}$ $\frac{\pm\infty}{\pm\infty}$

  → we use LHospital in step 1
  could not use L Hospital in step 2. (use MVT)

- If we use MVT in step 1

$$\frac{f(2x)-2f(x)}{x^2} = \frac{f(2x)-f(0)-2f(x)+2f(0)}{x^2} = \frac{f'(\xi)(\cancel{x})-2f'(\zeta)\cancel{x}}{x^{\cancel{2}}} = \frac{f'(\xi)-2f'(\zeta)}{x}\ \ldots ?$$

Jan

4 a) Let $f^4 \in \Re$ ($f^4$ is integrable in some closed interval). Prove or disprove $f \in \Re$.

b) Let $f^5 \in \Re$. Prove or disprove $f \in \Re$. Note : in here $f^4(x) = [f(x)]^4$ ~~does not mean~~ $~~f^4 = f(f(f(f(x))))~~$,

a) $f^k \in \Re$ on $[a,b] \Leftrightarrow \forall \varepsilon > 0, \exists$ a partition $P = \{x_0 = a, \leq x_1 \leq \dots \leq x_n = b\}$

$$U(P, f^k) - L(P, f^k) < \varepsilon$$

$$U(P, f) - L(P, f) < \varepsilon \Leftrightarrow \sum \Big[\sup_{x \in [x_{i-1}, x_i]} (f^k(x)) - \inf_{x \in [x_{i-1}, x_i]} (f^{(k)}(x))\Big] \Delta x \quad < \varepsilon$$

Do we have $U(P_1^*, f) - L(P_1^*, f) < \varepsilon$ for some $P^*$ ?

a) Let $f^4 \in \Re$. Now we give an example that $f \notin R$.

- Let $f(x) \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \; (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$

Then for all partition $P$, $U(P, f) = \sum_{i=1}^{n} 1 \Delta x_i = b - a$

$$L(P, f) = \sum -1 \Delta x_i = a - b$$

So $U(P, f) - L(P, f) = (b-a) - (a-b) = 2(b-a) > \varepsilon$

$\Rightarrow f \notin \Re$

- However $f^4 = 1, \forall x \in \mathbb{R}$ is integrable

In fact with this example, we have $f \notin \Re$ $f^k$, $k$ even is $\in \Re$.

* However in case $f$ is $\boxed{\text{positive}}$ or ~~non-negative~~, bounded function

then $f^k$, $k$ even $\in \Re \Longrightarrow f \in \Re$ because $\sqrt[\text{even}]{f}$ is continuous on $[0, +\infty)$

b) Let $f^5 \in \Re$. Prove that $f \in \Re$

We have $\phi(x) = \boxed{\sqrt[5]{x}}$ $\boxed{\text{is a continuous function in } \mathbb{R}}$

So $\sqrt[5]{f^5} = f \in \Re$ according to the theorem :

$$\left.\begin{array}{l} \text{Let } f(x) \in (m, M) \\ \quad \phi \text{ is continuous in } [m, M] \\ \quad \text{Put } h = \phi(f) \end{array}\right\} \rightarrow \text{Then If } f \in \Re \text{ then } h(=\phi(f)) \in \Re$$

Jan2001 1, P5.

$f(x,y)$ be a real continuous function on the rectangle $[0,1]\times[0,2]$

Given $\varepsilon > 0$, show that there exists $n$ and real continuous function $g_i(x)$ on $[0,1]$ for $i=1,$
$h_i(x)$ on $[0,2]$

so that $\left| f(x,y) - \sum_{i=1}^{n} g_i(x)\cdot h_i(y) \right| < \varepsilon$

for all $(x,y)$ in the rectangle.

Jan 2017 P6

Given the equations $x - f(u,v) = 0$

$y - g(u,v) = 0$

a) Give conditions that ensure you can solve for $(x,y)$ in term of $(u,v)$

b) Similarly that you can solve for $(u,v)$ in terms of $(x,y)$.

c) Assume that these conditions are satisfied, Prove that

$$\frac{\partial x}{\partial u}(u,v)\,\frac{\partial u(x,y)}{\partial x} = \frac{\partial y(u,v)}{\partial v}\cdot\frac{\partial v(x,y)}{\partial y}$$

---

Put $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$

$(x,y,u,v) \longmapsto (F_1 = x - f(u,v),\ F_2 = y - g(u,v))$

So we have $DF = \left[\begin{array}{cc|cc} 1 & 0 & -f_u & -f_v \\ 0 & 1 & -g_u & -g_v \end{array}\right]$.

We need DF to be ~~a~~ continuously differentiable $\Leftrightarrow$ we need all $f_u, f_v, g_u, g_v$ exist and continuous.

a) because $A_{xy} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has determinant $\neq 0$, then ~~the~~ above condition is enough to have $(x,y)$ can be solved in term of $(x,y)$.

b) We can solve for $(u,v)$ in term of $(x,y)$ when $\det A_{uv} \neq 0 \Leftrightarrow f_u g_v - f_v g_u \neq \phi$.

c) when above condition are hold.

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = -[A_{xy}]^{-1}\left[A_{uv} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -f_u & -f_v \\ -g_u & -g_v \end{bmatrix}\right.$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = -[A_{uv}]^{-1}[A_{xy}] = -\begin{bmatrix} -f_u & -f_v \\ -g_u & -g_v \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} \end{bmatrix} \qquad \begin{bmatrix} f_u & -f_v \\ g_u & -g_v \end{bmatrix}\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ ~~$f_u u_x - f_v v_x = 1$~~.

$\boxed{f_u u_y - f_v v_y = 0}$ $\Rightarrow$ What we need to prove

$g_u u_x - g_v v_x = 0$

Aug 2007) Show that any set $E$ in a connected metric space $X$ with no boundary in $X$ is either $X$ or $\phi$

Note the boundary of $E$: $\partial E = \overline{E} \cap \overline{E^c}$

We have $E$ has no boundary in $X$ $\Leftrightarrow$ $\partial E = \overline{E} \cap \overline{E^c} = \phi$

then we have
$$E \cap \overline{E^c} \subset \overline{E} \cap \overline{E^c} = \phi$$
$$\overline{E} \cap E^c \subset \overline{E} \cap \overline{E^c} = \phi .$$

this means $E$ and $E^c$ are seperated.

we have $X = E \cup E^c$ but $X$ is connected

(a connected set can't be written as a union of 2 seperated sets).

Then $\begin{cases} E = \phi \\ E^c = \phi \Rightarrow E = X \end{cases}$

$X$ is connected
$X = A \cup B$, when $A$ and $B$ are seperated $\Big\} \Rightarrow \begin{cases} A = \phi \\ B = \phi \end{cases}$

Aug 2007, P2. see Jan 2004, 7 Aug 1994.

Hard need to review

Suppose that a function $f$ is defined on $[0,+\infty)$, bounded on any interval $[0,a]$, $a<+\infty$. $\lim\limits_{x\to\infty}[f(x+1)-f(x)]$ exists.

Show that $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to+\infty}[f(x+1)-f(x)]$.

* With this problem $f$ is defined on $[0,+\infty)$ and we need to prove that $\lim\limits_{x\to+\infty}A(x)=\lim\limits_{x\to\infty}B(x)$ } NTP

We prove by way $\varepsilon < D(x)-L<\varepsilon$

$\boxed{\varepsilon < A(x)-L<\varepsilon}$ or $\boxed{A(x+1)}$ ↓ also work because $x\to+\infty$

We have

$\lim\limits_{x\to+\infty}[f(x+1)-f(x)]=L \overset{dy}{\Leftrightarrow} \forall\varepsilon>0, \exists A>0, \forall x\geq A,\ L-\varepsilon< f(x+1)-f(x)<L+\varepsilon$

We want to prove that $\exists B>0, \forall x>B,\ L-\varepsilon<\dfrac{f(x)}{x}<L+\varepsilon$

(The key in here is we consider $f(x+n)$ and let $n\to\infty$.

* We have $f(x+1)-f(x)\approx L \Rightarrow f(x+n)-f(x)=\underbrace{[f(x+n)-f(x+(n-1))]+\cdots+[f(x+1)-f(x)]}_{\approx nL}$

so we have $\dfrac{n(L-\varepsilon)}{x+n}\leq\dfrac{f(x+n)-f(x)}{x+n}\leq\dfrac{n(L+\varepsilon)}{x+n}$

Note that because $f$ is bounded for all $[0,A+1]$, we have $\dfrac{|f(x)|}{x+n}\leq\dfrac{M}{x+n}$.

so we have.

$$\underset{\downarrow\atop L-\varepsilon}{\frac{n(L+\varepsilon)}{x+n}}+\underset{\downarrow\atop 0}{\frac{M}{x+n}}\leq\frac{f(x+n)}{x+n}\leq\underset{\downarrow\atop L-\varepsilon.}{\frac{n(L+\varepsilon)}{x+n}}+\underset{\downarrow\atop 0}{\frac{M}{x+n}}$$

So we have $L-\varepsilon<\dfrac{f(x+n)}{x+n}<L+\varepsilon$

For $y>A+n$, $L-\varepsilon<f(y)<L+\varepsilon$

This means $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to+\infty}[f(x+1)-f(x)]$ ☐.

(Relating to Aug 2004, Y2). Stolz-Cesaro theorem ($(\frac{\infty}{\infty})$ form)

If $\{a_n\}$ $\{b_n\}$ are 2 sequence

If $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L$

$\{b_n\}$ strictly increasing, $\lim\limits_{n\to\infty} b_n = +\infty$

$\Bigg\}$ Then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$.

- $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} \xrightarrow{n\to\infty} L$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \left|\frac{a_{n+1}-a_n}{b_{n+1}-b_n} - L\right| < \varepsilon.$

We need to prove that $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ $\left|\frac{a_n}{b_n} - L\right| < \varepsilon$.

$$\Rightarrow \quad L-\varepsilon < \left|\frac{a_{n+1}-a_n}{b_{n+1}-b_n}\right| < L+\varepsilon$$

note that $\{b_n\}$ strictly increasing, we have

$$(L-\varepsilon)(b_{n+1}-b_n) < |a_{n+1}-a_n| < (L+\varepsilon)(b_{n+1}-b_n), \forall n \geq N.$$

Let $\ell$ be a natural number, $\ell > N$, we have

$$(L-\varepsilon)\sum_{i=N}^{\ell}(b_{i+1}-b_i) < \sum_{i=N}^{\ell} a_{i+1}-a_i < (L+\varepsilon)\sum_{i=N}^{k} b_{i+1}-b_i$$

$$\frac{(L-\varepsilon)(b_{\ell+1}-b_N)}{b_{\ell+1}} < \frac{a_{\ell+1}-a_N}{b_{\ell+1}} < \frac{(L+\varepsilon)(b_{\ell+1}-b_N)}{b_{\ell+1}}$$

$$\Rightarrow (L-\varepsilon)\underbrace{\left(1-\frac{b_N}{b_{\ell+1}}\right)}_{\to 1} + \underbrace{\frac{a_N}{b_{\ell+1}}}_{\to 0} < \frac{a_{\ell+1}}{b_{\ell+1}} < (L+\varepsilon)\left(1-\underbrace{\frac{b_N}{b_{\ell+1}}}_{\to 0}\right) + \underbrace{\frac{a_N}{b_{\ell+1}}}_{\to 0}$$

$$(L-\varepsilon) < \frac{a_{\ell+1}}{b_{\ell+1}} < (L+\varepsilon)$$

So we have $\lim\limits_{\ell\to\infty} \frac{a_\ell}{b_\ell} = L$. $\square$

Stolz-Cesaro theorem $\left(\frac{0}{0}\right)$ case

Given that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

and that $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L$

$\{b_n\}$ strictly monotonic.

Prove that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

We have

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L.$$

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, \left| \frac{a_{n+1}-a_n}{b_{n+1}-b_n} - L \right| < \varepsilon$$

$$\Rightarrow (L-\varepsilon) < \frac{a_{n+1}-a_n}{b_{n+1}-b_n} < (L+\varepsilon)$$

Wlog, assuming that $\{b_n\}$ strictly increasing $(b_{n+1}-b_n > 0, \forall n.)$

$$\Rightarrow (L-\varepsilon)(b_{n+1}-b_n) < a_{n+1}-a_n < (L+\varepsilon)(b_{n+1}-b_n), \forall n \geqslant N$$

Consider $\ell \in \mathbb{N}$, $\ell \geqslant N$, we have

$$(L-\varepsilon)\sum_{i=N}^{\ell}(b_{i+1}-b_i) < \sum_{i=N}^{\ell}(a_{i+1}-a_i) < (L+\varepsilon)\sum_{i=N}^{\ell}(b_{i+1}-b_i), \text{ for } n \geqslant N.$$

$$\Rightarrow (L-\varepsilon)(b_{\ell+1}-b_n) < a_{\ell+1}-a_n < (L+\varepsilon)(b_{\ell+1}-b_n)$$

Let $\boxed{\ell\to\infty}$, so we have

$$(L-\varepsilon)-b_n < -a_n < (L+\varepsilon)-b_n$$

(when $b_n$ strictly increasing, $\longrightarrow 0$, this means $b_n \leq 0, \forall n$.

$$\rightarrow (L-\varepsilon) \leq \frac{a_n}{b_n} \leq (L+\varepsilon)$$

So we have $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ $\square$.

Aug 2007.

3) Suppose that $\sum a_n$, $\sum b_n$ are series with $\boxed{\text{non-negative}}$ terms.

$\sum b_n$ converges.

$\cancel{\exists}\ \frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$, $\forall n \geq n_0$

a) Prove that $\sum a_n$ also converges.

b) Derive that $\sum a_n$ converges if $\begin{cases} a_n > 0 \\ \exists p > 1 \text{ s.t } \frac{a_{n+1}}{a_n} < 1 - \frac{p}{n} \text{ for all } n. \end{cases}$ Hint use $b_n = n^{-p}$

---

a) We have $\frac{\cancel{a_{n+1}}}{a_n} \le \frac{\cancel{b_{n+1}}}{b_n}$, $\forall n \geq n_0$

$$\frac{a_{n+2}}{\cancel{a_{n+1}}} \le \frac{b_{n+2}}{\cancel{b_{n+1}}} \Rightarrow \boxed{\frac{a_{n+2}}{a_n} \le \frac{b_{n+2}}{b_n}}$$

by induction, we have $\boxed{\begin{matrix} \frac{a_{n+k}}{a_n} \le \frac{b_{n+k}}{b_n} \\ \Rightarrow \frac{a_{n_0+k}}{a_{n_0}} \le \frac{b_{n_0+k}}{b_{n_0}} \end{matrix}}$ $\forall n \geq n_0$.

$$\Rightarrow a_n \le b_n \underbrace{\left(\frac{a_{n_0}}{b_{n_0}}\right)}_{\text{constant}}, \forall n \geq n_0 .$$

and because $\sum b_n$ converges.

$\Rightarrow \sum a_n$ converges.

b). Prove that if $a_n > 0$
$\exists p > 1$ s.t $\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n}$, $\forall n$ $\Big\}$ then $a_n$ converges.

we have $$\frac{a_{n+1}}{a_n} < \left(1 - \frac{p}{n}\right) \underset{\substack{\uparrow \\ \text{Binomial} \\ \text{inequality.}}}{<} \left(1 + \frac{1}{n}\right)^{-p} = \left(\frac{1+n}{n}\right)^{-p} = \frac{\left(\frac{1}{n+1}\right)^p}{\frac{1}{n^p}}$$

maybe the correct assumption is $p < -1$ because

$(1 + nx) \le (1+x)^n$ when $\boxed{n \geq 1}$

sth's wrong with the question!.

Then consider $b_n = \frac{1}{n^p}$, we have $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$, $\forall n$

$\sum b_n = \sum \frac{1}{n^p}$ converges when $\boxed{p > 1}$ ☐.

Aug 200-17

I47 Let $f(x)$ be continuous on $[0,1]$ and $\int_0^1 f(x)x^n\,dx = \frac{1}{n+1}$, for all $n = 0, 1, 2, \cdots$ What can we say about $f(x)$? Prove your answer.

* We note that $\frac{1}{n+1} = \int_0^1 x^n\,dx$.

So we have.

$$\int_0^1 f(x)\,x^n\,dx = \frac{1}{n+1} \Leftrightarrow \int_0^1 f(x)\,x^n\,dx - \int_0^1 x^n\,dx = 0 \Leftrightarrow \int_0^1 [f(x)-1]\;x^n\,dx = 0,\ \forall n = 0,1,2,3$$

Put $g(x) = f(x) - 1$, we have $g$ continuous on $[0,1]$

$$\left.\begin{array}{l} g \text{ continuous on } [0,1] \\ \int_0^1 g(x)\,x^n = 0,\ \forall n = 1,2,3 \end{array}\right\} \text{then it's easy to prove that } g \equiv 0 \text{ on } [0,1]$$

$\rightarrow\ f = \underline{1}$ on $[0,1]$.

Aug2004-7 I57.

Prove that the only function $f(x)$ satisfying $f^2(x)$ is Riemann integrable on $[0,1]$ and $f(x) = \int_0^x f^2(t)\,dt$ for $x \in [0,1]$ is the function $f(x) \equiv 0$.

Tiara

Weird.

Really stucks.

* A different, but really interesting way learned from Koji

We let $\varepsilon \in (0,1)$ st ~~$\exists x, |f(x)| > \varepsilon$~~ but does not work in this problem.

and ~~let $x_0 = \inf\{x \,/\, f(x) > \varepsilon\}$~~, we prove that $x_0 > 1$

$\Rightarrow f(x) \equiv 0$ on $[0,1]$.

* Note that $f(x) = \int_0^x f^2(t)\,dt$, where $f^2$ is Riemann integrable on $[0,1]$.

$\Rightarrow$ $f$ is continuous on $[0,1]$ $\Rightarrow$ $f^2(t)$ is continuous on $[0,1]$.

this means $f$ is differentiable and $f'(x) = f^2(x)$.

we have $f^2(x) \geq 0, \forall x$ $\Big\} \Rightarrow f'(x) \geq 0 \ \forall x$.

this means $f$ is ~~increasing~~ nondecreasing on $[0,1]$.

(1)

* We have $f(0) = 0$. (2).

* Then because of (1)+(2), if we can prove that $f(1) = 0$, then we're done.

Now assume $f(1) = c$, because $f$ is nondecreasing on $[0,1]$ $\Big\}$ $\Rightarrow c \geq 0$.

$f(0) = 0$

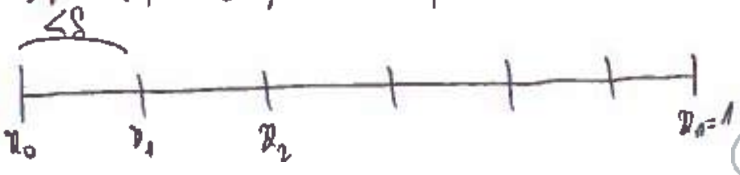
A useful trick (with problem giving $f(0)=0$ and some hypothesis including $f$ is (uniformly) continuous in a compact set) is partition $f$ into parts with length $\delta$)

* We have $f$ is continuous on $[0,1]$ $\Rightarrow$ uniformly cont.

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

So, now we divide $[0,1]$ into partition $\{x_0 = 0 <= x_1 \leq x_2 \leq \cdots \leq x_n = 1\}$.

with $x_i - x_{i-1} < \delta, \forall i = \overline{1,n}$



Aug2007 725 cont. 107 K

* ~~Note that we have~~

Now we first consider segment $[x_0, x_1]$, we have $|\xi - x_0| < \delta$ $\forall \xi \in [x_0, x_1]$, because ~~$|f(\xi)-$~~ then $|f(\xi) - \underset{0}{f(0)}| < \epsilon$ $\Rightarrow f$

$\Rightarrow f(\xi) < \varepsilon < 1$.

So we have

$$\underset{0}{|f(x_1) - f(x_0)|} \overset{f \text{ differentl}}{=\!=\!=} |f'(\xi)\underbrace{[x_1 - x_0]}_{\leq \delta}| \leq |f'(\xi)| = |f^2(\xi)| \underset{\substack{\text{because} \\ f(\xi) < \varepsilon < 1}}{\overset{\uparrow}{\leq}} |f(\xi)|$$

So we have $|f(x_1)| \leq f(\xi)$
but from above $f$ is nondecreasing $\Big\} \Rightarrow f(x_1) = f(\xi)$.

$\Rightarrow$ equality hold for all above means $f^2(\xi) = f(\xi)$ $\Rightarrow$ $f(\xi)\underbrace{(f(\xi) - 1)}_{\neq 0} = 0$
because $f(\xi) < \varepsilon < 1$

$\Rightarrow f(\xi) = 0$

$\Rightarrow$ $f(x_1) = f(\xi) = 0$ + the fact that $f$ is non decreasing
$\Downarrow$
$\Rightarrow f = 0$, $\forall x \in [x_0, x_1]$

+ Do the same thing for each segment $[x_i, x_{i+1}]$, $i = \overline{1, n-1}$

$\rightarrow$ we prove that $f \equiv 0$ $\forall x \in [0, 1]$.

* Review Cesaro theorem.

* Note that, when we have $\begin{cases} f(0) = 0 \\ f \text{ is (uniform) continuous on } [0, 1] \end{cases}$ Want to prove $f \equiv 0$

we can divide $[0,1]$ into partition with the leng of each segment $< \delta$.

Allgxw7

P6) Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $u = x^2 + y^2$
$(x,y) \longmapsto (u,v) = f(x,y)$ $\quad v = x^2 + y^2 - y$

a) Find all the point $(x,y)$ so that $f(x,y) = (1, 1/2)$

b) Choose one of the points you found in a) and call it $\vec{a} = (x_0, y_0)$
What does the IFT say about $f$ near a. State your answer carefully.

c) Why a) is not a contradiction to (b)?

a) Find all the point $(x,y)$ so that $f(x,y) = (1, 1/2)$

We consider
$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 - y = 1/2 \end{cases} \Leftrightarrow \begin{cases} R_1 - R_2 : \ y = 1/2 . \\ x^2 + 1/4 = 1 \Rightarrow x^2 = 3/4 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = \frac{\sqrt{3}}{4} \\ y = \frac{1}{2} \end{cases} \\ \begin{cases} x = -\frac{\sqrt{3}}{4} \\ y = \frac{1}{2} \end{cases} \end{array}\right.$$

b)
$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & 2y - 1 \end{bmatrix}$$

$$\det(DF) = 2x[2y-1] - 2x \cdot 2y = 2x[2y - 1 - 2y] = -2x .$$

Then $\det(DF)_{(x,y) = (\frac{\sqrt{3}}{4}, \frac{1}{2})} \neq 0$.

- We note that $f$ is a $C^1$ function (because $\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}$ exist and are continuous)

Then by IFT, there is an open neighborhood $V$ of $(\frac{\sqrt{3}}{4}, \frac{1}{2})$ and a open neighborhood $W$ of $(1, 1/2)$ such that $f : V \longrightarrow W$ is a bijection.

This means $\exists g : W \longrightarrow V$ is a $C^1$ bijection such that
$$(u,v) \longmapsto g(u,v) = f^{-1}|_V (x,y).$$

c) a) is not a contradiction to b because the IFT only state in a neighborhood of $(\frac{\sqrt{3}}{4}, \frac{1}{2})$.
($f$ is locally bijection). □

Hard .

## Analysis Preliminary Exam
## August, 2008

~~1.~~ Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by the formula

See Rudin 9.6

Almost Same with Jan 2004, P6.

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that $f$ is continuous at $(0,0)$.

(b) Prove that the first order partial derivatives of $f$ at $(0,0)$ exist.

(c) Prove that $f$ is not differentiable at $(0,0)$.

NTR

~~2.~~ Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the equation

Same Jan 2011

$$|f(x) - f(y)| \geqslant |x - y| \quad \text{for all } x, y \in \mathbb{R}. \quad \Rightarrow |f'(x)| \geqslant 1, \forall x \in \mathbb{R}.$$

Prove that $f(\mathbb{R}) = \mathbb{R}$.

NTR.

~~3.~~ Suppose the boundary of a set in $\mathbb{R}^2$ is a graph of a bounded function. Prove that the function is continuous.

~~4.~~ Prove or give a counterexample: Let $f : \underline{(0,1)} \to \mathbb{R}$ and $g : \underline{(0,1)} \to \mathbb{R}$ be continuously differentiable; that is, $f, g \in C^1(0,1)$. Suppose that

NTR.

$$\lim_{x\to 0+} f(x) = \lim_{x \to 0+} g(x) = 0$$

When we see a function cont (0,1) → can extend it to a cont on [0,1]

and $g$ and $g'$ never vanish on $(0,1)$. If

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = c \qquad \text{for some } c \in \mathbb{R},$$

then

$$\lim_{x\to 0+} \frac{f'(x)}{g'(x)} = c.$$

NTR

~~5.~~ Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of non-negative Riemann integrable functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 x^k \varphi_n(x)\, dx$$

exists for $k = 0, 1, 2, \ldots$ Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(x) \varphi_n(x)\, dx$$

exists for every continuous function $f$ on $[0,1]$.

6. For $n = 1, 2, 3, .., $ let

$$f_n(x) = \begin{cases} 1 & \text{if } \; x \in \{1, \frac{1}{2}, ..., \frac{1}{n}\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $\mathbb{R}$? Justify your answer.

(b) Assume that $\alpha : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function, prove or disprove the following identity

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\, \mathrm{d}\alpha(x) = \int_{-1}^{1} \lim_{n\to\infty} f_n(x)\, \mathrm{d}\alpha(x)\,.$$

Aug 2008
1 $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by the formula

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a) Show that $f$ is continuous at $(0,0)$

b) Prove that the first order partial derivatives of $f$ at $(0,0)$ exist.

c) Prove that $f$ is not differentiable at $(0,0)$.

a) Prove that $f$ is continuous at $(0,0)$ $\Leftrightarrow$ we want to prove that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

- Way 1: (Use comparison):

We have $\lim\limits_{(x,y)\to(0,0)} |f(x,y)| = \lim\limits_{(x,y)\to(0,0)} \left|\dfrac{x^2y}{x^2+y^2}\right| \circled{\leq} \lim\limits_{(x,y)\to(0,0)} \left|\dfrac{x^2y}{x^2}\right| = \lim\limits_{(x,y)\to(0,0)} |y| = 0 = f(0,0)$

$\Rightarrow f$ continuous at $(0,0)$.

$\vec{f}$ is differentiable at $\vec{x}$ $\Rightarrow$ all partial derivative exist at $\vec{x}$

but

all partial derivative exist $\not\Rightarrow$ $f$ is differentiable at $x$.

(See this problems gives an example).

- Way 2: Use polar coordinates:

Put $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ then $(x,y) \xrightarrow[r\to 0]{} 0$

Then $\lim\limits_{(x,y)\to 0} |f(x,y)| = \lim\limits_{r\to 0} \left|\dfrac{r^3\cos^2\varphi\sin\varphi}{r^2}\right| = \lim\limits_{r\to 0} |r\cos^2\varphi\sin\varphi| \circled{\leq} \lim\limits_{r\to 0} |r| = 0$

b) Prove that the first order partial derivaties of $f$ at $(0,0)$ exist

- $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$

- $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$.

c) Prove that $f$ is $\boxed{\text{not}}$ differentiable at $(0,0)$

If $f(x,y)$ were differentiable at $(0,0)$, then the follow limit exist and is $0$.

$$\lim_{(x,y)\to 0} \frac{f(x,y) - [f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)]}{\sqrt{x^2+y^2}} = 0$$

~~But~~ this is precisely equivalent with.

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^{3/2}} = 0$$

- But take $x = \frac{1}{n}$ $y = \frac{1}{n}$ for $n \in \mathbb{N}$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^{3/2}} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right)}{\left(\frac{1}{n^2}+\frac{1}{n^2}\right)^{3/2}} = \lim \frac{1}{\sqrt{8}} \neq 0$$

contradiction.

+ Review for e7 .

Let : $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$x \longmapsto f(x)$ .

Then def : $f$ is differentiable at $\boxed{a \in \mathbb{R}^n}$ iff $\exists\, T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t

$$\lim_{x \to a} \frac{\| f(x) - f(a) - T.(x-a) \|}{\|x-a\|} = 0$$

where $\boxed{T = Df(a)} = ( f_{x_1}(a),\ f_{x_2}(a) \dots f_{x_n}(a$

(where $x = (x_1, x_2, \dots, x_n)$) !

$$\Longrightarrow \lim_{x \to a} \frac{\| f(x) - f(a) - D_f(a) \cdot (x-a) \|}{\|x-a\|} = 0$$

In this case $\| (x,y) - (0,0) \| =$ ?

Aug 2008 / 2   See Jan 2011, 3

NTR. $\#$

Let $f: \mathbb{R} \to \mathbb{R}$ continuous function. $|f(x) - f(y)| \geq |x-y|$, $\forall x, y \in \mathbb{R}$ (*) } Prove that $f(\mathbb{R}) = \mathbb{R}$ (From this, we also prove $f^{-1}$ is continuous).

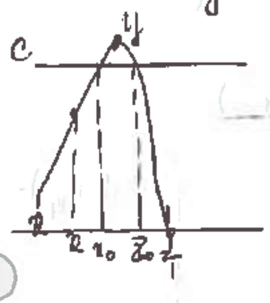
* Note that even this question only requires proving that $f$ is onto, we need to prove that $f$ is { one-to-one / strictly monotone } before proving $f(\mathbb{R}) = \mathbb{R}$.

* Prove $f$ is one-to-one ⟺ Prove that if $x \neq y$ in $\mathbb{R}$, then $f(x) \neq f(y)$.

we consider $x, y \in \mathbb{R}$, $x \neq y \Rightarrow |x-y| > 0$, then because $|f(x) - f(y)| \geq |x-y|$ } $\Rightarrow |f(x) - f(y)| > 0 \Rightarrow f(x) \neq f(y)$.

~~* Now prove that $f$ is one-to-one, $f$ is continuous } ⇒ then $f$ strictly monotone in $\mathbb{R}$.~~

Assume $\exists\, x < y < z$ such that $\begin{cases} f(x) < f(y) \\ f(z) < f(y) \end{cases}$



Then $\exists c$, $f(x) < c < f(y)$, $f(z) < c < f(y)$ because $f$ is continuous on $\mathbb{R}$ } By IVT, we have $\exists\, x_0 \in (x, y)$, $c = f(x_0)$; $z_0 \in (y, z)$, $c = f(z_0)$

This means $f(x_0) = f(z_0)$ while $x_0 \neq z_0 \Rightarrow f$ is not one-to-one (contradict)

$\Rightarrow f$ has to be strictly monotone

* Now prove that $f$ is onto ($f(\mathbb{R}) = \mathbb{R}$)

Note that this question is harder than Jan 2011, 3 since we don't have $f$ is differentiable.

W.l.o.g, assume $f$ strictly increasing.

We want to prove that $\forall z \in \mathbb{R}$, $\exists c \in \mathbb{R}$ such that $f(c) = z$.

+ Case 1 ($z > f(0)$)

Then choose $x \in \mathbb{R}$ s.t $f(0) + x > z$ (so we have $x > 0$).

So until now we have $f(0) + x > z > f(0)$ (1)

• From (*): $|f(x) - f(0)| \geq |x - 0| = |x|$, note that $x > 0$, $f$ strictly increasing } $\Rightarrow f(x) - f(0) \geq x \Rightarrow f(x) \geq x + f(0)$ (2)

(1)+(2) ⇒ $f(x) \geq x + f(0) > z > f(0)$, $f$ is continuous } by IVT, $\exists\, c \in (0, x)$, $f(c) = z$.

+ Case 2 $z < f(0)$ Choose $x \in \mathbb{R}$ such that $f(0) + x < z$ (this means $x < 0$)

Until now we have $f(0) + x < z < f(0)$

Similarly, ~~$f(0)$~~ $-f(x) + f(0) \geq -x \Rightarrow f(x) \leq f(0) + x$.

$\Rightarrow f(x) \leq f(0) + x < z < f(0) \Rightarrow \exists\, c \in (x, 0)$, $f(c) = z$.

+ Case $z = f(0)$, $z$ is the image of 0 through $f$

3 cases ⇒ done □ :)

37 Suppose the boundary of a set in $\mathbb{R}^2$ is a graph of a bounded function. Prove that the function is continuous.

Weird. *

Let $\Gamma = \{(x, f(x))\}$ is the graph of $f(x)$ and also is a boundary of a set $S$ in $\mathbb{R}^2$.

- We have because $\mathbb{R}^2$ is a connected set and the set $S$ have boundary on $\mathbb{R}^2$ $\Big\} \Rightarrow S \neq \phi$ and $S \neq \mathbb{R}$.

(In fact we have a result that a set has no boundary in $\mathbb{R}^2$ is either $\phi$ or $\mathbb{R}^2$).

This means $\Gamma = \overline{S} \cap \overline{S^c}$, thus $\Gamma$ is a closed set

- We have $\Gamma\{(x, f(x))\}$ is closed.

Assume that $f$ is not continuous at

Then $\exists (x_n) \longrightarrow x_0$ but $f(x_n) \not\longrightarrow f(x_0)$ (I)

because $(x_n) \longrightarrow x_0$, chose $x_{n_k} \longrightarrow x_0$ such that $f(x_{n_k}) \longrightarrow L$.

then because $(x_{n_k}, f(x_{n_k})) \subset \Gamma$, $\Gamma$ is closed $\Big\} \Longrightarrow (x_0, L) \in \Gamma$

$\Longrightarrow L = f(x_0)$.

$\Big\} \Rightarrow f(x_{n_k}) \rightarrow f(x_0)$ (II)

(I)+(II) $\Rightarrow$ contradiction.

$\Rightarrow f$ has to be continuous $\square$.

Aug 2008, P5.

Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of non negative Riemann integrable function on $[0,1]$ s.t. NTR.

$\lim_{n\to\infty}\int_0^1 x^l \varphi_n(x)\,dx$, exists for all $l = 0, 1, 2, \dots$

Show that the limit $\lim_{n\to\infty}\int_0^1 f(x)\varphi_n(x)\,dx$ exist for every continuous function $f$ on $[0,1]$.

Note: In here, we want $\exists \lim \int_0^1 f(x)\varphi_n(x)$ $\Rightarrow$ consider it as a normal sequence (even we have $f, \int$

$\Rightarrow$ we want to prove that $\left|\int_0^1 f(x)\varphi_n(x) - \int_0^1 f(x)\varphi_n(x)\right| < \varepsilon$

Note: (From Kogi)
because $\exists P_n \rightrightarrows f$, we can just use: choose $P$, s.t $|P(x) - f(x)| < \varepsilon$.

* We have $\lim \int_0^1 x^l \varphi_n(x)\,dx$ exists for all $l = 0, 1, 2, \dots$ (1)

So, in special case $\lim_{n\to\infty}\int_0^1 \varphi_n(x)\,dx$ exist, this means $\left\{\int_0^1 |\varphi_n(x)|\,dx\right\} \le M$, $\forall n \geq N$

(this is because $\exists \lim \int_0^1 \varphi_n(x)\,dx = L \Leftrightarrow \forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}, \forall n \geq n_0$, $\left|\int_0^1 \varphi_n(x)\,dx - L\right| < \varepsilon$

$\Rightarrow L - \varepsilon < \left|\int_0^1 \varphi_n(x)\right| < \underbrace{L+\varepsilon}_{M} \overset{\uparrow}{\Rightarrow} \int_0^1 |\varphi_n(x)| < M$.

$\varphi$ is nonnegative

$\rightarrow$ and this also means, $\int_0^1 |\varphi_n(x) - \varphi_m(x)|\,dx < \varepsilon$, $\forall m, n \geq \max\{N_0, N\}$ (2)

* Because $f$ is continuous on $[0,1]$, then by Stone - Weierstrass theorem,

$\exists P$ polynomial, s.t $\exists N_0 \in \mathbb{N}, \forall n \geq N_0$, $|P - f| < \varepsilon$, $\forall x \in [0,1]$.

This means we have, $\forall m, n > \max\{N, N_0\}$

$$\left|\int_0^1 f\varphi_n(x)\,dx - \int_0^1 f\varphi_m(x)\,dx\right| = \left|\int_0^1 \underbrace{(f-P)}_{<\varepsilon}\varphi_n(x) + \int_0^1 \underbrace{(f-P)}_{<\varepsilon}\varphi_m(x) + \int_0^1 \underbrace{P}_{<L}\underbrace{(\varphi_n - \varphi_m)}\,dx\right|$$

because $P$ bounded on $[$

$$\le \varepsilon \underbrace{\int_0^1 |\varphi_n(x)|\,dx}_{< M \text{ (by (1))}} + \varepsilon \underbrace{\int_0^1 |\varphi_m(x)|\,dx}_{\le M \text{ by (1)}} + L \underbrace{\int_0^1 |\varphi_n(x) - \varphi_m(x)|}_{\le \varepsilon \text{ beas (2}}$$

$$\le M\varepsilon + M\varepsilon + L\varepsilon.$$

This means $\lim \int_0^1 f\varphi_n(x)\,dx$ exists for $n \longrightarrow \infty$

For $n=1,2,3,\dots$ let $f_n(x)=\begin{cases}1 & , \text{ if } x=\{1,\frac{1}{2},\dots,\frac{1}{n}\}\\ 0 & , \text{ otherwise}\end{cases}$ NTR. ✱

a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $\mathbb{R}$? Justify your answer.

b) Assume that $\alpha:\mathbb{R}\longrightarrow\mathbb{R}$ is an increasing continuous function, prove or disprove the following identity.

$$\lim_{n\to\infty}\int_{-1}^{1} f_n(x)\,d\alpha(x) = \int_{-1}^{1}\lim_{n\to\infty} f_n(x)\,d\alpha(x).$$

* Note Put $M_n = \sup_{x\in\mathbb{R}} |f_n(x) - f_m(x)|$ then if $M_n \not\to 0$ then $f_n \not\rightrightarrows$

a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $\mathbb{R}$?. Justify.

Put $M_n = \sup_{x} |f_n(x) - f_m(x)|$

Now we have $|f_n(x)-f_m(x)| \underset{\text{assume } m>n}{\overset{\text{wlog}}{=}} |g_m(x)|$, where $g_m(x)=\begin{cases}1, & x\in\{\frac{1}{n+1},\dots,\frac{1}{m}\}\\ 0, & \text{otherwise}\end{cases}$

so we have $M_n = \sup_x |f_n(x) - f_m(x)| = 1 \not\to 0$

$\Rightarrow$ this means $f_n \not\rightrightarrows$

b) We first consider $\lim_{n\to\infty}\int_{-1}^{1} f_n(x)\,d\alpha(x)$ :

We have $f_n$ has finitely many points of discontinuity. $\Big\}\Rightarrow f_n\in\mathscr{R}(\alpha)$ on $[-1,1]$ (1)

$\alpha$ is continuous on $[-1,1]$ $\Rightarrow$ continuous at those points

and we have $\int_{-1}^{1} f_n(x)\,d\alpha(x) = \int_{0}^{1} f_n(x)\,d\alpha(x)$ because $f_n(x)\equiv 0$ on $[-1,0]$.

* Now consider any partition $P$, we have $\inf_{x\in[x_{i-1},x_i]} f(x) = 0$.

so we have $L(P,f,\alpha)=0$ , $\forall P$ (2)

(1)+(2) $\Rightarrow \int_{0}^{1} f_n(x)\,d\alpha(x) = \sup_{P} L(P,f,\alpha) = 0$.

$\Rightarrow \lim_{n\to\infty}\int_{-1}^{1} f_n(x)\,d\alpha(x) = 0$.

We note that we already have $f\in\mathscr{R}(\alpha)$ then we have

$\int f\,d\alpha = \underline{\int} f\,d\alpha = \sup L(P,f,\alpha)$

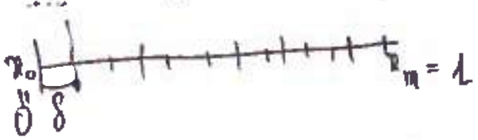
we don't have to care about $U(P,f,\alpha)$.

* Now we consider $\int_0^1 \lim_{n \to \infty} f_n(x) \, d\alpha_{\infty}(x)$

- We have $\frac{1}{n} \to 0$, then a neighborhood of 0 contains all but finitely many points of the sequence $\{\frac{1}{n}\}$.
- We also have that $\alpha$ is continuous on $[0,1]$ $\Rightarrow$ uniformly continuous.

$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1]$, $|x-y| < \delta$, then $|\alpha(x) - \alpha(y)| < \varepsilon/N$

Now consider neighborhood $N_\delta(0)$ contains all but finitely many point of $\{\frac{1}{n}\}$



In $\delta$ to 1, contains finitely many points of discontinuity of $f_n$.

So we have for any partition with $\{x_0 = 0, x_1 = \delta, \leq x_2 \leq \cdots \leq x_n = 1\}$, where $x_i - x_{i-1} < \delta$ for $i = \overline{2, m}$

we have

Not clear, need to check.

$$) = \sum_{i=1}^{m} M_i \Delta \alpha_i = \sum \sup_{x \in [x_{i-1}, x_i]} f(x) \, |\alpha(x_i) - \alpha(x_{i-1})| < \frac{N\varepsilon}{\varepsilon} = \varepsilon .$$

$$\Rightarrow \int_0^1 \lim f_n(x) \, d\alpha_n(x) = 0 \quad \square$$

Jan 2003 7

NTR

P↳ Let $C$ be the Cantor set on the interval $[0,1]$
Let $A = C^c$ be its complement on the real line. $(A = \mathbb{R} \setminus C)$
Identify the set of all limit point $A'$ of $A$, explaning your answer

The set of all limit point $A'$ of $A$ is $A' = \mathbb{R}$.

Now we will prove that $\forall p \in \mathbb{R}$, $p$ is a limit point of $A$.

Assumpt a contradiction that $\exists p \in \mathbb{R}$; $p$ is not a limit point of $A$,

this means, $\exists N_r(p)$, $N_r(p) \cap A = \phi$

$$\Rightarrow \exists N_r(p),\ N_r(p) \subset (\mathbb{R} \setminus A) = C$$

this contracts with the fact that Cantor set contains no interval.

Jan 2009?

P2 a) Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

b) Let $\{a_n\}$ be a sequence with limit $L$. Define $b_n = \frac{1}{n^2}\sum_{k=1}^{n} k a_k$ | Prove that $\lim_{n\to\infty} b_n = \frac{L}{2}$

a) Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Put $S = \sum_{k=1}^{n} k$, we have

$$S = 1 + 2 + 3 + 4 + \cdots + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \cdots + 2 + 1$$

$$2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ pairs}} = n(n+1).$$

So $\sum_{k=1}^{n} k = S = \frac{n(n+1)}{2}$ □ a)

b) Notice that $\underbrace{\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n}(kL)}_{\text{put } c_n :=} = L \underbrace{\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} k}_{\int_0^1 x\,dx = \frac{1}{2}} = \frac{L}{2}$

Then we apply: if $\lim c_n = l$ and $\exists n_0 \in \mathbb{N}, \forall n \geq n_0, |c_n - b_n| < \varepsilon$ or $\lim(b_n - c_n) \to 0$ $\Big\}$ then $\lim_{n\to\infty} b_n = l$

Now we want to prove that $|b_n - c_n|$

$$|b_n - c_n| = \left| \frac{1}{n^2}\sum_{k=1}^{n} k a_k - \frac{1}{n^2}\sum_{k=1}^{n} kL \right| = \left| \frac{1}{n^2}\sum_{k=1}^{n} k(a_k - L) \right| \leq \frac{1}{n^2}\sum_{k=1}^{n} |k||a_k - L|.$$

because $\lim_{k\to\infty} a_k = L \Leftrightarrow \forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, |a_k - L| < \varepsilon$

then $|b_n - c_n| \leq \underbrace{\frac{1}{n^2}\underbrace{\sum_{k=1}^{k_0 - 1}|k||a_k - L|}_{\text{bounded}}}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\frac{1}{n^2}\sum_{k_0}^{n}|k|\underbrace{|a_k - L|}_{<\varepsilon}}_{\text{sth} < \frac{1}{2}} \longrightarrow \text{sth}$

So we have $|b_n - c_n| \xrightarrow[n\to\infty]{} 0$
from above $\lim_{n\to\infty} c_n = \frac{L}{2}$ $\Bigg\} \Rightarrow \lim_{n\to\infty} b_n = \frac{L}{2}$ □.

IV, Let $f$ be a continuous real valued function on $[a, b]$ and differentiable on $(a, b)$

Prove that
$$\boxed{\max_{a\le x\le b} |f(x)| \le \frac{1}{(b-a)} \int_a^b |f(x)|dx + (b-a) \sup_{a<x<b} |f'(x)|}$$

It's hard to control the sign So we consider $|sth|$ first

(or we can add an assumption that $f$ is $\underbrace{\text{non negative}}_{f(x)\geq 0}$, $\underbrace{\text{increasing on } [a,b]}_{f'(x)\geq 0}$.

* According to Intermediate value theorem for integral, we have :
$$\left.\begin{array}{l} f \text{ is a continuous on } [a,b] \\ \alpha : \text{monotonic increasing on } [a,b] \end{array}\right\} \text{ then } \int_a^b f(x)dx = f(c)\,(b-a)$$
for some $c \in [a,b]$.

So we have $\int_a^b |f(x)|dx \geq \left|\int_a^b f(x)\,dx\right| = \left| f(c)(b-a)\right| = |f(c)|(b-a)$ for some $c \in [a,b]$.

* Because $f$ is continuous on $[a,b]$ $\Rightarrow$ attains maximum value at some $y_0$ in $[a,b]$.

Let $|f(y_0)| = \max_{a\le x\le b} |f(x)|$

So we need to prove $|f(y_0)| \leq |f(c)| + (b-a) \sup_{a<x<b} |f'(x)|$.

$\Leftrightarrow$ NTP $|f(y_0)| - |f(c)| \leq \sup_{a<x<b} |f'(x)|\,(b-a)$.

We have $|f(y_0)| - |f(c)| \leq |f(y_0) - f(c)| = |f'(\xi)|\,|y_0 - c|$ for some $\xi$ between $y_0$ and $c$

$$\leq \sup_{a<x<b} |f'(x)|\,(b-a) \qquad \square.$$

notice that
$y_0, c \in [a,b]$
$\Rightarrow |y_0 - c| \leq (b-a)$

Q7 Given any $\varepsilon > 0$ prove that

$$\boxed{\max_{a \leq x \leq b} |f(x)| \leq \frac{1}{\varepsilon} \int_a^b |f(x)|\, dx + \frac{\varepsilon}{2} \sup_{x \in (a,b)} |f''(x)|.}$$

Suppose $f(x+1) = f(x)$ for all real $x$.
$f$ is real value
$f$, is Riemann integrable on every compact interval
$\int_0^1 f\,dx = 0$.

a) Prove that $\exists\, x_0$ such that $F(x) = \int_{x_0}^{x} f(t)\,dt \geq 0$, $\forall x \in [0, 1]$ ← the question

b) Show by example that $F'(x_0) = 0$ need not be true.

* We first prove that $\left.\begin{array}{l} f(x+1) = f(x),\ \forall x \in \mathbb{R} \\ \int_0^1 f(x)\,dx = 0 \end{array}\right\}$ Then $G(x) := \int_0^x f(t)\,dt$ is periodic ~~for $x \in [0,1]$~~ actually for $x \in \mathbb{R}$.

• We have

$$G(x+1) - G(x) = \int_0^{x+1} f(t)\,dt - \int_0^{x} f(t)\,dt = \int_x^{x+1} f(t)\,dt \xlongequal[\text{$f$ is periodic}]{\text{note that}} \int_0^1 f(t)\,dt = 0$$

(because $G$ is first defined on $[0, 1]$ and then extended to $\mathbb{R}$ since $G$ is period)

(Another (better) way to explain this is bcs with frequency 1

$$G(x+1) = \int_0^{x+1} f(t)\,dt = \underbrace{\int_0^1 f(t)\,dt}_{=0} + \int_1^{x+1} f(t)\,dt = \int_1^{x+1} f(t)\,dt \xlongequal[\substack{t=1 \Rightarrow u=0 \\ t=x+1 \Rightarrow u=x}]{\text{put } u=t-1} \int_0^x f(u+1)\,du = \int_0^x f(u)\,du = G(x)$$

* Second, we prove that $\left.\begin{array}{l} G \text{ is continuous on } (\mathbb{R}) \\ G \text{ is periodic } G(x+p) = G(x) \end{array}\right\} \Rightarrow G$ attain min/max in $\mathbb{R}$.

(Example $\left.\begin{array}{l} G \text{ is not continuous} \\ G \text{ is periodic} \end{array}\right\}$ $G$ not attain min/max:

We will prove that if $G$ is periodic $G(x+p) = G(x)$ (in this specific case, $p = 1$).
and because $G$ is continuous on $\mathbb{R}$ $\Rightarrow$ continuous on $[0, p]$
$\Rightarrow$ $G$ attain local (on $[0,p]$) min/max in this.
and because $G(x+p) = G(x)$ in fact this min/max is also global min/max in $\mathbb{R}$

* So because $G$ attains min/max in $\mathbb{R}$.
Assume $G$ attains min at $x_0 \in [0,1]$, then $\forall x \in \mathbb{R}$, $G(x_0) \leq G(x)$

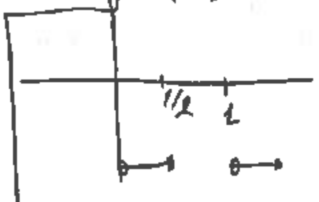
Then $G(x) - G(x_0) \geq 0 \Leftrightarrow \int_0^x f(t)\,dt - \int_0^{x_0} f(t)\,dt \geq 0 \Leftrightarrow \int_{x_0}^x f(t)\,dt \geq 0$ □ a)

b) Show by example that $F'(x_0) = 0$ need not be true.

We note one important result of $F$ and $f$ is that if $f$ is continuous at $x$, then $F'(x) = f(x)$.

$\Rightarrow$ We will find a periodic function such that
$f(x) \neq 0$, $\forall x$
$f$ is periodic with frequency 1
$\int_0^1 f\,dx = 0$

Let $f(x) = \begin{cases} -1, & x \in [0, 1/2] \\ 1, & x \in (1/2, 1) \end{cases}$ and .....



Then $F(x) = \begin{cases} 1-x, & x \leq \frac{1}{2} \\ x, & x \geq \frac{1}{2} \end{cases}$ □

Jan 2004, P5. NIA.

Let $f_n(x) = n\left(e^{\frac{x^2}{n}} - 1\right), \forall x \in \mathbb{R}$.

a) Prove that $\lim\limits_{n\to\infty} f_n(x) = x^2, \forall x$.

b) Prove $\{f_n\}$ is equicontinuous on $[0, M], \forall M > 0$

c) Prove that $\lim\limits_{n\to\infty}\int_0^1 [f_n(x)]^{1/3}\,dx$ exists and equals $3/5$

a) Prove that $\lim\limits_{n\to\infty} n\left(e^{\frac{x^2}{n}} - 1\right) = x^2, \forall x \in \mathbb{R}$.

We have
$$\lim_{n\to\infty} n\left(e^{\frac{x^2}{n}} - 1\right) = \lim_{n\to\infty}\frac{e^{\frac{x^2}{n}} - 1}{\frac{1}{n}} \underset{\text{L'Hopital}}{\overset{\frac{0}{0}}{=}} \lim \frac{-\frac{x^2}{n^2}e^{\frac{x^2}{n}}}{-\frac{1}{n^2}} = \lim_{n\to\infty} x^2 e^{\frac{x^2}{n}} = x^2 \quad \square a)$$

note $\lim n \to \infty$ (consider $n$: variable, $x$: constant.

b) Prove that $\{f_n\}$ is equicontinuous on $[0, M], \forall M > 0$

**Result 1:** $\{f_n\}$ sequence of differentiable on $[a, b] \subseteq \mathbb{R}^n$, $\{f_n'\}$ uniformly bounded $(\exists M > 0, |f_n'(x)| \leq M, \forall n, \forall x \in [a, b])$ $\Big\} \Rightarrow \{f_n\}$ equicontinuous on $[a, b]$.

(Prove this result and another related results next page.

We have that $f_n(x)$ is differentiable on $[0, M]$.
$$|f_n'(x)| = \left|n e^{\frac{x^2}{n}} \cdot \frac{2x}{n}\right| = \left|2x e^{\frac{x^2}{n}}\right| \leq 2M e^{\frac{M^2}{n}} \leq 2M e^{M^2}$$

From the above result, $\{f_n\}$ equicontinuous.

c) Prove that $\lim \int_0^1 [f_n(x)]^{1/3} dx$ exists and equals $3/5$

**Result 2:** $K$ compact, $\{f_n\}$ equicontinuous, $f_n \longrightarrow f$ pointwise $\Big\} \Rightarrow f_n \rightrightarrows f$ on $K$ (Also see this problem in Aug 2015, P5)

**Result 3:** $f_n \rightrightarrows f$ on $K$ compact, $h$ uniformly continuous $\Big\} \Rightarrow h(f_n) \rightrightarrows h(f)$.

Let $f(x) = x^2$, from (a) $f_n \longrightarrow f$ pointwise, from b) $\{f_n\}$ equicontinuous, $K$ compact $\Big\} \xRightarrow[\text{result 2}]{} f_n \rightrightarrows f$ on $K$.

- Now consider $h(x) = x^{1/3}$ is differentiable in $\mathbb{R}$, $h'(x) = \frac{1}{3}x^{-2/3}$ ~~is uniformly cont~~ bounded $\Big\} \Rightarrow$ uniformly cont.

$\Rightarrow (f_n)^{1/3} \rightrightarrows (f)^{1/3}$ So we have $\int_0^1 f^{1/3} dx = \lim \int_0^1 [f_n(x)]^{1/3} dx \Rightarrow$

$$\lim \int_0^1 [f_n(x)]^{1/3} dx = \int_0^1 [x^2]^{1/3} dx = \left.\frac{x^{5/3}}{5/3}\right|_0^1 = \frac{3}{5}$$

* Prove the results used in Problem ☺.

* **Result 1**: $\{f_n\}$: sequence of differentiable functions on $[a,b]\in\mathbb{R}$, $\{f_n'\}$ uniformly bounded $\Big\}$ $\{f_n\}$ equicontinuous

NTP $\{f_n\}$ equicontinuous $\Leftrightarrow$ NTP $\forall \varepsilon>0, \exists \delta>0, \forall x,y\in[a,b], |f_n(x)-f_n(y)|<\varepsilon$, $\boxed{\forall n}$

We have $\{f_n'\}$ uniformly bounded $\Leftrightarrow \exists M, |f_n'(x)|\leq M, \forall n, \forall x$

So consider $|f_n(x)-f_n(y)| = |f'(\xi)||y-x|$ for some $\xi\in(x,y)$.

$$\leq M|y-x|$$

Then $\forall\varepsilon>0$, choose $\delta$ s.t. $M\delta<\varepsilon$, then $\forall x,y\in[a,b]$, $|x-y|<\delta$, then

$$|f_n(x)-f_n(y)|\leq M\delta<\varepsilon$$

$\Rightarrow \{f_n\}$ equicontinuous $\square$

* **Result 2**. (Also in Aug 2015, P5)

K compact
$\{f_n\}$ equicontinuous
$f_n\longrightarrow f$ pointwise $\Bigg\}\Longrightarrow$ Prove that $f_n\rightrightarrows f$

(Until now, not sure if we can prove this (maybe we need $f_n, f\in C(K)$)

* Now we prove the question in Prelim Aug 2015, P5.

K compact $\subseteq\mathbb{R}$
$\{f_n\}$ equicontinuous
$f_n\longrightarrow$ in K $\Bigg\}\Longrightarrow$ Prove that $f_n\rightrightarrows$ in K

not conclude to $f$ yet.

- $\{f_n\}$ equicontinuous

$\Leftrightarrow \forall\varepsilon>0, \exists\delta>0, \forall x,y\in K, |x-y|<\delta, |f_n(x)-f_n(y)|<\varepsilon/3$ (1)

- $f_n$ converges pointwise in K.

$\Leftrightarrow \forall x\in K, \cancel{\forall\varepsilon}>0, \exists n_{x\varepsilon}\in\mathbb{N}, \forall m,n\geq n_{x\varepsilon}, |f_m(x)-f_n(x)|<\varepsilon/3$ (2)

- K is compact $\Leftrightarrow$ every open cover contains a finite subcover

We have $K\subset\bigcup_{x\in K}B(x,\delta)$ then $\exists\{x_1,\dots,x_k\}$, $K\subseteq\bigcup_{i=1}^{k}B(x_i,\delta)$, $x_i\in K$.

important step!

We need to prove $f_n\rightrightarrows$ in $K\subseteq\mathbb{R}$
$\Rightarrow$ NTP $\forall\varepsilon>0, \exists N\in\mathbb{N}$, $\forall m,n\geq N, \forall x\in K$, $|f_m(x)-f_n(x)|<\varepsilon$.

Now we choose $N=\max\{n_{x_1}, n_{x_2}, n_{x_3},\dots,n_{x_k}\}$

Then from (2), $\forall x_i, i=\overline{1,k}$, $\forall\varepsilon>0, \exists N, \forall m,n\geq N$, $|f_m(x_i)-f_n(x_i)|<\varepsilon/3$ (*)

* Now consider every $x\in K$, because $K\subseteq\bigcup_{i=1}^{k}B(x_i,\delta)$

then $\exists i_0$, $x\in B(x_{i_0},\delta)$

So we have $\forall m,n\geq N$

$$|f_m(x)-f_n(x)|\leq \underbrace{|f_m(x)-f_m(x_{i_0})|}_{\leq\varepsilon/3 \text{ because } x\in B(x_{i_0},\delta) \text{ and } (1)} + \underbrace{|f_m(x_{i_0})-f_n(x_{i_0})|}_{<\varepsilon/3 \text{ because of } (*)} + \underbrace{|f_n(x_{i_0})-f(x)|}_{\leq\varepsilon/3 \text{ because } x\in B(x_{i_0},\delta) \text{ and } (1).}$$

be

$\leq \cancel{3}\varepsilon \Rightarrow \square$

* f is continuously differentiable $\Leftrightarrow$ f is differentiable ($\exists f'$) and $f'$ is continuous.

$\frac{f}{x} \to 0$

$E = \{(0,1)\} \cup \{2\}$.

## Analysis Preliminary Exam, August 2016

$f: E \to \mathbb{R}$.

1. Consider the following proposition: *Every bounded continuous real-valued function $f$ on $\mathbb{R}$ attains its maximum.* The following argument which attempts to prove this has an error. (a) Find where the error occurs and (b) provide a counterexample, with details, to show that the argument indeed fails at that point:

   Let $M = \sup\{f(x) : x \in \mathbb{R}\}$, and let $x^*, x_n \in \mathbb{R}$ such that $x_n \to x^*$ and $f(x_n) \to M$. Since $f$ is continuous, $f(x_n) \to f(x^*)$, which implies $f(x^*) = M$. Hence, $x^*$ is where $f$ attains its maximum.

(See Jan E1)

* If $E \subset \mathbb{R}$, If E is an bound $\to \exists (x_n)$ in E, $x_n \to$ sup

2. Prove: there exists $c > 0$ and continuous functions $f, g$ on $(-c, c)$ such that $f(0) = g(0) = 0$ and

$$\sin(f(z)) + \cos(g(z)) = z^2 + 1, \text{ and}$$
$$(f(z))^2 + 2e^{2g(z)} = 2\cos z$$

   for all $z \in (-c, c)$.

3. Let $f$ be continuously differentiable, and suppose that $f(0) < -1$, $f(1) > 0$, and $f(2) < 0$. Prove that for each $c \in [0, 1]$ there exists $x_c \in (0, 2)$ such that $f'(x_c) = c$.

4. Let $(X, d)$ be a metric space. Prove or provide a counterexample:

   (a) The intersection of finitely many dense subsets of $X$ is dense.

   (b) The intersection of finitely many open dense subsets of $X$ is open and dense.

5. Let $f, g$ be continuous functions on $\mathbb{R}$ such that $f$ is differentiable everywhere and let $f(1) = 0$. Prove that $fg$ is differentiable at 1.

6. Let $(f_n)$ be a sequence of functions on $[0, 1]$ with continuous first and second derivatives, such that for all $n \geq 1$,

$$1 \leq f_n(0) \leq 2, \qquad 3 \leq f_n'(0) \leq 4, \qquad \sup_{0 \leq x \leq 1} |f_n''(x)| \leq 12$$

   Prove that $(f_n)$ has a subsequence which converges uniformly on $[0, 1]$.

$f(x) = +\frac{1}{|x|+1}$ $f: ~~(1, +\infty)~~ \to \mathbb{R}$ $f: \mathbb{R} \to [0, 1]$.

~~sup~~ $= \emptyset$

$x_n \to \infty$.

, Let $f, g$ be continuous function in $\mathbb{R}$. such that $f$ is differentiable everywhere
$f(1) = 0$
ve that $(fg)$ is differentiable at 1

e want to prove that $(fg)$ is differentiable at 1
$\Leftrightarrow$ we want to consider $\lim\limits_{t \to 1} \dfrac{(fg)(t) - (fg)(1)}{t-1}$

have $\dfrac{(fg)(t) - (fg)(1)}{t-1} = \dfrac{f(t)g(t) - f(1)g(1)}{t-1} \overset{f(1)=0}{=\!=} \dfrac{f(t)\,g(t)}{t-1}$.

then $\lim\limits_{t \to 1} \dfrac{f(t)\,g(t)}{t-1} \underset{f(1)=0}{=} \lim\limits_{t \to 1} \dfrac{[f(t) - f(1)]}{t-1} g(t) = \lim f'(t)\, g(1)$

Then $(fg)$ is differentiable at 1 □

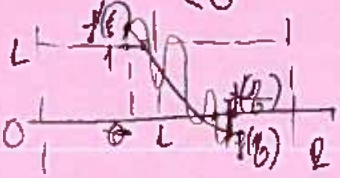
, Let $f$ be continuously differentiable, suppose that $f(0) < -1$, $f(1) > 0$, $f(2) < 0$
ove that for each $c \in [0,1]$, the exists $x_c \in (0,2)$ such that $f'(x_c) = c$.

We have $\underbrace{\underbrace{f(1)}_{>0} - \underbrace{f(0)}_{<-1}}_{>1} = f'(\xi)(1-0)$. $\Rightarrow \exists\, \xi \in (0,1)$, $f'(\xi) > 1$

$\underbrace{\underbrace{f(2)}_{<0} - \underbrace{f(1)}_{>0}}_{<0} = f'(b)$ $\Rightarrow \exists\, b \in (1,2)$, $f'(b) < 0$

$\Rightarrow [0,1] \subset f'((0,2))$

$f'$ is continuous.



$\Rightarrow [0,1] \subseteq f'[\xi, b] \subset f'(0,2)$.

$\Rightarrow$

Aug 2016 >

1> Consider the following proposition:

"Every bounded continuous real-valued function f on $\mathbb{R}$ attains its maximum".

The following argument which attempts to prove this has an error.

a> Find where the error occurs

b> Provide a counter example, with details, to show that the argument indeed fails at that po

Let $M = \sup\{f(x), x \in \mathbb{R}\}$.

Let $x^*, x_n \in \mathbb{R}$ such that $x_n \longrightarrow x^*$ and $f(x_n) \longrightarrow M$.

Since f is continuous $f(x_n) \longrightarrow f(x^*)$ which implies $f(x^*) = M$. ⇒ Hence $x^*$ is where f attail its m

---

a> The error of the assumption is that we can have $f(x_n) \longrightarrow M$ but we don't always have $x_n \longrightarrow x^*$

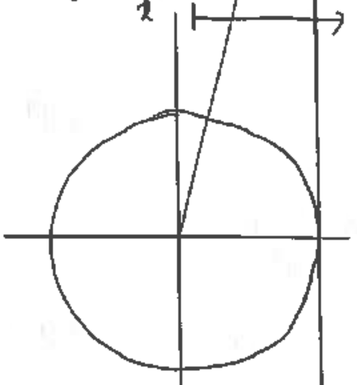
b> For example

Let $f : \mathbb{R} \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$, $x \longmapsto \arctan x$, then we have this is a continuous function & bounded function.

$M = \sup\{f(x), x \in \mathbb{R}\}$

then we can have $y_n = f(x_n) \longrightarrow \frac{\pi}{2}$.

but $x \longrightarrow \overset{x^*}{\infty}$ where $\arctan(x^*) = \frac{\pi}{2}$ )



* Another example is $f(x) : \mathbb{R} \longrightarrow [0, 1]$
$~~x~~ \longmapsto ~~|x|~~ \frac{1}{|x|+1}$.

then $\frac{1}{|x|+1} \longrightarrow 1$ when $x \longrightarrow \infty$

$M = \sup f(x) = 1$
but $f(x) \longrightarrow 1$ when $x \longrightarrow \infty$

In here the meaning of $x_n \longrightarrow x^*$ is $\left| \begin{array}{l} x_n \text{ converges to } x^* \\ x^* < +\infty \end{array} \right.$.

* The idea of this problem is $M = \sup\{f(x)\}, x \in \mathbb{R}\} \notin \mathrm{Im}[f(x)]$ and only be attained at $\infty$.

Aug 2016 - P2:

ove there exists $c>0$, and continuous function $f, g$ on $(-c,c)$ s.t

$$f(0)=g(0)=0$$

$$\begin{cases} \sin(f(z))+\cos(g(z)) = z^2+1 \\ [f(z)]^2+2e^{2g(z)} = 2\cos z \end{cases} \quad \text{for } z\in(-c,c).$$

nalyze the problem:

onsider $F:(x,y,z): \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x,y,z) \longmapsto F(x,y,z) = \{ F_1 = \sin x + \cos y - z^2 - 1, \quad F_2 = x^2 + 2e^{2y} - 2\cos z)$$

nd we want to prove that, there exist a neighborhood $(-c,c)$ of $0$, s.t $(x,y) = (f(z), g(z))$ nd $F(x,y,z) = \vec{0}_{\mathbb{R}^2}$.

So consider DF, we have

$$DF = \left[\begin{array}{cc|c} \cos x & -\sin y & -2z \\ 2x & 4e^{2y} & -2\sin z \end{array}\right]$$

We have $F$ is a $C^1$ function (all partial derivative exist and continuous)

$A_{xy}$ have $\det(A_{xy}) = \cos x \, 4e^{2y} + 2x \sin y \overline{\overline{\;_{\text{at } f(z)=0}\;}} 4 > 0$, at $(0,0,0)$

then $x=y=0$

because $f(0)=g(0)=0$

o by Implicit function theorem, ✓

There is a open neighborhood $V$ of $(0,0,0)$ in $\mathbb{R}^3$ and an open neighborhood $W=(-c,c)$ of $0$

such that $\forall z$ in $(-c,c)$, $\exists!\,(x,y)$ st. $\begin{cases} (x,y,z) \subset V \\ F(x,y,z)=0 \end{cases}$

and so we can find $f, g$ continuous (also stated in IFT)

satisfies above required. □

4) Let $(X,d)$ be a metric space. Prove or provide a counter example.

a) The finitely many dense subsets of $X$ is dense

The finitely many dense subsets of $X$ may not be dense :

EX: the set of rational numbers is dense in $\mathbb{R}$. (call $\mathbb{Q}$)

(by theorem $\forall a, b \in \mathbb{R}$, $a<b$ then $\exists q \in \mathbb{Q}$, $a<q<b$.)

the set of irrational numbers is dense in $\mathbb{R}$ (call $F$)

but $\mathbb{Q} \cap F = \phi$ (not dense in $\mathbb{R}$).

b) The intersection of finitely open and dense subsets of $X$ is open and dense.

* **Way 1**: In this prove we use the definition that $E \subseteq X$ is dense in $X$

$\iff \forall x \in X$, $\forall \lambda > 0$, $N_\lambda(x) \cap E \neq \phi$.

- Let $E, F$ open dense in $X$

$E$ is open, dense in $X$

$\iff \begin{cases} \forall e \in E, \exists N_\lambda(e) \subset E \\ \forall x \in X, N_\lambda(x) \cap E \neq \phi. \end{cases}$

$F$ open, dense in $X$

We NTP $E \cap F$ open and dense in $X$

$\iff$ NTP $\begin{cases} E \cap F \text{ open in } X \\ \forall x \in X, N_\lambda(x) \cap (E \cap F) \neq \phi. \end{cases}$

- We have $\left.\begin{matrix} E \text{ open in } X \\ F \text{ open in } X \end{matrix}\right\} \Rightarrow E \cap F$ open in $X$.

- Let $x \in X$, because $E$ dense in $X \Rightarrow \forall N_\lambda(x)$, $N_\lambda(x) \cap E \neq \phi$.

Assume $y \in N_\lambda(x) \cap E$. $\Rightarrow \exists N_{R_1}(y) \subset N_\lambda(x)$ (because $N_\lambda(x)$ open)

$\exists N_{R_2}(y) \subset E$ (because $E$ open.)

Choose $R = \min\{R_1, R_2\}$, then $N_R(y) \subset N_\lambda(x)$ and $N_R(y) \subset E$ (1)

We have $N_R(y)$ is a neighborhood of $y$, $F$ dense in $X$

$\Rightarrow N_R(y) \cap F \neq \phi$. (2).

(1)+(2) $\Rightarrow N_\lambda(x) \cap (E \cap F) \neq \phi$.

This means for $E, F$ open + dense in $X \Rightarrow E \cap F$ open and dense in $X$.

Assume $G_1$ open + dense in $X \Rightarrow E \cap F \cap G_1$ open and dense in $X$

$\Rightarrow$ ... intersection of finitely open + dense subset of $X$ is open and dense.

* In case only 2 dense subsets, we only need one of them is open $\Rightarrow E \cap F$ is dense

* **Way 2**: We use the definition: $E$ is dense in $X \iff \forall U$ nonempty open in $X$ then $U \cap E \neq \phi$

Let $U \neq \phi$, $U$ open in $X$, we need to prove $U \cap (E_1 \cap E_2 \dots E_n) \neq \phi$.

- We have $\left.\begin{matrix} U \neq \phi, \text{ open in } X \\ E_1 \text{ dense in } X \end{matrix}\right) \Rightarrow \left.\begin{matrix} U \cap E_1 \neq \phi \\ \text{intersection of finite open sets is open} \end{matrix}\right\} \Rightarrow U \cap E_1 = U_1 \neq \phi$ and open

open

$\Rightarrow$ similarly $U_1 \cap E_2 = U_2$ nonempty open $\Rightarrow U_2 \cap E_3 = U_3$ nonempty, open

$\Rightarrow$

g2016 >

> $\{f_n\}$ be a sequence of function on $[0,1]$ with continuous first and second derivative

$\forall n \geq 1,\quad 1 \leq f_n(0) \leq 2,$
$3 \leq f_n'(0) \leq 4$
$\sup\limits_{0\leq x\leq 1} |f_n''(x)| \leq 12$

} Prove that $\{f_n\}$ has a subsequence which converges uniformly on $[0,1]$.

idea of this prove is using
$0,1]$ is compact (already have)
$f_n\}$ is a sequence with $f_n \in C([0,1])$ (already have).
$_n\}$ is pointwise bounded + equicontinuous. (need to prove)

} Then by Arzela-Ascoli theorem, we have.
$\{f_n\}$ is { uniformly bounded
{ contains a convergent subsequence in $[0,1]$.

First, we prove that $\{f_n\}$ is point wise bounded (In fact $\{f_n\}$ is uniformly bounded)

Applying Taylor theorem to find Taylor series (Lagrange form of $f$) we have.

$$f_n(x) = f_n(0) + \frac{f_n'(0)}{1!}(x) + \frac{f_n''(\xi)}{2!}(x^2) \quad , \text{ for some } \xi \in (0,x).$$

om this we have,

$$-11 \leq 1+3x-12x^2 \leq f_n(x) \leq 2+4x+12x^2 \leq 18 \quad \text{for } x\in[0,1]$$

This mean $\{f_n\}$ uniformly bounded in $[0,1]$.

Now we want to prove that $\{f_n\}$ equicontinuous.

( we have a result (from Jan 2009 P5).
$\{f_n\}$ sequence of differentiable on $[a,b]$
$\{f_n'\}$ uniformly bounded
} then $\{f_n\}$ equicontinuous.

e have for $\forall x$, $\quad f_n'(x) - f_n'(0) = f_n''(\xi)x \quad$ for some $\xi \in [0\, x]$.

$$|f_n'(x)| - |f_n'(0)| \leq x \cdot \sup_{x\in[0,1]}|f''(\xi)| = 12x \leq 12 \quad \text{for } x\in[0,1].$$

$$\Rightarrow |f_n'(x)| \leq 12 + |f_n'(0)| < 12 + 4.$$

$\Rightarrow \{f_n'(x)\}$ uniformly bounded $\Rightarrow$

$\Rightarrow$ from above result $\{f_n\}$ equicontinuous.

From the idea stated at the begining of the proof, $\{f_n\}$ contains a convergent subsequence.

About continuous, uniformly continuous and compactness.

| | |
|---|---|
| * X compact<br>f is continuous on X | Prove that f is uniformly continuous on X |

(X compact ⇔ every sequence has a convergent subsequence

f continuous at $x$ on X ⇔ $\forall (x_n)$ in X, $x_n \longrightarrow x$ then $f(x_n) \longrightarrow f(x)$.)

* Assume f is not uniformly continuous on X

$\Leftrightarrow \exists \varepsilon_0 > 0,\ \exists (x_n)(y_n)$ s.t $\left(\lim\limits_{n\to\infty} |x_n - y_n| = 0\right.$ but $|f(x_n) - f(y_n)| > \varepsilon_0$ (*

because $(x_n)$ in X compact, then $\exists (x_{n_\ell})$, subsequence of $(x_n)$, $x_{n_\ell} \to x \in X$. (1)

- then because $|x_n - y_n| \longrightarrow 0$

we have $\lim\limits_{\ell\to\infty} y_{n_\ell} = \lim\limits_{\ell\to\infty} \left| x_{n_\ell} - (x_{n_\ell} - y_{n_\ell}) \right| = \lim x_{n_\ell} + \lim |x_{n_\ell} - y_{n_\ell}| = x$.

this means $(y_{n_\ell}) \longrightarrow x$ (2)

| note from this we learn that if
$|x_n - y_n| \longrightarrow 0$
subsequence $(x_{n_\ell}) \longrightarrow x$ } then $(y_{n_\ell}) \longrightarrow x$.

- (1)+(2)+ f is continuous, we have

$\lim\limits_{\ell\to\infty} |f(x_{n_\ell}) - f(y_{n_\ell})| = \left| f\left(\lim\limits_{\ell\to\infty} x_{n_\ell}\right) - f\left(\lim\limits_{\ell\to\infty} y_{n_\ell}\right) \right| = |f(x) - f(x)| = 0$.

but this contacts with the non-uniform continuity condition(*) $|f(x_{n_\ell}) - f(y_{n_\ell})| > \varepsilon_0$

therefore, f is uniformly continuous.

* $\boxed{\text{Does series sequence } \{c_n\} \text{ where } c_n = \frac{9^n}{n!} \text{ converges or diverges ?}}$

For $n > 9$, we write

$$0 \leq c_n = \underbrace{\frac{9 \cdot 9 \cdot 9 \cdots 9}{1 \cdot 2 \cdot 3 \cdots 9}}_{c :=} \frac{9 \cdots\cdots 9 \cdot 9}{10 \cdot 11 \cdot 12 \cdots n} = \leq c \cdot \frac{9}{n}.$$

each factor is less than 1

then

$$0 \leq \lim c_n = \lim \frac{9^n}{n!} \leq \lim_{n \to \infty} c \cdot \frac{9}{n} = 0$$

then $\lim_{n\to\infty} \frac{9^n}{n!} = 0$

Some common continuous function that we can apply $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$ :

$e^x$, $\ln x$, $x^k$, ...

* $\boxed{\text{Does } d_n = \ln 5^n - \ln(n!) \text{ converges or diverges?}}$

We have $d_n = \ln 5^n - \ln(n!) = \ln\left(\frac{5^n}{n!}\right)$

we have $e^{d_n} = \frac{5^n}{n!}$ $\lim e^{d_n} = \lim \frac{5^n}{n!} = 0$

We have ~~$e^x$~~ $f(x) = e^x$, is a continuous function, if $d_n$ converged, then

$$\lim_{n\to\infty} e^{d_n} = \underbrace{e^{\lim_{n\to\infty} d_n} = 0}_{\text{impossible}} \Rightarrow d_n \text{ diverges.}$$

* $\boxed{\text{Does}_{\text{ a sequence}} \; a_n = \left(2 + \frac{4}{n^2}\right)^{1/3} \text{ diverge or converge}}$

we have $f(x) = x^{1/3}$ continuous, put $c_n = 2 + \frac{4}{n^2}$

so $\lim_{n\to\infty} (c_n^{1/3}) = \lim_{n\to\infty} f(c_n) = f(\lim_{n\to\infty} c_n) = \left(\lim_{n\to\infty} c_n\right)^{1/3} = 2^{1/3}$

$\lim c_n = \lim\left(2 + \frac{4}{n^2}\right) = 2$

* $\boxed{\text{Sequence } \{d_n\} \text{ with } d_n = \ln\left(\frac{2n+1}{3n+4}\right)}$

Because $f(x) = \ln x$ is a continuous function on $(0, +\infty)$

$$\lim_{n\to\infty} \ln\left(\frac{2n+1}{3n+4}\right) = \ln\left(\lim_{n\to\infty} \frac{2n+1}{3n+4}\right) = \ln \frac{2}{3}$$

Jan 2007, 2

Prove that the series $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ is convergent and find its sum.

* Prove that the series converges

- We have $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{(n+1)^2}{(n+1)!} \frac{(n)!}{n^2}\right| = 0 < 1$.

Then the series converges "absolutely"

* Find its sum

We have $\sum_{n=1}^{\infty} \frac{n^2}{(n!)} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=1}^{\infty} \left[\frac{(n-1)}{(n-1)!} + \frac{1}{(n-1)!}\right]$

$= \sum_{n=1}^{\infty} \frac{n-1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$ (because the 2 series converge "absolutely".

$= \cancel{\sum_{n=1}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}}$

→ this is wrong because n from 1 ⟶ ∞.
~~we can~~
(k)! when k<0 is undefined.

* We have $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ $= \sum_{n=0}^{\infty} \frac{n}{n!} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{(n)!}}_{=e}$

= [illegible]

$\sum_{n=0}^{\infty} \frac{n}{n!} \neq \sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Then $\sum_{n=1}^{\infty} \frac{n^2}{n!} = 2e$.

Aug 2016 >

P6 Consider the mapping $f = (f_1, f_2, f_3)$ of $\mathbb{R}^3$ into $\mathbb{R}^3$ given by

$f_1(x_1, x_2, x_3) = x_1$ a) Is $f$ continuously differentiable? Why / why not

$f_2(x_1, x_2, x_3) = x_1^2 + x_2$ b) Find a point at which $f$ satisfies the assumptions of the Inverse Function Theorem

$f_3(x_1, x_2, x_3) = x_1 + x_2^2 + x_3^3$

c) Is $f$ injective?

a) $f$ is ~~diff~~ continuous differentiable $\overset{def}{\iff}$ all partial derivatives exist and continuous.

We have
$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2x_1 & 1 & 0 \\ 1 & 2x_2 & 3x_3^2 \end{bmatrix}$$

we have all partial derivative exists and continuous

$\iff$ $f$ is continuous differentiable

b) Find a point at which $f$ satisfies the assumptions of the IFT.

We have $\det[f'] = 3x_3^2$

We also have the assumption so that $f$ satisfies the assumption of IFT ~~at~~ the point $(x_1^0, x$

$$\begin{cases} f \text{ is } C_1 \text{ in an open } U \subseteq \mathbb{R} \\ (x_1^0, x_2^0, x_3^0) \\ f'(x_1^0, x_2^0, x_3^0) \text{ is invertible.} \end{cases}$$

→ $f$ is satisfies the assumption of IFT if $x_3 \neq 0$.

c) Is $f$ injective.

$\mathrm{Ker}\, f = \{ (x_1, x_2, x_3) \subseteq \mathbb{R}^3 \mid f_1(x_1, x_2, x_3) = 0,\ x_1^2 + x_2 = 0,\ x_1 + x_2^2 + x_3^3 = 0 \}$.

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_1^2 + x_2 = 0 \\ x_1 + x_2^2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

→ $\mathrm{Ker} f = \bar{0} \in \mathbb{R}^3 \implies f$ is injective.

# Analysis Preliminary Exam, May 2017

1. Let $f : \mathbb{Q} \to \mathbb{R}$ where $\mathbb{Q}$ is the set of all rational numbers.

   (a) If $f$ is uniformly continuous prove it has an extension to a continuous function $F : \mathbb{R} \to \mathbb{R}$, i.e. there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $f(q) = F(q)$ for all $q \in \mathbb{Q}$.

   (b) Give an example of a continuous $f : \mathbb{Q} \to \mathbb{R}$ that has no continuous extension $F : \mathbb{R} \to \mathbb{R}$.

See Aug 2003 P 57

2. Let $X$ denote the collection of all bounded functions $f : \mathbb{R} \to \mathbb{R}$. For $f, g \in X$ define
$$d(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}.$$
Then $(X, d)$ is a metric space. Let
$$E = \{f \in X : \text{there exists } K \text{ such that } f(x) = 0 \text{ for all } x > K\}.$$
Find the closure of $E$ in $X$.

3. For $p \geq 0$, find
$$\lim_{n\to\infty} n^{-(p+1)} \sum_{k=1}^{n} k^p.$$

4. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ and $\dfrac{\partial f}{\partial x} : \mathbb{R}^2 \to \mathbb{R}$ are both continuous. Let
$$g(x) = \int_0^1 f(x,t)\,dt.$$
Prove $g$ is differentiable and that
$$g'(x) = \int_0^1 \frac{\partial f}{\partial x}(x,t)\,dt.$$

5. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that
$$|f^{(n)}(x)| \leq n!|a_n|$$
for all $n$ and all $x \in \mathbb{R}$. Prove that the Taylor series about $x = 0$ for $f$ converges uniformly to $f$ on every closed and bounded interval $[-M, M]$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function such that $f(0) = 0$. Let $g : [0,1] \to \mathbb{R}$ be a continuous function such that
$$\int_0^1 f^n(x) g(x)\,dx = 0, \; n = 0, 1, 2, \ldots.$$
Prove that $g$ is identically *zero*.

# AUGUST 2017 PRELIMINARY EXAMINATION IN ANALYSIS

**1.** Let $X$ be a metric space. Consider a family of subsets of $X$, denoted $\{E_i : i \in A\}$ where $A$ is an uncountable index set. Suppose that for every finite or countable set $B \subset A$ the intersection

$$\bigcap_{i \in B} E_i$$

is open. Prove that the set

$$E = \bigcap_{i \in A} E_i$$

is also open.

**2.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that for every compact set $K \subset \mathbb{R}$ the inverse image $f^{-1}(K)$ is also compact. Prove that

$$\lim_{x \to +\infty} |f(x)| = +\infty$$

**3.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders and satisfies $f(0) = f'(0) = f''(0) = 0$. Prove that the function $g(x) = f(x)^{1/3}$ is differentiable at 0.

**4.** ~~Let $f$ and $g$ be Riemann-Stieltjes integrable on $[a, b]$ with respect to a non-decreasing function $\alpha$. Suppose that given any partition $P$ of $[a, b]$ there exists a partition $Q$ of $[a, b]$ such that~~

$$L(f, P, \alpha) \le L(g, Q, \alpha) \quad \text{and} \quad L(g, P, \alpha) \le L(f, Q, \alpha)$$

~~Prove that~~

$$\int_a^b f\,d\alpha = \int_a^b g\,d\alpha$$

**5.** Determine all positive continuous functions $f$ on $[1, \infty)$ such that

$$\ln\left(1 + \int_0^\theta f(e^x)\,dx\right) = \theta$$

for all real numbers $\theta > 0$. h

**6.** Prove that the image of any open set containing the unit disk $\{(x, y) : x^2 + y^2 \le 1\}$ under the mapping $f(x, y) = (x^4 + y^4, 2xy)$ is *not* a subset of the unit disk.

y2017, 22.

$X$ denotes the collection of all bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

r $f, g \in X$, define $d(f,g) = \sup\{|f(x) - g(x)|, x \in \mathbb{R}\}$. } Then we have $X$ is a metric space

$:= \{f \in X$, there exist $K$ such that $f(x) = 0$ for all $x > K\}$

ind the closure of $E$ in $X$.

Idea: Now we first try to analyze $\bar{E}$

$$\bar{E} = \{g : \mathbb{R} \to \mathbb{R} \text{ bounded}, \exists (f_n) \in E, f_n \to g \text{ on } X\}.$$

$$= \left\{ \begin{aligned} &g : \mathbb{R} \to \mathbb{R} \text{ bounded}, \exists (f_n) \in \mathbb{R} \to \mathbb{R} \text{ bounded}, \exists K_n, f_n(x) = 0, \forall x > K_n, \\ &\qquad \text{such that } \exists n \geq N, \forall n \geq N, \sup_{x \in \mathbb{R}} \{|f_n(x) - g(x)|, ~~x \in \mathbb{R}~~\} < \varepsilon \end{aligned} \right\}$$

* Claim $F = \{g : \mathbb{R} \to \mathbb{R}$ bounded and $\lim_{x \to \infty} g(x) = 0\}$ is the closure of $E$.

- First, we have $E \subset F$.
  This is obvious since $\exists K, \forall x > K, f(x) = 0$ is the definition of $\lim_{x \to \infty} f(x) = 0$.
- Second we will prove that $F$ is closed.
  We need to prove that if $g : \mathbb{R} \to \mathbb{R}$ bounded, $\exists g_n \in F$, $g_n \xrightarrow{\text{in } X} g$, then $\lim_{x \to \infty} g(x) = 0$.
  we have $g_n \to g \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \sup|g_n(x) - g(x)| < \varepsilon$.

$$\left.\begin{aligned} &\text{this mean } g_1(x \rightrightarrows g \\ &\text{and because } \lim_{n \to \infty} g_n(x) = 0 \end{aligned}\right\} \Rightarrow \lim_{x \to \infty} g(x) = 0.$$

$$\text{because } \left.\begin{aligned} &E \subset F \\ &F \text{ is closed} \end{aligned}\right\} \Rightarrow \bar{E} \subseteq F \qquad (1)$$

- Now we need to prove that $F \subseteq \bar{E}$ (2)

Let $g \in F$, so we have $\lim_{x \to \infty} g(x) = 0 \Leftrightarrow \forall \varepsilon > 0, \exists M > 0, \forall x > M, |g(x) = 0| < \varepsilon$

$-\varepsilon \leq g(x) < \varepsilon$

Then put $f(x) = g(x)$, for $x < M$

$f(x) = 0$, for $x \geq M + 1$ so we have $f(x) \in E$

and $d(f, g) < \varepsilon$.

Hence $g \in \bar{E}$

$$(1)+(2) \Rightarrow \bar{E} = F = \{g : \mathbb{R} \to \mathbb{R}, \lim_{x \to \infty} g(x) = 0\}.$$

May2017 > P4

Assume $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\frac{\partial f}{\partial x}:\mathbb{R}^2 \longrightarrow \mathbb{R}$ are both continuous.

Let $g(x) = \int_0^1 f(x,t)\,dt$.

Prove that $g(x)$ is differentiable and that $g'(x) = \int_0^1 \frac{\partial f}{\partial x}(x,t)\,dt$.

Now consider

$$\frac{g(y+h)-g(y)}{h} = \frac{\int_0^1 f(y+h,t)\,dt - \int_0^1 f(y,t)\,dt}{h} = \frac{\int_0^1 [f(y+h,t)-f(y,t)]\,dt}{h}$$

Note that $f(y+h,t)-f(y,t) = \frac{\partial}{\partial x}(\xi_h,t)\boxed{\cdot h}$ for some $\xi_h \in (y, y+h)$

note that $\xi(h) \longrightarrow \xi(0)$ when $h \to 0$.

and $|\xi(h)-\xi(0)| = |\xi(h)-y| \le h \longrightarrow 0$.

So we have $\frac{\partial}{\partial x}(\xi(h),t) \longrightarrow \frac{\partial}{\partial x}(\xi(0),t) = \frac{\partial}{\partial x}(y,t)$.

So we have $\lim_{h\to 0}\frac{g(y+h)-g(y)}{h} = \lim \int_0^1 \frac{[f(y+h,t)-f(y,t)]}{h}\,dt =$

$$= \int_0^1 \lim_{h\to 0} \frac{f(y+h,t)-f(y,t)}{h}\,dt =$$

$$= \int_0^1 \frac{\partial f}{\partial x}(y,t)\,dt \quad \square.$$

2017 P 5:

ppose that $\sum a_n x^n$ converges for all $x \in \mathbb{R}$.

t $f:\mathbb{R} \longrightarrow \mathbb{R}$ be an infinitely differentiable function such that

$$|f^n(x)| \le n!\, a_n \quad \forall n,\ \forall x \in \mathbb{R}.$$

ve that the Taylor series about $x=0$ for $f$ converges uniformly to $f$ on every closed and nded interval $[-M, M]$.

idea of this prove is we have $f(x)$

and we have $f(x) = P_d(x) + \frac{f^{(d+1)}(\xi)}{(d+1)!}(x-0)^{d+1}$

and we want to prove that $P_d(x) \rightrightarrows f(x) \Leftrightarrow$ NTP $|P_d(x) - f(x)| = \left| \frac{f^{(d+1)}(\xi)}{(d+1)!} x^{d+1} \right| \rightrightarrows 0$

$\uparrow$

( Taylor series about $x=0$
Taylor polynomial about $x=0$ of $f$ )

Now we will prove that $\left| \frac{f^{(d+1)}(\xi)}{(d+1)!} x^{d+1} \right| \rightrightarrows 0$.

we have $\left| \frac{f^{(d+1)}(\xi)}{(d+1)!} x^{d+1} \right| \le \left| \frac{(d+1)!}{(d+1)!} a\, a_{d+1} x^{d+1} \right| = |a_{d+1} x^{d+1}| \le M^{d+1} |a_{d+1}|$ -

note that $\sum a_n x^n$ converges $\Rightarrow \lim\limits_{n\to\infty} a_n x^n = 0 \to M^n a_n \underset{n\to\infty}{\rightrightarrows} 0$

This means $\left| \frac{f^{(d+1)}(\xi)}{(d+1)!} x^{d+1} \right| \rightrightarrows 0 \Rightarrow$ thus $P_d(x) \rightrightarrows f(x)$ $\square$.

May 2017

P67 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly increasing function such that $f(0)=0$

Let $g: [0,1] \longrightarrow \mathbb{R}$ be a continuous function such that $\int_0^1 f^n(x)\, g(x)\, dx = 0$, $n = 0, 1, 2, \ldots$

Prove that $g$ is identically zero.

---

Put $u = f(x)$ Then

$$\begin{cases} x = f^{-1}(u) \\ du = f'(x)\, dx \Rightarrow dx = \dfrac{du}{f'(x)} = \dfrac{du}{f'[f^{-1}(u)]} \\ x = 0 \Rightarrow u = f(0) = 0 \\ x = 1 \Rightarrow u = f(1). \end{cases}$$

This is the key step that helps solve the problem.
whenever see a problem relating to
that can't be solve by another way,
try to use integration by part to see if
can change to

So we have

$$\int_0^1 f^{(n)}(x)\, g(x)\, dx = 0, \quad n = 0, 1, 2, \ldots$$

$$\Leftrightarrow \int_0^{f(1)} \frac{u^n \; g(f^{-1}(u))}{f'[f^{-1}(u)]}\, du = 0$$

note that $f$ strictly increasing $\Rightarrow f'(f^{-1}(u)) > 0$

$$\Rightarrow \int_0^{f(1)} u^n\, g(f^{-1}(u))\, du = 0, \ \forall n.$$

This means $g(f^{-1}(u)) \equiv 0$, $\forall u \Rightarrow g(x) \equiv 0$, $\forall x$.

Poted problem ( from Math Stack.)

Find a continuous function $f$ such that $\int_a^{a^2+1} f(x)\,dx = 0$, $\forall a \in \mathbb{R}$.
and $f \in C^\infty$

ote that because $f$ is continuous, $f$ can be approximated by a Polynomial $P_n$.

$$P_n(x) = \sum_{\ell=1}^{\infty} c_\ell x^\ell \rightrightarrows f(x).$$

$$\int_a^{a^2+1} f(x)\,dx = \lim_{n\to\infty} \int_a^{a^2+1} P_n(x)\,dx = \lim_{n\to\infty} \sum_{\ell=1}^{n} \int_a^{a^2+1} c_\ell x^\ell = \sum_{\ell=1}^{\infty} \int_a^{a^2+1} c_\ell x^\ell\,dx \underline{\overset{\text{by asspt}}{=}} 0.$$

nd so $\displaystyle\sum_{\ell=1}^{\infty} c_\ell \frac{1}{(\ell+1)} x^{\ell+1}\Big|_a^{a^2+1} = 0$, .

$$\Leftrightarrow \quad \frac{c_\ell}{\ell+1}\Big[\underbrace{(a^2+1)^{\ell+1} - a^{\ell+1}}_{\geq 0.}\Big] = 0, \ \forall \ell = 1, \infty.$$

ote that $(a^2+1)^{\ell+1} - a^{\ell+1} = \underbrace{[a^2+1-a]}_{= \underbrace{~~(a^2+1)^2\ a~~}_{\underbrace{~~\text{long a}>0~~}_{>0,\ \forall a.}}}\Big[\underbrace{(a^2+1)^\ell a + \cdots + (a^2+1)a^\ell)}_{\neq 0 \text{ if } a \neq 0.}\Big]$

- So in case $a = 0$, $\int_0^1 f(x) = 0$, one of the case satisfies this is $f \equiv 0$
- And when $a \neq 0$ because $\underbrace{(a^2+1)^{\ell+1} - a^{\ell+1}}_{>0}$, $\forall \ell \Rightarrow \frac{c_\ell}{\ell+1} = 0 \Rightarrow c_\ell \equiv 0, \forall \ell$.

This means $f(x) \equiv 0$, $\forall x$.

checked

**Analysis preliminary exam** Jan. 8, 2009

NTR

1. Let $C$ be the standard Cantor set on the interval $[0, 1]$ and let $A = C^c$ be its complement on the real line. Identify the set of all limit points $A'$ of $A$, explaining your answer.

See Jan 2004 (even Jan 2004 P 4 is about integral.

See Aug 2008 P5

~~2.~~ (a) Prove

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

~~(b)~~ Let $\{a_n\}$ be a sequence with limit $L$. Define a sequence

$$b_n = \frac{1}{n^2}\sum_{k=1}^{n} k a_k$$

Prove $\lim_{n\to\infty} b_n = L/2$.

Apply $\lim a_n = L$, $|a_n - b_n| < \epsilon, \forall n \geq$ $\Rightarrow \lim b_n = L$

3. Let $f$ be a continuous real valued function on $[a, b]$ and differentiable on $(a, b)$.

~~(a)~~ Prove

$$\max_{a\leq x\leq b} |f(x)| \leq \frac{1}{b-a}\int_a^b |f(x)|dx + (b-a)\sup_{a<x<b} |f'(x)|$$

(b) Given any $\epsilon > 0$ prove

$$\max_{a\leq x\leq b} |f(x)| \leq \frac{1}{\epsilon}\int_a^b |f(x)|dx + \frac{\epsilon}{2}\sup_{a<x<b} |f'(x)|$$

Compare with May 2001 G P 4

~~4.~~ Suppose $f(x+1) = f(x)$ for all real $x$, $f$ is real valued, $f$ is Riemann integrable on every compact interval, and $\int_0^1 f(x)dx = 0$.

(a) Prove there exists $x_0$ such that $F(x) = \int_{x_0}^x f(t)dt \geq 0$ for all $x$.

(b) Show by example that $F'(x_0) = 0$ need not be true.

periodic function + continuous $\Rightarrow$ attain min/max in $\mathbb{R}$.

Some useful results needed to remember in this problem

NTR ~~5.~~ Let $f_n(x) = n(e^{x^2/n} - 1)$ for all real $x$.

(a) Prove $\lim_{n\to\infty} f_n(x) = x^2$ for each $x$.

(b) Prove $\{f_n\}$ is equicontinuous on $[0, M]$ for all positive $M$.

(c) Prove that $\lim_{n\to\infty}\int_0^1 (f_n(x))^{1/3}dx$ exists and equals $\frac{3}{5}$.

Review useful results in this problem ✱

Something need to be review when applying chain rule.

~~6.~~ The map $(x, y) \mapsto (e^x \sin x - x^2 y, y\cos x - e^x + 1)$ maps the *origin* to the *origin*. Show that the inverse map $G$ exists in a neighborhood of the *origin* and compute

$$\frac{d}{dt}\Big|_{t=0} f\circ G(-t, t^2) \text{ and } \frac{d}{dt}\Big|_{t=0} f\circ G(-t, t)$$

when $f(x, y) = x + 2y$.

put $g(t): \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (-t, t^2)$.

chain rule.

## Jan 2009 :

2G) The map $(x,y) \longmapsto (e^x \sin x - x^2 y,\ y\cos x - e^x + 1)$ maps the origin to the origin.
Show that a) the inverse map $G$ exists in a neighborhood of the origin and compute

ⓑ $\frac{d}{dt}_{t=0} f \circ G(-t,t^2)$ and ⓒ $\frac{d}{dt} f \circ G(-t,t)$ when $f(x,y) = x+2y$

* Let $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (e^x \sin x - x^2 y,\ y\cos x - e^x + 1)$

- We have

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \sin x + e^x \cos x - 2xy & -x^2 \\ -\sin x - e^x & \cos x \end{pmatrix}$$

So we have $DF_{(0,0)} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \Rightarrow \det D_{F(0,0)} = 1 \neq 0.$ (1)

- Note that $F$ is a $C^1$ function (2)

and $F(0,0) = (0,0)$ (3).

From (1)+(2)+(3) $\Rightarrow \exists$ open neighborhood $U$ of $(0,0)$ and a open neighborhood $V$ of $F(0,0)=(0,0)$
s.t $F: U \longrightarrow V$ is a bijective.
then $\exists\ G: V \longrightarrow U$ a $C^1$ function is a inverse map of $F$.

* b) Compute $\frac{d}{dt}_{t=0} f \circ G(-t,t^2)$ where $f(x,y) = (x+2y)$.

(note this with this kind of question, we easily can consider $g: \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (-t,t^2)$

* The key step that can make our life much easier

- Put $g: \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto g(t) = (-t,t^2)$.

So now we have $\mathbb{R} \xrightarrow{g} \mathbb{R}^2 \xrightarrow{G} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$.

$g'(t) = \begin{pmatrix} -1 \\ 2t \end{pmatrix}$ $\quad DG_{t=0} = [D_F(0)]^{-1}$ $\quad Df = [1\ \ 2],\ \forall x,y \in \mathbb{R}^2$.

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (we get luck here when $Df = [1, 2]$ $\forall x,y \in \mathbb{R}^2$

$g'_{t(0,0)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ (from above

(we don't need to compute $D f'[G(-t,t^2)]$ ~~when~~ at $t=0$

So we have

$$\frac{d}{dt}_{(t=0)} f \circ G(-t,t^2) = \frac{d}{dt}_{at\ t=0}\left[f \circ G \circ g(t)\right] \underset{\substack{\uparrow \\ \text{Chain rule}}}{=} \underbrace{f'[G[g(0)]]}_{at\ t=0}\ \underbrace{G'[g(t)]}_{at\ t=0}\ \underbrace{g'(t)}_{at\ t=0}$$

$= f'(0,0)$ $\quad G'[0,0]$ $\quad = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

because $F$ maps origin to origin

$$= [1\ \ 2]\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -3.$$

$$\boxed{\text{Compute } \left.\frac{d}{dt}\right|_{t=0}\left[ f\circ G(-t,t)\right]}$$

nt $h :=$ $h : \mathbb{R} \longrightarrow \mathbb{R}^2$

$t \longmapsto h(t) = (-t, t)$.

So we have

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] & \mathbb{R}^2 \arrow[r, "G"] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

we have

$$\left.\frac{d}{dt}\right|_{t=0} f\circ G\circ h(t) = f'[G(h(0))] \quad G'[h(0)] \quad h'(0)$$

$$= f'[G(0,0)] \quad G'[0,0] \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= f'[0,0] \quad \underbrace{G'[0,0]}_{=[DF(0,0)]^{-1}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

note that
F map (0,0) to (0,0)
So $G = F^{-1}$ also map (0,0) to (0,0)

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \qquad \square \text{ done.}$$

---

Outside of this problem :

n fact, the result of IFT states for all $x \in$ neighborhood of $x_0$.
$y \in$ neighborhood of $y_0$ :

$$G'[y] = [F'(x)]^{-1}$$

So from the formula of $F'(x)$, we can compute $G'(y)$ for all $y \in$ neighborhood of $y_0$.
but in this case, it takes time to compute that.

In this problem, because of the assumption : F maps origin to origin,
we save a lot of time to compute what we need when applying Chain rule

$\boxed{\text{The key step here}}$ is putting $g(t) : \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (t, t^2 \ldots)$ ....

* Some basic examples about equicontinuous

1) $\{f_n(x) = \sin nx\}_{n=1}^{\infty}$ is not equicontinuous on $[-1,1]$ (in fact in any nontrivial compact inter

$\{f_n\}$ is equicontinuous (in E) $\iff \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in E, |x-y| < \delta, \forall n, |f_n(x) - f_n(y)| < \epsilon$

$\{f_n\}$ is not equicontinuous in $E \iff \exists \epsilon > 0, \forall \delta > 0, \exists n_\delta, \exists x, y \in E, |x-y| < \delta, |f_{n_\delta}(x) - f_{n_\delta}(y)$

Choose $\epsilon = \frac{1}{2}$, then $\forall \delta > 0, \exists n$ s.t $\frac{\pi}{2n} < \delta$, then $\exists x = 0, y = \frac{\pi}{2n}$ $(|x-y| < \delta)$,
Then with $n_\delta$

$|f_{n_\delta}(x) - f_{n_\delta}(y)| = |\sin n_\delta 0 - \sin n_\delta y| = \sin n_\delta y = \sin\left(n_\delta \frac{\pi}{2n_\delta}\right) = \sin\frac{\pi}{2} = 1 > \frac{1}{2} =$

Then by def of equicontinuous, $\{f_n\}$ is not equicontinuous on $[-1,1]$.

2) $f_n(x) = x^n$ is not equicontinuous in $[0,1]$. (Note that $f_n(x) = x^n \not\rightrightarrows$ in $[0,1]$.)

**Analysis Preliminary Exam** **August 24, 2009**

1. If $F_1$ and $F_2$ are closed subsets of $R^1$ and $\text{dist}(F_1, F_2) = 0$ then $F_1 \cap F_2 \neq \emptyset$. Prove or give a counterexample. HW4.3.

Not done.

2. Newton's method for finding zeroes of a function $f : R^1 \to R^1$ is based on the recursion formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 1.$$

Show that if $f \in C^1$, $f(a) = 0$ and $f'(a) \neq 0$, then there exists a $\delta > 0$ such that if $|x_1 - a| < \delta$ then $x_n \to a$. (Suggestion: use the Mean Value Theorem.)

3. Let $f : [0, \infty) \to [0, \infty)$ and for $h > 0$ and $k \geq 1$ set

$$M_k(h) = \sup_{(k-1)h \leq x < kh} f(x), \quad m_k(h) = \inf_{(k-1)h \leq x < kh} f(x).$$

Let

$$U(h) = \sum_{k=1}^{\infty} M_k(h)h, \quad L(h) = \sum_{k=1}^{\infty} m_k(h)h.$$

We say $f$ is *directly Riemann integrable* if $U(h) < \infty$ for all $h > 0$ and

$$\lim_{h \downarrow 0}(U(h) - L(h)) = 0.$$

Recall $f$ is *improperly Riemann integrable on* $[0, \infty)$ if $f$ is Riemann integrable on $[0, a]$ for every $a > 0$, and

$$\lim_{a \to \infty} \int_0^a f(t)\, dt < \infty.$$

(a) Show that if $f$ is continuous and nonincreasing, then $f$ is directly Riemann integrable whenever $f$ is improperly Riemann integrable on $[0, \infty)$.

(b) Give an example of a continuous function $f$ which is improperly Riemann integrable on $[0, \infty)$ but not directly Riemann integrable.

4. Suppose $f : [0, \infty) \to [0, \infty)$ is such that for any sequence $a_n$ of nonnegative terms we have

$$\sum_{n=1}^{\infty} a_n < \infty \implies \sum_{n=1}^{\infty} f(a_n) < \infty$$

Prove that

$$\limsup_{x \to 0^+} \frac{f(x)}{x} < \infty$$

5. Let $f$ be a continuous real valued function defined on the unit square and for each $0 \leq x \leq 1$ let $f_x$ be the function on the unit interval defined by $f_x(y) = f(x, y)$. Prove that for any sequence $x_n$ in [0,1] there is a subsequence $n_k$ such that $f_{x_{n_k}}$ converges uniformly on [0,1].

6. If $c$ is a real parameter prove that $x^7 + x + c = 0$ has a unique real root and that this root is a differentiable function of $c$.

→ If $F_1$ and $F_2$ are closed subset of $\mathbb{R}^1$, $\mathrm{dist}(F_1, F_2) = 0$ ) then $F_1 \cap F_2 \neq \phi$
Prove or ~~disprove~~ give a counter example.

See HW 4.3.

Let $F_1 = \mathbb{N} = \{1, 2, 3, 4, 5, \dots, n, \dots\}$.

$F_2 = \{\mathbb{N} + \frac{1}{2^n}\} = \{1+\frac{1}{2}, 2+\frac{1}{4}, 3+\frac{1}{8}, \dots, n+\frac{1}{2^n}, \dots\}$.

Then we have $F_1$ and $F_2$ are closed in $\mathbb{R}^1$:

Indeed, a set $E$ is closed in $\mathbb{R}$ if every limit point of $E$ is belonged to $E$.

a point $p$ is a limit point of $E$ if $\forall N_\lambda(p)$ a neighborhood of $p$ in $\mathbb{R}$,
$N_\lambda(p) \setminus \{p\} \cap E \neq \phi$.

But $F_1, F_2$ contains isolated point in $\mathbb{R}$ $\exists N_R(p) \setminus \{p\} \cap F_1 = \phi$
$N_{R_1}(p) \setminus \{p\} \cap F_2 = \phi$.

$F_1, F_2$ contain no isolated point ⇒ $F_1, F_2$ closed in $\mathbb{R}^1$.

$\mathrm{dist}(F_1, F_2) = 0$ Remind $\mathrm{dist}(F_1, F_2) = \inf\{d(x,y),\ x \in F_1,\ y \in F_2\}$.

we consider $d(n, n+\frac{1}{2^n}) = \frac{1}{2^n}$ ⇒ $\mathrm{dist}(F_1, F_2) \leq \frac{1}{2^n}$

because $n$ is arbitrary large

$\Rightarrow \mathrm{dist}(F_1, F_2) \Longrightarrow 0$

But clearly, $F_1 \cap F_2 = \phi$.

Note: In case $A$ compact
$B$ closed
$A \cap B$ disjoint
$\Big\} \Rightarrow d(A, B) > \cancel{\varepsilon} > 0$

Aug 24, 2009 ⟶ (See MAT601, HW 4.3).

Related problem next p

1 ⟶ If $F_1, F_2$ are closed subset of $R^2$, $\mathrm{dist}(F_1, F_2) = 0$ $\Big\}$ Then $F_1 \cap F_2 \neq \phi$

(Prove or give a counterexample.

$\mathrm{dist}(F_1, F_2) = \inf\{d(x,y), x \in F_1, y \in F_2\}$ | $d(x, A) = \inf\{d(x,a), a \in A\}$

* Some important results needed to remember:

ℕ is closed in ℝ., not open in ℝ.

ℚ is not closed, not open in ℝ.

$\{n, n \in \mathbb{N}\}$, $\{n + \frac{1}{n}, n = 2, 3, 4, \dots\}$ closed in ℝ.

* Let $F_1 = \{n, n = 1, 2, 3, \dots\}$ we have $F_1$ is closed in ℝ because ~~it is a countable union of~~ discrete points which are closed

it contains no limit point.

$F_2 = \{n + \frac{1}{n}, n = 2, 3, 4 \dots\}$. closed in ℝ

$= \{2 + \frac{1}{2}, 3 + \frac{1}{3}, 4 + \frac{1}{4}, \dots\}$

$\mathrm{dist}(F_1, F_2) = \inf\{d(x,y), x \in F_1, y \in F_2\} = 0$

But $F_1 \cap F_2 = \phi$

So above statement is false ☐ ☐

---

* Note that we can explain $F_1$ and $F_2$ are closed in ℝ by:

- $F_1$ is closed because

$F_1^c = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \dots \cup (n-1, n) \cup (n, n+1) \cup \dots$ a countable union of open sets in ℝ is

$\Rightarrow F_1^c$ is open in ℝ.

- $F_2$ is close in ℝ because $F_2^c = (-\infty, 2 + \frac{1}{2}) \cup (2 + \frac{1}{2}, 3 + \frac{1}{3}) \cup \dots$ is open in ℝ.

A problem related to Q1:

Prove that $\left.\begin{array}{l} A \text{ compact} \\ B \text{ is closed} \\ A \cap B = \phi \end{array}\right\}$ Then $d(A,B) \geqslant \varepsilon > 0$

(Online) Let $A \subset X$ be a nonempty subset of a metric space $X$

▷ Show that $d(x,A) = 0$ iff $x \in \bar{A}$

▷ Show that if $A$ compact, then $d(x,A) = d(x,a)$ for some $a \in A$ ).

Aug 2009, ②

Newton's method for finding zeroes of a function $f: R^1 \to R^1$ is base on the recursion formula of
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , n \geq 1$$

Show that if $f \in C^1$, $f(a) = 0$, $f'(a) \neq 0$
then $\exists \, \delta > 0$ such that if $|x_1 - a| < \delta$ then $x_n \to a$.

Suggestion:
Use MVT

Weir

(Need

remen

(A way to prove this is use Contraction mapping principle.

Aug 2009, P5 → *

Let $f:[0,+\infty) \longrightarrow [0,+\infty)$

$h>0,\ k \geq 1$

Set $M_k(h) = \sup\limits_{(k-1)h \leq x \leq kh} f(x)$ $\qquad m_k(h) = \inf\limits_{(k-1)h \leq x \leq kh} f(x)$

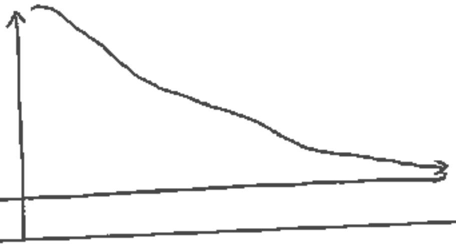
Let $U(h) = \sum\limits_{k=1}^{\infty} M_k(h)\, h$ $\qquad L(h) = \sum\limits_{k=1}^{\infty} m_k(h)\, h$ .

We say $f$ is directly Riemann integrable if $\begin{cases} U(h) < +\infty, \ \forall h > 0 \\ \lim\limits_{h \to 0} \left( U(h) - L(h) \right) = 0 \end{cases}$

Recall $f$ is improperly Riemann integrable on $[0,\infty)$ if

$\begin{cases} f \text{ is Riemann integrable on } [0,a], \forall a > 0 \\ \lim\limits_{a \to \infty} \int_0^a f(t)\,dt < +\infty \end{cases}$ .

a) $\left.\begin{array}{l} f \text{ is continuous + decreasing} \\ f \text{ is improperly Riemman integrable on } [0,+\infty) \end{array}\right\}$ Then $f$ is directly Riemann integrable on $[0,+\infty)$



b) Give an example of a continuous function $f$ which

$\begin{cases} \text{is improperly Riemann integrable} \\ \text{is not directly Riemann integrable.} \end{cases}$

Aug 2009, P 4

Suppose $f : [0, +\infty) \longrightarrow [0, +\infty)$ is a function s.t.

for any $\{a_n\}$; $a_n \geqslant 0, \forall n$; we have $\sum_{n=1}^{\infty} a_n < +\infty \rightarrow \sum_{n=1}^{\infty} f(a_n) < +\infty$

Prove that $\limsup_{x \to 0^+} \frac{f(x)}{x} < +\infty$

g2009, PG7

f $c$ is a real parameter, prove that $x^7+x+c$ has a unique real root and that this ot is a differentiable function of $c$.

ow we put $F(x,c): \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,c) \longmapsto F(x,c) = x^7+x+c$$

$F = [7x^6+1 \ \vdots \ 1]$ we ~~have all~~ $\Rightarrow$ $F$ is a $C^1$ function.

e also have $A_x = \underbrace{7x^6}_{\geq 0} + \underbrace{1}_{\geq 1} \geq 1, \forall x \in \mathbb{R}$.

hen by implicit function theorem, $\exists x = g(c)$ such that $F(g(c),c)=0$ and $g$ is a $C^1$ nction.

e also note that $A_x > 0, \forall x$ $\Rightarrow$ the function is increasing and the root is unique.

# JANUARY 2010 PRELIMINARY EXAM IN ANALYSIS.

**1.** Let $X$ be a connected metric space. Given two points $p, q \in X$ and a number $\epsilon > 0$, prove that there exist an integer $n \geqslant 0$ and points $a_0, a_1, \dots, a_n \in X$ such that $a_0 = p$, $a_n = q$, and

$$d(a_j, a_{j-1}) < \epsilon \quad \text{for all } j = 1, 2, \dots, n.$$

**2.** Suppose that $f\colon (0,1] \to \mathbb{R}$ is a bounded continuous function such that for every $t \in \mathbb{R}$ the set $\{x \in (0,1]\colon f(x) = t\}$ is finite. Prove that $f$ is uniformly continuous on $(0,1]$.

**3.** Prove or disprove the following: if a function $f\colon (-1,1) \to \mathbb{R}$ is differentiable on $(-1,1)$ and $f'(0) = 0$, then for every $\delta > 0$ there exists $\epsilon > 0$ such that

$$\left|\frac{f(t) - f(s)}{t - s}\right| < \delta \quad \text{whenever } -\epsilon < s < t < \epsilon.$$

**4.** Let $f$ be a bounded real-valued function on $[a,b]$ with a discontinuity at $c \in (a,b)$. Let $\alpha(x)$ be monotonically increasing on $[a,b]$ with $\alpha(c-) < \alpha(c) < \alpha(c+)$. Prove that $f$ is not Riemann-Stieltjes integrable with respect to $\alpha$ on $[a,b]$.

**5.** Give examples of sequences of functions $\{f_n\}$ and $\{g_n\}$ on $\mathbb{R}$ such that $\{f_n\}$ converges uniformly, $\{g_n\}$ converges uniformly but $\{f_n g_n\}$ does not converge uniformly on $\mathbb{R}$.

**6.** Let $\phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable functions and define $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) = (\phi(x,y,z), \psi(x,y,z), \phi^2(x,y,z) + \psi^2(x,y,z))$$

**(a)** Check whether or not the inverse function theorem applies to $F$ at any point $(x_0, y_0, z_0)$, i.e., check if $F$ satisfies the hypothesis of the inverse function theorem at any point $(x_0, y_0, z_0)$.

**(b)** Suppose that $F(\vec{a}) = \vec{b}$ for some points $\vec{a}, \vec{b} \in \mathbb{R}^3$. Explain geometrically why $F$ does not have an inverse function from an open set $V \subset \mathbb{R}^3$ containing $\vec{b}$ to an open set $U \subset \mathbb{R}^3$ containing $\vec{a}$.

# AUGUST 2010 PRELIMINARY EXAM IN ANALYSIS

1. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $f(f(x)) = x$ for all $x \in \mathbb{R}$. Prove that there exists an irrational number $t$ such that $f(t)$ is also irrational.

2. Find three subsets $A, B, C$ of the real line $\mathbb{R}$ such that $A \cap B = A \cap C = B \cap C = \varnothing$ and $\overline{A} = \overline{B} = \overline{C} = \mathbb{R}$. Prove that your sets satisfy these properties.

3. Let $X$ and $Y$ be metric spaces. Suppose that $f\colon X \to Y$ has the following property: for any continuous function $g\colon Y \to \mathbb{R}$ the composition $g \circ f$ is a continuous function from $X$ to $\mathbb{R}$. Prove that $f$ is continuous.

4. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $f'(x)$ exists for all $x \in \mathbb{R}$ and $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$. Prove that $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

5. Give an example of a bounded function $f\colon [0,1] \to \mathbb{R}$ such that

   - $f$ is not Riemann integrable on $[0,1]$
   - The function $g$ defined by $g(x) = \sin f(x)$ is Riemann integrable on $[0,1]$

   Prove your claims using the definition of the Riemann integral.

6. Let $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping defined by

$$y_1 = x_1 + x_2$$
$$y_2 = x_2 - x_1$$
$$y_3 = x_3^5$$

   (a) Determine all points $\mathbf{a} \in \mathbb{R}^3$ at which $\mathbf{f}$ satisfies the assumptions of the Inverse Function Theorem.

   (b) Is $\mathbf{f}$ an open mapping? Prove or disprove.

   *Reminder. A mapping* $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ *is open if* $\mathbf{f}(W)$ *is an open subset of* $\mathbb{R}^3$ *for every open set* $W \subset \mathbb{R}^3$.

Jan 2010 7

17 Let $X$: connected metric space

Give 2 points $p, q \in X$ and a number $\epsilon > 0$

Prove that there exist an integer $n \geq 0$, and points $(a_0 = p, a_1, a_2, \dots, a_n = q$ in $X$

and $d(a_i, a_{i-1}) < \epsilon$, $\forall i = \overline{1, n}$

$$* \text{ Let } S = \left\{ q \in X \;\middle|\; \exists n \geq 0, \text{ such that } a_0 = p, a_1, \dots, a_{n-1}, a_n = q \atop d(a_i, a_{i+1}) < \epsilon \right\} \subseteq X$$

(1)* We now prove that $S$ is open in $X$ $\Leftrightarrow$ NTP for $q \in S$, $\exists r > 0$, $N_r(q) \subset S$

Let $r = \epsilon$, Now consider $N_\epsilon(q)$

Let $a \in N_\epsilon(q)$, we have $d(q, a) \leqslant \epsilon$

So we have $a_0 = p, a_1 = \dots, a_n = q, a_{n+1} = a$, where $d(a_i, a_{i+1}) < \epsilon$

$\Rightarrow a \in S$ this means $N_\epsilon(q) \subset S$.

(2)* We now prove that $S$ is closed in $X$

$\Leftrightarrow$ NTP $\forall a$ is a limit point of $S$, then $a \in S$.

- We have $a$ is a limit point of $S \Leftrightarrow \exists (q_k) \subseteq S$, $q_k \to a$

$\Leftrightarrow \forall \epsilon > 0, \exists K \in \mathbb{N}, \forall k \geqslant K, d(q_k, a) <$

So we have $d(q_K, a) < \epsilon$

- because $q_K \in K \Rightarrow \exists n$, $a_0 = q, \dots, a_n = q_K$ $\Big\} \Rightarrow a_0 = q, \dots, a_n = q_K, a_{n+1} = a$

$\Rightarrow a \in S$.

(3)* We now prove $S \neq \phi$

This is because Let $p \in X$, then choose $n = 1$

$a_0 = p$

$a_1 = p$

$d(a_0, a_1) = 0 < \epsilon$

From (1)+(2)+(3)

+ the fact that $X$ is connected (the only 2 subsets that are both open and closed are $\phi$ and $X$ $\Big\}$

$\Rightarrow S = X$

This means For fixed $p$, then for any $q \in X$, $\exists n$ .. s.t ...

This is what we need to prove $\square$

in 2010

> Suppose that $f : (0,1] \longrightarrow \mathbb{R}$ is bounded continuous function .
> $\forall t \in \mathbb{R}$, the set $\{x \in (0,1] , f(x) = t\}$ is finite .
> Prove that $f$ is uniformly continuous on $(0,1]$ .

Jan 2010, P3

Prove or give a counterexample.

If $f:(-1,1)\longrightarrow \mathbb{R}$ is differentiable.

$f'(0)=0$

Then $\forall \varepsilon>0, \exists \delta>0$ s.t $\left|\frac{f(t)-f(s)}{t-s}\right|<\varepsilon$ when $\underbrace{-\delta<s<t<\delta}_{|t-s|<\delta}$.

This statement is wrong when $f'$ is not cont $\ast$.

$\Rightarrow$, $\exists \varepsilon>0, \forall \delta>0, \left|\frac{f(t)-f(s)}{t-s}-f'(0)\right|>\varepsilon$.

(Result from Ex Rudin 5.18.

If $f'$ is continuous on $[a,b]$

Then $\forall \varepsilon>0, \exists \delta>0, \left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|<\varepsilon$, $\forall t,x$ st $a\le t,x\le b$, $0<|t-x|<\delta$

$\overset{def}{\Longleftrightarrow}$ ~~f is uniformly differentiable on [a,b]~~

We want to give a counter example for above statement.

We want to find a function $f$ differentiable on $(-1,1)$, $f'$ is not continuous at $0$

$$f'(0)=0$$

and want to prove that $\exists \varepsilon>0, \forall \delta>0, \exists \underset{-\delta<s<t<\delta}{t,s}, \left|\frac{f(t)-f(s)}{t-s}\right|>\varepsilon$. (1)

* Let $f(x)=\begin{cases} x^2\sin\frac{1}{x} & , x\neq 0 \\ 0 & , x=0\end{cases}$

Then $f'(x)=\begin{cases} 2x\sin\frac{1}{x}+x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x} = 2x\sin\frac{1}{x}-\cos\frac{1}{x} & , x\neq 0 \\ 0 & , x=0.\end{cases}$

* Now we prove (1). $\forall \delta>0$, $\exists n$ s.t $\frac{1}{n}<\delta$

Set $t_n=\frac{1}{2\pi n}<\delta$ $s_n=\frac{1}{2\pi n+\frac{1}{n}}<t_n<\delta$,

note that because we wan

$\delta$ small

$t_n \sim s_n$ ($\Rightarrow$ think abo

$\frac{1}{n}$

and we have

$$\left|\frac{\overset{=0}{f(t_n)}-f(s_n)}{t_n-s_n}\right| = \left|\frac{\left(\frac{1}{2\pi n+\frac{1}{n}}\right)^2\sin\left(\frac{1}{n}\right)}{\frac{1}{2\pi n}-\frac{1}{(2\pi n+\frac{1}{n})}}\right| = \left|\frac{\frac{n^2}{(2\pi n^2+1)^2}\sin\frac{1}{n}}{\frac{\frac{1}{n}}{4\pi^2n^2+2\pi}}\right|$$

$$= \left|\underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{\xrightarrow[n\to\infty]{} 1}\cdot\underbrace{\frac{(4\pi^2n^2+2\pi)n^2}{(2\pi n^2+1)^2}}_{\longrightarrow 1}\right| \xrightarrow[n\to\infty]{} 1>\frac{1}{2}$$

Choose $\varepsilon=\frac{1}{2}$, then $\forall \delta>0, \exists t,s\ldots,$ s.t $|\cdots|$ $\Rightarrow$ done $\square$.

Jan 2010

47> Let $f$ be a bounded, real value function on $[a,b]$ with a discontinuity at $c \in (a,b)$

Let $\alpha(x)$ be a monotonically increasing on $[a,b]$ $\alpha(c^-) < \alpha(c) < \alpha(c^+)$

Prove that $f$ is not Riemann - Stieltjes integrable wrt $\alpha$ on $[a,b]$.

---

We NTP $f \notin \mathcal{R}(\alpha) \Leftrightarrow$ NTP $\exists \epsilon > 0$, $\forall$ partition $P$, $|U(P,f,\alpha) - L(P,f,\alpha)| > \epsilon$

Consider all partition $P = \{x_0 = a \leq x_1 \leq x_2 \leq \cdots \leq x_n = b\}$.

we have $c \in [x_k, x_{k+1}]$ for some $k = 0, n-1$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=0}^{n-1} \underbrace{(M_i - m_i)}_{\geq 0, \forall i} \underbrace{\Delta\alpha_i}_{\geq 0, \forall i \text{ cause } \alpha \text{ increas}}$$

cause equals sup - inf

$$\geq (M_k - m_k)(\alpha(x_{k+1}) - \alpha(x_{k+1}))$$

$$\geq \underbrace{[f(c^+) - f(c^-)]}_{>0} \underbrace{[\alpha(c^+) - \alpha(c^-)]}_{>0} > 0$$

Choose $\epsilon$ s.t ~~$f(c^+)$~~

$$[f(c^+) - f(c^-)][\alpha(c^+) - \alpha(c^-)] > \epsilon \Rightarrow \square.$$

---

? Prove by contradiction? ( Kopse solut ?

Jan 2010

5) Give an example of sequence of function $\{f_n\}$ and $\{g_n\}$ on $\mathbb{R}$.

$f_n \rightrightarrows$

$g_n \rightrightarrows$

but $f_n g_n \not\rightrightarrows$ on $\mathbb{R}$.

We have when $f_n \rightrightarrows f$

$g_n \rightrightarrows g$

$\{f_n\}, \{g_n\}$ ~~bounded~~ sequence of bounded functions } Then $f_n g_n \rightrightarrows$

* In here, we need to find $f_n, g_n$ such that $f_n$ or $g_n$ does not a sequence of bounded functions.

Let $f_n = x, \forall n$ then we have $f_n \rightrightarrows f(x)$, where $f(x) = x, \forall x$

$g_n = \frac{1}{n}, \forall n = \overline{1,\infty}$ $g_n \rightrightarrows g(x)$, where $g(x) = 0, \forall x$

But $f_n g_n = \frac{x}{n} = h_n(x)$ does not converge uniformly

(We NTP $\exists \epsilon > 0, \forall n$ large, $\exists x_n, |h_n(x)| > \epsilon$ (because we have $h_n(x) = \frac{x}{n} \xrightarrow[n\to\infty]{\text{pointwise}}$

Choose $\epsilon = \frac{1}{2}$, then $\forall n, \exists x = n, |h_n(x) = 1| > \epsilon$

$\Rightarrow h_n(x)$ does not converge uniformly $\square$

n2010, 7

Let $\phi, \psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be continuously differentiable functions and define

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longmapsto F(x,y,z) = \left(\phi(x,y,z), \psi(x,y,z), \phi^2(x,y,z) + \psi^2(x,y,z)\right)$$

Check whether or not the IFT applies to F at any point $(x_0, y_0, z_0)$, ie, check if F tisfies the hypothesis of the IFT at any point $(x_0, y_0, z_0)$

Suppose that $F(\vec{a}) = \vec{b}$ for some point $\vec{a}, \vec{b} \in \mathbb{R}^3$.

xplain geometrically why F does not have an inverse of T from an open set $V \subseteq \mathbb{R}^3$ containing $\vec{b}$ an open set $U \subseteq \mathbb{R}^3$ containing $\vec{a}$.

e have

$$F = \begin{bmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \\ 2\phi \frac{\partial \phi}{\partial x} + 2\psi \frac{\partial \psi}{\partial x} & 2\phi \frac{\partial \phi}{\partial y} + 2\psi \frac{\partial \psi}{\partial y} & 2\phi \frac{\partial \phi}{\partial z} + 2\psi \frac{\partial \psi}{\partial z} \end{bmatrix}$$

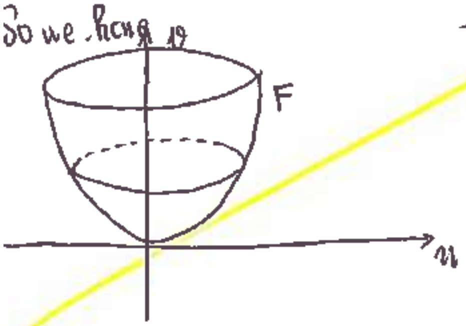
Notice that row 3 = $2\phi$ row 1 + $2\psi$ row 2

So we have $\det[DF] = 0$, $\forall (x,y,z)$ ⇒ F does not satisfies the hypothesis of IFT

Note $F = (\phi, \psi, \phi^2 + \psi^2)$

$\Rightarrow \forall (x,y,z) \in \mathbb{R}^3$, $F(x,y,z) = (u, v, u^2 + v^2)$ where $u = \phi(x,y,z)$
$v = \psi(x,y,z)$

So we have



The image of $F = \left\{ (u, v, u^2+v^2) \,\middle|\, \begin{array}{l} u \in \operatorname{Im}\phi \\ v \in \operatorname{Im}\psi \end{array} \right\}$ is a equation of a paraboloid, which is a surface

has no interior point

⇒ does not contain any open subset

⇒ what we need to prove □.

Aug2010/1

Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a function such that $f(f(x)) = x, \forall x\in\mathbb{R}$.
Prove that there exists an irrational number $t$ such that $f(t)$ is also irrational

* We have $f(f(x))=x, \forall x\in\mathbb{R}$ ⇒ f has itself as a inverse function.
⇒ f is bijection.

and because f is bijection.

then it is impossible to have $f: \underset{\text{uncountable}}{R\setminus Q} \longrightarrow \underset{\text{countable}}{Q}$ because (in this case f is not injective ⇒ not bijective.

~~we can have another explaination that:~~
~~because f is bijection then $f: Q \longrightarrow f(Q) = Q$.~~

⇒ then f map a uncountable set to uncountable set.

so $f: R\setminus Q \longrightarrow R\setminus Q$.

⇒ $\exists t$ irrational st $f(t)$: irrational. □

---

* We learn: If f has itself as a inverse function ($f^{-1} = f$)
⇒ f is a bijection

j2010 ②

Tricky*

ind 3 subsets $A, B$ and $C$ of $\mathbb{R}$ such that.

$$\begin{cases} A\cap B = B\cap C = C\cap A = \phi \\ \overline{A} = \overline{B} = \overline{C} = \mathbb{R} \end{cases}$$

've that your sets satisfy those properties.

In this problem, we use a lemma: if $p$ and $q$ are distinct primes, then $\sqrt{pq}$ irrational

Now we prove the lemma:

Assume a contradiction that $\sqrt{pq}$ rational $\Leftrightarrow \exists\, m, n \in \mathbb{Z}, n \neq 0, \sqrt{pq} = \frac{m}{n}$, $\gcd(m,n) = 1$

$\Rightarrow pq = \frac{m^2}{n^2} \Rightarrow n^2 pq = m^2$

From the above lemma, we have there is no $\begin{cases} m, n \in \mathbb{Q} \text{ such that } m+\sqrt{p} = n+\sqrt{q} \\ p, q \text{ distinct primes} \end{cases}$ (1)

This $m+\sqrt{p} \neq n+\sqrt{q} \Leftrightarrow m-n = \sqrt{q} - \sqrt{p}$

$$\Rightarrow \underbrace{(m-n)^2}_{\text{rational}} = \underbrace{q+p}_{\text{rational}} + \underbrace{2\sqrt{pq}}_{\text{irrational}} \Rightarrow \nexists\, m, n.$$

---

So because there are no $m, n \in \mathbb{Q}$ s.t $\forall p, q$ distinct prime $m+\sqrt{p} = n+\sqrt{q}$

Then let $A = \mathbb{Q}+\sqrt{p} = \{m+\sqrt{p},\ m\in\mathbb{Q}\}$

$B = \mathbb{Q}+\sqrt{q} = \{n+\sqrt{q},\ n\in\mathbb{Q}\}$ where $p, q, r$ distinct primes.

$C = \mathbb{Q}+\sqrt{r} = \{a+\sqrt{r},\ a\in\mathbb{Q}\}$

Then because of (1), we have $A\cap B = B\cap C = C\cap A = \phi$.

Now we want to prove that $A = \mathbb{Q}+\sqrt{p}$) dense in $\mathbb{R}$

(In fact, we see that $\mathbb{Q}+\sqrt{p} \cong \mathbb{Q}$ (dense in $\mathbb{R}$)

nt, now we prove directly, We want to prove that $\forall x\in\mathbb{R}, \exists\, a_n+\sqrt{p} \longrightarrow x$.

e have $\forall x\in\mathbb{R}, \Rightarrow (x-\sqrt{p})\in\mathbb{R} \underset{\substack{\uparrow \\ \text{because } \mathbb{Q} \text{ dense in } \mathbb{R}}}{\Rightarrow} \exists\, a_n\in\mathbb{Q}, a_n \longrightarrow x-\sqrt{p}$

$\Rightarrow a_n+\sqrt{p} \to x \Rightarrow \square$

③ Aug 2010

Let $X, Y$: metric spaces.

Suppose $f: X \longrightarrow Y$ has the following property:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\text{continuous.}"'] & Y \arrow[r, "g"', "\uparrow \text{continuous}"'] & \mathbb{R} \end{tikzcd}$$

$\forall g: Y \longrightarrow \mathbb{R}$ continuous function, the component $g \circ f: X \longrightarrow \mathbb{R}$ is continuous.

Prove that $f$ is continuous.

*

* First way: The idea of this way is because the assumption $\forall g$ cont ... then ... $g \circ f$ cont
⇒ we chose a special case when $g(y) = d(y, f(x))$
then $g \circ f(y) = d(f(y), f(x))$ cont ⇒ $f$ continuous.

* Eisst, put $g(y) = d(y, f(x))$, we now prove that $g$ is a continuous function.

Let $y_n \to y$ in $Y$. We need to prove that $g(y_n) \to g(y)$ in $\mathbb{R}$

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, d(y_n, y) < \varepsilon$ | NTP: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |g(y_n)-$

NTP: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N,$

$|d(y_n, f(x)) - d(y, f(x))|$

NTP, ... $\begin{cases} d(y_n, f(x)) - d(y, f(x)) < \varepsilon \\ d(y, f(x)) - d(y_n, f(x)) < \varepsilon \end{cases}$

We have

$\begin{cases} d(y_n, f(x)) \leq d(y_n, y) + d(y, f(x)) \\ d(y, f(x)) \leq d(y, y_n) + d(y_n, f(x)) \end{cases} \Rightarrow \begin{cases} d(y_n, f(x)) - d(y, f(x)) < d(y_n, y) \\ d(y, f(x)) - d(y_n, f(x)) < d(y, y_n) \end{cases}$

⇒ $g(y)$ is a a continuous function.

* Because assumption that $g \circ f: X \longrightarrow \mathbb{R}$ is a continuous function

So we have $g \circ f$ continuous $\forall x \in X$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_{\varepsilon, x} > 0, \forall y \in X, d_X(y, x) < \delta$ then $|\cancel{d}(g \circ f(y)) - \cancel{d}(g \circ f(x))| <$

$\Leftrightarrow |d(f(y), f(x)) - d(f(x), f(x))| <$

$\Leftrightarrow |d(f(y), f(x))| < \varepsilon$

⇒ $f$ is a continuous function □

* Another way next page! ⟶

ug2010/4

ppose $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a function such that

$$\begin{cases} f'(x) \text{ exist for all } x\in\mathbb{R} \\ f'(-x)=-f'(x) \quad \forall x\in\mathbb{R} \quad . (1) \end{cases}$$

Prove that $f(-x)=f(x)$, $\forall x\in\mathbb{R}$.

Put $g(x)=f(x)-f(-x)$

We want to prove that $g(x)=0$, $\forall x\in\mathbb{R}$.

We have $g(0)=0$

$$\left.\begin{array}{l} g'(x)=f'(x)+f'(-x)\underset{\text{by }(1)}{=}f'(x)-f'(x)=0,\ \forall x\in\mathbb{R}\Rightarrow g \text{ is a constant function} \end{array}\right\}\Rightarrow g(x)=0\ \forall x\in\mathbb{R}.$$

$f(-x)=f(x)$, $\forall x\in\mathbb{R}$ $\square$.

---

ther way for p3:

et $f:X\longrightarrow Y$ has the following property:

$\forall g:Y\longrightarrow\mathbb{R}$ continuous function, the component $g\circ f:X\longrightarrow\mathbb{R}$ is continuous.

rove that $f$ is a continuous function.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "cont"'] & Y \arrow[r, "g", "cont"'] & \mathbb{R} \end{tikzcd}$$

nportant result used in this prove

ndin 4.3/98: $f$ is a cont function on $X\longrightarrow Y$

then $\mathrm{Ker}f=\{x\in X,\ f(x)=0_Y\}$ is a closed set in $X$

e NTP $f$ is a continuous function

NTP that $\forall E$ closed in $Y$, $f^{-1}(E)$ is closed in $X$

We have $(g\circ f)$ is a continuous function from $X\longrightarrow\mathbb{R}$.

from above result, $(g\circ f)^{-1}(0_R)$ is closed in $X$.

$\Rightarrow$ We want to find a cont $g$ s.t $f^{-1}(E)=(g\circ f)^{-1}(0_R)$ $\Rightarrow$

We have $(g\circ f)^{-1}(0)=f^{-1}\circ g^{-1}(0)$

$\Rightarrow$ We want to find a continuous $g$ s.t $g^{-1}(0)=E$.

Put $g=\rho_E(y)=\inf\{d(x,y),\ x\in E\}$ then $\begin{cases} g \text{ is a cont function} \\ g^{-1}(0)=E. \end{cases}$

So our proof is done $\square$.

Aug 2010 ⑤

Give an example of bounded function $f : [0,1] \longrightarrow \mathbb{R}$ such that

- $f$ is not Riemann integrable on $[0,1]$.
- $g(x) = \sin(f(x))$ is Riemann integrable on $[0,1]$

$f$ is Riemann integrable on $[0,1]$ $\Leftrightarrow$ $\forall \varepsilon > 0, \exists$ partition $P$, $U(P,f) - L(P,f) < \varepsilon$.

$f$ is not Riemann integrable on $[0,1]$ $\Leftrightarrow$ $\exists \varepsilon > 0, \forall$ partition $P$, $U(P,f) - L(P,f) \geq \varepsilon$.

+ Put $f(x) = \begin{cases} \pi & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]. \end{cases}$

+ $g(x) = \sin(f(x)) = 0$ on $[0,1]$ $\Rightarrow$ $g$ is Riemann integrable on $[0,1]$.

- Now prove $f$ is bounded on $[0,1]$ : obvious
- Prove $f$ is not Riemann integrable.

We have for all partition $P$ in $[0,1]$, because $\mathbb{Q}$ is dense in $\mathbb{R}$, then.

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x) = \pi$$

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x) = 0$$

$$\Rightarrow U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \pi \Delta x_i = \pi \underbrace{\sum_{i=1}^{n} \Delta x_i}_{=1} = \pi > \overset{\overset{1}{\shortparallel}}{\varepsilon}$$

$\Rightarrow$ $f$ is not Riemann integrable.

Aug 2010/6

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a mapping defined by

$y_1 = x_1 + x_2$ a) Determine all point $a \in \mathbb{R}^3$ at which $f$ satisfies the assumption

$y_2 = x_2 - x_1$ of the Inverse function theorem.

$y_3 = x_3^5$ b) Is $f$ an open mapping? Prove or disprove.

a) a point $\vec{a} = (a_1, a_2, a_3)$ at which $f$ satisfies the assumption of the Inverse function theorem is when $f'(\vec{a})$ is invertible. (note that we already have $f$ is $C^1$ function.

we have

$$f'(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5x_3^4 \end{bmatrix}$$

$f'(x)$ invertible when $\det[f'(x)] \neq 0 \Leftrightarrow 5x_3^4 \cdot 2 \neq 0 \Leftrightarrow 10x_3^4 \neq 0 \Leftrightarrow x_3 \neq 0$.

$\Rightarrow$ A point $\vec{a}(a_1, a_2, a_3)$ at which $f$ satisfies the assumption of the inverse function theorem is when $a_3 \neq 0$

b) In order to be an open mapping

~~$f'(\vec{x})$ invertible $\forall \vec{x} \in \mathbb{R}^3$ but it is not when $x_3 = 0$ $\Rightarrow$ $f$ is not an open mapping~~

No, this is not true, The theorem says:

$$\left.\begin{array}{l} f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ Df \neq 0, \forall \vec{x} \in \mathbb{R}^n \end{array}\right\} \Rightarrow \text{then } f \text{ is an open mapping}$$

This does $\boxed{\text{not}}$ mean $Df = 0$ for some $\vec{x} \in \mathbb{R}$ then $f$ is not an open mapping.

* Another way to prove that $f$ is an open map:

$$\left.\begin{array}{l} \text{We find } g = f^{-1} \\ \text{prove that } g \text{ is continuous on } \mathbb{R}^n \end{array}\right\} \text{Then because } g \text{ is continuous} \Leftrightarrow \forall\ V \text{ open in } \mathbb{R}^n, \underset{f(V)}{\underset{\|}{g^{-1}(V)}} \text{ open in} \Leftrightarrow f \text{ is open map}$$

Now consider

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_2 - x_1 \\ y_3 = x_3^5 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 - y_1 = \frac{y_1 + y_2}{2} - \frac{y_1}{2} = \frac{y_2 - y_1}{2} \\ x_2 = \frac{y_1 + y_2}{2} \\ x_3 = \sqrt[5]{y_3} \end{cases}$$

Thus put: $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(y_1, y_2, y_3) \longmapsto (x_1, x_2, x_3) = \left( \frac{y_2 - y_1}{2}, \frac{y_1 + y_2}{2}, \sqrt[5]{y_3} \right)$ ) we have $g$ is continuous in $\mathbb{R}^3$ and $g = f^{-1}$

this means $\forall\ V$ open in $\mathbb{R}^3$, $\underset{f(V)}{\underset{\|}{g^{-1}(V)}}$ is open in $\mathbb{R}^3$

$\rightarrow$ $f$ is an open map.

## Preliminary Examination in Analysis, January 2011

1. Let $X, Y$ be metric spaces and $f : X \to Y$ be a function. Prove that $f$ is continuous on $X$ if and only if $\overline{f^{-1}(E)} \subset f^{-1}(\overline{E})$ for every $E \subset Y$. See 4.2/98 Rudin

② Prove that the sequence $x_n = n \sin(2\pi e n!)$, $n \geq 1$, is convergent and find its limit.
Hint: Use the fact that $e = \sum_{k=0}^{n} \frac{1}{k!} + r_n$, where $r_n < \frac{1}{n \cdot n!}$, $n \geq 1$.

ee Aug 200 ⑨ 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \geq 1$ for all $x \in \mathbb{R}$. Prove that $f$ is one-to-one and onto $\mathbb{R}$, and that the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is differentiable.

4. Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous. Show that

$$\int_0^1 f(x)x^2\,dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0, 1]$.

5. Let $f_1: [0, 1] \to \mathbb{R}$ be a continuous function. Consider the sequence of functions defined on the interval $[0, 1]$ as follows: for $n = 1, 2, \ldots,$

$$f_{n+1}(x) = \cos f_n(x).$$

Prove that $\{f_n\}$ contains a uniformly convergent subsequence.

6. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Suppose that none of the derivatives $f'$, $D_1 g$, $D_2 g$ attains the value 0. Define $\mathbf{h} = (h_1, h_2)$ by

$$\begin{aligned} h_1(x, y, z) &= f(x) + g(y, z) \\ h_2(x, y, z) &= f(y) - g(x, z). \end{aligned}$$

Prove that $\mathbf{h}(W)$ is an open subset of $\mathbb{R}^2$ for every open set $W \subset \mathbb{R}^3$.

# AUGUST 2011 PRELIMINARY EXAMINATION IN ANALYSIS

1. Suppose $A$ is an infinite bounded subset of the real line $\mathbb{R}$. Prove that there exists a set $B \subset A$ which is neither open nor closed in $\mathbb{R}$.

2. Let X be a metric space. Suppose that $f\colon [0,1] \to X$ is continuous. Prove that there exists an integer $n$ such that for any choice of the partition $0 = t_0 < t_1 < \cdots < t_n = 1$ we have

$$\min_{1 \leq i \leq n} \operatorname{diam} f([t_{i-1}, t_i]) \leq 1$$

Reminder: $\operatorname{diam} E = \sup\{d(a,b) : a, b \in E\}$.

3. Let $f\colon [1,e] \to \mathbb{R}$ be a continuous function. Prove that

$$\left(\int_1^e f(x)\,dx\right)^2 \leq \int_1^e x f(x)^2\,dx$$

4. Let $\{f_n\}$ be a sequence of Riemann integrable (with respect to $dx$) real-valued functions defined on $[0,1]$. Suppose that the functions $g_n(x) = \sqrt{x} f_n(x)$ form a uniformly convergent sequence. Prove that the limit

$$\lim_{n \to \infty} \int_0^1 f_n(x)\,dx$$

exists.

5. Let $f : (0,\infty) \to \mathbb{R}$ be everywhere differentiable with $|f'(x)| \leq \dfrac{1}{x^2}$, $0 < x < \infty$. Prove that the improper integrals

$$\int_{2y}^{\infty} (f(x) - f(x-y))\,dx, \qquad 0 < y < \infty$$

are well defined and in absolute value not greater than 1.

6. Let $g\colon \mathbb{R} \to \mathbb{R}$ be strictly increasing differentiable function. Define $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(x_1, x_2) = (x_1 + g(x_1 - x_2), x_2 + \sin x_2 - g(x_1 - x_2)).$$

Does it follow that $\mathbf{f}$ satisfies the conditions of the Inverse Function Theorem at every point of $\mathbb{R}^2$? Prove or give a counterexample.

Jan 2011, Q 1 (See 4.2 Rudin):

1) Let $X, Y$: metric spaces . $f: X \longrightarrow Y$ be a functions } Prove that $f$ is continuous on $X \Longleftrightarrow \overline{f^{-1}(E)} \subseteq f^{-1}(\overline{E})$, $\forall$

$(\Longleftrightarrow)$: Let $f$ continuous on $X$. Prove that $\overline{f^{-1}(E)} \subset f^{-1}(\overline{E})$, $\forall E \subseteq Y$.

- We have $f$ is continuous on $X \overset{def}{\Longleftrightarrow} \forall$ $V$ closed in $Y$, then $f^{-1}(V)$ closed in $X$.
  $\Rightarrow f^{-1}(\overline{E})$ closed in $X$.
- We have $E \subseteq \overline{E} \Rightarrow f^{-1}(E) \subseteq f^{-1}(\overline{E})$
- We also have $\overline{f^{-1}(E)}$ is the smallest closed subset of $X$ that contains $f^{-1}(E)$

$\Rightarrow \overline{f^{-1}(E)} \subseteq$ , $\forall$ □

$(\Longleftarrow)$: Let $\overline{f^{-1}(E)} \subset f^{-1}(\overline{E})$, $\forall E \subseteq Y$. Prove that $f$ is continuous on $X$.

We want to prove that $f$ is continuous on $X$

$\Leftrightarrow$ We Need to prove $\forall$ $E$ closed in $Y$, then $f^{-1}(E)$ closed in $X$

$\Leftrightarrow$ We need to prove $\forall$ $E$ closed in $Y$, then $f^{-1}(E) = \overline{f^{-1}(E)}$

we always have $f^{-1}(E) \subseteq \overline{f^{-1}(E)}$, so we need to prove $\forall E$ closed in $Y$, $\overline{f^{-1}(E)} \subseteq f^{-1}(E$

From $\overline{f^{-1}(E)} \subseteq f^{-1}(\overline{E}) \Rightarrow$

$E$ is closed $\Rightarrow f^{-1}(\overline{E}) = f^{-1}(E)$ } $\Rightarrow \overline{f^{-1}(E)} \subseteq f^{-1}(E) \Rightarrow$ done □⇐

* Another way: By proving $f^{-1}(E)$ closed by definition

We want to prove $f$ is cont on $X \Leftrightarrow$ NTP $\forall E$ closed in $Y$, then $f^{-1}(E)$ closed in $X$.

Let $x$ is a limit point of $f^{-1}(E)$, we NTP $x \in f^{-1}(E)$.

Because $x$ is a limit ~~point~~ of $f^{-1}(E) \Rightarrow x \in \overline{f^{-1}(E)} \subseteq f^{-1}(\overline{E}) \underset{\substack{\uparrow \\ E \text{ closed}}}{=} f^{-1}(E) \Rightarrow$ done

) Jan 2011 — Tricky.

Prove that the sequence $x_n = n\sin(2\pi e n!)$, $n \geq 1$ is convergent. Find its limit. NOT R

Important limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

x: radian (not degree)

$$e = \sum_{k=0}^{n} \frac{1}{k!} + x_n \qquad \frac{1}{(n+1)!} < x_n = \frac{e^{\xi}}{(n+1)!} \leq \frac{1}{n(n)!} \text{ for } n \geq 1 \qquad (1)$$

$\xi \in (0,x)$

We have $\sin(2\pi e n!) = \sin\left(2\pi\left(\sum_{k=1}^{n}\frac{1}{k!} + x_n\right)n!\right) = \sin\left(2\pi\left(\sum\frac{1}{k!}\right)n! + 2\pi x_n n!\right)$

note that $\frac{n!}{k!} \in \mathbb{N}$ (because $k \leq n$) $\Rightarrow \left(\sum_{k=1}^{n}\frac{1}{k!}\right)n! \in \mathbb{N}$

$\sin(2\pi e n!) = \sin(2\pi x_n n!)$ $\quad (*)$

Because of (1)
$$\frac{1}{(n+1)!} \leq x_n \leq \frac{1}{n(n)!}$$
$$\frac{2\pi}{n+1} \leq 2\pi x_n n! \leq \frac{2\pi}{n}$$

note that sin is an increasing function in $[0, \frac{2\pi}{n}]$, for $n \geq 2$

Then we have $\underbrace{n\sin\left(\frac{2\pi}{n+1}\right)}_{=A} \leq n\sin(2\pi x_n n!) \leq \underbrace{n\sin\left(\frac{2\pi}{n}\right)}_{=B}$

- Now find $\lim_{n\to\infty} A = \lim_{n\to\infty} n\sin\left(\frac{2\pi}{n+1}\right) = \lim_{n\to\infty} n \frac{\sin\left(\frac{2\pi}{n+1}\right)}{\frac{2\pi}{(n+1)}}\left(\frac{2\pi}{n+1}\right) \underset{\lim_{y\to0}\frac{\sin y}{y}=1}{=} 2\pi$

$$\lim_{n\to\infty} B = \lim_{n\to\infty} n\sin\left(\frac{2\pi}{n+}\right) = \lim_{n\to\infty} n \cdot \frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)}\left(\frac{2\pi}{n}\right) = 2\pi$$

ier by Squeeze theorem, $\lim n\sin(2\pi x_n)n! = 2\pi \Rightarrow \lim n\sin(2\pi e n!) = 2\pi$ ☐

Now we prove the part that was used above $e = \sum_{k=1}^{n}\frac{1}{k!} + x_n$ where $\frac{1}{(n+1)!} \leq x_n \leq \frac{1}{n(n!)}$

We have $e = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \underbrace{\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots}_{x_n} \Rightarrow x_n \geq \frac{1}{(n+1)!}$

e ~~Calculate solution~~.

$$e = 1 + \cdots + \frac{1}{n!} + \underbrace{\frac{1}{n!(n+1)} + \frac{1}{(n!)(n+1)(n+2)} + \cdots + \frac{1}{n!(n+1)(n+2)(n+3)}}$$
$$\leq \frac{1}{n!}\sum_{k=0}^{\infty}\frac{1}{(n+1)^{k+1}} = \frac{1}{n!}\frac{1/n+1}{1-\frac{1}{n+1}} = \frac{1}{n!(n)}$$

Jan2011 : ③

Let $f: \mathbb{R} \to \mathbb{R}$ differentiable function $|f'(x)| \geq 1, \forall x \in \mathbb{R}$. } Prove that $f$ is one-to-one, onto that $f'$ is differentiable.

✱ See Aug2008/2.

Aug2008

2) Suppose $f: \mathbb{R} \to \mathbb{R}$ continuous function $|f(x) - f(y)| \geq |x-y|, \forall x, y \in \mathbb{R}$ } Prove $f(\mathbb{R}) = \mathbb{R}$.

* Prove that $f$ is one-to-one. NTP $\forall x, y \in \mathbb{R}, x \neq y$, then $f(x) \neq f(y)$

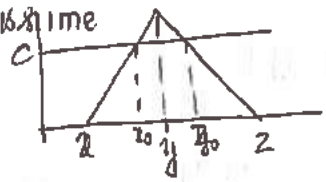
Because $f$ is differentiable in $\mathbb{R}$, then by MVT, $\exists c \in (x, y)$.

$f(x) - f(y) = \underbrace{f'(c)}_{\neq 0} \underbrace{(x-y)}_{\neq 0}$. then this means for $x \neq y$, then $f(x) \neq f(y) \Rightarrow f$ is one-to-one

* Observe $\left.\begin{array}{l} f \text{ is one-to-one} \\ f \text{ is continuous. } \mathbb{R} \end{array}\right\} \Rightarrow f$ has to be strictly monotone in $\mathbb{R}$ $\left(\begin{cases} f'(x) \geq 1, \forall x \in \mathbb{R} \\ f'(x) \leq 1, \forall x \in \mathbb{R} \end{cases}\right)$

(we can understand that if $f$ has local maximum (or minimum) at $x_0$, then $\begin{cases} f'(x_0) = 0 \\ f'(x_0) \text{ does not exist} \end{cases}$ > both of these contracts with $f$ differentiable on R $|f'(x)| \geq 1, \forall x$.)

Thus, assume



$\exists x < y < z$ such that $f(x) < f(y)$ 
$f(z) < f(y)$

then $\exists c$, $\left.\begin{array}{l} f(x) < c < f(y) \\ f(z) < c < f(y) \end{array}\right\} \Rightarrow \begin{array}{l} \exists x_0 \in (x, y) \quad c = f(x_0). \\ \exists z_0 \in (y, z) \quad c = f(z_0) \end{array}$
because $f$ is continuous

This means $f(x_0) = f(z_0)$ where $x_0 \neq z_0$ (this contracts with $f$ is one-to-one)

So we have $\begin{cases} f'(x) \geq 1, \forall x \in \mathbb{R} \\ f'(x) \leq -1, \forall x \in \mathbb{R}. \end{cases}$

* Now we will prove that $f$ is onto

Wlog assume $f'(x) \geq 1, \forall x \in \mathbb{R}$

- Put $g(x) = f(x) - x$, then we have $g$ is differentiable in $\mathbb{R}$ and $g'(x) = f'(x) - 1 \geq 0, \forall x \in \mathbb{R}$.

Then for $x > 0$: $[g(x) - g(0)] = \underbrace{g'(x)}_{\geq 0} \underbrace{[x - 0]}_{> 0}$

$\Rightarrow g(x) \geq g(0)$
$\Rightarrow f(x) - x \geq f(0)$
$\Rightarrow f(x) \geq x + f(0)$ $\Rightarrow \lim_{x \to +\infty} f(x) = +\infty$ (1)

for $x < 0$ $[g(x) - g(0)] = \underbrace{g'(x)}_{\geq 0} \underbrace{[x - 0]}_{< 0}$

$\Rightarrow g(x) \leq g(0)$
$\Rightarrow f(x) \leq x + f(0)$
$\Rightarrow \lim_{x \to -\infty} f(x) = -\infty$ (2)

(1)+(2) $\Rightarrow$ $f(R) = \mathbb{R}$

Now we prove that $f^{-1}$ is differentiable

ecause $f$ is bijective then $f^{-1}$ is welldefined $\forall y \in Y$

w consider

$$\lim_{\substack{\to y_0 \\ \neq y_0}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$$

Because $f^{-1}$ is welldefined

Let $x = f^{-1}(y)$

$x_0 = f^{-1}(y_0)$.

Note that we have $f^{-1}$ is continuous because $y_n \to y_0$, then $f^{-1}(y_n) \to f(y_0)$

This assume $f^{-1}(y_n) \not\to f^{-1}(y_0)$

$$\left.\begin{aligned} &\Leftrightarrow \quad x_n \not\to x_0 \\ &f \text{ is continuous} \end{aligned}\right\} \Rightarrow \ ?\ (*)$$

en
$$\lim_{\substack{y \to y_0 \\ y \neq y_0}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)}$$
$$= \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}.$$

we have $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$, $\forall y_0 \in \mathbb{R}$ where $y_0 = f(x_0)$, then $f^{-1}$ is differentiable (with $(f^{-1})'(y) = \frac{1}{f'(x)}$)

---

oe(*): Prove that
$$\left.\begin{aligned} &f \text{ is continuous, bijective } R \to R \\ &|f'(x)| \geq 1, \forall x \in \mathbb{R} \end{aligned}\right\} \Rightarrow f^{-1} \text{ is continuous}$$

we have $\forall x, y \in \mathbb{R}$, $\exists c$ s.t

$$|f(x) - f(y)| = \underbrace{|f'(c)|}_{\geq 1} |x - y| \Rightarrow |f(x) - f(y)| \geq |x - y|$$

$$\text{Let } \begin{aligned} z &= f(x) \\ l &= f(y) \end{aligned} \Rightarrow \quad |z - l| \geq |f^{-1}(z) - f^{-1}(l)|, \ \forall z, l \in R$$

$\Rightarrow f^{-1}$ is continuous.

47 Jan 2011

Suppose $f: \mathbb{R} \to \mathbb{R}$ continuous.
Show that $\int_0^1 f(x)x^2 dx = \frac{1}{3} f(\xi)$ for some $\xi \in [0,1]$.

* Mean value theorem for integral (The proof after theorem 6.13)

Let $f:[a,b] \longrightarrow \mathbb{R}$, $f$ is continuous on $[a,b]$
$\alpha$: monotonically increasing

$\Big\}$ Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$ (
and $\exists c \in [a,b]$ such that $\int_a^b f\, d\alpha = f(c)[\alpha$

Then apply above theorem, we have

$$\int_0^1 f(x)x^2 dx = \int_0^1 f(x)\,\alpha'(x)dx = \int_0^1 f(x) d\alpha$$

where $\alpha(x) = \left(\frac{1}{3}x^3\right)$

by above theorem $\exists c \in [a,b]$

$$\int_0^1 f\,d\alpha = f(c)\left[\frac{1}{3}1^3 - \frac{1}{3}0^3\right] = \frac{1}{3}f(c) \Rightarrow$$

---

Way 2* Now we prove directly (Prove the mean value theorem in case $f(x)$ continuous
$\alpha(x) = \frac{1}{3}x^3$.

- We have $f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous
$[0,1]$ compact in $\mathbb{R}$ $\Big\} \Rightarrow f$ attain maximum and min of $f([0,1])$ in $\mathbb{R}$

Let $m = \min_{x \in [0,1]} f(x)$ $\quad M = \max_{x \in [0,1]} f(x)$

This means $m \leq f(x) \leq M$, $\forall x \in [0,1]$

$$m x^2 \leq f(x)x^2 \leq M x^2$$

$$m \int_0^1 x^2 dx. \leq \int_0^1 f(x)x^2 dx \leq M \int_0^1 x^2 dx.$$

$$\Leftrightarrow \quad \frac{m}{3} \leq \int_0^1 f(x)x^2 dx \leq \frac{M}{3}$$

$$\Leftrightarrow \quad m \leq 3\int_0^1 f(x)x^2 dx \leq M.$$

Then by the Intermediate value theorem $\exists \xi \in [a,b]$ s.t $f(\xi) = 3\int_0^1 f(x)x^2 dx$

$$\Leftrightarrow \int_0^1 f(x)x^2 dx = \frac{1}{3} f(\xi) \quad \square.$$

Let $f_1 : [0,1] \longrightarrow \mathbb{R}$ be a continuous function.
onsider the sequence of functions defined on the interval $[0,1]$ as follows:
or $n = 1, 2, \dots$ $\quad f_{n+1}(x) = \cos f_n(x)$
ve that $\{f_n\}$ contains a uniformly convergent subsequence.

We have (1): $K = [0,1]$ is compact.

We have $f_1$ is continuous on a compact set $\Rightarrow f([0,1])$ is bounded $\Rightarrow \exists M, |f_1(x)| \le M, \forall x$.

$|f_k(x)| \leqslant 1, \forall x, \forall k = 2, 3, \dots$

then we have $\forall k = 1, 2, 3, \dots$ $|f_k(x)| \le \max\{M, 1\} \Rightarrow$ uniformly bounded $\Rightarrow$ pointwise bounded. (2)

We also have $f_k$ continuous $\forall k$.

Now we will prove (3): $\{f_n\}_{n=1}^{\infty}$ is a equicontinuous family on $[0,1]$.

First, we have $\left.\begin{array}{l} f_1 : [0,1] \longrightarrow \mathbb{R} \text{ continuous} \\ [0,1] \text{ compact} \end{array}\right\} \Rightarrow f_1$ is uniformly continuous

is means $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x - y| < \delta$, then $|f_1(x) - f_1(y)| < \varepsilon$ $(*)$

ow consider $f_2(x) = \cos f_1(x)$

 have $\forall x, y \in [0,1], |x - y| < \delta$, then (wlog assume $x > y$)

$$f_2(x) - f_2(y)| \underset{\text{FTofC}}{=} \left| \int_x^y f_2'(t)\,dt \right| \leqslant \int_x^y |f_2'(t)|\,dt = \int_x^y \underbrace{|-\sin f_1(t)|}_{\le 1} \cdot |f_1'(t)|\,dt$$

we couldn't use this way because we don't know if $f_1$ is differentiable or not

$$\leqslant |f_1'(y)| - |f_1(x)| \le |f_1(y) - f_1(x)| \underset{\text{by }(*)}{<} \varepsilon$$

en by induction, we have $\forall \varepsilon > 0, \exists \delta > 0, \forall |x - y| < \delta$

$$|f_{n+1}(x) - f_{n+1}(y)| \le |f_n(x) - f_n(y)| \le \dots \le |f_2(x) - f_2(y)| \le |f_1(x) - f_1(y)| < \varepsilon$$

This means $\{f_n\}$ equicontinuous family on $[0,1]$.

In conclusion, from (1) + (2) + (3) $\Longrightarrow$ $\{f_n\}$ contains a uniformly convergent subsequence.

$\longrightarrow$ we have

$$f_2(x) - f_2(y)| = |\cos f_1(x) - \cos f_2(y)| \underset{\text{MVT}}{=} \underbrace{|-\sin f(\xi)|}_{\le 1} \, |f_1(x) - f_2(y)|, \ \xi \in \min, \max\{f_1(x),$$

$$\le |f_1(x) - f_1(y)| < \varepsilon$$

Jan 2011 G7

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\Big\}$ be continuously differentiable

Suppose none of the derivative of $f'$, $D_1 g$, $D_2 g$ attains the value 0

Define $\vec{h}$ by $\vec{h} = (h_1, h_2)$ $\quad h_1(x,y,z) = f(x) + g(y,z)$

$h_2(x,y,z) = f(y) - g(x,z)$.

Prove that $h(W)$ is an open subset of $\mathbb{R}^2$ for every open set $W \subseteq \mathbb{R}^3$

different

have not done y

note that $g(y,z) \rightarrow$ we use

$g(x,z)$

$(x,y) = \alpha($

* Note that this problem looks like proving a map h is a open map.

We have the corollary of Inverse function theorem:

$$\left.\begin{array}{l} f: O \text{ open in } \boxed{\mathbb{R}^n} \longrightarrow \mathbb{R}^n \\ f \text{ is } C^1 \text{ function} \\ f'(\vec{x}) \text{ is invertible } \forall x \in \vec{O} \end{array}\right\} \Rightarrow f: V \longrightarrow f(V) \text{ is an open mapping}$$

but in this problem: $h: R^3 \longrightarrow \mathbb{R}^2 \Rightarrow$ we have to use implicit function theorem where $\boxed{(x,y) = \alpha(z)}$

* Put $\vec{H} = (h_1, h_2, z)$, then we have $\vec{H}: R^3 \longrightarrow \mathbb{R}^3$.

$$H' = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial z} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial z} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} f'_x & D_1 g(y,z) & D_2 g(y,z) \\ -D_1 g(x,z) & f'_y & -D_2 g(y,z) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det[H'] = f'_x f'_y + D_1 g(x,z) D_1 g(y,z) =$$

$$A_{xy} = \begin{bmatrix} f'_x & D_1 g(y,z) \\ -D_1 g(x,z) & g'_y \end{bmatrix} \qquad A_z = \begin{bmatrix} D_2 g(y,z) \\ -D_2 g(y,z) \end{bmatrix}$$

$$\det A_{xy} = f'_x f'_y + D_1 g($$

Aug 2011

P17 Suppose A is an infinite bounded subset of the real line $\mathbb{R}$.
Prove that $\exists$ a set $B \subset A$ which is neither open nor closed in $\mathbb{R}$.

NTR

A is an infinite bounded subset of the real line $\mathbb{R}$

$\Rightarrow$ A contains some limit point $a_0 \in \mathbb{R}$.

$\Rightarrow \exists (a_n) \subset A$, $a_n \to a_0$.

then $B = \{a_n, n \in \mathbb{N}\}$ is the set that neither open nor closed in $\mathbb{R}$.

B is not closed in $\mathbb{R}$, because it does not contains the limit point $a_0$

B is not open because it is a union of ~~d~~ single points in $\mathbb{R}$ $\square$

ig2011 *6

) Let $\mathcal{X}$ be a metric space

uppose that $f : [0,1] \longrightarrow \mathcal{X}$ continuous

ove that $\exists$ an integer $n$ st. $\forall$ choice of the partition $0 = t_0 < t_1 < \cdots < t_n = 1$,

we have $\min\limits_{1 \leq i \leq n} \operatorname{diam} f([t_{i-1}, t_i]) \leq 1$

Reminder $\operatorname{diam} E = \sup\{d(a,b), a, b \in E\}$

---

: $[0,1] \longrightarrow \mathcal{X}$ continuous, $[0,1]$ capact in $\mathbb{R}$, $\mathcal{X}$ : metric space $\Rightarrow$ $f$ is uniformly continuous on $[0,1]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta$ then $d_{\mathcal{X}}(f(x), f(y)) < \varepsilon$

hoose $\varepsilon = 1$, then we have $\forall \delta$, $\forall x, y \in [0,1]$, $|x-y| < \delta$ then $d_x(f(x), f(y)) < \varepsilon$ (*

ow we choose $n$ such that $\frac{1}{n} < \delta$

ecause $0 \leq t_0 < t_1 < \cdots < t_n = 1$ ($t_i \neq t_j$) for $i \neq j$

$\Rightarrow \exists$ at least one segment $[t_{i-1}, t_i]$ such that.

~~$|t_{i-1}, t_i| <$~~

$|t_i - t_{i-1}| < \delta$.

hen $\exists$ segment st $\operatorname{diam} f([t_{i-1}, t_i]) < 1$ (because of (*)).

$$\Rightarrow \min_{1 \leq i \leq n} \operatorname{diam} f([t_{i-1}, t_i]) \leq 1 \quad \square.$$

Aug 2011

P3, Let $f:[1,e] \longrightarrow \mathbb{R}$ be a continuous function.

Prove that $\left(\int_1^e f(x)\,dx\right)^2 \leqslant \int_1^e x\, f^2(x)\,dx$.

Holder inequality $\left(\int fg\,dx\right) \leq \left(\int f^p dx\right)^{1/p} \left(\int g^q dx\right)^{1/q}$ where $\frac{1}{p}+\frac{1}{q}=1$

Let $p=q=2$ $\left(\int fg\,dx\right)^2 \leq \left(\int f^2 dx\right)\left(\int g^2 dx\right)$

We put $F(x) = \sqrt{x}\, f(x)$

$G(x) = \frac{1}{\sqrt{x}}$ $\quad (x \in [1, e] \Rightarrow$ well defined$)$

Then apply Holder inequality, we have:

$$\left(\int FG\,dx\right)^2 \leqslant \int F^2 dx \int G^2 dx.$$

$$\Leftrightarrow \int_1^e f(x)\,dx \leqslant \int_1^e x f^2(x)\,dx \underbrace{\int_1^e \frac{1}{x}\,dx}_{= \ln x \Big|_1^e = \ln e - \ln 1 = \ln e = 1} \Rightarrow \text{WWNTP} \quad \square.$$

g 2011 →

1) Let $\{f_n\}$ : be a Riemann integrable (wrt $dx$) real-valued function defined on $[0,1]$.
Suppose $g_n(x) = \sqrt{x}\, f_n(x)$ form a uniformly convergent sequence.
Prove that the limit $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx$ exist.

We have $g_n(x) \rightrightarrows \Leftrightarrow \{g_n\}$ uniformly Cauchy

$$\Leftrightarrow \quad \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, |g_m(x) - g_n(x)| < \varepsilon$$

$$\Rightarrow |\sqrt{x}\,(f_m(x) - f_m(x))| < \varepsilon$$

$$\Rightarrow |f_m(x) - f_n(x)| < \frac{\varepsilon}{\sqrt{x}} \quad x > 0$$

Note that from here, we don't have $\{f_n\}$ uniformly Cauchy

So now we prove the limit $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx$ exist by proving that sequen $\boxed{F_n (= \int_0^1 f_n(x)dx}$ is a Cauchy sequence.

For all $m, n \geq n_0$, we have

$$F_m - F_n| = \left| \int_0^1 f_m(x)dx - \int_0^1 f_n(x)dx \right| \leq \int |f_m(x) - f_n(x)| dx$$

$$< \int_0^1 \frac{\varepsilon}{\sqrt{x}} dx = 2 \int_0^1 \frac{\varepsilon}{2\sqrt{x}} dx = 2\sqrt{x}\,\varepsilon \Big|_0^1$$

$$= 2\varepsilon$$

So $\{F_n\}$ Cauchy $\Leftrightarrow \int_0^1 f_n(x)dx$ Cauchy $\Rightarrow$ the limit exists $\square$.

Aug 2011

15> Let $f:(0,\infty) \longrightarrow \mathbb{R}$ be everywhere differentiable

$|f'(x)| \leq \frac{1}{x^2}$ , $0 < x < +\infty$ .

Prove that the improper integrals

$\int_{2y}^{\infty} [f(x) - f(x-y)]\, dx \quad 0 < y < +\infty$ are well-defined.

and in absolute value not greater than $1$

---

* We have $f$ is everywhere differentiable in $(0, \infty)$ } we have

Then because $2y > 0$, $x > 2y \Rightarrow x - y > 0$, $|f(x) - f(x-y)| = |f'(\xi)|\, y \quad \xi \in (x-y$

* So we have

$$\left| \int_{2y}^{\infty} [f(x) - f(x-y)]\, dx \right| \leq \int_{2y}^{\infty} |f(x) - f(x-y)|\, dx \leq \int_{2y}^{\infty} |f'(\xi)|\, y\, dx.$$

$x-y < \xi < x$

$\frac{1}{x} < \frac{1}{\xi} < \frac{1}{x}$

(Note that $|f'(\xi)| \leq \frac{1}{\xi^2}$)

$$\leq \int_{2y}^{\infty} \frac{1}{(x-y)^2}\, y\, dx.$$

$$= -y \frac{1}{(x-y)} \Big|_{2y}^{\infty} = \frac{y}{2y - y} = 1$$

So the above improper integrals are well defined and $|\;| \leq 1 \Rightarrow \square$ .

g2011

26) Let $g: \mathbb{R} \longrightarrow \mathbb{R}$ be strictly increasing differentiable function.
Define $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$(x_1, x_2) \longmapsto f(x_1, x_2) = \left( x_1 + g(x_1 - x_2),\ x_2 + \sin x_2 - g(x_1 - x_2) \right)$$

Does it follow that $f$ satisfies the condition of IFT at every point of $R^2$.
Prove or give a counterexample.

We have $Df = \begin{bmatrix} 1 + g'(x_1 - x_2) & -g'(x_1 - x_2) \\ -g'(x_1 - x_2) & 1 + \cos x_2 + g'(x_1 - x_2) \end{bmatrix}$

$$\begin{aligned} \mathrm{Det}(Df) &= [1 + g'(x_1 - x_2)][1 + \cos x_2 + g'(x_1 - x_2)] - [g'(x_1 - x_2)]^2 \\ &= 1 + 2g'(x_1 - x_2) + \cos x_2 + \cos x_2\, g'(x_1 - x_2) \\ &= (1 + \cos x_2) + (2 + \cos x_2) \underbrace{g'(x_1 - x_2)}_{>0 \text{ because } g \text{ is a strictly increasing differentiable function}}, \end{aligned}$$

Choose $g'(x_1 - x_2) = 1$, then we have $(1 + \cos x_2) + 2 + \cos x_2 = 0$

$$\Leftrightarrow \quad 2\cos x_2 = -3$$

$$\cos x_2 = -3/2$$

Then $\exists x_2$ st $\cos x_2 = -\frac{3}{2}$, at that point $\det(D_f) = 0$

$\Rightarrow$ $f$ does not satisfy IFT

$\Rightarrow$ $f$ does not satisfy the condition of IFT at every point of $\mathbb{R}^2$ $\square$.

Hard.

# AUGUST 2012 PRELIMINARY EXAMINATION IN ANALYSIS

**1.** Let $X$ be a metric space. Suppose that $A_n$, $n = 1, 2, 3, \dots$ are nonempty compact subsets of $X$ such that $A_{n+2} \subset A_n \cup A_{n+1}$ for every $n \geq 1$. Prove that there exists a point $x \in X$ such that $x \in A_n$ for infinitely many values of $n$.

**2.** Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to (0, \infty)$ are continuous functions. For $x \in \mathbb{R}$ define

$$h(x) = \sup_{0<t<g(x)} f(t)$$

(a) Prove that $h\colon \mathbb{R} \to \mathbb{R}$ is continuous.

(b) Give an example in which $f$ is uniformly continuous on $\mathbb{R}$ but $h$ is not.

**3.** Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x+1) = f'(x)$ for all $x \in \mathbb{R}$. Prove that the limit $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x}$ exists and is finite.

**4.** Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \dots$, be $C^1$-functions; that is, continuously differentiable functions such that, for all $n$,

$$|f_n'(x)| \leqslant \frac{1}{\sqrt{x}} \quad (0 < x \leqslant 1) \quad \text{and} \quad \int_0^1 f_n(x)\,\mathrm{d}x = 0\,.$$

Prove that the sequence $\{f_n\}$ has a subsequence that converges uniformly on $[0, 1]$.

**5.** Suppose that $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^1$-mapping with $\det \mathbf{f}'(x) > 0$ for all $x \in \mathbb{R}^2$. Assume that $\mathbf{f}^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact. Prove that $\mathbf{f}(\mathbb{R}^2) = \mathbb{R}^2$.

**6.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a $C^1$-function with $f'(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $f$ takes the interval $[0, 1]$ onto itself. Prove that there is a sequence of polynomials $p_n\colon [0, 1] \to [0, 1]$ such that $p_n \to f$ uniformly on $[0, 1]$ and each $p_n$ is a strictly increasing function on $[0, 1]$.

# Analysis Preliminary Exam, January 2012

*1. Let $\{c_n\}$ be a sequence so that $c_n > 0$ for all $n \geq 1$ and $\lim_{n\to+\infty} c_n = 0$. Show that there exists a sequence $\{a_n\}$ so that $a_n > 0$ for all $n \geq 1$, $\sum_{n=1}^{\infty} a_n$ is divergent and $\sum_{n=1}^{\infty} c_n a_n$ is convergent.

2. Let $f : \mathbf{R} \longrightarrow \mathbb{R}$ be a uniformly continuous function. Show that there exist positive constants $A, B$, so that $|f(x)| \leq A|x| + B$ for every $x \in \mathbb{R}$.

3. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function which is differentiable at 0 and so that $f(0) = 0$. Show that the following limit exists and find it:

$$\lim_{x\to 0} \frac{f(x) - \sin f(x)}{x^3}.$$

4. Does the improper integral $\int_0^{\infty} \cos(x^2)\, dx$ converge or diverge? Prove your answer.

5. Given that

$$(1+t)^{-1/2} = 1 - \frac{1}{2}t + \frac{1\cdot 3}{2\cdot 4}t^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3 + \cdots$$

has radius of convergence 1 about $t = 0$, and that

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{for } |x| < 1,$$

find the Taylor series expansion for $\arcsin(x)$ at 0 and its radius of convergence. Justify your reasoning.

6. Given the real valued function $g(x, y, z) = z - x^2 - y^2$ on $\mathbb{R}^3$, find $Dg(0)$.
Define the mapping $F(x, y, z) = (x^3, y^3, g(x, y, z))$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ with $F(0) = 0$. What does the inverse function theorem say about $F$ in a neighborhood of the origin?
Does $F$ have a continuous inverse in neighborhood of the origin?

Jan 2012 : See Aug 1993

P1 Let $\{c_n\}$ be a sequence so that $c_n > 0$, $\forall n \geq 1$.
$\lim_{n\to\infty} c_n = 0$
Show that $\exists$ a sequence $\{a_n\}$, $a_n > 0$, $\forall n \geq 1$
$\sum a_n$ diverges. and $\sum c_n a_n$ converges.

+ We have $\lim_{n\to\infty} c_n = 0 \overset{dg}{\Leftrightarrow} \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon, |c_n| < \varepsilon$

Chose $\varepsilon = \frac{1}{2^k}$, then $\exists n_k$, $\forall n \geq n_k$, $|c_n| < \frac{1}{2^k}$

(this means, $\exists \{n_k\}$, $n_k \longrightarrow \infty$ s.t $c_{n_k} < \frac{1}{2^k}$).

(for each $\varepsilon$, $\exists n_\varepsilon$)
$\rightarrow$ there is a sequence of $n_k$, and we only cni
case when $n = n_k$.

+ So now put $a_n = \begin{cases} 1, & n = n_k,\ k \geq 1. \\ 0, & \text{for } n \neq n_k. \end{cases}$

Then we have, $\sum a_n$ diverges.

and $\sum c_n a_n =$ where $c_n a_n = \begin{cases} 1 c_{n_k}, & n = n_k \\ 0 & , n \neq n_k \end{cases} \neq \begin{cases} 1 c_{n_k} | < \frac{1}{2^k}, & n = n_k \\ 0 & , n \neq n_k \end{cases}$

, so $\sum c_n a_n$ converges. $\square$

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+ Note $\sum \frac{1}{2^n}$ converges when $2 > 1$.

*Another ~~easy~~ problem.

One example when $\sum a_n$ diverges, then $\sum c_n$ such that $\sum a_n c_n$ converges.
$\sum 1$ diverges, then $\sum \frac{1}{2^n}$ s.t $\sum 1 \frac{1}{2^n}$ converges.

2012 →

2: Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ uniformly continuous function.
how that $\exists$ positive constants $A, B$ so that $|f(x)| \leq A|x| + B \quad , \forall\, x \in R$.

This problem requires to prove that $\forall x \in \mathbb{R}, \exists A_x, D_x > 0$ s.t $|f(x)| \leq A|x| + B$.

One useful strategy

have $f: \mathbb{R} \to \mathbb{R}$ uniformly continuous

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x, y \in \mathbb{R}, |x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

it's important to add 1 here so that $|x_i - x_{i-1}| = \frac{1}{n} = \frac{\delta}{|x|+1} <$

ien $\forall x \in \mathbb{R}$, we choose $n = \frac{|x|}{\delta} \bigcirc\!\!\!\!\!+1$ so $\forall x_i$ in $\{\overset{0}{x_0}, x_1, \ldots, \overset{x}{\overset{\shortparallel}{x_n}}\}$. $|x_i - x_{i-1}| \leq \frac{1}{n} < \delta$

$\Rightarrow |f(x_i) - f(x_{i-1})| < \varepsilon$

hen $|f(x) - f(x_0)| \leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

$$\leq n\varepsilon = \left(\frac{|x|}{\delta} + 1\right)\varepsilon = \frac{|x|}{\delta}\varepsilon + \varepsilon$$

Then put $A = \frac{\varepsilon}{\delta} > 0$ and $B = f(0) + \varepsilon > 0$

$\Rightarrow |f(x)| \leq A|x| + B \quad \square$.

Jan 2012

*

P3 Let $f:\mathbb{R}\to\mathbb{R}$ be a function differentiable at 0, $f(0)=0$. Show that the following limit exists and find it

$$\lim_{x\to 0}\frac{f(x)-\sin f(x)}{x^3}$$

We can't not use L'Hospital here because we don't have f is differentiable in a neighborhood of 0

Taylor series can be use to find limit of a function,

But note that: in here we just consider $f(x)=y$, and expand $\sin y$.

* We have $\sin y = y-\frac{y^3}{3!}+\frac{y^5}{5!}-\frac{y^7}{7!}+\frac{y^9}{9!}$

Then $\sin f(x) = f(x)-\frac{[f(x)]^3}{3!}+\frac{[f(x)]^5}{5!}-\frac{[f(x)]^7}{7!}+\frac{[f(x)]^9}{9!}-\cdots$

So we have:

$$\frac{f(x)-\sin f(x)}{x^3}=\frac{\frac{[f(x)]^3}{3!}-\frac{[f(x)]^5}{5!}+\frac{[f(x)]^7}{7!}-\frac{[f(x)]^9}{9!}}{x^3}$$

$$\lim_{x\to 0}\frac{f(x)-\sin f(x)}{x^3}=\lim_{x\to 0}\left|\frac{1}{3!}\left[\frac{f(x)-f(0)}{x-0}\right]^3-\frac{1}{5!}\left(\frac{f(x)}{x}\right)^3[f(x)]^2+\frac{1}{7!}\left(\frac{f(x)}{x}\right)^3[f\right.$$

$$\overset{}{\underset{\substack{\text{note}\\ f(0)=0}}{=\!=}}\frac{1}{3!}[f'(0)]^3-0\cdots$$

So we have $\lim_{x\to 0}\frac{f(x)-\sin f(x)}{x^3}=\frac{1}{3!}[f'(0)]^3$

m2012

17 Does the integral $\int_0^\infty \cos(x^2)dx$ converges or diverges ? Prove

Similar problems next pages.

Another way next page.

Problem here $d(x^2) = 2x\,dx$ and $\overset{a=0}{b=\infty}$.

ssume put $u = x^2$, then $du = 2x\,dx \Rightarrow dx = \frac{du}{2x} = \frac{du}{\sqrt{u}}$ (only well defined when $u \neq 0, x \neq 0$

So we can only do this when $x > 0$.

we have $\int_0^\infty \cos(x^2)dx = \int_0^1 \cos(x^2)dx + \int_1^\infty \cos(x^2)dx$.

a problem turns into considering if the~~t~~ improper integral $\int_1^\infty \cos(x^2)dx$ converges or diverg

~~We have~~

ow consider $\int_1^\infty \cos(x^2)dx = \lim_{A\to\infty}\int_1^A \cos(x^2)dx$

(Sometimes, we need to consider this first when instead of considering $\int_1^\infty$ directly)

Put $\begin{cases} u = x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}} \\ x = 1 \Rightarrow u = 1 \\ x = A \to u = A^2 \end{cases}$

* Some ways to investigate the convergence / divergence of an improper integral.

- comparison test
- Dirichlet test $\int_1^\infty f(x)dx$ & $\sum f(n)$ both converge/diverge
- Changing variable.

hen $\int_1^A \cos(x^2)dx = \int_1^{A^2} \cos u \frac{du}{2\sqrt{u}}$

Put $\begin{cases} w = \frac{1}{\sqrt{u}} \Rightarrow dw = \frac{-1}{2(u)^{3/2}} \\ dv = \cos u\,du \to v = \int \cos u\,du = \sin u + C. \end{cases}$

$$\lim_{A\to\infty}\int_1^A \cos(x^2)dx = \lim_{A\to\infty}\int_1^{A^2} \cos u \frac{du}{2\sqrt{u}} = \frac{1}{2}\frac{\sin u}{\sqrt{u}}\Big|_1^{A\to\infty} + \lim_{A\to\infty}\int_1^{A^2}\frac{1}{4}\frac{\sin u}{u^{3/2}}du$$

$$= \underbrace{\lim_{A\to\infty}\frac{1}{2}\frac{\sin A^2}{A} - \frac{\sin 1}{2\sqrt{1}}}_{\text{converges.}} + \lim_{A\to\infty}\int_1^{A^2}\frac{\sin u}{2u^{3/2}}du$$

• $\left|\int_1^{A^2}\frac{\sin u}{2u^{3/2}}du\right| \le \int_1^{A^2}\left|\frac{\sin u}{2u^{3/2}}\right|du \le \int_1^{A^2}\frac{1}{2u^{3/2}}du = \int_1^A\left(\frac{1}{\sqrt{u}}\right)'du = \underbrace{2\frac{1}{\sqrt{u}}\Big|_1^A}_{\text{converges when } A\to\infty}$

So $\int_0^\infty \cos(x^2)dx$ converges.

$$\boxed{* \text{ Prove that } \int_0^\infty \sin(x^2)\,dx \text{ converges.}}$$

Similar problems next pages.

We have $\sin(x^2)$ integrable on $[0, 1]$, so we have

$$\int_0^\infty \sin(x^2)\,dx = \int_0^1 \sin(x^2)\,dx + \int_1^\infty \sin(x^2)\,dx.$$

Our problem turns into considering the convergence/divergence of $\int_1^\infty \sin(x^2)\,dx$.

* Now consider $\int_1^\infty \sin(x^2)\,dx = \lim_{A\to\infty}\int_1^A \sin(x^2)\,dx.$

- Put $\begin{cases} u = x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}} \\ x = 1 \Rightarrow u = 1 \\ x = A \Rightarrow u = A^2 \end{cases}$

So $\int_1^\infty \sin(x^2)\,dx = \lim_{A\to\infty}\int_1^{A^2} \sin(u)\frac{1}{2\sqrt{u}}\,du = (*)$

- Put $\begin{cases} u = \frac{1}{\sqrt{u}} \Rightarrow du = \frac{-1}{2(u)^{3/2}} \\ dv = \sin(u)\,du \Rightarrow v = \int \sin u\,du = -\cos u + C \end{cases}$

$$\Rightarrow (*) = \underbrace{\lim_{A\to\infty}\left(\frac{-1}{2\sqrt{u}}\cos u\right)\Bigg|_1^A}_{\text{converges.}} - \underbrace{\int_1^\infty \frac{1}{4u^{3/2}}\cos u\,du}_{=(I)}$$

$$|(I)| \leq \int_1^\infty \left|\frac{\cos u}{4u^{3/2}}\right| du \leq \int_1^\infty \left|\frac{1}{4u^{3/2}}\right| du = \int_1^\infty \frac{1}{2} - \left(\frac{1}{\sqrt{u}}\right)' du = \underbrace{\frac{1}{2}\frac{1}{\sqrt{u}}\Bigg|_1^\infty}_{\text{converges.}}$$

$$\Rightarrow \int_0^\infty \sin(x^2)\,dx \text{ converges.}$$

Does the improper integral converge? $\int_0^\infty \underbrace{\sin x \ \sin(x^2)}\, dx$ ?

think about $\int u v' dx \Rightarrow$

oitce that $\sin(x^2) = \frac{1}{2x}(\cos x^2)'$ → does not exist at $x = 0$ so $\int_0^\infty = \int_0^1 + \int_1^\infty$

e have $\int_0^\infty \sin x \ \sin(x^2)dx = \int_0^1 (\sin x)\sin(x^2)dx + \int_1^\infty \sin x \ \sin(x^2)dx$.

, problem turns into investigating the convergence of $\int_1^\infty \sin x \ \sin(x^2)dx$

The ideal is to reach $\int \frac{g(x)}{x^n}dx$ where $\begin{cases} g(x) \text{ is bounded} \\ n > 1 \end{cases}$

Consider $\int_1^\infty \sin x \ \sin(x^2) dx$.

Notice that $\sin(x^2) = -(\cos x^2)' \frac{1}{2x}$

$$\left(\frac{1}{2x}\sin x\right)' = \frac{\cos x \cdot 2x - \sin x \cdot 2}{4x^2}$$

hen $\int_1^\infty \sin x \ \sin x^2 dx = -\int_1^\infty \underbrace{\frac{1}{2x}\sin x}_{v} \underbrace{(\cos x^2)'}_{u'}dx = -uv| + \int v'u\, dx = \underbrace{-\frac{\cos(x^2)\sin x}{2x}\Big|_1^\infty}_{\text{converges.}} + \underbrace{\int_1^\infty \frac{(2x\cos x - 2\sin x)}{4x^2}(\mathrm{co}}_{\text{put} = I}$

- note that $|\cos(x^2)\sin x| < 1$
- ~~Notice that $\left|\int_1^\infty \frac{(2x\cos x - 2\sin x)\cos(x^2)}{4x^2}dx\right| \le \int_1^\infty \frac{|2x\cos x - 2\sin x|}{4x^2}dx \le \int_1^\infty \frac{|2x|}{4x^2}dx = \int_1^\infty \frac{1}{2x}dx$.~~

* Now consider $I := \int_1^\infty \frac{2x\cos x \ \cos x^2}{4x^2} - \frac{1}{2}\frac{\sin x \ \cos x^2}{x^2}dx$

We have $\int_1^\infty \frac{1}{2}\frac{\sin x \cos x^2}{x^2}$ converges because $\left|\int_1^\infty \frac{1}{2}\frac{\sin x \cos x^2}{x^2}dx\right| < \underbrace{\frac{1}{2}\int_1^\infty \left|\frac{1}{x^2}\right|dx}_{\text{converges.}}$

Jow - we consider $\int_1^\infty \frac{\cancel{2x}\cos x \cos x^2}{4x^{\cancel{2}}}dx = \int_1^\infty \frac{\cos x \ \cos x^2}{2x}dx = \int_1^\infty \underbrace{\frac{\cos x}{4x^2}}_{v}\underbrace{(\sin x)'}_{u'}dx =$

(Notice $(\cos x^2) = (\sin x^2)'\frac{1}{2x}$)

$$= uv| - \int uv'dx = \underbrace{\frac{\sin x \ \cos x}{4x^2}\Big|_2^\infty}_{\text{converges}} - \int_1^\infty \sin x \ \frac{4x^2\sin x - \cos x \cdot 8x}{16x^4}dx$$

$$= \underbrace{\int_1^\infty \frac{1}{4}\frac{\sin^2 x}{x^2} - \frac{\cancel{8}\ \sin x \ \cos x}{2x^3}dx}_{\text{converges.}} \quad \square.$$

⇒ So the integral converges.

* Prove that the improper integral $\int_0^\infty \frac{\sin x}{x}\,dx$ converges

* Easier problem Investigate the convergence of $\int_1^\infty \frac{\sin x}{x}\,dx$.

$$\text{Put}\begin{cases} u = \frac{1}{x} \\ dv = \sin x \, dx \end{cases} \Rightarrow \begin{cases} du = -\frac{1}{x^2}\,dx \\ v = \int \sin x \, dx = -\cos x + C \end{cases}$$

$$\text{So } \int_1^\infty \frac{\sin x}{x}\,dx = \int u\,dv = uv\Big|_1^\infty - \int v\,du = \underbrace{-\frac{\cos x}{x}\Big|_1^\infty}_{= -\left(0 - \frac{\cos 1}{1}\right)} + \underbrace{\int_1^\infty \frac{\cos x}{x^2}\,dx}_{< \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{converges}}} \quad \square .$$

* Come back to our problem

Does $\displaystyle\int_0^\infty \frac{x}{1+x^2\sin^2 x}\,dx$. converge or diverge ?

e have ~~$|1+x^2\sin^2 x| \geqslant 2|x||\sin x|$~~

$$\Rightarrow 1+x^2\sin^2 x \leqslant 1+x^2$$

$$\Rightarrow \int_0^\infty \frac{x}{1+x^2\sin^2 x} \geqslant \int_0^\infty \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{d(1+x^2)}{1+x^2} = \frac{1}{2}\ln(1+x^2)\Big|_0^\infty \longrightarrow \infty$$

So the above integral diverges. □

Jan 2012

P5) Given that $(1+t)^{-1/2} = \frac{1}{(1+t)^{1/2}} = 1 - \frac{1}{2}t + \frac{1\cdot3}{2\cdot4}t^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}t^3 + \cdots$

has radius of convergence 1 about $t = 0$,

and that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$,

Find the Taylor series expansion for arcsin(x) at 0 and its radius of convergence. Justify

---

* We have from above assumption

$$\frac{1}{\sqrt{(1+t)}} = 1 - \frac{1}{2}t + \frac{1\cdot3}{2\cdot4}t^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}t^3 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}t^5 = \Sigma$$

Substitude $t = -x^2$, we have

$$\frac{1}{\sqrt{1-x^2}} = 1 - \frac{1}{2}(-x^2) + \frac{1\cdot3}{2\cdot4}(-x^2)^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}(-x^2)^3 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}(-x^2)^5$$

$$= 1 + \frac{1}{2}x^2 + \frac{1\cdot3}{2\cdot4}x^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}x^{10}.$$

Because

$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$, then we have

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}dt = \int_0^x 1 + \frac{1}{2}t^2 + \frac{1\cdot3}{2\cdot4}t^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}t^6 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}t^{10} + \cdots dt$$

$$= 1\frac{x^2}{2} + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\frac{x^{1}}{11}$$

not done yet

, Jan 2012

Given the real valued function $g(x,y,z) = z - x^2 - y^2$ on $\mathbb{R}^3$ a) Find $Dg(\bar{0})$.

Define the mapping $F(x,y,z) = (x^3, y^3, g(x,y,z))$ from $\mathbb{R}^3$ to $\mathbb{R}^3$

$F(\bar{0}) = \bar{0}$

, What does the IFT say about F in a neighborhood of the origin ?

Does F have a continuous inverse in neighborhood of the origin ?

---

, We have $g: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x,y,z) \longmapsto g(x,y,z) = z - x^2 - y^2$$

~~$Dg(0) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix}_{(x,y,z)=(0,0,0)} = \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix}_{(x,y,z)=(0,0,0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$~~

$$Dg(0) = [g_x \quad g_y \quad g_z]_{(x,y,z)=(0,0,0)}$$
$$= (-2x \quad -2y \quad 1)_{(x,y,z)=(0,0,0)}$$
$$= (0, 0, 1)$$

, ~~$DF = \begin{bmatrix} 3x^2 & 3y & -2x \\ 0 & 3y^2 & -2y \\ 0 & 0 & 1 \end{bmatrix} = 3x^2 \cdot 3y^2 = 9x^2y^2$~~

$$[DF] = \begin{pmatrix} 3x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \\ -2x & -2y & 1 \end{pmatrix}$$

~~So $DF(0) = 0$~~

$$\det(DF(0,0,0)) = (3x^2 \cdot 3y^2)_{x,y=0,0} = 0.$$

So the IFT says nothing about F in a neighborhood of the origin .

a continuous inverse of F

, Does F have a continuous inverse in a neighborhood of the origin ?

onsider ~~2 points in a neighborhood of (0,0,0) : $(\varepsilon, y_0, z_0) \neq (-\varepsilon, y_0, z_0)$~~

~~then we have $F(\varepsilon, y_0, z_0) = 1$~~

$$\begin{cases} x^3 = w \\ y^3 = v \\ z - x^2 - y^2 = u \end{cases} \Rightarrow \begin{cases} x = w^{1/3} \\ y = v^{1/3} \\ z = u + x^2 + y^2 = u + w^{2/3} + v^{2/3} \end{cases}$$

So we have $F^{-1}(w, v, u) = (w^{1/3}, v^{1/3}, u + w^{2/3} + v^{2/3})$ is a continuous inverse of F

(thus, also a continuous inverse of F in a neighborhood of origin □.

Aug 2012 ✱✱

P1 Let X be a metric space
$\{A_n\}$, $n=1,2,3,\dots$ are nonempty compact subset of X such that $A_{n+2} \subseteq A_n \cup A_{n+1}$ $\forall n \geq 1$
Prove that $\exists$ a point $x \in X$ such that $x \in A_n$ for infinitely many values of $n$

Theorem 236:
$\{K_\alpha\}$ is a collection of compact subset
the intersection of every finite subcollection $\{K_\alpha\}$ is nonempty $\Big\} \Rightarrow \bigcap K_\alpha$ is nonempty

Corollary
$\{K_\alpha\}$ is collection of nonempty, nested compact subset ~~set~~ $\Rightarrow \bigcap_\alpha K_\alpha$ is nonempty

(In here, we can't use theorem 236 directly because we need
intersection of every finite subcollection $\{K_\alpha\}$ is nonempty )
but $A_1 \cap A_2$ may be $= \phi$. )

✱ We have
$A_1 \cup A_2 \supset A_3 \Rightarrow A_1 \cup A_2 \supset A_2 \cup A_3$.
$A_2 \cup A_3 \supset A_4 \Rightarrow A_2 \cup A_3 \supset A_3 \cup A_4$.
$\vdots$
$A_n \cup A_{n+1} \supset A_{n+2} \Rightarrow A_n \cup A_{n+1} \supset A_{n+1} \cup A_{n+2}$

- Put $K_1 = A_1 \cup A_2$ then we have $\{K_n\}$ is a sequence of compact subsets.
$K_2 = A_2 \cup A_3$ (finite union of compact is compact).
$\vdots$ and $\{K_n\} \neq \phi$ and nested sequence.
$K_n = A_n \cup A_{n+1}$.

$\Rightarrow$ Then we have $\bigcap_{n=1}^{\infty} K_n$ is non empty.

This means. $\exists x \in \bigcap_{n=1}^{\infty} K_n \Rightarrow \exists K_{n_0}$, $x \in K_{n_0}$, and because $K_n \supset K_{n+1} \supset K_n$
$\Rightarrow x \in K_n, \forall n \geq n_0$.

(not clear): and so, because of the property, $A_n \cup A_{n+1} \supset A_{n+2}$
$A_{n+2} \cup A_{n+3} \supset A_{n+4} \dots$

$\Rightarrow \exists n_1$ such that $x \in A_n$, $\forall n \geq n_1 \Rightarrow \square$.

ig 2012 >

12 > Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$
$g : \mathbb{R} \longrightarrow (0, +\infty)$ } are continuous functions.

For $x \in \mathbb{R}$, define $h(x) = \sup\limits_{0<t<g(x)} f(t)$

a) Prove that $h : \mathbb{R} \longrightarrow \mathbb{R}$ continuous.

b) Give an example in which $f$ is uniformly on $\mathbb{R}$ but $h$ is not

* NTR.

Put $F(x) = \sup\limits_{0<t<x} f(t)$ Then we have $h(x) = F(g(x))$

This is a very useful trick to use.

~~o we~~ have $g(x)$ is continuous.

⇒ It suffices to show that $F$ is a continuous function.

Note that $F$ is an increasing function, then it suffices to prove that
$F(x_0^-) = F(x) = F(x_0^+)$.

Here, we understand that, we need to prove $F$ continuous for all $x_0 \in \mathbb{R}$.
because $f$ is continuous ⇒ $f$ continuous at $x_0$.
⇒ $\forall \varepsilon > 0, \exists \delta_0$, $\forall y \in \mathbb{R}, |y - x_0| < \delta_0$, $|f(y) - f(x_0)| < \varepsilon$

Then $F(y) - F(x_0) = \sup\limits_{0<t<y} f(t) - \sup\limits_{0<t<x_0} f(t) \underset{|y-x_0|<\delta_0}{\overline{\overline{\quad\quad}}} < \varepsilon$.

> Give an example in which $f$ is uniformly on $\mathbb{R}$ but $h$ is not

et $f(t) = t$, then we have $f$ is uniformly continuous.
$g(x) = x^2$

Aug 2012 → P3 →

Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function

$$f'(x+1) = f'(x) \text{ for all } x \in \mathbb{R}.$$

Prove that the limit $\lim\limits_{x\to\infty} \frac{f(x)}{x}$ exists and is finite.

We have $\lim\limits_{x\to\infty} \frac{f(x)}{x} \underset{\text{L'Hospital}}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x\to\infty} \frac{f'(x)}{1}$

So it suffices to prove that $\exists \underset{x\to\infty}{\text{limit}}\, f'(x)$ and the limit is finite.

ug 2012, P4

el $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \dots$ be $C^1$ functions
t $\forall n$, $|f_n'(x)| \le \frac{1}{\sqrt{x}}$ $(0 < x \le 1)$
$\int_0^1 f_n(x)\,dx = 0$

Prove that the sequence has a subsequence that converges uniformly on $[0,1]$.

), We have $[0,1]$ is compact.

)) By assumption $\Rightarrow f_n \in C([0,1])$, $\forall n$.

). We now want $f_n$ pointwise bounded.
This is true because $\int_0^1 f_n\,dx = 0$.

) We now want $\{f_n\}$ equicontinuous.
We NTP $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon, \forall n$

really good trick to prove $\{f_n\}$ equicontinuous when we have $f_n'(x) \le g(x)$. is

ong $|f_n(x) - f_n(y)| = \int_x^y f_n'(t)\,dt \le \int_x^y |g(t)|\,dt \le \dots < \varepsilon$ when ....

le have
$$|f_n(x) - f_n(y)| \le \int_x^y |f_n'(t)|\,dt \le \int_x^y \left|\frac{1}{\sqrt{t}}\right| dt = \frac{1}{2}\sqrt{t}\Big|_x^y = \frac{2}{2}(\sqrt{y} - \sqrt{x})$$

ote that $h(x) = \sqrt{x}$ is a continuous function on $[0,1] \Rightarrow$ uniformly cont
$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta, |\sqrt{y} - \sqrt{x}| < \varepsilon$

iuse $\{f_n\}$ equicontinuous.

hen from (1)+(2)+(3)+(4) + apply ~~Asa~~ Arzela theorem, we have

$\exists$ ..... $\square$.

Aug 2012 7

P5 7 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^1$ function

det $f'(\vec{x}) > 0$, $\forall x \in \mathbb{R}^2$

Assume ~~f maps~~ $f^{-1}(K)$ is compact for all $K \subseteq \mathbb{R}^2$ is compact } Prove that $f(R^2) = \mathbb{R}^2$

---

See May 2016 : Let $X, Y$ metric space

$f : X \to Y$ continuous function such that

$\forall K$ compact $\subset Y$, $f^{-1}(K)$ compact $\subset X$ } Prove that for every $F \subset X$ closed then $f(F$ ... closed

* With this question, we are required to prove that $f(R) = R$, there are 2 ways to solve.

Way 1: Prove that $f$ is bijective.

Way 2: Prove that $f(R^2) \neq 0$, closed + open in $\mathbb{R}^2$ + fact $R^2$ is connected } $\Rightarrow f(R^2) = \mathbb{R}^2$.

* We have because det $f'(x) > 0$, $\forall x \in \mathbb{R}^2$, Then by inverse function theorem, we have $f(\mathbb{R}^2)$ is open in $\mathbb{R}^2$ (1)

* We now need to prove that $f(R^2)$ is closed in $\mathbb{R}^2$.

We will redo the prove for May 2016 7.

If $f^{-1}(K)$ is compact for all $K \subseteq \mathbb{R}^2$ compact

$f$ is continuous, differentiable } then $\forall E$ closed in $\mathbb{R}^2$, $f(E)$ is closed in $\mathbb{R}$

---

Now let $y \in \mathbb{R}^2$ such that $\exists (y_n) \subset f(E)$, $y_n \to y$. NTP $y \in f(E)$

NTP $\exists x_0 \in E$, $y = f(x_0)$.

We have $\{y_n\} \cup \{y\}$ is compact.

Then because of the assumption, $f^{-1}(\{y_n\} \cup \{y\})$ is compact.

this means $\{x_n\} \cup f^{-1}(y)$ is compact.

- $\{x_n\}$: sequence in a compact set $\Rightarrow \exists x_{n_\ell} \longrightarrow x_0 \in E$

because $f$ continuous, $f(x_{n_\ell}) \to f(x_0)$

because $f(x_n) \longrightarrow y$. } $\Rightarrow y = f(x_0)$

So Then we have $\forall E$ closed in $R^2$, $f(E)$ is closed in $\mathbb{R}^2$.

$R^2$ closed $\Rightarrow f(R^2)$ closed in $\mathbb{R}^2$ (2)

we have $f(R^2)$ open + closed in $\mathbb{R}^2$.

ug 2012, P 6

t $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a $C^1$ function; $f'>0$, $\forall x\in\mathbb{R}$

$f$ takes $[0,1]$ onto itself

Prove that there is a sequence of polynomials $P_n:[0,1]\longrightarrow[0,1]$ such that

$P_n \rightrightarrows f$ on $[0,1]$

each $P_n$ is strictly increasing function for $[0,1]$.

NT R.

Weird.

Note that some time we apply SW for $f'$, not just $f$

in this case $f$ is a $C^1$ function $\Longrightarrow$ $f'$ cont

Note that $f'$ is a continuous theorem,

then by S.W theorem, $\exists P_n$, $P_n \rightrightarrows f'$

So because $f'>0$, $\forall x\in\mathbb{R}$, $P_n>0$.

We create $Q_n(x):=\int_0^x P_n(t)\,dt$

$f(x)=\int_0^x f'(t)\,dt$

$\Rightarrow |Q(x)-f(x)| \leq x\,|P(t)-f'(t)| \leq x\cdot\varepsilon \leq \varepsilon$ on $[0,1]$

$\Rightarrow$ ~~$1-\frac{1}{n}\leq Q(1)\leq 1+1/n$~~ This means

$|Q(1)-1|<\varepsilon$.

$\Rightarrow Q_n(x)=\dfrac{Q_n(x)}{Q_1(x)}$ $\Rightarrow$ $Q_n(1)$

**MAT 601-2** **Prelim Exam** **January 2013**

**Problem 1.** Let $f_n$ be non-negative differentiable functions on [0,1] such that for every $x$ the sequence $f_n'(x)$ is non-increasing, and such that $f_n(0)$ is also non-increasing. Prove that the $f_n$ converge point-wise on [0,1].

**Problem 2.** Let $(\mathcal{M}, d)$ be a non-empty compact metric space and $f : M \to M$ a continuous mapping such that $d(f^{(n)}(x), f^{(n)}(y)) \to 0$ uniformly in $x, y$, where $f^{(n)}(x)$ denotes n-fold composition of $f$ with itself (for example, $f^{(3)}(x) = f(f(f(x)))$.) Prove that $f$ has a fixed point $x$, i.e., there exists an $x \in \mathcal{M}$ such that $f(x) = x$.

**Problem 3.** Let $f$ be a continuous function such that $\lim_{x\to\infty} f(x) = c \in \mathbb{R}$. Prove that for any $\alpha > 0$ we have

Jan 2015 ⑤

$$\lim_{N\to\infty} \frac{\alpha+1}{N^{\alpha+1}} \int_0^N x^\alpha f(x)\,dx = c.$$

**Problem 4.** The Dirichlet function $D(x)$ on $[0,1]$ is the function equal to 1 when $x$ is rational and 0 when $x$ is irrational. Show that $D(x) \notin \mathcal{R}(\alpha)$ for any monotonically increasing non-constant function $\alpha$. (Recall that $\mathcal{R}(\alpha)$ is the space of functions on $[0,1]$ integrable with respect to $\alpha$ in the Riemann sense.)

**Problem 5.** Let $f$ be a differentiable function on $\mathbb{R}$ and its derivative $f'$ is continuous there. Show that the functions

$$f_n(x) = n\left(f(x+\frac{1}{n}) - f(x)\right)$$

converge uniformly to $f'$ on any interval $[a,b]$, $-\infty < a < b < \infty$.

**Problem 6.** Is the function $f(x,y) = (x^3+y^3)^{1/3}$ differentiable at $(0,0)$?

✱ Jan2013 / 1 $f_n(x) \geq 0, \forall x, \forall n$

Let $\{f_n\}$: non negative differentiable functions on $[0,1]$

$\forall x \in [0,1]$, $f_n'(x) \geq f_{n+1}'(x)$

$f_n(0) \geq f_{n+1}(0)$

Prove that $\{f_n\}$ converges pointwise on $[0,1]$.

Note: It's a really good trick to remember:

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)\,dt.$$

$$\& \; f(x) - f(y) = \int_x^{y} {}^{0} f'(t)\,dt.$$

* At $x = 0$:

We have $\{f_n(0)\}$ is a decreasing sequence, bounded by 0 $\Rightarrow \{f_n(0)\}$ converges

* At $x > 0$, ($x \in (0,1]$):

We have

$$\left.\begin{array}{l} f_n(x) - f_n(0) = \int_0^x f_n'(t)\,dt \\ f_{n+1}(x) - f_{n+1}(0) = \int_0^x f_{n+1}'(t)\,dt \\ \text{we also have assumption } f_n'(t) \geq f_{n+1}'(t), \forall t \end{array}\right\} \Rightarrow f_n(x) - f_n(0) \geq f_{n+1}(x) - f_{n+1}(0)$$

$$\Rightarrow f_n(x) - f_{n+1}(x) \geq \underbrace{f_n(0) - f_{n+1}(0)}_{\geq 0}$$

$$\Rightarrow f_n(x) - f_{n+1}(x) \geq 0$$

This means for each $x \in X$, $\left.\begin{array}{r}\{f_n(x)\} \text{ is a decreasing sequence} \\ \text{bounded by } f_n(x) \geq 0\end{array}\right\} \Rightarrow \{f_n(x)\}$ converg

In conclusion, $\{f_n\}$ converges pointwise on $[0,1]$

---

* Something learned from this problem.

- When proving $\{f_n(x)\}$ converges pointwise in $[a,b]$
  Then for each fixed $x$, we can consider $\{s_n\} = \{f_n(x)\}$, then we need to prove that $\{s_n\}$ converges by using $\left[\begin{array}{l}\text{def} \\ \text{cauchy criterion} \\ \text{monotonic + bounded} \ldots\end{array}\right.$
- When see $f(a)$, $f(b)$ and $f'$ → think about:
  $\left[\begin{array}{l}\text{MTV } f(a) - f(b) = f'(\xi)\,[a-b] \\ \text{MVT (integration form) } \& \; f(b) - f(a) = \int_a^b f'(x)\,dx.\end{array}\right.$

n 2013, P 2

n, d) be a non empty compact metric space

$f: M \to M$ is a continuous mapping s.t $d(f^{(n)}(x), f^{(n)}(y)) \xrightarrow{\text{in } x,y} 0$

(when $f^{(n)}$ denotes the n fold composition of f with itself $f^{(3)}(x) = f(f(f(x)))$)

Prove that f has a fixed point i.e, there exists an $x \in M$ s.t $f(x) = x$.

ssume that we have $f(x) \neq x$, $\forall x \in M$.

this means $d(f(x), x) > 0$, $\forall x \in M$.

Then put $a := \inf_{x \in M} d(f(x), x)$

Now consider $y = f(x)$.

Then we have $d(f^n(x), f^n(y)) \Longrightarrow 0$

This means $d(f^n(x), f^{n+1}(x)) \Longrightarrow 0$

Put $z = f^n(x)$, we have $d(z, f(z)) \longrightarrow 0$ $\Big\}$ $\Rightarrow$ contradiction

but we have $d(z, f(z)) \geq a$

So $\exists x \in M$ such that $f(x) = x$ $\square$ ☺

Jan 2013 ③

$f$ be a continuous function s.t. $\lim_{x \to +\infty} f(x) = c \in \mathbb{R}$.

Prove that for any $\alpha > 0$, we have $\lim_{N \to \infty} \frac{\alpha+1}{N^{\alpha+1}} \int_0^N x^\alpha f(x)\,dx = c$

* Notice that with a problem having a $\int$ and a constant, we want to use $c = \frac{c}{b-a}\int_a^b dx$ as we want to have $c = c\underbrace{\left(\int_0^N \cdots\right)}_{=1}$ so that we can compute ....

* We want to prove that

$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}, \forall N \geq N_0$, $\left| \frac{\alpha+1}{N^{\alpha+1}} \int_0^N x^\alpha f(x)\,dx - c \right| < \epsilon$

- We notice that

we don't know $f(x)$ ⇒ just try with $\int x^\alpha$

$$\int_0^N x^\alpha \cancel{f(x)}\,dx = \frac{1}{(\alpha+1)} \int_0^N (\alpha+1) x^\alpha \cancel{f(x)}\,dx = \frac{1}{\alpha+1}\int_0^N d\left(x^{\alpha+1}\right) = \frac{1}{\alpha+1} N^{\alpha+1}$$

Then we have $\frac{\alpha+1}{N^{\alpha+1}}\int_0^N x^\alpha\,dx = 1$

* So we have $\left|\frac{\alpha+1}{N^{\alpha+1}}\int_0^N x^\alpha f(x)\,dx - c\right| = \left|\frac{\alpha+1}{N^{\alpha+1}}\int_0^N \left(x^\alpha f(x) - c\,x^\alpha\right) dx\right|$.

$$= \left|\frac{\alpha+1}{N^{\alpha+1}}\int_0^N \left(f(x) - c\right) x^\alpha\,dx\right| \quad (*)$$

* Note that we have $\lim_{x \to \infty} f(x) = c \iff \exists K, \forall x > K, |f(x) - c| < \epsilon$.

$$\text{So } (*) \leq \underbrace{\frac{\alpha+1}{N^{\alpha+1}} \underbrace{\int_0^K |f(x)-c|\, x^\alpha\,dx}_{\text{bounded since K is finite}} + \frac{\alpha+1}{N^{\alpha+1}} \underbrace{\int_K^N \underbrace{|f(x)-c|}_{<\epsilon} x^\alpha\,dx}_{\text{bounded}}}$$

So $\lim_{N \to \infty} = 0$   So we have what we need to prove.

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* Think ! : can use L Hospital ? _ _

12023 /4 $d: [0,1] \longrightarrow \mathbb{R}$

richlet function $d(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \text{ irrational } \mathbb{R}\setminus\mathbb{Q} \end{cases}$ defin of dirichlet function $d(x) = \begin{cases} c, & x \in \mathbb{Q} \\ d, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ $c \neq d$

ow that $d(x) \notin \mathcal{R}(\alpha)$ for any monotonically increas is discontinuous everywhe $d(x) = \begin{cases} \frac{1}{b} & x = \frac{a}{b} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$
on $[0,1]$ non constant $\alpha$.

have $f \in \mathcal{R}(\alpha)$ on $[0,1]$ $\Longleftrightarrow \begin{cases} f \text{ is bounded} \\ \forall \varepsilon > 0, \exists \text{ partition } P, \ U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \end{cases}$.

need to prove $d \notin \mathcal{R}(\alpha) \Rightarrow$ NTP $\exists \varepsilon > 0$, $\forall$ partition $P$, $U(P,d,\alpha) - L(P,d,\alpha) > \varepsilon$.

because $\alpha$ non constant increasing $\Rightarrow \exists \varepsilon > 0$, $\alpha(1) - \alpha(0) > \varepsilon$ =

'e have for all partition $P = \{x_0 = a \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b\}$, then because

and $R/\mathbb{Q}$ dense in $\mathbb{R}$

$$\left.\begin{aligned} M_i &= \sup_{x \in [x_{i-1}, x_i]} f(x) = 1 \\ m_i &= \inf_{x \in [x_{i-1}, x_i]} f(x) = 0 \end{aligned}\right\} \Rightarrow \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = \sum_{i=1}^{n} \Delta\alpha_i = \alpha(1) - \alpha(0)$$

incase $\alpha$ is nonconstant and monotonically increasing

$\Rightarrow \alpha(1) - \alpha(0) > \varepsilon$

$$\Rightarrow U(P,d,\alpha) - L(P,d,\alpha) > \varepsilon, \ \forall P.$$

---

other way Using def $f \in \mathcal{R}(\alpha) \Longleftrightarrow \underline{\int} f d\alpha = \overline{\int} f d\alpha$

$d \notin \mathcal{R}(\alpha) \Longleftrightarrow \underline{\int} d\, d\alpha \neq \overline{\int} d\, d\alpha$, $\forall P$.

ve have for all partition $P$, $U(P,d,\alpha) = \sum M_i \Delta\alpha_i = \sum 1 \Delta\alpha_i = \alpha(1) - \alpha(0) \neq 0$

$L(P,d,\alpha) = \sum m_i \Delta\alpha_i = 0$

$$\Rightarrow \left.\begin{aligned} \underline{\int} d\, d\alpha &= \inf_{P} U(P,d,\alpha) = \alpha(1) - \alpha(0) \\ \overline{\int} d\, d\alpha &= \sup L(P,d,\alpha) = 0 \end{aligned}\right\} \Rightarrow \forall P, \ \underline{\int} d\, d\alpha \neq \overline{\int} d\, d\alpha$$

$\rightarrow d \notin \mathcal{R}(\alpha)$.

Jan2013/P5 ✱✱

Let $f$ be a differentiable function on $\mathbb{R}$

$f'$ is continuous on $\mathbb{R}$

Show that the function $f_n(x) = n\left(f(x+\frac{1}{n}) - f(x)\right) \rightrightarrows f'(x)$ on any interval $[a,b]$

$-\infty < a < b < +\infty$

---

We need to prove that $f_n(x) \rightrightarrows f'(x)$

$\Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in [a,b], |f_n(x) - f'(x)| < \varepsilon$

When see $f(b) - f(a)$ and $f'(x)$

$\Rightarrow$ think about MVT

* We have    This is the key step in this solution

$$f_n(x) = n\left[f(x+\tfrac{1}{n}) - f(x)\right] \underset{\substack{\text{MVT} \\ \text{for } f}}{=\!=} n f'(y_n)\left[x + \tfrac{1}{n} - x\right] = f'(y_n) \text{ for some } y_n \in \left(x, x+\tfrac{1}{n}\right) \quad (1)$$

* We have $f'$ continuous on $\mathbb{R}$ $\Rightarrow$ uniformly continuous on $[a,b]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$, for all $x,y \in [a,b]$, $|x-y| < \delta$, then $|f'(x) - f'(y)| < \varepsilon$    (2)

Then we choose $n_0$ such that $\frac{1}{n_0} < \delta$, then $\forall n \geq n_0$, we have $\frac{1}{n} \leq \frac{1}{n_0} < \delta$

Then by (1), $y_n \in (x, x+\frac{1}{n}) \subset (x, x+\delta) \underset{(2)}{\Rightarrow} |f'(x) - \underset{\substack{\| \text{ by } (1) \\ f_n(x)}}{f'(y_n)}| < \varepsilon$

$\Rightarrow |f'(x) - f_n(x)| < \varepsilon$

This means $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in [a,b], |f'(x) - f_n(x)| < \varepsilon$.

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* We can $\boxed{\text{not}}$ use def (without using MVT like above) because

- $f_n(x) \xrightarrow[\text{pointw}]{} f'(x)$, mean $\forall x \in [a,b]$

$\forall \varepsilon > 0, \exists n_{\varepsilon,x}, \forall n \geq n_{\varepsilon,x}, |f_n(x) - f'(x)| < \varepsilon$

- $f_n(x)$ continuous ? not sure

NTP

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in [a,b]$,

$|f_n(x) - f'(x)| < \varepsilon$

because $|f_n(x) - f_n(y)| = n\left[f(x+\frac{1}{n}) - f(x) - f(y+\frac{1}{n}) + f(y)\right] \overset{?}{<} \varepsilon$

$\uparrow$ but in here we consider when $n$ large n2013, PG7 Aug 1997/6/ Aug 2008/1/

Is the function $f(x,y) = (x^3 + y^3)^{1/3}$ differentiable at $(0,0)$.

We first compute the partial derivative of $f$ at $(0,0)$.

$$_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{(h^3)^{1/3}}{h} = 1$$

ilarly $f_y(0,0) = 1$.

Then we consider

$$\frac{|f(h_1,h_2) - f(0,0) - f_x(0,0)h_1 - f_y(0,0)h_2|}{\sqrt{h_1^2+h_2^2}} = \frac{(h_1^3+h_2^3)^{1/3} - (h_1+h_2)}{\sqrt{h_1^2+h_2^2}} = (*)$$

ole $h_1 = h_2$, then

$$*) = \frac{|(2h_1^3)^{1/3} - 2h_1|}{\sqrt{2h_1^2}} = \frac{|\sqrt[3]{2}\,h_1 - 2h_1|}{\sqrt{2}|h_1} = \frac{|(\sqrt[3]{2} - 2)||h_1|}{\sqrt{2}\,|h_1|} = \frac{\sqrt[3]{2} - 2}{\sqrt{2}} \neq 0$$

$\Rightarrow f(x,y)$ is not differentiable at $(0,0)$. ■

---

ote $f$ is differentiable at $\vec{x}_0$ $\quad f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\Leftrightarrow \exists A \in L(\mathbb{R}^n, \mathbb{R}^m), \quad \lim_{\vec{h}\to\vec{0}} \frac{\| f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - A\vec{h} \|}{\|\vec{h}\|} = 0.$$

hen in case $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^1$.

$f$ is differentiable at $(x_{01}, x_{02})$ iff $\lim_{(h_1,h_2)\to(0,0)} \frac{f(x_{01}+h_1, x_{02}+h_2) - f(x_{01}, x_{02}) - D_{x_1} f(x_{01}, x_{02}) h_1 - D_{x_2} f(x_{01}, x_{02})|}{\sqrt{h_1^2+h_2^2}}$

Aug 2013

See MAT601 HW5.3/4

P1 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$

$f$ has three derivatives in an open interval containing the point $a$.

a) Show that $\lim\limits_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a)$

b) Show that $\lim\limits_{h\to 0} \frac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3} = f'''(a)$

HW 5.3/4 Derivative and limit

$f: \mathbb{R} \to \mathbb{R}$ is a function such that $f''(a)$ exists for some $a \in \mathbb{R}$.

Prove that $\lim\limits_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a)$.

Very important: In here, we consider $\lim\limits_{h\to 0}$ ? if we take derivative, we consider $f$ as a function of $h$.

+ We have if $f''(x)$ exists, $f''(x) = \lim\limits_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \underset{h=2h}{=} \lim\limits_{h\to 0} \frac{f'(x+2h) - f'(x)}{2h}$

- $f^{(n)}(x)$ exist $\Longrightarrow \begin{cases} f^{(n-1)}(x) \text{ exists in a neighborhood of } x \Rightarrow f^{(n-2)} \text{ differentiable in a neighb of } x \\ f^{(n-1)} \text{ differentiable at } x \Rightarrow f^{(n-1)} \text{ continuous at } x. \end{cases}$

a) We have

$$\text{LHS} = \lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} \xrightarrow[\text{wrt } h]{\text{L Hospital}} \lim_{h\to 0} \frac{\cancel{2}f'(a+2h) - \cancel{2}f'(a+h)}{\cancel{2}h} = (*)$$

(Note that $f''(a) = \lim\limits_{h\to 0} \frac{f'(a+h) - f'(a)}{h}$ (or =) $= \lim\limits_{h\to 0} \frac{f'(a) - f'(a-h)}{h}$)

Then LHS
Way 1 $(*) = 2\lim \frac{f'(a+2h) - f'(a)}{2h} - \lim\limits_{h\to 0} \frac{f'(a+h) - f'(a)}{h} = 2f''(a) - f''(a) = f''($

Way 2: Note that because $f'''(a)$ exist $\Rightarrow f''(a)$ is differentiable c

$\Rightarrow f''$ continuous at $a$.

Then LHS $\underset{*}{}\xrightarrow[\text{Mean Value T}]{} \lim\limits_{h\to 0} \frac{f''(\xi)(\cancel{a}+2h-\cancel{a}-h)}{\cancel{h}} = \lim\limits_{\substack{h\to 0 \\ \xi \in (a+h, a+2h)}} f''(\xi) = f''(a)$

$f$ has three derivatives in an open interval containing $a$.

Show that $\lim\limits_{h\to 0} \dfrac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3} = f'''(a)$

le that $f'''$ exists at $a$ $\Rightarrow \begin{cases} f'' \text{ exist at neighborhood of } a \\ f'' \text{ differentiable} \Rightarrow \text{continuous at } a \end{cases}$

$\Rightarrow \{$ $f''$ differentiable in a neighborhood of $a$. $\Rightarrow$ actually $\exists f'(x)$
$x \in$ neighborhood of $a$.

have LHS $= \left(\lim \frac{0}{0}\right)$ and notice that $\left| \begin{array}{l} (h^3)' = 3h^2 \neq 0 \text{ if } h \neq 0. \\ f' \text{ exists in a neighborhood of } a. \end{array} \right)$ we have to check this carefully to

n LHS $\underset{\text{wrt } h}{\overset{\text{LHospital}}{=\!=}} \lim\limits_{h\to 0} \dfrac{\cancel{3} f'(a+3h) - 2\cancel{3} f'(a+2h) + \cancel{3} f'(a+h)}{\cancel{3} h^2} =$

$\underset{\text{wrt } h}{\overset{\text{LHospital}}{=\!=}} \lim\limits_{h\to 0} \dfrac{3f''(a+3h) - 4f''(a+2h) + f''(a+h)}{2h}$

form $\frac{0}{0}$ again
$(h^2)' = 2h \neq 0$ for $h \neq 0$
$f''$ exist in a neighborhood of

(Now we def of $f'''(a) = \lim\limits_{k\to 0} \dfrac{f(a+k) - f(a)}{k}$ where $k = 3h, 2h, h$.

$$\lim \frac{3f''(a+3h) - 3f''(a)}{2h} - \frac{4f''(a+2h) - 4f''(a)}{2h} + \frac{f''(a+h) - f''(a)}{2h}$$

$$\lim \frac{\frac{1}{2}\cdot 3\cdot 3\,[f''(a+3h) - f''(a)]}{\frac{1}{\cancel{2}}\,2\;3h} - 4\,\frac{f''(a+2h) - f''(a)}{2h} + \frac{1}{2}\,\frac{f''(a+h) - f''(a)}{h} =$$

$$\frac{9}{2} f'''(a) - 4f'''(a) + \frac{1}{2} f'''(a) = f'''(a) \quad \square \ .$$

Aug 2013

NTR

Q7 Let the sequence $\{x_n\}$ given by

$$x_n = \prod_{l=1}^{n}\left(1-\frac{1}{2^l}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^n}\right)$$

Prove that the sequence $x_n$ converges and that the limit is not 0.

* Prove that the sequence $x_n$ converges.

⊕ • $x_1 = \left(1-\frac{1}{2}\right)$

• $x_2 = \left(1-\frac{1}{2}\right)\underbrace{\left(1-\frac{1}{4}\right)}_{=3/4<1} < x_1$ $\Rightarrow x_{n+1} = x_n\underbrace{\left(1-\frac{1}{2^{n+1}}\right)}_{<1} < x_n$

$\Rightarrow \{x_n\}$ is a decreasing sequence.
⊕ $x_n > 0, \forall n$ $\Bigg\}\Rightarrow \{x_n\}$ converges.

* Prove that the limit is not 0

• We have $x = e^{\ln x}$ so we want to use this to solve the problem by using.

$$\boxed{\lim_{n\to\infty} x_n = e^{\lim_{n\to\infty}\ln x_n}}$$

*Consider $\ln x_n = \ln\prod_{l=1}^{n}\left(1-\frac{1}{2^l}\right) = \sum_{l=1}^{n}\ln\left(1-\frac{1}{2^l}\right) = s_n$, this means $s_n$ is partial of $\sum_{l=1}^{\infty}\ln\left(1-\frac{1}{2^l}\right)$, and so $s_n$ converges $\Leftrightarrow$ the series $\sum_{l=1}^{\infty}\ln\left(1-\frac{1}{2^l}\right)$ converges.

• We have $\lim_{x\to 0}\frac{\ln(1+x)}{x} \overset{\text{LHopital}}{=} \lim_{x\to 0}\frac{\cancel{\ln(1}\ \frac{1}{1+x}}{1} = 1$

So we have $\lim_{l\to\infty}\frac{\ln\left(1-\frac{1}{2^l}\right)}{-\frac{1}{2^l}} = 1 \neq +\infty$ and $1>0$

So by limit comparison test, $\sum_{l=1}^{\infty} l\left(1-\frac{1}{2^l}\right)$ converges since $\sum_{l=1}^{n}\left(\frac{-1}{2^l}\right)$ converges.

• Assume $\ln x_n = \sum_{l=1}^{\infty}\ln\left(1-\frac{1}{2^l}\right)$ converges to some number, say c

Then $\lim_{n\to\infty} x_n = e^c > 0$ □.

Note that we want to prove the limit is not 0 but in here, we just need to prove that $\lim \ln(g(x_n))$ converges to some $c$ and we $e^c > 0, \forall c$.

ug2013 $(f:\mathbb{R} \longrightarrow \mathbb{R})$

3] Let $f$ be a real valued function on $\mathbb{R}$ that satisfies $A\{x \mid |f(x)| > \varepsilon\}$ is compact
r all $\varepsilon$

Prove or give a counterexample to the statement $f$ has limit at $|x| \to \infty$

$_\varepsilon = \{x, |f(x)| > \varepsilon\}$ is compact

$A_\varepsilon$ is closed + bounded, $\forall \varepsilon$

s means $\forall \varepsilon > 0, \exists r > 0, A_\varepsilon \subseteq N_r(0)$ because $\mathbb{R} = \bigcup_{r=1}^{\infty} N_r(0)$

(note that $f:\mathbb{R} \longrightarrow \mathbb{R}$)

s means, $\forall \varepsilon > 0, \exists r > 0, \forall x$ s.t $|f(x)| > \varepsilon$, then $|x| < r$.

his means, $\forall \varepsilon > 0, \exists r > 0, \forall x$ s.t $|x| > r$ then $|f(x)| < \varepsilon$

This is the definition of $\lim_{x \to \infty} |f(x)| = 0$, thus, f has limit as $|x| \to \infty$ □

Aug 2013 : P 4. (Almost same) See Jan ~~2005~~ P5 ✱ (≠ ways) Jan 2013 P 3.

Let $f:[0,1]\longrightarrow\mathbb{R}$ be a continuous function

$\alpha_n(x)=x^n,\ n=1,2,\ldots$ $= \lim_{n\to\infty} n\int_0^1 f\,x^{n-1}\,dx$.

Prove that the limit $\lim_{n\to\infty}\int_0^1 f\,d\alpha_n$ exists and determined its value.

---

* We have $f:[0,1]\longrightarrow\mathbb{R}$ continuous ([0,1] compact) $\Big\}\rightarrow P_\ell(x)\rightrightarrows f(x)$ (1)

The idea of this problem is because $P_\ell(x)\rightrightarrows f(x)$ so we want to consider $\int P_\ell(x)\,n$

* ~~Now we compute~~

chỉ số ≠

$$n\int_0^1 P_\ell(x)\,x^{n-1}\,dx = n\int_0^1 \left(a_\ell x^\ell + a_{\ell-1}x^{\ell-1}+\cdots+a_1x+a_0\right)x^{n-1}\,dx \quad \left(\int_0^1\left(\sum_{i=1}^{\ell}C_ix^{i}\right)x\right)$$

Note that $\int x^n\,dx=\frac{1}{n+1}$

note that we consider still $P_\ell$ and $nx^{n-1}$.

$$= \int_0^1\left(a_\ell x^{\ell+n-1}+a_{\ell-1}x^{(\ell-1)+(n-1)}+\cdots+a_1x^n+a_0x^{n-1}\right)dx.$$

$$= a_\ell\frac{n}{\ell+n}x^{\ell+n}\Big|_0^1 + a_{\ell-1}\frac{n}{\ell+n-1}x^{\ell+n-1}\Big|_0^1+\cdots+a_1\frac{n\,x^{n+1}}{n+1}\Big|_0^1+a_0\frac{n}{n}x^n\Big|_0^1$$

$$= a_\ell\frac{n}{n+\ell}+a_{\ell-1}\frac{n}{n+\ell-1}+\cdots+a_1\frac{n}{n+1}+a_0\frac{n}{n}.$$

Then $\boxed{\lim_{n\to\infty} n\int_0^1 P_\ell(x)\,x^{n-1}\,dx} = a_\ell+a_{\ell-1}+\cdots+a_1+a_0=\boxed{P_\ell(1)}.$ (✱)

This means $\forall\varepsilon>0,\ \exists n_0\in\mathbb{N},\ \forall n\geq n_0,\ \forall \ell\in\mathbb{N},\ \left|n\int_0^1 P_\ell(x)x^{n-1}dx - P_\ell(1)\right|<\varepsilon$ (I

* $(1)\left(\Leftrightarrow P_\ell\rightrightarrows f\right)\Leftrightarrow\forall\varepsilon>0,\ \exists \ell_0\in\mathbb{N},\ \forall\ell\geq\ell_0,\ \forall x\in[0,1],\ |P_\ell(x)-f(x)|<\varepsilon$

So we have $\forall\ell\geq\ell_0$,

$$\left|n\int_0^1 f\,x^{n-1}dx - n\int_0^1 P_\ell(x)\,x^{n-1}dx\right| \leq \left|n\int_0^1|f-P_\ell(x)|\,|x^{n-1}|\,dx\right| < \varepsilon\left(\int_0^1 n\,x^{n-1}dx\right)=\varepsilon \ (I$$

- And also from $P_\ell\rightrightarrows f$ in $[0,1]\Rightarrow P_\ell(1)\xrightarrow[\ell\to\infty]{\text{pt}} f(1)$

$\Leftrightarrow\forall\varepsilon>0,\ \exists\ell_1\in\mathbb{N},\ \forall\ell\geq\ell_1,\ |P_\ell(1)-f(1)|<\varepsilon$ (III)

Then choose $N=\max\{n_0,\ell_0,\ell_1\}$, we have

$$\left|n\int_0^1 f\,x^{n-1}dx - f(1)\right| \leq \underbrace{\left|n\int f\,x^{n-1}dx - n\int_0^1 P_n(x)\,x^{n-1}\,dx\right|}_{<\varepsilon\ \text{by (II)}} + \underbrace{\left|n\int P_n(x)\,x^{n-1}dx - P_n(1)\right|}_{<\varepsilon\ \text{(by I)}} + \underbrace{\left|P_n(1)-\right.}_{<\varepsilon}$$

$\leq 3\varepsilon$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f\,d\alpha_n = \lim_{n\to\infty} n\int_0^1 f\,x^{n-1}\,dx = f(1)\quad\square.$$

ug 2013 :  $\in$ Stone Weierstrass theorem * *

5) Let $f : [1, +\infty) \longrightarrow \mathbb{R}$ be a continuous function st
$\lim_{x\to\infty} f(x) = 0$
Prove that $\forall \varepsilon > 0, \exists n$ and $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that $\left| f(x) - \sum_{k=1}^{n} c_k e^{-kx} \right| < \varepsilon$ for all $x \in [1,$

ut $P_n(x) := \sum_{k=1}^{n} c_k x^k$

hen we have $P_n(e^{-x}) = \sum_{k=1}^{n} c_k (e^{-x})^k$

ut $u = e^{-x}$, we want to prove that $\forall \varepsilon > 0$ $\exists n$ and $c_1, \dots, c_n$ s.t
$|f(x) - P_n(u)| < \varepsilon$

Because $u = e^{-x}$, we have $u = \frac{1}{e^x} \Rightarrow e^x = \frac{1}{u} \Rightarrow x = \ln(u^{-1}) = -\ln u$.

when $x = 1$, $u = e^{-1} = 1/e$

$x \to \infty$, then $u \longrightarrow 0$

ow consider ~~$|f(x) - P_n(u)| = |f(-\ln u) - P_n(u)|$~~

Put $g(u) = f(-\ln u)$ then we have $g : [0, 1/e] \longrightarrow \mathbb{R}$ and $g$ is continuous.

then by Stone Weierstrass theorem, $\exists P_n(u) \Longrightarrow g(u)$

$$|g(u) - P_n(u)| < \varepsilon$$

$$\Leftrightarrow |f(x) - P_n(e^{-x})| < \varepsilon$$

$$\Leftrightarrow \left| f(x) - \sum_{k=1}^{n} c_k e^{-kx} \right| < \varepsilon$$

Aug 2013) PG 7.

Consider the mapping $\vec{f}:(f_1, f_2, f_3)$ of $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x_1, x_2, x_3) \longmapsto \vec{f}(x_1, x_2, x_3) = (f_1, f_2, f_3)$

$f_1(x_1, x_2, x_3) = x_1$

$f_2(x_1, x_2, x_3) = x_1^2 + x_2$ .

$f_3(x_1, x_2, x_3) = x_1 + x_2^2 + x_3^3$

a) Is $f$ continuously differentiable. Why

b) Find all point at which $f$ satisfies the assumptions of the IFT

c) Is $f$ injective ?

a) The function $f$ has Jacobian :

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2x_1 & 1 & 0 \\ 1 & 2x_2 & 3x_3^2 \end{pmatrix} \Rightarrow \det(D_f) = 3x_3^2$$

$f$ is continuously differentiable because all $\frac{f_i}{\partial x_j}$ $i = \overline{1,3}, j = \overline{1,3}$ exist and continuous .

b) ~~$f$ satis~~

we have from a) $f$ is $C^1$ function .

$\Rightarrow$ $f$ satisfies the assumption of IFT when $\det(D_f) \neq 0$, which means when $x_3 \neq 0$ .

c) Is $f$ an injection ?

$f$ is an injection, because we can have unique $(x_1, x_2, x_3)$ from each $(f_1, f_2, f_3)$ .

$$\begin{aligned} f_1 &= x_1 \\ f_2 &= x_1^2 + x_2 \\ f_3 &= x_1 + x_2^2 + x_3^3 \end{aligned} \quad \Rightarrow \quad \begin{cases} x_1 = f_1 \\ x_2 = f_2 - x_1^2 = f_2 - f_1^2 \\ x_3 = \sqrt{f_3 - x_1 - x_2^2} \quad = \ldots \end{cases}$$

# Analysis Preliminary Exam, January 2014

1. Show that the following limit exists and find it:

$$\lim_{n\to+\infty}\left(\frac{(3n)!}{(n!)^3}\right)^{1/n}.$$

Not yet

2. Let $f : X \longrightarrow Y$ be a continuous function, where $X$, $Y$ are metric spaces and $X$ is compact. Assume that $y_0 \in Y$ is a point which has a unique preimage $x_0 \in X$, i.e. $f^{-1}(y_0) = \{x_0\}$. Prove that for every open neighborhood $U$ of $x_0$ in $X$ there exists an open neighborhood $V$ of $y_0$ in $Y$ such that $f^{-1}(V) \subset U$. Give an example to show that this conclusion is false if $X$ is not compact.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $\lim_{x\to+\infty} f'(x) = 1$, and let $a \in \mathbb{R}$. Prove that the following limit exists and find it:

$$\lim_{x\to+\infty}\frac{e^{f(x+a)}}{e^{f(x)}}.$$

4. For each $s \in [0,1]$ there is a function $f_s(x)$ defined for $x \in [a,b]$ and $f_s \in \mathcal{R}(\alpha)$ on $[a,b]$, where $\alpha$ is a monotonically increasing function on $[a,b]$. Suppose that

$$f_{s_j} \to f_{\frac{1}{2}} \text{ uniformly on } [a,b] \text{ as } j \to \infty$$

for any sequence $\{s_j\}_{j=1}^{\infty}$ from $[0,1]$ that converges to $\frac{1}{2}$. Show that

$$\lim_{s\to\frac{1}{2}}\int_a^b f_s(x)\,d\alpha(x) = \int_a^b f_{\frac{1}{2}}(x)\,d\alpha(x).$$

5. Let $f$ be a real valued continuous function on $[0,1]$, with $\|f\| \leq 1$ (sup norm less than or equal 1) and $f(0) = 0$. Show that the sequence of powers of $f$, $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if $\|f\| < 1$.

6. Let $\mathbf{f} = (f_1, f_2)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ be given by $f_1(x,y) = 2x + |x| - |x+1|$, $f_2(x,y) = (y-1)^3$.
(a) At which points $(x,y)$ does the inverse function theorem provide the existence of a $C^1$ inverse in a neighborhood? Check the conditions of the theorem!
(b) At which points is $\mathbf{f}$ not invertible?

checked.

# AUGUST 2013 PRELIMINARY EXAMINATION IN ANALYSIS

1. Let $f$ be a real valued function on $\mathbb{R}$ and suppose that $f$ has three derivatives in an open interval containing the the point $a$. Show

$$\lim_{h\to 0}\frac{f(a+2h)-2f(a+h)+f(a)}{h^2}=f''(a)$$

and

$$\lim_{h\to 0}\frac{f(a+3h)-3f(a+2h)+3f(a+h)-f(a)}{h^3}=f'''(a)$$

2. Let the sequence $x_n$ be given by

$$x_n=\prod_{k=1}^{n}\left(1-\frac{1}{2^k}\right)=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^n}\right)$$

Prove that the sequence $x_n$ converges and that the limit is **not** 0.

3. Let $f$ be a real valued function on $\mathbb{R}$ that satisfies $\{x \mid |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$. Prove or provide a counterexample to the statement: $f$ has a limit as $|x| \to \infty$.

4. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. For $n = 1,2,\dots$ let $\alpha_n(x) = x^n$. Prove that the limit

$$\lim_{n\to+\infty}\int_0^1 f\,d\alpha_n$$

exists and determine its value.

5. Let $f\colon [1,\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x) = 0$. Prove that for every $\epsilon > 0$ there exists an integer $n$ and real numbers $c_0,\dots,c_n$ such that

$$\left|f(x)-\sum_{k=0}^{n}c_k e^{-kx}\right|<\epsilon \quad \text{for all } x\in[1,\infty)$$

6. Consider the mapping $\mathbf{f} = (f_1, f_2, f_3)$ of $\mathbb{R}^3$ into $\mathbb{R}^3$ given by

$$f_1(x_1,x_2,x_3)=x_1$$
$$f_2(x_1,x_2,x_3)=x_1^2+x_2$$
$$f_3(x_1,x_2,x_3)=x_1+x_2^2+x_3^3$$

(a) Is f continuously differentiable? Why or why not?

(b) Find all point at which f satisfies the assumptions of the Inverse Function Theorem.

(c) Is f injective?

Jan 2014

1) Show that the following limit exists and find it.

$$\lim \left[\frac{(3n)!}{(n!)^3}\right]^{1/n}$$ — 1/n here → have to use the below theorem

Use a theorem in Rudin's book (and also one of prelim problems.)

$$\liminf \frac{c_{n+1}}{c_n} \le \liminf \sqrt[n]{c_n} \le \limsup \sqrt[n]{c_n} \le \limsup \frac{c_{n+1}}{c_n}).$$

* We have

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim \frac{(3(n+1))!\ (n!)^3}{((n+1)!)^3\ (3n)!} = \lim \frac{(3n+3)!\,(n!)^3}{(n!)^3(n+1)^3\,(3n)!} = \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{27n^3+\cdots}{n^3+3n^2+3n+1} = 27.$$

then by above theorem

$$27 \le \liminf \sqrt[n]{c_n} = \limsup \sqrt[n]{c_n} \le 27 \Rightarrow \lim\left[\frac{(3n)!}{(n!)^3}\right]^{1/n} = 27 \qquad \square$$

↑ by Squeeze theorem.

---

* Now, prove $\liminf \frac{C_{n+1}}{C_n} \le \lim \sqrt[n]{C_n}$

2014.

7 $f: X \longrightarrow Y$ continuous

$X, Y$ metric spaces , $X$ compact

Assume $y_0 \in Y$ is a point which has a unique image $x_0 \in X$ ie $f^{-1}(y_0) = \{x_0\}$.

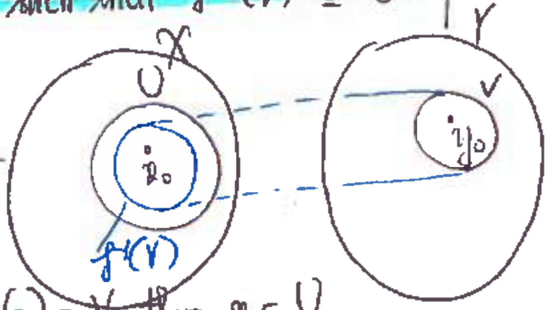
rove that for every open neighborhood $U$ of $x_0$ in $X$,

there exists an open neighborhood $V$ of $y_0$ such that $f^{-1}(V) \subseteq U$

Give an example to show that this conclusion is

lse if $X$ is not compact

Hard



e need to prove $\forall U$ open $, x_0 \in U, \exists V$ open $, y_0 \in V$

$f^{-1}(V) \subseteq U$.

le NTP, $\forall U$ open $, x_0 \in U, \exists V$ open $, y_0 \in V, f(x) \in V$ then $x \in U$

NTP $\forall x \in X \setminus U$, then $\forall V$ open neighbor of $y_0$, then $f(x) \in Y \setminus V$.

NTP $\forall x \in X \setminus U$, $\forall V$ open neighbor of $y_0$, $\exists r, \ d(f(x), y_0) > r$.

Now put $g(x) = d(f(x), y_0))$, we want to prove that $\exists r, \ g(x) > r$.

have $g$ is a continuous function
and because $x \in X \setminus U$, a compact set $\Big\} \Rightarrow g$ attains min on $X \setminus U$.

~~this~~ Put $r := \min d(f(x), y_0))$

Therefore have $r > 0$ because if $d(f(x), y_0) = 0$, then $f(x) = y_0$, contracts with the assumption that $x_0$ is a unique point that $f(x_0) = y_0$.

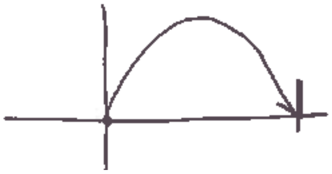
So we have $\exists r > 0, \ g(x) \geq r$. $\Rightarrow$ ~~dt~~ done.

7 Let $X = [0 \ 1)$.

$f(x) = -(x-1)^2$

Let $y_0 = 0$

Then



Jan2014

3) Let $f: \mathbb{R} \to \mathbb{R}$ differentiable function, $\lim_{x \to \infty} f'(x) = 1$. Let $a \in \mathbb{R}$ } Prove that the following exists and find it $\lim_{x\to\infty} \frac{e^{f(x+a)}}{e^{f(x)}}$

* We have

$$\frac{e^{f(x+a)}}{e^{f(x)}} = e^{f(x+a) - f(x)} = e^{f'(\xi)a} \quad \text{for some } \xi \in [x, x+a].$$

Then because exp is a continuous function,

$$\lim_{x\to\infty} \frac{e^{f(x+a)}}{e^{f(x)}} = e^{\lim_{x\to\infty} f(x+a) - f(x)} = e^{\lim_{x\to\infty} f'(\xi)a} = e^{1 \cdot a} = e^{a} \quad \square$$

n2014

) For each $s \in [0,1]$, there is a function $f_s(x)$ defined for $x \in [a,b]$.

$f_s \in \mathcal{R}(\alpha)$ on $[a,b]$.

where $\alpha$:, monotonically increasing funct on $[a,b]$.

Suppose that $f_{s_j} \underset{j\to\infty}{\rightrightarrows} f_{1/2}$ on $[a,b]$ for any $\{s_j\}_{j=1}^{\infty}$ from $[0,1]$, $s_j \xrightarrow{j\to\infty} \frac{1}{2}$

ow that $\displaystyle \lim_{s\to\frac{1}{2}} \int_a^b f_s(x)\,d\alpha(x) = \int_a^b f_{1/2}(x)\,d\alpha(x)$.

$f_{s_n} \overset{n\to\infty}{\rightrightarrows} f_{1/2} \overset{\text{for } s_n \to 1/2}{\Leftrightarrow} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in [a,b], |f_{s_n}(x)\, f_{1/2}(x)| < \varepsilon$

Put $f_n = f_{s_n}$, we have $f_n \rightrightarrows f$ on $[a,b]$

ve have $\displaystyle \int_a^b f_{1/2}(x)\,d\alpha(x) = \lim \int_a^b f_n(x)\,d\alpha x = \lim_{s_n\to\infty} \int_a^b f_{s_n}(x)\,d\alpha(x) = \lim_{s\to 1/2} \int_a^b f_s(x)\,d\alpha(x)$

In line question: See 79/166 Rudin sol

t $f_n$ be a continuous, real value function on $[0,1]$

$f_n \rightrightarrows f$

} Prove that $\lim_{n\to\infty} f_n(x_n) = f(1/2)$ for any sequence $x_n \to \frac{1}{2}$

b) Must the conclusion still hold if the convergence is only pointwise? Explain

Jan 2014

5) Let $f$: real valued, continuous function on $[0,1]$

$\|f\| \leq 1$, (sup norm less than or equal 1) and $f(0) = 0$

Show that the sequence of power of $f$, $\{f^n\}_{n=1}^{\infty}$, is equicontinuous $\Leftrightarrow$ $\|f\| < 1$

*not done

(Note: sequence of power of $f$: $[f(x)]^n$

($\Leftarrow$): Give $\|f\| < 1$, Prove that sequence of power of $f$ is equicontinuous.

$\Leftrightarrow$ NTP $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta$, then $|f^n(x) - f^n$

$\forall n.$

- We will prove this by introduction.

$f$ continuous on $[0,1] \Rightarrow f$ uniformly continuous on $[0,1]$.

$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

- Introduction hypothesis:

$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta$ then $|[f(x)]^{n-1} - [f(y)]^{n-1}| < \epsilon$.

⊕ Notice that we have $\|f\| = \sup_{x \in [0,1]} |f(x)| \leq 1 \Rightarrow$ means $0 \leq |f(x)| \leq 1$

$\Rightarrow 0 \leq f^2(x) = [f(x)]^2 \leq 1$

$\Rightarrow \sup_{x \in [0,1]} |[f(x)]^2| \leq 1$

By induction, we have $\|f^n\| \leq 1, \forall n$.

So now in case $n$, we NTP $\forall \epsilon > 0, \exists \forall x, y \in [0,1], |x-y| < \delta$, then $|[f(x)]^n - [f(y)]^n|<$

We have $|[f(x)]^n - [f(y)]^n| = n \underbrace{[f(c)]^{n-1}}_{\leq 1} [f(x) - f(y)]$

not done

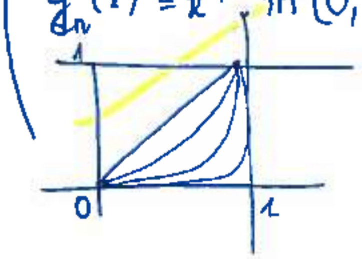
($\Rightarrow$): $\{f^n\}$ equicontinuous. Prove that $\|f\| < 1$

(We already know $\|f\| \leq 1$)

So now we prove that if $\|f\| = 1$, then $\{f^n\}$ is not equicontinuous.

(Note: this part base on an important example in uniformly cont

$g_n(x) = x^n$ in $[0,1)$



$g_n(x)$ (when $n \to \infty$)

$$g_n(x) \longrightarrow \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$$

$$g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$$

an 2014

et $f=(f_1, f_2): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \longmapsto f=[f_1(x,y), f_2(x,y)] \qquad f_1(x,y)=2x+|x|-|x+1|$$

$$f_2(x,y)=(y-1)^3.$$

At which points $(x,y)$ does the IFT provide the existence of a $C^1$ inverse in a neighborhood? Check the conditions of the theorem?

At which points is $f$ not invertible.

irst, we compute$_x$ $f_1(x,y)$.

| | | -1 | | 0 | |
|---|---|---|---|---|---|
| $f_1(x,y)$ | $2x$<br>$-x$<br>$x+1$ | $2x$<br>$-x$<br>$-x-1$ | $2x$<br>$-x$<br>$-x-1$ | $2x$<br>$x$<br>$-x-1$ | $2x$<br>$x$<br>$-x-1$ |
| $f_1(x,y)=$ | $2x+1$ | $-1$ | $-1$ | $-1$ | $2x-1$ |

~~we have~~ By def $f$ is a $C^1$ function $\overset{def}{\Longleftrightarrow}$ partial derivatives exist and continuous,

have $f$ is a $C^1$ function ~~when $x<-1$ and $x>0$~~ for all $x$.

$$[f'] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} (*) & 0 \\ 0 & 3(y-1)^2 \end{bmatrix}$$

e can apply IFT, when $\det[f'] \neq 0$, so we have

when $x \in [-1,0]$, $\det[f'] = 0 \cdot 3(y-1)^2 = 0 \quad \forall y$ ⇒ could not apply IFT

$x<-1$, $\det[f'] = 2 \cdot 3(y-1)^2 \neq 0$ when $y \neq 1$

x > ①, $\det[f'] = 2 \cdot 3(y-1)^2 \neq 0$, when $y \neq 1$ ) can apply IFT

we can apply IFT when $x<-1$ or $x>0$ and $y \neq 1$.

A which point $f$ is not invertible.

ote that when we have $f$ satisfies IFT, we can know that $f$ is invertible.

n case $f$ does not satisfies IFT, we need to check if $f$ is not invertible,

(the easy way is to check that $f$ is not injective).

When $y=1$, then $f$ is not ~~big~~ injective ⇒ not invertible.

n case $x \in (-1,0)$, $f(x,y) = (-1, (y-1)^3)$ ⇒ not injective ⇒ not invertible.

is not invertible when $x \in [-1,0]$ and when $y=1$

## AUGUST 2014 PRELIMINARY EXAMINATION IN ANALYSIS

1. Suppose $f$ is positive, twice differentiable, and log-concave, i.e., the graph of the composite function $\ln(f)$ is everywhere concave down. Prove that the function
$$g(x) = f(x)\left(\frac{1}{f(x)}\right)'$$
is non-decreasing.

2. Let $X$ be a compact metric space with metric $d$, and let $x_0 \in X$. Prove that $K = \{d(x_0, x) : x \in X\}$ is a closed subset of the real numbers.

3. Let $A$ be a subset of the natural numbers whose elements have been arranged into a sequence $a_1, a_2, \ldots$. Call the set *petite* if it is finite, or if it is infinite and
$$\sum_{j=1}^{\infty} \frac{1}{a_j} < \infty.$$
A set which is not petite is called *husky*. Prove that the complement of a petite set is husky, but that the complement of a husky set is not necessarily petite.

4. Suppose that $\{f_n\}$, $n = 1, 2, \ldots$, are continuous functions defined on the interval $[0,1]$, and
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$$
Suppose also that for each $n$, the function $f_n$ is increasing, and $f_n(0) = 0$. Prove that $f_n$ converges to 0 uniformly on the interval $[0, 1/2]$.

5. Let $f \colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[0,1]$, and $p_n(x) > p_{n+1}(x)$ for every $x \in [0,1]$ and every $n = 1, 2, \ldots$.

6. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable nondecreasing function. Define $\mathbf{g} \colon \mathbb{R}^2 \to \mathbb{R}^2$ by
$$\mathbf{g}(x_1, x_2) = (x_2 + f(2x_1 + x_2), 2x_1 + f(2x_1 + x_2))$$
Show that $\mathbf{g}$ satisfies the conditions of the Inverse Function Theorem at every point of $\mathbb{R}^2$.

Aug 2014 7

⑰ Suppose $f$ is positive, twice differentiable, and log concave,
(a function is log concave means $\ln(f)$ is everywhere concave down, means $(\ln f)'' < 0, \forall x$
Prove that the function ~~f(x)~~ $g(x) = f(x)\left(\frac{1}{f(x)}\right)'$ is non decreasing .

---

We want to prove that $g(x)$ is non-decreasing $\Leftrightarrow$ NTP $g'(x) \geq 0, \forall x$.

+ We have $\left(\frac{1}{f'(x)}\right)' = \frac{-f''(x)}{(f'(x))^2}$

Because $g(x) = f(x)\left(\frac{1}{f(x)}\right)'$ and $f$ is twice differentiable, then $g$ is differentiable and :

$$g'(x) = f'(x)\frac{1}{f'(x)} + f(x)\left(\frac{1}{f(x)}\right)' = 1 - f(x)\frac{f''(x)}{[f'(x)]^2}$$ and we want to prove that this ≥

(1)

+ We now use the assumption that $(\ln f)'' < 0, \forall x$.

$$(\ln f)' = \frac{1}{f(x)} f'(x)$$

then $$(\ln f)'' = \left[\frac{f'(x)}{f(x)}\right]' = \frac{f''(x)f(x) - f'(x)f'(x)}{f^2(x)} = \frac{f''(x)f(x) - [f'(x)]^2}{f^2(x)}.$$

we have because $(\ln f)'' \leq 0$, $f''(x)f(x) \leq [f'(x)]^2$

$$\Rightarrow \frac{f(x)f''(x)}{[f'(x)]^2} \leq 1 \quad (2)$$

(1)+(2) $\Rightarrow g'(x) \geq 0, \forall$ so $g$ is non decreasing □ .

g2014 > P2 >

t X be a compact metric space with metric d.
Let $x_0 \in X$.
ove that $K = \{ d(x_0, x),\ x \in X \}$ is a closed subset of the real numbers

Ve consider $f : X \longrightarrow \mathbb{R}$

$$x \longmapsto f(x) = d(x, x_0)$$

›we have $f$ is a continuous function.

hen we have $\left.\begin{array}{l} f : X \longrightarrow \mathbb{R} \text{ continuous} \\ X \text{ compact} \end{array}\right\}$ then $f(X)$ compact in $\mathbb{R}$.

$$\{ d(x_0, x),\ x \in X \} = K$$

$\rightarrow K$ is closed + bounded in $\mathbb{R}$. □

---

Now we prove what we used above:

Fix $x_0$ in $X$, prove that $f : X \longrightarrow \mathbb{R}$, $x \longmapsto \overset{f(x) =}{d(x, x_0)}$ is continuous function.

e NTP that $_{\wedge}\forall \varepsilon > 0$, $\exists \delta_x$, $\forall y \in X$, $d(y, x)| < \delta_x$, then $|f(x) - f(y)| < \varepsilon$
$\forall x$

we have $|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq |d(x, y)|$

So $\forall \varepsilon > 0$, choose $\delta_x \leq \varepsilon$, we have $f$ is continuous.

Let $\left.\begin{array}{l} f : X \longrightarrow Y \text{ continuous} \\ X \text{ compact} \end{array}\right\}$ Prove that $f(X)$ compact in $Y$.

We prove this by proving that for every open cover of $f(X)$ contains a finite subcover ---

Let $U$

Aug 2014, P5.

Let $f:[0,1]\longrightarrow\mathbb{R}$ be a continuous function.

Prove that there exists a sequence of polynomials $\{P_n\}$, $P_n\rightrightarrows f$ on $[0,1]$.

$P_n(x) > P_{n+1}(x)$, $\forall x\in[0,1]$, $n=1,2,\dots$

We have $f:[0,1]\longrightarrow\mathbb{R}$ is a continuous function, then by Stone-Weierstrass theorem, $\exists P_n(x)\rightrightarrows f$ on $[0,1]$, which means:

$\forall\varepsilon>0,\ \exists N_\varepsilon\in\mathbb{N},\ \forall n\geqslant N_\varepsilon,\ \forall x\in[0,1],\ |P_n(x)-f(x)|<\varepsilon.$

2014 > PG >

t $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable nondecreasing function.

ine $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x_1, x_2) \longmapsto g(x_1, x_2) = \left(x_2 + f(2x_1 + x_2),\ 2x_1 + f(2x_1 + x_2)\right)$$

e that g satisfies the conditions of the Inverse function Theorem at every point of $\mathbb{R}^2$.

We have

$$Dg = \begin{bmatrix} 2 f'(2x_1 + x_2) & 1 + f'(2x_1 + x_2) \\ 2 + 2f'(2x_1 + x_2) & f'(2x_1 + x_2) \end{bmatrix}$$

) We have g is a $C^1$ function because all of the partial derivative exists and continuous (cause f is continuously differentiable function)

) We have $\det(Dg) = 2[f']^2 - 2 - 2f' - 2f' - 2[f']^2 = -2[1 + f']$

we have because f is nondecreasing $\Rightarrow f' \geq 0$

so $\det(Dg) \neq 0$, $\forall (x_1, x_2) \in \mathbb{R}^2$

we have g satisfies the conditions of IFT. $\forall (x_1, x_2) \in \mathbb{R}^2$

Aug 2014 7.

3) Let A be the set of the natural numbers whose elements have been arranged into a sequence
Call the set "petile" if it is finite, or it is infinite and $\sum_{j=1}^{\infty}\frac{1}{a_j} < +\infty$
A set which is not "pehle" is called husky.
Prove that the complement of a pehle is husky
the complement of a husky set is not neccessary pehle.

j2014 Q 4.

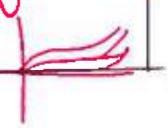
pose that $\{f_n\}$, $n = 1, 2, \dots$ are continuous functions, define on the interval $[0,1]$.

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0$$

ppose also that for each $n$, the function $f_n$ is increasing and $f_n(0)=0$.

ve that $f_n \rightrightarrows 0$ on $[0, 1/2]$.

In here it is importal to have that $f_n$ increas and $f_n(0)=0$



Note that in here we have $f_n$ is increasing and $f_n(0)=0$ $\Big\} \Rightarrow f_n(x) \geq 0, \ \forall n, \forall x \in [0,1]$.

is means we have
$$0 \leq \frac{1}{2} f_n(1/2) \leq \int_{1/2}^{1} f_n(x)\,dx \leq \underbrace{\int_0^1 f_n(x)\,dx}_{\longrightarrow 0}$$

so we have $\lim_{n\to\infty} f_n(1/2) = 0$

we also know that $f_n(x) \leq \underbrace{f_n(1/2)}_{\longrightarrow 0} \ \forall x \in [0,1/2]$ $\Big\} \ f_n(x) \rightrightarrows 0$ on $[0, 1/2]$ □.

L

# Analysis Preliminary Exam, ~~January~~ 2015

1. (i) If $x > 0$ and $y > 0$ show that $x + \frac{1}{y^2 x} \geq \frac{2}{y}$.

(ii) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all $n \geq 1$. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 a_n}$ diverges.

2. Let $f : [0, +\infty) \longrightarrow \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty}(f(x) - x) = 0$. Prove or provide a counterexample to the statement: $f$ is uniformly continuous on $[0, +\infty)$.

3. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions such that $f$ is differentiable and for every $x, h \in \mathbb{R}$ one has $f(x + h) - f(x - h) = 2hg(x)$. Prove that $f$ is a polynomial of degree at most 2.

Want to prove that $f$ is a polynomial of degree at most 2
$\Rightarrow$ NTP $f''$ is a constant

4.

(a) Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative $f'$ is not continuous. Prove that your example works.

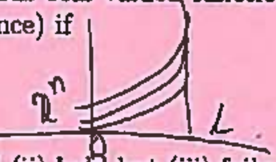
(b) Let $f$ be as in Part (a). If $f'(0) < 2 < f'(1)$, prove that $f'(x) = 2$ for some $x \in [0, 1]$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n \to \infty} (n + 1) \int_0^1 x^n f(x)\, dx = f(1).$$

6. The Arzelá-Ascoli theorem asserts that a sequence $\{f_n\}$ of continuous real valued functions on a metric space $\Omega$ is precompact (i.e. has a uniformly convergent subsequence) if

(i) $\Omega$ is compact,
(ii) $\sup\{|f_n(x)| : x \in \Omega \text{ and } n \in \mathbb{N}\} < \infty$,
(iii) the sequence is equicontinuous.



Give examples of sequences which are not precompact such that: (i) and (ii) hold but (iii) fails; (i) and (iii) hold but (ii) fails; (ii) and (iii) hold but (i) fails. Take $\Omega$ to be a subset of the real line.

$f_n(x) = n$.

Jan 2015

1 i) If $x>0$ and $y>0$, show that $x+\frac{1}{y^2x} \geq \frac{2}{y}$

ii) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges, $a_n>0, \forall n \geq 1$ } Show that $\sum_{n=1}^{\infty} \frac{1}{n^2 a_n}$ diverges.

i) If $x>0$, $y>0$. Show that $x+\frac{1}{y^2x} \geq \frac{2}{y}$

We have $\underbrace{x}_{a}+\underbrace{\frac{1}{y^2x}}_{b} \underset{\substack{\uparrow \\ a+b\geq 2\sqrt{ab} \\ (a,b>0)}}{\geq} 2\sqrt{x\frac{1}{y^2x}} = \frac{2}{y}$ (note that $x,y>0$).

ii) Suppose $\sum a_n$ converges, $a_n>0, \forall n\geq 1$ } Show that $\sum_{n=1}^{\infty} \frac{1}{n^2 a_n}$ diverges

We have from a) $a_n + \frac{1}{n^2 a_n} \geq \frac{2}{n} > 0$

then because $\sum a_n$ converges

assume that $\sum \frac{1}{n^2 a_n}$ converges

} by comparison test $\sum_{n=1}^{\infty} \frac{2}{n}$ converges (infact $\sum \frac{2}{n}$ diverges)

$\Rightarrow \sum \frac{1}{n^2 a_n}$ has to be divergent.

---

* One thing learned from this problem

$a_n + b_n \geq c_n \geq 0$

Then if $\sum c_n$ diverges $\Rightarrow$ [ $\sum a_n$ diverges

$\sum b_n$ diverges

{ $\sum a_n$ converges

$\sum b_n$ converges } $\Rightarrow \sum c_n$ converges.

n 2015 ②

t $f:[0,+\infty)\longrightarrow\mathbb{R}$ be a continuous function:

$\lim\limits_{x\to+\infty}(f(x)-x)=0$

Prove or provide a counter example to the statement: $f$ is uniformly continuous on $[0,+\infty)$

We have $\lim\limits_{x\to+\infty} f(x)=x \Leftrightarrow \exists N\in\mathbb{R}^+,\ \forall x>N,\ |f(x)-x|<\varepsilon/3$

Now we consider $f(x)$ in $[0,N]$ and $[N,+\infty)$

<u>In $[0,N]$</u>, $f$ continuous in $\mathbb{R}$ $\Rightarrow$ continuous in $[0,N]$ $\Rightarrow$ uniformly continuous in $[0,N]$.

$\Leftrightarrow \forall\varepsilon>0, \exists\delta_1>0,\ \forall x,y\in[0,N],\ |x-y|<\delta_1,\ |f(x)-f(y)|<\varepsilon$.

<u>In $[N,+\infty)$</u>: (Note that $\forall x>N,\ |f(x)-x|<\varepsilon/3$).

$\Rightarrow \forall\varepsilon>0$, chose $\delta_2=\varepsilon/3$, $\forall x,y\in[N,+\infty)$, $|x-y|<\frac{\varepsilon}{3}$, then

$$|f(x)-f(y)|\leqslant \underbrace{|f(x)-x|}_{<\frac{\varepsilon}{3}}+\underbrace{|x-y|}_{<\frac{\varepsilon}{3}}+\underbrace{|f(y)-y|}_{<\varepsilon/3}<\varepsilon.$$

conclusion, $\forall\varepsilon>0$, Chose $\delta=\min\{\delta_1,\delta_2\}$, then $\forall x,y\in[0,+\infty)$, $|x-y|<\delta$ then $|f(x)-f(y)|<\varepsilon$

$\Rightarrow f$ is ~~equicontinuous~~ uniformly continuous in $[0,+\infty)$.

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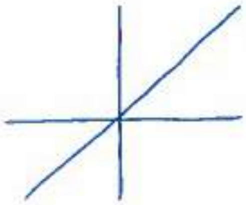
Something learned from this problem:

uniformly continuous $\not\Rightarrow$ bounded.

EX: $f(x)=x$ is uniformly continuous



$\forall\varepsilon>0, \exists\delta=\varepsilon$,

$\forall x,y\in\mathbb{R}$ (or $[0,+\infty)$)

$|x-y|<\varepsilon$, then $|f(x)-f(y)|<\varepsilon$

bounded in $[0,1]$ $\not\Rightarrow$ uniformly continuous

Just choose any function that is <u>not</u> continuous on $[0,1]$.

Jan 2015 / ③ ueir

Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions
$f$ is differentiable
$\forall x, h \in \mathbb{R}$, $f(x+h) - f(x-h) = 2h\,g(x)$.
} Prove that $f$ is a polynomial of degree at most 2

* We want to prove that $f$ is a polynomial of degree at most 2
⟺ We want to prove that $\boxed{f''(x) = \text{constant}}$

* We have $f$ is differentiable $\boxed{\text{wrt } x}$
$2h\,g(x) = f(x+h) - f(x-h)$
$\Longrightarrow$ $g$ is differentiable $\boxed{\text{wrt to } x}$ (1)

* Consider when $h \neq 0$, we have

$$\frac{f(x+h) - f(x-h)}{2h} = g(x) \Longrightarrow \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0} g(x) = g(x)$$

$$\Rightarrow f'(x) = g(x) \quad (2).$$

- We have (1) : $g$ differentiable wrt to $x$
$f'(x) = g(x)$
$\Rightarrow$ $f$ is twice differentiable $(\exists f''(x))$.

* Let (differentiate (wrt $h$))

$$f'(x+h) - f'(x-h) = 2g(x).$$

$$f''(x+h) - f''(x-h) = 0$$

Let $x = h$, then $f''(2x) = f''(0)$ $\Rightarrow$ ($f''$ is a constant $\Rightarrow$ $f$ is a polynomial of degree at most 2)

---

* Learn from this problem :

- $f$ is differentiable in both ⓧ and ⓗ → This means we can take derivative of $f$ wrt to $x$ or wrt to $h$.

⇒ need to consider take derivative wrt $h$ when needed.

* ($f(x) = g(x+c)$, $f$ is differentiable wrt $x$ } → $g$ differentiable wrt to $x$)

* we want to prove that $f$ is a polynomial of degree at most 2

⟺ we NTP that $\boxed{f'' = \text{constant}}$

⟺ This means NTP $\begin{cases} \exists f', \exists f'' \\ f'' = \text{constant} \end{cases}$.

n2015 .47

) Give an example of a differentiable function $f: \mathbb{R} \longrightarrow \mathbb{R}$ whose derivative $f'$ is not continuous. Prove your example work.

- Let $f$ be in part a7. If $f'(0) < 2 < f'(1)$. Prove that $f'(x) = 2$ for some $x \in [0,1]$.

Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & , x \neq 0 \\ 0 & x = 0 \end{cases}$

Now we prove that $f$ is differentiable in $\mathbb{R} \Leftrightarrow$ NTP $\exists f'(x), \forall x \in \mathbb{R}$.

For $x \neq 0$, $f'(x) = 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

At $x = 0$: We want to compute $\lim\limits_{t \to 0} \frac{f(t) - f(0)}{t - 0}$

We have $0 < \left|\frac{f(t) - f(0)}{t - 0}\right| = \left|\frac{t^2 \sin\frac{1}{t}}{t}\right| = \left|t \sin\frac{1}{t}\right| \leq |t|$

$\lim\limits_{t \to 0} |t| = 0$

By Squeeze theorem. $\lim\limits_{t \to 0} \left|\frac{f(t) - f(0)}{t - 0}\right| = 0$

$\Rightarrow \lim\limits_{t \to 0} \frac{f(t) - f(0)}{t - 0} = 0$

This means $f'(0) = 0$

So $f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ ☐ part 1.

Now we prove that $f'$ is not continuous, we will prove that $\nexists \lim\limits_{x \to 0} f'(x)$

We have $\lim\limits_{x \to 0} f'(x)$ does not exist because $\nexists \lim\limits_{x \to 0} \cos\frac{1}{x}$ $\Rightarrow$ ☐a7

- If $f'(0) < 2 < f'(1)$ Prove that $f'(x) = 2$ for some $x \in [0,1]$.

apply theorem Intermediate value theorem (for $f'()$):

$f$ is differentiable in $[a,b]$ s.t. $f'(a) < \lambda < f'(b)$ $\Big\}$ Then $\exists x \in (a,b)$ $f'(x) = \lambda$

Apply this theorem with $a = 0$ $b = 1$ $\lambda = 2$

note that in here ~~we don't apply~~ we need $g'(x) = 0$ for some $x$

important trick with Intermediate value theorem

Now we prove the theorem:

Let $g(x) = f(x) - \lambda x$

$\Rightarrow g'(x) = f'(x) - \lambda$

$g'(b) = f'(b) - \lambda > 0$

$g'(a) = f'(a) - \lambda < 0$

$\Rightarrow$ Neither of a or b are the point where g attain min

$\Rightarrow \exists x \in (a,b)$, g attains min at $x$

$g'(x) = 0$

$\Leftrightarrow \exists x \in (a,b)$ $f'(x) = \lambda$ $\Rightarrow$ ☐ b.

Jan 2015 / ⑤ See Jan 2013 / 3 (harder) Aug 2013

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function

Show that $\lim_{n\to\infty}\left((n+1)\int_0^1 x^n f(x)\,dx\right) = f(1)$

When we see $\lim \int = \textcircled{a}$ $*$

Way 1: use $\textcircled{a} = \frac{1}{b-a}\int_a^b 1\,dx$ (can't w the p

Way 2: If $f$ cont in $[a,b]$ $\Rightarrow \exists P_n \rightrightarrows f$ on

$*$ First, we have $\left.\begin{matrix} f:\mathbb{R}\to\mathbb{R} \text{ continuous} \\ [0,1] \text{ compact}\end{matrix}\right\} \underset{\text{Weierstrass}}{\Rightarrow} \exists \{P_\ell(x)\}, \; P_\ell(x) \rightrightarrows f(x)$ on $[0,1]$ $(*)$.

$*$ Now we ~~will prove~~ have that: for $P_\ell(x) = a_\ell x^\ell + a_{\ell-1}x^{\ell-1} + \cdots + a_1 x + a_0$.

- We have

$$\int_0^1 x^n P_\ell(x)\,dx = \int_0^1 a_\ell x^{n+\ell} + a_{\ell-1}x^{n+(\ell-1)} + \cdots + a_1 x^{n+1} + a_0 x^n\,dx = \frac{a_\ell}{n+\ell+1} + \frac{a_{\ell-1}}{n+\ell} + \cdots + \frac{a_1}{n+2} +$$

Then

$$\lim_{n\to\infty}\left((n+1)\int_0^1 x^n P_\ell(x)\,dx\right) = \lim_{n\to\infty}\left((n+1)\left(\frac{a_\ell}{n+\ell+1} + \cdots + \frac{a_1}{n+2} + \frac{a_0}{n+1}\right)\right) = a_\ell + a_{\ell-1} + \cdots + a_1 + a_0 = P_\ell(1) \quad \forall \ell$$

This means $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \; \left|(n+1)\int_0^1 x^n P_\ell(x)\,dx - P_\ell(1)\right| < \varepsilon$ $\forall \ell$ (1)

$*$ By $(*)$ $P_\ell(x) \rightrightarrows f(x)$ on $[0,1]$

$\Leftrightarrow \forall \varepsilon > 0, \exists \ell_0 \in \mathbb{N}, \forall \ell \geq \ell_0, \forall x \in [0,1], \; |P_\ell(x) - f(x)| < \varepsilon$ $(**)$

Then consider

$$\left|(n+1)\int_0^1 x^n P_\ell(x)\,dx - (n+1)\int_0^1 x^n f(x)\,dx\right| \leq (n+1)\int_0^1 x^n \underbrace{|P_\ell(x) - f(x)|}_{<\varepsilon,\ \forall \ell \geq \ell_0} dx$$

$$\leq \varepsilon \underbrace{(n+1)\int_0^1 x^n\,dx}_{=1}$$

$$= \varepsilon.$$

This means $\forall \ell \geq \ell_0$, $\left|(n+1)\int_0^1 x^n P_\ell(x)\,dx - (n+1)\int_0^1 x^n f(x)\,dx\right| < \varepsilon$ (2)

$*$ Also because of $(**)$, $\forall \ell \geq \ell_0$, $|P_\ell(1) - f(1)| < \varepsilon$ (3).

Because of (1)+(2)+(3), Choose $N = \max\{n_0, \ell_0\}$, then for all $n \geq N$.

$$\left|(n+1)\int_0^1 x^n f(x)\,dx - f(1)\right| \leq \left|(n+1)\int x^n f(x)\,dx - (n+1)\int x^n P_n(x)\,dx\right| +$$

$$\left|(n+1)\int_0^1 x^n P_n(x)\,dx - P_n(1)\right| + |P_n(1) - f(1)|$$

$= 3\varepsilon$

This is what we need to do.

$*$ Note:

Some things learned from this problem.

This problem can't be solved by use $f(1) = f(1)\underbrace{(n+1)\underbrace{\int_0^1 x^n\,dx}_{=\frac{1}{n+1}}}_{=1}$

cause consider

$$\left|(n+1)\int_0^1 x^n f(x)\,dx - f(1)\right| = \left|(n+1)\int_0^1 x^n f(x) - (n+1)\int_0^1 x^n f(1)\,dx\right|$$

$$\le (n+1)\int_0^1 x^n \underbrace{|f(x)-f(1)|}\,dx$$

we only know this $\le M$ on $[0,1]$

$$\le M\underbrace{(n+1)\int_0^1 x^n}_{=1} \not< \varepsilon.$$

$>$ we use $P_n(x) \rightrightarrows f(x)$.

Jan 2015 6)

The Ascoli theorem asserts that a sequence $\{f_n\}$ of continuous real valued functions on a metric space $\Omega$ is precompact (i.e. has a uniformly convergent subsequence) if :

i) $\Omega$ compact

ii) $\sup\{|f_n(x)\}, x \in \Omega, n \in \mathbb{N}\} < \infty$

iii) the sequence is equicontinuous in $\Omega$

Note that with $\Longrightarrow$ equicontinuous we have to say in where

Give examples of sequences which are not precompact such that

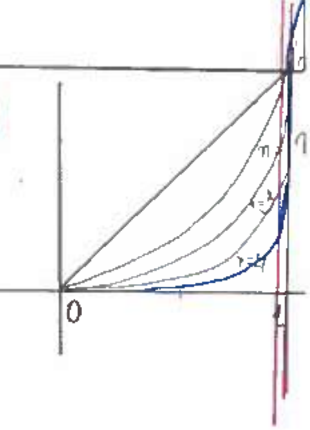
a) (i), (ii) hold (iii) fails. c) (ii) and (iii) holds (i) fail.

b) (i)(iii) holds (ii) fails

a) Example of $\begin{cases} \Omega \text{ compact} \\ \sup\{|f_n(x)\}, x \in \Omega, n \in \mathbb{N}\} < +\infty \\ \{f_n\} \text{ is not equicontinuous in } \Omega \end{cases}$

$\{f_n(x) = x^n$ in $\Omega = [0,1]$

(bounded $\not\Longrightarrow$ equicontinuous



- Actually $\Omega = [0,1]$ compact in $\mathbb{R}$.
- $\sup\{x^n, \forall x \text{ on } [0,1], n \in \mathbb{N}\} = 1$.
- Now prove $\{f_n\}$ is $\boxed{\text{not}}$ equicontinuous.

We want to prove $\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in [0,1], |x-y| < \delta, \exists n_0, |f_{n_0}(x) - f_{n_0}(y)|$

Choose $\varepsilon = 1/2$

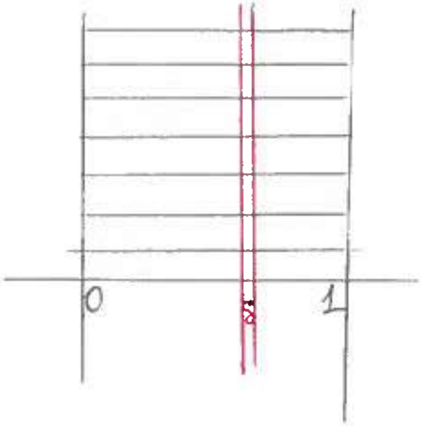
Then $\forall \delta > 0$, choose $x = 1$, $y = 1 - \frac{\delta}{2}$, then $|x-y| = |1 - 1 + \frac{\delta}{2}| = |\frac{\delta}{2}| < \delta$.

and $\forall \delta > 0$, $\exists n_0$ big enough st $(1 - \frac{\delta}{2})^n < \frac{1}{2}$

Then $|f_n(x) - f_n(y)| = \left|1^n - (1 + \frac{\delta}{2})^n\right| > \left|1 - \frac{1}{2}\right| = \frac{1}{2} = \varepsilon$

This means $\{f_n(x)\}$ is not a equicontinuous family in $[0,1]$

b) Example when $\begin{cases} \Omega \text{ compact} \\ \sup\{f_n(x), x \in \Omega, n \in \mathbb{N}\} \text{ is not bounded} = \infty \\ \{f_n\} \text{ is equicontinuous in } \Omega. \end{cases}$ (equicontinuous $\not\Longrightarrow$ bounded)



*Let $f_n(x) = n$, $\forall x \in \Omega = [0,1]$.

- $\Omega = [0,1]$ compact
- $\sup\{n, x \in [0,1], n \in \mathbb{N}\} = +\infty$.
- $\{f_n\}$ is equicontinuous in $[0,1]$.

$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta, |f_n(x) - f_n(y)| = |n - n| = 0 < \varepsilon, \forall$

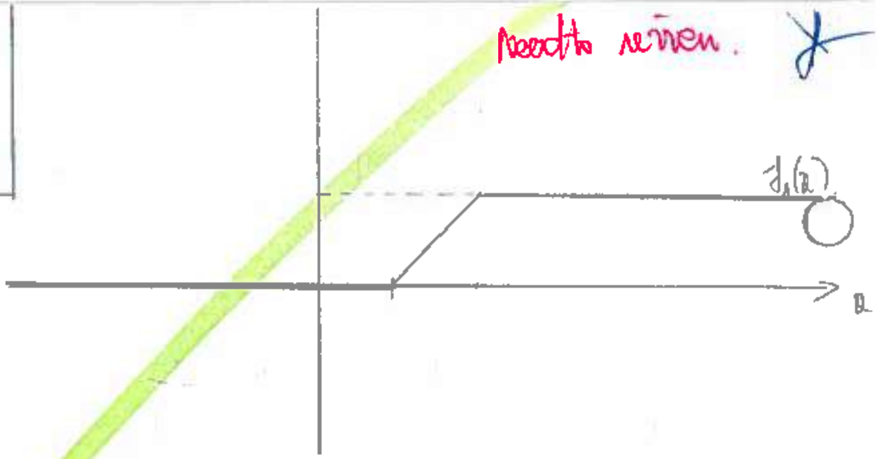
$\Rightarrow \square$.

Example when $\Omega$ is not compact

$$\begin{cases} \sup\{f_n(x),\ x \in R,\ n \in N\} < +\infty \\ \{f_n(x)\} \text{ equicontinuous in } \Omega. \end{cases}$$

$$f_n(x) = \begin{cases} 0 & x \leq n \\ x-n & n < x \leq n+1 \\ 1 & x > L. \end{cases}$$

Need to review.



# Analysis Preliminary Exam, August 2015

$f_n\ X \longrightarrow \mathbb{C}$, X is countable.
$f_n$ is pointwise bounded
$\Longrightarrow$ $\{f_n\}$ contains a pointwise convergent subsequence.

① Assume $f_n$ is a sequence of functions mapping $R$ into $[0,1]$. Prove there is a subsequence $n_k$ along which $f_{n_k}(q)$ converges for all rational $q$

Same with Aug 2003.

② Prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$$

exists.

Note that $\ln n = \int_1^n \frac{1}{x}\,dx$.

Prove that $\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln n \longrightarrow \gamma$

See Jan 2012 $\int_0^\infty \cos(x^2)\,dx$.

~~3.~~ Is

$$\int_1^\infty \frac{\sin x}{x}\,dx$$

a convergent integral?

Template: with this kind of question just use integration by part and then use $\sum_{n=1}^{\infty} f(n)$ and $\int_1^\infty f(x)\,dx$ both converge and diverge if f is eventually decreasing... + comparison.

④ If $p_k \geq 0$ and $\sum_{k=1}^{\infty} p_k = 1$, show that

$$\left(\sum_{k=1}^{\infty} kp_k\right)^2 \leq \sum_{k=1}^{\infty} k^2 p_k.$$

Use Cauchy Suatch inequality in case

$a_n = k\sqrt{p_\ell}$

$b_n = \sqrt{p_\ell}$.

$|\langle u,v\rangle| \leq \|u\| \cdot \|v\|$

⑤ Let $\{f_n\}$ be equicontinuous on the compact set $K$. Assume that $\{f_n\}$ converges pointwise. Prove that $\{f_n\}$ converges uniformly on $K$.

~~6.~~ Let

$$f(x) = \begin{cases} x + 2x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

with $f : \mathbb{R} \to \mathbb{R}$.

~~(a)~~ Show that $f'(0) = 1$. Show that $f'$ is not continuous at $x = 0$.

~~(b)~~ Write $y = f(x)$, what does the inverse function theorem say or not say about the inverse of $f$ in a neighborhood of $y = 0$. Explain.

~~(c)~~ Show that $f$ is not 1-1 in any neighborhood of $x = 0$.

In case $f : \mathbb{R} \longrightarrow \mathbb{R}$ we can prove $f$ is not 1-1 by proving that

Aug 2015

Need to review

(P1) → Assume $f_n$: sequence of function mapping $\mathbb{R} \longrightarrow [0,1]$

Prove that there is a subsequence $n_\ell$ along which $f_{n_\ell}(q)$ converges for all rational $q$

Because we only consider $f_n(q)$ when $q \in \mathbb{Q}$ $\Rightarrow$ We can consider $\{f_n\}$ as a sequence from $\mathbb{Q}$ to $[0,1]$

$f_n : \mathbb{Q} \longrightarrow [0,1]$.

So we have $\forall q \in \mathbb{Q}$, $|f_n(q)| \leq 1$ $\Rightarrow$ $\{f_n\}$ pointwise bounded.

By the theorem :

$$\left.\begin{array}{l} f_n : X \text{ (countable)} \longrightarrow \mathbb{C} \\ \{f_n\} \text{ pointwise bounded} \end{array}\right\} \Rightarrow \begin{array}{l} \{f_n\} \text{ contains a pointwise convergent subsequence} \\ \exists \{f_{n_\ell}(x)\} \text{ pointwise bounded.} \end{array}$$

$\Rightarrow$ $f_{n_\ell}(q)$ converges pointwise for all rational $q$ $\Rightarrow \square$

ug2015, P2

rove that $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$ exists.

e consider $s_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n$ and we want to prove that $\lim_{n\to\infty} s_n$ exists.

> NTP $\{s_n\}$ is a monotonic and bounded sequence.

When $n=1$, $s_n=1-\ln 1=1$

hen $n=2$, $s_n=1+\frac{1}{2}-\underbrace{\ln 2}_{0.3}>1$.

Aug 2015 - same with Aug 2003.

$$\boxed{\text{Prove that } \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n \longrightarrow \gamma \text{ for some } \gamma \in (1/2, 1).}$$

Note that with problem requiring to prove $\exists \lim_{n\to\infty} f(s_n)$, we first consider $s_n = f(n)$ before trying another way!

* Put $a_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$.

* First, considering:

$$a_n - a_{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n+1) = \ln(n+1) - \ln(n) - \frac{1}{n+1}.$$

Note that $\ln(n+1) - \ln(n) = \int_{n}^{n+1} \frac{1}{x}\,dx$

and note that

$\frac{1}{x}$ is decreasing in $(n, n+1)$ for any $n > 0$, so we have

$$\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n}, \ \forall x \in (n, n+1).$$

so $\frac{1}{n+1} \le \int_{n}^{n+1} \frac{1}{x}\,dx \le \frac{1}{n}$.

so we have $a_n - a_{n+1} \geqslant \frac{1}{n+1} - \frac{1}{n+1} = 0$ ⇒ This is an decreasing function.

This two things are extremely important in this problem, help solve the problem

* Now we have

$$a_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n = \sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n} \frac{1}{x}\,dx$$

note that $\int_{1}^{n} \frac{1}{x}\,dx = \int_{1}^{2} \frac{1}{x}\,dx + \int_{2}^{3} \frac{1}{x}\,dx + \cdots + \int_{n-1}^{n} \frac{1}{x}\,dx = \underbrace{\sum_{k=1}^{n-1} \int_{k}^{k+1} \frac{1}{x}}_{}$

$$\sum_{k=1}^{n-1} \frac{1}{k+1} \le \sum_{k=1}^{n-1} \int_{1}^{k+1} \frac{1}{x}\,dx \le \sum_{k=1}^{n-1} \frac{1}{k} < \sum_{k=1}^{n} \frac{1}{k}$$

so $a_n \geqslant 0, \forall n$.

So we have $a_n$ decreasing and bounded below by 0 ⇒ converges.

Aug 2015 P3 See Rudin 6.9/139

Is the following integral convergent $\int_1^\infty \frac{\sin x}{x}\,dx$.

By integration by part, we have

$$\int_1^\infty \frac{\sin x}{x}\,dx = -\int_1^\infty (\cos x)'\frac{1}{x}\,dx = -\frac{1}{x}\cos x\Big|_1^\infty + \int_1^\infty \cos(x)\frac{1}{x^2}\,dx =$$

$$= \underbrace{-\lim_{x\to\infty}\frac{\cos x}{x}}_{=0} + \frac{\cos 1}{1} + \int_1^\infty \frac{\cos x}{x^2}\,dx.$$

- we have (*) $\left|\int_1^\infty \frac{\cos x}{x^2}\,dx\right| \le \int_1^\infty \left|\frac{\cos x}{x^2}\right| dx \le \int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -0+1 = 1$. See (**)

$\Rightarrow \int_1^\infty \frac{\cos x}{x^2}\,dx$ converges $\Rightarrow$ the integral $\int_1^\infty \frac{\sin x}{x}\,dx$ converges.

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*Learn from this problem

$$\int \frac{\sin x}{x} = \left[\begin{array}{l} -\int (\cos x)'\frac{1}{x}\,dx \\ \int \sin x\,(\ln x)'\,dx = \underbrace{\sin x \ln x\Big|_1^\infty}_{\sin\infty\,\ln\infty\,?} - \int \ln x\cos x\,dx \end{array}\right.$$

$\Rightarrow$ don't use this way.

- We want to prove $\int_1^\infty f(x)\,dx$ converges. (in this problem $f(x)=\frac{\cos x}{x^2}$)

we prove $\left|\int_1^\infty f(x)\,dx\right| < \dots$

(**): Consider $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\left.\begin{array}{l} \text{We have } f(x)=\frac{1}{x^2} \text{ is a decreasing, continuous, } \geq 0 \\ \sum_{n=1}^\infty \frac{1}{n^2} \text{ converges} \end{array}\right\} \Rightarrow \int_1^\infty \frac{1}{x^2}\,dx \text{ converges by integral test}$$

g 2015, 4

$p_\ell \geq 0$

$\sum_{\ell=1}^{\infty} p_\ell = 1$

Show that $\left(\sum_{\ell=1}^{\infty} k p_\ell\right)^2 \leq \sum_{\ell=1}^{\infty} k^2 p_k$

With this kind We need to prove $\sum_{n=1}^{n} a_n \leq \sum_{n=1}^{n} b_n$ and take $n \longrightarrow \infty$

e have Cauchy-S inequality

$$\left(\sum_{\ell=1}^{n} a_\ell b_\ell\right)^2 \leq \left(\sum_{\ell=1}^{n} a_\ell^2\right)\left(\sum_{\ell=1}^{n} b_\ell^2\right)$$

ly this inequality with $a_\ell = k\sqrt{p_k}$, $b_\ell = \sqrt{p_\ell}$ (note that $p_\ell \geq 0 \Rightarrow \sqrt{p_\ell}$ well defined)

$$\left(\sum_{\ell=1}^{n} k p_\ell\right)^2 \leq \sum_{\ell=1}^{n} (k^2 p_\ell) \sum_{\ell=1}^{n} p_\ell .$$

t $n \longrightarrow \infty$, note that $\sum_{\ell=1}^{\infty} p_\ell = 1$

$$\Rightarrow \left(\sum_{\ell=1}^{\infty} \ell p_\ell\right)^2 \leq \sum_{\ell=1}^{\infty} \ell^2 p_\ell \quad \square .$$

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eview: Cauchy Suatch inequality

$$|\langle \vec{x}, \vec{v} \rangle| \leq \|\vec{x}\| \, \|\vec{v}\|$$

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Aug 2015 / ⑤

Let $\{f_n\}$ equicontinuous on $K$ compact } Prove that $f_n \rightrightarrows$ on $K$ (to $f$)
$\{f_n\}$ converges pointwise (to $f$)

Very useful result + proof *

○ • $\{f_n\}$ equicontinuous on $K$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in K, |x-y| < \delta, |f_n(x) - f_n(y)| < \varepsilon$ (1)

NTP $f_n \rightrightarrows$ on $K$

NTP $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall \underset{n}{m} \geq N$ (∀x

$|f_m(x) - f_n(x)| < \varepsilon$

• $\{f_n\}$ converges pointwise

$\Rightarrow \forall x \in K, \forall \varepsilon > 0, \exists n_x \in \mathbb{N}, \forall m, n \geq n_x, |f_m(x) - f_n(x)| < \varepsilon$ (2)

• $K$ is compact
Let $\{B(x, \delta)\}_{x \in K}$ is a open cover of $K$ } $\Rightarrow \exists$ finite subcover $K \subseteq \bigcup_{i=1}^{k} B(x_i, \delta)$ (3)

This is a really go
trick can be use
in problems relat
to uniformly contin
or equicontinuous

From (2) and (3) We chose $N = \max\{n_{x_1}, n_{x_2}, \dots, n_{x_k}\}$

Now we have $\forall \varepsilon > 0, \exists N, \forall m, n \geq N, |f_m(x_i) - f_n(x_i)| < \varepsilon, \forall x_i \in \{x_1, \dots, x_k\}$ (*)

and also $\forall x \in K$, because of (3), $x \in B(x_{i0}, \delta)$ for some $x_{i0} \in \{x_1, \dots, x_k\}$

this means $|x - x_{i0}| < \delta \xRightarrow[(1)]{} \forall n \in \mathbb{N}, |f_n(x) - f_n(x_{i0})| <$ (

○ In conclusion, $\forall \varepsilon > 0, \exists N, \forall m, n \geq N, \forall x \in K$,

$$|f_m(x) - f_n(x)| \leq \underbrace{|f_m(x) - f_m(x_{i0})|}_{< \varepsilon \text{ (by **)}} + \underbrace{|f_m(x_{i0}) - f_n(x_{i0})|}_{< \varepsilon \text{ (by *)}} + \underbrace{|f_n(x_{i0}) - f_n(x)|}_{< \varepsilon \text{ (by **)}} <$$

this is what we need to do.

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* Somethings learned from this problem:

• We have $K$ is compact.
Then Let $B(x_\alpha, \delta_\alpha)$ is a open cover } $\Rightarrow \exists$ subcover $K \subseteq \bigcup_{i=1}^{k} (x_i, \delta_i)$

But in here, $\{f_n\}$ equicontinuous (1) and from what we NTP,
we should chose $\delta_i = \delta_j = ⓓ$ (all ~~of~~ $K$ is covered by open cover of the same radius)

* Question: $\{f_n\}$ equicontinuous, $K$ compact, $f_n \longrightarrow f$ pointwise } $f_n \rightrightarrows f$ on $K$.

○

g2015/6 $f: \mathbb{R} \longrightarrow \mathbb{R}$

$x) = \begin{cases} x + 2x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

a) Show that $f'(0) = 1$

Show that $f'$ is not continuous at 0

b) Write $y = f(x)$, what does inverse function theorem say/not say about the inverse of $f$ in a neighborhood of $y = 0$. Explain.

Show that $f$ is not 1-1 in any neighborhood of $x = 0$.

Show that $f'(0) = 1$

$0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{t + 2t^2 \sin\frac{1}{t}}{t} = \lim_{t\to 0} 1 + 2t \sin\frac{1}{t}$

e have $0 \leq |2t \sin\frac{1}{t}| \leq |2t|$, $\lim_{t\to 0} |2t| = 0$ $\Rightarrow \lim_{t\to 0} |2t \sin\frac{1}{t}| = 0 \Rightarrow \lim_{t\to 0} 2t \sin\frac{1}{t} = 0$ $\Rightarrow f'(0) = 1$.

Show that $f'$ is not continuous at 0

ere $x \neq 0$, $f'(x) = 1 + 2 \cdot 2x \sin\frac{1}{x} + 2x^2 \left(-\frac{1}{x^2}\right) \cos\frac{1}{x} = 1 + 4x \sin\frac{1}{x} - 2\cos\frac{1}{x}$.

$\underset{0}{n}\cos\frac{1}{x}$ does not exist $\Rightarrow \nexists \lim_{x\to 0} f'(x) \Rightarrow f'$ is not continuous at 0.

) What does inverse function say/not say about inverse of $f$ in a neighborhood of $y = f(x) = 0$.

We have to be satisfies the inverse function theorem.

$f: U$ open in $\mathbb{R} \longrightarrow \mathbb{R}$

$f$ has to be a $C^1$ function ( $\exists f'$ and $f'$ is continuous in $U$ )

$x_0$ where $y_0 = x_0$ ($x_0$ has to be in $U$) and $f'(x_0) \neq 0$

We have $f'$ is not continuous at 0

Then even $f'(x_0) \neq 0$, and $y = f(0) = 0$, but because $f'$ is not a $C^1$ function in a neighborhood of C

$\Rightarrow$ does not satisfy condition of the theorem $\Rightarrow$ say nothing about inverse of $y = 0$ at neighborhood of 0.

If $x_0 \in R$, $x_0 \neq 0$, $f(x_0) = 0 = y$

Then ~~for any~~ neighborhood E of $x_0$ and a neighborhood V of $y = 0$ such that

$f: E \longrightarrow V$ bijective

And let $g = f^{-1}$, then we have $g'(0) = \frac{1}{[f'(x_0)]^{-1}}$

) Show that $f$ is not one to one in a neighborhood of 0

We have $f$ continuous in $\mathbb{R}$.

~~$f'(0) = 1 > 0$~~ then if we can prove that $\exists x_0$, $f'(x_0) < 0$ in a neighborhood of 0, $\exists x_1$, $f'(x_1) > 0$ } by the figure $f$ is not ~~no~~ one to one.

$x_n = \frac{1}{2n\pi} \Rightarrow f'(x_n) = -1 < 0$ → done.

$x_m = \frac{1}{(2n+1)\pi}$ $f'(x_m) = 3 > 0$.

# JANUARY 2016 PRELIMINARY EXAMINATION IN ANALYSIS

1. Let $E \subset \mathbb{R}$ be a nonempty set.

(a) What does it mean to say that $E$ has an upper bound?

(b) When $E$ has an upper bound define $\sup E$, the supremum of $E$.

(c) Give an example of a bounded set $E$ such that $\sup E \notin E$.

(d) If $E$ has an upper bound prove there is a sequence $\{x_n\}$, $x_n \in E$, such that $\lim_{n\to\infty} x_n = \sup E$.

2. Let $f$ be a real valued function defined on a metric space $X$ with distance $d(x,y)$, $x, y \in X$. Prove or disprove the following assertions.

(a) If $f$ is uniformly continuous on $X$ and if $\{x_n\}$, $x_n \in X$, is a Cauchy sequence, then $\{f(x_n)\}$ is Cauchy.

(b) If $f$ is continuous on $X$ and if $\{x_n\}$, $x_n \in X$, is a Cauchy sequence, then $\{f(x_n)\}$ is Cauchy.

3. Let $f$ be a real valued continuous function on the interval $[0,1]$.

(a) If $0 < p < 1$ and $f(x) = x^p \sin(x^{1-p})$, $x \in (0,1]$, compute (the one-sided derivative) $f'(0)$.

(b) Give an example of an $f$ with $f'(x)$ uniformly bounded on $(0,1]$ such that $f'(0)$ does not exist.

(c) Suppose $f'(x)$ is uniformly bounded and nondecreasing for $x \in (0,1]$. Prove $f'(0) = \lim_{x\to 0} f'(x)$.

4. Suppose a non-negative function $f$ has maximum equal to 1 and vanishes on a dense set of points in [0,1]. Let $\beta$ be a nondecreasing continuous function such that $\beta(0) = 0$ and $\beta(1) = 1$. Show that any number $0 < \alpha < 1$ can be obtained as the value of some Riemann sum for the integral $\int_0^1 f\,d\beta$.

5. Let $\mathcal{F}$ be an equicontinuous family of non-negative continuous functions on a metric space $(M, d)$. Let $S$ be dense in $M$ and suppose that for each $x \in S$ we have $f(x) = 0$ for some $f \in \mathcal{F}$. Prove that for any $y \in M$ we have $\inf\{f(y) : f \in \mathcal{F}\} = 0$.

6. Let $f$ and $g$ be $C^1$ real-valued functions such that $f(0) = g(0) = 0$ and $f'(0) = g'(0) = 1$. Show that for any $\epsilon > 0$ there are numbers $x, y$ such that $|x| + |y| < \epsilon$ and $f(x) = g(y) > 0$. Hint: consider the mapping $F(x,y) = (f(x), g(y))$.

Jan2016

1) Let E be a nonempty set, $E \subseteq \mathbb{R}$.

a) What does it mean to say that E has an upper bound.

b) When E has an upper bound, define sup E.

c) Give an example of a bounded E that $\sup E \notin E$.

d) If E has an upper bound prove $\exists (x_n)$, $x_n \in E$, $x_n \longrightarrow \sup E$.

a) E has an upper bound $\Longleftrightarrow$ $\exists\, a$ such that $\forall x \in E$, $x \leq a$.

b) $$\sup E = \alpha \Leftrightarrow \begin{cases} \forall x \in E, x \leq \alpha \\ \forall \varepsilon > 0, \exists x_0 \in E, \ \alpha - \varepsilon < x_0 \end{cases}$$

sup E is the least upper bound of E.

c) <u>Example 1</u>: $E = (0, 1)$, $\sup E = 1$ but $1 \notin E$.

<u>Example 2</u>: $E = \{-\frac{1}{n}, n \in \mathbb{Z}\}$ $\sup E = 0$ but $0 \notin E$.

d) <u>If E has an upper bound prove that $\exists (x_n)$, $x_n \in E$, $x_n \longrightarrow \sup E$.</u>

Let $\alpha = \sup E$, then by def of supremum

$\forall \varepsilon > 0, \exists x_0 \in E, \ \alpha - \varepsilon < x$

Let $\varepsilon = \frac{1}{n}$, then $\exists x_n \in E$, $\alpha - \frac{1}{n} < x_n$

$$\Leftrightarrow x_n - \frac{1}{n} \underset{\substack{\uparrow \\ \text{because } \alpha \text{ is an upper bound.}}}{<} \alpha < x_n + \frac{1}{n}$$

$$\Leftrightarrow |x_n - \alpha| < \frac{1}{n}$$

This means for $\varepsilon$ is given, choose $n_0 \in \mathbb{N}$, s.t $\frac{1}{n_0} < \varepsilon$, then $\forall n \geq n_0$, $|x_n - \alpha| < \frac{1}{n} \leq$

this means $\exists \{x_n\}$, $x_n \longrightarrow \alpha$.

* <u>Note</u>:
$$\left.\begin{array}{l} \text{We have E has an upper bound, } E \subseteq \mathbb{R}, E \neq \phi \\ \mathbb{R} \text{ is an ordered set with least-upper-bound-property} \end{array}\right\} \Rightarrow \begin{array}{l} \exists \alpha = \sup E \\ \text{and } \alpha \in \mathbb{R} \end{array}$$

- Let $f(X,d) \longrightarrow \mathbb{R}$. Prove or disprove

$f$ is uniformly continuous on $X$
$\{x_n\} \in X$, ~~X~~ $\{x_n\}$ is a Cauchy sequence $\Big\}$ $\overset{?}{\Rightarrow}$ $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$

- $f$ is continuous on $X$
$x_n \in X$, $\{x_n\}$ cauchy sequence $\Big\}$ $\overset{?}{\Rightarrow}$ $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$.

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$f$ uniformly continuous on $X$
$\{x_n\} \subset X$, $\{x_n\}$ is a cauchy sequence $\Big\}$ $\Rightarrow$ Prove $\{f(x_n)\}$ Cauchy in $\mathbb{R}$.

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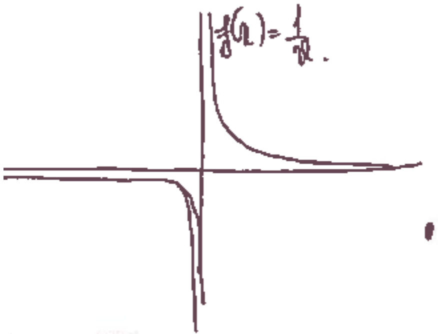
$f$ is uniformly continuous on $X$
$\forall \varepsilon > 0, \exists \delta > 0, \forall x,y \in X, |x-y| < \delta$, then
$|f(x) - f(y)| < \varepsilon$ (1)

NTP $\{f(x_n)\}$ Cauchy in $\mathbb{R}$
$\Leftrightarrow$ NTP $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n \geq N$
$|f(x_n) - f(x_m)| < \varepsilon$

$\{x_n\}$ Cauchy in $X$
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m,n \geq n_0, |x_n - x_m| < \varepsilon$
$\varepsilon = \delta$, then we have $\forall m,n \geq n_0$, $|x_m - x_n| < \delta$ (2)
(1) + (2) $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m,n \geq n_0, |x_m - x_n| < \delta \Rightarrow |f(x_m) - f(x_n)| < \varepsilon$
$\rightarrow$ Da

- An example to show that ~~∃~~ $f$ is continuous in $X$
$\{x_n\}$ Cauchy in $X$ $\Big\}$ but $f\{x_n\}$ is not Cauchy in $\mathbb{R}$.



+ Let $f\ [0,+\infty) \longrightarrow \mathbb{R}$
$f(x) = \frac{1}{x}$ continuous in $[0,+\infty)$

- Let $\{x_n\} = \frac{1}{n}$, then we have $\frac{1}{n} \xrightarrow{n\to\infty} 0$
converges $\rightarrow$ Cauchy

but $f(x_n) = n$, we have $\{f(x_n)\}$ is not Cauchy in $\mathbb{R}$
because $\forall n_0 \in \mathbb{N}$, $\forall m > n \geq n_0$, $|m-n| > 1 > \varepsilon$

Jan2016 ③

Let $f:[0,1]\longrightarrow\mathbb{R}$, $f$ continuous.

ⓐ If $0<p<1$

$f(x)=x^p\sin(x^{1-p})$, $x\in(0,1]$ } Compute the one-sided derivative $f'(0)$

not sure

ⓑ Give an example of an $f$ with { $f'(x)$ uniformly bounded on $(0,1]$

$f'(0)$ does not exist

ⓒ Suppose $f'(x)$ uniformly bounded

$f'(x)$ increasing for $x\in(0,1]$ } Prove $f'(0)=\lim_{x\to0}f'(x)$

a) Because $f$ is continuous on $[0,1]$

$$\Rightarrow f(0)=\lim_{x\to0}f(x)=\lim_{x\to0}x^p\underbrace{\sin(x^{1-p})}_{|\sin(x^{1-p})|<1}=\lim_{x\to0}x^p=0.$$

$$\text{then } f'(0)=\lim_{t\to0^+}\frac{f(t)-f(0)}{t-0}=\lim_{t\to0}\frac{t^p\sin(t^{1-p})}{t}=\lim_{t\to0}\frac{\sin(t^{1-p})}{t^{1-p}}=0.$$

b) Let $f(x)=\begin{cases}x^2\sin\frac{1}{x}, & x\in(0,1]\\ 0, & x=0\end{cases}$ (Note that $g(x)=\begin{cases}x\sin\frac{1}{x}, & x\in(0,1]\\ 0, & x=0\end{cases}$ does not have $g'(x)$ bounded in $(0,1]$.

- For $x\neq0$, $f'(x)=2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)=2x\sin\frac{1}{x}-\cancel{\frac{1}{x}}\cos\frac{1}{x}$
  (means $x\in(0,1]$

  Then we have $|f'(x)|=\left|2x\sin\frac{1}{x}-\cos\frac{1}{x}\right|\leq\left|2x\sin\frac{1}{x}\right|+\left|\cos\frac{1}{x}\right|\leq|2x|+1\leq2+1=$
  $\forall x$.

  This means $f'(x)$ bounded on $(0,1]$.

- $\lim_{x\to0}f'(x)$ does not exist because $\lim_{x\to0}\cos\frac{1}{x}$ does not exist $\Rightarrow\nexists f'(0)$.

c) Suppose $f'(x)$ uniformly bounded, $f'(x)$ increasing for $x\in(0,1]$ } Prove that $f'(0)=\lim_{x\to0}f'(x)$.

?

(From Kor: If $f$ is continuous in $I$, $\exists\lim_{x\to p}f'(x)$ for some $p\in I$ } $\to\exists f'(p)$ and $f'(p)=\lim_{x\to p}f'(x)$)

- We have $x_0=0$ is a limit of $(0,1]$

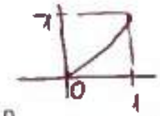
  then $\exists\{x_n\}$, $x_n\subseteq(0,1]$, $x_n\longrightarrow0$

  because $\{f'(x_n)\}$ uniformly bnd, $\{f'(x_n)\}$ increasing } $\to\exists\lim_{x_n\to0}f'(x_n)\Rightarrow\exists\lim_{x\to0}f'(x)$

} $\to\exists f'(0)$

n2016 / 24 $f : [0,1] \longrightarrow \mathbb{R}$

uppose $\begin{cases} f : \text{nonnegative function} . \; f(x) \geq 0, \forall x \in [0,1] \\ \max\limits_{x\in[0,1]} f(x) = 1 . \\ f \text{ vanishes on a dense set of points in } [0,1] . \end{cases}$



et $\beta$ : increasing continuous function, $\beta(0) = 0$, $\beta(1) = 1$

how that any number $0 < \alpha < 1$ can be obtained as the value of some Rieman sum for the integral $\int_0^1 f \, d\beta$.

e need to prove that any number $0<\alpha<1$ can be obtained as the value of some Rieman sum.

Note that upper Rieman sum $\sum (M_i) \Delta\beta_i$

lower Rieman sum $\sum m_i \Delta\beta_i$

o we want to prove that $\begin{cases} 0 < \sum m_i \Delta\beta_i < M_i \Delta\beta_i < 1 \\ (2). \end{cases}$

First, note that for any partition $P = \{x_0 = 0 \leq x_2 \leq \cdots \leq x_n = 1\}$. notations is noted.

because $f$ vanishes on a dense set A of [0, 1].

so in any segment $[x_i, x_{i+1}]$, we have $[x_i, x_{i+1}] \cap A \neq 0$.

this means $\inf\limits_{x\in[x_i,x_{i+1}]} f(x) = 0$ $\forall$ segment.

$\max\limits_{x\in[0,1]} f(x) = 1 . \Rightarrow M_i \leq 1, \forall i$.

This means
$$0 = \underbrace{\sum m_i \Delta\beta_i}_{=0} \leq \sum M_i \Delta\beta_i \leq 1 \sum_{i=1}^{n} \Delta\beta_i = 1 (\overset{=1}{\beta(1)} - \overset{=0}{\beta(0)}] \\ = 1 .$$

Now we need (2),

~~this~~ we means there are many ways to chose partition P such that $\sum_{i=1}^{n} M_i \Delta\beta_i$ can attain

This question is not clear, it would be more suitable to require to prove that some $0<\alpha<1$ can be attained by the Rieman sum of the $\int_0^1 f \, d\beta$.

* In case we only care about Riemann sum $\sum M_i \Delta\beta_i$ (don't care that $f \in \mathcal{R}(\beta)$)

we have we can chose parition s.t $\sum (M_i \Delta\beta_i)$ can attain any value on $[0,1)$.

by using $\sum_{i=1}^{n} M_i \Delta\beta_i \leq \sum_{k=1}^{m} M^*_k \Delta\beta^*_k$ for $n < m$.

Jan 2016 # 5:

Let $\mathcal{F}$ = equicontinuous family of nonnegative functions on a metric space $(M, d)$ means $f: M \longrightarrow$

Let $S$ dense in $M$

Suppose that for each $x \in S$, we have $f(x) = 0$ for some $f \in \mathcal{F}$

Prove that for any $y \in M$, we have $\inf_{f \in \mathcal{F}} \{ f(y), f \in \mathcal{F} \} = 0$

- $\mathcal{F}$ equicontinuous

$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in M$, $d(x,y) < \delta$

then $|f(x) - f(y)| < \epsilon$ (1)

- $S$ dense in $M$

$\Leftrightarrow \forall y \in M, \forall \delta > 0, N_\delta(y) \cap S \neq \phi$

$\Leftrightarrow \forall y \in M, \forall \delta > 0, \exists x \in S, d(x,y) < \delta$ (2)

We NTP, for each $y \in M$

$\inf_{f \in \mathcal{F}} \{ f(y), f \in \mathcal{F} \} = 0$

$\Leftrightarrow$ NTP $\begin{cases} \forall f \in \mathcal{F}, f(y) \geq 0 & \text{done (by } \mathcal{F} \text{ family of nonnegative f} \\ \forall \epsilon > 0, \exists f_0 \in \mathcal{F}, \epsilon > f_0(y) \end{cases}$

- Assumption $\forall x \in S, \exists f_0 \in \mathcal{F}, f_0(x) = 0$ (3)

It's suffices to prove that $\overset{\forall y \in M}{\forall \epsilon > 0}$, $\exists f_0 \in \mathcal{F}$ so that, $\epsilon > f_0(y)$.

- We have $\forall y \in M$, by (2), $\forall \delta > 0, \exists x \in S, d(x,y) < \delta$

then by (1) $\Longrightarrow |f(y) - f(x)| < \epsilon$, $\forall f \in \mathcal{F}$.

- But from (3), $\forall x \in S, \exists f_0 \in \mathcal{F}, f_0(x) = 0$

$\Bigg\} \Rightarrow |f_0(y)| < \epsilon$

$\rightarrow -\epsilon < f_0(y) < \epsilon$

This is what we need to prove $\square$.

2016 P6:

******* Weird.

f and g be $\boxed{C^1}$ $\boxed{\text{real-value}}$ functions such that $f(0) = g(0) = 0$

$f'(0) = g'(0) = 1$.

ow that for any $\varepsilon > 0$, there are numbers $x, y$ such that $\begin{cases} |x| + |y| < \varepsilon \\ f(x) = g(y) > 0 \end{cases}$

nsider $F(x,y) = (f(x), g(y))$ $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto F(x,y) = \big(f(x), g(y)\big)$

hen we have because f and g are $C^1 \Rightarrow F$ is a $C^1$ function (1).

$(0,0) \in \mathbb{R}^2$ and $F(0,0) = (f(0), g(0)) = 0$.

$$F' = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 > 0, \ \forall (x,y) \in \mathbb{R}^2$$

$\Rightarrow F'(0,0) > 0$.

hen by inverse function theorem, $\exists$ an (open) neighborhood $U$ of $(0,0)$ and a open neighborhood $V$ of $(0,0)$ such that $F : U \longrightarrow V$ is bijective

d $\exists$ a $C^1$ bijective function $G : V \longrightarrow U$

$\vec{z} \longmapsto G(\vec{z}) = F^{-1}\big|_U(\vec{z})$ note that $\boxed{c > 0}$

Because $V$ is open, then $\exists N_c(0) \subseteq V$, this means $(c,c) \in V$ such that $\exists, x, y \in U, G(c,c) = F^{-1}(c,c) = (x,y)$ this means $F(x,y) = \big(f(x), g(y)\big) = (c,c)$ and $c > 0$

ote that with a problem requiring $\exists\ f(x) \neq f(y)$ or some other requirement about $f(x)$ and $f(y)$

e only need to consider $(c,d)$ in the domain and prove that $\exists (x,y)$ s.t $\big(c = f(x), d = g(y)\big)$

## Analysis Preliminary Exam, May 2016

1. (i) Give an example of a sequence of real numbers $\{a_n\}_{n\geq 1}$ such that the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} a_n^2$ diverges.

(ii) If $a_n \geq 0$ for all $n \geq 1$ and the series $\sum_{n=1}^{\infty} a_n$ converges show that the series $\sum_{n=1}^{\infty} a_n^2$ must converge.

2. Let $X, Y$ be metric spaces and $f : X \longrightarrow Y$ be a continuous function such that for every compact $K \subset Y$, $f^{-1}(K)$ is a compact subset of $X$. If $F \subset X$ is closed, prove that $f(F)$ is closed in $Y$.

! compare with : $\left. \begin{matrix} f: X \longrightarrow Y \text{ continuous} \\ X \text{ compact} \end{matrix} \right\} \Rightarrow f(X)$ compact in $Y$.

3. Let $f, g : \mathbb{R} \longrightarrow (0, +\infty)$ be differentiable functions such that $g'(x) > 0$ for all $x$, $\lim_{x\to+\infty} g(x) = +\infty$, and $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L$ for some number $L > 0$. Show that $\lim_{x\to+\infty} \frac{\log f(x)}{\log g(x)} = 1$.

Compare with Jan 2009 II/4.

4. Let $f \colon [0,1] \to \mathbb{R}$ be an integrable function. Prove that there exists $a \in (0,1)$ such that $\int_0^a |f(x)|\, dx \leq \int_a^1 |f(x)|\, dx$.

5. Let $K$ be a compact subset of a metric space $X$. Given a bounded sequence $\{x_n\}$ in $X$, define $f_n(x) = d(x, x_n) - d(x, x_1)$ for $n = 1, 2, \ldots$ Prove that there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on $K$.

6. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous differentiable function such that $f(0) = 0$ and $f(1) = 1$. Prove that there exists a point in $\mathbb{R}^2$ where the map

$$\mathbf{F}(x_1, x_2) = (x_1 + x_2^3,\ f(x_1) + x_2)$$

does not satisfy the assumptions of the Inverse Function Theorem.

① : i) Give an example of sequence real numbers $\{a_n\}_{n\geq 1}$ s.t $\sum a_n$ converges but $\sum a_n^2$ diverges

- Let $a_n = \frac{(-1)^n}{\sqrt{n}} = c_n b_n$ where $c_n = (-1)^n$, $\forall n \geq 1$
$b_n = \frac{1}{\sqrt{n}}$, $\forall n \geq 1$

Then we have $\sum c_n$ has bounded partial sum $s_n = \left[\begin{matrix}1\\0\end{matrix}\right.$ (bounded)

$b_n \geq 0, \forall n$, $b_n$ decreasing and $b_n \downarrow 0$

$\Bigg\} \Rightarrow \sum \frac{(-1)^n}{\sqrt{n}}$ converges.

but $\sum a_n^2 = \sum \frac{1}{n}$ diverges (geometric series ~~with p = 1~~ $\sum \frac{1}{n^p}$ with $p = 1$).

ii) If $a_n \geq 0$, $\forall n \geq 1$, $\sum a_n$ converges $\Rightarrow \sum a_n^2$ converges.

Let $X, Y$ be a metric spaces

$f: X \longrightarrow Y$ be a continuous function such that for every compact $K \subseteq Y$,

$f^{-1}(K)$ is a compact subset of $X$.

If $F \subset X$ is closed in $X$. Prove that $f(F)$ is closed in $Y$.

* Need to review

Let $F \subset X$ is closed. We NTP $f(E)$ is closed in $Y$

NTP if $\forall (y_n) \subseteq f(E)$ and $y_n \longrightarrow y$, then $y \in f(E)$.

his means let $(y_n) \subset f(E)$ and $y_n \to y$. Then need to prove $\exists x_0$ such that, $y = f(x_0)$

ecause $y_n \subset f(E)$, this means $\exists x_n \subset E$, and $f(x_n) = y_n$

this means $f(x_n) \to y$.

Note that we have $(\{f(x_n)\} \cup \{y\})$ is a compact set.

So we have because of the assumption that $f^{-1}(K)$ is compact for every $K$ compact in $Y$.

we have $f^{-1}[\{f(x_n)\} \cup \{y\}]$ is compact in $X$.

This means $\{x_n\} \cup \{f^{-1}(y)\}$ is compact in $X$.

we have because $\{x_n\}$ bounded (because $\{x_n\}$ in a compact set in $X$)

then $\exists x_{n_l} \longrightarrow x_0$, because $f$ continuous $f(x_{n_l}) \to f(x_0)$.

from above $f(x_n) \longrightarrow y$.

So we have $y = f(x_0)$.

this means □ ♡

---

ne thing we learn from this problem is that. (See more in Aug 2006).

ven $f$ is continuous and $f(x_n) \to y$

does not mean $\exists$ ~~$x_0 = f^{-1}(y)$ s.t. $x_n \to x_0$~~

$x_n \longrightarrow x_0$, then $y = f(x_0)$.

For example: $f(x): \mathbb{R} \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$x \longmapsto f(x) = \arctan x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

May 2016

P3

ay 2016, P4.

t $f:[0,1] \longrightarrow \mathbb{R}$ be an integrable function

rove that there exists $a \in (0,1)$ s.that $\int_0^a |f(x)|dx \leq \int_a^1 |f(x)|dx$.

e need to prove that $\exists a \in (0,1)$ $\int_a^1 |f(x)|dx - \int_0^a |f(x)|dx \geqslant 0$

$$\Leftrightarrow \int_0^1 |f(x)|dx - 2\int_0^a |f(x)|dx \geqslant 0.$$

t $F(y) := \int_0^y |f(x)|dx$, then we have
$F(y) > 0$ since $|f(x)| \geqslant 0$, $\forall x \in [0,1]$
$F(y)$ is a continuous
$F(y)$ is an increasing function.

Then we have $\exists a \in (0,1)$ such that

$$F(1) - 2F(a) \geqslant 0,$$

which means $\int_0^1 |f(x)|dx - 2\int_0^a |f(x)|dx \geqslant 0$ □.

May 2016 >

P5 > Let K be a compact subset of a metric space $X$

Given a bounded sequence $\{x_n\}$ in K,

define $f_n(x) = d(x, x_n) - d(x, x_1)$, for $n = 1, 2, \dots$

Prove that there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on K

We have :

(1) : K is compact set .

(2) : $\{f_n\}$ is a sequence of continuous function on K because it's a subtraction of 2 continuous functions.

Thus $d(x,a)$ is a continuous function because $|d(x,a) - d(y,a)| \leq d(x,y)$ (Liptchiz).

(3) $\{f_n\}$ is a sequence of pointwise bounded function because (thus uniformly bounded)

$|f_n(x)| = |d(x, x_n) - d(x, x_1)| \leq |d(x_1, x_n)| \leq M$ because $\{x_n\}$ bounded .

(4) Prove that $\{f_n\}$ equicontinuous :

NTP $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in K, d(x,y) < \delta$ then $|f_n(x) - f_n(y)| < \varepsilon, \forall n$.

We have

$$|f_n(x) - f_n(y)| = |d(x, x_n) - d(x, x_1) - d(y, x_n) + d(y, x_1)|$$
$$\leq |d(x, x_n) - d(y, x_n)| + |d(x, x_1) - d(y, x_1)|$$
$$\leq 2d(x,y)$$

So for all $\varepsilon > 0$, choose $\delta$ s.t $2\delta < \varepsilon$, so we have

$$|f_n(x) - f_n(y)| \leq 2(d(x,y)) = 2\delta < \varepsilon$$

Then from (1)(2)(3)(4) + applying Azela Ascoli theorem

$\exists \{f_{n_k}\}$ converges uniformly on K.

y 2016 7 P6 7

uppose $f:\mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable function such that .

$$f(0)=0 \quad f(1)=1$$

Prove that there exists a point in $\mathbb{R}^2$ where the map

$$F(x_1,x_2) = (x_1 + x_2^3 \, , \, f(x_1) + x_2)$$

does not satisfies the assumption of the Inverse Function theorem .

$$)F = \begin{bmatrix} 1 & 3x_2^2 \\ f'(x_1) & 1 \end{bmatrix} = \quad \Rightarrow \det(DF) = 1 - f'(x_1)\,3x_2^2$$

ote that because $f(0)=0$, $f(1)=1$ $\Rightarrow$ ~~1 =~~ $f(1) - f(0) = f'(\xi)\,1 = 1$ for some $\xi \in (0,1)$

So $\exists\, \xi \in (0\,1)$ s.t $\det(DF) = 1 - 3x_2^2 = 0$ when $x_2^2 = \frac{1}{3} \Rightarrow x_2 = \frac{1}{\sqrt{3}}$ or $\frac{-1}{\sqrt{3}}$

his means $\exists$ point $(\xi, \frac{1}{\sqrt{3}})$ or $(\xi, \frac{-1}{\sqrt{3}})$ in $\mathbb{R}^2$ where $\det DF = 0$ does not satisfy the assumption of Inverse function theorem .