To Remember:

- "order" on a set has trichotomy
- supremum = least upper bound; infimum = greatest lower bound
  - in $\mathbb{R}$, every nonempty set w/ an upper bound has a sup (similar for inf, by pf.)
  - if $z>0$, $\sup E$ exists, then $\sup(zE) = z \cdot \sup E$
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$
  - in $\mathbb{R}$, if E has no u.b., $+\infty$ is sup in $\overline{\mathbb{R}}$
  - $\sup \emptyset = -\infty$ in $\overline{\mathbb{R}}$
- modulus ≈ abs value - for $a+bi$, $|a+bi| = \sqrt{a^2+b^2}$
- Cauchy-Schwarz (on $\mathbb{C}^n$): $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ (equality in $\mathbb{R}$)
  - Δ ineq: $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$
- finite means $\cong \{1, \ldots, n\}$ for some $n \in \mathbb{N}$
  - if B c'ble and $A \subseteq B$, A at most c'ble
- metric space req's:
  - $d(x,y) \geq 0$, eq iff $x=y$
  - $d(x,y) = d(y,x)$
  - $d(x,z) \leq d(x,y) + d(y,z)$
- <u>interior point</u>: can find nbhd of the pt still inside the set
- E <u>dense</u> in X: E intersects every nbhd of every $x \in X$; $\overline{E} = X$
- <u>limit point</u> of E: every $N^*(x)$ intersects E
- <u>closed</u>: contains all limit points; complement open
- <u>open</u>: every point is interior; complement closed

- a set is <u>bounded</u> if it is contained in some nbhd
- <u>compact</u>: every open cover has finite subcover
  - closed subsets are compact
  - compact $\Rightarrow$ closed
  - in $\mathbb{R}$, compact $\Leftrightarrow$ closed & bounded
  - finite union of compacts is compact
  - nonempty, nested compact sets: intersection nonempty
- <u>perfect</u>: closed w/ no isolated points; $E = E'$
  - nonempty perfect set in $\mathbb{R}$ is unc'ble
  - E no iso. pts. and G open, $E \cap G$ has no isolated pts.
  - if E has no isolated pts, $\bar{E}$ does not either
- <u>connected</u>: can't be written as $A \cup B$ for separated A&B
  - $E \subseteq \mathbb{R}$ connected $\Rightarrow$ E is an interval
  - intersection of compact, connected sets is connected
- <u>limit</u> of a <u>sequence</u>: $\lim_{n\to\infty} (p_n) = p$ if
  $\forall \varepsilon > 0, \exists N$ st $\forall n \geq N$, $p_n \in N_\varepsilon(p)$; or $d(p_n, p) < \varepsilon$
  - if $\{x_n\} \subseteq E$, $x_n \to x$, then $x \in E'$
  - if $x \in E'$, $\exists \{x_n\} \subseteq E$ st $x_n \to x$
- <u>subsequence</u>: $\{p_{n_k}\} \subset \{p_n\}$
  - S = set of subseq. lim's is closed
  - $\forall$ closed $E \subseteq \mathbb{R}$, $\exists \{p_n\}$ st $S = E$
  - X compact, $\forall \{p_n\}$, $S \neq \emptyset$
  - (Bolzano-Weierstrass) every bounded seq. in $\mathbb{R}^k$ has a conv. subseq.
- <u>Cauchy</u>: $\forall \varepsilon > 0, \exists N$ st $\forall m, n \geq N$, $d(p_n, p_m) < \varepsilon$
  - convergent $\Rightarrow$ Cauchy
  - Cauchy <u>and</u> $\exists$ conv. subseq $\Rightarrow$ convergent
  - Cauchy $\Rightarrow$ bounded
  - in $\mathbb{R}^n$, Cauchy $\Rightarrow$ convergent

- complete: every Cauchy seq. converges
  - every closed ball in X is compact $\Rightarrow$ X is complete
  - closed subset of complete set is complete
  - complete subsets of metric space are closed

- diameter: $\operatorname{diam} E = \sup\{d(x,y) \mid x, y \in E\}$
  - $\{p_n\}$ Cauchy $\Rightarrow \operatorname{diam}\{p_n \mid n \geq m\} \to 0$ as $m \to \infty$

- nested, closed, bounded, nonempty subsets of complete space w/ diam $\to 0$ have intersection of exactly one pt.

- Baire Category Thm: $G_1, G_2, \ldots$ dense, open subsets of complete space $\Rightarrow \bigcap_{n=1}^{\infty} G_n$ dense.

- complete space w/o isolated pts is unc'ble

- monotone & bounded $\Rightarrow$ convergent

- infinite limits: $x_n \to +\infty$ if $\forall M \in \mathbb{R}, \exists N$ st $n \geq N \Rightarrow x_n \geq M$
  - $+\infty \in S \Leftrightarrow \{x_n\}$ has no upper bound

- upper limit: $\limsup_{n\to\infty} x_n = \sup S = \inf\{\sup T_m\}$

- lower limit: $\liminf_{n\to\infty} x_n = \inf S = \sup\{\inf T_m\}$
  - lim sup, lim inf $\in S$ (S is closed)

- to analyze limits, we have squeeze thm, comparison thm (no strict ineq.'s) and Bernoulli's ineq: $(1+x)^n \geq 1 + nx$ for $x > -1, n \in \mathbb{N}$

- Cauchy criterion: $\sum a_n$ conv. $\Leftrightarrow \forall \varepsilon > 0, \exists N$ st $m \geq n \geq N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$

- Comparison: if $|a_n| \leq b_n \; \forall n$, $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

- Cauchy condensation: $a_n \geq 0$, $a_n \geq a_{n+1}$; $\sum_{n=1}^{\infty} a_n$ conv. $\Leftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k}$ conv.

- remainder estimate: if $\exists\, 0<b<1$ ST $\left|\frac{a_{n+1}}{a_n}\right| \leq b \ \forall\, k>n$, then $\left|\sum_{k=n+1}^{\infty} a_k\right| \leq \frac{|a_{n+1}|}{1-b}$ (n^{th} remainder, "r_n")
  - if $\sum p_n/q_n \subseteq \mathbb{Q}$ and $r_n \cdot \mathrm{lcm}(q_1, \ldots, q_n) \to 0$, sum is irrational.

- root test: $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$
  - $\alpha<1$, $\sum a_n$ conv.
  - $\alpha>1$, $\sum a_n$ div.
  - $\alpha=1$, no conclusion

- ratio test: $\beta = \limsup_{n\to\infty} |a_{n+1}/a_n|$, $\gamma = \liminf_{n\to\infty} |a_{n+1}/a_n|$
  - $\beta<1$, $\sum a_n$ conv.
  - $|a_{n+1}/a_n| \geq 1$ for $\forall n \geq$ some N, div.
  - $\beta \leq 1 \leq \gamma$, inconclusive

- power series: $\sum c_n z^n$
  - $\alpha = \limsup_{n\to\infty} |c_n|^{1/n}$
  - radius of conv. $R = 1/\alpha$
  - conv. if $|z|<R$; div if $|z|>R$; inconclusive if $|z|=R$

- Summation by Parts:
  - $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$

    where $A_n = a_0 + a_1 + \cdots + a_n$

- Dirichlet Test: if $\sum a_n$ has bounded partial sums and $b_n \to 0$ is decreasing, then $\sum a_n b_n$ converges.

- absolute conv $\Rightarrow$ conv

- if $\sum a_n$ conv. abs., any rearrangement conv. (to same sum)
  - if non-abs., can get any sum we want

- Cauchy product: $\sum a_n b_n = \sum c_n$ where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$
  - if $\sum a_n$ conv. abs. and $\sum b_n$ conv., then $\sum c_n$ conv.

- <u>limit of a function</u>: $\lim_{x \to p} f(x) = q$ if
  $\forall \varepsilon > 0, \exists \delta > 0$ ST $d_x(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$
  - topologically: $\forall \varepsilon > 0, \exists \delta > 0$ ST $f(E \cap N^*_\delta(p)) \subseteq N_\varepsilon(q)$
  - seq. characterization:
    $\lim_{x \to p} f(x) = q \Leftrightarrow \forall$ seq ST $x_n \to p, x_n \neq p$, $\lim_{n \to \infty} f(x_n) = q$
    - must all have <u>same</u> limit

- if $\lim_{x \to p} f(x) = q$, then $\bigcap_{n=1}^{\infty} \overline{f(E \cap N^*_{1/n}(p))} = \{q\}$
  - converse is true iff Y is compact

- limits do not behave well under composition

- f is <u>continuous</u> at p (in domain):
  $\forall \varepsilon > 0, \exists \delta > 0$ ST $\forall x \in E$, $d_x(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$
  - topologically: $\forall \varepsilon > 0, \exists \delta > 0$ ST $f(E \cap N_\delta(p)) \subseteq N_\varepsilon(f(p))$
  - if $p \in E'$, f cont @ $p \Leftrightarrow \lim_{x \to p} f(x) = f(p)$
  - if $p \notin E'$, f automatically cont. @ p
  - * $\forall f: \mathbb{Z} \to Y$ is cont.
  - f cont @ $p \Leftrightarrow$ if $f(p) \in \text{int}(B)$, $p \in \text{int}(f^{-1}(B))$

- global continuity (cont @ all pts)
  - $\forall$ open $B \subseteq Y$, $f^{-1}(B)$ open in X // $\forall$ closed $C \subseteq Y$, $f^{-1}(C)$ closed in X

- open map: $\forall$ open U in X, $f(U)$ open in Y

- continuity behaves well under fcn arithmetic

- compactness pushes forward under continuity
  - extreme value thm: if X compact, $f: X \to Y$ cont
    - $f(X)$ bounded in Y (f is bounded)
    - if $Y = \mathbb{R}$ then f attains its sup & inf
      ($\exists a, b \in X$ ST $f(a) \leq f(x) \leq f(b)$)
  - X compact, $f: X \to Y$ cont. bij $\Rightarrow f^{-1}$ cont.

- connectedness pushes forward under continuity
  - intermediate value thm: if X connected, $f: X \to Y$ cont, $f(a) < t < f(b) \Rightarrow \exists c$ ST $f(c) = t$
- if $f: \mathbb{R}^n \to \mathbb{R}^n$ is a cont. bijection, f^{-1} is cont.
- $\exists$ cont. bij. $f: (0,1) \to \mathbb{R}$, but not from $[0,1]$ or $[0,1)$

- Lebesgue Number Lemma: if $\{U_\alpha\}$ open cover of compact X, then there is some minimum radius r ST $N_r(x) \subseteq U_\alpha$ $\forall x, \forall \alpha$

- uniformly continuous: $\forall \varepsilon > 0, \exists \delta > 0$ ST if $d_X(p,q) < \delta$, $d_Y(f(p), f(q)) < \varepsilon$

- X compact $\Rightarrow$ $f: X \to Y$ cont $\Leftrightarrow$ unif. cont.

- $f(p-) = \sup\{f(x) \mid x < p\}$; $f(p+) = \inf\{f(x) \mid x > p\}$
- if f is monotone, set of discontinuities is at most c'ble

- derivative: $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ if it exists.

- remainder prop: f' exists $\Leftrightarrow \exists A \in \mathbb{R}$ ST $f(t) = f(x) + A(t-x) + r(t)(t-x)$
  where $r(t) \to 0$ as $t \to \infty$. $f'(x) = A$.
  - $\lim_{t \to x} f(t) = f(x)$; $\exists f'(x) \Rightarrow f$ is cont. at x.
  - product, chain rule, etc. apply.

- if f has a local max or min at $c \in (a,b)$, then either $f'(c) = 0$ or $f'(c)$ DNE.

- Rolle's Thm: if $f: [a,b] \to \mathbb{R}$ is diff' on (a,b) and $f(a) = f(b)$ then $\exists c \in (a,b)$ ST $f'(c) = 0$.

- MVT: $f: [a,b] \to \mathbb{R}$ cont. and diff' on (a,b) then $\exists c \in (a,b)$ ST
  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- if $f: \mathbb{R} \to \mathbb{R}$ is diff' and f' is bounded then f is unif. cont.

- <u>Generalized MVT:</u> $g, f: [a,b] \to \mathbb{R}$ cont + diff on (a,b). $\exists c$ ST

$$\begin{vmatrix} f'(c) & g'(c) \\ f(b)-f(a) & g(b)-g(a) \end{vmatrix} = 0 \qquad \text{(determinant)}$$

- f' might not be cont., but if it <u>is</u> cont. (on an interval I) and $\lim_{x \to p} f'(x)$ exists for $p \in I$, $f'(p)$ exists and is said limit.

- If f' exists on an interval, it attains intermediate values.
  - that is, $f'(a) < y < f'(b) \Rightarrow \exists c \in (a,b)$ ST $f'(c) = y$.

- <u>L'Hopital's:</u> for $\lim_{x \to a} \frac{f(x)}{g(x)}$, $a \in \overline{\mathbb{R}}$.
  - need: $\lim f(x) = \lim g(x) = 0$ <u>or</u> $\lim g(x) = \infty$ and f', g' exist; $g' \neq 0$ in nbhd of a.

- <u>Taylor's:</u>
  - Peano form: Let $f^{(d)}(\alpha)$ exist. Then $f(t) \approx \sum_{k=0}^{d} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k + r(t)(t-\alpha)^k$ where $r(t) \to 0$ as $d \to \infty$.
  - Lagrange form: Let $f^{(d)}$ be cont. on $[\alpha, \beta]$ and diff' on (α, β) (ok)

$$f(\beta) = \sum_{k=0}^{d} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{f^{(d+1)}(x)}{(d+1)!}(\beta-\alpha)^{d+1} \quad \text{for an } x \text{ btwn } \alpha + \beta$$

- Derivatives in $\mathbb{C}$ or $\mathbb{R}^n$
  - def'n same: $\vec{f}'(x) = \lim_{h \to 0} \frac{\vec{f}(x+h) - \vec{f}(x)}{h}$
  - treat $z \in \mathbb{C}$ as 2-comp. vector if it helps
  - dot product $\cong$ product rule
  - no MVT, but have <u>MVT inequality</u>:
    $\vec{f}$ cont on $[a,b]$, diff on (a,b) $\Rightarrow |\vec{f}(b) - \vec{f}(a)| \leq \sup_{a \leq t \leq b} |f'(t)| \cdot (b-a)$
    - if $|\vec{f}'|$ bounded, it's unif. cont. here as well
  - if $\vec{f}$ is diff in $N_r(\vec{p}) \setminus \{\vec{p}\}$ and $\lim_{x \to p} \vec{f}'(x)$ exists, then $\vec{f}'(p)$ exists and is said limit.
  - restricted L'H:
    if $f'(x) \to A$, $g'(x) \to B \neq 0$ as $x \to p$
    and if $f, g \to 0$ as $x \to p$; $f(p), g(p) = 0$
    then $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{A}{B}$

Ch. 6

Integral defn's

partition P of $[a,b]$: $\{x_0, x_1, \ldots, x_n\}$ st $x_0 = a \le \cdots \le x_n = b$.

$M_i = \sup f(x)$ for $x_{i-1} \le x \le x_i$

$m_i = \inf f(x)$ for $x_{i-1} \le x \le x_i$

$U(P,f) = \sum_1 M_i \Delta x_i$

$L(P,f) = \sum m_i \Delta x_i$

$\overline{\int_a^b} f dx = \inf U(P,f)$

$\underline{\int_a^b} f dx = \sup L(P,f)$

$= \int_a^b f dx$ if equal

fact: $\forall P$,
$m|a-b| \le L(P,f) \le U(P,f) \le M|a-b|$

With α's:

α always monotonically increasing on $[a,b]$

$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$U(P,f,\alpha) = \sum M_i \Delta \alpha_i$

$L(P,f,\alpha) = \sum_1 m_i \Delta \alpha_i$

$\overline{\int_a^b} f d\alpha = \inf U(P,f,\alpha)$

$\underline{\int_a^b} f d\alpha = \sup L(P,f,\alpha)$

def: P^* a refinement of P if it has more points:
$P \subset P^*$

Thm 6.4: If P^* a refinement of P then

$$\mathcal{L}(P, f, \alpha) \leq \mathcal{L}(P^*, f, \alpha) \text{ and}$$

$$\mathcal{U}(P^*, f, \alpha) \leq \mathcal{U}(P, f, \alpha).$$

Thm 6.5: $\underline{\int_a^b} \leq \overline{\int_a^b}$.

Thm 6.6: $f \in \mathcal{R}(\alpha)$ iff $\forall \varepsilon > 0, \exists P$ st

$$\mathcal{U}(P, f, \alpha) - \mathcal{L}(P, f, \alpha) < \varepsilon.$$

Thm 6.7: if the above holds for some P and some ε, it holds for every refinement of P w/ same ε.

Guaranteeing $f \in \mathcal{R}(\alpha)$

Thm 6.8: f continuous on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,b]$

Thm 6.9: f monotonic on $[a,b]$ and α continuous on $[a,b]$ $\Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,b]$

Thm 6.10: f bounded on $[a,b]$ w/ finitely many discontinuities and α continuous where f is not $\Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,b]$

Thm 6.11: $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ cont. on $[m,M]$ (bounded)
and $h(x) = \phi(f(x))$ on $[a,b]$ $\Rightarrow$ $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

Properties

$f_1, f_2 \in \mathcal{R}(\alpha) \Rightarrow f_1 + f_2 \in \mathcal{R}(\alpha)$ and $cf \in \mathcal{R}(\alpha)$

$$\int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

$$c \int_a^b f \, d\alpha = \int_a^b cf \, d\alpha$$

$$f_1(x) \leq f_2(x) \ \forall x \in [a,b] \Rightarrow \int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$$

$$|f(x)| \leq M \ \forall x \in [a,b] \Rightarrow \left| \int_a^b f \, d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

$$\int_a^b = \int_a^c + \int_c^b$$

$f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2) \Rightarrow f \in \mathcal{R}(\alpha_1 + \alpha_2)$
and for constant c, $f \in \mathcal{R}(c\alpha)$

More properties:

$f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha) \Rightarrow fg \in \mathcal{R}(\alpha)$

$\Downarrow$

$|f| \in \mathcal{R}(\alpha)$ and $\boxed{\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha}$

Unit step fcn:

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

α $I(x-s)$ gives $\begin{cases} 0 & x \leq s \\ 1 & x > s \end{cases}$

Thm 6.15: If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then $\boxed{\int_a^b f\,d\alpha = f(s)}$

Thm 6.17: α incr. monotonic and $\alpha' \in \mathcal{R}$ on $[a,b]$.
Let f be a real, bounded fcn on $[a,b]$.
Then $f \in \mathcal{R}(\alpha) \Leftrightarrow f\alpha' \in \mathcal{R} \Rightarrow \int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$

Int. by Parts:

F, G differentiable on $[a,b]$, $F' = f \in \mathcal{R}$, $G' = g \in \mathcal{R}$.

Then $$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

u-dv version:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ch. 7

def: $\{f_n\}$ a seq. of fcn's on a set E.
Suppose the seq. of numbers $\{f_n(x)\}$ converges $\forall x \in E$. Then we define

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E)$$

and say that $\{f_n\}$ converges pointwise to f on E.

def: if $\sum_{n=1}^{\infty} f_n(x)$ converges $\forall x \in E$ and $f(x) = \sum f_n(x)$, f is called the sum of the series.

def: f is continuous at a limit point x if

$$\lim_{t\to x} f(t) = f(x).$$

So when we want to know if limit fcn f is continuous, we want

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$$

def: $f_n \rightrightarrows f$ if $\forall \varepsilon > 0$, $\exists N$ ST $n \geq N$
$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in E$.

Guarantee uniform convergence

Thm 7.8: $\{f_n\}$ defined on E converges uniformly on E iff $\forall \varepsilon, \exists N$ ST $m,n \geq N \Rightarrow |f_n(x) - f_m(x)| \leq \varepsilon \ \forall x \in E$.

* Cauchy criterion for uniform convergence *

Thm 7.9: Suppose $f_n \to f$ (need pointwise) and $\boxed{M_n = \sup_{x \in E} |f_n(x) - f(x)|}$.
Then $f_n \rightrightarrows f$ iff $M_n \to 0$ as $n \to \infty$.

Series conv.

Thm 7.10: (M-Test) if $|f_n(x)| \leq M_n$,
then $\sum M_n$ converges $\Rightarrow \sum f_n$ converges.

Thm 7.11: If $f_n \rightrightarrows f$ and each $f_n \to A_n$,
then $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

Thm 7.12: The uniform limit of continuous functions is continuous.

Thm 7.13: If
- K compact
- $\{f_n\}$ is a seq. of cont. fcn's on K
- $\{f_n\}$ converges pointwise to a continuous f on K
- $f_n \geq f_{n+1}$

Then $f_n \rightrightarrows f$ on K.

$\mathcal{C}(X)$:= set of all bounded, continuous fcns on the metric space X

$$\boxed{\|f\| = \sup_{x \in X} |f(x)|}$$ (makes $\mathcal{C}(X)$ a metric space)

Thm 7.15: This metric makes $\mathcal{C}(X)$ complete.

* $\{f_n\}$ converges to f w/r to the metric of $\mathcal{C}(X)$ iff $f_n \rightrightarrows f$ on X.

Thm 7.16: (Integration) $f_n \in \mathcal{R}(\alpha)$ and $f_n \rightrightarrows f$ on $[a,b]$ $\Rightarrow f \in \mathcal{R}(\alpha)$ and $\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$

Cor: if $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha$.

Thm 7.17: (Differentiation) $\{f_n\}$ diff. on $[a,b]$ and $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$, then if $\{f_n'\}$ converges unif. on $[a,b]$ then $\{f_n\}$ converges unif. on $[a,b]$ to a fcn f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$.

def: $\{f_n\}$ is pointwise banded if $\{f_n(x)\}$ is banded $\forall x \in E$.

$\Leftrightarrow$ exists a finite-valued fcn ϕ ST $|f_n(x)| < \phi(x)$ $\forall x \in E$.

def: a family $\mathcal{F}$ of complex fcn's in a metric space X is equicontinuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ ST

$$|f(x) - f(y)| < \varepsilon$$

for any $d(x,y) < \delta$, $x, y \in E$, $f \in \mathcal{F}$.

* small input distance $\Rightarrow$ small output distance *

Thm 7.23: If $\{f_n\}$ is a pointwise banded seq of complex fcn's on cantable E, then $\{f_n\}$ has a subseq $\{f_{n_k}\}$ ST $\{f_{n_k}(x)\}$ converges $\forall x \in E$.

Thm 7.24: K a compact metric space, $f_n \in \mathcal{C}(K)$ $\forall n$, and $f_n \rightrightarrows f$ on K $\Rightarrow \{f_n\}$ is equicontinuous on K.

Thm 7.25: K compact, $f_n \in \mathcal{C}(K)$, and $\{f_n\}$ is pointwise banded and equicontinuous on K, then

- $\{f_n\}$ is uniformly banded on K
- $\{f_n\}$ contains a unif. convergent subseq.

Thm 7.26: (Stone-Weierstrass)

If f continuous on $[a,b]$, $\exists$ seq of polynomials P_n
ST $P_n \rightrightarrows f$ on $[a,b]$.

Cor: for every interval $[-a,a]$, $\exists$ seq of real P_n
ST $P_n(0) = 0$ and $P_n \rightrightarrows |x|$ on $[-a,a]$

Algebras

def: a family $\mathcal{A}$ of complex fcn's is an algebra
if it is closed under add'n, mult, and scalar mult.

def: if $(\{f_n\} \in \mathcal{A}$ and $f_n \rightrightarrows f) \Rightarrow f \in \mathcal{A}$, it is uniformly closed

def: uniform closure $\mathcal{B} := \{f \mid \exists \{f_n\} \in \mathcal{A}$ ST $f_n \rightrightarrows f\}$. $(\mathcal{A} \subseteq \mathcal{B})$

ex: $\mathcal{A}$ = set of polynomials, $\mathcal{B}$ = all cont. fcn's on $[a,b]$
(by Weierstrass)

Thm 7.29: if $\mathcal{A}$ is algebra of bounded fcn's, corresponding
$\mathcal{B}$ is a uniformly closed algebra.

(Algebras cont.)

def: <u>separates points</u> if $\forall x_1 \neq x_2$, $\exists f^{\in \mathcal{A}}$ ST $f(x_1) \neq f(x_2)$

def: <u>vanishes at no point</u> if $\forall x$, $\exists g \in \mathcal{A}$ ST $g(x) \neq 0$

<u>Thm 7.31</u> If $\mathcal{A}$ algebra that separates pts, and vanishes at no pt, then for $x_1 \neq x_2$ in domain and constants c_1, c_2, $\exists f \in \mathcal{A}$ ST

$$f(x_1) = c_1 \text{ and } f(x_2) = c_2.$$

<u>Thm 7.32</u>: If $\mathcal{A}$ algebra on $\underline{\mathbb{R}}$ (not $\mathbb{C}$!) on a compact set K, separates pts on K, vanishes at no pt of K, then $\mathcal{B}$ (unif. closure) consists of all real, cont. fcn's on K.

<u>Pf facts</u>:

- if $f \in \mathcal{B}$ then $|f| \in \mathcal{B}$.
- if $f, g \in \mathcal{B}$ then $\max(f,g), \min(f,g) \in \mathcal{B}$.
- given real fcn $f \in \mathcal{C}(K)$, an $x \in K$, and $\varepsilon > 0$, $\exists g_x \in \mathcal{B}$ ST $g_x(x) = f(x)$ and $g_x(t) > f(t) - \varepsilon \quad \forall t \in K$.
- given real fcn $f \in \mathcal{C}(K)$ and $\varepsilon > 0$, $\exists h$ ST $|h(x) - f(x)| < \varepsilon \quad \forall x \in K$

$\max/\min(f,g) = \frac{f+g}{2} \pm \frac{|f-g|}{2}$

def: complex $\mathcal{A}$ is <u>self-adjoint</u> if $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$.

<u>Thm 7.33</u>: if $\mathcal{A}$ is self-adjoint, 7.32 holds (in $\mathbb{C}$).

$$\int a^u du = \frac{a^u}{\ln(a)} \qquad a \neq 1 \quad a > 0$$

$$\int \ln(u) du = u \ln(u) - u$$

Ch. 9

Recall definitions of:

- vector space (add'n & scalar mult.)
- linear combination
- spanning
- independent/dependent
- dimension of a v.s.
- basis
- linear transformation

New: if $\vec{x} = \sum c_j x_j$ for a basis $\mathcal{B} = \{x_1, \ldots, x_n\}$, the numbers $c_1, \ldots, c_n$ are called the <u>coordinates</u> of x w/r to the basis $\mathcal{B}$.

<u>Thm</u>: if X is spanned by $r \geq 1$ vectors, $\dim X \leq r$. (9.2)

<u>Thm</u>: (9.3) X a v.s. w/ dim n

(a) n vectors span $X \iff$ they are ind.

(b) every basis has n vectors (and one exists)

(c) every independent set of $m \leq n$ vectors is contained in a basis

<u>Thm</u>: (9.5) A linear operator A on a <u>finite dimensional</u> v.s. X is 1-1 $\iff$ range of A is all of X (onto).

Def'n: <u>$L(X,Y)$</u> is the set of all lin. transformations from X to Y, or $L(X)$ for X to X.

Def'n: If $A \in L(X,Y)$ and $B \in L(Y,Z)$, the <u>product</u> BA is defined $(BA)x = B(Ax)$.
So $BA \in L(X,Z)$.

Defn: for $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ the <u>norm</u> of A is

$\|A\| = \sup_{|x| \leq 1} |Ax|$, where $|\cdot|$ is the vector norm.

Note that $|Ax| \leq \|A\|\,|x| \quad \forall x \in \mathbb{R}^n$ and that if $|Ax| \leq \lambda |x| \quad \forall x \in \mathbb{R}^n$ then $\|A\| < \lambda$.

<u>Thm:</u> (a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.

(b) $\|A+B\| \leq \|A\| + \|B\|$, $\|cA\| = |c|\,\|A\|$

and $d(A,B) := \|A-B\|$ makes $L(\mathbb{R}^n, \mathbb{R}^m)$ a metric space.

(c) $\|BA\| \leq \|B\|\,\|A\|$

Recall: unif. cont. means $\forall \varepsilon > 0, \exists \delta > 0$ ST

$d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

Here, $\underbrace{|x-y| < \delta}_{\text{in } \mathbb{R}^n} \Rightarrow \underbrace{|Ax - Ay| = |A(x-y)| < \varepsilon}_{\text{in } \mathbb{R}^m}$

<u>Important!</u>

<u>Thm:</u> (9.8)

Let Ω be the set of all invertible (1-1 + onto) lin. op's on $\mathbb{R}^n$.

(a) If $A \in \Omega$ (is invertible) and $B \in L(\mathbb{R}^n)$ (just a lin. op.)

and $\|B-A\| \cdot \|A^{-1}\| < 1$,

then $B \in \Omega$ (is also invertible).

(b) Ω is an open subset of $L(\mathbb{R}^n)$ (we can always find an invertible matrix close to a known inv. one) and the mapping $A \mapsto A^{-1}$ is continuous on Ω

Matrices via Rudin

If $\{x_i\}_n$, $\{y_j\}_m$ are bases of X & Y, every $A \in L(X,Y)$ determines $\{a_{ij}\}_{n,m}$ st $Ax_j = \sum_{i=1}^{m} a_{ij} y_i \quad (1 \le j \le n)$.
those make the mx $[A] := [a_{ij}]$.
So the range of A is spanned by the col. vectors of $[A]$.
This gives a 1-1 corresp. between $L(X,Y)$ and $M_{m\times n}$.

Observation:
If S is a metric space, if $a_{11}, \dots, a_{mn}$ are real continuous functions on S, and if, $\forall p \in S$, A_p is the lin. trans. of $\mathbb{R}^n \to \mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, then the mapping $p \mapsto A_p$ is a cont. mapping of $S \to L(\mathbb{R}^n, \mathbb{R}^m)$.

Multi-Var. Differentiation

Familiar:
- $f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ if the limit exists.

or $f(x+h) - f(x) = f'(x)h + r(h)$ where $\lim_{h\to 0} r(h) = 0$.

Consider:
$f : (a,b) \subset \mathbb{R}^1 \to \mathbb{R}^m$
Then $f'(x) := y \in \mathbb{R}^m$ st $\lim_{h\to 0} \left\{ \frac{f(x+h) - f(x)}{h} - y \right\} = 0$

$\iff f(x+h) - f(x) = hy + r(h)$

The map $h \mapsto hy \in L(\mathbb{R}^1, \mathbb{R}^m)$, so we can think of f' as in $L(\mathbb{R}^1, \mathbb{R}^m)$

That is, $f'(x)$ is the lin. trans st
$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0.$$

Def'n: Suppose E is an open subset of $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, and $x \in E$.

If there exists a lin. trans. $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ ST

$$\lim_{h \to \vec{0}} \frac{|f(x+h) - f(x) - Ah|_m}{|h|_n} = 0$$

then we say f is <u>differentiable at x</u> and we write $f'(x) = A$ or say that A is the <u>differential</u> of f at x.

If f is diff'ble $\forall x \in E$, say f is diff'ble in E.

<u>Thm</u> (9.12): the differential is unique, if it exists.

Remarks: $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a function
$x \mapsto f'(x)$

while $f'(x)$ itself is also a fcn (lin. trans.).

<u>Note</u>: if $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ then $A'(x) = A$.

<u>Thm</u>: (9.15 / chain rule) $\left(g(f(x_0))\right)' = g'(f(x_0))\, f'(x_0)$.

<u>Partials</u>:

Let $\{e_i\}_n$, $\{u_j\}_m$ be std bases for $\mathbb{R}^n + \mathbb{R}^m$.

The <u>components of f</u> are the real fcn's $f_1, \ldots, f_m$ defined by

$$f(x) = \sum_{i=1}^{m} f_i(x) u_i \iff f_i(x) = f(x) \cdot u_i \quad 1 \le i \le m.$$

Then for $x \in dom(f)$, $1 \le i \le m$, $1 \le j \le n$, we define the <u>partial derivative</u>:

$$(D_j f_i)x = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t} \quad \text{if it exists.}$$

We see $D_j f_i$ is the deriv. of f_i w/r to x_j, $\frac{\partial f_i}{\partial x_j}$.

Warning:

existence of all partials $\not\Rightarrow$ differentiability

but the converse is true and the partials completely determine $f'(x)$ in that case.

Thm: (9.17) Suppose f maps an open set $E \subset \mathbb{R}^n \to \mathbb{R}^m$ and f is differentiable at a point $x \in E$.
Then the partial derivatives $(D_j f_i)(x)$ exist and

$$f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x)\, u_i \qquad 1 \le j \le n.$$

So $f'(x)e_j$ is the j^{th} column of the matrix

$$\begin{bmatrix} (D_1 f_1)(x) & \cdots & (D_n f_1)(x) \\ \vdots & & \\ (D_1 f_m)(x) & \cdots & (D_n f_m)(x) \end{bmatrix}$$

col   row

Thm: (9.19) Suppose f maps a convex open set $E \subset \mathbb{R}^n \to \mathbb{R}^m$, f is differentiable in E, and $\exists M \in \mathbb{R}$ st $\|f'(x)\| \le M$ $\forall x \in E$.
Then, $\forall a, b \in E$,

$$|f(b) - f(a)|_m \le M\,|b-a|_n.$$

Cor: If, in addition, $f'(x) = 0$ $\forall x \in E$, then f is constant.

Defn: diff'ble $f: E \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable in E if $f': E \to L(\mathbb{R}^n, \mathbb{R}^m)$, $x \mapsto f'(x)$ is continuous. Say $f \in \mathscr{C}'(E)$.
That is, $\forall x \in E$, $\forall \varepsilon > 0$, $\exists \delta > 0$ st $y \in E$ and $|x-y|_n < \delta \Rightarrow \|f'(x) - f'(y)\| < \varepsilon$

Thm: (9.21) continuous deriv $(\in \mathscr{C}'(E)) \Leftrightarrow$ continuous/existant partials.

The Contraction Principle

Def'n: For metric space X, $\varphi: X \to X$, if $\exists\, c<1$ ST

$$d(\varphi(x), \varphi(y)) \le c\, d(x,y) \qquad \forall x, y \in X$$

then φ is a contraction of X into X.

Thm (9.23) If X is complete and φ is a contraction on X, then there exists one and only one fixed point ($\varphi(x) = x$).

* contraction $\Rightarrow$ unif. continuous.

Inverse Function Theorem

Thm (9.24) Suppose $f \in \mathscr{C}'(E)$, $f: E \subset \mathbb{R}^n \to \mathbb{R}^m$, $f'(a)$ is invertible for some $a \in E$, and $b = f(a)$.

(a) there exist open $U, V \subset \mathbb{R}^n$ ST
$a \in U$, $b \in V$, f is 1-1 on U, and $f(U) = V$.

(b) if g is the inverse of f defined in V by
$$g(f(x)) = x, \quad x \in U$$
then $g \in \mathscr{C}'(V)$.

Writing $\vec{y} = \mathbf{f}(\vec{x})$ in component form, we see: the system of n eq's $y_i = f_i(x_1, \dots, x_n)$ $1 \le i \le n$ can be solved for $x_1, \dots, x_n$ in terms of $y_1, \dots, y_n$ if we restrict x and y to small enough nbhds of a & b. The sol'ns are unique & cont. diff'ble.

<u>Thm:</u> (9.25) If $f \in \mathscr{C}'(E)$, $f: E \subset \mathbb{R}^n \to \mathbb{R}^m$, and if $f'(x)$ is invertible for every $x \in E$, then $f(W)$ is an open for every open $W \subseteq E$ $\iff$ f is an open mapping.

③ STUDY THIS!

$$\lim_{x\to 0} \frac{f(2x)-2f(x)}{x^2}$$

$$\frac{f(2\cdot 0)-2f(0)}{0} = \frac{0-0}{0} = \frac{0}{0}$$

L'H valid ✓

So we get $\lim_{x\to 0} \frac{2f'(2x)-2f'(x)}{2x}$ ~~$= \frac{f'(2x)-f'(x)}{x}$~~

Since $f''(0)$ exists, f' is cont. @ 0

$$= \lim_{x\to 0} \frac{2f'(2x)-2f'(0)}{2x} + \frac{\cancel{2}f'(0)-\cancel{2}f'(x)}{\cancel{2}x}$$

$$= 2\lim_{x\to 0} \frac{f'(2x)-f'(0)}{2x} + \lim_{x\to 0}\frac{f'(0)-f'(x)}{x}$$

$= 2f''(0) - f''(0)$

$= f''(0)$

which exists ✓