1.1: Complex Numbers

$|z+w| \le |z| + |w|$

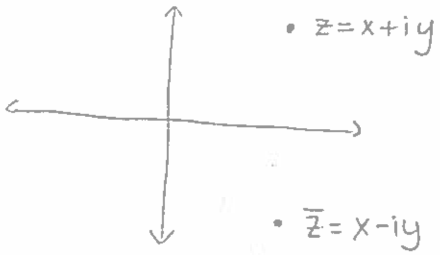
$\Rightarrow |z-w+w| \le |z-w| + |w|$

$\Rightarrow |z-w| \ge |z| - |w|$

Why are $\mathbb{C}$ and $\mathbb{R}^2$ different (property-wise)?

We can multiply in $\mathbb{C}$.

$$\frac{1}{z} = \frac{1}{(x+iy)} \cdot \frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2}$$



Complex Conjugation Properties

1.) $\overline{z+w} = \bar{z} + \bar{w}$

2.) $\overline{zw} = \bar{z}\,\bar{w}$

3.) $|z| = |\bar{z}|$

$|z| = \sqrt{x^2+y^2} = \sqrt{x^2+(-y)^2} = |\bar{z}|$

4.) $|z|^2 = z\bar{z}$

$|z|^2 = x^2+y^2 = (x+iy)(x-iy) = z\bar{z}$

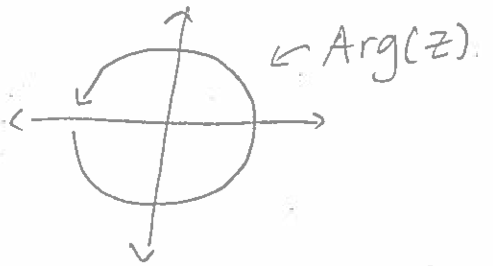
$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$ since $\frac{x+iy+x-iy}{2} = \frac{2x}{2} = x = \mathrm{Re}(z)$

$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$ since $\frac{x+iy-x+iy}{2i} = \frac{2iy}{2i} = y =$ ~~Re~~ $\mathrm{Im}(z)$

Chapter 1, Section 2:

Def'n: arg(z) is multivalued

Principal argument: Arg(z) is b/t $-\pi$ & π



(Your principal is bigger than you)

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$z = re^{i\theta}, \quad |z| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

☆ $$e^{i(\theta+2\pi m)} = \cos(\theta+2\pi m) + i\sin(\theta+2\pi m)$$
$$= \cos(\theta) + i\sin(\theta) = e^{i\theta}$$

$$e^{2\pi mi} = \cos(2\pi m) + i\sin(2\pi m) - 1$$
$$(e^{6\pi i} = 1, \ e^{4\pi i} = 1, \text{ etc})$$

$$|e^{i\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}$. can use this to prove the sine and cosine addition formulae

$$\cos(\theta+\varphi) + i\sin(\theta+\varphi) = (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi).$$

$$\Rightarrow \cos(\theta+\varphi) + i\sin(\theta+\varphi) = (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi).$$

$\Rightarrow \cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

$z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

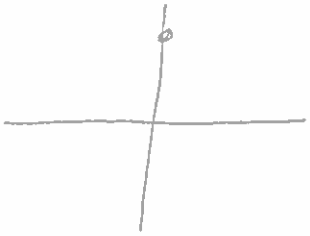
$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n.$

example) Find and plot the square roots of $4i$.

First, write $4i$ in polar form:

$r = \sqrt{0^2+4^2} = 4$

$\theta = \arg(4i) = \frac{\pi}{2}$



$\Rightarrow 4i = 4e^{i\frac{\pi}{2}}$

So $4e^{\frac{i\pi}{2}} = (re^{i\theta})^2$

So $4 = r^2 \Rightarrow r = 2$

$e^{\frac{i\pi}{2}} = e^{2i\theta} \Rightarrow 2\theta = \frac{\pi}{2} + 2\pi k,\ k = 0, 1$

$\Rightarrow \theta = \frac{\pi}{4} + \frac{2\pi k}{2},\ k = 0, 1$

$\Rightarrow \theta = \frac{5\pi}{4}$

$\pm(\sqrt{2} + \sqrt{2}i)$

example) Find and plot the cube root of $1+i$.

$r = \sqrt{1+1} = \sqrt{2}$

$\theta = \pi/4$

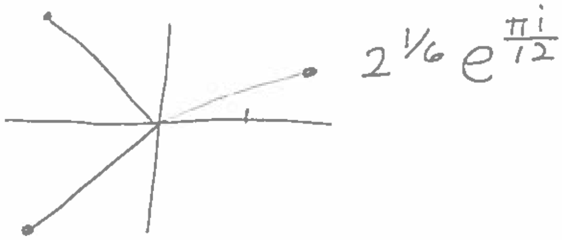
$\Rightarrow r = 2^{1/6}, 3\theta = \frac{\pi}{4} + \frac{2\pi k}{},\ k = 0, 1, 2$

$1+i = \sqrt{2}e^{i\pi/4} = w^3.$

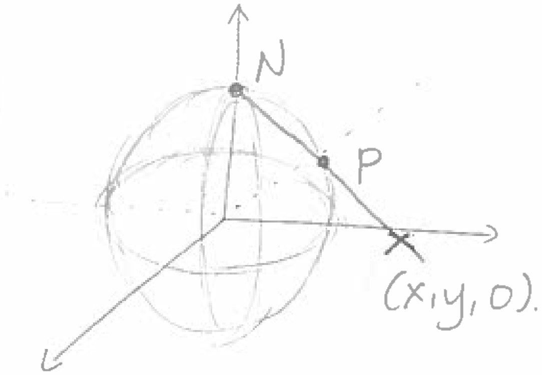
$r^3e^{3i\theta} = w^3 = 2^{\frac{1}{2}}e^{\frac{i\pi}{4}}$

$\Rightarrow r = 2^{1/6},\ \theta = \frac{\pi}{12} + \frac{2\pi k}{3},\ \cancel{\theta}\ k = 0, 1, 2$

$\Rightarrow \theta = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}$



Chapter 1, Section 3: Stereographic Projection:



Line through P & N:

$$N + t(P - N)$$

N-Pole

$$(x,y,0) = \cancel{(x} \; (0,0,1) + t[(X,Y,Z) - (0,0,1)].$$

Theorem: Under Stereographic projection, circles ↦ circles & lines.

Call straight lines in the complex plane "circles through ∞".

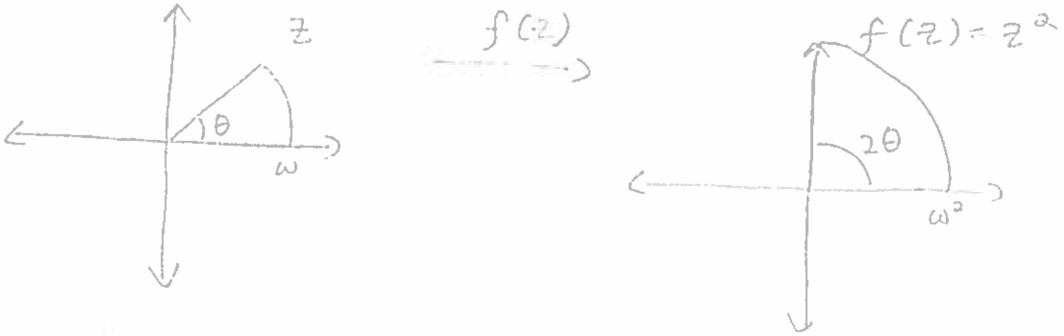
Chapter 1, Section 4: The Square and Square Root Fnc

If we graph from $\mathbb{C} \longrightarrow \mathbb{C}$ we would have
$(x,y) \mapsto (x,y)$

to graph in 4-D (which is impossible by hand)

Or, we can draw one plane as the domain and one plane as the range.

example) $f(z) = z^2$

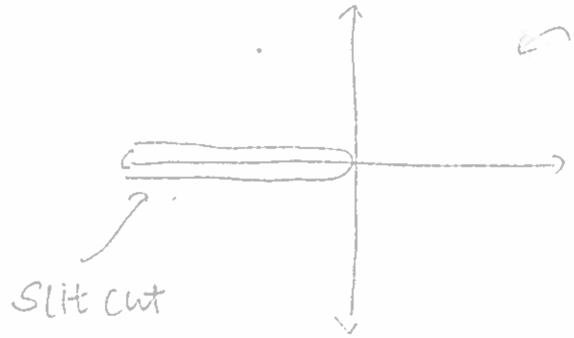


$$f(z) = z^2 = r^2 e^{2i\theta}$$

$$\Rightarrow |f(z)| = |z|^2 = r^2$$

$$\arg(f(z)) = 2\theta = 2\arg(z).$$

example) We want to find an inverse function for $f(z) = z^2$. We need to restrict our domain, since $\forall w \in \mathbb{C}, \exists \pm\sqrt{w}$ s.t. $f(\pm\sqrt{w}) = w$



$\mathbb{C} \setminus (-\infty, 0]$

↑ Slit Plane

Chapter 1, Section 5: The Exponential Function

$$\boxed{e^z =} e^{x+iy} = \boxed{e^x e^{iy}} = e^x(\cos y + i \sin y)$$

Where does this identity come from?

$$|e^z| = \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2} = \sqrt{e^{x2}} = e^x$$

$$\arg(e^z) = y.$$

1.) e^z is periodic since cosine & sine are periodic.

2.) $e^{z+w} = e^z e^w$ since

$$e^{z+w} = e^{x+iy+a+ib} = e^{\overbrace{x+a}^{\in\mathbb{R}}} e^{i\overset{\in\mathbb{R}}{(y+b)}} = e^x e^a e^{iy} e^{ib}$$

$$= e^x e^{iy} e^a e^{ib} = e^{x+iy} e^{a+ib} = e^z e^w.$$

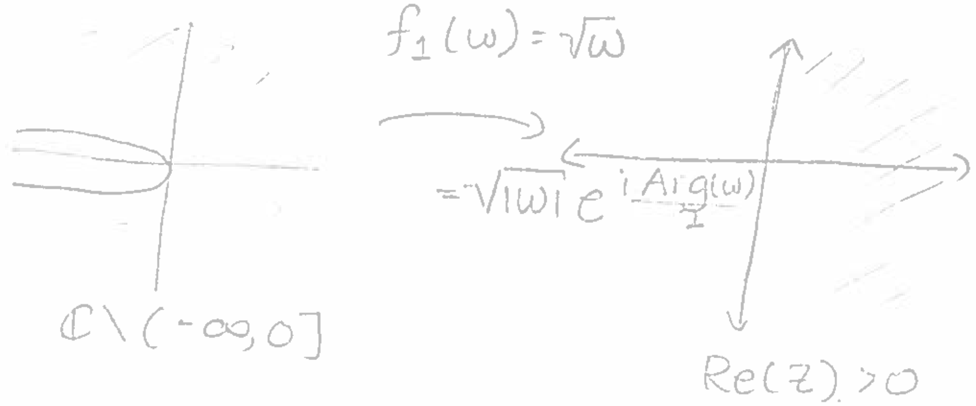
3.) $e^{-z} = \frac{1}{e^z}$, $z \in \mathbb{C}$ since

$$e^z e^{-z} = e^{z-z} = e^0 = 1$$

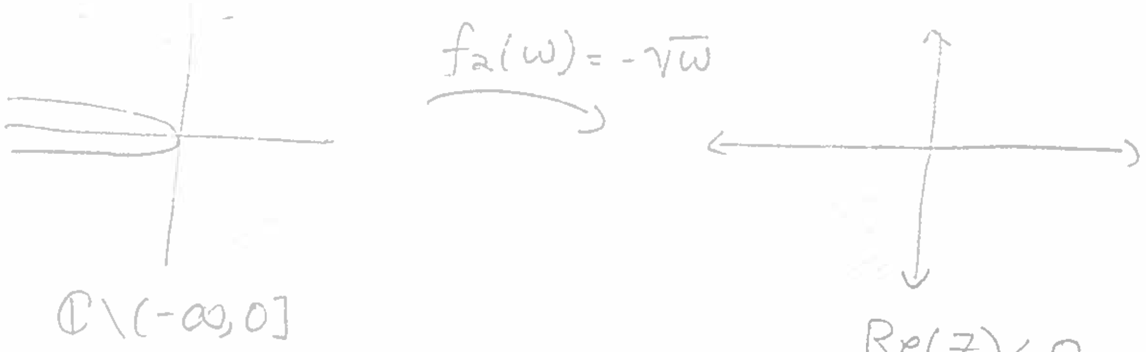
$$\Rightarrow e^{-z} = \frac{1}{e^z}$$

We're going to define our inverse function on $\mathbb{C}\setminus(-\infty,0]$

$f_1(w)$ [called the Principal Branch]



$f_2(w)$ [why doesn't this have a name?]



We glue together these 2 $Re(z)<0$ and $Re(z)>0$

and call $f(w) = f_1(w), f_2(w)$

Chapter 1, Section 6: The Logarithm Function

For $z \neq 0$, define

$$\log(z) = \log|z| + i \arg z$$

$$= \log|z| + i \operatorname{Arg} z + 2\pi i m, \quad m = 0, \pm 1, \pm 2, \ldots$$

WTS $e^{\log z} = z$

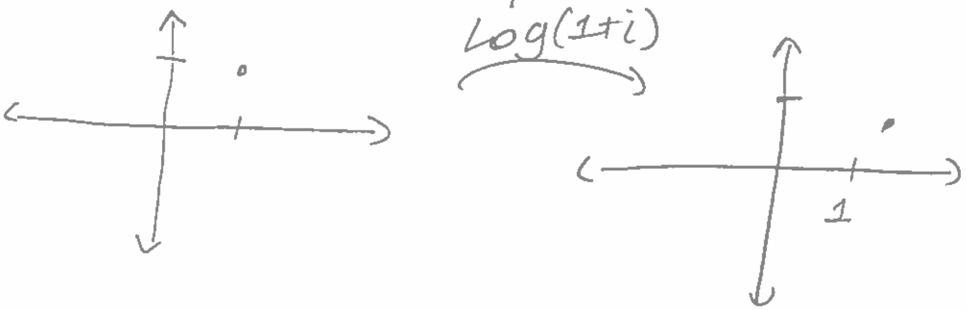
$$e^{\log z} = e^{\log|z| + i \operatorname{Arg} z + 2\pi i m} = e^{\log|z| + i \operatorname{Arg} z} = |z| e^{i \operatorname{Arg} z}$$

$$= |z|(\cos\theta + i \sin\theta) = z.$$

Principal Value of $\log z$: $\operatorname{Log}(z) = \log|z| + i \operatorname{Arg} z.$

Example) $\log(1+i) = \operatorname{Log}\sqrt{2} + \frac{i\pi}{4} + 2\pi i m, \quad m = 0, \pm 1, \pm 2, ..$

$\operatorname{Log}(1+i) = \operatorname{Log}\sqrt{2} + \frac{i\pi}{4}$



Chapter 1, Section 7: Power Functions and Phase Factors

Define $z^{\alpha} = e^{\alpha \log z}, \quad z \neq 0$

$$\Rightarrow z^{\alpha} = e^{\alpha(\log|z| + i\arg z)}$$

example) $i^{i} = e^{i(\log|i| + i\arg(i))}$

$$= e^{i(i\pi/2)} e^{i2\pi m i} = e^{-\pi/2} e^{-2\pi m}, \quad m = 0, \pm 1, \pm 2, ..$$

example) $i^{-i} = e^{-i\log(i)} =$

Chapter 1, Section 8: Trigonometric and Hyperbolic Functions

$$\cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

$2\pi i$-periodic

$\Rightarrow \cosh(z+2\pi i) = \cosh(z)$

$\sinh(z+2\pi i) = \sinh(z).$

$\cosh(iz) = \cos z, \quad \cos(iz) = \cosh(z)$

$\sinh(iz) \equiv i\sin z, \quad \sin(iz) = i\sinh(z).$

why? $$\cosh(iz) = \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$$

$$\sinh(iz) = \frac{e^{iz}-e^{-iz}}{2} = i\sin(z).$$

$$\sin(iz) = \frac{e^{iiz}-e^{-iiz}}{2i} = \frac{e^{-z}-e^{z}}{2i} = -\frac{1}{i}\left(\frac{e^{z}-e^{-z}}{2}\right)$$

$$= \frac{-1}{i}\sinh(z) = \frac{-i}{-1}\sinh(z) = i\sinh(z).$$

$$\cos(iz) = \frac{e^{iiz}+e^{-iiz}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh(z)$$

$\sin(z).$

$\Rightarrow \overset{\shortparallel}{\sin}(x+iy) = \cos(x)\sin(iy) + \sin(x)\cos(iy)$

$= \cos(x)\, i\sinh(y) + \sin(x)\cosh(y)$

$\Rightarrow |\sin z|^2 = \cos^2(x)\sinh^2(y) + \sin^2(x)\cosh^2 y$

$$\Rightarrow |\sin z|^2 = \sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 y$$

$$= \sin^2 x + \sinh^2(y)(\sin^2 x + \cos^2 x)$$

$$= \sin^2 x + \sinh^2(y).$$

$$\Rightarrow \sin z = 0 \Leftrightarrow \sin x = 0 \ \& \ \sinh(y) = 0.$$

$$\tan(z) = \frac{\sin z}{\cos z}, \quad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}.$$

$$\tanh(iz) = i\tan(z).$$

Now we want to look at the inverse trig. functions:

Let $w = \sin^{-1}(z)$.

$$\Rightarrow \sin(w) = z.$$

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i} = z.$$

$$\Rightarrow e^{iw} - e^{-iw} = 2iz$$

$$\Rightarrow e^{iw} - e^{-iw} - 2iz = 0$$

$$\Rightarrow e^{2iw} - e^{iw} 2iz - 1 = 0$$

$$\Rightarrow e^{iw} = \frac{+2iz \pm \sqrt{(2iz)^2 - 4(1)(-1)}}{2}$$

$$= \frac{2iz \pm \sqrt{-4z^2+4}}{2} = \frac{2}{2}\left(iz \pm \sqrt{-z^2+1}\right)$$

$$\Rightarrow iw = \log(iz \pm \sqrt{1-z^2})$$

$$\Rightarrow w = -i\log(iz \pm \sqrt{1-z^2}).$$

$$\Rightarrow \sin^{-1}(z) = -i\log(iz \pm \sqrt{1-z^2}).$$

Chapter 2, Section 1:

Convergent Sequence: $\{S_n\} \to s$ if $\forall \varepsilon > 0, \exists N \geq 1$ s.t. $|S_n - S| < \varepsilon$ $\forall n \geq N$.

$\frac{1}{n^p} \to 0$

$|z|^n \to 0$ if $|z| < 1$

$\sqrt[n]{n} \to 1$

Theorem: A convergent sequence is bounded. Further, if $S_n \to s$, $t_n \to t \Rightarrow$

(i) $S_n + t_n \to s + t$

(ii) $S_n t_n \to st$

(iii) $\frac{S_n}{t_n} \to \frac{s}{t}$ if $t \neq 0$

example) $$\lim_{n\to\infty} \frac{3n^2+2n-1}{5n^2-4n+8} = \lim_{n\to\infty} \frac{3+\frac{2}{n}-\frac{1}{n^2}}{5-\frac{4}{n}+\frac{8}{n^2}} = \frac{3}{5}.$$

Theorem: If $r_n \leq S_n \leq t_n$ and $r_n \to L$ and $t_n \to L \Rightarrow S_n \to L$.

Theorem: A bounded monotone sequence of real numbers converges

$\liminf(S_n) = -\limsup(-S_n)$.

$\liminf(S_n) + \limsup(-S_n) = 0$.

Theorem: A sequence $\{S_K\}$ of complex numbers converges

Theorem: A sequence of complex numbers converges $\Leftrightarrow$ it is a Cauchy sequence. (i.e. the complex numbers = complete)

Lemma: $f(z)$ has limit L as $z \to z_0$ $\Leftrightarrow$ $f(z_n) \to L$ for any sequence $\{z_n\}$ in the domain of $f(z)$ s.t. $z_n \neq z_0$ and $z_n \to z_0$.

Theorem: If a function has a limit at z_0, then the function is bounded near z_0. Further, if $f(z) \to L$ and $g(z) \to M$ as $z \to z_0$, then as $z \to z_0$ we have

(i) $f(z) + g(z) \to L+M$

(ii) $f(z)g(z) \to LM$

(iii) $\dfrac{f(z)}{g(z)} \to \dfrac{L}{M}$ if $M \neq 0$.

Def'n: $f(z)$ is cts. at z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$

$\Rightarrow$ If f & g are cts,

$$(f+g)(z) = f(z) + g(z)$$
$$\downarrow$$
$$f(z_0) + g(z_0) = (f+g)(z_0)$$

$$(fg)(z) = f(z)g(z) \to f(z_0)g(z_0) = (fg)(z_0)$$

$$\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)} \to \frac{f(z_0)}{g(z_0)} = \left(\frac{f}{g}\right)(z_0).$$

$$f(g(z)) \to f(g(z_0))$$

example) $|z-z_0| = |Re(z-z_0) + i Im(z-z_0)|$

$$|x-y+y| \leq |x-y| + |y|$$
$$\Rightarrow |x| - |y| \leq |x-y|$$
$$\Rightarrow |Re(z-z_0)| - |Im(z-z_0)| \leq |Re(z-z_0) - i Im(z-z_0)|$$

??

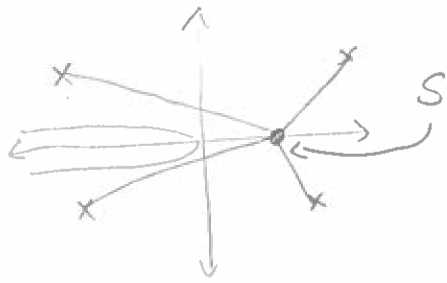
$$|Re(z-z_0)| \leq |z-z_0|$$
$$|Im(z-z_0)| \leq |z-z_0|$$
$$||z| - |z_0|| \leq |z-z_0|$$

Theorem: If $h(x,y)$ is a continuously differentiable function on a domain D so that $\nabla h = 0$ on D, then h is constant.

Def'n. A set is convex if whenever 2 points belong to the set, then the straight line joining the two points are contained in the set.

Def'n: Star-shaped wrt z_0: If $z_1 \in$ Set then the line joining z_0 to z_1 is in the domain

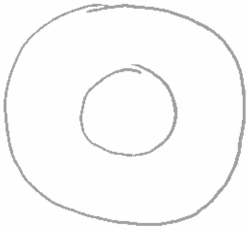
Def'n: Star-shaped domain: $\exists$ point so that the set is star-shaped wrt that pt example) $\mathbb{C} \setminus (-\infty, 0]$



Star-shaped wrt this point.

convex $\Rightarrow$ Star-shaped.

example)



Not star-shaped.

<u>Def'n</u> $\left(\text{If } s_n \in E \Rightarrow s_n \to \underset{\overset{\cap}{E}}{s}\right) \Rightarrow$ closed

Chapter 2, Section 2: Analytic Functions

Defn: $f(z)$ is differentiable if

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) \text{ exists.}$$

example) $f(z)=z^m \Rightarrow f'(z)=mz^{m-1}$.

$$(z+\Delta z)^n = z^m + mz^{m-1}\Delta z + \frac{m(m-1)}{2}z^{m-2}(\Delta z)^2 + \dots + (\Delta z)^m$$

$$\Rightarrow \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{z^m + mz^{m-1}\Delta z + \dots + (\Delta z)^m - z^m}{\Delta z}$$

$$= mz^{m-1} + \frac{m(m-1)}{2}z^{m-2}\Delta z + \dots + (\Delta z)^{m-1}$$

$$\Rightarrow \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = mz^{m-1}.$$

example) Show that $f(z)=\bar{z}$ is not diff-ble at any point.

$$\frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

If $\Delta z = \varepsilon \Rightarrow \overline{\Delta z} = \varepsilon \Rightarrow \dfrac{\overline{\Delta z}}{\Delta z} = 1$

If $\Delta z = i\varepsilon \Rightarrow \overline{\Delta z} = -i\varepsilon \Rightarrow \dfrac{\overline{\Delta z}}{\Delta z} = \dfrac{-i\varepsilon}{i\varepsilon} = -1.$

So no limit.

Theorem: If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

$$f(z) = f(z_0) + f(z) - f(z_0) = f(z_0) + \left(\frac{f(z)-f(z_0)}{z-z_0}\right)(z-z_0).$$

$\Rightarrow$ Letting $z \to z_0$, $f(z) \to f(z_0) + f'(z_0)\underbrace{(z-z_0)}_{\searrow 0}$

So $f(z) \to f(z_0)$.

Rules for differentiation:

(i) $(cf)'(z) = cf'(z)$

(ii) $(f+g)'(z) = f'(z) + g'(z)$

(iii) $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$

(iv) $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$

Chain Rule: suppose $g(z)$ is diff'ble at z_0, and $f(w)$ is diff'ble at $g(z_0)$. Then $f(g(z))$ is diff'ble at z_0 and $f\circ g'(z_0) = f'(g(z_0))g'(z_0)$

example) $f(w) = \frac{1}{w}$, $g(z) = z^2 - 1$ $\Rightarrow$ $f(g(z)) = \frac{1}{z^2-1}$

$$\Rightarrow \frac{d}{dz}\left(\frac{1}{z^2-1}\right) = -\frac{1}{(z^2-1)^2}\cdot(2z) = \frac{-2z}{(z^2-1)^2}$$

Def'n: A function $f(z)$ is analytic on the open set $\mathcal{U}$ if $f(z)$ is diff'ble at each point of $\mathcal{U}$ and ($f'(z)$ cts. on $\mathcal{U}$)

So $f(z)$ is analytic on U if $f(z)$ is diff'ble on U.

So Analytic $(U) = \{ f(z) : f'(z)$ exists on $U \}$

Chapter 2, Section 3: The Cauchy-Riemann Equations

Suppose we have $f: D \subset \mathbb{C} \longrightarrow \mathbb{C}$

$$((x,y)=)z \longmapsto (u,v)$$

Fix $z \in D$. $\Rightarrow f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$

But we want to take into account the fact that $f(z) = \underset{\overset{\uparrow}{\mathbb{R}}}{u(z)} + i\underset{\overset{\uparrow}{\mathbb{R}}}{v(z)}$.

We're going to first look at $\Delta z = \Delta x \in \mathbb{R}$

Then $\Delta z = i\Delta y \in I$.

1.) $$\frac{f(z+\Delta x)-f(z)}{\Delta z} = \frac{u(x+\Delta x,y)+iv(x+\Delta x,y)-u(x,y)-iv(x,y)}{\Delta x}$$

$$= \underset{\swarrow \atop \frac{\partial u}{\partial x}}{\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}} + i\,\underset{\swarrow \atop +i\frac{\partial v}{\partial x}(x,y)}{\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}}$$

2.) $$\frac{f(z+i\Delta y)-f(z)}{i\Delta y} = \frac{u(x,y+\Delta y)+iv(x,y+\Delta y)-u(x,y)-iv(x,y)}{i\Delta y}$$

$$= \frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + i\,\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}$$

$$= \underset{-i\,\partial u/\partial y}{-i\left(\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}\right)} + \underset{+\frac{\partial v}{\partial y}}{\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}}$$

✯ Cauchy Riemann Eq'ns ✯

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

✯✯

Theorem: Let $f = u + iv$ be defined on D in the complex plane, where u & v are real-valued. Then $f(z)$ is analytic on $D \iff u$ & v have continuous first-order partial partial derivatives that satisfy the CR-eq'ns. ✯✯ (So if satisfy CR $\Rightarrow f(z)$ is analytic!!)

– skip proof –

example) The functions $u(x,y) = x$ and $v(x,y) = y$ satisfy the CR eq'ns.

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 1 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow x + iy = f(z)$ is analytic.

example) $f(z) = x - iy = \bar{z}$

$\Rightarrow u(x,y) = x$
$v(x,y) = -y$

$\Rightarrow \frac{\partial u}{\partial x} = 1, \ \frac{\partial v}{\partial y} = -1 \Rightarrow$ Doesn't satisfy CR

$\Rightarrow \bar{z}$ is not analytic

example) $f(z) = e^z = e^x \cos y + i e^x \sin y$

$$\Rightarrow u(x,y) = e^x \cos y$$
$$v(x,y) = e^x \sin y$$

$$\Rightarrow \frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial v}{\partial y} = e^x \cos y \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -e^x \sin y \quad \frac{\partial v}{\partial x} = e^x \sin y \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \checkmark$$

$\Rightarrow e^z$ is analytic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^x$$

(OR)

$$f'(x) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = +e^x \cos y + i e^x \sin y. \checkmark$$

We can get derivatives of trig functions from this

<u>Theorem</u>: If $f(z)$ is analytic on D and if $f'(z) = 0$ on $D \Rightarrow f(z)$ is constant.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$$

$$\left(\begin{array}{l} \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (0,0) \Rightarrow u \text{ is constant.} \\ \nabla v = (0,0) \Rightarrow v \text{ is constant} \end{array} \right)$$

$\Rightarrow f(z)$ is constant.

Theorem: If $f(z)$ is analytic and real-valued on a domain D, then $f(z)$ is constant.

$f(z) = u(z) + i\,0$, $f(z)$ analytic $\Rightarrow$ CR-Holds

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \quad \Rightarrow \frac{\partial u}{\partial x} = 0.$$

$\Rightarrow \quad \nabla u = 0 \Rightarrow u$ is constant

$\nabla v = 0 \Rightarrow v$ is constant.

$\Rightarrow f(z)$ is constant.

Chapter 2, Section 4: Inverse Mappings and the Jacobian

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

$$\text{Jacobian}(f) = \begin{array}{cc|c} \overset{x}{\downarrow} & \overset{y}{\downarrow} & \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \leftarrow u \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \leftarrow v \end{array}$$

$$\det \text{Jacobian}(f) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

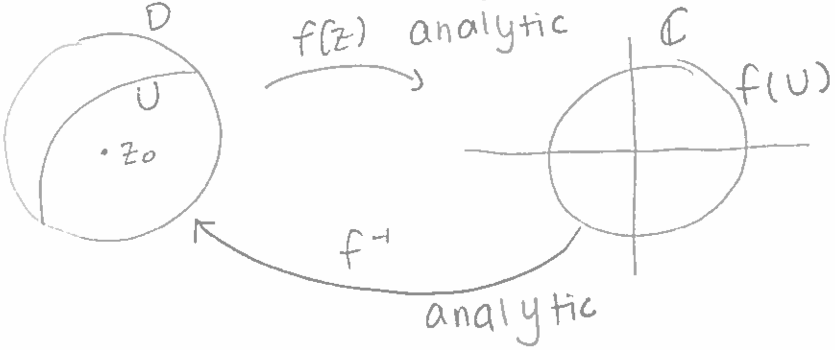
$$= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

Theorem: If $f(z)$ is analytic, then its Jacobian matrix J_f has determinant:

$$\det J_f(z) = |f'(z)|^2$$

Theorem: Suppose $f(z)$ is analytic on a domain D, $z_0 \in D$, and $f'(z_0) \neq 0$. Then there is a small disk $U \subset D$ containing z_0 s.t. $f(z)$ is one-to-one on U, $f(U)$ is open, and $f^{-1} : f(U) \to U$ is analytic and

Satisfies $$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$



We use the inverse function thm. etc etc

But where does our formula come from?

$$f^{-1}(f(z)) = z$$

Taking derivative

$$\Rightarrow (f^{-1})'(f(z)) \cdot f'(z) = 1$$

$$\Rightarrow (f^{-1})'(f(z)) = \frac{1}{f'(z)}$$

example) $z = e^{\text{Log}(z)}$

Taking derivative

$$\Rightarrow \quad 1 = e^{\text{Log}(z)} \frac{d}{dz}(\text{Log} z) = z \frac{d}{dz}(\text{Log}(z))$$

$$\Rightarrow \quad \frac{d}{dz} \text{Log}(z) = \frac{1}{z}$$

example) WTS $\frac{d}{dz}\sqrt{z} = \frac{1}{2\sqrt{z}}$

$$w^2 = z \quad \Rightarrow w = \sqrt{z}$$

$$\Rightarrow 2w \frac{dw}{dz} = 1$$

$$\Rightarrow \frac{dw}{dz} = \frac{1}{2w} = \frac{1}{2\sqrt{z}}$$

Chapter 2, Section 5: Harmonic Functions

Sum of the second partial derivatives set equal to 0 is called Laplace's Equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

$$\Delta u = 0$$

(Reminder, gradient is the coordinates of the first partial derivatives:

$$\nabla u = \frac{\partial u}{\partial x_1}\hat{i} + \frac{\partial u}{\partial x_2}\hat{j} + \frac{\partial u}{\partial x_3}\hat{k}$$

/

Def'n: $u(x,y)$ is harmonic if all its partial & second partial derivatives exist and are continuous and satisfy Laplace's equation.

Theorem: If $f=u+iv$ is analytic, and the functions u & v have continuous second-order partial derivatives, then u & v are harmonic.

(Analytic ⇒ Harmonic)

Since analytic, CR-eq'ns are true.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \overset{\text{by cty of partials}}{=} \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic.}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Defn: If u is harmonic on D, and v is a harmonic function such that $u+iv$ is analytic, then v is a harmonic conjugate of u.

(Does a harmonic conjugate always exist though?) yes!

Idea: v is unique up to adding a constant.

\$ v_0 is harmonic & $u+iv_0$ is analytic.

$$\Rightarrow i(v-v_0) \text{ is analytic}$$

and $v-v_0 \in \mathbb{R}$ $\Rightarrow$ $v-v_0$ is constant.

example) Show that $u(x,y) = xy$ is harmonic, and find a harmonic conjugate for u.

$$\frac{\partial u}{\partial x} = y \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = x \Rightarrow \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{harmonic}$$

$$y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = y \Rightarrow v(x,y) = \frac{y^2}{2} + \overset{\substack{\text{constant wrt } y\\ \swarrow}}{h(x)}$$

$$\Rightarrow \frac{\partial v}{\partial x} = h'(x) = -x$$

$$\Rightarrow h(x) = -\frac{x^2}{2} + C$$

$$\Rightarrow v(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + c$$

Chapter 2, Section 6: Conformal Mappings

Let $\gamma(t)=x(t)+iy(t)$, $0\leq t\leq 1$, be a smooth parametrized curve terminating at $z_0=\gamma(0)$. We refer to

$$\gamma'(0)=\lim_{t\to 0}\frac{\gamma(t)-\gamma(0)}{t}=x'(0)+iy'(0)$$

as the tangent vector to the curve γ at z_0.

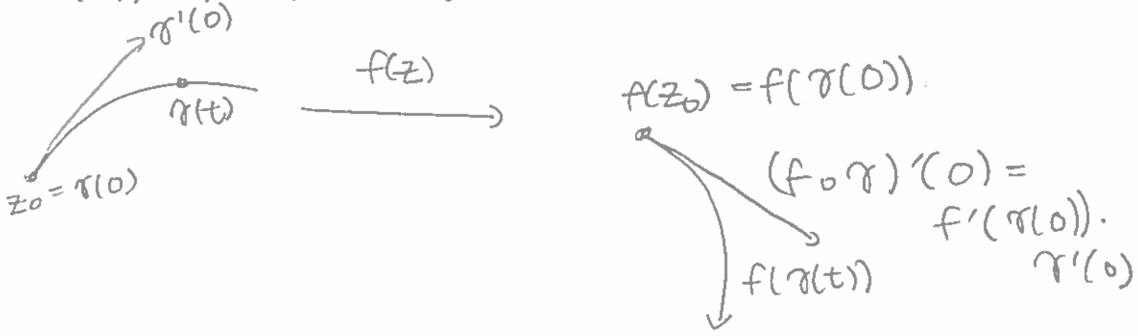
We define the angle b/t two curves at z_0 to be the angle b/t their tangent vectors at z_0.

Theorem: If $\gamma(t)$, $0\leq t\leq 1$, is a smooth parametrized curve terminating at $z_0=\gamma(0)$, and $f(z)$ is analytic at z_0, then the tangent to the curve $f(\gamma(t))$ terminating at $f(z_0)$ is

$$(f\circ\gamma)'(0)=f'(z_0)\gamma'(0).$$

$$\lim_{t\to 0}\frac{f(\gamma(t))-f(\gamma(0))}{t-0}=\lim_{t\to 0}\frac{f(\gamma(t))-f(\gamma(0))}{\gamma(t)-\gamma(0)}\cdot\frac{\gamma(t)-\gamma(0)}{t-0}$$

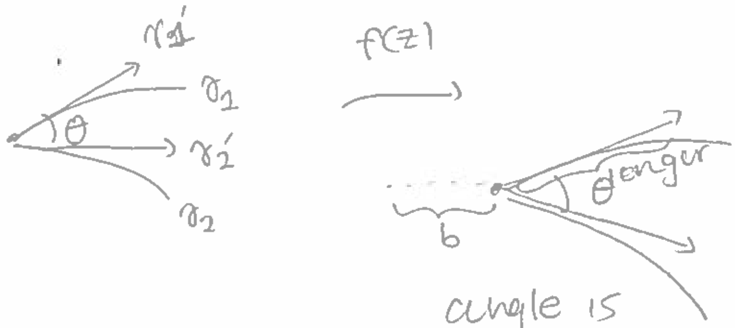
$$=f'(\gamma(0))\cdot\gamma'(0)=f'(z_0)\gamma'(0)\ \checkmark$$



Def'n: A function is conformal if it preserves angles.

Def'n: A conformal mapping of one domain D onto V is a continuously differentiable function that is conformal at each point of D and that maps D one-to-one onto V.

example) $f(z)=az+b$



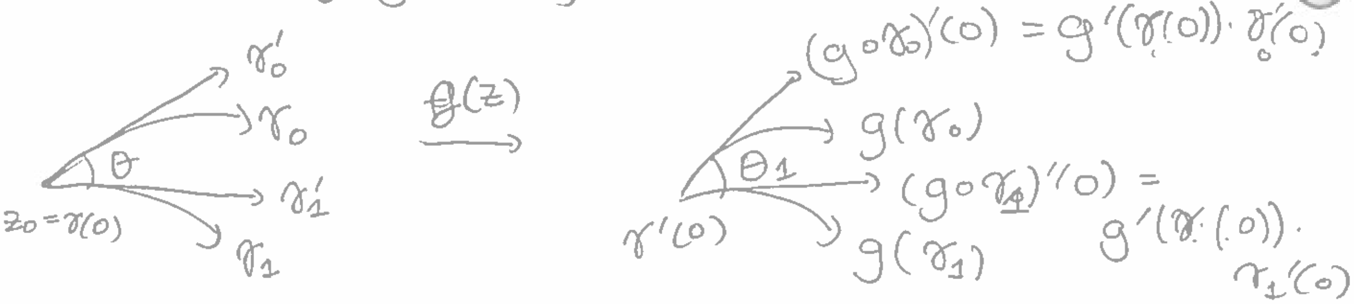
angle is the same.

So linear maps are conformal.

z^n, $\bar{z}$ not conformal.
↑ not conformal at 0.
conformal everywhere else

Theorem: If $f(z)$ is analytic at z_0 and $f'(z_0)\neq 0$, then $f(z)$ is conformal at z_0.

(Actually get conformal $\Leftrightarrow$ analytic!)



$\Rightarrow$

$$\Rightarrow \theta = \arg(\gamma_0') - \arg(\gamma_1').$$

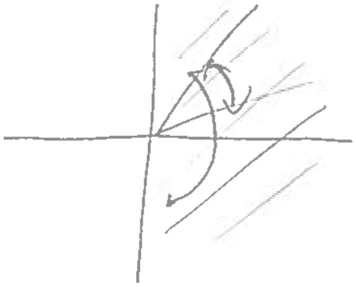
$$\Rightarrow \theta_1 = \arg(g'(\gamma(0))\gamma_0'(0)) - \arg(g'(\gamma(0))\gamma_1'(0))$$

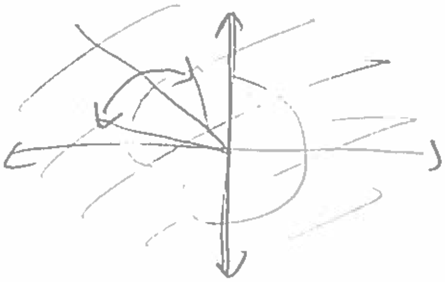
$$\Rightarrow \theta_1 = \arg(g'(\gamma(0))) + \arg(\gamma_0'(0)) - \arg(g'(\gamma(0))) - \arg(\gamma_1'(0))$$

$$= \theta.$$

$\Rightarrow$ conformal.

example) The function $w=z^2$ maps the right half-plane $\{\operatorname{Re} z>0\}$ conformally onto the slit-plane $\mathbb{C}\setminus(-\infty,0]$. For any fixed θ_0, $0<\theta_0\leq\frac{\pi}{2}$: confor





$\{|\arg z|<\theta_0\} \longrightarrow \{|\arg z|<2\theta_0\}$

Any conformal mapping carries orthogonal curves to orthogonal curves.

Chapter 2, Section 7: Fractional Linear Transformations
(möbius transformation)

<u>Defn</u>: A fractional linear transformation is a function of the form $w = f(z) = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$.

$$a, b, c, d \in \mathbb{C}.$$

We get $f'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{da-bc}{(cz+d)^2}$.

so if $ad-bc \neq 0 \Rightarrow f'(z) \neq 0$.

☆example) A function of the form $f(z) = az+b$, where $a \neq 0$ is called an <u>affine transformation</u>.
(dis just linear.)

special cases:

1.) $z \mapsto z+b$ (translations)
2.) $z \mapsto az$ (dilations)

example) The FLT $f(z) = \frac{1}{z}$ is called an inversion.

<u>Weird idea:</u> we regard FLT's as a map from $\mathbb{C}^*$ to $\mathbb{C}^*$.

If $f(z)$ is affine, $f(\infty) = a \cdot \infty + b = \infty$

If $f(z)$ is not affine, $f(\infty) = \lim_{z \to \infty} f(z) = \lim_{z \to \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}}$
$= \frac{a}{c}$

$\Rightarrow$ If $f(z) = \frac{1}{z} \Rightarrow f(\infty) = \lim_{z \to \infty} \frac{1}{z} = 0$

Want to find the inverse of FLT.

$$z = \frac{af^{-1}(z)+b}{cf^{-1}(z)+d}$$

$$\Rightarrow z[cf^{-1}(z)+d] = af^{-1}(z)+b.$$

$$\Rightarrow zcf^{-1}(z) - af^{-1}(z) = b - zd$$

$$\Rightarrow f^{-1}(z)[zc-a] = b-zd$$

$$\Rightarrow f^{-1}(z) = \frac{b-zd}{zc-a} \qquad (-d)(-a)-(b)(c) \neq 0$$

$\Rightarrow f^{-1}(z)$ is also an FLT.

What about FLT composed with FLT?

$$f(z) = \frac{az+b}{cz+d} \qquad g(z) = \frac{a_1z+b_1}{c_1z+d_1}.$$

$$f(g(z)) = \frac{a\left(\frac{a_1z+b_1}{c_1z+d_1}\right)+b}{c\left(\frac{a_1z+b_1}{c_1z+d_1}\right)+d} = \frac{\frac{aa_1z+ab_1+bc_1z+bd_1}{c_1z+d_1}}{\frac{ca_1z+cb_1+dzc_1+d_1d}{c_1z+d_1}}$$

$$= \frac{aa_1z+ab_1+bc_1z+bd_1}{ca_1z+cb_1+dzc_1+d_1d} = \frac{(aa_1+bc_1)z+(ab_1+bd_1)}{(ca_1+dc_1)z+(cb_1+dd_1)}$$

where $(aa_1+bc_1)(cb_1+dd_1)-(ab_1+bd_1)(ca_1+dc_1) \neq 0$

where $aa_1cb_1 + aa_1dd_1 + bc_1cb_1 + bdc_1d_1$
$- aa_1cb_1 - ab_1dc_1 - bd_1ca_1 - bd_1dc_1 \neq 0$

where $ad(a_1d_1-b_1c_1) - bc(a_1d_1-b_1c_1)$
$= (ad-bc)(a_1d_1-b_1c_1) \neq 0$ ✓

So $f(g(z))$ is an FLT!

Theorem : Given any 3 distinct points z_0, z_1, z_2 in the tended complex plane, and given any 3 distinct values w_0, w_1, w_2 in the extended complex plane, $\exists !$ fractional linear transformation $w = w(z)$ s.t. $w(z_0) = w_0$, $w(z_1) = w_1$ & $w(z_2) = w_2$

-skip proof-

example) Find the FLT mapping
$$-1 \mapsto 0$$
$$\infty \mapsto 1$$
$$i \mapsto \infty$$

$$w(z) = a\,\frac{z+1}{z-i}$$

$$= \lim_{z\to\infty} w(z) = a$$

$$\Rightarrow w(z) = \frac{z+1}{z-1}$$

Theorem: Every FLT is a composition of dilations, translations, and inversions.

Theorem: An FLT maps circles in $\mathbb{C}^*$ to circles.

(By Section 1.3: Stereographic Projection)

example) Find the eq'n of the FLT mapping
$$0 \mapsto -1$$
$$i \mapsto 0$$
$$\infty \mapsto 1.$$

$$f(z) = \frac{a(z+\frac{b}{a})}{cz+d} = \overset{a}{\frac{z-i}{cz+d}} \Rightarrow f(0) = \frac{-i}{d} = -1$$
$$\Rightarrow d = i$$

$$\Rightarrow f(z) = \frac{z-i}{cz+i}$$

Ideas: 3 pts. determine a circle

FLTs are conformal.

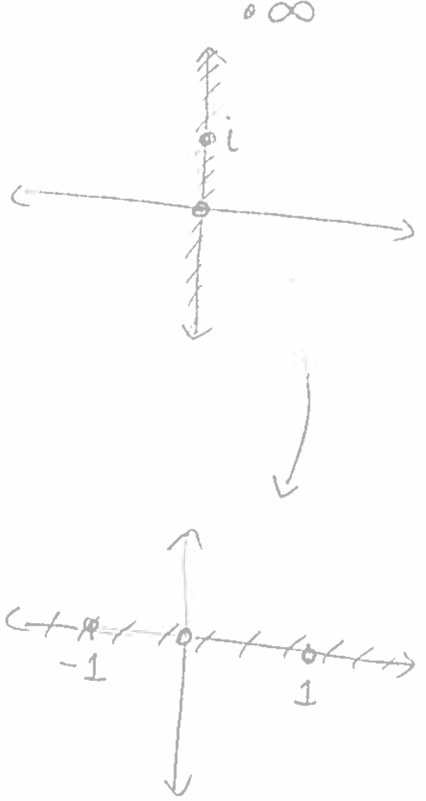
$$f(\infty) = \lim_{z \to \infty} \left(\frac{z-i}{cz+d} \right) = \frac{1}{c} = 1$$

$$\Rightarrow c = 1$$

$$\Rightarrow f(z) = \frac{z-i}{z+i}$$

example) Determine the images of the following sets under the above FLT:

(a) The imaginary axis:



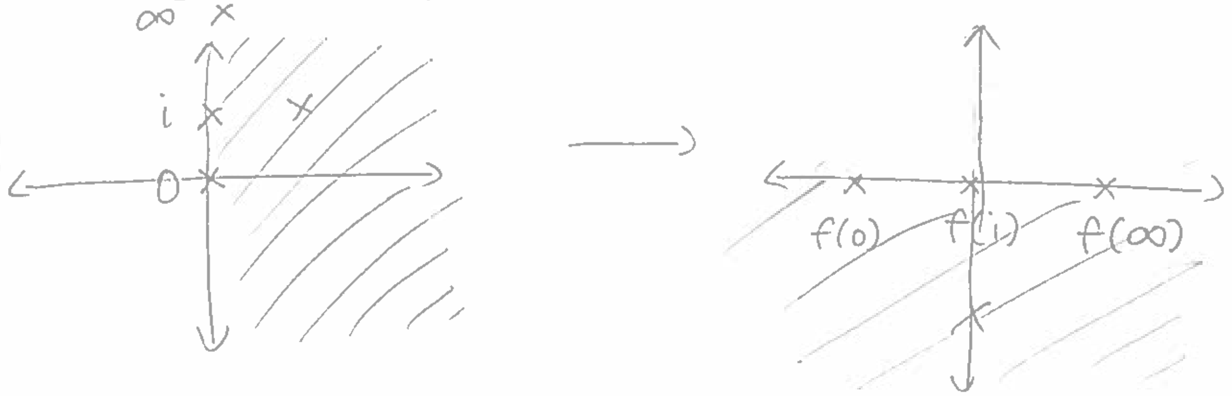
$$\frac{0-i}{0+i} = -1$$

$$\frac{i-i}{i+i} = 0$$

$$\frac{\infty-i}{\infty+i} = 1$$

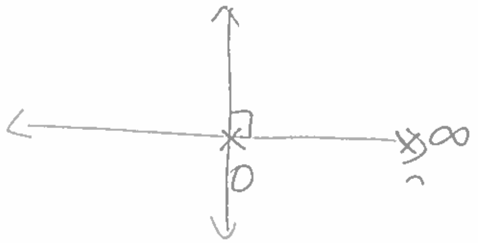
Real line.

(b) The right half-plane.



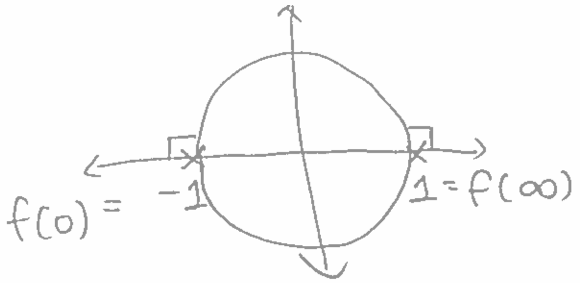
(Like, flip this so that f(0), f(i), f(∞) look like the picture on the left).

(c) The real axis



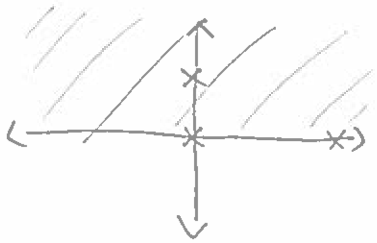
f(0) = −1

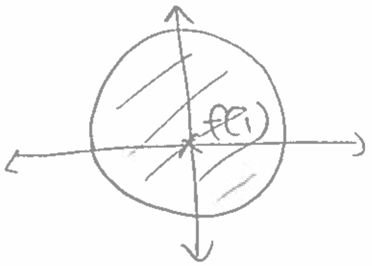
f(∞) = 1



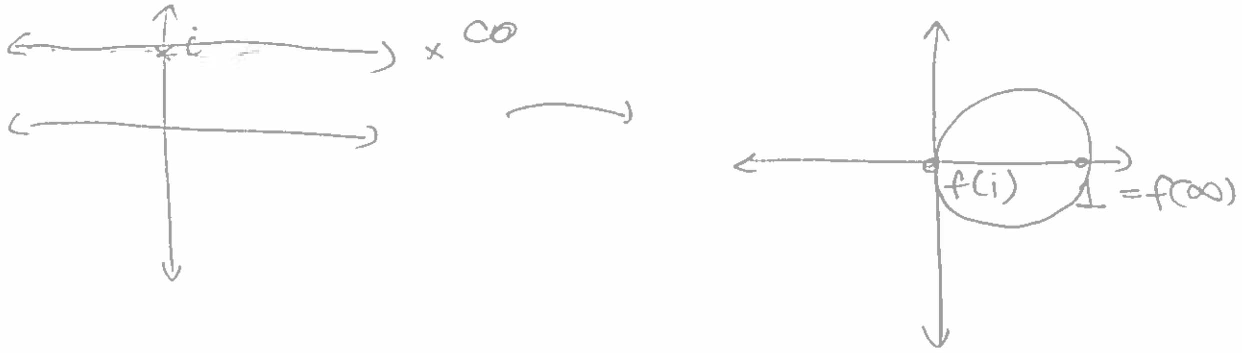
orthogonal to ℝ.

(d) The upper half-plane.





(e) The horizontal line through i.



should be parallel to $f(\mathbb{R})$

orthogonal to $\mathbb{R} = f(i\mathbb{R})$.

Chapter 3, Section 1: Line Integrals and Harmonic Functions
Green's Theorem

<u>Defin</u>: A <u>path</u> in the plane from $A = \gamma(a)$ to $B = \gamma(b)$ is a s. function $t \longmapsto \gamma(t)$ on some parameter interval $a \le t \le b$.

<u>Defin</u>: The path is <u>closed</u> if $\gamma(a) = \gamma(b)$.

<u>Defin</u>: A <u>simple closed path</u> is a closed path γ s.t. $\gamma(s) \neq \gamma(t)$ for $a \le s < t < b$.

simple path

not simple path

simple closed path

Reparametrizations:

If $\phi(s), \alpha \le s \le \beta$ is a strictly increasing cts. fnc. s.t. $\phi(\alpha) = a$ and $\phi(\beta) = b$, then $\gamma(\phi(s))$ is also a path from $\gamma(a)$ to $\gamma(b)$. $\gamma(\phi)$ is a reparametrization of γ. We consider γ and its reparametrizations as "the same".

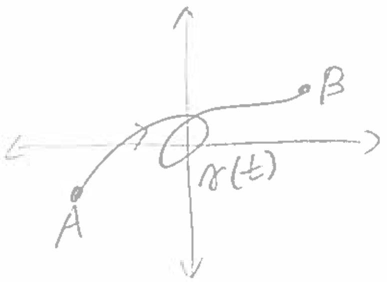
<u>Defn</u>. The trace of the path γ is $\gamma([a,b])$

☆ <u>Def'n</u>: A smooth path is $\gamma(t) = (\underset{\uparrow}{x(t)}, \underset{\nearrow}{y(t)})$, $a \le t \le b$.

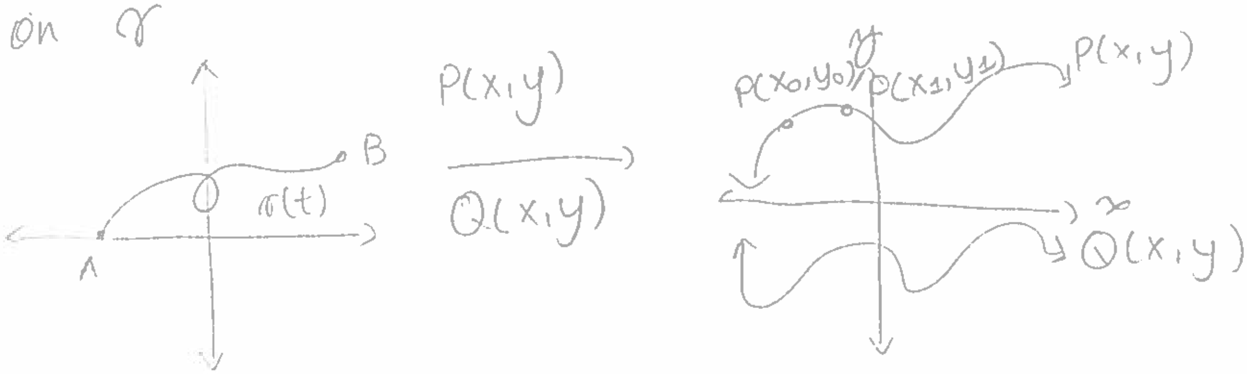
<u>Def'n</u>: A <u>piecewise smooth path</u> is a $\overset{\text{smooth}}{}$ concatenation of smooth paths.

<u>Defin</u>. A <u>curve</u> is either a smooth or a piecewise smooth path.

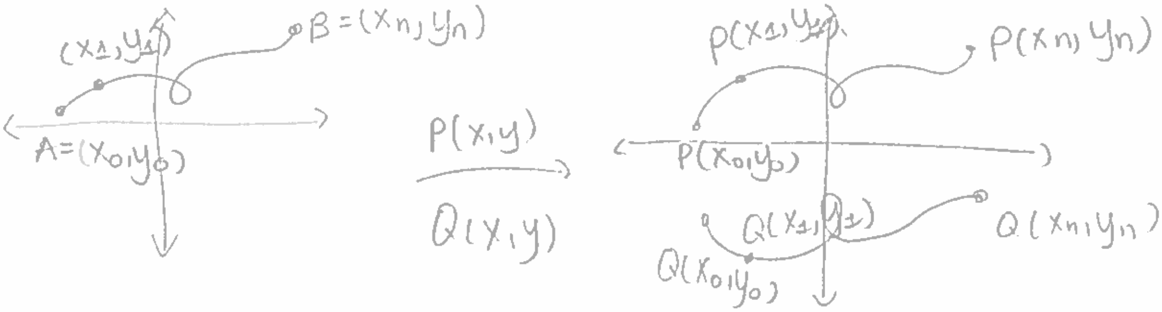
Let γ be a path in the plane from A to B.



Let $P(x,y)$ and $Q(x,y)$ be continuous complex fncs, on γ



We consider successive points on the path,



We call $\lim\limits_{\substack{d(x_i,x_{i+1})\to 0\\ d(y_i,y_{i+1})\to 0}} \sum P(x_j,y_j)(x_{j+1}-x_j) + \sum Q(x_j,y_j)(y_{j+1}-y_j) = \int_\gamma P\,dx + Q\,dy$

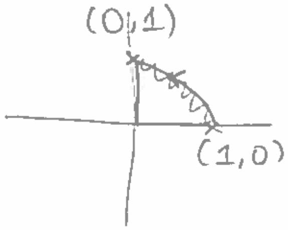
~~We get $\int_\gamma Pdx+Qdy = \int_a^b P(x(t),y$~~

example) To evaluate $\int_\gamma xy\,dx$, where γ is the quarter-circle from $(1,0)$ to $(0,1)$ on the unit circle, we parametrize γ by $(x(\theta),y(\theta))=(\cos\theta,\sin\theta)$, $0\le\theta\le\frac{\pi}{2}$

$\int$

// we get $\int_\gamma P dx + Q dy = \int_a^b P(x(t), y(t)) \frac{dx}{dt} dt + \int_a^b Q(x(t), y(t)) \frac{dy}{dt}c$

x.) Evaluate $\int_\gamma xy\,dx$, where γ is the quarter-circle from (1,0) to (0,1) on the unit circle.



$$\gamma(t) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le \tfrac{\pi}{2}$$

$$\int_0^{\pi/2} \cos\theta \sin\theta \, d(\cos\theta) = \int_0^{\pi/2} \cos\theta \sin\theta (-\sin\theta) d\theta$$

$$= \int_0^{\pi/2} -\cos\theta \sin^2\theta \, d\theta = \int_0^1 -u^2 du = -\frac{u^3}{3}\Big]_0^1 = -\frac{1}{3}$$

$u = +\sin\theta$

$du = \cos\theta \, d\theta$

The orientation of ∂D is chosen so that D lies to the left

ex.) To evaluate $\int_{\partial D} xy\,dx$, where D is the quarter-disk in the first quadrant, we divide the integral into 3 pieces,

$$\int_{\partial D} xy = \int_\gamma xy\,dx + \int_{(0,1)}^{(0,0)} xy\,dx + \int_{(0,0)}^{(1,0)} xy\,dx$$

$$= -\frac{1}{3} + \int_{(0,1)}^{(0,0)} 0 \cdot y\,dx + \int_{(0,0)}^{(1,0)} x \cdot 0\,dx = -\frac{1}{3}$$

Green's Theorem: Let D be a bounded domain in the plane whose boundary ∂D consists of a finite number of disjoint piecewise smooth closed curves. Let P & Q be ctsly diff'ble fncs. on D ∪ ∂D.

$$\Rightarrow \int_{\partial D} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

ex.) $$\int_{\partial D} xy\,dx = \iint_{D} (0 - x)\,dx\,dy = \int_{0}^{\pi/2}\int_{0}^{1} -r\cos\theta\, r\,dr\,d\theta$$

$P = xy \quad Q = 0$

$\frac{\partial P}{\partial y} = x$

$$= \int_{0}^{\pi/2}\int_{0}^{1} -r^2\cos\theta\,dr\,d\theta$$

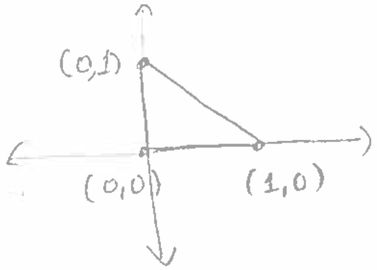
$$= \int_{0}^{\pi/2} \cos\theta\,d\theta \int_{0}^{1} -r^2\,dr$$

$$= \sin\theta\Big]_{0}^{\pi/2} \left(-\frac{r^3}{3}\right)_{0}^{1}$$

$$= (1)\left(-\frac{1}{3}\right) = -\frac{1}{3}.$$

Proof of Green's Theorem:

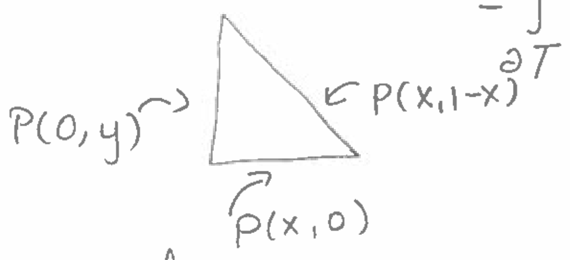
First, we'll establish the formula with the triangle



We need $\int_{\partial T} P\,dx = -\iint_{T} \frac{\partial P}{\partial y}\,dx\,dy$ & $\int_{\partial T} Q\,dy = \iint_{T} \frac{\partial Q}{\partial x}\,dx\,dy$

$$\iint_T \frac{\partial P}{\partial y}\,dx\,dy = \int_0^1 \left[\int_0^{1-x} \frac{\partial P}{\partial y}\,dy \right] dx = \int_0^1 \underbrace{\left[P(x,1-x) - P(x,0) \right]}_{-\int_{\partial T} P\,dx} dx$$

Why?



Next, we'll prove the formula for any domain that can be obtained from the triangle.

Finally, we'll cut up any domain into triangles and add all the triangles together.

Chapter 5, Section 2: Independence of Path

FTOC:

Part I: If $F(t)$ is an antiderivative for the continuous fnc. $f(t)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

Part II: If $f(t)$ is a continuous function on $[a,b]$, then

$$\frac{d}{dt}\int_a^t f(s)\,ds = f(t), \quad a \leq t \leq b.$$

Defn: If $h(x,y)$ is a continuously complex-valued fnc, we define the differential

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$$

Defn. We say a differential $P\,dx + Q\,dy$ is exact if $P\,dx + Q\,dy = dh$ for some fnc. h.

Analogue of FTOC:

Part I: If γ is a piecewise smooth curve from A to B, and if $h(x,y)$ is continuously differentiable on γ, then:

$$\gamma : t \mapsto (x(t), y(t)),\ a \leq t \leq b$$

$$\boxed{\int_\gamma dh} = \int_\gamma \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy. = \int_a^b \frac{\partial h}{\partial x}\frac{dx}{dt}\,dt + \int_a^b \frac{\partial h}{\partial y}\frac{dy}{dt}\,dt$$

$$= \int_a^b \frac{d}{dt}h(x(t),y(t))\,dt = h(x(t),y(t))\Big]_a^b$$

$$= \boxed{h(B) - h(A)}$$

example) $\int_{\gamma} 2xy\,dx + (x^2+2y)dy$, where γ is the quarter circle $\gamma(\theta)=(\cos\theta, \sin\theta)$, $0 \le \theta \le \frac{\pi}{2}$

Way 1 : $\int_0^{\pi/2} 2\cos\theta\sin\theta(-\sin\theta)d\theta + (\cos^2\theta + 2\sin\theta)\cos\theta\, d\theta$

$= \int_0^{\pi/2} (2(-\sin^2\theta)\cos\theta + \cos^3\theta + 2\sin\theta\cos\theta)d\theta$

etc.

Way 2: $dh = 2xy\,dx + (x^2+2y)dy$

$= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$

$\frac{\partial h}{\partial x} = 2xy \Rightarrow h(x,y) = \frac{x^2y}{2} + h(y)$ ↙ constant wrt x.

$\frac{\partial h}{\partial y} = \frac{2x^2}{2} + h'(y)$

$\Rightarrow h'(y) = 2y$

$\Rightarrow h(y) = \frac{2y^2}{2} + c$

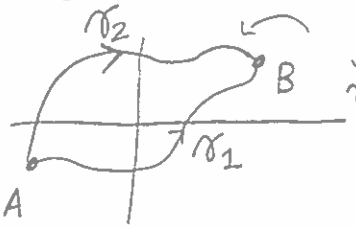
$\Rightarrow h(x,y) = x^2y + y^2 + c.$

$\Rightarrow \int dh = h(x,y)\Big]_{(1,0)}^{(0,1)}$

$= x^2y + y^2\Big]_{(1,0)}^{(0,1)} = 1 - [0] = 1$

But how do we know when a differential is exact?

Def'n: We say the line integral $\int Pdx + Qdy$ is independent of path in D if for any two points of D, the integrals $\int_\gamma Pdx + Qdy$ are the same for any path γ in D from A to B.



$\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_2} Pdx + Qdy.$

This is equivalent to saying ~~on~~ $\int_\gamma Pdx + Qdy = 0$ ∀ closed paths in D from A to B.

Def'n: $Pdx + Qdy$ is closed on D if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Lemma: Let P and Q be continuous complex-valued functions on a domain D. Then $\int Pdx + Qdy$ is independent of path in D $\Leftrightarrow$ $Pdx + Qdy$ is exact.

($\Rightarrow$) Difficult

($\Leftarrow$) By FTOC, $\int dh = h(B) - h(A) \Rightarrow$ indep. of path.

Lemma: Exact differentials are closed

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\Rightarrow \int Pdx + Qdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0.$$

$\Rightarrow$ Closed.

FTOC analogue Part II: Let P & Q be continuously differentiab complex-valued functions on a domain D. Suppose

(i) D is a star-shaped domain (as a disk or rectangle), &

(ii) $Pdz + Qdy$ is closed on D.

$\Rightarrow$ $Pdx + Qdy$ is exact on D.

example) Let's look at $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, $x+iy \in \mathbb{C}\setminus\{0\}$.

WTS not independent of path:

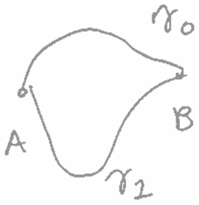
$$\oint_{|z|=1} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \frac{+\sin^2\theta + \cos^2\theta}{\cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \theta\Big]_0^{2\pi} \boxed{= 2\pi} \neq 0$$

$\Rightarrow$ Not independent of path.

$\Rightarrow$ The differential is not exact on $\mathbb{C}\setminus\{0\}$.

Theorem: Let D be a domain, and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \le t \le b$, be two paths in D from A to B. Suppose that γ_0 can be continuously deformed to γ_1, in the sense that for $0 \le s \le 1$ there are paths $\gamma_s(t)$, $a \le t \le b$, from A to B s.t. $\gamma_s(t)$ depends continuously on s & t for $0 \le s \le 1$, $a \le t \le b$. Then $\int_{\gamma_0} Pdx + Qdy = \int_{\gamma_1} Pdx + Qdy$ for any closed differential $Pdx + Qdy$



Theorem: Let D be a domain, and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \le t \le b$ be two closed paths in D. Suppose γ_0 can be continuously deformed to γ_1, in the sense that for $0 \le s \le 1$ there are closed paths $\gamma_s(t)$, $a \le t \le b$, s.t $\gamma_s(t)$ depends ctsly on s & t for $0 \le s \le 1$, $a \le t \le b$. Then $\int_{\gamma_0} Pdx + Qdy = \int_{\gamma_1} Pdx + Qdy$ for any closed differential $Pdx + Qdy$ on D

on any Domain:

Independent of Path $\Longleftrightarrow$ exact $\Rightarrow$ closed

on Star shaped Domain:

Independent of Path $\Longleftrightarrow$ Exact $\Longleftrightarrow$ closed.

Chapter 3, Section 2: Harmonic Conjugates

Lemma: If $u(x,y)$ is harmonic, then the differential

$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$ is closed.

Proof:

$$P = -\frac{\partial u}{\partial y}$$

$$Q = \frac{\partial u}{\partial x}.$$

WTS $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

WTS $-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$

WTS $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \leftarrow$ Laplace's Eq'n which is true since $u(x,y)$ is harmonic.

Theorem: Any harmonic function $u(x,y)$ on a star-shaped domain D (as a disk or rectangle) has a harmonic conjugate fnc. $v(x,y)$ on D.

ex.) To find a harmonic conjugate $v(z)$ for $u = \log|z|$ on the star-shaped domain $\mathbb{C} \setminus (-\infty, 0]$,

$$u(x,y) = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{1}{2(x^2+y^2)} \cdot 2x\,dx + \frac{y}{x^2+y^2}dy$$

$$= \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy$$

$$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -\frac{y}{x^2+y^2} dy + \frac{x}{x^2+y^2} dx$$

$$\Rightarrow v = \int \frac{-y}{x^2+y^2} dy + \frac{x}{x^2+y^2} dx$$

? Where are we integrating though?

$$= \int \frac{-r\sin\theta \, r\cos\theta}{r^2\sin^2\theta + r^2\cos^2\theta} + \frac{r\cos\theta(-r\sin\theta)}{r^2\sin^2\theta + r^2\cos^2\theta}$$

$$= \int \frac{-r^2\sin\theta\cos\theta - r^2\cos\theta\sin\theta}{r^2}$$

$$= \int \frac{-2r^2\sin\theta\cos\theta}{r^2} = -2\int \sin\theta\cos\theta \, d\theta =$$

$$u = \sin\theta$$
$$du = \cos\theta \, d\theta$$

$$= -2\int u \, du = \frac{-2u^2}{2} + C = -\sin^2\theta$$

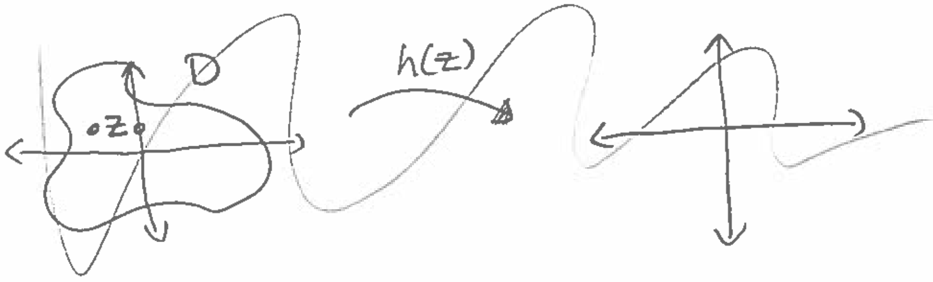
Chapter 3, Section 4: The Mean Value Property

Let $h(z)$ be a continuous real-valued function on a domain D.

Let $z_0 \in D$, and suppose D contains the disk $\{|z-z_0|<\rho\}$.

Def'n: The average value of $h(z)$ on the circle $\{|z-z_0|=r\}$ is:

$$A(r)=\int_0^{2\pi} h(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \quad 0<r<\rho.$$

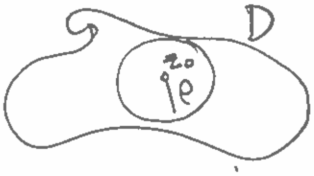


WTS $\lim_{r\to 0} A(r) = h(z_0)$.

$$|A(r)-h(z_0)| = \left|\int_0^{2\pi}[h(z_0+re^{i\theta})-h(z_0)]\frac{d\theta}{2\pi}\right|$$

$$\le \int_0^{2\pi}|h(z_0+re^{i\theta})-h(z_0)|\frac{d\theta}{2\pi} \longrightarrow 0$$ as $r \to 0$ since h is cts.

Theorem: If $u(z)$ is a harmonic function on a domain D, and if the disk $\{|z-z_0|<\rho\}$ is contained in D, then $u(z_0)=\int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}$, $0<r<\rho$.



$$\Rightarrow \quad u(z_0)=\int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}$$

$\forall\, r<\rho$.

We say that a continuous function $h(z)$ on a domain D has the mean value property if $\forall z_0 \in D$, $h(z_0)$ is the average of its values over any small circle centered at z_0.

Theorem Restated: Harmonic $\Rightarrow$ Has MVP.

Chapter 3, Section 5: The Maximum Principle

Let $u(z)$ be a real-valued harmonic function on a domain D such that $u(z) \leq M$ $\forall z \in D$. If $u(z_0) = M$ for some $z_0 \in D$, then $u(z) = M$ $\forall z \in D$.

Suppose $u(z_1) = M$.

$$\Rightarrow u(z_1) = \int_0^{2\pi} u(z_1 + re^{i\theta}) \frac{d\theta}{2\pi}$$

$$\Rightarrow \int_0^{2\pi} [u(z_1) - u(z_1 + re^{i\theta})] \frac{d\theta}{2\pi} = 0. \quad , \quad 0 < r < \rho$$

Since $u(z_1) = M$, $u(z_1 + re^{i\theta}) \leq M$

$$\Rightarrow u(z_1) - u(z_1 + re^{i\theta} \geq 0.$$

$$\Rightarrow u(z_1) - u(z_1 + re^{i\theta}) = 0$$

$$\Rightarrow u(z_1) = u(z_1 + re^{i\theta}) = M. \quad \text{for } 0 \leq \theta \leq 2\pi, \ 0 < r < \rho.$$

$$\Rightarrow \underbrace{z_1 + re^{i\theta}}_{\text{disk}} \in \{z : u(z) = M\}.$$

$\Rightarrow$ $\{z : u(z) = M\}$ is open.

$\Rightarrow$ $\{z : u(z) < M\}$ is also open since $u(z)$ is cts.

$\Rightarrow$ One of these is ~~ope~~ D & the other is empty.

☆ Analytic ⇒ Harmonic ☆.

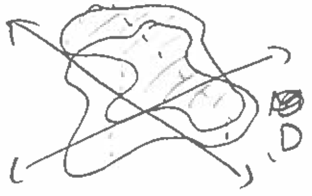
~~Exam~~

__Strict Maximum Principle__ (complex): Let h be $\overset{a}{\vee}$ bounded complex-valued harmonic function on a domain D. If $|h(z)| \leq M$ $\forall z \in D$, and $|h(z_0)| = M$ for some $z_0 \in D$, then $h(z)$ is constant on D

__Maximum Principle__: Let $h(z)$ be a complex-valued harmonic function on a bounded domain D s.t. $h(z)$ extends continuously to the boundary ∂D of D. If $|h(z)| \leq M$ $\forall z \in \partial D$, then $|h(z)| \leq M$ $\forall z \in D$.

Chapter 3, Section 5: The Maximum Principle

Strict Maximum Principle (Real Version): Let $u(z)$ be a al-valued harmonic function on a domain D s.t. $u(z) \leq M$ $\forall z \in D$. If $u(z_0) = M$ for some $z_0 \in D$, then $u(z) = M$ $\forall z \in D$.

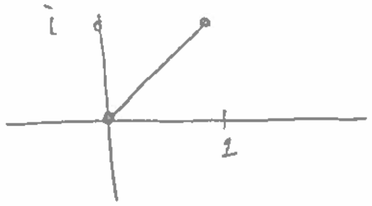


Chapter 4, Section 1: Complex Integration & Analyticity.

$dz = dx + i\,dy$. (I guess $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = dx + i\,dy$.

$$\Rightarrow \int_\gamma h(z)\,dz = \int_\gamma h(z)\,dx + i\int_\gamma h(z)\,dy.$$

example) Compute $\int_0^{1+i} z^2\,dz$ along the straight line segment from 0 to $1+i$



$$z(t) = 0(1-t) + (1+i)t$$

$$= t + it \quad , \quad 0 \leq t \leq 1.$$

$$\Rightarrow x(t) = t \Rightarrow dx = 1\,dt$$

$$y(t) = t \Rightarrow dy = 1\,dt.$$

$$\Rightarrow dz = dx + i\,dy = (1+i)dt$$

$$\int_0^{1+i} z^2 dz = \int_0^1 (t+it)^2 (1+i)\,dt = \int_0^1 (t^2 + 2it^2 - t^2)(1+i)dt$$

$$= \int_0^1 2it^2(1+i)dt = \int_0^1 (2it^2 - 2t^2)dt$$

$$= \int_0^1 2t^2(i-1) = \frac{2t^3}{3}(i-1)\Big]_0^1 = \frac{2}{3}(i-1).$$

example) $\oint\limits_{|z|=1} \frac{dt}{z}$

$z(\theta) = \cos\theta + i\sin\theta \quad , \quad 0 \le \theta \le 2\pi$

$x(\theta) = \cos\theta \qquad dx = -\sin\theta \, d\theta$

$y(\theta) = \sin(\theta) \qquad dy = \cos\theta \, d\theta$

$$= \int_0^{2\pi} \frac{-\sin\theta d\theta + i\cos\theta d\theta}{\cos\theta + i\sin\theta} = \int_0^{2\pi} \frac{i\,[\cos\theta + \cancel{i}\sin\theta]\,d\theta}{\cos\theta \cancel{+} i\sin\theta}$$

$$= 2\pi i$$

example) For $m \in \mathbb{Z}$ and $R > 0$,

$$\int\limits_{|z-z_0|=R} (z-z_0)^m dz = \int_0^{2\pi} R^m Re^{i\theta} i \, d\theta = \int_0^{2\pi} R^{m+1} e^{i\theta} i \, d\theta$$

$z = z_0 + Re^{i\theta}$

$dz = Re^{i\theta} i \, d\theta$

$$= \frac{iR^{m+1} e^{i\theta}}{i} \Big]_0^{2\pi} = 0 \text{ if } m \neq -1$$

$$= 2\pi i \text{ if } m = -1$$

<u>Def'n</u>: Arclength: $\overset{ds}{\overset{\shortparallel}{|dz|}} = \sqrt{(dx)^2 + (dy)^2}$

How to parametrize by arclength:

$z(t) = x(t) + iy(t)$

$dz = [x'(t) + iy'(t)]\,dt$

$$\int_\gamma h(z)\,|dz| = \int_\gamma h(z)\,ds = \int_a^b h(z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

example) The parametrization $z(\theta) = z_0 + Re$ of the circle $|z - z_0| = R$ can be used to derive the infinitesimal arclength of the circle in terms of θ.

$$x(\theta) = x_0 + R\cos\theta \quad \Rightarrow \quad dx = -R\sin\theta \, d\theta$$
$$y(\theta) = y_0 + R\sin\theta \qquad dy = R\cos\theta \, d\theta$$

$$\Rightarrow \quad |dz| = \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2}$$
$$= \sqrt{R^2(\sin^2\theta) + R^2(\cos^2\theta)} = R.$$

$$\Rightarrow \int_0^{2\pi} |dz| = 2\pi R = \text{length of circle}$$

Theorem: Suppose γ is a piecewise smooth curve. If $h(z)$ is a continuous function on γ, then

$$\left|\int_\gamma h(z)\,dz\right| \leq \int_\gamma |h(z)|\,|dz|.$$

So if $|h(z)| \leq M$ & $\int_\gamma |dz| = L \Rightarrow$

$$\left|\int_\gamma h(z)\,dz\right| \leq ML.$$

(ML- Estimate).

example) $\left|\int_0^{1+i} z^2 dz\right| \leq 2\sqrt{2}$

$|z^2| \leq |(1+i)^2| =$, $\int_0^{1+i} |dz| = \sqrt{2}$

$= |1+2i-1| = 2$

example) $\displaystyle\oint_{|z-z_0|=R} \frac{1}{z-z_0}\,dz$

$$\left|\frac{1}{z-z_0}\right| \le \frac{1}{R}$$

~~since max is on the boundary~~

$$\int_{|z-z_0|=R} |dz| = 2\pi R$$

$$\Rightarrow \left|\oint_{|z-z_0|=R} \frac{1}{z-z_0}\,dz\right| \le \frac{2\pi R}{R} = 2\pi$$

Chapter 4, Section 2: Fundamental Theorem of Calculus for Analytic Functions

Let $f(z)$ be a continuous function on a domain D. A function $F(z)$ on D is a "primitive" for $f(z)$ if $F(z)$ is analytic and $F'(z) = f(z)$.

"Primitive" is ~~the~~ an antiderivative?

Analogue to F ToC:

Theorem: If $f(z)$ is continuous on a domain D, and if $F(z)$ is a primitive for $f(z)$, then

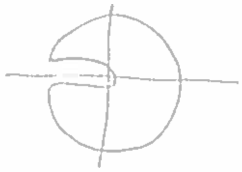
$$\int_A^B f(z)\,dz = F(B) - F(A),$$

where the integral can be taken over any path in D from A to B.

example) $\int_0^{1+i} z^2\,dz = \left.\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$.

example) The function $\frac{1}{z}$ does not have an analytic primitive on any domain ctg the unit circle.

But on $\int_{\mathbb{C}\setminus(-\infty,0]\,|z|=1} \frac{dz}{z} = \left.\operatorname{Log}(z)\right]_{z=-1-0i}^{z=\cancel{0}\,-1+0i} = i\pi - i(-\pi) = 2\pi i$



Theorem: Let D be a star-shaped domain, and let $f(z)$ be analytic on D. Then $f(z)$ has a primitive on D, and the primitive is unique up to adding a constant.

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta, \quad z \in D.$$

Chapter 4, Section 3:

Let $f(z) = u + iv$ be smooth

$$f(z)dz = (u+iv)(dx+idy) = udx + iudy + ivdx - vdy$$
$$= (u+iv)dx + (-v+iu)dy$$

When is $f(z)dz$ closed? Need $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$P = u + iv \qquad \frac{\partial P}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$
$$Q = -v + iu \qquad \frac{\partial Q}{\partial x} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}$$

So need $\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}$.

So need $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$.

So need CR-Eq'ns to hold.

So need analytic

Morera's Theorem: A ctsly diff'ble fnc. $f(z)$ on D is analytic $\Leftrightarrow$ $f(z)dz$ is closed.

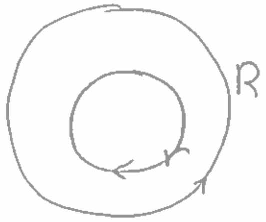
Cauchy's Theorem: Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic fnc. on D that extends smoothly to ∂D

$\Rightarrow \int_{\partial D} f(z) = 0$.

Why? Analytic $\Rightarrow$ $f(z)dz$ is closed

$$\Rightarrow \int_{\partial D} f(z)dz = \iint_D \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\, dx\, dy = 0$$

example) Let $f(z)$ be analytic on the annulus $D = \{r < |z| < R\}$.



By Cauchy's Theorem,

$$0 \overset{CT}{=} \int_{\partial D} f(z)dz = \oint_{|z|=R} f(z)dz - \oint_{|z|=r} f(z)dz$$

$$\Rightarrow \oint_{|z|=R} f(z)dz = \oint_{|z|=r} f(z)dz$$

Chapter 4, Section 4: The Cauchy Integral Formula

The Cauchy Integral Formula: Let D be a bounded domain With piecewise smooth boundary. If $f(z)$ is analytic on D, and $f(z)$ extends smoothly to ∂D $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw, \; z \in D.$$

What's the difference b/t this & CT?

CIF says: $f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw$

CT says $\int_{\partial D} f(z) = 0.$

heorem: Let D be bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to the boundary on D, then $f(z)$ has complex derivatives of all orders on D,

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\, dw.$$

Corollary: If $f(z)$ is analytic on a domain D, then $f(z)$ is infinitely diff'ble, and $f'(z), f''(z), \ldots$ are all analytic on D.

example) CIF for z^2.

$$\int_{|z|=2} \frac{z^2}{z-1}\,dz = 2\pi i \, z^2\Big]_{z=1} = 2\pi i$$

ex.) $$\oint_{|z|=2\pi} \frac{z^2\sin(z)}{(z-\pi)^3}\,dz = \frac{d^2}{dz^2}\left(z^2\sin(z)\right)\Big]_{z=\pi} \frac{2\pi i}{2!}$$

$f(z) = z^2\sin(z)$

$$= \frac{d}{dz}\left(2z\sin(z) + z^2\cos(z)\right)\Big]_{z=\pi} \frac{2\pi i}{2!}$$

$z = \pi$

$m = 2$

$$= \Big[2\sin(z) + 2z\cos(z) + 2z\cos(z) - z^2\sin(z)\Big]_{z=\pi} \cdot \frac{2\pi i}{2!}$$

$$= \Big[2\sin(z) + 4z\cos(z) - z^2\sin(z)\Big]_{z=\pi} \frac{2\pi i}{2!}$$

$$= 2\sin(\pi) + 4\pi\cos(\pi) - \pi^2\sin(\pi) \; \frac{2\pi i}{2!}$$

$$= \frac{-8\pi^2 i}{2} = -4\pi^2 i$$

example) $$\oint_{|z|=2} \frac{e^z}{z^2(z-1)}\,dz$$

By CT, $$\oint_{|z|=2} \frac{e^z}{z^2(z-1)}\,dz - \oint_{|z|=\varepsilon} \frac{e^z}{z^2(z-1)}\,dz - \oint_{|z-1|=\varepsilon} \frac{e^z}{z^2(z-1)} = 0.$$

$$= \oint_{|z|=2} \frac{e^z}{z^2(z-1)}\,dz - \oint_{|z|=\varepsilon} \frac{\frac{e^z}{z-1}}{z^2} - \oint_{|z-1|=\varepsilon} \frac{\frac{e^z}{z^2}}{z-1} = 0.$$

$$\Rightarrow \oint_{|z|=2} \frac{e^z}{z^2(z-1)}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{e^z}{z-1}\right)\Big]_{z=0} + \frac{2\pi i}{0!}\left(\frac{e^z}{z^2}\right)_{z=1} = \frac{2\pi i \cdot 1}{1} + 2\pi i e$$

$$= \pi i \left[\frac{(z-1)e^z - e^z}{(z-1)^2} \right]_{z=0} + 2\pi i e$$

$$2\pi i \left(\frac{-e^0 - e^0}{1} \right) + 2\pi i e = -4\pi i + 2\pi i e$$

4.5: Liouville's Theorem

Suppose $f(z)$ is analytic on the closed disk $\{|z-z_0|\leq \rho$

[i.e. $f(z)$ is analytic on some domain that contains $\{|z-z_0|\leq \rho\}$].

By the CIF,

$$f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{m+1}}\,dz$$

Let $z = z_0 + \rho e^{i\theta}$ $\quad 0 \leq \theta \leq 2\pi$

$dz = \rho i e^{i\theta} d\theta$

$$\Rightarrow \frac{m!}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{(\rho e^{i\theta})^{m+1}}\,\rho i e^{i\theta}\,d\theta = f^{(m)}(z_0).$$

$$\Rightarrow f^{(m)}(z_0) = \int_0^{2\pi} \frac{m!\,\rho^{-m} e^{-i\theta m} f(z_0+\rho e^{i\theta})\,d\theta}{2\pi}$$

$$= \int_0^{2\pi} \frac{m!\, f(z_0+\rho e^{i\theta})\,d\theta}{\rho^m e^{im\theta}\; 2\pi}$$

$$\Rightarrow |f^{(m)}(z_0)| \leq \frac{m!}{\rho^m}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,\frac{d\theta}{2\pi}$$

Theorem: (Cauchy's Estimates)

Suppose $f(z)$ is analytic for $|z-z_0| \le \rho$. If $|f(z)| \le M$ for $|z-z_0| = \rho$, then

$$|f^{(m)}(z_0)| \le \frac{m!}{\rho^m} M \;, \quad m \ge 0$$

Liouville's Theorem: Let $f(z)$ be an analytic function on the complex plane. If $f(z)$ is bounded, then $f(z)$ is constant.

Proof: Suppose $|f(z)| \le M$ $\forall z \in \mathbb{C}$. $f(z)$ is analytic on $\{|z-z_0| \le \rho\}$ $\forall z_0, \rho$.

$$\Rightarrow |f'(z_0)| \le \frac{M}{\rho}$$

Let $\rho \to \infty$ $\Rightarrow f'(z_0) = 0$

Since z_0 is arbitrary, $f'(z) = 0$ $\forall z \in \mathbb{C}$

$\Rightarrow f$ is constant.

Defn: Entire function is a function that's analytic on $\mathbb{C}$

ex.) Polynomials

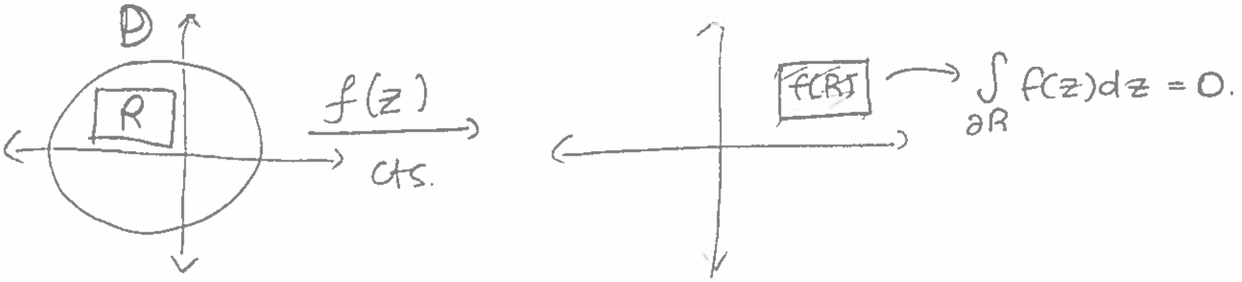
ex.) e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$

Non ex.) $\frac{1}{z}$, $\log z$, $\sqrt{z}$

4.6 : Morera's Theorem

Morera's Theorem: Let $f(z)$ be a continuous function on a domain

If $\int_{\partial D} f(z)dz = 0$ for every closed rectangle R contained in D w/ sides parallel to the coordinate axes, then $f(z)$ is analytic on D.



$\Rightarrow f(z)$ analytic in D

Proof: Assume D is a disk with center z_0. Define

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta.$$

$$\Rightarrow F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta = \int_{z}^{z+\Delta z} f(\zeta)d\zeta$$

TRICK

$$\overset{\downarrow}{=} \int_{z}^{z+\Delta z} [f(\zeta) + f(z) - f(z)]d\zeta = \int_{z}^{z+\Delta z} f(z)\,d\zeta + \int_{z}^{z+\Delta z} (f(\zeta) - f(z))d\zeta.$$

$$= f(z)\Delta z + \int_{z}^{z+\Delta z} (f(\zeta) - f(z))d\zeta$$

$$\Rightarrow \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z} \int_{z}^{z+\Delta z} (f(\zeta) - f(z))d\zeta \right| \leq 2M\varepsilon, \ |\Delta z| < \varepsilon$$

$\int_{z}^{z+\Delta z} |d\zeta| \leq 2|\Delta z|$ and since $f(z)$ is cts, $\max(|f(\zeta) - f(z)|)$ exists on the disk

Since f cts, $M_\varepsilon \to 0$ as $\varepsilon \to 0$

$\Rightarrow F'(z) = f(z)$.

$\Rightarrow$ Since $f(z)$ is cts $\Rightarrow F(z)$ is analytic.

$\Rightarrow$ Since $f(z)$ is the derivative of an analytic fnc, $f(z)$ is analytic

Theorem: Suppose $h(t,z)$ is a cts complex-valued fnc defined for $a \le t \le b$ and $z \in D$. $\forall t$ fixed, $h(t,z)$ is an analytic function of $z \in D$. Then $H(z) = \int_a^b h(t,z)\,dt$, $z \in D$ is analytic on D.

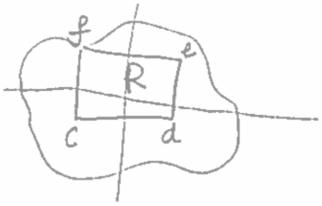
Proof:

First, we wts that $H(z)$ is continuous on D.

$\$$ that $z_n \to z$ $\Rightarrow$ since $h(t,z)$ is cts, $h(t,z_n) \to h(t,z)$ uniformly for $a \le t \le b$, so $\int_a^b h(t,z_n) \to \int_a^b h(t,z)$

$$\Rightarrow H(z_n) \to H(z).$$

Let R be a closed rectangle in D.



By CT, since $h(t,z)$ is analytic, $\int_{\partial R} h(t,z)dz = 0$

$$\Rightarrow \int_a^b \int_{\partial R} h(t,z)\,dz\,dt = 0.$$

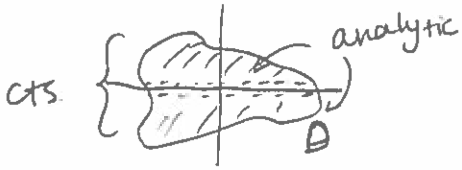
$$\Rightarrow \int_{\partial R} h(t,z)\,dz = \underbrace{\int h(\text{stuff})\,d\text{stuff}}_{\text{Side 1}} + \underbrace{\int h(\text{stuff})\,d\text{stuff}}_{\text{Side 2}} + \underbrace{\int h(\text{stuff})\,d\text{stuff}}_{\text{Side 3}} + \underbrace{\int h(\text{stuff})\,d\text{stuff}}_{\text{Side 4}}$$

Then since $h(t,z)$ cts. we can interchange integrals.

$$\Rightarrow 0 = \int_{\partial R}\int_a^b h(t,z)\,dt\,dz = \int_{\partial R} H(z)\,dz$$

$\Rightarrow$ By Morera's Theorem, $\int_{\partial R} H(z)dz$ is analytic.

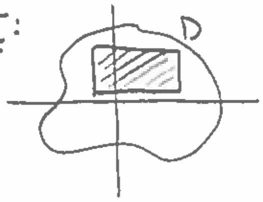
<u>Theorem</u>: Suppose that $f(z)$ is a continuous function on a domain D that is analytic on $D\setminus\mathbb{R}$, that is, on the part of D not lying on the real axis. Then $f(z)$ is analytic on D.



$\Big\} \Rightarrow f(z)$ is analytic on D.

<u>Proof</u>: We want to use Morera's Theorem again. Let R be a closed rectangle contained in D, w/ sides parallel to the coordinate axes.

<u>Case I</u>:

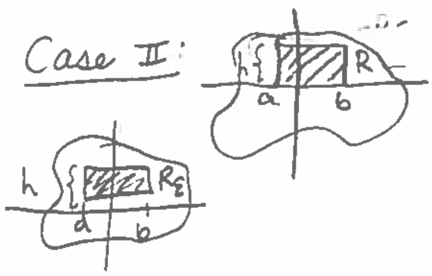


R does not meet $\mathbb{R}$.

$\Rightarrow f(z)$ is analytic on R.

$\Rightarrow$ By CT, $\int_{\partial R} f(z)dz = 0$.

<u>Case II</u>:



one Edge lies on $\mathbb{R}$.

For $\varepsilon > 0$ small, let R_ε be the closed rectangle in the UHP consisting of $z \in R$ s.t. $\text{Im} z \geq \varepsilon$.

By CT, $\int_{\partial R_\varepsilon} f(z)dz = 0$.

WTS $\int_{\partial R_\varepsilon} f(z)dz \longrightarrow \int_{\partial R} f(z)dz$.

The bottom edge of R_ε integrates $\int_a^b f(t+i\varepsilon)\,dt$.

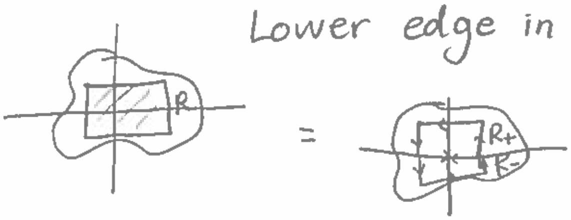
$f(t+i\varepsilon) \to f(t)$ uniformly, so $\int_a^b f(t+i\varepsilon)dt \to \int_a^b f(t)dt$

Top edge of R = Top edge of R_ε

Vertical sides of R = Vertical sides of $R_\varepsilon + \varepsilon$

$$\Rightarrow \int_{\partial R} f(z)dz = 0.$$

<u>Case III</u> : Top edge in UHP
Lower edge in LHP



$$\int_{\partial R} f(z)dz = \int_{\partial R_+} f(z)dz + \int_{\partial R_-} f(z)dz$$

$$= 0 \quad + \quad 0 = 0 \quad \checkmark$$

Chapter 4, Section 7: Goursat's Theorem

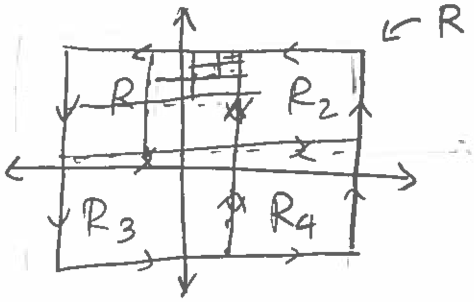
Goursat's Theorem: If $f(z)$ is a complex-valued function on a domain D s.t.

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$ exists at each point $z_0 \in D$,

then $f(z)$ is analytic on D

—useless thm—

Proof: Let R be a closed rectangle in D. we subdivide R into 4 equal subrectangles



Since $\int_{\partial R} f(z) = \int_{\partial R_1} f(z) + \int_{\partial R_2} f(z) + \int_{\partial R_3} f(z) + \int_{\partial R_4} f(z)$

⇒ ∃ at least one rectangle (R_1, WLOG) s.t.

$$\left|\int_{\partial R_1} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)\,dz\right|.$$

Now cut R_1 into 4 equal subrectangles and repeat the procedure.

⇒ we have a nested sequence of rectangles R_n s.t. $\left|\int_{\partial R_n} f(z)\right| \geq \frac{1}{4^n}\left|\int_{\partial R_{n-1}} f(z)\,dz\right| \geq \cdots \geq \frac{1}{4^n}\left|\int_{\partial R} f(z)\,dz\right|$

Since the R_n's are decreasing and have diameters $\downarrow 0$ the R_n's converge to some point $z_0 \in D$.

Since $f(z)$ is diff'ble at z_0, we have

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \le \varepsilon_n, \quad z \in R_n, \text{ where } \varepsilon_n \to 0 \text{ as } n \to \infty.$$

Let $L = |\partial R|$.

$\Rightarrow$ The length of ∂R_n is $\frac{L}{2^n}$

$$\Rightarrow |f(z) - f(z_0) - f'(z_0)(z-z_0)| \le \varepsilon_n |z - z_0| \le \frac{2\varepsilon_n L}{2^n}$$

$$\Rightarrow \left|\int_{\partial R_n} f(z)\,dz\right| = \left|\int_{\partial R_n} |f(z) - f(z_0) - f'(z_0)(z-z_0)\right| \le \frac{2\varepsilon_n L}{2^n} \cdot \frac{L}{2^n}$$

$\uparrow$

Idk why.

$$= \frac{\varepsilon_n L^2}{2^{2n-1}}$$

$$\Rightarrow \left|\int_{\partial R} f(z)\,dz\right| \le 4^n \left|\int_{\partial R^n} f(z)\,dz\right| \le \frac{2L^2}{\varepsilon^n} \to 0.$$

$\Rightarrow$ By Morera, $f(z)$ is analytic.

Chapter 4, Section 8: Complex Notation & Pompeiu's Formula

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right].$$

$$\frac{\partial f}{\partial x} - \frac{1}{2}\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial(iy)}\right] = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right].$$

In Chapter 2 we saw that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -if'(z). = \frac{\partial f}{\partial(iy)}$$

$$\Rightarrow f'(z) = \frac{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial(iy)}}{2} = \frac{\partial f}{\partial z}$$

(if $f(z)$ is analytic).

What about $\frac{\partial}{\partial \bar{z}}$? $f = u + iv$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} + ii\frac{\partial v}{\partial y}\right]$$

$$- \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

If $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ CR-Eq'ns!

So $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow$ CR $\Leftrightarrow$ Analytic

Thm: Let $f(z)$ be a ctsly differentiable function on a domain D. Then $f(z)$ is analytic $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$

If $f(z)$ is analytic, then $f'(z) = \frac{\partial f}{\partial z}$.

Rules for $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial \bar{z}}$

1) Linear: $\frac{\partial}{\partial z}(af + bg) = a\frac{\partial f}{\partial z} + b\frac{\partial g}{\partial z}$

$\frac{\partial}{\partial \bar{z}}(af + bg) = a\frac{\partial f}{\partial \bar{z}} + b\frac{\partial g}{\partial \bar{z}}$

2.) Leibniz: $\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z}g + \frac{\partial g}{\partial z}f$

$\frac{\partial}{\partial \bar{z}}(fg) = f\frac{\partial g}{\partial \bar{z}} + g\frac{\partial f}{\partial \bar{z}}$

3.) $\frac{\partial f}{\partial \bar{z}} = \overline{\frac{\partial \bar{f}}{\partial z}}$, $\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$

We'll derive the complex form of the formula for the tangent vector to a curve.

Let $f(z)$ be a smooth function, and let $\gamma(t)$ be a smooth curve terminating at $\gamma(0) = z_0$.

By Taylor's series, $f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)(z-z_0) + \frac{\partial f}{\partial \bar z}(z_0)(\overline{z-z_0}) + O(|z-z_0|^2)$

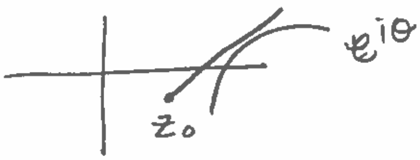
So $$f(\gamma(t)) = f(\gamma(0)) + \frac{\partial f}{\partial z}(\gamma(0))(\gamma(t)-\gamma(0)) + \frac{\partial f}{\partial \bar z}(\gamma(0))(\overline{\gamma(t)-\gamma(0)}) + O(|\gamma(t)-\gamma(0)|^2)$$

$$\Rightarrow \frac{f(\gamma(t))-f(\gamma(0))}{t-0} = \frac{\frac{\partial f}{\partial z}(\gamma(0))(\gamma(t)-\gamma(0))}{t-0} + \frac{\partial f}{\partial \bar z}\frac{(\gamma(0))(\overline{\gamma(t)-\gamma(0)})}{t-0} + \frac{O(|\gamma(t)-\gamma(0)|^2)}{t-0}$$

$$\Rightarrow \boxed{(f\circ\gamma)'(0) = \frac{\partial f}{\partial z}(\gamma(0))\cdot\gamma'(0) + \frac{\partial f}{\partial \bar z}(\gamma(0))\,\overline{\gamma'(0)}}$$

Theorem: Let $f(z)$ be a ctsly diff'ble function on a domain D. Suppose that the gradient of $f(z)$ does not vanish at any point of D, and that $f(z)$ is conformal. Then $f(z)$ is analytic on D, and $f'(z) \neq 0$ on D.

Proof: Fix $z_0 \in D$. Consider $\gamma(t) = z_0 + te^{i\theta}$, $0 \le t \le \varepsilon$



$$(f\circ\gamma)'(0) = \frac{\partial f}{\partial z}(z_0)e^{i\theta} + \frac{\partial f}{\partial \bar{z}}(z_0)e^{-i\theta}$$

Since $f(z)$ is conformal,

$$\arg(f\circ\gamma'(0)) - \arg(\gamma'(0)) = C$$ constant, independent of θ.

$$\Rightarrow \frac{f\circ\gamma'(0)}{\gamma'(0)} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)e^{-2i\theta}$$ is indep. of θ.

This only occurs when $\frac{\partial f}{\partial \bar{z}} = 0$.

So $f(z)$ is analytic on D.

Since $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \neq (0,0)$

$$\Rightarrow f'(z) = \frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] \neq 0.$$

Theorem: If D is a bounded domain in the complex plane with piecewise smooth boundary, and if $g(z)$ is a smooth function on $D\cup\partial D$, then

$$\int_{\partial D} g(z)\,dz = 2i\iint_D \frac{\partial g}{\partial \bar{z}}\,dx\,dy.$$

Proof.

$$\int_{\partial D} g(z)\,dz = \int_{\partial D} g(z)\,dx + i\int_{\partial D} g(z)\,dy \overset{\text{Green's Theorem}}{=} \iint_D \left(i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\right)dx\,dy$$

$P = g(z)$

$Q = g(z)i$

$$\int P\,dx + Q\,dy = \iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$\frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right)$ (why? $\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + \cancel{i}\frac{\partial g}{\partial (iy)}\right)$

$\Rightarrow \frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - \frac{i\partial g}{\partial (iy)}\right)$

$\Rightarrow \frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right)$)

$$\Rightarrow \iint_D \left(i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy = \iint_D 2i\frac{\partial g}{\partial \bar{z}}\,dx\,dy.$$

Note: If $g(z)$ analytic on $D \Rightarrow \frac{\partial g}{\partial \bar{z}} = 0 \Rightarrow$

$\int_{\partial D} g(z)\,dz = 0$ which is CT.

Pompeiu's Formula: Suppose D is a bounded domain w/ piecewise smooth boundary. If $g(z)$ is a smooth complex-valued function on $D \cup \partial D$, then

$$g(w) = \frac{1}{2\pi i}\int_{\partial D}\frac{g(z)}{z-w}\,dz - \frac{1}{\pi}\iint_D \frac{\partial g}{\partial \bar{z}}\,\frac{1}{z-w}\,dx\,dy,\ w \in D.$$

-Proof omitted-

Chapter 5: Power Series

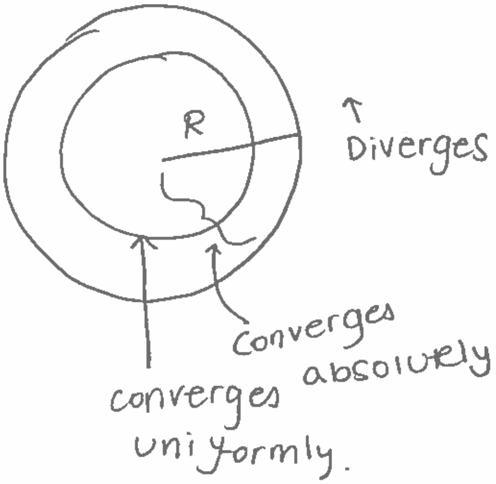
Section 3: Power Series

A power series centered at z_0 is a series of the form

$$\sum_{k=0} a_k(z-z_0)^k.$$

Theorem: Let $\sum_k a_k z^k$ be a power series. Then there is R, $0 \leq R \leq \infty$, s.t. $\sum a_k z^k$ does not converge for $|z| > R$
converges abs. for $|z| < R$

For each fixed r satisfying $r < R$, the series $\sum a_k z^k$ converges uniformly for $|z| \leq r$.



General Case (i.e. not centered at zero):

Power series is $\sum a_k(z-z_0)^k$

Domain of convergence: $|z-z_0| < R$.

Diverges if: $|z-z_0| > R$.

Unknown: $|z-z_0| = R$.

example) $\sum z^k$.

If $|z|<1 \Rightarrow$ our series converges.

If $|z|\geq 1 \Rightarrow$ our series diverges.

$\Rightarrow R = 1$.

example) $\sum \frac{z^k}{k^2}$ converges uniformly for $|z|\leq 1$. Why?

[Reminder: Weierstrass M-Test

$\$$ $M_k \geq 0$ and $\sum M_k$ converges.

If $|g_k(x)|\leq M_k \ \forall x\in E$, then $\sum g_k(x)$ converges unif. on E.]

$\left|\frac{z^k}{k^2}\right| \leq \frac{1}{k^2}$, where $\sum\frac{1}{k^2}$ converges, so $\frac{z^k}{k^2}$ converges.

If $r>1$, then $\frac{r^k}{k^2}\to\infty$ as $k\to\infty$, so the series does not converge for $|z|>1$.

$\Rightarrow R=1$.

example) The series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}z^{2k} = 1 + \frac{-1}{2}z^2 + \frac{1}{4}z^4 + \frac{-1}{8}z^6 \dots$

Geometric series looks like $\sum_{k=0}^{\infty} z^k$

So let $w = -\frac{z^2}{2}$. $\Rightarrow \sum_{k=0}^{\infty}\left(\frac{-z^2}{2}\right)^k$.

So converges when $\left|-\frac{z^2}{2}\right|<1$

when $|z^2|<2$

when $|z|<\sqrt{2}$.

So $R=\sqrt{2}$.

The series converges to $\frac{1}{1-\left(-\frac{z^2}{2}\right)} = \frac{1}{1+\frac{z^2}{2}} = \frac{2}{2+z^2}$

example) The Series $\Sigma k^k z^k$ has R.O.C. $R=0$.

If $z=0 \Rightarrow \Sigma k^k z^k \to 0.$

If $z \neq 0 \Rightarrow k^k z^k \to \infty$

example) $\Sigma \frac{z^k}{k^k}$ has r.o.c $R=\infty$.

It converges everywhere.

<u>Theorem</u>: Suppose $\Sigma a_k z^k$ is a power series w/ R.O.C $R>0$.
Then $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $|z|<R$ is analytic.
The derivatives of $f(z)$ are obtained by differentiating the series term by term.

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2}, \ |z|<R.$$

The coefficients of the series are:

$$a_0 = \frac{1}{0!} f^{(0)}(0) = 1 \cdot \sum_{k=0}^{\infty} a_k (0)^k = a_0 \cdot 0^0 + 0$$
$$= a_0$$

$$a_1 = \frac{1}{1!} \sum_{k=1}^{\infty} k a_k 0^{k-1} = 1(1 \cdot a_1 0^{1-1} + 0)$$
$$= a_1$$

$$a_2 = \frac{1}{2!} \sum_{k=2}^{\infty} k(k-1) a_k 0^{k-2} = \frac{1}{2}(2 a_2 0^0 + 0)$$
$$= a_2$$

$\vdots$

So $a_k = \frac{1}{k!} f^{(k)}(0)$, $k \geq 0$.

example) By differentiating $\frac{1}{1-z}$ as a geometric series, we obtain a power series representation of $\frac{1}{(1-z)^2}$.

Since $\sum z^k = \frac{1}{1-z}$

$$\Rightarrow \frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1}$$

Let $m=k-1$.

$\Rightarrow k=m+1$

$$= \sum_{m=0}^{\infty} (m+1) z^m, \quad |z|<1$$

Since power series converge uniformly on $|z|<r<R$, a power series can be integrated term by term too!!

So if $\sum a_k z^k$ has radius of convergence R, then

$$\int_0^z \sum (a_k \zeta^k)\, d\zeta \underset{\substack{\uparrow \\ \text{can} \\ \text{switch} \\ \Sigma \,\&\, \int}}{=} \sum a_k \int_0^z \zeta^k\, d\zeta = \sum a_k \left(\frac{\zeta^{k+1}}{k+1} \right)\Big]_0^z = \sum a_k \frac{z^{k+1}}{k+1}$$

for $|z|<R$.

example) We want to integrate $\sum z^k = \frac{1}{1-z}$ term by term.

$$\int_0^z \frac{1}{1-\zeta} = \int_0^z \sum_{k=0}^{\infty} \zeta^k\, d\zeta \text{ for } |z|<1.$$

$$\Rightarrow -\text{Log}(1-z) = \sum_{k=0}^{\infty} \int_0^z \zeta^k\, d\zeta \quad \text{for } |z|<1$$

$$= \sum_{k=0}^{\infty} \frac{\zeta^{k+1}}{k+1}\Big]_0^z \quad \text{for } |z|<1$$

$$= \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = z + \frac{z^2}{2} + \dots \quad \text{for } |z|<1.$$

Let $w=1-z$. $\Rightarrow$ $z=1-w$

$$\Rightarrow -\mathrm{Log}(w) = \sum_{k=0}^{\infty} \frac{(1-w)^{k+1}}{k+1}$$

$$\Rightarrow \mathrm{Log}(w) = \sum_{k=0}^{\infty} \frac{(-1)(1-w)^{k+1}}{k+1} \underset{\substack{\uparrow \\ \text{Let} \\ a=k+1}}{=} \sum_{a=1}^{\infty} \frac{(-1)(1-w)^{a}}{a} = \sum_{a=1}^{\infty} \frac{(-1)(-1)^{a}(w-1)^{a}}{a}$$

$$= \sum_{a=1}^{\infty} \frac{(-1)^{a+1}(w-1)^{a}}{a}$$

$$, |w-1| < 1.$$

Theorem: $R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$ if this limit exists.

example) $\sum kz^k$, find the radius of convergence

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{k}{k+1} \rightarrow 1$$

So R.O.C. = 1

example) $\sum \frac{z^k}{k!}$, find the radius of convergence.

$$\left|\frac{a_k}{a_{k+1}}\right| = \left|\frac{\frac{1}{k!}}{\frac{1}{(k+1)!}}\right| = \left|\frac{(k+1)!}{k!}\right| = k+1 \rightarrow \infty$$

So r.o.c. is ∞

Theorem: $R = \dfrac{1}{\lim \sqrt[k]{|a_k|}}$ if the limit exists.

example) $\sum kz^k$

$$\Rightarrow \sqrt[k]{k} \rightarrow 1$$

$$\Rightarrow \frac{1}{\lim \sqrt[k]{k}} = 1 \quad \Rightarrow \text{R.O.C.} = 1$$

*More general Root Test ☆ Cauchy-Hadamard

Every sequence has a limsup, so

$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}$. Should always be able to find this (in theory).

example) $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k} = 1 - \frac{z^2}{2} + \frac{z^4}{2^2} - \frac{z^6}{2^3} + \dots$

$$= \begin{cases} 0 & \text{if } a \text{ is odd} \\ \frac{(-1)^{\frac{a}{2}} z^a}{2^{a/2}} & \text{if } a \text{ is even.} \end{cases}$$

$$\Rightarrow \sqrt[a]{|a_k|} = \begin{cases} 0 & \text{if } a \text{ is odd} \\ \frac{1^{1/2}}{2^{1/2}} & \text{if } a \text{ is even} \end{cases}$$

$$= \begin{cases} 0 & \text{if } a \text{ is odd} \\ \frac{1}{\sqrt{2}} & \text{if } a \text{ is even.} \end{cases}$$

$\Rightarrow \quad R = \sqrt{2}$.

5.4: Power Series Expansion of an Analytic Function

☆ **Theorem:** Suppose that $f(z)$ is analytic for $|z-z_0|<\rho$. Then $f(z)=\sum_{k=0}^{\infty} a_k(z-z_0)^k$, $|z-z_0|<\rho$, where

$$a_k=\frac{1}{k!}f^{(k)}(z_0),\ k\geq 0$$

and where the power series has R.O.C. $R\geq\rho$.

For any fixed r, $0<r<\rho$, we have

$$a_k=\frac{1}{2\pi i}\oint_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta,\ k\geq 0$$

Further, if $|f(z)|\leq M$ for $|z-z_0|=r$, then

$$|a_k|\leq\frac{M}{r^k},\ k\geq 0.$$

example) The function e^z is analytic everywhere, so can write as power series.

$$a_k=\frac{1}{k!}\frac{d^k}{dz^k}(e^z)\Big]_{z=0}=\frac{1}{k!}e^z\Big]_{z=0}=\frac{1}{k!}$$

$$\Rightarrow e^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$$

R.O.C. $=\infty$

example) sinz is also entire, and cosz is entire.

$$\sin(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\cos(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}$$

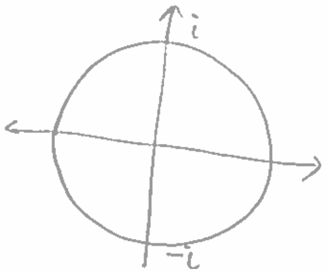
Corollary: Suppose $f(z)$ and $g(z)$ are analytic for $|z-z_0|<r$. If $f^{(k)}(z_0) = g^{(k)}(z_0)$ for $k \geq 0$, then $f(z) = g(z)$ for $|z-z_0|<r$

Corollary: Suppose $f(z)$ is analytic at z_0, with po series expansion $f(z) = \sum a_k (z-z_0)^k$. Then the r.o.c. is the largest number R s.t. $f(z)$ extends to be analytic on $\{|z-z_0|<R\}$.

example) $\frac{1}{1+x^2}$ around $x=0$:

$\frac{1}{1+z^2}$ has singularities at $z = \pm i$

$\Rightarrow$ R.O.C. = 1

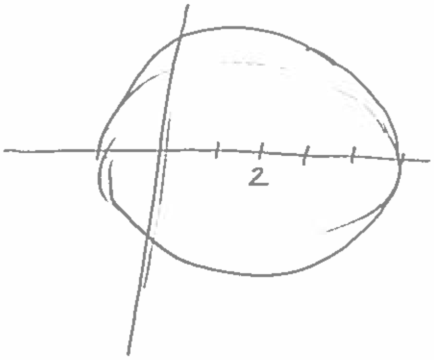


example) Consider the power series expansion of $f(z) = \frac{z^3-1}{z^2-1}$ about $z=2$, $f(z) = \sum a_k (z-2)^k$

$f(z)$ is analytic everywhere except at $z = \pm 1$.

But actually, $f(z) = \frac{(z^2+z+1)\cancel{(z-1)}}{(z+1)\cancel{(z-1)}} = \frac{z^2+z+1}{z+1}$

So actually analytic everywhere except $z=-1$.



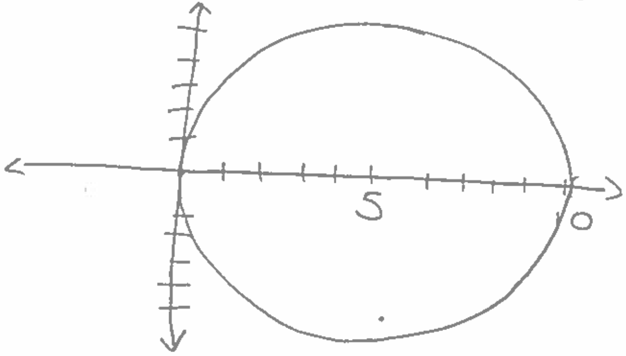
$\Rightarrow$ R.o.C. $= 3$

example) The r.o.c. of $\sum a_k (z-5)^k$ of the fnc $\frac{\operatorname{Log} z}{z-1}$ about $z=5$ is $R=5$.

$$\operatorname{Log} z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \dots, \quad |z-1| < 1$$

$$\Rightarrow \frac{\operatorname{Log} z}{z-1} = 1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{3} - \frac{(z-1)^3}{4} + \dots$$

As $z \to 0$, $\frac{\operatorname{Log} z}{z-1} \to \infty$, so this is the only genuine singularity.



Chapter 5, Section 5: Power Series Expansion at Infinity

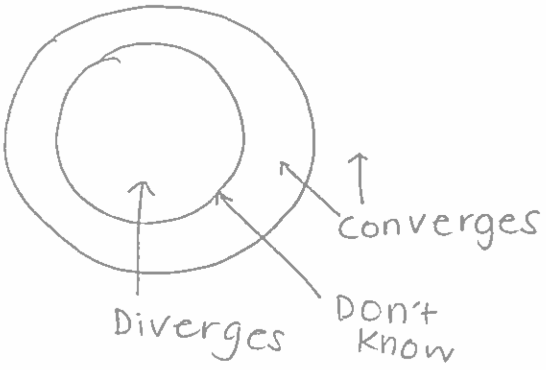
We say $f(z)$ is analytic at $z=\infty$ if $g(w)=f(\frac{1}{w})$ is analytic at $w=0$.

If $f(z)$ is analytic at ∞, then $g(w)=f(\frac{1}{w})$ has a power series expansion centered at $w=0$:

$$g(w) = \sum_{k=0}^{\infty} b_k w^k = b_0 + b_1 w + b_2 w^2 + \dots \quad , |w|<\rho$$

$$\Rightarrow f(w) = g(\tfrac{1}{w}) = \sum_{k=0}^{\infty} \frac{b_k}{w^k} = b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \dots , |w| > \frac{1}{\rho}$$

The series converges absolutely for $|w| > \frac{1}{\rho}$, and for any $r > \frac{1}{\rho}$, it converges uniformly for $|z| \geq r$.



example) If $n \geq 0$, the function $f(z) = \frac{1}{z^n}$ is analytic at ∞, since $g(w) = w^n$ is analytic at $w=0$.

example) $f(z) = \frac{1}{z^2+1}$

$$g(w) = \frac{1}{\frac{1}{w^2}+1} = \frac{w^2}{1+w^2}$$

Is this analytic at 0? $g'(w) = \frac{(1+w^2)2w - w^2 2w}{(1+w^2)^2}$

$= \frac{2w}{(1+w^2)^2}$ is analytic at $w=0$.

Remember, $\frac{1}{1+w^2} = \sum_{k=0}^{\infty}(-1)^k w^{2k}$

$$g(w) = w^2 \sum_{k=0}^{\infty}(-1)^k w^{2k} = w^2 - w^4 + w^6 - w^8 + \ldots$$

$$\Rightarrow f(z) = \left(\frac{1}{z}\right)^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{z^{2k}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{z^{2k+2}} = \sum_{a=1}^{\infty} \frac{(-1)^{a-1}}{z^{2a}} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8} + \ldots,$$

Let $a = k+1$

$|z| > 1$.

OR: $\frac{1}{1+z^2} = \frac{1}{z^2}\left(\frac{1}{1+\frac{1}{z^2}}\right) = \frac{1}{z^2}\left(\sum_{k=0}^{\infty}(-1)^k z^{-2k}\right)$

The Zeroes of an Analytic Function

Let $f(z)$ be analytic at z_0, and suppose $f(z_0)=0$, but $f(z)\neq 0$.

We say $f(z)$ has a zero of order N at z_0 if

$$f(z_0)=f'(z_0)=\ldots=f^{(N-1)}(z_0)=0,\ f^{(N)}(z_0)\neq 0.$$

If our function is a power series, then $f(z)$ has zero of order $N \iff$

$$f(z)=a_N(z-z_0)^N+a_{N+1}(z-z_0)^{N+1}+\ldots,$$

where $a_N\neq 0$.

$$\Rightarrow f(z)=(z-z_0)^N\left[a_N+a_{N+1}(z-z_0)+\ldots\right]$$

$$=(z-z_0)^N h(z),$$ where $h(z)$ is analytic at z_0 and $h(z_0)\neq 0$, then the leading term in the power series for $f(z)$ is $h(z_0)(z-z_0)^N$, and $f(z)$ has a zero of order N at z_0.

<u>Def'n</u>: Zero of order one is called a "simple zero"

<u>Def'n</u>: Zero of order two is called a "double zero"

example) The zeroes of $\sin(z)$ are the points $n\pi, n\in\mathbb{Z}$ but $\cos(n\pi)=1\ \forall n\in\mathbb{Z}$, so all the zeroes of $\sin(z)$ are simple.

example) $f(z) = (z-z_0)^n$

Has one zero, $z=z_0$ of order n.

example) $\sin(z) = -(z-\pi) + \frac{1}{3!}(z-\pi)^3 - \frac{1}{5!}(z-\pi)^5 + \dots$

$$\Rightarrow \sin(z) + z - \pi = \frac{1}{3!}(z-\pi)^3 - \frac{1}{5!}(z-\pi)^5 + \dots$$

$$= (z-\pi)^3 \left[\frac{1}{3!} - \frac{1}{5!}(z-\pi)^5 + \dots \right]$$

So $\sin(z)+z-\pi$ has a zero of order 3 at $z=\pi$.

$$\Rightarrow \frac{\sin(z)}{z-\pi} = -1 + \frac{1}{3!}(z-\pi)^2 - \frac{1}{5!}(z-\pi)^4 + \dots$$

$$\Rightarrow \frac{\sin(z)}{z-\pi} + 1 = \frac{1}{3!}(z-\pi)^2 - \frac{1}{5!}(z-\pi)^4 + \dots$$

has zero of order 2 at $z=\pi$.

Idea: Suppose $f(z) = a_n(z-z_0)^n + \dots$ has zero of order n at z_0 & $g(z) = b_m(z-z_0)^m + \dots$ has zero of order m at z_0 $\Rightarrow$ $f(z)g(z) = a_n b_m (z-z_0)^{n+m} + \dots$ has a zero of order $m+n$ at z_0.

Idea at ∞: If $f(z)$ is analytic at ∞ and $f(\infty)=0$, $f(z)$ has a zero of order N at $z=\infty$ if $g(w) = f(\frac{1}{w})$ has a zero of order N at $w=0$.

$\Rightarrow f(z) = \frac{b_N}{z^N} + \frac{b_{N+1}}{z^{N+1}} + \dots$, $|z|>R$, where $b_N \neq 0$

example) The function $\frac{1}{1+z^2}$ has a double zero at ∞.

$$= \frac{1}{z^2} - \frac{1}{z^4} + \dots = \frac{1}{z^2}\left(1 - \frac{1}{z^2} + \dots\right)$$

example) $\frac{1}{(z-z_0)^n}$ has a zero of order n at ∞.

Weird

Theorem: If D is a domain, and $f(z)$ is an analytic function on D that isn't zero (i.e. $f(z) \neq 0$), then the zeroes of $f(z)$ are isolated.

Theorem (Uniqueness Principle): If $f(z)$ and $g(z)$ are analytic on a domain D, and if $f(z)=g(z)$ for z belonging to a set that has a nonisolated point, then $f(z)=g(z)$ $\forall z \in D$.

example) Let $f(z) = \sin^2 z + \cos^2(z)$

$g(z) = 1$.

$\sin^2 z + \cos^2(z) = 1$ $\forall z \in \mathbb{R}$, where $\mathbb{R}$ consists of nonisolated pts.

$\Rightarrow \sin^2(z) + \cos^2(z) = 1$ $\forall z \in \mathbb{C}$ since f & g are entire fncs.

Theorem: Let D be a domain, and let $E \subset D$ that has a non-isolated point. Let $F(z,w)$ be a function defined for $z, w \in D$ such that $F(z,w)$ is analytic in z for each fixed $w \in D$ and analytic in w for each fixed $z \in D$.
If $F(z,w) = 0$ whenever $z, w \in E$, then $F(z,w) = 0$ $\forall z, w \in D$.

example) $F(z,w) = e^{z+w} - e^z e^w$ is entire.

When $z, w \in \mathbb{R}$, then $F(z,w) = 0$.

$\Rightarrow F(z,w) = 0 \quad \forall z, w \in \mathbb{C}$

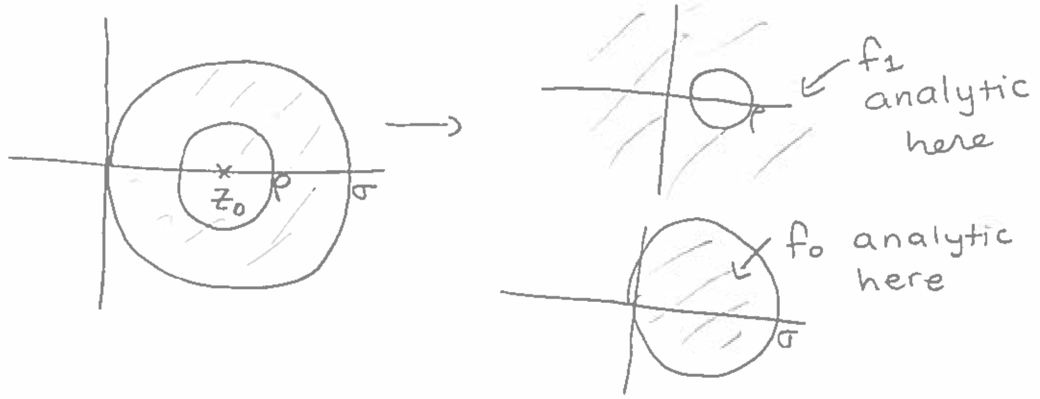
$\Rightarrow e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$.

Chapter 6, Section 1: Laurent Decomposition

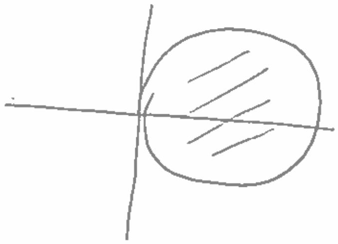
Theorem: (Laurent Decomposition): Suppose $0 \le \rho < \sigma \le \infty$, and suppose $f(z)$ is analytic for $\rho < |z-z_0| < \sigma$. Then

$$f(z) = f_0(z) + f_1(z), \text{ where } f_0(z) \text{ is analytic}$$

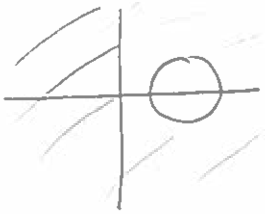
for $|z-z_0| < \sigma$ and $f_1(z)$ is analytic for $|z-z_0| > \rho$ and at ∞. If we normalize the decomposition so that $f_1(\infty) = 0$, then the decomposition is unique.



If $f(z)$ is analytic for $|z-z_0| < \sigma$, then $f(z) = f_0(z) + 0$.



If $f(z)$ is analytic for $|z-z_0| > \rho$ and vanishes at ∞, $f(z) = 0 + f_1(z)$



Now we want to show uniqueness.

Liouville's Theorem says that if $f(z)$ is an analytic function on $\mathbb{C}$ and if $f(z)$ is bounded, then $f(z)$ is constant.

Suppose $f(z) = g_0(z) + g_1(z)$
$= f_0(z) + f_1(z)$.

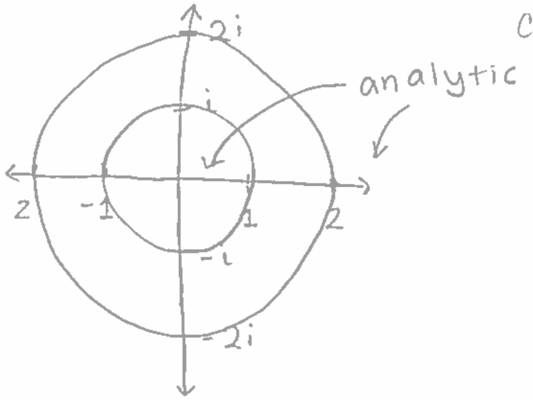
$\Rightarrow g_0(z) - f_0(z) = f_1(z) - g_1(z)$, $\rho < |z - z_0| < \sigma$.

Define $h(z) = \begin{cases} g_0(z) - f_0(z) & \text{in } \{|z - z_0| < \sigma\} \\ f_1(z) - g_1(z) & \text{in } \{|z - z_0| > \rho\} \end{cases}$

$h(z)$ is entire & $h(z) \to 0$ as $z \to \infty$.

By Liouville, $h(z) = 0$ ✓

example) $f(z) = \dfrac{1}{(z-1)(z-2)}$ has 3 Laurent decompositions centered at zero.



Only have to worry about $\{1 < |z| < 2\}$.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

$\Rightarrow Az - 2A + Bz - B = 1$

$\Rightarrow z(A+B) + (-2A - B) = 1$

$\Rightarrow A+B=0 \quad \Rightarrow \quad -A=1 \quad \Rightarrow \quad B=1$

$-2A-B=1 \qquad \Rightarrow A=-1$

$\frac{1}{z-2} \cancel{+} \frac{-1}{z-1}$ in the annulus

$\swarrow$ analytic for $|z|<2$

$\uparrow$ analytic for $|z|>1$.

$\Rightarrow f_0(z) = \frac{1}{z-2}, \quad f_1(z) = \frac{-1}{z-1}$

Suppose $f(z) = f_0(z) + f_1(z)$ is the Laurent decomposition for a function analytic for $\rho < |z-z_0| < \sigma$.

$$f_0(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k, \quad |z-z_0| < \sigma.$$

& f_0 converges absolutely, and for $s < \sigma$ it converges uniformly for $|z-z_0| \le s$

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k, \quad |z-z_0| > \rho.$$

again, converges abs, unif, ...

$$\Rightarrow f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k, \quad \rho < |z-z_0| < \sigma$$

$\nearrow$ Laurent series expansion of $f(z)$ wrt $\rho < |z-z_0| < \sigma$.

which converges abs.

〃 unif for $r \le |z-z_0| \le S$.

Theorem: (Laurent Series Expansion):

Suppose $f(z)$ is analytic for $0 \le \rho < |z-z_0| < \sigma \le \infty$. Then $f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ that converges absolutely at each point of the annulus, uniformly on each subannulus $r \le |z-z_0| \le s$, where $\rho<r<s<\sigma$. The coefficients are uniquely determined by $f(z)$, and they are given by $a_n = \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ for any fixed $r, \rho<r<\sigma$

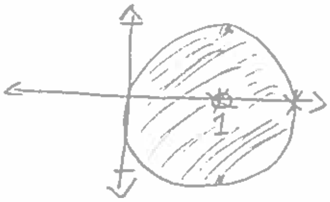
ex.) We want to expand $f(z)=\frac{1}{(z-1)(z-2)}$ in a Laurent series centered at $z=0$ and converging in $\{1<|z|<2\}$

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-z/2} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k = -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\dots\right)$$

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}+\frac{1}{z^2}+\dots$$

$$\Rightarrow f(z) = \sum_{k=-\infty}^{-1} -z^k + \sum_{k=0}^{\infty}\frac{-1}{2^{k+1}}\,z^k.$$

example) The function $f(z)=\frac{1}{(z-1)(z-2)}$ can also be expanded in a Laurent series centered at $z=1$, convergent in the punctured disk $\{0<|z-1|<1\}$.



$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\frac{1}{z-2} = \frac{1}{1-(z-1)} = -\sum_{k=0}^{\infty}(z-1)^k, \quad |z-1|<1$$

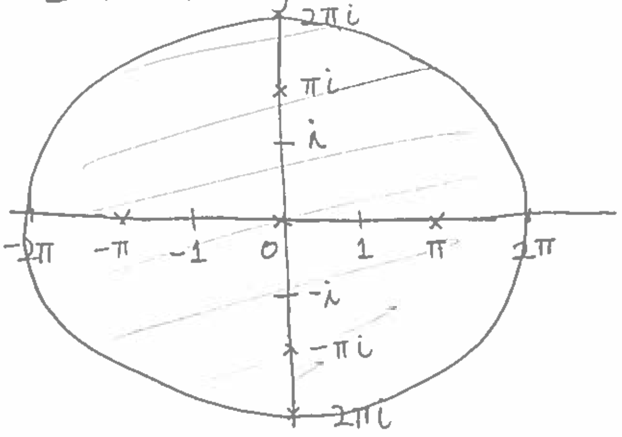
$$\Rightarrow \frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - (z-1)^3 - \cdots$$

$$= -\sum_{k=-1}^{\infty} (z-1)^k \quad , \quad 0<|z-1|<1$$

exercise) Consider the Laurent series for $f(z) = \frac{z^2-\pi^2}{\sin z}$ that is centered at 0 and that converges for $|z|=1$. What is the largest open set on which the series converges?

$$f(z) = \frac{(z-\pi)(z+\pi)}{\sin(z)}$$

zeroes of $\sin(z)$: $n\pi$, $n \in \mathbb{Z}$



π & -π are removable.

Largest open disk: $\{0<|z|<2\pi\}$

Chapter 6, Section 2: Isolated Singularities of an Analytic Function

A point z_0 is an isolated singularity of $f(z)$ if $f(z)$ is analytic in $\{0<|z-z_0|<r\}$

non-examples) $\sqrt{z}$, $\log(z)$ don't have isolated singularities at $z=0$.

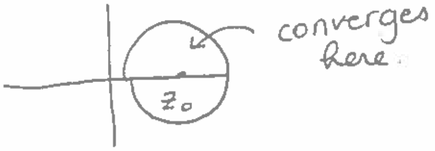
Idea: Suppose $f(z)$ has an isolated singularity at z_0
Then $f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$, $0<|z-z_0|<r$

Types of Isolated Singularities:

1.) Removable Singularity

$$f(z)=\sum_{k=0}^{\infty} a_k(z-z_0)^k \quad , 0<|z-z_0|<r$$

If we define $f(z_0)=a_0$, the function $f(z)$ becomes analytic on the entire disk $\{|z-z_0|<r\}$



example) $\frac{\sin(z)}{z}$ has an isolated singularity at $z=0$.

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\Rightarrow \frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

If we let $\frac{\sin(0)}{0} = 1 \Rightarrow \frac{\sin(z)}{z}$ becomes entire.

What is removable then? Pretty much if we can get a limit

$$\lim_{z\to 0} \frac{\sin(z)}{z} = \lim_{z\to 0} \cos(z) = 1$$

Theorem: (Riemann's Theorem on Removable Singularities)

Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bdd near z_0, then $f(z)$ has a removable singularity at z_0.

2.) The isolated singularity at z_0 is called a pole if $f(z) = \sum_{k=-N}^{\infty} a_k(z-z_0)^k$ for $N>0$.

$$= \underbrace{\sum_{k=-N}^{-1} a_k(z-z_0)^k}_{\text{Principal part}} + \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

ex.) $\frac{1}{z}$ has simple pole at $z=0$

$\frac{1}{(z-i)^2}$ has double pole at $z=i$

Theorem: Let z_0 be an isolated singularity of $f(z)$ Then z_0 is a pole of $f(z)$ of order $N \Leftrightarrow f(z) = \frac{g(z)}{(z-z_0)^N}$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Theorem: Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order $N \Leftrightarrow \frac{1}{f(z)}$ is analytic at z_0 and has a zero of order N.

☆ Basically, if you get $\frac{1}{0}$, then it's a pole. ☆

example) The function $\frac{1}{\sin z}$ has poles at each of the zeroes of $\sin(z)$ are simple, they are simple poles for $\frac{1}{\sin z}$.

Def'n: $f(z)$ is meromorphic on a domain D if $f(z)$ is analytic on D except possibly at poles.

★ Sums & Quotients of meromorphic functions are meromorphic ← as long as denominator $\neq 0$.

example) $\frac{1}{\sin(z)}$ is meromorphic on $\mathbb{C}$.

(only poles at $z = n\pi$.)

example) Let $R(z)$ be any rat'l fnc.

$$R(z) = c\,\frac{(z-\zeta_1)^{m_1}\cdots(z-\zeta_k)^{m_k}}{(z-z_1)^{n_1}\cdots(z-z_\ell)^{n_\ell}}$$

where $\zeta_1, \ldots, \zeta_k \,\&\, z_1, \ldots, z_\ell$ all distinct

$R(z)$ meromorphic on entire plane, since $z_1, \ldots, z_\ell$ are all poles.

Theorem: Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole $\iff |f(z)| \to \infty$ as $z \to z_0$.

3.) The isolated singularity of $f(z)$ at z_0 is defined to be an essential singularity if $a_k \neq 0$ for infinitely many $k < 0$.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k |z - z_0|^k.$$

example) The Laurent expansion of $e^{\frac{1}{z}}$ at $z=0$ is

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots , \quad z \neq 0$$

Since there are infinitely many negative powers this is essential

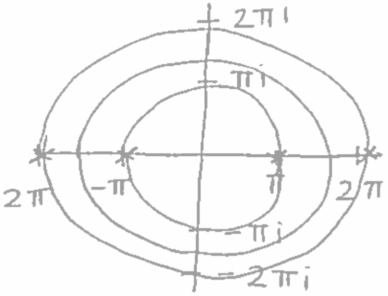
$$\left.\begin{array}{l} e^{\frac{1}{x}} \to \infty \quad \text{as} \quad x \to 0 \\ e^{\frac{1}{iy}} \to \infty \quad \text{as} \quad iy \to 0 \end{array}\right\} \Rightarrow \begin{array}{l} \text{not removable} \\ \text{not pole} \end{array}$$

Theorem : (Casorati-Weierstrass)

Suppose z_0 is an essential isolated singularity of $f(z)$. Then $\forall w_0 \in \mathbb{C}$, $\exists z_n \to z_0$ s.t. $f(z_n) \to w_0$

example) $\frac{1}{\sin z}$, consider the Laurent series expansion that converges on $\{|z|=4\}$. Find $a_0, a_{-1}, a_{-2}, a_{-3}$ of $1, \frac{1}{z}, \frac{1}{z^2}, \frac{1}{z^3}$

Poles at $n\pi$, $n \in \mathbb{Z}$.



$f_0(z)$ is Laurent expansion for $|z| < 2\pi$.

$$f_0(z) = \frac{1}{\sin z} - f_1(z)$$

$f_1(z)$ is LE for $|z| > \pi$.

We know the order of each pole is 1.

$\sin(z) = z + \Theta(z^3)$ at $z=0$

$$\Rightarrow \frac{1}{\sin(z)} = \frac{1}{z} + \text{Analytic} \Rightarrow \frac{1}{\sin z} - \frac{1}{z} = \text{Analytic}$$

$\sin(z) = -(z-\pi) + \Theta((z-\pi)^3)$ at $z=\pi$

$$\Rightarrow \frac{1}{\sin z} = -\frac{1}{z-\pi} + \text{Analytic}$$

$$\Rightarrow \frac{1}{\sin z} + \frac{1}{z-\pi} = \text{Analytic at } z=\pi.$$

$\sin(z) = -(z+\pi) + \Theta((z+\pi)^3)$ at $z=-\pi$

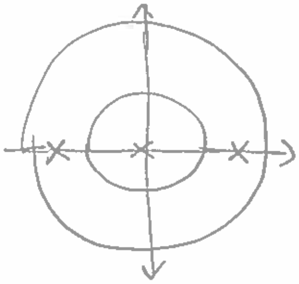
$$\Rightarrow \frac{1}{\sin z} = \frac{-1}{z+\pi} + \text{Analytic}$$

$$\Rightarrow \frac{1}{\sin z} + \frac{1}{z+\pi} = \text{Analytic.}$$

$\Rightarrow \frac{1}{\sin(z)} = \frac{g(z)}{z}$, where $g(z)$ is analytic

$\frac{1}{\sin(z)} = \frac{g_1(z)}{z-\pi}$, where $g(\pi)$ is analytic

$\frac{1}{\sin(z)} = \frac{g_2(z)}{z+\pi}$, where g



So centered at 0.

$\frac{1}{\sin(z)} =$ $f_1(z)$ is for $|z|>\pi$.

$f_1(z)$ is the stuff that's "bad".

Chapter 6, Section 3 : Isolated Singularity at Infinity.

Defn: $f(z)$ has an isolated singularity at ∞ if $f(z)$ is analytic outside some $\{|z|>R\}$.

(i.e. if $g(w) = f(\frac{1}{w})$ has an isolated singularity at $w=0$)

(Plug in ∞ and get ∞).

ex.) Poly of degree $N \geq 1$ has a pole of order N at ∞.

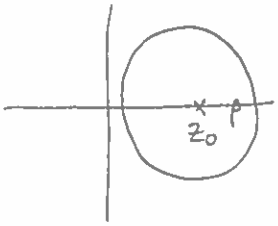
Principal part = the poly.

ex.) $e^z = 1 + z + \frac{z^2}{2!} + \cdots$ has essential singularity at ∞

Chapter 7, Section 1: The Residue Theorem

Suppose z_0 is an isolated singularity of $f(z)$ and that $f(z)$ has Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n , \qquad 0 < |z - z_0| < \rho .$$



Def'n: We define the residue of $f(z)$ at z_0 to be the coefficient a_{-1} of $\frac{1}{z-z_0}$ in this Laurent expansion.

$$\text{Res}\left[f(z), z_0\right] = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\, dz$$

$$\left(\text{Remember, } a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right)$$

where r is any fixed radius satisfying $0 < r < \rho$.

example) $\text{Res}\left[\frac{1}{z}, 0\right] = a_{-1} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{1}{z}\, dz \overset{??}{=} 1$

$$\text{Res}\left[\frac{1}{(z-z_0)^2}, z_0\right] \overset{??}{=} 0$$

example) $\text{Res}\left[\frac{1}{z^2+1}, i\right]$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$\Rightarrow 1 = A(z-i) + B(z+i)$$

$$1 = (A+B)z + (-Ai+Bi)$$

$$\Rightarrow A+B=0 \qquad \Rightarrow i\,(B+A)=0$$

$$Bi-Ai=1 \quad \Rightarrow \underline{i\,(B-A)=1}$$

$$2Bi = 1$$

$$\Rightarrow B = \frac{1}{2i}$$

$$A = -\frac{1}{2i}$$

$$\Rightarrow \frac{1}{z^2+1} = -\frac{1}{2i(z+i)} + \frac{1}{2i(z-i)}$$

$$\Rightarrow \frac{1}{z^2+1} = \underbrace{\frac{1}{2i}\cdot\frac{1}{z-i}}_{\text{stuff that's not analytic at } i} + \left\{\text{stuff that's analytic at } i\right\}$$

$$\Rightarrow \operatorname{Res}\left[\frac{1}{z^2+1}, i\right] = \frac{1}{2i}$$

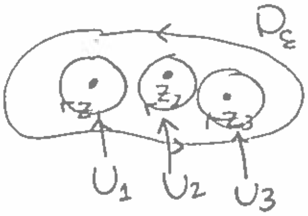
☆☆ So Residue is the coefficient of stuff that's not analytic at a point. ☆☆

<u>Residue Theorem:</u> Let D be a bounded domain in $\mathbb{C}$ w/ piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$, except for a finite # of isolated singularities $z_1, \dots, z_n$ in D. Then

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}\left[f(z), z_j\right].$$

Reminder of CT: If $f(z)$ analytic on D, extends smoothly to ∂D, then $\int_{\partial D} f(z)dz = 0 = 2\pi i \operatorname{Res}(f(z), \emptyset)$

Thm.



$$\Rightarrow \quad \operatorname{Res}[f(z), z_j] = \frac{1}{2\pi i} \oint_{|z-z_j|=\varepsilon} f(z)dz$$

By CT, $$0 = \int_{\partial D_\varepsilon} f(z)dz = \int_{\partial D} f(z)dz - \sum_{j=1}^{m} \int_{\partial U_j} f(z)dz$$

$$= \int_{\partial D} f(z)dz - \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j]$$

Rules for Calculating Residues:

If $f(z)$ has a simple pole at $f(z) \Rightarrow$

$$\operatorname{Res}[f(z), z_0] \overset{?}{=} \lim_{z \to z_0} (z-z_0) f(z).$$

Why? $$f(z) = \frac{a_{-1}}{z-z_0} + [\text{analytic at } z_0].$$

$$\Rightarrow \lim_{z \to z_0} (z-z_0) f(z) = a_{-1} + [\text{analytic at } z_0](z-z_0)$$

$$= a_{-1}.$$

example) $\operatorname{Res}\left[\frac{1}{z^2+1}, i\right]$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} \text{ has simple pole at } i$$

$$\Rightarrow \frac{1}{(z+i)(z-i)} = \frac{a_{-1}}{z-i} + [\text{analytic at } i]$$

$$\Rightarrow \operatorname{Res}\left[\frac{1}{z^2+1}, i\right] = \lim_{z \to i} \left[a_{-1} + [\text{analytic at } i][z-i]\right]$$

$$= \lim_{z \to i} \left[\frac{1}{z+i}\right] = \frac{1}{2i}$$

Rule 2: If $f(z)$ has a double pole at z_0, then

$$\text{Res}\,[f(z), z_0] \overset{?}{=} \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$$

Why? $$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + [\text{analytic at } z_0]$$

$$\Rightarrow (z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + (z-z_0)^2[\text{stuff}]$$

$$\Rightarrow \frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1} + (z-z_0)[\text{stuff}]$$

$$\Rightarrow \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1}.$$

ex.) $\text{Res}\left[\frac{1}{(z^2+1)^2}, i\right]$

Has 2 poles at $z=i$

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2} = \frac{a_{-2}}{(z-i)^2} + \frac{a_{-1}}{(z-i)} + a_0 + \dots$$

$$\Rightarrow \frac{1}{(z+i)^2} = a_{-2} + a_{-1}(z-i) + a_0(z-i)^2 + \dots$$

$$\Rightarrow \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] = a_{-1} + a_0\,[\text{stuff with } (z-i) \text{ in it}]$$

$$\frac{-2(z+i)}{(z+i)^4} = \frac{-2}{(z+i)^3}$$

$$\Rightarrow \lim_{z \to i}\left[\frac{-2}{(z+i)^3}\right] = a_{-1}$$

$$\frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{2}{8i} = \frac{1}{4i}$$

Rule 3: If $f(z)$ and $g(z)$ are analytic at z_0, and if $g(z)$ has a simple zero at z_0, then

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

So either $f(z_0) = \overset{=0}{g(z_0)} \Rightarrow$ Removable $\Rightarrow$ No residue?
$f(z_0) \neq g(z_0) = 0 \Rightarrow$ simple pole

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \lim_{z \to z_0} (z - z_0)\frac{f(z)}{g(z)} \overset{\text{since } g(z_0)=0}{=} \lim_{z \to z_0} \frac{f(z)(z-z_0)}{g(z) - g(z_0)}$$

$$= \lim_{z \to z_0} \frac{f(z)}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{f(z_0)}{g'(z_0)}$$

example) $\text{Res}\left[\frac{z^3}{z^2+1}, i\right]$

There's a simple pole at $z = i$

$$\frac{z^3}{z^2+1} = \frac{a_{-1}}{(z-i)} + [\text{analytic at } z = i].$$

$$\Rightarrow \frac{z^3}{(z+i)(z-i)} = \frac{a_{-1}}{(z-i)} + [\text{analytic at } z = i]$$

$$\Rightarrow \frac{z^3}{z+i} = a_{-1} + [\text{analytic at } z = i](z-i)$$

$$\lim_{z \to i}\left[\frac{z^3}{z+i}\right] = a_{-1}$$

$$\| $$

$$\frac{-i}{2i} = -\frac{1}{2}$$

or $\quad \text{Res}\left[\frac{z^3}{z^2+1}, i\right] = \left.\frac{z^3}{2z}\right]_{z=i} = \frac{-i}{2i} = -\frac{1}{2}$

Rule 4 If $g(z)$ is analytic and has a simple zero at z_0, then $\text{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$.

example) $\text{Res}\left[\frac{1}{z^2+1}, i\right] = \left.\frac{1}{2z}\right]_{i} = \frac{1}{2i}$

– I kind ~~of think~~ we only need Rules 1 & ~~2.~~

~~If $\frac{f(z)}{g(z)}$ ← has simple zero,~~

~~then $h(z) = \frac{f(z)}{g(z)}$ has simple pole~~ –

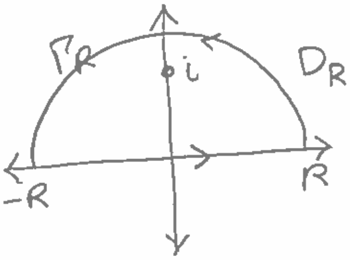
– maybe Rules 2 & 3 better —

If $\frac{1}{f(z)}$ has simple pole $\Rightarrow$ $\frac{1}{f'(z)}$ = Res.

If double pole $\Rightarrow$ Take limit $\left.\frac{d}{dz}((z-z_0)^2 f(z))\right]_{z=z_0}$

Chapter 7, Section 2: Integrals Featuring Rational Functions

example) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$



$\dfrac{1}{1+z^2}$ has one pole in D_R

By the residue theorem,

$$\int_{\partial D_R} \frac{dz}{1+z^2} = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^2}, i\right] = \frac{2\pi i}{2i} = \pi .$$

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{a_{-1}}{z-i} + \text{Eko}\left[\text{Analytic at } z=i\right]$$

$$\Rightarrow \frac{1}{z+i} = a_{-1} + (\quad)(z-i)$$

$$\Rightarrow \lim_{z \to i} \underset{\substack{\parallel \\ \frac{1}{2i}}}{\frac{1}{z+i}} = a_{-1}$$

$$\int_{\partial D_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{1}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}$$

Now, $\displaystyle\left|\int_{\Gamma_R} \frac{dz}{1+z^2}\right| \leq \frac{\pi R}{R^2-1}$ by ML-Estimate.

$$|z^2+1| \geq R^2 - 1$$

$$|x+y-y| \leq |x+y| + |y|$$

$$\Rightarrow |x+y| \geq |x| - |y|$$

$$\Rightarrow \left|\int_{\Gamma_R} \frac{dz}{1+z^2}\right| \leq \frac{\pi R}{R^2-1} \longrightarrow 0 \text{ as } R \to \infty$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \lim_{R\to\infty}\left(\pi - \int_{\Gamma_R} \frac{dz}{1+z^2}\right) = \pi.$$

We can do the same thing for integrals

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$ ✡ if $Q(z)$ has no zeroes on the real axis ✡

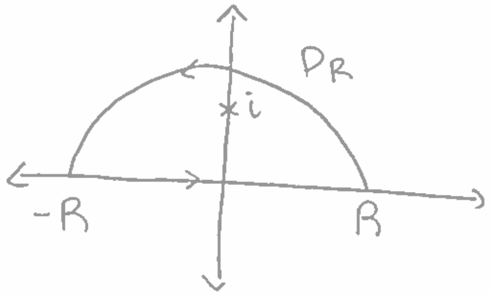
For the integral to converge, $\deg Q(z) \geq \deg P(z) + 2$.
(B/c of the $\int_{\Gamma_R} \frac{P}{Q}$ term).

$$\Rightarrow \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j\right].$$

example) WTS $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx = \frac{\pi}{e^a}$, $a>0$.

(we'll use $|e^{iz}| \leq 1$ for $\operatorname{Im}(z) \geq 0$)

$$\left[\begin{aligned} |e^{iz}| = |e^{ix-y}| &= |e^{-y}||\cos x + i\sin x| \\ &= |e^{-y}| = \frac{1}{|e^{y}|} \leq \frac{1}{e^0} = 1 \end{aligned}\right]$$



$\displaystyle\int_{\partial D_R} \frac{e^{iaz}}{1+z^2}$ has one pole in D_R

$$\Rightarrow \int_{\partial D_R} \frac{e^{iaz}}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left[\frac{e^{iaz}}{1+z^2}, i\right]$$

$$= 2\pi i \left(\frac{e^{iaz}}{2z}\right)_{z=i} = \frac{2\pi i\, e^{-a}}{2i} = \frac{\pi}{e^a}$$

$$\int_{\partial D_R} \frac{e^{iaz}}{1+z^2}\,dz = \int_{-R}^{R} \frac{e^{iax}}{1+x^2}\,dx + \int_{\Gamma_R} \frac{e^{iaz}}{1+z^2}\,dz$$

$$\left| \int_{\Gamma_R} \frac{e^{iaz}}{1+z^2}\,dz \right| \leq \frac{\pi R \cdot e^{\text{[illegible]}}}{R^2-1} \to 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{iaz}}{1+z^2}\,dz = \frac{\pi}{e^a}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{iaz}}{1+z^2}\,dz \right) = \mathrm{Re}\left(\frac{\pi}{e^a}\right) = \frac{\pi}{e^a}.$$

Note: $\int_{-\infty}^{\infty} \frac{\sin(ax)}{1+x^2}\,dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{e^{iaz}}{1+z^2}\,dz \right) - \mathrm{Im}\left(\frac{\pi}{e^a}\right) = 0$

Chapter 7, Section 3: Integrals of Trigonometric Fnc

example) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$, $a > 1$

$$z = e^{i\theta}$$

$$\Rightarrow dz = ie^{i\theta}\,d\theta \Rightarrow d\theta = \cancel{i z\, d\theta} = \frac{dz}{iz}$$

$$= iz\,d\theta$$

$\Rightarrow$ On the unit circle,

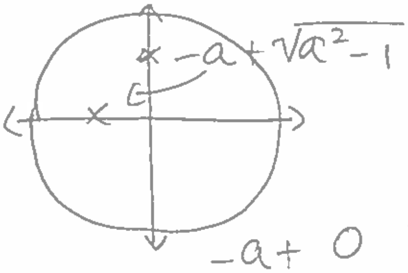
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$$

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \oint_{|z|=1} \frac{dz}{iz\left(a+\frac{z+\frac{1}{z}}{2}\right)} = \oint_{|z|=1} \frac{2\,dz}{iz\left(2a+z+\frac{1}{z}\right)}$$

$$= \oint_{|z|=1} \frac{2}{i}\ \frac{dz}{2az+z^2+1}$$

Poles: $z = \dfrac{-2a \pm \sqrt{4a^2-4\cancel{a}}}{2} = -a \pm \sqrt{a^2-1}$



If $-1 < z < 1$, then in the unit circle.

$-a+0 < -a+\sqrt{a^2-1} < -1+\sqrt{1-1}$

$= -1$

$a > 1 \Rightarrow -a < -1$???

$$\text{Res}\left[\frac{1}{z^2+2az+1}, z_0\right] = \left[\frac{1}{2z+2a}\right]_{z=-a+\sqrt{a^2-1}} = \frac{1}{2\sqrt{a^2-1}}$$

$\Rightarrow$ By Res. Thm, $\int_0^{2\pi} = \frac{2}{i}\, 2\pi i \,\frac{\mathrm{Res}}{i} = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$.

Now, consider $\int_0^{2\pi} \frac{d\theta}{\omega + \cos\theta}$, $\omega \in \mathbb{C} \setminus [-1, 1]$.

$$\overset{?}{=} \frac{2\pi}{\sqrt{\omega^2 - 1}}$$

I don't get this.

Chapter 7, Section 4: Integrands with Branch Points

example) $\int_0^\infty \frac{x^a}{(1+x)^2}\,dx \overset{?}{=} \frac{\pi a}{\sin(\pi a)}$, $-1<a<1$.

Case 1: $a=0$

$$\int_0^\infty \frac{1\,dx}{(1+x)^2} = \int_0^\infty \frac{1}{u^2}\,du = \frac{-1}{1+x}\Big]_{x=0}^{x=\infty} = 0+1=1$$

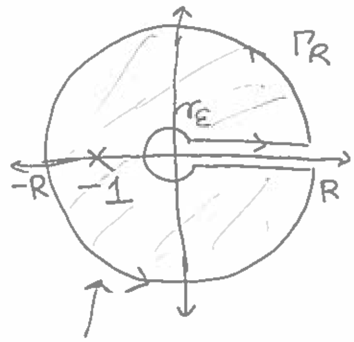
$u=1+x$

$du=dx$

$$\lim_{a\to 0}\frac{\pi a}{\sin(\pi a)} = \lim_{a\to 0}\frac{\pi}{\pi\cos(\pi a)} = 1.$$

Case 2: $a\neq 0$

Let's look at $\frac{z^a}{(1+z)^2}$ on $\mathbb{C}\setminus[0,\infty)$.



We call this whole thing D.

$$f(z) = \frac{r^a e^{ia\theta}}{(1+z)^2}$$

$f(z)$ has a double pole at $z=-1$.

$$\int_{\partial D} f(z) = 2\pi i\,\mathrm{Res}\left[\frac{z^a}{(1+z)^2}, -1\right] = 2\pi i\left[\frac{d}{dz}z^a\right]_{z=-1}$$

$$= 2\pi i\, a z^{a-1}\Big]_{z=-1} = 2\pi i a e^{(a-1)\log(1)+(a-1)\pi i}$$

$$= 2\pi i a e^{a\pi i}e^{-\pi i} = -2\pi i a e^{a\pi i}$$

$$\int_{\partial D} f(z) = \int_\varepsilon^R \frac{x^a e^0}{(1+x)^2}\,dx + \int_{\Gamma_R}\frac{z^a}{(1+z)^2}\,dz + \int_R^\varepsilon \frac{x^a e^{2\pi i a}}{(1+x)^2}\,dx + \int_{\gamma_\varepsilon}\frac{z^a}{(1+z)^2}\,dz$$

$$\left|\int_{\Gamma_R} \frac{z^a}{(1+z)^2}\,dz\right| \leqslant \frac{2\pi R\, R^a}{(R-1)^2} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty$$

$$0 < a+1 \leqslant 2$$

$$\left|\int_{\gamma_\varepsilon} \frac{z^a}{(1+z)^2}\,dz\right| \leq \frac{\varepsilon^a\, 2\pi\varepsilon}{(1-\varepsilon)^2} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0$$

$$\Rightarrow -2\pi i\, a e^{\pi i a} = \int_0^\infty \frac{x^a}{(1+x)^2}\,dx - \int_0^\infty \frac{e^{2\pi i a}\, x^a}{(1+x)^2}\,dx$$

$$\Rightarrow \int_0^\infty \frac{x^a}{(1+x)^2}\,dx = \frac{-2\pi i a e^{\pi i a}}{1-e^{2\pi i a}} = \frac{2\pi i a}{-e^{-\pi i a}+e^{\pi i a}}$$

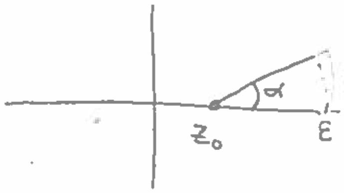
$$= \frac{\pi a}{\sin(\pi a)} \quad ✓$$

$$\left[\sin(\pi a) = \frac{e^{\pi i a} - e^{-\pi i a}}{2i}\right]$$

Chapter 7, Section 5: Fractional Residues

§. z_0 is an isolated singularity of $f(z)$. For $\varepsilon > 0$ small

$$\int_{C_\varepsilon} f(z)dz$$



"Aperture" = α

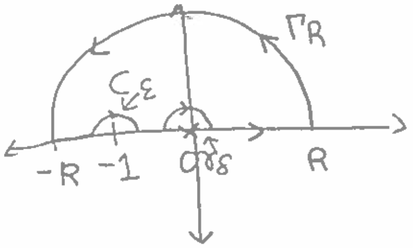
I guess aperture = angle.

Fractional Residue Theorem: If z_0 is a simple pole of $f(z)$ & C_ε is an arc of $\{|z-z_0|=\varepsilon\}$ of angle α,

then $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)dz = \alpha i \, \text{Res}[f(z), z_0]$.

In particular, $\int_{\partial ●} f(z)dz = 2\pi i \, \text{Res}[f(z), z_0]$

example) $\int_0^\infty \frac{\log(x)}{x^2-1} dx = \frac{\pi^2}{4}$



$f(z)$ is analytic on D.

$$\Rightarrow \int_{\partial D} f(z)dz = 0$$

$$\|$$

$$\int_\delta^R \frac{\log(x)}{x^2-1} + \int_{\Gamma_R} \frac{\log(z)}{z^2-1} + \int_{-R}^{-1-\varepsilon} \frac{\log x}{x^2-1} + \int_{C_\varepsilon} \frac{\log z}{z^2-1} + \int_{-1+\varepsilon}^{-\delta} \frac{\log x}{x^2-1}$$

$$+ \int_{\gamma_\delta} \frac{\log z}{z^2-1}$$

$$\left|\int_{\Gamma_R} \frac{\log z}{z^2-1}\,dz\right| \le \frac{\sqrt{\log^2 R+\pi^2}}{R^2-1}\pi R \to 0 \quad \text{as } R\to\infty$$

$$\left|\int_{\gamma_\delta} \frac{\log z}{z^2-1}\,dz\right| \le \frac{\sqrt{\log^2\delta+\pi^2}}{1-\delta^2}\pi\delta \to 0 \quad \text{as } \delta\to 0$$

$$\int_0^\infty \frac{\log x}{x^2-1}\,dx + \int_{-\infty}^{-1-\varepsilon}\frac{\log|x|+i\pi}{x^2-1}\,dx + \int_{-1+\varepsilon}^{0}\frac{\log|x|+\pi i}{x^2-1}\,dx$$

$$+\int_{C_\varepsilon}\frac{\log z}{z^2-1}\,dz = 0$$

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\log z}{z^2-1}\,dz = -\pi i \,\mathrm{Res}\left[\frac{\log z}{z^2-1}, -1\right] = -\pi i \left.\frac{\log z}{2z}\right]_{z=-1}$$

$$= -\pi i\,(0) + \frac{-\pi i\, i\pi}{2(-1)} = -\frac{\pi^2}{2}$$

$$\Rightarrow \int_0^\infty \frac{\log x}{x^2-1}\,dx + \int_{-\infty}^{-1}\frac{\log|x|}{x^2-1} + i\pi\int_{-\infty}^{-1}\frac{dx}{x^2-1} + \int_{-1}^{0}\frac{\log|x|}{x^2-1}$$

$$+ i\pi\int_{-1}^{0}\frac{\log|x|}{x^2-1} = \frac{\pi^2}{2}.$$

(Taking real parts)

$$\Rightarrow \int_0^\infty \frac{\log x}{x^2-1} + \int_{-\infty}^{0}\frac{\log|x|}{x^2-1} = \frac{\pi^2}{2}$$

Let $x = -x$

$$\int_0^\infty \frac{\log x}{x^2-1} + \int_0^\infty \frac{\log x}{x^2-1} = \frac{\pi^2}{2}$$

$$\Rightarrow \int_0^\infty \frac{\log x}{x^2-1} = \frac{\pi^2}{4}.$$

Chapter 8, Section 1: The Argument Principle

Reminder: A function is meromorphic if its analytic (except at finitely many poles)

Suppose $f(z)$ is analytic on D. For a curve $\gamma \in D$ s.t. $f(z) \neq 0$ on γ, we call

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)\,dz}{f(z)} = \frac{1}{2\pi i}\int_{\gamma} d\log f(z)$$

the logarithmic integral of $f(z)$ along γ.

Theorem: Let D be a bounded domain w/ piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, s.t. $f(z) \neq 0$ on ∂D. Then $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty$.

where N_0 is # of zeroes in D, counting multiplicities

N_∞ # of poles, counting multiplicities

Pf: By residue thm,

$$\int_{\partial D} \frac{f'(z)}{f(z)}\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}\left(\frac{f'(z_j)}{f(z_j)}\right)$$

$$f(z) = (z-z_0)^N g(z)$$

$$\Rightarrow f'(z) = N(z-z_0)^{N-1} g(z) + (z-z_0)^N g'(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{N}{z-z_0} + \text{analytic}$$

↓

has simple pole at z_0 w/ residue N.

...

Let's look at $\frac{1}{2\pi i}\int_\gamma d\log f(z)$

$$= \frac{1}{2\pi i}\int_\gamma d\log|f(z)| + \frac{1}{2\pi \cancel{i}}\int_\gamma \cancel{i}\, d\arg(f(z))$$

Parametrize:

$$\int_\gamma d\log|f(z)| = \log|f(\gamma(b))| - \log|f(\gamma(a))|$$

If γ is closed, then $\int_\gamma d\log|f(z)| = 0$

$\int_\gamma d\arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$ is called the increase in arg. of $f(z)$ along γ.

8.2: Rouche's Theorem: f

Rouche's Theorem: Let D be a bounded domain w/ piecewise smooth boundary ∂D. Let $f(z)$ & $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ & $f(z) + h(z)$ have the same # of zeroes in D, counting multiplicities.

example) How many zeroes are there for $p(z) = z^6 + 9z^4 + z^3 + 2z + 4$ in the unit circle?

$$f(z) = 9z^4$$
$$h(z) = z^6 + z^3 + 2z + 4$$
$$|h(z)| = |z^6 + z^3 + 2z + 4| \leq |z|^6 + |z|^3 + 2|z| + 4$$
$$= 8 < |9z^4| = 9|z|^4$$

$\Rightarrow 9z^4$ & $p(z)$ have same # of zeroes in unit circle. i.e. 4

example) Find all sol'ns to $e^z = 1 + 2z$ that satisfy $|z| < 1$.

$$e^z - 2z - 1 = 0$$

$$f(z) = -2z$$
$$h(z) = e^z - 1$$

$$|e^z - 1| = \left| 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right| \leq |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \dots$$

-1

$$= 1 + \frac{1}{2!} + \frac{1}{3!} + \dots < 2 = |-2z| = 2|z|$$

$\Rightarrow$ One zero

Chapter 8, Section 3: Hurwitz's Theorem

Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$, $f(z)$ has a zero of order N at z_0. Then $\exists\ \rho>0$ s.t. for k large, $f_k(z)$ has exactly N zeroes in $\{|z-z_0|<\rho\}$, counting multiplicity, and these zeroes converge to z_0 as $k \to \infty$

Defn: Univalent : A fnc. is univalent if it's analytic & one-to-one on D.

Other Hurwitz's Thm: Suppose $f_k(z)$ is a sequence of univalent fncs. on D that converges normally on D to $f(z)$. Then $f(z)$ is univalent/constant.

Pf:

S $f(z)$ not constant. WTS one-to-one.

S $f(z_0) = f(\zeta_0) = w_0$

⇒ Let $f(z)-w_0$ be the fnc. we're looking at.

$f(z_0) - w_0 = f(\zeta_0) - w_0 = 0.$

⇒ z_0 & ζ_0 are zeroes (of finite order?)

⇒ $\exists\ z_k, \zeta_k$ s.t. $z_k \to z_0$, $\zeta_k \to \zeta_0$

$f_k(z_k) - w_0 = 0$, $f_k(\zeta_k) - w_0 = 0$.

Since f_K is univalent,

$$f_K(z_K) = f_K(\zeta_K) \Rightarrow \underset{\substack{\downarrow \\ z_0}}{z_K} = \underset{\substack{\downarrow \\ \zeta_0}}{\zeta_K}$$

$\Rightarrow f(z)$ is univalent.

example) $f_K(z) = \frac{z}{K}$. , $K \geq 1$

$f_K(a) = f_K(b)$

$\Rightarrow \frac{a}{K} = \frac{b}{K} \Rightarrow a = b \Rightarrow$ univalent

$f_K(z) = \frac{z}{K} \to 0$ as $K \to \infty$ (uniformly on every closed disk in D)

$\Rightarrow 0$ is constant ✓

Chapter 8, Section 4: Open Mapping and Inverse Function Theorem

Def'n:

Let $f(z)$ be meromorphic on D. $f(z)$ attains the value w_0 m times at z_0 if $f(z) - w_0 = 0$ m times.

$f(z)$ attains $w_0 < \infty$ m times at $z_0 = \infty$ if

$f(\frac{1}{z}) - w_0 = 0$ m times.

$f(z)$ attains ∞ m times at z_0 if

$f(z_0) = \infty$ m times.

example) $z^m + 1$ attains $w = 1$

$z^m + 1 = 1$

$z^m = 0$

m times at $z = 0$

$z^m + 1$ attains $w = \infty$

$z^m + 1 = \infty$

$z^m = \infty$ m times at $z = \infty$

Open Mapping Theorem for Analytic Functions: If $f(z)$ is analytic on a domain D, and $f(z)$ is not constant, then $f(z)$ maps open sets to open sets.

# of zeroes in $\{|z - z_0| < \rho\}$: $\displaystyle \frac{1}{2\pi i} \int_{|z - z_0| = \rho} \frac{f'(z)}{f(z)}\, dz$

IFT: If $f(z)$ is analytic for $|z-z_0| \leq \rho$ and satisfies $f(z_0)=w_0$, $f'(z_0) \neq 0$, and $f(z) \neq w_0$ for $0<|z-z_0| \leq \rho$.

Let $\delta > 0$ s.t. $|f(z)-w_0| \geq \delta$ for $|z-z_0|=\rho$.

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0|=\rho} \frac{\zeta f'(\zeta)}{f(\zeta)-w}\, d\zeta \quad , |w-w_0|<\delta$$

Chapter 9, Section 1: The Schwarz Lemma:

Thm (Schwarz Lemma): Let $f(z)$ be analytic for $|z|<1$ & $|f(z)| \le 1$ $\forall\ |z|<1$ and $f(0)=0$. Then

$$|f(z)| \le |z|, \quad |z|<1$$

Further, if equality holds, at $z_0 \neq 0$, then

$$f(z) = \lambda z \quad , \quad |\lambda| = 1.$$

Proof: $f(0)=0 \Rightarrow f(z) = z\,g(z)$, where $g(z)$ is analytic

Let $r<1$. If $|z|=r$, then $|g(z)| = \dfrac{|f(z)|}{r} \le \dfrac{1}{r}$

on $|z|=r$

By Maximum principle, $|g(z)| \le \frac{1}{r}$ $\forall\ |z| \le r$.

Let $r \to 1 \Rightarrow |g(z)| \le 1$ $\forall\ |z| \le 1$

$$\Rightarrow \left|\frac{f(z)}{z}\right| \le 1 \Rightarrow |f(z)| \le |z|. \checkmark$$

Suppose $|f(z_0)| = |z_0|$ for $z_0 \neq 0$.

$\Rightarrow |g(z_0)| = 1$

$\Rightarrow$ By strict maximum principle,

$g(z) = \lambda$ constant.

$$\Rightarrow \frac{f(z)}{z} = \lambda \quad \checkmark$$

What about on a disk?

Let $f(z)$ be analytic for $|z-z_0|<R$

$|f(z)| \le M$

$f(z_0)=0$

$\Rightarrow f(z) = (z-z_0)g(z)$

$\Rightarrow$ Let $r<R$ If $|z-z_0|=r$

$\Rightarrow \cancel{|f(z)|} \; |g(z)| = \frac{|f(z)|}{r} \le \frac{M}{r}$ on $|z-z_0|=r$.

$\Rightarrow$ By Maximum principle, $|g(z)| \le \frac{M}{r}$ on $|z-z_0| \le r$

Letting $r \to R$, $|g(z)| \le \frac{M}{R}$

$\Rightarrow |f(z)| \le \frac{M}{R}|z-z_0|$.

If $|f(a)| = \frac{M}{R}|a-z_0|$.

$\Rightarrow \frac{|f(a)|}{|a-z_0|} = \frac{M}{r} \Rightarrow g(a) = \frac{M}{r}$.

By the strict maximum principle,

$g(z) = \lambda$.

$\Rightarrow \frac{f(z)}{|z-z_0|} = \lambda \Rightarrow f(z) = \lambda|z-z_0|$

Theorem: Let $f(z)$ be analytic for $|z|<1$. If $|f(z)| \le 1$ for $|z|<1$, and $f(0)=0$, then

$$|f'(0)| \le 1$$

w/ equality $\Leftrightarrow$ $f(z) = \lambda z$, $|\lambda| = 1$.

$$f'(0) = \cancel{\lim_{h\to 0} \frac{f(h)-f(0)}{h}} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} f'(h) =$$

$$\Rightarrow |f'(0)| = \left|\lim_{h\to 0} \frac{f(h)}{h}\right| \overset{=}{\le} \lim_{h\to 0} \left|\frac{f(h)}{h}\right| \le \lim_{h\to 0} 1$$

$$= 1.$$

✓

For equality, $f(z) = z g(z)$

$$g(0) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= f'(0)$$

If $|f'(0)| = 1 \Rightarrow |g(0)| = 1$

$\Rightarrow$ By strict maximum principle,

$g(z) = \lambda$

$\Rightarrow f(z) = \lambda z.$

Chapter 9, Section 2: Conformal Self-Maps on the Unit Disk.

Def'n: A conformal self-map of the unit disk is an analytic fnc. from $\mathbb{D} \to \mathbb{D}$ that is one-to-one & onto.

ex.) φ fixed angle.

$z \mapsto e^{i\varphi}z$ is conformal.

rotate by φ.

Lemma If $g(z)$ is a conformal self-map of $\mathbb{D}$ s.t. $g(0)=0$, then $g(z)=e^{i\varphi}z$ rotation for fixed $0 \le \varphi \le 2\pi$.

$|g(z)|<1$, $g(0)=0$, so $|g(z)| \le |z|$ by Schwarz Lemma.

$|g^{-1}(w)| \le |w|$ ~~for~~ by Schwarz lemma to inverse,

so $|z| \le |g(z)|$

$\Rightarrow |g(z)| = |z|. \Rightarrow \left|\frac{g(z)}{z}\right| = 1 \Rightarrow \frac{g(z)}{z} = \lambda.$

by Strict maximum principle.

$\Rightarrow g(z) = \lambda z.$

Theorem: The conformal self-maps of $\mathbb{D}$ open are the FLTs of the form

$$f(z) = e^{i\varphi}\left(\frac{z-a}{1-\bar{a}z}\right), \quad |z|<1$$

$a \in \mathbb{C}$, $|a|<1$, $0 \le \varphi \le 2\pi$.

Let $g(z) = \frac{z-a}{1-\bar{a}z}$. Since

$$|g(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-e^{+i\theta}\bar{a}}\right| = \frac{|\overline{e^{i\theta}-a}|}{|1-e^{i\theta}\bar{a}|} = \frac{|e^{-i\theta}-\bar{a}|}{|1-e^{i\theta}\bar{a}|}$$

$$= \frac{|\cancel{e^{-i\theta}}||1-\bar{a}e^{i\theta}|}{|1-e^{i\theta}\bar{a}|} = 1. \qquad g(a)=0.$$

So $g(z)$ maps $|z|=1$ to $|z|=1$.

So $f(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z}$ is rotation of $|z|<1$

Pick's Lemma. If $f(z)$ is analytic and $|f(z)|<1$ for $|z|<1$, then

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$$

If $f(z)$ is a conformal self-map of $\mathbb{D}$, then equality. Otherwise, strict inequality.

Chapter 10, Section 1: The Poisson Integral Formula

Poisson kernel function : $P_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$

Where does this function converge uniformly?

For each fixed $\rho < 1$, $r \leq \rho$, $-\pi \leq \theta \leq \pi$

$$|r^{|k|} e^{ik\theta}| \leq \rho^{|k|}, \quad \text{where } \sum \rho^{|k|} = \frac{1}{1-\rho}$$

$\Rightarrow$ The harmonic extension is

$$\boxed{\begin{aligned} \tilde{h}(re^{i\theta}) &= \int_{-\pi}^{\pi} h(e^{i\varphi}) P_r(\theta-\varphi)\frac{d\varphi}{2\pi}, \quad re^{i\theta} \in \mathbb{D} \\ &\varphi \longmapsto \theta - \varphi \\ &= \int_{-\pi}^{\pi} h(e^{i(\theta-\varphi)}) P_r(\varphi)\frac{d\varphi}{2\pi} \end{aligned}}$$

Thm : $\lim_{\substack{z \to \zeta \\ z \in \mathbb{D}}} \tilde{h}(z) = h(\zeta)$. Thus, $\tilde{h}$ extends asly to $\overline{\mathbb{D}}$ by setting $\tilde{h} := h$ on $\partial\mathbb{D}$.

$$P_r(\theta) = \sum_{k=-\infty}^{-1} r^{|k|} e^{ik\theta} + r^0 e^0 + \sum_{k=1}^{\infty} r^{|k|} e^{ik\theta}$$

$$= 1 + \sum_{k=1}^{\infty} r^{\sigma} e^{-ik\theta} + \sum_{k=1}^{\infty} r^k e^{ik\theta} = 1 + \sum_{j=1}^{\infty} \bar{z}^{j}$$

$j = -k$

$$\sum_{k=1}^{\infty} z^k$$

$$= 1 + \frac{1}{1-z} - 1 + \frac{1}{1-\bar{z}} - 1$$

$$= 1 + \frac{1}{1-z} - \frac{1-z}{1-z} + \frac{1}{1-\bar{z}} - \frac{1-\bar{z}}{1-\bar{z}}$$

$$= 1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{|1-z|^2 + z(1-\bar{z}) + \bar{z}(1-z)}{|1-z|^2}$$

Let $z = re^{i\theta}$

$$= \frac{|1-z|^2 + z - |z|^2 + \bar{z} - |z|^2}{|1-z|^2} = \frac{(1-re^{i\theta})(1-re^{-i\theta}) + re^{i\theta} - r^2 + re^{-i\theta} - r^2}{1 - re^{i\theta} - re^{-i\theta} + r^2}$$

$$= \frac{1 - 2r\cos\theta + r^2 + re^{i\theta} - r^2 + re^{-i\theta} - r^2}{1 + r^2 - 2r\cos\theta}$$

$$= \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

Let $h(e^{i\varphi}) = 1$.

$$\Rightarrow \tilde{h}(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\varphi)\frac{d\varphi}{2\pi} = 1.$$

(why? This $\tilde{h}$ is what's happening as $\lim_{e^{i\varphi}\to\infty} h(e^{i\varphi})$)

Properties:

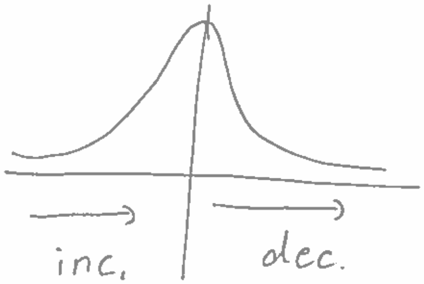
1.) $P_r(\theta) > 0$. Why?

$$P_r(\theta) = \frac{1-|z|^2 > 1-1 = 0.}{|1-z|^2 > 0}$$ Since $z \in \mathbb{D}$.

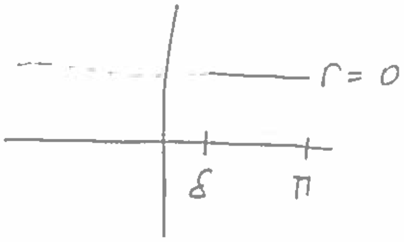
2.) $P_r(\theta)$ is 2π-periodic.

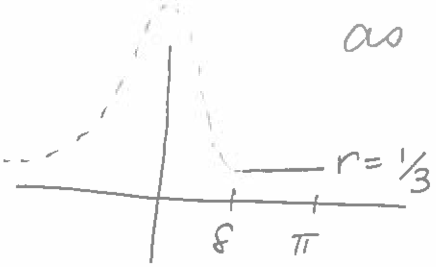
3.) $P_r(-\theta) = P_r(\theta)$ [even fnc]

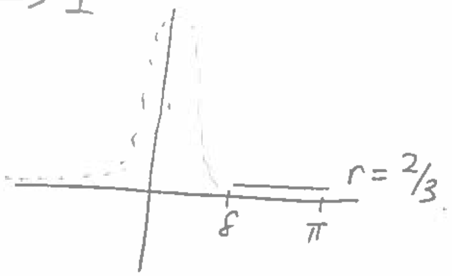
4.)



5.) For fixed $\delta > 0$, $\max\{P_r(\theta) : \delta \le |\theta| \le \pi\} \to 0$ as $r \to 1$







6.) $$\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 2\pi.$$

7.) $$P_r(\theta) = 1 + 2\,\mathrm{Re}\left(\frac{z}{1-z}\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right)$$

Defn: If $h(e^{i\theta})$ is cts, complex-valued,

$$\tilde{h}(z) = \tilde{h}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta-\varphi)\, h(e^{i\varphi})\, d\varphi$$ ← Poisson int. of h

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, h(e^{i\varphi})\, d\varphi \qquad z\in\mathbb{D}$$

Lemma: $\tilde{h}$ is harmonic on $\mathbb{D}$.

$h = u+iv \Rightarrow \tilde{h} = \tilde{u} + i\tilde{v}$.

$$\tilde{u}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1+z}{1-z}\right) u(e^{i\varphi})$$

$$= \mathrm{Re}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})\, \frac{1+re^{i(\theta-\varphi)}}{1-re^{i(\theta-\varphi)}}\right]$$

$$= \mathrm{Re}\left[\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})\, \frac{e^{i\varphi}+z}{e^{i\varphi}-z}}_{\text{analytic}}\right]$$

$\Rightarrow \tilde{u}$ is harmonic in $\mathbb{D}$.

Similarly, $\tilde{v}$ is harmonic in $\mathbb{D}$

$\Rightarrow \tilde{h} = \tilde{u} + i\tilde{v}$ is harmonic in $\mathbb{D}$.

so what are we really talking about?

$$\tilde{h}(z) = \begin{cases} h(z) & \text{on } |z|<1 \\ \lim\limits_{z\to\partial\mathbb{D}} h(z) & \text{on } |z|=1 \end{cases}$$

Chapter 10, Section 2: Characterization of Harmonic Fnc

Lemma: Let $h:\bar{D}\to\mathbb{R}$, D bdd domain, h cts. on $\bar{D}$.
It has MVP on D: $\forall z_0\in D$, $\exists\ \bar{\Delta}(z_0,\rho)\subset D$ s.t.

$$h(z_0)=\frac{1}{2\pi}\int_0^{2\pi} h(z_0+re^{it})dt$$

Thm: Let h be cts. on D. Then h is harmonic on $D \Leftrightarrow h$ has MVP.

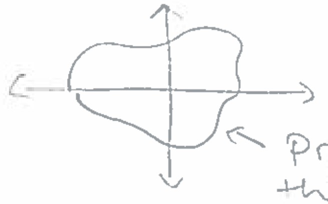
Corollary: If u_n are harmonic on D & converge normally to u on D, then u is harmonic on D.

10.3: Schwarz Reflection Principle ☆☆☆

Defn/Notation: $u^*(z) := u(\bar{z})$

If $u(z)$ is harmonic on D, then $u^*(z)$ is harmonic in $D^* = \{\bar{z} : z \in D\}$

Thm: Let D be a domain that's symmetric wrt $\mathbb{R}$.



Pretend this is symmetric.

Let $D^+ = D \cap \{\operatorname{Im} z > 0\}$. Let $u(z)$ be a real-valued harmonic fnc. on D^+ s.t. $u(z) \to 0$ as $z \in D^+ \to$ any pt. of $D \cap \mathbb{R}$ $\Rightarrow$ $u(z)$ extends to be harmonic on D, and the extension satisfies $u(\bar{z}) = -u(z)$, $z \in D$.

Proof: Define $u(z) = \begin{cases} u(z) & \text{on } D^+ \\ 0 & \text{on } D \cap \mathbb{R} \\ -u(\bar{z}) & \text{on } D^- \end{cases}$

First, let's show u is harmonic.

If $z \in D^+ \Rightarrow u$ harmonic ✓

If $z \in D^- \Rightarrow$ fix $\{|z - z_0| < \rho\} \subset D^-$. For $r < \rho$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt = \frac{1}{2\pi}\int_{+\theta}^{2\pi} -u(\bar{z}_0 + re^{-it})\,dt$$

$\theta = 2\pi - t$

$d\theta = -dt$

$$= -\frac{1}{2\pi}\int_0^{2\pi} u(\bar{z}_0 + re^{i(\theta - 2\pi)})\,d\theta = -\frac{1}{2\pi}\int^{2\pi} u(\bar{z}_0 + re^{i\theta})\,d\theta = -u(\bar{z}_0)$$

3. If $\zeta \in D \cap \mathbb{R}$, fix $\bar{\Delta}(\zeta, \rho) \subset D$

$$u(\zeta) = \int_0^{2\pi} u(\zeta + re^{it})dt = \int_0^{\pi} \underbrace{u(\zeta + re^{it})}_{\text{in upper half plane}} dt + \int_{\pi}^{2\pi} - \underbrace{u(\zeta + re^{-it})}_{\text{in lower half plane}} dt$$

Let $\theta = 2\pi - t$
$d\theta = -dt$

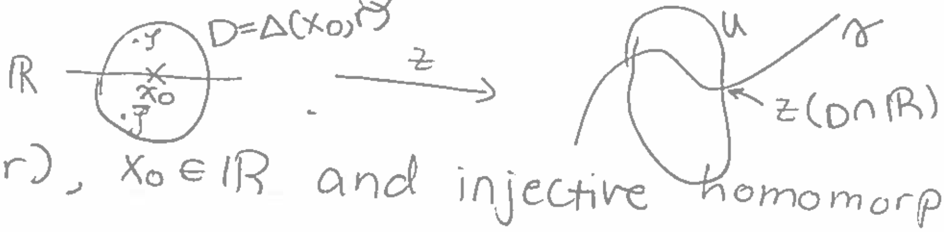
$$= \int_0^{\pi} u(\zeta + re^{it})dt + \int^{0} u(\zeta + re^{-i(2\pi - \theta)})d\theta$$

$$= \int_0^{\pi} u(\zeta + re^{it})dt - \int_0^{\pi} u(\zeta + re^{i\theta})d\theta = 0$$

Theorem: Let $f(z) = u(z) + iv(z) : D^+ \longrightarrow \mathbb{C}$ be holomorphic on D^+ such that $v(z) \to 0$ as $z \to \zeta$, $z \in D^+$ $\forall \zeta \in D \cap \mathbb{R}$. Then f extends to a holomorphic fnc. on D which satisfies

$$F(\bar{z}) = \overline{F(z)}, \; z \in D$$

Defn: γ is an analytic curve if every point of γ has an open U for which there is a conformal map $\zeta \mapsto z(\zeta)$ of a disk D centered on $\mathbb{R}$ onto U s.t. the image of $D \cap \mathbb{R}$ coincides w/ $U \cap \gamma$



i.e. $\exists D = \Delta(z_0, r)$, $x_0 \in \mathbb{R}$ and injective homomorphic map, $z = z(\zeta)$, $\zeta \in D$, mapping D onto U & $D \cap \mathbb{R}$ onto $U \cap \gamma$

We refer to the map $z \mapsto z^*$ as reflection across γ.

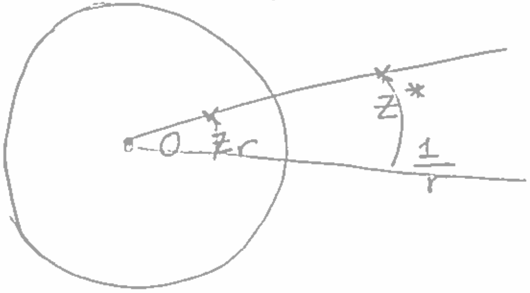
ex.) $z(\zeta) = \frac{\zeta - i}{\zeta + i}$ sends $\mathbb{R}$ into the unit circle

why? $\left|\frac{\zeta - i}{\zeta + i}\right|_{\zeta \in \mathbb{R}} = \frac{\sqrt{\zeta^2+1}}{\sqrt{\zeta^2+1}} = 1$

$\Rightarrow$ sends $\mathbb{R} \mapsto$ ◯

$$z^* = \dot{z}(\bar{\zeta}) = \frac{\bar{\zeta} - i}{\bar{\zeta} + i} = \overline{\left(\frac{\zeta + i}{\zeta - i}\right)} = \frac{1}{\bar{z}} = \frac{z}{|z|^2} = z\underset{\substack{\nearrow \\ \#}}{\left(\frac{1}{|z|^2}\right)}$$

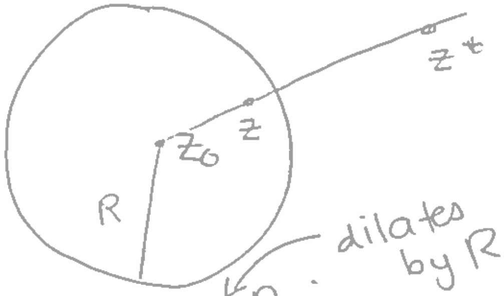
From this formula, we see that z^* lies on the same ray through the origin as z.



If $|z| = r$

$\Rightarrow |z^*| = \frac{1}{r}$

example) Reflection into circle $|z - z_0| = R$.



$$z = \underset{\substack{\uparrow \\ \text{moves to } z_0}}{z_0} + \overset{\text{dilates by } R}{R\frac{\zeta - i}{\zeta + i}}$$

If $\zeta \in \mathbb{R} \rightarrow |z| = \left|z_0 + R\frac{\zeta - i}{\zeta + i}\right|$

$$\Rightarrow z^* = z_0 + R\left(\frac{\overline{\xi}-i}{\overline{\xi}+i}\right) = z_0 + R\,\overline{\left(\frac{\xi+i}{\xi-i}\right)}$$

We need to find $R\,\overline{\left(\frac{\xi+i}{\xi-i}\right)}$ wrt z, z_0.

Have $$\frac{z-z_0}{R} = \frac{\xi-i}{\xi+i}$$

$$\Rightarrow \frac{\xi+i}{\xi-i} = \frac{R}{z-z_0}$$

$$\Rightarrow \overline{\left(\frac{\xi+i}{\xi-i}\right)} = \overline{\left(\frac{R}{z-z_0}\right)}$$

Plugging in $\Rightarrow$ $$z^* = z_0 + R\left(\frac{R}{\overline{(z-z_0)}}\right) \cdot \frac{(z-z_0)}{(z-z_0)}$$

$$= z_0 + \frac{R^2(z-z_0)}{|z-z_0|^2} \checkmark$$

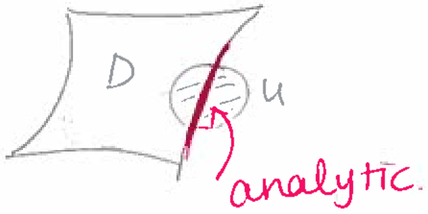
$$h(z) = \overline{z_0} + \frac{R^2}{z-z_0}$$ is a Möbius map!

That sends circles in $\mathbb{C}^*$ to circles in $\mathbb{C}^*$

Let D be a domain, $\gamma \subset \partial D$ is called a free analytic boundary arc of D if γ is an analytic arc and every $z_0 \in \gamma$ is contained in a disk $\mathcal{U}$ s.t. $\mathcal{U} \setminus \gamma$ has two connected components:

one in D, the other in $\mathbb{C} \setminus \overline{D}$.

ex.)



4 free analytic boundary arcs.

Lemma: D is simply connected domain, g is holomorphic function on D, $g(z) \neq 0$ $\forall z \in D$. ☆ ☆

$\Rightarrow \exists f$ holomorphic on D w/ $e^f = g$.

Proof: $\left(\text{Want } g' = e^f f', \; f' = \frac{g'}{e^f} = \frac{g'}{g} \right)$

By primitive stuff, since $\frac{g'}{g}$ is analytic on D,

$\exists f$ s.t. $f' = \frac{g'}{g}$

$\Rightarrow$ Given $z_0 \in D$, choose f so that $f(z_0) \in \log g(z_0)$.

$[\, e^{f(z_0)} = g(z_0) \,]$

Let $h(z) = e^{-f(z)} g(z)$) $\Rightarrow h(z_0) = 1$

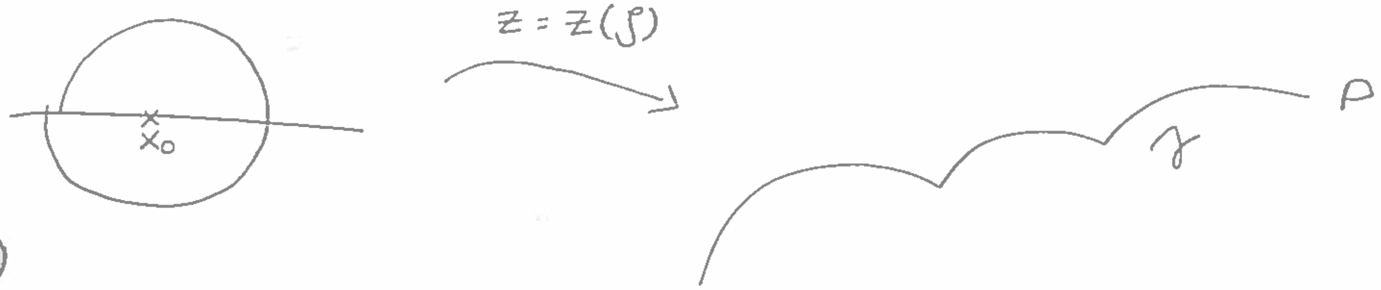
$$\Rightarrow h'(z) = -e^{-f(z)} g(z) f'(z) + e^{f(z)} g'(z)$$

$$= -e^{-f(z)} g'(z) + e^{f(z)} g'(z) = 0.$$

$\Rightarrow 1 = e^{-f(z)} g(z)$

$\Rightarrow e^{f(z)} = g(z)$ ✓

Theorem: Let $D \subseteq \mathbb{C}$ domain, γ a free analytic boundary arc of D. Suppose f is holomorphic in D and $|f(z)| \longrightarrow R$ as $z \xrightarrow{\in D} \gamma$. Then f extends analytically to nbhd(γ) and extension verifies $f(z^*) = \frac{R^2}{\overline{f(z)}}$, when z is near γ, where $z \longmapsto z^*$ is reflection across γ.



Chapter 11, section 1: Mappings to the Unit Disk and the Upper Half-Plane

Reminder: $\varphi : D \to V$ is ~~a~~ conformal if $\varphi(z)$ is one-to-one & onto

Properties: Composition of 2 conformal maps is conform &
Inverse is conformal.

If $\psi : D \to \mathbb{D}$ conformal
$\varphi : D \to \mathbb{D}$
$\Rightarrow \psi \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}.$

Conformal self maps of the open unit disk.

$$g(z) = \lambda\left(\frac{z-a}{1-\bar{a}z}\right), \quad z \in \mathbb{D}$$

$$|a| < 1, \ |\lambda| = 1.$$

ex.) $f(z) = \dfrac{z-i}{z+i}$ maps ~~the~~ $\mathbb{H}$ onto $\mathbb{D}$ open

why? $\left|\dfrac{z-i}{z+i}\right|_{z \in \mathbb{R}} = \dfrac{\sqrt{z^2+1}}{\sqrt{z^2+1}} = 1$

$\Rightarrow \mathbb{R} \longmapsto$ unit circle.

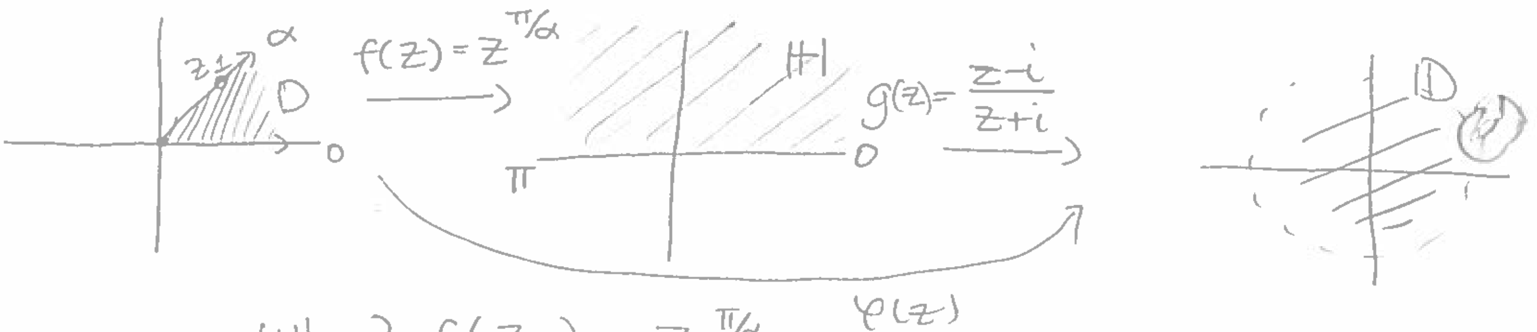
~~If~~ owe

$$z = \frac{f^{-1}(z) - i}{f^{-1}(z) + i} \Rightarrow z f^{-1}(z) + zi = f^{-1}(z) - i$$

$$zi + i = f^{-1}(z)(1 - z)$$

$$\Rightarrow f^{-1}(z) = \frac{zi + i}{1 - z}$$

Mapping Sector → $\overset{\text{Upper}}{\wedge}$ Half-plane → open unit disk



Why? $f(z_1) = z_1^{\pi/\alpha} \notin$
has
angle π.

$$\Rightarrow \varphi(z) = g(f(z)) = \frac{z^{\pi/\alpha}-i}{z^{\pi/\alpha}+i}, \quad z \in D.$$

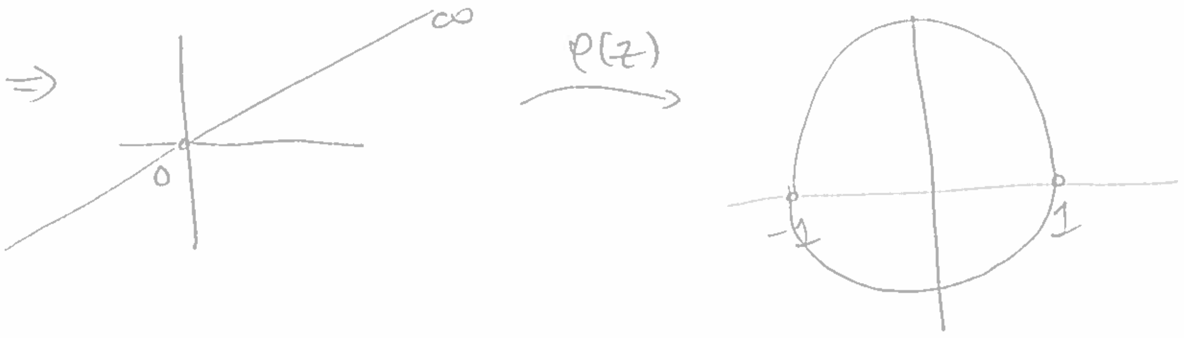
under this map, the vertex $z=0$:

$\varphi(0) = g(f(0)) = -1$

Other vertex. $z=\infty$

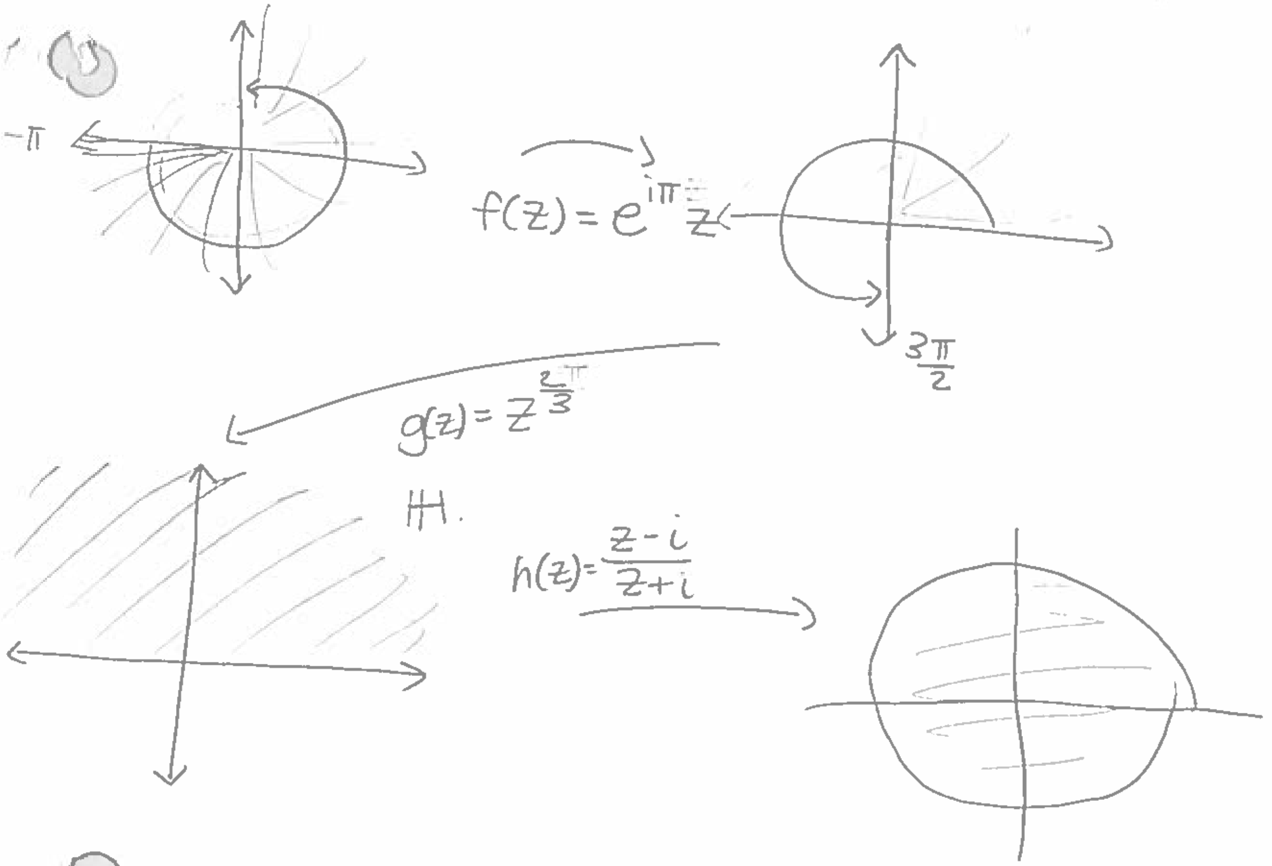
$\varphi(\infty) = g(f(\infty)) = 1$.

⇒



⇒ Rays emanating through the center ⇒ arcs through 1 & -1.

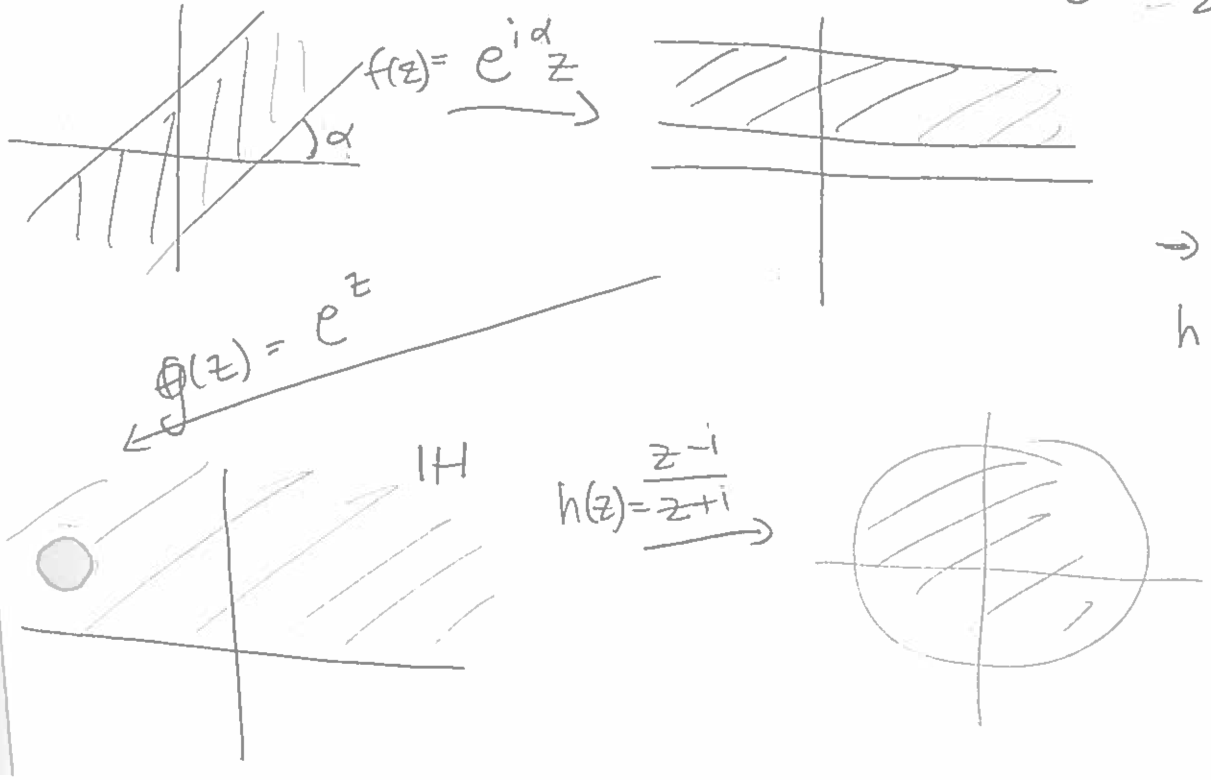
example) Find a conformal map of $\{-\pi < \arg z < \pi/2\}$ onto the open unit disk:



$$\varphi(z) = h(g(f(z))) = h(g(e^{i\frac{\pi}{z}})) = h(e^{\frac{2\pi i}{3}} z^{2/3})$$

$$= \frac{e^{\frac{2\pi i}{3}} z^{2/3} - i}{e^{2\pi i/3} z^{2/3} + i}$$

Strips:



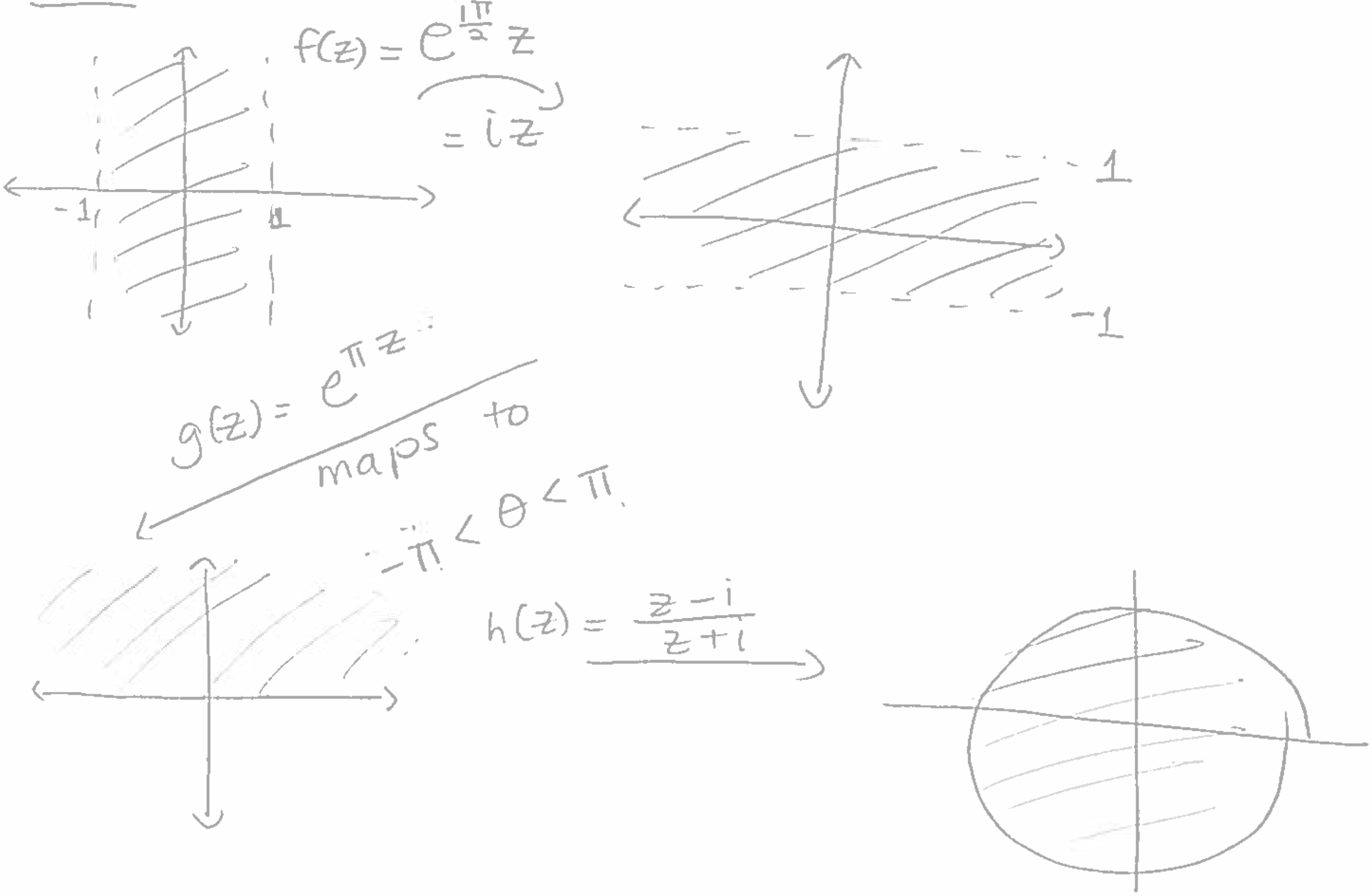
$\rightarrow \varphi(z) =$

$h(e$

ex.) $\{-1 < \text{Re}\,z < 1\} \longrightarrow \mathbb{D}$

What are the images of vertical & horizontal lines in the strip under the map?

Sol'n:



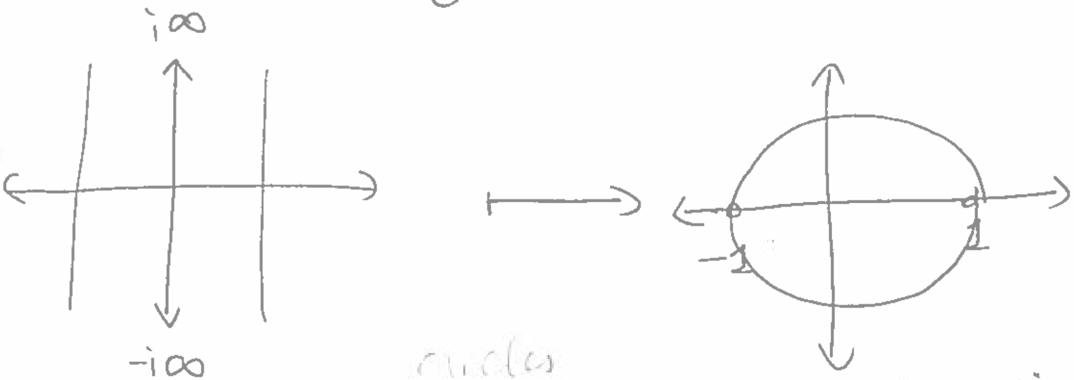
$$\varphi(z) = h(g(f(z))) = h(g(e^{\frac{i\pi}{2}}z) = h(e^{\pi(iz)})$$

$$= \frac{e^{\pi i z} - i}{e^{\pi i z} + i}$$

$$= \frac{(e^{\pi i z}-i)(e^{\pi i z}-i)}{(e^{\pi i z}+i)(e^{\pi i z}-i)} = \frac{e^{2\pi i z} - 2ie^{\pi i z} - 1}{e^{2\pi i z} + 1}$$

$$z = -i\infty \longmapsto \frac{e^{+\pi\infty}-i}{e^{\pi\infty}+i} = 1$$

$$z = +i\infty \longmapsto \frac{e^{-\pi\infty}-i}{e^{-\pi\infty}+i} \rightarrow \frac{-i}{i} = -1$$



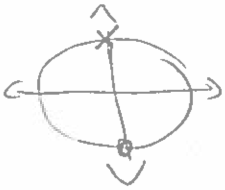
circles in the unit disk passing through ±1

ex.) Find a conformal map of $\{-1 < \operatorname{Re} z < 1\}$ to the open unit disk $\mathbb{D}$ that maps $-i\infty$ to -1 & $i\infty$ to i.

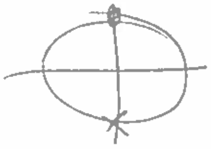
Take the map we just made, $\frac{e^{\pi i z}-i}{e^{\pi i z}+i} = f(z)$

$-i\infty \longmapsto 1$
$i\infty \longmapsto -1$

Need: $\{-1 < \operatorname{Re} z < 1\} \xrightarrow{f(z)}$



$\downarrow \quad g(w) = \frac{\lambda(w-a)}{1-\bar{a}w}$



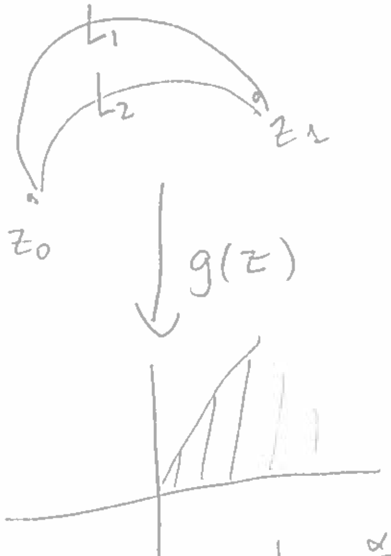
$g(1) = -1$
$g(-1) = i$

$$\left.\begin{aligned} &\Rightarrow \lambda\left(\frac{1-a}{1-\bar{a}}\right) = -1 \\ &\lambda\left(\frac{-1-a}{1+\bar{a}}\right) = i \end{aligned}\right\} \Rightarrow \begin{aligned} &\lambda - \lambda a = -1 + \bar{a} \\ &-\lambda - \lambda a = i + \bar{a} i \end{aligned}$$

$$\Rightarrow -2\lambda a =$$

Basically can solve for a

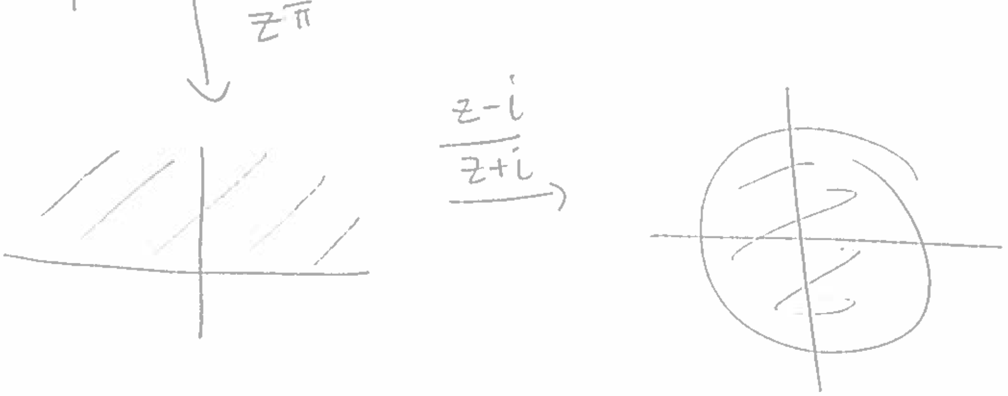
Lunar Domains: Suppose D has a boundary consisting of 2 curves, each of which is an arc of a circle or a straight line segment.



⇒ ∃ FLT mapping $z_0 \mapsto 0$
$z_1 \mapsto \infty$.

⇒ L_1 = ray from $0 \to \infty$
L_2 = ray from $0 \to \infty$

⇒ $g(z)$ maps to a sector!

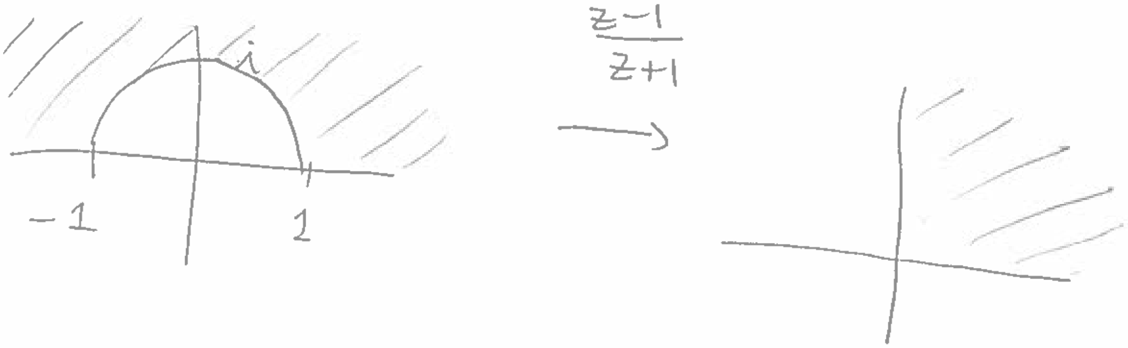


ex.) Find a conformal map ~~to~~ from the UHP outside the unit circle onto the entire UHP mapping

$-1 \to -1$

$i \to 0$

$1 \to 1$



11.2: Riemann Mapping Theorem:

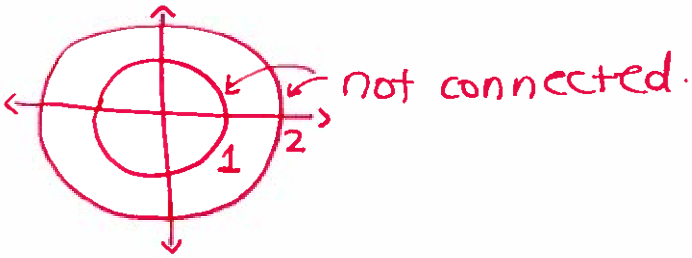
$D \subseteq \mathbb{C}$ is simply connected if every closed path in D is homotopic to 0.

$D \subseteq \mathbb{C}$ is simply connected $\iff$ $\mathbb{C}^* \setminus D$ is connected

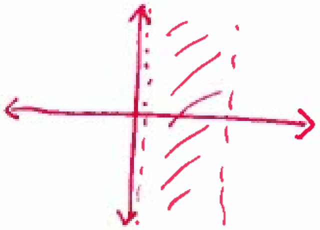
$\iff$ $\partial D \subset \mathbb{C}^*$ is connected

example).

1.) $D = \{1 < |z| < 2\}$ is not simply connected.



2.) $D = \{0 < Re(z) < 1\}$ is simply connected.



actually looks like

So ∂D is connected.

Theorem: Let $D \subset \mathbb{C}$ be a domain, $D \neq \mathbb{C}$.

TFAE:

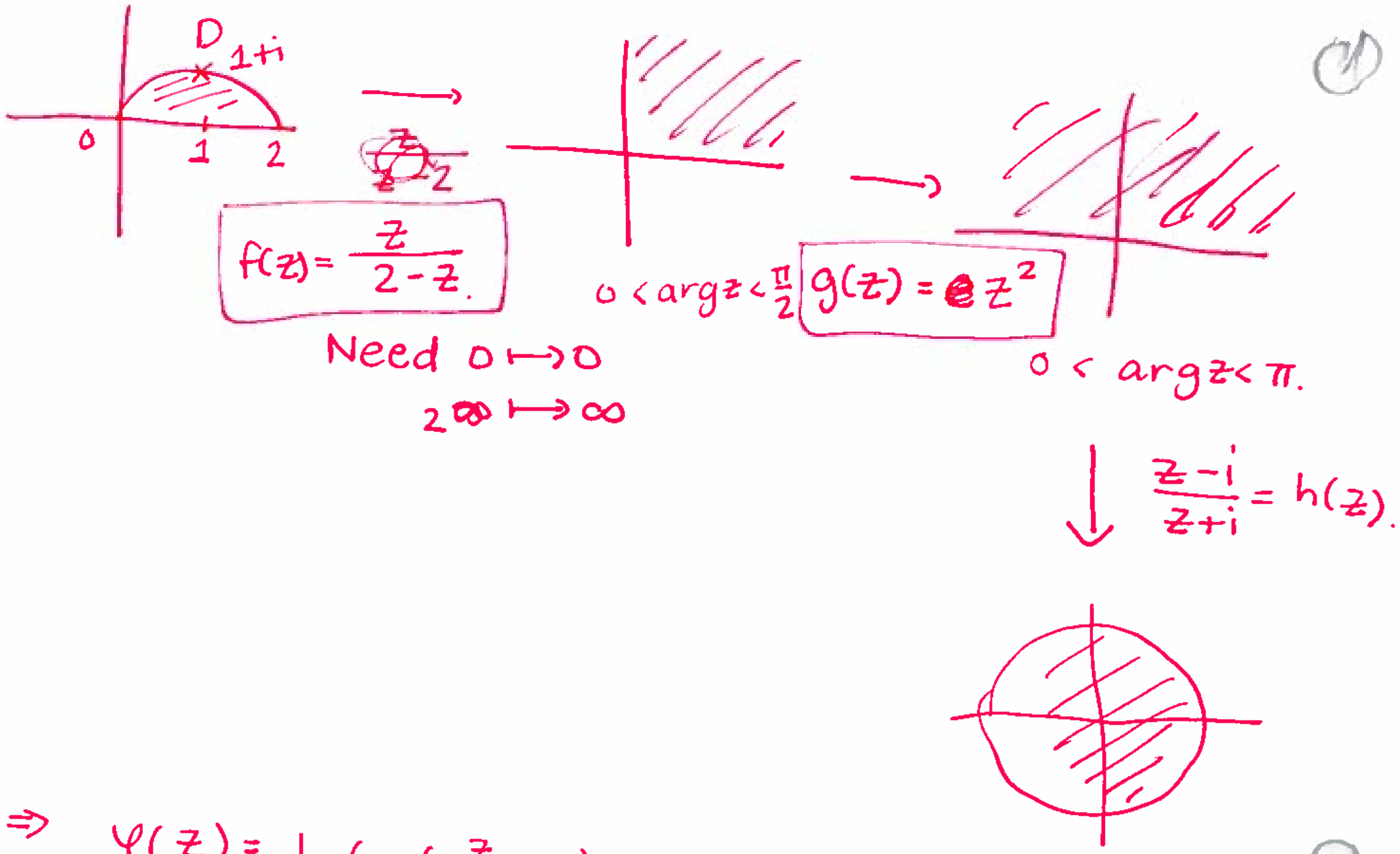
1.) D is simply connected.

2.) Every closed form ($\omega = Pdx + Qdy$, $P, Q \in C^1(D)$)
$Q_x = P_y$
in D is exact. ($\exists f \in C^2(D)$, $\omega = df$).

3.) $\forall a \notin D$, $\exists f$ holomorphic on D s.t. $e^{f(z)} = z - a$.

4.) $\exists\ \varphi : D \to \mathbb{D}$ conformal.

ex.) Find a conformal map from $D = \{|z-1|<1,\ \mathrm{Im}\,z>0\}$ onto $\mathbb{D}$.



$\boxed{f(z) = \frac{z}{2-z}}$

Need $0 \mapsto 0$

$2 \mapsto \infty$

$0 < \arg z < \frac{\pi}{2}$ $\boxed{g(z) = z^2}$

$0 < \arg z < \pi$.

$\frac{z-i}{z+i} = h(z)$.

$$\Rightarrow \quad \varphi(z) = h\left(g\left(\frac{z}{2-z}\right)\right) = h\left(\frac{z^2}{(2-z)^2}\right) = \frac{\frac{z^2}{(2-z)^2} - i}{\frac{z^2}{(2-z)^2} + i}$$

$$= \frac{z^2 - i(2-z)^2}{z^2 + i(2-z)^2}$$

Pf:

$1 \Rightarrow 2$: Ch. 3

$2 \Rightarrow 3$: Let $\omega = \frac{1}{z-a} dz$.

ω is closed in D (?)

$\Rightarrow \omega = df$ is exact, so f is holomorphic on D

$\Rightarrow f' = \frac{1}{z-a}$.

Choose f s.t. $f(z_0) \in \log(z_0 - a)$.

~~$\Rightarrow e^{f(z_0)} = z_0 - a.$~~

$\Rightarrow e^{f(z)} = z - a$?

I guess

$f(z) = \log(z-a) + c$.

$3 \Rightarrow 4$: Proof of Riemann Mapping thm.

$4 \Rightarrow 1$: $\exists\, \varphi : D \to \mathbb{D}$ homeomorphism s.t. D is simply connected. Let γ be a closed path in D.

Let $\Gamma : I \times I \longrightarrow \mathbb{D}$

$\varphi \circ \gamma$ is path homotopic to 0.

$\Rightarrow \gamma$ is homotopic to $\varphi^{-1}(0)$ by $\varphi^{-1} \circ \Gamma : I \times I \to D$.

Riemann Mapping Thm: If $D \subset \mathbb{C}$ is a simply connected domain, $D \neq \mathbb{C}$, then D is conformally equivalent to $\mathbb{D}$, i.e. $\exists \varphi : D \to \mathbb{D}$ conformal.

Note: If $\varphi : \mathbb{C} \to \mathbb{D}$ holomorphic, then

$$|\varphi(z)| \leq 1, \ \varphi \text{ analytic}$$

$$\Rightarrow \text{By Liouville, } \varphi(z) = c.$$

Corollary: If $D \subseteq \mathbb{C}^*$ is simply connected domain, then either $D = \mathbb{C}^*$ or D is conformally equivalent to $\mathbb{C}$ or $\mathbb{D}$.

Proof: Suppose $D \neq \mathbb{C}^*$.

Case 1: $D = \mathbb{C}^* \setminus \{a\}$. If $a = \infty$, $D = \mathbb{C}$. ✓

If $a \in \mathbb{C} \Rightarrow \varphi : \mathbb{C}^* \setminus \{a\} \to \mathbb{C}$

$$z \longmapsto \frac{1}{z-a}$$

$$\infty \longmapsto 0. \ ✓$$

Case 2: $D \subset \mathbb{C}^* \setminus \{a, b\}$

$$\varphi : D \overset{\subset \mathbb{C}^* \setminus \{a,b\}}{} \longrightarrow \varphi(D)$$

$$z \longmapsto \frac{1}{z-a}$$

$$a \not\mapsto \varphi(a)$$

$$b \not\mapsto \varphi(b)$$

$$\Rightarrow \varphi : D \to \varphi(D) \subset \mathbb{C}^* \setminus \{\varphi(a), \varphi(b)\}$$

$\therefore \varphi(D)$ is simply connected in $\mathbb{C}$...

$\Rightarrow$ By RMT, $\varphi(D) \neq \mathbb{C}$.
$\exists \psi : \varphi(D) \to \mathbb{D}$ conformal.

$\Rightarrow \psi \circ \varphi : D \to \mathbb{D}$ is conformal.

$\Rightarrow$ D conformally equivalent to $\mathbb{D}$.

Suppose γ_1, γ_2 are disjoint free analytic boundary arcs in ∂D. Then φ extends analytically across γ_1 & γ_2 and maps γ_1 & γ_2 one-to-one & onto arcs Γ_1 & Γ_2 resp. in $\partial\mathbb{D}$, $\Gamma_1 \cap \Gamma_2 = \emptyset$.

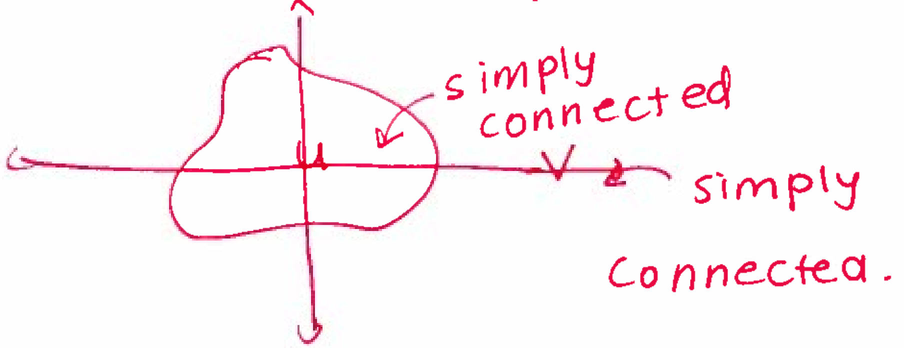
Jordan Domains:

$\gamma : [0,1] \to \mathbb{C}$ is continuous, $\gamma(0) = \gamma(1)$ and $\gamma(s) \neq \gamma(t)$ if $0 \leq s < t < 1$ is called a simply closed curve or a Jordan curve.

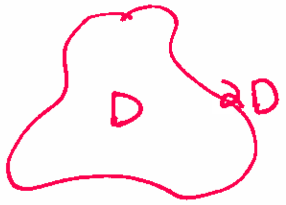
Equivalently, $\exists\, S^1 = \partial\mathbb{D} \xrightarrow[\text{homeo}]{\text{cts., bijection}} \Gamma = \gamma([0,1])$

Since S^1, Γ are compact, map is open, hence inverse is continuous.

Jordan Curve Theorem: If Γ is a Jordan curve in $\mathbb{C}$, then $\mathbb{C} \setminus \Gamma$ has two connected components, $\mathbb{C} \setminus \Gamma = U \cup V$, U bounded, V unbounded and $\partial U = \partial V = \Gamma$, U is simply connected, $V \cup \{\infty\}$ is simply connected in $\mathbb{C}^*$.



A bounded domain $D \subset \mathbb{C}$ whose boundary is a Jordan curve is called a Jordan domain.



Carathēdory Extension Theorem:

Let D be a Jordan domain, $\varphi : D \to \mathbb{D}$ conformal. Then φ has a homeomorphic extension $\varphi : \overline{D} \longrightarrow \mathbb{D}_{\text{closed}}$ which maps $\overline{D}$ onto $\overline{\mathbb{D}}$ and ∂D onto $\partial \mathbb{D}$.

11.5: Compactness of families of functions.

1.) Equicontinuity: (E,d), (K,d') metric spaces

$\mathcal{F}$ family of fncs. $f: E \to K$

- $\mathcal{F}$ equicontinuous at $z_0 \in E$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $d(z, z_0) < \delta$

  $\Rightarrow d^1(f(z), f(z_0)) < \varepsilon \quad \forall f \in \mathcal{F}$

- $\mathcal{F}$ equicontinuous on E if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $d(z, z^1) < \delta \Rightarrow d^1(f(z), f(z')) < \varepsilon \quad \forall f \in \mathcal{F}$.

<u>Lemma</u>: If E compact and $\mathcal{F}$ is equicontinuous at each $z_0 \in E$, then $\mathcal{F}$ is equicontinuous on E.

2.) $\mathcal{F}$ family of fncs. $f: (E,d) \to \mathbb{C}$

- $\mathcal{F}$ is pointwise bdd if $\forall z \in E$, $\exists M_z > 0$ s.t. $|f(z)| \leq M_z \quad \forall f \in \mathcal{F}$.
- $\mathcal{F}$ is uniformly bounded on E if $\exists M > 0$ s.t. $|f(z)| \leq M \quad \forall z \in E, \forall f \in \mathcal{F}$.

<u>Ascoli-Arzela Theorem</u>: If (E,d) is a compact metric space, $\mathcal{F}$ family of functions $f: E \to \mathbb{C}$. Assume $\mathcal{F}$ is pointwise bdd and equicts. on E. Then $\forall \{f_n\} \subset \mathcal{F}$, $\exists \{f_{n_k}\}$ which converges uniformly on E.

Montel's Theorem: Let $\mathcal{F}$ be a family of holomorphic fncs. on a domain $D \subseteq \mathbb{C}$ s.t. $\mathcal{F}$ is unif bdd. on each compact subset of D.

$\Rightarrow$ Any sequence in $\mathcal{F}$ has a normally convergent subsequence on D.

ex.) Let D be a domain, and fix $z_0 \in D$. Let $\mathcal{F}$

$= \{ f(z) : D \to f(D)$ s.t. $|f(z)| \leq 1 \}$.

We want to find $|\mathcal{F}'(z_0)|$.

Let $A = \sup \{ |f'(z_0)| : f \in \mathcal{F} \}$.

If $f_n(z) \in \mathcal{F}$ s.t. $|f_n'(z_0)| \to A$,

$\Rightarrow$ By Montel, the f_n's have a subsequence $f_{n_k}(z)$ that converges normally to $f(z)$ analytic on D.

By Thm. on pg. 137,

~~$|(f_n)'(z_0)| \to f'(z_0)$~~

$$f_{n_k}(z) \longrightarrow f(z)$$

$$\begin{tikzcd} {|f_{n_k}'(z_0)|} \arrow[r] \arrow[d] & {|f'(z_0)|} \\ A & \end{tikzcd}$$

$\Rightarrow |f'(z_0)| = A$, $g(z) \in A \Rightarrow |g(z)| \leq 1$.

Theorem: suppose $D \subset \mathbb{C}$ is a domain so that $\exists$ a nonconstant bounded holomorphic function h on D. Then $\exists G : D \to \mathbb{D}$ holomorphic s.t.

$$G(z_0) = 0$$
$$G'(z_0) \neq 0$$
$$|f'(z_0)| \leq |G'(z_0)|$$

$\forall$ holomorphic f on D w/ $|f(z)| < 1$ $\forall z \in D$

1.) $\mathcal{F} := \{ f \text{ holomorphic on } D,\ |f(z)| \leq 1\ \forall z \in D \}$

$h(z) = h(z_0) + a(z - z_0)^N +$ higher order terms.

$$|z - z_0| \leq r,\ \{|z - z_0| < r\} \subset D,\ N \geq 1,\ a \neq 0$$

$$N = \text{order}(h - h(z_0), z_0)$$

$$g(z) = \begin{cases} \varepsilon \left[\dfrac{h(z) - h(z_0)}{(z - z_0)^{N-1}} \right., & z \in D,\ z \neq z_0 \\ 0, & z = z_0 \end{cases}$$

g holomorphic on D:

$$|z - z_0| \leq r, \quad \left| \frac{h(z) - h(z_0)}{(z - z_0)^{N-1}} \right| \leq M$$

$$|g(z)| \leq \varepsilon M.$$

$$|z - z_0| > r \quad |g(z)| \leq \varepsilon \frac{2\|h\|_\infty}{r^{N-1}}$$

where $\|h\|_\infty = \sup\{|h(z)| : z \in D\} < \infty$.

$$\|g\|_\infty \leq \varepsilon \max\left\{M, \frac{2\|h\|_\infty}{r^{N-1}}\right\} < 1.$$

$g \in \mathcal{F}$, $g(z_0) = 0$, $g'(z_0) = a\varepsilon \neq 0$.

$\Rightarrow \mathcal{F} \neq \emptyset$.

2.) $A = \sup\{|f'(z_0)| : f \in \mathcal{F}\} \overset{?}{<} \infty$

$\cancel{|f'(z_0)|}$

$\exists f_n \in \mathcal{F}$ s.t. $|f_n'(z_0)| \to A$ as $n \to \infty$.

$\exists$ normal convergent subsequence $f_{n_k} \to G$.

$G : D \to \overline{\mathbb{D}}$ holomorphic   Montel's Thm.

$f_n'(z_0) \to G'(z_0) \Rightarrow A = |G'(z_0)| > 0$

By pg. 187.

$\therefore$ G non constant.

$|G'(z_0)| = A \geq |f'(z_0)| \quad \forall f \in \mathcal{F}$

Let's look at $\dfrac{G(z) - G(z_0)}{1 - \overline{G(z)}\,G(z)}$.

$$\Rightarrow \frac{d}{dz}\left(\frac{G(z)-G(z_0)}{1-\overline{G(z)}G(z)}\right) = \frac{(1-\overline{G(z)}G(z))G'(z) - \cancel{G(z)G(z)G(z)}\ (G(z)-G(z_0))\overline{G(z)}}{(1-\overline{G(z)}G(z))^2}$$

$$= \frac{G'(z) - |\cancel{\overline{G(z)}}|^2 G'(z) - |G(z)|^2 - G(z_0)\overline{G(z)}}{(1-|G(z)|^2)^2}$$

$$\left|\frac{d}{dz}\left(\frac{G(z)-G(z_0)}{1-|G(z)|^2}\right)_{z=z_0}\right| = \left|\frac{G'(z_0) - |G(z_0)|^2 G'(z_0)}{(1-|G(z_0)|^2)^2}\right|$$

$$= \frac{|G'(z_0)|(1-|G\cancel{'}(z_0)|^2)}{(1-|G(z_0)|^2)^{\cancel{2}}} = \frac{|G'(z_0)|}{1-|G(z_0)|^2} = \frac{A}{1-|G(z_0)|^2}$$

$\leq A.$

$\Rightarrow \quad 1 - |G(z_0)|^2 \geq 1$

$\Rightarrow |G(z_0)|^2 \leq 0$

$\Rightarrow |G(z_0)| = 0.$

11.6 : Riemann Mapping Theorem

If $D \subsetneq \mathbb{C}$ is a simply connected domain, then $\exists f : D \to \mathbb{D}$ conformal.

Proof: Fix $z_0 \in D$. Construct $f : D \to \mathbb{D}$ conformal w/ $f(z_0) = 0$.

$$\mathcal{F} = \{ f : D \to \mathbb{D} : f \text{ univalent}, f(z_0) = 0 \}.$$

1.) $\mathcal{F} \neq \emptyset$

2.) $M := \sup \{ |f'(z_0)| : f \in \mathcal{F} \} \in [0, \infty]$

$\exists f_0 \in \mathcal{F}$ so that $|f_0'(z_0)| = M$

3.) $f_0(D) = \mathbb{D}$.

– skip. –

Gamelin Exercises

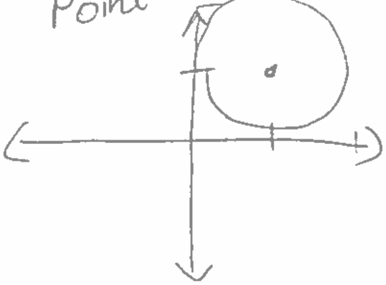
Chapter 1, Section 1

1) Identify and sketch the set of points satisfying:

(a) $|z-1-i|=1$

$|z-(1+i)|=1$

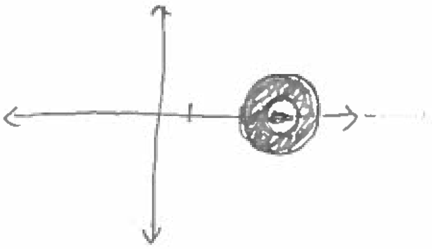
Point Radius



(b) $1 < |2z-6| < 2$

$1 < |2(z-3)| < 2$

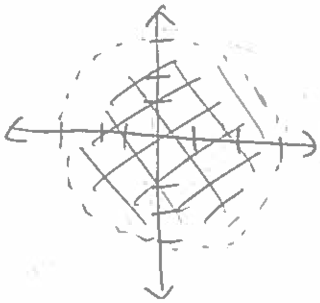
$\frac{1}{2} < |z-3| < 1$



(c) $|z-1|^2 + |z+1|^2 < 8$

$|(x-1)+iy|^2 + |(x+1)+iy|^2 < \infty$.

$|z^2-2z+1| + |z^2+2z+1| < 8$



Z

$|z-1| < \sqrt{8-|z+1|^2}$

$|z-1|^2 + |z+1|^2 - 8 < 0$

$|z|^2 - 2|z|$

$(x-1)^2 + y^2 + (x+1)^2 + y^2 < 8$

$x^2 - 2x + 1 + x^2 + 2x + 1 + 2y^2 < 8$

$2x^2 + 2 + 2y^2 < 8$

$x^2 + y^2 + 1 < 4$

$|z|^2 < 3$

(d) $|z-1| + |z+1| \leq 2.$

$$|(x-1)+iy| + |(x+1)+iy| \leq 2.$$

$$\sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} \leq 2.$$

$$(x-1)^2+y^2+2\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2} + (x+1)^2+y^2 \leq 4.$$

$$x^2-2x+1+y^2+2\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}$$
$$+x^2+2x+1 \leq 4.$$

$$2x^2+2+2|z-1||z+1| \leq 4$$

$$2|z^2-1| \leq 2-2x^2$$

$$|z^2-1| \leq 1-x^2 = -x^2+1$$

$$\Rightarrow (x-1)^2+y^2 \leq (-x^2+1)^2 = (-(x^2-1))^2$$
$$= (x^2-1)^2$$
$$= (x-1)^2(x+1)^2$$

$$\Rightarrow (1+(x+1))^2+y^2 \leq 0$$

2.)

(a) $\overline{z+w} = \bar{z} + \bar{w}$

$z = x + iy$
$w = a + ib$

$z + w = (x+a) + i(y+b)$
$\overline{z+w} = (x+a) - i(y+b) = (x - iy) + (a - ib) = \bar{z} + \bar{w}$

(b) $\overline{zw} = \overline{(x+iy)(a+ib)} = \overline{(xa + i(xb+ya) - yb)} = (xa - yb) - i(xb+ya)$
$\bar{z}\,\bar{w} = (x-iy)(a-ib) = (xa - yb) - i(xb + ya)$ ✓

(c) $|\bar{z}| = |z|$

$\bar{z} = x - iy$
$|\bar{z}| = \sqrt{x^2 + y^2}$
$|z| = \sqrt{x^2 + y^2}$

(d) $|z|^2 = z\bar{z}$

$|z|^2 = x^2 + y^2$
$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2.$

3.) Show that the equation $|z|^2 - 2\operatorname{Re}(\bar{a}z) + |a|^2 = \rho^2$ represents a circle centered at a with radius ρ.

$$(z-a)(\bar{z}-\bar{a}) = |z|^2 - \bar{a}z - a\bar{z} + |a|^2$$

$$= |z|^2 - \bar{a}z - \overline{\bar{a}z} + |a|^2$$

$$= |z|^2 - (\bar{a}z + \overline{\bar{a}z}) + |a|^2$$

$$\left[\quad \operatorname{Re}(\bar{a}z) = \frac{(\bar{a}z) + (\overline{\bar{a}z})}{2} \qquad \operatorname{Re}(z) = \frac{x+iy + x-iy}{2} \qquad \Rightarrow 2\operatorname{Re}(\bar{a}z) = \bar{a}z + (\overline{\bar{a}z}). \quad \right]$$

$$\Rightarrow \quad |z|^2 - 2\operatorname{Re}(\bar{a}z) + |a|^2 = |z-a|^2 = \rho^2$$

$$|(x-a_x) + (y-a_y)|^2 = \rho^2$$

$$(x-a_x)^2 + (y-a_y)^2 = \rho^2$$

↗

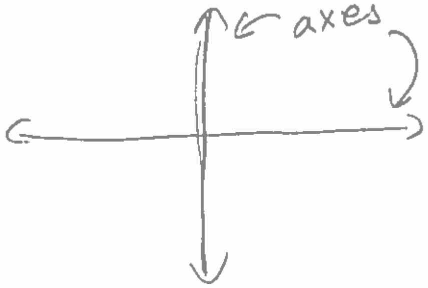
Circle centered at a with radius ρ.

4.) Show that $|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$, and sketch the set of points for which equality holds

$$|z| = |Re(z) + iIm(z)| \le |Re(z)| + |Im(z)|.$$

Where does $|z| = |Re z| + |Im(z)|$?

$$|x+iy| = |x| + |y|$$
$$x^2 + y^2 = |x|^2 + 2|x||y| + |y|^2$$
$$= x^2 + 2|x||y| + y^2$$
$$\Rightarrow |x||y| = 0.$$
$$\Rightarrow |x| = 0 \text{ or } |y| = 0$$
$$\Rightarrow x = 0 \text{ or } y = 0.$$



5.) Show that $|\mathrm{Re}\, z| \le |z|$ and $|\mathrm{Im}\, z| \le |z|$. Show that

$$|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}).$$

Use this to prove the triangle inequality

$$|z+w| \le |z| + |w|.$$

$x^2 \le x^2 + y^2$ $\qquad$ $y^2 \le x^2 + y^2$

$|\mathrm{Re}\, z|^2 \le |z|^2$ $\qquad$ $|\mathrm{Im}\, z|^2 \le |z|^2$

$\Rightarrow |\mathrm{Re}\, z| \le |z|.$ $\qquad$ $|\mathrm{Im}\, z| \le |z|$

$$\begin{aligned}
|z+w|^2 = (z+w)(\bar{z}+\bar{w}) &= |z|^2 + z\bar{w} + \bar{z}w + |w|^2 \\
&= |z|^2 + (z\bar{w}) + \overline{(z\bar{w})} + |w|^2 \\
&= |z|^2 + 2\left(\frac{z\bar{w} + \overline{z\bar{w}}}{2}\right) + |w|^2 \\
&= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2
\end{aligned}$$

WTS $\quad |z+w|^2 \le |z|^2 + |w|^2 + 2|z||w|.$

$$\begin{aligned}
|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) &\le \\
|z|^2 + |w|^2 &+ 2|z\bar{w}| \\
= |z|^2 + |w|^2 &+ 2|z||w|. \checkmark
\end{aligned}$$

$\Rightarrow |z+w| \le |z| + |w|$

6.) For fixed $a \in \mathbb{C}$, show that $\dfrac{|z-a|}{|1-\bar{a}z|} = 1$ if $|z|=1$ and $1-\bar{a}z \neq 0$.

WTS $|z-a| = |1-\bar{a}z|$.

Let $z = x+iy$
$a = m+in$.

$$\Rightarrow z-a = (x-m) + i(y-n).$$

$$|z-a| = \sqrt{(x-m)^2+(y-n)^2} = \sqrt{x^2-2xm+m^2+y^2-2yn+n^2} = \sqrt{1-2xm+m^2+n^2-2yn}$$

$$1-\bar{a}z = 1-(m-in)(x+iy)$$

$$= 1-(mx+imy-ixn+ny)$$

$$|1-\bar{a}z| = \sqrt{(1-mx-ny)^2+(-my+xn)^2} = \sqrt{(1-(mx+ny))^2+(xn-my)^2}$$

$$= \sqrt{1-2(mx+ny)+(mx+ny)^2+x^2n^2-2xnmy+m^2y^2}$$

$$= \sqrt{1-2mx-2ny+m^2x^2+\cancel{2mxny}+n^2y^2+x^2n^2-\cancel{2xnmy}+m^2y^2}$$

$$= \sqrt{1-2mx-2ny+m^2(x^2+y^2)+n^2(x^2+y^2)}$$

$$= \sqrt{1-2mx-2ny+(m^2+n^2)(x^2+y^2)}$$

$$= \sqrt{1-2mx-2ny+|a|^2|z|^2}$$

$$= \sqrt{1-2xm-2yn+m^2+n^2} = |z-a|.$$

$$\Rightarrow \frac{|z-a|}{|1-\bar{a}z|} = 1.$$

7.) Fix $\rho > 0$, $\rho \neq 1$, and fix $z_0, z_1 \in \mathbb{C}$. Show that the set of z satisfying $|z - z_0| = \rho|z - z_1|$ is a circle. Sketch it for $\rho = \frac{1}{2}$ and $\rho = 2$, with $z_0 = 0$ and $z_1 = 1$. What happens when $\rho = 1$?

$z_0 = x_0 + i y_0$

$z_1 = x_1 + i y_1.$

$$(x - x_0)^2 + (y - y_0)^2 = \rho^2[(x - x_1)^2 + (y - y_1)^2]$$

$$x^2 - 2x x_0 + x_0^2 + y^2 - 2y y_0 + y_0^2$$
$$= \rho^2[x^2 - 2x_1 x + x_1^2 + y^2 - 2y y_1 + y_1^2]$$

$$\Rightarrow x^2[1 - \rho^2] - 2x[x_0 - \rho^2 x_1] + y^2[1 - \rho^2]$$
$$-2y[y_0 - y_1\rho^2] + y_0^2 + x_0^2 - \rho^2 x_1^2 + \rho^2 y_1^2 = 0$$

$$\Rightarrow (x^2 + y^2)(1 - \rho^2) - 2x[x_0 - \rho^2 x_1] - 2y[y_0 - y_1\rho^2]$$
$$+ y_0^2 + x_0^2 - \rho^2 x_1^2 + \rho^2 y_1^2 = 0$$

Toyota

8) Let $p(z)$ be a polynomial of degree $n \geq 1$ and let $z_0 \in \mathbb{C}$. Show that there is a polynomial $h(z)$ of degree $n-1$ so that $p(z) = (z-z_0)h(z) + p(z_0)$. In particular if $p(z_0)=0$, then $p(z) = (z-z_0)h(z)$.

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$.

$h(z) = b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + \dots + b_1 z + b_0$.

Let $$a_n z^n + \dots + a_1 z + a_0 = (z-z_0)(b_{n-1}z^{n-1} + \dots + b_0) + a_n z_0^n + a_{n-1} z_0^{n-1} + \dots + a_1 z_0 + a_0$$

$$= b_{n-1} z^n + \dots + b_1 z^2 + b_0 z - b_{n-1} z^{n-1} z_0 - \dots - b_1 z z_0 - b_0 z_0 + a_n z_0^n + a_{n-1} z_0^{n-1} + \dots + a_1 z_0 + a_0$$

$$\Rightarrow a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + (\cancel{a_0} - a_n z_0^n - a_{n-1} z_0^{n-1} - \dots - \cancel{a_0})$$

$$= b_{n-1}(z^n - z^{n-1} z_0) + \dots + b_1(z^2 - z z_0) + b_0(z - z_0).$$

$$= z^{n-1}(b_{n-1} z - z_0) + \dots + z^1(b_1 z - z_0) + (b_0 z - z_0)$$

$$\Rightarrow b_{n-1} z - z_0 = a_{n-1}$$

$$\Rightarrow b_{n-1} = \frac{a_{n-1} + z_0}{z}$$

$\vdots$

$$b_1 z - z_0 = a_1 \Rightarrow b_1 = \frac{a_1 + z_0}{z}$$

$$b_0 z - z_0 = -a_n z_0^n - a_{n-1} z_0^{n-1} - \dots - a_1 z_0$$

Gamelin Exercises

Section 1.2:

1. Express all values of the following expressions in both polar and cartesian coordinates, and plot them.

(a) $\sqrt{i} = i^{1/2}$

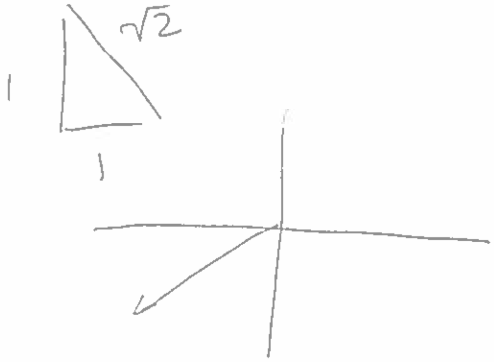
$i = e^{\frac{\pi i}{2}} = r^2 e^{2i\theta}$

$\Rightarrow \quad r = 1^{1/2}$

$2\pi k + \frac{\pi}{2} = 2\theta \quad , k = 0, 1$

$\Rightarrow \theta = \frac{2\pi k}{2} + \frac{\pi}{4} , k = 0, 1$

$\Rightarrow \theta = \pi k + \frac{\pi}{4} , k = 0, 1.$



$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}.$

$\Rightarrow \sqrt{i} = e^{\pi/4 i}, e^{\frac{5\pi}{4} i}$

$= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$

$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$

(c) $\sqrt[4]{-1}$

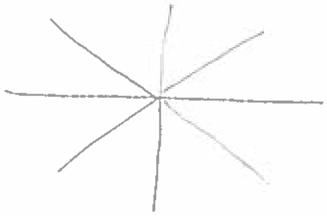
$-1 = e^{\pi i} = r^4 e^{\theta i 4}$

$\Rightarrow r = 1$

$2\pi k + \pi = 4\theta$, $k = 0, 1, 2, 3$

$\theta = \frac{2\pi k}{4} + \frac{\pi}{4} = \frac{2\pi k}{4} + \frac{\pi}{4}$, $0, 1, 2; 3 = k.$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \overset{-\frac{3\pi}{4}}{\frac{5\pi}{4}}, \overset{-\frac{\pi}{4}}{\frac{7\pi}{4}}.$

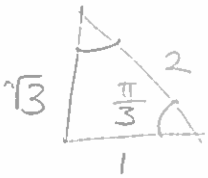


$\Rightarrow \sqrt[4]{-1} = e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{-3\pi i}{4}}, e^{-\frac{\pi i}{4}}$

$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$

(e) $(-8)^{\frac{1}{3}}.$

$-8 = 8e^{\pi i} = r^3 e^{3i\theta}$

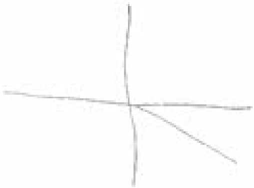


$r = 2$

$3\theta = \pi + 2\pi k$, $k = 0, 1, 2$

$\Rightarrow \theta = \frac{\pi}{3} + \frac{2\pi k}{3}$

$\Rightarrow (-8)^{1/3} = 2e^{\pi/3 i}, 2e^{\pi i}; 2e^{\frac{5\pi i}{3}} = 2e^{-\frac{\pi i}{3}}$



$\Rightarrow (-8)^{1/3} = \frac{2}{2} + i\frac{2\sqrt{3}}{2}, -2, \frac{2}{2} - i\sqrt{3}$

$= 1 + \sqrt{3}i, -2, 1 - \sqrt{3}i$

Gamelin Chapter 1, Section 5 Exercises:

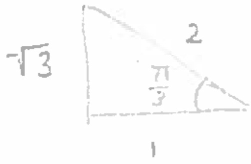
1. Calculate and plot e^z for the following points z:

(a) 0. $e^0 = 1$

(b) $\pi i + 1$

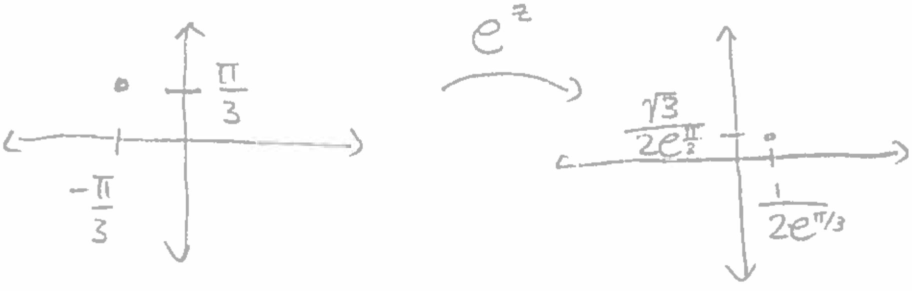
$$e^{\pi i+1} = e(\cos(\pi)+i\sin(\pi)) = e(-1+i0) = -e$$

(c) $\frac{\pi(i-1)}{3} = \frac{\pi i}{3} - \frac{\pi}{3}$



$$e^{-\frac{\pi}{3}+\frac{\pi i}{3}} = e^{-\frac{\pi}{3}}\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$$

$$= e^{-\frac{\pi}{3}}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \frac{1}{2e^{\frac{\pi}{3}}} + i\left(\frac{\sqrt{3}}{2e^{\frac{\pi}{3}}}\right)$$



(d) $37\pi i$

$$e^{37\pi i} = \cos(37\pi) + i\sin(37\pi)$$

$$= \cos(2\cdot18\pi+\pi) + i\sin(2\cdot18\pi+\pi)$$

$$= \cos(\pi) + i\sin(\pi) = -1$$

18
2) 36
2
16

(e) $\frac{\pi i}{m}$, $m = 1, 2, 3, \ldots$

$$e^{\frac{\pi}{m} i} = \cos\left(\frac{\pi}{m}\right) + i\sin\left(\frac{\pi}{m}\right), \quad m = 1, 2, \ldots$$

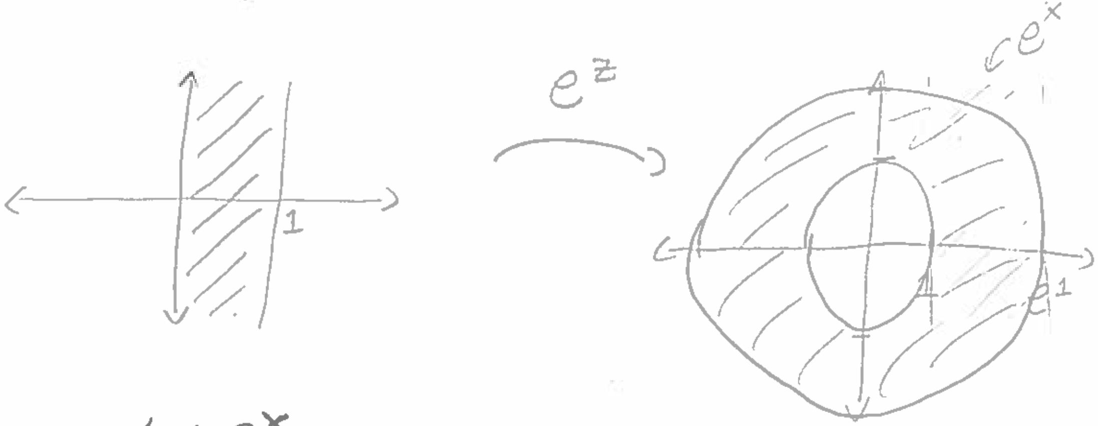
(f) $m(i-1)$, $m = 1, 2, 3, \ldots$

$$e^{m(i-1)} = e^{mi} e^{-m} = e^{-m}(\cos(m) + i\sin(m))$$

2.) Sketch each of the following figures and its images under the exponential map $w=e^z$. Indicate the images of horizontal and vertical lines in your sketch

(a) The vertical strip $0<\text{Re}(z)<1$.

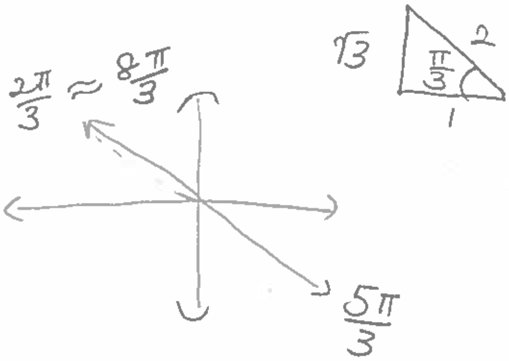
$e^0 < e^{\text{Re}(z)} < e^1$. The angle varies.



$1 < e^x < e$

e^{iy} varies.

(b) The horizontal strip

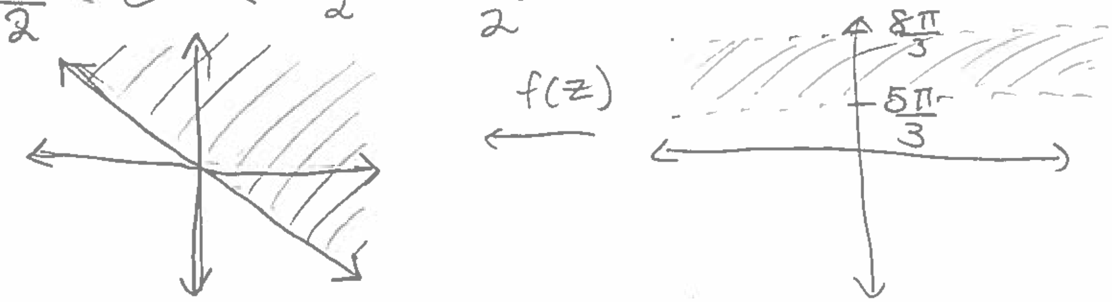


$$\frac{5\pi}{3} < \text{Im}(z) < \frac{8\pi}{3}$$

$$e^{\frac{5\pi i}{3}} < e^{iy} < e^{i\frac{8\pi}{3}}$$

$$\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) < e^{iy} < \cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right)$$

$$\frac{1}{2} - i\frac{\sqrt{3}}{2} < e^{iy} < -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$



3.) Show that $e^{\bar{z}} = \overline{e^{z}}$

$\bar{z} = x - iy$

$e^{\bar{z}} = e^{x}e^{-iy}$

$$e^{z} = e^{x+iy} = e^{x}e^{iy}$$
$$= e^{x}(\cos y + i\sin(y))$$
$$= e^{x}\cos y + i(e^{x}\sin(y)).$$
$$\Rightarrow \overline{e^{z}} = e^{x}\cos y - i(e^{x}\sin y).$$
$$= e^{x}(\cos y - i\sin y)$$
$$= e^{x}e^{-iy}$$
✓

4.) Show that the only periods of e^{z} are the integral multiples of $2\pi i$, that is, if $e^{z+\lambda} = e^{z}$ $\forall z$, then λ is an integer times $2\pi i$

Suppose $e^{z+\lambda} = e^{z}$ $\forall z$

$\Rightarrow z = x+iy, \lambda = a+ib.$

$\Rightarrow e^{x+iy+a+ib} = e^{x+iy}$

$\Rightarrow e^{z}e^{\lambda} = e^{z}.$

$e^{z} \neq 0$ since $e^{x} \neq 0$, $e^{iy} = \cos y + i\sin y \neq 0$

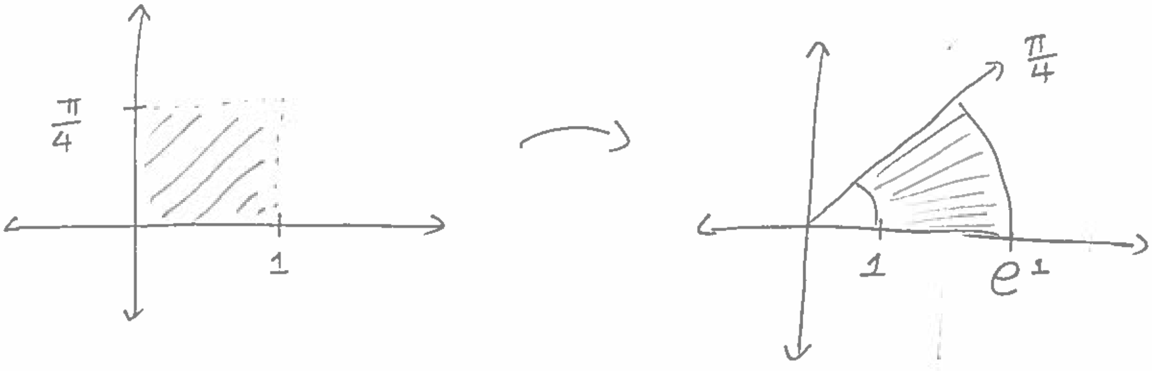
$\Rightarrow e^{\lambda} = 1.$

$\lambda = 2\pi k i$, $k \in \mathbb{Z}.$

since $e^{0} = 1$ and e is $2\pi i$-periodic.

2. ctd.)

(c) The rectangle $0<x<1$, $0<y<\frac{\pi}{4}$



$$e^0 < e^x < e^1$$
$$e^{i0} < e^{iy} < e^{i\frac{\pi}{4}}$$

(d) The disk $|z| \leq \frac{\pi}{2}$

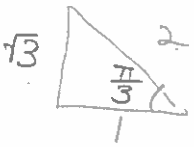
$$0 \leq \sqrt{x^2+y^2} \leq \frac{\pi}{2}$$

$$|r\cos\theta + ri\sin\theta| \leq \frac{\pi}{2}$$
$$\Rightarrow \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \leq \frac{\pi}{2}$$

$$\Rightarrow r^2 \leq \frac{\pi^2}{4} \Rightarrow r \leq \pm\frac{\pi}{2}$$

$\log(1+i) = \log(\sqrt{2}) + \frac{i\pi}{4} + 2\pi i m$, $m = 0, \pm 1, \pm 2, \ldots$

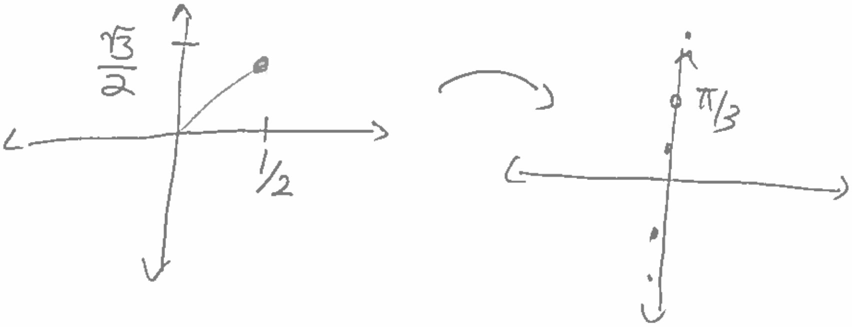
(d) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$



$$\mathrm{Log}\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) = \log\sqrt{\tfrac{1}{4} + \tfrac{3}{4}} + i\,\mathrm{Arg}\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)$$

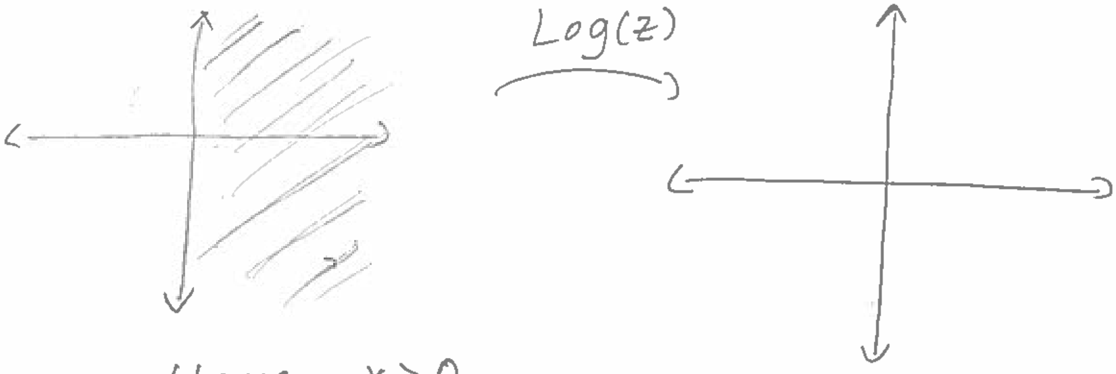
$$= \log\sqrt{1} + \frac{i\pi}{3} = i\frac{\pi}{3}$$

$\Rightarrow \log\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = i\frac{\pi}{3} + 2\pi i m$, $m = 0, \pm 1, \pm 2, \ldots$



2.) Sketch the image under the map $w = \mathrm{Log}(z)$ of each of the following figures.

(a) The right half-plane : $\mathrm{Re}(z) > 0$.



Have $x > 0$.

$\Rightarrow \mathrm{Log}(z) = \log|z| + i\arg(z)$.

$-\frac{\pi}{2} <$ angle $< \frac{\pi}{2}$

$-\infty < y < \infty$

$+\infty > x > 0$

$|z| = \sqrt{x^2 + y^2} > \sqrt{y^2}$

Chapter 1, Section 6: Exercises:

1.) Find and plot $\log z$ for the following complex numbers z. Specify the principal value.

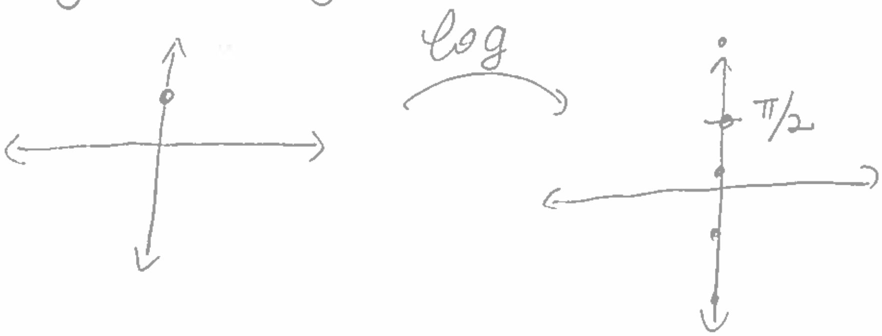
(a) 2

$$\mathrm{Log}(2) = \log|2| + i\,\mathrm{Arg}(2) = \log(2) + i(0) = \underset{\substack{\uparrow \\ \mathrm{PV}}}{\log(2)}$$

$$\log(2) = \log(2) + i2\pi m\ , \quad m = 0, \pm 1, \pm 2, \ldots$$

(b) i

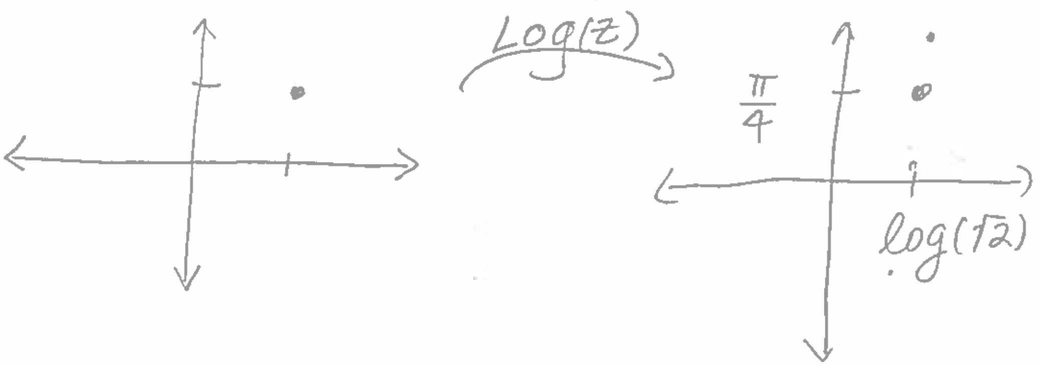
$$\mathrm{Log}(i) = \log|i| + i\,\mathrm{Arg}(i) = 0 + \frac{i\pi}{2} \leftarrow \mathrm{PV}$$



$$\log(i) = \frac{i\pi}{2} + i\,2\pi m\ , \quad m = 0, \pm 1, \pm 2, \ldots$$

(c) $1+i$

$$\mathrm{Log}(1+i) = \log\sqrt{1+1} + i\,\mathrm{Arg}(1+i)$$
$$= \log\sqrt{2} + \frac{i\pi}{4}$$



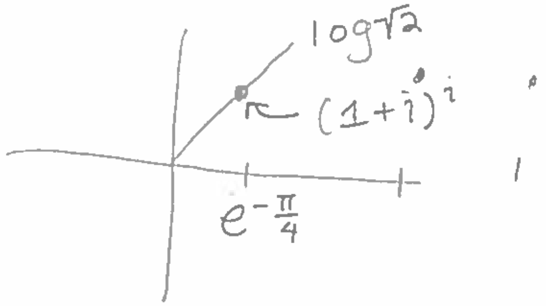
Chapter 1, Section 7 Exercises:

1.) Find all values and plot:

(a) $(1+i)^i$ = (think = $e^{\log(1+i)^i} = e^{i\log(1+i)}$)

$= e^{i(\log\sqrt{1+1} + i\,\mathrm{Arg}(1+i) + 2\pi i m)} = e^{i\log\sqrt{2} - \frac{\pi}{4} - 2\pi m}$

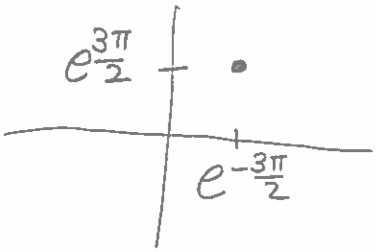
$$= \underbrace{e^{i\log\sqrt{2}}}_{\text{angle}} \underbrace{e^{-\frac{\pi}{4}} e^{-2\pi m}}_{\text{real part}}$$



(b) $(-i)^{1+i}$ ($= e^{\log(-i)^{1+i}}$) $= e^{(1+i)\log(-i)}$

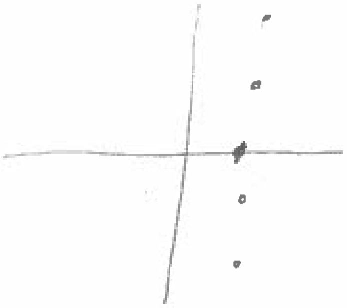
$= e^{(1+i)(\log|-1| + i\arg(-i))} = e^{(1+i)(i(\frac{3\pi}{2} + 2\pi m))}$

$= e^{i\frac{3\pi}{2} + i2\pi m - \frac{3\pi}{2} - 2\pi m} = e^{i(\frac{3\pi}{2} + 2\pi m)} e^{-(\frac{3\pi}{2} + 2\pi m)}$



(c) $2^{-1/2}$ ($= e^{\log(2^{-1/2})}$) $= e^{-\frac{1}{2}\log(2)}$

$= e^{-\frac{1}{2}(\log 2 + 2\pi m i)} = e^{-\frac{1}{2}\log 2} e^{-\pi m i}$



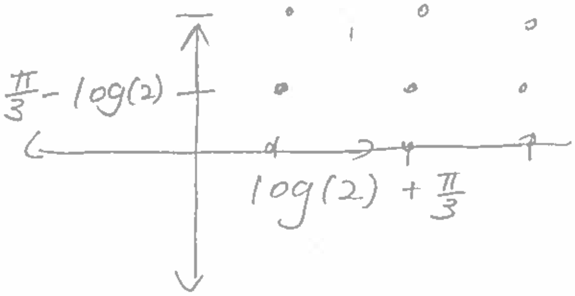
(d) $(1+i\sqrt{3})^{(1-i)} = e^{\log(1+i\sqrt{3})^{(1-i)}} = e^{(1-i)\log(1+i\sqrt{3})}$

$= e^{(1-i)(\log|1+i\sqrt{3}| + i\arg(1+i\sqrt{3}))}$ ○

$= e^{(1-i)(\log\sqrt{1+3} + i(\frac{\pi}{3}+2\pi m))}$

$= e^{\log 2 + i\frac{\pi}{3} + i2\pi m - i\log 2 + \frac{\pi}{3} + 2\pi m}$

$= e^{(\log(2)+\frac{\pi}{3}+2\pi m)} e^{i(\frac{\pi}{3}-\log(2)+2\pi m)}$



2.) Compute and plot $\log[(1+i)^{2i}]$

$(1+i)^{2i} = e^{\log(1+i)^{2i}} = e^{2i\log(1+i)} = e^{2i(\log\sqrt{2} + \frac{i\pi}{4} + 2\pi}$ ○

$= e^{2i\log\sqrt{2}} e^{-\frac{\pi}{2}} e^{-4\pi m}$

$\Rightarrow \log(1+i)^{2i} = \log|e^{2i\log\sqrt{2} - \frac{\pi}{2} - 4\pi m}| + i\arg(e^{2i\log\sqrt{2} - \frac{\pi}{2} - 4\pi m})$

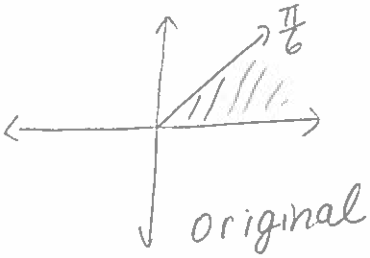
$= \cancel{2i\log\sqrt{2}} - \frac{\pi}{2} - 4\pi m + i(2\log\sqrt{2}) + i2\pi m$

$= \cancel{i\log(2)} - \frac{\pi}{2} - 4\pi m + i\log(2) + 2\pi i m$

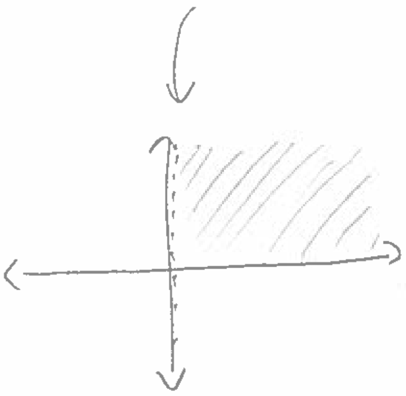
since $\log|e^{iy}| = \log\sqrt{\cos^2 y + \sin^2 y}$ ○

(3.) Sketch the image of the sector $\{0 < \arg z < \frac{\pi}{6}\}$ under the map $f(z) = z^a$ for

(a) $a = 3/2.$



$$z^{3/2} = e^{\frac{3}{2}(\log|z| + i\arg z)}$$
$$= e^{\frac{3}{2}\log|z|} e^{i\frac{3}{2}\arg z}$$



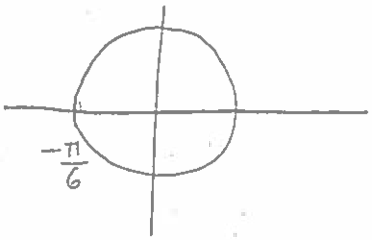
$$0 \cdot \frac{3}{2} < \frac{3}{2}\arg z < \frac{3}{2}\frac{\pi}{6}$$
$$0 < \frac{3}{2}\arg z < \frac{\pi}{4}$$
$$\frac{3}{2}\cdot 0 < \frac{3}{2}\log|z| < \frac{3}{2}\cdot\infty.$$

(b) $a = i$

$$z^i = e^{\log(z^i)} = e^{i\log(z)} = e^{i(\log|z| + i\operatorname{Arg}(z))} =$$
$$e^{i\log|z|} e^{-\operatorname{Arg}(z)}$$

$-\frac{\pi}{6} <$ radius $< 0.$

$0 <$ angle $< \infty$?



4.) Show that $(zw)^a = z^a w^a$, where on the right we take all possible products.

$(zw)^a = e^{a\log(zw)} = e^{a(\log|zw| + i\arg(zw))}$

(Remember, $\arg(zw) = \arg(z) + \arg(w)$)

$= e^{a(\log|z| + \log|w| + i\arg(z) + i\arg(w))}$

$= e^{a(\log|z| + i\arg z)} e^{a(\log|w| + i\arg(w))}$

$= z^a w^a$

5.) Find i^{i^i}. Show that it does not coincide with $i^{i \cdot i} = i^{-1}$

$i^{i^i} = (i^i)^i = e^{i\log(i^i)} = e^{i\log|i^i| + i\arg(i^i)}$

$\left[i^i = e^{i(\log|i| + i\arg(i))} = e^{-\frac{\pi}{2} - 2\pi n}. \right.$

$= e^{i(\log|e^{-\frac{\pi}{2}-2\pi n}| + i\arg(e^{-\frac{\pi}{2}-2\pi n}))} = e^{i(-\frac{\pi}{2}-2\pi n)} e^{i\,\mathrm{Arg}(e^{-\frac{\pi}{2}-2\pi n})} e^{2\pi i}$

$\mathrm{Arg}(e^{-\frac{\pi}{2}-2\pi n}) = 0$!!! so

$= e^{i(-\frac{\pi}{2}-2\pi n)} e^{-2\pi m} = -i e^{2\pi m}.$

Chapter 1, Section 8:

1.) Establish the following addition formulae:

(a) $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.

$$\cos(z+w) = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw}}{2}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(z)\cos(w) = \frac{(e^{iz}+e^{-iz})}{2} \cdot \frac{(e^{iw}+e^{-iw})}{2}$$

$$= \frac{e^{iz}e^{iw} + e^{iz}e^{-iw} + e^{-iz}e^{iw} + e^{-iz}e^{-iw}}{4}$$

$$\sin(z)\sin(w) = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right)$$

$$= \frac{e^{iz}e^{iw} - e^{iz}e^{-iw} - e^{-iz}e^{iw} + e^{-iz}e^{-iw}}{-4}$$

$$\Rightarrow \cos(z)\cos(w) - \sin(z)\sin(w) = \frac{2e^{iz}e^{iw} + 2e^{-iz}e^{-iw}}{4}$$

$$= \cos(z+w).$$ ✓

(b) $\sin(z+w) = \sin z \cos(w) + \cos z \sin w$

$$\sin(z+w) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \frac{e^{iz}e^{iw} - e^{-iz}e^{-iw}}{2i}$$

$$\sin(z)\cos(w) = \frac{(e^{iz} - e^{-iz})}{2i}\frac{(e^{iw} + e^{-iw})}{2} = \frac{e^{iz}e^{iw} + e^{iz}e^{-iw} - e^{-iz}e^{iw} - e\,e^{-iw}}{4i}$$

$$\cos(z)\sin(w) = \frac{e^{iz} + e^{-iz}}{2} \cdot \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{iz}e^{iw} - e^{iz}e^{-iw} + e^{-iz}e^{iw} - e^{-iz}e^{-i}}{4i}$$

$$\Rightarrow \sin(z)\cos(w) + \cos(z)\sin(w) = \frac{2e^{iz}e^{iw} - 2e^{-iz}e^{-iw}}{4i} = \frac{e^{iz}e^{iw} - e^{-iz}e^{-i}}{2i}$$

$$= \sin(z+w) \checkmark$$

(c) $\cosh(z+w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$

$$\cosh(z+w) = \cos(i(z+w)) = \cos(iz+iw) =$$

$$\cos(iz)\cos(iw) - \sin(iz)\sin(iw)$$

$$= \cosh(z)\cosh(w) - i\sinh(z)\, i\sinh(w)$$

$$= \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$$

✓

(d) $\sinh(z+w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$

$$\sinh(z+w) = -i\sin(i(z+w)) = -i[\sin(iz)\cos(iw) + \sin(iw)\cos(iz)]$$

$$= -i[i\sinh(z)\cosh(w) + i\sinh(w)\cosh(z)]$$

$$= \sinh(z)\cosh(w) + \sinh(w)\cosh(z)$$

✓

2.) Show that $|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$, where $z = x + iy$. Find all zeros and periods of $\cos z$.

$$\cos(z) = \cos(x+iy) = \cos(x)\cos(iy) - \sin(iy)\sin(x)$$
$$= \cos(x)\cosh(y) - i\sinh(y)\sin(x).$$

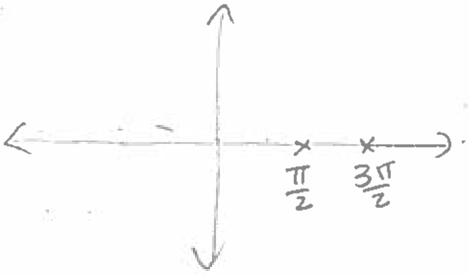
$$\Rightarrow |\cos(z)|^2 = \cos^2(x)\cosh^2(y) + \sinh^2(y)\sin^2(x).$$
$$= \cos^2(x)[1+\sinh^2(y)] + \sinh^2(y)\sin^2(x)$$
$$= \cos^2(x) + \sinh^2(y)[\cos^2(x)+\sin^2(x)]$$
$$= \cos^2(x) + \sinh^2(y). \quad ✓$$

Zeroes are when $\cos(x) = 0$, $\sinh(y) = 0$.

$x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n$

$y = 0.$

$$\frac{e^y - e^{-y}}{2} = 0 \Leftrightarrow e^y - e^{-y} = 0 \Leftrightarrow e^{2y} - 1 = 0$$
$$\Leftrightarrow e^{2y} = 1$$
$$\Leftrightarrow 2y = 0$$
$$\Leftrightarrow y = 0.$$



$2\pi n$ period

4.) Show that $\tan^{-1}(z) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$, where both sides of the identity are to be interpreted as subsets of the complex plane. In other words, show that $\tan w = z$ $\Leftrightarrow$ $2iw$ is one of the values of the logari m featured on the right.

Let $w = \tan^{-1}(z) = \frac{1}{2i}\log\left(\frac{1+zi}{1-iz}\right)$

$$\Rightarrow 2iw = \log\left(\frac{1+iz}{1-iz}\right)$$

$$z = \tan(w) = \frac{e^{iw}-e^{-iw}}{2i} \cdot \frac{2}{e^{iw}+e^{-iw}} = \frac{-i(e^{iw}-e^{-iw})}{(e^{iw}+e^{-iw})}$$

$$\Rightarrow e^{iw}z + e^{-iw}z = -ie^{iw} + ie^{-iw}$$

$$\Rightarrow e^{iw}z + e^{-iw}z + ie^{iw} - ie^{-iw} = 0$$

$$\Rightarrow e^{2iw}z + z + e^{2iw}i - i = 0$$

$$\Rightarrow e^{2iw}(z+i) + (z-i) = 0.$$

$$\Rightarrow e^{2iw} = \frac{i-z}{i+z}$$

$$\Rightarrow 2iw = \log\left(\frac{i-z}{i+z}\right) = \log\left(\frac{1-\frac{z}{i}}{1+\frac{z}{i}}\right) = \log\left(\frac{1+iz}{1-iz}\right)$$ ✓

7.) Let $w = \cos(z)$ and $\zeta = e^{iz}$. Show that $\zeta = w \pm \sqrt{w^2-1}$. Show that $\cos^{-1}(w) = -i\log[w \pm \sqrt{w^2-1}]$, where both sides of the identity are to be interpreted as subsets of the complex plane.

~~$\zeta = e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y} = e^{-y}(\cos x + i\sin y)$~~

~~$b = -w$~~

~~$4ac = 1$~~

$$w = \frac{e^{iz}+e^{-iz}}{2} = \frac{\zeta + \frac{1}{\zeta}}{2} = \frac{\zeta^2+1}{2\zeta}$$

$$\Rightarrow \zeta^2 + 1 = 2w\zeta$$

$$\Rightarrow \zeta^2 = 2w\zeta - 1$$

$$\Rightarrow \zeta^2 - 2w\zeta + 1 = 0$$

$$\Rightarrow \zeta = \frac{+2w \pm \sqrt{4w^2-4}}{2} = w \pm \sqrt{w^2-1} \checkmark$$

Show that $\cos^{-1}(w) = -i\log[w \pm \sqrt{w^2-1}]$.

want to find z.

$$e^{iz} = w \pm -\sqrt{w^2-1}.$$

$$\Rightarrow iz = \log[w \pm \sqrt{w^2-1}]$$

$$\Rightarrow z = -i\log[w \pm \sqrt{w^2-1}]$$

Chapter 2, Section 2: Exercises:

1.) Find the derivatives of the following functions:

(a) $z^2-1=f(z)$

$f'(z)=2z$

(b) $f(z)=z^n-1$

$f'(z)=nz^{n-1}$

(c) $f(z)=(z^2-1)^n$

$a(z)=z^n \qquad a'(z)=nz^{n-1}$

$b(z)=z^2-1 \qquad b'(z)=2z$

$f'(z)=n(z^2-1)^{n-1}(2z)$

(d) $\frac{1}{1-z}=f(z)$

$a(z)=\frac{1}{z} \qquad a'(z)=-\frac{1}{z^2}$

$b(z)=1-z \qquad b'(z)=-1$

$f'(z)=\frac{-1}{(1-z)^2}\cdot(-1)=\frac{1}{(1-z)^2}$

(e) $\frac{1}{z^2+3}=f(z)$

$a(z)=\frac{1}{z} \qquad a'(z)=-\frac{1}{z^2}$

$b(z)=z^2+3 \qquad b'(z)=2z$

$f'(z)=\frac{-1}{(z^2+3)^2}\cdot 2z$

(f) $f(z)=\frac{z}{z^3+5}$

$f'(z)=\frac{(z^3+5)-z(3z^2)}{(z^3+5)^2}$

(g) $\frac{az+b}{cz+d} = f(z)$

$$\Rightarrow f'(z) = \frac{(cz+d)a - (az+d)c}{(cz+d)^2}$$

(h) $\frac{1}{(cz+d)^2} = f(z)$

$a(z) = \frac{1}{z^2} \qquad a'(z) = -2z^{-3} = \frac{-2}{z^3}$

$b(z) = cz+d \qquad b'(z) = c$

$$f'(z) = \frac{-2}{(cz+d)^3} \cdot c$$

2.) Show that

$$1+2z+3z^2+ \ldots + nz^{n-1} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}.$$

$$1+z+z^2+z^3+ \ldots + z^n = \frac{1-z^{n+1}}{1-z}$$

$$\Rightarrow 1+2z+3z^2+ \ldots + nz^{n-1} = \frac{(1-z)(-(n+1)z^n) + (1-z^{n+1})}{(1-z)^2}$$

$$= \frac{-nz^n - z^n}{(1-z)} + \frac{1-z^{n+1}}{(1-z)^2}$$

$$= \frac{-nz^n}{1-z} + \frac{-z^n(1-z) + 1 - z^{n+1}}{(1-z)^2} = \frac{-nz^n}{1-z} + \frac{-z^n + \cancel{z^{n+1}} + 1 - \cancel{z^{n+1}}}{(1-z)^2}$$

$$= \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}$$ ✓

$$(1+z+z^2+z^3+\ldots+z^n)(1-z)$$

$$= 1+z+z^2+\ldots+z^n$$

$$- \quad z - z^2 - \ldots - z^n - z^{n+1}$$

$$= 1 - z^{n+1}$$ ✓

3.) Show from the defin of the functions $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ are not complex diff'ble at any point.

$$x = \mathrm{Re}(z) = \frac{x+iy+x-iy}{2} = \frac{z+\bar{z}}{2}$$

$$\Rightarrow \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\frac{(z+\Delta z)+(\bar{z}+\overline{\Delta z})}{2} - \frac{z+\bar{z}}{2}}{\Delta z}$$

$$= \frac{\cancel{\frac{z}{2}-\frac{z}{2}} + \frac{\Delta z}{2} + \frac{\overline{\Delta z}}{2} + \cancel{\frac{\bar{z}}{2}-\frac{\bar{z}}{2}}}{\Delta z} = \frac{\Delta z + \overline{\Delta z}}{2\Delta z}$$

If $\Delta z = \varepsilon \Rightarrow \dfrac{\varepsilon+\varepsilon}{2\varepsilon} = 1$

If $\Delta z = i\varepsilon \Rightarrow \dfrac{i\varepsilon - i\varepsilon}{2i\varepsilon} = 0$

So not diff'ble.

$$y = \operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$$

$$\Rightarrow \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\dfrac{(z+\Delta z)-\overline{(z+\Delta z)}}{2i} - \dfrac{z-\bar{z}}{2i}}{\Delta z}$$

$$= \frac{\cancel{\dfrac{z}{2i}} + \dfrac{\Delta z}{2i} - \cancel{\dfrac{\bar{z}}{2i}} - \dfrac{\overline{\Delta z}}{2i} - \cancel{\dfrac{z}{2i}} + \cancel{\dfrac{\bar{z}}{2i}}}{\Delta z} = \frac{\dfrac{\Delta z}{2i} - \dfrac{\overline{\Delta z}}{2i}}{\Delta z}$$

$$= \frac{1}{2i} - \frac{\overline{\Delta z}}{2i\,\Delta z}$$

⇒ Not diff'ble.

4) Suppose $f(z) = az^2 + bz\bar{z} + c\bar{z}^2$, where a, b, c are fixed complex numbers. By differentiating by hand, show that $f(z)$ is complex differentiable at $z \Leftrightarrow bz + 2c\bar{z} = 0$.

Where is $f(z)$ analytic?

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{a(z+\Delta z)^2 + b(z+\Delta z)(\bar{z}+\overline{\Delta z}) + d(\bar{z}+\overline{\Delta z})^2 - az^2 - bz\bar{z} - c\bar{z}^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{a(z^2 + 2z\Delta z + (\Delta z)^2) + b(z\bar{z} + z\overline{\Delta z} + \Delta z\,\bar{z} + \Delta z\,\overline{\Delta z}) + c(\bar{z}^2 + 2\bar{z}\,\overline{\Delta z} + (\overline{\Delta z})^2) - az^2 - bz\bar{z} - c\bar{z}^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[\cancel{az^2} + 2az\Delta z + a(\Delta z)^2 - \cancel{az^2}] + [\cancel{bz\bar{z}} + bz\overline{\Delta z} + b\Delta z\,\bar{z} + b\Delta z\,\overline{\Delta z} - \cancel{bz\bar{z}}] + [\cancel{c\bar{z}^2} + c\,2\bar{z}\,\overline{\Delta z} + d(\overline{\Delta z})^2 - \cancel{c\bar{z}^2}]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2az + a\Delta z + \frac{bz\overline{\Delta z}}{\Delta z} + b\bar{z} + b\overline{\Delta z} + \frac{2c\bar{z}\,\overline{\Delta z}}{\Delta z} + \frac{d(\overline{\Delta z})^2}{\Delta z}$$

$$= 2az + b\bar{z} + b\overline{\Delta z} + \lim_{\Delta z \to 0} \frac{bz\overline{\Delta z} + 2c\bar{z}\overline{\Delta z} + c(\overline{\Delta z})^2}{\Delta z}$$

Why does $\frac{(\overline{\Delta z})^2}{\Delta z} \to 0$?

OH. $\overline{\Delta z} \to 0$ as $\Delta z \to 0$. Why?

$\Delta z \to 0 \Rightarrow x, y \to 0$ $\frac{x^2+y^2-2xiy}{x+iy} = \bar{z} -$

The only issue we had was $\frac{\overline{\Delta z}}{\Delta z} \to \frac{0}{0}$.

So we get $2az + b\bar{z} + \lim_{\Delta z \to 0} \frac{bz(\overline{\Delta z}) + 2c\bar{z}(\overline{\Delta z})}{\Delta z}$

So then differentiable $\Longleftrightarrow bz + 2c\bar{z} = 0$

$f(z)$ is analytic where

$$f(z) = az^2 + \bar{z}(-2c\bar{z}) + c\bar{z}^2$$

$$= az^2 - 2c\bar{z}^2 + c\bar{z}^2$$

$$= az^2 - c\bar{z}^2$$

If $z = 0 \Rightarrow$ ok $\forall a, b, c$.

If $z \neq 0 \Rightarrow b, c = 0$

5.) Show that if f is analytic on D, then $g(z) = \overline{f(\bar{z})}$ is analytic on the reflected domain $D^* = \{\bar{z} : z \in D\}$ and $g'(z) = \overline{f'(\bar{z})}$.

If $g(z) = \overline{f(\bar{z})} \Rightarrow \overline{g(z)} = f(\bar{z})$. WTS $\bar{g}'(z) = f'(\bar{z})$

$$\Rightarrow \quad \bar{g}'(z) = \lim_{\Delta z \to 0} \frac{\bar{g}(z+\Delta z) - \bar{g}(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(\overline{z+\Delta z}) - f(\bar{z})}{\Delta z} = f'(\bar{z}) \checkmark$$

$$f'(\bar{z}) = \lim_{\Delta z \to 0} \frac{f(\overline{z+\Delta z}) - f(\bar{z})}{\Delta z}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ (not sure...)

6.) Let $h(t)$ be a continuous complex-valued function on the unit interval $[0,1]$, and define

$$H(z) = \int_0^1 \frac{h(t)}{t-z}\,dt, \qquad z \in \mathbb{C} \setminus [0,1]$$

Show that $H(z)$ is analytic and compute its derivative

$$H'(z) = \lim_{\Delta z \to 0} \frac{H(z+\Delta z) - H(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\int_0^1 \frac{h(t)}{t-(z+\Delta z)} - \int_0^1 \frac{h(t)}{t-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \int_0^1 \frac{h(t)(t-z) - h(t)(t-z-\Delta z)}{\Delta z[t-z][t-z-\Delta z]} =$$

$$= \lim_{\Delta z \to 0} \int_0^1 \frac{h(t)t - h(t)z - h(t)t + h(t)z + h(t)\Delta z}{\Delta z[t-z][t-z-\Delta z]}$$

$$= \lim_{\Delta z \to 0} \int_0^1 \frac{h(t)}{[t-z][t-z-\Delta z]} \overset{\text{by cty}}{=} \int_0^1 \frac{h(t)}{(t-z)^2}$$

Chapter 2, Section 3 Exercises:

1.) Find the derivatives of the following functions:

(a) $\tan z = \frac{\sin z}{\cos z}$

$$\Rightarrow \frac{d}{dz}(\tan z) = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

(b) $\tanh(z) = \frac{\sinh z}{\cosh z}$

$$\Rightarrow \frac{d}{dz} \tanh z = \frac{\cosh z \cosh z - \sinh z \sinh z}{\cosh^2 z}$$

$$= \frac{1}{\cosh^2 z} = \operatorname{sech}^2 z$$

(c) $\sec(z) = \frac{1}{\cos(z)}$

$$\Rightarrow \frac{d}{dz}(\sec(z)) = \frac{\cos z(0) - 1(-\sin z)}{\cos^2 z} = \frac{\sin z}{\cos^2 z}$$

$$= \sec z \tan z$$

2.) Show that $u = \sin x \sinh y$ and $v = \cos x \cosh y$ satisfy the CR-equations. Do you recognize the analytic function $f = u + iv$?

$$\frac{\partial u}{\partial x} = \cos x \sinh y \qquad \frac{\partial v}{\partial y} = \cos x \sinh y \quad ✓$$

$$\frac{\partial u}{\partial y} = \sin x \cosh y \qquad -\frac{\partial v}{\partial x} = +\sin x \cosh y. \quad ✓$$

$$f = \sin x \sinh y + i \cos x \cosh y$$

$$= i\cos(z).$$

3.) Show that if f and $\bar{f}$ are both analytic on a domain D, then f is constant.

$f = u + iv$, $\bar{f} = u - iv$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

$\Rightarrow \nabla u = 0 \Rightarrow u$ is constant

$\nabla v = 0 \Rightarrow v$ is constant

$\Rightarrow f$ is constant.

4.) Show that if f is analytic on a domain D, and if $|f|$ is constant, then f is constant.

Hint: Write $\bar{f} = \frac{|f|^2}{f}$

$|f| = c$

$\Rightarrow u^2 + v^2 = c^2$

$$\Rightarrow 2u \cdot \frac{\partial u}{\partial x} + 2v \cdot \frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$$

$$2u \cdot \frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow u\frac{\partial u}{\partial y} - v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow -v.\left(v\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right) \neq 0$$

Either $v = 0$ or $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v$

$\Rightarrow u = 0 \Rightarrow f$ is constan

5.) If $f = u + iv$ is analytic, then $|\nabla u| = |\nabla v| = |f'|$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} = \sqrt{\left(\frac{\partial v}{\partial y}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2} = |\nabla v|$$

$$= \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} = |f'|$$

6.) If $f = u + iv$ is analytic on D, then ∇v is obtained by rotating ∇u by 90°. In particular, ∇u and ∇v are orthogonal.

$$\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle \cdot \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = \left(\frac{\partial u}{\partial x}\right)\left(\cancel{\partial x} \cdot \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x} = 0.$$

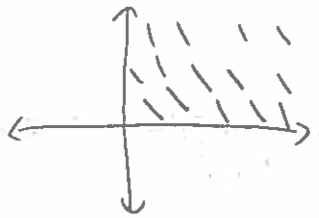
7.) Sketch the vector fields ∇u and ∇v for the following functions $f = u + iv$

(a) $iz = ix - y$

$u = -y$
$v = x$

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = -1$$

$(0, -1)$



$$\frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = 0$$

(b) z^2.

etc. I don't really remember how to draw these.

8.) Derive the polar form of the Cauchy-Riemann eq'ns for u and v:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Check that for any integer m, the function

$u(re^{i\theta}) = r^m \cos(m\theta)$

$v(re^{i\theta}) = r^m \sin(m\theta)$

satisfy the CR-equations.

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = -\sin\theta\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\cos\theta$$

$$= +\sin\theta\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\cos\theta = \frac{\partial u}{\partial r} \checkmark$$

$$-r\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

$$= \frac{\partial u}{\partial x}\sin\theta - \frac{\partial u}{\partial y}\cos\theta$$

$$\Rightarrow -r\frac{\partial v}{\partial r} = r\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}r\sin\theta = \frac{\partial u}{\partial \theta} \checkmark$$

$$\frac{\partial u}{\partial r} = mr^{m-1}\cos(m\theta)$$

$$\frac{\partial v}{\partial \theta} = r^{m}\cos(m\theta)m \Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = mr^{m-1}\cos(m\theta) = \frac{\partial u}{\partial r} \checkmark$$

$$\frac{\partial u}{\partial \theta} = -r^{m}\sin(m\theta)\cdot m$$

$$\frac{\partial v}{\partial r} = mr^{m-1}\sin(m\theta) \Rightarrow -r\frac{\partial v}{\partial r} = -r^{m}\sin(m\theta).m = \frac{\partial u}{\partial \theta} \checkmark$$

Chapter 2, Section 4: Exercises:

2.) Let $a \in \mathbb{C}, a \neq 0$, and let $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Show that $f'(z) = \frac{af(z)}{z}$.

$$f(z) = z^a = e^{a \log z}$$

$$\Rightarrow \log f(z) = a \log z$$

$$\Rightarrow \frac{1}{f(z)} \cdot f'(z) = \frac{a}{z}$$

$$\Rightarrow f'(z) = \frac{a f(z)}{z}$$

3.) Consider the branch of $f(z) = \sqrt{z(1-z)}$ on $\mathbb{C} \setminus [0,1]$ that has positive imaginary part at $z=2$. What is $f'(z)$?

$$f(z)^2 = z(1-z)$$

$$\Rightarrow 2f(z)f'(z) = z(-1) + (1-z) = -z + 1 - z$$

$$\Rightarrow 2f(z)f'(z) = -2z + 1$$

$$\Rightarrow f'(z) = \frac{-2z+1}{2f(z)} = \frac{-2z+1}{2\sqrt{z(1-z)}}$$

4.) $\mathrm{Tan}^{-1}(z) = \frac{1}{2i} \mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$, $z \notin (-i\infty, -i] \cup [i, i\infty)$

Find the derivative

$$\frac{d}{dz}\mathrm{Tan}^{-1}(z) = \frac{1}{2i} \, \frac{1-zi}{1+zi} \cdot \frac{(1-iz)(i) - (1+iz)(-i)}{(1-iz)^2}$$

$$= \frac{1}{2i}\left[\frac{i + z + i - z}{(1+iz)(1-iz)}\right] = \frac{2i}{2i(1+z^2)} = \frac{1}{1+z^2}$$

5.) Recall that $\cos^{-1}(z) = -i\log[z \pm \sqrt{z^2-1}]$. Suppose $g(z)$ is an analytic branch of $\cos^{-1}(z)$, defined on a domain D. Find $g^{-1}(z)$.

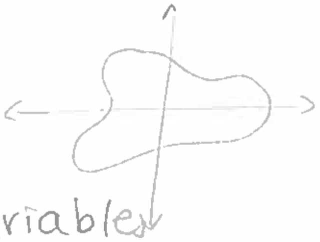
$$\frac{d}{dz}[\cos^{-1}(z)] = \frac{-i}{z \pm \sqrt{z^2-1}} \cdot \left(1 \pm \frac{1}{\cancel{2}}(z^2-1)(\cancel{2}z)\right)$$

$$= \frac{-i\left(1 \pm (z^3-1)\right)}{z \pm \sqrt{(z^2-1)}} \cdot \frac{z \mp \sqrt{z^2-1}}{z \mp \sqrt{z^2-1}}$$

7.) Let $f(z)$ be a bounded analytic function, defined on a bounded domain D in the Complex plane, and suppose that $f(z)$ is one-to-one. Show that the area of $f(D)$ is given by

$$\text{Area}(f(D)) = \iint_D |f'(z)|^2 \, dx\,dy$$

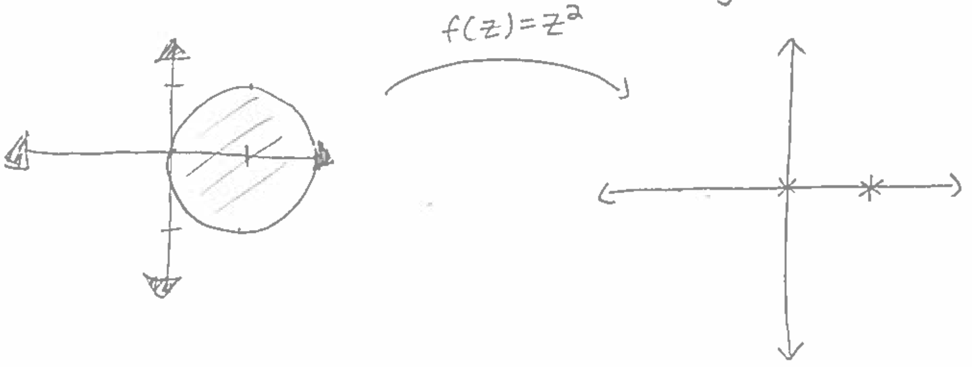
$$\text{Area}(f(D)) = \iint_{f(D)} du\,dv.$$



We need to change variables

$$= \iint_D \det J_f{}^{dxdy} = \iint_D |f'(z)|^2 \, dx\,dy.$$

8.) Sketch the image of the circle $\{|z-1| \leq 1\}$ under the map $w = z^2$. Compute the area of the image.



$$z^2 = e^{2\log z} = e^{2\log|z|} e^{2i\arg z}$$

$$\text{Area}(f(D)) = 2\iint_{\{|z-1|\leq 1\}} \sqrt{x^2+y^2}\,dx\,dy$$

$$f'(z) = 2z \Rightarrow |f'(z)| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$$
$$= 2x+2iy$$

$$= 2\iint_{\{|r\cos\theta-1+r\sin\theta|\leq 1\}} r^2\,ar\,d\theta = \quad = 2\iint_{\{|r\cos\theta-1+r\sin\theta|\leq 1\}} dr\,d\theta = 8\pi.$$

Chapter 2, Section 5: Exercises

1.) Show that the following functions are harmonic, and find harmonic conjugates:

(a) $x^2-y^2=u(x,y)$.

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2\\ \frac{\partial u}{\partial y} &= -2y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2\end{aligned}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

constant wrt y

$$\Rightarrow \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 2x \Rightarrow v(x,y) = 2xy + h(x)$$

$$\Rightarrow -\frac{\partial u}{\partial y} = 2y = \frac{\partial v}{\partial x} \qquad \Rightarrow \frac{\partial v}{\partial x} = 2y + h'(x)$$

$$\Rightarrow h'(x) = 0$$

$$\Rightarrow h(x) = c$$

$$\Rightarrow v(x,y) = 2xy + c$$

(b) $u(x,y) = xy + 3x^2y - y^3$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= y + 6xy \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6y\\ \frac{\partial u}{\partial y} &= x + 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6y\end{aligned}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0.$$

constant wrt y

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = y + 6xy \qquad \Rightarrow v(x,y) = \frac{y^2}{2} + \frac{6xy^2}{2} + h(x)$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 3y^2 - x - 3x^2 \qquad \Rightarrow \frac{\partial v}{\partial x} = \frac{6y^2}{2} + h'(x)$$

$$= 3y^2 + h'(x)$$

$$\Rightarrow h'(x) = -x - 3x^2$$

$$\Rightarrow h(x) = -\frac{x^2}{2} - \frac{3x^3}{3} + c$$

$$= -\frac{x^2}{2} - x^3 + c.$$

$$\Rightarrow v(x,y) = \frac{y^2}{2} + 3xy^2 - \frac{x^2}{2} - x^3 + c.$$

(c) $u(x,y) = \sinh(x)\sin(y)$

$$\frac{\partial u}{\partial x} = \cosh(x)\sin(y) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \sinh(x)\sin(y)$$

$$\frac{\partial u}{\partial y} = \sinh(x)\cos(y) \Rightarrow \frac{\partial^2 u}{\partial y^2} \overset{-}{=} \sinh(x)\sin(y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \cosh(x)\sin(y) \Rightarrow v(x,y) = -\cosh(x)\cos(y) + \overset{\substack{\text{constant}\\ \text{wit } y\\ \downarrow}}{h(x)}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\sinh(x)\cos(y) \qquad \frac{\partial v}{\partial x} = -\sinh(x)\cos(y) + h'(x)$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = c$$

$$\Rightarrow v(x,y) = -\cosh(x)\cos(y) + c.$$

(d) $e^{x^2-y^2}\cos(2xy) = u(x,y) = e^{x^2}e^{-y^2}\cos(2xy)$

$$\frac{\partial u}{\partial x} = e^{x^2}e^{-y^2}\,2x\cos(2xy) + e^{x^2-y^2}(-\sin(2xy))\,2y$$

$$= 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)]$$

2) Show that if v is a conjugate for u, then $-u$ is a harmonic conjugate for v.

v is harmonic and $u+iv$ is analytic.

u is harmonic. Is $-u$ harmonic?

Have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\Rightarrow -\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

$\Rightarrow v - iu = -i(u+iv)$ is analytic.

3.) Define $u(z) = \mathrm{Im}\left(\frac{1}{z^2}\right)$ for $z \neq 0$, and set $u(0) = 0$.

(a) Show that all partial derivatives of u wrt x exist at all points of $\mathbb{C}$, as do all partial derivatives of u wrt y.

$$u(z) = \begin{cases} \mathrm{Im}\left(\frac{\bar{z}^2}{|z|^4}\right) = \mathrm{Im}\left(\frac{x^2 - 2xiy + y^2}{(x^2+y^2)^2}\right) = \frac{-2xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$$\frac{-2xy}{(x^2+y^2)} \in C^\infty, \quad 0 \in C^\infty \ \checkmark$$

(b) Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)^2(-2y) + 2xy \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4} = \frac{-2y(x^2+y^2) + 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y - 2y^3 + 8x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{6x^2y-2y^3}{(x^2+y^2)^3}\right] = \frac{(x^2+y^2)^3(12xy)-(6x^2y-2y^3)(3(x^2+y^2)^2 \cdot 2x)}{(x^2+y^2)^6}$$

$$= \frac{(x^2+y^2)(12xy) - 6x(6x^2y-2y^3)}{(x^2+y^2)^4}$$

$$= \frac{12x^3y+12xy^3-36x^3y+12xy^3}{(x^2+y^2)^4}$$

$$= \frac{-24x^3y+24xy^3}{(x^2+y^2)^4} = \frac{24xy(y^2-x^2)}{(x^2+y^2)^4}$$

By symmetry, $\frac{\partial^2 u}{\partial y^2} = \frac{24xy(x^2-y^2)}{(x^2+y^2)^4}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(c) Show that u is not harmonic on $\mathbb{C}$.

At $(0,0)$, $\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{0}{0}$ undef., so not cts.

(d) Skipped lol.

4.) Show that if $h(z)$ is a complex-valued harmonic function (sol'n of Laplace's eq'n) such that $zh(z)$ also harmonic, then $h(z)$ is analytic.

$$zh = (x+iy)h(z) = xh(z) + iyh(z)$$

$$\frac{\partial(zh)}{\partial x} = h(z) + x\frac{\partial h}{\partial x} + iy\frac{\partial h}{\partial x}$$

$$= h(z) + z\frac{\partial h}{\partial x}$$

$$\Rightarrow \frac{\partial^2(zh)}{\partial x^2} = \frac{\partial h}{\partial x} + x\frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x} + iy\frac{\partial^2 h}{\partial x^2}$$

$$= z\frac{\partial^2 h}{\partial x^2} + 2\frac{\partial h}{\partial x}$$

$$\frac{\partial(zh)}{\partial y} = x\frac{\partial h}{\partial y} + i\left[y\frac{\partial h}{\partial y} + \frac{\partial h}{\partial y}\right]$$

$$= x\frac{\partial h}{\partial y} + iy\frac{\partial h}{\partial y} + ih$$

$$\Rightarrow \frac{\partial^2(zh)}{\partial y^2} = x\frac{\partial^2 h}{\partial y^2} + i\frac{\partial h}{\partial y} + iy\frac{\partial^2 h}{\partial y^2} + i\frac{\partial h}{\partial y}$$

$$= z\frac{\partial^2 h}{\partial y^2} + 2i\frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial^2(zh)}{\partial y^2} + \frac{\partial^2(zh)}{\partial x^2} = 0$$

$$\Rightarrow z\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) + 2\left[\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y}\right] = 0$$

$$\Rightarrow 2\left[\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y}\right] = 0 \quad \Rightarrow \quad \frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y} = 0$$

$$\Rightarrow h'(z) = 0. \Rightarrow h \text{ is analytic.}$$

5.) Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial r},$$

$$\frac{\partial U}{\partial r} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial U}{\partial r} = \frac{\partial U}{\partial x}\cos\theta + \frac{\partial U}{\partial y}\sin\theta$$

∂

—Not finished—

Chapter 2, Section 7 Exercises:

1.) Compute explicitly the fractional linear transformations determined by the following correspondences of triples:

(a) $(1+i, 2, 0) \mapsto (0, \infty, i-1)$

$$f(z) = a\frac{(z-1-i)}{z-2}$$

$$\Rightarrow f(2) = \frac{a(2-1-i)}{2-2} \to \infty \quad ✓$$

$$f(1+i) = \frac{1+i-1-i}{1+i-2}\cdot a = 0 \quad ✓$$

Need $f(0) = i-1$

$$i-1 = f(0) = a\left(\frac{-1-i}{-2}\right) = a\left(\frac{1+i}{2}\right)$$

$$\Rightarrow a + ai = 2i-2$$

$$\Rightarrow a = \frac{2i-2}{1+i}$$

$$\Rightarrow f(z) = \left(\frac{2i-2}{1+i}\right)\frac{(z-(1+i))}{z-2}$$

$$\frac{2(i-1)}{1+i}\cdot\frac{1-i}{1-i} = \frac{2(i+1-1+i)}{1\cancel{-i+i}+1} = 2i$$

$$= 2i\left[\frac{z-(1+i)}{z-2}\right]$$

(b) $(0, 1, \infty) \mapsto (1, 1+i, 2)$

$0 \mapsto 1$
$1 \mapsto 1+i$
$\infty \mapsto 2$

$(z, 0, 1, \infty) = (f(z), 1, 1+i, 2)$
$(z, z_0, z_1, z_2) = (f(z), w_0, w_1, w_2)$

$$\frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)} = \frac{(f(z)-w_0)(w_1-w_2)}{(f(z)-w_2)(w_1-w_0)}$$

$$\frac{(z-0)(1-\infty)}{(z-\infty)(1-0)} = \frac{(f(z)-1)(1+i-2)}{(f(z)-2)(1+i-1)}$$

$$\frac{z(0-1)}{(\frac{z}{\infty}-1)\,1} = \frac{(f(z)-1)(-1+i)}{(f(z)-2)\,i}$$

$$zf(z)i - 2iz = -f(z) + if(z) + 1 - i$$

$$zf(z)i + f(z) - if(z) = 1 - i + 2iz$$

$$f(z)\left[iz + 1 - i\right] = 1 - i + 2iz$$

$$f(z) = \frac{2iz + (1-i)}{iz + (1-i)}$$

(c) $(\infty, 1+i, 2) \mapsto (0, 1, \infty)$

$(z, \infty, 1+i, 2) \mapsto (f(z), 0, 1, \infty)$

$(z, z_0, z_1, z_2) \quad (f(z), w_0, w_1, w_2)$

$$\frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)} = \frac{(f(z)-w_0)(w_1-w_2)}{(f(z)-w_2)(w_1-w_0)}$$

$$\frac{(z-\infty)(1+i-2)}{(z-2)(1+i-\infty)} = \frac{(f(z)-0)(1-\infty)}{(f(z)-\infty)(1-0)}$$

$$\frac{(0-1)(-1+i)}{(z-2)(0-1)} = \frac{f(z)(0-1)}{(0-1)\,1}$$

$$\frac{1-i}{2-z} = +f(z)$$

(d) $(-2, i, 2) \longmapsto (1-2i, 0, 1+2i)$

$(z, -2, i, 2) \quad (f(z), 1-2i, 0, 1+2i)$

$(z, z_0, z_1, z_2) \quad (f(z), w_0, w_1, w_2)$

$$\frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)} = \frac{(f(z)-w_0)(w_1-w_2)}{(f(z)-w_2)(w_1-w_0)}$$

$$\frac{(z+2)(i-2)}{(z-2)(i+2)} = \frac{(f(z)-1+2i)(0-1-2i)}{(f(z)-1-2i)(0-1+2i)}$$

$$\frac{iz-2z+2i-4}{iz+2z-2i-4} = \frac{-f(z)-2if(z)+1+\cancel{2i-2i}+4}{-f(z)+2if(z)+1\cancel{-2i+2i}+4}$$

$$[f(z)(2i-1)+5][iz-2z+2i-4] = [f(z)[-2i-1]+5](iz+2z-2i-4)$$

(e) $(1,2,\infty) \mapsto (0,1,\infty)$

$(z,1,2,\infty) : (f(z),0,1,\infty)$

$$\frac{(z-1)(2-\infty)}{(z-\infty)(2-1)} = \frac{(f(z)-0)(1-\infty)}{(f(z)-\infty)(1-0)}$$

$$\frac{(z-1)(0-1)}{(0-1)(1)} = \frac{f(z)(0-1)}{(0-1)\,1}$$

$$(z-1) = f(z).$$

(f) $(0,\infty,1) \longmapsto (0,1,\infty)$

$(z,0,\infty,i)$ $(f(z),0,1,\infty)$

(z,z_0,z_1,z_2) $(f(z),w_0,w_1,w_2)$

$$\frac{(z-0)(\infty-i)}{(z-i)(\infty-0)} = \frac{(f(z)-0)(1-\infty)}{(f(z)-\infty)(1-0)}$$

$$\frac{z\,(1)}{(z-i)(1-\infty)} = \frac{f(z)(0-1)}{(0-1)\cdot 1}$$

$$\frac{z}{z-i} = f(z)$$

(g) $(0,1,\infty) \longmapsto (0,\infty,i)$

$$\frac{(z-0)(1-\infty)}{(z-\infty)(1-0)} = \frac{(f(z)-0)(\infty-i)}{(f(z)-i)(\infty-0)}$$

$$\frac{z\,(0-1)}{(0-1)} = \frac{f(z)(1-0)}{(f(z)-i)(\ -\ )}$$

$$z = \frac{f(z)}{f(z)-i}$$

$\Rightarrow zf(z) - zi = f(z) \Rightarrow f(z)\,(1-z) = -zi$

$$\Rightarrow f(z) = \frac{-zi}{1-z}$$

(h) $(1, i, -1) \longmapsto (1, 0, -1)$

$$\frac{(z-1)(i+(-1))}{(z-(-1))(i-1)} = \frac{(f(z)-1)(0-(-1))}{(f(z)-(-1))(0-1)}$$

$$\frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{-f(z)}{f(z)+1}$$

$$(zi+z-i-1)(f(z)+1) = -f(z)(zi-z+i-1)$$

$$\underline{zif(z)}+zi+\cancel{zf(z)}+z\cancel{-if(z)}-i-\underline{f(z)}-1$$

$$= \underline{-f(z)zi}\cancel{+zf(z)}\cancel{-if(z)}+\underline{f(z)}$$

$$2f(z)zi+zi+z-i-2f(z)-1=0$$

$$f(z)[2zi-2] = -zi-z+i+1$$

$$\Rightarrow f(z) = \frac{z(-i-1)+(i+1)}{z(2i)-2}$$

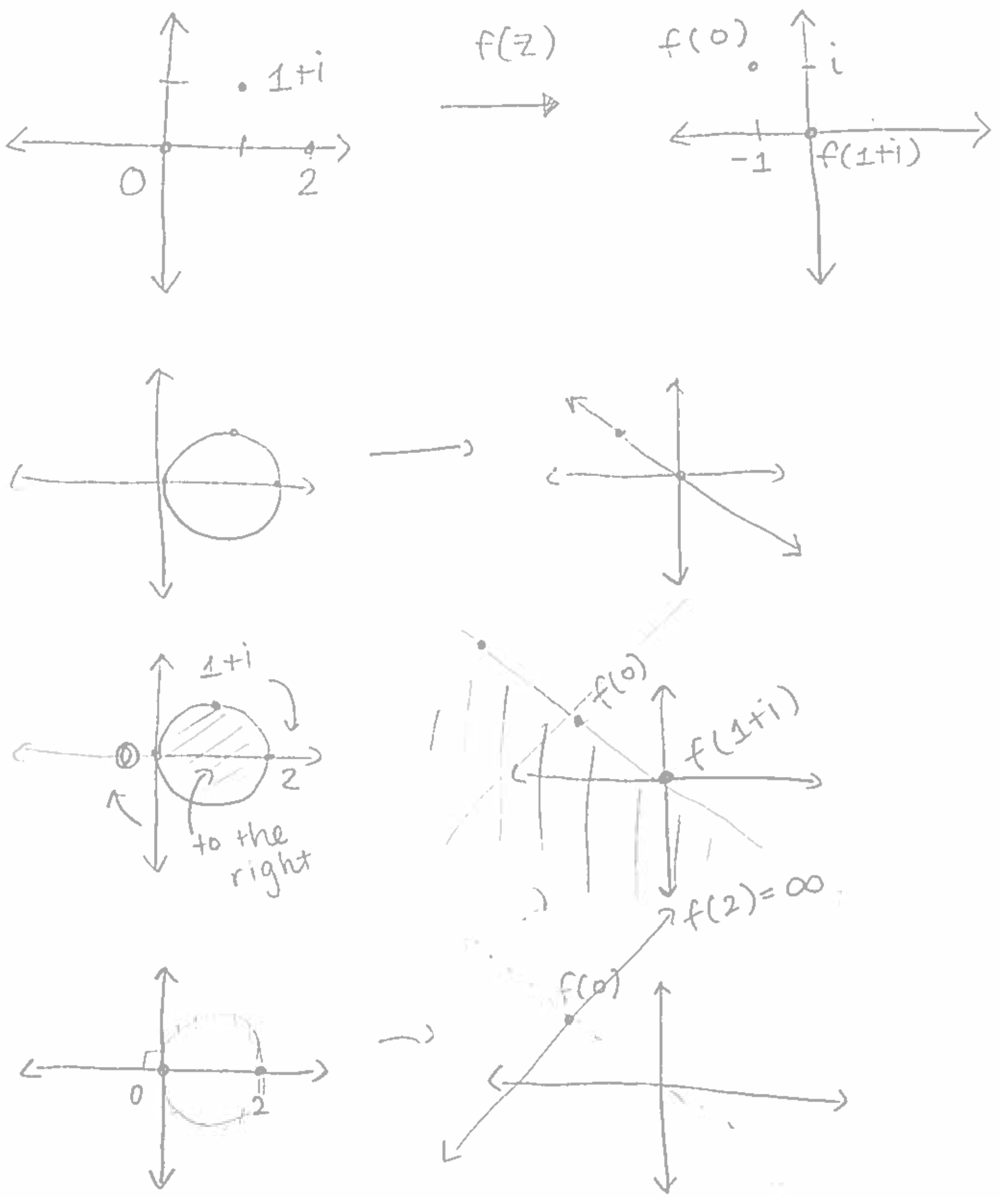
2.) Consider the FLT $(1+i, 2, 0) \mapsto (0, \infty, i-1)$.

Determine the image of the circle $\{|z-1|=1\}$,

the disk $\{|z-1|<1\}$,

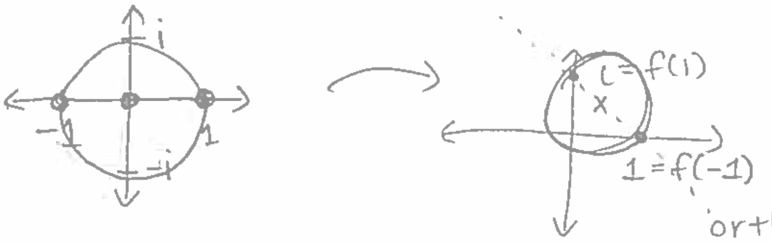
the real axis.

$\circ\ f(2)=\infty$

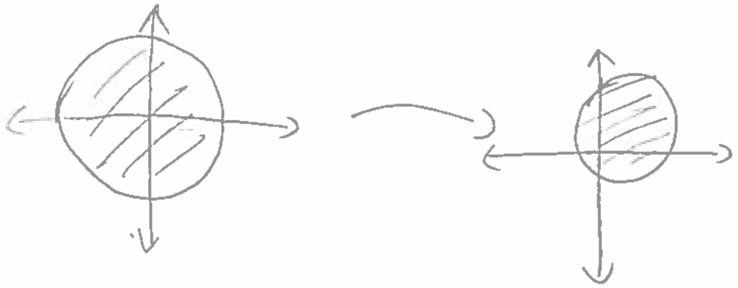


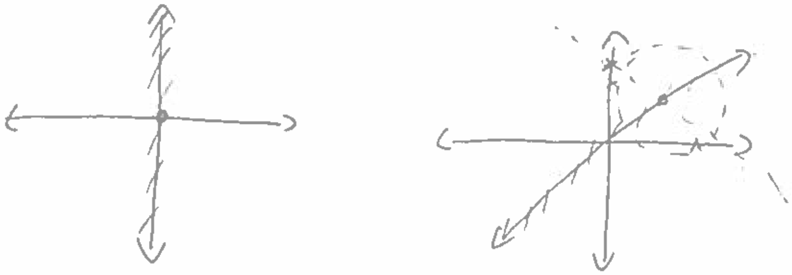
3.) Consider the FLT that maps $1 \mapsto i$
$0 \mapsto 1+i$
$-1 \mapsto 1.$

Determine the image of the unit circle $\{|z|=1\}$
$\{|z|<1\}$
imaginary axis.



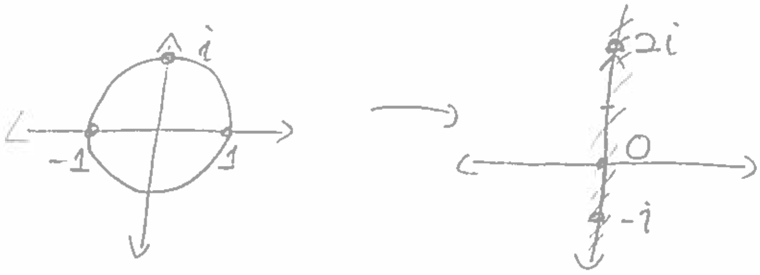
orthogonal
to this line.

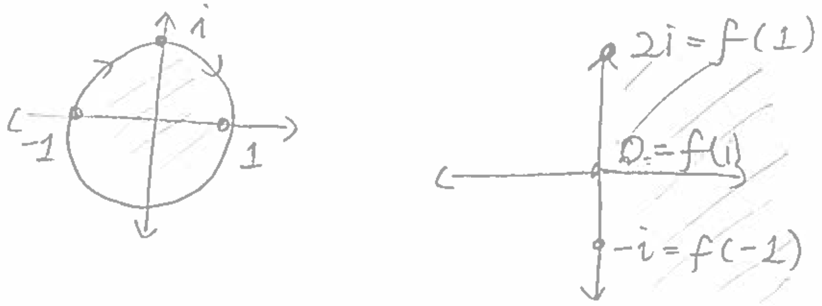


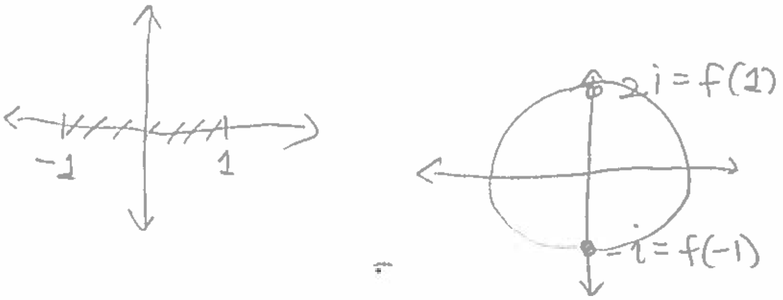


4.) Consider the FLT that maps $-1 \mapsto -2$
$1 \mapsto 2i$
$i \mapsto 0$

Determine the image of $\{|z|=1\}$
$\{|z|<1\}$
the interval $[-1, 1]$

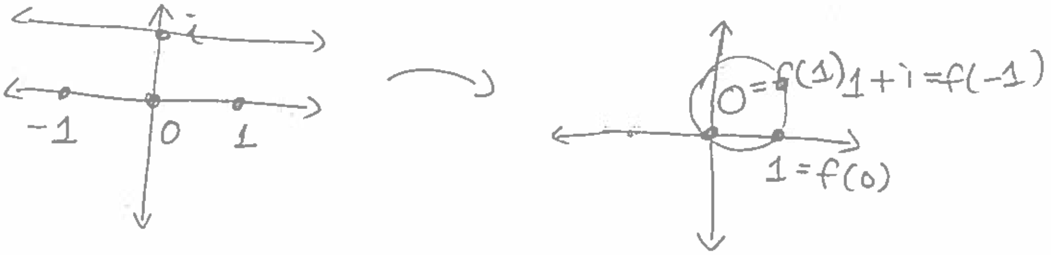






5.) What is the image of the horizontal line through i under the FLT that interchanges 0 & 1 and maps -1 to $1+i$? Illustrate with a sketch.

$0 \mapsto 1$
$1 \mapsto 0$
$-1 \mapsto 1+i$



$(z, 0, 1, -1)$ $\quad (f(z), 1, 0, 1+i)$

$$\frac{(z-0)(1-(-1))}{(z-(-1))(1-0)} = \frac{(f(z)-1)(0-1-i)}{(f(z)-1-i)(0-1)}$$

$$\frac{2z}{(z+2)} = \frac{f(z)(-1-i)+(1+i)}{-f(z)+1+i}$$

$$-2zf(z) + 2z + 2zi = (z+2)(-f(z)-if(z)+1+i)$$
$$= -f(z)z - if(z)z + z + \cancel{2}iz$$
$$-2f(z) - 2if(z) + 2 + 2i$$

$$z + iz - 2 - 2i = f(z)z - if(z)z - 2f(z) - 2if(z)$$

$$\Rightarrow f(z) = \frac{z(1+i)-(2+2i)}{z(1-i)-(2+2i)}$$

$$\Rightarrow f(i) = \frac{i(1+i)-(2+2i)}{i(1-i)-(2+2i)} = \frac{i-1-2-2i}{i+1-2-2i} = \frac{-i-3}{-i-1}$$

$$f(1+i) = \frac{(1+i)(1+i)-(2+2i)}{(1+i)(1-i)-(2+2i)} = \frac{1+2i-1-2-2i}{1+1-2-2i} = -i$$

$$f(-1+i) = \frac{(-1+i)(1+i)-(2+2i)}{(-1+i)(1-i)-(2+2i)} = \frac{-1-i+i-1-2-2i}{-1+2i+1-2-2i}$$
$$= \frac{-4-2i}{-2} = 2+$$

7.) Show that the FLT $f(z)=\dfrac{az+b}{cz+d}$ is the Id mapping $z \iff b=c=0$ and $a=d\neq 0$

$(\Leftarrow)$ If $b=c=0$

$$\Rightarrow f(z)=\frac{az}{d}$$

If $a=d\neq 0 \Rightarrow f(z)=\dfrac{az}{a}=z$

$(\Rightarrow)$ If $f(z)=\dfrac{az+b}{cz+d}=z \Rightarrow$

$$az+b=cz^2+dz$$

$$\Rightarrow cz^2+(d-a)z-b=0$$

$$\Rightarrow c=0,\ -b=0$$

$$d-a=0$$

$$\Rightarrow d=a$$

Suppose $d=0, a=0. \Rightarrow \dfrac{az}{d}=\dfrac{0}{0}$ #.

So $a=d\neq 0$. ✓

8) Show that any FLT can be represented in the form $f(z)=\dfrac{az+b}{cz+d}$, where $ad-bc=1$. Is this representation unique?

$$a=\frac{1+bc}{d}$$

$f(z)=\dfrac{\left(\frac{1+bc}{d}\right)z+\frac{bd}{d}}{cz+d} \Rightarrow \left(\dfrac{1+bc}{(d)}\right)d-bc=1$. ✓

Not unique.

Now $\frac{(1+bc)d^2 - bddc}{d^2} = \frac{d^2+bcd^2 - bd^2c}{d^2} = \frac{d^2}{d^2}$

So $f(z) = \frac{z\left(\frac{1+bc}{d^2}\right) + \frac{b}{d}}{z\left(\frac{c}{d}\right) + 1}$

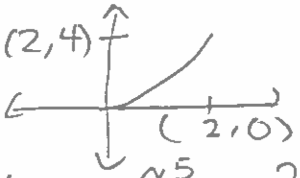
$$\Rightarrow \left(\frac{1+bc}{d^2}\right)\cdot 1 - \left(\frac{b}{d}\right)\left(\frac{c}{d}\right) = \frac{1+bc}{d^2} - \frac{bc}{d^2} = \frac{1}{d^2}$$

$$\left(\frac{1+bc}{d}\right)(d) - (b)(c) = 1+bc-bc = \underline{1}.$$

Chapter 3, Section 1: Exercises

1.) Evaluate $\int y^2dx + x^2dy$ along the following paths γ from $(0,0)$ to$^{\gamma}$ $(2,4)$.

(a) the arc of the parabola $y = x^2$

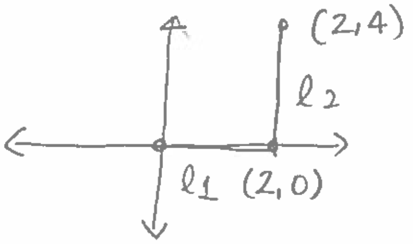


$$\int_{x=0}^{x=2} x^4 dx + x^2 \cdot 2x dx = \int_{x=0}^{x=2} (x^4 + 2x^3) dx = \frac{x^5}{5} + \frac{2x^4}{4}\Bigg]_{x=0}^{x=2}$$

$$= \frac{2^5}{5} + \frac{2^5}{4} = \frac{32}{5} + \frac{32}{4} = \frac{4\times 32 + 5\times 32}{20} = \frac{128 + 160}{20}$$

$$= \frac{288}{20} = \frac{72}{5}$$

) the horizontal interval from $(0,0)$ to $(2,0)$, followed by the vertical interval from $(2,0)$ to $(2,4)$.



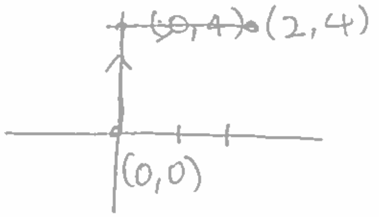
l_1: $y = 0$, $x: 0 \to 2$

l_2: $y: 0 \to 4$, $x = 2$

$$\int_\gamma y^2dx + x^2dy = \int 0 + x^2 \cdot 0 + \int_{y=0}^{y=4} y^2 \cdot 0 + 4dy$$

$$= 4y\Big]_0^4 = 16.$$

(c) The vertical interval from (0,0) to (0,4), followed by the horizontal interval from (0,4) to (2,4).



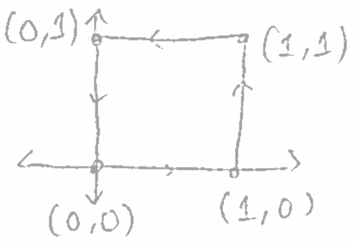
ℓ_1: $x=0$, $y: 0 \to 4$ ○

ℓ_2: $y=4$, $x: 0 \to 2$

$$\int_\gamma y^2 dx + x^2 dy = \int y^2 \cdot 0 + 0^2 dy + \int_{x=0}^{x=2} 16 dx + x^2 \cdot 0$$

$$= 16x \Big]_{x=0}^{x=2} = 32.$$

2.) Evaluate $\int xy\,dx$ both directly and using Green's Theorem, where γ γ is the boundary of the square with vertices at (0,0), (1,0), (1,1) and (0,1). ○



ℓ_1: $x: 0 \to 1$, $y=0$, ~~$dy=0$~~

ℓ_2: $x=1$, $dx=0$, $y: 0 \to 1$

ℓ_3: $x: 1 \to 0$, $y=1$

ℓ_4: $x=0$, $dx=0$, $y: 1 \to 0$

Directly:

$$\int_\gamma xy\,dx = \int 0\,dx + \int xy \cdot 0 + \int_{x=1}^{x=0} x\,dx + \int 0y\,0 = \frac{x^2}{2}\Big]_{x=1}^{x=0}$$

$$= \boxed{-\frac{1}{2}}$$

w/ Green's Theorem: $P(x,y) = xy \Rightarrow \frac{\partial P}{\partial y} = x\,dy$

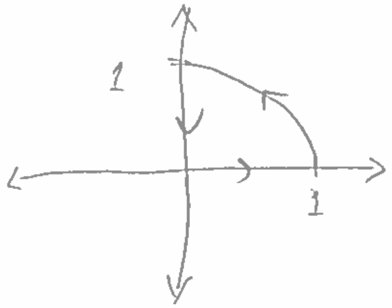
$Q(x,y) = 0$ ○

$\frac{\partial Q}{\partial x} = 0$

$$= \iint_\square (0-x)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1} -x\,dx\,dy = \int_{y=0}^{y=1} \frac{-x^2}{2}\Big]_{x=0}^{x=1} = \frac{-1}{2}y\Big]_{y=}^{y=}$$

$$= \boxed{-\frac{1}{2}}$$

3.) Evaluate $\int_{\partial D} x^2 dy$ both directly and using Green's Theorem where D is the quarter-disk in the first quadrant ounded by the unit circle and the two coordinate axes.



Directly: l_1: $x: 0 \to 1$, $y = 0$

l_2: $x = \cos\theta$, $y = \sin\theta$, $dy = \cos\theta \, d\theta$, $0 \le \theta \le \frac{\pi}{2}$

l_3: $x = 0$, $y: 1 \to 0$

$$\int_{\partial D} x^2 dy = \int x^2 \cdot 0 + \int_0^{\pi/2} \cos^3\theta \, d\theta \; d + \int 0 \, dy$$

$$= \int_0^{\pi/2} \cos\theta (1 - \sin^2\theta) = \int_0^{\pi/2} \cos\theta - \cos\theta \sin^2\theta = \sin\theta \Big]_0^{\pi/2} - \int_0^1 u^2 du$$

$u = \sin\theta$
$du = \cos\theta d\theta$

$$= 1 - \frac{u^3}{3}\Big]_0^1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}.$$

Using Green's Theorem:

$\int_{\partial D} x^2 dy$

$P = 0$ $\qquad \frac{\partial P}{\partial y} = 0$

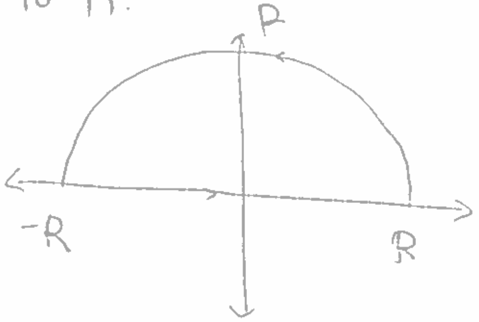
$Q = x^2$ $\qquad \frac{\partial Q}{\partial x} = 2x \, dx$

$x^2 + y^2 = 1$
$x = \sqrt{1-y^2}$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} 2x \, dx dy = \int_0^1 \left[x^2\right]_0^{\sqrt{1-y^2}} dy = \int_0^1 1 - y^2 \, dy$$

$$= y - \frac{y^3}{3}\Big]_0^1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

4.) Evaluate $\int_\gamma y\,dx$ both directly and using Green's Theorem! where γ is the semicircle in the upper half-plane from $-R$ to R.



Directly:

ℓ_1: $x: -R \to R \quad y=0$

ℓ_2: $x=R\cos\theta,\ y=R\sin\theta \quad dx=-R\sin\theta\,d\theta$

$0 \leq \theta \leq \pi$

$$\int_\gamma y\,dx = \int 0 + \int_0^\pi R\sin\theta(-R\sin\theta\,d\theta)$$

$$= \int_0^\pi -R^2\sin^2\theta\,d\theta =$$

$$\cos(2\theta) \overset{0}{=} \cos^2\theta - \sin^2\theta = 1-2\sin^2\theta$$

$$= \int_0^\pi -R^2\left(\frac{\cos(2\theta)-1}{-2}\right) = \frac{R^2}{2}\int_0^\pi (\cos(2\theta)-1)\,d\theta$$

$$= \frac{R^2}{2}\left[\frac{\sin(2\theta)}{2} - \theta\right]_0^\pi = \frac{R^2}{2}\left[\frac{\sin(2\pi)}{2} - \pi - \frac{\sin(0)}{2} + 0\right]$$

$$= \frac{-\pi R^2}{2}$$

4 c+d.) $P = y \Rightarrow \frac{\partial P}{\partial y} = 1$

$$= \iint_{\frown} -1\,dx\,dy = \int_0^R \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} -1\,dx\,dy = \int_0^R [-x]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\,dy =$$

$$\int_0^R [-\sqrt{1-y^2} - \sqrt{1-y^2}\,]\,dy = \int_0^R -2\sqrt{1-y^2}\,dy = -2\int_0^R \sqrt{1-y^2}\,dy$$

$$u = \sqrt{1-y^2} \qquad v = y$$

$$du = \frac{1}{2\sqrt{1-y^2}}(-2y) \quad dv = dy$$

$$= \frac{-y}{\sqrt{1-y^2}}$$

$$-2\left(y\sqrt{1-y^2}\,\Big]_0^R - \int_0^R \frac{-y^2}{\sqrt{1-y^2}} \right)$$

$$\left[\int_0^R \frac{-y^2}{\sqrt{1-y^2}} = \int_0^R \frac{1-y^2}{\sqrt{1-y^2}} - \int \frac{1}{\sqrt{1-y^2}} \right.$$

$$\left. = \int_0^R \sqrt{1-y^2} - \arcsin(y)\,\Big]_0^R \right]$$

$$= -2\left(R\sqrt{1-R^2}\right) - 2\int_0^R \sqrt{1-y^2} + 2\arcsin(R) - 2\arcsin(0) \Big)$$

$\Rightarrow$

5.) Show that $\int_{\partial D} x\,dy$ is the area of D, while $\int_{\partial D} y\,dx$ is minus the area of D.

$$x = Q \Rightarrow \frac{\partial Q}{\partial x} = 1$$

$$\Rightarrow \int_{\partial D} x\,dy = \iint_D 1\,dx\,dy$$

$$y = P \Rightarrow \frac{\partial P}{\partial y} = 1$$

$$\Rightarrow \int_{\partial D} y\,dx = \iint_D -1\,dx\,dy.$$

Chapter 3, Section 2: Exercises

1.) Determine whether each of the following line integrals is independent of path. If it is, find a function h s.t. $dh = Pdx + Qdy$. If it is not, find a closed path γ around which the integral is not zero.

(a) $xdx + ydy$.

$$xdx + ydy = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy.$$

$$\frac{\partial h}{\partial x} = x \Rightarrow h(x,y) = \frac{x^2}{2} + h(y)$$

$$\Rightarrow \frac{\partial h}{\partial y} = h'(y) = y \Rightarrow h(y) = \frac{y^2}{2}$$

$$\Rightarrow h(x,y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

$\Rightarrow$ Exact

$\Rightarrow$ Independent of path.

(b) $x^2 dx + y^5 dy$.

$$x^2 dx + y^5 dy = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy.$$

$$\frac{\partial h}{\partial x} = x^2 \Rightarrow h(x,y) = \frac{x^3}{3} + h(y)$$

$$\Rightarrow \frac{\partial h}{\partial y} = h'(y) = y^5$$

$$\Rightarrow h(y) = \frac{y^6}{6}$$

$$\Rightarrow h(x,y) = \frac{x^3}{3} + \frac{y^6}{6}$$

$\Rightarrow$ Exact

$\Rightarrow$ Independent of path.

(c) $y\,dx + x\,dy$

$$y\,dx + x\,dy = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$$

$$\Rightarrow \frac{\partial h}{\partial x} = y \Rightarrow h(x,y) = xy + h(y)$$

$$\Rightarrow \frac{\partial h}{\partial y} = x + h'(y)$$

$$\Rightarrow h'(y) = 0. \Rightarrow h(y) = c$$

$$\Rightarrow h(x,y) = xy$$

$\Rightarrow$ Exact

$\Rightarrow$ Independent of path.

(d) $y\,dx - x\,dy$.

$$y\,dx - x\,dy = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

$$\frac{\partial h}{\partial x} = y \Rightarrow h(x,y) = xy + h(y)$$

$$\Rightarrow \frac{\partial h}{\partial y} = x + h'(y)$$

$\Rightarrow$ Not exact.

$$\int_0^{2\pi} -\sin^2\theta - \cos^2\theta = -\theta\Big]_0^{2\pi} = -2\pi \neq 0.$$

So around the unit circle

$\gamma = (\cos\theta, \sin\theta)\ 0<\theta<2\pi.$

2.) Show that the differential

$$\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \quad , (x,y) \neq (0,0)$$

is closed. Show that it is not independent of path on any annulus centered at 0.

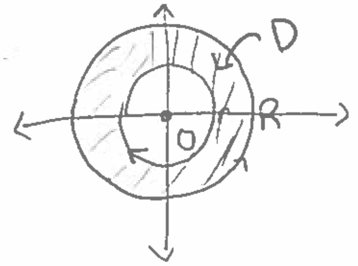
WTS $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$P(x,y) = \frac{-y}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$Q(x,y) = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial Q}{\partial x} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \text{closed}$$



$$\int_{\partial D} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_{|z|=R} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy - \int_{|z|=r} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

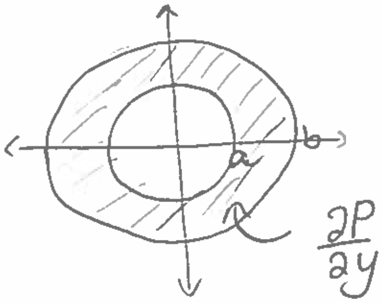
$$= \int_0^{2\pi} \frac{+R^2\sin^2\theta + R^2\cos^2\theta}{R^2\cos^2\theta + R^2\sin^2\theta} = 2\pi - 2\pi \text{ ?}$$

$$- \int_0^{2\pi} \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$y = R\sin\theta$ $\quad dy = R\cos\theta\, d\theta$

$x = R\cos\theta$ $\quad dx = R\sin\theta\, d$

3.) Suppose that P & Q are smooth functions on the annulus $\{a<|z|<b\}$ that satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Show directly using Green's Theorem that $\oint_{|z|=r} Pdx+Qdy$ is independent of the radius r, for $a<r<b$.



$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\oint_{|z|=r} Pdx+Qdy = \iint_{|z|<r} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_{a<|z|<r} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \iint_{a>|z|} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= 0 + \iint_{|z|<a} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_{|z|=a} Pdx + Qdy$$

Oh. which is independent of r.

4.) Let P & Q be smooth functions on D satisfying $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Let γ_0 & γ_1 be two closed paths in D s.t. the straight line segment from $\gamma_0(t)$ to $\gamma_1(t)$ lies in D for every parameter value t. Then $\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$. Use this to give another sol'n to the preceding exercise.

—not finished—

Chapter 3, Section 3 : Exercises

1.) For each of the following harmonic functions u, find du, find dv, and find v, the conjugate harmonic function of u.

(a) $u(x,y) = x - y$

$$\Rightarrow \frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = y + h(x)$$

$$\frac{\partial u}{\partial y} = -1 = \frac{-\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 1 \Rightarrow \frac{\partial v}{\partial x} = h'(x) = 1$$

$$\Rightarrow h(x) = x$$

$$\Rightarrow v(x,y) = y + x + c$$

(b) $u(x,y) = x^3 - 3xy^2$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 6xy$$

$$\Rightarrow v(x,y) = 3x^2y - y^3 + h(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 6xy + h'(x) \Rightarrow h'(x) = 0$$

$$\Rightarrow v(x,y) = 3x^2y - y^3 + c.$$

(c) $u(x,y) = \sinh(x)\cos(y)$.

$$\Rightarrow \frac{\partial u}{\partial x} = \cosh(x)\cos(y) = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\sinh(x)\sin(y) = -\frac{\partial v}{\partial x}$$

$$\Rightarrow v(x,y) = \sin(y)\cosh(x) + h(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = \sinh(x)\sin(y) + h'(x) \Rightarrow h(x) = c. \Rightarrow v(x,y) = \sin(y)\cosh(x) + c.$$

(d) $u(x,y) = \frac{y}{x^2+y^2}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2) - y\,2y}{(x^2+y^2)^2} = \frac{-y^2+x^2}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}dx + \frac{-2xy}{(x^2+y^2)^2}dy$$

$$\Rightarrow v = \int \frac{y^2-x^2}{(x^2+y^2)^2}dx + \frac{-2xy}{(x^2+y^2)^2}dy$$

$x = r\cos\theta \quad dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \quad dy = r\cos\theta\, d\theta$

$$= \int \frac{r^2\sin^2\theta - r^2\cos^2\theta}{r^4}(-r\sin\theta\, d\theta) + \frac{-2r^3\cos^2\theta\sin\theta}{r^4}d\theta$$

$$= \int \frac{-r^3\sin^3\theta + r^3\cos^2\theta\sin\theta - 2r^3\cos^2\theta\sin\theta}{r^4}d\theta$$

$$= \int \frac{r^3\sin\theta}{r^4}\left[-\sin^2\theta + \cos^2\theta - 2\cos^2\theta\right]d\theta$$

$$= -\int \frac{\sin\theta}{r}d\theta = \frac{1}{r}\cos\theta = \frac{x}{x^2+y^2} \left(= \frac{r\cos\theta}{r^2}\right)$$

(2.) Show that a complex-valued function $h(z)$ on a star-shaped domain D is harmonic $\Leftrightarrow$ $h(z) = f(z) + \overline{g(z)}$, where $f(z)$ and $g(z)$ are analytic on D.

($\Leftarrow$) Suppose $h(z) = f(z) + \overline{g(z)}$.

$f(z)$ and $g(z)$ are analytic, so

$f(z) = u + iv \quad \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$g(z) = a + ib \quad \Rightarrow \overline{g(z)} = a - ib \quad \frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}, \quad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$

$h(z) = (u+a) - i(v+b)$

$$\Rightarrow \frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 a}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} + i\frac{\partial^2 b}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 a}{\partial x^2} - i\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - i\frac{\partial}{\partial x}\left(\frac{\partial a}{\partial y}\right)$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 a}{\partial y^2} - i\cdot\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 b}{\partial y^2}$$

By cty, $\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 a}{\partial x^2} - i\frac{\partial^2 v}{\partial y^2} - i\frac{\partial^2 b}{\partial y^2}$

$=$

Chapter 3, Section 4: Exercises:

1.) Let $f(z)$ be a continuous function on a domain D. Show that if $f(z)$ has the MVP wrt circles, as defined above, then $f(z)$ has the MVP wrt disks, that is, if $z_0 \in D$ and D_0 is a disk centered at z_0 w/ area A and contained in D, then

$$f(z_0) = \frac{1}{A} \iint_{D_0} f(z)\,dx\,dy.$$

$$f(z_0) = \int_{\theta=0}^{\theta=2\pi} f(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} =$$

$$D_0 = \{z \in \mathbb{C} : |z - z_0| \leq R\}.$$

$$A = \pi R^2.$$

$$\left.\begin{aligned} x &= x_0 + r\cos\theta \\ y &= y_0 + r\sin\theta \end{aligned}\right\} \Rightarrow z = z_0 + re^{i\theta}$$

$$\frac{1}{A}\iint_{D_0} f(z)\,dx\,dy = \frac{1}{\pi R^2} \int_0^{2\pi}\int_0^R f(z_0 + re^{i\theta})\, r\,dr\,d\theta$$

$$= \frac{R^2}{2\pi R^2} \int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

$$= f(z_0).$$

(2.7) Derive

$$r\int_0^{2\pi}\frac{\partial u}{\partial r}(z_0+re^{i\theta})\,d\theta$$

$$\|$$

$0 = r\int_0^{2\pi}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]d\theta$" from the polar form of the CR-eq'ns.

<u>CR-eq'ns:</u>

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Have $\int_0^{2\pi} r\frac{\partial u}{\partial r}(z_0+re^{i\theta})\,d\theta = \int_0^{2\pi}\frac{\partial v}{\partial \theta}(z_0+re^{i\theta})\,d\theta$

$= 2\pi \cdot \frac{\partial v}{\partial \theta}(z_0) = 0$. ✓

$x = x_0 + r\cos\theta$

$y = y_0 + r\sin\theta$

And $\int_0^{2\pi} r\frac{\partial u}{\partial r}(z_0+re^{i\theta}) = r\int_0^{2\pi}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]d\theta$

✓

3.) A function $f(t)$ on an interval $I = (a,b)$ has the <u>mean value property</u> if:

$$f\left(\frac{s+t}{2}\right) = \frac{f(s)+f(t)}{2}, \quad s,t \in I.$$

Show that any affine function $f(t) = At+B$ has the mean value property.

$$f\left(\frac{s+t}{2}\right) = A\left(\frac{s+t}{2}\right) + B \quad \checkmark$$

$$\frac{f(s)+f(t)}{2} = \frac{As+B + At+B}{2} = A\left(\frac{s+t}{2}\right) + B. \quad \checkmark$$

3 ☆ Interesting Idea

A function $f(z)$ on the complex plane is doubly periodic if there are two periods ω_0 & ω_1 of $f(z)$ that don't lie on the same line through the origin

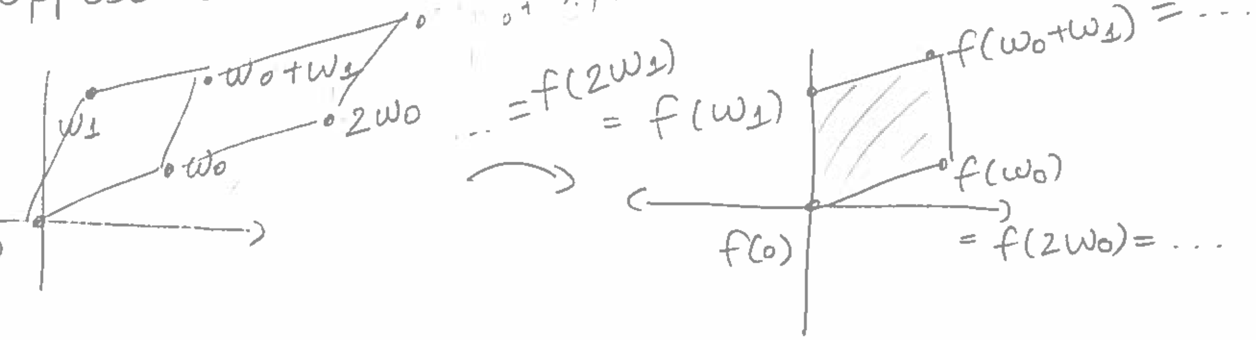
[i.e. ω_0 & ω_1 are LI over the reals and

$$f(z + \omega_0) = f(z+\omega_1) = f(z) \quad \forall z \in \mathbb{C}]$$

Prove that the only entire functions that are doubly periodic are the constants

WTS entire & doubly periodic $\Leftrightarrow$ constant

($\Rightarrow$) Suppose entire & doubly periodic.



So ~~~~ since $f(z)$ analytic on a compact set, $\Rightarrow$ cts.

$\Rightarrow$ bounded.

$\Rightarrow$ By Liouville, $f(z)$ is constant.

($\Leftarrow$) Clear.

4.) Suppose that $f(z)$ is an entire function s.t. $\frac{f(z)}{z^n}$ is bounded for $|z| \geq R$. Show that $f(z)$ is a poly of degree at most n. What can be said ~~about~~ if $\frac{f(z)}{z^n}$ is bounded on the entire complex plane.

Proof: If $|z-z_0|<R$. for some $z_0 \neq \pm\infty$.

$\Rightarrow |z|-|z_0| \leq |z-z_0| < R$.

$\Rightarrow |z| < R+|z_0|$.

$\Rightarrow \left|\frac{f(z)}{z^n}\right| \leq M$ for $|z| \geq R$

$\Rightarrow |f(z)| \leq M|z|^n$ for $|z| \geq R$.

$\Rightarrow |f(z)| \leq M|z|^n$ for $|z| \geq R$

$\Rightarrow |f(z)| \leq M(R+|z_0|)^n$ for $|z-z_0|<R$.

$f(z)$ analytic everywhere.

$\Rightarrow$ By Cauchy's estimate,

$$|f^{(n+1)}(z_0)| \leq \frac{(n+1)!}{R^{n+1}} M(R+|z_0|)^n \rightarrow 0$$

$\Rightarrow f(z)$ is a polynomial of degree at most n.

If $\frac{f(z)}{z^n}$ is bounded on the entire complex plane

$\Rightarrow \left|\frac{f(z)}{z^n}\right| \leq M \Rightarrow |f(z)| \leq M|z|^n$

If $f(z)=z^n g(z) \Rightarrow \left|\frac{f(z)}{z^n}\right| = |g(z)|$, where $g(z)$ is analytic $\Rightarrow$ By Liouville, $\frac{f(z)}{z^n}=c \Rightarrow f(z)=cz^n$.

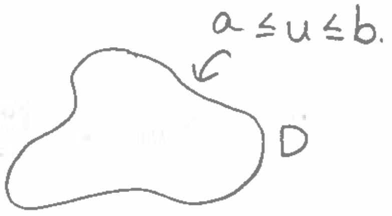
Otherwise, let $\varepsilon>0$, $|z|<\varepsilon$. $\Rightarrow \left|\frac{f(z)}{z^n}\right| \leq M$

$\Rightarrow |f(z)| \leq M|z|^n < M\varepsilon^n \rightarrow 0 \Rightarrow f(z)=0$.

Chapter 3, Section 5 : Exercises

☆ Weird thinking! ☆

1.) Let D be a bounded domain, and let u be a real-valued harmonic function on D that extends continuously to the boundary ∂D. Show that if $a \le u \le b$ on ∂D, then $a \le u \le b$. on D.



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since $D \cup \partial D$ is compact, u must attain a maximum and minimum value on $D \cup \partial D$

$\Rightarrow u(z) \le M \quad \forall z \in D$

Suppose $\overset{\exists z_0 \in D \text{ s.t.}}{\wedge}$ $u(z_0) = M > b$ on D

$\Rightarrow u(z) = M \quad \forall z \in D$

But then the function doesn't extend continuously to the boundary #

$\Rightarrow u(z) \le b$ on D.

Now suppose $u(z_0) = \overset{\text{min. on } D}{M} < a$ $\Rightarrow -u(z) \le -M \quad \forall z \in D \cup \partial D$

$\Rightarrow -u(z) \le -a \quad \forall z \in \partial D$.

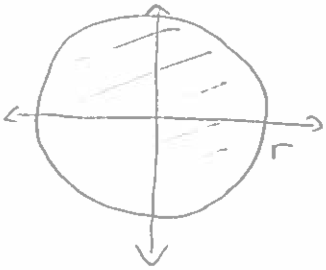
$\Rightarrow -u(z) = -M \quad \forall z \in D$

$\Rightarrow u(z) = M \quad \forall z \in D$.

$\Rightarrow$ # since function doesn't extend ctsly onto boundary.

$\Rightarrow u(z) \ge a$.

2.) Fix $n \geq 1, r > 0, \lambda = \rho e^{i\varphi}$. What is the maximum modulus of $z^n + \lambda$ over the disk $\{|z| \leq r\}$? Where does $z^n + \lambda$ attain its maximum modulus over the disk?



$$|z^n + \lambda| = |R^n\cos(n\theta) + \rho\cos(\varphi) + i R^n\sin(n\theta) + i\rho\sin\varphi|$$

$$|z^n + \lambda| \leq |z|^n + |\lambda| \leq r^n + |\rho e^{i\varphi}| = r^n + \rho.$$

~~☆~~ ☆ In all cases, the function attains its max modulus on ∂D (Assuming bounded D).

(Is the triangle inequality the max??)

When does $z^n + \rho e^{i\varphi} = r^n + \rho$?

$$z^n - r^n - \rho(e^{i\varphi} - 1) = 0.$$

$$\varphi = 2\pi k n.$$

$$z = r e^{i\varphi}$$

$$\Rightarrow z = r e^{i\varphi}$$

3.) Use the maximum principle to prove the Fundamental Theorem of algebra, that any poly $p(z)$ of degree ≥ 1 has a zero, by applying the maximum principle to $\frac{1}{p(z)}$ on a disk of large radius.

Suppose FTSOC that $\frac{1}{p(z)} \neq 0$ everywhere.

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ on disk of radius R.

Since $D \cup \partial D$ is compact, $\frac{1}{p(z)}$ attains maximum on

$D \cup \partial D$.

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right).$$

If $|z| \geq R \Rightarrow \left|\frac{1}{p(z)}\right| = \left|\frac{1}{z^n\left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right)}\right| \leq$

$|z|^n \geq R \Rightarrow \frac{1}{|z|^n} \leq \frac{1}{R^n} \Rightarrow \left|z^n\left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right)\right|$

$\geq R^n\left(|a_n| - \left|\frac{a_{n-1}}{R}\right| \dots - \left|\frac{a_0}{R^n}\right|\right)$

$\Rightarrow \frac{1}{|z|^n}$ is bounded.

$\Rightarrow$ when $|z| \leq R \Rightarrow \frac{1}{p(z)}$ must be bounded ~~on~~

Since continuous function on compact sets attains its maximum modulus on ∂D.

$\Rightarrow \frac{1}{p(z)}$ is analytic & bdd $\Rightarrow$ constant $\#$

4.) Let $f(z)$ be an analytic function on a domain D that has no zeroes on D.

(a) Show that if $|f(z)|$ attains its minimum on D, then $f(z)$ is constant.

Suppose $|f(z)| \geq M = \min(f(z))$. $f(z_0) = M \; z_0 \in D$

$$\Rightarrow \frac{1}{|f(z)|} \leq \frac{1}{M}. \quad \Rightarrow \frac{1}{f(z_0)} = \frac{1}{M} \text{ on } D.$$

$$\Rightarrow \frac{1}{f(z)} = \frac{1}{M} \quad \forall z \in D$$

$$\Rightarrow f(z) = M \; \forall z \in D.$$

(b) Show that if D is bounded, and if $f(z)$ extends continuously to ∂D of D, then $|f(z)|$ attains its minimum on ∂D.

If $|f(z)| \geq M \; \forall z \in \partial D$

$$\Rightarrow \frac{1}{|f(z)|} \leq \frac{1}{M} \quad \forall z \in \partial D.$$

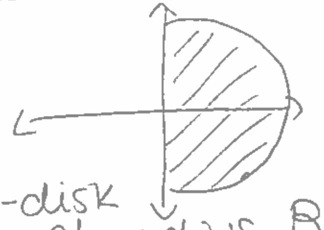
$$\Rightarrow \frac{1}{|f(z)|} \leq \frac{1}{M} \quad \forall z \in D.$$

$$\Rightarrow |f(z)| \geq M \quad \forall z \in D.$$

5.) Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose that $f(z)$ extends ctsly to the imaginary axis and $|f(iy)| \le M$ $\forall iy \in i\mathbb{R}$. Show that $|f(z)| \le M$ $\forall z$ in the right-half plane.

Hint: For $\varepsilon > 0$ small, consider $\frac{f(z)}{(1+z)^{\varepsilon}}$ on a large

semidisk



On the semi-disk of radius R,

$$\left|\frac{f(z)}{(1+z)^{\varepsilon}}\right| \le \frac{M}{(1-R)^{\varepsilon}} \quad \forall z \in D$$

$$\left|\frac{f(z)}{(1+z)^{\varepsilon}}\right| \le \frac{M}{\quad}$$

$$|z+w| \le |z|+|w|$$

$$\Rightarrow |z+w-w| \le |z-w|+|w|$$

$$|z-w| \ge |z|-|w|$$

So $\left|\frac{f(z)}{(1+z)^{\varepsilon}}\right| \le \max\left(\frac{M}{(1-R)^{\varepsilon}},\right.$

$\Rightarrow \ |\ f(z)$

$\Rightarrow$ Let R be so large that $\left|\frac{f(z)}{(1+z)^{\varepsilon}}\right| \le M.$

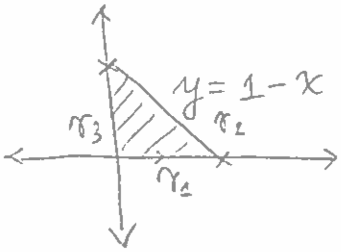
$\Rightarrow$ Letting $\varepsilon \to 0$,

$|f(z)| \le M.$

Chapter 4, Section 1: Exercises:

1.) Let γ be the boundary of the triangle $\{0<y<1-x,\ 0<x<1\}$ with the usual ccw orientation. Evaluate the following integrals:

(a) $\int_\gamma \operatorname{Re} z\, dz$



$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \operatorname{Re} z\, dz = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x\,(dx+i\,dy) = \int_{x=0}^{x=1}$$

$$= \int_{\triangle} \operatorname{Re} z\, dz = \int_0^1$$

$$\gamma_1(t) = t; \quad 0 \le t \le 1 \Rightarrow \begin{matrix} x=t \\ y=0 \end{matrix} \Rightarrow \begin{matrix} dx = dt \\ dy = 0 \end{matrix}$$

$$\gamma_2(t) = (1-t) + ti, \quad 0 \le t \le 1 \Rightarrow \begin{matrix} dx = -dt \\ dy = i\,dt \end{matrix}$$

$$\gamma_3(t) = (1-t)i$$

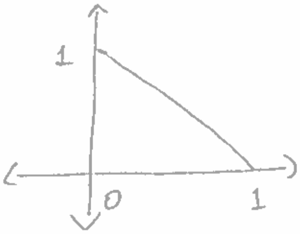
$$\int_0^1 t\,dt + \int_0^1 (1-t)[-1\,dt + i\,dt] + \int_0^1 0$$

$$= \frac{t^2}{2}\Big]_0^1 + \int_0^1 (-1+i+t-it)\,dt$$

$$= \frac{1}{2} + -t+it+\frac{t^2}{2}-\frac{it^2}{2}\Big]_0^1$$

$$= \frac{1}{2} - 1 + i + \frac{1}{2} - \frac{i}{2} = \boxed{\frac{i}{2}}$$

(b) $\int_{\gamma} \operatorname{Im} z \, dz$



$=$

$\gamma_1 = t \qquad x = t, \quad y = 0 \Rightarrow dx = dt$

$\gamma_2 = (1-t) + it \qquad \Rightarrow dx = -1\,dt, \quad dy = dt$

$\gamma_3 = (1-t)i$

$dx = 0$

$dy = -1dt$

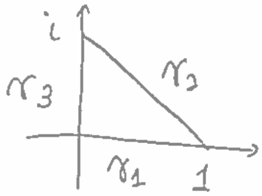
$$\int_0^1 0 + \int_0^1 t(-1+i)dt + \int_0^1 (1-t)(-i)dt$$

$$= \int_0^1 (-t+ti)dt + \int_0^1 (-i+it)dt$$

$$= \frac{-t^2}{2} + \frac{it^2}{2}\Big]_0^1 + -it + \frac{it^2}{2}\Big]_0^1$$

$$= -\frac{1}{2} + \frac{i}{2} - i + \frac{i}{2} = -\frac{1}{2}$$

(c) $\int_{\gamma} z\, dz$



$\gamma_1 = t \Rightarrow dx = dt$
$dy = 0$

$\gamma_2 = (1-t) + ti \Rightarrow dx = -dt$
$dy = dt$

$\gamma_3 = (1-t)i \Rightarrow dx = 0$
$dy = -dt$

$$\int_0^1 t\, dt + \int_0^1 ((1-t) + ti)(-1+i)dt + \int_0^1 (1-t)i\,(-i)dt)$$

$$= \frac{t^2}{2}\Big]_0^1 + \int_0^1 \Big[(-1+i) - t(-1+i) + ti(-1+i)\Big]dt$$

$$+ \int_0^1 (1-t)dt$$

$$= \frac{1}{2} + (-1+i)t - \frac{t^2}{2}(-1+i) + \frac{t^2}{2}i(-1+i) + t - \frac{t^2}{2}\Big]_0^1$$

$$= \cancel{\frac{1}{2}} + \cancel{-1} + i - \frac{1}{2}(-1+i) + \frac{i(-1+i)}{2} + 1 - \cancel{\frac{1}{2}}$$

$$= i + \cancel{\frac{1}{2}} - \frac{i}{2} - \frac{i}{2} - \frac{1}{2} = 0$$

2.) Let γ be the unit circle $\{|z|=1\}$, with the usual ccw orientation. Evaluate the following integrals, for $m = 0, \pm 1, \pm 2, \ldots$

(a) $\int_\gamma z^m dz$

$z = \cos\theta + i\sin\theta$ $\quad x = \cos\theta \quad dx = -\sin\theta\, d\theta$

$0 \le \theta \le 2\pi$ $\quad y = \sin\theta \quad dy = \cos\theta\, d\theta$

$$= \int_0^{2\pi} (\cos\theta + i\sin\theta)^m (-\sin\theta + i\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} (\cos\theta + i\sin\theta)^m\, i(\cos\theta + i\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} i(\cos\theta + i\sin\theta)^{m+1}\, d\theta = \int_0^{2\pi} i(\cos((m+1)\theta) + i\sin((m+1)\theta))\, d\theta$$

$$= \int_0^{2\pi} ie^{(m+1)i\theta} = \begin{cases} 2\pi i & \text{if } m = -1 \\ \left[\dfrac{ie^{(m+1)i\theta}}{i(m+1)}\right]_0^{2\pi} = 0 & \text{if } m \neq -1 \end{cases}$$

(b). $$\int_\gamma \bar{z}^m dz = \int_0^{2\pi} (\cos\theta - i\sin\theta)^m (-\sin\theta + i\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} e^{-im\theta}\, ie^{i\theta}\, d\theta = \int_0^{2\pi} ie^{i\theta(1-m)}\, d\theta = \begin{cases} 2\pi i \text{ if } m = 1 \\ \left[\dfrac{ie^{i\theta(1-m)}}{i(1-m)}\right]_0^{2\pi} \end{cases}$$

$$= \left\{\begin{matrix} 2\pi i \text{ if } m = 1 \\ \left[\dfrac{e^{i\theta(1-m)}}{1-m}\right]_0^{2\pi} \end{matrix}\right\} = \begin{cases} 2\pi i \text{ if } m = 1 \\ 0 \quad \text{if } m \neq 1 \end{cases}$$

(c) $\int_{\gamma} z^m |dz| = \int_{0}^{2\pi} e^{im\theta}\sqrt{dx^2+dy^2}\, d\theta = \int_{0}^{2\pi} e^{im\theta} d\theta$

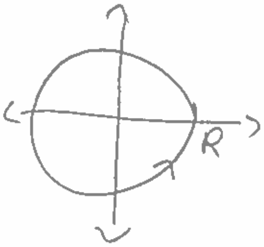
$x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

$$= \begin{cases} 2\pi \text{ if } m = 0 \\ \left[\frac{e^{im\theta}}{im}\right]_0^{2\pi} \text{ if } m \neq 0 \end{cases} = \begin{cases} 2\pi \text{ if } m = 0 \\ 0 \text{ if } m \neq 0 \end{cases}$$

3.) Let γ be the circle $\{|z| = R\}$, with the usual ccw orientation. Evaluate the following integrals, for $m = 0, \pm 1, \pm 2, \ldots$

(a) $\int_{\gamma} |z^m| dz$



$z = Re^{i\theta} \Rightarrow z^m = R^m e^{mi\theta} \Rightarrow |z^m| = \sqrt{R^{2m}\cos^2(m\theta) + R^{2m}\sin^2(m}$

$= R^m\cos(m\theta) + R^m\sin(m\theta)i$

$\Rightarrow dz = \frac{Re^{i\theta}}{i} d\theta$

$$\int_0^{2\pi} \frac{R^m Re^{i\theta}}{i} d\theta = \begin{cases} \left[\frac{R^{m+1}}{i} \frac{e^{i\theta}}{i}\right]_0^{2\pi} \text{ if } m \neq -1 \\ \left[\frac{e^{i\theta}}{-1}\right]_0^{2\pi} \text{ if } m = -1 \end{cases}$$

$$= \begin{cases} -R^{m+1}\left[e^{2\pi i} - e^0\right] \text{ if } m \neq -1 \\ 0 \quad \text{ if } m = -1 \end{cases} = 0$$

(b) $\int_{\gamma} |z^m|\,|dz|.$

$$z = Re^{i\theta} \Rightarrow z^m = R^m e^{mi\theta} \Rightarrow |z^m| = R^m$$

$$dz = iRe^{i\theta} = iRe^{i\theta}d\theta \Rightarrow |dz| = |\,iRe^{i\theta}d\theta|$$

$$= Rd\theta$$

$$\Rightarrow \int_0^{2\pi} R^m R\,d\theta = \int_0^{2\pi} R^{m+1}\,d\theta = R^{m+1}2\pi.$$

(c) $\int_{\gamma} \bar{z}^m\,dz$

$$\bar{z}^m = (Re^{-i\theta})^m = R^m e^{-im\theta}$$

$$dz = iRe^{i\theta}d\theta$$

$$\int_0^{2\pi} R^m e^{-im\theta}\,iRe^{i\theta}d\theta = \int_0^{2\pi} R^{m+1}\,i\,e^{i\theta(1-m)}d\theta$$

$$= iR^{m+1}\int_0^{2\pi} e^{i\theta(1-m)}\,d\theta = \begin{cases} iR^{m+1}\,2\pi & \text{if } m=1 \\ \left.\dfrac{\cancel{i}R^{m+1}e^{i\theta(1-m)}}{\cancel{i}(1-m)}\right]_0^{2\pi} & \text{if } m\neq 1 \end{cases}$$

$$= \begin{cases} 2\pi i R^2 & \text{if } m=1 \\ 0 & \text{if } m\neq 1 \end{cases}$$

4.) Show that if D is a bounded domain with smooth boundary, then

$$\int_{\partial D} \bar{z}\,dz = 2i\,\text{Area}(D).$$

$$\int_{\partial D} \bar{z}\,dz = \int_{\partial D} (x-iy)(dx+idy) = \int_{\partial D} xdx + xi\,dy - iy\,dx + y\,dy$$

$$= \int_{\partial D} (x-iy)dx + (xi+y)\,dy = \iint_{D} 2i\,dx\,dy = 2i\,\text{Area}(D)$$

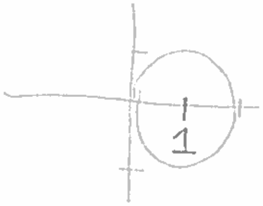
$$P = x - iy \qquad \frac{\partial P}{\partial y} = -i$$

$$Q = xi + y \qquad \frac{\partial Q}{\partial x} = i$$

(5.) Show that $\left| \oint_{|z-1|=1} \frac{e^z}{z+1}\,dz \right| \leq 2\pi e^2.$

$$\oint_{|z-1|=1} |dz| = 2\pi$$

$$\left|\frac{e^z}{z+1}\right| \leq e^2$$



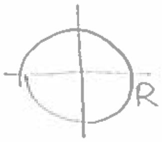
wts $|z+1| = |1+e^{i\theta}+1| = |2+e^{i\theta}| \geq 2 - |e^{i\theta}| = 2-1 = 1$

$|e^z| = |e^x e^{iy}| = |e^x \cos y + ie^x \sin y|$

$= \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y} = |e^x| \leq e^2$

(6.) Show that

$$\left| \oint_{|z|=R} \frac{\mathrm{Log} z}{z^2} \, dz \right| \leq 2\sqrt{2}\pi \frac{\log R}{R} \quad , \quad R > e^{\pi}.$$



$$\oint_{|z|=R} |dz| = 2\pi R$$

WTS $\left| \frac{\mathrm{Log} z}{z^2} \right| \leq \frac{\sqrt{2} \log R}{R^2}$

Since $-\pi \leq \theta \leq \pi$

$z = Re^{i\theta}$

$$\Rightarrow |\mathrm{Log} z| = \sqrt{(\log|z|)^2 + \theta^2} = \sqrt{(\log R)^2 + \theta^2} \leq \sqrt{(\log R)^2 + \pi^2}$$

$$\log(e^{\pi}) < \log(R)$$

$$\Rightarrow \pi < \log R$$

$$< \sqrt{(\log R)^2 + (\log R)^2} < \sqrt{2(\log R)^2} = \sqrt{2} \log R$$

$$|z^2| = R^2.$$

$$\Rightarrow \left| \oint_{|z|=R} \frac{\mathrm{Log} z}{z^2} dz \right| \leq \frac{2\sqrt{2}\pi \log R}{R}$$

by M-L Estimate.

7) Show that there is a strict inequality

$$\left|\oint_{|z|=R} \frac{z^n}{z^m-1}\,dz\right| < \frac{2\pi R^{n+1}}{R^m-1}, \quad R>1, \; m\ge 1, \; n\ge 0.$$

$$\oint_{|z|=R} |dz| = 2\pi R$$

$$\left|\frac{z^n}{z^m-1}\right| = \frac{R^n}{R^m-1}$$

$$|z^m-1| \ge |z^m| - 1 = R^m-1$$

$$|x-y+y| \le |x-y| + |y|$$

$$\Rightarrow |x|-|y| \le |x-y|$$

$$|z|^n = R^n$$

$\Rightarrow$ By M-L Estimate, $\left|\oint_{|z|=R} \frac{z^n}{z^m-1}\,dz\right| \le \frac{2\pi R^{n+1}}{R^m-1}$

$$\oint_{|z|=R} \frac{z^n}{z^m-1}\,dz = \int_0^{2\pi} \frac{R^n e^{in\theta}}{R^m e^{im\theta}-1}\, iRe^{i\theta}\,d\theta = \int_0^{2\pi} \frac{iR^{n+1}e^{i\theta(n+1)}}{R^m e^{im\theta}-1}\,d\theta$$

$z = Re^{i\theta}$

$\Rightarrow dz = iRe^{i\theta}\,d\theta$

$$= iR^{n+1} \int_0^{2\pi} \frac{e^{i\theta(n+1)}}{R^m e^{im\theta}-1}\,d\theta$$

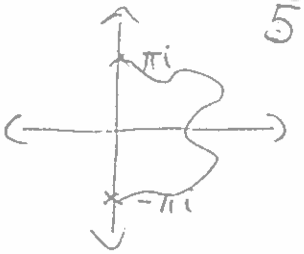
$$\frac{iR^{n+1}}{R^m e^{im\theta}-1} \left.\frac{e^{i\theta(n+1)}}{i(n+1)}\right]_0^{2\pi} = R^{n+1}e$$

—not sure why strict—

Chapter 4 Section 2: Exercises

1.) Evaluate the following integrals, for a path γ that travels from $-\pi i$ to πi in the right half-plane, and also for a path γ from $-\pi i$ to πi in the left half-plane.

(a) $\int_\gamma z^4 dz = \frac{z^5}{5}\Big]_{-\pi i}^{\pi i} = \frac{(\pi i)^5}{5} - \frac{(-\pi i)^5}{5} = \frac{2(\pi i)^5}{5}$



On left-half-plane the same why? Indep. of path

~~$dt=$~~

(b) $\int_\gamma e^z dz = e^z\Big]_{-\pi i}^{\pi i} = e^{\pi i} - e^{-\pi i} = \cos(\pi) + i\sin(\pi)$
$- \cos(-\pi) - i\sin(-\pi)$

$= -1 + 0 + \underline{1} - 0$

$=$ ~~-2~~. 0.

(c) $\int_\gamma \cos z \, dz = \sin z\Big]_{-\pi i}^{\pi i} = \sin(\pi i) - \sin(-\pi i)$

$=$

5.) Show that an analytic function $f(z)$ has a primitive in $D \Leftrightarrow \int_{\gamma} f(z)dz = 0$ for every closed path γ in D.

$(\Rightarrow)$ If $f(z)$ has a primitive D

$$\Rightarrow F'(z) = f(z).$$

Let γ be a closed path in D.

$$\Rightarrow \int_{\gamma} f(z)dz = \int_{a}^{a} f(\gamma(t))\,dt = 0.$$

$(\Leftarrow)$ Let $\int_{\gamma} f(z)\,dz = 0$ for every closed path γ in D.

$$F(z) = \int f(z)$$

3.) Show that if $m \neq -1 \Rightarrow z^m$ has a primitive on $\mathbb{C}\setminus\{0\}$

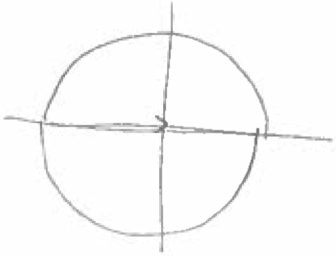
$$\frac{z^{m+1}}{m+1}$$ is a primitive on $\mathbb{C}\setminus\{0\}$.

Chapter 4, Section 3 Exercises:

3.) Let $f(z) = c_0 + c_1 z + \dots + c_n z^n$ be a polynomial.

(a) If the c_k's are real, show that

$$\int_{-1}^{1} f(x)^2 dx \le \pi \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \pi \sum_{k=0}^{n} c_k^2$$



$f(z)^2$ is analytic on ⌓, a bdd. domain, so

$$0 = \int_{\triangle} f(z)^2 dz = \int_{-1}^{1} f(x)^2 dx + \int_0^{\pi} f(e^{i\theta})^2 d\theta$$

$$0 = \int_{\smile} f(z)^2 dz = \int_{1}^{-1} f(x)^2 dx + \int_{\pi}^{2\pi} f(e^{i\theta})^2 d\theta$$

$$\Rightarrow -\int_{-1}^{1} f(x)^2 dx = \int_0^{\pi} f(e^{i\theta})^2 d\theta$$

$$\Rightarrow \int_{-1}^{1} f(x)^2 dx = \int_{\pi}^{2\pi} f(e^{i\theta})^2 d\theta$$

$$\Rightarrow 2\left|\int_{-1}^{1} f(x)^2 dx\right| \le \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$$

$$\Rightarrow \left|\int_{-1}^{1} f(x)^2 dx\right| \le \pi \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

$f(e^{i\theta}) = c_0 + c_1 e^{i\theta} + \dots + c_n e^{in\theta}$

$\Rightarrow |f(e^{i\theta})|^2 = (c_0 + c_1 e^{i\theta} + \dots + c_n e^{in\theta})(\overline{c_0} + \overline{c_1} e^{-i\theta} + \overline{c_2} e^{-2i\theta} + \dots + \overline{c_n} e^{-ni\theta})$

$= |c_0|^2 + c_0 \overline{c_1} e^{-i\theta} + \dots + c_0 \overline{c_n} e^{-ni\theta}$

$+ |c_1|^2 + c_1 \overline{c_0} e^{i\theta} + c_1 \overline{c_2} e^{-2ni\theta + i\theta} + \dots + c_1 e^{-ni\theta + i\theta} \overline{c_n}$

$+$

$+ |c_n|^2 + \overline{c_0} c_n e^{ni\theta} + \dots + c_n e^{ni\theta} \overline{c_{n-1}} e^{-(n-1)i\theta}$

~~$= |c_0|^2 + |c_1|^2 + \dots + |c_n|^2 +$~~

OH. c_k's are real, so $c_0 c_1 e^{-i\theta} + c_1 c_0 e^{i\theta}$

$= 2 c_0 c_1$

$e^{-i\theta} + e^{i\theta} = \cos(-\theta) + i\sin(-\theta)$

$+ \cos\theta + i\sin\theta$

$= 2\cos\theta = 2\,\mathrm{Re}(\theta)$

$\frac{c_0 \overline{c_1}}{e^{i\theta}} + \frac{c_1 c_0 e^{i\theta}}{}$

$f(e^{i\theta})\overline{f(e^{i\theta})} = (c_0 + c_1 e^{i\theta} + \dots + c_n e^{in\theta})\left(c_0 + \frac{c_1}{e^{i\theta}} + \dots + \frac{c_n}{e^{ni\theta}}\right)$

$= \frac{1}{e^{in\theta}} (c_0 e^{in\theta} + c_1 e^{i(n-1)\theta} + \dots + c_n)$

$(c_0 + c_1 e^{i\theta} + \dots + c_n e^{in\theta})$

$$\int_0^{2\pi} \left(\sum_{k=0}^{n} c_k e^{ik\theta}\right)\left(\sum_{m=0}^{n} c_m e^{-im\theta}\right) d\theta$$

$$= \sum_{k=0}^{n} \sum_{m=0}^{n} \int_0^{2\pi} c_k c_m e^{i\theta(k-m)} \, d\theta$$

$$= \left\{ \sum_{k=0}^{n} \sum_{m=0}^{n} c_k c_m \frac{e^{i\theta(k-m)}}{i(k-m)} \right]_0^{2\pi} = 0 \quad \text{when } k \neq m.$$

$$= \begin{cases} ~~0 \text{ when } k \neq m~~ \\ \sum_{k=0}^{n} \sum_{m=0}^{n} c_k c_m 2\pi \quad \text{when } k = m \end{cases}$$

$$= \begin{cases} 0 \quad \text{when } k \neq m \\ 2\pi \sum_{k=0}^{n} |c_k|^2 \quad \text{when } k = m. \end{cases}$$

(b) If the c_k's are complex, show that

$$\int^{1} |f(x)|^2 dx \leq \pi \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \pi \sum_{k=0}^{n} |c_k|^2.$$

Write $c_k = a_k + i b_k$

$$\Rightarrow f(z) = a_0 + ib_0 + (a_1 + ib_1)z + \ldots + (a_n + ib_n)z^n$$

$$= a_0 + a_1 z + \ldots + a_n z^n + i(b_0 + b_1 z + \ldots + b_n z^n)$$

$$|f(x)|^2 = (a_0 + a_1 x + \ldots + a_n x^n + i(b_0 + b_1 x + \ldots + b_n x^n))$$

$$(a_0 + a_1 x + \ldots + a_n x^n - i(b_0 + b_1 x + \ldots + b_n x^n))$$

$$= (a_0 + a_1 x + \ldots + a_n x^n)^2 + (b_0 + b_1 x + \ldots + b_n x^n)^2$$

$$\Rightarrow \int_{-1}^{1} |f(x)|^2 = \int_{-1}^{1} (a_0 + a_1 x + \ldots + a_n x^n)^2 + \int_{-1}^{1} (b_0 + b_1 x + \ldots + b_n x^n)$$

$$\leq \frac{\pi}{2\pi} \int_0^{2\pi} |a_0 + a_1 e^{i\theta} + \ldots + a_n e^{in\theta}|^2 + \frac{\pi}{2\pi} \int_0^{2\pi} | b_0 + b_1 e^{i\theta} + \ldots + b e^{in\theta}$$

$$= \frac{\pi}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \pi \sum_{k=0}^{n} a_k^2 + \pi \sum_{k=0}^{n} b_k^2 = \pi \sum_{k=0}^{n} |c_k|^2.$$

(c) Establish the following variant of Hilbert's inequality, that $\left|\sum_{j,k=0}^{n} \frac{c_j c_k}{j+k+1}\right| \le \pi \sum_{k=0}^{n} |c_k|^2$ with strict inequality unless $c_0, \dots, c_n = 0$.

$\int_0^1 \cancel{f(x)^2}\,dx =$ Let $f(x) = \frac{c_k c_j}{j+k+1}$

6.) Hard Problem. Suppose $f(z)$ is continuous in $\{|z| \leq R\}$ and analytic on $\{|z|<R\}$. Show that $\oint_{|z|=R} f(z)dz = 0$.

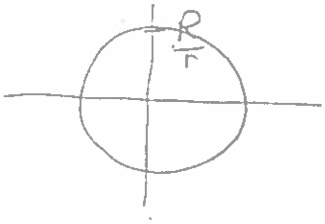
$\{|z| \leq R\}$ is compact $\Rightarrow f(z)$ is unif. cts. on $\{|z| \leq R\}$.

Let $r<1$

$f(rz)$ is analytic on $|rz|<R \Rightarrow z < \frac{R}{r}$.

Since $R > \frac{R}{r} > z$,

z is analytic on $|z| \leq R$.



so by CT,

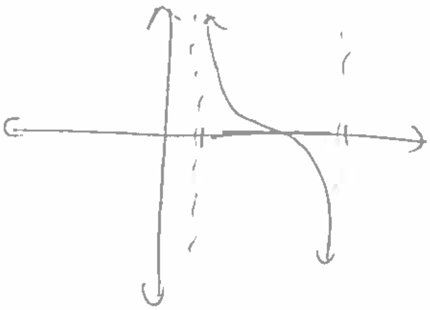
$$\oint_{|z|=R} f(rz) = 0.$$

$$\Rightarrow \left|\oint_{|z|=R} f(z)dz\right| = \left|\oint_{|z|=R} f(z)-f(rz)\right| \leq \oint_{|z|=R} |f(z)-f(rz)|\,dz$$

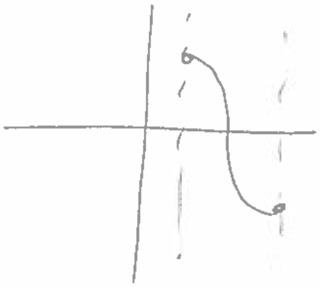
$$\underset{\uparrow}{\leq} \varepsilon 2\pi R$$

By unif. cty & ML-Estimate.

example of "cts." but not "extends ctly"



we need



Chapter 4, Section 4: Exercises

1.) Evaluate the following integrals, using the Cauchy integral formula:

(a) $\oint_{|z|=2} \frac{z^n}{z-1}\, dz = \frac{2\pi i}{0!} z^n \Big]_{z=1} = 2\pi i$

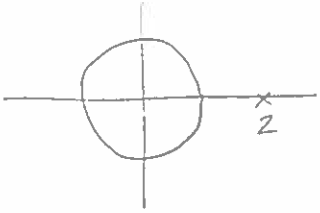
$f(z) = z^n$

$z = 1$

(b) $\oint_{|z|=1} \frac{z^n}{z-2}\, dz, \; n \geq 0$

$f(z) = z^n$

$z = 2$



$= 2\pi i\, z^n \Big]_{z=2} = 2\pi i\, 2^n = 2^{n+1}\pi i$

Just kidding. our function is analytic in $|z| \leq 1$, so $\oint \frac{z^n}{z-2}\, dz = 0$.

(c) $\oint_{|z|=1} \frac{\sin(z)}{z}\, dz = \frac{2\pi i \sin(z)}{0!} \Big]_{z=0} = 2\pi i \sin(0) = 0.$

$f(z) = \sin(z)$

$z = 0$

(d) $\oint_{|z|=1} \frac{\cosh(z)}{z^3}\, dz = \frac{d^3}{dz^3} \cosh(z) \Big]_{z=0} \cdot \underline{2\pi i}$

$= \cosh(0) \cdot 2\pi i$

(e) $\displaystyle\oint_{|z|=1} \frac{e^z}{z^m}\, dz, \quad -\infty < m < \infty$

$$= \frac{d^{m-1}}{dz^{m-1}}(e^z)\Big]_{z=0} = \frac{2\pi i e^0}{(m-1)!} = 1\,\frac{2\pi i}{(m-1)!} = \frac{2\pi i}{(m-1)!}$$

(f) $\displaystyle\int_{|z-(1+i)|=5/4} \frac{\operatorname{Log} z}{(z-1)^2}\, dz = \frac{2\pi i}{1!}\,\frac{d}{dz}\operatorname{Log}(z)\Big]_{z=1}$

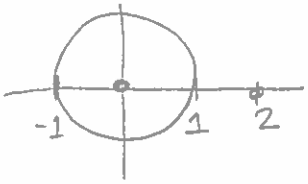
$f(z) = \operatorname{Log}(z)$

$m=1$

$z=1$

$$= \frac{2\pi i}{z}\Big]_{z=1} = \frac{2\pi i}{1} = 2\pi i$$

(g) $\displaystyle\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} = \int_{|z|=1} \frac{\frac{dz}{(z^2-4)e^z}}{z^2} = 2\pi i\,\frac{d}{dz}\left[\frac{1}{(z^2-4)e^z}\right]_{z=0}$



$f(z) = \dfrac{1}{(z^2-4)e^z}$

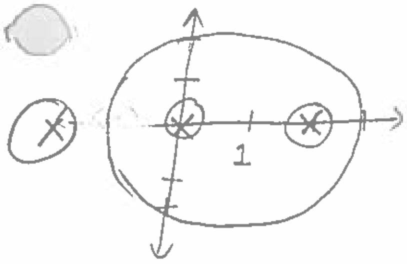
$m=1$

$z=0$

$$= 2\pi i\left[\frac{(z^2-4)e^z(0) - 1(2ze^z + (z^2-4)e^z)}{(z^2-4)^2 e^{2z}}\right]_{z=0}$$

$$= 2\pi i\left[\frac{-(-4)}{16\,e^0}\right] = \frac{4\cdot 2\pi i}{16} = \frac{\pi i}{2}$$

(h) $\oint\limits_{|z-1|=2} \frac{dz}{z(z^2-4)e^z}$



$$\Rightarrow \oint\limits_{|z-1|=2} \frac{dz}{z(z^2-4)e^z} - \overset{①}{\int\limits_{|z|=\varepsilon}} \frac{dz}{z(z^2-4)e^z} - \overset{②}{\int\limits_{|z-2|=\varepsilon}} \frac{dz}{z(z^2-4)e^z} - \overset{③}{\int\limits_{|z+2|=\varepsilon}} \frac{dz}{z(z^2-4)e^z} = 0$$

by CT

① $\int\limits_{|z|=\varepsilon} \frac{\frac{dz}{(z^2-4)e^z}}{z} = \frac{2\pi i}{0!} \frac{1}{(z^2-4)e^z}\Big]_{z=0} = \frac{2\pi i}{-4e^0} = \frac{-\pi i}{2}$

$f(z) = \frac{1}{(z^2-4)e^z}$

$m=0$

$z=0$

② $\int\limits_{|z-2|=\varepsilon} \frac{\frac{dz}{z(z+2)e^z}}{z-2} = \frac{2\pi i}{0!} \frac{1}{z(z+2)e^z}\Big]_{z=2} = \frac{\cancel{2}\pi i}{\cancel{2}(4)(e^2)} = \frac{\pi i}{4e^2}$

$f(z) = \frac{1}{z(z+2)e^z}$

$m=0$

$z=2$

③ $\int \frac{\frac{dz}{z(z-2)e^z}}{z+2} = \frac{2\pi i}{0!} \frac{1}{z(z-2)e^z}\Big]_{z=-2} = \frac{\cancel{2}\pi i}{-\cancel{2}(-4)e^{-2}} = \frac{\pi i e^2}{4}$

$f(z) = \frac{1}{z(z-2)e^z}$

$m=0$

$z=-2$

$$\Rightarrow \oint\limits_{|z-1|=2} \frac{dz}{z(z^2-4)e^z} = \frac{-\pi i}{2} + \frac{\pi i}{4e^2} + \frac{\pi i e^2}{4}$$

2.) Show that a harmonic function is C^∞ i.e. show that it has partial derivatives of all orders.

~~$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$~~

~~$= \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$~~

Harmonic.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$f''(z) =$

~~$\frac{\partial u}{\partial x} = Re(f'(z))$~~ ~~$\frac{\partial v}{\partial x} = Re(-if')$~~

Let u be a harmonic function.

$\Rightarrow \exists v$ harmonic s.t. $f = u + iv$ is analytic

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = Re(f'(z))$$

$$(-if'(z)) = \frac{\partial u}{\partial y} - i\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial y} = Re(-if'(z)).$$

$$\Rightarrow f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = Re(f''(z)), \quad \frac{\partial^2 u}{\partial y^2} = Re(-if''(z)).$$

$$\Rightarrow \frac{\partial^m u}{\partial x^m} = Re(f^{(m)}(z)), \quad \frac{\partial^m u}{\partial y^m} = Re(-if^{(m)}(z))$$

$$\frac{\partial^2 u}{\partial x \partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = +\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$$

$$\Rightarrow f''(z) = \frac{\partial^2 v}{\partial y^2} - i\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial}{\partial y}\frac{\partial u}{\partial x} - i\frac{\partial}{\partial x}\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \mathrm{Re}(f''(z))$$

$$\frac{\partial^2 u}{\partial x \partial y} = \mathrm{Re}(+i f''(z))$$

So $$\frac{\partial^m u}{\partial x^a \partial y^b} = \mathrm{Re}(i f^{(a)}(z)) + \mathrm{Re}(f^{(b)}(z))$$

(or something)

Chapter 4, Section 5: Exercises

1.) Show that if u is a harmonic function on $\mathbb{C}$ that is bounded above, then u is constant.

Hint: Express u as the real part of an analytic fnc. & exponentiate.

$f = u + iv$

$e^f = e^u \cos v + e^u i \sin v = e^u e^{iv}$

$u \leq M$, [illegible].

Want to use Liouville.

e^f is analytic on the complex plane.

$|e^f| = |e^u e^{iv}| = |e^u| \sqrt{\cos^2 v + \sin^2 v} = |e^u|$

$\leq e^M \Rightarrow$ Bounded.

$\Rightarrow e^f$ is constant.

$\Rightarrow e^f = K$

$\Rightarrow f = \log(K)$

$\Rightarrow u + iv = \log(K)$.

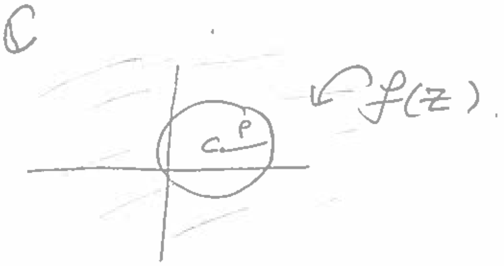
$\Rightarrow u = \text{Re}(\log(K))$ which is a constant.

② Show that if $f(z)$ is an entire function, and there is a nonempty disk such that $f(z)$ does not attain any values in the disk, then $f(z)$ is constant.

☆(WHAT DOES "$f(z)$ does not attain any values in the disk" MEAN?)☆

⇒ for some $\rho > 0$, some $c \in \mathbb{C}$

$$|f(z) - c| \ge \rho.$$



$$\Rightarrow \quad \frac{1}{f(z)-c} \le \frac{1}{\rho}.$$

$$\left(\frac{1}{f(z)-c}\right)' = \frac{(f(z)-c)(0) - f'(z)}{(f(z)-c)^2} = \frac{-f'(z)}{(f(z)-c)^2}$$

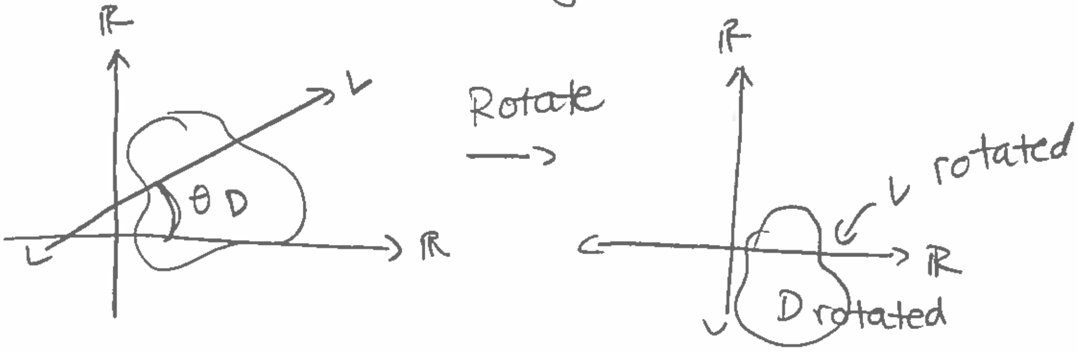
where $f(z) - c > \rho > 0$, $f'(z)$ exists by $f(z)$ entire

⇒ By Liouville, $\frac{1}{f(z)-c}$ is constant

⇒ $f(z)$ is constant ✓

4.6 Exercises:

#1: Let L be a line in the complex plane. Suppose $f(z)$ is a cts complex-valued function on a domain D that is analytic on $D \setminus L$. Show that $f(z)$ is analytic on D.



Let $z = re^{i\varphi}$, where r = radius of D.

$\Rightarrow f(re^{i\varphi})$ is a cts. complex-valued fnc. on D that is analytic on $D \setminus L$.

$\Rightarrow f(re^{i(\varphi-\theta)})$ is still a cts. complex-valued function on a domain D that is analytic on $D \setminus \mathbb{R}$.

$\Rightarrow$ By Thm, $f(re^{i(\varphi-\theta)})$ is analytic on D. (rotated)

$\Rightarrow f(z)$ is analytic on D.

#2 : Let $h(t)$ be a cts function on $[a,b]$. Show that $H(z) = \int_a^b h(t) e^{-itz} dt$ is an entire function that satisfies

$|H(z)| \le C e^{A|y|}$, $z = x+iy \in \mathbb{C}$.

First, to show entire, we need for each fixed t, $\frac{h(t)}{e^{itz}}$ is analytic on $\mathbb{C}$.

Let $f(z) = h(t)e^{-itz}$

$\Rightarrow f'(z) = h(t)(-it)e^{-itz}$

$\Rightarrow$ Analytic. GOSH. This is a fnc. wrt z...

So, by Thm, $H(z) = \int_a^b h(t)e^{-itz}dt$ is entire.

Now, we'll use M-L estimate

$|H(z)| \le (b-a)|h(t)e^{-itz}|$.

since cts on compact set

$|h(t)e^{-itz}| = |h(t)|\,|e^{-itz}| \le M|e^{-it(x+iy)}|$

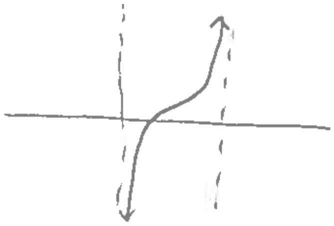
$= M|e^{-itx}e^{ty}| = M|e^{ty}| \le Me^{\max(|a|,|b|)|y|}$ ✓

#3: Let $h(t)$ be a continuous function on a subinterval $[a,b]$ of $[0,\infty)$. Show that the Fourier transform $H(z) = \int_a^b h(t)e^{-itz}\,dt$ is bounded in the LHP.

Let $z \in LHP$

$\Rightarrow x \in \mathbb{R},\ y \leq 0$

$\Rightarrow |y| \geq 0.$



$$\Rightarrow |H(z)| \leq \overset{<\infty}{(b-a)}\,|h(t)|\,|e^{-itz}|$$

$\max(|a|,|b|) < \infty.$

$$e^{-it(x+iy)} = e^{-itx}e^{y} \leq e^{-itx}e^{0} = e^{-itx}$$

$$\Rightarrow |e^{-it(x+iy)}| = |e^{-itx}| = 0.$$

$$\Rightarrow |H(z)| \leq (b-a)|h(t)| \cdot 0 = 0 < \infty.$$

Chapter 4, Section 8: Exercises.

#1: Show from the definition that

- $\frac{\partial}{\partial z}(z) = 1$

$$\frac{\partial(z)}{\partial z} = \frac{1}{2}\left(\frac{\partial z}{\partial x} + \frac{\partial(z)}{\partial(iy)}\right) = \frac{1}{2}(1+1) = 1$$

- $\frac{\partial}{\partial \bar{z}}(z) = 0$

$$\frac{\partial(z)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial(z)}{\partial x} - \frac{\partial(z)}{\partial(iy)}\right) = \frac{1}{2}(1-1) = 0$$

- $$\frac{\partial(\bar{z})}{\partial z} = \frac{1}{2}\left(\frac{\partial \bar{z}}{\partial x} + \frac{\partial z}{\partial(iy)}\right) = \frac{1}{2}(1-1) = 0$$

- $$\frac{\partial(\bar{z})}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \bar{z}}{\partial x} - \frac{\partial \bar{z}}{\partial(iy)}\right) = \frac{1}{2}(1-(-1)) = 1$$

#2: Compute $\frac{\partial}{\partial \bar{z}}(az^2 + bz\bar{z} + c\bar{z}^2)$. Use the result to determine where $az^2 + bz\bar{z} + c\bar{z}^2$ is complex-differentiable and where it is analytic.

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial(iy)}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$a(x+iy)^2 + b(x+iy)(x-iy) + c(x-iy)^2$$

$$= a(x^2 + 2xyi - y^2) + b(x^2 - xiy + xiy + y^2) + c(x^2 - 2xiy - y^2)$$

$$= ax^2 + (2xya)i - ay^2 + bx^2 + by^2 + cx^2 - 2xiyc - cy^2$$

$$= (ax^2 - ay^2 + bx^2 + by^2 + cx^2 - cy^2) + i(2xya - 2xyc)$$

$$\Rightarrow \frac{\partial}{\partial \bar{z}}(az^2 + bz\bar{z} + c\bar{z}^2) = \frac{1}{2}(2ax + 2yai + 2bx + 2cx - 2iyc + i\,2xai + i(-2ay) + i(2by + i(-2xic) +$$

$+ i(-2ay)\cdot \overset{x}{+} + i(2by) + i(-2xic) + i(-2cy))$

$= \cancel{\bigcirc\!\!\!\!\!ax} + \cancel{ya i} + bx + cx - iyc - xa - iay + biy + xc - icy$

$= 2cx - 2icy + b(x+iy) = 2c(x-iy) + b(x+iy)$

$= 2c\bar{z} + bz$

Let $f'(z) = \frac{\partial f}{\partial x}$

Analytic when $\frac{\partial f}{\partial \bar{z}} = 0$. So when $2c\bar{z} + bz = 0$

$2cx + bx + i(by - 2cy) = 0$

$2cx + bx = 0 \qquad x(2c+b) = 0$

$by - 2cy = 0. \qquad y(b-2c) = 0$

$x = 0$ or $2c = -b \Rightarrow c = -\frac{b}{2}$

$y = 0$ or $2c = b \Rightarrow c = \frac{b}{2}$

If $2c = -b \Rightarrow c = -\frac{b}{2} = \frac{b}{2} \Rightarrow \boxed{b = c = 0}$

Otherwise, $z = 0$.

From II.2.5, complex diff'ble at $2c\bar{z} + bz = 0$.

3.) Show that the Jacobian of a smooth function is given by: $\det J_f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2$

$$f(z)=(u(x,y),v(x,y))$$

$$\det\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$f=u+iv$$

$$\left|\frac{\partial f}{\partial z}\right|^2 = \left|\frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right]\right|^2 = \frac{1}{4}\left|\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]\right|^2$$

$$= \frac{1}{4}\left|\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + i\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\right|^2$$

$$= \frac{1}{4}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]^2 + \frac{1}{4}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]^2$$

$$\left|\frac{\partial f}{\partial \bar{z}}\right|^2 = \left|\frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right]\right|^2 = \frac{1}{4}\left|\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]\right|^2$$

$$= \frac{1}{4}\left|\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right|^2$$

$$= \frac{1}{4}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2$$

$$\Rightarrow \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 = \frac{1}{4}\left[\cancel{\left(\frac{\partial u}{\partial x}\right)^2} + \cancel{\left(\frac{\partial v}{\partial y}\right)^2} + \boxed{\cancel{2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}} + \cancel{\left(\frac{\partial v}{\partial x}\right)^2} + \cancel{\left(\frac{\partial u}{\partial y}\right)^2} \boxed{- 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} - \cancel{\left(\frac{\partial u}{\partial x}\right)^2} - \cancel{\left(\frac{\partial v}{\partial y}\right)^2} + \boxed{\cancel{2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}} - \cancel{\left(\frac{\partial v}{\partial x}\right)^2} - \cancel{\left(\frac{\partial u}{\partial y}\right)^2} \boxed{- 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}\right] = \frac{1}{4}\left[4\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - 4\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$ ✓

4.) Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{1}{4}i\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial x \partial y}\right) + \frac{1}{4}i\left(\frac{\partial^2}{\partial y \partial x} - i\frac{\partial^2}{\partial y^2}\right)$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} \cancel{- i\frac{\partial^2}{\partial x \partial y}} \cancel{+ i\frac{\partial^2}{\partial y \partial x}} + \frac{\partial^2}{\partial y^2}\right]$$

$$\Rightarrow 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Deduce the following for a smooth complex-valued ~~f~~ h.

(a) h is harmonic $\Leftrightarrow \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0$

($\Rightarrow$) h is harmonic $\Rightarrow \frac{\partial h}{\partial \bar{z}} = 0 \Rightarrow \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0$.

($\Leftarrow$) $\frac{\partial^2 h}{\partial z \partial \bar{z}} = 0 \Rightarrow h = u + iv$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

Since $0 = \frac{\partial^2 h}{\partial z \partial \bar{z}} = \frac{\partial^2 u}{\partial z \partial \bar{z}} + i\frac{\partial^2 v}{\partial z \partial \bar{z}}$ ✓

(b) h is harmonic $\Leftrightarrow$ $\frac{\partial h}{\partial z}$ is analytic.

($\Rightarrow$) If h is harmonic $\Rightarrow$ need $\frac{\partial h}{\partial \bar{z}}$ analytic.

$\Rightarrow$ need $\frac{\partial^2 h}{\partial z \partial \bar{z}} = 0.$

$\Rightarrow$ need $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$ ✓

($\Leftarrow$) Suppose $\frac{\partial h}{\partial z}$ is analytic. $\Rightarrow$ CR eq'ns hold

$\Rightarrow \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0$

$\Rightarrow \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$ $\Rightarrow$ Harmonic.

(c) h is harmonic $\Longleftrightarrow$ $\frac{\partial h}{\partial \bar{z}}$ is conjugate-analytic

5.) With $d\bar{z} = dx - i dy$, show for a smooth function $f(z)$ that

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial f}{\partial z} dz = \frac{\partial f}{\partial z}(dx + i dy) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(dx + i dy)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} dx + \underline{i\frac{\partial f}{\partial x} dy} \underline{- i\frac{\partial f}{\partial y} dx} + \frac{\partial f}{\partial y} dy\right)$$

$$\frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(dx - i dy).$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} dx \underline{- i\frac{\partial f}{\partial x} dy} + \underline{i\frac{\partial f}{\partial y} dx} + \frac{\partial f}{\partial y} dy\right)$$

$$\Rightarrow \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df \checkmark$$

Exercises: 5.3 ☆

<u>#4</u>: Show that the function defined by $f(z)=\sum z^{n!}$ is analytic on the open unit disk $\{|z|<1\}$. Show that $|f(r\lambda)|\to\infty$ as $r\to 1$ whenever λ is a root of unity.

$$f(z)=z^{0!}+z^{1!}+z^{2!}+z^{3!}+z^{4!}+\dots$$

$$=z^{1}+z^{1}+z^{2}+z^{6}+z^{24}+\dots$$

$$=\sum_{k=0}^{\infty}a_k z^k,\text{ where } a_k=\begin{cases}0 & \text{if } k\neq n! \text{ for } n\in\mathbb{N}\\ 1 & \text{if } k=n! \text{ for } n\in\mathbb{N}.\end{cases}$$

$$\Rightarrow \sup\sqrt[k]{|a_k|}=1$$

$$\Rightarrow \limsup\sqrt[k]{|a_k|}=1$$

$$\Rightarrow R=1.$$

$\Rightarrow f(z)$ is analytic in $|z|<1$.

$$f(r\lambda)=\sum_{n=0}^{\infty}(r\lambda)^{n!}=\sum_{n=0}^{a-1}r^{n!}\lambda^{n!}+\sum_{n=a}^{\infty}r^{n!}\lambda^{n!}$$

$\lambda^{a!}=1$ for all $k\geq a$.

$$=\sum_{n=0}^{a-1}r^{n!}\lambda^{n!}+\underset{\substack{\downarrow\\ \infty}}{\sum_{n=a}^{\infty}r^{n!}}$$

$$\Rightarrow |f(r\lambda)|=|r\lambda+r^2\lambda^2+\dots+r^{(a-1)!}\lambda^{(a-1)!}+r^{a!}+\dots+r^{\infty!}|$$

$$\geq \underset{\substack{\swarrow\\ \infty}}{\left|\sum_{n=a}^{\infty}r^{n!}\right|}-\underset{\substack{\downarrow\\ 0}}{\left|\sum_{n=0}^{a-1}r^{n!}\lambda^{n!}\right|}\quad\Rightarrow |f(r\lambda)|\to\infty.$$

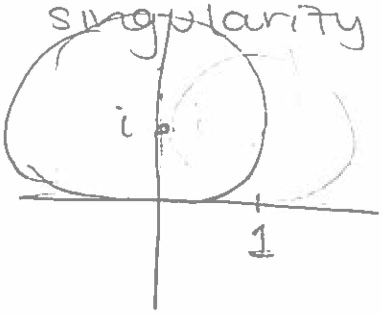
I think

Chapter 5, Section 4, Exercises

1. Find the R.O.C. of the power series expanded about the given pt:

(a) $\frac{1}{z-1}$, about $z=i$

Only singularity is at $z=1$



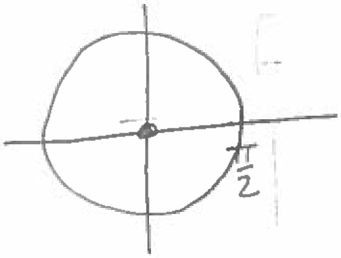
$d((0,1),(1,0)) =$

$\sqrt{1^2+1^2} = \sqrt{2}$

$R = \sqrt{2}$

(b) $\frac{1}{\cos z}$ about $z=0$.

$\cos z = 0$ when $z = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

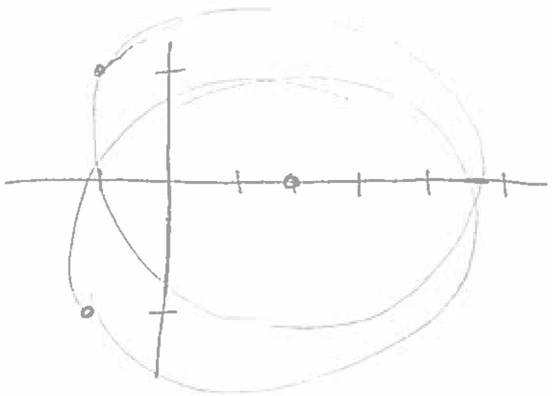


$\Rightarrow R = \frac{\pi}{2}$

2.) Show that the R.O.C. of the power series expansion of $\frac{z^2-1}{z^3-1}$ about $z=2$ is $\sqrt{7}$

$$\frac{z^2-1}{z^3-1} = \frac{z+1}{z^2+z+1}$$

So only singularity (non-removable) is



$$z^2+z+1=0$$

$$\frac{-1\pm\sqrt{1-4(1)(1)}}{2} =$$

$$-\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$= -\frac{1+\sqrt{3}i}{2} \ \& \ \frac{-1-\sqrt{3}i}{2}$$

$$d\left((2,0),\left(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right)\right) = \sqrt{\frac{3}{4}+\frac{25}{4}} = \sqrt{\frac{28}{4}} = \sqrt{7}.$$

$$d\left((2,0),\left(-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}\right)\right) = \sqrt{\frac{3}{4}+\frac{25}{4}} = \sqrt{7} \ \checkmark$$

12.) Suppose that $f(z)$ has power series expansion $\sum a_n z^n$. Show that if $f(z)$ is an even function, then $a_n = 0$ for n odd. Show that if $f(z)$ is an odd function, then $a_n = 0$ for n even.

1.) $f(z)$ is even, then $f(-z) = f(z)$.

$$\Rightarrow \sum_{n=0}^{\infty} a_n(-z)^n = \sum_{n=0}^{\infty} a_n z^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n((-z)^n - z^n) = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{2n}((-z)^{2n} - z^{2n}) + \sum_{n=0}^{\infty} a_{2n+1}((-z)^{2n+1} - z^{2n+1}) = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{2n+1}((-z)^{2n+1} - z^{2n+1}) = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{2n+1} 2 z^{2n+1} = 0.$$

$$a_{2n+1} = \frac{g^{(2n+1)}(0)}{(2n+1)!}, \text{ where } g = \sum_{n=0}^{\infty} a_{2n+1} 2 z^{2n+1}$$

$$= 0.$$

$$\Rightarrow a_{2n+1} = 0.$$

(Power Series $= 0 \Leftrightarrow$ Each $a_k = 0$)

Not sure how to conclude.

2.) Odd is similar.

14.) Let f be a continuous function on $T = \{|z|=1\}$.
Show that f can be approximated uniformly on T [illegible] a sequence of polynomials in $z \Leftrightarrow f$ has an extension F that is cts. on the closed disk $\{|z| \leq 1\}$ and analytic on $\{|z|<1\}$.

Hint: To approximate such an F, consider $F_r(z) = F(rz$

Chapter 5, Section 5 Exercises:

1.) Expand the following functions in power series about ∞:

(a) $\frac{1}{z^2+1} = f(z)$

$$g(z) = f\left(\frac{1}{z}\right) = \frac{1}{1+\frac{1}{z^2}} = \frac{z^2}{1+z^2} = z^2\left(\frac{1}{1+z^2}\right) = z^2\sum_{k=0}^{\infty}(-z^2)^k$$

$$= z^2\sum_{k=0}^{\infty}(-1)^k z^{2k}$$

$$\Rightarrow f(z) = \frac{1}{z^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^{2k}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{2k+2}}$$

(b) $f(z) = \frac{z^2}{z^3-1}$

$$g(\omega) = f\left(\frac{1}{\omega}\right) = \frac{\frac{1}{\omega^2}}{\frac{1}{\omega^3}-1} = \frac{\omega}{1-\omega^3} = \omega\left(\frac{1}{1-\omega^3}\right)$$

$$= \omega\sum_{k=0}^{\infty}\omega^{3k}$$

$$\Rightarrow f(z) = \frac{1}{\omega}\sum_{k=0}^{\infty}\omega^{-3k} = \sum_{k=0}^{\infty}\omega^{-3k-1}$$

(c) $f(z) = e^{1/z^2}$

$$g(\omega) = f\left(\frac{1}{\omega}\right) = e^{\omega^2} = \sum_{k=0}^{\infty}\frac{(\omega^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{\omega^{2k}}{k!}$$

$$\Rightarrow f(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{z^{2k}}$$

2.) Suppose $f(z)$ is analytic at ∞, with series expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{b_k}{z^k} = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots, \quad |z| > \frac{1}{\rho}.$$

With the notation $f(\infty) = b_0$
$f'(\infty) = b_1$, show that

$$f'(\infty) = \lim_{z \to \infty} z\,[f(z) - f(\infty)].$$

$$z[f(z)-f(\infty)] = z\,f(z) - z f(\infty) = \begin{array}{l} b_0 z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \dots \\ - b_0 z \end{array}$$

$$= b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \dots$$

$$\Rightarrow \lim_{z \to \infty} \left(z[f(z) - f(\infty)] \right) = b_1 = f'(\infty).$$

Chapter 5, Section 7: Exercises

1.) Find the zeroes and orders of zeroes of the following functions.

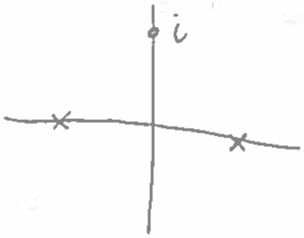
(a) $\frac{z^2+1}{z^2-1} = \frac{(z+i)(z-i)}{(z-1)(z+1)}$

$z = \pm i$, both of order 1.

(b) $\frac{1}{z} + \frac{1}{z^5} = f(z)$

$= \frac{z^4+1}{z^5}$

$z^4 = -1$



$\pm e^{\pi i} = -1$

$\pm e^{3\pi i} = -1$

$\Rightarrow \pm e^{\frac{\pi i}{4}}, \pm e^{\frac{3\pi i}{4}}$ are zeroes of order 1.

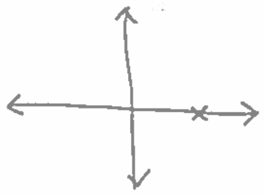
(c) $z^2 \sin(z)$.

$z = 0$ is order 3 zero.

$z = n\pi$, $n \in \mathbb{Z}/\{0\}$ is zero of order one.

(d) $\cos(z) - 1$

$f(z) = \cos(z) - 1 = 0$ when $\cos(z) = 1$



$\Rightarrow z = 2\pi n$

$f'(z) = -\sin(z) = 0$ for $z = 2\pi n$

$f''(z) = -\cos(z) \neq 0$ for $z = 2\pi n$.

So double zero at $z = 2\pi n$.

6.) Suppose $f(z)$ is analytic on a domain D and $z_0 \in D$. Show that if $f^{(m)}(z_0)=0$ for $m \geq 1$, then $f(z)$ is constant on D.

☆ weird ☆

Let $g(z) = f(z) - f(z_0)$. This is analytic on D

z_0 is a zero of $g(z)$.

WTS $g(z) = 0$. If z_0 is not isolated, then by the uniqueness principle, $g(z)=0 \Rightarrow f(z)=f(z_0$

$\Rightarrow f(z)$ is constant on D.

If z_0 is isolated, $\$$ FTSOC that $g(z) \neq 0$

$$g'(z) = f'(z)$$
$$g'(z_0) = 0$$
$$g''(z_0) = 0$$
$$\vdots$$
$$\Rightarrow g^{(m)}(z_0) = 0 = \sum_{k=0}^{\infty} 0|z-z_0|^k \quad \text{for } |z-z_0|<R.$$

So z_0 is <u>not</u> isolated. #

2.) Determine which of the functions in the preceding exercise are analytic at ∞, and determine the orders of any zeroes at ∞.

(a) $f(z) = \dfrac{z^2+1}{z^2-1}$

$$g(w) = f\left(\tfrac{1}{w}\right) = \frac{\frac{1}{w^2}+1}{\frac{1}{w^2}-1} = \frac{w^2+1}{1-w^2}$$

$g(0) = 1$, so not zero at ∞.

(b) $f(z) = \dfrac{1}{z} + \dfrac{1}{z^5}$

$$g(w) = f\left(\tfrac{1}{w}\right) = \frac{1}{\frac{1}{w}} + \frac{1}{\frac{1}{w^5}} = w + w^5 = w(1+w^4)$$

$g(0) = 0$ of order 1

$\Rightarrow f(\infty) = 0$ of order 1

(c) $f(z) = z^2 \sin(z)$

$$g(w) = \frac{1}{w^2}\sin\left(\tfrac{1}{w}\right)$$

$$g(0) = \lim_{w\to 0} \frac{\cos\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\right)}{2w} = \lim_{w\to 0} \frac{-\cos\left(\frac{1}{w}\right)}{2w^3}$$

Not analytic at ∞

(d) $f(z) = \cos z - 1$

$g(w) = \cos\left(\tfrac{1}{w}\right) - 1$

$g(0) \neq 0.$

Not analytic at ∞

3.) Show that the zeroes of $\sin(z)$ & $\tan(z)$ are all simple.

Zeroes of $\sin(z)$ are $z = n\pi,\ n \in \mathbb{Z}$.

$f'(z) = \cos(z)$, but $f'(n\pi) = \cos(n\pi) = \pm 1 \ne 0$

Zeroes of $\tan(z)$ are $z = n\pi$

$f'(z) = \sec^2(z) = \frac{1}{\cos^2(z)}$, so no zeroes.

9.) Show that if the analytic function $f(z)$ has a zero of order N at z_0, then $f(z) = g(z)^N$ for some function $g(z)$ analytic near z_0 and satisfying $g'(z_0) \ne 0$.

$f(z) = (z - z_0)^N h(z)$, where $h(z)$ is analytic and $h(z_0) \ne 0$

Let $g(z) = (z - z_0)\, e^{\frac{1}{N}\log(h(z))}$.

$\Rightarrow g(z)^N = (z - z_0)^N e^{\log(h(z))} = (z - z_0)^N h(z)$

analytic.

$\Rightarrow g'(z_0) = (z - z_0) e^{\frac{1}{N}\log(h(z))} \cdot \frac{1}{Nh(z)} h'(z) + e^{\frac{1}{N}\log(h(z))}$.

$\Rightarrow g'(z_0) = e^{\frac{1}{N}\log h(z_0)} \ne 0.$

11.) Show that if $f(z)$ is a nonconstant analytic fnc. on a domain D, then the image under $f(z)$ of any open set is open. ☆ Confusing ☆

If $f'(z) \neq 0$. $\Rightarrow$ $\det J_f(z) = |f'(z)|^2$

$\Rightarrow f^{-1} : \overset{f(U)}{\overset{=}{V}} \longrightarrow \overset{\subset D}{U}$ open

s.t. $(f^{-1})'(f(z)) = \dfrac{1}{f'(z)}$.

So open $\Rightarrow$ open.

If $f'(z)=0$

$\Rightarrow$ If $f(z_0)=0$ $\Rightarrow$ $f(z)=g(z)^N$ s.t. $g(z)$ analytic near z_0.

$\Rightarrow$ $g(z) = \sum a_k (z-z_0)^N$ on a disk of positive radius near z_0

$\Rightarrow$ open set

$\Rightarrow$ $f(z)$ also covers open set.

Chapter 6, Section 1, Exercises:

1.) i) Find all possible Laurent expansions centered at 0 of the following functions:

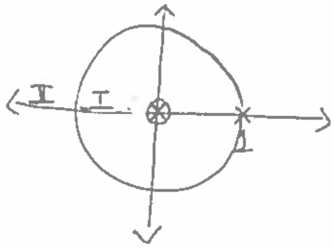
(a) $\dfrac{1}{z^2-z} = \dfrac{1}{z(z-1)}$

REMINDER: what are we looking for in each region?

$f(z) = f_0(z) + f_1(z)$

where f_0 is analytic for $|z| < \sigma$

f_1 analytic for $|z| > \rho$.



I: $0 < |z| < 1$

Need f_0 analytic for $|z| < 1$

f_1 analytic for $|z| > 0$.

$$f_0(z) = -\frac{1}{z} + \frac{1}{z-1} = \frac{-1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{k=0}^{\infty} z^k$$

Analytic for $|z|>0$

Analytic for $|z|<1$

$$= -\frac{1}{z} - 1 - z - z^2 - \dots$$

$$= -\sum_{k=-1}^{\infty} z^k$$

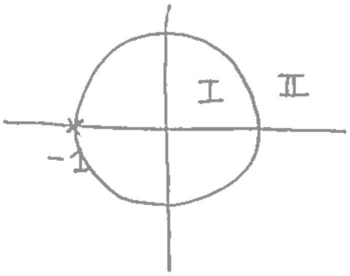
II: $1 < |z| < \infty$

$f(z)$ is analytic for $|z| > 1$.

So $f(z) = \dfrac{1}{z(z-1)} + 0$

$$= \frac{1}{z^2} \frac{1}{1-\frac{1}{z}} = \frac{1}{z^2} \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty} \frac{1}{z^{k+2}}$$

(b) $\frac{z-1}{z+1} = f(z).$



I: $\{0<|z|<1\}.$

$f_0 =$ $|z|>0$

$f_1 =$ $|z|<1$

If $|z|<1 \Rightarrow$ f is analytic, so

$$f(z) = \frac{z-1}{z+1} = \frac{(z+1)-2}{z+1} = 1 - \frac{2}{1-(-z)} = 1 - 2\sum_{k=0}^{\infty}(-z)^k$$

why? Our geometric series only works for $|z|<1$!!!

II: $\{1<|z|<\infty\}$

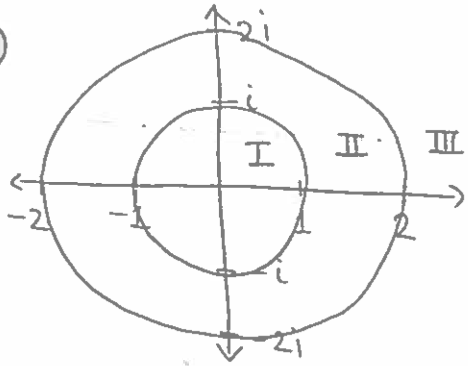
f(z) is analytic for $|z|>1$

$\Rightarrow \left|\frac{1}{z}\right|<1 \Rightarrow$ Rewrite in terms of $\frac{1}{z}.$

$$f(z)=\frac{z-1}{z+1} = 1 - \frac{2}{1+z} = 1 - \frac{1}{z}\left(\frac{2}{1+\frac{1}{z}}\right)$$

$$= 1 - \frac{2}{z}\sum_{k=0}^{\infty}\left(-\frac{1}{z}\right)^k = 1 - 2\sum_{k=0}^{\infty}(-1)^k\frac{1}{z^{k+1}}$$

(c) $\frac{1}{(z^2-1)(z^2-4)} = \frac{1}{(z-1)(z+1)(z-2)(z+2)}$



I: $\{|z|<1\}$.

our function is analytic there.

So $f(z) = f_0(z) =$

$1 = A(z+1)(z-2)(z+2) + B(z-1)(z-2)(z+2)$
$+ C(z-1)(z+1)(z+2) + D(z-1)(z+1)(z-2)$

$\Rightarrow 1 = A(z^2-z-2)(z+2) + B(z^2-3z+2)(z+2)$
$+ C(z^2-1)(z+2) + D(z^2-1)(z-2)$.

$\Rightarrow 1 = A(z^3 + z^2 - 4z - 4)$
$+ B(z^3 - z^2 - 4z + 4)$
$+ C(z^3+2z^2-z-2)$
$+ D(z^3-2z^2-z+2)$

$\Rightarrow A+B+C+D=0$
$A - B + 2C - 2D = 0$ $\Rightarrow 2A + 3C - D = 0$
$-4A - 4B - C - D = 0$
$-4A + 4A - 2C + 2D = 0$.

whatever. $\frac{1}{(z^2-1)(z^2-4)} = -\frac{1}{3}\frac{1}{z^2-1} + \frac{1}{3}\frac{1}{z^2-4}$

$\Rightarrow +\frac{1}{3}\frac{1}{1-z^2} - \frac{1}{12}\frac{1}{1-\frac{z^2}{4}} = \frac{1}{3}\sum_{k=0}^{\infty} z^{2k} - \frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{z^2}{4}\right)^k$

II: $\{1 < |z| < 2\}$.

$|z| > 1$ not analytic

$|z| < 2$ not analytic.

$$\text{So} \quad f(z) = -\frac{1}{3}\,\frac{1}{z^2-1} + \frac{1}{3}\,\frac{1}{z^2-4}$$

Need analytic for $|z|>1$.

Can do geometric series for $|\frac{1}{z}|<1$.

Can do geometric series for $\frac{1}{2}|z|<1$.

$$= -\frac{1}{3}\,\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{12}\,\frac{1}{1-\frac{z^2}{4}}$$

$$= -\frac{1}{3}\,\frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{1}{z^2}\right)^k - \frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{z^2}{4}\right)^k$$

III: $\{|z|>2\}$.

$f(z)$ analytic. $f(z) = f_1(z)$.

But can do geometric series only if $\frac{2}{|z|} < 1$

$$f(z) = -\frac{1}{3}\,\frac{1}{z^2-1} + \frac{1}{3}\,\frac{1}{z^2-4} \qquad 4z^2\left(\frac{z^2}{4}-1\right)$$

$$= -\frac{1}{3z^2}\,\frac{1}{1-\frac{1}{z^2}} + \frac{1}{3z^2}\cdot\frac{1}{1-\frac{4}{z^2}}$$

$$= -\frac{1}{3z^2}\sum_{k=0}^{\infty}\left(\frac{1}{z^2}\right)^k - \frac{1}{3z^2}\sum_{k=0}^{\infty}\left(\frac{4}{z^2}\right)^k$$

4.) Suppose that $f(z) = f_0(z) + f_1(z)$ is the Laurent decompositio of an analytic function $f(z)$ on the annulus $\{A<|z|<B\}$. Show that if $f(z)$ is an even function, then $f_0(z)$ & $f_1(z)$ are even functions, and the Laurent series expansion of $f(z)$ has only even powers of z. Show that if $f(z)$ is an odd function, then $f_0(z)$ & $f_1(z)$ are odd functions and the Laurent series expansion has only odd powers of z.

$f_0(z)$ is analytic for $|z|>A$

$f_1(z)$ is analytic for $|z|<B$

$$\left.\begin{aligned} f_0(z) &= \sum_{k=0}^{\infty} a_k z^k \\ f_1(z) &= \sum_{k=-\infty}^{-1} a_k z^k \end{aligned}\right\} \Rightarrow f(z) = \sum_{k=-\infty}^{+\infty} a_k z^k$$

$$f(-z) = \sum_{k=-\infty}^{\infty} a_k(-z)^k = \sum_{k=-\infty}^{\infty} a_k z^k = f(z)$$

$$= f_0(-z) + f_1(-z) = \sum_{k=0}^{\infty} a_k(-z)^k + \sum_{k=-\infty}^{-1} a_k(-z)^k$$

$$= f_0(z) + f_1(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=-\infty}^{-1} a_k z^k$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} a_k(z^k - (-z)^k) = 0$$

$$\Rightarrow z^k = -z^k \quad \forall k.$$

$$\Rightarrow f_0(-z) = f_0(z)$$

$$f_1(-z) = f_1(z)$$ Only even powers, otherwise $z^k - (-z)^k = 2z$

odd similar.

5.) Suppose $f(z)$ is analytic on the punctured plane $D = \mathbb{C} \setminus \{c\}$. Show that there is a constant c s.t. $f(z) - \frac{c}{z}$ has a primitive in D. Give a formula for c in terms of an integral of $f(z)$.

Chapter 6, Section 2: Exercises

1.) Find the isolated singularities of the following fncs and determine whether they are removable, essential, or poles. Determine the order of any pole, and find the principal part at each pole.

(a) $\dfrac{z}{(z^2-1)^2} = \dfrac{z}{(z+1)^2(z-1)^2}$

$z = \pm 1$ are isolated singularities.

$$\lim_{z\to 1} \frac{z}{(z+1)^2(z-1)^2} = \frac{1}{0} = \infty$$

$\Rightarrow$ Pole of order 2.

$$\lim_{z\to -1} \frac{z}{(z+1)^2(z-1)^2} = -\frac{1}{0} = \infty$$

$\Rightarrow$ Pole of order 2.

$$\frac{z}{(z^2-1)^2} = \frac{z}{(z-1)(z+1)}$$

(b) $\dfrac{ze^z}{z^2-1}$

$z = \pm 1$ both simple poles.

(c) $\dfrac{e^{2z}-1}{z}$

$z=0$ $\quad \displaystyle\lim_{z\to 0} \frac{e^{2z}-1}{z} = \frac{0}{0}$

$$= \lim_{z\to 0} 2e^{2z} = 2.$$ $\Rightarrow$ Removable.

(d) $\tan z = \frac{\sin z}{\cos z}$

$z = \frac{\pi}{2} + 2\pi n$

$\Rightarrow$ Pole, simple

(e) $z^2 \sin(\frac{1}{z})$

$z = 0.$

$$\lim_{z\to 0} \frac{\sin(\frac{1}{z})}{z^{-2}} = \lim_{z\to 0} \frac{-\frac{1}{z^2}\sin(\frac{1}{z})}{-2z} =$$

$$\lim_{z\to 0} \frac{1}{2z^3}\sin(\frac{1}{z})$$

Essential Singularity.

(f) $\frac{\cos z}{(z-\frac{\pi}{2})(z+\frac{\pi}{2})}$

$z = \pm\frac{\pi}{2}$

$$\lim_{z\to \frac{\pi}{2}} \frac{\cos(z)}{(z-\frac{\pi}{2})(z+\frac{\pi}{2})} = \frac{0}{0}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{-\sin(z)}{(z+\frac{\pi}{2})+(z-\frac{\pi}{2})} = \frac{-1}{\pi}$$

$\Rightarrow$ Both Removable

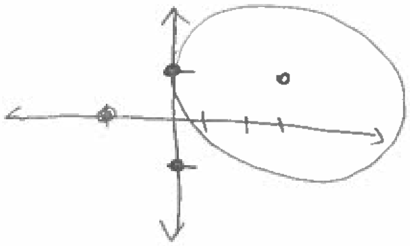
2.) Find the radius of convergence of the power series for the following functions, expanded about the indicated point.

(a) $\frac{z-1}{z^4-1}$ about $z=3+i$

$$\frac{z-1}{(z^2-1)(z^2+1)} = \frac{\cancel{z-1}}{(\cancel{z-1})(z+1)(z^2+1)}$$

Non removable singularities:

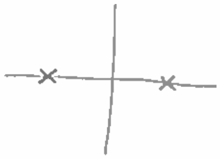
$z=\pm i, -1$



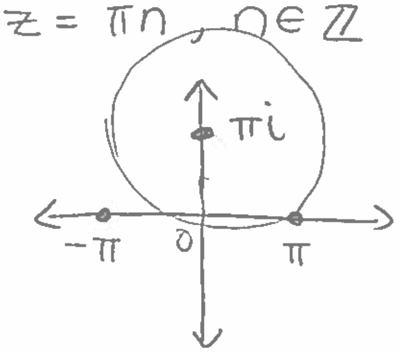
$d(i, 3+i) = \sqrt{0^2+3^2} = 3.$

$R=3.$

(c) $\frac{z}{\sin z}$, about $z=\pi i$



$z=\pi n, n \in \mathbb{Z}$, not removable. EXCEPT $z=0$.



$d(\pi i, \pi) = \sqrt{(\pi-0)^2 + (\pi-0)^2}$

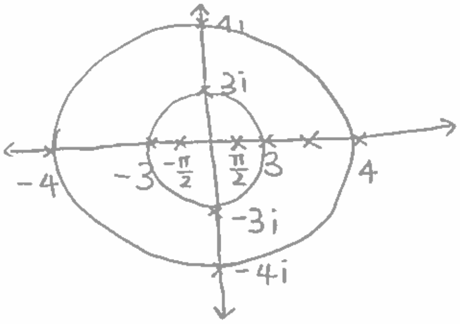
$= \sqrt{2\pi^2} = \sqrt{2}\pi$

$\Rightarrow R=\sqrt{2}\pi.$

3.) Consider the function $f(z) = \tan z$ in the annulus $\{3 < |z| < 4\}$. Let $f(z) = f_0(z) + f_1(z)$ be the Laurent decomposition of $f(z)$, so that $f_0(z)$ is analytic for $|z| < 4$, and $f_1(z)$ is analytic for $|z| > 3$ and vanishes at ∞.

(a) Obtain an explicit expression for $f_1(z)$.

$f(z) = \dfrac{\sin z}{\cos z}$ w/ poles at $z = \frac{\pi}{2} + n\pi$



$|z| > 4$

$$f_1(z) = \frac{1}{z - \frac{\pi}{2}} + \frac{1}{z + \frac{\pi}{2}} = \frac{1}{z}\frac{1}{1 - \frac{\pi}{2z}} + \frac{1}{z}\frac{1}{1 + \frac{\pi}{2z}}$$

$$= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\pi}{2z}\right)^k + \frac{1}{z}\sum_{k=0}^{\infty}\left(-\frac{\pi}{2z}\right)^k$$

(b) Write down the series expansion for $f_1(z)$ & determine the largest domain on which it converges.

$$f_1(z) = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\pi}{2z}\right)^k + \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{-\pi}{2z}\right)^k$$

converges for $|z| > \frac{\pi}{2}$

Chapter 6, Section 3 : Exercises

#2) Suppose that $f(z)$ is an entire function that is not polynomial. What kind of singularity can $f(z)$ have at ∞?

$g(w) = f(\frac{1}{w})$ can be ~~principal~~ or essential.

Pretty much will have infinitely many negative exponents.

Ah. Since ~~if~~ ~~these~~ ~~are~~ entire, there ~~are~~ ~~no~~ $f(z) \neq \frac{1}{z} + \text{stuff}$

since $\frac{1}{z}$ not entire.

So $f(z) = \sum_{k=0}^{\infty} a_k (z \cancel{- z_0})^k$.

$\Rightarrow f(\frac{1}{w}) = \sum a_k (\frac{1}{w} - z_0)^k$

3.) Show that if $f(z)$ is a nonconstant entire function, then $e^{f(z)}$ has an essential singularity at $z = \infty$.

Clearly singularity?

~~e~~ Suppose removable. $\Rightarrow e^{f(z)} = \sum_{k=0}^{\infty} b_k z^k$

$|z| > R$.

Chapter 6, Section 3: Isolated Singularity at Infinity.

Chapter 7, Section 1 Exercises

1. Evaluate the following residues

(a) $\text{Res}\left[\frac{1}{z^2+4}, 2i\right]$

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

So simple pole at $2i$

$$\frac{1}{z^2+4} = \frac{a_{-1}}{z-2i} + [\text{Analytic at } 2i]$$

$$\frac{1}{z+2i} = a_{-1} + [\text{Analytic at } 2i](z-2i]$$

$$\lim_{z\to 2i}\left[\frac{1}{z+2i}\right] = a_{-1}$$

$$\overset{\shortparallel}{\frac{1}{4i}}$$

(c) $\text{Res}\left[\frac{1}{z^5-1}, 1\right]$

$$\begin{array}{r|l} & z^4+z^3+z^2+z+1 \\ z-1 & z^5-1 \\ & z^5-z^4 \\ \hline & z^4-1 \\ & z^4-z^3 \\ \hline & z^3-1 \\ & z^3-z^2 \\ \hline & z^2-1 \\ & z^2-z \\ \hline & z-1 \end{array}$$

$$\frac{1}{z^5-1} = \frac{1}{(z-1)(z^4+z^3+z^2+z+1)}$$

$$\Rightarrow \frac{1}{z^4+z^3+z^2+z+1} = a_{-1} + [\text{analytic at } z=1]$$

$$\Rightarrow \lim_{z\to 1}\frac{1}{z^4+z^3+z^2+z+1} = a_{-1} \Rightarrow \frac{1}{5} = a_{-1}$$

(d) $\text{Res}\left[\frac{\sin z}{z^2}, 0\right]$

$\frac{\sin z}{z^2}$ = Pole of order 1

Removable

$$\frac{\sin z}{z^2} = \frac{a_{-1}}{z} + (\text{analytic at } 0).$$

$$\frac{\sin z}{z} = a_{-1} + (\text{analytic at } 0)(z).$$

$$\lim_{z \to 0} \frac{\sin z}{z} = a_{-1}$$

$$1 = a_{-1}$$

(e) $\text{Res}\left[\frac{\cos z}{z^2}, 0\right].$

$z=0$ is pole of order 2.

$$\frac{\cos z}{z^2} = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots$$

$$\cos z = a_{-2} + a_{-1} z + a_0 z^2 + \cdots$$

$$\Rightarrow \underset{-\sin z}{\overset{}{\underset{\shortparallel}{\frac{d}{dz}\cos z}}} = a_{-1} + 2a_0 z + \cdots$$

$$\Rightarrow \lim_{z \to 0} \underset{0}{\underset{\shortparallel}{(-\sin z)}} = a_{-1}$$

2.) Calculate the residue at each isolated singularity in the complex plane of the following functions.

$e^{1/z}$

$z = 0$: is an essential singularity.

$$\text{Res}\left[e^{\frac{1}{z}}, 0\right] =$$

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{z^n} \cdot \frac{1}{n!} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots$$

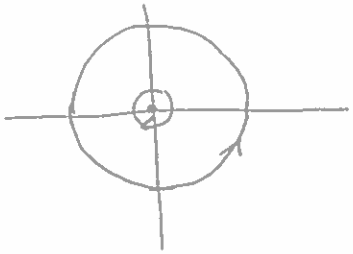
$$= a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

$$\Rightarrow \boxed{a_{-1} = 1}$$

3.) Evaluate the following Integrals, using the residue theorem.

(a) $\oint\limits_{|z|=1} \frac{\sin z}{z^2} dz$

Pole at $z = 0$.

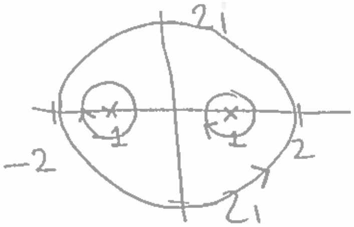


$$0 = \oint\limits_{|z|=1} \frac{\sin z}{z^2} dz - 2\pi i \operatorname{Res}\left[\frac{\sin z}{z^2}, 0\right]$$

$$\Rightarrow \oint\limits_{|z|=1} \frac{\sin z}{z^2} dz = 2\pi i$$

(b) $\oint\limits_{|z|=2} \frac{e^z}{z^2-1}$

Simple poles at $z = \pm 1$.



$$\Rightarrow \oint\limits_{|z|=2} \frac{e^z}{z^2-1} = 2\pi i \operatorname{Res}\left[\frac{e^z}{z^2-1}, 1\right] + 2\pi i \operatorname{Res}\left[\frac{e^z}{z^2-1}, -1\right]$$

$$\left[\begin{array}{l} \operatorname{Res}\left[\frac{e^z}{z^2-1}, 1\right] = \left.\frac{e^z}{2z}\right]_{z=1} = \frac{e}{2} \\ \operatorname{Res}\left[\frac{e^z}{z^2-1}, -1\right] = \left.\frac{e^z}{2z}\right]_{z=-1} = \frac{1}{-2e} \end{array}\right]$$

$$= \frac{2\pi i e}{2} + \frac{2\pi i}{-2e} = \boxed{\pi i e - \frac{\pi i}{e}}$$

5.) By estimating the coefficients of the Laurent series, prove that if z_0 is an isolated singularity of f, and if $(z-z_0)f(z) \to 0$ as $z \to z_0$, then z_0 is removable.

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

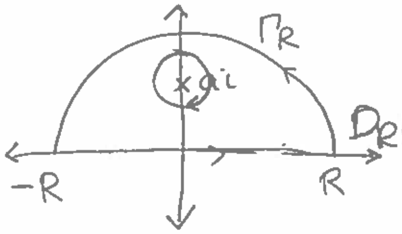
$L = 2\pi r$

$M =$

Chapter 7, Section 2: Exercises

1.) Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}, \quad a>0$$



$$\frac{1}{z^2+a^2} = \frac{1}{(z-ia)(z+ai)}$$ ai is only pole in D_R

$$\int_{\partial D_R} \frac{dz}{z^2+a^2} = 2\pi i \,\mathrm{Res}\left[\frac{1}{z^2+a^2}, ai\right] = 2\pi i \lim_{z\to ai} \frac{1}{z+ai}$$

$$= \frac{2\pi i}{2ai} = \frac{\pi}{a}$$

$$\int_{\partial D_R} \frac{dz}{z^2+a^2} = \int_{-R}^{R} \frac{1}{x^2+a^2}\,dx + \int_{\Gamma_R} \frac{dz}{z^2+a^2}$$

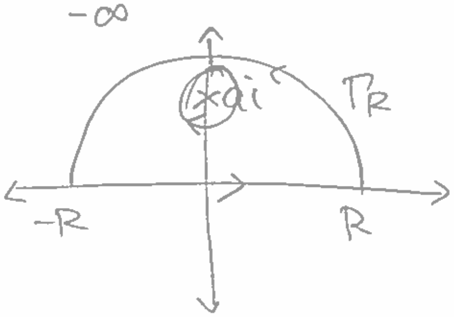
$$\left|\int_{\Gamma_R} \frac{dz}{z^2+a^2}\right| \le \frac{\pi R}{R^2-a^2} \to 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx = \lim_{R\to\infty}\left[\int_{\partial D_R} \frac{dz}{z^2+a^2} - \int_{\Gamma_R} \frac{dz}{z^2+a^2}\right]$$

$$= \frac{\pi}{a} \quad ✓$$

2.) Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$



$$\int_{\partial D_R} \frac{dz}{(z^2+a^2)^2} = 2\pi i \, \text{Res}\left[\frac{1}{(z^2+a^2)^2}, ai\right]$$

$$\frac{1}{(z^2+a^2)^2} = \frac{a_{-2}}{(z-ai)^2} + \frac{a_{-1}}{(z-ai)} + a_0 + \cdots$$

$$\Rightarrow \frac{1}{(z+ai)^2} = a_{-2} + a_{-1}(z-ai) + a_0(z-ai)^2$$

$$\Rightarrow -2(z+ai)^{-3} = a_{-1} + (\text{stuff})$$

$$\Rightarrow \left.\frac{-2}{(z+ai)^3}\right]_{z=ai} = a_{-1}$$

$$\Rightarrow \frac{-2}{(2ai)^3} = a_{-1} \Rightarrow a_{-1} = \frac{+2}{8a^3 i} = \frac{1}{4ia^3}$$

$$\Rightarrow \int_{\partial D_R} \frac{dz}{(z^2+a^2)^2} = \frac{-2\pi i}{ai} = \frac{2\pi i}{4ia^3} = \frac{\pi}{2a^3}$$

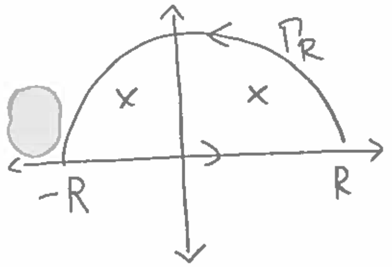
$$\parallel$$

$$\int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{\Gamma_R} \frac{dz}{(z^2+a^2)^2}$$

$$\left| \int_{\Gamma_R} \frac{dz}{(z^2+a^2)^2} \right| \le \frac{\pi R}{(R^2-a^2)^2} \longrightarrow 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

4.) Using residue theory, show that $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$



$$\frac{1}{z^4+1} = \frac{1}{(z^2+i)(z^2-i)} =$$

$$\frac{1}{(z-\sqrt{i})(z+\sqrt{i})(z+}$$

$z^4+1=0 \Rightarrow z^4=-1$

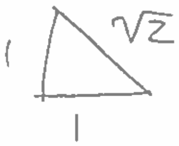
$$\frac{1}{}$$

$z^4 = \cos(4\theta) + i\sin(4\theta) = 1e^{i(\pi\epsilon)}$

$\Rightarrow 4\theta = \pi + 2\pi n$

$\Rightarrow \theta = \frac{\pi}{4} + \frac{\pi n}{2}$

$\Rightarrow z = \underbrace{e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}}_{\text{are in } D_R.}, e^{\frac{7\pi}{4}i}, e^{\frac{5\pi}{4}i}$



$\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4}) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$$\int_{\partial D_R} \frac{1}{z^4+1} = 2\pi i\left(\frac{1}{(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})}\right.$$

$$\left.+ \frac{1}{(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}+\frac{i}{\sqrt{2}})(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})}\right)$$

$$= 2\pi i\left(\frac{1}{\frac{4i(2+2i)}{2\sqrt{2}}} + \frac{1}{\frac{-4i(-2+2i)}{2\sqrt{2}}}\right) = \left(\frac{1}{\frac{8i-8}{2\sqrt{2}}} + \frac{1}{\frac{8i+8}{2\sqrt{2}}}\right)2\pi i$$

$$= \left(\frac{2\sqrt{2}}{2(4i-4)} + \frac{2\sqrt{2}}{2(4i+4)}\right)2\pi i = \left(\frac{\sqrt{2}}{4i-4} + \frac{\sqrt{2}}{4i+4}\right)2\pi i$$

$$= \left(\frac{\sqrt{2}(4i+4) + \sqrt{2}(4i-4)}{-16 -16}\right)2\pi i = 2\pi i\left(\frac{4i\sqrt{2} + 4\sqrt{2} + 4\sqrt{2}i - 4\sqrt{2}}{-16\cdot 2}\right)$$

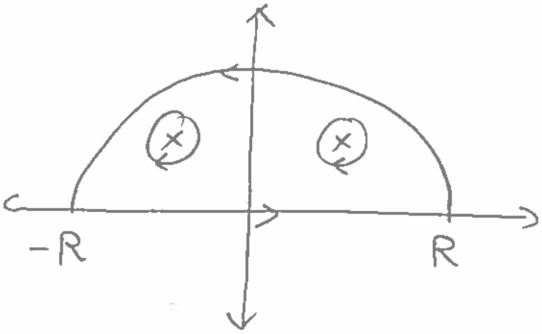
$$= \frac{(\cancel{8}i\sqrt{2})\cancel{2}\pi i}{-\cancel{8}\cdot\cancel{2}\cdot 2} = \frac{-\pi i^2\sqrt{2}}{2} = \frac{+\pi}{\sqrt{2}}.$$ (Thank God)

$$\Rightarrow \left|\int_{\Gamma_R} \frac{dz}{z^4+1}\right| \leq \frac{\pi R}{R^4-1} \longrightarrow 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

5.) Using residue theory, show that $\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$.

$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\int_0^\infty \frac{x^2}{x^4+1}$ Since even function.



$\frac{z^2}{z^4+1} = \frac{z^2}{(z - \frac{1+i}{\sqrt{2}})(z-(-\frac{1+i}{\sqrt{}}}$

-omg-

$\text{Res}\left[\frac{z^2}{z^4+1}, \frac{1+i}{\sqrt{2}}\right] = \left.\frac{z^2}{4z^3}\right]_{z=\frac{1+i}{\sqrt{2}}} = \frac{\sqrt{2}}{8}(1-i)$

$\text{Res}\left[\frac{z^2}{z^4+1}, \frac{-1+i}{\sqrt{2}}\right] = \left.\frac{z^2}{4z^3}\right]_{z=\frac{-1+i}{\sqrt{2}}} = \left.\frac{1}{4z}\right]_{z=\frac{-1+i}{\sqrt{2}}}$

$= \frac{1}{4\left(\frac{-1+i}{\sqrt{2}}\right)} = \frac{1}{\frac{-4+4i}{\sqrt{2}}} = \frac{\sqrt{2}}{-4+4i}$

$2\pi i \sum \text{Res} = \pi i$

$= \frac{\sqrt{2}(-1-i)}{-4(-1+i)(-1-i)} = \frac{\sqrt{2}(-1-i)}{-4(2)} = \frac{\sqrt{2}(-1-i)}{8}$

$\cancel{2}\pi i\left(\frac{\sqrt{2} - i\sqrt{2} + -\sqrt{2} - \sqrt{2}i}{\cancel{8}\,4}\right)$

$= \frac{-\cancel{2}i\sqrt{2}\pi i}{\cancel{4}\,2} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}$

$\int_{\partial D_R} \frac{z^2}{z^4+1}\,dz = \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx + \int_{\Gamma_R} \frac{z^2}{z^4+1}\,dz$

$$\left| \int_{\Gamma_R} \frac{z^2}{z^4+1} \right| \leq \frac{R^2 \pi}{R^4-1} \longrightarrow 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^2}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

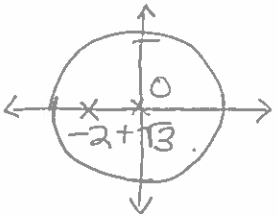
Chapter 7, Section 3: Exercises

1.) Show using residue theory that

$$\int_0^{2\pi} \frac{\cos(\theta)}{2+\cos(\theta)}\,d\theta = 2\pi\left(1-\frac{2}{\sqrt{3}}\right)$$

$$\cos(\theta) = \frac{z+\frac{1}{z}}{2}$$

$$= \oint_{|z|=1} \frac{\frac{z+\frac{1}{z}}{2}}{2+\frac{z+\frac{1}{z}}{2}}\,\frac{dz}{iz} = \oint_{|z|=1}\frac{z+\frac{1}{z}}{4+z+\frac{1}{z}}\,\frac{dz}{iz} = \oint_{|z|=1}\frac{z^2+1}{4z+z^2+1}\,\frac{dz}{iz}$$



$$= \oint_{|z|=1}\frac{(z^2+1)\,dz}{iz(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))}$$

$$= \oint_{|z|=1}\frac{(z^2+1)\,dz}{4iz^2+iz^3+iz}$$

$$= 2\pi i\cdot\left[\frac{z^2+1}{8iz+3iz^2+i}\right]_{z=-2+\sqrt{3}} + 2\pi i\left[\frac{z^2+1}{8iz+3iz^2+i}\right]_{z=0}$$

$$= 2\pi i\left[\frac{(-2+\sqrt{3})^2+1}{8i(-2+\sqrt{3})+3i(-2+\sqrt{3})^2+i}+\frac{1}{i}\right]$$

$$= 2\pi i\left[\frac{4-4\sqrt{3}+3+1}{-16i+8i\sqrt{3}+3i(4-4\sqrt{3}+3)+i}+\frac{1}{i}\right]$$

$$\begin{array}{r} -16 \\ +12 \\ +9 \\ +1 \\ \hline \end{array}$$

$$\begin{array}{r} 22 \\ -16 \\ \hline 6 \end{array}$$

$$= 2\pi i\left[\frac{8-4\sqrt{3}}{-16i+8i\sqrt{3}+12i-12i\sqrt{3}+9i+i}+\frac{1}{i}\right]$$

$$= 2\pi\left[\frac{8-4\sqrt{3}}{-6+4\sqrt{3}}+1\right] = 2\pi\left[\frac{-2\sqrt{3}}{3}+1\right]$$ ✓

$$\Rightarrow \int_0 \frac{\cos\theta}{2+\cos\theta}\,d\theta = 2\pi\left(1-\frac{2}{\sqrt{3}}\right)$$

2.) Show using residue theory that

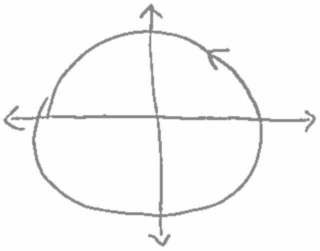
$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad , a>b>0.$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}.$$

$$\left.\begin{array}{l} z = e^{i\theta} \\ dz = ie^{i\theta}d\theta \end{array}\right\} \Rightarrow \quad d\theta = \frac{dz}{iz}.$$

$$\frac{1}{i}\oint_{|z|=1}\frac{dz}{z\left(a+b\left(\frac{z-\frac{1}{z}}{2i}\right)\right)} = \frac{1}{i}\oint_{|z|=1}\frac{dz}{z\left(\frac{2ia}{2i}+\frac{bz-\frac{b}{z}}{2i}\right)} = \frac{1}{i}\oint_{|z|=1}\frac{dz}{\left(\frac{2iaz+bz^2-b}{2i}\right)}$$

$$= \frac{1}{\cancel{i}}\oint_{|z|=1}\frac{2\cancel{i}dz}{bz^2+2iaz-b} = \oint_{|z|=1}\frac{2dz}{bz^2+2iaz-b}$$



isolated singularities:

$$\frac{-2ia\pm\sqrt{(2ia)^2+4b^2}}{2b} = \frac{2ia\pm\sqrt{-4a^2+4b^2}}{2b}$$

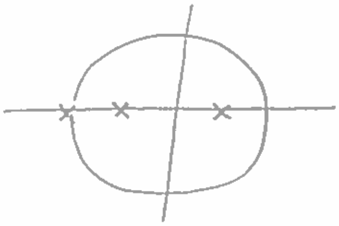
$$= \frac{-\cancel{2}ia\pm 2i\sqrt{a^2-b^2}}{\cancel{2}b} = i\left[\frac{-a\pm\sqrt{a^2-b^2}}{b}\right]$$

$$=2\pi i\,\mathrm{Res}\left[\frac{2}{bz^2+2iaz-b},\left[\frac{-a+\sqrt{a^2-b^2}}{b}\right]i\right] = 2\pi i\left(\frac{2}{2bz+2ia}\right)_{\left[\frac{-a+\sqrt{a^2-b^2}}{b}\right]}$$

$$= \frac{2\cancel{4}\pi i}{\cancel{2b}\left[\frac{-a+\sqrt{a^2-b^2}}{\cancel{b}}\right]i+\cancel{2}ia} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad ✓$$

4.) Show using residue theory that

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2(\theta)} = \pi\sqrt{2}.$$



$\theta = \pi - t$

$d\theta = -dt$

$$= \int_{2\pi}^{0} \frac{-dt}{1+\sin^2(\pi - t)} = \int_{0}^{2\pi} \frac{dt}{1+\sin^2 t}$$

$$\sin t = \frac{z - \frac{1}{z}}{2i} \qquad dt = \frac{dz}{iz}$$

$$= \oint_{|z|=1} \frac{dz}{iz\left(1 + \frac{z^4-2z^2+1}{-4z^2}\right)} \qquad \sin^2 t = \frac{z^2 - 2 + \frac{1}{z^2}}{-4} = \frac{z^4-2z^2+1}{-4z^2}$$

$$= \frac{1}{i}\oint_{|z|=1} \frac{dz}{z + \left(\frac{z^4-2z^2+1}{-4z}\right)} = \frac{1}{i}\oint_{|z|=1} \frac{dz}{\frac{-4z^2}{-4z} + \frac{z^4-2z^2+1}{-4z}}$$

$$= \frac{1}{i}\oint_{|z|=1} \frac{dz}{\frac{z^4-6z^2+1}{-4z}} = \frac{1}{i}\oint_{|z|=1} \frac{-4z\,dz}{z^4-6z^2+1}$$

Poles: $\pm\sqrt{3-2\sqrt{2}}$

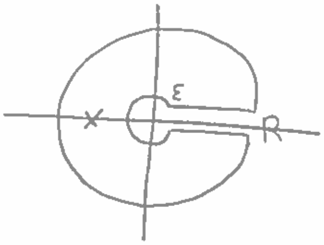
$$= \frac{2\pi i}{i}\,\mathrm{Res}\left[\frac{-4z}{z^4-6z^2+1}, \sqrt{3-2\sqrt{2}}\right] + \frac{2\pi i}{i}\,\mathrm{Res}\left[\frac{-4z}{z^4-6z^2+1}, -\sqrt{3-2\sqrt{2}}\right]$$

$$= \frac{2\pi i}{i}\left(\frac{-4z}{4z^3-12z}\right)_{z=\sqrt{3-2\sqrt{2}}} + \frac{2\pi i}{i}\left(\frac{-1}{z^2-3}\right)_{z=-\sqrt{3-2\sqrt{2}}}$$

$$= \frac{-2\pi}{3-2\sqrt{2}-3} \quad \frac{+2\pi}{+2\sqrt{2}} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2} \quad ✓$$

Chapter 7, Section 4: Exercises:

1.) By integrating around the keyhole contour, show that $\displaystyle\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin(\pi a)}$, $0 < a < 1$.



$$\int \frac{r^{-a}e^{-ia\theta}}{1+z} dz$$

$$\int_{\partial D} \frac{z^{-a}}{1+z} dz = 2\pi i\, z^{-a}\Big]_{z=-1} = 2\pi i\, e^{-a\log(+1) + -a\pi i}$$

$z=-1$ is only singularity in D.

$$= 2\pi i\, e^{-a\pi i}$$

$$\int_\varepsilon^R \frac{x^{-a}}{1+x} dx + \int_{\Gamma_R} \frac{z^a}{1+z} dz + \int_R^\varepsilon \frac{x^{-a}e^{-ia2\pi}}{1+x} dx$$

$$+ \int_{\gamma_\varepsilon} \frac{z^a}{1+z} dz$$

$$\left|\int_{\Gamma_R} \frac{z^{-a}}{1+z} dz\right| \le \frac{2\pi R\, R^a}{R-1} \longrightarrow 0 \text{ as } R \to \infty.$$

$$0 > -a > -1$$

$$1 > -a+1 > 0$$

$$\left|\int_{\gamma_\varepsilon} \frac{z^{-a}}{1+z} dz\right| \le \frac{2\pi\varepsilon \cdot \varepsilon^{-a}}{1-\varepsilon} \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

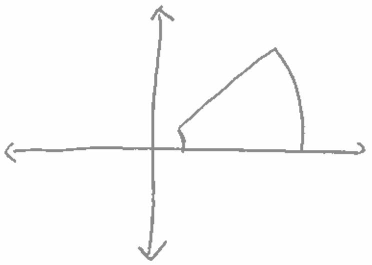
$$\Rightarrow \frac{2\pi i e^{-\pi i a}}{1-e^{-2\pi i a}} = \int_0^\infty \frac{x^{-a}}{1+x}\,dx$$

$$\sin(\pi a) = \frac{e^{+\pi i a} - e^{-\pi i a}}{2i}$$

$$\Rightarrow \pi\left[\frac{2i}{e^{\pi i a} - e^{-\pi i a}}\right] = \frac{\pi}{\sin(\pi a)} \quad \checkmark$$

2.) By integrating around the boundary of a pie-slice domain of aperture $\frac{2\pi}{b}$, show that

$$\int_0^\infty \frac{dx}{1+x^b} = \frac{\pi}{b\sin(\frac{\pi}{b})}, \quad b>1$$



3.) By integrating around the keyhole contour, show that

$$\int_0^\infty \frac{\log x}{x^a(x+1)} dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}$$

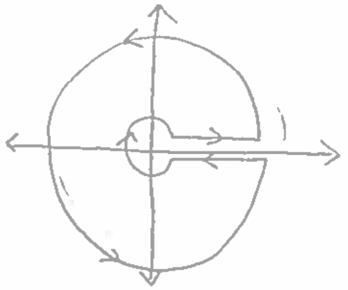
$$\left(\int_0^\infty \frac{-x^{-a}\log(x)}{x+1} = \frac{-\pi \cos(\pi a)\pi}{\sin^2(\pi a)} \right) \checkmark$$

$$\frac{d}{dx} a^x = ?$$

$$\left(\frac{d}{dx} e^{\log(a^x)} = \frac{d}{dx} e^{x\log(a)} = e^{x\log(a)} \log(a) = a^x \log(a) \right)$$

$$\Rightarrow \frac{d}{dx} a^x = e$$

Also,



$$f(z) = \frac{\log(z)}{z^a(z+1)} = \frac{\log(re^{i\theta})}{r^a e^{ia\theta}(z+1)}$$

$$z = -1.$$

$$\int_{\partial D} \frac{\log(z)}{z^a(z+1)} dz = 2\pi i \frac{\log(z)}{z^a}\Big]_{z=-1} = \frac{2\pi i [\log|1| + i\pi]}{e^{a\log(1) + ai\pi}}$$

$$= \frac{2\pi i(\cdot\pi)}{e^{a\pi i}} = 2\pi^2 i e^{-a\pi i}$$

$$= \int_\varepsilon^R \frac{\log x}{x^a(x+1)} dx + \int_{\Gamma_R} \frac{\log z \, dz}{z^a(z+1)} + \int_R^\varepsilon \frac{\log(x) + \log(e^{2\pi i})}{x^a e^{2\pi i a}(x+1)}$$

$$+ \int_{\gamma_\varepsilon} \frac{\log z \, dz}{z^a(z+1)} = \int_\varepsilon^R \frac{\log x}{x^a(x+1)} + \int_{\Gamma_R} \frac{\log z}{z^a(z+1)} dz - \int_\varepsilon^R \frac{\log(x) + 2\pi i}{x^a e^{2\pi i a}(x+1)}$$

$$+ \int_{\gamma_\varepsilon} \frac{\log z \, dz}{z^a(z+1)}$$

$$\left|\int_{\Gamma_R} \frac{\log z}{z^a(z+1)}\right| \leq \frac{2\pi R\sqrt{\log^2 R + 4\pi^2}}{R^a(R-1)} \to 0 \text{ as } R \to \infty$$

$$\left|\int_{\gamma_\varepsilon} \frac{\log z \, dz}{z^a(z+1)}\right| \leq \frac{2\pi\varepsilon\sqrt{\log^2 \varepsilon + 4\pi^2}}{\varepsilon^a(1-\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

Have $2\pi^2 i e^{a\pi i} = \int_0^\infty \frac{\log x}{x^a(x+1)} dx - \int_0^\infty \frac{\log x}{x^a e^{2\pi i a}(x+1)}$

$$-i\int_0^\infty \frac{2\pi}{x^a e^{2\pi i a}(x+1)}$$

$$\Rightarrow 2\pi^2 i e^{-a\pi i} = \left(1 - \frac{1}{e^{2\pi i a}}\right)\left[\int_0^\infty \frac{\log x}{x^a(x+1)}\right] - i\int_0^\infty \frac{2\pi}{x^a e^{2\pi i a}(x+1)}$$

$$\Rightarrow 2\pi^2 i \cos(\pi a) + 2\pi^2 \sin(\pi a) = \left(1 - \frac{1}{e^{2\pi i a}}\right)\left[\int_0^\infty \frac{\log x}{x^a(x+1)}\right] - i\int_0^\infty \frac{2\pi}{x^a e^{2\pi i a}(x+1)}$$

$$2\pi^2 \cos(\pi a) = -\int_0^\infty \frac{2\pi}{x^a e^{2\pi i a}(x+1)}$$

$$(+2\pi^2 \sin(\pi a))(1 - e^{2\pi i a}) = \int_0^\infty \frac{\log x}{x^a(x+1)}$$

$$-e^{-2\pi i a}\int_0^\infty \frac{\pi^2}{x^a(x+1)} = \pi^2 \cos(\pi a)$$

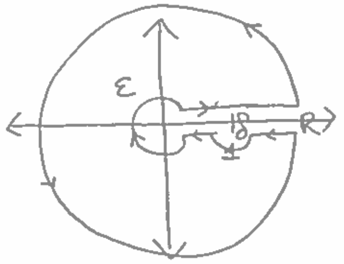
$\Rightarrow$ After computation, get

$$\int_0^\infty \frac{\log(x)}{x^a(x+1)} dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$ Gosh

Chapter 7, Section 5: Exercises:

1.) Use the keyhole contour indented on the lower edge of the axis at $x = 1$ to show that

$$\int_0^{\infty} \frac{\log x}{x^a(x-1)}\,dx = \frac{2\pi^2}{1-\cos(2\pi a)} \quad 0<a<1$$



$$f(z) = \frac{\log z}{z^a(z-1)} = \frac{\log|z| + i\arg(z)}{e^{a\log|z| + ai\arg z}(z-1)}$$

$$0 = \int_{\partial D} \frac{\log z}{z^a(z-1)}\,dz = -\pi i \operatorname{Res}\left[\frac{\log z}{z^a(z-1)}, z=1\right]$$

ok. I think. $\displaystyle\int_{\varepsilon}^{R} \frac{\log x}{x^a(x-1)} + \int_{\Gamma_R} + \int_{R}^{1+\delta} \frac{\log|x| + 2\pi}{x^a(x-1)}$

$\displaystyle + \int_{\gamma_\delta} + \int$

Chapter 8, Section 2: Exercises:

1) Show that $2z^5+6z-1$ has one root in $0<x<1$ & four roots in $\{1<|z|<2\}$.

First, we'll show 4 roots in $\{1<|z|<2\}$.

$f(z) = 2z^5$

$h(z) = 6z-1.$

On $|z|=2$

$$|6z-1| \leq 12-1 = 11 \leq 2^6 = 2|z|^5.$$

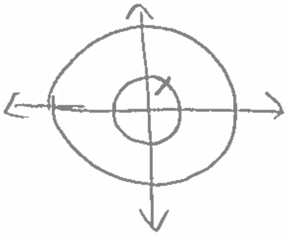
So on $|z|\leq 2$, our function has 5 roots.

On $|z|=1$

$$|2z^5-1| < 2|z|^5-1 = 1 < 6 = 6|z|$$

$\Rightarrow$ On $|z|\leq 1$, 1 root.

So then there are 4 roots in $\{1<|z|<2\}$.



Theres one root in the unit circle.

b/t $-1<x<1$.

But what if it's negative?

$p(0) = -1$

$p(1) = 7.$

$\Big\} \Rightarrow$ since cts, $0<x<1$

We're accounted for all 5 roots, so only 1 root in $0<x<1$.

3.) Show that if m & n are positive integers, then

$$p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$$

has exactly n integers in the unit disk.

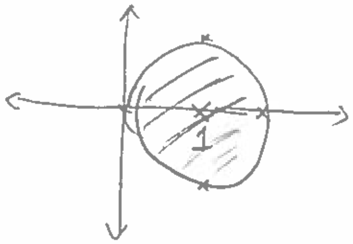
$$\left|1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!}\right| \leq 1 + |z| + \frac{|z|^2}{2!} + \cdots + \frac{|z|^m}{m!}$$

$$\leq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} \leq |3z^n| = 3 \quad \text{on } |z| = 1$$

$\Rightarrow$ Has n integers in $\{|z| \leq 1\}$.

4.) Fix a complex number λ s.t. $|\lambda|<1$. For $n\geq 1$, show that $(z-1)^n e^z-\lambda$ has n zeroes satisfying $|z-1|<1$ and no other zeroes in the right-half-plane. Determine the multiplicity of the zeroes.

$$(z-1)^n e^z-\lambda = (z-1)^n\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)-\lambda$$



$|-\lambda|<1$ On $|z-1|=1$

$$|(z-1)^n e^z| = |e^z| = \left|1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right|$$

$|z-1|=1 \Rightarrow z=1+e^{i\theta}$

$$=|e^{1+e^{i\theta}}| = |e^{1+\cos\theta+i\sin\theta}| = e^{1+\cos\theta}$$

$$\geq 1 > |-\lambda|.$$

$\Rightarrow$ n zeroes satisfying $|z-1|<1$

$$f'(z) = n(z-1)^{n-1}e^z + (z-1)^n e^z$$

$$= e^z(z-1)^{n-1}[n+(z-1)]$$

only one in right-half-plane

Zeroes of $f'(z)$: $z=1$, $z=1-n$

multiplicity $n-1$

$f(1)=-\lambda\neq 0$

5.) For a fixed λ satisfying $|\lambda| < 1$, show that $(z-1)^n e^z + \lambda(z+1)^n$ has n zeroes in the right half-plane, which are all simple if $\lambda \neq 0$.

7.) Let $f(z)$ & $g(z)$ be analytic on D bdd that extends ctsly to ∂D & satisfy $|f(z)+g(z)| < |f(z)| + |g(z)|$ on ∂D.

Show that f & g have same # of zeroes in D, counting multiplicity.

WTS $|f(z)-g(z)| < |g(z)|$ on ∂D.

B/c $\Rightarrow$ $f(z)-g(z)+g(z)$ has same # of zeroes as $g(z)$.

$$|f(z)-g(z)| = |f(z)+g(z)-2g(z)|$$

$$\leq |f(z)+g(z)| - 2|g(z)| < |f(z)| + |g(z)| - 2|g(z)| = |f(z)| - |g(z)|$$

$<$

$$|g(z)| > |f(z)+g(z)| - |f(z)| >$$

Chapter 8, Section 3: Exercises

2.) Let S be the family of univalent fncs $f(z)$ defined on $\{|z|<1\}$ that satisfy $f(0)=0$ & $f'(0)=$ Show that if a sequence in S converges normally to $f(z)$, then $f \in S$.

Let $f_K(z) \in S$. Then $\forall K$, $f_K(0)=0$

$f_K'(0)=1$

$f_K(z)$ univalent in $\{|z|<1\}$

$\Rightarrow$ By Hurwitz, $f(z)$ is univalent or constant.

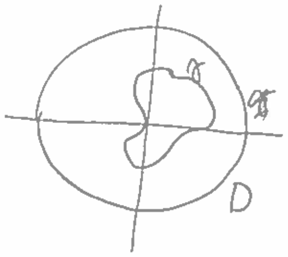
If $f(z)$ constant, 'then'

$$\underset{0}{\overset{\shortparallel}{f_K(0)}} \to f(0) \neq \qquad \Rightarrow f(0)=0$$

$$\Rightarrow f(z)=0.$$

$$f(z)=a$$

$$\Rightarrow f(z)-a = f(0)$$



$f_n \to f$ normally on γ

$$\int_\gamma f = \int_\gamma \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\gamma f_n = 0$$

f is analytic

$$f(0) = \lim_{n\to\infty} f_n(0) = 0 \checkmark$$

~~$f'(0)=$~~

$f_n \xrightarrow{unif} f$ on $|z|=r<1$ compact.

~~$f(z)=\frac{1}{2\pi i}$~~

$$f'(0) = \frac{0!}{2\pi i} \int_{|z|=r} \frac{f(w)}{(w)^2}\,dw$$

$$= \lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} \int_{|z|=r} \frac{1!}{2\pi i} \frac{f_n(\omega)}{\omega^2} dw = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(\omega)}{\omega^2} dw$$

$$= f'(0)$$

Smart!

$\Rightarrow f'(0) = 1. \Rightarrow$ not constant.

$\Rightarrow f$ is univalent by Hurwitz

on any compact subset of $\{|z| < 1\}$.

Chapter 8, Section 4: Exercises

1.) S D bdd w/ piecewise smooth bdry.
Let $f(z)$ be meromorphic & $g(z)$ analytic on D.
S f & g extend analytically across ∂D, & $f(z)\neq 0$ on ∂D. Show that

$$\frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} m_j\, g(z_j)$$

z_j are zeroes & poles of $f(z)$
m_j is order of $f(z)$ @ z_j.

Let $\frac{g(z)f'(z)}{f(z)}$ be our function.

$$\oint_{\partial D}\frac{g(z)f'(z)}{f(z)} = 2\pi i\sum_{j=1}^{n}\operatorname{Res}\left[\frac{g(z)f'(z)}{f(z)}, z_j\right]$$

Say z_j is a zero of $f(z)$.

$$\Rightarrow \frac{g(z)\,f'(z)}{f(z)} \qquad h(z_j)\neq 0$$

$$= \frac{g(z)\,m_j(z-z_j)^{m_j-1}}{(z-z_j)^{m_j}} + \frac{h'(z)g(z)}{h(z)}$$

$$= \frac{g(z)m_j}{(z-z_j)} + \frac{h'(z)g(z)}{h(z)} = \frac{\dfrac{g(z)m_j h(z) + h'(z)g(z)(z-}{h(z)}}{(z-z_j)}$$

$$\Rightarrow \operatorname{Res}\left[\frac{g(z)f'(z)}{f(z)}, z_j\right] = \frac{g(z_j)m_j h(z_j)}{h(z_j)} = g(z_j)m_j$$

Phew.

If z_j is a pole of $f(z)$.

$$\Rightarrow f(z) = \frac{h(z)}{(z-z_j)^{m_j}} \quad , \quad h(z_j) \neq 0.$$

$$\Rightarrow \frac{g(z) f'(z)}{f(z)} = \frac{g(z) \left[\frac{(z-z_j)^{m_j} h'(z) - h(z) m_j (z-z_j)^{m_j-1}}{(z-z_j)^{2m_j}} \right]}{\frac{h(z)}{(z-z_j)^{m_j}}}$$

$$= \frac{g(z)(z-z_j)^{m_j} h'(z) - g(z) h(z) m_j (z-z_j)^{m_j-1}}{h(z)(z-z_j)^{m_j}}$$

$$= \frac{\frac{g(z) h'(z)(z-z_j) - g(z) m_j h(z)}{h(z)}}{(z-z_j)}$$

$$\Rightarrow \text{Res}\left[\frac{g(z) f'(z)}{f(z)}, z_j\right] = m_j\, g(z). \quad \checkmark$$

3.) Let $\{f_K(z)\}$ be a sequence of analytic functions on a domain D that converges normalley to $f(z)$. \$ $f_K(z)$ attains each value w at most m tim (counting multiplicity) in D. Show that either $f(z)$ is constant, or $f(z)$ attains each value w at most m times in D.

$\Rightarrow$ $f_K(z) - \omega = 0$ at most m times.

\$ $f(z)$ not constant.

Let $z_0 \in D$ s.t. $f(z_0) = \omega_0$.

$$N(\omega_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)-\omega_0}\, dz =$$

$$\Rightarrow \quad 2\pi i * N(\omega_0) = \oint_{|z-z_0|=\rho} \frac{f'(z)}{f(z)-\omega_0} = \oint_{|z-z_0|=\rho} \lim_{k\to\infty} \frac{f_K'(z)}{f_K(z)-\omega_0}$$

$$= \lim_{K\to\infty} \oint_{|z-z_0|=\rho} \frac{f_K'(z)}{f_K(z)-\omega_0} = 2\pi i\, N_K(\omega) < 2\pi i\, m$$

$\Rightarrow$ $f(z)$ attains ω_0 at most m times.

5.) Let $f(z)$ be a meromorphic function on the omplex plane, and suppose there is an integer m s.t. $f^{-1}(w)$ has at most m points $\forall w \in \mathbb{C}$. Show that $f(z)$ is a rat'l function.

By thm, (Ch. 6.4) meromorphic fncs. on $\mathbb{C}^*$ are rat'l.

We know analytic except for some poles on $\mathbb{C}$. Need to show that ∞ is <u>not</u> an essential singularity.

& FTSOC, ∞ is an essential singularity.

we have $f(z)-w=0$ has at most m solns.

By Casorati-Weierstrass, $\forall w_0 \in \mathbb{C}, \exists z_n \to \infty$

$$\begin{tikzcd} \text{s.t. } f(z_n) \arrow[r] \arrow[d] & w_0 \arrow[d] \\ f(z) \arrow[r, phantom, "(=)"] & w_0 \end{tikzcd}$$

$\Rightarrow f(z)=w_0$ has infinitely many sol'ns. ⨳

$\Rightarrow$ ∞ is removable/pole

$\Rightarrow$ $f(z)$ is meromorphic on $\mathbb{C}^*$

8.) Let D be a bdd. domain, and let $f(z)$ be a cts. fnc. on $D \cup \partial D$ that is analytic on D. Show that $\partial(f(D)) \subseteq f(\partial D)$.

$f(D)$ is open since f is cts.

Let $w_0 \in \partial(f(D))$, not open

Take $z_n \in D$ s.t. $f(z_n) \to w_0$.

Assume $z_n \longrightarrow z_0 \in D \cup \partial D$.

$\Rightarrow f(z_0) = w_0 \in f(D \cup \partial D) = f(D) \cup f(\partial D)$.

But $w_0 \notin f(D)$ since $f(D)$ is open.

$\Rightarrow w_0 \in f(\partial D)$.

Chapter 9, Section 1 : Exercises

1.) Let $f(z)$ be analytic, $|f(z)| \leq M$ for $|z-z_0|<R$. If $f(z)$ has a zero of order m at z_0, then

$$|f(z)| \leq \frac{M}{R^m} |z-z_0|^m \quad , \quad |z-z_0|<R.$$

Show that equality holds at $z \neq z_0$ only when $f(z)$ is a constant multiple of $(z-z_0)^m$.

$$f(z) = (z-z_0)^m g(z) \; , \; g(z_0) \neq 0, \; g(z) \text{ analytic.}$$

$\Rightarrow$ For $|z-z_0| = r < R$,

$$|g(z)| = \frac{|f(z)|}{r^m} \leq \frac{M}{r^m} \longrightarrow \frac{M}{R^m}$$

$$\Rightarrow |f(z)| \leq \frac{M}{R^m} |z-z_0|^m \quad , \quad |z-z_0|<R.$$

If $|f(a)| = \frac{M}{R^m} |a-z_0|^m$

$\Rightarrow g(a) = \frac{M}{R^m}$. By strict maximum principle,

$g(z) = \lambda$ constant.

$\Rightarrow f(z) = \lambda (z-z_0)^m$.

2.) \$ $f(z)$ is analytic and satisfies $|f(z)| \leq 1$ for $|z| < 1$. Show that if $f(z)$ has a zero of order m at z_0, then $|z_0|^m \geq |f(0)|$.

$$\psi : \mathbb{D} \to \mathbb{D}$$

$$z \longmapsto \frac{z - z_0}{1 - \bar{z}_0 z}$$ is univalent.

$$|\psi(z)| \leq 1 \quad \text{for } |z| < 1$$

$$\psi(0) = -z_0$$

$$|f(z)| = |(z - z_0)^m g(z)|$$

$$|\psi(z)|^m = \frac{|z - z_0|^m}{(1 - \bar{z}_0 z)^m}$$

$$\frac{|f(z)|}{|\psi(z)|^m} = \frac{|g(z)|}{(1 - \bar{z}_0 z)^m}$$

$$f(z_0) = 0$$

8.) S $f(z)$ analytic for $|z| < 1$ and satisfies $|f(z)| < 1$, $f(0)=0$, $|f'(0)| < 1$. Let $r < 1$. Show that $\exists\, c < 1$ s.t. $|f(z)| \leq c|z|$. Show that $f_n(z) = f(f(\ldots f(z)\ldots)) = f(f_{n-1}(z))$ of $f(z)$ satisfies $|f_n(z)| \leq c^n|z|$ for $|z| \leq r$. Deduce that $f_n(z) \to 0$ normal on $\mathbb{D}$.

$f(z) = z g(z)$. Let $r < 1$, $|z| < r < 1$

$$\Rightarrow |g(z)| \leq \frac{|f(z)|}{r} < \frac{1}{r}$$

$$\Rightarrow \left|\frac{f(z)}{z}\right| < \frac{1}{r} \longrightarrow 1 \quad \text{for } |z| < 1.$$

$$\Rightarrow \left|\frac{f(z)}{z}\right| < \max_{z \in \mathbb{D} \cup \partial\mathbb{D}} \left(\left|\frac{f(z)}{z}\right|\right)$$ (which exists since $\partial\mathbb{D} \cup \mathbb{D}$ is compact.)

$$|f_n(z)| \leq c\,|f^{(n-1)}(z)| \leq c^2 |f^{(n-2)}(z)| \leq \ldots \leq c^n |z|$$

for $|z| \leq r$.

On $|z| \leq 1$, $|f_n(z)| \leq c^n < 1$

9.1 #2: § $f(z)$ is analytic, $|f(z)| \leq 1$ for $|z| < 1$.
Show that if $f(z)$ has a zero of order m at z_0, then $|f(0)| \leq |z_0|^m$ ☆ Weird ☆

Let $\psi(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$ $\Rightarrow$ WTS $|f(z)| \leq |\psi(z)|^m$

Idea: Want to find $\underbrace{f(z_0)}_{= \psi^{-1}(0)} = 0$.

$$\frac{\psi^{-1}(z) - z_0}{1 - \bar{z}_0 \psi^{-1}(z)} = z \Rightarrow \psi^{-1}(z) = \frac{z + z_0}{1 + \bar{z}_0 z}$$

$$\Rightarrow \psi^{-1}(0) = z_0$$

$\Rightarrow f(\psi^{-1}(0)) = 0$.

$|f(\psi^{-1}(z))| \leq 1$ for $|z| < 1$.

$(\psi^{-1} : \mathbb{D} \to \mathbb{D}$ & $|f| \leq 1$ for $|z| < 1)$

$\Rightarrow$ By Schwarz,

$|f(\psi^{-1}(z))| \leq |z| \leq |z|^m$ for $|z| < 1$.

$\Rightarrow |f(\psi^{-1}(z_0))| \leq |z_0|^m$

$\Rightarrow |f(0)| \leq |z_0|^m$ ✓

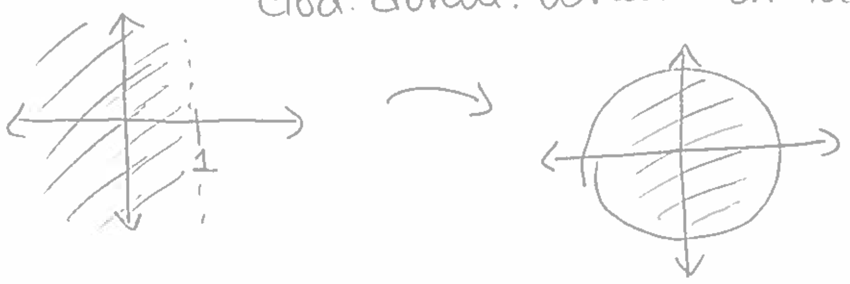
) $\$$ $f(z)$ is analytic for $|z|<1$ and satisfies $f(0)=0$ &

$\operatorname{Re} f(z) < 1$.

(a) Show that $|f(z)| \leq \dfrac{2|z|}{1-|z|}$ ○

Hint: Consider the composition of $f(z)$ & the FLT mapping $\{\operatorname{Re} w < 1\}$ onto the unit disk.

God. donut. what in tarnation.



$\psi(z) = \dfrac{z}{z-2}$. (How?)

$\Rightarrow \psi \circ f(0) = \psi(0) = 0.$ ○

$|\psi \circ f(0)| \leq 1$ ✓

$\Rightarrow$ By Schwarz, $|\psi \circ f(z)| \leq |z|$.

$$\Rightarrow \left|\frac{f(z)}{f(z)-2}\right| \leq |z|.$$

$$\Rightarrow |f(z)| \leq |z|\,(|f(z)-2|)$$

$$\leq |z|.|f(z)| + 2|z|$$

$$\Rightarrow |f(z)|(1-|z|) \leq 2|z|$$

$$\Rightarrow |f(z)| \leq \frac{2|z|}{1-|z|}.$$ ○

(b) Show that $|f'(0)| \le 2$

$$|(\psi \circ f)'(0)| \le 1$$

$$\Rightarrow |\psi'(f(0))f'(0)| \le 1$$

$$\Rightarrow |\psi'(0)||f'(0)| \le 1$$

$$\rightarrow \psi'(z) = \frac{(z-2)-z}{(z-2)^2} \Rightarrow \psi'(0) = -\frac{1}{2}$$

$$\Rightarrow |f'(0)| \le 2.$$

(c) For fixed z_0 with $0<|z_0|<1$, determine for which fncs. $f(z)$ there is equality in (a)

$|\psi \circ f(z)| = |z|$ if

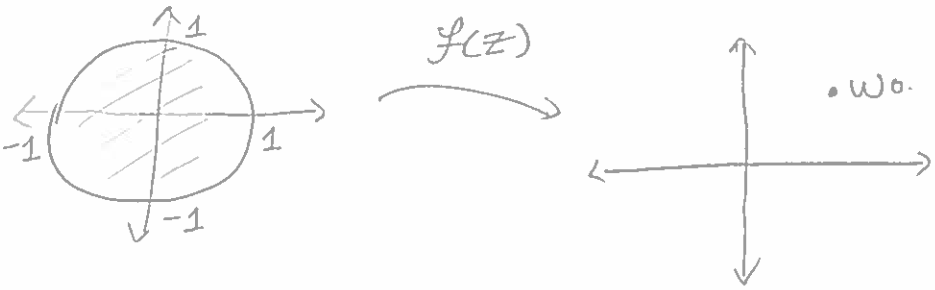
$$\psi \circ f(z) = \lambda z$$

if $\dfrac{f(z)}{f(z)-2} = \lambda z$

$$f(z) = \lambda z f(z) - 2\lambda z$$

$$f(z) = \frac{-2\lambda z}{1-\lambda z}.$$

.) Let $f(z)$ be a conformal map of $\{|z|<1\}$ onto domain D. Show that $d(f(0), \partial D) \leq |f'(0)|$.



Say $f(0)=w_0$

Let $f^{-1}(w)=(w-w_0)g(w)$, so that

$f^{-1}(w_0)=0$.

$$|g(w)| = \frac{|f^{-1}(w)|}{r} \quad \text{for } |w-w_0|=r$$

$$< \frac{1}{r} \longrightarrow \frac{1}{d(f(0),\partial D)} \quad \text{if } r \to d(f(0),\partial D)$$

$\Rightarrow$ By max. principle, $|g(w)| \leq \dfrac{1}{d(f(0),\partial D)}$ for $|w-w_0| \leq r$.

$$|g(w)| \overset{?}{\geq} \frac{1}{f'(0)}$$

$$|g(w)| = \left| \frac{f^{-1}(w) - f^{-1}(w_0)}{w-w_0} \right|$$

$$\Rightarrow \lim_{w\to w_0} |g(w)| = |(f^{-1})'(w_0)| = \left| \frac{1}{f'(0)} \right| \neq$$

$$\Rightarrow |f'(0)| \geq d(f(0),\partial D).$$

Chapter 9, Section 2: Exercises.

1.) A finite Blaschke product is a rat'l function $B(z) = e^{i\varphi}\left(\frac{z-a_1}{1-\overline{a}_1 z}\right)\cdots\left(\frac{z-a_n}{1-\overline{a}_n z}\right)$, $a_1, \dots, a_n \in \mathbb{D}$ $0 \leq \varphi \leq 2\pi$. Show that if $f(z)$ is cts. for $|z| \leq 1$ & analytic for $|z| < 1$ and $|f(z)| = 1$ for $|z| = 1$, then $f(z)$ is a finite Blaschke product.

$$|B(z)| = |e^{i\varphi}|\left|\frac{z-a_1}{1-\overline{a}_1 z}\right|\cdots\left|\frac{z-a_n}{1-\overline{a}_n z}\right| = 1.$$

$$\Rightarrow \left|\frac{f(z)}{B(z)}\right| = 1 \text{ for } |z| = 1$$

$$\Rightarrow \left|\frac{f(z)}{B(z)}\right| \leq 1 \text{ for } |z| \leq 1 \text{ by maximum principle.}$$

$$\Rightarrow |f(z)| \leq |B(z)| \text{ for } |z| \leq 1$$

Similarly, $|f(z)| \geq |B(z)|$ for $|z| \leq 1$.

$$\Rightarrow |f(z)| = |B(z)| \text{ for } |z| \leq 1.$$

$$\Rightarrow f(z) = \lambda B(z).$$

) Suppose $f(z)$ is analytic for $|z|<3$. If $|f(z)|\leq 1$, and $f(\pm i) = f(\pm 1) = 0$, what is the maximum value of $|f(0)|$? For which functions is the maximum attained?

$|f(3z)|$ is analytic for $|3z|<3 \Rightarrow |z|<1$

$\downarrow$

zeroes are at $z = \pm \frac{i}{3}, \pm \frac{1}{3}$.

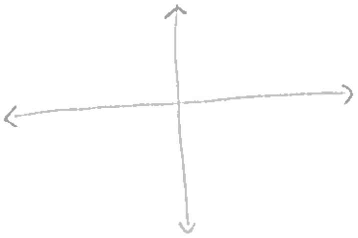
maximum value of $f(z)$ is if

$|f(3z)| = 1$ for $|z| = 1$

$\Rightarrow f(3z) = B(z)$

$\Rightarrow |f(0)| = |B(0)| = \left|(\frac{i}{3})(-\frac{i}{3})(\frac{1}{3})(-\frac{1}{3})\right| = \frac{\pm 1}{81}$

When does $f(0) = \frac{\pm 1}{81}$?



13.) $f(z)$ is analytic from $\mathbb{D} \to \mathbb{D}$, $f(z) \not\equiv z$. Show that $f(z)$ has at most one fixed pt. in $\mathbb{D}$.

Suppose $f(z_0)=z_0$; $f(z_1)=z_1$.

Let $\phi(z) = \dfrac{z-z_0}{1-\bar{z}_0 z_0}$ so that $\phi(z_0)=0$.

$\Rightarrow \phi^{-1}(z) = \dfrac{z+z_0}{1-\bar{z}_0 z}$ so that $\phi^{-1}(0)=z_0$.

$\Rightarrow \quad \phi \circ f \circ \phi^{-1}(0) = 0$

$|\phi \circ f \circ \phi^{-1}(z)| < 1$ for $|z|<1$

$\Rightarrow$ By Schwarz, $|\phi \circ f \circ \phi^{-1}(z)| \leq |z|$

Also have $|\phi \circ f \circ \phi^{-1} \circ \phi(z_1)| = |\phi(z_1)|$.

$\Rightarrow \phi \circ f \circ \phi^{-1}(z) = \lambda z.$

$\phi \circ f \circ \phi^{-1}(\phi(z_1)) = \lambda \phi(z_1)$

$\Rightarrow \phi \circ f(z_1) = \lambda \phi(z_1)$

$\Rightarrow \phi(z_1) = \lambda \phi(z_1)$

$\Rightarrow \lambda = 1.$

$\Rightarrow \phi(f(\phi^{-1}(z))) = z.$

$\Rightarrow f(\phi^{-1}(z)) = \phi^{-1}(z)$

$\Rightarrow f(z) = z$ #.

Chapter 10, Section 1: Exercises.

2.) Let $R>0$, $h(Re^{i\theta})$ cts. on $\{|z|=R\}$. Show that

$$\tilde{h}(z) = \int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-\varphi)} h(Re^{i\varphi}) \frac{d\varphi}{2\pi}, \quad |z|<R, \text{ is}$$

harmonic on $\{|z|<R\}$ and has boundary values

$h_1(Re^{i\theta})$ on the bdy circle

$$\tilde{h}(z) = \int_{-\pi}^{\pi} \frac{1-(\frac{r}{R})^2}{1+(\frac{r}{R})^2-2r\cos(\theta+\varphi)} h(e^{i\varphi}) \frac{d\varphi}{2\pi} \quad \text{on} \quad |z|<R$$

$$= \int_{-\pi}^{\pi} P_{\frac{r}{R}}(\theta-\varphi) h(e^{i\varphi}) \frac{d\theta}{2\pi} = \tilde{h}(\tfrac{r}{R}e^{i\varphi}) \quad \text{for } \tfrac{r}{R}e^{i\varphi}\in$$

$\rightarrow$ harmonic

bdy values on $h(e^{i\varphi})$.

) Suppose $f(z)=u(z)+iv(z)$ is analytic for $|z|<1$ and hat $u(z)$ extends to be cts. on $\{|z|\le 1\}$. Show hat

$$f(z)=\int_0^{2\pi} u(e^{i\varphi})\,\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,\frac{d\varphi}{2\pi}+iv(0),\quad |z|<1.$$

Darn.

$$u(z)=\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right)u(e^{i\varphi})\,\frac{d\varphi}{2\pi}\quad \text{on } |z|<1.$$

$$\tilde{f}(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,u(e^{i\varphi})\,d\varphi$$

$$\Rightarrow \mathrm{Re}(\tilde{f}(z))=\tilde{u}(z)=u(z)\quad \text{on } |z|<1$$

$$\left.\begin{array}{l}\tilde{f}(z)=u(z)+i\tilde{v}(z)\\ f(z)=u(z)+iv(z)\end{array}\right\}\ \text{on } |z|<1.$$

Since $\tilde{v}(z)-v(z)$ analytic, and real-valued

$\Rightarrow$ constant.

$$\Rightarrow \tilde{f}(z)-f(z)=ic$$

$$\begin{aligned}\Rightarrow ic=\tilde{f}(0)-f(0)&=i(\tilde{v}(0)-v(0))\\ &=\frac{1}{2\pi}\int_0^{2\pi}u(e^{i\varphi})\,d\varphi-u(0)-iv(0)\\ &=u(0)-u(0)-iv(0)\\ &=-iv(0)\Rightarrow c=-v(0).\end{aligned}$$

$$\Rightarrow f(z)=\tilde{f}(z)+iv(0).\ \checkmark$$

4.) Let $\{f_n = U_n(z) + i V_n(z)\}$ analytic on $\mathbb{D}$ open s.t. $U_n(z)$ extends ctsly to $\partial\mathbb{D}$, $U_n(e^{i\theta})$ converges unif. on $\partial\mathbb{D}$ to $u(e^{i\theta})$, $V_n(0)$ converges.

Show that $f_n(z)$ converges normally to $f(z)$ analytic, $\operatorname{Re} f(z) = \tilde{U}(z)$.

10.2 Exercises:

2.) Assume that $u(x,y)$ is a 2x ct'ly diff'ble fnc. on D.

(a) For $(x_0,y_0) \in D$, let $A_\varepsilon(x_0,y_0)$ be the avg. of $u(x,y)$ on the circle centered at (x_0,y_0) of radius ε.

Show that $\lim_{\varepsilon \to 0} \frac{A_\varepsilon(x_0,y_0) - u(x_0,y_0)}{\varepsilon^2} = \frac{1}{4}\Delta u(x_0,y_0).$

$$\Delta u(x_0,y_0) = \frac{\partial^2 u}{\partial x^2}\Big]_{(x_0,y_0)} + \frac{\partial^2 u}{\partial y^2}\Big]_{(x_0,y_0)}$$

$$A_\varepsilon(x_0,y_0) = \int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta}) \frac{d\theta}{2\pi}$$

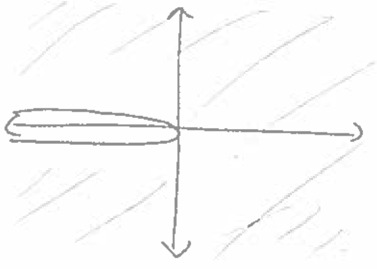
$u(x,y) =$

Chapter 11, Section 1: Exercises:

2.) Find a conformal map of $\mathbb{C} \setminus (-\infty, 0]$ onto the open unit disk satisfying $w(0) = i$

$w(-1+0i) = 1$

$w(-1-0i) = -1.$



$-\pi < \arg z < \pi$

sector

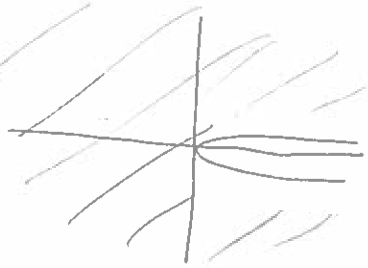
I guess

$\longrightarrow$

$z\bar{e}^{\pi i}$

$\shortparallel$

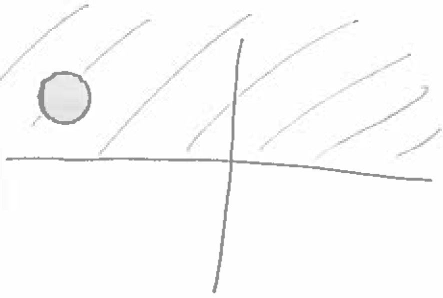
$f(z)$



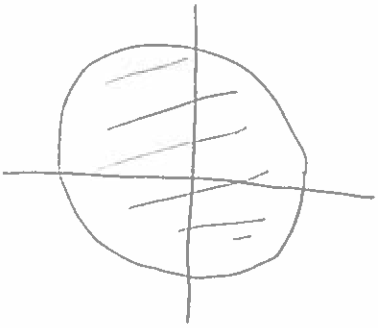
$0 < \arg z < 2\pi$

$\swarrow$

$z^{\pi/2} = g(z)$



$\xrightarrow{\quad h(z) = \frac{z-i}{z+i} \quad}$



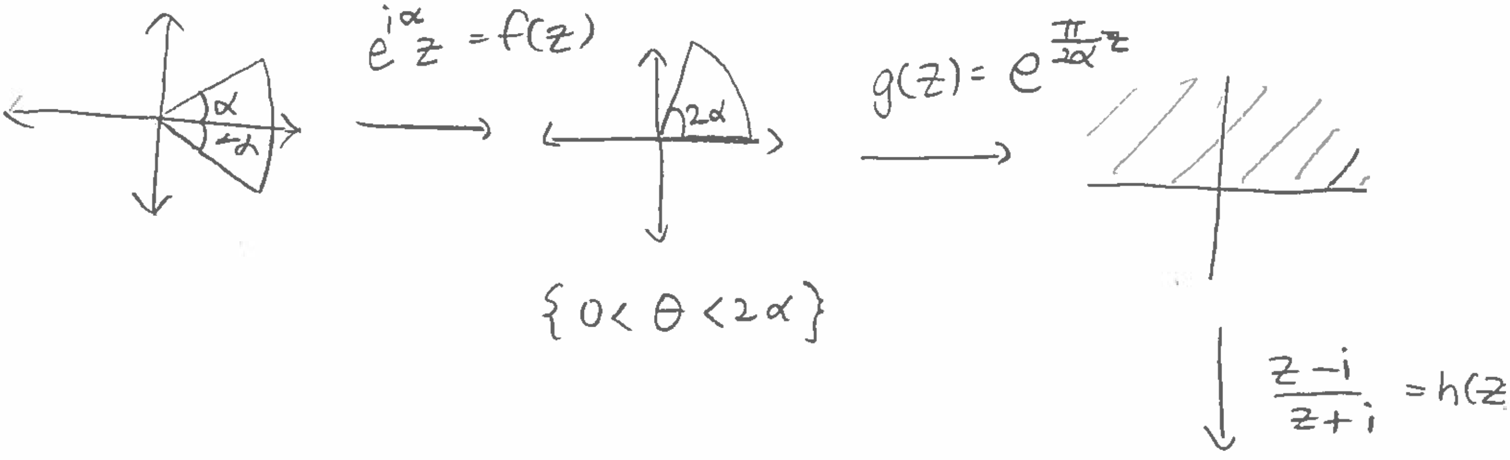
$$\varphi(z) = h(g(-z)) = h((-z)^{\pi/2}) = \frac{(-z)^{\pi/2} - i}{(-z)^{\pi/2} + i}$$

$0 \longmapsto -1$

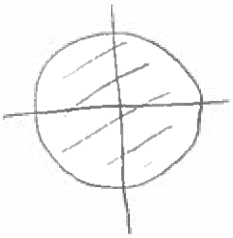
$-1 \longmapsto \frac{1-i}{1+i}$

$\Rightarrow$ FLT

7.) Find the conformal map of $\{|\arg z|<\alpha, |z|<1\}$ onto the open unit disk such that $w(0)=-1$, $w(1)=1$, $w(e^{i\alpha})=i$

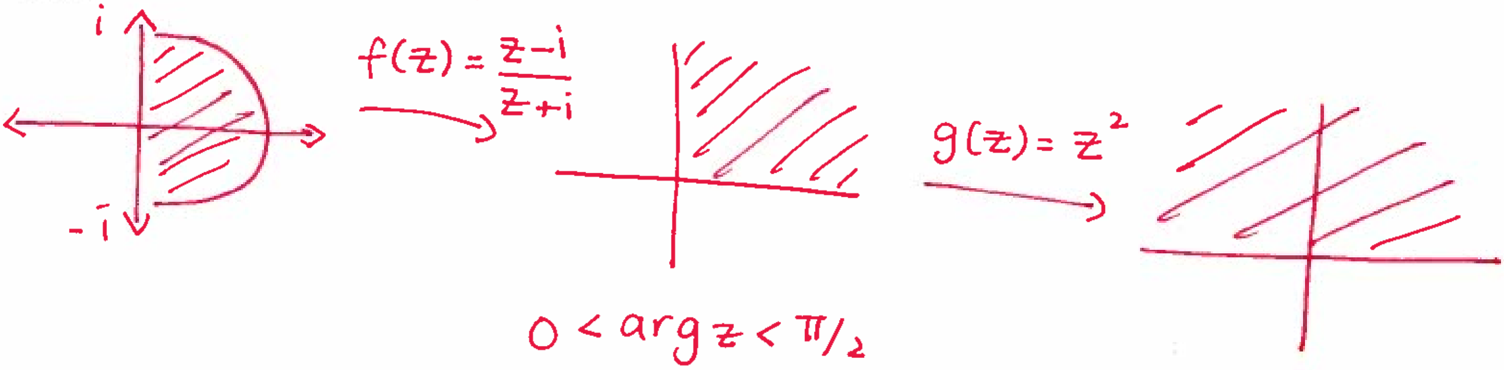


$$h(g(e^{i\alpha}z)) = h(\, e^{\frac{\pi}{2\alpha}}e^{i\alpha}z\,)$$



5.) Find a conformal map $w(z)$ of the right half-disk $\{\mathrm{Re}\, z > 0, |z| < 1\}$ onto the UHP that maps $-i \mapsto 0$, $i \mapsto \infty$, $0 \mapsto -1$

What is $w(1)$?



$0 < \arg z < \pi/2$

$0 < \arg z < \pi$

$\varphi(z) = h(g(f(z))) =$ ~~h(g(f(~~ ~~$\left(\frac{z+i}{z-i}\right)^2$~~

$\left(\frac{z-i}{z+i}\right)^2$

$\varphi(i) = 0$

$\varphi(0) = 1$

$\varphi(-i) = \infty.$

Need FLT $a(z)$: $\infty \mapsto 0$, $0 \mapsto \infty$, $1 \mapsto -1$

$h(z) = \frac{\lambda}{z} = \boxed{-\frac{1}{z}}$

$\Rightarrow h(1) = \lambda = -1$

$\Rightarrow \lambda = -1$

$\Rightarrow$ Answer: $-\left(\frac{z+i}{z-i}\right)^2$

$\Rightarrow w(1) = -\left(\frac{1+i}{1-i}\right)^2 = 1.$

REAL ANALYSIS AND MEASURE THEORY
QUALIFYING EXAM
AUGUST 20, 2019

Notation: L^p spaces are with respect to the Lebesgue measure m.

1. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Let E be the set of all numbers $x \in \mathbb{R}$ such that the sequence

$$\{f(x), f(f(x)), f(f(f(x))), \dots\}$$

is bounded. Prove that E is a measurable set.

2. Consider the sequence of functions $f_k(x) = kx^k$ on the set $[0,1]$ equipped with the Lebesgue measure. Prove that this sequence (a) converges in measure; (b) does not converge in $L^1([0,1])$.

3. Suppose that $f\colon [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$\int_{[0,1]} xf(x)\, dx = 1$$

Prove that

$$\int_{[0,1]} f(x)^4\, dx \geq 8$$

4. Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \in L^1(\mathbb{R})$ but the limit $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x}$ does not exist.

Qualifying Exam, Complex Analysis, August 20, 2019

1. Find all the functions $g : \mathbb{R} \longrightarrow \mathbb{R}$ of class C^2 such that the function $f(z) = x^2 - y^2 + ig(xy)$, $z = x + iy$, is entire.

2. Find all the possible Laurent expansions centered at 2 of the function $f(z) = \dfrac{z^2}{z+2}$.

3. Assume that f is holomorphic on $D = \{z \in \mathbb{C} : |z| > 1\}$, $\lim_{z\to\infty} f(z) = 1$, and $r \in (0,1)$. Find

$$\int_{|z|=r} f\left(\frac{1}{z}\right) \frac{dz}{z}.$$

4. Suppose that f is an entire function with the property that there exist positive constants C, N, R_0, such that

$$\max_{|z|=R} |e^{f(z)}| \leq CR^N, \ \forall R > R_0.$$

Prove that f is constant.

Mathematics 440 & 508

Homework #10

XI.5-1. Let $\{f_n(z)\}$ be a uniformly bounded sequence of analytic functions on a domain D and let $z_0 \in D$. Suppose that for each $m \geq 0$, $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ normally on D.

Ans: Let $D(z_0, r)$ be the largest open disk with centre z_0 contained in D. We first show that f_n converges uniformly to 0 on any closed subdisk $\bar{D}(z_0, r_1) \subset D(z_0, r)$. Let M be the uniform bound on $f_n(z)$ on D. Then by homework exercise V.2.12, $|f_n^{(m)}(z_0)| \leq m!M/r^m$ and hence

$$|f_n(z) - \sum_{m=0}^{N} \frac{f_n^{(m)}(z_0)}{m!}(z-z_0)^m| \leq \sum_{m=N+1}^{\infty} M\left(\frac{|z-z_0|}{r}\right)^m.$$

For $|z - z_0|/r \leq r_1/r < 1$ the bound on the RHS may be made smaller than $\epsilon/2$ for sufficiently large N, uniformly in n. And then since $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$, the finite sum $\sum_{m=0}^{N} \frac{f_n^{(m)}(z_0)}{m!}(z-z_0)^m$ may be made smaller than $\epsilon/2$ for n sufficiently large uniformly in z for $z \in D(z_0, r_1)$. So $f_n(z) \to 0$ uniformly on $D(z_0, r_1)$.

By Montel's theorem, p.308, f_n has a subsequence f_{n_k} that converges normally on D to some f analytic in D. By what we have just shown, f is identically zero on $D(z_0, r)$ and hence 0 in all of D by the uniqueness principle (p.156). Since this limit is independent of the subsequence, it is clear that this means that the full sequence f_n converges normally to 0 on D. Otherwise, it would have a subsequence that does not converge to 0 for all z and, by Montel's theorem, this would in turn have a normally convergent subsequence that converges to a limit that is not identically zero But this would be a contradiction of what we have just proved.

(One does not really need to use Montel's theorem to prove this result. To give a direct argument, notice that one can connect z_0 to any $z_1 \in D$ by a finite chain of overlapping disks, with the centre of each being in the previous disk. Then one inductively proves that $f_n(z) \to 0$ uniformly on each of these disks. It is then easy to conclude that f_n converges uniformly to 0 on each closed subdisk of D and hence normally to 0 on D.)

XI.5-8. Let $\{f_n(z)\}$ be a sequence of analytic functions on a domain D. Suppose that $\int\int_D |f_n(z)|\,dx\,dy \leq 1$ for $n \geq 1$.

(a) Show that $\{f_n(z)\}$ has a subsequence that converges normally to an analytic function $f(z)$ on D. *Hint.* To estimate $f(z)$ use the mean value property with respect to area (see Exercise III.4.1).

(b) Show that $\int\int_D |f(z)|\,dx\,dy \leq 1$.

(c) Show that if $\int\int_D |f_n(z) - f_m(z)|\,dx\,dy \to 0$ as $m, n \to \infty$, then $\int\int_D |f_n(z) - f(z)|\,dx\,dy \to 0$ as $n \to \infty$.

Ans: Suppose that f is analytic in the disk $|z - z_0| \leq a$, with $a > 0$. By Cauchy's formula for the circle $\gamma(t) = z_0 + re^{i\theta}$, one has

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

Multiplying this by $r\,dr$, integrating from 0 to a, and dividing by $\int_0^a r\,dr = a^2/2$, we have

$$f(z_0) = \frac{1}{\pi a^2}\int\int_D f(z)\,dA,$$

where $dA = dx\,dy = r\,dr\,d\theta$ is the element of area. This is the mean-value property referred to in the hint.

(a) Using the above, we show that $\{f_n(z)\}$ is locally uniformly bounded in D. If K is any compact subset of D, let $\delta = \text{dist}(K, \partial D)$. Then each point $z \in K$ is contained in the closed disk $\bar{D}(z,\delta) \subset D$ so $f_n(z) = \frac{1}{\pi\delta^2}\int\int_{\bar{D}(z,\delta)} f_n(\zeta)\,dA$. So

$$|f_n(z)| \leq \frac{1}{\pi\delta^2}\int\int_{\bar{D}(z,\delta)} |f_n(\zeta)|\,dA \leq \frac{1}{\pi\delta^2}\int\int_D |f_n(\zeta)|\,dA \leq \frac{1}{\pi\delta^2}.$$

Since this bound is independent of n, this proves the local uniform boundedness of the family $\{f_n\}$. By Montel's theorem, a subsequence of $\{f_n\}$ converges normally to an f analytic in D. For convenience, denote this subsequence again by $\{f_n\}$ below.

(b) Let K_k be the standard sequence of compact subsets of D whose union is D, defined by

$$K_k = \{z \in D | \text{dist}(z, \partial D) \leq 1/k \text{ and } |z| \leq k\}.$$

Then $\int\int_{K_k} |f_n(z)|\,dA \leq \int\int_D |f_n(z)|\,dA \leq 1$ since the integrand is non-negative. Since K_k is compact, we have uniform convergence of f_n to f on K_k and hence $\int\int_{K_k} |f(z)|\,dA = \lim_{n\to\infty}\int\int_{K_k} |f_n(z)|\,dA \leq 1$. Now, by a fundamental property of integrals (monotone convergence), we have

$$\int\int_D |f(z)|\,dA = \lim_{k\to\infty}\int\int_{K_k} |f(z)|\,dA \leq 1.$$

(c) If $\int\int_D |f_n(z) - f_m(z)|\,dx\,dy \to 0$, then given $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $\int\int_D |f_n(z) - f_m(z)|\,dx\,dy < \epsilon$, for $m, n \geq N$. By the same argument as in (b), letting $m \to \infty$, we can conclude from this that $\int\int_D |f_n(z) - f(z)|\,dx\,dy \leq \epsilon$ for $n \geq N$. This shows that $\int\int_D |f_n(z) - f(z)|\,dx\,dy \to 0$ as $n \to \infty$.

XI.6–2. Let $\phi(z)$ be the Riemann map of a simply connected domain D onto the open unit disk, normalized by $\phi(z_0) = 0$ and $\phi'(z_0) > 0$. Show that if $f(z)$ is any analytic function on D such that $|f(z)| \leq 1$ for $z \in D$, then $|f'(z_0)| \leq \phi'(z_0)$, with equality only when $f(z)$ is a constant multiple of $\phi(z)$. *Remark* This shows that $\phi(z)$ is the Ahlfors function of D corresponding to z_0.

Ans: Suppose that $f(z_0) = \alpha$. Let $\phi_\alpha(z) = (z-\alpha)/(1-\bar{\alpha}z)$ and consider $g = \phi_\alpha \circ f \circ \phi^{-1}$. Then ϕ^{-1} maps $\mathbb{D}$ one-one onto D with $\phi^{-1}(0) = z_0$, f maps D into $\bar{\mathbb{D}}$ with $f(z_0) = \alpha$ and ϕ_α maps $\bar{\mathbb{D}}$ one-one onto $\bar{\mathbb{D}}$ with $\phi_\alpha(\alpha) = 0$, so g maps $\mathbb{D}$ into $\bar{\mathbb{D}}$ with $g(0) = 0$. Thus Schwarz' lemma applies to g implying $|g'(0)| \leq 1$. Computing $g'(0)$ by the chain rule, we have

$$g'(0) = \phi_\alpha'(\alpha)f'(z_0)(\phi^{-1})'(0) = \frac{1}{1-|\alpha|^2}f'(z_0)\frac{1}{\phi'(z_0)}.$$

Hence $|f'(z_0)| \leq |g'(0)|(1-|\alpha|^2)\phi'(z_0) \leq \phi'(z0)$, since both $|g'(0)| \leq 1$ and $1-|\alpha|^2 \leq 1$.

For equality to hold, we must have both $|g'(0)| = 1$ and $1-|\alpha|^2 = 0$. The first implies that $g(z) = cz$ for a constant c with $|c| = 1$, and the second that $\alpha = 0$. Thus $f \circ \phi^{-1}(z) = cz$ and so $f(z) = c\phi(z)$ for all $z \in D$.

Remark: Note that the book's hint only applies to the case $\alpha = 0$.

XI.6–3. Let $\phi(z)$ be the Riemann map of a simply connected domain D onto the open unit disk, normalized by $\phi(z_0) = 0$ and $\phi'(z_0) > 0$. Show that if $f(z)$ is any analytic function on D such that $|f(z)| \leq 1$ for $z \in D$, then $\text{Re}\, f'(z_0) \leq \phi'(z_0)$, with equality only when $f(z) = \phi(z)$.

Ans: Since $\text{Re}\, f'(z_0) \leq |f'(z_0)|$ and $|f'(z_0)| \leq \phi'(z_0)$ by the previous problem, we clearly have $\text{Re}\, f'(z_0) \leq \phi'(z_0)$. If equality holds then by the first inequality, we also must have $|f'(z_0)| = \phi'(z_0)$ and hence by the previous problem we have $f(z) = c\phi(z)$ with $|c| = 1$ a constant. But then we have $\text{Re}\, f'(z_0) = c\phi'(z_0)$ and $\text{Re}\, f'(z_0) = \phi'(z_0)$ so $c = 1$.

11.5.1. Let $\{f_n(z)\}$ be a uniformly bounded sequence of holomorphic functions on a domain D, and let $z_0 \in D$. Suppose that for each $m \geq 0$, $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ normally on D.

Solution: Let $\{f_{n,k}\}$ be a subsequence of $\{f_n\}$. Because $\{f_n\}$ is uniformly bounded on D, Montel's theorem implies that $\{f_{n,k}\}$ contains a subsequence $\{f_{n,k,j}\}$ which converges normally on D to some function f. Moreover, (by previous theorems) f is holomorphic on D since $f_{n,k,j}$ are holomorphic on D and $f_{n,k,j}^{(m)} \to f^{(m)}$ normally on D for $m \geq 1$.

In particular, $f^{(m)}(z_0) = 0$ for $m \geq 0$ since, by assumption, $f_{n,j,k}^{(m)}(z_0) \to 0$. Thus the Taylor expansion of f centered at z_0 has coefficients $f^{(m)}(z_0)/m! = 0$ for $m \geq 0$, so $f \equiv 0$ on a disk $\Delta(z_0, r) \subset D$ for some $r > 0$. And by the identity principle, $f \equiv 0$ on D.

Since $\{f_{n,k}\}$ was arbitrary, every subsequence of $\{f_n\}$ has a subsequence which converges normally to the zero function on D. Therefore, by the "fact" mentioned in the hint, $\{f_n\}$ also converges normally to the zero function on D.

☐

11.5.7. Let D be a bounded domain, and let $f(z)$ be a holomorphic function from D into D. Show that if $z_0 \in D$ is a fixed point for $f(z)$, then $|f'(z_0)| \leq 1$.

Solution: The functions $f_n := f \circ \cdots \circ f$ (n times) for $n \geq 0$ are well-defined, since $f : D \to D$; holomorphic, since (by induction) $(f_n)' = (f')^n$ for $n \geq 0$; and uniformly bounded, since D is bounded and $f_n(D) \subseteq D$ for $n \geq 0$. Thus, by Montel's theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges normally on D. By a previous theorem, $\{f'_{n_k}\}$ converges normally on D as well. In particular, $\{|f'_{n_k}(z_0)|\}$ converges, so $|f'(z_0)| \leq 1$; otherwise, $|f'(z_0)| > 1$ implies $|f'_{n_k}(z_0)| = |f'(z_0)|^{n_k} \to +\infty$ as $k \to +\infty$; a contradiction.

☐