

Green's

thm.  $D$  - bounded with piecewise smooth  $\partial D$ .

$P, Q \in C^1(D \cup \partial D)$ .

$$\text{Then } \int_{\partial D} P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

ex.  $D =$  quarter disk in 1st quadrant

$$\int_{\partial D} xy dx$$

$P(x,y) = xy, Q(x,y) = 0$

$$\rightarrow \int_{\partial D} xy dx = - \iint_D x dx dy = - \iint_D r \cos \theta r dr d\theta$$

$$= - \int_0^{\pi/2} \cos \theta d\theta \int_0^1 r^2 dr = -\frac{1}{3}$$

defn. A differential  $P dx + Q dy$  is exact if

$$P dx + Q dy = dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

for some function  $h$ .

lemma.  $P, Q$  continuous,  $C$ -valued on domain  $D$

$\int P dx + Q dy$  is ind of path on  $D$

$\Leftrightarrow$

$P dx + Q dy$  is exact

( $h \in C^1$  and unique up to constant)

defn.  $P dx + Q dy$  is closed if  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

lemma. Exact  $\Rightarrow$  closed

thm.  $P, Q$  cont diff,  $\mathbb{C}$ -valued on star-shaped domain  $D$  and  $Pdx + Qdy$  closed on  $D$   
 $\Rightarrow Pdx + Qdy$  is exact.

ex. Consider  $\frac{-y dx + x dy}{x^2 + y^2}$ ,  $x + iy \in \mathbb{C} \setminus \{0\}$

$\hookrightarrow$  closed on  $\mathbb{C} \setminus \{0\}$

but not ind of path or exact on  $\mathbb{C} \setminus \{0\}$ .

BUT is exact on  $\mathbb{C} \setminus (-\infty, 0]$   
 on  $\mathbb{C} \setminus (-\infty, 0]$ ,

$$\frac{-y dx + x dy}{x^2 + y^2} = d(\text{Arg } z)$$

thm. If  $Pdx + Qdy$  closed on domain  $D$ ,  
 continuous deformation of a path  
 does not change value of integral.

lemma. If  $u(x, y)$  harmonic, then the  
 differential  $-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

is closed

thm.  $u \in \text{Harm}(D)$  and  $\{|z - z_0| < \rho\} \subset D$ ,  
 $\text{mean } u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{\partial \theta}{2\pi}$   
 $0 < r < \rho$

thm. (Strict Max princ - Real)

$u(z)$  -  $\mathbb{R}$ -valued harmonic function  
on domain  $D$

$$u(z) \leq M \quad \forall z \in D$$

If  $u(z_0) = M$  for some  $z_0 \in D$ ,

$$\text{then } u(z) = M \quad \forall z \in D$$

thm. (Strict Max princ - Complex)

$h$  - bounded,  $\mathbb{C}$ -valued harmonic  
function on domain  $D$

$$|h(z)| \leq M \quad \forall z \in D$$

If  $|h(z_0)| = M$  for some  $z_0 \in D$ ,

$$\Rightarrow h(z) \text{ is constant on } D$$

thm. (Max Principle)

$h$  -  $\mathbb{C}$ -valued, harmonic function on  
bounded domain  $D$

continuous on  $\partial D$

If  $|h(z)| \leq M \quad \forall z \in \partial D$ , then

$$|h(z)| \leq M \quad \forall z \in D$$

ML

thm.  $\gamma$  - piecewise smooth curve.

$h(z)$  - cont on  $\gamma$

$$\left| \int_{\gamma} h(z) dz \right| \leq \int_{\gamma} |h(z)| |dz|$$

If  $\text{length}(\gamma) = L$  and  $|h(z)| \leq M$  on  $\gamma$

$$\text{then } \left| \int_{\gamma} h(z) dz \right| \leq ML$$



thm.  $D$  - bounded domain with piecewise smooth boundary  
If  $f(z)$  analytic on  $D$  and extends smoothly to  $\partial D$ ,  
then  $\int_{\partial D} f(z) dz = 0$

thm.  $D$  - bounded domain with piecewise smooth bdy  
 $f(z)$  - analytic on  $D$ , extends smoothly to boundary,  
then  $f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw, z \in D$

thm.  $D, f$  as above  
 $\Rightarrow f$  has complex derivs of all orders  
and  $f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw, z \in D$

collary.  $f \in A(D) \Rightarrow f$  is infinitely differentiable  
and  $f^{(m)}(z)$  is analytic on  $D$   
 $\downarrow m$

thm.  $f(z)$  analytic for  $|z-z_0| \leq p$   
If  $|f(z)| \leq M$  for  $|z-z_0| = p$ ,  
then  $|f^{(m)}(z_0)| \leq \frac{m!}{p^m} M$

Liouville

thm.  $f$  - entire

If  $f$  bounded,  $f$  is constant.

Morera

thm.  $f$  - cont on domain  $D$

If  $\int_{\partial R} f(z) dz = 0$  for every closed rectangle

$R \subset D$  with sides parallel to coordinate axes,

then  $f$  is analytic on  $D$

thm.  $h(t, z)$  - cont,  $\mathbb{C}$ -valued

$a \leq t \leq b, z \in D$

If  $\forall$  fixed  $t$ ,  $h(t, z)$  is an analytic function of  $z$ , then

$H(z) = \int_a^b h(t, z) dt$  is analytic on  $D$ .

thm.  $f$  - continuous on domain  $D$ ,

analytic on  $D \setminus \mathbb{R}$

$\Rightarrow f$  is analytic on  $D$

Goursat

thm.  $f$  - complex valued funct on domain  $D$

st  $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists

$\forall z_0 \in D,$

Then  $f$  is analytic on  $D$

defn.  $\frac{\partial}{\partial z} = \frac{1}{2} \left[ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right]$

defn.  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right]$

thm.  $f \in \mathcal{L}'(D)$

$$f \text{ analytic} \iff \frac{\partial f}{\partial \bar{z}} = 0.$$

$$\text{If } f \text{ analytic, } f'(z) = \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

thm.  $f \in \mathcal{L}'(D)$ ,  $\nabla f \neq 0$ ,  $f(z)$  conformal.  
 $\Rightarrow f \in \mathcal{A}(D)$  and  $f'(z) \neq 0$ .

thm.  $D$  - bounded domain w/ piecewise smooth boundary

$g \in \mathcal{L}'(D \cup \partial D)$

$$\text{Then } \int_{\partial D} g(z) dz = 2i \iint_D \frac{\partial g}{\partial \bar{z}} dx dy$$

thm.  $D, g$  as above

$$g(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(z)}{z-w} dz = \frac{1}{\pi} \iint_D \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w} dx dy$$



# 712 Exam I

1 (a)  $(i^i)^i$

$$i^i = e^{i[\ln|i| + i\text{Arg}(i) + 2\pi k]} \quad k \in \mathbb{Z}$$

$$= e^{-\pi/2 - 2\pi k}$$

$$(i^i)^i = e^{i \log(i^i)} = e^{i \log(e^{-\pi/2 - 2\pi k})}$$

$$= e^{i(-\pi/2 - 2\pi k + i \text{Arg}(e^{-\pi/2 - 2\pi k}) + 2\pi i m)}$$

$$= e^{-\pi i/2} e^{-2\pi i k} e^{0} e^{2\pi i k} = 1$$

(b)  $\log((1-i)^{2i})$

$$(1-i)^{2i} = e^{2i \log(1-i)}$$

$$= e^{2i(\log|1-i| + i \text{Arg}(1-i) + 2\pi i k)}$$

$$= e^{2i(\log\sqrt{2} - i\pi/4 + 2\pi i k)}$$

$$= e^{2i \ln\sqrt{2} + \pi/2 - 4\pi k} = e^{2i \ln\sqrt{2} + \pi/2} e^{-4\pi k}$$

$$\log(1-i)^{2i} = \ln\left( e^{2i \ln\sqrt{2} + \pi/2 - 4\pi k} \right)$$

$$+ i \text{Arg}(e^{2i \ln\sqrt{2} + \pi/2 - 4\pi k}) + 2\pi i m$$

$$= \ln(e^{\pi/2 - 4\pi k}) + i(0 + 2\pi m)$$

$$= \pi/2 - 4\pi k + 2\pi i m$$

$k, m \in \mathbb{Z}$

Transcript

19/11/21

1. The first part of the transcript is about the

importance of the

document

and the way it is used

in the

process

of the

Principal

The principal is the person who is responsible for the

management of the

19/11/21

document and the way it is used

in the process of the

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2 (a) Suppose that each of  $f$  and  $\bar{f}$  are analytic on domain  $D$ . Show that  $f$  must be constant.

$f$  analytic —  $f = u + iv$  st  $u_x = v_y, u_y = -v_x$

$\bar{f}$  analytic —  $f = u - iv$  st  $u_x = -v_y, u_y = v_x$

$$\Rightarrow u_x \equiv 0 \equiv u_y \quad \text{and} \quad v_x \equiv 0 \equiv v_y$$

$u, v$  real valued functions with deriv  $\equiv 0$  on  $D \Rightarrow u, v$  constant.

$\Rightarrow f$  is constant.

(b) Suppose that  $f$  is analytic on a domain  $D$  and that  $|f|$  is constant. Show that  $f$  must be constant.

$$\bar{f} = \frac{|f|^2}{f}$$

If  $|f| \equiv 0, f \equiv 0$  — done.

Assume  $f \neq 0$ .

Then  $\frac{|f|^2}{f}$  is analytic.

By (a)  $f$  is constant.

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3 (a) Suppose that  $f$  has the mean value property w.r.t circles at  $z_0 \in D$ . Show  $f$  has the mean value property w.r.t discs at  $z_0 \in D$ .

$\exists \varepsilon > 0$  s.t.  $D_\varepsilon(z_0) \subset D$  and  
 $f(z_0) = \int_0^{2\pi} f(z_0 + te^{i\theta}) \frac{d\theta}{2\pi}$  for any  $0 < t < \varepsilon$ .

$$\begin{aligned} \frac{1}{\pi \varepsilon^2} \iint_{D_\varepsilon(z_0)} f(z) dx dy &= \frac{1}{\pi \varepsilon^2} \int_0^+ \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \cdot r dr \\ &= \frac{2}{\pi \varepsilon^2} \int_0^+ \left( \int_0^{2\pi} f(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \right) r dr \\ &= \frac{2f(z_0)}{\pi \varepsilon^2} \int_0^+ r dr = f(z_0) \end{aligned}$$

(b) Suppose that  $f$  is analytic in  $D$  and has no zeroes in  $D$ .

(b.1) Show that if  $|f(z)|$  attains its minimum in  $D$ , then  $f$  is constant.

Let  $g(z) = \frac{1}{f(z)}$ . — analytic, well defined.  
 $|g(z)| = \frac{1}{\min |f(z)|}$

Suppose  $|f(z^*)| = \min |f(z)|$ ,  $z^* \in D$ .

Then  $|g(z^*)| = \max |g(z)|$ .

Since  $g$  is analytic and attains its max in  $D$ ,

by the max principle,  $g$  is constant.  
 $\Rightarrow f$  is constant



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(b.2). Show that if  $D$  is bounded and  $f$  is analytic in  $D$  and continuous on  $\bar{D}$ , then  $|f|$  attains its minimum on  $\partial D$ .

Let  $g(z) = \frac{1}{f(z)}$  as before.

Then  $|g(z)| \leq \frac{1}{\min |f(z)|}$ .

By the max principle,  $|g|$  attains its max on the boundary.

$\Rightarrow |f|$  attains its minimum on the boundary.

(b.3) Give an example to show that the hypothesis that  $f(z) \neq 0 \forall z \in D$  is necessary.

$f(z) = z^2$ ,  $D = \mathbb{D}_1(0)$

$\min |f(z)| = 0$  achieved in  $D$  (not on  $\partial D$ )  
but  $f$  not constant.

1. The first step in the process of...  
is to identify the...  
of the...  
to the...

2. The second step is to...  
the...  
of the...  
to the...

3. The third step is to...  
the...  
of the...  
to the...

4. The fourth step is to...  
the...  
of the...  
to the...



4 (a) Use the ML theorem to find an upper bound for  $\left| \int_{|z|=R} \frac{z^3}{z^5-1} dz \right|$ , where  $R > 1$ .

$$\left| \int_{|z|=R} \frac{z^3}{z^5-1} dz \right| \leq \int_{|z|=R} \underbrace{\left| \frac{z^3}{z^5-1} \right|}_{\downarrow} |dz| \leq \frac{2\pi R^4}{R^5-1}$$

$$\left| \frac{z^3}{z^5-1} \right| \leq \frac{R^3}{R^5-1}$$

$$|dz| = 2\pi R$$

(b) Compute the following integrals.

$$\int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} : \text{CFD, } z_0 = 0, f(z) = \frac{1}{(z^2-4)e^z}$$

$$D = \{D, (0)\}$$

$$m=1, f'(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z^2} dz$$

$$f'(z) = -[(z^2-4)e^z]^{-2} \cdot (2ze^z + (z^2-4)e^z)$$

$$f'(0) = -[(-4)]^{-2} \cdot (0 + (-4))$$

$$= \frac{4}{16} = \frac{1}{4}$$

$$\int = 2\pi i \cdot \frac{1}{4} = \frac{\pi i}{2} \quad \text{sign?}$$

1. The first part of the problem is to find the region between the curves  $y = \sqrt{x}$  and  $y = x^2$  from  $x = 0$  to  $x = 1$ .

$$A = \int_0^1 (\sqrt{x} - x^2) dx = \left[ \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$\begin{aligned} \frac{2}{3}x^{3/2} &= \frac{2}{3} \cdot 1^{3/2} = \frac{2}{3} \\ \frac{1}{3}x^3 &= \frac{1}{3} \cdot 1^3 = \frac{1}{3} \end{aligned}$$

2. The second part of the problem is to find the volume of the solid generated by revolving the region about the y-axis.

$$V = \int_0^1 \pi (x^2 - x^4) dx = \pi \left[ \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 = \pi \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}$$

$$\pi \left( \frac{1}{3} - \frac{1}{5} \right) = \pi \left( \frac{5}{15} - \frac{3}{15} \right) = \frac{2\pi}{15}$$

$$\begin{aligned} \frac{1}{3}x^3 &= \frac{1}{3} \cdot 1^3 = \frac{1}{3} \\ \frac{1}{5}x^5 &= \frac{1}{5} \cdot 1^5 = \frac{1}{5} \end{aligned}$$

$$\frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$$

$$\int_{|z-1|=2} \frac{dz}{z(z^2-4)} e^z \quad \text{forma } D = D_2(1)$$

$$z_0 = 2, 0$$



$$\frac{1}{z(z+2)(z-2)}$$

$$z(z-2)$$

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$A(z-2) + Bz = 1$$

$$A+B=0$$

$$-2A=1$$

$$A = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

$$f(z) = \frac{1}{(z+2)e^z}$$

$$g(z) = \frac{1}{(z-2)e^z}$$

$$-\frac{1}{2} \int_{\partial D} \frac{f(z)}{z} dz + \frac{1}{2} \int_{\partial D} \frac{g(z)}{z-2} dz$$

$$2(-2\pi i f(0) + 2\pi i g(2))$$

$$m=0 \downarrow$$

$$2(-2\pi i \cdot \frac{1}{2} + 2\pi i \cdot \frac{1}{4e^2})$$

$$-2\pi i + \frac{\pi i}{2e^2}$$

sign?



(1)  $CI = 2$

$\frac{1}{2} = \frac{1}{2}$

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## 712 Exam II

1 What functions are represented by the following power series and for what values of  $z$  are they valid?

$$\sum_{k=1}^{\infty} k z^k \quad - \quad \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad |z| < 1$$

Derivatives can be taken termwise within the radius of convergence so:

$$\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1}$$

Multiplying both sides by  $z$ :

$$\frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k z^k \quad |z| < 1$$

$$\sum_{k=1}^{\infty} k^2 z^k \quad - \quad \text{Compute derivative of (a):}$$
$$\frac{(1-z)^2 + z \cdot 2(1-z)}{(1-z)^4} = \sum_{k=2}^{\infty} k^2 z^{k-1}$$

Multiply by  $z$  again:

$$\frac{z(1-z)^2 + 2z^2(1-z)}{(1-z)^4} = \sum_{k=1}^{\infty} k^2 z^k, \quad |z| < 1$$



2. (a) Explain what it means for  $f(z)$  to be analytic at  $\infty$ . First check that  $f(z)$  is analytic at  $\infty$ . Then find the power series expansion of  $f(z)$  at  $\infty$ :

$$f(z) = \frac{z^2}{z^3 - 1}$$

$f(z)$  analytic at  $\infty \iff g(w) = f(1/w)$  is analytic at 0.

$$g(w) = \frac{(1/w)^2}{(1/w)^3 - 1} = \frac{1/w^2}{\frac{1 - w^3}{w^3}} = \frac{w}{1 - w^3}$$

$$\sum_{k=0}^{\infty} w^{3k+1} = w \sum_{k=0}^{\infty} w^{3k} \quad \text{GS}, |w| < 1$$

$\frac{1}{2}$   $g(w)$  analytic at 0 so  $f$  analytic at  $\infty$

$$f(z) = g(1/z) = \sum_{k=0}^{\infty} (1/z)^{3k+1}, \quad |z| > 1.$$

- (b) Find all zeroes and the order of zeroes of  $f(z)$ . Explain.

$$f(z) = \frac{(z^2 + 1)(e^z - 1)^2}{(z^2 - 1)}$$

$$z^2 + 1 = 0 \iff z = \pm i$$

$$e^z = 1 \iff z = 0, 2\pi i, \dots, z = 2k\pi i$$

$$f'(z) = \frac{[(2z)(e^z - 1)^2 + (z^2 + 1)2(e^z - 1)e^z](z^2 - 1) - 2z(z^2 + 1)(e^z - 1)^2}{(z^2 - 1)^2}$$

$f'(\pm i) \neq 0$  so  $z = \pm i$  simple zeroes.

$f'(2k\pi i) = 0 = f''(2k\pi i) \Rightarrow z = 2k\pi i \quad k > 0, 1, 2, \dots$   
is a double zero for each  $k$ .



20.  $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx$   
 Let  $f(z) = \frac{e^{-z^2}}{1+z^2}$   
 Poles at  $z = \pm i$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \text{Res}(f, i)$   
 $\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \frac{e^{-z^2}}{1+z^2} = \frac{e^{-i^2}}{2i} = \frac{e^{-1}}{2i}$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}$

21.  $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx$   
 Let  $f(z) = \frac{e^{-z^2}}{1+z^2}$   
 Poles at  $z = \pm i$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \text{Res}(f, i)$   
 $\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \frac{e^{-z^2}}{1+z^2} = \frac{e^{-i^2}}{2i} = \frac{e^{-1}}{2i}$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}$

22.  $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx$   
 Let  $f(z) = \frac{e^{-z^2}}{1+z^2}$   
 Poles at  $z = \pm i$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \text{Res}(f, i)$   
 $\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \frac{e^{-z^2}}{1+z^2} = \frac{e^{-i^2}}{2i} = \frac{e^{-1}}{2i}$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}$

23.  $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx$   
 Let  $f(z) = \frac{e^{-z^2}}{1+z^2}$   
 Poles at  $z = \pm i$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \text{Res}(f, i)$   
 $\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \frac{e^{-z^2}}{1+z^2} = \frac{e^{-i^2}}{2i} = \frac{e^{-1}}{2i}$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}$

24.  $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx$   
 Let  $f(z) = \frac{e^{-z^2}}{1+z^2}$   
 Poles at  $z = \pm i$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \text{Res}(f, i)$   
 $\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \frac{e^{-z^2}}{1+z^2} = \frac{e^{-i^2}}{2i} = \frac{e^{-1}}{2i}$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}$

25.  $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx$   
 Let  $f(z) = \frac{e^{-z^2}}{1+z^2}$   
 Poles at  $z = \pm i$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \text{Res}(f, i)$   
 $\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \frac{e^{-z^2}}{1+z^2} = \frac{e^{-i^2}}{2i} = \frac{e^{-1}}{2i}$   
 $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}$

(c) Find and classify all singularities of  $f(z)$  (including singularities at  $\infty$ ). Explain.

$$f(z) = \frac{e^{3z} - 1}{z^2}$$

$f(z)$  has a pole at 0  
 $e^{3z} = \sum_{k=0}^{\infty} \frac{(3z)^k}{k!}$

$$e^{3z} - 1 = \sum_{k=1}^{\infty} \frac{(3z)^k}{k!}$$

$$\frac{e^{3z} - 1}{z^2} = \sum_{k=1}^{\infty} \frac{(3z)^{k-2}}{k!} \quad \text{only for } k=1$$

$\Rightarrow$  0 is a simple pole of the function.

$$f(1/z) = (e^{3/z} - 1)z^2$$

$e^{3/z}$  has an essential singularity at 0.

$\Rightarrow f(z)$  has an essential singularity at  $\infty$ .

(d) Suppose that  $f \in A(D, (0))$  and that  $f(k_n) = k_n$ ,  $n=1, 2, 3, 4, \dots$ . What can you say about  $f$ ?

Consider  $f(z) - z$  on  $D, (\mathbb{C})$ .

This function is clearly analytic and  
 $f(k_n) - k_n = 0 \quad \forall n$

But zeros of analytic functions are isolated and  $\{k_n\}$  has a limit point in the disc.

$\Rightarrow f(z) - z \equiv 0$  everywhere.

$\Rightarrow f(z) = z$ .

... to ... ..

$$\frac{1}{2} = \frac{1}{2}$$

... ..

$$\frac{1}{2} = \frac{1}{2}$$

... ..

$$\frac{1}{2} = \frac{1}{2}$$

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... ..

$$\frac{1}{2} = \frac{1}{2}$$

3 Let  $f(z) = \frac{1}{z^2 + z}$

(a) First find all annuli centered at  $z_0 = 0$  where  $f$  is analytic.

Then compute the Laurent decomposition for  $f$  relative to each such annulus.

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} + \frac{-1}{z+1}$$

$$1 = A(z+1) + Bz$$

$$A+B=0 \Rightarrow A=-B$$

$$A=1 \Rightarrow B=-1$$



$A_{0,1}(0)$

$A_{1,\infty}(0)$

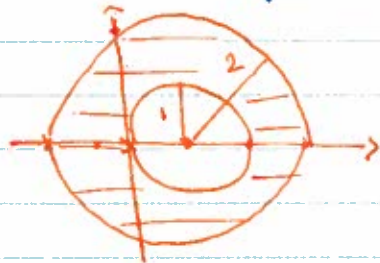
$0 < |z| < 1$ :  $\frac{1}{z} \in A(|z| > 0)$ , vanishes at  $\infty$   
 $-\frac{1}{z+1} \in A(|z| < 1)$

$1 < |z| < \infty$ :  $f(z) \in A(A_{1,\infty}(0))$   
 and vanishes at  $\infty$





(b) Find the Laurent expansion centered at  $z_0 = 1$  and converges at  $z = -\frac{1}{2}$ .



$$1 < |z-1| < 2$$

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} + \frac{-1}{z+1}$$

$$\frac{-1}{z+1} = \frac{-1}{z-1+2} = \frac{-1}{2-(1-z)} = \frac{-1}{1-\left(\frac{1-z}{2}\right)}$$

$$\frac{-1}{\frac{z-1}{2}+1} = \frac{-1}{\frac{2}{(1-z)}-1} = \frac{1}{1-\frac{2}{1-z}}$$

$$\stackrel{\text{GS}}{=} \sum_{k=0}^{\infty} \left(\frac{2}{1-z}\right)^k \quad \text{oops} \quad \leftarrow \text{114}$$

$$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{1+\frac{1}{z-1}} = \frac{1}{1-\left(-\frac{1}{z-1}\right)}$$

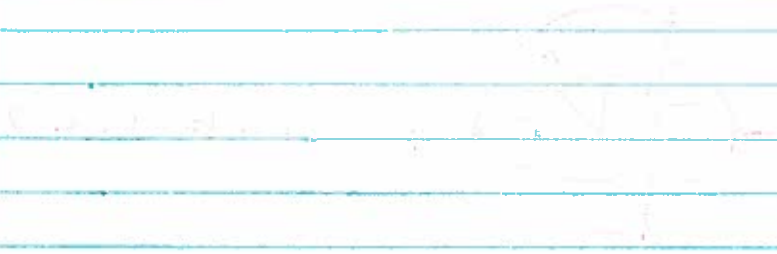
$$\stackrel{\text{GS}}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^k} \quad \text{OK}$$

$$-\frac{1}{z+1} = -\sum_{k=0}^{\infty} \left(\frac{1-z}{2}\right)^k$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^k} + -\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k (1-z)^k$$

~~114~~

In a diagram showing a triangle with vertices A, B, and C. The interior angles are labeled as follows: angle A is  $2x$ , angle B is  $3x$ , and angle C is  $4x$ .



$$2x + 3x + 4x = 180^\circ$$

$$9x = 180^\circ$$

$$x = \frac{180^\circ}{9}$$
$$x = 20^\circ$$

$$\angle A = 2x = 2(20^\circ) = 40^\circ$$

$$\angle B = 3x = 3(20^\circ) = 60^\circ$$

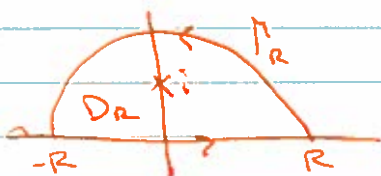
$$\angle C = 4x = 4(20^\circ) = 80^\circ$$

$$40^\circ + 60^\circ + 80^\circ = 180^\circ$$

✓

4 Evaluate:  $I = \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2} dx$

Let  $D_R$  be the upper semidisk of radius  $R$  centered at  $O$ .



$$\int_{\partial D_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz$$

$$= 2\pi i \operatorname{Res} \left[ \frac{z^3 e^{iz}}{(z^2+1)^2}, i \right]$$

$$\operatorname{Res} \left[ \frac{z^3 e^{iz}}{(z^2+1)^2}, i \right] =$$

$$\lim_{z \rightarrow i} \frac{d}{dz} \left( \frac{z^3 e^{iz}}{(z+i)^2} \right) = \frac{1}{4e}$$

← Calculations on test

$$\int_{\partial D_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz = \frac{\pi i}{2e}$$

$$\left| \int_{\Gamma_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz \right| \leq \frac{R^3}{R^4} \int_{\Gamma_R} |e^{iz}| |dz| \leq \frac{\pi}{R}$$

Jordan's Lemma.

$$\rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\int_{-R}^R \frac{x^3 e^{ix}}{(x^2+1)^2} dx \rightarrow \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)^2} dx = \frac{\pi i}{2e}$$

To get  $\sin(x)$  take imaginary part.

$$I = \frac{\pi}{2e}$$



$$\text{Let } \frac{1}{1+x^2} = \frac{A}{1+ix} + \frac{B}{1-ix}$$

multiply both sides by  $(1+ix)(1-ix)$

$$1 = \frac{A(1-ix) + B(1+ix)}{(1+ix)(1-ix)}$$

$$1 = \frac{A(1-ix) + B(1+ix)}{1-x^2}$$

$$1 = \frac{A(1-ix) + B(1+ix)}{1-x^2}$$

$$1 = \frac{A(1-ix) + B(1+ix)}{1-x^2}$$

$$1 = \frac{A(1-ix) + B(1+ix)}{1-x^2}$$

$$1 = \frac{A(1-ix) + B(1+ix)}{1-x^2}$$

(b) Evaluate:  $I = \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}$

Replace  $\sin \theta$  by  $\frac{z - \frac{1}{z}}{2i}$  and  $d\theta$  by  $\frac{dz}{iz}$

$$I = \frac{1}{i} \int_{|z|=1} \frac{4z}{6z^2 - z^4 - 1} dz$$

Apply residue theorem: poles are  $\pm \sqrt{3 - 2\sqrt{2}}$

$$I = \pi\sqrt{2}$$

$$z = e^{i\theta}$$

$$\sin \theta = \frac{z - \frac{1}{z}}{2i}$$

$$\cos \theta = \frac{z + \frac{1}{z}}{2}$$

$$\frac{d^2 y}{dx^2} = T + \sin(x) + T \cos(x)$$

$$T \sin(x) + T \cos(x) = 0$$

$$\sin(x) + \cos(x) = 0$$

$$\tan(x) = -1$$

$$x = \frac{3\pi}{4}$$

$$x = \frac{3\pi}{4}$$

5 Suppose  $f$  is continuous on a domain  $D$ .

Define  $g(z) = f^3(z)$ ,  $z \in D$ , and suppose  $g$  is analytic in  $D$ .

(a) Show that  $f$  is analytic in  $D$ .

$$g'(z) = \lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0} \text{ exists for all } z \in D.$$

$$\begin{aligned} g'(z) &= \lim_{z \rightarrow z_0} \frac{f^3(z) - f^3(z_0)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \left[ \frac{f(z) - f(z_0)}{z - z_0} \cdot (f^2(z_0) + f(z_0)f(z) + f^2(z)) \right] \\ &= \lim_{z \rightarrow z_0} \left( \frac{f(z) - f(z_0)}{z - z_0} \right) \cdot 3f^2(z_0) \end{aligned}$$

since  $f$  is continuous.

$$\Rightarrow \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists } \forall z \in D.$$

(b) Can you give an example  $f$  and domain  $D$  that show that the hypothesis that  $f$  be continuous on  $D$  is necessary?

$$f = \sqrt[3]{z}, \quad D = \Delta ?$$

$$g(z) = z \in A(\Delta)$$

But  $f$  is not continuous (I think).



3. When  $\lambda = 0$ , the matrix  $A$  is singular.  
The eigenvalues are  $\lambda = 0, 1, 1$ .  
The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ .

4. When  $\lambda = 1$ , the matrix  $A - I$  is singular.

The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

5. When  $\lambda = -1$ , the matrix  $A + I$  is singular.  
The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

6. When  $\lambda = 2$ , the matrix  $A - 2I$  is singular.  
The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

7. When  $\lambda = -2$ , the matrix  $A + 2I$  is singular.  
The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

8. When  $\lambda = 3$ , the matrix  $A - 3I$  is singular.  
The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

9. When  $\lambda = -3$ , the matrix  $A + 3I$  is singular.  
The eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

## 712 Final Exam

- 1 (a) Suppose that both  $f(z)$  and  $h(z) = f(\bar{z})$  are analytic on  $D_1(0)$ . Show that  $f$  must be constant.

$$f(z), h(z) \text{ analytic} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} \equiv 0 \equiv \frac{\partial h}{\partial z}$$

$$\text{But } \frac{\partial h}{\partial z} = \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} \equiv 0$$

$$\Rightarrow f \text{ is constant on } D_1(0)$$

- (b) Suppose that  $u$  is harmonic on the entire complex plane and that  $|u(z)| \leq M$  for any  $z \in \mathbb{C}$ . Show that  $u$  must be constant.

$u \in \text{Harm}(\mathbb{C}) \Rightarrow u$  has a harmonic conjugate  $v$   
st  $f = u + iv$  is entire.

Consider  $e^{f(z)}$ .

$f$  entire  $\Rightarrow e^{f(z)}$  entire.

$$|e^{f(z)}| = |e^u| \leq e^M \quad \forall z$$

By Liouville's theorem,  $e^{f(z)}$  is constant.

$\Rightarrow f$  is constant

$\Rightarrow u$  constant

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2 Compute  $I = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$

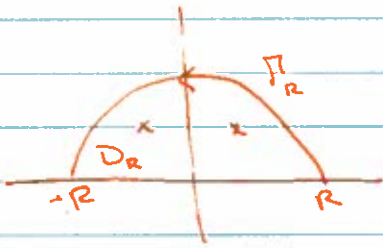
$$x^4 + 1 = (x^2 - i)(x^2 + i)$$

$$x^2 = i, x^2 = -i$$

$$x = \pm \sqrt{-4(16i)} \sim$$

$$x_1 = \sqrt[4]{2}(-1+i)$$

$$x_2 = \sqrt[4]{2}(1+i)$$



$$\int_{\partial D_R} \frac{z^2}{z^4+1} dz = \int_{\Gamma_R} \frac{z^2}{z^4+1} dz + \int_{-R}^R \frac{x^2}{x^4+1} dx$$

By the Residue theorem,

$$\int_{\partial D_R} \frac{z^2}{z^4+1} dz = 2\pi i \sum \text{Res} \left[ \frac{z^2}{z^4+1}, z_i \right]$$

$$\text{Res} \left[ \frac{f}{g}, a \right] = \lim_{z \rightarrow a} \frac{f}{g'}$$

$$\text{Res} \left[ \frac{z^2}{z^4+1}, \frac{1}{\sqrt{2}}(-1+i) \right] = \lim_{z \rightarrow z_1} \frac{z^2}{4z^3} = \lim_{z \rightarrow z_1} \frac{1}{4z}$$

$$= \frac{1}{4 \frac{1}{\sqrt{2}}(-1+i)}$$

$$= \frac{\sqrt{2}}{4(-1+i)(-1-i)}$$

$$= \frac{\sqrt{2}(-1-i)}{4(\sqrt{2})} = \frac{-1-i}{4}$$

$$\text{Res} \left[ \frac{z^2}{z^4+1}, \frac{1}{\sqrt{2}}(1+i) \right] = \lim_{z \rightarrow z_2} \frac{1}{4z}$$

$$= \frac{1}{4 \frac{1}{\sqrt{2}}(1+i)} = \frac{\sqrt{2}}{4(1+i)(1-i)}$$

$$= \frac{\sqrt{2}(1-i)}{\sqrt{2} \cdot 4} = \frac{1-i}{4}$$

$$2\pi i \left( \frac{-1-i + 1-i}{4} \right) = 2\pi i \left( \frac{-2i}{4} \right) = \pi \quad ? \leftarrow \text{should be } \frac{\pi}{\sqrt{2}} ?$$

Use ML to get rid of  $\int_{\Gamma_R}$  part



2.  $\sqrt{2} \leq \sqrt{2} < \sqrt{3}$  (by definition)

$\Rightarrow \sqrt{2} < \sqrt{2} < \sqrt{3}$   
 $\Rightarrow \sqrt{2} < \sqrt{2} < \sqrt{3}$   
 $\Rightarrow \sqrt{2} < \sqrt{2} < \sqrt{3}$

Let  $x = \sqrt{2}$  then  $x^2 = 2$   
 $\Rightarrow x^2 - 2 = 0$

$\Rightarrow x = \pm \sqrt{2}$

Since  $x > 0$   
 $\Rightarrow x = \sqrt{2}$

Let  $x = \sqrt{3}$  then  $x^2 = 3$   
 $\Rightarrow x^2 - 3 = 0$

$\Rightarrow x = \pm \sqrt{3}$

Since  $x > 0$   
 $\Rightarrow x = \sqrt{3}$

$\Rightarrow \sqrt{2} < \sqrt{2} < \sqrt{3}$

Let  $x = \sqrt{2}$  then  $x^2 = 2$   
 $\Rightarrow x^2 - 2 = 0$

$\Rightarrow x = \pm \sqrt{2}$

Since  $x > 0$   
 $\Rightarrow x = \sqrt{2}$

Let  $x = \sqrt{3}$  then  $x^2 = 3$   
 $\Rightarrow x^2 - 3 = 0$

$\Rightarrow x = \pm \sqrt{3}$

3 (a) Show that the equation

$$2e^z - 7z + 1 = 0$$

has exactly one solution in  $D_1(0)$ . Also, show that this solution is real. Explain and show work.

$$\text{On } |z|=1, |2e^z| \leq 2e < 6 \\ | -7z + 1 | \geq 6$$

By Rouché's theorem, since  $-7z+1$  has a zero in  $D_1(0)$ ,  $2e^z - 7z + 1$  has exactly one zero in  $D_1(0)$  as well.

Furthermore, the fact that the root of  $-7z+1$  lies in the interval  $0 < x < 1$  implies that the root of  $2e^z - 7z + 1$  is real as well.

(b) How many roots does  $p(z) = z^9 + z^5 - 8z^3 + 2z + 1$  have inside the annulus  $A_{1,2}(0)$ ? Explain and show work.

In  $|z| < 1$ , Big =  $-8z^3$  and little =  $z^9 + z^5 + 2z + 1$   
On  $|z|=1$ ,  $| -8z^3 | = 8$  and  $| z^9 + z^5 + 2z + 1 | = 5$   
 $-8z^3$  has 3 zeroes in  $|z| < 1$ , so  $p(z)$  does as well.

In  $|z| < 2$ , Big =  $z^9$  and little =  $z^5 - 8z^3 + 2z + 1$   
On  $|z|=2$ ,  $|z^9| = 512$ , and  $|z^5 - 8z^3 + 2z + 1| \leq 101$ ?  
 $z^9$  has 9 zeroes in  $|z| < 2$ , so  $p(z)$  does as well.

$\Rightarrow$  The # of roots in  $A_{1,2}(0)$  is  $9 - 3 = 6$ .

horribly  
written



4 (a) Let  $S(0)$  denote a square with center at the origin. Suppose that  $F: \mathbb{D}(0) \rightarrow S(0)$  is analytic, one-to-one, onto, and  $F(0) = 0$ . Show that  $F(iz) = iF(z)$  for any  $z \in \mathbb{D}(0)$ . Explain and show work.

Let  $M =$  side length of  $S(0)$ .

$$\Rightarrow |F(z)| \leq \frac{\sqrt{2}}{2} M \quad \forall |z| < 1.$$

By Schwarz lemma,  $|F(z)| \leq \frac{\sqrt{2}}{2} M |z|$ .

$F$  biholomorphic  $\Rightarrow F$  conformal.

$\Rightarrow F$  extends continuously to the boundary  
(boundary is Jordan curve for both)  
and furthermore  $z \in \partial\mathbb{D}(0) \mapsto w \in \partial S(0)$ .

For  $|z| = 1$ ,  $|F(z)| = \frac{\sqrt{2}}{2} M$ .

~~not 100% sure~~

$$\Rightarrow F(z) = \frac{\sqrt{2}}{2} M a z, \quad |a| = 1$$

$$F(iz) = i \frac{\sqrt{2}}{2} M a z = \frac{\sqrt{2}}{2} M a (iz) = iF(z)$$

← not 100% sure





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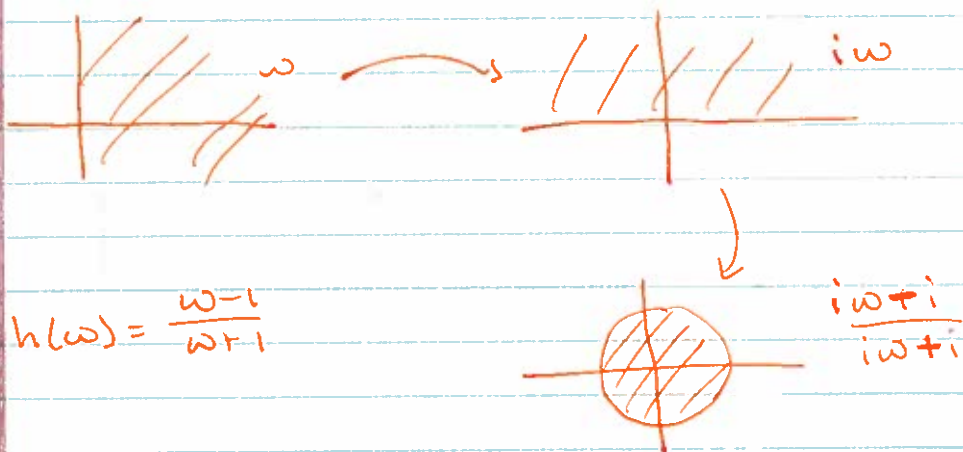
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4 (b)  $\mathcal{F} = \{f: D_1(0) \rightarrow \{\operatorname{Re} w > 0\} \mid f \in A, f(0) = 1\}$ .

Let  $M = \sup \{|f'(0)| \mid f \in \mathcal{F}\}$ .

Compute  $M$ . Also: can you find  $f \in \mathcal{F}$  such that  $|f'(0)| = M$ ? Explain and show work.

Following hint, find analytic, 1-1 map of  $\{\operatorname{Re} w > 0\}$  onto  $D_1(0)$ .



$h$  is analytic and 1-1 ~~map~~ (check). Consider  $h \circ f$  where  $f \in \mathcal{F}$ .

$h \circ f: D_1(0) \rightarrow D_1(0)$  is analytic and  $h \circ f(0) = h(1) = 0$ .

By the Schwarz Lemma,

$$|(h \circ f)'(0)| \leq 1.$$

$$(h \circ f)'(z) = h'(f(z)) \cdot f'(z).$$

$$h'(w) = \frac{w+1 - (w-1)}{(w+1)^2} = \frac{2}{(w+1)^2} \quad (\text{note: } h' \neq 0)$$

$$h'(f(0)) = h'(1) = \frac{2}{2^2} = \frac{1}{2}.$$

$$\Rightarrow \frac{1}{2} \cdot |f'(0)| \leq 1 \Rightarrow |f'(0)| \leq 2$$

So  $M = 2$ .

$f = h^{-1}$  should work

$$f(z) = \frac{z+1}{1-z}, \quad f'(z) = \frac{2}{(1-z)^2}, \quad f'(0) = 2 \quad \checkmark$$

1.  $\frac{1}{x^2} = x^{-2}$   
Derivative:  $\frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}$

$$\frac{d}{dx} \frac{1}{x^2} = -\frac{2}{x^3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

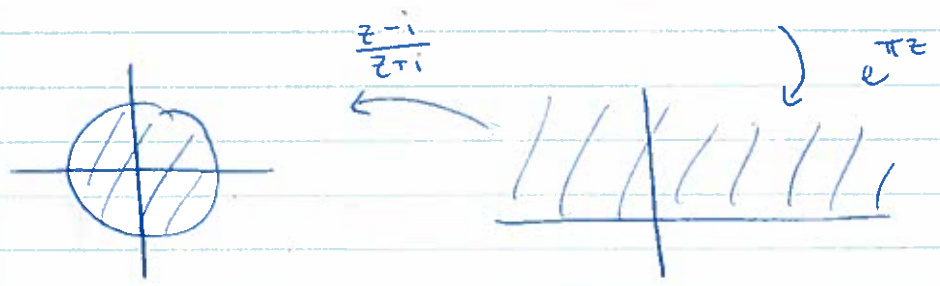
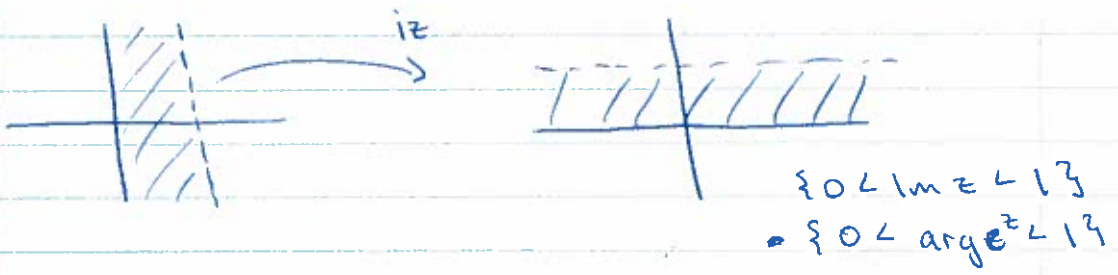
$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

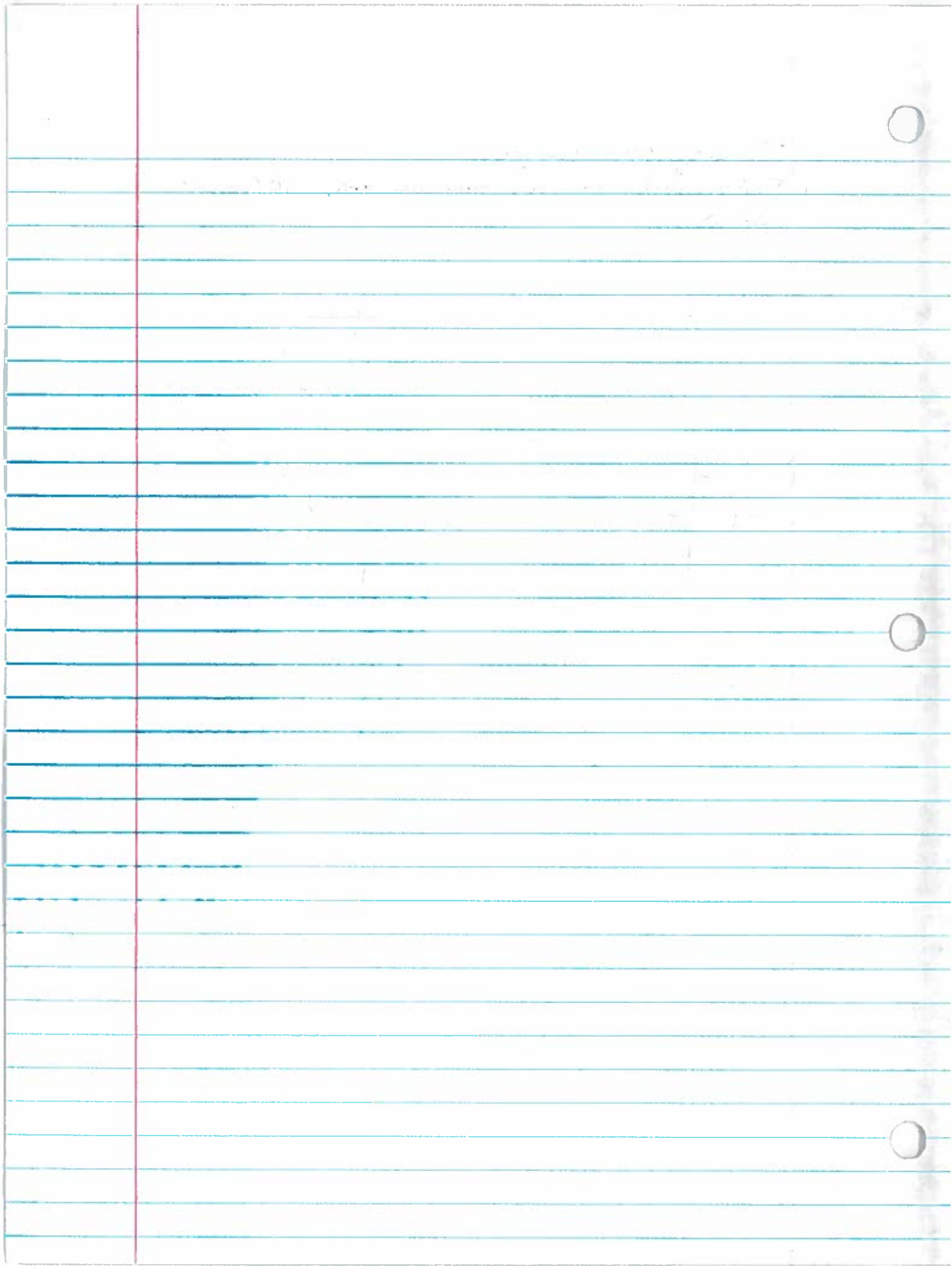
January 2013, Complex

1. Find a conformal map from the strip  $\{0 < \operatorname{Re} z < 1\}$  onto  $\Delta$ .



$$\frac{e^{\pi iz} - i}{e^{\pi iz} + i}$$



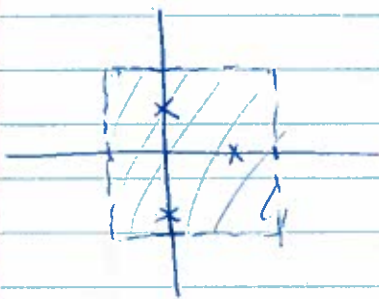


2 Let  $C$  denote the positively oriented boundary of the domain

$$D = \{z \in \mathbb{C} : -\frac{1}{2} < \operatorname{Re} z < 2, |\operatorname{Im} z| < 2\}$$

Find  $\int_C \frac{z^n}{z^4-1} dz$ , where  $n \neq 0$  is an integer.

Write your answer in algebraic form  $a+bi$ .



$$\begin{aligned} z^4 - 1 &= (z^2 + 1)(z^2 - 1) \\ &= (z+i)(z-i)(z+1)(z-1) \end{aligned}$$

poles in  $D = \pm i, 1$

$$\int_C \frac{z^n}{z^4-1} dz = 2\pi i \sum \operatorname{Res}$$

$$\operatorname{Res}\left[\frac{z^n}{z^4-1}, -i\right] = \lim_{z \rightarrow -i} \frac{z^n}{(z-i)(z^2-1)} = \frac{(-i)^n}{(-2i)(-2)} = \frac{(-1)^n i^n}{4i}$$

$$\operatorname{Res}\left[\frac{z^n}{z^4-1}, 1\right] = \lim_{z \rightarrow 1} \frac{z^n}{(z+i)(z-i)(z+1)} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

$$\operatorname{Res}\left[\frac{z^n}{z^4-1}, i\right] = \lim_{z \rightarrow i} \frac{z^n}{(z+i)(z^2-1)} = \frac{i^n}{(2i)(-2)} = \frac{i^n}{-4i}$$

$$\begin{aligned} 2\pi i \left( \frac{(-1)^n i^n}{4i} + \frac{1}{4} + \frac{i^n}{-4i} \right) \\ = \frac{(-1)^n \pi i^n}{2} + \frac{\pi i}{2} - \frac{\pi i^n}{2} \end{aligned}$$

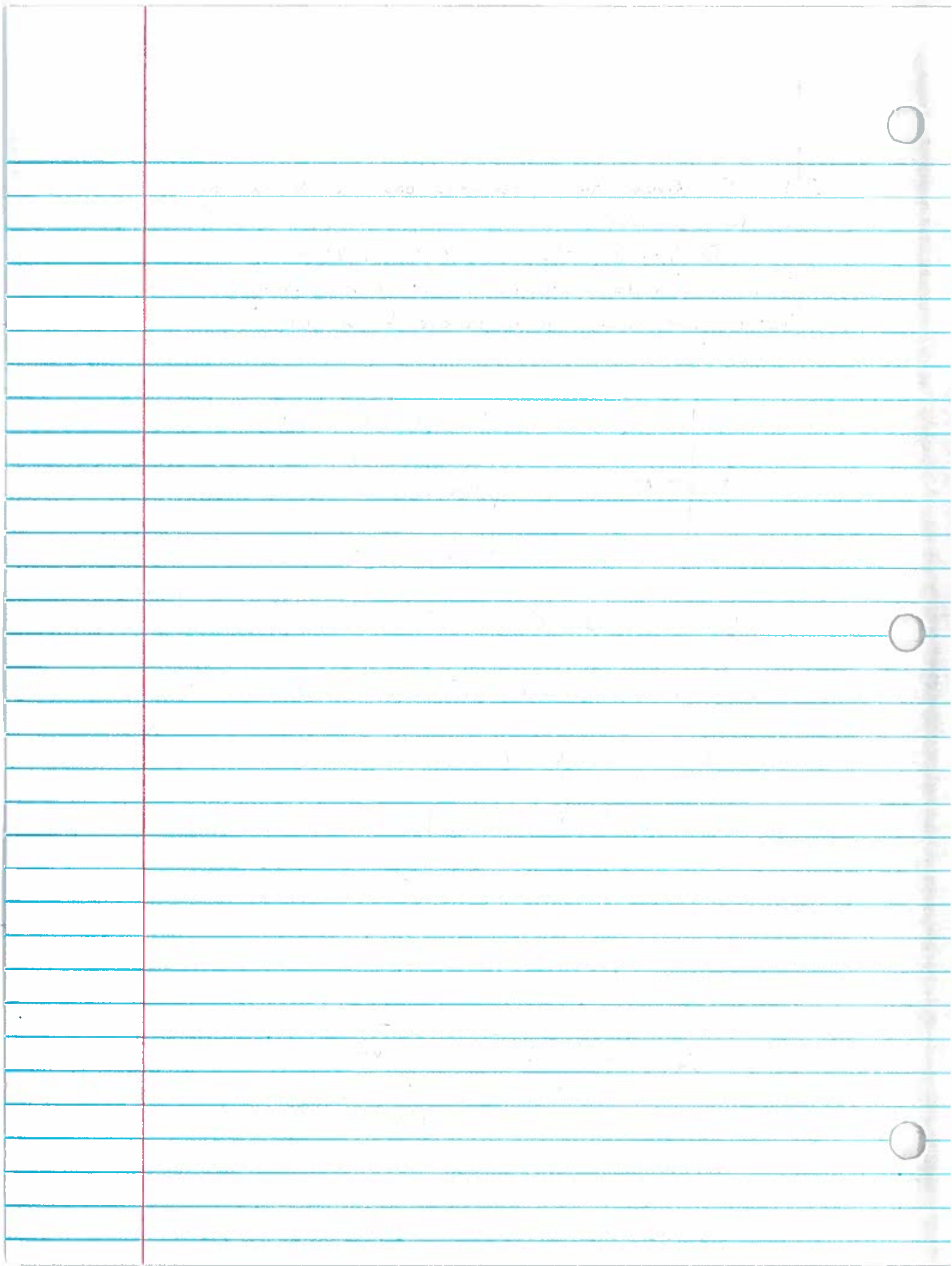
$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$$

$$\text{If } n = 0 \pmod{4}, \quad \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{If } n = 1 \pmod{4}, \quad -\frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} = \frac{\pi i}{2}$$

$$\text{If } n = 2 \pmod{4}, \quad -\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{If } n = 3 \pmod{4}, \quad -\frac{\pi(-i)}{2} + \frac{\pi i}{2} - \frac{\pi(-i)}{2} \\ \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} = \frac{3\pi i}{2}$$



3 Is there an entire function  $f(z)$  such that  $e^{f(z)}$  has a pole at  $\infty$ ?

No.

Assume there is.

Then  $e^{f(z)}$  is entire.

$e^{f(z)}$  has a pole at  $\infty \iff e^{-f(z)}$  has a zero at  $\infty$ .

But  $e^{-f(z)}$  is also entire.

(\*) Entire functions have essential singularities or poles at  $\infty$ .

This is a contradiction.

(\*)  $g(z)$  entire  $\implies g(z) = \sum_{k=0}^{\infty} a_k z^k$   
If  $a_k = 0$  for  $k \geq N$ ,  $g$  has a pole of order  $N$  at  $\infty$ .

o/w it has an essential singularity (or is constant)

entire  $f(z)$  st  $\|f(z)\| \leq M \|z\|^n \quad \forall z$  with  $\|z\| \geq R$  is a polynomial of degree at most  $n$ .

$n=0$  — Liouville's Theorem.



1. The first part of the document is a list of names and addresses.

2. The second part of the document is a list of names and addresses.

3. The third part of the document is a list of names and addresses.

4. The fourth part of the document is a list of names and addresses.

5. The fifth part of the document is a list of names and addresses.

6. The sixth part of the document is a list of names and addresses.

7. The seventh part of the document is a list of names and addresses.

8. The eighth part of the document is a list of names and addresses.

4 Suppose that  $f, g$  are holomorphic functions in  $\Delta$  so that  $f(0) = g(0) = 1$  and  $(f'g - fg')(z_n) = 0$  for all integers  $n \geq 2$ . Show that  $f = g$  on  $\Delta$ .

$f, g$  holomorphic  $\Rightarrow f', g'$  holomorphic  
 $\Rightarrow f'g, fg'$  holomorphic  
 $\Rightarrow f'g - fg'$  holomorphic.

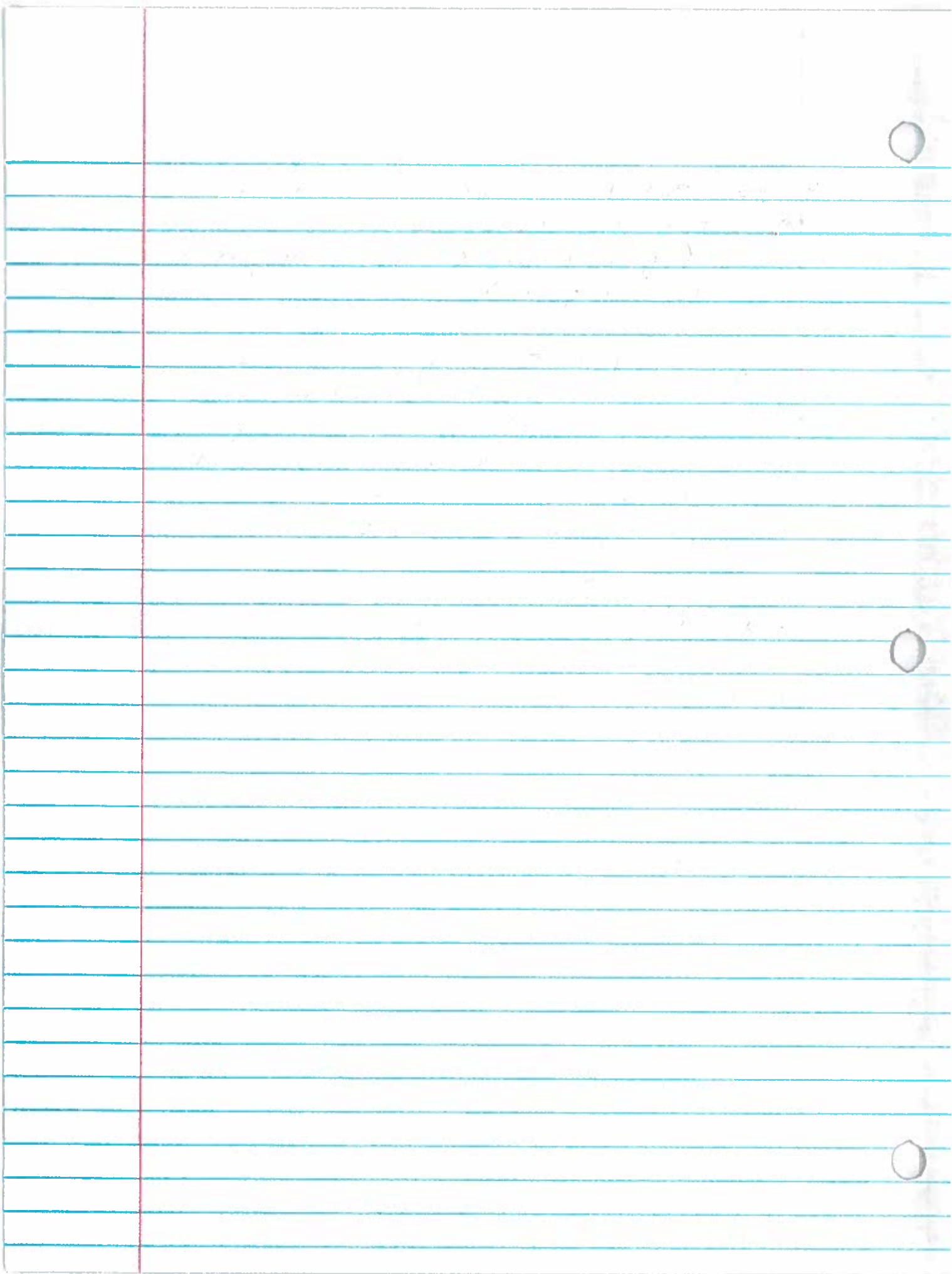
Since  $f'g - fg' = 0$  on a set with an accumulation

point,  $f'g - fg' \equiv 0$  in  $\Delta$ .  
 $\Rightarrow \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \equiv 0$  in  $\Delta$   $g \neq 0?$

$\Rightarrow \frac{f}{g}$  is constant in  $\Delta$

$$\frac{f(0)}{g(0)} = \frac{1}{1} \Rightarrow \frac{f(z)}{g(z)} = 1 \quad \forall z$$

Finally,  $f(z) = g(z) \quad \forall z$ .



Complex, August 2014

1 Find a conformal map from the half-disc

$$D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z > 0\} \text{ onto } \Delta$$

Rotate  $D$  to the upper half disc:  $z \mapsto iz$

Map the half disc to the 1st quadrant:

$$\begin{array}{l} 1 \rightarrow \infty \\ -1 \rightarrow 0 \end{array} \quad \tilde{z} \mapsto \frac{\tilde{z} + 1}{\tilde{z} - 1} = \frac{(iz) + 1}{(iz) - 1} = \frac{i(z-i)}{i(z+i)}$$

Map the 1st quadrant to the upper half plane:

$$w \mapsto w^2 = \frac{(z-i)^2}{(z+i)^2}$$

Map the upper half plane to the disc  $\Delta$ :

$$\tilde{w} \mapsto \frac{\tilde{w} - i}{\tilde{w} + i}$$

Composing these maps gives:

$$z \mapsto \frac{\left(\frac{z-i}{z+i}\right)^2 - i}{\left(\frac{z-i}{z+i}\right)^2 + i}$$



1. The first part of the document is a list of names and dates. The names are: John Doe, Jane Smith, and Bob Johnson. The dates are: 1/1/2020, 2/1/2020, and 3/1/2020.

2. The second part of the document is a list of items and their prices. The items are: Apples, Bananas, and Oranges. The prices are: \$1.00, \$0.50, and \$0.75.

3. The third part of the document is a list of tasks and their due dates. The tasks are: Complete report, Review documents, and Prepare presentation. The due dates are: 4/1/2020, 4/15/2020, and 4/30/2020.

4. The fourth part of the document is a list of locations and their coordinates. The locations are: New York, Los Angeles, and Chicago. The coordinates are: (40.71, -87.63), (34.05, -118.24), and (41.88, -87.63).

5. The fifth part of the document is a list of events and their dates. The events are: Christmas, New Year, and Valentine's Day. The dates are: 12/25/2019, 1/1/2020, and 2/14/2020.

6. The sixth part of the document is a list of countries and their capitals. The countries are: United States, Canada, and Mexico. The capitals are: Washington D.C., Ottawa, and Mexico City.

7. The seventh part of the document is a list of sports and their teams. The sports are: Football, Basketball, and Baseball. The teams are: Manchester United, Los Angeles Lakers, and New York Yankees.

8. The eighth part of the document is a list of movies and their directors. The movies are: The Godfather, E.T., and Star Wars. The directors are: Francis Ford Coppola, Steven Spielberg, and George Lucas.

9. The ninth part of the document is a list of books and their authors. The books are: The Catcher in the Rye, To Kill a Mockingbird, and The Great Gatsby. The authors are: J.D. Salinger, Harper Lee, and F. Scott Fitzgerald.

2 Let  $D$  be a domain in  $\mathbb{C}$  containing  $0$  and  $f: D \rightarrow \mathbb{R}$  be a continuous function such that  $f(0) = 0$  and  $\int_{\partial R} f(z) dz = 0$ .

Prove that  $f(z) = 0$  for every  $z \in D$ .

$\int_{\partial R} f(z) dz = 0 \quad \forall R \rightarrow f$  is holomorphic (Morera)

$f$  holomorphic and real-valued  $\Rightarrow f$  is constant.

$f(0) = 0 \Rightarrow f(z) = 0 \quad \forall z \in \mathbb{C}$ .



The first part of the paper is a
   
 description of the problem. It
   
 is a problem of finding the
   
 maximum value of a function
   
 of two variables. The function
   
 is given by the equation
   

$$z = x^2 + y^2 + 2x - 4y + 3$$



3 Let  $D \subset \mathbb{C}$  be a bounded domain,  $z_0 \in D$  and  $f: D \rightarrow D$  be a holomorphic function such that  $f(z_0) = z_0$ . Show that  $|f'(z_0)| \leq 1$ .

$D$  bounded  $\Leftrightarrow D \subset \{ |z| \leq R \}$  for some  $R > 0$ .

Define  $f_n = \underbrace{f \circ \dots \circ f}_{n\text{-times}}$

$f_n: D \rightarrow D$ , holomorphic, and  $f_n(z_0) = z_0 \forall n$ .

Furthermore  $f'_n(z_0) = (f'(z_0))^n$ .

$f_n(D) \subset D \Rightarrow |f_n(z)| \leq R \forall n$ .

$\Rightarrow |f'_n(z)| \leq M$  for some  $M \forall n$ .

by the Cauchy estimates.

$|f'_n(z_0)| = |f'(z_0)|^n \leq M$ .

Suppose  $|f'(z_0)| > 1$ .

Then  $\lim_{n \rightarrow \infty} |f'(z_0)|^n = \infty$ .

Therefore,  $|f'(z_0)| \leq 1$ .

$f_n \rightarrow z_0$  important in any way?



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4 Let  $f_n: \Delta \rightarrow \Delta$ ,  $n \geq 1$ , be a sequence of holomorphic functions such that  $f_n$  has a zero of order  $m_n$  at  $0$ , where  $\lim_{n \rightarrow \infty} m_n = \infty$ .  
 Show that  $\{f_n\}$  converges locally uniformly to zero on  $\Delta$ .

$f_n$  holomorphic with a zero of order  $m_n$  at  $0$   
 $\Rightarrow f_n(z) = \sum_{k=m_n}^{\infty} \frac{f_n^{(k)}(0)}{k!} z^k$  (1)

By Schwarz lemma,  $|f_n(z)| \leq |z| \quad \forall z \in \Delta$   
 and  $|f_n'(0)| \leq 1 \quad \forall n$ .

From (1) it is clear that  $f_n \rightarrow 0$  pointwise as  $n \rightarrow \infty$ .

$f_n(z) = z^{m_n} g_n(z)$ ,  $g_n: \Delta \rightarrow \Delta$  holomorphic and  $g_n(0) \neq 0$ .

$|f_n(z)| = |z|^{m_n} |g_n(z)| \leq |z|^{m_n} \quad |g_n(z)| \leq 1$   
 $m_n \rightarrow \infty$  and  $|z| < 1 \Rightarrow |z|^{m_n} \rightarrow 0$   
 $\forall \varepsilon > 0 \exists N \text{ s.t. } n \geq N \Rightarrow |z|^{m_n} < \varepsilon$   
 $\Rightarrow |f_n(z)| < \varepsilon$

This is just uniform — not sure



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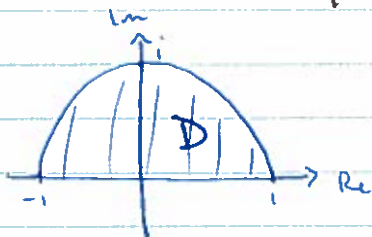
Complex, August 2015

1. Find the image of the half disc

$$D = \{z : |z| < 1, \operatorname{Im} z > 0\}$$

by the Möbius map

$$f(z) = \frac{1+z}{1-z}$$



Consider the image of the boundary of  $D$  under  $f$ .

$$-1 < x < 1, y=0 : f(x) = \frac{1+x}{1-x}$$

$$f(-1) = 0$$

$$f(1) = \infty$$

For all other  $x$  in this interval  $f(x)$  is positive, real.

$$z = e^{i\theta}, 0 < \theta < \pi : f(z) = \frac{1+e^{i\theta}}{1-e^{i\theta}}$$

$$f(i) = \frac{1+i}{1-i} = i$$

$$|f(z)| = \left| \frac{1+re^{i\theta}}{1-re^{i\theta}} \right| < \frac{1+r}{1-r}$$

not helpful

$$z = \frac{1}{2} + \frac{i}{2} : f(z) = 1 + 2i$$

$$z = -\frac{1}{2} + \frac{i}{2} : f(z) = \frac{1}{5} + \frac{2}{5}i$$

$\operatorname{Im} f(z)$  = first quadrant





*[Faint, illegible handwriting is visible in the upper portion of the page, primarily in the right-hand column.]*

2 Let  $f$  be a holomorphic function on  $\Delta \setminus \{0\}$  such that  $|f(z)| > 1$  for all  $z \in \Delta \setminus \{0\}$ . Show that  $0$  is an isolated singularity of  $f$  which is either removable or a pole.

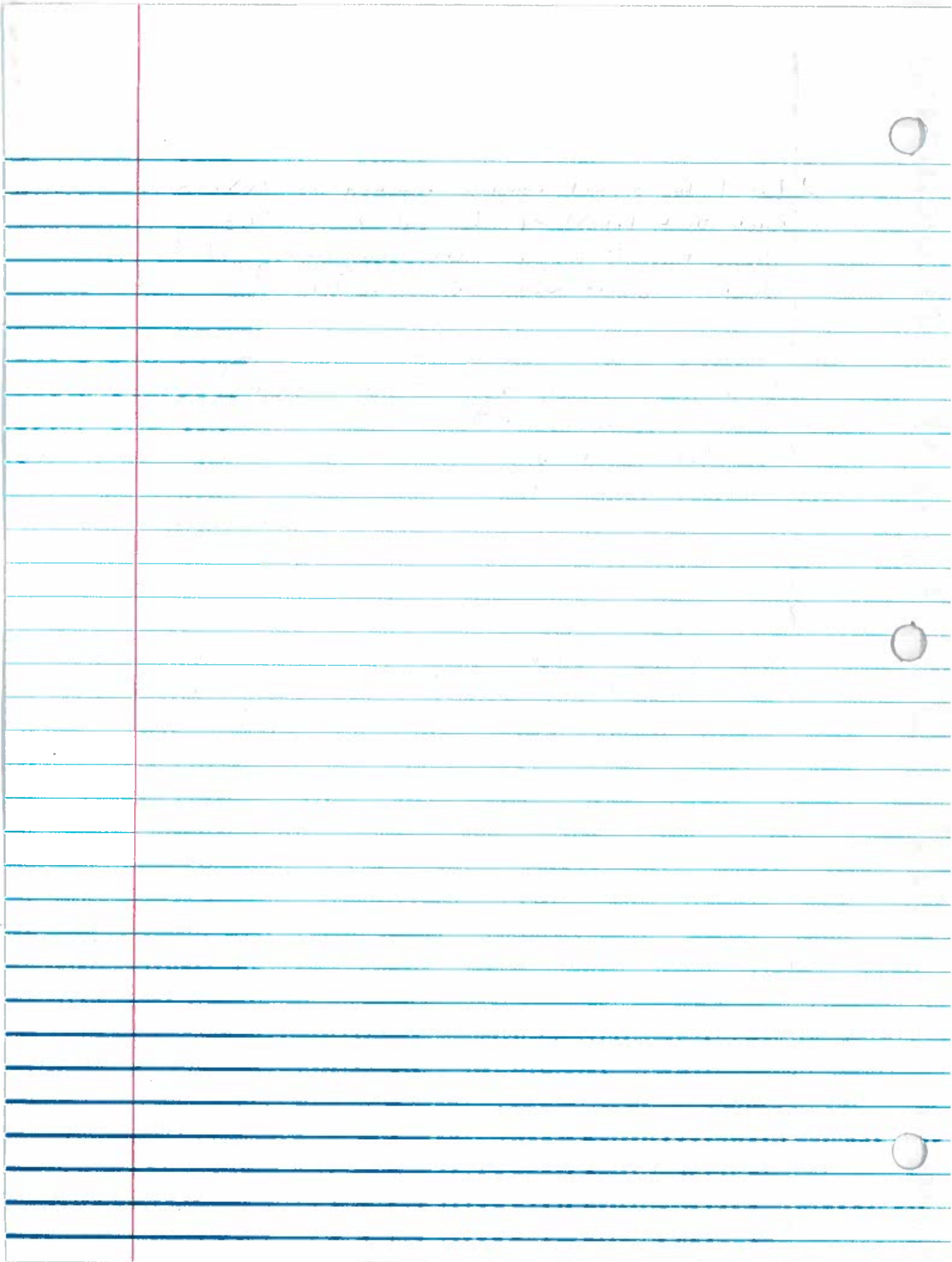
Suppose  $z=0$  is an essential singularity of  $f(z)$ . Then  $\exists$  sequence  $z_n \rightarrow 0$  such that  $f(z_n) \rightarrow 0$  (Casorati - Weierstrass).

But  $|f(z_n)| > 1 \forall n$ .

This is a contradiction.

Therefore  $z=0$  is a removable singularity or a pole.

Note: It is an isolated singularity by definition since  $f$  is holomorphic on the punctured disk centered at  $z=0$ .



## Schwarz lemma!

3 Let  $D \subset \mathbb{C}$  be a simply connected domain,  $z_0 \in D$ , and  $f: D \rightarrow \Delta$  be a conformal map such that  $f(z_0) = 0$ .

If  $g: D \rightarrow \Delta$  is a holomorphic map such that  $g(z_0) = 0$ , show that  $|g'(z_0)| \leq |f'(z_0)|$ , and that equality holds iff  $g$  is a conformal map.

$f$  conformal  $\Leftrightarrow f$  holomorphic and  $f'(z) \neq 0 \forall z \in D$   
Furthermore,  $f$  is 1-1 onto  $\Delta$ .

Therefore  $f^{-1}$  is defined and by the inverse function theorem  $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$ . ( $f^{-1}$  hol.)

Let  $h(w) = g \circ f^{-1}(w)$ ,  $h: \Delta \rightarrow \Delta$ .

As a composition of hol. functions,  $h$  is hol., and  $h(0) = g \circ f^{-1}(0) = g(z_0) = 0$ .

By the Schwarz lemma,

$$|h'(0)| = \left| \frac{g'(z_0)}{f'(z_0)} \right| \leq 1$$

$$\Rightarrow |g'(z_0)| \leq |f'(z_0)|.$$

If  $g$  is conformal, apply the same argument to get  $|g'(z_0)| \geq |f'(z_0)|$  so equality holds.

Suppose  $|g'(z_0)| = |f'(z_0)|$ , i.e.  $|h'(0)| = 1$ .

Again by the Schwarz lemma,  $h(w) = aw$ , where  $|a| = 1$ .

$$\Rightarrow h'(w) = a = \frac{g'(f^{-1}(f(z)))}{f'(z)}, \text{ where } w = f(z).$$

$$\Rightarrow g'(z) = a f'(z) \quad \forall z \in D.$$

$$\Rightarrow g'(z) \neq 0 \quad \forall z \in D \text{ since } f'(z) \neq 0.$$

Since  $g$  is holomorphic with nonzero derivative,  $g$  is conformal.

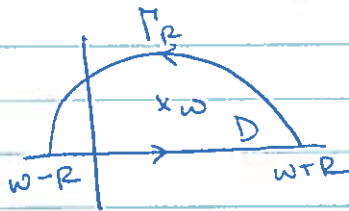




4 Compute  $F(\omega) = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2} dx$ , where  $\omega \in \mathbb{C} \setminus \mathbb{R}$ .

Hint: Consider  $\text{Im } \omega > 0$  and  $\text{Im } \omega < 0$  separately.

$\text{Im } \omega > 0$ :



$$\int_{\partial D} \frac{e^{iz}}{(z-\omega)^2} dz =$$

$$2\pi i \text{Res} \left[ \frac{e^{iz}}{(z-\omega)^2}, \omega \right]$$

$$= 2\pi i \cdot ie^{i\omega} = -2\pi e^{i\omega}$$

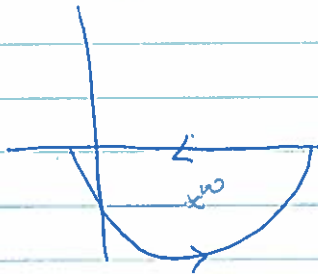
$$\int_{\partial D} = \int_{\Gamma_R} + \int_{\omega-R}^{\omega+R}$$

$$\left| \int_{\Gamma_R} \frac{e^{iz}}{(z-\omega)^2} dz \right| \leq \frac{\pi R}{R^2} = \frac{\pi}{R} \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\int_{\omega-R}^{\omega+R} \frac{e^{ix}}{(x-\omega)^2} dx \rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2} dx \text{ as } R \rightarrow \infty.$$

$$\text{So } \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-\omega)^2} dx = -2\pi e^{i\omega}$$

$\text{Im } \omega < 0$ :



Sign will be reversed

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Complex. January 2015

1 Show that  $\sum_k z^{k!}$  converges absolutely for  $|z| < 1$ .

Also show that there are infinitely many  $z$  with  $|z|=1$  for which the series diverges.

$$|\sum_k z^{k!}| \leq \sum_k |z|^{k!}$$

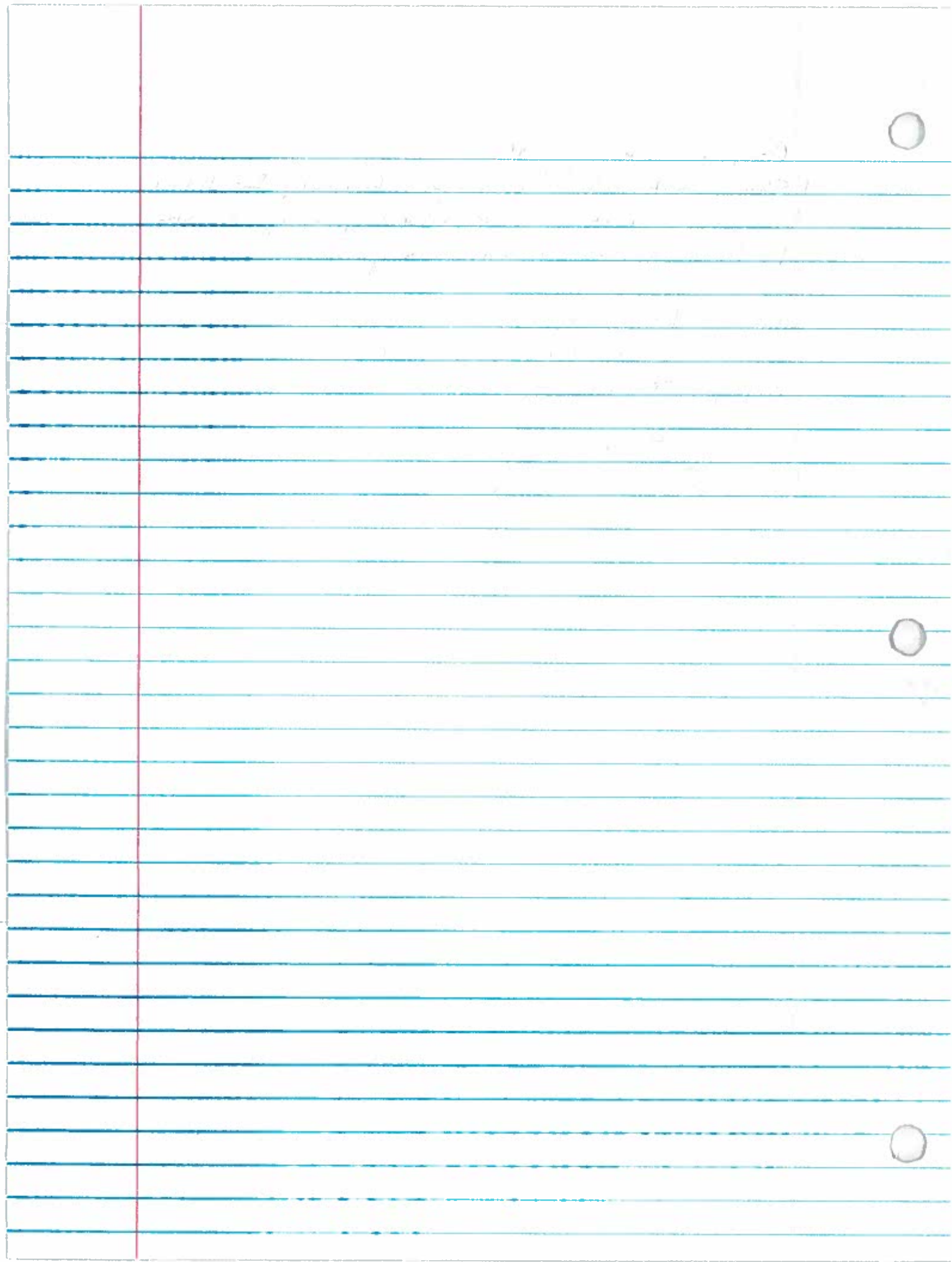
$$\leq \sum_k |z|^k \text{ when } |z| < 1$$

$$\limsup \sqrt[n]{|z|^n} = |z| < 1 \Rightarrow \text{convergence}$$

$$\text{Let } z = e^{2\pi i \frac{k}{m}}$$

$$\text{Then } \sum_{n=m}^{\infty} \frac{1}{n} z^{n!} = \sum_{n=m}^{\infty} \frac{1}{n}, \text{ which diverges.}$$





2 Let  $f(z)$  be holomorphic on  $\mathbb{C}$  except for poles.  
 At  $\infty$  assume that  $f$  has a removable singularity  
 or a pole.

(a) Show that  $f$  has finitely many poles on  $\mathbb{C} \cup \{\infty\}$ .

Suppose not.

Since  $\mathbb{C} \cup \{\infty\}$  is compact  $\Rightarrow$  the set of poles  
 has an accumulation point in  $\mathbb{C} \cup \{\infty\}$ . (at  $\infty$ )

But poles are isolated points.

This is a contradiction.

(b) Let  $p_j(z)$  be the principal part of  $f$  at the  $j$ th  
 pole,  $1 \leq j \leq N$ . Show that

$$f(z) - \sum_{j=1}^N p_j(z)$$

is constant.

$f(z) - \sum_{j=1}^N p_j(z)$  is entire:

If  $z_i$  is a pole of  $f$ ,

$$f(z_i) - p_i(z_i) = \sum_{j \neq i} p_j(z_i)$$

$$= \underbrace{\sum_{k=0}^{\infty} a_k (z_i - z_i)^k}_{\text{analytic}} = \underbrace{\sum p_j(z_i)}_{\text{analytic at } z_i}$$

Because the singularity at  $\infty$  is also removed,

we see that  $f(z) - \sum p_j(z)$  is bounded

for  $|z| > R$  (In fact I think it must  $\rightarrow 0$ )

But  $f(z) - \sum p_j(z)$  is clearly bounded on  
 compact sets  $|z| \leq R$ .

$\Rightarrow f(z) - \sum p_j(z)$  is bounded everywhere.

By Liouville's theorem, it is constant.



3 Let  $f$  be continuous on  $\mathbb{C}$  and analytic except possibly on the unit circle  $|z|=1$ . Assume there is an entire function  $g$  st  $f(z)=g(z)$  for  $|z|=1$ .  
Prove that  $f=g$ .

On ~~the~~ August 2016.





4) Let  $f_n$  be analytic in the unit disc  $D$  and have positive real part:  $\operatorname{Re}(f_n(z)) > 0$ . Assume that  $f_n$  converges pointwise on  $D$  to a function  $f$  having  $\operatorname{Re}(f(z)) \leq 0$  on  $D$ . Prove that  $f$  is constant.

Let  $f_n = u_n + i v_n$

$f_n$  converges pointwise  $\Leftrightarrow u_n$  and  $v_n$  do.

$u_n(x, y) \rightarrow u(x, y)$  where  $u_n(x, y) \geq 0$  and  $u(x, y) \leq 0$ .

$\Rightarrow u(x, y) = 0$  for each  $x, y$ .

$\Rightarrow f$  is purely imaginary.

By Osgood's theorem,  $\exists$  an open dense subset  $V$  of  $D$  st  $f$  is holomorphic on  $V$ .

$f$  holomorphic and purely imaginary

$\Rightarrow f$  is constant on  $V$ .

Solutions use Montel's theorem and Open mapping theorem

$e^{-f_n} = h_n$  — normal family

$h_n: D \rightarrow D$  analytic

Some subsequence converges uniformly on compact subsets to  $h = e^{-f}$  — analytic

$|h(z)| = 1$  since  $\operatorname{Re} f = 0$

$\Rightarrow h$  is constant by the open mapping theorem

$\Rightarrow f$  constant

Open mapping thm:  $U$  domain,  $f: U \rightarrow \mathbb{C}$  nonconst. and hol; then  $f$  is an open map



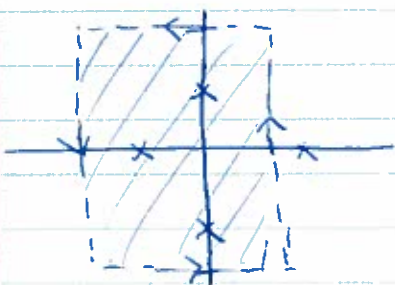
Complex, August 2016

1. Let  $C$  denote the boundary of the domain

$$D = \{z \in \mathbb{C} : -2 < \operatorname{Re} z < \frac{1}{2}, | \operatorname{Im} z | < 2\}.$$

Find  $I = \int_C \frac{z^n}{z^4 - 1} dz$  where  $n \geq 0$  is an integer.

Write your answer in algebraic form  $a+ib$ .



$$\frac{z^n}{z^4 - 1} = \frac{z^n}{(z+1)(z-1)(z+i)(z-i)}$$

By the Residue theorem

$$\int_C \frac{z^n}{z^4 - 1} dz = 2\pi i \sum \operatorname{Res}$$

$$\operatorname{Res} \left[ \frac{z^n}{z^4 - 1}, i \right] = \lim_{z \rightarrow i} \frac{z^n}{(z^2 - 1)(z + i)} = -\frac{i^n}{4i}$$

$$\operatorname{Res} \left[ \frac{z^n}{z^4 - 1}, -i \right] = \lim_{z \rightarrow -i} \frac{z^n}{(z^2 - 1)(z - i)} = \frac{(-i)^n i^n}{4i}$$

$$\operatorname{Res} \left[ \frac{z^n}{z^4 - 1}, 1 \right] = \lim_{z \rightarrow 1} \frac{z^n}{(z+1)(z^2 + 1)} = -\frac{(-1)^n}{4}$$

$$n = 0 \pmod{4} : I = -\frac{\pi i}{2}$$

$$n = 1 \pmod{4} : I = -\frac{\pi i}{2}$$

$$n = 2 \pmod{4} : I = -\frac{\pi i}{2}$$

$$n = 3 \pmod{4} : I = \frac{3\pi i}{2}$$



10/18 - 1/10

Answer: The following is a list of the

names of the people who were

involved in the project.

The names are listed in

alphabetical order.

The names are listed in

alphabetical order.

The names are listed in

alphabetical order.

The names are listed in

alphabetical order.

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alphabetical order.

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alphabetical order.

The names are listed in

2. Let  $f$  be continuous on  $\mathbb{C}$  and analytic except possibly on the unit circle  $\{|z|=1\}$ . Suppose that there is an entire function  $g$  such that  $f(z)=g(z)$  for  $|z|=1$ . Prove that  $f=g$ .

Note: can't apply Cauchy integral formula directly since we don't know  $f$  extends smoothly just that it's continuous (might be able to show though)

Take in  $\nearrow 1$ .

$$f \in A(\Delta) \text{ so } f(z) = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta}) r e^{i\theta}}{r e^{i\theta} - z} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta}) e^{i\theta}}{e^{i\theta} - z} d\theta \quad \text{BCT, } f \text{ cont.}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{g(e^{i\theta}) e^{i\theta}}{e^{i\theta} - z} d\theta = g(z) \quad |z| < 1$$

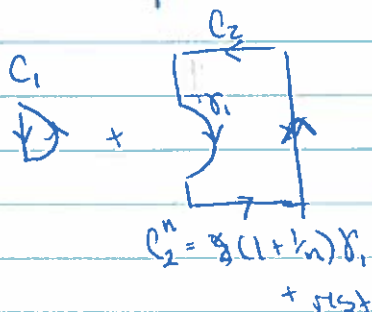
since  $g$  entire

$\Rightarrow f=g$  for  $|z| \leq 1$ .

Let  $R$  be a rectangle in  $\mathbb{C}$  with sides parallel to the coordinate axes which intersects  $\Delta$ .



$\int_{\partial R} f dz$  can be split into closed paths in  $|z| \leq 1$  and  $|z| \geq 1$  by including the intersected arc of  $|z|=1$ .



$$\int_{C_1} f(z) dz = \int_{C_1} g(z) dz = 0 \text{ by Cauchy}$$

$$\int_{C_2} f(z) dz = \lim_{n \rightarrow \infty} \int_{C_2^n} f(z) dz = 0 \text{ by Cauchy since } f \text{ analytic outside of } \Delta$$

of rectangle

$\Rightarrow f$  is analytic in  $\mathbb{C}$  by  $\xi$  Morera.

Since  $f=g$  on  $\Delta$ , by the uniqueness principle,  
 $f \equiv g$  on  $\mathbb{C}$ .

3 Let  $S$  be a square with center at the origin.  
Suppose that  $F: \Delta \rightarrow S$  is analytic, one-to-one,  
and onto, and furthermore, that  $F(0) = 0$ .  
Show that  $F(iz) = iF(z)$  for all  $z \in \Delta$ .

On final





The first part of the paper is a
   
 very short introduction. It
   
 is followed by a section on
   
 the history of the subject.



4 Prove that there are no nonconstant polynomials of the form  $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$  that satisfy  $|p(z)| \leq 1$  when  $|z| = 1$ .

Suppose  $p$  is such a polynomial.

Then  $p^{(n)}(z) = n!$  for all  $z$ .

However, by the Cauchy estimates

$$|p^{(n)}(0)| < n! \quad |z| < 1$$

This is a contradiction.

$\Rightarrow p$  must be constant.

Handwritten notes in the top section of the page, including a title and several lines of text.

