Green's

thm. D — bounded with piecewise smooth ∂D.
$P, Q \in \mathcal{C}^1(D \cup \partial D)$.
Then $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

ex. D = quarter disk in 1st quadrant
$\int_{\partial D} xy\,dx$

$P(x,y) = xy$, $Q(x,y) = 0$
$\rightarrow \int_{\partial D} xy\,dx = -\iint_D x\,dx\,dy = -\iint r\cos\theta\; r\,dr\,d\theta$

$= -\int_0^{\pi/2} \cos\theta\,d\theta \int_0^1 r^2\,dr = -\frac{1}{3}$

defn. A differential $P\,dx + Q\,dy$ is exact if
$P\,dx + Q\,dy = dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$
for some function h.

Lemma. P, Q continuous, $\mathbb{C}$-valued on domain D
$\int P\,dx + Q\,dy$ is ind of path on D
$\Updownarrow$
$P\,dx + Q\,dy$ is exact
($h \in \mathcal{C}^1$ and unique up to constant)

defn. $P\,dx + Q\,dy$ is closed if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

lemma. Exact $\Rightarrow$ closed

thm. P, Q cont diff, $\mathbb{C}$-valued on star-shaped domain D and $P\,\partial x + Q\,\partial y$ closed on D
$\Rightarrow P\,\partial x + Q\,\partial y$ is exact.

ex. Consider $\frac{-y\,\partial x + x\,\partial y}{x^2+y^2}$, $x+iy \in \mathbb{C}\setminus\{0\}$

↳ closed on $\mathbb{C}\setminus\{0\}$

but not ind of path or exact on $\mathbb{C}\setminus\{0\}$.

BUT is exact on $\mathbb{C}\setminus(-\infty, 0]$

on $\mathbb{C}\setminus(-\infty,0]$,

$$\frac{-y\,\partial x + x\,\partial y}{x^2+y^2} = \partial(\operatorname{Arg} z)$$

thm. If $P\,\partial x + Q\,\partial y$ closed on domain D, continuous deformation of a path does not change value of integral.

lemma. If u(x,y) harmonic, then the differential $-\frac{\partial u}{\partial y}\partial x + \frac{\partial u}{\partial x}\partial y$

is closed

thm. u ∈ Harm(D) and $\{|z-z_0| < \rho\} \subset D$,
then $u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\,\frac{\partial\theta}{2\pi}$
$0 < r < \rho$

thm. (Strict Max princ — Real)

$u(z)$ — $\mathbb{R}$-valued harmonic function on domain D

$u(z) \leq M \;\; \forall \; z \in D$

If $u(z_0) = M$ for some $z_0 \in D$,

then $u(z) = M \;\; \forall \; z \in D$

thm. (Strict Max princ — Complex)

h — bounded, $\mathbb{C}$-valued harmonic function on domain D

$|h(z)| \leq M \;\; \forall \; z \in D$

$|h(z_0)| = M$ for some $z_0 \in D$

$\implies h(z)$ is constant on D

thm. (Max Principle)

h — $\mathbb{C}$-valued, harmonic function on bounded domain D

continuous on ∂D

If $|h(z)| \leq M \;\; \forall \; z \in \partial D$, then

$|h(z)| \leq M \;\; \forall \; z \in D$

ML thm. γ — piecewise smooth curve.

$h(z)$ — cont on γ

$$\left| \int_\gamma h(z)\, dz \right| \leq \int_\gamma |h(z)|\, |dz|$$

If $\text{length}(\gamma) = L$ and $|h(z)| \leq M$ on γ

then $\left| \int_\gamma h(z)\, dz \right| \leq ML$

thm. D – bounded domain with piecewise smooth boundary

If $f(z)$ analytic on D and extends smoothly to ∂D,

then $\oint_{\partial D} f(z)\,dz = 0$

thm. D – bounded domain with piecewise smooth bdry.

$f(z)$ – analytic on D, extends smoothly to boundary,

then $f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw$, $z \in D$

thm. D & f as above

$\Longrightarrow$ f has complex derivs of all orders

and $f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dw$, $z \in D$.

corollary. $f \in A(D) \rightarrow$ f is infinitely differentiable and $f^{(m)}(z)$ is analytic on D $\forall\, m$

thm. $f(z)$ analytic for $|z - z_0| \le \rho$

If $|f(z)| \le M$ for $|z - z_0| = \rho$,

then $|f^{(m)}(z_0)| \le \frac{m!}{\rho^m} M$

Liouville thm. f — entire
If f bounded, f is constant.

Morera thm. f — cont on domain D
If $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle $R \subset D$ with sides parallel to coordinate axes,
then f is analytic on D

thm. $h(t,z)$ — cont, $\mathbb{C}$-valued
$a \le t \le b$, $z \in D$
If $\forall$ fixed t, $h(t,z)$ is an analytic function of z, then
$H(z) = \int_a^b h(t,z)\,dz$ is analytic on D.

thm. f-continuous on domain D,
analytic on $D \setminus \mathbb{R}$
$\Rightarrow f$ is analytic on D

Goursat thm. f — complex valued funct on domain D
st $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists
$\forall z_0 \in D$,
Then f is analytic on D

defn. $\partial/\partial z = \frac{1}{2}\left[\partial/\partial x - i\,\partial/\partial y\right]$

defn $\partial/\partial \bar{z} = \frac{1}{2}\left[\partial/\partial x + i\,\partial/\partial y\right]$

thm. $f \in \mathcal{C}'(D)$

f analytic $\iff \partial f / \partial \bar{z} = 0$.

If f analytic, $f'(z) = \partial f / \partial x = -i \, \partial f / \partial y$.

thm. $f - \mathcal{C}'(D)$, $\nabla f \neq 0$, $f(z)$ conformal.

$\Rightarrow f \in A(D)$ and $f'(z) \neq 0$.

thm. D - bounded domain w/ piecewise smooth boundary

$g \in \mathcal{C}'(D \cup \partial D)$

Then $\int_{\partial D} g(z)\, dz = 2i \iint_D \frac{\partial g}{\partial \bar{z}}\, dx\, dy$

thm. D, g as above

$$g(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(z)}{z-w}\, dz - \frac{1}{\pi} \iint_D \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w}\, dx\, dy$$

712 Exam I

1 (a) $(i^i)^i$

$$i^i = e^{i[\ln|i| + i\,Arg(i) + 2i\pi k]} \qquad k \in \mathbb{Z}$$
$$= e^{-\pi/2 - 2\pi k}$$

$$(i^i)^i = e^{i\log(i^i)} = e^{i\log(e^{-\pi/2 - 2\pi k})}$$
$$= e^{i(-\frac{\pi}{2} - 2\pi k + i\,Arg(e^{-\pi/2 - 2\pi k}) + 2\pi i m)}$$
$$Arg(\dots) = 0$$
$$= e^{-\pi i/2}\, e^{-2\pi i k} \qquad e^{2\pi i k} = 1$$

(b) $\log((1-i)^{2i})$

$$(1-i)^{2i} = e^{2i\log(1-i)}$$
$$= e^{2i(\log|1-i| + i\,Arg|1-i| + 2\pi i k)}$$
$$\log\sqrt{2} \quad -\frac{i\pi}{4} \quad +2\pi i k$$
$$= e^{2i\ln\sqrt{2} + \pi/2 - 4\pi k} = e^{2i\ln\sqrt{2} + \frac{\pi}{2}}\, e^{-4\pi k}$$

$$\log(1-i)^{2i} = \ln\left(\left|e^{2i\ln\sqrt{2} + \frac{\pi}{2} - 4\pi k}\right|\right)$$
$$+ i\,Arg(\ln(2)) + 2\pi i m$$
$$Arg(\ln(2)) = 0$$
$$= \ln(e^{\pi/2 - 4\pi k}) + i(0 + 2\pi m)$$
$$= \pi/2 - 4\pi k + 2\pi i m$$
$$k, m \in \mathbb{Z}$$

2 (a) Suppose that each of f and $\bar{f}$ are analytic on domain D. Show that f must be constant.

f analytic — $f = u + iv$ st $u_x = v_y$, $u_y = -v_x$

$\bar{f}$ analytic — $\bar{f} = u - iv$ st $u_x = -v_y$, $u_y = v_x$

$\Longrightarrow u_x \equiv 0 \equiv u_y$ and $v_x \equiv 0 \equiv v_y$

u, v real valued functions with deriv $\equiv 0$ on D $\Longrightarrow$ u, v constant.

$\Rightarrow$ f is constant.

(b) Suppose that f is analytic on a domain D and that $|f|$ is constant. Show that f must be constant.

$\bar{f} = \frac{|f|^2}{f}$

If $|f| \equiv 0$, $f \equiv 0$ — done.

Assume $f \not\equiv 0$.

Then $\frac{|f|^2}{f}$ is analytic.

By (a) f is constant.

3 (a) Suppose that f has the mean value property wrt circles at $z_0 \in D$. Show f has the mean value property wrt discs at $z_0 \in D$.

$\exists\, \varepsilon > 0$ st $\mathbb{D}_\varepsilon(z_0) \subset D$ and
$$f(z_0) = \int_0^{2\pi} f(z_0 + te^{i\theta}) \frac{d\theta}{2\pi} \quad \text{for any } 0 < t < \varepsilon.$$

$$\frac{1}{\pi t^2} \iint_{\mathbb{D}_t(z_0)} f(z)\,dx\,dy = \frac{1}{\pi t^2} \int_0^t \int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta \cdot r\,dr$$

$$= \frac{2}{t^2} \int_0^t \underbrace{\left(\int_0^{2\pi} f(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \right)}_{= f(z_0)} r\,dr$$

$$= \frac{2 f(z_0)}{t^2} \int_0^t r\,dr = f(z_0)$$

(b) Suppose that f is analytic in D and has no zeroes in D.

(b.1) Show that if $|f(z)|$ attains its minimum in D, then f is constant.

Let $g(z) = \frac{1}{f(z)}$. — analytic, well defined.
$|g(z)| \leq \frac{1}{\min|f(z)|}$
~~too~~

Suppose $|f(z^*)| = \min|f(z)|$, $z^* \in D$.
Then $|g(z^*)| = \max|g(z)|$.
Since g is analytic and $|g|$ attains its max in D, by the max principle, g is constant.
$\Rightarrow f$ is constant

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(6.2). Show that if D is bounded and f is analytic in D and continuous on $\bar{D}$, then $|f|$ attains its minimum on ∂D.

Let $g(z) = 1/f(z)$ as before.
Then $|g(z)| \leq \frac{1}{\min|f(z)|}$.
By the max principle, $|g|$ attains its max on the boundary.
$\Rightarrow$ $|f|$ attains its minimum on the boundary

(6.3) Give an example to show that the hypothesis that $f(z) \neq 0 \ \forall z \in D$ is necessary

$f(z) = z^2$, $D = \mathbb{D}_1(0)$
$\min |f(z)| = 0$ achieved in D (not on ∂D)
but f not constant.

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4 (a) Use the ML theorem to find an upper bound for $\left|\int_{|z|=R} \frac{z^3}{z^5-1}\,dz\right|$, where $R>1$.

$$\left|\int_{|z|=R} \frac{z^3}{z^5-1}\,dz\right| \le \int_{|z|=R} \left|\frac{z^3}{z^5-1}\right| |dz| \le \frac{2\pi R^4}{R^5-1}$$

$$\left|\frac{z^3}{z^5-1}\right| \le \frac{R^3}{R^5-1}$$

$$|dz| = 2\pi R$$

(b) Compute the following integrals.

$\int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$: CFD, $z_0 = 0$, $f(z) = \frac{1}{(z^2-4)e^z}$

$D = \mathbb{D}_1(0)$

$m = 1$, $f'(z_0) = \frac{1!}{2\pi i} \int_{\partial D} \frac{f(z)}{z^2}\,dz$

$$f'(z) = -\left[(z^2-4)e^z\right]^{-2} \cdot \left(2z e^z + (z^2-4)e^z\right)$$

$$f'(0) = -\left[(-4)\right]^{-2} \cdot (0 + (-4))$$

$$= \frac{4}{16} = 1/4$$

$$\int = 2\pi i \cdot 1/4 = \pi i/2 \qquad \text{sign?}$$

$$\int_{|z-1|=2} \frac{dz}{z(z^2-4)e^z}$$

: $D = \mathbb{D}_2(1)$

$z_0 = 2, 0$

fine

$z(z+2)(z-2)$

$z(z-2)$

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$A(z-2) + Bz = 1$

$A + B = 0$

$-2A = 1$

$A = -1/2$

$B = 1/2$

$f(z) = \frac{1}{(z+2)e^z}$ $\quad g(z) = \frac{1}{(z+2)e^z}$

$$-\frac{1}{2}\int_{\partial D} \frac{f(z)}{z}\,dz + \frac{1}{2}\int_{\partial D} \frac{g(z)}{z-2}\,dz$$

|| ||

$2(-2\pi i\, f(0) + 2\pi i\, g(2))$

$n = 0$ ↓

$2(-2\pi i \cdot \frac{1}{2} + 2\pi i \cdot \frac{1}{4e^2})$

$-2\pi i + \frac{\pi i}{2e^2}$ sign?

712 Exam II

1 What functions are represented by the following power series and for what values of z are they valid?

$\sum_{k=1}^{\infty} k z^k$ — $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $|z| < 1$

Derivatives can be taken termwise within the radius of convergence so:

$$\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1}$$

Multiplying both sides by z:

$$\frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k z^k \qquad |z| < 1$$

$\sum_{k=1}^{\infty} k^2 z^k$ — Compute derivative of (a):

$$\frac{(1-z)^2 + z \cdot 2(1-z)}{(1-z)^4} = \sum_{k=2}^{\infty} k^2 z^{k-1}$$

Multiply by z again:

$$\frac{z(1-z)^2 + 2z^2(1-z)}{(1-z)^4} = \sum_{k=1}^{\infty} k^2 z^k , \quad |z| < 1$$

2 (a) Explain what it means for $f(z)$ to be analytic at ∞. First check that $f(z)$ is analytic at ∞. Then find the power series expansion of $f(z)$ at ∞:

$$f(z) = \frac{z^2}{z^3 - 1}$$

$f(z)$ analytic at ∞ $\Longleftrightarrow$ $g(w) = f(1/w)$ is analytic at 0.

$$g(w) = \frac{(1/w)^2}{(1/w)^3 - 1} = \frac{\frac{1}{w^2}}{\frac{1 - w^3}{w^3}} = \frac{w}{1 - w^3}$$

$$= \sum_{k=0}^{\infty} w^{3k+1} = w \sum_{k=0}^{\infty} w^{3k} \quad (\text{GS}), \quad |w| < 1$$

$\therefore$ $g(w)$ analytic at 0 so f analytic at ∞

$$f(z) = g(1/z) = \sum_{k=0}^{\infty} (1/z)^{3k+1}, \quad |z| > 1.$$

(b) Find all zeroes and the order of zeroes of $f(z)$. Explain.

$$f(z) = \frac{(z^2+1)(e^z - 1)^2}{(z^2 - 1)}$$

$z^2 + 1 = 0 \Longleftrightarrow z = \pm i$

$e^z = 1 \Longleftrightarrow z = 0, 2\pi i, \ldots, \quad z = 2k\pi i$

$$f'(z) = \frac{\left[(2z)(e^z - 1)^2 + (z^2+1)2(e^z-1)e^z\right](z^2-1) - 2z(z^2+1)(e^z-1)^2}{(z^2-1)^2}$$

$f'(\pm i) \neq 0$ so $z = \pm i$ simple zeroes.

$f'(2k\pi i) = 0 = f''(2k\pi i)$ so $z = 2k\pi i$, $k = 0, 1, \ldots$ is a double zero for each k.

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(c) Find and classify all singularities of $f(z)$ (including singularities at ∞). Explain.

$$f(z) = \frac{e^{3z} - 1}{z^2}$$

$f(z)$ has a pole at 0:

$$e^{3z} = \sum_{k=0}^{\infty} \frac{(3z)^k}{k!}$$

$$e^{3z} - 1 = \sum_{k=1}^{\infty} \frac{(3z)^k}{k!}$$

$$\frac{e^{3z}-1}{z^2} = \sum_{k=1}^{\infty} \frac{3^k z^{k-2}}{k!} \qquad k-2 < 0 \text{ only for } k=1$$

$\Rightarrow$ 0 is a simple pole of the function.

$f(1/z) = (e^{3/z} - 1) z^2$

$e^{3/z}$ has an essential singularity at 0.

$\Rightarrow$ $f(z)$ has an essential singularity at ∞.

(d) Suppose that $f \in A(D_1(0))$ and that $f(1/n) = 1/n$, $n = 1, 2, 3, 4, \ldots$ What can you say about f?

Consider $f(z) - z$ on $D_1(0)$.

This function is clearly analytic and

$$f(1/n) - 1/n = 0 \quad \forall n$$

But zeros of analytic functions are isolated and $\{1/n\}$ has a limit point in the disc.

$\Rightarrow$ $f(z) - z \equiv 0$ everywhere.

$\Rightarrow$ $f(z) = z$.

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3 Let $f(z) = \frac{1}{z^2 + z}$

(a) First find all annuli centered at $z_0 = 0$ where f is analytic.
Then compute the Laurent decomposition for f relative to each such annulus.

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} + \frac{-1}{z+1}$$

$$1 = A(z+1) + Bz$$
$$A + B = 0 \Rightarrow A = -B$$
$$A = 1 \Rightarrow B = -1$$

$A_{0,1}(0)$
$A_{1,\infty}(0)$

$0 < |z| < 1$: $\frac{1}{z} \in A(|z| > 0)$, vanishes at ∞
$-\frac{1}{z+1} \in A(|z| < 1)$

$1 < |z| < \infty$: $f(z) \in A(A_{1,\infty}(0))$
and vanishes at ∞

(b) Find the Laurent expansion centered at $z_0 = 1$ and converges at $z = -\frac{1}{2}$.

$1 < |z-1| < 2$

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} + \frac{-1}{z+1}$$

$$\frac{-1}{z+1} = \frac{-1}{z-1+2} = \frac{-1}{2-(1-z)} \qquad \frac{-1}{1-\left(\frac{1-z}{2}\right)}$$

$$= \frac{-1}{\frac{z-1}{2}+1} = \frac{-1}{\frac{2}{(1-z)}-1} = \frac{1}{1-\frac{2}{1-z}} \leftarrow |\,| < 1$$

GS = oops

$$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{1+\frac{1}{z-1}} = \frac{1}{1-\left(\frac{-1}{z-1}\right)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^k} \quad \text{ok}$$

$$-\frac{1}{z+1} = -\sum_{k=0}^{\infty} \left(\frac{1-z}{2}\right)^k$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^k} + -\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k (1-z)^k$$

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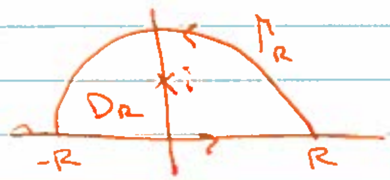
4 Evaluate : $I = \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2} dx$

Let D_R be the upper semidisk of radius R centered at O.



$$\int_{\partial D_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz = 2\pi i \operatorname{Res}\left[\frac{z^3 e^{iz}}{(z^2+1)^2}, \pm i\right]$$

$$\operatorname{Res}\left[\frac{z^3 e^{iz}}{(z^2+1)^2}, i\right] =$$

$$\lim_{z \to i} \frac{\partial}{\partial z}\left(\frac{z^3 e^{iz}}{(z+i)^2}\right) = \frac{1}{4e}$$ ← Calculations on test

$$\int_{\partial D_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz = \frac{\pi i}{2e}$$

$$\left|\int_{\Gamma_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz\right| \le \frac{R^3}{R^4} \int_{\Gamma_R} |e^{iz}||dz| \le \frac{\pi}{R}$$

↑ Jordan's Lemma.

" $\longrightarrow 0$ as $R \to \infty$

$$\int_{-R}^{R} \frac{x^3 e^{ix}}{(x^2+1)^2} dx \longrightarrow \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)^2} dx = \frac{\pi i}{2e}$$

To get $\sin(x)$ take imaginary part.

$I = \frac{\pi}{2e}$.

[illegible]

(b) Evaluate : $I = \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$

Replace $\sin\theta$ by $\frac{z - \frac{1}{z}}{2i}$ and $d\theta$ by $\frac{dz}{iz}$

$$I = \frac{1}{i} \int_{|z|=1} \frac{4z}{6z^2 - z^4 - 1}\, dz$$

Apply residue theorem : poles are $\pm\sqrt{3-2\sqrt{2}}$

$I = \pi\sqrt{2}$

$z = e^{i\theta}$

$\sin\theta = \frac{z - \frac{1}{z}}{2i}$

$\cos\theta = \frac{z + \frac{1}{z}}{2}$

5 Suppose f is continuous on a domain D. Define $g(z) = f^3(z)$, $z \in D$, and suppose g is analytic in D.

(a) Show that f is analytic in D.

$g'(z) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}$ exists for all $z \in D$.

$$g'(z) = \lim_{z \to z_0} \frac{f^3(z) - f^3(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \cdot \left(f^2(z_0) + f(z_0)f(z) + f^2(z) \right) \right]$$

$$= \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \cdot 3f^2(z_0)$$

since f is continuous.

$\Rightarrow \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists $\forall z \in D$.

(b) Can you give an example f and domain D that show that the hypothesis that f be continuous on D is necessary?

$f = \sqrt[3]{z}$, $D = \Delta$?

$g(z) = z \in A(\Delta)$

But f is not continuous (I think).

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712 Final Exam

1 (a) Suppose that both $f(z)$ and $h(z) = f(\bar{z})$ are analytic on $\mathbb{D}_1(0)$. Show that f must be constant.

$f(z)$, $h(z)$ analytic $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} \equiv 0 \equiv \frac{\partial h}{\partial \bar{z}}$

But $\frac{\partial h}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{\bar{z}}} = \frac{\partial f}{\partial z}$..

$\Rightarrow \frac{\partial f}{\partial z} \equiv 0$.

$\Rightarrow f$ is constant on $\mathbb{D}_1(0)$.

(1b) Suppose that u is harmonic on the entire complex plane and that $|u(z)| \leq M$ for any $z \in \mathbb{C}$. Show that u must be constant.

$u \in \text{Harm}(\mathbb{C}) \Rightarrow u$ has a harmonic conjugate v st $f = u + iv$ is entire.

Consider $e^{f(z)}$.

f entire $\Rightarrow e^{f(z)}$ entire.

$|e^{f(z)}| = |e^u| \leq e^M \quad \forall z$.

By Liouville's theorem, $e^{f(z)}$ is constant.

$\Rightarrow f$ is constant

$\Rightarrow u$ constant

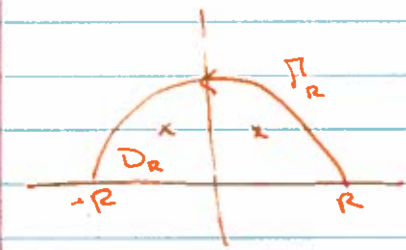
2 Compute $I = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$

$x^4 + 1 = (x^2 - i)(x^2 + i)$ $\quad x = \pm\sqrt{-4(1)(i)}$ ~

$x^2 = i,\ x^2 = -i$

$x_1 = \frac{1}{\sqrt{2}}(-1+i)$

$x_2 = \frac{1}{\sqrt{2}}(1+i)$



$$\int_{\partial D_R} \frac{z^2}{z^4+1}\,dz = \int_{\Gamma_R} \frac{z^2}{z^4+1}\,dz + \int_{-R}^{R} \frac{x^2}{x^4+1}\,dz$$

By the Residue theorem.

$$\int_{\partial D_R} \frac{z^2}{z^4+1}\,dz = 2\pi i \sum \text{Res}\left[\frac{z^2}{z^4+1}, z_i\right]$$

$$\text{Res}\left[\frac{f}{g}, a\right] = \lim_{z \to a} \frac{f}{g'}$$

$$\text{Res}\left[\frac{z^2}{z^4+1}, \tfrac{1}{\sqrt{2}}(-1+i)\right] = \lim_{z \to z_1} \frac{z^2}{4z^3} = \lim_{z \to z_1} \frac{1}{4z}$$

$$= \frac{1}{\frac{4}{\sqrt{2}}(-1+i)}$$

$$= \frac{\sqrt{2}}{4(-1+i)} \frac{(-1-i)}{(-1-i)}$$

$$\frac{\sqrt{2}(-1-i)}{4(\sqrt{2})} = \frac{-1-i}{4}$$

$$\text{Res}\left[\frac{z^2}{z^4+1}, \tfrac{1}{\sqrt{2}}(1+i)\right] = \lim_{z \to z_2} \frac{1}{4z}$$

$$= \frac{1}{\frac{4}{\sqrt{2}}(1+i)} = \frac{\sqrt{2}}{4(1+i)} \frac{(1-i)}{(1-i)}$$

$$= \frac{\sqrt{2}(1-i)}{\sqrt{2} \cdot 4} = \frac{1-i}{4}$$

$2\pi i \left(\frac{-1-i+1-i}{4}\right) = 2\pi i \left(\frac{-2i}{4}\right) = \pi$? ← should be $\frac{\pi}{\sqrt{2}}$?

Use ML to get rid of $\int_{\Gamma_R}$ part

3 (a) Show that the equation

$$2e^z - 7z + 1 = 0$$

has exactly one solution in $\mathbb{D}_1(0)$. Also, show that this solution is real. Explain and show work.

On $|z|=1$, $|2e^z| \leq 2e < 6$

$|-7z+1| \geq 6$

By Rouche's theorem, since $-7z+1$ has a zero in $\mathbb{D}_1(0)$, $2e^z - 7z + 1$ has exactly one zero in $\mathbb{D}_1(0)$ as well.

Furthermore, the fact that the root of $-7z+1$ lies in the interval $0 < x < 1$ implies that the root of $2e^z - 7z + 1$ is real as well

horribly written

(b) How many roots does $p(z) = z^9 + z^5 - 8z^3 + 2z + 1$ have inside the annulus $A_{1,2}(0)$? Explain and show work.

In $|z|<1$, Big $= -8z^3$ and little $= z^9 + z^5 + 2z + 1$

On $|z|=1$, $|-8z^3| = 8$ and $|z^9 + z^5 + 2z + 1| = 5$

$-8z^3$ has 3 zeroes in $|z|<1$, so $p(z)$ does as well.

In $|z|<2$, Big $= z^9$ and little $= z^5 - 8z^3 + 2z + 1$

On $|z|=2$, $|z^9| = 512$, and $|z^5 - 8z^3 + 2z + 1| \leq 101$?

z^9 has 9 zeroes in $|z|<2$, so $p(z)$ does as well.

$\Rightarrow$ The # of roots in $A_{1,2}(0)$ is $9-3=6$.

4 (a) Let $S(0)$ denote a square with center at the origin. Suppose that $F: \mathbb{D}_1(0) \to S(0)$ is analytic, one-to-one, onto, and $F(0)=0$. Show that $F(iz) = iF(z)$ for any $z \in \mathbb{D}_1(0)$. Explain and show work.

Let M = side length of $S(0)$.

$\Rightarrow |F(z)| \le \frac{\sqrt{2}}{2} M \quad \forall\ |z| < 1$.

By Schwarz lemma, $|F(z)| \le \frac{\sqrt{2}}{2} M |z|$.

F biholomorphic $\Rightarrow$ F conformal.

$\Rightarrow$ F extends continuously to the boundary (boundary is Jordan curve for both) and furthermore $z \in \partial\mathbb{D}_1(0) \mapsto w \in \partial S(0)$.

For $|z| = 1$, $|F(z)| = \frac{\sqrt{2}}{2} M$. ← not 100% sure

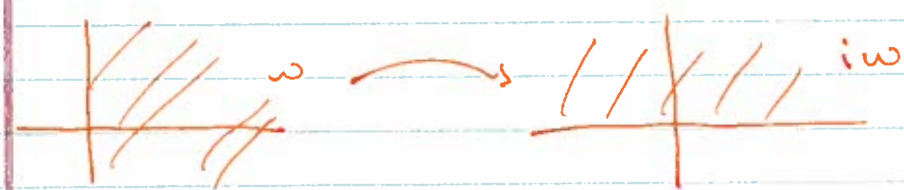
~~[illegible]~~

$\Rightarrow F(z) = \frac{\sqrt{2}}{2} M a z, \quad |a| = 1$

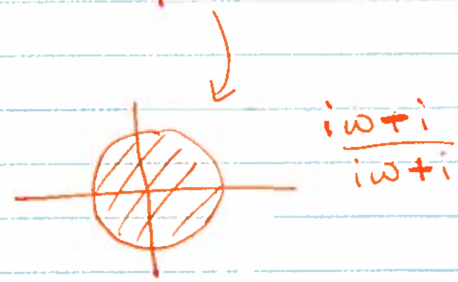
$F(iz) = i\frac{\sqrt{2}}{2} M a z = \frac{\sqrt{2}}{2} M a (iz) = iF(z)$

4 (b) $\mathcal{F} = \{f : \mathbb{D}_1(0) \longrightarrow \{\mathrm{Re}(w) > 0\} \mid f \in A,\ f(0) = 1\}$.
Let $M = \sup\{|f'(0)| \mid f \in \mathcal{F}\}$.
Compute M. Also: can you find $f \in \mathcal{F}$ such that $|f'(0)| = M$? Explain and show work.

Following hint, find analytic, 1-1 map of $\{\mathrm{Re}\, w > 0\}$ onto $\mathbb{D}_1(0)$.



$h(w) = \dfrac{w-1}{w+1}$



h is analytic and 1-1 (check)
Consider $h \circ f$ where $f \in \mathcal{F}$.
$h \circ f : \mathbb{D}_1(0) \longrightarrow \mathbb{D}_1(0)$ is analytic and
$h \circ f(0) = h(1) = 0$.
By the Schwarz lemma,
$$|(h \circ f)'(0)| \le 1.$$
$(h \circ f)'(z) = h'(f(z)) \cdot f'(z)$.
$h'(w) = \dfrac{w+1 - (w-1)}{(w+1)^2} = \dfrac{2}{(w+1)^2}$ (note: $h' \neq 0$)

$h'(f(0)) = h'(1) = \dfrac{2}{2^2} = \dfrac{1}{2}$.

$\Rightarrow |\tfrac{1}{2} \cdot f'(0)| \le 1 \Rightarrow |f'(0)| \le 2$
So $M = 2$.
$f = h^{-1}$ should work
$f(z) = \dfrac{z+1}{1-z}$, $f'(z) = \dfrac{2}{(1-z)^2}$, $f'(0) = 2$ ✓

[illegible]

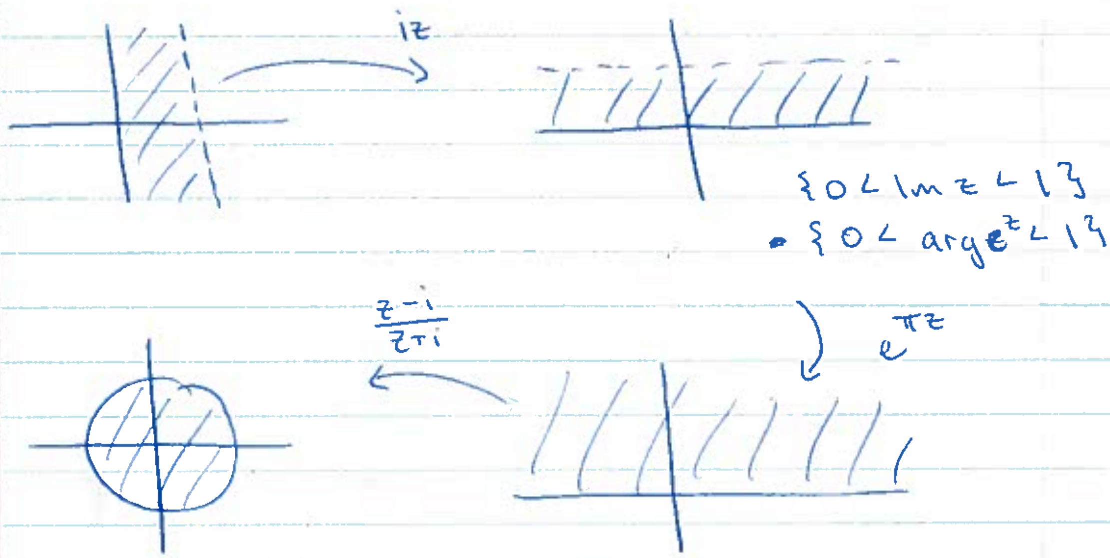
[illegible]

[illegible]

[illegible]

January 2013, Complex

1 Find a conformal map from the strip $\{0 < \operatorname{Re} z < 1\}$ onto Δ.

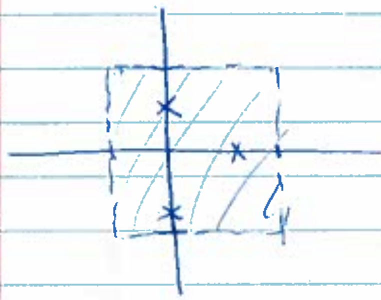


$$\frac{e^{\pi i z} - i}{e^{\pi i z} + i}$$

2 Let C denote the positively oriented boundary of the domain

$$D = \{z \in \mathbb{C} : -\tfrac{1}{2} < \operatorname{Re} z < 2, |\operatorname{Im} z| < 2\}$$

Find $\int_C \frac{z^n}{z^4-1}\,dz$, where $n \geq 0$ is an integer. Write your answer in algebraic form $a+bi$.



$$z^4 - 1 = (z^2+1)(z^2-1)$$
$$= (z+i)(z-i)(z+1)(z-1)$$

poles in D = $\pm i, 1$

$$\int_C \frac{z^n}{z^4-1}\,dz = 2\pi i \sum \text{Res}$$

$$\text{Res}\left[\frac{z^n}{z^4-1}, -i\right] = \lim_{z\to -i} \frac{z^n}{(z-i)(z^2-1)} = \frac{(-i)^n}{(-2i)(-2)} = \frac{(-1)^n i^n}{4i}$$

$$\text{Res}\left[\frac{z^n}{z^4-1}, 1\right] = \lim_{z\to 1} \frac{z^n}{(z+i)(z-i)(z+1)} = \frac{1}{2\cdot 2} = 1/4$$

$$\text{Res}\left[\frac{z^n}{z^4-1}, i\right] = \lim_{z\to i} \frac{z^n}{(z+i)(z^2-1)} = \frac{i^n}{(2i)(-2)} = \frac{i^n}{-4i}$$

$$2\pi i\left(\frac{(-1)^n i^n}{4i} + \frac{1}{4} + \frac{i^n}{-4i}\right)$$
$$= \frac{(-1)^n \pi i^n}{2} + \frac{\pi i}{2} - \frac{\pi i^n}{2}$$

$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$

If $n = 0 \bmod 4$, $\frac{\pi}{2} + \frac{\pi i}{2} - \frac{\pi}{2} = \frac{\pi i}{2}$

If $n = 1 \bmod 4$, $-\pi i/2 + \frac{\pi i}{2} + \pi i/2 = \pi i/2$

If $n = 2 \bmod 4$, $-\pi/2 + \pi i/2 + \pi/2 = \pi i/2$

If $n = 3 \bmod 4$, $-\frac{\pi(-i)}{2} + \frac{\pi i}{2} - \frac{\pi(-i)}{2}$

$\pi i/2 + \pi i/2 + \pi i/2 = \frac{3\pi i}{2}$

3 Is there an entire function $f(z)$ such that $e^{f(z)}$ has a pole at ∞?

No.
Assume there is.
Then $e^{f(z)}$ is entire.
$e^{f(z)}$ has a pole at $\infty \Longleftrightarrow e^{-f(z)}$ has a zero at ∞.
But $e^{-f(z)}$ is also entire.
(*) Entire functions have essential singularities or poles at ∞.
This is a contradiction.

(*) $g(z)$ entire $\Rightarrow g(z) = \sum_{k=0}^{\infty} a_k z^k$
If $a_k = 0$ for $k \gneq N$, g has a pole of order N at ∞.
o/w it has an essential singularity (or is constant)

entire $f(z)$ st $\|f(z)\| \leq M\|z\|^n$ $\forall z$ with $\|z\| \geq R$ is a polynomial of degree at most n.
$n=0$ — Liouville's theorem.

4. Suppose that f, g are holomorphic functions in Δ so that $f(0) = g(0) = 1$ and

$$(f'g - fg')(1/n) = 0 \quad \text{for all integers } n \geq 2.$$

Show that $f = g$ on Δ.

f, g holomorphic $\Rightarrow f', g'$ holomorphic
$\Rightarrow f'g, fg'$ holomorphic
$\Rightarrow f'g - fg'$ holomorphic.

Since $f'g - fg' = 0$ on a set with an accumulation point, $f'g - fg' \equiv 0$ in Δ.

$\Rightarrow (f/g)' = \dfrac{f'g - fg'}{g^2} \equiv 0$ in Δ $\qquad g \neq 0$?

$\Rightarrow f/g$ is constant in Δ

$\dfrac{f(0)}{g(0)} = \dfrac{1}{1} \Rightarrow \dfrac{f(z)}{g(z)} = 1 \quad \forall z$

Finally, $f(z) = g(z) \quad \forall z$.

Complex, August 2014

1. Find a conformal map from the half-disc

$$D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z > 0\} \text{ onto } \Delta$$

Rotate D to the upper half disc: $z \longmapsto iz$

Map the half disc to the 1st quadrant:

$$1 \to \infty \qquad \tilde{z} \longmapsto \frac{\tilde{z}+1}{\tilde{z}-1} = \frac{(iz)+1}{(iz)-1} = \frac{i(z-i)}{i(z+i)}$$

$$-1 \to 0$$

Map the 1st quadrant to the upper half plane:

$$w \longmapsto w^2 = \frac{(z-i)^2}{(z+i)^2}$$

Map the upper half plane to the disc Δ:

$$\tilde{w} \longmapsto \frac{\tilde{w}-i}{\tilde{w}+i}$$

Composing these maps gives:

$$z \longmapsto \frac{\left(\frac{z-i}{z+i}\right)^2 - i}{\left(\frac{z-i}{z+i}\right)^2 + i}$$

2 Let D be a domain in $\mathbb{C}$ containing 0 and
$f: D \to \mathbb{R}$ be a continuous function such that
$f(0) = 0$ and $\int_{\partial R} f(z)\,dz = 0$ ________.

Prove that $f(z) = 0$ for every $z \in D$.

$\int_{\partial R} f(z)\,dz = 0 \;\; \forall R \;\; \to \;\; f$ is holomorphic (Morera)

f holomorphic and real-valued $\Rightarrow f$ is constant.
$f(0) = 0 \Rightarrow f(z) = 0 \;\; \forall z \in \mathbb{C}$.

3 Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in D$ and $f: D \to D$ be a holomorphic function such that $f(z_0) = z_0$. Show that $|f'(z_0)| \leq 1$.

D bounded $\iff$ $D \subset \{|z| \leq R\}$ for some $R > 0$.

Define $f_n = \underbrace{f \circ \cdots \circ f}_{n\text{-times}}$

$f_n : D \to D$, holomorphic, and $f_n(z_0) = z_0 \ \forall n$.

Furthermore $f_n'(z_0) = (f'(z_0))^n$.

$f_n(D) \subseteq D \implies |f_n(z)| \leq R \ \forall n$.

$\implies |f_n'(z)| \leq M$ for some $M \ \forall n$ by the Cauchy estimates.

$|f_n'(z_0)| = |f'(z_0)|^n \leq M$

Suppose $|f'(z_0)| > 1$.

Then $\lim_{n \to \infty} |f'(z_0)|^n = \infty$.

Therefore, $|f'(z_0)| \leq 1$

$f_n \to z_0$ important in any way?

4. Let $f_n : \Delta \to \Delta$, $n \geq 1$, be a sequence of holomorphic functions such that f_n has a zero of order m_n at 0, where $\lim_{n\to\infty} m_n = \infty$.
Show that $\{f_n\}$ converges locally uniformly to zero on Δ.

f_n holomorphic with a zero of order m_n at 0

$$\implies f_n(z) = \sum_{k=m_n}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \qquad (1)$$

By Schwarz lemma, $|f_n(z)| \leq |z| \ \forall z \in \Delta$
and $|f_n'(0)| \leq 1 \ \forall n$.

From (1) it is clear that $f_n \to 0$ pointwise as $n \to \infty$.

$f_n(z) = z^{m_n} g_n(z)$, $g_n : \Delta \to \Delta$ holomorphic and $g_n(0) \neq 0$.

$|f_n(z)| = |z^{m_n}||g(z)| \leq |z|^{m_n}$ $\qquad |g(z)| \leq 1$

$m_n \to \infty$ and $|z| < 1 \implies |z|^{m_n} \to 0$.

$\forall \varepsilon > 0 \ \exists N$ st $n \geq N \implies |z|^{m_n} < \varepsilon$

$\implies |f_n(z)| < \varepsilon$

This is just uniform — not sure

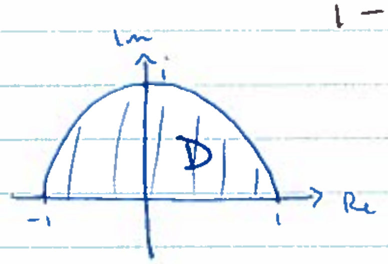
Complex, August 2015

1 Find the image of the half disc

$$D = \{z : |z| < 1, \operatorname{Im} z > 0\}$$

by the Möbius map

$$f(z) = \frac{1+z}{1-z}$$



Consider the image of the boundary of D under f.

$-1 < x < 1,\ y = 0$: $f(x) = \dfrac{1+x}{1-x}$

$f(-1) = 0$

$f(1) = \infty$

For all other x in this interval $f(x)$ is positive, real.

$z = e^{i\theta},\ 0 < \theta < \pi$: $f(z) = \dfrac{1+e^{i\theta}}{1-e^{i\theta}}$

$f(i) = \dfrac{1+i}{1-i} = i$

$$|f(z)| = \left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right| \leq \frac{1+r}{1-r}$$

not helpful

$z = \frac{1}{2} + \frac{i}{2}$: $f(z) = 1 + 2i$

$z = -\frac{1}{2} + \frac{i}{2}$: $f(z) = \frac{1}{5} + \frac{2}{5}i$

$\operatorname{Im} f(z)$ = first quadrant

2 Let f be a holomorphic function on $\Delta \setminus \{0\}$ such that $|f(z)| > 1$ for all $z \in \Delta \setminus \{0\}$. Show that 0 is an isolated singularity of f which is either removable or a pole.

Suppose $z=0$ is an essential singularity of $f(z)$. Then $\exists$ sequence $z_n \to 0$ such that

$f(z_n) \to 0$ (Casorati – Weierstrass).

But $|f(z_n)| > 1$ $\forall$ n.

This is a contradiction.

Therefore $z=0$ is a removable singularity or a pole.

Note: It is an isolated singularity by definition since f is holomorphic on the punctured disk centered at $z=0$.

Schwarz lemma!

3 Let $D \subsetneq \mathbb{C}$ be a simply connected domain, $z_0 \in D$, and $f: D \to \Delta$ be a conformal map such that $f(z_0) = 0$.
If $g: D \to \Delta$ is a holomorphic map such that $g(z_0) = 0$, show that $|g'(z_0)| \le |f'(z_0)|$, and that equality holds iff g is a conformal map.

f conformal $\iff$ f holomorphic and $f'(z) \neq 0 \ \forall z \in D$
Furthermore, f is 1-1 onto Δ.
Therefore f^{-1} is defined and by the inverse function theorem $(f^{-1})'(f(z)) = \dfrac{1}{f'(z)}$. ($f^{-1}$ hol.)

Let $h(w) = g \circ f^{-1}(w)$, $h: \Delta \to \Delta$.
As a composition of hol. functions, h is hol, and $h(0) = g \circ f^{-1}(0) = g(z_0) = 0$.
By the Schwarz lemma,

$$|h'(0)| = \left| \frac{g'(z_0)}{f'(z_0)} \right| \le 1$$

$\Rightarrow |g'(z_0)| \le |f'(z_0)|$.

If g is conformal, apply the same argument to get $|g'(z_0)| \ge |f'(z_0)|$ so equality holds.

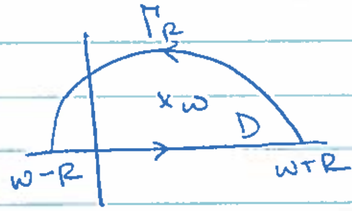
Suppose $|g'(z_0)| = |f'(z_0)|$, ie. $|h'(0)| = 1$.
Again by the Schwarz lemma, $h(w) = aw$, where $|a| = 1$

$\Rightarrow h'(w) = a = \dfrac{g'(f^{-1}(f(z)))}{f'(z)}$, where $w = f(z)$.

$\Rightarrow g'(z) = a f'(z) \quad \forall z \in D$.
$\Rightarrow g'(z) \neq 0 \quad \forall z \in D$ since $f'(z) \neq 0$.
Since g is holomorphic with nonzero derivative, g is conformal.

4 Compute $F(\omega) = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2}\,dx$, where $\omega \in \mathbb{C}\setminus\mathbb{R}$.
Hint: Consider $\operatorname{Im}\omega > 0$ and $\operatorname{Im}\omega < 0$ separately.

$\operatorname{Im}\omega > 0$:



$$\int_{\partial D} \frac{e^{iz}}{(z-\omega)^2}\,dz = 2\pi i \operatorname{Res}\left[\frac{e^{iz}}{(z-\omega)^2}, \omega\right] = 2\pi i \cdot i e^{i\omega} = -2\pi e^{i\omega}$$

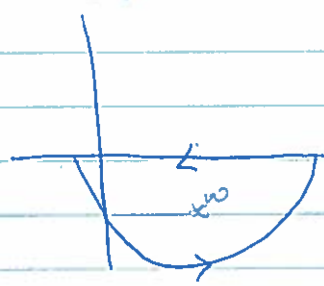
$$\int_{\partial D} = \int_{\Gamma_R} + \int_{\omega-R}^{\omega+R}$$

$$\left|\int_{\Gamma_R} \frac{e^{iz}}{(z-\omega)^2}\,dz\right| \le \frac{\pi R}{R^2} = \frac{\pi}{R} \longrightarrow 0 \text{ as } R \to \infty$$

$$\int_{\omega+R}^{\omega-R} \frac{e^{ix}}{(x-\omega)^2}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2}\,dx \text{ as } R \to \infty.$$

So $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2}\,dx = -2\pi e^{i\omega}$

$\operatorname{Im}\omega < 0$:



Sign will be reversed

Complex, January 2015

1 Show that $\sum \frac{1}{n} z^{n!}$ converges absolutely for $|z| < 1$. Also show that there are infinitely many z with $|z| = 1$ for which the series diverges.

$|\sum \frac{1}{n} z^{n!}| \le \sum \frac{1}{n} |z|^{n!}$

$\le \sum \frac{1}{n} |z|^n$ when $|z| < 1$

lim sup $\sqrt[n]{\frac{1}{n}|z|^n} = |z| < 1 \Rightarrow$ convergence

Let $z = e^{2\pi i \frac{k}{m}}$

Then $\sum_{n=m}^{\infty} \frac{1}{n} z^{n!} = \sum_{n=m}^{\infty} \frac{1}{n}$, which diverges.

2 Let $f(z)$ be holomorphic on $\mathbb{C}$ except for poles.
At ∞ assume that f has a removable singularity or a pole.

(a) Show that f has finitely many poles on $\mathbb{C} \cup \{\infty\}$.

Suppose not.
Since $\mathbb{C} \cup \{\infty\}$ is compact $\Rightarrow$ the set of poles has an accumulation point in $\mathbb{C} \cup \{\infty\}$. (at ∞)
But poles are isolated points.
This is a contradiction.

(b) Let $p_j(z)$ be the principal part of f at the jth pole, $1 \le j \le N$. Show that

$$f(z) - \sum_{j=1}^{N} p_j(z)$$

is constant.

$f(z) - \sum_{j=1}^{N} p_j(z)$ is entire:

If z_i is a pole of f,

$$f(z) - p_i(z) - \sum_{j \neq i}^{N} p_j(z)$$

$$= \underbrace{\sum_{k=0}^{\infty} a_k (z - z_i)^k}_{\text{analytic}} - \underbrace{\sum p_j(z)}_{\text{analytic at } z_i}$$

Because the singularity at ∞ is also removed, we see that $f(z) - \sum p_j(z)$ is bounded for $|z| > R$ (In fact I think it must $\to 0$)
But $f(z) - \sum p_j(z)$ is clearly bounded on compact sets $|z| \le R$.
$\Rightarrow f(z) - \sum p_j(z)$ is bounded everywhere
By Liouville's theorem, it is constant.

3 Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle $|z|=1$. Assume there is an entire function g st $f(z)=g(z)$ for $|z|=1$. Prove that $f=g$.

On [illegible] August 2016.

4 Let f_n be analytic in the unit disc D and have positive real part: $Re(f_n(z)) > 0$. Assume that f_n converges pointwise on D to a function f having $R(f(z)) \leq 0$ on D. Prove that f is constant.

Let $f_n = u_n + iv_n$

f_n converges pointwise $\Leftrightarrow$ u_n and v_n do.

$u_n(x,y) \longrightarrow u(x,y)$ where $u_n(x,y) \geq 0$ and $u(x,y) \leq 0$.

$\Rightarrow u(x,y) = 0$ for each x,y

$\Rightarrow f$ is purely imaginary.

By Osgood's theorem, $\exists$ an open dense subset V of D st f is holomorphic on V.

f holomorphic and purely imaginary

$\Longrightarrow f$ is constant on V.

Solutions use Montel's theorem and Open mapping theorem

$e^{-f_n} = h_n$ — normal family

$h_n : D \to D$ analytic

Some subsequence converges uniformly on compact subsets to $h = e^{-f}$ — analytic

$|h(z)| = 1$ since $\Re f = 0$

$\Rightarrow h$ is constant by the open mapping theorem

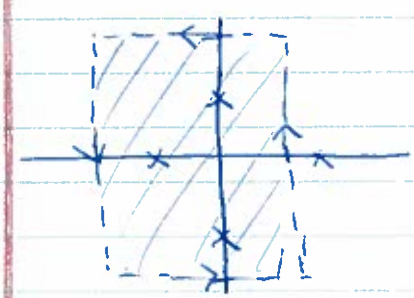
$\Rightarrow f$ constant

Open mapping thm: U domain, $f : U \to \mathbb{C}$ nonconst. and hol, then f is an open map

Complex, August 2016

1 Let C denote the boundary of the domain
$$D = \{z \in \mathbb{C} : -2 < \text{Re}\, z < \tfrac{1}{2},\ |\text{Im}\, z| < 2\}.$$
Find $I = \int_C \frac{z^n}{z^4 - 1}\, dz$ where $n \geq 0$ is an integer.

Write your answer in algebraic form $a + ib$.



$$\frac{z^n}{z^4-1} = \frac{z^n}{(z+1)(z-1)(z+i)(z-i)}$$

By the Residue theorem
$$\int_C \frac{z^n}{z^4-1}\, dz = 2\pi i \sum \text{Res}$$

$$\text{Res}\left[\frac{z^n}{z^4-1}, i\right] = \lim_{z \to i} \frac{z^n}{(z^2-1)(z+i)} = -\frac{i^n}{4i}$$

$$\text{Res}\left[\frac{z^n}{z^4-1}, -i\right] = \lim_{z \to -i} \frac{z^n}{(z^2-1)(z-i)} = \frac{(-1)^n i^n}{4i}$$

$$\text{Res}\left[\frac{z^n}{z^4-1}, -1\right] = \lim_{z \to -1} \frac{z^n}{(z+1)(z^2+1)} = -\frac{(-1)^n}{4}$$

$n = 0 \mod 4$: $I = -\pi i/2$

$n = 1 \mod 4$: $I = -\pi i/2$

$n = 2 \mod 4$: $I = -\pi i/2$

$n = 3 \mod 4$: $I = 3\pi i/2$.

2. Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle $\{|z|=1\}$. Suppose that there is an entire function g such that $f(z)=g(z)$ for $|z|=1$. Prove that $f=g$.

Note: can't apply Cauchy integral formula directly since we don't know f extends smoothly just that it's continuous (might be able to show though)

Take $r_n \nearrow 1$.

$f \in A(\Delta)$ so $f(z) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{f(r_n e^{i\theta})\, r_n e^{i\theta}}{r_n e^{i\theta} - z}\, d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta}-z}\, d\theta$ BCT, f cont.

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{g(e^{i\theta})e^{i\theta}}{e^{i\theta}-z}\, d\theta = g(z)$ $|z|<1$

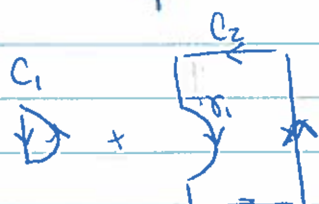
since g entire

$\Rightarrow f = g$ for $|z| \leq 1$.

Let R be a rectangle in $\mathbb{C}$ with sides parallel to the coordinate axes which intersects Δ.

eg.

$\int_{\partial R} f\, dz$ can be split into closed paths in $|z| \leq 1$ and $|z| \geq 1$ by including the intersected arc of $|z|=1$.



$\int_{C_1} f(z)\,dz = \int_{C_1} g(z)\,dz = 0$ by Cauchy

$C_2^n = \gamma(1+1/n)\gamma_1$ + rest of rectangle

$\int_{C_2} f(z)\,dz = \lim_{n\to\infty} \int_{C_2^n} f(z)\,dz = 0$ by Cauchy since f analytic outside of Δ

$\Rightarrow$ f is analytic in $\mathbb{C}$ by Morera.
Since $f = g$ on Δ, by the uniqueness principle,
$f \equiv g$ on $\mathbb{C}$.

3 Let S be a square with center at the origin. Suppose that $F: \Delta \to S$ is analytic, one-to-one, and onto, and furthermore, that $F(0) = 0$. Show that $F(iz) = iF(z)$ for all $z \in \Delta$.

On final

4 Prove that there are no nonconstant polynomials of the form $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ that satisfy $|p(z)| < 1$ when $|z| = 1$.

Suppose p is such a polynomial.
Then $p^{(n)}(z) = n!$ for all z.
However, by the Cauchy estimates $|z| < 1$

$$|p^{(n)}(0)| < n!$$

This is a contradiction.
$\Longrightarrow$ p must be constant.