DEFINITIONS...

1. Ordered Set: An ordered set is a set E on which is defined a relation R $\ni$
 1) Transitive ie xRy and $yRz \Rightarrow xRz$
 2) Trichotomy Law ie $\forall x,y \in E$ either $x=y$, xRy, or yRx
2. Bounded Above: Let S be an ordered set and $E \subseteq S$. E is bounded above if $\exists y \in S \ni \forall x \in E$, $x \le y$. In this case y is an upper bound of E
3. Least Upper Bound: Let S be an ordered set and $E \subseteq S$. If E has an upperbound y, then y is a least upper bound for E if $y \le z$ for any upperbound z.
4. Least Upper Bound Property: An ordered set has the least upper bound property if every nonempty subset having an upper bound has a least upper bound
5. Field: A field is a set F with two operations $+, \cdot$ $\ni$ $(F,+)$ is an abelian group, $(F \setminus \{0\}, \cdot)$ is an abelian group and $a \cdot (b+c) = a \cdot b + a \cdot c$
6. Ordered Field: Let $F(+,\cdot)$ be a field which is also an ordered set. Then F is an ordered field if
 a) $x<y \Rightarrow x+z<y+z$ $\forall z \in F$
 b) $x>0$ and $y>0 \Rightarrow xy>0$
7. Complex Modulus: if $z \in \mathbb{C}$, $|z| = \sqrt{z \cdot \bar{z}}$ is the complex modulus of z
8. Norm: Let $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. The norm of x is $|x| = \sqrt{x \cdot x} = \sqrt{\sum x_i^2}$
9. Algebraic: A complex number x is algebraic if it is a root of a polynomial $a_n x^n + \dots + a_0$, $a_i \in \mathbb{Z}$, $a_n \ne 0$
10. Cardinality: Two sets A, B have the same cardinality, $A \sim B$, if $\exists f: A \to B$ bijective
11. Finite: Let $J_n = \{1, \dots, n\}$ for some $n \in \mathbb{Z}$. A is finite if $A \sim J_n$ for some n
12. Infinite: A is infinite if A is not finite
13. Countable: A is countable if $A \sim \mathbb{N}$
14. At Most Countable: A is at most countable if it is either finite or countable
15. Uncountable: A is uncountable if it is neither finite or countable
16. Metric Space: A metric space is a pair (X, d) where X is a set and $d: X \times X \to \mathbb{R}^+ \ni$
 1) $d(x,y) = d(y,x)$ 2) $d(x,y) = 0 \Rightarrow x=y$ 3) $d(x,z) \le d(x,y) + d(y,z)$

17. Convex: A set $E \subseteq \mathbb{R}^k$ is convex if $\lambda x + (1-\lambda)y \in E$ whenever $x, y \in E$ and $0<\lambda<1$

18. Neighborhood: The neigborhood of x radius r is $N_r(x) = \{y \in X \mid d(x,y) < r$

19. Limit Point: If $A \subseteq X$, then $x \in X$ is a limit point of A if for each $r>0$ $\exists y \in N_r(x) \cap A \ni y \neq x$

20. Isolated Point: If $x \in A$ and x is not a limit point of A, then x is an isolated point of E

21. Closed: A is closed if every limit point of A is contained in A

22. Interior Point: A point x is an interior point of A if $\exists r>0 \ni N_r(x) \subseteq E$

23. Open: A is open if every point of A is an interior point of A

24. Perfect: A is perfect if A is closed and every point of A is a limit point of A ie. A has no isolated points

25. Bounded: A is bounded if $\exists M \in \mathbb{R}$ and $y \in X \ni d(x,y) < M \ \forall x \in A$

26. Dense: A is dense in X if every point of X is a limit point of A or a point of A, or both

27. $E' = \{x \mid x$ limit point of $E\}$

28. Closure: The closure of E is $\overline{E} = E \cup E'$

29. Interior: The interior of E is $E^\circ = \{x \in E \mid x$ interior point of $E\}$

30. Open Cover: If $K \subseteq X$ is a set, then an open cover of K is a collection $\{V_\alpha\}_{\alpha \in I}$ of open sets $\ni K \subseteq \bigcup_{\alpha \in I} V_\alpha$

31. Compact: K is compact if every open cover $\{V_\alpha\}_{\alpha \in I}$ has a finite subcover ie. $\exists J \subseteq I$ finite $\ni K \subseteq \bigcup_{\alpha \in J} V_\alpha$

32. Sequentially Compact: A set K is sequentially compact if every sequence $x_n \in K$ has a convergent subsequence

33. Diameter: If $K \subseteq X$, the diameter of K is $\operatorname{diam}(K) = \sup\{d(x,y) : x, y \in K$

34. Bounded: A set is bounded if it has finite diameter

35. Separated: Two subsets $A, B \subseteq X$ are separated if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$

36. Connected: A set $E \subseteq X$ is connected if E is not a union of two nonempty separated sets. OR: E is connected if $\nexists A \subseteq X \ni A \neq X, \emptyset \ni A$ both open and closed

37. Converge: A sequence $\{x_n\} \in X$ converges if $\exists x \in X \ni$ for each $\epsilon > 0$ $\exists N \ni n \geq N \Rightarrow d(x_n, x) < \epsilon$, denoted $x_n \to x$ or $\lim_{n\to\infty} x_n = x$

If this is not the case, $\{x_n\}$ diverges

38. Subsequence: Let $\{x_n\}$ be a sequence and $\{n_k\}$ a sequence of positive integers $\ni n_1 < n_2 < \ldots$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$

39. Subsequential Limit: If $\{x_{n_k}\}$ converges, its limit is a subsequential limit of $\{x_n\}$

40. Cauchy sequence: A sequence $\{x_n\} \in X$ is a cauchy sequence if for each $\epsilon > 0$ $\exists N \ni d(x_n, x_m) < \epsilon$ if $n, m \geq N$

41. Complete: A metric space in which every cauchy sequence converges is complete

42. Nondecreasing: $\{x_n\} \in \mathbb{R}$ is nondecreasing if $x_n \leq x_{n+1}$, $n = 1, 2, \ldots$

43. Nonincreasing: $\{x_n\} \in \mathbb{R}$ is nonincreasing if $x_n \geq x_{n+1}$, $n = 1, 2, \ldots$

44. Increasing: $\{x_n\} \in \mathbb{R}$ is increasing if $x_n < x_{n+1}$, $n = 1, 2, \ldots$

45. Decreasing: $\{x_n\} \in \mathbb{R}$ is decreasing if $x_n > x_{n+1}$, $n = 1, 2, \ldots$

46. Monotone: $\{x_n\} \in \mathbb{R}$ is monotone if it is nondecreasing or nonincreasing

47. $x_n \to +\infty$ if for each $M \in \mathbb{R}$ $\exists N \ni n \geq N \Rightarrow x_n \geq M$

48. $x_n \to -\infty$ if for each $M \in \mathbb{R}$ $\exists N \ni n \geq N \Rightarrow x_n \leq M$

50. $\overline{\lim}\, x_n = \sup\{r \in \mathbb{R} \mid r = \lim_{k \to \infty} x_{n_k}\} = \lim_{n \to \infty} \sup\{x_k : k \geq n\}$

51. $\underline{\lim}\, x_n = \inf\{r \in \mathbb{R} \mid r = \lim_{k \to \infty} x_{n_k}\} = \lim_{n \to \infty} \inf\{x_k : k \geq n\}$

52. Partial Sum: The n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \sum_{j=1}^{n} a_j$

53. Converges: If $s_n \to s$, $\sum_{n=1}^{\infty} a_n$ converges and its sum is s. If $\{s_n\}$ diverges, the series diverges.

54. $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \to \infty} (1 + \frac{1}{n})^n$

55. Power Series: Let $\{c_n\} \in \mathbb{C}$, then $\sum_{n=0}^{\infty} c_n z^n$ is a power series

56. Absolute Convergence: The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

57. Conditional Convergence: If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ converges conditionally

58. Cauchy Product: If $\sum a_n$, $\sum b_n$ are series, and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $\sum c_n$ is the product of $\sum a_n$, $\sum b_n$

59. Rearrangement: If $\varphi : \mathbb{N} \to \mathbb{N}$ is bijective then $\sum_{j=1}^{\infty} x_{\varphi(j)}$ is a rearrangement of $\sum_{j=1}^{\infty} x_j$

60. Limit of Function: Let X, Y be metric spaces, $E \subseteq X$, $f : E \to Y$, p limit point of E

Then $\lim_{x \to p} f(x) = q$ if $\exists q \in Y \ni$ for each $\epsilon > 0$ $\exists \delta > 0 \ni \forall x \in E$ with $d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \epsilon$

61. Continuous: f is continuous at x_0 if for each $\epsilon > 0$ $\exists \delta > 0 \ni d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \epsilon$ OR: f is continuous at x_0 if $f(N_\delta(x_0)) \subseteq N_\epsilon(f(x_0))$
OR: f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$

62. Continuous: If $f: X \to Y$, $E \subseteq X$, then f is continuous on E if f is continuous at each $x_0 \in E$

63. Bounded: $f: E \to \mathbb{R}^k$ is bounded if $\exists M \in \mathbb{R} \ni |f(x)| \leq M \ \forall x \in E$

64. Lipschitz: $f: X \to Y$ is Lipschitz if $d(f(x), f(y)) \leq M d(x,y)$

65. Uniformly Continuous: $f: X \to Y$ is uniformly continuous on $A \subseteq X$ if for each $\epsilon > 0$ $\exists \delta > 0 \ni d(x_1, x_2) < \delta$, $x_1, x_2 \in A \Rightarrow d(f(x_1), f(x_2)) < \epsilon$ where δ depends only on ϵ

66. Pathwise Connected: X is pathwise connected if for any $x_1, x_2 \in X$ $\exists \gamma: [0,1] \to X$ continuous $\ni \gamma(0) = x_1$ and $\gamma(1) = x_2$

67. Right-Hand Limit: $f(x+) = \lim_{y \to x^+} f(y) = L$ if for any $\epsilon > 0$ $\exists \delta > 0 \ni$ $x < y < x + \delta \Rightarrow |f(y) - L| < \epsilon$

68. Left-Hand Limit: $f(x-) = \lim_{y \to x^-} f(y) = L$ if for any $\epsilon > 0$ $\exists \delta > 0 \ni$ $x - \delta < y < x \Rightarrow |f(y) - L| < \epsilon$

69. Type I Discontinuity: If f is discontinuous at x and if $f(x+)$ and $f(x-)$ exist, then f has a type I discontinuity

70. Type II Discontinuity: If f is discontinuous at x and at least one of $f(x+)$, $f(x-)$ fails to exist then f has a type II discontinuity

71. Increasing: f is increasing if $x < y \Rightarrow f(x) \leq f(y)$

72. Decreasing: f is decreasing if $x < y \Rightarrow f(x) \geq f(y)$

73. Monotone: f is monotone if f is either increasing or decreasing

74. $D_f = \{x : f \text{ discontinuous at } x\}$

75. Differentiable: f is differentiable at x_0 if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. In this case $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ OR: f is differentiable at x_0 if $\exists \varphi_{x_0}: \mathbb{R} \to \mathbb{R}$ continuous at $x_0 \ni \varphi_{x_0}(x_0) = 0$ and $\exists \alpha \in \mathbb{R} \ni f(x) = f(x_0) + \alpha(x - x_0) + \varphi_{x_0}(x)(x-$ and in this case $f'(x_0) = \alpha$

76. Local Maximum: Let $f: X \to \mathbb{R}$. f has a local maximum at a point $p \in X$ if $\exists \delta > 0 \ni f(q) \leq f(p)$ $\forall q \in X$ with $d(p,q) < \delta$

77. Local Minimum: Let $f: X \to \mathbb{R}$, f has a local minimum at a point $p \in X$ if $\exists \delta > 0 \ni f(q) \geq f(p)$ $\forall q \in X$ with $d(p,q) < \delta$

78. Lower Riemann Integral: Let $f: [a,b] \to \mathbb{R}$ bounded, $P = \{a = x_0 < \ldots < x_n = b\}$ a partition, $m_i = \inf\{f(x): x_{i-1} \leq x \leq x_i\}$, $M_i = \sup\{f(x): x_{i-1} \leq x \leq x_i\}$, $\Delta x_i = x_i - x_{i-1}$, $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$, $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$. Then the lower Riemann integral of f is $\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f): P \text{ partition}\}$

79. Upper Riemann Integral: The upper Riemann integral of f is $\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f): P \text{ partition}\}$

80. Riemann Integrable: f is Riemann integrable on $[a,b]$, denoted $f \in \mathcal{R}$, if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ and in this case $\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

81. Rieman-Stieltjes Integrable: Let $\alpha: [a,b] \to \mathbb{R}$ be increasing, $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$, $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$, $L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$. f is Riemann Stieltjes integrable wrt α if $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

82. Refinement: P^* is a refinement of P if $P \subseteq P^*$

83. Common Refinement: If P_1, P_2 are partitions, $P_1 \cup P_2$ is a common refinement of P_1, P_2

84. $I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$, $I(x-5) = \begin{cases} 0 & x \leq 5 \\ 1 & x > 5 \end{cases}$

85. Integration of Vector Valued Functions: Let $f = (f_1, \ldots, f_k): [a,b] \to \mathbb{R}^k$, α increasing on $[a,b]$. If $f_j \in \mathcal{R}(\alpha)$ $\forall j$ then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f\,d\alpha = \left(\int_a^b f_1\,d\alpha, \ldots, \int_a^b f_k\,d\alpha\right)$

86. Curve: If $\gamma: [a,b] \to \mathbb{R}^k$ is continuous, γ is a curve

87. Closed Curve: If $\gamma(a) = \gamma(b)$, γ is a closed curve

88. Arc: If γ injective, γ is an arc

89. Length: Let $P = \{a = x_0 < \ldots < x_n = b\}$ be a partition of $[a,b]$, $\gamma: [a,b] \to \mathbb{R}^k$, $\Lambda(P,\gamma) = \sum_{j=1}^{n} |\gamma(x_j) - \gamma(x_{j-1})|$. The length of γ is $\Lambda(\gamma) = \sup\{\Lambda(P,\gamma): P \text{ partition}\}$

90. Rectifiable: γ is rectifiable if $\Lambda(\gamma) < +\infty$

91. Pointwise Convergence: Let $f_n, f: E \to \mathbb{C}$. $\{f_n\}$ converges pointwise to f if $\forall x \in E$ $\forall \epsilon > 0$ $\exists N = N(\epsilon, x) \ni n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$ ie $f_n(x) \to f(x)$ as $n \to \infty$ $\forall x \in E$

92. Uniform Convergence: $\{f_n\}$ converges uniformly to f on E, $f_n \rightrightarrows f$, if $\forall \epsilon > 0$ $\exists N = N(\epsilon) \ni n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$ $\forall x \in E$

93. Uniform Convergence: Let $f_n: E \to \mathbb{C}$. $\sum f_n$ converges uniformly on E if

$S_n = \sum_{k=1}^{n} f_k$ converges uniformly on E

94. $\mathscr{C}(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous and bounded}\}$ where X metric space

95. Sup Norm: If $f \in \mathscr{C}(X)$, the sup norm of f is $\|f\| = \sup\{|f(x)| : x \in X\}$

96. Pointwise Bounded: $\{f_n\}$ is pointwise bounded if $\exists \varphi : E \to [0, +\infty) \ni$ $|f_n(x)| \le \varphi(x)$ $\forall n \ge 1$ $\forall x \in E$

97. Uniformly Bounded: $\{f_n\}$ is uniformly bounded if $\exists M > 0 \ni |f_n(x)| \le M$ $\forall n \ge 1$ $\forall x \in E$

98. Equicontinuous: Let $E \subseteq (X, d)$, $\mathcal{F}$ family of functions $f : E \to \mathbb{C}$. $\mathcal{F}$ is equicontinuous if $\forall \epsilon > 0$ $\exists \delta = \delta(\epsilon) \ni x, y \in E$, $d(x,y) < \delta \Rightarrow$ $|f(x) - f(y)| < \epsilon$ $\forall f \in \mathcal{F}$

99. Uniformly closed: $\mathcal{A} \subseteq \mathcal{F}(E, \mathbb{C})$ is uniformly closed if $f \in \mathcal{F}(E, \mathbb{C})$ and $\exists f_n \in \mathcal{A} \ni f_n \rightrightarrows f$ on $E \Rightarrow f \in \mathcal{A}$

100. Uniform closure: $\overline{\mathcal{A}} = \{f \in \mathcal{F}(E, \mathbb{C}) \mid \exists f_n \in \mathcal{A} \ni f_n \rightrightarrows f \text{ on } E\}$ is the uniform closure of $\mathcal{A}$

101. Algebra: $\mathcal{A} \subseteq \mathcal{F}(E, \mathbb{C})$ is an algebra of functions if $f, g \in \mathcal{A}$, $c \in \mathbb{C}$ $\Rightarrow f+g, fg, cf \in \mathcal{A}$

102. Separates Points: An algebra $\mathcal{A} \subseteq \mathcal{F}(E, \mathbb{C})$ separates points on E if for each $x_1 \ne x_2$ in E, $\exists f \in \mathcal{A}$ with $f(x_1) \ne f(x_2)$

103. Does Not Vanish: An algebra $\mathcal{A} \subseteq \mathcal{F}(E, \mathbb{C})$ vanishes at no point of E if for each $x \in E$, $\exists f \in \mathcal{A} \ni f(x) \ne 0$

104. Self Adjoint: An algebra $\mathcal{A} \subseteq \mathcal{F}(E, \mathbb{C})$ is self adjoint if $\overline{f} \in \mathcal{A}$ whenever $f \in \mathcal{A}$

105. Power Series: If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges $\forall x \in (-R, R)$ for some $R > 0$, f is expanded in a power series about $x = 0$

106. Power Series: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x - a| < R$, f is expanded in a power series about $x = a$

107. Analytic: $f : (\alpha, \beta) \to \mathbb{R}$ is analytic on (α, β) if $\forall a \in (\alpha, \beta)$ $\exists \delta > 0 \ni$ $(a - \delta, a + \delta) \subseteq (\alpha, \beta)$ and $c_n = c_n(a) \in \mathbb{R} \ni f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $(a - \delta, a + \delta)$

108. $e^x = \sup\{e^r : r \le x, r \in \mathbb{Q}\} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

109. Trigonometric Polynomial: $f(x) = a_0 + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx = \sum_{n=-N}^{N} c_n e^{inx}$ is a trigonometric polynomial

110. Trigonometric Series: $\sum_{-\infty}^{+\infty} c_n e^{inx} = \lim_{N \to \infty} \sum_{-N}^{N} c_n e^{inx}$ is a trigonometric series

111. Fourier Series: $\sum_{-\infty}^{+\infty} c_n e^{inx}$ is the Fourier series of f with coefficients $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$

112. Orthogonal: Let $\{\varphi_n\}_{n\geq 1}$, $\varphi_n \in \mathcal{R}[a,b]$, $\varphi_n : [a,b] \to \mathbb{C}$. If $\int_a^b \varphi_n \overline{\varphi_m} = 0 \ \forall n \neq m$, $\{\varphi_n\}_{n\geq 1}$ is orthogonal

113. Orthonormal: Let $\{\varphi_n\}_{n\geq 1}$, $\varphi_n \in \mathcal{R}[a,b]$, $\varphi_n : [a,b] \to \mathbb{C}$. If $\{\varphi_n\}_{n\geq 1}$ is orthogonal and $\int_a^b \varphi_n \overline{\varphi_n} = |\varphi_n|^2 = 1$, $\{\varphi_n\}_{n\geq 1}$ is orthonormal

114. Dirichlet Kernel: The Dirichlet Kernel is $D_N(x) = \sum_{-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$

115. Gamma Function: The gamma function is $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\,dt$, $x > 0$

116. Beta Function: The beta function is $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, $x,y > 0$

117. Range: The range of A is $\mathcal{R}(A) = A(X)$

118. Nullspace/Kernel: The nullspace/Kernel of A is $\mathcal{N}(A) = \{x \in X \mid A(x) = 0\}$

119. $L(X,Y) = \{A : X \to Y \mid A \text{ linear}\}$

120. Norm: $|\cdot|$ is a norm if $|x| = 0 \Leftrightarrow x = 0$, $|cx| = |c||x| \ \forall c \in \mathbb{R}$, $|x+y| \leq |x| + |y|$

121. $\|A\| = \sup_{|x|\leq 1} |Ax|$ for $A \in L(\mathbb{R}^n, \mathbb{R}^m)$

122. $\Omega = \{A \in L(\mathbb{R}^n) \mid A \text{ invertible}\}$

123. Differentiable: Let $f : E \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$, $x \in E$. f is differentiable at x if $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m) \ni \lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$. In this case $A = f'(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$ is the differential of f at x

124. Partial Derivative: The partial derivative of f_i wrt x_j is $D_j f_i(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t} = \lim_{t\to 0} \frac{f_i(x_1, \ldots, x_j + t, \ldots, x_n) - f_i(x_1, \ldots, x_n)}{t}$

125. Gradient: Let $f : \mathbb{R}^n \to \mathbb{R}$. The gradient of f is $\nabla f(x) = (D_1 f(x), \ldots, D_n f(x))$

126. Directional Derivative: Let $f : E \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}$, $x \in E$, $f'(x)$ exists, u unit vector in $\mathbb{R}^n$. The directional derivative is $D_u f(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t}$

127. Continuously Differentiable: $f : E \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ differentiable on E is continuously differentiable on E, $f \in \mathcal{C}'(E)$, if $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous ie $\forall x \in E \ \forall \epsilon > 0 \ \exists \delta = \delta(x, \epsilon) > 0 \ni$ if $y \in E$, $|y - x| < \delta \Rightarrow \|f'(y) - f'(x)\| < \epsilon$

128. Contraction: Let (X,d) be a metric space. $\varphi : X \to X$ is a contraction if $\exists c \in (0,1) \ni d(\varphi(x), \varphi(y)) < c\, d(x,y) \ \forall x,y \in X$

129. $(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$

130. Jacobian: Let $f : E \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ differentiable at $x \in E$. The Jacobian is $J_f(x) = \det f'(x) = \left| \frac{\partial f_i(x)}{\partial x_j} \right|$

131. Second Order Partials: Let $f : E \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}$, $\exists D_1 f, \ldots, D_n f : E \to \mathbb{R}$. The second order

partial derivatives are $D_{ij}f = D_iD_jf$

132. $f \in \mathcal{C}''(E)$ if all $D_{ij}f$ are continuous on E

133. $f \in \mathcal{C}^r(E)$ if all partial derivatives of order $\le r$ exist and are continuous

134. $f \in \mathcal{C}^\infty(E)$ if $f \in \mathcal{C}^r(E)$ $\forall r \geq 0$

135. Partial Derivative of Order $|\alpha|$: Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ be the length of α. The partial derivative of f of order $|\alpha|$ is
$D^\alpha f = \underbrace{D_1 \dots D_1}_{\alpha_1} \dots \underbrace{D_n \dots D_n}_{\alpha_n} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$

136. Taylor Polynomial: The Taylor Polynomial of f at x_0 of order $r-1$ is $T^{x_0}_{r-1}(x) = \sum_{|\alpha| = r-1} \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha$ where $\alpha! = \alpha_1! \dots \alpha_n!$ and
$(x - x_0)^\alpha = (x_1 - x_0)^{\alpha_1} \dots (x_n - x_0)^{\alpha_n}$

137. Hessian: The Hessian of f at a is $H(a) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) \end{bmatrix}$

THEOREMS...

1. $a, b \in \mathbb{Q}$ and $a<b \Rightarrow \exists c \in \mathbb{Q} \ni a<c<b$
2. $\exists \mathbb{R}$ an ordered field which has the least upper bound property containing $\mathbb{Q}$ as an ordered subfield
3. Archimedean Property $x, y \in \mathbb{R}$ and $x>0 \Rightarrow \exists n \in \mathbb{N} \ni nx>y$
4. $\mathbb{Q}$ dense in $\mathbb{R}$ $x, y \in \mathbb{R}$ and $x<y \Rightarrow \exists p \in \mathbb{Q} \ni x<p<y$
5. $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z} \ni x, n>0 \ \exists! y \in \mathbb{R} \ni y>0$ with $y^n = x$
6. $z, w \in \mathbb{C} \Rightarrow$
   a) $\overline{z+w} = \bar{z} + \bar{w}$
   b) $\overline{zw} = \bar{z}\,\bar{w}$
   c) $z+\bar{z} = 2\mathrm{Re}(z) \qquad z-\bar{z} = 2i\,\mathrm{Im}(z)$
   d) $|z|>0$ unless $z=0$ in which case $|z|=0$
   e) $|\bar{z}| = |z|$
   f) $|zw| = |z||w|$
   g) $|\mathrm{Re}(z)| \le |z|$
   h) $|z+w| \le |z|+|w|$
7. Schwarz Inequality $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C} \Rightarrow \left|\sum_{j=1}^{n} a_j \bar{b_j}\right| \le \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$
8. $x, y, z \in \mathbb{R}^k \Rightarrow |x-z| \le |x-y| + |y-z|$
9. X countable, $Y \subseteq X$ infinite $\Rightarrow Y$ countable
10. A at most countable, $f: A \to B$ surjective $\Rightarrow B$ at most countable
11. B at most countable, $f: A \to B$ injective $\Rightarrow A$ at most countable
12. $\{E_n\}_{n \ge 1}$ sequence of countable sets $\Rightarrow \bigcup_{n=1}^{\infty} E_n$ countable
13. A countable, $B_n = \{(a_1, \dots, a_n) : a_k \in A\} \Rightarrow B_n$ countable
14. $\mathbb{Q}$ countable, $\mathbb{R}$ uncountable, $\mathbb{R} \setminus \mathbb{Q}$ uncountable
15. $N_r(x)$ open
16. p limit point of $E \Rightarrow \forall r>0$, $N_r(p)$ contains infinitely many points of E
17. E set containing finitely many points $\Rightarrow E$ has no limit points
18. $\emptyset, X$ both open and closed
19. C closed $\Leftrightarrow C^c$ open
20. U open $\Leftrightarrow U^c$ closed
21. $\{G_\alpha\}_{\alpha \in I}$ open sets $\Rightarrow \bigcup_{\alpha \in I} G_\alpha$ open
22. $\{F_\alpha\}_{\alpha \in I}$ closed sets $\Rightarrow \bigcap_{\alpha \in I} F_\alpha$ closed
23. $G_1, \dots, G_n$ open sets $\Rightarrow \bigcap_{i=1}^{n} G_i$ open

24. $F_1, \dots, F_n$ closed sets $\Rightarrow \bigcup_{i=1}^{n} F_i$ closed

25. X metric space, $E \subseteq X \Rightarrow$

a) $\bar{E}$ closed

b) $E = \bar{E} \iff E$ closed

c) $\bar{E} \subseteq F \quad \forall F \subset X$ closed $\ni E \subset F$

26. $\phi \neq E \subseteq \mathbb{R}$ bounded above, $y = \sup E \Rightarrow y \in \bar{E}$ ie E closed $\Rightarrow y \in E$

27. $\bar{E} = \cap \{C \mid C$ closed and $E \subseteq C\}$ ie $\bar{E}$ smallest closed set containing E

28. $Y \subseteq X$. E open in $Y \iff E = Y \cap G$ for some open $G \subseteq X$

29. $K \subseteq Y \subseteq X$. K compact in $X \iff K$ compact in Y

30. $K \subseteq X$ compact $\Rightarrow K$ closed

31. $C \subseteq K$, C closed, K compact $\Rightarrow C$ compact

32. F closed, K compact $\Rightarrow F \cap K$ compact

33. $\{K_\alpha\}_{\alpha \in I}$ compact $\Rightarrow \bigcap_{\alpha \in I} K_\alpha$ compact

34. Finite Intersection Property $\{K_\alpha\}_{\alpha \in I}$ compact, intersection of every finite subcollection nonempty $\Rightarrow \bigcap_{\alpha \in I} K_\alpha \neq \phi$

35. $\{K_n\}_{n \geq 1}$ nonempty, compact sets $\ni K_n \supseteq K_{n+1} \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \phi$

36. $E \subseteq K$, E infinite, K compact $\Rightarrow E$ has a limit point in K

37. $\{I_n\}_{n \geq 1}$ intervals in $\mathbb{R}$ $\ni I_n \supseteq I_{n+1} \Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$

38. Heine-Borel Theorem $E \subseteq \mathbb{R}^k$. TFAE:

a) E closed and bounded

b) E compact

c) Every infinite subset of E has a limit point in E

39. Weierstrauss Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$

40. Subsequence Principle $x_n \to x \iff$ every subsequence of x_n has a further subsequence converging to x

41. $x_n \to x \iff$ every subsequence $x_{n_k} \to x$

42. X metric space. K compact $\Rightarrow K$ sequentially compact

43. $x_n \in K \subseteq X$ compact. x_n converges $\iff \exists!$ subsequential limit

44. K compact $\Rightarrow K$ bounded

45. $\phi \neq P \subseteq \mathbb{R}^k$, P perfect $\Rightarrow P$ uncountable

46. $E \subseteq \mathbb{R}$ connected $\iff$ If $x, y \in E$ and $x < z < y$, then $z \in E$

47. $\{p_n\} \in X$ metric space $\Rightarrow$

a) $p_n \to p \Leftrightarrow$ every nbd of p contains p_n for all but finitely n

b) $\{p_n\}$ converges $\Rightarrow \{p_n\}$ bounded

c) $E \subseteq X$, p limit point of $E \Rightarrow \exists \{p_n\} \in E \ni p_n \to p$

48. $\{s_n\}, \{t_n\} \in \mathbb{C} \ni s_n \to s,\ t_n \to t \Rightarrow$

a) $s_n + t_n \to s + t$

b) $cs_n \to cs \qquad c + s_n \to c + s \qquad \forall c$

c) $s_n t_n \to st$

d) $\frac{1}{s_n} \to \frac{1}{s}$ if $s_n, s \neq 0$

49. $\{p_n\} \in X$ compact $\Rightarrow$ some subsequence of $\{p_n\}$ converges to a point of X

50. $\{p_n\} \in X$ metric space $\Rightarrow \{$subsequential limits of $\{p_n\}\}$ closed subset of X

51. $\operatorname{diam} \overline{E} = \operatorname{diam} E$

52. $\{K_n\}$ sequence of compact sets in $X \ni K_n \supset K_{n+1}$ and $\lim\limits_{n\to\infty} \operatorname{diam} K_n = 0$

$\Rightarrow \bigcap\limits_{n=1}^{\infty} K_n$ consists of exactly one point

53. X metric space. $\{x_n\}$ convergent $\Rightarrow \{x_n\}$ cauchy

54. X complete metric space, $\{x_n\}$ cauchy in $X \Rightarrow \{x_n\}$ converges to some point in X

55. In $\mathbb{R}^k$, every cauchy sequence converges

56. X compact $\Rightarrow X$ complete

57. $\{x_n\}$ monotonic. $\{x_n\}$ converges $\Leftrightarrow \{x_n\}$ bounded

58. $\{x_n\}$ monotonic $\Rightarrow \lim\limits_{n\to\infty} x_n$ exists (possibly $\pm\infty$)

59. $\overline{\lim}\, x_n = +\infty \Leftrightarrow \{x_n\}$ not bounded above

60. $\overline{\lim}\, x_n = -\infty \Leftrightarrow \lim\limits_{n\to\infty} x_n = -\infty$

61. $\underline{\lim}\, x_n = -\infty \Leftrightarrow \{x_n\}$ not bounded below

62. $\underline{\lim}\, x_n = +\infty \Leftrightarrow \lim\limits_{n\to\infty} x_n = +\infty$

63. $\lim\limits_{n\to\infty} x_n$ exists $\Leftrightarrow \overline{\lim}\, x_n = \underline{\lim}\, x_n$ and in this case $\lim\limits_{n\to\infty} x_n = \overline{\lim}\, x_n = \underline{\lim}\, x_n$

64. $\overline{\lim}\, x_n = l \Rightarrow \exists x_{n_k} \to l$

65. $\underline{\lim}\, x_n = -\overline{\lim}\,(-x_n)$

66. $\overline{\lim}\,(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$

67. $\underline{\lim}\,(x_n + y_n) \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$

68. $\overline{\lim}\,(x_n - y_n) \leq \overline{\lim}\, x_n - \underline{\lim}\, y_n$

69. $s_n \leq t_n$ for $n \geq N \Rightarrow \underline{\lim}\, s_n \leq \underline{\lim}\, t_n$, $\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$

70. $p>0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$

71. $p>0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$

72. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

73. $p>0,\ \alpha \in \mathbb{R} \Rightarrow \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

74. $|x|<1 \Rightarrow \lim_{n\to\infty} x^n = 0$

75. Cauchy Criterion $\sum a_n$ converges $\Leftrightarrow \forall \epsilon > 0\ \exists N \ni \left|\sum_{k=n}^{m} a_k\right| < \epsilon$ for $m \geq n \geq N$

76. $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

77. $\sum_{j=1}^{\infty} x_j \ni x_j \geq 0$ converges $\Leftrightarrow S_n$ bounded above

78. Comparison Test

a) $|a_n| \leq c_n$ for $n \geq N_0$ and $\sum c_n$ converges $\Rightarrow \sum a_n$ converges

b) $a_n \geq d_n \geq 0$ for $n \geq N_0$ and $\sum d_n$ diverges $\Rightarrow \sum a_n$ diverges

79. $\sum_{n=0}^{\infty} x^n$ converges for $x \in [0,1)$ with sum $\frac{1}{1-x}$, but diverges for $x \geq 1$

80. Cauchy Condensation Test $x_1 \geq x_2 \geq \dots \geq 0$. $\sum_{n=1}^{\infty} x_n$ converges $\Leftrightarrow$
$\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges

81. $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$

82. n^{th} Term Test $\lim_{n\to\infty} x_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} x_n$ diverges

83. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p>1$ and diverges if $p \leq 1$

84. $\sum a_n$ converges, $a_n \geq 0 \Rightarrow \exists b_n \geq 0 \ni$

1) $\sum b_n < \infty$

2) $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$

85. Root Test $\alpha = \overline{\lim} \sqrt[n]{|a_n|}$

a) $\alpha < 1 \Rightarrow \sum a_n$ converges

b) $\alpha > 1 \Rightarrow \sum a_n$ diverges

c) $\alpha = 1 \Rightarrow$ test fails

86. Ratio Test $\sum a_n$ converges if $\overline{\lim} \left|\frac{a_{n+1}}{a_n}\right| < 1$ and diverges if
$\left|\frac{a_{n+1}}{a_n}\right| \geq 1\ \forall n \geq n_0$

87. $\alpha = \overline{\lim} \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha} \Rightarrow \sum c_n z^n$ converges if $|z|<R$ and diverges
if $|z|>R$ (R radius of convergence)

88. Summation By Parts $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$, $A_{-1} = 0 \Rightarrow$ for $0 \leq p \leq q$
$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$ OR: $\sum_{j=1}^{n} x_j \Delta t_j = t_n x_n - \sum_{j=1}^{n} t_{j-1}\Delta$

89. $A_n = \sum_{k=0}^{n} a_k$ bounded, $b_0 \geq b_1 \geq \dots$, $\lim_{n\to\infty} b_n = 0 \Rightarrow \sum a_n b_n$ converges

90. $|c_1| \geq |c_2| \geq \dots$, $c_{2m-1} \geq 0$, $c_{2m} \leq 0$, $\lim_{n\to\infty} c_n = 0 \Rightarrow \sum c_n$ converges

91. Alternating Series Test $a_1 \geq a_2 \geq \dots$, $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

92. Dirichlet Theorem $t_n = \sum_{j=1}^{n} y_j$ bounded, $x_1 \geq x_2 \geq \dots$, $\lim_{n\to\infty} x_n = 0 \Rightarrow$ $\sum_{j=1}^{\infty} x_j y_j$ converges (possibly conditionally)

93. $\sum c_n z^n$, $R = 1$, $c_0 \geq c_1 \geq \dots$, $\lim_{n\to\infty} c_n = 0 \Rightarrow \sum c_n z^n$ converges at every point on circle $|z| = 1$ except possibly at $z = 1$

94. $\sum a_n$ converges absolutely $\Longrightarrow$ $\sum a_n$ converges

95. $\sum a_n = A$, $\sum b_n = B \Rightarrow \sum a_n + b_n = A + B$ and $\sum c a_n = cA \ \forall c$

96. Mertens $\sum a_n$ converges absolutely, $\sum a_n = A$, $\sum b_n = B$, and $c_n = \sum_{k=0}^{n} a_k b_{n-k} \Rightarrow \sum_{n=0}^{\infty} c_n = AB$

97. Abel $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C, and $c_n = a_0 b_n + \dots + a_n b_0$ $\Rightarrow C = AB$

98. Riemann $\sum x_j$ converges conditionally, $-\infty < \alpha \leq \beta < \infty \Rightarrow \exists$ rearrangement $\ni$ $\underline{\lim} \sum_{j=1}^{n} x_{\varphi(j)} = \alpha$ and $\overline{\lim} \sum_{j=1}^{n} x_{\varphi(j)} = \beta$

99. $\sum a_n$ converges absolutely $\Rightarrow$ every rearrangement of $\sum a_n$ converges to the same sum

100. $f: E \subseteq X \to Y$, p limit point of E, $\lim_{x\to p} f(x) = q \Leftrightarrow \lim_{n\to\infty} f(p_n) = q \ \forall \{p_n\} \in E \ni$ $p_n \neq p$ and $p_n \to p$

101. $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B \Rightarrow$
a) $\lim_{x\to p} (f+g)(x) = A + B$
b) $\lim_{x\to p} (fg)(x) = AB$
c) $\lim_{x\to p} \frac{f}{g}(x) = \frac{A}{B}$ if $B \neq 0$

102. $f: X \to Y$, $g: Y \to Z$, f continuous at x_0, g continuous at $y_0 = f(x_0) \Rightarrow g \circ f: X \to Z$ is continuous at x_0

103. $f: X \to Y$ continuous on $X \Longleftrightarrow f^{-1}(V)$ is open in X $\forall$ open $V \subseteq Y$

104. $f: X \to Y$ continuous on $X \Longleftrightarrow f^{-1}(C)$ is closed in X $\forall$ closed $C \subseteq Y$

105. $f, g: X \to \mathbb{C}$ continuous $\Rightarrow f+g, fg, \frac{f}{g}$ continuous on X

106. $f = (f_1, \dots, f_k): X \to \mathbb{R}^k$ continuous $\Leftrightarrow f_1, \dots, f_k$ continuous

107. $f, g: X \to \mathbb{R}^k$ continuous $\Rightarrow f+g, f \cdot g$ continuous

108. $f: X \to Y$ continuous, $K \subseteq X$ compact $\Rightarrow f(K)$ compact

109. $f: X \to \mathbb{R}^k$ continuous, X compact $\Rightarrow f$ bounded

110. $f: X \to \mathbb{R}$, X compact, $M = \sup_{p\in X} f(p)$, $m = \inf_{p\in X} f(p) \Rightarrow \exists p, q \in X \ni f(p) = M$ and $f(q) = m$

111. $f: X \to Y$ continuous bijection, X compact $\Rightarrow f^{-1}: Y \to X$ continuous bijection

112. f Lipschitz $\Rightarrow$ f uniformly continuous

113. $f: X \to Y$ continuous, X compact $\Rightarrow$ f uniformly continuous on X

114. $f: X \to Y$ continuous, $E \subseteq X$ connected $\Rightarrow f(E)$ connected

115. Intermediate Value Theorem $f: [a,b] \to \mathbb{R}$, $f(a) < c < f(b) \Rightarrow \exists x \in (a,b) \ni f(x) = c$ (where f is continuous)

116. X pathwise connected $\Rightarrow$ X connected

117. $\lim_{y \to x} f(y)$ exists $\Leftrightarrow f(x-) = f(x+) = \lim_{y \to x} f(y)$

118. f has type I discontinuity at $x \Rightarrow$ either $f(x+) \neq f(x-)$ or $f(x+) = f(x-) \neq f(x$

119. f increasing on $(a,b) \Rightarrow f(x+), f(x-)$ exist $\forall x \in (a,b)$ and $\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$. And if $a<x<y<b$, $f(x+) \leq f(y-)$

120. f monotonic $\Rightarrow$ f has no discontinuities of type II

121. f monotonic $\Rightarrow D_f$ at most countable

122. f differentiable at $x_0 \Rightarrow$ f continuous at x_0

123. f, g differentiable at $x \Rightarrow f+g, fg, \frac{f}{g}$ differentiable at x and

a) $(f+g)'(x) = f'(x) + g'(x)$

b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

124. Chain Rule g differentiable at x_0, f differentiable at $y_0 = g(x_0) \Rightarrow$ $f \circ g$ differentiable at x_0 and $(f \circ g)'(x_0) = f'(y_0)g'(x_0)$

125. f has local maximum at x, $f'(x)$ exists $\Rightarrow f'(x) = 0$

126. Rolle's Thm f continuous on $[a,b]$, differentiable on (a,b), $f(a) = f(b)$ $\Rightarrow f'(x) = 0$ for at least one $x \in (a,b)$

127. Mean Value Theorem $f, g: [a,b] \to \mathbb{R}$ continuous on $[a,b]$, differentiable on $(a,b) \Rightarrow \exists x \in (a,b) \ni [f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$

128 The Mean Value Theorem $f: [a,b] \to \mathbb{R}$ continuous, differentiable on $(a,b) \Rightarrow \exists x \in (a,b) \ni f(b) - f(a) = (b-a)f'(x)$

129. f differentiable in (a,b), $f'(x) \geq 0 \quad \forall x \in (a,b) \Rightarrow$ f increasing

130. f differentiable in (a,b), $f'(x) = 0 \quad \forall x \in (a,b) \Rightarrow$ f constant

131. f differentiable in (a,b), $f'(x) \leq 0 \quad \forall x \in (a,b) \Rightarrow$ f decreasing

132. $f: [a,b] \to \mathbb{R}$ differentiable, $f'(a) < \lambda < f'(b) \Rightarrow \exists x \in (a,b) \ni f'(x) = \lambda$

133. f differentiable on $[a,b] \Rightarrow f'$ has no type I discontinuities

134. L'Hospital's Rule $f,g:(a,b)\to\mathbb{R}$ differentiable, $g'(x)\neq 0\ \forall x\in(a,b)$, $-\infty\le a<b\le+\infty$, $\frac{f'(x)}{g'(x)}\to A$ as $x\to a$. If $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ or if $g(x)\to+\infty$ as $x\to a$ then $\frac{f(x)}{g(x)}\to A$ as $x\to a$

135. Taylor's Theorem $f:[a,b]\to\mathbb{R}$, $n\in\mathbb{Z}^+$, $f^{(n-1)}$ continuous on $[a,b]$, $f^{(n)}(t)$ exists $\forall t\in(a,b)$, $\alpha,\beta\in[a,b]\ \ni \alpha<\beta$, $P(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \Rightarrow \exists x\in(\alpha,\beta)\ni$ $f(\beta)=P(\beta)+\frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$

136. Mean Value Thm for vector valued functions $f:[a,b]\to\mathbb{R}^k$ continuous on $[a,b]$, differentiable on $(a,b)\Rightarrow\exists x\in(a,b)\ni|f(b)-f(a)|\le(b-a)|f'(x)|$

137. $m\le f(x)\le M\ \ \forall x\in[a,b]\Rightarrow m(b-a)\le L(P,f)\le U(P,f)\le M(b-a)\ \forall$ partitions P

138. $P\subseteq P^*\Rightarrow L(P,f,\alpha)\le L(P^*,f,\alpha)\le U(P^*,f,\alpha)\le U(P,f,\alpha)$

139. P_1,P_2 partitions of $[a,b]\Rightarrow L(P_1,f,\alpha)\le\underline{\int_a^b} f\,d\alpha\le\overline{\int_a^b} f\,d\alpha\le U(P_2,f,\alpha)$

140. $f\in\mathcal{R}(\alpha)\Leftrightarrow\forall\epsilon>0\ \exists P$ partition of $[a,b]\ni 0\le U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$

141. Given $\epsilon>0$, P partition $\ni U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$
(i) $P\subseteq P^*\Rightarrow U(P^*,f,\alpha)-L(P^*,f,\alpha)<\epsilon$
(ii) $s_i,t_i\in[x_{i-1},x_i]\Rightarrow\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i<\epsilon$
(iii) $t_i\in[x_{i-1},x_i]\Rightarrow\left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|<\epsilon$

142. f continuous on $[a,b]\Rightarrow f\in\mathcal{R}(\alpha)$ on $[a,b]$

143. f continuous on $[a,b]$ except at finitely many points $s_1,\dots,s_N$, α continuous at each $s_i\Rightarrow f\in\mathcal{R}(\alpha)$

144. f monotonic, α continuous $\Rightarrow f\in\mathcal{R}(\alpha)$

145. $f\in\mathcal{R}(\alpha)$ on $[a,b]$, $m\le f\le M$, $\phi:[m,M]\to\mathbb{R}$ continuous, $h=\phi\circ f\Rightarrow h\in\mathcal{R}(\alpha)$

146. $f_1,f_2\in\mathcal{R}(\alpha)$ on $[a,b]$, $c\in\mathbb{R}\Rightarrow f_1+f_2\in\mathcal{R}(\alpha)$, $cf_1\in\mathcal{R}(\alpha)$ and $\int_a^b(f_1+f_2)\,d\alpha=\int_a^b f_1\,d\alpha+\int_a^b f_2\,d\alpha$, $\int_a^b(cf_1)\,d\alpha=c\int_a^b f_1\,d\alpha$

147. $f\in\mathcal{R}(\alpha_1)\cap\mathcal{R}(\alpha_2)$, $c\ge 0\Rightarrow f\in\mathcal{R}(c\alpha_1)$, $f\in\mathcal{R}(\alpha_1+\alpha_2)$ and $\int_a^b f\,d(c\alpha)=c\int_a^b f\,d\alpha$, $\int_a^b f\,d(\alpha_1+\alpha_2)=\int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2$

148. $f_1\le f_2\Rightarrow\int_a^b f_1\,d\alpha\le\int_a^b f_2\,d\alpha$

149. $|f(x)|\le M\Rightarrow\left|\int_a^b f\,d\alpha\right|\le M(\alpha(b)-\alpha(a))$

150. $f\in\mathcal{R}(\alpha)$ on $[a,b]$, $a<c<b\Rightarrow\int_a^b f\,d\alpha=\int_a^c f\,d\alpha+\int_c^b f\,d\alpha$

151. $f\in\mathcal{R}(\alpha)$ on $[a,c]$ and $[c,b]\Rightarrow f\in\mathcal{R}(\alpha)$ on $[a,b]$

152. $f,g\in\mathcal{R}(\alpha)$ on $[a,b]\Rightarrow fg\in\mathcal{R}(\alpha)$ on $[a,b]$

153. $|f|\in\mathcal{R}(\alpha)\Rightarrow\left|\int_a^b f\,d\alpha\right|\le\int_a^b|f|\,d\alpha$

154. $a<s<b$, f continuous at s, $\alpha(x)=I(x-s) \Rightarrow \int_a^b f\,d\alpha = f(s)$

155. $c_n \ge 0$, $\sum_{n=1}^{\infty} c_n < +\infty$, $s_n \in (a,b)$ $n=1,2,\dots$ distinct, f continuous on $[a,b]$, $\alpha(x)=\sum_{n=1}^{\infty} c_n I(x-s_n) \Rightarrow f \in \mathcal{R}(\alpha)$ and $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$

156. α differentiable on $[a,b]$, $\alpha' \in \mathcal{R}$, f bounded on $[a,b]$. $f \in \mathcal{R}(\alpha) \Leftrightarrow f\alpha' \in \mathcal{R}$ on $[a,b]$ and in this case $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$

157. Change of Variables $f,\alpha:[a,b]\to\mathbb{R}$ bounded, $\varphi:[A,B]\to[a,b]$ strictly increasing, continuous, surjective, $\beta=\alpha\circ\varphi:[A,B]\to\mathbb{R}$ increasing, $g=f\circ\varphi:[A,B]\to\mathbb{R}$. Then $f\in\mathcal{R}(\alpha)\Rightarrow g\in\mathcal{R}(\beta)$ and $\int_a^b f\,d\alpha = \int_A^B g\,d\beta$

158. $f\in\mathcal{R}$ on $[a,b]$, $F(x)=\int_a^x f(t)\,dt$, $a\le x\le b \Rightarrow F$ continuous on $[a,b]$. And f continuous at $x_0\in[a,b] \Rightarrow F$ differentiable at x_0 and $F'(x_0)=f(x_0)$

159. Fundamental Theorem of Calculus $f\in\mathcal{R}$ on $[a,b]$, $\exists F$ differentiable on $[a,b] \ni F'=f \Rightarrow \int_a^b f(x)\,dx = F(b)-F(a)$

160. Integration by Parts F,G differentiable on $[a,b]$, $f=F'$, $g=G'\in\mathcal{R}$ on $[a,b]$ $\Rightarrow FG', F'G \in \mathcal{R}$ on $[a,b]$ and $\int_a^b FG'\,dx = F(b)G(b)-F(a)G(a)-\int_a^b F'G\,dx$

161. $f:[a,b]\to\mathbb{R}^k$, $f=(f_1,\dots,f_k)$, $f_j:[a,b]\to\mathbb{R}$, $\alpha:[a,b]\to\mathbb{R}$ increasing. $f\in\mathcal{R}(\alpha)\Leftrightarrow f_j\in\mathcal{R}(\alpha)$ $\forall j=1,\dots,k$ and in this case $\int_a^b f\,d\alpha = \left(\int_a^b f_1\,d\alpha,\dots,\int_a^b f_k\,d\alpha\right)$ (FTC holds, $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$)

162. $\gamma\in\mathcal{C}'([a,b]) \Rightarrow \Lambda(\gamma)=\int_a^b |\gamma'(t)|\,dt$

163. $M_n=\sup\{|f_n(x)-f(x)| : x\in E\}$. Then $f_n \rightrightarrows f$ on $E \Leftrightarrow M_n\to 0$ as $n\to\infty$

164. Cauchy's criterion ① $f_n\rightrightarrows f$ on $E \Leftrightarrow \forall\epsilon>0\ \exists N(\epsilon) \ni m,n\ge N \Rightarrow$ $|f_m(x)-f_n(x)|<\epsilon\ \forall x\in E$

② $\sum_{n=0}^{\infty} f_n$ converges uniformly on $E \Leftrightarrow \forall\epsilon>0\ \exists N(\epsilon) \ni N\le n\le m \Rightarrow$ $|f_n(x)+\dots+f_m(x)|<\epsilon\ \forall x\in E$

165. Weierstrass M-test $f_n:E\to\mathbb{C}$, $|f_n(x)|\le M_n\ \forall x\in E$, $\sum_{n=1}^{\infty} M_n<+\infty$ $\Rightarrow \sum_{n=1}^{\infty} f_n$ converges uniformly on E

166. $f_n, f: E\to\mathbb{C}$, $E\subseteq X$ metric space, $f_n\rightrightarrows f$ on E, $x\in E'$, $A_n := \lim_{t\to x} f_n(t)$ exists $\Rightarrow \{A_n\}$ convergent and $\exists \lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$ ie $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$

167. $f_n:E\to\mathbb{C}$ continuous, $f_n\rightrightarrows f$ on $E \Rightarrow f$ continuous on E

168. Dini's Theorem $f_n, f: X\to\mathbb{R}$, X compact metric space, f_n, f continuous, $\forall x, f_n(x)\downarrow f(x) \Rightarrow f_n\rightrightarrows f$

169. $f_n, f\in\mathcal{C}(X)$, $d(f,g)=\|f-g\|$. Then $f_n\to f$ in $(\mathcal{C}(X),d) \Leftrightarrow f_n\rightrightarrows f$ on X

170. $\mathcal{A} = \overline{\mathcal{A}} \Leftrightarrow \mathcal{A}$ uniformly closed

171. $(\mathcal{C}(X), d)$ complete

172. $\alpha : [a,b] \to \mathbb{R}$ increasing, $f_n \in \mathcal{R}(\alpha)$, $f_n \rightrightarrows f$ on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ and $\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$

173. $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$, $f = \sum_{n=1}^{\infty} f_n$ converges uniformly on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ and $\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$

174. $f_n : [a,b] \to \mathbb{R}$ differentiable on $[a,b]$, $\exists x_0 \in [a,b] \ni \{f_n(x_0)\}$ converges, $\{f_n'\}$ converges uniformly on $[a,b] \Rightarrow f_n$ converges uniformly on $[a,b]$ to a differentiable function $f : [a,b] \to \mathbb{R}$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$

175. <u>Lebesgue's Dominated Convergence Theorem</u> $h_n, h : [a,b] \to \mathbb{R}$, $h_n, h \in \mathcal{R}$, $h_n \to h$, $\{h_n\}$ uniformly bounded $\Rightarrow \int_a^b h(x)dx = \lim_{n\to\infty} \int_a^b h_n(x)dx$

176. $f_n : K \to \mathbb{C}$, K compact, f_n continuous, $\{f_n\}$ converges uniformly on K $\Rightarrow \{f_n\}$ equicontinuous

177. K compact metric space $\Rightarrow \exists E \subseteq K$ at most countable and dense

178. <u>Diagonalization</u> E countable, $f_n : E \to \mathbb{C}$ pointwise bounded $\Rightarrow \exists \{f_{n_k}\}$ subsequence which converges pointwise on E

179. <u>Ascoli-Arzela</u> K compact metric space, $f_n : K \to \mathbb{C}$, $\{f_n\}$ equicontinuous and pointwise bounded $\Rightarrow$
1) $\{f_n\}$ uniformly bounded
2) $\exists \{f_{n_k}\}$ subsequence which converges uniformly on K

180. $g_i : K \to \mathbb{C}$ equicontinuous, pointwise bounded, g_i converges pointwise on a dense subset $E \subseteq K$ compact $\Rightarrow g_i$ converges uniformly on K

181. <u>Weierstrass</u> $f : [a,b] \to \mathbb{C}$ continuous $\Rightarrow \exists \{P_n\}$ sequence polynomials $\ni P_n \rightrightarrows f$ on $[a,b]$. If f real valued we can choose P_n to be real valued

182. For each $[-a,a]$, $a > 0$, $\exists$ sequence of real valued polynomials $\{P_n\} \ni P_n(x) \rightrightarrows |x|$ on $[-a,a]$ and $P_n(0) = 0$

183. $\mathcal{A} \subseteq \mathcal{B}(E, \mathbb{C})$ algebra $\Rightarrow \overline{\mathcal{A}}$ algebra

184. $\mathcal{A} \subseteq \mathcal{F}(E, \mathbb{C})$ algebra that separates points and vanishes at no point on E, $x_1 \neq x_2 \in E$, $c_1, c_2 \in \mathbb{C} \Rightarrow \exists f \in \mathcal{A}$ with $f(x_1) = c_1$ and $f(x_2) = c_2$

185. <u>Stone-Weierstrauss</u> $\mathcal{A} \subseteq \mathcal{C}(K, \mathbb{R})$ algebra which separates points and vanishes at no point on K, K compact $\Rightarrow \overline{\mathcal{A}} = \mathcal{C}(K, \mathbb{R})$: $\forall f \in \mathcal{C}(K, \mathbb{R})$ $\exists f_n \in \mathcal{A} \ni f_n \rightrightarrows f$ on K

186. $\mathcal{A} \subseteq \mathcal{C}(K, \mathbb{C})$ selfadjoint, separates points, vanishes at no point of K, K compact $\Rightarrow \overline{\mathcal{A}} = \mathcal{C}(K, \mathbb{C})$

187. $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$ converges absolutely in $(a-R, a+R)$ and diverges on $(-\infty, a-R) \cup (a+R, +\infty)$

188. $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, $R > 0$. Then $\epsilon > 0 \Rightarrow f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges absolutely and uniformly on $[-R+\epsilon, R-\epsilon]$ and f is continuous and differentiable on $(-R, R)$ with $f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$ converging on $|x| < R$

189. radius of convergence of $\sum_{n=1}^{\infty} n c_n x^{n-1}$ and $\sum_{n=0}^{\infty} c_n x^n$ are $R_0, R \Rightarrow R_0 = R$

190. $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R > 0 \Rightarrow f \in C^{\infty}(-R, R)$ and $f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n x^{n-k}$, $|x| < R$

191. $c_k = \frac{f^{(k)}(0)}{k!}$

192. $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges on $|x| < 1$, $\sum_{n=0}^{\infty} c_n$ converges $\Rightarrow \exists \lim_{x \nearrow 1} f(x) = \sum_{n=0}^{\infty} c_n$

193. $\{a_{ij}\}_{j \geq 1}^{i \geq 1}$, $\sum_{j=1}^{\infty} |a_{ij}| = b_i$, $\sum_{i=1}^{\infty} b_i < +\infty \Rightarrow \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$

194. $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $|x| < R$, $a \in (-R, R) \Rightarrow f(x)$ can be expanded in a power series centered at a, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, $|x-a| < R - |a|$

195. Weak Identity Theorem $\sum a_n x^n = \sum b_n x^n$, $-R < x < R$, $R > 0 \Rightarrow a_n = b_n \; \forall n \geq$

196. Identity Theorem $\sum a_n x^n$, $\sum b_n x^n$ convergent in $(-R, R)$, $R > 0$, $E = \{x \in (-R, R) : \sum a_n x^n = \sum b_n x^n\}$. If $E' \cap (-R, R) \neq \emptyset$ then $a_n = b_n$ $\forall n \geq 0$, $\sum a_n x^n = \sum b_n x^n$ $\forall x \in (-R, R)$

197. $E(z+w) = E(z)E(w)$

198. $x > 0 \Rightarrow E(x) > 1$

199. $x < 0 \Rightarrow E(x) \in (0, 1)$

200. $x > 0 \Rightarrow E(x) > 1 + x$

201. $E(x) \to +\infty$ as $x \to +\infty$, $E(x) \to 0$ as $x \to -\infty$

202. $E(x)$ strictly increasing, bijective

203. $E \in C^{\infty}(\mathbb{R})$

204. $\lim_{x \to +\infty} x^n e^{-x} = 0 \quad \forall n \in \mathbb{N}$

205. $E^{-1} = L : (0, +\infty) \to \mathbb{R}$ bijective strictly increasing

206. $L(1) = 0$, $\lim_{x \searrow 0} L(x) = -\infty$, $\lim_{x \to +\infty} L(x) = +\infty$

207. $L'(y) = \frac{1}{y}$, $L(y) = \int_1^y \frac{dt}{dt}$, $y > 0$

208. $L(uv) = L(u) + L(v)$, $L\left(\frac{1}{u}\right) = -L(u)$

209. $x^{\alpha} = E(\alpha L(x))$

210. $\forall \alpha > 0$, $\lim_{x\to+\infty} \frac{\log x}{x^{\alpha}} = 0$

211. $z = x + iy \Rightarrow \bar{z} = x - iy$, $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, $|z|^2 = z\bar{z} = x^2 + y^2$

212. $\overline{E(z)} = E(\bar{z})$

213. $E(ix) = C(x) + iS(x)$, $C(x) = \frac{E(ix)+E(-ix)}{2}$, $S(x) = \frac{E(ix)-E(-ix)}{2i}$

214. $C(x)^2 + S(x)^2 = 1$

215. $C(-x) = C(x)$ even, $S(-x) = -S(x)$ odd

216. $C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, $S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $x \in \mathbb{R}$

217. $E(z + 2\pi i) = E(z)\ \forall z \in \mathbb{C}$

218. $0 < t < 2\pi \Rightarrow E(it) \neq 1$

219. $\{e^{inx}\}_{n\in\mathbb{Z}}$ linearly independent over $\mathbb{C}$

220. $f(x) = \sum_{-N}^{N} c_n e^{inx}$, $c_n \in \mathbb{C}$ real valued $\Leftrightarrow c_n = \overline{c_{-n}}\ \forall n$

221. $\{\varphi_n\}_{n\geq 1}$ orthonormal system on $[a,b]$, $f \in \mathcal{R}[a,b] \sim \sum_{n=1}^{\infty} c_n \varphi_n$, $s_n = \sum_{k=1}^{n} c_k \varphi_k$

$t_n = \sum_{k=1}^{n} \gamma_k \varphi_k$ for $\gamma_k \in \mathbb{C}$ $\Rightarrow$

1) $\int_a^b |s_n|^2 = \sum_{k=1}^{n} |c_k|$, $\int_a^b |t_n|^2 = \sum_{k=1}^{n} |c_k|$, $\int_a^b |f|^2 - \int_a^b |s_n|^2 = \int_a^b |f - s_n|^2$

2) Bessels Inequality $\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f|^2$

3) Minimal Property $\int_a^b |f - s_n|^2 \leq \int_a^b |f - t_n|^2$

222. $\|f\|_2 = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\right]^{1/2}$, $\|f-g\|_2 = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-g|^2\right]^{1/2}$

223. Holders Inequality $\left|\int_{-\pi}^{\pi} f\bar{g}\right| \leq \int_{-\pi}^{\pi} |f||g| \leq \left[\int_{-\pi}^{\pi} |f|^2\right]^{1/2} \left[\int_{-\pi}^{\pi} |g|^2\right]^{1/2}$

224. $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$, $\|f-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2$

225. $f: [-\pi,\pi] \to \mathbb{C}$, $f \in \mathcal{R}$, $\epsilon > 0 \Rightarrow \exists g: \mathbb{R} \to \mathbb{C}$ continuous, 2π periodic $\ni$

$\|f-g\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-g|^2\right)^{1/2} < \epsilon$

226. $f: [-\pi,\pi] \to \mathbb{C}$, $f \in \mathcal{R}$, $\epsilon > 0 \Rightarrow \exists P(x) = \sum_{-N}^{N} \alpha_n e^{inx}$ trig polynomial $\ni \|f(x) - P(x)\|_2 < \epsilon$

227. Parseval's Theorem $f, g: [-\pi,\pi] \to \mathbb{C}$, $f, g \in \mathcal{R}$, $f(x) \sim \sum_{-N}^{N} c_n e^{inx} := S_N(f)$,

$g(x) \sim \sum_{-N}^{N} \gamma_n e^{inx}$. Then:

(i) $\|f - S_N(f)\|_2 \to 0$ as $N \to \infty$

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g} = \sum_{-\infty}^{\infty} c_n \overline{\gamma_n}$

(iii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \sum_{-\infty}^{\infty} |c_n|^2$

228. $f: [a,b) \to \mathbb{R}$, $b \leq +\infty$, $f \in \mathcal{R}[a,c]\ \forall c \in [a,b) \Rightarrow \int_a^b f(x)\,dx = \lim_{c\to b} \int_a^c f(x)\,dx$ if it exists

229. Cauchy Criterion $\int_a^b f$ converges $\Leftrightarrow \forall \epsilon > 0\ \exists c = c(\epsilon) < b \ni$ if $c < t_1 < t_2 < b$

$\Rightarrow \left|\int_{t_1}^{t_2} f\right| < \epsilon$

230. $f \geq 0$, $\int_a^b f$ converges $\Leftrightarrow \exists M > 0 \ni \int_a^c f \leq M\ \forall c < b$

231. $|f(x)| \le g(x)$, $x \in [a,b]$, $g \in \mathcal{R}[a,c]$, $\int_a^b g$ converges $\Rightarrow \int_a^b f$ converges absolutely

232. $x>0 \Rightarrow \Gamma(x+1) = x\Gamma(x)$

233. $\Gamma(n+1) = n!$, $n = 0, 1, \ldots$

234. $\log\Gamma$ is convex on $(0,+\infty)$

235. $f:(\alpha,\beta) \to \mathbb{R}$ convex, $x_1 < x_2 < x_3$ in $(\alpha,\beta) \Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_1)}{x_3-x_1}$ and $\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_2)}{x_3-x_2}$

236. If $f:(0,+\infty) \to (0,+\infty)$ satisfies $f(1)=1$, $f(x+1) = xf(x)$ $\forall x>0$, $\log f$ is convex on $(0,+\infty) \Rightarrow f = \Gamma$

237. $\forall x>0$, $\Gamma(x) = \lim_{n\to+\infty} \frac{n!\,n^x}{x(x+1)\cdots(x+n)}$

238. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x,y>0$

239. $\int_{-\infty}^{\infty} e^{-s^2}\,ds = \Gamma(\frac{1}{2}) = \sqrt{\pi}$

240. $\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2})$

241. Sterling's Formula – $\lim_{x\to+\infty} \frac{\Gamma(x+1)}{(\frac{x}{e})^x\sqrt{2\pi x}} = 1$ ie $\lim_{n\to\infty} \frac{n!}{(\frac{n}{e})^n\sqrt{2\pi n}} = 1$

242. $\dim X = k$, $\{x_1,\ldots,x_r\}$ linearly independent, $\{y_1,\ldots,y_s\}$ spans X $\Rightarrow r \le k \le s$. And if $r=k \Rightarrow \{x_1,\ldots,x_r\}$ basis, if $s=k \Rightarrow \{y_1,\ldots,y_s\}$ basis

243. $\{x_1,\ldots,x_k\}$ basis of $X \Rightarrow A$ completely determined by $A(x_1),\ldots,A(x_k)$

244. A injective $\Leftrightarrow \mathcal{N}(A) = \{0\}$

245. A surjective $\Leftrightarrow \mathcal{R}(A) = Y$

246. A bijective $\Leftrightarrow A^{-1}\ Y \to X$ linear transformations

247. $A: X \to Y$, $\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = \dim X$

248. $A: X \to X$ linear operator. A injective $\Longleftrightarrow A$ surjective

249. $A \in L(X,Y)$, $B \in L(Y,Z) \Rightarrow BA = B\circ A \in L(X,Z)$

250. $|Ax| \le \|A\|\,|x|$, $\forall x \in \mathbb{R}^n$

251. $|Ax| \le \gamma|x|$, $\forall x \in \mathbb{R}^n \Rightarrow \|A\| \le \gamma$

252. $A \in L(\mathbb{R}^n,\mathbb{R}^m) \Rightarrow \|A\| < +\infty$ (ie A Lipschitz $\Rightarrow A$ uniformly continuous)

253. $\|\cdot\|$ is a norm on the vector space $L(\mathbb{R}^n,\mathbb{R}^m)$

254. $A \in L(\mathbb{R}^n,\mathbb{R}^m)$, $B \in L(\mathbb{R}^m,\mathbb{R}^p) \Rightarrow \|BA\| \le \|B\|\cdot\|A\|$

255. $A \in \Omega$, $B \in L(\mathbb{R}^n)$, $\|B-A\| \le \frac{1}{\|A^{-1}\|} \Rightarrow B \in \Omega$

256. B invertible $\Longleftrightarrow \exists\, \gamma>0 \ni \gamma|x| \le |Bx|$ $\forall x \in \mathbb{R}^n$. in this case $\|B^{-1}\| = \frac{1}{\gamma}$

257. $\lim_{h\to 0} \frac{|f(x+h)-f(x)-Ah|}{|h|} = 0$ where $A = f'(x) \in L(\mathbb{R}^n,\mathbb{R}^m) \Leftrightarrow f(x+h)-f(x) = f'(x)h + r(h)$ where $\frac{|r(h)|}{|h|} = 0$ ($|r(h)| = \epsilon(h)|h|$, $\epsilon(h) \to 0$ as $h \to 0$)

258. f differentiable at $x \Rightarrow f$ continuous at x

259. Chain Rule $f: \underset{\text{open}}{E} \subseteq \mathbb{R}^n \to \underset{\text{open}}{X} \subseteq \mathbb{R}^m$, $g: X \to \mathbb{R}^k$, f differentiable at x_0, g differentiable at $f(x_0) \Rightarrow g(f(x_0))' = g'(f(x_0))f'(x_0) \in L(\mathbb{R}^n, \mathbb{R}^k)$

260. $f: \underset{\text{open}}{E} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$, f differentiable at $x \in E \Rightarrow$ All partials exist and $f'(x)e_j = \sum_{i=1}^{m} (D_j f_i(x)) u_i$

261. $[f'(x)] = \begin{bmatrix} D_1 f_1(x) & \dots & D_n f_1(x) \\ D_1 f_m(x) & \dots & D_n f_m(x) \end{bmatrix}$

262. $f = (f_1, \dots, f_2)$ differentiable $\Leftrightarrow f_i$ differentiable at x $\forall i$

263. $\gamma: (a,b) \subseteq \mathbb{R} \to \mathbb{R}^n$, $\gamma = (\gamma_1, \dots, \gamma_n) \Rightarrow [\gamma'(t)] = \begin{bmatrix} \gamma_1'(t) \\ \gamma_n'(t) \end{bmatrix}$. If also $f: \mathbb{R}^n \to \mathbb{R}$, $g = f \circ \gamma: (a,b) \to \mathbb{R} \Rightarrow g'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = \sum_{j=1}^{n} D_j f(\gamma(t)) \gamma_j'(t)$

264. f differentiable at $x \Rightarrow \exists D_u f(x) = \nabla f(x) \cdot u$ where u unit vector

265. $E \subseteq \mathbb{R}^n$ convex, open, $f: E \to \mathbb{R}^m$ differentiable, $\exists M \geq 0 \ni \|f'(x)\| \leq M$ $\forall x \in E$ $\Rightarrow |f(y) - f(x)| \leq M|y - x|$ $\forall x, y \in E$

266. $f: \underset{\text{open+connected}}{E} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\exists f'(x) = 0$ $\forall x \in E \Rightarrow f$ constant

267. $f: \underset{\text{open}}{E} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f \in \mathscr{C}'(E) \Leftrightarrow$ All partials exist and are continuous on E

268. Contraction Principle X complete metric space, $\varphi: X \to X$ contraction $\Rightarrow$ φ has a unique fixed point: $\exists! x \in X \ni \varphi(x) = x$

269. Inverse Function Theorem $f: \underset{\text{open}}{E} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathscr{C}'(E)$, $a \in E$, $b = f(a)$, $f'(a)$ bijective (ie $f'(a) \in \Omega$ ie $f'(a)$ invertible). Then:
(i) $\exists U$ open $\ni a \in U \subseteq E \ni V = f(U)$ open and $f: U \to V$ bijective
(ii) The inverse function $g: V \to U$ $(g(f(x)) = x$ $\forall x \in U)$ is in $\mathscr{C}'(V)$

270. $f = (f_1, \dots, f_n)$, $f_1(x_1, \dots, x_n) = y_1, \dots, f_n(x_1, \dots, x_n) = y_n \Rightarrow$ for every y sufficiently close to b, the system has a unique solution in a nbd of a

271. $f: \underset{\text{open}}{E} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathscr{C}'(E)$, $f'(x)$ invertible $\forall x \in E$. Then:
(i) f is locally injective: $\forall x \in E$ $\exists U \underset{\text{open}}{\subseteq} E$, $x \in U \ni f|_U$ injective
(ii) f is open: If $W \subseteq E$ open then $f(W)$ is open

272. $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n) \ni A_x$ invertible $\Rightarrow$ for each $k \in \mathbb{R}^m$ $\exists! h \in \mathbb{R}^n \ni A(h,k) = 0$ and $h = -A_x^{-1} A_y k$

273. Implicit Function Theorem Let $f: \underset{\text{open}}{E} \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n$, $f \in \mathscr{C}'$ $\ni f(a,b) = 0$ for some $(a,b) \in E$. Let $A = f'(a,b) \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$. Assume $A_x \in L(\mathbb{R}^n)$ is invertible. Then $\exists$ open $U \subseteq E \ni (a,b) \in U$ and open $W \subseteq \mathbb{R}^m \ni b \in W \ni \forall y \in W$ $\exists! x$, $(x,y) \in U$, $f(x,y) = 0$. Let $g: W \subseteq \mathbb{R}^m \to \mathbb{R}^n$, $g(y) = x$. Then: $g(b) = a$, $f(g(y), y) = 0$ for $y \in W$, $g \in \mathscr{C}'(W)$, $g'(b) = -A_x^{-1} A_y$

i.e. $\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{(a,b)}$ is invertible $\Rightarrow g'(b) = -\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{x_n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_m} \end{bmatrix}\Bigg|_{(a,b)}$

274. $\det(x_1,\dots,x_{j-1}, ay+bz, x_{j+1},\dots,x_n) = a\det(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n)$
$+ b\det(x_1,\dots,x_{j-1},z,x_{j+1},\dots,x_n)$

275. $\det[A] = \det[A]^T$ where A is $n\times n$

276. If $[A]_1$ is obtained from $[A]$ by interchanging 2 columns then $\det[A]_1 = -\det[A]$

277. $[A],[B]$ $n\times n$ $\Rightarrow \det([A][B]) = \det[A]\det[B]$

278. $A\in L(\mathbb{R}^n)$ invertible $\Leftrightarrow \det[A]\neq 0$

279. $f: \underset{open}{E\subseteq}\mathbb{R}^n \to \mathbb{R}^m$, $f=(f_1,\dots,f_m)$, $f\in\mathcal{C}' \Leftrightarrow f_i\in\mathcal{C}'(E)\ \forall i$

280. Mean value Thm $f: \underset{open}{E\subseteq}\mathbb{R}^2\to\mathbb{R}$, $D_1f, D_{21}f$ exist in E, $Q\subseteq E$ closed
rectangle with vertices $(a,b),(a+h,b+k)$, $h,k\neq 0$,
$\Delta(f,Q) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b) \Rightarrow \exists (x,y)\in \operatorname{Int} Q$
$\ni \Delta(f,Q) = hk\,D_{21}f(x,y)$

281. $f: \underset{open}{E\subseteq}\mathbb{R}^2\to\mathbb{R}$, $D_1f, D_2f, D_{21}f$ exist on E, $D_{21}f$ continuous at $(a,b)\in E$
$\Rightarrow \exists D_{12}f(a,b) = D_{21}f(a,b)$

282. Clairaut's Thm $f: \underset{open}{E\subseteq}\mathbb{R}^n\to\mathbb{R}$, $D_if, D_{ij}f: E\to\mathbb{R}$ exist, $D_{ij}f$ are continuous
on E, $1\leq i,j\leq n \Rightarrow D_{ij}f = D_{ji}f$ on E

283. $(x,t)\to\varphi(x,t)\in\mathbb{R}$, $a\leq x\leq b$, $c\leq t\leq d$, $\alpha:[a,b]\to\mathbb{R}$ increasing, $\forall t\in[c,d]$
$\varphi(\cdot,t)\in\mathcal{R}(\alpha)$ on $[a,b]$, $\exists \frac{\partial\varphi}{\partial t}(x,t)$ continuous on $[a,b]\times[c,d] \Rightarrow$
$f(t) = \int_a^b \varphi(x,t)\,d\alpha(x)$ is differentiable on $[c,d]$ and $f'(t) = \int_a^b \frac{\partial\varphi}{\partial t}(x,t)\,d\alpha(x)$

284. $a(t), b(t)$ differentiable on $\mathbb{R}$, $\varphi(x,t), \frac{\partial\varphi}{\partial t}(x,t)$ are continuous on $\mathbb{R}^2$
$\Rightarrow f(t) = \int_{a(t)}^{b(t)}\varphi(x,t)\,dx$, $t\in\mathbb{R}$ is differentiable and
$f'(t) = \int_{a(t)}^{b(t)} \frac{\partial\varphi(x,t)}{\partial t}\,dx + \varphi(b(t),t)\,b'(t) - \varphi(a(t),t)\,a'(t)$

285. $\varphi:\mathbb{R}^2\to\mathbb{R} \ni \varphi(x,t), \frac{\partial\varphi}{\partial t}(x,t)$ continuous on $\mathbb{R}^2$, $\int_{-\infty}^{+\infty}\varphi(x,t)\,dx$ converges
$\forall t$, $\exists h:\mathbb{R}\to[0,+\infty) \ni \left|\frac{\partial\varphi}{\partial t}(x,t)\right| \leq h(x)\ \forall (x,t)\in\mathbb{R}^2$ and $\int_{-\infty}^{+\infty} h(x)\,dx < +\infty$
$\Rightarrow f(t) = \int_{-\infty}^{+\infty}\varphi(x,t)\,dx$ is differentiable on $\mathbb{R}$ and $f'(t) = \int_{-\infty}^{+\infty}\frac{\partial\varphi}{\partial t}(x,t)\,dx$

286. $f\in\mathcal{C}^{(r)}(E)$, $2\leq k\leq r$, $\{i_1,\dots,i_k\}\subset\{1,\dots,n\} \Rightarrow D_{i_1\dots i_k}$ is independent of
order of $i_1,\dots,i_k$

287. # of ordered k-tuples $\{j_1,\dots,j_k\}\subset\{1,\dots,n\} \ni D_{j_1\dots j_k}f = D^\alpha f$ is
$\frac{k!}{\alpha!}$ where $\alpha! = \alpha_1!\dots\alpha_n!$ and $|\alpha| = k$

288. Taylor's Thm $E\subseteq\mathbb{R}^n$ open, convex, $f\in\mathcal{C}^{(r)}(E,\mathbb{R})$, $x_0\in E$, $h\in\mathbb{R}^n \ni$
$x_0+h\in E \Rightarrow \exists\theta\in(0,1) \ni f(x) = \sum_{|\alpha|\leq r-1}\frac{D^\alpha(x_0)}{\alpha!}h^\alpha + \sum_{|\alpha|=r}\frac{D^\alpha(x_0+\theta h)}{\alpha!}h^\alpha$

289. $Q_a(h) = \frac{1}{2} H(a)h \cdot \frac{1}{2} < H(a)h, h >$, Then:

1) $Q_a > 0 \Leftrightarrow$ eigenvalues of $H(a) > 0$

2) $Q_a > 0 \Rightarrow f$ has local min at a

3) $Q_a < 0 \Rightarrow f$ has local max at a

Chapter 1

1. If $0 \neq r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, prove that $r+x, rx \notin \mathbb{Q}$

$r \in \mathbb{Q} \Rightarrow r = \frac{m}{n}$ for $m, n \in \mathbb{Z}$

Suppose $r+x \in \mathbb{Q}$

Then $r+x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$

So $\frac{m}{n} + x = \frac{p}{q} \Rightarrow x = \frac{p}{q} - \frac{m}{n} = \frac{pn-qm}{qn} \in \mathbb{Q}$

Contradiction

$\therefore r+x \notin \mathbb{Q}$

Now suppose $rx \in \mathbb{Q}$

Then $rx = \frac{p}{q}$ for $p, q \in \mathbb{Z}$

So $\frac{m}{n} x = \frac{p}{q} \Rightarrow x = \frac{pn}{qm} \in \mathbb{Q}$

Contradiction

$\therefore rx \notin \mathbb{Q}$

4. Let $E \neq \emptyset$ be a subset of an ordered set; suppose α is a lowerbound of E and β is an upperbound of E. Prove that $\alpha \leq \beta$.

α lowerbound of $E \Rightarrow \alpha \leq x \quad \forall x \in E$

β upperbound of $E \Rightarrow x \leq \beta \quad \forall x \in E$

Since $E \neq \emptyset$, let $x \in E$

Then $\alpha \leq x \leq \beta$

$\therefore \alpha \leq \beta$

7. Fix $b>1, y>0$, and prove that $\exists! x \in \mathbb{R} \ni b^x = y$ by completing the outline:

a. For any $n \in \mathbb{Z}^+$, $b^n - 1 \geq n(b-1)$

For $n=1$, we have that $b-1 \geq b-1$

$\therefore$ True for $n=1$

Assume true for $n=k$ ie $b^k - 1 \geq k(b-1)$

Then $b^{k+1} - 1 = b^{k+1} - b + b - 1 = b(b^k - 1) + b - 1 \geq bk(b-1) + b - 1 = (b-1)(bk+1)$

$\geq (k+1)(b-1)$

$\therefore b^n - 1 \geq n(b-1) \quad \forall n \in \mathbb{Z}^+$

b. Hence $b-1 \geq n(b^{1/n}-1)$

Note that $b^{1/n} > 1$ since $b>1$

Then by (a) we have $(b^{1/n})^n - 1 \geq n(b^{1/n}-1)$

$\therefore b-1 \geq n(b^{1/n}-1)$

c. If $t>1$ and $n > \frac{b-1}{t-1}$, then $b^{\frac{1}{n}} < t$.

$n > \frac{b-1}{t-1} \Rightarrow n(t-1) > b-1$ since $t>1$

So $n(b^{1/n}-1) \leq b-1 < n(t-1)$

$\therefore b^{1/n}-1 < t-1$

$\therefore b^{1/n} < t$

d. If ω is $\ni b^w < y$, then $b^{w+\frac{1}{n}} < y$ for sufficiently large n.

Note $t = y \cdot b^{-\omega} = y \cdot (b^{\omega})^{-1} = \frac{y}{b^{\omega}} > \frac{b^{\omega}}{b^{\omega}} = 1$ since $b^{\omega} < y$

Then by (c) we have $b^{1/n} < y \cdot b^{-\omega}$ for $n > \frac{b-1}{t-1}$

$\therefore b^{\omega} b^{1/n} < y$

$\therefore b^{w+\frac{1}{n}} < y$ for sufficiently large n

e. If $b^{\omega} > y$, then $b^{\omega - \frac{1}{n}} > y$ for sufficiently large n.

Now let $t = y^{-1} b^{\omega} = \frac{b^{\omega}}{y} > \frac{y}{y} = 1$

Then by (c) we have $b^{\frac{1}{n}} < y^{-1} b^{\omega}$ for $n > \frac{b-1}{t-1}$

So $y < b^{-\frac{1}{n}} b^{\omega} = b^{\omega - \frac{1}{n}}$

$\therefore b^{\omega - \frac{1}{n}} > y$ for sufficiently large n

f. Let $A = \{w \mid b^{\omega} < y\}$ and show that $x = \sup A$ satisfies $b^x = y$

Suppose that $b^x < y$

Then for sufficiently large n, $b^{x+\frac{1}{n}} < y$ by (d)

Then $x + \frac{1}{n} \in A$

$\therefore x + \frac{1}{n} \leq x$ since $x = \sup A$

Contradiction since $x + \frac{1}{n} > x$

Now suppose that $b^x > y$

Then for sufficiently large n, $b^{x-\frac{1}{n}} > y$ by (e)

Then $b^{x-\frac{1}{n}} > y > b^{\omega}$ $\forall \omega \in A$

$\therefore x-\frac{1}{n} > \omega \quad \forall \omega \in A$

$\therefore x-\frac{1}{n}$ is an upper bound of A

But $x-\frac{1}{n} < x$

Contradiction since $x = \sup A$

$\therefore b^x = y$

g. Prove that x is unique

Take $z \neq x$

WLOG say $z > x$

Then $b^z = b^{x+z-x} = b^x b^{z-x} > b^x$ since $z > x$

$= y$

Then $b^z > y$

$\therefore x$ unique

10. Suppose $z = a+bi$, $w = u+iv$ and $a = \left(\frac{|w|+u}{2}\right)^{1/2}$, $b = \left(\frac{|w|-u}{2}\right)^{1/2}$. Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number with one exception has 2 complex square roots.

$z^2 = (a+bi)^2 = a^2 + 2abi - b^2 = \frac{|w|+u}{2} + 2abi - \frac{|w|-u}{2} = u + 2abi$

$= u + 2\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2} i = u + 2i\left(\frac{1}{4}(|w|^2 - u^2)\right)^{1/2} = u + i(|w|^2 - u^2)^{1/2}$

$= u + i(u^2 + v^2 - u^2)^{1/2} = u + iv = \omega$ as long as $v \geq 0$

$\therefore z^2 = \omega$ for $v \geq 0$

$(\bar{z})^2 = (a-bi)^2 = a^2 - 2abi - b^2 = \frac{|w|+u}{2} - 2abi - \frac{|w|-u}{2} = u - 2abi$

$= u - 2i\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2} = u - 2i\left(\frac{1}{4}(|w|^2 - u^2)\right)^{1/2} = u - i(|w|^2 - u^2)^{1/2}$

$= u - i(u^2 + v^2 - u^2)^{1/2} = u - iv = \omega$ as long as $v \leq 0$

$\therefore (\bar{z})^2 = \omega$ for $v \leq 0$

$\therefore$ Every complex number has atleast 2 complex square roots except 0

13. If $x, y \in \mathbb{C}$, prove that $\big||x| - |y|\big| \leq |x-y|$

Note that $|x| = |x-y+y| \leq |x-y| + |y|$

$\therefore |x| - |y| \leq |x-y|$

Similarly $|y| = |y-x+x| \leq |y-x| + |x| = |x-y| + |x|$

$\therefore |y| - |x| \leq |x-y|$

And $||x|-|y|| = |x|-|y|$ or $||x|-|y|| = |y|-|x|$

$\therefore ||x|-|y|| \leq |x-y|$

16. Suppose $k \geq 3$, $x, y \in \mathbb{R}^k$, $|x-y| = d > 0$ and $r > 0$. Prove:

a. If $2r > d$, $\exists$ infinitely many $z \in \mathbb{R}^k \ni |z-x| = |z-y| = r$

$2r > d \Rightarrow 2\left(\sum_{i=1}^{k}(z_i - x_i)^2\right)^{1/2} > \left(\sum_{i=1}^{k}(x_i - y_i)^2\right)^{1/2}$ and $2\left(\sum_{i=1}^{k}(z_i - y_i)^2\right)^{1/2} > \left(\sum_{i=1}^{k}(x_i - y_i)^2\right)^{1/2}$

so $4\sum_{i=1}^{k}(z_i - x_i)^2 > \sum_{i=1}^{k}(x_i - y_i)^2$ and $4\sum_{i=1}^{k}(z_i - y_i)^2 > \sum_{i=1}^{k}(x_i - y_i)^2$

$\Rightarrow 4\sum_{i=1}^{k} z_i^2 - 2z_ix_i + x_i^2 > \sum_{i=1}^{k} x_i^2 - 2x_iy_i + y_i^2$ and $4\sum_{i=1}^{k} z_i^2 - 2z_iy_i + y_i^2 > \sum_{i=1}^{k} x_i^2 - 2x_iy_i + y_i^2$

so $4\sum_{i=1}^{k} y_i^2 - 2z_iy_i + 2z_ix_i - x_i^2 > 0$

$\Rightarrow 8\sum_{i=1}^{k} z_ix_i - z_iy_i > 4\sum_{i=1}^{k} x_i^2 - y_i^2$

$\Rightarrow 2\sum_{i=1}^{k} z_i \sum_{i=1}^{k} x_i - y_i > \sum_{i=1}^{k} x_i^2 - y_i^2$

$\Rightarrow 2\sum_{i=1}^{k} z_i > \sum_{i=1}^{k} x_i + y_i$

$\Rightarrow \sum_{i=1}^{k} z_i > \frac{1}{2}\sum_{i=1}^{k} x_i + y_i$

$\therefore |z| > \frac{|x+y|}{2}$

$\therefore \exists$ infinitely many $z \in \mathbb{R}^k \ni |z-x| = |z-y| = r$, namely any $|z| > \frac{|x+y|}{2}$

b. If $2r = d$, $\exists$ exactly one such z

Similarly, $2r = d \Rightarrow z = \frac{x+y}{2}$

$\therefore \exists$ exactly one $z \ni |z-x| = |z-y| = r$, namely $z = \frac{x+y}{2}$

c. If $2r < d$, $\nexists$ such a z

Similarly $2r < d \Rightarrow |z| < \frac{|x+y|}{2} < |x+y|$

But $z = x+y$

$\therefore \nexists z \ni |z-x| = |z-y| = r$

19. Suppose $a, b \in \mathbb{R}^k$. Find $c \in \mathbb{R}^k$ and $r > 0 \ni |x-a| = 2|x-b| \Leftrightarrow |x-c| = r$

$|x-a| = 2|x-b| \Rightarrow \left(\sum_{i=1}^{k}(x_i - a_i)^2\right)^{1/2} = 2\left(\sum_{i=1}^{k}(x_i - b_i)^2\right)^{1/2}$

$\Rightarrow \sum_{i=1}^{k} x_i^2 - 2x_ia_i + a_i^2 = 4\sum_{i=1}^{k} x_i^2 - 8x_ib_i + 4b_i^2$

$\Rightarrow \sum_{i=1}^{k} a_i^2 - 2x_ia_i + 8x_ib_i - 4b_i^2 = 3\sum_{i=1}^{k} x_i^2$

$|x-c| = r \Rightarrow \left(\sum_{i=1}^{k}(x_i - c_i)^2\right)^{1/2} = r \Rightarrow \sum_{i=1}^{k} x_i^2 - 2x_ic_i + c_i^2 = r^2$

$\Rightarrow \sum_{i=1}^{k} x_i^2 = r^2 + 2x_ic_i - c_i^2$

so $\sum_{i=1}^{k} a_i^2 - 2x_ia_i + 8x_ib_i - 4b_i^2 = 3(r^2 + 2x_ic_i - c_i^2)$

So $\sum_{i=1}^{k} a_i^2 - 2x_i a_i + 8x_i b_i - 4b_i^2 - 6x_i c_i + 3c_i^2 = 3r^2$

$\Rightarrow \sum_{i=1}^{k} a_i^2 - 4b_i^2 + 3c_i^2 + 2\sum_{i=1}^{k}(-a_i + 4b_i - 3c_i)x_i = 3r^2$

So $2\sum_{i=1}^{k}(-a_i + 4b_i - 3c_i) = 0 \Rightarrow 3\sum_{i=1}^{k} c_i = 4\sum_{i=1}^{k} b_i - \sum_{i=1}^{k} a_i$

$\therefore 3c = 4b - a$

Then $9c^2 = 16b^2 - 8ab + a^2 \Rightarrow c^2 = \frac{16}{9}b^2 - \frac{8}{9}ab + \frac{1}{9}a^2$

And we have that $\sum_{i=1}^{k} a_i^2 - 4b_i^2 + 3c_i^2 = 3r^2$

So $r^2 = \sum_{i=1}^{k} \frac{1}{3}a_i^2 - \frac{4}{3}b_i^2 + c_i^2$

$\therefore r^2 = \sum_{i=1}^{k} \frac{1}{3}a_i^2 - \frac{4}{3}b_i^2 + \frac{16}{9}b_i^2 - \frac{8}{9}a_i b_i + \frac{1}{9}a_i^2 = \sum_{i=1}^{k} \frac{4}{9}a_i^2 + \frac{4}{9}b_i^2 - \frac{8}{9}a_i b_i$

So $9r^2 = \sum_{i=1}^{k} 4a_i^2 + 4b_i^2 - 8a_i b_i = 4\sum_{i=1}^{k}(a_i - b_i)^2$

So $3r = 2\left(\sum_{i=1}^{k}(a_i - b_i)^2\right)^{1/2}$

$\therefore 3r = 2|a-b|$

$\therefore c = \frac{4}{3}b - \frac{1}{3}a, \; r = \frac{2}{3}|a-b|$

Chapter 2

2. Prove that $\mathbb{A}$ is countable.

Let $\mathbb{P}_n = \{a_0x^n + \dots + a_n \mid a_0, \dots, a_n \in \mathbb{Z} \text{ not all zero}\}$, $\Pi_n = \{(a_0, \dots, a_n) \mid a_0, \dots, a_n \in \mathbb{Z}\}$

Define $\varphi : \mathbb{P}_n \to \Pi_n \ni \varphi(a_0x^n + \dots + a_n) = (a_0, \dots, a_n)$

Assume $\varphi(a_0x^n + \dots + a_n) = \varphi(b_0x^n + \dots + b_n)$

Then $(a_0, \dots, a_n) = (b_0, \dots, b_n) \Rightarrow a_0x^n + \dots + a_n = b_0x^n + \dots + b_n$

$\therefore \varphi$ injective

Also Π_n, the set of all n-tuples is countable

$\therefore \mathbb{P}_n$ countable

So $\mathbb{P}_n = \{P_1, P_2, \dots\}$

Let the set of roots for the polynomial P_i be denoted as R_i for each i

Note that each polynomial has at most n solutions

$\therefore R_i$ finite for each i ie R_i at most countable

Then $\mathbb{A} = \bigcup_{i=1}^{\infty} R_i$ is countable

$\therefore \mathbb{A}$ countable

5. Construct a bounded set of real numbers with exactly 3 limit points.

Let $S_1 = \{0 + \frac{1}{n} \mid n = 1, 2, \dots\}$, $S_2 = \{2 + \frac{1}{n} \mid n = 1, 2, \dots\}$, $S_3 = \{4 + \frac{1}{n} \mid n = 1, 2, \dots\}$

Note the only limit points of S_1, S_2, S_3 are $0, 2, 4$ respectively

And each set is bounded

$\therefore S_1 \cup S_2 \cup S_3$ is bounded since it is the union of finitely many bounded sets and $S_1 \cup S_2 \cup S_3$ has 3 limit points: $0, 2, 4$

8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? What about closed sets in $\mathbb{R}^2$?

Let $E \subseteq \mathbb{R}^2$ be open and let $(x,y) \in E$

The since E open $\exists r > 0 \ni N_r(x,y) \subseteq E$

And for each $r' > 0$, $\exists (x', y') \ni d((x,y),(x',y')) = \frac{1}{2}\min(r, r')$

So $\forall r' > 0 \; \exists (x', y') \in N_{r'}(x,y) \cap E \ni (x',y') \neq (x,y)$ since $(x',y') \in N_r(x,y) \subseteq E$

$\therefore (x,y) \in E'$

$\therefore$ Every point of E open is a limit point of E

Now take $E = \{(0,0)\}$

Note that E closed since E singleton set
But also E has no limit points, so (0,0) is not a limit point of E
$\therefore$ If E is closed, every point of E need not be a limit point

11. For $x, y \in \mathbb{R}$, define $d_1(x,y) = (x-y)^2$, $d_2(x,y) = \sqrt{|x-y|}$, $d_3(x,y) = |x^2-y^2|$, $d_4(x,y) = |x-2y|$, $d_5(x,y) = \frac{|x-y|}{1+|x-y|}$. Determine whether each is a metric or not

Consider $d_1(x,y) = (x-y)^2$
$d_1(0,2) = 4$, but $d_1(0,1) + d_1(1,2) = 2$
$\therefore d_1(0,2) > d_1(0,1) + d_1(1,2)$
$\therefore d_1$ not metric

Consider $d_2(x,y) = \sqrt{|x-y|}$
Clearly $d_2(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y,x)$
And $d_2(x,y) = \sqrt{|x-y|} \geq 0$ with $d_2(x,y) = 0 \Rightarrow \sqrt{|x-y|} = 0 \Rightarrow x = y$
Finally $|x-y| = |x-z+z-y| \leq |x-z| + |z-y| = (\sqrt{|x-z| + |z-y|})^2$
$\therefore \sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$
$\therefore d_2(x,y) \leq d_2(x,z) + d_2(z,y)$
$\therefore d_2$ metric

Consider $d_3(x,y) = |x^2-y^2|$
Note that $d_3(x,-x) = |x^2-(-x)^2| = |x^2-x^2| = 0$ but $x \neq -x$
$\therefore d_3$ not a metric

Consider $d_4(x,y) = |x-2y|$
$d_4(1,3) = |1-6| = 5$, but $d_4(3,1) = |3-2| = 1$
$\therefore d_4(1,3) \neq d_4(3,1)$
$\therefore d_4$ not metric

Consider $d_5(x,y) = \frac{|x-y|}{1+|x-y|}$
Clearly $d_5(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d_5(y,x)$
And $d_5(x,y) = \frac{|x-y|}{1+|x-y|} \geq 0$ with $d_5(x,y) = 0$ iff $x = y$
Note that for $a \leq b$, $a + ab \leq b + ab \Rightarrow a(1+b) \leq b(1+a) \Rightarrow \frac{a}{1+a} \leq \frac{b}{1+b}$
So $d_5(x,z) = \frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} = \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|}$
$\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} = d_5(x,y) + d_5(y,z)$

$\therefore d_5$ metric

14. Give an example of an open cover of $(0,1)$ which has no finite subcover

Let $V_n = (\frac{1}{n}, \frac{n-1}{n})$, $n = 3, 4, \ldots$

Clearly $\bigcup_{n=3}^{\infty} V_n$ open cover of $(0,1)$

But take $\bigcup_{n=3}^{N} V_n$ for some N

$\bigcup_{n=3}^{N} V_n = (\frac{1}{k}, \frac{k-1}{k})$ for some k

But $x = \frac{1}{2k} \notin (\frac{1}{k}, \frac{k-1}{k})$

$\therefore \bigcup_{n=3}^{N} V_n$ doesnot cover $(0,1)$

$\therefore \nexists$ finite subcover

17. Let $E = \{x \in [0,1] \mid$ decimal expansion contains only $4, 7\}$. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Suppose E countable

Then $E = \{x_1, x_2, \ldots\}$

where $x_1 = 0.a_{11}a_{12}\ldots$

$x_2 = 0.a_{21}a_{22}\ldots$

$\vdots$

Take $x = 0.x_1x_2\ldots$ where $x_n = \begin{cases} 4 & a_{nn} = 7 \\ 7 & a_{nn} = 4 \end{cases}$

Then $x \notin E$ since it does not appear in the list

But $x \in [0,1]$ and decimal expansion contains only $4, 7$

Contradiction

$\therefore E$ uncountable

E not dense in $[0,1]$ since $E \subseteq [0.4, 0.8]$

Clearly E bounded so show E closed

Take $x \in [0,1] \ni x \notin E$ and show $x \notin E'$

$x \notin E \Rightarrow x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$ where $a_n \notin \{4, 7\}$

Choose $k \ni a_k \notin \{4, 7\}$ and $r = \frac{1}{10^{k+1}}$

Then $N_r(x) \cap E = \phi$

$\therefore x \notin E'$

$\therefore E$ closed

$\therefore E$ compact

Let $x \in E \ni x = 0.p_1p_2\ldots$

Let $\{x_k\}$ be defined $\ni$ $x_k = 0.y_1y_2\ldots$ where $y_n = \begin{cases} p_k & k \neq n \\ 4 & p_n = 7 \\ 7 & p_n = 4 \end{cases}$

Then $|x_k - p| \longrightarrow 0$ as $k \to \infty$ and $x_k \neq p$ $\forall k$

$\therefore$ $p \in E'$

$\therefore$ Every point of E is a limit point of E and E closed

$\therefore$ E perfect

20. Are closures and interiors of connected sets always connected?

Let E be connected

Suppose $\bar{E}$ not connected

Then $\bar{E} = A \cup B \ni A, B \neq \phi$ and $A \cap \bar{B} = \bar{A} \cap B = \phi$

Note that $E = E \cap \bar{E} = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$

Suppose $E \cap A = \phi$

Then $E = \phi \cup (E \cap B) = E \cap B \subseteq B \subseteq \bar{B}$

But $\bar{B}$ closed so $\bar{E} \subseteq \bar{B}$

Also $B \subseteq A \cup B = \bar{E}$

And $\bar{E}$ closed so $\bar{B} \subseteq \bar{E}$

$\therefore$ $\bar{B} = \bar{E}$

Then $\phi = A \cap \bar{B} = A \cap \bar{E} = A \cap (A \cup B) = A$

$\therefore$ $A = \phi$

contradiction

$\therefore$ $E \cap A \neq \phi$

Similarly $E \cap B \neq \phi$

And $(E \cap A) \cap \overline{E \cap B} \subseteq A \cap \bar{B} = \phi$, $(E \cap B) \cap \overline{E \cap A} \subseteq B \cap \bar{A} = \phi$

$\therefore$ E not connected

contradiction

$\therefore$ $\bar{E}$ connected

But take $E = A \cup B = \{(x,y) \mid d((0,0),(x,y)) \leq 1\} \cup \{(x,y) \mid d((2,0),(x,y)) \leq 1\}$

Note that $A, B \neq \phi$ and $(1,0) \in A \cap \bar{B}$, $\bar{A} \cap B$

$\therefore$ E connected

But $E^\circ = \{(x,y) \mid d((0,0),(x,y)) < 1\} \cup \{(x,y) \mid d((2,0),(x,y)) < 1\}$

Then $A \cap \bar{B} = \bar{A} \cap B = \phi$ and $A, B \neq \phi$

$\therefore E^\circ$ not connected

$\therefore$ interiors of connected sets not always connected

23. Prove that every separable metric space has a countable base.

Let X be a separable metric space.

Then X has a countable dense subset S by definition

Let $\{V_\alpha\} = \{N_r(x) \mid r \in \mathbb{Q}, x \in S\}$

Let $x \in X$ and $G \subseteq X$ open $\ni x \in G$

Then $\exists r > 0 \ni N_r(x) \subseteq G$ since G open $\Rightarrow x \in G^\circ$

But S dense in $X \Rightarrow \exists \{s_n\} \to x$

So for $r_n < \frac{r}{2}$, $N_{r_n}(s_n) \subseteq N_r(x)$ for n large

so $x \in V_\alpha \subseteq G$ since $N_{r_n}(s_n) \in V_\alpha$

$\therefore \{V_\alpha\}$ base for X

And $\{V_\alpha\}$ countable since $\mathbb{Q}$ countable

$\therefore X$ has a countable base

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact.

since X metric space in which every infinite subset has a limit point, X is separable

Then S has a countable base

Then every open cover has a countable subcover

Let $\{V_\alpha\}_{\alpha\in I}$ be an open cover with countable subcover $\{V_n\}_{n=1}^{\infty}$

suppose $\{V_\alpha\}$ has no finite subcover

Then $\bigcup_{n=1}^{N} V_n$ does not cover X $\forall N$

so $W_N = (\bigcup_{n=1}^{N} V_n)^c \neq \phi$ for each N

But $\bigcap_{N=1}^{\infty} W_N = \phi$ since $X \in \bigcup_{n=1}^{\infty} V_n$

And $\bigcap_{N=1}^{\infty} W_N$ infinite subset so has a limit point p

But also $\bigcap_{N=1}^{\infty} W_N$ closed since each W_N closed

so $p \in \bigcap_{N=1}^{\infty} W_n = \phi$

contradiction

$\therefore \{V_\alpha\}$ has a finite subcover

$\therefore X$ compact

29. Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Let $O \subseteq \mathbb{R}$ open

Then for each $x \in O$, $\exists r>0 \ni N_r(x) \subseteq O$ ie $(x-r, x+r) \subseteq O$

let $a_x = \inf\{x-r : x-r \in O\}$, $b_x = \sup\{x+r : x+r \in O\}$ for each $x \in O$

And let $S_x = (a_x, b_x)$ for each $x \in O$

Let $x \in O \Rightarrow x \in S_x \Rightarrow O \subseteq \bigcup_{x \in O} S_x$

Let $y \in \bigcup_{x \in O} S_x \Rightarrow y \in S_x$ for some $x \Rightarrow (y-r, y+r) \subseteq O$ for some $r>0$

$\therefore y \in O$

$\therefore \bigcup_{x \in O} S_x \subseteq O$

$\therefore O = \bigcup_{x \in O} S_x$

Let $(a,b), (c,d) \in \bigcup_{x \in O} S_x \ni x \in (a,b), (c,d)$

Then $a<x<b$, $c<x<d$

So $a<d$, $c<b$

Suppose $b \in O$

Then $\exists r>0 \ni N_r(b) \subseteq O$ ie $(b-r, b+r) \subseteq O$

Contradiction since b upper bound $\{x+r \mid x+r \in O\}$

$\therefore b \notin O$

$\therefore b \geq d$ since $c<b$

Similarly $d \notin O$

So $d \geq b$ since $a<d$

$\therefore d=b$

And thus $a=c$

$\therefore (a,b) = (c,d)$

$\therefore \bigcup_{x \in O} S_x$ is a union of disjoint segments

Now since $\mathbb{Q}$ dense in $\mathbb{R}$, choose $q_x \in S_x$ for each $x \in O$

Since $\mathbb{Q}$ countable, $\{S_x : x \in O\}$ at most countable

$\therefore O = \bigcup_{x \in O} S_x$ a union of at most countable collection of disjoint segments

Chapter 3

3. If $s_1=\sqrt{2}$ and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$, prove that $\{s_n\}$ converges and that $s_n<2$ $\forall n$.

First show that $s_n<2$ $\forall n$

Note $s_1=\sqrt{2}<2$

$\therefore$ True for $n=1$

Assume true for $n=k$ ie $s_k<2$

Then $s_{k+1}=\sqrt{2+\sqrt{s_k}} < \sqrt{2+2} = 2$

$\therefore s_{k+1}<2$

$\therefore s_n<2$ $\forall n$

$\therefore \{s_n\}$ bounded above

Now show $s_n \le s_{n+1}$ $\forall n$

$s_1=\sqrt{2} \le \sqrt{2+\sqrt{2}} = \sqrt{2+\sqrt{s_1}} = s_2$

$\therefore$ True for $n=1$

Assume true for $n=k-1$ ie $s_{k-1}\le s_k$

Then $s_k=\sqrt{2+\sqrt{s_{k-1}}} \le \sqrt{2+\sqrt{s_k}} = s_{k+1}$

$\therefore s_n \le s_{n+1}$ $\forall n$

$\therefore \{s_n\}$ nondecreasing

$\therefore \{s_n\}$ converges

6. Does Σa_n converge or diverge

a. $a_n=\sqrt{n+1}-\sqrt{n}$

$a_n=\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \ge \frac{1}{2\sqrt{n+1}}$ diverges by p-test, $p=\frac{1}{2}$

$\therefore \Sigma a_n$ diverges by comparison test

b. $a_n=\frac{\sqrt{n+1}-\sqrt{n}}{n}$

$a_n=\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{n\cdot 2\sqrt{n}} = \frac{1}{2n^{3/2}}$

And $\Sigma \frac{1}{2n^{3/2}}$ converges by p-test, $p=\frac{3}{2}$

$\therefore \Sigma a_n$ converges by comparison test

c. $a_n=(\sqrt[n]{n}-1)^n$

$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n}-1 = 0$ since $\lim_{n\to\infty} \sqrt[n]{n}=1$

$\therefore \alpha<1$

$\therefore \sum a_n$ converges by root test

d. $a_n = \frac{1}{1+z^n}$ for $z \in \mathbb{C}$

If $|z| \le 1$, then $|a_n| \ge \frac{1}{2}$

$\therefore a_n \not\to 0$

$\therefore \sum a_n$ diverges for $|z| \le 1$

If $|z| > 1$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1+z^n}{1+z^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{z^{-n}+1}{z^{-n}+z}\right| = \left|\frac{1}{z}\right| < 1$

$\therefore$ if $|z|>1$, $\sum a_n$ converges by ratio test

9. Find the radius of convergence:

a. $\sum n^3 z^n$

Let $a_n = n^3$

Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{n^3}\right| = 1$

$\therefore$ Radius of convergence is 1

b. $\sum \frac{2^n}{n!} z^n$

Let $a_n = \frac{2^n}{n!}$

Then $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0$

And $R = \frac{1}{\alpha} = \infty$

$\therefore$ Radius of convergence is ∞

c. $\sum \frac{2^n}{n^2} z^n$

Let $a_n = \frac{2^n}{n^2}$

Then $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}\right| = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2$

And $R = \frac{1}{\alpha} = \frac{1}{2}$

$\therefore$ Radius of convergence is $\frac{1}{2}$

d. $\sum \frac{n^3}{3^n} z^n$

Let $a_n = \frac{n^3}{3^n}$

Then $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3}$

$\therefore$ Radius of convergence is 3

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{m=n}^{\infty} a_m$

a. Prove that $\frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$ if $m<n$ and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Note that $r_n = \sum_{m=n}^{\infty} a_m = a_n + \sum_{m=n+1}^{\infty} a_m = a_n + r_{n+1}$

$\therefore r_n - r_{n+1} = a_n > 0$

$\therefore r_n > r_{n+1} \ \forall n$

$\therefore r_m > r_n$ since $m<n$

So $\frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \dots + \frac{a_n}{r_m} = \frac{a_m + \dots + a_n}{r_m} \cdot \frac{r_m - r_{n+1}}{r_m} > \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}$

$\therefore \frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$ for $m<n$

Then $\sum_{k=m}^{n} \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m}$

Suppose $\sum \frac{a_n}{r_n}$ converges

Then by cauchy criterion, $\forall \epsilon > 0 \ \exists N \ni n > m \geq N \Rightarrow \left| \sum_{k=m}^{n} \frac{a_k}{r_k} \right| < \epsilon$

Take $\epsilon = 1 - \frac{r_n}{r_m} > 0$ since $r_m > r_n \Rightarrow \frac{r_n}{r_m} < 1$

Then $n > m \geq N \Rightarrow \left| \sum_{k=m}^{n} \frac{a_k}{r_k} \right| < 1 + \frac{r_n}{r_m}$

contradiction

$\therefore \sum \frac{a_n}{r_n}$ diverges

b. Prove that $\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$ and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges

$\frac{a_n}{\sqrt{r_n}}(\sqrt{r_n} + \sqrt{r_{n+1}}) = a_n + \frac{a_n\sqrt{r_{n+1}}}{\sqrt{r_n}} < a_n + \frac{a_n\sqrt{r_{n+1}}}{\sqrt{r_{n+1}}} = 2a_n = 2(r_n - r_{n+1})$

$\therefore \frac{a_n}{\sqrt{r_n}} < \frac{2(r_n - r_{n+1})}{\sqrt{r_n} + \sqrt{r_{n+1}}} = \frac{2(r_n - r_{n+1})(\sqrt{r_n} - \sqrt{r_{n+1}})}{r_n - r_{n+1}} = 2(\sqrt{r_n} - \sqrt{r_{n+1}})$

$\therefore \frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$

So $\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < \sum_{n=1}^{N} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_N})$

$\therefore \sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}}$ bounded

$\therefore \sum \frac{a_n}{\sqrt{r_n}}$ converges since $\frac{a_n}{\sqrt{r_n}} > 0 \ \forall n$

15. Show that for $a_n \in \mathbb{R}^k$, if $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$

$a_n \in \mathbb{R}^k \Rightarrow a_n = (a_{n1}, \dots, a_{nk})$

$\sum a_n$ converges $\Rightarrow \sum a_{ni}$ converges $\forall i$ since cauchy criterion holds

$\therefore \lim_{n\to\infty} a_{ni} = 0 \ \forall i$

$\therefore \lim_{n\to\infty} a_n = 0$

21. Prove that if $\{E_n\}$ sequence of closed nonempty bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$ and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$ then

$\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

choose $x_n \in E_n$ since $E_n \neq \emptyset$

Note that $\{x_n\}$ cauchy since $\lim_{n\to\infty} \operatorname{diam} E_n = 0$

since X complete, $\exists x \in X \ni x_n \to x$

And $x \in E_n$ since E_n closed

And since $E_{n+1} \subseteq E_n$, $x_m \in E_n \ \forall m \geq n$

$\therefore x \in E_n \ \forall n$

$\therefore x \in \bigcap_{n=1}^{\infty} E_n$

But $\nexists y \in \bigcap_{n=1}^{\infty} E_n \ni y \neq x$ since for any $y \neq x$ $\operatorname{diam} E_n < d(x,y)$

$\therefore \bigcap_{n=1}^{\infty} E_n$ consists of exactly one point

Chapter 4

1. Suppose $f: \mathbb{R} \to \mathbb{R} \ni \lim_{h\to 0}[f(x+h) - f(x-h)] = 0 \ \forall x \in \mathbb{R}$. Does this imply that f is continuous?

Take $f(x) = \begin{cases} 1 & x \in \mathbb{Z} \\ 0 & x \notin \mathbb{Z} \end{cases}$

Then either $x+h, x-h \in \mathbb{Z}$ or $x+h, x-h \notin \mathbb{Z}$

In either case $f(x+h) - f(x-h) = 0$

so $\lim_{h\to 0}[f(x+h) - f(x-h)] = \lim_{h\to 0} 0 = 0$

But f not continuous

4. Let $f, g: X \to Y$ be continuous and let $E \subseteq X$ be dense. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p) \ \forall p \in E$, prove that $g(p) = f(p) \ \forall p \in X$.

Let $y_0 \in f(X) \Rightarrow y_0 = f(x_0)$ for some $x_0 \in X$

Since E dense in X, every point of X is in E, a limit point of E, or both

$\therefore x_0 \in E \cup E' = \overline{E}$

$\therefore y_0 = f(x_0) \in f(\overline{E}) \subseteq \overline{f(E)}$ since f continuous

$\therefore y_0 \in f(E) \cup (f(E))'$

$\therefore f(E)$ dense in $f(X)$

Now let $x \in E^c$

Since E dense in X, $\exists \{x_n\} \in E \ni x_n \to x$

Then $f(x_n) = g(x_n)$ since $x_n \in E$

But since f, g continuous $f(x_n) \to f(x)$, and $g(x_n) \to g(x)$

$\therefore f(x) = g(x) \ \forall x \in E^c$

$\therefore f(x) = g(x) \ \forall x \in X$

7. If $E \subseteq X$ and if f is defined on X, the restriction of f to E is the function defined on $E \ni g(p) = f(p) \ \forall p \in E$. Define f, g on $\mathbb{R}^2$ $\ni f(0,0) = g(0,0) = 0$, $f(x,y) = \frac{xy^2}{x^2+y^4}$, $g(x,y) = \frac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$. Prove that f is bounded on $\mathbb{R}^2$, g is unbounded in every nbd of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous.

Note that $(x-y^2)^2 \geq 0 \Rightarrow x^2 - 2xy^2 + y^4 \geq 0 \Rightarrow x^2 + y^4 \geq 2xy^2$

So $|f(x,y)| = \left|\frac{xy^2}{x^2+y^4}\right| \le \left|\frac{2xy^2}{x^2+y^4}\right| \le \left|\frac{x^2+y^4}{x^2+y^4}\right| = 1$

$\therefore$ f bounded on $\mathbb{R}^2$

Note that $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y^3}} g(x,y) = \lim\limits_{y\to 0} g(y^3,y) = \lim\limits_{y\to 0} \frac{y^5}{2y^6} = \lim\limits_{y\to 0}\frac{1}{2y} = \infty$

$\therefore$ g unbounded in every nbd of $(0,0)$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y^2}} f(x,y) = \lim\limits_{y\to 0} f(y^2,y) = \lim\limits_{y\to 0}\frac{y^4}{2y^4} = \frac{1}{2}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=2y^2}} f(x,y) = \lim\limits_{y\to 0} f(2y^2,y) = \lim\limits_{y\to 0}\frac{2y^4}{5y^4} = \frac{2}{5}$

$\therefore$ f not continuous at $(0,0)$

Clearly f,g continuous $\forall (x,y) \neq (0,0)$

So look at $f(x,ax) = \frac{a^2x^3}{x^2+a^4x^4} = \frac{a^2x}{1+a^4x^2}$ continuous

And $g(x,ax) = \frac{a^?x^3}{x^2+a^6x^6} = \frac{a^2x}{1+a^4x^4}$ continuous

$\therefore$ restrictions of f,g to every straight line in $\mathbb{R}^2$ are continuous

10. Let $f: X \to Y$ be continuous and X compact. Prove that f is uniformly continuous on X.

Suppose f not uniformly continuous

Then for some $\epsilon > 0$ $\exists \{p_n\}, \{q_n\} \in X \ni d_X(p_n,q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$

But $\exists \{p_{n_k}\}, \{q_{n_k}\} \ni p_{n_k} \to p$ and $q_{n_k} \to q$

Since $d_X(p_n,q_n) \to 0$, $p = q$

But f continuous $\Rightarrow f(p_{n_k}), f(q_{n_k}) \to f(p)$

And $d_Y(f(p_{n_k}), f(q_{n_k})) \le d_Y(f(p_{n_k}), f(p)) + d_Y(f(p), f(q_{n_k})) \to 0$

$\therefore$ $d_Y(f(p_{n_k}), f(q_{n_k})) \to 0$

Contradiction since $d_Y(f(p_n), f(q_n)) > \epsilon$

$\therefore$ f uniformly continuous

13. Let $E \subseteq X$ be dense and let $f: E \to \mathbb{R}$ be uniformly continuous. Prove that f has a continuous extension from E to X.

For each $p \in X$ and $n \in \mathbb{Z}^+$, let $V_n(p) = \{q \in E \mid d(p,q) < \frac{1}{n}\}$

$\therefore$ $V_n(p) = N_{\frac{1}{n}}(p) \cap E$

Since f uniformly continuous, for $\epsilon = 1$ $\exists \delta > 0 \ni \frac{1}{n} < \delta$ with $\operatorname{diam}(f(V_n(p))) < 1$

$\therefore$ $f(V_n(p))$ bounded

$\therefore \overline{f(V_n(p))}$ bounded

Also $\overline{f(V_n(p))}$ closed

$\therefore \overline{f(V_n(p))}$ compact

And $\overline{f(V_1(p))} \supseteq \overline{f(V_2(p))} \supseteq \ldots$

$\therefore \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} \neq \phi$

Let $x, y \in \bigcap_{n=1}^{\infty} \overline{f(V_n(p))}$

Choosing n large enough $\ni \frac{1}{n} < \delta \Rightarrow \text{diam}(f(V_n(p))) \longrightarrow 0$

$\therefore x = y := x_p$

Define $g(p) = \begin{cases} f(p) & p \in E \\ x_p & p \notin E \end{cases}$

Since f uniformly continuous, for $\frac{\epsilon}{3} > 0$ $\exists \delta > 0 \ni d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \frac{\epsilon}{3}$

Let $x, y \in X \ni d(x,y) < \frac{\delta}{3} \Rightarrow d(f(x), f(y)) < \frac{\epsilon}{3}$

And take $x', y' \in E$ with $d(x, x') < \frac{\delta}{3}$, $d(y, y') < \frac{\delta}{3}$

Then $d(f(x), g(x')) < \frac{\epsilon}{3}$ and $d(f(y), g(y')) < \frac{\epsilon}{3}$

Then $d(g(x'), g(y')) \leq d(g(x'), f(x)) + d(f(x), f(y)) + d(f(y), g(y')) = \epsilon$

$\therefore$ g uniformly continuous

$\therefore$ f has continuous extension from E to X namely g

Clearly this holds for $\mathbb{R}^k$, compact sets, and complete sets

However this does not hold for any metric space

Take $X = \mathbb{R}$, $Y = E = \mathbb{Q}$, $f: E \to Y \ni f(x) = x$

Then there is no possible extension from X to Y because $X \notin Y$

16. Let $[x] \in \mathbb{Z} \ni x - 1 < [x] \leq x$, and let $(x) = x - [x]$. What discontinuities do $[x], (x)$ have?

Let $x \in \mathbb{Z}$

Suppose $[x]$ continuous at x

Then for $\epsilon = \frac{1}{2}$, $\exists 0 < \delta < \epsilon \ni |x - y| < \delta \Rightarrow |[x] - [y]| < \frac{1}{2}$

And $|(x - \frac{\delta}{2}) - x| < \delta$ so $|[x - \frac{\delta}{2}] - [x]| < \frac{1}{2}$

But $|[x - \frac{\delta}{2}] - [x]| = |x - 1 - x| = 1 > \frac{1}{2}$

Contradiction

$\therefore [x]$ discontinuous on $\mathbb{Z}$

Suppose (x) continuous at x

Then for $\epsilon = \frac{1}{2}$, $\exists 0 < \delta < \epsilon \ni |x - y| < \delta \Rightarrow |[x] - [y]| < \frac{1}{2}$

And $\left|x-\frac{\delta}{2}-x\right|<\delta$ so $\left|(x-\frac{\delta}{2})-(x)\right|<\frac{1}{2}$

But $\left|(x-\frac{\delta}{2})-(x)\right| = \left|1-\frac{\delta}{2}\right| > \left|1-\frac{\epsilon}{2}\right| = \frac{3}{4} > \frac{1}{2}$

Contradiction

$\therefore$ (x) discontinuous on $\mathbb{Z}$

Clearly $[x], (x)$ both continuous $\forall x \notin \mathbb{Z}$

$\therefore$ The set of discontinuities for both functions in $\mathbb{Z}$

19. Suppose $f: \mathbb{R} \to \mathbb{R}$ satisfies intermediate value property. And $\forall r \in \mathbb{Q}$, $\{x \mid f(x)=r\}$ is closed. Prove that f continuous.

Let $x_n \to x_0$

Suppose $f(x_0) < f(x_n)$ $\forall n$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $r \ni f(x_0) < r < f(x_n)$ $\forall n$

Then by IVP $\exists t_n$ between x_n, x_0 $\ni f(t_n)=r$

But $x_n \to x_0$, so $t_n \to x_0$ by squeeze principle

$\therefore$ x_0 is a limit point of $\{t_n\}$

But $t_n \in \{x \mid f(x)=r\}$, so x_0 is a limit point of this closed set

$\therefore$ $x_0 \in \{x \mid f(x)=r\}$

$\therefore$ $f(x_0)=r$

Contradiction since $f(x_0)<r$

Similarly if $f(x_0) > f(x_n)$ $\forall n$ we come to a contradiction

$\therefore$ $f(x_n) \to f(x)$

$\therefore$ f continuous

Chapter 5

1. Let f be defined $\forall x \in \mathbb{R}$ and suppose that $|f(x)-f(y)| \le (x-y)^2 \;\; \forall x,y \in \mathbb{R}$.
Prove that f is constant

Note that $|f(x)-f(y)| \le (x-y)^2 \Rightarrow \frac{|f(x)-f(y)|}{|x-y|} \le |x-y|$

So $|f'(x)| = \left| \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \right| = \lim_{t\to x} \left| \frac{f(t)-f(x)}{t-x} \right| = \lim_{t\to x} \frac{|f(t)-f(x)|}{|t-x|} \le \lim_{t\to x} |t-x| = 0$

$\therefore |f'(x)| \le 0$

$\therefore |f'(x)| = 0$

$\therefore f'(x) = 0 \;\; \forall x$

$\therefore f$ constant

4. If $C_0 + \frac{C_1}{2} + \dots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$ where $C_0, \dots, C_n \in \mathbb{R}$ prove that
$C_0 + C_1 x + \dots + C_{n-1}x^{n-1} + C_n x^n = 0$ has at least one real root
between $0, 1$.

Let $f(x) = C_0 x + \frac{C_1}{2}x^2 + \dots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$

Note that $f(1) = C_0 + \frac{C_1}{2} + \dots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$ by hypothesis

Also $f(0) = 0$

$\therefore f(0) = f(1)$

Then by Rolles Thm, $f'(x) = 0$ for at least one $x \in (0,1)$

And $f'(x) = C_0 + C_1 x + \dots + C_{n-1}x^{n-1} + C_n x^n$

$\therefore C_0 + C_1 x + \dots + C_{n-1}x^{n-1} + C_n x^n = 0$ for at least one $x \in (0,1)$

$\therefore C_0 + C_1 x + \dots + C_{n-1}x^{n-1} + C_n x^n$ has at least one root $x \in (0,1)$

7. Suppose $f'(x), g'(x)$ exist, $g'(x) \ne 0$ and $f(x) = g(x) = 0$. Prove that
$\lim_{t\to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$

$\lim_{t\to x} \frac{f(t)}{g(t)} = \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-g(x)}$ since $f(x) = g(x) = 0$

$= \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-g(x)} \cdot \frac{t-x}{t-x} = \lim_{t\to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{\lim_{t\to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t\to x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$

$\therefore \lim_{t\to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$

10. Suppose $f, g: (0,1) \to \mathbb{C} \ni f(x), g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$
where $A, B \in \mathbb{C}$, $B \ne 0$. Prove that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}$.

$f = f_1 + i f_2$, $g = g_1 + i g_2$

Note $\lim_{x\to 0} \frac{f_1(x)}{x} = \frac{0}{0} \Rightarrow \lim_{x\to 0} \frac{f_1(x)}{x} = \lim_{x\to 0} f_1'(x)$ by L'Hospitals Rule

And $\lim_{x\to 0}\frac{f_2(x)}{x} = \frac{0}{0} \Rightarrow \lim_{x\to 0}\frac{f_2(x)}{x} = \lim_{x\to 0} f_2'(x)$ By L'Hospital's Rule

Similarly $\lim_{x\to 0}\frac{g_1(x)}{x} = \lim_{x\to 0} g_1'(x)$ and $\lim_{x\to 0}\frac{g_2(x)}{x} = \lim_{x\to 0} g_2'(x)$

So $\lim_{x\to 0}\frac{f_1(x)}{x} + c\lim_{x\to 0}\frac{f_2(x)}{x} = \lim_{x\to 0} f_1'(x) + c\lim_{x\to 0} f_2'(x) = \lim_{x\to 0} f'(x)$

But also $\lim_{x\to 0}\frac{f_1(x)}{x} + c\lim_{x\to 0}\frac{f_2(x)}{x} = \lim_{x\to 0}\frac{f_1(x)+cf_2(x)}{x} = \lim_{x\to 0}\frac{f(x)}{x}$

$\therefore \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0} f'(x) = A$

Similarly $\lim_{x\to 0}\frac{g(x)}{x} = \lim_{x\to 0} g'(x) = B$

$\therefore \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\left(\frac{f(x)}{x} - A\right)\frac{x}{g(x)} + A\frac{x}{g(x)} = \left(\lim_{x\to 0}\frac{f(x)}{x} - A\right)\lim_{x\to 0}\frac{x}{g(x)} + \lim_{x\to 0} A\frac{x}{g(x)}$

$= (A-A)\frac{1}{B} + A\cdot\frac{1}{B} = \frac{A}{B}$

$\therefore \lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{A}{B}$

13. Suppose $a, c \in \mathbb{R}$, $c>0$, and f defined on $[-1,1]$ by $f(x) = \begin{cases} x^a \sin(|x|^{-c}) & x\neq 0 \\ 0 & x=0 \end{cases}$

Prove:

a. f is continuous iff $a>0$

Case 1 $a>0$

Show f continuous

Note that f continuous $\forall x\neq 0$ so show f continuous at $x=0$

$-x^a \le x^a\sin(|x|^{-c}) \le x^a \Rightarrow \lim_{x\to 0} -x^a \le \lim_{x\to 0} x^a \sin(|x|^{-c}) \le \lim_{x\to 0} x^a$

$\therefore 0 \le \lim_{x\to 0} x^a \sin(|x|^{-c}) \le 0$

$\therefore \lim_{x\to 0} x^a \sin(|x|^{-c}) = 0$ by squeeze principle

$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$

$\therefore f$ continuous at $x=0$

$\therefore f$ continuous $\forall x$

Case 2 $a=0$

Show f discontinuous at $x=0$

Define $x_n = \left(\frac{1}{2n\pi + \frac{\pi}{2}}\right)^{\frac{1}{c}}$

Then $x_n \to 0$ as $n\to\infty$

But $f(x_n) = \sin(2n\pi + \frac{\pi}{2}) = 1$

$\therefore f(x_n) \to 1$ as $n\to\infty$

$\therefore f(x_n) \not\to f(0) = 0$

$\therefore f$ not continuous at $x=0$

Case 3 $a<0$

Show f discontinuous at $x=0$

Define $x_n = \left(\frac{1}{2n\pi + \frac{\pi}{2}}\right)^{\frac{1}{c}}$

$x_n \to 0$ as $n \to \infty$

But $f(x_n) = \left(\frac{1}{2n\pi + \frac{\pi}{2}}\right)^{\frac{a}{c}} \sin(2n\pi + \frac{\pi}{2}) = (2n\pi + \frac{\pi}{2})^{-\frac{a}{c}}$

$\therefore f(x_n) = (2n\pi + \frac{\pi}{2})^{-\frac{a}{c}} \to \infty$ as $n \to \infty$ since $\frac{a}{c} < 0 \Rightarrow -\frac{a}{c} > 0$

$\therefore f(x_n) \not\to f(0) = 0$

$\therefore f$ discontinuous at $x = 0$

$\therefore f$ continuous iff $a > 0$

b. $f'(0)$ exists iff $a > 1$

$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^a \sin(|h|^{-c})}{h} = \lim_{h\to 0} h^{a-1} \sin(|h|^{-c})$

Case 1 $a > 1$

Show $f'(0)$ exists

$-h^{a-1} \le h^{a-1}\sin(|h|^{-c}) \le h^{a-1} \Rightarrow \lim_{h\to 0} -h^{a-1} \le \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) \le \lim_{h\to 0} h^{a-1}$

$\therefore$ $0 \le \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) \le 0$ since $a > 1 \Rightarrow a - 1 > 0$

$\therefore \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) = 0$ by squeeze principal

$\therefore f'(0) = \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) = 0$

$\therefore f'(0)$ exists

Case 2 $a = 1$

Show $f'(0)$ does not exist

Let $x_n = \left(\frac{1}{2n\pi + \frac{\pi}{2}}\right)^{\frac{1}{c}}$, $y_n = \left(\frac{1}{2n\pi + \frac{3\pi}{2}}\right)^{\frac{1}{c}}$

Clearly $x_n, y_n \to 0$ as $n \to \infty$

So $f'(0) = \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) = \lim_{n\to\infty} \sin(|x_n|^{-c})$ since $a = 1 \Rightarrow a - 1 = 0$

$= \lim_{n\to\infty} \sin(2n\pi + \frac{\pi}{2}) = \lim_{n\to\infty} 1 = 1$

Also $f'(0) = \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) = \lim_{n\to\infty} \sin(|y_n|^{-c}) = \lim_{n\to\infty} \sin(2n\pi + \frac{3\pi}{2})$

$= \lim_{n\to\infty} -1 = -1$

$\therefore f'(0)$ does not exist

Case 3 $a < 1$

Show $f'(0)$ does not exist

$f'(0) = \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) = \lim_{n\to\infty} x_n^{a-1}\sin(|x_n|^{-c}) = \infty$

And $f'(0) = \lim_{h\to 0} h^{a-1}\sin(|h|^{-c}) = \lim_{n\to\infty} y_n^{a-1}\sin(|y_n|^{-c}) = -\infty$

$\therefore f'(0)$ does not exist

16. Suppose f is twice differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$ and $f(x)\to 0$ as $x\to\infty$. Prove that $f'(x)\to 0$ as $x\to\infty$.

Note that $\sup\limits_{x>a}|f'(x)|^2 \le 4 \sup\limits_{x>a}|f(x)||f''(x)|$

so $\lim\limits_{a\to\infty}\sup\limits_{x>a}|f'(x)|^2 \le 4\lim\limits_{a\to\infty}\sup\limits_{x>a}|f(x)||f''(x)|$

$= 4\cdot 0\cdot M$ since f'' bounded and $f'(x)\to 0$

$\therefore \lim\limits_{a\to\infty}\sup\limits_{x>a}|f'(x)|^2 \le 0$

$\therefore \lim\limits_{x\to\infty}|f'(x)|^2 \le 0$

$\therefore \lim\limits_{x\to\infty} f'(x)=0$

$\therefore f'(x)\to 0$ as $x\to\infty$

19. Suppose f is defined on $(-1,1)$ and $f'(0)$ exists. Suppose $-1<\alpha_n<\beta_n<1$, $\alpha_n\to 0$, and $\beta_n\to 0$ as $n\to\infty$. Define $D_n=\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}$. Prove:

a. If $\alpha_n<0<\beta_n$, then $\lim\limits_{n\to\infty} D_n=f'(0)$

Let $\epsilon>0$ be given

Choose $\delta \ni \left|\frac{f(x)-f(0)}{x}-f'(0)\right|<\epsilon$ for $|x|<\delta$ since $f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}$

Then choose $N \ni -\delta<\alpha_n<0<\beta_n<\delta$ for $n>N$

Then for $n>N$ we have $|\alpha_n|, |\beta_n|<\delta$

So $|D_n-f'(0)| = \left|\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)\right| = \left|\frac{f(\beta_n)-f(0)+f(0)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)\right|$

$= \left|\frac{f(\beta_n)-f(0)}{\beta_n-\alpha_n}-\frac{f(\alpha_n)-f(0)}{\beta_n-\alpha_n}-f'(0)\right| = \left|\frac{\beta_n}{\beta_n-\alpha_n}\frac{f(\beta_n)-f(0)}{\beta_n}-\frac{\alpha_n}{\beta_n-\alpha_n}\frac{f(\alpha_n)-f(0)}{\alpha_n}-f'(\right.$

$\le \frac{\beta_n}{\beta_n-\alpha_n}\left|\frac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right| - \frac{\alpha_n}{\beta_n-\alpha_n}\left|\frac{f(\alpha_n)-f(0)}{\alpha_n}-f'(0)\right|$

$< \frac{\beta_n}{\beta_n-\alpha_n}\epsilon - \frac{\alpha_n}{\beta_n-\alpha_n}\epsilon = \epsilon$

$\therefore$ For $n>N$, $|D_n-f'(0)|<\epsilon$

$\therefore \lim\limits_{n\to\infty} D_n=f'(0)$

b. If $0<\alpha_n<\beta_n$ and $\left\{\frac{\beta_n}{\beta_n-\alpha_n}\right\}$ is bounded, then $\lim\limits_{n\to\infty} D_n=f'(0)$

$\left\{\frac{\beta_n}{\beta_n-\alpha_n}\right\}$ bounded $\Rightarrow \frac{\beta_n}{\beta_n-\alpha_n}\le M \;\; \forall n$

And since $\alpha_n<\beta_n$, $\frac{\alpha_n}{\beta_n-\alpha_n}<\frac{\beta_n}{\beta_n-\alpha_n}\le M$

$\therefore \frac{\alpha_n}{\beta_n-\alpha_n}\le M \;\; \forall n$

Let $\epsilon>0$ be given

choose $\delta \ni \left|\frac{f(x)-f(0)}{x}-f'(0)\right|<\frac{\epsilon}{2M}$ for $|x|<\delta$

And choose $N \ni \beta_n<\delta$ for $n>N$

Then for $n > N$, $|\beta_n| < \delta \Rightarrow |\alpha_n| < \delta$

So $|D_n - f'(0)| \le \frac{\beta_n}{\beta_n - \alpha_n} \left| \frac{f(\beta_n) - f(0)}{\beta_n} \right| + \frac{\alpha_n}{\beta_n - \alpha_n} \left| \frac{f(\alpha_n) - f(0)}{\alpha_n} \right| < M \cdot \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} = \epsilon$

$\therefore$ for $n > N$, $|D_n - f'(0)| < \epsilon$

$\therefore \lim_{n \to \infty} D_n = f'(0)$

C. If f' is continuous on $(-1,1)$, then $\lim_{n\to\infty} D_n = f'(0)$

Since f' exists on $(-1,1)$, f continuous on $(-1,1)$

$\therefore f$ continuous on $[\alpha_n, \beta_n]$ since $-1 < \alpha_n < \beta_n < 1$

And f differentiable on (α_n, β_n)

Then by MVT $\exists\, \gamma_n \in (\alpha_n, \beta_n) \ni \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(\gamma_n)$

$\therefore D_n = f'(\gamma_n)$

And $\alpha_n < \gamma_n < \beta_n \Rightarrow \lim_{n\to\infty} \alpha_n \le \lim_{n\to\infty} \gamma_n \le \lim_{n\to\infty} \beta_n \Rightarrow 0 \le \lim_{n\to\infty} \gamma_n \le 0$

$\therefore \gamma_n \to 0$ as $n \to \infty$ by squeeze principle

So $\lim_{n\to\infty} D_n = \lim_{n\to\infty} f'(\gamma_n) = f'(0)$ since f' continuous

$\therefore \lim_{n\to\infty} D_n = f'(0)$

Now give an example in which f differentiable in $(-1,1)$ but f' not continuous at 0 and $\alpha_n, \beta_n \to 0$ $\ni \exists \lim_{n\to\infty} D_n \neq f'(0)$

Take $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Clearly f differentiable for $x \neq 0$

And $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h \sin\frac{1}{h}$

And $-h \le h\sin\frac{1}{h} \le h \Rightarrow \lim_{h\to 0} -h \le \lim_{h\to 0} h\sin\frac{1}{h} \le \lim_{h\to 0} h \Rightarrow 0 \le \lim_{h\to 0} h\sin\frac{1}{h} \le 0$

$\therefore \lim_{h\to 0} h\sin\frac{1}{h} = 0$ by squeeze principle

$\therefore f'(0) = 0$

So $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$\therefore f'$ differentiable on $(-1,1)$ but not continuous at 0

Take $\alpha_n = \frac{1}{2\pi n}$, $\beta_n = \frac{1}{2\pi n + \frac{\pi}{2}}$

Clearly $\alpha_n, \beta_n \to 0$

But $\lim_{n\to\infty} D_n = \lim_{n\to\infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2\pi n + \frac{\pi}{2}}\right)^2}{\frac{\frac{\pi}{2}}{4\pi^2 n^2 + n}} = \lim_{n\to\infty} \frac{4\pi^2 n^2 + n}{2\pi^3 n^2 - \pi^3 n - \frac{\pi^3}{8}}$

$= \frac{2}{\pi}$

$\therefore \exists \lim_{n\to\infty} D_n = \frac{2}{\pi} \neq 0 = f'(0)$

22. Suppose $f : (-\infty, \infty) \to \mathbb{R}$.

a. If f is differentiable and $f'(t) \neq 1 \ \forall t \in \mathbb{R}$, prove that f has at most one fixed point

Suppose f has 2 fixed points, $x_1 \neq x_2$. WLOG take $x_1 < x_2$

Then $f(x_1) = x_1$, $f(x_2) = x_2$

Since f differentiable on $\mathbb{R}$, f continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2)

Then by MVT $\exists x \in (x_1, x_2) \ni f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$

$\therefore f'(x) = 1$

contradiction since $f'(t) \neq 1 \ \forall t \in \mathbb{R}$

$\therefore f$ has at most one fixed point

b. Show that $f(t) = t + \frac{1}{1+e^t}$ has no fixed point, although $0 < f'(t) < 1$ $\forall t \in \mathbb{R}$.

Suppose f has a fixed point x

Then $f(x) = x$

Then $x = x + \frac{1}{1+e^x} \Rightarrow \frac{1}{1+e^x} = 0$

contradiction

$\therefore f$ has no fixed point

But $f'(t) = 1 - \frac{e^t}{(1+e^t)^2}$

Then clearly $0 < f'(t) < 1$

c. If $\exists A < 1 \ni |f'(t)| \leq A \ \forall t \in \mathbb{R}$, prove that a fixed point x of f exists and that $x = \lim_{n\to\infty} x_n$ where $x_1 \in \mathbb{R}$ and $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$

By MVT we have $|f(x_n) - f(x_{n-1})| = |f'(t_n)||x_n - x_{n-1}| \leq A|x_n - x_{n-1}|$

$\therefore |x_{n+1} - x_n| \leq A|x_n - x_{n-1}| \ \forall n$

so $|x_{n+1} - x_n| \leq A|x_n - x_{n-1}| \leq A^2|x_{n-1} - x_{n-2}| \leq \ldots \leq A^{n-1}|x_2 - x_1|$

$\therefore |x_{n+k} - x_n| = |(x_{n+k} - x_{n+k-1}) + (x_{n+k-1} - x_{n+k-2}) + \ldots + (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)$

$\leq |x_{n+k} - x_{n+k-1}| + |x_{n+k-1} - x_{n+k-2}| + \ldots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n$

$\leq A^{n+k-2}|x_2 - x_1| + A^{n+k-3}|x_2 - x_1| + \ldots + A^n|x_2 - x_1| + A^{n-1}|x_2 - x_1|$

$= (A^{n+k-2} + \ldots + A^{n-1})|x_2 - x_1|$

$\leq k A^{n-1}|x_2 - x_1|$

so $\lim_{n\to\infty} |x_{n+k} - x_n| \le \lim_{n\to\infty} kA^{n-1}|x_2 - x_1| = 0$ since $A<1$

$\therefore \{x_n\}$ converges by cauchy criterion

$\therefore \exists x \in \mathbb{R} \ni x_n \to x$

$\therefore f(x_{n-1}) \to x$

But f continuous so $f(x_{n-1}) \to f(x)$

$\therefore f(x) = x$

$\therefore \exists x \in \mathbb{R} \ni x = \lim_{n\to\infty} x_n$ and x fixed point of f

25. Suppose f twice differentiable on $[a,b]$, $f(a) < 0 < f(b)$, $f'(x) \ge \delta > 0$, and $0 \le f''(x) \le M$ $\forall x \in [a,b]$. Let ξ be the unique point in (a,b) $\ni f(\xi) = 0$.

a. Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Interpret this geometrically, in terms of a tangent to the graph of f.

The tangent line to the graph of f at x_n has the equation

$y - f(x_n) = f'(x_n)(x - x_n)$

Setting $y = 0$ and solving for x, we get $x = x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$

so x_{n+1} is the point at which the tangent line at $(x_n, f(x_n))$ intersects the x axis

b. Prove that $x_{n+1} < x_n$ and that $\lim_{n\to\infty} x_n = \xi$

since $x_1 \in (\xi, b)$ and ξ only point in $(a,b) \ni f(\xi) = 0$ and $0 < f(b)$,

$f(x_n) > 0$ since f continuous

And $f'(x_n) > 0$ by assumption

$\therefore$ $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} < 0$

$\therefore x_{n+1} < x_n$

$\therefore \{x_n\}$ decreasing

Also $\{x_n\}$ bounded below by ξ

$\therefore \{x_n\}$ converges

Suppose $x_n \to \eta \ni \eta \ge \xi$

So $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow \eta = \eta - \frac{f(\eta)}{f'(\eta)} \Rightarrow f(\eta) = 0$

$\therefore \eta = \xi$

$\therefore \lim_{n\to\infty} x_n = \xi$

c. Use Taylor's theorem to show that $x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$ for some $t_n \in (\xi, x_n)$

By Taylor's thm, $\exists\, t_n \in (\xi, x_n) \ni f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$

$\therefore\ 0 = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$

$-f'(x_n)(\xi - x_n) - f(x_n) = \frac{f''(t_n)}{2}(\xi - x_n)^2$

$\therefore\ x_n - \xi - \frac{f(x_n)}{f'(x_n)} = \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2$

$\therefore\ x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$ for some $t_n \in (\xi, x_n)$

d. If $A = \frac{M}{2\delta}$ deduce that $0 \le x_{n+1} - \xi \le \frac{1}{A}[A(x_1 - \xi)]^{2^n}$.

We know $0 \le x_{n+1} - \xi$ since $x_{n+1} \ge \xi$

And by (c) $x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \le \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2$

$\therefore\ x_{n+1} - \xi \le A(x_n - \xi)^2$

So $x_{n+1} - \xi \le A(x_n - \xi)^2 \le A^3(x_{n-1} - \xi)^4 \le \dots \le A^{2^n - 1}(x_1 - \xi)^{2^n}$

$\therefore\ 0 \le x_{n+1} - \xi \le \frac{1}{A}[A(x_1 - \xi)]^{2^n}$

e. Show that Newton's method amounts to finding a fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$. How does $g'(x)$ behave for x near ξ?

Let x_0 be a fixed point of g

Then $g(x_0) = x_0$

So $x_0 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow \frac{f(x_0)}{f'(x_0)} = 0 \Rightarrow f(x_0) = 0$

Hence $x_0 = \xi$

Newton's method is used to compute $\xi \ni f(\xi) = 0$

$\therefore$ Newton's method amounts to finding a fixed point for g

$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$

So $\lim_{x \to \xi} g'(x) = \lim_{x \to \xi} \frac{f(x)f''(x)}{(f'(x))^2} = \lim_{x \to \xi} f(x) \lim_{x \to \xi} \frac{f''(x)}{(f'(x))^2} = 0$ since f continuous and $f(\xi) = 0$

$\therefore \lim_{x \to \xi} g'(x) = 0$

f. For $f(x) = x^{\frac{1}{3}}$ on $(-\infty, \infty)$ try Newton's Method. What happens?

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = x_n - 3x_n = -2x_n \quad \forall n$

$\therefore\ x_n = -2x_{n-1} = (-2)^2 x_{n-2} = (-2)^3 x_{n-3} = \dots = (-2)^{n-1} x_1$

$\therefore x_n = (-2)^{n-1} x_1$

$\therefore \{x_n\}$ diverges for any x_1

$\therefore$ We cannot use Newtons method to find $\xi \ni f(\xi) = 0$

Chapter 6

3. Define $\beta_1(x)=\begin{cases}0 & x\le 0\\ 1 & x>0\end{cases}$, $\beta_2(x)=\begin{cases}0 & x<0\\ 1 & x\ge 0\end{cases}$, $\beta_3(x)=\begin{cases}0 & x<0\\ \frac{1}{2} & x=0\\ 1 & x>0\end{cases}$. Let f be bounded on $[-1,1]$.

a. Prove that $f\in\mathcal{R}(\beta_1)$ iff $f(0+)=f(0)$ and that then $\int_{-1}^{1} f\,d\beta_1 = f(0)$

($\Rightarrow$) Assume $f\in\mathcal{R}(\beta_1)$

Let $\epsilon>0$ be given

Then $\exists P$ partition of $[-1,1]\ni U(P,f,\beta_1)-L(P,f,\beta_1)<\epsilon$

We can assume $0\in P$ since any refinement of P will work

But $U(P,f,\beta_1)-L(P,f,\beta_1) = \Sigma(M_i-m_i)\Delta x_i = M_j-m_j$ where $x_{j-1}\le 0<x_j$

$\therefore M_j-m_j<\epsilon$

Then for $0\le x\le x_j$, $m_j\le f(x)\le M_j$

$\therefore |f(x)-f(0)|\le M_j-m_j<\epsilon$

$\therefore f(0+)=f(0)$

($\Leftarrow$) Assume that $f(0+)=f(0)$

Let $\epsilon>0$ be given

Then $\exists\delta>0 \ni 0\le x\le\delta \Rightarrow |f(x)-f(0)|<\frac{\epsilon}{2}$

Let $P=\{-1,0,\delta,1\}$

Then $U(P,f,\beta_1)-L(P,f,\beta_1)=\sup_{0\le x\le\delta} f(x)-\inf_{0\le x\le\delta} f(x) < \frac{\epsilon}{2}+f(0)+\frac{\epsilon}{2}-f(0)=\epsilon$

$\therefore f\in\mathcal{R}(\beta_1)$

And $\int_{-1}^{1} f\,d\beta_1 = \overline{\int_{-1}^{1}} f\,d\beta_1 \le U(P,f,\beta_1)=\sup_{0\le x\le\delta} f(x)=\frac{\epsilon}{2}+f(0)$

But ϵ arbitrary $\Rightarrow \int_{-1}^{1} f\,d\beta_1\le f(0)$

Similarly $\int_{-1}^{1} f\,d\beta_1 = \underline{\int_{-1}^{1}} f\,d\beta_1 \ge L(P,f,\beta_1)=\inf_{0\le x\le\delta} f(x) = f(0)-\frac{\epsilon}{2}$

$\therefore \int_{-1}^{1} f\,d\beta_1\ge f(0)$

$\therefore \int_{-1}^{1} f\,d\beta_1 = f(0)$

b. State and prove a similar result for β_2.

Show $f\in\mathcal{R}(\beta_2)$ iff $f(0-)=f(0)$ and then $\int_{-1}^{1} f\,d\beta_2 = f(0)$

($\Rightarrow$) Assume $f\in\mathcal{R}(\beta_2)$

Let $\epsilon>0$ be given

Then $\exists P$ partition of $[-1,1]\ni U(P,f,\beta_2)-L(P,f,\beta_2)<\epsilon$

But $U(P,f,\beta_2)-L(P,f,\beta_2)=M_j-m_j$ where $x_{j-1}<0\le x_j$

$\therefore M_j-m_j<\epsilon$

Then for $x_{j-1} \le x \le 0$, $m_j \le f(x) \le M_j$

So $|f(x) - f(0)| \le M_j - m_j < \epsilon$

$\therefore f(0-) = f(0)$

$(\Leftarrow)$ Assume $f(0-) = f(0)$

Let $\epsilon > 0$ be given

Then $\exists \delta > 0 \ni -\delta \le x \le 0 \Rightarrow |f(x) - f(0)| < \frac{\epsilon}{2}$

Let $P = \{-1, -\delta, 0, 1\}$

Then $U(P, f, \beta_2) - L(P, f, \beta_2) = \sup_{-\delta \le x \le 0} f(x) - \inf_{-\delta \le x \le 0} f(x) < \frac{\epsilon}{2} + f(0) + \frac{\epsilon}{2} - f(0)$

$= \epsilon$

$\therefore f \in \mathcal{R}(\beta_2)$

And by same argument as (a), $f(0) \le \int_{-1}^{1} f \, d\beta_2 \le f(0)$

$\therefore \int_{-1}^{1} f \, d\beta_2 = f(0)$

c. Prove that $f \in \mathcal{R}(\beta_3)$ iff f continuous at 0

$(\Rightarrow)$ Assume $f \in \mathcal{R}(\beta_3)$

Let $\epsilon > 0$ be given

Then $\exists P$ partition of $[-1, 1] \ni U(P, f, \beta_3) - L(P, f, \beta_3) < \frac{\epsilon}{2}$

But $U(P, f, \beta_3) - L(P, f, \beta_3) = \frac{1}{2}\left(\sup_{x_{i-1} \le x < 0} f(x) - \inf_{x_{i-1} \le x \le 0} f(x) + \sup_{0 \le x \le x_{i+1}} f(x) - \inf_{0 \le x \le x_i} f(x \right.$

$\therefore \frac{1}{2}(M_i - m_i + M_{i+1} - m_{i+1}) < \frac{\epsilon}{2} \Rightarrow M_i - m_i + M_{i+1} - m_{i+1} < \epsilon$

Take $\delta = \min\{-x_{i-1}, x_{i+1}\}$

Then for $|x| < \delta$, $|f(x) - f(0)| \le M_i - m_i + M_{i+1} - m_{i+1} < \epsilon$

$\therefore f$ continuous 0

$(\Leftarrow)$ Assume f continuous at 0

Then $f(0) = f(0+) = f(0-)$

Then $f \in \mathcal{R}(\beta_1), \mathcal{R}(\beta_2)$ by (a), (b)

$\therefore f \in \mathcal{R}(\frac{1}{2}(\beta_1 + \beta_2)) = \mathcal{R}(\beta_3)$

$\therefore f \in \mathcal{R}(\beta_3)$

d. If f continuous at 0 prove that $\int f \, d\beta_1 = \int f \, d\beta_2 = \int f \, d\beta_3 = f(0)$

By (a), (b), f continuous at $0 \Rightarrow \int f \, d\beta_1 = \int f \, d\beta_2 = f(0)$

And by (c) f continuous at $0 \Rightarrow f \in \mathcal{R}(\beta_3)$

So show $\int f \, d\beta_3 = f(0)$

$\int_{-1}^{1} f\, d\beta_3 = \int_{-1}^{1} f\, d\left(\frac{1}{2}(\beta_1+\beta_2)\right) = \frac{1}{2}\left[\int_{-1}^{1} f\, d\beta_1 + \int_{-1}^{1} f\, d\beta_2\right] = \frac{1}{2}\left[f(0)+f(0)\right] = f(0)$

$\therefore \int f\, d\beta_1 = \int f\, d\beta_2 = \int f\, d\beta_3 = f(0)$

6. Let P be the Cantor set. Let $f: [0,1] \to \mathbb{R}$ be bounded on $[0,1]$ and continuous on P^c. Prove that $f \in \mathcal{R}$ on $[0,1]$.

First note P can be covered by by finitely many open intervals, O_α, whose total length can be as small as desired

$\mathcal{O} = \bigcup_{\alpha \in I} O_\alpha$ open since each O_α is open

Then $\mathcal{O}^c = [0,1] \setminus \bigcup_{\alpha \in I} O_\alpha$ closed

$\therefore \mathcal{O}^c$ compact since closed subset of $[0,1]$ compact

And f continuous on $\mathcal{O}^c$ since f continuous on P^c

$\therefore f$ uniformly continuous on $\mathcal{O}^c$

Then for $x, y \in \mathcal{O}^c$ $\exists \delta > 0 \ni |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

Choose a partition of $[0,1] \ni$ each point of the partition are elements of $\mathcal{O}^c$ and the distance between consecutive points is less than δ

Then $U(f,P) - L(f,P) = \sum (M_i - m_i)\Delta x_i < \epsilon(b-a)$

$\therefore f \in \mathcal{R}$

9. Show that integration by parts can sometimes be applied to improper integrals. For instance, show that $\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx$

Show that one of these converge absolutely but the other does not

Claim Let $f, g \in \mathcal{C}'$ defined on $[a,\infty) \ni \lim_{c\to\infty} f(c)g(c)$ exists and $\int_a^\infty f(x)g'(x)\,dx$ converges. Then $\int_a^\infty f'(x)g(x)\,dx$ converges and $\int_a^\infty f'(x)g(x)\,dx = \lim_{c\to\infty}[f(c)g(c) - f(a)g(a)] - \int_a^\infty f(x)g'(x)\,dx$

Proof of Claim Note that $\forall b \ni a < b < \infty$,

$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx$ exists by integration by parts

So $f'g \in \mathcal{R}$ on $[a,b]$ $\forall b$

And $\lim_{c\to\infty} \int_a^c f'(x)g(x)\,dx$ exists since $\lim_{c\to\infty} f(c)g(c)$ exists and $\int_a^\infty f(x)g'(x)\,dx$ converges

$\therefore \int_a^\infty f'(x)g(x)\,dx$ converges and $\int_a^\infty f'(x)g(x)\,dx = \lim_{c\to\infty}[f(c)g(c) - f(a)g(a)] - \int_a^\infty f(x)g'(x)$

Now $\int_0^\infty \frac{\cos x}{1+x}\,dx = \lim_{c\to\infty}\int_0^c \frac{\cos x}{1+x}\,dx$ $\quad u=\frac{1}{1+x}$ $\quad dv=\cos x\,dx$
$du=-\frac{1}{(1+x)^2}dx$ $\quad v=\sin x$

So $\int_0^\infty \frac{\cos x}{1+x}\,dx = \lim_{c\to\infty}\left[\left(\frac{\sin x}{1+x}\right)_0^c + \int_0^c \frac{\sin x}{(1+x)^2}\,dx\right]$

$= \lim_{c\to\infty}\left[\frac{\sin c}{1+c} + \int_0^c \frac{\sin x}{(1+x)^2}\,dx\right]$

And $-\frac{1}{1+c} \le \frac{\sin c}{1+c} \le \frac{1}{1+c} \Rightarrow \lim_{c\to\infty} -\frac{1}{1+c} \le \lim_{c\to\infty}\frac{\sin c}{1+c} \le \lim_{c\to\infty}\frac{1}{1+c}$

$\therefore 0 \le \lim_{c\to\infty}\frac{\sin c}{1+c} \le 0$

$\therefore \lim_{c\to\infty}\frac{\sin c}{1+c} = 0$ by squeeze principle

$\therefore \int_0^\infty \frac{\cos x}{1+x}\,dx = \lim_{c\to\infty}\left[\frac{\sin c}{1+c} + \int_0^c \frac{\sin x}{(1+x)^2}\,dx\right] = \lim_{c\to\infty}\int_0^c \frac{\sin x}{(1+x)^2}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$

$\therefore \int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$

$\int_0^\infty \left|\frac{\sin x}{(1+x)^2}\right| dx \le \int_0^\infty \frac{1}{(1+x)^2}\,dx = \lim_{c\to\infty}\int_0^c \frac{1}{(1+x)^2}\,dx = \lim_{c\to\infty}\left[\left(-\frac{1}{1+x}\right)_0^c\right]$

$= \lim_{c\to\infty}\left[1 - \frac{1}{1+c}\right] = 1$

$\therefore \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ absolutely convergent

12. Suppose $f\in\mathcal{R}(\alpha)$ and $\epsilon>0$. Prove that $\exists g$ continuous on $[a,b]$ $\ni$ $\|f-g\|_2 < \epsilon$.

$f\in\mathcal{R}(\alpha) \Rightarrow f$ bounded ie $m \le f(x) \le M$ $\forall x$ and $m_i \le m$, $M \ge M_i$ $\forall i$

And $\exists P$ partition $\ni$ $U(f,P,\alpha) - L(f,P,\alpha) < \frac{\epsilon^2}{M-m}$

Define $g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$ if $x_{i-1} \le t \le x_i$

Clearly g continuous on $[a,b]$ since g linear on each $[x_{i-1}, x_i]$

And $g(x_i) = f(x_i)$, $g(x_{i-1}) = f(x_{i-1})$

so g is madeup of straight lines connecting $f(x_i), f(x_{i-1})$ and

is hence continuous on $[a,b]$

Also note that $f(x_{i-1}) \le g(t) \le f(x_i)$ for $t\in[x_{i-1}, x_i]$

$\therefore m_i \le g(t) \le M_i$ for $t\in[x_{i-1}, x_i]$

So $\|f-g\|_2^2 = \int_a^b |f-g|^2\,d\alpha = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f-g|^2\,d\alpha \le \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |M_i - m_i|^2\,d\alpha$

$= \sum_{i=1}^n |M_i - m_i|^2 \int_{x_{i-1}}^{x_i} d\alpha = \sum_{i=1}^n |M_i - m_i|^2 \left(\alpha(x_i) - \alpha(x_{i-1})\right)$

$\le (M-m)\sum_{i=1}^n (M_i - m_i)\Delta\alpha(x_i) < (M-m)\cdot\frac{\epsilon^2}{M-m} = \epsilon^2$

$\therefore \|f-g\|_2 < \epsilon$

15. Suppose $f:[a,b]\to\mathbb{R}$, $f\in\mathcal{C}'$, $f(a)=f(b)=0$ and $\int_a^b f^2(x)\,dx = 1$.
Prove that $\int_a^b x f(x) f'(x)\,dx = -\frac{1}{2}$ and that $\int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x)\,dx > \frac{1}{4}$

Let $F = x f(x)$, $G' = f'(x)$

So $F' = x f'(x) + f(x)$ and $G = f(x)$

So F, G both differentiable on $[a,b]$

And $F', G' \in \mathcal{R}$ on $[a,b]$ since $f \in \mathcal{C}'$

So integrating by parts gives:

$\int_a^b x f(x) f'(x) dx = [x f^2(x)]_a^b - \int_a^b x f(x) f'(x) + f^2(x)\, dx$

$= b f^2(b) - a f^2(a) - \int_a^b x f(x) f'(x) dx - \int_a^b f^2(x) dx$

$= -\int_a^b x f(x) f'(x) dx - 1$

$\therefore 2\int_a^b x f(x) f'(x) dx = -1$

$\therefore \int_a^b x f(x) f'(x) dx = -\frac{1}{2}$

And by Holder's Inequality we have $\left(\int_a^b x f(x) f'(x) dx\right)^2 \le \int_a^b x^2 f^2(x) dx \int_a^b (f'(x))^2 dx$

So $\frac{1}{4} \le \int_a^b x^2 f^2(x) dx \cdot \int_a^b (f'(x))^2 dx$

18\. Let $\gamma_1, \gamma_2, \gamma_3 : [0, 2\pi] \to \mathbb{C} \ni \gamma_1(t) = e^{it}$, $\gamma_2(t) = e^{2it}$, $\gamma_3(t) = e^{2\pi i t \sin\frac{1}{t}}$.

Show that $\gamma_1, \gamma_2, \gamma_3$ have same range, γ_1, γ_2 are rectifiable, the length of γ_1 is 2π, the length of γ_2 is 4π, and γ_3 is not rectifiable.

Since $\gamma_1, \gamma_2 \in \mathcal{C}'$ on $[0, 2\pi]$, $\Lambda(\gamma_1) = \int_0^{2\pi} |\gamma_1'(t)| dt = \int_0^{2\pi} |i e^{it}| dt$

$= \int_0^{2\pi} 1\, dt = 2\pi$

$\therefore \Lambda(\gamma_1) = 2\pi$ and hence γ_1 rectifiable

And $\Lambda(\gamma_2) = \int_0^{2\pi} |\gamma_2'(t)| dt = \int_0^{2\pi} |2i e^{2it}| dt = \int_0^{2\pi} 2\, dt = 4\pi$

$\therefore \Lambda(\gamma_2) = 4\pi$ and hence γ_2 rectifiable

Now suppose γ_3 rectifiable

Note that $f(t) = \begin{cases} t\sin\frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$ continuous, $g(t) = e^{2\pi i t}$ continuous

So $\gamma_3(t) = g(f(t))$ continuous

And $\gamma_3'(t) = 2\pi i e^{2\pi i t \sin\frac{1}{t}} \left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right)$ continuous

$\therefore \gamma_3 \in \mathcal{C}'$

So $\Lambda(\gamma_3) = \int_0^{2\pi} |\gamma_3'(t)| dt = \int_0^{2\pi} 2\pi \left|\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right| dt$

But $\lim_{t\to 0} \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t} = \lim_{u\to\infty} \sin 2\pi u - 2\pi u \cos 2\pi u = \lim_{u\to\infty} -2\pi u = -\infty$

$\therefore \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}$ not bounded

$\therefore \Lambda(\gamma_3) = \int_0^{2\pi} 2\pi \left|\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right| dt$ does not exist

Contradiction

$\therefore \gamma_3$ not rectifiable

Chapter 7

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

$f_n \rightrightarrows f$ on $E \Rightarrow$ For $\epsilon = 1$ $\exists N = N(\epsilon) \ni m, n \geq N \Rightarrow |f_n(x) - f_m(x)| < 1 \quad \forall x \in E$.
by cauchy criterion

And $\{f_n\}$ bounded $\Rightarrow$ For each n $\exists M_n \ni |f_n(x)| \leq M_n \ \forall x \in E$

Let $M = \max\{M_1, \dots, M_{N-1}, 1 + M_N\}$

Then for $n \geq N$, $|f_n(x)| = |f_n(x) - f_N(x) + f_N(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)|$
$< 1 + M_N \leq M$

$\therefore |f_n(x)| \leq M \ \forall n, \forall x$

$\therefore \{f_n\}$ uniformly bounded

4. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Clearly for $x = -\frac{1}{n^2}$, f does not converge

And for $x = 0$ $f(x) = \sum_{n=1}^{\infty} 1$ which diverges

If $x > 0$, $\sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right| = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\therefore \sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right|$ converges by comparison test

$\therefore$ If $x > 0$, f converges absolutely

If $-\frac{1}{n^2} \neq x < 0$, $\sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right| = \sum_{n=1}^{\infty} \frac{1}{1+n^2|x|} \leq \sum_{n=1}^{\infty} \frac{1}{n^2|x|} = |x| \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\therefore \sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right|$ converges by comparison test

$\therefore$ If $-\frac{1}{n^2} \neq x < 0$, f converges absolutely

$\therefore f$ converges absolutely $\forall x \neq -\frac{1}{n^2}, 0$

Note that since f diverges at $x = -\frac{1}{n^2}, 0$, f does not converge uniformly on any interval $[a,b] \ni -\frac{1}{n^2}, 0 \in [a,b]$

Now consider $[a,b] \ni a > 0$

$\left|\frac{1}{1+n^2x}\right| = \frac{1}{1+n^2x} \leq \frac{1}{n^2x} \leq \frac{1}{an^2}$

And $\sum_{n=1}^{\infty} \frac{1}{an^2}$ converges

So f converges uniformly on $[a,b]$ by Weierstrauss M-test

Consider $[a,b] \ni b < 0$ with $-\frac{1}{n^2} \notin [a,b]$

$\left|\frac{1}{1+n^2x}\right| = \frac{1}{1+n^2|x|} \le \frac{1}{n^2|x|} \le \frac{1}{|a|n^2}$

And $\sum_{n=1} \frac{1}{|a|n^2}$ converges

so f converges uniformly on $[a,b]$ by Weierstrass M-test

$\therefore$ f converges on all intervals $[a,b]$ $\ni$ $0, -\frac{1}{n^2} \notin [a,b]$

Note that f is continuous $\forall x \neq -\frac{1}{n^2}$ and f diverges for $x = -\frac{1}{n^2}$

$\therefore$ f is continuous where it converges

Now look at $f\left(\frac{1}{k}\right) = \sum_{n=1}^{\infty} \frac{1}{1+n^2\frac{1}{k}} = k\sum_{n=1}^{\infty} \frac{1}{k+n^2} \longrightarrow \infty$ as $k \to \infty$

$\therefore$ f not bounded

7. Show that $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to a function f $\ni$ $f'(x) = \lim_{n\to\infty} f_n'(x)$ is true if $x \neq 0$, but false if $x = 0$.

Note that for $x=0$, $f_n(x) = 0 \longrightarrow 0$ as $n \longrightarrow \infty$

And for $x \neq 0$, $f_n(x) = \frac{x}{1+nx^2} \longrightarrow 0$ as $n \longrightarrow \infty$

$\therefore$ $f_n(x) \longrightarrow f(x) = 0$ as $n \longrightarrow \infty$ $\forall x$

Now $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$

And setting $1-nx^2 = 0$ we see that f_n has critical points $\pm\frac{1}{\sqrt{n}}$

So $|f_n(x)| \le f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$

Now choose $N \ni \frac{1}{2\sqrt{N}} < \epsilon$

Then for $n \ge N$, $|f_n(x)| = \frac{1}{2\sqrt{n}} \le \frac{1}{2\sqrt{N}} < \epsilon$

$\therefore$ for $n \ge N$, $|f_n(x)| < \epsilon$ $\forall x$

$\therefore$ $f_n \rightrightarrows f(x) = 0$

Now note that $f'(x) = 0$ $\forall x$

And for $x \neq 0$, $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} = 0$

$\therefore$ $f'(x) = \lim_{n\to\infty} f_n'(x)$ for $x \neq 0$

But for $x=0$, $\lim_{n\to\infty} f'_n(x) = \lim_{n\to\infty} 1 = 1$

$\therefore$ For $x=0$, $f'(x) \neq \lim_{n\to\infty} f_n'(x)$

10. Consider $f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$, $x \in \mathbb{R}$. Find all discontinuities of f and show that they form a countable dense set. Show that $f \in \mathcal{R}$ on every bounded interval.

Define $f_m(x) = \sum_{n=1}^{m} \frac{(nx)}{n^2}$, Let $\epsilon > 0$ be given, and let $[a,b]$ be our bounded interval

Then $\exists N \ni m \geq N \Rightarrow |f_m - f| = \left| \sum_{n=1}^{m} \frac{(nx)}{n^2} - \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \right| = \left| \sum_{n=m+1}^{\infty} \frac{(nx)}{n^2} \right| \leq \sum_{n=m+1}^{\infty} \left| \frac{(nx)}{n^2} \right|$

$< \sum_{n=m+1}^{\infty} \frac{1}{n^2} < \epsilon$ since $\sum \frac{1}{n^2}$ converges

$\therefore f_m \rightrightarrows f$ on $[a,b]$

Let $x \in \mathbb{Q} \Rightarrow x = \frac{p}{q} \ni \gcd(p,q) = 1$

$\lim_{t \to x^+} \frac{(nt)}{n^2} = \lim_{t \to x^+} \frac{nt - \lfloor nt \rfloor}{n^2} = \frac{1}{n^2}$

$\lim_{t \to x^-} \frac{(nt)}{n^2} = \lim_{t \to x^-} \frac{nt - \lfloor nt \rfloor}{n^2} = 0$

And since $f_m \rightrightarrows f$, $\sum_{n=1}^{\infty} f_n(x+) = f(x+)$ and $\sum_{n=1}^{\infty} f_n(x-) = f(x-)$

So we have $|f(x+) - f(x-)| = \left| \sum_{n=1}^{\infty} f_n(x+) - \sum_{n=1}^{\infty} f_n(x-) \right| = \left| \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} 0 \right| = \left| \sum_{n=1}^{\infty} \frac{1}{n^2} \right| \neq 0$

$\therefore f(x+) \neq f(x-)$

$\therefore f$ discontinuous at x

$\therefore f$ discontinuous on $\mathbb{Q}$

Now let $x \notin \mathbb{Q}$ and choose $N \ni \sum_{n \geq N} \frac{1}{n^2} < \frac{\epsilon}{3}$

Note that (nx) continuous at x, so f_N continuous at x

so $\exists \delta > 0 \ni |f_N(t) - f_N(x)| < \frac{\epsilon}{3}$ for $|t - x| < \delta$

$\therefore$ For $|t - x| < \delta$, $|f(t) - f(x)| = \left| f_N(t) + \sum_{n=N+1}^{\infty} \frac{(nt)}{n^2} - f_N(x) - \sum_{n=N+1}^{\infty} \frac{(nx)}{n^2} \right|$

$\leq |f_N(t) - f_N(x)| + \left| \sum_{n=N+1}^{\infty} \frac{(nt)}{n^2} \right| + \left| \sum_{n=N+1}^{\infty} \frac{(nx)}{n^2} \right|$

$\leq \frac{\epsilon}{3} + \sum_{n=N+1}^{\infty} \frac{1}{n^2} + \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

$\therefore f$ continuous at x

$\therefore f$ continuous on $\mathbb{Q}^c$

$\therefore$ The set of discontinuities of f form a countable dense set since $\mathbb{Q}$ is countable and dense

Now f_m has only finitely many discontinuities on $[a,b]$

$\therefore f_m \in \mathcal{R}$

Then since $f_m \in \mathcal{R}$ and $f_m \rightrightarrows f$, $f \in \mathcal{R}$ on $[a,b]$

16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

$\{f_n\}$ equicontinuous $\Rightarrow \forall \epsilon > 0\ \exists \delta = \delta(\epsilon) \ni x, y \in K$ and $|x - y| < \delta \Rightarrow$ $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}\ \forall n \geq 1$

Note that $K \subseteq \bigcup_{x \in K} N_\delta(x)$ is an open cover

Then since K compact, $K \subseteq N_\delta(x_1) \cup \ldots \cup N_\delta(x_n)$ is a finite subcover

Then for each $x \in K$ $\exists x_j \ni x \in N_\delta(x_j)$
so $|x-x_j|<\delta \Rightarrow |f_n(x)-f_n(x_j)|<\frac{\epsilon}{3}$ by equicontinuity
$\{f_n\}$ converges pointwise $\Rightarrow \{f_n(x_i)\}$ cauchy sequence
so $\exists N_i \ni |f_n(x_i)-f_m(x_i)|<\frac{\epsilon}{3}$ for $n,m \geq N_i$
Take $N = \max\{N_1, \dots, N_n\}$
Then $n,m \geq N \Rightarrow |f_m(x)-f_n(x)| = |f_m(x)-f_m(x_i)+f_m(x_i)-f_n(x_i)+f_n(x_i)-f_n$
$\leq |f_m(x)-f_m(x_i)|+|f_m(x_i)-f_n(x_i)|+|f_n(x_i)-f_n(x)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon$
$\therefore$ $\{f_n\}$ converges uniformly by Cauchy Criterion

19. Let K be a compact metric space and let $S \subseteq \mathscr{C}(K)$. Prove that S is compact iff S is uniformly closed, pointwise bounded, and equicontinuous.
($\Rightarrow$) Assume S compact
Then S closed and bounded
Now suppose S not equicontinuous
So $\exists \epsilon>0 \ni \forall \delta>0$ $\exists x,y \in S \ni |x-y|<\delta \Rightarrow \exists g_n \in S \ni |g_n(x)-g_n(y)| \geq \epsilon$
Hence no subsequence of $\{g_n\}$ is equicontinuous since $|g_{n_k}(x)-g_{n_k}(y)| \geq$
So since S compact and $g_n \in S \subseteq \mathscr{C}(K)$, no subsequence of $\{g_n\}$ converges on S
$\therefore$ S not compact
contradiction
$\therefore$ S closed, bounded, equicontinuous
($\Leftarrow$) Assume S closed, bounded, and equicontinuous
Then by Ascoli-Arzela, every $\{g_n\}$ has a uniformly convergent subsequence
And since S closed, the limit belongs to S
$\therefore$ S compact

22. Assume $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and prove that $\exists \{P_n\}$ polynomials $\ni \lim_{n\to\infty} \int_a^b |f-P_n|^2 d\alpha = 0$.
Let $\epsilon>0$ be given
By previous problem, since $f \in \mathscr{R}(\alpha)$ $\exists g$ continuous on $[a,b]$

$\ni \|f-g\|_2 < \sqrt{\frac{\epsilon}{2}}$

so $\left(\int_a^b |f-g|^2 d\alpha\right)^{1/2} < \sqrt{\frac{\epsilon}{2}} \Rightarrow \int_a^b |f-g|^2 d\alpha < \frac{\epsilon}{2}$

And since g continuous on $[a,b]$ $\exists \{P_n\} \ni P_n \rightrightarrows g$ by Weierstrass

so $\exists N \ni n \geq N \Rightarrow |P_n - g| < \sqrt{\frac{\epsilon}{2(\alpha(b)-\alpha(a))}}$

so for $n \geq N$, $\left|\int_a^b |f-P_n|^2 d\alpha\right| = \left|\int_a^b |f-g+g-P_n|^2 d\alpha\right|$

$\leq \left|\int_a^b |f-g|^2 + |g-P_n|^2 d\alpha\right| = \left|\int_a^b |f-g|^2 d\alpha + \int_a^b |g-P_n|^2 d\alpha\right|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2(\alpha(b)-\alpha(a))} \int_a^b d\alpha = \frac{\epsilon}{2} + \frac{\epsilon}{2(\alpha(b)-\alpha(a))}(\alpha(b)-\alpha(a)) = \epsilon$

$\therefore \lim_{n\to\infty} \int_a^b |f-P_n|^2 d\alpha = 0$

Chapter 8

6. Suppose $f(x)f(y) = f(x+y) \ \forall x, y \in \mathbb{R}$

a. Assuming that f is differentiable and not zero, prove that $f(x) = e^{cx}$ where c is constant

Note that $f(x)f(0) = f(x)$, so $f(0) = 1$ since $f \not\equiv 0$

And f differentiable $\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h}$

$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} = f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f(x) f'(0)$

$\therefore f'(x) = f(x) f'(0) \ \ \forall x \in \mathbb{R}$

Let $g(x) = f(x) e^{-x f'(0)}$

Then $g'(x) = -f'(0) f(x) e^{-x f'(0)} + f'(x) e^{-x f'(0)}$

$= -f'(0) f(x) e^{-x f'(0)} + f'(0) f(x) e^{-x f'(0)} = 0$

$\therefore g'(x) = 0 \ \forall x \in \mathbb{R}$

$\therefore g$ constant

And $g(0) = f(0) = 1$

$\therefore 1 = f(x) e^{-x f'(0)} \ \ \forall x \in \mathbb{R}$

$\therefore f(x) = e^{f'(0) x}$

$\therefore f(x) = e^{cx}$ where $c = f'(0)$ constant

b. Prove the same thing, assuming only that f is continuous.

Suppose $\exists z \in \mathbb{R} \ni f(z) = 0$

Then $f(z)f(y) = f(z+y) \ \ \forall y \in \mathbb{R}$

$\therefore 0 = f(z+y) \ \forall y \in \mathbb{R}$

$\therefore f \equiv 0$

Contradiction since f nonzero

$\therefore f(x) \neq 0 \ \ \forall x \in \mathbb{R}$

And since $f(0) = 1$ and f continuous, $f(x) > 0 \ \ \forall x \in \mathbb{R}$

So we can define $g(x) = \log f(x)$

And $g(x+y) = \log f(x+y) = \log(f(x)f(y)) = \log f(x) + \log f(y) = g(x) + g(y)$

$\therefore g(x+y) = g(x) + g(y) \ \ \forall x, y \in \mathbb{R}$

It suffices to show that $g(x) = cx$ for c constant

Note that $g(0) = \log f(0) = \log 1 = 0$

so $g(0) = 0$

And $g(-x) = g(0) - g(x) = -g(x)$

so $g(-x) = -g(x) \quad \forall x \in \mathbb{R}$

Now assume that $g((n-1)x) = (n-1)g(x)$

Then $g(nx - x) = ng(x) - g(x) \Rightarrow g(nx) - g(x) = ng(x) - g(x)$

$\therefore g(nx) = ng(x) \quad \forall n \in \mathbb{Z}$

Now consider $S = \{x \in \mathbb{R} \mid g(x) = g(1)x\}$

Note that $g(0) = 0 \Rightarrow 0 \in S$

And $g(1) = g(1) \cdot 1 \Rightarrow 1 \in S$

Now assume $a \in S$ and let $n \in \mathbb{Z}$

$g(na) = ng(a) = n \cdot g(1)a = g(1)na$

$\therefore na \in S$

$\therefore \mathbb{Z} \subseteq S$

Again assume $a \in S$ and let $0 \neq n \in \mathbb{Z}$

$g(a) = g(n \cdot \frac{a}{n}) = ng(\frac{a}{n})$

so $g(\frac{a}{n}) = \frac{1}{n}g(a) = \frac{1}{n}g(1)a = g(1)\frac{a}{n}$

$\therefore \frac{a}{n} \in S$

$\therefore \mathbb{Q} \subseteq S$

$\therefore g(x) = g(1)x \quad \forall x \in \mathbb{Q}$

But $\mathbb{Q}$ dense in $\mathbb{R}$ and g continuous

$\therefore g(x) = g(1)x$

$\therefore \log f(x) = \log f(1) x$

$\therefore f(x) = e^{f(1)x}$

$\therefore f(x) = e^{cx}$ where $c = f(1)$ constant

9. a. Put $s_N = 1 + \frac{1}{2} + \dots + \frac{1}{N}$. Prove that $\lim_{N\to\infty}(s_N - \log N)$ exists.

Claim $1 - \frac{1}{x} < \log x < x - 1 \quad \forall x > 0$

Let $f(x) = x - 1 - \log x$

Then $f'(x) = 1 - \frac{1}{x}$

Setting $f'(x) = 0 \Rightarrow 1 - \frac{1}{x} = 0 \Rightarrow 1 = \frac{1}{x} \Rightarrow x = 1$

And $f'(x) < 0 \quad \forall x < 1$ and $f'(x) > 0 \quad \forall x > 1$

$\therefore f$ decreasing on $(0,1)$ and increasing on $(1,\infty)$

$\therefore f$ has a minimum at $x = 1$

$\therefore f(x) > f(1)\ \forall x > 0$

$\therefore f(x) > 0\ \forall x > 0$

$\therefore \log x < x-1\ \forall x > 0$

Let $g(x) = \log x - 1 + \frac{1}{x}$

$g'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$

setting $g'(x) = 0 \Rightarrow \frac{x-1}{x^2} = 0 \Rightarrow x = 1$

And $g'(x) < 0\ \forall x < 1$ and $g'(x) > 0\ \forall x > 1$

so g decreasing on $(0,1)$ and increasing on $(1,\infty)$

$\therefore g$ has a minimum at $x = 1$

so $g(x) > g(1)\ \forall x > 0$

$\therefore g(x) > 0\ \forall x > 0$

$\therefore 1 - \frac{1}{x} < \log x\ \forall x > 0$

$\therefore 1 - \frac{1}{x} < \log x < x - 1\ \forall x > 0$

But $\frac{x+1}{x} > 0\ \forall x > 0$, so $1 - \frac{1}{\left(\frac{x+1}{x}\right)} < \log \frac{x+1}{x} < \frac{x+1}{x} - 1$

$\therefore 1 - \frac{x}{x+1} < \log(x+1) - \log x < \frac{x+1}{x} - 1$

$\therefore \frac{1}{x+1} < \log(x+1) - \log x < \frac{1}{x}\ \forall x > 0$

Now show $S_N - \log N$ bounded below

For $k \le x$, $\frac{1}{k} \ge \frac{1}{x} \Rightarrow \int_k^{k+1} \frac{1}{k} dx \ge \int_k^{k+1} \frac{1}{x} dx \Rightarrow \frac{1}{k} \ge \int_k^{k+1} \frac{1}{x} dx$

$\Rightarrow \sum_{k=1}^{N} \frac{1}{k} \ge \sum_{k=1}^{N} \int_k^{k+1} \frac{1}{x} dx = \int_1^{N+1} \frac{1}{x} dx = \log(N+1)$

$\therefore S_N - \log N \ge \log(N+1) - \log N > \frac{1}{N+1}$ by above claim

> 0

$\therefore \{x_n\}$ bounded below where $x_n = S_N - \log N$

Now show $S_N - \log N$ decreasing

$x_n - x_{n+1} = S_N - \log N - S_{N+1} + \log(N+1) = \log(N+1) - \log N - \frac{1}{N+1} > 0$ by above claim

$\therefore \{x_n\}$ decreasing

$\therefore \{x_n\}$ converges

$\therefore \lim_{N\to\infty} (S_N - \log N)$ exists

12. Suppose $0 < \delta < \pi$, $f(x) = \begin{cases} 1 & |x| < \delta \\ 0 & \delta < |x| \le \pi \end{cases}$, $f(x + 2\pi) = f(x)\ \forall x$

a. Compute the fourier coefficients of f

$C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$

$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1\,dx = \frac{1}{2\pi}\cdot 2\delta = \frac{\delta}{\pi}$

$c_k = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-ikx}\,dx = \frac{1}{2\pi}\left[\left(-\frac{1}{ik}e^{-ikx}\right)_{-\delta}^{\delta}\right] = -\frac{1}{2\pi ik}\left(e^{-ik\delta} - e^{ik\delta}\right)$

$= \frac{i}{2\pi k}(\cos k\delta - i\sin k\delta - \cos k\delta - i\sin k\delta) = \frac{\sin k\delta}{\pi k}$

b. Conclude that $\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = \frac{\pi-\delta}{2}$

$f(x) \sim c_0 + \sum_{-\infty}^{\infty} c_k e^{ikx}$ on $[-\delta,\delta]$

$= \frac{\delta}{\pi} + \sum_{k=1}^{\infty}\frac{\sin k\delta}{\pi k}(e^{ikx} + e^{-ikx}) = \frac{\delta}{\pi} + \sum_{k=1}^{\infty}\frac{\sin k\delta}{\pi k}(\cos kx + i\sin kx + \cos kx - i\sin$

$= \frac{\delta}{\pi} + \sum_{k=1}^{\infty}\frac{2\sin k\delta\cos kx}{\pi k}$

$x=0 \Rightarrow f(0) = \frac{\delta}{\pi} + \sum_{k=1}^{\infty}\frac{2\sin k\delta}{\pi k} \Rightarrow \sum_{k=1}^{\infty}\frac{2\sin k\delta}{\pi k} = 1 - \frac{\delta}{\pi} = \frac{\pi-\delta}{\pi}$

$\therefore \sum_{k=1}^{\infty}\frac{\sin k\delta}{k} = \frac{\pi-\delta}{2}$

c. Deduce from Parseval's Theorem that $\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi-\delta}{2}$

$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1\,dx = \frac{\delta}{\pi}$

so by parseval, $\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{k=1}^{\infty}\frac{\sin^2 k\delta}{\pi^2 k^2}$

so $\pi = \delta + 2\sum_{k=1}^{\infty}\frac{\sin^2 k\delta}{k^2\delta}$

$\therefore \sum_{k=1}^{\infty}\frac{\sin^2 k\delta}{k^2\delta} = \frac{\pi-\delta}{2}$

e. Put $\delta = \frac{\pi}{2}$ in (c). What do you get?

$\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2\frac{\pi}{2}} = \frac{\pi}{4}$

$\therefore \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$

15. With $D_n = \sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$ the Dirichlet Kernel, put $K_n(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x)$

Prove that $K_n(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}$ and that

$(N+1)(1-\cos x)\cdot K_n(x) = (1-\cos x)\sum_{n=0}^{N}\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}} = 2\sin^2\frac{x}{2}\sum_{n=0}^{N}\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}$

$= \sum_{n=0}^{N} 2\sin\frac{x}{2}\sin(n+\frac{1}{2})x$

$= \sum_{n=0}^{N}\cos nx - \cos((n+1)x) = 1 - \cos((N+1)x)$

$\therefore K_n(x) = \frac{1}{N+1}\cdot\frac{1-\cos((N+1)x)}{1-\cos x}$

a. $K_n \geq 0$

since $K_n(x) = \frac{1}{N+1}\cdot\frac{1-\cos((N+1)x)}{1-\cos x}$

And $\cos x \leq 1\ \forall x \Rightarrow 1-\cos x \geq 0\ \forall x$

$\therefore K_n(x) \geq 0$

b. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$

We know that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1$

So $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{n=0}^{N} D_n(x)\,dx = \frac{1}{2\pi}\frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(x)\,dx$

$= \frac{1}{N+1}\sum_{n=0}^{N} 1 = \frac{N+1}{N+1} = 1$

$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$

c. $K_n(x) \le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ for $0<\delta\le|x|\le\pi$

$K_n(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \le \frac{1}{N+1}\cdot\frac{2}{1-\cos x}$ since $\cos(N+1)x \ge -1$

$\le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ since $\cos x$ decreasing on $[0,\pi]$

$\therefore K_n(x) \le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ for $0<\delta\le|x|\le\pi$

If $S_N = S_N(f;x)$ is the nth partial sum of the fourier series of f consider $\sigma_N = \frac{S_0+\dots+S_N}{N+1}$. Prove that $\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)\,dt$ and prove that if f is continuous with period 2π, then $\sigma_N(f;x) \rightrightarrows f(x)$ on $[-\pi,\pi]$

We know that $S_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt$

So $\sigma_N = \frac{S_0+\dots+S_N}{N+1} = \frac{1}{N+1}\sum_{n=0}^{N} S_n(f;x) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cdot\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)\,dt$

Now since f continuous on $[-\pi,\pi]$, we have f uniformly continuous on $[-\pi,\pi]$

But since f 2π-periodic, f uniformly continuous on $\mathbb{R}$

Let $\epsilon>0$ be given

Then $\exists\,\delta>0 \ni$ for $x\in\mathbb{R} \ni |t|<\delta$, we have $|f(x-t)-f(x)|<\frac{\epsilon}{2}$

And since f continuous on $[-\pi,\pi]$ and f 2π-periodic, f bounded on $\mathbb{R}$ ie $|f(x)|\le M\ \forall x$

Choose $N \ni \frac{2M}{N+1}\cdot\frac{2}{1-\cos\delta} < \frac{\epsilon}{2}$

Then for $n\ge N$, $|\sigma_n(f;x)-f(x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)K_n(t)\,dt\right|$

$= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t)-f(x))K_n(t)\,dt\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-t)-f(x)|K_n(t)\,dt$

$= \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(x-t)-f(x)|K_n(t)\,dt + \frac{1}{2\pi}\int_{\delta\le|t|\le\pi}|f(x-t)-f(x)|K_n(t)\,dt$

$< \frac{\epsilon}{2}\cdot 1 + \frac{1}{2\pi}\cdot 2M\cdot\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}\cdot 2\pi < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$

$\therefore$ For $n\ge N$, $|\sigma_n(f;x)-f(x)|<\epsilon\ \ \forall x\in[-\pi,\pi]$

$\therefore \sigma_n(f;x) \rightrightarrows f(x)$ on $[-\pi,\pi]$

30. use Sterling's formula to prove that $\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1 \ \forall c \in \mathbb{R}$

Claim $\lim_{x\to\infty} \left(1+\frac{c}{x}\right)^x = e^c$

$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = \lim_{y\to 0}(1+y)^{\frac{c}{y}} = \left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)^c$

And $\lim_{y\to 0}(1+y)^{\frac{1}{y}} = \lim_{y\to 0} e^{\frac{1}{y}\log(1+y)}$

So find $\lim_{y\to 0} \frac{\log(1+y)}{y}$

Let $f(y) = \log(1+y)$

Then $f'(0) = \lim_{y\to 0} \frac{f(y)-f(0)}{y-0} = \lim_{y\to 0} \frac{\log(1+y)}{y}$

$\therefore \lim_{y\to 0} \frac{\log(1+y)}{y} = f'(0) = 1$

$\therefore \lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = \left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)^c = \left(\lim_{y\to 0} e^{\frac{1}{y}\log(1+y)}\right)^c = \left(e^1\right)^c = e^c$

$\therefore \lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = e^c$

Now note that sterling's formula is $\lim_{x\to+\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}} = 1$

But as $x \to +\infty$, $x+c \to +\infty \ \forall c \in \mathbb{R}$

so $\lim_{x\to+\infty} \frac{\Gamma(x+c+1)}{\left(\frac{x+c}{e}\right)^{x+c}\sqrt{2\pi(x+c)}} = 1$

Then $\lim_{x\to+\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} = \lim_{x\to+\infty} \frac{\Gamma(x+1+c)}{(x+1)^c\Gamma(x+1)}$ since $x+1 \to +\infty$ as $x \to +\infty$

$= \lim_{x\to+\infty} \frac{\Gamma(x+1+c)}{(x+1)^c\Gamma(x+1)} \cdot \frac{\left(\frac{x+c}{e}\right)^{x+c}\sqrt{2\pi(x+c)}}{\left(\frac{x+c}{e}\right)^{x+c}\sqrt{2\pi(x+c)}} \cdot \frac{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}$

$= \lim_{x\to+\infty} \frac{\Gamma(x+1+c)}{\left(\frac{x+c}{e}\right)^{x+c}\sqrt{2\pi(x+c)}} \cdot \frac{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}{\Gamma(x+1)} \cdot \frac{\left(\frac{x+c}{e}\right)^{x+c}\sqrt{2\pi(x+c)}}{(x+1)^c\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}$

$= \lim_{x\to+\infty} \frac{\left(\frac{x+c}{e}\right)^{x+c}\sqrt{2\pi(x+c)}}{(x+1)^c\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}$ by sterling's formula

$= \lim_{x\to+\infty} \left(\frac{x+c}{x}\right)^x e^{-c}\left(\frac{x+c}{x+1}\right)^c\sqrt{\frac{x+c}{x}} = e^{-c}\lim_{x\to+\infty}\left(1+\frac{c}{x}\right)^x\left(\frac{x+c}{x+1}\right)^c\sqrt{\frac{x+c}{x}}$

$= e^{-c}e^c$ by claim

$= 1$

$\therefore \lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} = 1 \ \forall c \in \mathbb{R}$