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Complex Analysis Review Sheet

Chapter 1–The Complex Plane and Elementary Functions

Definition. Basic Definitions

- $z = x + iy$
- $x = Rez = \boxed{\frac{z+\bar{z}}{2}}$
- $y = Imz = \boxed{\frac{z-\bar{z}}{2i}}$
- $|z| = \sqrt{x^2+y^2}$ = **modulus** of z
- $z\bar{z} = |z|^2$
- $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ So on $|z|=1$, $\bar{z} = \frac{1}{z}$
- $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$
- $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$ sin has i and $-$
- $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

forms of trig functions

Definition. $G \subset \mathbb{C}$ is **connected** if for every two points in F there exists a broken line segment connecting them contained completely in G.

Definition. A **domain** is both open and connected.

closure, ϕ, boundary

Proposition. *$z_0 \in \bar{E}$ if and only if for all $r > 0$, $\Delta(z_0, r) \cap E \neq \phi$. $z_0 \in \delta E$ if and only if for all $r > 0$, $\Delta(z_0, r) \cap E \neq \phi$ and $\Delta(z_0, r) \setminus E \neq \phi$.*

Definition. $(x, y) \longrightarrow (r, \theta)$ where $r = |z|$ and $tan\theta = \frac{y}{x}$. θ is the **argument** of z. $argz = \{\theta + 2k\pi\}$ and $Argz = \theta \in [-\pi, \pi]$.

Proposition. *(DeMoivre's Formula)* $(cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$

Definition. ***n roots of** ω*: $z_k = |\omega|^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}$ *for* $n = 0, 1, \ldots, n-1$.   n roots of 1 $z_k = e^{\frac{i2k\pi}{n}}$

A **generalized circle** in $\mathbb{C}^*$ means a line or a circle.

Chapter 2–Analytic Functions

Definition. $U_{open} \subset \mathbb{C}$, $f : U \to \mathbb{C}$, $z_0 \in U$. f is **differentiable** if there exists $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$.   or check $\frac{\partial}{\partial \bar{z}} = 0$

Theorem. *f differentiable at z_0 means that f is continuous at z_0.*

~~so don't need to check continuity for~~

analytic means differentiable! w/ continuous derivative.

Definition. $U_{open} \subset \mathbb{C}$, $f : U \to \mathbb{C}$, f is **holomorphic** or **analytic** if f is differentiable at each $z \in U$ and $f'(z)$ is continuous on U.

don't need to check f' cont by Goursat

Theorem. *(Cauchy Reimann Equations) $f : D_{open} \to \mathbb{C}$ be holomorphic on D if and only if $u, v \in C'(D)$ and they verify*

$$\boxed{\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}}$$

$u_x = v_y$

$u_y = -v_y$

or

$$\boxed{\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial u}{\partial \theta} &= -r \cdot \frac{\partial v}{\partial r}\end{aligned}}$$

$u_r = \frac{1}{r} v_\theta$ or $r u_r = v_\theta$

$u_\theta = -r v_r$ $\quad$ $u_\theta = -r v_r$

Moreover, if f holomorphic then

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

i.e. f is holomorphic if and only if Df is $\mathbb{C}-$linear.

Theorem. *$f : D_{open} \to \mathbb{C}$ be holomorphic on D and $f'(z) = 0$ for all $z \in D$ then f is constant.*

Theorem. *If f is holorphic on domain and real valued then f is constant. (Also holds for constant imaginary functions)*

holomorphic + real valued $\Rightarrow$ f constant

Definition. Jf=jacobian= $\boxed{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}}$

$\det Jf = |f'|^2 \geq 0$

Proposition. *$f : D \to f(D)$, holomorphic, 1-to-1, D bounded domain in $\mathbb{C}$. $f(D)$ bounded, h continuous on $\overline{f(D)}$. Then*

$$\iint_{f(D)} h d\lambda = \iint_D h \circ f \, |f'|^2 \, d\lambda$$

Theorem. *(Inverse Function Theorem) f holomorphic on domain D, $f'(z_0) \neq 0$ for some $z_0 \in D$. Then there exists $U \subset D$ open, $z_0 \in U$ so that $V = f(U)$ is open, $f : U \to V$ bijectie, $f^{-1} : V \to U$ is holomorphic and $(f^{-1})(f(z)) = \frac{1}{f'(z)}$, $z \in U$.*

$f'(z_0) \neq 0$
$\Rightarrow f$ is bijective in some neighborhood of z_0.

Definition. $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ =the Laplacian of u

$= 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$= 4\frac{\partial^2}{\partial \bar{z} \partial z}$

Definition. $u : D_{open} \to \mathbb{R}$, $u \in C^2(D)$ is harmonic if $\Delta u = 0$

$\Rightarrow u = $ ref for analytic f.

$f = u + iv : D_{open} \to \mathbb{C}$, $f \in C^2(D)$ is **harmonic** if $\Delta f = \Delta u + i\Delta v = 0$.

Theorem. *If $f = u + iv$ is holomprhic on D and $f \in C^2(D)$ then u, v are harmonic on D.*

Definition. If $u : 0 \to \mathbb{R}$ is harmonic then we call $v : D \to \mathbb{R}$ a **harmonic conjugate** of u in D if $u + iv$ is holomorphic in D

*v is harmonic

*conjugates differ by a constant

Proposition. *If $D = \Delta(z_0, r)$ then $u : D \to \mathbb{R}$ has a harmonic conjugate in D.*

Remark. To find harmonic conjugates

(1) v_x, v_y
(2) $v(x, y) = v_x + h(y)$
(3) $v_y(x, y)$ and compare to find $h'(y)$
(4) integrate to find $h(y)$

steps to find harmonic conjugate.

or compare $v(x, y) = v_y + g(x)$ and $v(x, y) = v_x + h(y)$

Theorem. *(Chain Rule) f holomorphic on D, $\gamma : [0, 1] \to D$ a smooth C' path, then $\frac{d}{dt} f \circ \gamma(t) = f'(\gamma(t))\gamma'(t)$*

Definition. The complex Cauchy Reimman Equation:

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}$$

Definition. A C' map f defined near z_0 is **conformal at z_0** if it preserves angles at z_0.

Definition. A C' map $f : 0 \to \mathbb{C}$ is **conformal on D** if f is injective and conformal at each $z_0 \in D$. Then $D, f(d)$ are **conformally equivalent.**

Conformal $\Rightarrow$ injective and holomorphic.

Theorem. *If f is differentiable at z_0 with $f'(z_0) \neq 0$ then f is conformal at z_0.*

Conformal and nonzero derivative go together.

Theorem. *If f is of class C' near z_0 and conformal at z_0 then f is differentiable at z_0 and $f'(z_0) \neq 0$.*

Converse of thm above.

Definition. Let $z_0, z_1, z_2, z_3 \in \mathbb{C}$ be distinct. The cross ratio

$$\boxed{(z_0, z_1, z_2, z_3) = \frac{z_0 - z_1}{z_0 - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}}$$

evens and odds flip flops

then the cross ratio in $\mathbb{C}*$ of z_0, z_1, z_2, z_3 distinct is

$$(z_0, z_1, \infty, z_3) = \frac{z_0 - z_1}{z_0 - z_3}$$

Proposition. *If f is a mobius map and $z_0, z_1, z_2, z_3 \in \mathbb{C}$ are distinct then $(f(z_0), f(z_1), f(z_2), f(z_3)) = (z_0, z_1, z_2, z_3)$.*

Corollary. *If $z_1, z_2, z_3 \in \mathbb{C}^*$ are distinct and $w_1, w_2, w_3 \in \mathbb{C}^*$ are distinct then there exists a unique mobius map f such that $f(x_j) = w_j$ for $j = 1, 2, 3$.*

Proposition. *Mobius maps map circles in $\mathbb{C}^*$ onto circles in $\mathbb{C}^*$.*

$$\boxed{w = \frac{az + b}{cz + d}, z = \frac{dw + b}{-cw + a}}$$

$ad - bc \neq 0$

the image of $Az\bar{z} + Bz + \bar{B}\bar{z} + c = 0$

Proposition. *$z_0, z_1, z_2, z_3 \in \mathbb{C}^*$ distinct lie on a line or cirlce if and only if $(z_0, z_1, z_2, z_3) \in \mathbb{R}$.*

Remark. Γ a circle in $\mathbb{C}^*$, f a mobius map, pole at z_0. Then if $z_0 \in \Gamma$ then $f(\Gamma)$ is a line. If $z_0 \notin \Gamma$ then $f(\Gamma)$ is a circle. f maps circles to circles (or lines)

if pole is on circle its mapped to line.

Chapter 3–Line Integrals and Harmonic Functions

- A **path** in Cfrom z_0 to z_1 is a continuous $\gamma : [0, 1] \to$ Csuch that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.
- A **simple path** is injective doesnt cross
- A **closed path** has $\gamma(0) = \gamma(1)$ starts + ends at same place.
- A path is **simple and closed** if $0 \leq s < t < 1$ then $\gamma(s) \neq \gamma(t)$ and $\gamma(0) = \gamma(1)$
- $\gamma = x + iy$ is **smooth** if and only if x, y are smooth
- γ is **piecewise smooth** if and only if it is a concatenation of finely make smooth paths
- $L(\gamma) = \sup_{\Pi} \sum_1^n |\gamma(t_j) - \gamma(t_{j-1})| \in [0, \infty)$. **Length** of γ.
- γ is **rectifiable** if $L(\gamma) < \infty$
- $\int_\gamma P dx + Q dy = \int_0^1 (P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t))\, dt$ line integral formula
  - constant over reparametrixation
  - $\int_{\gamma_1 + \gamma_2} = \int_{\gamma_1} + \int_{\gamma_2}$ and $\int_{-\gamma} = -\int_\gamma$

Theorem. *$\gamma_n, \gamma : [0,1] \to D$ rectifiable and γ_n converges uniformly to γ on $[0,1]$ and $L(\gamma_n) \leq M$ for some M and all n, $P, Q \in C(D)$. Then $\lim\limits_{n\to\infty} \int_{\gamma_n} Pdx + Qdy = \int_\gamma Pdx + Qdy$*

Theorem. *(Green's Theorem) Let D be a bounded domain with piecewise smooth boundary with positive orientation. Let P, Q be complex valued of class C' defned in a neighborhood of $\bar{D}$. Then $\int_{\gamma\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.*

- Let $h : D \subset \mathbb{C} \to \mathbb{C}$ in C'. Then the **differential** of h is $\partial h = \frac{\partial h}{\partial x}dx - \frac{\partial h}{\partial y}dy$. $\partial h(x_0, y_0)(u,v) = \frac{\partial h}{\partial x}(x_0,y_0)u + \frac{\partial h}{\partial y}(x_0,y_0)v$.
- $w = Pdx + Qdy$ is a differential 1-form.
  - w is **exact** in F if there exists $h \in C'(D)$ such that $w = dh$
    - $P = \frac{\partial h}{\partial x}, Q = \frac{\partial h}{\partial y}$
  - w is **closed** if $P, Q \in C'(D)$ and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ i.e. $dw = 0$

Theorem. *$D \subset \mathbb{C}$ a domain. $P, Q : D \to \mathbb{C}$ continuous.*

(1) If $P, Q \in C'(D)$ and w is exact them w is closed
(2) $\int Pdx + Qdy$ is path independent if and only if w is exact. If $w = dh$, $h \in C'(D)$ then h is unique up to adding a constant
(3) For γ rectifiable, $\int_\gamma w = \int_\gamma dh = h(\gamma(1)) - h(\gamma(0))$
(4) D a disk then w closed implies w exact.
(5) $w = Pdx + Qdy$, $\gamma = \delta u$. If $u \subset D$ then $\int_{du} w = 0$. $\leftarrow$ means w is independent of path.

Definition. A domain D is called **star shaped** if there exists $z_0 \in D$ such that for all $z \in D$, $[z_0, z] \subset D$.

Vowels

$$\begin{tikzcd} \text{exact} \arrow[rr, Leftrightarrow] \arrow[dr] & & \text{independent of path} \arrow[dl] \\ & \text{Closed} & \end{tikzcd}$$

Proposition. *Star shaped domains are simply connected.*

Theorem. *Let $w = Pdx + Qdy$ be a closed form on D.*

(1) *If $\gamma_0, \gamma_1 : I \to D$. $\gamma_0(0) = \gamma_1(0) = z_0$, $\gamma_0(1) = \gamma_1(1) = z_1$, are rectifiable and path homotopic in D then $\int_{\gamma_0} w = \int_{\gamma_1} w$.*
(2) *If $\gamma_0, \gamma_1 : I \to D$ are closed paths homotopic in D then $\int_{\gamma_0} w = \int_{\gamma_1} w$.*

Lemma. *$u \in C^2(D, \mathbb{R})$ a domain D. $w = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$*

(1) *u is harmonic in $D \Leftrightarrow w$ is closed* harmonic = closed
(2) *If u is harmonic in D then w is exact $\Leftrightarrow u$ has a harmonic conjugate v and in this case $w = dv$*

Theorem. *A harmonic function $u : D \to \mathbb{R}$ on a simply connected domain D has a harmonic conjugate v given $v(z) = \int_{z_0}^{z} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ integral is along any rectifiable path in D joining z_0 to z.*

Remark.

$$\begin{aligned}
\text{u has harmonic conjugate} &\Leftrightarrow \int_{\gamma_r} -\tfrac{\partial u}{\partial y}dx + \tfrac{\partial u}{\partial x}dy = 0\\
&\Leftrightarrow w = \tfrac{\partial u}{\partial y}dx + \tfrac{\partial u}{\partial x}dy \text{ exact}\\
&\Leftrightarrow \int_{\gamma} -\tfrac{\partial u}{\partial y}dx + \tfrac{\partial u}{\partial x}dy \text{ is path independent in } D\\
&\Leftrightarrow \int_{\gamma} -\tfrac{\partial u}{\partial y}dx + \tfrac{\partial u}{\partial x}dy = 0 \text{ for all } \gamma \text{ closed rectifiable path in } D
\end{aligned}$$

Theorem. *Closed differential forms in simply connected domains are exact.*

Definition. **Average Value** of $h(z)$ on the circle $\{|z - z_0| = r\}$ is $\boxed{A(r) = \int_0^{2\pi} h\left(z_0 + re^{i\theta}\right) \frac{d\theta}{2\pi}.}$

Theorem. *If $u(z)$ is a harmonic function on a domain D and if the disk $\{|z - z_0| < \rho\}$ is contained in D then $u(z_0) = \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) \frac{d\theta}{2\pi}$. i.e. the average value of a harmonic function on the boundary circle of any disk contained in D is its value at the center of the disk.*

Definition. $h(z)$ has the **Mean Value Property** if for each point $z_0 \in D$, $h(z_0)$ is the average of its values over any small circle centered at z_0.

Proposition. *Harmonic functions have the mean value property.*

Theorem. ***ML-estimate*** $\left|\int_\gamma h(z)dz\right| \le \int_\gamma |h(z)|\,|dz| \le M \cdot L(\gamma)$ *if* $|h| \le M$

Theorem. *$f : D \to \mathbb{C}$ continuous. Then f has a primitive if and only if $\int_\gamma f(z)dz = 0$ for every closed rectifiable path in D.*
an antiderivative.

needs to be holomorphic across bdr

Theorem. ***Cauchy's Theorem.*** *Let D be a bounded domain with piecewise smooth boundary. Let f be in class C' in a neighborhood of $\bar{D}$, f holomorphic in D. Then $\int_{\partial D} f(z)dz = 0$*

Theorem. ***Strict Maximum Principle.*** *Let h be a bounded complex valued harmonic function on a domain D. If $|h(z)| \le M$ for all $z \in D$ and $|h(z_0)| = M$ for some $z_0 \in D$ then $h(z)$ is constant on D. i.e. If a function is bounded on a domain and the function attains that maximum on the interior of the domain then the function is constant on the domain.* so non constant functions attain the max on the boundary of the domain.

.

Theorem. *(Cauchy's Integral Formula.* *D bounded domain with piecewise smooth boundary, Let f be in class C' in a neighborhood of $\bar{D}$, f holomorphic in D. (i.e. If $f(z)$ is analytic on D and $f(z)$ extends smoothly to the boundary of D) Then*

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta}$$

for all $z \in D$.

very useful!

Corollary. *If f and D as above, then f has all complex derivatives given by*

$$\boxed{f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta - z)^{m+1}}d\zeta}$$

Corollary. *If f is holomprhic on domain D then all compelx derivatives exist and are holomorphic.*

Corollary. *If $f = u + iv$ is holomprhic on domain D then u, v are harmonic in D.*

Corollary. *Holomorphic functions satisfy both maximum principles for complex valued functions.*

Theorem. *(Cauchy's Estimate) If f is holomprhic in $\overline{\Delta}(z_0, \rho)$ and $|f(z)| \le M$ on $|z - z_0| = \rho$ then $|f^{(m)}(z_0)| \le \frac{m!M}{\rho^m}$*

Definition. A function holomporhic on $\mathbb{C}$ is called *entire.*

Theorem. *(Louiville's Theorem) A bounded entire function is constant.*

Remark. If $|f(z)| \le CR^n$ with $|z| = R$ and $R \ge R_0$ then f is a polynomial of degree of at most n.

Theorem. *(Morera's Theorem) Let $f(z)$ be a continuous function on a domain D. If $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in D with sides parallel to the coordinate axes then $f(z)$ is analytic on D.*

Theorem. *$D \subseteq \mathbb{C}$. $[a, b] \subset \mathbb{R}$. If $h : D \times [a, b] \to \mathbb{C}$ is continuous and $h(\cdot, t) : D \to \mathbb{C}$ is holomorphic for $t \in [a, b]$ then for $t \in [a, b]$, $H(z) = \int_a^b h(z, t)dt$ is holomorphic on D.*

can rotate omitting one
so line is ok.

Theorem. *If f is continuous on D and holomorphic on $D \setminus \mathbb{R}$ then f is holomorphic on D.*

Theorem. *(Goursat's Theorem) Let $f : D \to \mathbb{C}$ is complex differentiable at each point of D. $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ for all $z_0 \in D$. Then f is holomorphic on D.*

Proposition. ~~(Pompein's Formula)~~ $\boxed{\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}$ no i's.

not called this
but useful

Theorem. *If $f \in C'(D)$. Then f is holomorphic if and only if $\frac{\delta f}{\delta \bar z} = 0$ on D.* Good way to check differentiable

Definition. *f is **antiholomorphic** if $\bar f$ is holomorphic.*

Proposition. *$f \in C'$ antiholomorphic if and only if $\frac{\delta \bar f}{\delta \bar z} = \overline{\frac{\delta f}{\delta z}} = 0$ if and only if $\frac{\delta f}{\delta z} = 0$. i.e. f depends only on $\bar z$.*

obvious opposite of theorem above.

Remark.

$$\begin{aligned}
\delta f(0) \text{ is } \mathbb{C}\text{-linear} &\Leftrightarrow df(0)(iz) = idf(0)(z)\\
&\Leftrightarrow \tfrac{\delta f}{\delta \bar z}(0) = 0\\
&\Leftrightarrow f \text{ differentiable at } 0\\
&\Leftrightarrow f \text{ complex differentiable at } z_0
\end{aligned}$$

Theorem. *(Chain Rule) $f : D \to \mathbb{C}$, $\gamma : [0,1] \to D \in C^1$ with $\gamma(0) = z_0$. $\frac{d}{dt}f(\gamma(t)) = \frac{\delta f}{\delta z}(\gamma(t))\,\gamma'(t)\frac{\delta f}{\delta \bar z}\overline{\gamma'(t)}$.*

Theorem. *$f : D \to \mathbb{C}$, $D \in C^1$, $z_0 \in D$. If f is conformal at z_0 then f is complex differentiable at z_0 with $f'(z_0) \neq 0$.* conformal => $f'(z_0) \neq 0$ and differentiable.

~~**Theorem.** *(Green's Theorem) $f : D \to \mathbb{C}$, $\gamma : [0,1] \to D \in C^1$ with $\gamma(0) = z_0$. $\frac{d}{dt}f(\gamma(t)) = \frac{\delta f}{\delta z}(\gamma(t))\gamma'(t)\frac{\delta f}{\delta \bar z}\overline{\gamma'(t)}$.*~~

(Pompein's Formula) If $f : D \to \mathbb{C}$, $\gamma : [0,1] \to D \in C^1$ with $\gamma(0) = z_0$. Then

$$\boxed{g(z) = \frac{1}{2\pi i}\int_{\delta D} \frac{g(\zeta)}{\zeta - z}d\zeta + \frac{1}{\pi}\int\int_D \frac{\delta g}{\delta \zeta}\frac{1}{\zeta - z}dxdy}$$

Only continuous Not holomorphic needs correction term

CHAPTER 5–POWER SERIES

Definition. Let $f_j, f : E \subset \mathbb{C} \to \mathbb{C}$.

1. f_j **converegs pointwise** to f if for all $z \in E$, $f_j(z) \to f(z)$.
2. f_j **converegs uniformly** to f on E if $\forall \epsilon > 0$, $\exists j_0 = j_0(\epsilon)$ such that $j \geq j_0$, $z \in E$ $|f_j(z) - f(z)| < \epsilon$. Equivalently,

$$\|f_j - f\|_{L^\infty(E)} = \|f_j - f\|_\infty = sup\{|f_j(z) - f(z)| : z \in E\} < \epsilon$$

3. $\sum_{j=1}^{\infty} f_j$ is **convergent pointwise/uniformly** if the partial sums converge pointwise/uniformly.

Theorem. (Weierstrass M-test) If $|f_j(z)| \leq M_j$, $\forall z \in E$ and $\sum_{j=1}^{\infty} M_j < \infty$ then $\sum_{j=1}^{\infty} f_j$ converges uniformly on E.

useful result from 601

uniformly convergent cont. fcns converge to continuous fcns.

Theorem. *1. f_j continuous on E and $f_j \to f$ uniformly on E then f is continuous on E.*

2. $\gamma : [0,1] \to \mathbb{C}$ rectifiable, $K = \gamma([0,1])$, $f_j : K \to \mathbb{C}$ continuous. If $f_j \to f$ uniformly on K then $\lim_{j\to\infty} \int_\gamma f_j(z)dz = \int_\gamma f(z)dz$.

Definition. Let $f_j, f : D \to \mathbb{C}$, D an open set in $\mathbb{C}$. f_j **converges normally (or locally uniformly)** to f if $\forall K \subset D$ is compact then $f_j \to f$ uniformly on$K$.

uniformly on every compact subset. or just consider closed disk.

Montel

Uniformly bounded family of holomorphic functions defined on an open subset of $\mathbb{C}$ is normal

Lemma. *f_j converges normally to f on $D \Leftrightarrow \forall z \in D$, $\exists \bar{\Delta}(z, r_2) \subset D$, $r_2 > 0$ such that $f_j \to f$ uniformly on $\bar{\Delta}(z, r_2)$.*

Theorem. *Suppose f_j is holomorphic on D, $f_j \to f$ normally on D. Then f is holomorphic on D and for $m > 0$ $f_j^{(m)} \to f^{(m)}$ normally on D.* derivatives also converge normally.

normally convergent holomorphic fcns converge to a holomorphic fcn.

Lemma. *Let f_j be holomorphic in $\bar{\Delta}(z_0, R) = \{|z - z_0| \leq R\}$ and $f_j \to f$ uniformly on $\bar{\Delta}(z_0, R)$. Then f is continuous of $\bar{\Delta}(z_0, R)$ and holomorphic on $\Delta(z_0, R)$ and $\forall m > 0$, $f_j^{(m)} \to f^{(m)}$ normally i.e. uniformly on every $\Delta(\bar{z_0}, r)$ for $r \subset R$.*

Theorem. *Assume $\sum_{n=0}^{\infty} a_n (z - z_0)^n = f(z)$ has radius of convergence $R > 0$/ Then f is holomorphic in $\Delta(z_0, R) = \{|z - z_0| < R\}$ and*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)a_n (z - z_0)^{n-k}$$

Can differentiate and integrate term by term if holomorphic

in $\Delta(z_0, R)$, $a_k = \frac{f^{(k)}(z_0)}{k!}$. A primitive F of f is $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$ is unique with $F(z_0) = 0$.

Definition. A function $f : D \to \mathbb{C}$ is **analytic** in D if $\forall z_0 \in D$, $\exists \Delta(z_0, R) \subset D$, $R > 0$ such that $\sum_{n=0}^{\infty} a_n (z - z_0)^n = f(z)$ for $z \in \Delta(z_0, R)$.

analytic functions can be represented by a power series.

Theorem. *$f : D \to \mathbb{C}$. Then f is holomorphic on D if and only if f is analytic on D.*

alternate formula for a_n

formula for a_n

Theorem. *If f is holomorphic in $\{|z - z_0| < \rho\}$ then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$ for $r < \rho$ and power series has radius of convergence $R \geq \rho$ if $|f(z)| \leq M$ on $|z - z_0| = r$ then $|a_n| \leq r^n$ for $n \geq 0$.*
i.e. If f is holomorphic it can be represented by a power series.

Corollary. *If f, g are holomorphic in $\Delta(z_0, \rho)$ and $f^{(n)}(z_0) = g^{(n)}(z_0)$, $\forall n \geq 0$ then $f = g$ in $\Delta(z_0, \rho)$.*

Corollary. *If f is holomorphic at z_0 i.e. in a neighborhood of z_0 then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in small disk $\Delta(z_0, \rho)$ then the radius convergence of the power series is the largest R for which f extends holomorphically to $\Delta(z_0, R)$.* largest distance can go before reaching a problem area.

Definition. $f : \{|z| > R\} \to \mathbb{C}$ holomorphic is **holomorphic at ∞** if $\exists l = \lim_{z \to \infty} f(z) \in \mathbb{C}$. finite limit

Definition. If $f : D \to \mathbb{C}$ is holomorphic and $f \not\equiv 0$. Then say $z_0 \in D$ is a **zero** of f of order N is $f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ and $f^{(N)}(z_0) \neq 0$.

Definition. f is holomorphic at ∞ has a **zero of order N at ∞** $(ord(f, \infty) = N)$ if $g(w) = f(\frac{1}{w})$ has a zero of order N at 0. and $f \not\equiv 0$.

identity thm

Theorem. *Let f be holomorphic on a domain $D \subset \mathbb{C}$, $E = \{z \in D : f(z) = 0\}$. If E has a limit point in D then $f = 0$ on D. Equivalently, if $f \not\equiv 0$ on D then its zeros are isolated points of D. So there are at more countably many zeros and if infinitely many zeros then cluster at δD.*

Theorem. *(Identity Principle) Let f, g be holomorphic on domain D. Then if $E = \{z \in D : f(z) = g(z)\}$ and $E' \cap D \neq \phi$ then $f = g$.*

Theorem. *Let $D \subset \mathbb{C}$, a domain, $e \subset D$ that has a limit point in D. Let $F : D \times D \to \mathbb{C}$ be holomorphic in each variable seperately. Then $F(z,w) = 0\ \forall z,w \in D$ if $F(z,w) = 0,\ \forall z,w \in F$.*

Definition. A ***function element*** *at z_0 is a pair (D,f) where D is an open disc centered at z_0 and f is a function holomorphic on D.*

(1) Two function elements (D_1, f_1) and (D_2, f_2) at z_0 are **equivalent** if $f_1 = f_2$ on $D_1 \cap D_2$. The equivalent classes are called **germs.**
(2) (D_0, f_0) at z_0, (D_1, f_1) at z_1 are funtion elements. Say (D_1, f_1) is an **analytic continuation** of (D_0, f_0) if there exists function elements (V_0, g_0) at z_0, (V_1, g_1) at $z_1, \cdots, (V_n, g_n)$ at z_n so that $(V_0, g_0) \backsim (D_0, f_0)$ and $(V_n, g_n) \backsim (D_1, f_1)$ and $V_{j-1} \cap V_j \neq \phi$, $g_j = g_{j-1}$on $V_j \cap V_{j-1}$ for $j = 1, \ldots, n$.
(3) Let $\gamma : [0,1] \to \mathbb{C}$a path $\gamma(0) = z_0$, $\gamma(1) = z_1$, (D_0, f_0) a function element at z_0. Say (D_0, f_0) is a **continuable** along γ if there is a function element (D_1, f_1) at z_1, a chain $(V_j, g_j)_{0 \le j \le n}$ from (D_0, f_0) to (D_1, f_1) and a partition $0 = s_0 < s_1 < \cdots < s_{n+1} = 1$ such that $\gamma([s_j, s_{j+1}]) \subset V_j$ for $j = 0, \ldots n$.

Lemma. *If (D_1, f_1), $\left(\widetilde{D_1}, \widetilde{f_1}\right)$ are continuations of (D_0, f_0) along γ then $(D_1, f_1) \sim \left(\widetilde{D_1}, \widetilde{f_1}\right)$.*

Theorem. *(Monodromy Thm) Let $\Gamma : [0,1] \times [0,1] \to \mathbb{C}$ continuous. $\Gamma(0,s) = z_0$, $\Gamma(1,s) = z_1$, $\gamma_s(t) = \Gamma(t,s)$ be a path homotopy between γ_0 and $\gamma)1$. Assume the functione lement (V,f) at z_0 can be continued analytically along each path γ_s. Then its analytic and continuous at z_1, and (D_0, f_0) along γ_0 and (D_1, f_1) along γ_1 are equivalent.*

Corollary. *If D is a simply connected domain and a function (V,f) at $z_0 \in D$ can be continued analytically along each path in D starting at z_0 then f extends to a holomorphic F on D.*
i.e. $\exists F$ holomorphic on D such that $F = f$ on $D \cap V$

Chapter 6–Laurent Series and Isolated Singularities

Definition. **Laurent Series** centered at z_0.

has negative powers of z.

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Theorem. *(Laurent Series Expansion) Let $0 \le \rho < \sigma \le \infty$. Let f be holomorphic in $A = \{\rho < |z - z_0| < \sigma\}$.*

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n}$$

where the a_n's are uniquely determined by f.

$$\boxed{a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}} dz}$$

Same as for power series

for $n \in \mathbb{Z}$, $\rho < r < \sigma$ and the Laurent series converges absolutely and normally in A.

Theorem. *(Laurent Decomposition) If f is holomorphic in A. Then f can be written uniquely by $f = f_0 + f_1$ where f_0 is holomorphic in $\{|z - z_0| < \sigma\}$, f_1 is holomorphic on $\{|z - z_0| > \rho\}$. $\lim_{z\to\infty} f_1(z) = 0$.*

Definition. Call $z_0 \in \mathbb{C}$ an *isolated singular point of f if f is defined and holomorphic in $\{0 < |z - z_0| < r\}$ for some $r > 0$.*

Theorem. *If z_0 is an isolated singularity then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $0 < |z - z_0| < r$.*

(1) All $a_n = 0$ for $n < 0$. z_0 is ~~an isolated~~ removable **singularity.** no negatives powers of z
(2) For $n < 0$, all but finitely many a_n are zero. i.e. $\exists$N>0 such that $a_{-N} \neq 0$and $a_n = 0$ for all $n < -N$. z_0 is a **pole of order N.** finitely many negative powers
(3) For infinitely many $n < 0$, $a_n \neq 0$. z_0 is an **isolated essential singularity** of f. infinitely many negative powers.

Theorem. *(Riemann Extension Theorem) If z_0 is an isolated singularity of f and f is bounded in some region $\{0 < |z - z_0| < \rho\}$ then f extends holomorphically at z_0. i.e. z_0 is removable.*
if limit exists as you approach the singularity then its removable

Theorem. *Let z_0 is an isolated singularity of f. TFAE*

(1) z_0 is a pole of order N or f
(2) z_0 is a zero of order N of $\frac{1}{f}$

(3) $f(z) = \frac{g(z)}{(z-z_0)^N}$ *where g is holomorphic in a neighborhood of z_0 and $g(z_0) \neq 0$.*

Corollary. *Let z_0 be an isolated singularity of f. Then z_0 is a pole of f if and only if $\lim_{z\to z_0} f(z) = \infty$.*

Theorem. *Let $f : D \to \mathbb{C}^* = S^2$. If z_0 is a pole of f then f is continuous at z_0, $f(z_0) = \infty$ and $f$is a holomorphic function from D to S^2.*

Definition. A function $f : D \to \mathbb{C}$ where D is open is called **meromorphic** on D if it is holomorphic on D except at isolated singularities which are poles. only finitely many negative values
Call $M(D)$ = {meromorphic functions on D} and $O(D)$ = {holomorphic functions on D}

Remark. $f \in M(D)$ has at most countably many poles which cluster only on the boundary.
If $f \in M(D)$, $f \neq 0$ then $\frac{1}{f} \in M(D)$.
If $g, h \in O(D)$, $h = 0$ then $\frac{g}{h} \in M(D)$.

Theorem. *If $f \in M(D)$ then $f = \frac{g}{h}$ for $g, h \in O(D)$ then $h = 0$.*
meromorphic functions are the quotient of 2 analytic functions

Theorem. *(Picard's Theorem) Suppose z_0 is an isolated essential singularity of f. Then for $w \in \mathbb{C}$ with at most one exception there exists a sequence $z_n \to z_0$, $z_n \neq z_0$ so that for $f(z_n) = w$.* attains value for every pt in sequence

Theorem. *(Casorati-Weierstrass Thm) If f has an isolated essential singularity at z_0 and $w_0 \in \mathbb{C}$ then $\exists z_n \to z_0$, $z_n \neq z_0$ so that $f(z_n) \to w_0$.* $\forall w_0 \in \mathbb{C}$, there is $\{z_n\}$ approaching an essential singularity s.t. $f(z_n) \to w_0$

Definition. f has an ***isolated singularity at*** ∞ *if f is holomorphic in $\{|z| > R\}$ for some $R > 0$.*

Theorem. *Let $D \subset \mathbb{C}^*$ a domain. f is meromorphic on D if f is holomorphic on D except at isolated singularity which are poles or removable.*

Theorem. *Any meromorphic function on $\mathbb{C}^*$ is rational.*

Chapter 7–Residues

Definition. f has an isolated singularity at z_0. $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $0 < |z - z_0| < \rho$. The **residue of f at z_0** is $\text{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z)dz$ for $0 < r < \rho$.

Theorem. *(Residue Theorem) Let D be a bounded domain with a piecewise smooth boundary, f holomorphic in a neighborhood of $\bar{D}$ except at finitely many points $z_1, \dots, z_n \in D$. Then $\int_{\partial D} f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$.*

Remark. Finding Residues

(1) z_0 is a simple pole of f, $f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 + \dots$

$$\boxed{\text{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} (z - z_0) f(z)}$$

(2) z_0 is a simple pole of f, $f = \frac{g}{h}$ $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$.

$$\boxed{\text{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}}$$

(3) z_0 is a double pole of f,$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0 (z-z_0)^2 + \dots$

$$\boxed{\text{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]}$$

(4) z_0 is a pole of order n of f,$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{n-1} + a_0 (z-z_0)^n \dots$

$$\text{Res}(f, z_0) = a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z)\right]$$

$$\boxed{\text{Res}(f, z_0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z)\right] |_{z=z_0}}$$

only have to consider singularities in upper half plane if

Theorem. *$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ converges if deg$(Q(x)) \geq$deg$(P(x))+2$, in this case $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum Res\left[\frac{P(z)}{Q(z)}, z_j\right]$ where z_j are the poles of $\frac{P(z)}{Q(z)}$ is the upper half plane.*

Theorem. *(Fractional Residue Theorem) Suppose f has a simple pole at z_0, C_ϵ an arc of circle $|z-z_0| = \epsilon$ of angle α then*

$$\boxed{\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) dz = \alpha i Res(f, z_0)}$$

Theorem. *$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} sin(x) dx$, $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} cos(x) dx$, $Q(x) \neq 0$ for all $x \in \mathbb{R}$. If $degP \leq degQ - 2$ then the integrals are absolutely convergent. If $degP = degQ - 1$ then the integrals are convergent but not absolutely.*

Remark. $\int_0^{\infty} \frac{x^a}{(1-x)^2} dx = \frac{\pi a}{\sin(\pi a)}, -1 < a < 1$

$\int_0^{2\pi} \frac{d\theta}{a + cos\theta} = \frac{2\pi}{\sqrt{w^2-1}}, w \in \mathbb{C} \setminus [-1, 1]$

Theorem. *If f holomorphic on $|z| > R$ so $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ and $Res(f, \infty) = -a_{-1} = -\frac{1}{2\pi i} \int_{|z|=\rho} f(z) dz = Res\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right)$.*

at ∞ gets an extra negative

Theorem. *If f is holomorphic on $\mathbb{C} \setminus \{z_1, z_2, \dots, z_n\}$ then $\sum_{j=1}^{n} Res(f, z_j) + Res(f, \infty) = 0$.*

Theorem. *(Residue Theorem for Exterior Domains.) Let $D \subset \mathbb{C}$ be a domain, ∂D piecewise smooth with $\{|z| > R\} \subset D$, f holomorphic in neighborhood of $\bar{D}$ except at $z_1, z_2, \dots, z_n \in D$. Then*

$$\boxed{\int_{\partial D} f(z) dz = 2\pi i \left(Res(f, \infty) + \sum_{j=1}^{n} Res(f, z_j)\right)}$$

Chapter 8–The Logarithmic Integral

Definition. Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ for $0 \notin \{\gamma\}$ be closed and rectifiable. Then $W(\gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$ is the **Winding number of γ with respect to 0.** For $z_0 \notin \{\gamma\}$, $W(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}$.

Lemma. $W(\gamma, 0) \in \mathbb{Z}$. *If $Arg(\gamma(t))$ is a continuous branch of $arg(\gamma(t))$ on $[0,1]$ then* $W(\gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = \frac{Arg(\gamma(1)) - Arg(\gamma(0))}{2\pi}$.

Theorem. *(Argument Principle) Let D be a bounded domain with a piecewise smooth boundary, holomorphic on a neighborhood of $\bar{D}$ except at finitely many poles all inside D. If $f(z) \neq 0, \forall z \in \partial D$ then $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = N_0 - N_\alpha$. Where N_0 is the number of zeros of f in D, counted with multiplicity. and N_α is the number of poles of f in D, counted with multiplicity.*

Theorem. *(Ruche's Theorem) D is bounded domain with piecewise smooth boundary, f, h holomorphic in a neighborhood of $\bar{D}$. Assume $|h(z)| < |f(z)|, \forall z \in \partial D$. Then $f, f + h$ have the same number of zeros counted with multiplicity in D.*

a more usefull way to ~~cheat~~ count # of zeros.

Theorem. *(Hurwitz's Theorem) Let f_k be holomorphic on a domain D, converging normally to f on D, $z_0 \in D$ be a zero of order N of f (in particular $f \neq 0$) then $\exists \rho > 0$, $k_0 \in \mathbb{N}$ so that $\bar{\Delta}(z_0, \rho) \subset D$ and if $k \geq k_0$ then f_k has exactly N zeros (counted with multiplicity), These zeros converge to z_0 as $k \to \infty$.*

$f_k \rightrightarrows f$, f has zero of order N at z_0, then $\exists$ nbhd of z_0 s.t. f_k has N zeros for k large enough.

Corollary. *If f_k converges normally to f on domain D and each f_k has no zeros in D then either $f = 0$ or f has no zeros in D.*

Definition. A function f is called **univalent** *on D if f is holomorphic and injective on D.*

Corollary. *If f_k are univalent on a domain D and converge normally to f on D then either f is constant or it is univalent on D.* univalent, normally convergent sequence → f constant or univalent

Definition. Let f be holomorphic near $z_0 \in \mathbb{C}$. $w_0 = f(z_0)$. Say w_0 is **assumed m times at z_0**, If z_0 is a zero of order m for $f - w_0$ where $f(z) = w_0 + a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots$ where $a_m \neq 0$

Definition. f assumes value w_0 m times at ∞ if $f(\frac{1}{z}) - w_0$ has a zero of order m at the origin. Value ∞ is assumed by f m times at $z_0 \in \mathbb{C}^*$ if f is a pole of order m at z_0.

Definition. If w_0 is assumed $m > 1$ times by f at z_0 then f' has a zero of order $m-1>0$ at z_0. Call z_0 a **critical point** of order $m-1$ of f. $w_0 = f(z_0)$ is a **critical value of** f.

Theorem. *Let f be holomorphic on D, $z_0 \in D$, $f(z_0) = w_0$, f attains value w_0 m times at z_0 (in particular f is not constant). Then $\exists \rho, \delta > 0$ so that $\{|z - z_0| \le \rho\} \subset D$ and if $0 < |w - w_0| < \delta$ then the equation $f(z) = w$ has m distinct roots in the open disk $\{|z - z_0| < \rho\}$.*

for any w close to w_0

Corollary. *(Open Mapping Theorem) If f is holomorphic and non-zero on a domain D then f is open. i.e. If $U \subset D$ is open then $f(U)$ is open.*

Theorem. *If f is univalent on D then $f'(z) \neq 0$ for all $z \in D$.*

univalent $\Rightarrow f'(z) \neq 0$
conformal $\Rightarrow f'(z) \neq 0$ on disk

Theorem. *(Inverse Function Theorem) Let f be holomorphic in $\{|z - z_0| \le \rho\} \subset D$, $f(z_0) = w_0$, $f'(z_0) \neq 0$,* $f(z) \neq w_0$ for all z. $0 < |z - z_0| \le \rho$. Let $\delta = \min\{|f(z) - w_0| : |z - z_0| = \rho\} > 0$. For every w, $|w - w_0| < \rho$ there is a unique z. $|z - z_0| < \rho$. $f(z) = w$. (i.e. every such w has a unique preimage)

if $f'(z_0) \neq 0$ and $f(z_0) = w_0$ is only one that $= w_0$ in nbhd then there is a nbhd of w_0 s.t. every value in there has a preimage

$$\boxed{z = f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta}$$

for $|w - w_0| < \rho$.

Chapter 9–The Schwarz Lemma and Hyperbolic Geometry

Theorem. *(Schwartz Lemma) f holomorphic on $\mathbb{D}$. $|f(z)| \le 1$ for all $z \in \mathbb{D}$. $f(0) = 0$ then $|f(z)| \le |z|$ for all $z \in \mathbb{D}$. Equality at $z_0 \neq 0$ if and only if $f(z) = \lambda z$ for $\lambda \in \mathbb{C}$, $|\lambda| = 1$.*
In addition, $|f'(0)| \le 1$, $|f'(0)| = 1$ if and only if $f(z) = \lambda z$ for $|\lambda| = 1$.

Definition. $g : \mathbb{D} \to \mathbb{D}$ holomorphic, bijective is called an **automorphism** of $\mathbb{D}$. We denote the set of all automorphisms of $\mathbb{D}$ as $Aut\mathbb{D}$.

Send $\mathbb{D} \to \mathbb{D}$

Theorem. *If $f \in Aut\mathbb{D}$ then $\boxed{f(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}}$ where $|a| < 1$ and $\theta \in \mathbb{R}$.*

all automorphisms of $\mathbb{D}$ are of this form.

Theorem. *If $g(z)$ is a conformal self-map of the unit disk such that $g(0) = 0$ then $g(z)$ is a rotation, that is, $g(z) = e^{i\phi} z$ for some fixed $0 \le \phi \le 2\pi$.*

Lemma. *(invariant form of Schwartz Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ holomorphic (i.e. $|f(z)| < 1$ for $z \in \mathbb{D}$). Then $\left|\frac{f(z)-f(\zeta)}{1-\overline{f(\zeta)}f(z)}\right| \le \left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|$ for $z, \zeta \in \mathbb{D}$. Also, $|f'(\zeta)| \le \frac{1-|f(\zeta)|^2}{1-|\zeta|^2}$ for $\zeta \in \mathbb{D}$. Equality in both cases if and only if $f \in Aut\mathbb{D}$.*

Chapter 10–Harmonic Functions and the Reflection Principle

Definition. The **Poisson Kernel** in $\mathbb{D}$ for $z = re^{i\theta}$, $P(r,\theta) = P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos(\theta)} = \frac{1-|z|^2}{|1-z|^2} = Re\left(\frac{1+z}{1-z}\right)$.

- It is 2π periodic
- $P(r,\theta) = P(r,-\theta) > 0$
- $P(r,\cdot)$ increasing on $[-\pi, 0]$
- $P(r,\cdot)$ decreasing on $[0,\pi]$

Theorem. *If $h(e^{i\theta})$ is continuous the Poisson Integral of h is $\tilde{h}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\theta})P(r,\theta-\phi)d\phi$*

Theorem. *h is harmonic in the unit circle.*

Theorem. *If $\zeta \in \partial\mathbb{D}$ then $\lim\limits_{z\in\mathbb{D}\to\zeta} \tilde{h}(z) = h(\zeta)$. Thus $\tilde{h}$ extends continuously to $\bar{\mathbb{D}}$ by $\tilde{h} = h$ on $\partial\mathbb{D}$.*

Lemma. *Let $h : \bar{D} \to \mathbb{R}$ continuous, D a bounded domain, h has MVP on D. $\forall z \in D$, $\exists \Delta(\bar{z}_0, \rho) \subset D$ such that $h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{it})dt$ for $0 < r < \rho$.*

Theorem. *If $a \le h(z) \le b$ for $z \in \partial D$ then $a \le h(z) \le b$ for $z \in D$.*
bounded on boundary ⇒ same bounds for inside domain

Theorem. *Let h be continuous on D. Then h is harmonic on D if and only if h has MVP.*

Corollary. *If u_n are harmonic on D and u_n converge normally on D to u then u is harmonic.*
harmonic sequence of functions converge normally to a harmonic function

Definition. D a domain symmetric with respect to $\mathbb{R}^2$ ($z \in D \Leftrightarrow \bar{z} \in D$)

$D^+ = \{z \in D : Imz > 0\}$

$D^- = \{z \in D : Imz < 0\}$

$D = D^+ \cup D^- \cup (D \cap \mathbb{R})$

Theorem. *Let $u : D^+ \to \mathbb{R}$ be harmonic such that $u(z) \to 0$ as $z \to \zeta \in D \cap \mathbb{R}$, $\forall \zeta$. Then u extends to a harmonic function u on D which satisfies $u(\bar{z}) = -u(z)$ for $z \in D$.*
if goes to 0 where intersects w/ ℝ then it extends across it.

Theorem. *Let $f = u + iv$ be holomorphic on D^+ so that $v(z) \to 0$ as $z \in D^+ \to \zeta \in D \cap \mathbb{R}$ for all ζ. Then f extends to a holomorphic function F on D which verifies $F(\bar{z}) = \overline{F(z)}$ for $z \in D$.*

Definition. Let $\gamma \subset \mathbb{C}$ be an analytic arc. If $\forall z_o \in \bar{\gamma}$ and there exists U a neighborhood of $z_0$such that $\exists D = \Delta(x_0, r)$, $x_0 \in \mathbb{R}$ and injective holomorphic map $z = z(\zeta)$ for $\zeta \in D$ mapping D onto U and $D \cap \mathbb{R}$ onto $U \cap \gamma$.

Definition. $\gamma \subset \partial D$, D a domain is called a **free analytic boundary arc** if γ is an analytic arc and any $z_0 \in \bar{\gamma}$ has a neighborhood U so that $U \setminus \gamma$has 2 components connected, one in D and the other in $\mathbb{C} \setminus \bar{D}$.

Lemma. *If f is holomorphic on a simply connected domain D, $f(z) \neq 0, \forall z \in D$. Then $\exists g$ holomorphic on D so that $e^g = f$.* holomorphic nonzero functions can be expressed in exponential form

Theorem. *Let D be a domain, γa free analytic boundary arc of D, f a holomorphic function on D so that $|f(z)| \to R$ if $z \to \zeta \in \gamma$, $\forall \zeta \in \gamma$. Then f extends analytically to a neighborhood of γ and the extension f verifies $f(z^*) = \frac{R^2}{f(z)}$ for z near γ where $z \to z^*$ is the reflection across γ.*

Chapter 11–Conformal Mapping

Definition. $\mathbb{D} = \{z : |z| < 1\}$ **unit disc**
$\mathbb{H} = \{z : \text{im} z > 0\}$ **upper half plane**

Theorem. *D, V domains, $f : D \to V$ is conformal if f is holomorphic, injective and surjective. Then $f'(z) \neq 0$, $\forall z \in D$ and $f^{-1} : V \to D$ is holomorphic.*

Remark. Given $z_0 \in D$, $\exists! g : D \to \mathbb{D}$ conformal with $g(z_0) = 0$, $g'(z_0) > 0$.

Theorem. *$D \subset \mathbb{C}$ is simply connected $\Leftrightarrow$ $\mathbb{C}^* \setminus D$ connected $\Leftrightarrow$ $\partial D \subset \mathbb{C}^*$ is connected in* $\mathbb{C}$.

Theorem. *If $D \subset \mathbb{C}$ a domain, $D \neq \mathbb{C}$. The following are equivalent:*

Equivalent is $D \neq \mathbb{C}$, $D \subset \mathbb{C}$.

(1) D is simply connected.
(2) Every closed form on D is exact
(3) For all $a \in D$, $\exists f$ holomorphic on D such that $e^{f(z)} = z - a$, $z \in D$
(4) $\exists \phi : D \to \mathbb{D}$ conformal.

Theorem. *(Reimann Mapping Theorem) If $D \neq \mathbb{C}$ is a simply connected domain then D is conformally equivalent to $\mathbb{D}$: $\exists \phi : D \to \mathbb{D}$ conformal. All such ϕ are called* ***Riemann maps****. If $\phi : \mathbb{C} \to \mathbb{D}$ holomorphic then ϕ is bounded and hence constant.*

all simple connected domains that are not $\mathbb{C}$ are conformally equivalent to $\mathbb{D}$

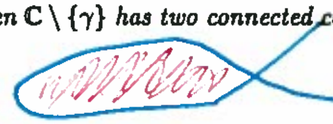
Corollary. *If $\phi : \mathbb{C} \to \mathbb{D}$* holomorphic *then ϕ is bounded and hence constant.*

any holomorphic map from $\mathbb{C} \to \mathbb{D}$ are constant.

Corollary. *If $D \subset \mathbb{C}^*$ is simply connected then either $D = \mathbb{C}^*$ or D is conformally equivalent to $\mathbb{C}$ or to $\mathbb{D}$.*

any simple connected domain that isn't $\mathbb{C}^*$ is either conformally equivalent to $\mathbb{C}$ or $\mathbb{D}$

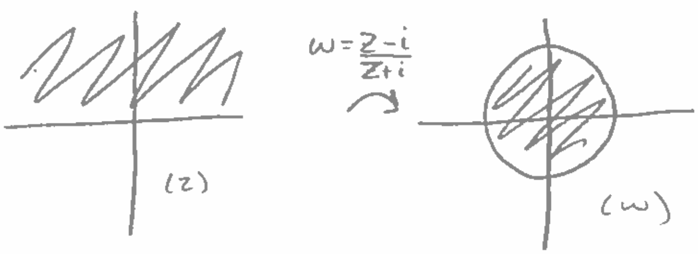
Theorem. *(Jordan Curve Theorem) If $\gamma \subset \mathbb{C}$ is a Jordan Curve then $\mathbb{C} \setminus \{\gamma\}$ has two connected components, a bounded one U and an unbounded one V and $\partial U = \partial V = \gamma$.*

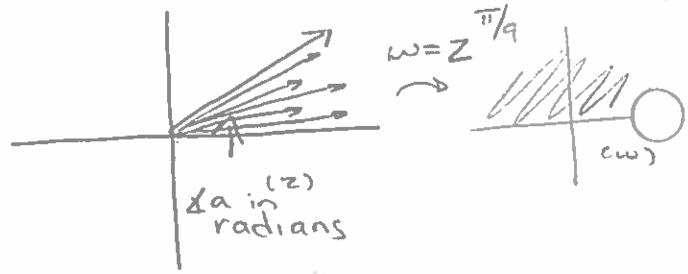


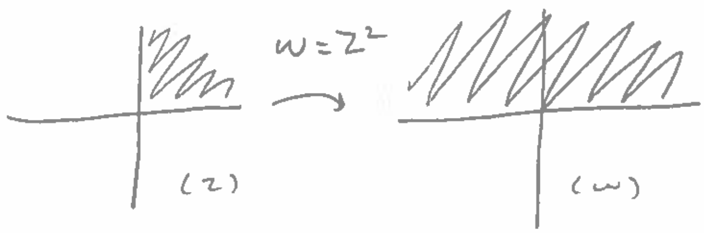
must cross.

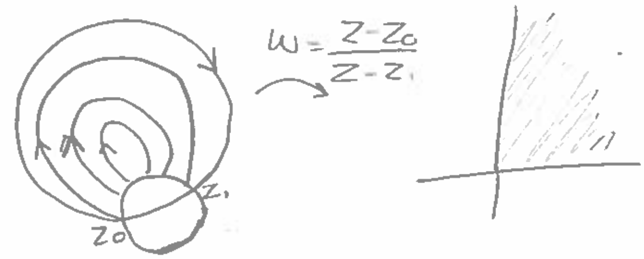
Definition. A **Jordan Domain** is a bounded simply connected domain whose boundary is a Jordan Curve.

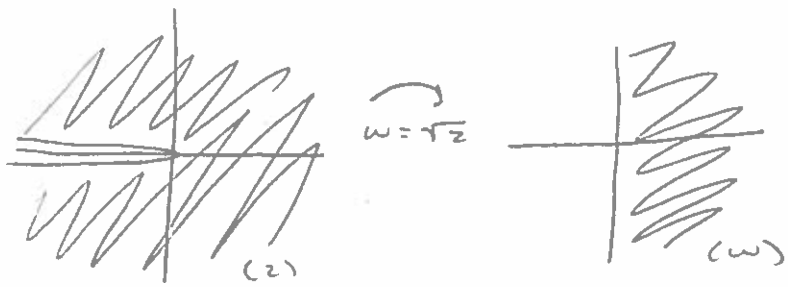
Theorem. *(Caratheodory Extension Theorem) If D is a Jordan Domain and $\phi : D \to \mathbb{D}$ is conformal then ϕ extends to a homeomorphic $\phi : \bar{D} \to \bar{\mathbb{D}}$.*

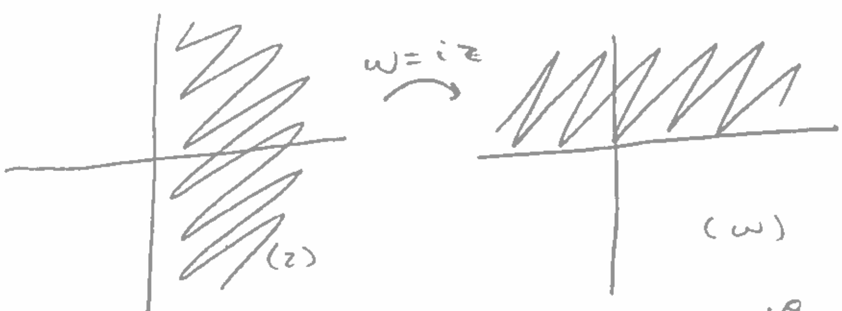




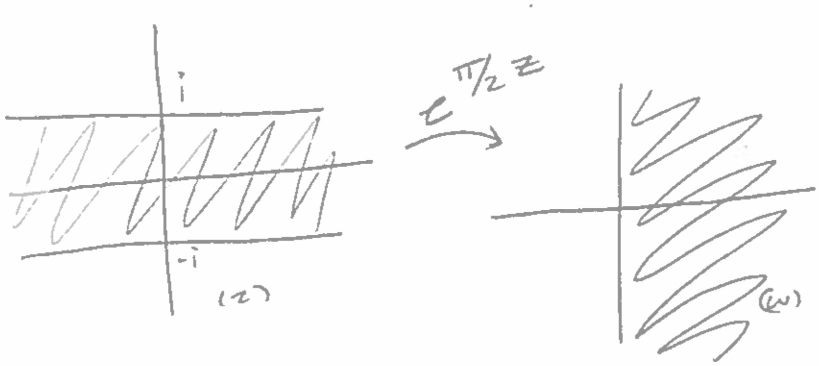






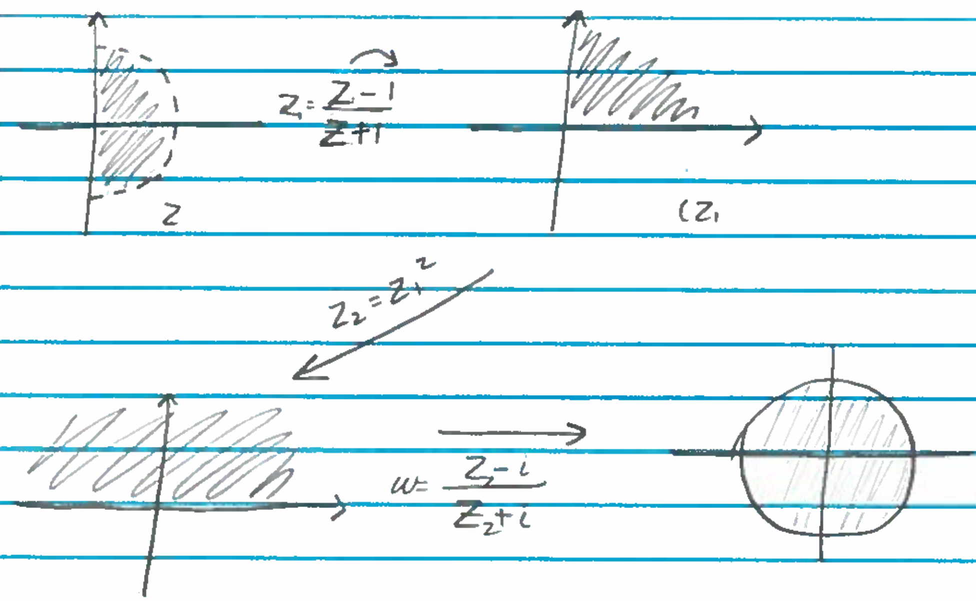


rotation otherwise $e^{i\theta}$



A14

1. Find a conformal map from $D = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Re}\, z > 0\} \to \Delta$



$$\boxed{w = \frac{\left(\frac{z-1}{z+1}\right)^2 - i}{\left(\frac{z-1}{z+1}\right)^2 + i}}$$

2. Let D be domain in $\mathbb{C}$ containing 0
and $f: D \to \mathbb{R}$ cont so $f(0)=0$ and $\int_{\partial R} f(z)dz = 0$
$\forall$ closed rectangle $R \subset D$ w/ sides parallel to
coordinate axes Prove $f(z)=0$ $\forall z \in D$

Pf By Morera's we know f is analytic.
Let $f = u + iv : D \to \mathbb{R}$.
Then by Cauchy
$u_x = v_y$ and $u_y = -v_x$
Since $f: D \to \mathbb{R}$ $v_y = v_x = 0$
$\Rightarrow u_x = u_y = 0$
$\Rightarrow u$ is constant
$\Rightarrow f(z) = c \in \mathbb{R}$ $\forall z \in \mathbb{C}$
$\Rightarrow f(z) = 0$ $\forall z \in \mathbb{C}$ Since $f(0)=0$

$\square$

3. Let $D \subset \mathbb{C}$ be a bounded domain $z_0 \in D$
and $f: D \to D$ be holomorphic s.t. $f(z_0) = z_0$
Show $|f'(z_0)| \le 1$.

Pf Let $f_n = f \circ f \circ \dots \circ f$ (n times)
f analytic $\Rightarrow$ $f_n : D \to D$ is analytic.
$f(z_0) = z_0$ $\Rightarrow$ $f_n(z_0) = z_0$

$$f_n'(z_0) = f'(f^{(n-1)}(z_0))\, f'(f^{(n-2)}(z_0)) \cdots f'(z_0)$$
$$= (f'(z_0))^n$$

D bdd $\Rightarrow$ f_n uniformly bdd on D
$\Rightarrow$ f_n' uniformly bdd on a compact subset of D
$\Rightarrow$ $|f_n'| \le M$ for some $M > 0$
$\Rightarrow$ $|f_n'(z_0)| = |f'(z_0)|^n \le M$
$\Rightarrow$ $|f_n'(z_0)| \le 1$
otherwise $|f_n'(z_0)|^n \to \infty$ as $n \to \infty$

$\square$

4. Let $f_n : \Delta \to \Delta$ $n \geq 1$ be sequence of holomorphic functions s.t. f_n has a zero of order m_n at 0 where $\lim m_n = \infty$. Show $\{f_n\}$ converges locally uniformly to 0 on D.

Pf Let $r \in (0,1)$ be fixed, let $\varepsilon > 0$

WTS $f_n \to 0$ uniformly on $\{|z| \leq r\} = \overline{B_r(0)}$

f_n has 0 of order m_n at 0

$\Rightarrow f_n = z^{m_n} g_n(z)$ where $g_n(0) \neq 0$

$\Rightarrow g_n(z) = \dfrac{f_n(z)}{z^{m_n}}$

$\Rightarrow |g_n(z)| = |f_n(z)| / |z|^{m_n}$

$\leq 1/r^{m_n}$ on $|z| = r$

$\Rightarrow |g_n(z)| \leq 1/r^{m_n}$ on $|z| \leq r$ by max

$\Rightarrow |g_n(z)| \leq 1$ as $r \to 1$

$\Rightarrow |g_n(z)| \leq 1$ on $|z| < 1$ by max

$\Rightarrow f_n(z) \leq r^{m_n}$ on $\{|z| < r\}$

$\Rightarrow f_n(z) \to 0$ uniformly as $n \to \infty$ on $\{|z| \leq r\}$ by Weierstrass M test

$\Rightarrow f_n(z)$ converges locally uniformly to 0 on D.

$\square$

Instructions: Please use the bluebooks provided for your solutions of the 4 problems below. You may use without proof any standard results from class.

Problem 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} z^{n!}$ converges absolutely for $|z| < 1$. Also show that there are infinitely many z with $|z| = 1$ for which the series diverges.

Answer: Thinking of $\sum_{n=1}^{\infty} \frac{1}{n} z^{n!} = \sum a_k z^k$ we have that $\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_n (\frac{1}{n})^{1/n!} = 1$ so the radius of convergence is 1. For the second part let $z = e^{i\frac{p}{q}\pi}$, where $p, q \geq 1$ are integers, then $z^{n!} = e^{2\pi i l} = 1$ for some integer l as soon as $n > q$. Therefore at angles that are rational multiples of π the series diverges, since all but finitely many terms agree with those of the divergent harmonic series.

Problem 2. Let $f(z)$ be holomorphic on $\mathbb{C}$ except for poles. At ∞ assume that f has a removable singularity or a pole.
(a) Show that f has finitely many poles on $\mathbb{C} \cup \{\infty\}$.
(b) Let $p_j(z)$ be the principal part of f at the jth pole, $1 \leq j \leq N$, show that

$$f(z) - \sum_{j=1}^{N} p_j(z)$$

is constant.

Answer: (a) Since f has a removable singularity or a pole at infinity there is an $R > 0$ so that for $|z| > R$, $z \in \mathbb{C}$ the function f is differentiable. For $|z| \leq R$ the function has only poles and there can only be a finite number of these (since otherwise the set of poles has a limit point inside $|z| \leq R$, but poles are isolated, by definition.) (b) Let the poles be $a_1, \ldots, a_N$ (where the last one is ∞ if there is a pole there), let $p_j(z)$ be the principal part of f at each pole (the negative indices in the laurent expansion, be careful at infinity the principal part is polynomial). Then $f - p_j$ has a removable singularity at a_j, so that $f - \sum p_j$ is bounded and analytic on $\mathbb{C}$, and therefore constant.

Problem 3. Let f be continuous on $\mathbb{C}$ and analytic except possibly on the unit circle, $|z| = 1$. Assume there is an entire function g such that $f(z) = g(z)$ for $|z| = 1$. Prove that $f = g$, and hence f is entire.

Answer: Let $r_n \nearrow 1$ strictly. Then by the Cauchy Integral Theorem, for

$|z| < 1$,

$$f(z) = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_n e^{i\theta}) r_n e^{i\theta}}{r_n e^{i\theta} - z}\, d\theta.$$

Using, say, the bounded convergence theorem (BCT), we have that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - z}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(e^{i\theta})e^{i\theta}}{e^{i\theta} - z}\, d\theta.$$

Reversing steps with g in place of f we conclude that $f(z) = g(z)$ for all $|z| \le 1$. To prove that f is entire we may use Morera's Theorem. If a triangular contour T meets both $|z| < 1$ and $|z| > 1$ we may decompose T into two closed contours as $T = \gamma + \gamma'$, with γ in $|z| \le 1$ and γ' in $|z| \ge 1$, by including suitable arcs of the unit circle in each. Then

$$\int_T f(z)\, dz = \int_\gamma f(z)\, dz + \int_{\gamma'} f(z)\, dz.$$

The first integral on the right vanishes since we may replace f by g and use Cauchy's Theorem. The second is easily seen to be zero by a limiting argument that involves deforming the contour outward slightly and using Cauchy's Theorem for f outside the unit disc. For example, replace γ' by $(1 + \frac{1}{n})\gamma'$ and use the BCT as $n \to \infty$.

Finally, we conclude the $f = g$ everywhere by the identity theorem.

Problem 4. Let f_n be analytic in the unit disc, D, and have positive real part: $\mathcal{R}(f_n(z)) > 0$ on D. Assume that the f_n converge pointwise on D to a function f having $\mathcal{R}(f(z)) \le 0$ on D. Prove that f is constant on D.

Answer: Let $h_n = e^{-f_n}$. Then the h_n map D to D and are analytic, hence they form a normal family by Montel's Theorem. It follows that some subsequence converges uniformly on compact subsets of D to $h = e^{-f}$, hence h is analytic on D. But we also have that $|h(z)| = 1$ on D since the function f necessarily has zero real part. Thus h must be constant by the Open Mapping Theorem, and therefore f is also constant.

Qualifying Exam, Complex Analysis, January 11, 2013

Notation: Throughout the exam Δ denotes the open unit disc in $\mathbb{C}$.

1. Find a conformal map from the strip $\{0 < \mathbf{Re}\, z < 1\}$ onto Δ.

2. Let C denote the positively oriented boundary of the domain

$$D = \{z \in \mathbb{C} : -1/2 < \mathbf{Re}\, z < 2\,,\ |\mathbf{Im}\, z| < 2\}\,.$$

Find $\displaystyle\int_C \frac{z^n}{z^4 - 1}\, dz$, where $n \geq 0$ is an integer. Write your answer in algebraic form, $a + bi$.

3. Is there an entire function $f(z)$ such that $e^{f(z)}$ has a pole at ∞ ?

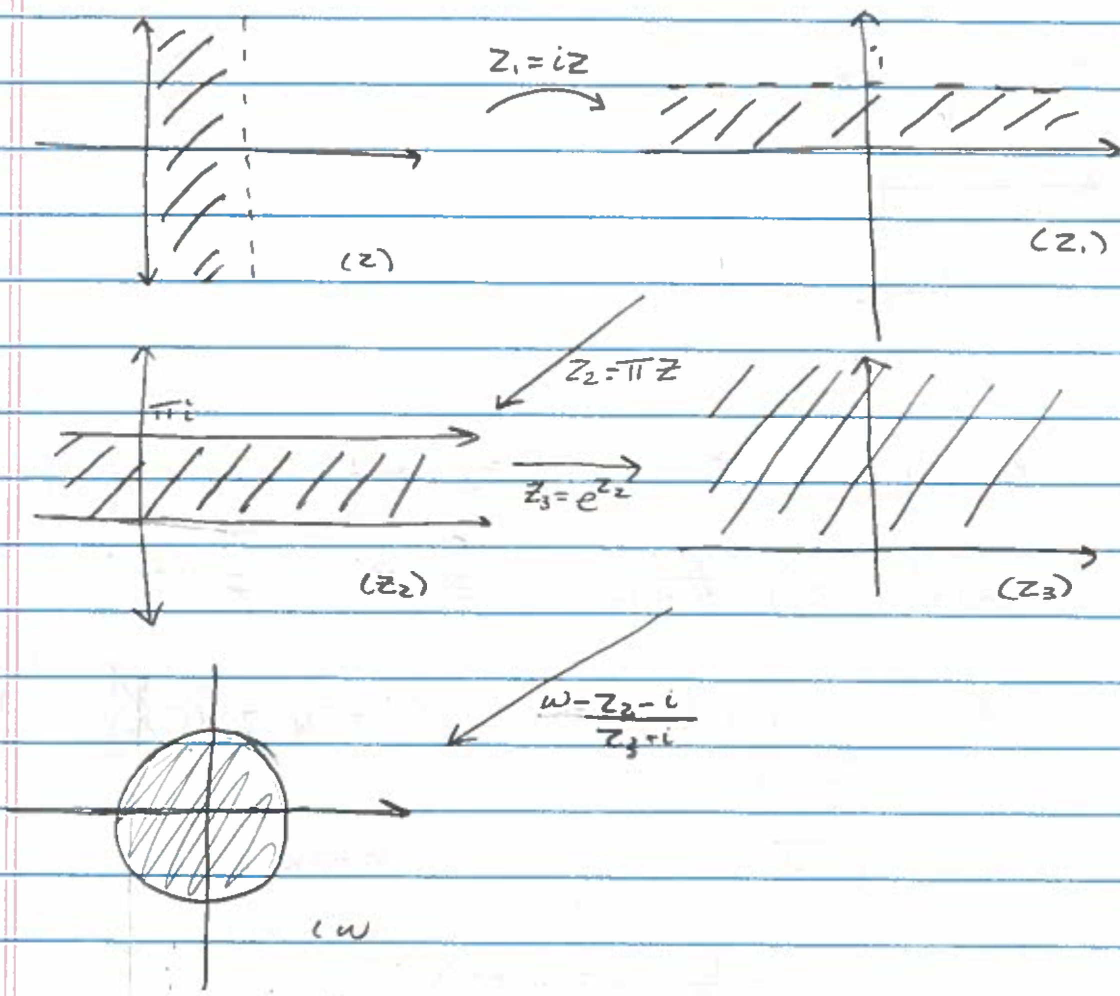
4. Suppose that f, g are holomorphic functions in Δ so that $f(0) = g(0) = 1$ and

$$(f'g - fg')(1/n) = 0$$

for all integers $n \geq 2$. Show that $f = g$ on Δ.

J13

1. Find a conformal map from the strip $\{0 < \mathrm{Re}\, z < 1\}$ onto Δ.

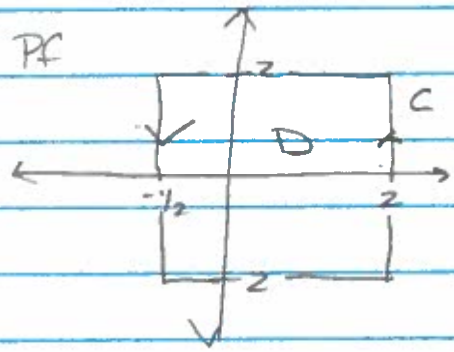


$$\boxed{w = \frac{e^{\pi i z} - i}{e^{\pi i z} + i} : \{0 < \mathrm{Re}\, z < 1\} \to \Delta}$$

2. Let $C = \partial D$ where $D = \{z \in \mathbb{C} : -\tfrac{1}{2} < \operatorname{Re} z < 2, \ |\operatorname{Im} z| < 2\}$

Find $\int_C \frac{z^n}{z^4-1}$ where $n \in \mathbb{N}$.

Pf



$z^4 - 1 = 0 \iff z^4 = 1 \iff z_j = \pm i, \pm 1$ (all simple poles)

$$\operatorname{Res}\left(\frac{z^n}{z^4-1}, z_j\right) = \frac{z^n}{4z^3}\bigg|_{z_j} = \frac{1}{4} z_j^{\,n-3} = \frac{1}{4} z_j^{\,n+1} \quad \text{since } z_j^4 = 1$$

$$\int_C \frac{z^n}{z^4-1} = 2\pi i\left(\frac{1}{4}(i)^{n+1} + \frac{1}{4}(-i)^{n+1} + \frac{1}{4} + \cancel{\frac{1}{4}(-1)^{n+1}}\right) \quad \text{not in } D$$

$$= \frac{\pi i}{2}\left(1 + i^{n+1}\left(1 + (-1)^{n+1}\right)\right)$$

$$= \begin{cases} \dfrac{\pi i}{2} & n = 2k \\[2mm] \dfrac{\pi i}{2}\left(1 + 2i^{2k+2}\right) & n = 2k+1 \end{cases}$$

$$= \frac{\pi i}{2}\left(1 + 2(-1)^{k+1}\right)$$

$$= \begin{cases} \dfrac{3\pi i}{2} & k \text{ odd} \\[2mm] -\dfrac{\pi i}{2} & k \text{ even} \end{cases}$$

$$\boxed{\int_C \frac{z^n}{z^4-1} = \begin{cases} \dfrac{\pi i}{2} & n = 0, 2 \bmod 4 \\[2mm] \dfrac{3\pi i}{2} & n = 3 \bmod 4 \\[2mm] \dfrac{-\pi i}{2} & n = 1 \bmod 4 \end{cases}}$$

J13

3 Is there an entire function $f(z)$ s.t. $e^{f(z)}$ has a pole at ∞?

Pf no

Suppose Bwoc there is such $f(z)$

Then $e^{-f(z)}$ has a zero at ∞

$\Rightarrow e^{-f(z)}$ is constant by Louiville's thm

since $f(z)$ entire $\Rightarrow e^{-f(z)}$ entire.

(b/c $e^{f(z)} > 0$ so $e^{-f(z)}$ is entire)

and $e^{f(z)} > 0 \Rightarrow e^{-f(z)}$ is bounded

$\Rightarrow e^{-f(z)} \equiv 0$ since $e^{-f(z)}$ has 0 at ∞.

This contradicts since $e^{-f(z)} \neq 0$.

$\square$

4. Suppose f, g are holomorphic in Δ s.t. $f(0)=g(0)=1$ and $\forall n\geq 2$ $(f'g-fg')(1/n)=0$. Show $f=g$ on Δ.

Pf Let $h=\frac{f}{g}$ which is holomorphic in $\{|z|<\varepsilon\}$ for $0<\varepsilon<1$.

$h'=\frac{f'g-g'f}{g^2}$ by quotient rule

$\Rightarrow h'(\frac{1}{n})=0$ for all $n\geq 2$, with $\frac{1}{n}<\varepsilon$.

$\Rightarrow h'\equiv 0$ by identity thm since $\frac{1}{n}$ has limit point in $\{|z|<\varepsilon\}$

$h(0)=\frac{f(0)}{g(0)}=1 \Rightarrow f=g$ in $\{|z|<\varepsilon\}$

$\Rightarrow f\equiv g$ on Δ by indetity thm again

$\square$

August 2012

Complex Part

1. Suppose that $f(z) = u(x,y) + iv(x,y)$ is a function on a domain D and $z_0 \in D$. Show that if: a) u and v are differentiable at z_0; b) the limit

$$\lim_{\Delta z \to 0} \left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right|$$

exists, then either $f(z)$ or $\bar{f}(z)$ are complex differentiable at z_0.

2. Suppose that f is an analytic function on a disk $\{|z| < 2r\}$ given by a series $\sum_{n=0}^{\infty} c_n z^n$. Show that the series

$$F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

converges on $\mathbb{C}$ and $|F(z)| \le M e^{|z|/r}$, where

$$M = \max_{|z|=r} |f(z)|.$$

3. Let $\mathcal{F}$ be a family of analytic functions on the open unit disk $\mathbb{D}$ such that $\Re f(z) \ge 0$ for each $f \in \mathcal{F}$ and $z \in \mathbb{D}$. Show that every sequence of functions in $\mathcal{F}$ contains a subsequence converging normally to a function in $\mathcal{F}$ or ∞.

4. Let f be a nonconstant analytic function on the unit disk $\mathbb{D}$ and let $U = f(\mathbb{D})$. Show that if ϕ is a function on U (not necessarily even continuous) and $\phi \circ f$ is analytic on $\mathbb{D}$, then ϕ is analytic on U.

August 2012

1 Suppose $f(z) = u(x,y) + i v(x,y)$ is a function on D

$z_0 \in D$. Show if a) u, v differentiable b)

$$\lim_{\Delta z \to 0} \left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| \quad \text{exists}$$

then either $f(z)$ or $\bar{f}(z)$ are complex diff at z_0

Pf We know $f(z_0 + \Delta z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}} \overline{\Delta z} + o(|z - z_0|^2)$

with $o(|z - z_0|^2) \to 0$ as $z \to z_0$

$$\lim_{\Delta z \to 0} \left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right|$$

$$= \lim_{\Delta z \to 0} \left| \frac{\frac{\partial f}{\partial z}(z_0)\, \Delta z + \frac{\partial f}{\partial \bar{z}} \overline{\Delta z}}{\Delta z} \right|$$

$$= \lim_{\Delta z \to 0} \left| \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}} \frac{\overline{\Delta z}}{\Delta z} \right| \qquad \text{let } \Delta z = e^{i\theta} \Rightarrow \frac{\overline{\Delta z}}{\Delta z} = e^{-2i\theta}$$

$$= \lim_{\Delta z \to 0} \left| \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} e^{-2i\theta} \right|$$

$$\approx \lim_{\Delta z \to 0} |A + R|$$ where A is center of circle and R is radius

if $\frac{\partial f}{\partial z} = 0$ then circle is at 0 so limit exists since modulus is same all the way around

If $\frac{\partial f}{\partial \bar{z}} = 0$ then radius $= 0$ so circle is just a point. So limit exists.

Otherwise we are going around a circle w/ positive radius and not at 0 so modulus won't be the same

$\square$

2. Suppose f is analytic on $\{|z|<2r\}$ given by $\sum_0^\infty c_n z^n$. Show $F(z) = \sum_0^\infty \frac{c_n}{n!} z^n$ converges on $\mathbb{C}$ and $|F(z)| \le Me^{|z|/r}$ where $M = \max_{|z|=r} |f(z)|$

Pf $|F(z)| \le \sum \frac{|c_n|}{n!}|z|^n$ and $c_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$

$$\Rightarrow |c_n| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z)|}{r^{n+1}} dz$$

$$\le \frac{1}{2\pi}\int \frac{M}{r^{n+1}} dz$$

$$\le \frac{1}{2\pi} M \frac{1}{r^{n+1}} 2\pi r$$

$$= \frac{M}{r^n}$$

$$\Rightarrow |F(z)| \le \sum \frac{M}{r^{n+1} n!}|z|^n$$

$$= \sum \frac{M}{n!}\left|\frac{|z|}{r}\right|^n$$

$$= Me^{|z|/r}$$

$\square$

$$\boxed{a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz}$$

3. Let $\mathcal{F}$ be family of analytic functions on $\mathbb{D}$ s.t. $\operatorname{re} f \geq 0$ $\forall f \in \mathcal{F}$ and $z \in \mathbb{D}$. Show every sequence in $\mathcal{F}$ contains a subsequence converging normally to $f \in \mathcal{F}$ or ∞

create a uniformly bdd family

Pf Consider $\hat{\mathcal{F}} = \left\{ \frac{1}{1+f} \mid f \in \mathcal{F} \right\}$

then for all $\hat{f} \in \hat{\mathcal{F}}$ we have $|\hat{f}| = \frac{1}{|1+f|} \leq 1$ since $\operatorname{re} f \geq 0$

$\Rightarrow$ every sequence in $\hat{\mathcal{F}}$ has a normally convergent subsequence in $\hat{\mathcal{F}}$

Montel's Thm

Let $(f_n) \subset \mathcal{F} \to \hat{f}_n = \frac{1}{1+f_n} \in \hat{\mathcal{F}}$

So $\exists \hat{f}_{n_k} \to \hat{f} \in \hat{\mathcal{F}}$

$$\frac{1}{1+f_{n_k}} \to \hat{f}$$

since $\frac{1}{1+f_{n_k}} \neq 0$ $\forall z \in \mathbb{D}$ then $\hat{f} \equiv 0$ or $\hat{f}(z) \neq 0$ $\forall z$

(i) $f_{n_k} \to \infty$

(ii) $\lim \frac{1}{1+f_{n_k}} = \hat{f} \Rightarrow \frac{1-\hat{f}}{\hat{f}} = \lim f_{n_k} \in \mathcal{F}$

or

$\varphi = \frac{z-1}{z+1} = w$

$\Rightarrow \varphi \circ f$ is uniformly bdd

$\Rightarrow$ every sequence in has a normally convergent subsequence

$\varphi^{-1}(w) = \frac{1+w}{1-w} \to \infty$ on bdry

4. Let f be a nonconstant analytic function on unit disk $\mathbb{D}$ and let $U = f(\mathbb{D})$. Show if ϕ is a fcn in U (maybe not even continuous) and $\phi \circ f$ is analytic on $\mathbb{D}$ then so is ϕ

Pf Let $z_0 \in \mathbb{D}$

If $f'(z_0) \neq 0$ then by open mapping theorem $\exists f^{-1}$ in a nbhd of $f(z_0) \Rightarrow \phi = (\phi \circ f) \circ f^{-1}$ in analytic in a nbhd of z_0 analytic analytic

If $f'(z_0) = 0$ then since f is nonconstant $\forall \varepsilon > 0 \ \exists z_\varepsilon \in \Delta_\varepsilon(z_0)$ s.t. $f'(z_\varepsilon) \neq 0$
by above ϕ is analytic in nbhd of z_ε.
and $z_\varepsilon \to z_0$
$\Rightarrow \phi(z_0) = \lim_{\varepsilon \to 0} (\phi \circ f) \circ f^{-1}(z_\varepsilon) = c$ so ϕ is

analytic in a nbhd of z_0 and bounded
So there is a removable singularity at z_0

$\Rightarrow \phi$ is analytic on U □

inverse fcn ~~open mapping thm~~
f nonconstant $\Rightarrow f$ open map $\Rightarrow$ inverse exists in nbhd and is analytic.

Riemann Removable singularity thm
if f bdd near a then a is a removable singularity

August 2011

1. Under what conditions on complex numbers a and b the linear function $ax + by$ is analytic as a function of $z = x_iy$?

2. Find the formula for entire analytic functions which have a simple 0 at 0. What entire analytic functions have simple zero at ∞?

3. Let f be a conformal mapping of a disk. Show that f' is never equal to 0.

4. Let $D \subset \mathbb{C}$ is a domain and $\{f_j\}$ is a sequence of analytic functions on D such that the functions

$$g_n(z) = \sum_{j=1}^{n} |f_j(z)|$$

converge normally on D. Show that the functions

$$h_n(z) = \sum_{j=1}^{n} |f_j'(z)|$$

also converge normally on D.

August 2011

1. Under what conditions on complex #'s a, b is the linear function $ax+by$ analytic as a function of $z = x+iy$

Pf. $z = x+iy$,

$$\Rightarrow x = \frac{z+\bar{z}}{2} \qquad y = \frac{z-\bar{z}}{2i}$$

$$\Rightarrow ax+by = \frac{az}{2} + \frac{a\bar{z}}{2} + \frac{bz}{2i} - \frac{b\bar{z}}{2i}$$

$$= \left(\frac{a}{2} + \frac{b}{2i}\right)z + \left(\frac{a}{2} - \frac{b}{2i}\right)\bar{z}$$

$$\Rightarrow \frac{d}{d\bar{z}}(ax+by) = \frac{a}{2} - \frac{b}{2i} = 0$$

$$\Rightarrow a = b/i$$

$$\Rightarrow ai = b$$

$$\Rightarrow f(z) = ax + aiy.$$

$\square$

2. Find formula for entire analytic functions which have simple 0 at 0. What entire functions have simple 0 at ∞?

Pf a. f has simple 0 at 0

$\Rightarrow f(z) = z\,g(z)$ where g is analytic and $g(0) \neq 0$.

b. Assume f is entire and has simple 0 at ∞

$\Rightarrow$ f bdd since cant have problem earlier

$\Rightarrow f \equiv c$ by Louisville

$\Rightarrow f \equiv 0$

$\Rightarrow$ not a simple 0 since $f'(\infty) = 0$

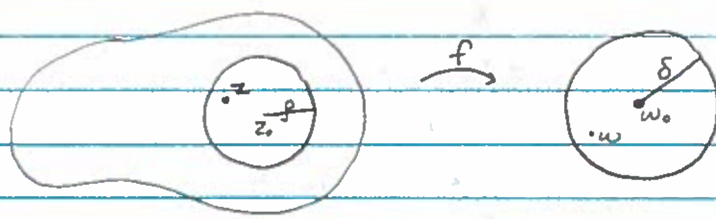
$\Rightarrow$ there is no such function

$\square$

3. Let f be a conformal mapping of disk
Show f' is never equal to 0.

Pf f conformal
$\Rightarrow f$ injective and holomorphic.

Assume $\exists z_0$ s.t. $f'(z_0)=0$ with $f(z_0)=w_0$
$\Rightarrow \exists \rho, \delta > 0$ s.t. $|f(z_0)-w| < \delta$ on $\{|z-z_0|<\rho\}=B$ s.t. $z \in B$ is attained at $m \geq 2$ distinct points in $|z-z_0|<\rho$.
$\Rightarrow$ Contradicts injectivity.



$\square$

Look at description of concepts on pg 233

4 Let $D \subset \mathbb{C}$ be a domain and $\{f_j\}$ a sequence of analytic functions on D s.t.

$$g_n(z) = \sum_1^n |f_j(z)|$$

converge normally on D. Show

$$h_n(z) = \sum_1^n |f_j'(z)|$$

also converge normally on D

Pf $g_n(z)$ converge normally

$\Rightarrow$ On any $\overline{B_r(0)}$ $\exists N$, s.t. $\sum_N^\infty |f_j(z)| < \varepsilon$ $\forall \varepsilon$

Let $\varepsilon > 0$.

$$\begin{aligned}
|h_\infty(z) - h_N(z)| &= \left|\sum_N^\infty |f_j'(z)|\right| \\
&\leq \sum_N^\infty |f_j'(z)| \\
&= \sum_N^\infty \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(w)}{(z-w)} dw \right| \\
&\leq \sum_N^\infty \frac{1}{2\pi} \int \frac{|f(w)|}{|z-w|} dw \\
&= \sum_N^\infty \frac{1}{2\pi} \int \frac{|f(w)|}{r} dw \\
&= \frac{1}{2\pi r} \int \sum_N^\infty |f(w)| \, dw \\
&< \frac{1}{2\pi r} \int \varepsilon \\
&= \varepsilon
\end{aligned}$$

$\Rightarrow$ $h_n(z)$ converges uniformly on any compact set

$\Rightarrow$ $h_n(z)$ converges normally

$\square$

Qualifying Exam, Complex Analysis, August 2010

1. Let $n > 0$ be an integer. How many solutions does the equation $3z^n = e^z$ have in the open unit disk? Justify your answer in full detail.

2. Let $f(z) = \sum_{n \geq 0} a_n z^n$ be holomorphic in the unit disk U such that

$$|f'(z)| \leq \frac{1}{1-|z|}\,, \ \forall\, z \in U.$$

Prove that $|a_n| \leq e$ for all $n \geq 1$.

3. Are there any entire functions f which satisfy $|f(z)| \geq \sqrt{|z|}$ for all $z \in \mathbb{C}$? Justify your answer in full detail.

4. Show that the function $I(z) = \displaystyle\int_{-\infty}^{+\infty} e^{-(t-z)^2} dt$, $z \in \mathbb{C}$, is constant.

410

1. Let $n \geq 0$ be an integer. How many solns to $3z^n = e^z$ in $\mathbb{D}$

Pf Let $g(z) = 3z^n - e^z$.

$f(z) = 3z^n$

$h(z) = -e^z$

$|h(z)| \leq |-e^z| \leq e^{|z|} \leq e$ where $|z|=1$

$|f(z)| = |3z^n| = 3|z|^n = 3$ where $|z|=1$

So $|h(z)| \leq |f(z)|$

So by Rouches thm f and g have same # of roots on $\mathbb{D}$. Thus g has n roots on $\mathbb{D}$

$\square$

2 Let $f(z) = \sum_{n \geq 0} a_n z^n$ be holomorphic in $\mathbb{D}$ s.t. $|f'(z)| \leq \frac{1}{1-|z|}$ $\forall z \in U$. Prove $|a_n| \leq e$ $\forall n \geq 1$

Pf $f'(z) = \sum_{n \geq 0} n a_n z^{n-1}$

Now consider $|na_n|$ for $n > 1$

$$|na_n| \leq \frac{\max_{|z|=r} |f'(z)|}{r^{n-1}}$$

$$\leq \frac{\frac{1}{1-|z|}}{r^{n-1}}$$

$$= \frac{\frac{1}{1-r}}{r^{n-1}} \qquad \text{let } r = 1 - \frac{1}{n}$$

$$= \frac{\frac{1}{\frac{1}{n}}}{(1-\frac{1}{n})^{n-1}}$$

$$= \frac{n}{(1-\frac{1}{n})^{n-1}}$$

So $|a_n| \leq \frac{1}{(1-\frac{1}{n})^{n-1}} \to e$

Now we WTS $(1-\frac{1}{n})^{n-1} \geq \frac{1}{e}$

$$\Leftrightarrow (n-1)\log\frac{n-1}{n} \geq -1$$

$$\Leftrightarrow (n-1)\log(n-1) - (n-1)\log(n) \geq -1$$

$$\Leftrightarrow (n-1)\log(n) - (n-1)\log(n-1) \leq 1$$

$$\Leftrightarrow (n-1)\frac{1}{c} \leq 1 \quad \text{for } c \in (n, n-1) \text{ by MVT}$$

The last statement is true so it follows that

$$\frac{1}{e} \leq \left(1 - \frac{1}{n}\right)^{n-1}$$

$$\Rightarrow e \geq \frac{1}{(1-1/n)^{n-1}} \geq |a_n|$$

$$\Rightarrow |a_n| \leq e \text{ for all } n.$$

$\square$

3. Are there any entire functions f which satisfy $|f(z)| \geq \sqrt{|z|}$ $\forall z \in \mathbb{C}$.

Pf $f(z)=0 \Rightarrow z=0$

Case 1 $f(0) \neq 0$.

$\Rightarrow 1/f(z)$ is entire since $f(z) \neq 0$ $\forall z$

$\Rightarrow 1/f(z)$ is bdd

$\Rightarrow 1/f(z) \equiv 1/c$ by Louiville for some constant c

$\Rightarrow f(z) \equiv c$

which contradicts $|f(z)| \geq \sqrt{|z|}$ $\forall z$.

Case 2 $f(0)=0$.

$\Rightarrow f(z) = z^n g(z)$ where g is analytic and $g(z) \neq 0$ $\forall z$

$\Rightarrow g \equiv M$ by Louiville.

$\Rightarrow f(z) = Mz^n$ where $n \in \mathbb{N} \setminus \{0\}$

$\Rightarrow |M||z|^n \geq \sqrt{|z|}$

$\Rightarrow |M| \geq |z|^{1/2 - n}$

contradicts as $z \to 0$

D

4) death.

A10

3. Are there any entire functions f s.t. $|f(z)| \geq \sqrt{|z|}$ $\forall z \in \mathbb{C}$.

Pf Let $g(z) = \frac{1}{f}$ assuming there is such an f.

If $z \neq 0$ then $f(z) \neq 0$ since $|f(z)| \geq \sqrt{|z|} \neq 0$.

If $z = 0$ then $f(0) \neq 0$.

Assume BWOC $f(0) = 0$ then $|f(z)| \leq M|z|$ near zero by Taylor series

$\Rightarrow M|z| \geq |z|^{1/2}$

$\Rightarrow |z|^{1/2} \geq \frac{1}{M}$

this contradicts near zero so $f(0) \neq 0$

Thus $f(z) \neq 0$ $\forall z \in \mathbb{C}$.

$\Rightarrow g(z)$ is entire and $\lim_{z \to \infty} g = 0$

$\Rightarrow g$ is constant since g is bdd and entire (Louiville)

$\Rightarrow g \equiv 0$

$\Rightarrow \frac{1}{f} = 0$ which contradicts since $\frac{1}{f} \neq 0$

$\Rightarrow \nexists$ such f

$\square$

step 1: $g = 1/f$

step 2: show $f(z) \neq 0$

step 3: g constant

step 4: $g \equiv 0$

step 5: no such f

$|e^z| = e^{\operatorname{Re} z}$

4 Show $I(z) = \int_{-\infty}^{\infty} e^{-(t-z)^2}\,dt \quad z \in \mathbb{C}$ is constant

Pf First let $z = x+iy$ and show $I(z)$ is a fcn of y.

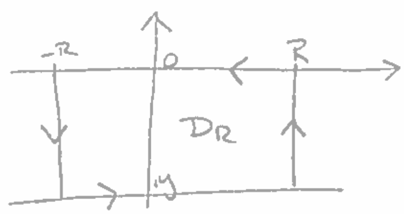
$$I(z) = \int_{-\infty}^{\infty} |e^{-(t-z)^2}|\,dt$$
$$= \int_{-\infty}^{\infty} e^{-\operatorname{Re}(t-z)^2}\,dt \qquad (t-x-y)^2 = (t-x)^2 - y^2$$
$$= e^{-y^2}\int_{-\infty}^{\infty} e^{-(t-x)^2}\,dt \qquad s = t-x$$
$$= e^{-y^2}\int_{-\infty}^{\infty} e^{-s^2}\,ds$$
$$= e^{-y^2}\sqrt{\pi}$$

$$I(z) = \int_{-\infty}^{\infty} e^{-(t-x-iy)^2}\,dt$$
$$= \int_{-\infty}^{\infty} e^{-(t-iy)^2}\,dt \quad t = t-x$$
$$= \tilde{I}(y)$$ so I does not depend on x

*

WLOG assume $y>0$.

Now WTS $\tilde{I}(y) = \tilde{I}(0)$



$\Gamma_R = \partial D_R$.

$$\int_{\partial D_R} e^{-z^2}\,dz = 0 = -\int_{-R}^{R} e^{-t^2}\,dt + \int_{-R}^{R} e^{-(t-iy)^2}\,dt + \int_{-y}^{0} e^{-(R+it)^2}\,dt + \int_{0}^{-y} e^{-(-R+it)^2}\,dt$$

Let $R\to\infty$ WTS $\int_{-y}^{0} e^{-(R+it)^2}\,dt + \int_0^{-y} e^{-(R+it)^2}\,dt \to 0$

Consider $\left|\int_{-y}^{0} e^{-(\pm R+it)^2}\,dt\right| \le y e^{-y^2-R^2} \to 0$ as $R\to\infty$.

Since $|e^{-(\pm R\pm it)^2}| \le e^{-(R^2-t^2)} \le e^{y^2-R^2}$.

So $I(z) = \sqrt{\pi}$

* Could also show $I(z)$ is holomorphic then $I'(z) = \frac{\partial I}{\partial x}(z) = 0$ since does not depend on x
$\therefore$ I is constant.

Complex Part January 2010

1. Show that the function $f(z) = 1/z$ has no a holomorphic anti-derivative on $\{ \quad 1 < |z| < 2\}$.

2. Suppose that f is an entire function and f^2 is a holomorphic polynomial. Show that f is also a holomorphic polynomial.

3. Suppose that a function f is meromorphic on the unit disk $\mathbb{D}$ and continuous in a neighborhood of its boundary $\partial\mathbb{D}$. Show that for any number A such that $|A| > \sup_{z\in\partial\mathbb{D}} |f(z)|$ the number of zeros of the function $f - A$ is equal to the number of poles of f in $\mathbb{D}$.

4. Suppose that f and g are entire functions such that $f \circ g(x) = x$ when $x \in \mathbb{R}$. Show that f and g are linear functions.

January 2010

1. Show $f(z) = 1/z$ has no holomorphic antiderivative on $\{1 < |z| < 2\}$.

Pf Suppose it did.

$\Rightarrow \int_\gamma f(z)\,dz = 0$ for $\gamma \in \{1<|z|<2\}$ by Cauchy

$\Rightarrow \int_{|z|=1.5} 1/z\,dz = 0$

Let $z = 3/2 e^{i\theta}$

$\Rightarrow \int_\gamma 1/z\,dz = \int_0^{2\pi} \frac{1}{3/2 e^{i\theta}}\,d\theta$ $\qquad z = e^{i\theta} \quad dz = ie^{i\theta}d\theta = iz\,d\theta$

$= \int_0^{2\pi} 2/3 \; \frac{iz}{z}\,dz$

$= 2i/3 \; 2\pi \; 3/2$

$= 2\pi i$ which contradicts.

or use Residue Theory!!

$\square$

2. Suppose f is an entire function and f^2 is a holomorphic polynomial. Show f is a holomorphic polynomial.

Pf f^2 a polynomial

$\Rightarrow f^2(z) = a_0 + a_1 z + \dots + a_n z^n$

$\Rightarrow |f^2(z)| \le |a_0| + |a_1||z| + \dots + |a_n||z^n|$

$\le C|z|^n$ where $C = |a_0| + \dots + |a_n|$

$\Rightarrow |f(z)| \le \sqrt{C}\,|z|^{n/2}$ $|z| \ge 1$

$\Rightarrow f$ is a polynomial of degree at most $n/2$ by hw #4 in 4.5.

$\square$

Alternatively

Pf Let $z_0, \dots z_n$ be zeros of f. s.t

$\Rightarrow f = (z-z_0)(z-z_1)\cdots(z-z_n)g(z)$ $g(z) \neq 0$

$\Rightarrow f^2 = (z-z_0)^2(z-z_1)^2 \cdots (z-z_n)^2 g^2(z)$

but $f^2 = (z-z_0)^2 \cdots (z-z_n)^2 h(z)$

show finite zeros

$\Rightarrow 1/g^2(z) = \dfrac{(z-z_0)^2 \cdots (z-z_n)^2}{g^2(z)}$ has no zeros

$\Rightarrow g^2(z)$ is bdd and entire

$\Rightarrow g^2(z)$ is constant by Liouville

$\Rightarrow g(z)$ is constant

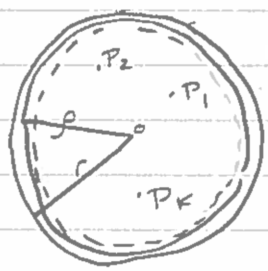
$\Rightarrow f$ is a polynomial

$\square$

JTO

3. Suppose f meromorphic on $\mathbb{D}$ and continuous in nbhd of $\partial\mathbb{D}$. Show $\forall A$ s.t. $|A| > \sup_{z\in\partial\mathbb{D}}|f(z)|$ the # of zeros of fcn $f-A$ is equal to # of poles of f in $\mathbb{D}$.

Pf



Let circle of radius ρ be just inside circle of radius r, then by continuity as $\rho \to 1$ we can apply Rouches thm

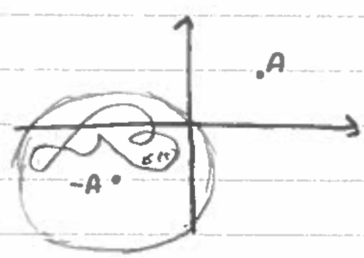
WTS $0 = \frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z)-A}dz = N_0(f-A) - N_\infty(f-A) = N_0(f-A) - N_\infty(f)$

# of poles are same since A merely shifts f.

Let $w = f(z) - A$
$dw = f'(z)dz$
$\gamma(t) = f(e^{it}) - A$

Then $\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)dz}{f(z)-A} = \frac{1}{2\pi i}\int_\gamma \frac{1}{w}dw$

$|\gamma(t) + A| = |f(e^{it})| < A$



$\frac{1}{w}$ has primitive in $\Delta(-A, |A|)$ and $0 \notin \Delta(-A, |A|)$

Thus $\frac{1}{2\pi i}\int_\gamma \frac{dw}{w} = 0$

$\Rightarrow \frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z)-A} = 0$

J10

4 Suppose f and g are entire s.t. $f\circ g(x) = x$ when $x \in \mathbb{R}$. Show f and g are linear.

Pf By identity thm $f\circ g(z) = z \quad \forall z \in \mathbb{C}$.

- $g(z_1) = g(z_2) \Rightarrow f(g(z_1)) = f(g(z_2)) \Rightarrow z_1 = z_2 \Rightarrow g$ injective.
- $g: \mathbb{C} \to \mathbb{C}$ bijective if g not surjective then $\exists g: \mathbb{C} \to g(\mathbb{C})$ conformal $\Rightarrow g(\mathbb{C})$ is simply connected
  $\Rightarrow g(\mathbb{C}) = \mathbb{C}$ by RMT
  $\Rightarrow g$ surjective.

$\Rightarrow g$ is a homeomorphism

WTS $\lim_{z\to\infty} g(z) \neq \infty$

Assume BWOC $\lim g(z) = \infty$

$\Rightarrow \exists \{z_n\} \to \infty$ with $|g(z_n)| \leq M$ for some M.

$\Rightarrow \exists w_{n_k} = g_{n_k} \to w$ since $g(z_n)$ bdd

$\Rightarrow g^{-1}(w_{n_k}) \to g^{-1}(w) \in \mathbb{C}$ by applying inverses.

$\Rightarrow z_{n_k} \to g^{-1}(w) \in \mathbb{C}$ contradicts since $z_{n_k} \to \infty$

$\therefore \lim_{z\to\infty} g(z) \neq \infty$

Only polynomials can have holes at ∞.

$\Rightarrow g$ is an injective polynomial

$\Rightarrow g$ is linear

$\Rightarrow g(z) = az + b \quad a \neq 0$

$\Rightarrow f(\underbrace{az+b}_{w}) = z$

$\Rightarrow f(w) = \frac{w-b}{a}$

$\Rightarrow f$ is also linear

$\square$

3 Suppose a function f is meromorphic on unit disk $\mathbb{D}$ and continuous in a nbhd of $\partial\mathbb{D}$. Show that for any number A s.t. $|A| > \sup_{z\in\partial\mathbb{D}} |f(z)|$ the number of zeros of the function $f-A$ is equal to # of poles of f in $\mathbb{D}$

Pf meromorphic => f has finitely many poles all in $\mathbb{D}$

$$N_0(f-A) - \underbrace{N_\infty(f-A)}_{=N_\infty(f)} = \frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{f(z)-A}\,dz$$

Let $w = f(z) - A$ $\quad \gamma = f(re^{it}) - A$

$dw = f'(z)\,dz$

$\Rightarrow |\gamma(t) + A| = |f(re^{it})| < |A|$

$\Rightarrow \gamma(t) \in \Delta(-A, |A|)$ Shift so 0 isnt in center of path

$$\Rightarrow \frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{f(z)-A} = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w}$$

$= 0$ since $1/w$ is holomorphic in $\Delta(-A, |A|)$

$= N_0(f-A) = N_\infty(f)$

4 Suppose f and g are entire functions s.t. $f \circ g(x) = x$ when $x \in \mathbb{R}$. Show f and g are linear functions.

Pf $f \circ g(x) = x \quad x \in \mathbb{R}$

$\Rightarrow f \circ g(z) = z \quad \forall z \in \mathbb{C}$ by identity principle

$\Rightarrow g$ is injective

$g(z_1) = g(z_2) \Rightarrow f(g(z_1)) = f(g(z_2)) = z_1 = z_2$

$\Rightarrow g$ conformal since g is entire and injective

$\Rightarrow g'(z) \neq 0 \quad \forall z$

$\Rightarrow \frac{1}{g'(z)}$ is bdd and entire

(g' exists since g is analytic)

$\Rightarrow \frac{1}{g'(z)}$ constant by Liouville

$\Rightarrow g'(z)$ is constant

$\Rightarrow g(z) = az+b$ for some $a, b \in \mathbb{C}$

$\Rightarrow f(az+b) = z$

$\Rightarrow f(w) = \frac{w-b}{a}$

$\Rightarrow f$ is linear. ✓

□

QUALIFYING EXAM COMPLEX ANALYSIS

Thursday, January 8, 2009

Show **ALL** your work. Write all your solutions in clear, logical steps. **Good luck!**

Your Name:

Problem	Score	Max
1		20
2		20
3		30
4		30
Total		100

Problem 1. Let $f = f(z)$ be analytic in the unit disk, $f(0) = 0$. Show that the infinite series

$$\sum_{n=1}^{\infty} f(z^n)$$

is converging and represents an analytic function in the unit disk.

Problem 2.

Consider an analytic function defined in the unit disk by the following power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n , \quad \text{where the coefficients are real numbers such that} \quad n^{-2009} \leqslant a_n \leqslant n$$

Show that f does not extend analytically near the point $z = 1$.

Problem 3. (Cauchy Formula)
Let $\mathbb{F}$ be a countable compact subset of a domain $\Omega \subset \mathbb{C}$. Suppose we are given a bounded holomorphic function

$$f : \Omega \setminus \mathbb{F} \to \mathbb{C}$$

Show that f extends holomorphically to the entire domain Ω.

a) First try a simple case when $\mathbb{F}$ is finite
b) Try the case when $\mathbb{F}$ has finite number of accumulation points
c) Try the general case.
d) The problem still remains valid if $\mathbb{F}$ is a compact set of zero length (1-dimensional Hausdorff measure), try to extend your proof to this general case. Recall that $\mathbb{F}$ has zero length if it can be covered by a finite number of disks whose diameters sum up to a number as small as we wish.

Problem 4.

Compute the following integral

$$\int_0^\infty \frac{\cos x}{(1+x^2)^2}\,dx$$

Hint. Consider the following complex function in the upper half plane

$$f(z) = \frac{e^{iz}}{(1+z^2)^2}$$

J09.1 Let $f = f(z)$ be analytic on $\mathbb{D}$. $f(0)=0$. Show $\sum_1^\infty f(z^n)$ converges to an analytic fcn on $\mathbb{D}$

Pf We WTS $\sum_1^\infty f(z^n)$ converges normally since analytic fcns converge normally to an analytic fcn

$\exists M$ s.t. $|f(z)| \leq M$ on $|z| \leq 1/2$ since f continuous and $f(0)=0$.

$\Rightarrow |f(z)/z| \leq 2M$ on $|z| = 1/2$

$\Rightarrow |f(z)/z| \leq 2M$ on $|z| \leq 1/2$ by Max principle

$\Rightarrow |f(z)| \leq 2M|z|$ on $|z| \leq 1/2$.

Fix $r>0$.

If $|z| \leq r$, $\exists N$ s.t. $n \geq N \Rightarrow |z|^n \leq r^n < 1/2$

$\Rightarrow |f(z^n)| \leq 2M|z|^n \leq 2Mr^n$ for $n \geq 0$

$\Rightarrow \sum 2Mr^n < \infty$

$\Rightarrow \sum |f(z^n)|$ converges uniformly

This holds for every $r \in (0,1)$

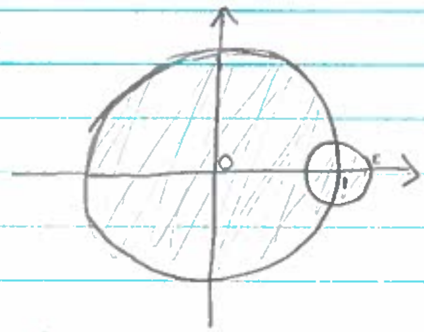
$\Rightarrow \sum_1^\infty f(z^n)$ converges normally

$\Rightarrow \sum_1^\infty f(z^n) = f$ where f is analytic

$\square$

J09.2 Let $f = \sum_1^\infty a_n z^n$ be analytic $a_n \in \mathbb{R}$ $n^{-2009} \le a_n \le n$
Show f does not extend analytically near $z=1$

Pf



Let $\varepsilon > 0$.
Let $D_\varepsilon = B_1(0) \cup B_\varepsilon(1)$
We WTS f does not extend to a holomorphic fcn on D_ε $\forall \varepsilon$.

Assume BWOC $\exists \varepsilon > 0$ s.t. $\exists F$ holomorphic on D_ε with $F|_{\mathbb{D}} = f$.
Then $f^{(2008)}(z) = \sum_{n=2008}^\infty n(n-1)\dots(n-2007)a_n z^{n-2008}$
and $\lim_{r \nearrow 1} f^{(2008)}(r) = F^{(2008)}(1)$ since derivatives converge

Now if $N > 2008$: since $a_n \ge n^{-2009}$

$$\sum_{2008}^N \frac{n(n-1)\dots(n-2007)}{n^{2009}} r^{n-2008} \le \sum_{2008}^N n(n-1)\cdots(n-2007)a_n r^{n-2008}$$
$$\le f^{(2008)}(r)$$
$$\to F^{(2008)}(1) < \infty$$

$\Rightarrow \sum_{2008}^\infty \frac{n\dots(n-2007)}{n^{2009}} < \infty$ since F has infinitely many derivatives
converges which contradicts the comparison test to $\sum_{2008}^\infty \frac{1}{n}$

$\square$

J.9.3 Let $\mathbb{F}$ be countable compact subset of $\Omega \subset \mathbb{C}$
$f: \Omega \setminus \mathbb{F} \to \mathbb{C}$ bdd holomorphic. Show f extends holomorphically to Ω

a) $\mathbb{F}$ finite

b) $\mathbb{F}$ finite # of accumulation points

c) general $\mathbb{F}$

d) $\mathbb{F}$ compact set of zero length

Pf a) Assume $\mathbb{F}$ finite.

$\Rightarrow f$ only has isolated singularities

$\Rightarrow f$ only has removable singularities since f bdd near each by Reimann's Thm on Removable Singularity.

$\Rightarrow f$ extends holomorphically to Ω

b) Assume $\mathbb{F}$ has finite # of accumulation points

Let $z_1, z_2, \dots z_n$ be accumulation points.

$\{z_{i_k}\} \to z_i$ z_{i_k} are isolated so removable.

$\Rightarrow \Omega \setminus \{z_i\}$ extends to z_i since once extends to the ones approaching the accumulation points become removable

c).

$N > 2008$:

$$\sum_{n=2008}^{N} \frac{n(n-1)\cdots(n-2007)}{n^{2009}} r^n \le \sum_{2008}^{N} n(n-1)\ldots(n-2007)\, a_n r^n$$
$$\le f^{(2008)}(r) \quad r \nearrow 1$$
$$\to \sum \frac{n(n-1)\ldots(n-2007)}{n^{2009}}$$
$$\le F^{(2008)}(1)$$

$\sum_{n=2008}^{\infty} \frac{n \cdots (n-2007)}{n^{2009}} < \infty$ converges which contradicts Comparison test to $\sum_{2008}^{\infty} \frac{1}{n}$.

$\square$

J2009.3 F countable compact subset of domain Ω
$f: \Omega \setminus F \to \mathbb{C}$ holomorphic and bounded.
$(|F(z)| \le M \quad \forall z \in \Omega \setminus F)$
$\Rightarrow f$ extends holomorphically to Ω

a) F finite $\Rightarrow f$ has only isolated singularities
$\Rightarrow f$ extends holomorphically (Riemman rem sing thm)

b) $z_n \to z_0$
$F = \{z_0, z_n : n \ge 1\}$
$\Omega \setminus \{z_0\}$ extends to Ω

c) $\forall \varepsilon > 0$, $\exists \Delta_j^\varepsilon$ $1 \le j \le N_\varepsilon$, open discs in Ω
s.t. $F \subset \bigcup_{j=1}^{N_\varepsilon} \Delta_j^\varepsilon$, $\sum_1^{N_\varepsilon} \operatorname{diam} \Delta_j^2 < \varepsilon$

$F = \{z_1, z_2, \ldots, z_n, \ldots\}$
$\sum_1^\infty \frac{\varepsilon}{2^n} = \varepsilon$

$\exists \Delta_n$ open disc centered at z_n, $\Delta_n \subset \Omega$ with $\operatorname{diam}(\Delta_n) < \frac{\varepsilon}{2^n}$

$F \subset \bigcup_{n=1}^{\infty} \Delta_n$, pick a finite subcover
$\Delta_1^\varepsilon, \ldots, \Delta_{N_\varepsilon}^\varepsilon$

F compact $\subset \Omega$, $\forall \varepsilon > 0$, $\exists$ open discs $\Delta_j^\varepsilon \subset \Omega$, $F \subset \bigcup_{j=1}^{N_\varepsilon} \Delta_j^\varepsilon$, $\sum_j \operatorname{diam} \Delta_j^\varepsilon < \varepsilon$

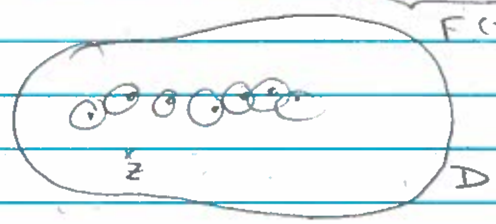
WLOG $F \cap \Delta_j^\varepsilon \neq \emptyset$ $\forall j$.

(Cantor set is uncountable and has these prop)

Fix D a domain w/ smooth boundary, bdd.
$F \subset D \subset \Omega$

Show $\underbrace{f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta}$ $\quad z \in D \setminus F$

$F(z)$ is hol on D hence F extends



$z \in D \setminus F$ $\quad \delta = \operatorname{dist}(z, F) > 0$ $\quad \varepsilon < \delta$
$\exists\, \Delta_j^\varepsilon$ $j = 1, \dots, N_\varepsilon$, $\overline{\Delta}_j \subset D$
$F \subset \bigcup \Delta_j^\varepsilon$, $\sum \operatorname{diam} \Delta_j^\varepsilon < \varepsilon$.

$\operatorname{dist}(z, \bigcup_{j=1}^{N_\varepsilon} \overline{\Delta}_j^\varepsilon) > \delta - \varepsilon$.

$|\operatorname{dist}(z, F) - \operatorname{dist}(\zeta, F)| \leq |z - \zeta|$

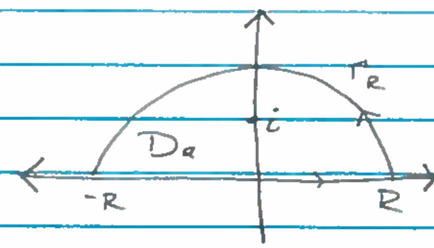
$\underbrace{D \setminus \bigcup \overline{\Delta}_j^\varepsilon}_{D_\varepsilon}$ piecewise smooth.

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$\left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq \frac{1}{2\pi} \frac{M}{\delta - \varepsilon} \rho(\partial D_\varepsilon)$$
$$\leq \frac{1}{2\pi} \frac{M}{\delta - \varepsilon} \sum_{j=1}^{N_\varepsilon} \ell(\partial \Delta_j^\varepsilon)$$
$$= \frac{M}{2(\delta - \varepsilon)} \varepsilon$$
$$\to 0 \text{ as } \varepsilon \to 0.$$

J.9.4 Compute $\int_0^\infty \frac{\cos x}{(1+x^2)^2}\,dx$.

Pf $\int_0^\infty \frac{\cos x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{(1+x^2)^2}\,dx$.



i is only singularity in upper half plane.

don't need this by thm

$$\int_{\partial D_R}\frac{\cos z}{(1+z^2)^2}\,dz = \int_{-R}^{R}\frac{\cos x}{(1+x^2)^2}\,dx + \int_{\Gamma_R}\frac{\cos z}{(1+z^2)^2}\,dz$$

$$\int_{\Gamma_R}\frac{\cos z}{(1+z^2)^2} \le \int_{\Gamma_R}\frac{1}{(1+z^2)^2} = \frac{\pi R^2}{2}\,\frac{1}{(1+R^2)^2} \to 0 \quad \text{as } R\to\infty$$

$$\int_{D_R}\frac{e^{iz}}{(1+z^2)^2} = 2\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{(1+z^2)^2}, i\right)$$

$$= 2\pi i \lim_{z\to i}\frac{d}{dz}\left(\frac{(z-i)^2 e^{iz}}{(z+i)(z-i)(z+i)(z-i)}\right)$$

$$= 2\pi i \lim_{z\to i}\frac{d}{dz}\,\frac{e^{iz}}{(z+i)^2}$$

$$= 2\pi i \lim_{z\to i}\frac{i(z+i)^2e^{iz} - e^{iz}(2)(z+i)}{(z+i)^4}$$

$$= 2\pi i\left(\frac{i(2i)^2e^{i^2} - e^{i^2}(2)(2i)}{(2i)^4}\right)$$

$$= \frac{2\pi i\,(-4ie^{-1} - e^{-1}4i)}{16}$$

$$= \frac{16\pi}{16e} = \frac{\pi}{e}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ix}}{(1+x^2)^2} = \int_{-\infty}^{\infty}\frac{\cos x}{(1+x^2)^2} \quad \text{since } \mathrm{re}(e^{iz}) = \cos x$$

$$= \frac{\pi}{2e}$$

Qualifying Exam, Complex Analysis, August 2008

1. Let f be an entire function, $a \in \mathbb{C}$ and $r > |a|$. Show that

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{f(1/z)}{z-a}\, dz = f(0) .$$

2. Find the image of the first quadrant $\{x > 0, y > 0\}$ under the Möbius map $w = \dfrac{z-i}{z+i}$.

3. Find all the continuous functions $v : \mathbb{C} \longrightarrow \mathbb{R}$ which have the property that for every rectangle $R \subset \mathbb{C}$ with sides parallel to the coordinate axes

$$\int_{\partial R} v\, dx = -area\ R \ , \quad \int_{\partial R} v\, dy = 0 ,$$

where ∂R is traversed counterclockwise. (Hint: Consider the function $f(z) = x + iv(x, y)$, where $z = x + iy$.)

4. Suppose that

$$f(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

is a holomorphic function on the closed unit disc $\overline{\Delta}$ such that $|f(z)| \leq M$ for $|z| = 1$. If $z_0 \in \Delta$ is a zero of f show that

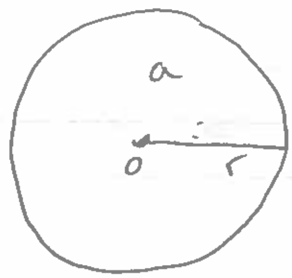
$$|z_0| \geq \frac{1}{M+1} .$$

A08

1. Let f be an entire function. $a \in \mathbb{C}$ and $r > |a|$. Show

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{f(1/z)}{z-a}\,dz = f(0).$$

Pf.



1. Apply residue thm on $\Delta(0,r)$
$a \neq 0$ and $a = 0$.

2. Let $g = f(1/z)/(z-a)$
g has a Laurent Series on $\{|z| > |a|\}$
which converges uniformly on $|z| = r$
so we integrate term by term

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$

Then $$\frac{f(1/z)}{z-a} = \frac{1}{z}\,\frac{1}{1-\frac{a}{z}}\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right)$$
$$= \left(1 + \frac{a}{z} + \frac{a^2}{z^2}\dots\right)\left(\frac{a_0}{z} + \frac{a_1}{z^2} + \dots\right)$$
$$= \frac{a_0}{z} + \frac{a_1 + a \cdot a_0}{z^2} + \dots$$

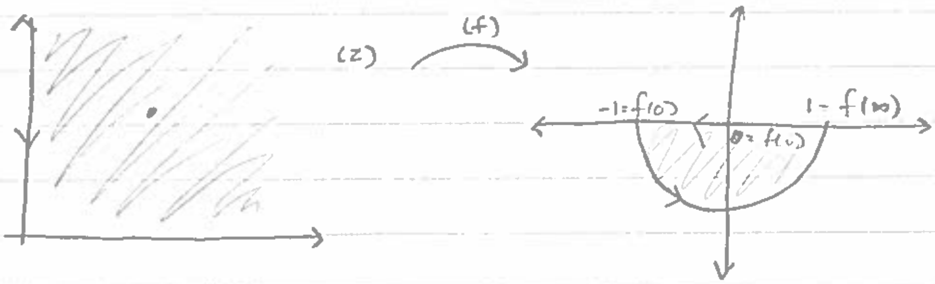
Thus $$\frac{\int_{|z|=r} g(z)}{2\pi i} = a_0 \underbrace{\frac{1}{2\pi i}\int_{|z|=r}\frac{dz}{z}}_{1} + 0 + 0 + \dots + 0$$
$$= a_0$$
$$= f(0).$$

$$\therefore \frac{1}{2\pi i}\int_{|z|=r}\frac{f(1/z)}{z-a}\,dz = f(0)$$

$\square$

2. Find the image of first quadrant under Möbius map $w = \frac{z-i}{z+i}$.

Pf



$f(0) = \frac{-i}{i} = -1$

$f(\infty) = 1$

$f(i) = 0$

$f(1+i) = \frac{1+i-i}{1+i+i} = \frac{1}{1+2i} = \frac{1-2i}{1+4} = \frac{1}{5} - \frac{2i}{5}$

$f(1) = -i$

So image is $A = \{z \,|\, |z| \leq 1,\ \operatorname{Im} z \leq 0\}$

$\square$

3. Find all continuous functions $v: \mathbb{C} \to \mathbb{R}$ which have property that $\forall$ rectangle $R \subset \mathbb{C}$ w/ sides parallel to coordinate axes

$$\int_{\partial R} v\,dx = -\text{area } R, \quad \int_{\partial R} v\,dy = 0$$

where ∂R is traversed counterclockwise.

Pf Let $f(z) = x + i\,v(x,y)$, $z = x+iy$

Then $\int_{\partial R} f(z)dz = \int_{\partial R} (x+iv)(dx+idy)$

$$= \int_{\partial R} x\,dx + i\,x\,dy + i\int_{\partial R} v\,dx - \int_{\partial R} v\,dy$$

$$= \underbrace{\iint_R (1-0)\,dx\,dy}_{\text{Green's}} \underbrace{- i\,\text{area } R - 0}_{\text{hyp}}$$

$$= 0$$

So f is entire.

Use CRE

$$\left.\begin{array}{l} \frac{\partial}{\partial x} x = \frac{\partial v}{\partial y} \\ \frac{\partial}{\partial y} x = -\frac{\partial v}{\partial x} \end{array}\right\} \Rightarrow \begin{array}{l} \frac{\partial v}{\partial y} = 1 \\ \frac{\partial v}{\partial x} = 0 \end{array}$$

So $v(x,y) = y + C$ for $C \in \mathbb{R}$.

$\square$

4 Suppose $f(z) = 1 + c_1 z + c_2 z^2 + \dots$ is a holomorphic function on the closed unit disc $\bar{\Delta}$ s.t. $|f(z)| \le M$ for $|z|=1$. If $z_0 \in \Delta$ is a zero of f s.t. $|z_0| \geq \frac{1}{M+1}$

Pf: First $|f(z)| \le M \Rightarrow |1 + c_1 z + c_2 z^2 + \dots| \le M$

$$\Rightarrow |z^n| \left| \frac{1}{z^n} + \frac{c_1}{z^{n-1}} + \frac{c_2}{z^{n-2}} + \dots \right| \le M$$

$$\Rightarrow |z^n| |1 + c_1 + c_2 + \dots| \le M$$

$$\Rightarrow |c_n| \le M/1^n = M$$

blah this makes no sense

Now since z_0 is a zero of $f(z)$ we have

$\Rightarrow 0 = f(z_0)$

$\Rightarrow 0 = 1 + c_1 z_0 + c_2 z_0^2 + c_3 z_0^3 + \dots$

$\Rightarrow -1 = c_1 z_0 + c_2 z_0^2 + \dots$

$\Rightarrow 1 \le |c_1||z_0| + |c_2||z_0|^2 + \dots$ by triangle inequaity.

$\Rightarrow 1 \le M(|z_0| + |z_0|^2 + \dots)$ Since $|c_n| \le M \ \forall n$

$\Rightarrow 1 \le \dfrac{M|z_0|}{1-|z_0|}$ Since $\frac{1}{1-z} = 1 + z + z^2 + \dots$

$\Rightarrow 1 - |z_0| \le M|z_0|$

$\Rightarrow |z_0| \geq \frac{1}{M+1}$

$\square$

4. Alternative solution.

Pf: $f(z_0) = 0 = 1 + c_1 z_0 + c_2 z_0^2 + \cdots$

$\Rightarrow -1 = c_1 z_0 + c_2 z_0^2 + \cdots$

$= \sum_0^\infty c_{n+1} z_0^{n+1}$

$= z_0 \sum_0^\infty c_{n+1} z_0^n$

$$\Rightarrow |z_0| = \frac{1}{|\sum_0^\infty c_{n+1} z_0^n|}$$

Now $|f(z)| \le M$ if $|z| = 1$

$\Rightarrow |f(z) - 1| \le |f(z)| + |-1|$

$\le M + 1$

$\Rightarrow |c_1 z + c_2 z^2 + \cdots| \le M + 1$

$\Rightarrow |z \sum_0^\infty c_{n+1} z^n| \le M+1$

$\Rightarrow |\sum_0^\infty c_{n+1} z^n| \le M+1$

$\Rightarrow |z_0| \geq \frac{1}{M+1}$

$\square$

Qualifying Exam, Complex Analysis, January 11, 2008

Notation: Throughout the exam U denotes the open unit disc in $\mathbb{C}$.

1. Show that a complex valued function $h(z)$ on U is harmonic if and only if
$$h(z) = f(z) + \overline{g(z)},$$
where $f(z)$ and $g(z)$ are analytic on U.

2. Find $\displaystyle\int_{|z|=1} z^n \cos z \, dz$, where $n \in \mathbb{Z}$.

3. Find all the possible Laurent expansions centered at 0 of the function
$$f(z) = \frac{4z^2}{(z+1)(z-3)}\,.$$
Specify the annulus of convergence for each such expansion.

4. (i) Show that the Möbius transformation $h(z) = \dfrac{z-a}{1-\bar{a}z}$, where $a \in U$, is a conformal self-map of U.

(ii) Let $f : U \to U$ be a holomorphic function and assume that $a_1, \ldots, a_n \in U$ are zeros of f. Prove that $|f(0)| \leq |a_1 \ldots a_n|$.

J08

1. Show complex valued $h(z)$ on $\mathbb{D}$ is harmonic $\Leftrightarrow$ $h(z) = f(z) + \overline{g(z)}$ where $f(z)$ and $g(z)$ are analytic on $\mathbb{D}$

Pf Let $h(z) = u + iv$ w/ h harmonic

$\Rightarrow u = \operatorname{Re} \varphi$ for some analytic φ

$v = \operatorname{Re} \psi$ for some analytic ψ

$$\Rightarrow u = \frac{\varphi + \overline{\varphi}}{2} \qquad v = \frac{\psi + \overline{\psi}}{2}$$

$$\Rightarrow u + iv = \left(\frac{\varphi}{2} + i\frac{\psi}{2}\right) + \left(\frac{\overline{\varphi}}{2} + i\frac{\overline{\psi}}{2}\right)$$

$$= \underbrace{\left(\frac{\varphi}{2} + i\frac{\psi}{2}\right)}_{\text{analytic}} + \underbrace{\left(\frac{\overline{\varphi - i\psi}}{2}\right)}_{\text{analytic}} \quad \checkmark$$

Now let $h = f(z) + \overline{g(z)}$

$$\Delta h = 4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar z} f\right) + 4\frac{\partial}{\partial \bar z}\left(\frac{\partial}{\partial z}\overline{g}\right)$$

$$= 4\frac{\partial}{\partial z}(0) + 4\frac{\partial}{\partial \bar z}(0)$$

$$= 0$$

$\Rightarrow h$ is harmonic

$\square$

2. Find $\int_{|z|=1} z^n \cos z \, dz$ where $n \in \mathbb{Z}$.

Pf

$\int_{|z|=1} z^n \cos z \, dz = -2\pi i \operatorname{Res}[z^n \cos z, 0]$ only possible problem is at 0.

$z^n \cos z = z^n \sum_{k=0}^{\infty} \frac{(-1) z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{z^{2k+n}}{(2k)!} (-1)^k$

Case 1 $\exists$ k s.t. $2k+n=-1$, i.e. $k = \frac{-1-n}{2}$

then $\operatorname{Res}[z^n \cos z, 0] = -\frac{1}{(-1-n)!}$

$\Rightarrow \int_{|z|=1} z^n \cos z \, dz = \frac{2\pi i}{(-1-n)!} (-1)^{(-1-n)/2}$

Case 2 $\nexists$ k s.t. $2k+n=-1$

then $\operatorname{Res}[z^n \cos z, 0] = 0$

$\Rightarrow \int_{|z|=1} z^n \cos z = 0$

$\square$

3. Find all possible Laurent expansions centered at 0 of $f(z) = \dfrac{4z^2}{(z+1)(z-3)}$

Pf

$$\begin{array}{r} 4 \\ z^2-2z-3 \overline{) \; 4z^2 \qquad\qquad} \\ \underline{4z^2 - 8z - 12} \\ 8z+12 \end{array} \qquad \Rightarrow f(z) = 4 + \frac{8z+12}{(z+1)(z-3)}$$

$$4 + \frac{8z+12}{(z+1)(z-3)} = 4 + \frac{A}{z+1} + \frac{B}{(z-3)}$$

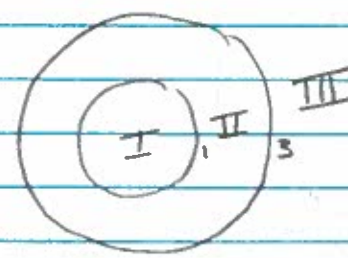
$\Rightarrow 8z+12 = (z-3)A + (z+1)B$

$\Rightarrow 36 = 4B \quad \Rightarrow B = 9 \qquad z=3$

$\Rightarrow 4 = -4A \qquad\qquad z=-1$

$\Rightarrow -1 = A$

$\therefore f(z) = 4 - \frac{1}{z+1} + \frac{9}{z+3}$



① $I = \{z \mid |z| < 1\}$

$$f(z) = 4 - \frac{1}{1-(-z)} + 3\left(\frac{1}{1-(-\frac{z}{3})}\right)$$
$$= 4 - \sum(-z)^n + 3\sum\left(\frac{-z}{3}\right)^n$$
$$= 4 - \sum(-1)^n z^n + \sum \frac{(-1)^n}{3^{n-1}} z^n$$
$$= 4 + \sum(-1)^n\left(\frac{1}{3^{n-1}} - 1\right) z^n$$

② $II = \{z \mid 1 < |z| < 3\}$

$$f(z) = 4 - \frac{1}{z}\left(\frac{1}{1-(-\frac{1}{z})}\right) + 3\left(\frac{1}{1-(-\frac{z}{3})}\right)$$
$$= 4 - \frac{1}{z}\sum(-1)^n\left(\frac{1}{z}\right)^n + 3\sum(-1)^n\left(\frac{1}{3}\right)^n z^n$$
$$= 4 - \sum(-1)^n\left(\frac{1}{z}\right)^{n+1} + 3\sum(-1)^n\left(\frac{1}{3}\right)^n (z)^n$$
$$= 4 - \sum(-1)^n\left[\left(\frac{1}{z}\right)^{n+1} + (-1)^n\left(\frac{1}{3}\right)^{n-1}(z)^n\right]$$

$\rightarrow$

③ $III = \{z \mid 3 > |z|\}$

$$f = 4 - \frac{1}{z}\left(\frac{1}{1-(-\frac{1}{z})}\right) + \frac{9}{z}\left(\frac{1}{1-(\frac{-3}{z})}\right)$$

$$= 4 - \frac{1}{z}\sum(-1)^n z^{-n} + \frac{9}{z}\sum(-3)^n (z^{-n})$$

$$= 4 - \sum(-1)^n z^{-n-1} + \sum(-1)^n (3)^{n+2} (z^{-n-1})$$

$$= 4 - \sum(-1)^n (1 + 3^{n+2}) z^{(-n-1)}$$

D

4 (i) Show $h(z) = \frac{z-a}{1-\bar{a}z}$ $a \in \mathbb{D}$ is a conformal self map of $\mathbb{D}$

(ii) Let $f: \mathbb{D} \to \mathbb{D}$ be holomorphic and $a_1, \dots, a_n \in \mathbb{D}$ are zeros of f. Prove $|f(0)| \le |a_1 \dots a_n|$

Pf (i) $h'(z) = \dfrac{(1-\bar{a}z)-(z-a)(-\bar{a})}{(1-\bar{a}z)^2}$

$$= \frac{1-\bar{a}z + z\bar{a} - a\bar{a}}{(1-\bar{a}z)^2}$$

$$= \frac{1-|a|^2}{(1-\bar{a}z)^2} \neq 0 \quad \text{for any } z \in \mathbb{D}$$

nonzero derivative means conformal

$\therefore$ h is conformal

$$|h(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| = \left|\frac{\frac{\bar{z}}{\bar{z}}(z-a)}{1-\bar{a}z}\right| = \left|\frac{1}{\bar{z}}\right|\left|\frac{z\bar{z}-a\bar{z}}{1-\bar{a}z}\right| \le \left|\frac{1}{z}\right|\left|\frac{|z|^2-a\bar{z}}{1-\bar{a}z}\right|$$

$$= \left|\frac{1}{z}\right|\left|\frac{1-a\bar{z}}{1-\bar{a}z}\right| = \frac{1}{|z|}\frac{|1-a\bar{z}|}{|1-\bar{a}z|} \le \left|\frac{1-a\bar{z}}{1-a\bar{z}}\right| = 1 \quad \text{when } |z| \le 1$$

$h(a) = 0$ and $|h(z)| = 1$ when $|z| = 1$.

$\Rightarrow h: \mathbb{D} \to \mathbb{D}$ is onto and conformal

(ii) Let $B(z)$ be finite Basche product w/ same zeros, where $B(z) = e^{i\varphi}\left(\frac{z-a_1}{1-\bar{a}_1 z}\right)\cdots\left(\frac{z-a_n}{1-\bar{a}_n z}\right)$

Since f, B have same zeros then f/B has no zeros on $\mathbb{D}$.

For $|z|=1$ $|f(z)/B(z)| = |f(z)| \le 1$ since $f: \mathbb{D} \to \mathbb{D}$

$\Rightarrow \left|\dfrac{f(z)}{B(z)}\right| \le 1$ on $\mathbb{D}$.

$\Rightarrow |f(0)| \le |B(0)| = \left|\prod_{i=1}^{n} a_i\right| = |a_1 \dots a_n|$

□

Qualifying Exam, Complex Analysis, August 22, 2006

1. Find a conformal map from the strip $\{0 < \mathbf{Im}\, z < 1\}$ onto the unit disk.

2. Find $\displaystyle\int_{|z|=2} \frac{\sin(\pi z)}{z^2(1-z)}\, dz$.

3. Let f be a holomorphic function on the closed disk $\Delta_R = \{z \in \mathbb{C} : |z| \leq R\}$. Show that

$$|f'(0)| \leq \frac{3}{2\pi R^3} \iint_{\Delta_R} |f(z)|\, dx\, dy\ .$$

4. Suppose that f_n are holomorphic functions on a domain D and $\displaystyle\sum_{n=1}^{\infty} |f_n|$ converges locally uniformly on D. Show that $\displaystyle\sum_{n=1}^{\infty} |f_n'|$ converges locally uniformly on D.

Real analysis qualifying exam Aug. 22, 2006

1. Let $E \subset \mathbb{R}$ denote a countable set.

(a) Compute the Lebesgue measure of E.

(b) Construct an E that is a G_δ set (countable intersection of open sets).

(c) Construct an E that is not a G_δ set.

2. Give an example of a sequence $\{f_n\}$ for each of the requirements below or show that no such sequence exists. L^1 denotes the Lebesgue integrable functions on $\mathbb{R}$.

(a) $0 \leq f_n \to 0$ in L^1, but $\{f_n\}$ does not converge pointwise a.e. to zero.

(b) $0 \leq f_n \to 0$ a.e., but $\{f_n\}$ does not converge in L^1 to zero.

(c) $0 \leq f_n \to f$ a.e. and $\int f_n \leq 1$, but $f \notin L^1$.

3. Given a $p \geq 1$ let $f \in L^p([0,1])$ with respect to Lebesgue measure m, and let $E \subset [0,1]$ be measurable. Put $\nu(E) = \int_E f dm$.

(a) Show that ν is a complex measure absolutely continuous with respect to m.

(b) Let $g(x) = \nu([0,x])$ for each $x \in [0,1]$. Prove

$$\|g\|_p \leq \left(\frac{1}{p}\right)^{\frac{1}{p}} \|f\|_p$$

4. For some $1 \leq p \leq \infty$ let $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be a continuous linear operator. Suppose $\|f\|_p \leq \|Tf\|_p$ for all $f \in L^p(\mathbb{R})$.

(a) Show there exists a real constant C independent of f so that

$$\|Tf\|_p \leq C\|f\|_p$$

for all f.

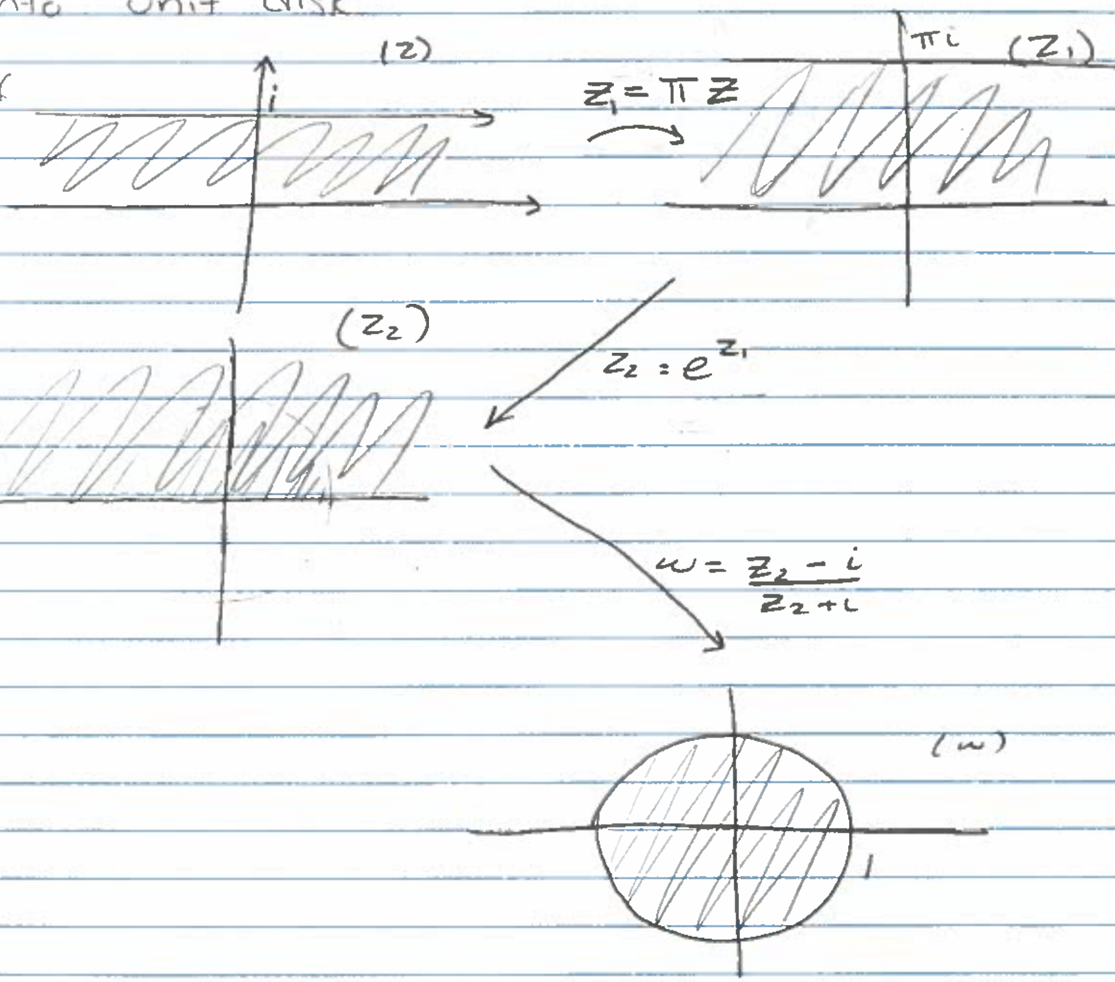
(b) Show T is $1:1$.

(c) Show T has closed range, i.e. whenever $Tf_j \to g$ in L^p there exists $f \in L^p$ such that $Tf = g$.

Aug 2006

1. Find conformal map from strip $\{0 < \operatorname{Im} z < 1\}$ onto unit disk

Pf



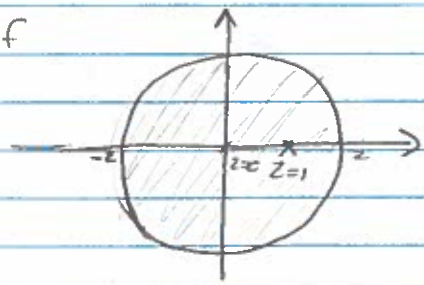
$$\boxed{w = \frac{e^{\pi z} - i}{e^{\pi z} + i}}$$

□

August 2006

2. Find $\displaystyle\int_{|z|=2} \frac{\sin(\pi z)}{z^2(1-z)}\,dz$

pf



Let $f(z) = \dfrac{\sin \pi z}{z^2(1-z)}$

$f(z)$ has isolated singularities at $z=0, z=1$

$$\frac{\sin \pi z}{z^2(1-z)} = \frac{1}{z^2 - z^3}\left(\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots\right)$$

$$\frac{1}{z^2(1-z)} = \frac{A+Bz}{z^2} + \frac{C}{1-z} = \frac{1+z}{z^2} + \frac{1}{(1-z)}$$

$$(A+Bz)(1-z) + C(z^2) = 1$$

$$A + Bz - Az - Bz^2 + Cz^2 = 1$$

$$C - B = 0$$

$$B - A = 0$$

$$A = 1 \Rightarrow B = 1 \Rightarrow C = 1$$

$$\frac{\sin \pi z}{z^2(1-z)} = \left(\frac{1}{z^2}\sin \pi z + \frac{1}{z}\sin \pi z + \frac{1}{1-z}\sin \pi z\right)$$

$$a_{-1} = \pi \qquad a_{-1} = 0 \qquad a_{-1} = 0$$

$$\int_{|z|=2} \frac{\sin(\pi z)}{z^2(1-z)} = -2\pi i \operatorname{Res}(f, \infty) = +2\pi^2 i$$

or
$z=0$ double pole $\operatorname{Res}(f, 0) = \pi$
$z=1$ simple pole $\operatorname{Res}(f, 1) = 0$

$$\Rightarrow \int_{|z|=2} \frac{\sin(\pi z)}{z^2(1-z)} = 2\pi i\left[\operatorname{Res}(f, 0) + \operatorname{Res}(f, 1)\right] = 2\pi^2 i$$

3 Let f be holomorphic on $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$

Show $|f'(0)| \leq \frac{3}{2\pi R^3} \iint_{\Delta_R} |f(z)| \, dx\, dy$.

Pf $|f'(0)| = \left| \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w-0)^2} \right| dw \quad \forall r \in (0,1)$ by CRE

$$\leq \frac{1}{2\pi} \int_{|w|=r} \frac{|f(w)|}{|w|^2} dw$$

$$= \frac{1}{2\pi} \int \frac{|f(w)|}{r^2} dw$$

$$= \frac{1}{2\pi r^2} \int |f(w)| dw$$

$$\Rightarrow r^2 |f'(0)| \leq \frac{1}{2\pi} \int_{|w|=r} |f(w)| dw$$

$$\Rightarrow \int_0^R r^2 |f'(0)| \leq \int_0^R \frac{1}{2\pi} \int_{|w|=r} |f(w)| dw\, dr$$

$$\Rightarrow \frac{R^3}{3} |f'(0)| \leq \frac{1}{2\pi} \int_0^R \int_{|w|=r} |f(w)| dw\, dr$$

$$= \frac{1}{2\pi} \iint_{\Delta_R} |f(w)| dx\, dy$$

$$\Rightarrow |f'(0)| \leq \frac{3}{R^3 2\pi} \iint_{\Delta_R} |f(z)| dx\, dy$$

$\square$

4 Suppose f holomorphic on a domain D and $\sum |f_n|$ converges locally uniformly,
$\Rightarrow \sum |f_n'|$ converges locally uniformly.

Pf $\sum |f_n'(z)| = \sum \left| \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f_n(w)}{(w-z)^2} dz \right|$

$$\leq \sum \frac{1}{2\pi} \int_{|w-z|=r} \frac{|f_n(w)|}{r^2} dz$$
$$= \frac{1}{2\pi} \int_{|w-z|=r} \frac{1}{r^2} \sum |f_n(w)|$$

Now $\left| \sum_1^{k-1} |f_n| - \sum_1^{\infty} |f_n| \right| \leq \varepsilon \quad \forall z$ s.t. $|z-z_0| < 2r \quad \forall k > k_0$
since f_n converges locally uniformly

$$\left| \sum_1^{k-1} |f_n'(z_0)| - \sum_1^{\infty} |f_n'(z_0)| \right| = \left| \sum_k^{\infty} f_n'(z_0) \right|$$
$$= \left| \sum_k^{\infty} \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f_n(z_0)}{(z-z_0)^2} dz \right|$$
$$\leq \sum_k^{\infty} \frac{1}{2\pi} \int_{|z-z_0|=r} \left| \frac{f_n(z_0)}{(z-z_0)^2} \right| dz$$
$$= \sum_k^{\infty} \frac{1}{2\pi} \frac{1}{r^2} \int_{|z-z_0|=r} |f_n(z_0)| dz$$
$$= \frac{1}{2\pi r^2} \int_{|z-z_0|=r} \sum_k^{\infty} |f_n(z_0)|$$
$$< \frac{1}{2\pi r^3} \int \varepsilon \, dz$$
$$= \varepsilon / r$$

Qualifying Exam, Complex Analysis, January 28, 2006

1. Find a conformal map from the half-disk $\{z : |z-1| < 1, \ \mathrm{Im}\, z > 0\}$ onto the upper half-plane $\{\mathrm{Im}\, w > 0\}$.

2. Find $\displaystyle\int_{|z|=1} z^n e^{1/z}\, dz$, where n is an integer.

3. Let f be a holomorphic function on $U \setminus \{0\}$, where U is the open unit disk, such that $f(1/2) = 2$ and the function

$$g(z) = \overline{z}\, |f(z)|^2$$

is holomorphic on $U \setminus \{0\}$. Find f.

4. Let f be a holomorphic function in $U \setminus \{0\}$, where U is the open unit disk, which satisfies

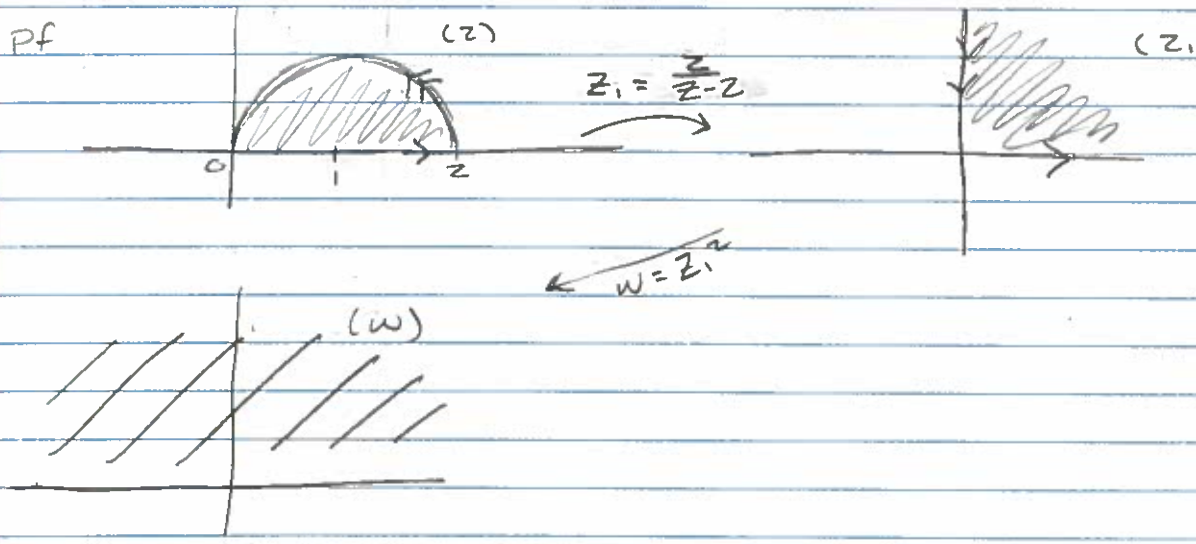
$$|f(z)| \le -\log|z|, \ \forall\, z \in U \setminus \{0\}.$$

Prove that $f = 0$.

January 2006

1. Find a conformal map from half disk $\{z : |z-1| < 1, \operatorname{Im}(z) > 0\}$ onto upper half plan

Pf



$$w = \left(\frac{z}{2-z}\right)^2 : \{z : |z-1| < 1, \operatorname{Im}(z) > 0\} \to \mathbb{H}.$$

January 2006

2. Find $\int_{|z|=1} z^n e^{1/z}\,dz$ where n is an integer.

Pf We use residue at ∞.

$$\int_{|z|=1} z^n e^{1/z}\,dz = 2\pi i \operatorname{Res}(f, \infty)$$
$$= 2\pi i \operatorname{Res}(z^n e^{1/z}, \infty)$$

$$z^n e^{1/z} = z^n \sum_{k=0}^{\infty} (1/z)^k \frac{1}{k!}$$
$$= \sum_{k=0}^{\infty} \frac{z^n}{z^k} \frac{1}{k!}$$
$$= \sum_{k=0}^{\infty} z^{n-k} \frac{1}{k!}$$

Case 1 $\exists\, k$ s.t. $n-k=-1 \Rightarrow k = n+1$

then $\operatorname{Res}(z^n e^{1/z}, \infty) = \frac{-1}{(n+1)!}$

$\Rightarrow \int_{|z|=1} z^n e^{1/z} = -2\pi i/(n+1)!$

Case 2 there is no such k, then

$\int_{|z|=1} z^n e^{1/z} = 0$.

$\square$

3 Let f be holomorphic on $U \setminus \{0\}$ s.t. $f(1/2) = 2$
and $g(z) = \bar{z}|f(z)|^2$ is holomorphic on $U \setminus \{0\}$
Find f.

Pf $\frac{\partial}{\partial \bar{z}} g(z) = \frac{\partial}{\partial \bar{z}}[\bar{z} f(z) \overline{f(z)}]$

$$= f(z) f(\bar{z}) + \bar{z} f(z) \frac{\partial f}{\partial \bar{z}}$$

$$= f(z) \overline{f(z)} + \bar{z} f(z) \overline{\frac{\partial f}{\partial z}}$$

$$= f(z)(\overline{f(z)} + \bar{z} \overline{f'(z)})$$

$= 0$ since g is holomorphic ?

$\Rightarrow \overline{f}(z) + \bar{z} \overline{f'(z)} = 0$ if $f \neq 0$

$\Rightarrow \overline{f(z)} = -\bar{z} \overline{f'(z)}$

$$\Rightarrow \frac{-1}{\bar{z}} = \frac{\overline{f'(z)}}{\overline{f(z)}}$$

$$\Rightarrow \overline{\left(\frac{-1}{\bar{z}}\right)} = \overline{\left(\frac{\overline{f'(z)}}{\overline{f(z)}}\right)}$$

$\Rightarrow \log f(z) = -\log(z) = \log z^{-1} = \log \frac{1}{z}$

$\Rightarrow f(z) = 1/z$

$\square$

4 Let f be holomorphic in $U \setminus \{0\}$ and
$|f(z)| \le -\log|z| \quad \forall z \in U \setminus \{0\}$
Prove $f = 0$

Pf f holomorphic $\Rightarrow$ $f = \sum a_n z^n$

$$a_n = \frac{n!}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz$$

$$\Rightarrow |a_n| \le \frac{n!}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} \, dz$$

$$= \frac{n!}{2\pi} \int \frac{-\log|z|}{r^{n+1}} \, dz \quad \to 0$$

$$= n!\, r \frac{\log \frac{1}{r}}{r^{n+1}} \to 0 \quad \text{as } r \to 1$$

$\Rightarrow a_n = 0 \quad \forall n \in \mathbb{Z}$
$\Rightarrow f = 0$

or

f holomorphic in $U \setminus \{0\}$
Let $g = z f(z)$. Then g has a removable singularity at 0.
Since $\lim_{z \to 0} g(z) = \lim_{z \to 0} z f(z)$

$$\le \lim_{z \to 0} -z \log|z| = 0$$

☺

So $f = \frac{g(z)}{z}$ is basically holomorphic and $|f(z)| \le -\log 1 = 0$ on $|z| = 1$.

So by Max principle $|f| \le 0$ on $U \setminus \{0\} \Rightarrow f = 0$ □

Complex Analysis
Fall 2004 & Spring 2005

1. Find all points where the polynomial $p(z,\bar{z}) = 1 + 2z + \bar{z} + z\bar{z}^2 + z^2\bar{z} + i\bar{z}^2$ is complex differentiable.

2. Find the maximal radius of the disks centered at 0, where the function $f(z) = \frac{z}{\sin z}$ can be represented by a Taylor series.

3. Suppose that a function f is holomorphic in a neighborhood of the origin and $f(z) = f(2z)$ whenever z and $2z$ are in this neighborhood. Show that f is constant.

4. Show that the function $f(z) = \bar{z}$ cannot be uniformly approximated on the unit circle by polynomials of z.

5. Show that an entire function $f(z)$ such that $|f(z)| \geq |z|^N$ for sufficiently large N is a polynomial.

6. If function f_j, $j = 1, 2, \ldots$, are holomorphic and uniformly bounded in the unit disk are not equal to 0 there and $f_j(0) \to 0$ as $j \to \infty$, then $f_j \to 0$ uniformly on compacta in the unit disk.

7. If f is holomorphic and bounded in $\{\mathbf{Im}\, z \geq 0\}$, real on the real axis, then f is constant.

Fall 2004

1. Find all points where $p(z,\bar{z}) = 1+2z+\bar{z}+z\bar{z}^2+z^2\bar{z}+i\bar{z}^2$ is complex differentiable.

Pf $p(z,\bar{z})$ is complex differentiable at $z_0 \iff \frac{\partial p}{\partial \bar{z}}(z_0)=0$

$$\frac{\partial p}{\partial \bar{z}} = 0+0+1+2z\bar{z}+z^2+2i\bar{z}$$
$$= 1+2z\bar{z}+z^2+2i\bar{z}$$
$$= 1+z^2+2(z+i)\bar{z}$$
$$= (z-i)(z+i)+2(z+i)\bar{z}$$
$$= (z+i)(z+2\bar{z}-i)$$

$(z+i)(z+2\bar{z}-i)=0 \iff z=-i$ or $z=i-2\bar{z}$

$$z=i-2\bar{z} \Rightarrow x+iy = i-2(x-iy)$$
$$\Rightarrow x+iy = i-2x+2iy = -2x+i(2y+1)$$
$$\Rightarrow x=2x, \quad y=1+2y$$
$$\Rightarrow x=0 \qquad y=-1$$
$$\Rightarrow x=0, \qquad y=-1$$
$$\Rightarrow z=-i$$

So $z_0=-i$ is the only such point where $p(z,\bar{z})$ is complex differentiable

$\square$

2. Find max radius of disks centered at 0 where $f(z) = \frac{z}{\sin z}$ can be represented by a Taylor series

$\sin z = 0 \Rightarrow z = k\pi$ for $k \in \mathbb{Z}$

$\Rightarrow$ Radius of Convergence is π
Since 0 is a removable singularity
Since $z = 0$ at 0 as well

$\square$

Fall 2004

3\. Suppose f is holomorphic in a neighborhood of the origin and $f(z) = f(2z)$ whenever z and $2z$ are in this nbhd. Show f is constant

Pf Let R be s.t. f is holomorphic in $\Delta(0,R)$ then if $|z|, |2z| < R$ we have $f(z) = f(2z)$

Let $z_n \to 0$ $|z_n| < R$ be s.t. $z_n = 2z_{n+1}$
then $f(z_n) = f(z_{n+1})$ for all $n \to \infty$
Then $f(z_1) = f(z_n)$ $\forall n$
Let $E = \{z \in |z| < R : f(z) = f(z_1)\}$
then by identity principle $f = f(z_1)$
So f is constant

□

i.e $x_n = 1/2^n$ then $2x_{n+1} = x_n$

This is wrong since need limit pt to use isolated point

Alternatively

Pf Let N be a nbhd of 0
$\Rightarrow \exists r > 0$ s.t. $\Delta(0,r) \subset N$
$\Rightarrow \overline{\Delta(0, r/2)} \subset N$
$\overline{\Delta(0, r/2)}$ compact
$\Rightarrow f$ attains max on $\overline{\Delta(0, r/2)}$
$\Rightarrow f(z_0) = M$ for some $z_0 \in \overline{\Delta(0, r/2)} \subset \Delta(0,r)$ where $|f| \leq M$ on $\overline{\Delta(0, r/2)}$
$f(z) = f(2z)$
$\Rightarrow |f| \leq M$ on $\overline{\Delta(0,r)}$
$\Rightarrow |f(z)| = M$ for some $z \in \Delta(0,R)$
$\Rightarrow f \equiv M$ on $\Delta(0,r)$ by Max principle
$\Rightarrow f$ constant on N be identity thm.

□

4. Show $f(z)=\bar{z}$ cannot be uniformly approximated on $\mathbb{D}$ by polynomials of z.

Pf Suppose BWOC $\exists \{P_n\} \to f$ uniformly on $\partial\mathbb{D}$.

$P_n(z)$ a polynomial so analytic

$\Rightarrow \int_{|z|=1} P_n(z) = 0$ since P is analytic

$\Rightarrow$ If $P_n(z) \xrightarrow{u} f(z)$ then $\int P_n(z) \to \int f$

$\Rightarrow \int f = 0$

However. $\int_{|z|=1} \bar{z}\,dz = \int_{|z|=1} 1/z\,dz = 2\pi i \neq 0$

since $z\bar{z} = |z|^2 = 1$

So $\bar{z}$ cannot be uniformly approximated on $\mathbb{D}$ by polynomials of z.

$\square$

5. Show $f(z)$ entire s.t. $|f(z)| \geq |z|^N$ for N large is a polynomial.

Pf $|f(z)| \geq |z|^n \Rightarrow f$ can only be 0 at 0.

Case 1 $f(0)=0$

$\Rightarrow f$ has a zero of order N at 0.

$\Rightarrow f(z) = z^n g(z)$ $g(0) \neq 0$.

$\Rightarrow g$ constant by $*$

$\Rightarrow g = c$

$\Rightarrow f = cz^n$

Case 2 $f(0) \neq 0$

$\Rightarrow \frac{1}{f}$ is entire.

$\Rightarrow |1/f| \leq 1$ outside of $\mathbb{D}$.

$1/f$ bdd

$\Rightarrow 1/f$ constant by Louiville

$\Rightarrow f$ is constant

$\Rightarrow f$ is a polynomial

$\square$

6. If f_j $j=1,2\ldots$ are holomorphic and uniformly bdd in $\mathbb{D}$ and never 0 in $\mathbb{D}$ and $f_j(0)\to 0$ as $j\to\infty$ then $f_j\to 0$ uniformly on a compact set in $\mathbb{D}$.

Pf Let f_{n_k} be a subsequence of f_n

$\Rightarrow f_{n_k}$ is holomorphic and uniformly bdd on $\mathbb{D}$

$\Rightarrow \exists f_{n_{k_\ell}} \rightrightarrows f$ by Montel.

Since $f_{n_{k_\ell}}\neq 0$ on $\mathbb{D}$

$\Rightarrow f$ is never 0 or $f\equiv 0$ by Hurwitz.

$f_{n_{k_\ell}}(0)\to 0$

$\Rightarrow f\equiv 0$

$\Rightarrow f_{n_{k_\ell}}\to 0$ $\Rightarrow$ every subseq has convergent subseq.

$\Rightarrow f_n\to 0$ $\square$

7. If f holomorphic and bdd in $\{\operatorname{Im} z\geq 0\}$ real on real axis $\Rightarrow f$ constant.

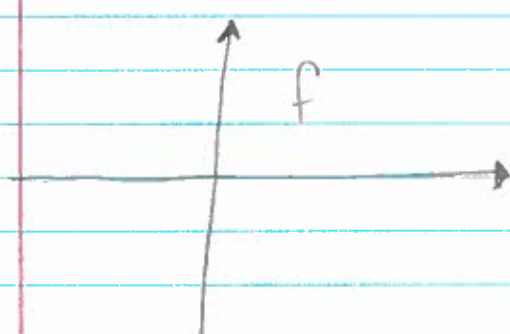
Pf Extend f to $\mathbb{C}$ s.t. $\tilde{f}(z)=\overline{f(\bar{z})}$ by reflection

$\Rightarrow \tilde{f}$ is entire and bdd

$\Rightarrow \tilde{f}$ constant by Liouville

$\Rightarrow f$ constant

$\square$



FALL 2005

Measure Theory Part

1. **1.** Let $\{r_n\}_{n=1}^{\infty}$ be the rationals, $f(x) = x^{-1/2}$ for $0 < x < 1$ and 0 otherwise, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Is $f(x)$ measurable? Why? Is $g(x)$ measurable? Why? What is the set of points of discontinuity of g? Is g integrable? Why? Show that g is not in L^2 on any interval.

2. Let μ be Lebesgue measure on the borel sets of the real line, and define $\nu(E)$ to be 1 if $0 \in E$ and 0 if $0 \notin E$ for all borel sets E. Is ν a measure? σ finite? Compute $\frac{d\nu}{d\mu}$.

3. Define L^p (Lebesgue measure). Is $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$? Why? Is $L^2(0,1) \subset L^1(0,1)$? Why?

4. Let $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise and $\|g_k\|_\infty \leq M$ for all k. Prove that $f_k g_k \to fg$ in L^p.

Complex Part

1. Let f be an analytic function on the unit disk and $f(z)$ is real when z is real. Show that $\bar{f}(\bar{z}) = f(z)$.

2. Let $\{f_n\}$ be a sequence of continuous functions on the closed unit disk that are analytic in the open unit disk. Suppose $\{f_n\}$ converges uniformly on the unit circle. Show that $\{f_n\}$ converges uniformly on the closed unit disk.

3. Suppose that f is an analytic function on an open set containing the closed unit disk, $|f(z)| = 1$ when $|z| = 1$ and f is not a constant. Prove that the image of f contains the closed unit disk.

4. Let $\mathcal{F}$ be a family of analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

on the open unit disk such that $|a_n| \leq n$ for each n. Show that $\mathcal{F}$ is normal, i.e. every sequence of functions in $\mathcal{F}$ contains a subsequence converging normally to a function in $\mathcal{F}$.

Fall 2005

1. Let f be an analytic function on $\mathbb{D}$ and $f(z)$ is real when z is real. Show $\bar{f}(\bar{z}) = f(z)$.

Pf First we have to show $\bar{f}(\bar{z})$ is analytic.

If $g(z) = \bar{f}(\bar{z})$ then $g'(z) = \lim_{\Delta z \to 0} \frac{g(z+\Delta z) + g(z)}{z_0}$

$$= \lim_{\Delta z \to 0} \frac{\bar{f}(\overline{z+\Delta z}) + \bar{f}(\bar{z})}{z_0}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\frac{f(\bar{z}+\bar{\Delta}z) + f(\bar{z})}{\Delta z}}}{}$$

$$= \overline{\lim_{\Delta z \to 0} \left(\frac{f(\bar{z}+\Delta z) + f(\bar{z})}{\Delta z} \right)}$$

$$= \overline{f'(\bar{z})}$$

f analytic $\Rightarrow f'$ cont

$\Rightarrow g'$ is comp of continuous fcns.

$\Rightarrow g'$ exists and is cont

$\Rightarrow g$ is analytic

So since $f(z)$ is real when z is real

$\Rightarrow \bar{f}(\bar{z}) = f(z)$ on $\mathbb{R}$

$\Rightarrow \overline{f}(\bar{z})$ on $\mathbb{D}$ by identity principle

or to show analytic:

$$\frac{\partial}{\partial \bar{z}} \bar{f}(\bar{z}) = \frac{1}{2}(\partial x + i\partial y)(u(x,-y) - iv(x,-y))$$

$$= \frac{1}{2}(u_x(x,-y) + i u_y(x,-y)(-1) - iv_x(x,-y) - (-1)v_y(x,-y)(-1))$$

$$= \frac{1}{2}((u_x - u_y)(\bar{z}) - i(v_x + u_y)(z))$$

$$= 1/2\ (0 - 0) \quad \text{since } f \text{ is analytic}$$

$\Rightarrow \bar{f}(\bar{z})$ is analytic

$\square$

2 Let $\{f_n\}$ be continuous fcns on closed unit disk, analytic on $\mathbb{D}$. f_n converges uniformly on unit circle. Show f_n converges uniformly on $\overline{\mathbb{D}}$.

Pf f is cauchy on unit circle.

$\Rightarrow \exists n,m$ s.t $|g(z)| = |f_n(z) - f_m(z)| < \varepsilon$

$\Rightarrow |g(z)| < \varepsilon$ inside $\mathbb{D}$ by max principle.

$\Rightarrow f_n$ converges uniformly on $\overline{\mathbb{D}}$.

$\square$

3. Suppose f analytic on open set containing $\overline{\mathbb{D}}$, $|f(z)|=1$ when $|z|=1$ and f not constant. Prove image contains closed unit disk.

Pf Case 1 Assume f has no zeros in $\mathbb{D}$.

$\Rightarrow |f| \leq 1$ when $|z|<1$ by max principle

$\Rightarrow f$ has a min (not 0) on $\overline{\mathbb{D}}$

$\Rightarrow 1/f$ has max on $\mathbb{D}$

$\Rightarrow 1/f$ is constant by strict max principle

$\Rightarrow f$ is constant on $\mathbb{D}$

Case 2 Assume f has a zero in $\mathbb{D}$

Assume BWOC $\exists w_0 \in \overline{\mathbb{D}}$ s.t. f does not attain w_0.

$\Rightarrow f(z) - w_0 \neq 0 \quad \forall z.$

But by Rouches since $f(z)$ has a zero in $\mathbb{D}$ then $f(z)-w_0$ has a zero since $|w_0| \leq |f(z)|$ on $z=1$

$\Rightarrow \exists z_0$ s.t. $f(z_0) = w_0$

$\Rightarrow$ every value in $\mathbb{D}$ is attained.

Now the boundary of an image is contained in image of boundary so the set $\{|z|=1\}$ is also attained.

$\square$

4. Let $\mathcal{F}$ be a family of analytic functions
$$f(z) = z + \sum_2^\infty a_n z^n$$
on $\mathbb{D}$. s.t. $|a_n| \le n$ $\forall n$. Show $\mathcal{F}$ is normal.

Pf Let $r = \max_{z \cdot f(w)} |z| < 1$ since $f(z)$ is on $\mathbb{Z}$

Then
$$\begin{aligned}
|f(z)| &= |z + \textstyle\sum_2^\infty a_n z^n| \\
&\le |z| + \textstyle\sum_2^\infty |a_n z^n| \\
&\le r + \textstyle\sum_2^\infty n r^n \\
&= r(1 + \textstyle\sum_2^\infty n r^{n-1}) \\
&= r(1 + \tfrac{\partial}{\partial r} \textstyle\sum r^n) \\
&= r(1 + \tfrac{\partial}{\partial r} \tfrac{1}{1-r}) \\
&= r(1 + (1-r)^{-2}) \\
&= \frac{r((1-r)^2 + 1)}{(1-r)^2} \\
&= \frac{r(1 - 2r + r^2 + 1)}{1 - 2r + r^2} \\
&= \frac{r^3 - 2r^2 + 2r}{r^2 - 2r + 1} = r\left(\frac{r^2 - 2r + 2}{r^2 + 2r - 1}\right) \\
&< \infty \quad \text{since } r \in (0,1).
\end{aligned}$$

So $f(z)$ is uniformly bounded
$\Rightarrow \mathcal{F}$ is normal by Montel's Thm.

$\square$

Montel's Thm: Any uniformly bdd family of holomorphic functions is normal

Gamelin Chapter 1

1.1 # 2,4,5,6
1.2 # 5,6,7
1.5 # 4
1.6 # 1
1.7 # 2,4,5
1.8 # 2,3,4

1.1.2 Verify from definitions

a) $\overline{z+w} = \bar z + \bar w$

b) $\overline{zw} = \bar z\,\bar w$

c) $|\bar z| = |z|$

d) $|z|^2 = z\bar z$

Pf Let $z = x+iy$, $w = u+iv$

a) $\overline{z+w} = \overline{(x+iy)+(u+iv)} = \overline{(x+u)+i(y+v)} = (x+u) - i(y+v) = x - iy + u - iv = \bar z + \bar w$

b) $\overline{zw} = \overline{(x+iy)(u+iv)} = \overline{xu+iyu+ivx-yv} = xu-yv-iyu-ivx = \overline{(x+iy)}\,\overline{(u+iv)} = \bar z\bar w$

c) $|\bar z| = |x-iy| = \sqrt{(x)^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$

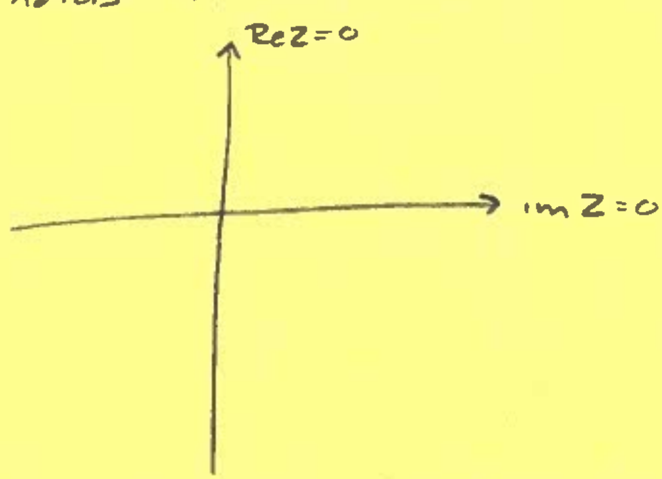
d) $|z|^2 = |x+iy|^2 = x^2+y^2 = (x+iy)(x-iy) = z\bar z$ □

1.1.4 Show $|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$ and sketch where equality holds

Pf $|z|^2 = x^2+y^2 = (\mathrm{Re}\, z)^2 + |\mathrm{Im}\, z|^2 \le |\mathrm{Re}\, z| + 2|\mathrm{Re}\, z||\mathrm{Im}\, z| + |\mathrm{Im}\, z| = (|\mathrm{Re}\, z| + |\mathrm{Im}\, z|)^2$

$\Rightarrow |z| = |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$

Equality holds if $\mathrm{Re}\, z = 0$ or $\mathrm{Im}\, z = 0$



1.1.5 Show (a) $|\mathrm{Re}\, z| \le |z|$, $|\mathrm{Im}\, z| \le z$

(b) $|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w})$

(c) $|z+w| \le |z| + |w|$

Pf (a) $|z|^2 = |\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2 \ge |\mathrm{Re}\, z|^2 \quad \Rightarrow |z| \ge |\mathrm{Re}\, z|$

$|z| = |\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2 \ge |\mathrm{Im}\, z|^2 \quad \Rightarrow |z| \ge |\mathrm{Im}\, z|$

(b)
$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \qquad \text{from } 1.2 \\
&= (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} \\
&= |z|^2 + w\bar{z} + z\bar{w} + |w|^2 \\
&= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2
\end{aligned}$$

Note:

$$\begin{aligned}
\mathrm{Re}(z\bar{w}) &= \mathrm{Re}(x+iy)(u-iv) \\
&= xu + vy \\
w\bar{z} + z\bar{w} &= (u+iv)(x-iy) \\
&\quad + (u-iv)(x+iy) \\
&= ux + ivx - iyu + vy \\
&\quad + ux - ivx + iyu + vy \\
&= 2ux + 2vy
\end{aligned}$$

(c)
$$\begin{aligned}
|z+w|^2 &= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2 \\
&\le |z|^2 + 2|\mathrm{Re}(z\bar{w})| + |w|^2 \\
&\le |z|^2 + 2z\bar{w} + |\bar{w}| \\
&= (|z| + |w|)^2 \qquad \Rightarrow |z+w| \le |z| + |w|
\end{aligned}$$

$\square$

1.1.6 For fixed $a \in \mathbb{C}$ Show $\dfrac{|z-a|}{|1-\bar{a}z|} = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$

Pf $a = 0 \Rightarrow \dfrac{|z|}{|1-0|} = |z| = 1$

$$\begin{aligned}
a \neq 0 \Rightarrow \frac{|z-a|}{|1-\bar{a}z|} &= \frac{\left|\frac{z}{z}(z-a)\right|}{|1-\bar{a}z|} \\
&= \frac{\left|\frac{1}{z}\right| |\bar{z}z - a\bar{z}|}{|1-\bar{a}z|} \\
&= \frac{\big||z|^2 - a\bar{z}\big| \left|\frac{1}{z}\right|}{|1-\bar{a}z|} \\
&= \frac{|1-\bar{a}z|}{|1-\bar{a}z|} \quad \text{since } |z| = 1 \\
&= 1
\end{aligned}$$

$\square$

$|z| = 1 \Rightarrow z\bar{z} = 1 \Leftrightarrow |\bar{z}| = 1$

<u>1.2.5</u> For $n \geq 1$ show

(a) $1 + z + z^2 + \dots + z^n = (1 - z^{n+1}) / (1 - z)$

(b) $1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + 1/2)\theta}{2 \sin \theta/2}$

<u>Pf</u> (a) $(1 + z + \dots + z^n)(1 - z) = 1 + \cancel{z} + \cancel{/} \dots + \cancel{z^n} - \cancel{z} - \cancel{z^2} - \dots \cancel{/} - z^{n+1}$

$= 1 - z^{n+1}$ since it all cancels out.

(b) Let $z = e^{i\theta} \Rightarrow 1 + e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta} = \dfrac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$

Let $z = e^{-i\theta} \Rightarrow 1 + e^{-i\theta} + e^{-2i\theta} + \dots + e^{-ni\theta} = \dfrac{1 - e^{-(n+1)i\theta}}{1 - e^{-i\theta}}$

add them and divide by 2:

$$1 + \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{2i\theta} + e^{-2i\theta}}{2} + \dots + \frac{e^{ni\theta} + e^{-ni\theta}}{2} = \frac{1}{2}\,\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} + \frac{1}{2}\,\frac{1 - e^{-(n+1)i\theta}}{1 - e^{-i\theta}}$$

$$\Rightarrow 1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta = \frac{1 - e^{(n+1)i\theta} - e^{-i\theta} + e^{ni\theta} + 1 - e^{i\theta} + e^{-ni\theta} - e^{-(n+1)i\theta}}{2(1 - e^{i\theta})(1 - e^{-i\theta})}$$

$= 2 -$

<u>1.2.6</u> Fix $R > 1$ and $n \geq 1$, $m \geq 0$. Show that

$\left| \dfrac{z^m}{z^n + 1} \right| \leq \dfrac{R^m}{R^n - 1}$ $|z| = R$. sketch where equality holds.

?

<u>Pf</u> $\left| \dfrac{z^m}{z^n + 1} \right| = \dfrac{|z^m|}{|z^n + 1|} \leq \dfrac{|z|^m}{|z|^n - 1} = \dfrac{R^m}{R^n - 1}$

$|z^n + 1| \leq |z^n| + 1$

1.5.4 Show the only periods of e^z are the integral multiples of $2\pi i$ that is $e^{z+\lambda}=e^z$ then λ is an integer times $2\pi i$.

Pf Assume $e^{z+\omega}=e^z$ for $z\in\mathbb{C}$

$\Rightarrow e^{\omega}=e^{0+\omega}=e^0=1$

$\Leftrightarrow 1=e^{\omega}=e^{u+iv}=e^u(\cos v+i\sin v)$ if $\omega=u+iv$

$\Leftrightarrow u=0$ and $\cos v+i\sin v=1$

$\Leftrightarrow u=0$ and $\cos v=1$ $\sin v=1$

$\Leftrightarrow u=0$ and $v=2\pi i k$ for $k\in\mathbb{Z}$

$\Leftrightarrow \omega=2\pi i k$ $k\in\mathbb{Z}$ $\square$

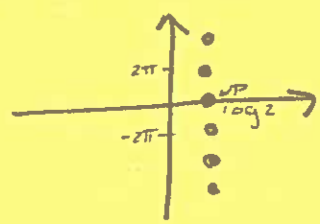
1.6.1 Find and plot $\log z$ for following z. Specify principle valu

(a) 2 (b) i (c) $i+1$ (d) $(1+i\sqrt{3})/2$

Pf (a) $\log 2=\log|2|+i\arg z$

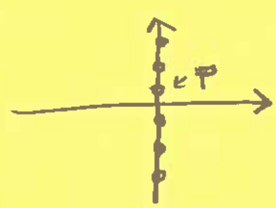
$=\log 2+i(0)+2\pi i m$

$=\log 2+2\pi i m$



(b) $\log i=\log 1+i\,\pi/2+2\pi i m$

$=i(\pi/2+2\pi m)$



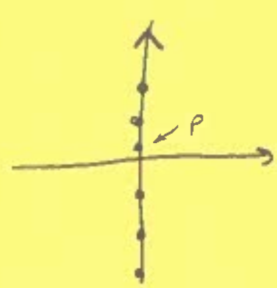
$\log z=\log|z|+i\arg z$

(c) $\log(1+i)=\log\sqrt{2}+i\,\pi/4+2\pi m i$

$=\log\sqrt{2}+i(\pi/4+2\pi m)$

(d) $\log\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)=\log 1+i\,\pi/3+2\pi i m$

$=i(\pi/3+2\pi m)$



1.7.2 Compute and plot $\log[(1+i)^{2i}]$

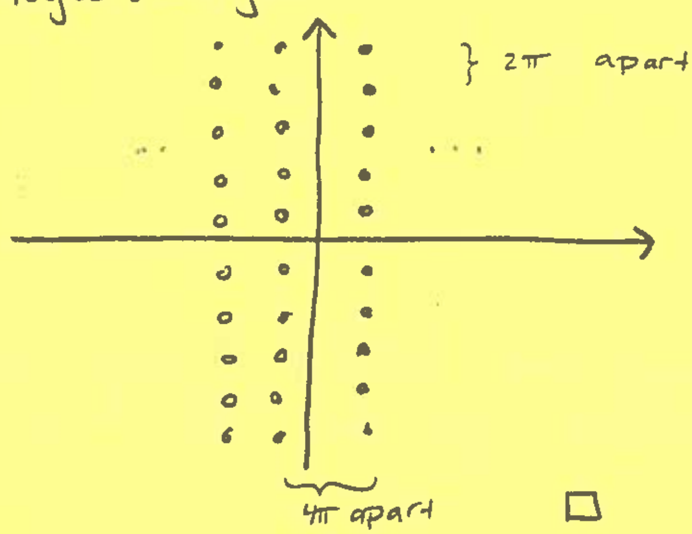
Pf Let $(1+i)^{2i} = z^{\alpha}$ where $z = 1+i$, $\alpha = 2i$

$\Rightarrow (1+i)^{2i} = e^{2i\log(1+i)}$ since $(1+i)^{2i}$

$= e^{2i(\log\sqrt{2} + i\pi/4 + 2\pi i n)}$

$= e^{i\log 2 - \pi/2 - 4\pi m}$

$\Rightarrow \log(z^{\alpha}) = \log(e^{i\log 2 - \pi/2 - 4\pi m}) = i\log 2 - \pi/2 - 4\pi m + 2\pi i n$



$\square$

1.7.4 Show $(zw)^{\alpha} = z^{\alpha}w^{\alpha}$ where on right we take all possible products.

Pf $(zw)^{\alpha} = e^{\alpha\log|zw|}$

$= e^{\alpha\log|z||w| + i\operatorname{Arg} zw + 2\pi i n}$

$= e^{\alpha(\log|z| + i\operatorname{Arg} z + 2\pi i k)} e^{\alpha(\log w + i\operatorname{Arg} w + 2\pi i l)}$

$= z^{\alpha}w^{\alpha}$

$\square$

1.7.5 Find i^{i^i}. Show i^{i^i} doesnt coincide w/ i^{-1}

Pf $i^i = e^{i \log i} = e^{i(i\pi/2 + 2\pi i m)} = e^{-\pi/2 - 2\pi m}$

$$(i^i)^i = e^{e^{(-\pi/2 - 2\pi m)\log i}}$$
$$= e^{e^{(-\pi/2-2\pi m)} e^{i(\pi/2 + 2\pi m)}}$$
$$= e^{e^{(-\pi/2 - 2\pi m) + i(\pi/2 + 2\pi m)}}$$

$z^\alpha = e^{\alpha \log z}$

However $i^{-1} = 1/i = -i$

$\square$

1.8.2 Show $|\cos z|^2 = \cos^2 x + \sinh^2 y$ where $z = x+iy$
Find all periods and zeros of $\cos z$.

Pf $\cos z = \cos(x+iy)$
$$= \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - \sin x \sinh y$$
$$= \cos x \cosh y - i \sin x \sinh y$$

$$|\cos z|^2 = \cos^2 \underbrace{\cosh^2 y}_{\sinh^2 y + 1} + \underbrace{\sin^2 x}_{1-\cos^2 x} \sinh^2 y$$
$$= \cos^2 x \sinh^2 y + \cos^2 x + \sinh^2 y - \cos^2 x \sinh^2 y$$
$$= \cos^2 x + \sinh^2 y$$

zeros: $\cos z = 0 \iff \cos^2 x + \sinh^2 y = 0 \iff \cos^2 x = -\sin^2 y$

1.8.3 Find all periods of cosh z and sinh z

Pf $\cosh z = 0 \iff \cos iz = 0$

$\iff \sin(\pi/2 + iz) = 0$

$\iff \pi/2 - iz = k\pi$

$\iff z = \dfrac{\pi/2 - k\pi}{i}$

$\iff z = i\pi/2 - ik\pi$ period is $2\pi i$

$\sinh z = 0 \iff i \sin i z = 0$

$\iff \sin iz = 0$

$\iff iz = k\pi$

$\iff z = ik\pi$ $k \in \mathbb{Z}$ period $2\pi i$

$\square$

$\cos(iy) = \cosh(y)$

$\cosh(iy) = \cos(y)$

$\sinh(y) = i\sin(iy)$

$\sin(y) = i\sinh(iy)$

1.8.4 Show $\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$ where both sides are interpreted as subsets in $\mathbb{C}$. In other words show $\tan w = z$ iff $2iw$ is one of the values of log.

Pf $z = \tan w \iff z = \dfrac{\sin w}{\cos w}$

$$\iff z = \frac{1}{i}\,\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$$

$$\iff iz = \frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$$

$$\iff iz = \frac{\cancel{e^{-iw}}(e^{2iw}-1)}{\cancel{e^{-iw}}(e^{2iw}+1)}$$

$$\iff iz\,e^{2iw} + iz = e^{2iw} - 1$$

$$\iff e^{2iw}(1-iz) = 1+iz$$

$$\iff e^{2iw} = \frac{1+iz}{1-iz}$$

$$\iff 2iw \in \log\left(\frac{1+iz}{1-iz}\right)$$

$\Rightarrow \tan z = w \iff 2iw \in \log\left(\frac{1+iz}{1-iz}\right)$

$\therefore \tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

Gamelin Chapter 2

2.2 # 4,5,6

2.3 # 3, 4, 5, 6, 8

2.5 # 1b, 5

2.7 # 1a, 2, 3, 4, 7, 9

2.2.4 Suppose $f(z) = az^2 + bz\bar{z} + c\bar{z}^2$, $a, b, c \in \mathbb{C}$

Show f is complex diff $\Leftrightarrow bz + 2c\bar{z} = 0$ where analytic

Pf Assume f is complex diff

$\Rightarrow$ no $\frac{dz}{d\bar{z}}$ term

analytic $\Rightarrow$ no $\frac{\partial}{\partial \bar{z}}$ term

Now $f'(z) = 2az + bz\frac{\partial f}{\partial \bar{z}} + b\bar{z} + 2c\bar{z}\frac{\partial f}{\partial \bar{z}}$

$= 2az + b\bar{z} + \frac{\partial f}{\partial \bar{z}}(bz + 2c\bar{z})$

$\Rightarrow bz + 2c\bar{z} = 0$

Next we want to find where f is analytic

i.e where $bz + 2c\bar{z} = 0$

Let $U = \{z \in \mathbb{C} : bz + 2c\bar{z} = 0\}$

$= \{x + iy \in \mathbb{C} : b(x+iy) + 2c(x-iy) = 0\}$

$= \{x + iy \in \mathbb{C} : x = \frac{b - 2c}{b + 2c} iy\}$

which is the interior of a line so $U = \emptyset$

$\Rightarrow f$ is nowhere analytic

$\Rightarrow f$ is complex differentiable on $\emptyset$.

$\square$

2.2.5 Show if f analytic on D then $g(z)=\overline{f(\bar{z})}$ is analytic on reflected domain $D^* = \{\bar{z} : z \in D\}$ and $g'(z) = \overline{f'(\bar{z})}$.

Pf f analytic $\Rightarrow$ $f'(z)$ exists. and is continuous

Now $\lim \frac{g(z+h)-g(z)}{h} = \lim \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h}$

$= \overline{\left(\lim \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}}\right)}$

$= \overline{f'(\bar{z})}$

or use Cauchy Riemann equations to prove analytic.

f analytic $\Rightarrow$ $f'(\bar{z})$ exists and is continuous

$\Rightarrow g'$ is continuous & exists since g' is composition of continuous functions

$\Rightarrow g'$ exists on D^*

$\Rightarrow g$ is analytic $\square$

2.2.6 $h(t)$ is continuous in $\mathbb{C}$ on $[0,1]$. $H(z) = \int_0^1 \frac{h(t)}{t-z}\,dt$ $z \in \mathbb{C}\setminus[0,1]$ Show H analytic and find H'.

Pf $H'(z) = \lim_{r\to 0} \frac{H(z+r) - H(z)}{r}$

$= \lim_{r\to 0} \frac{\int_0^1 \frac{h(t)}{t-z-r} - \int_0^1 \frac{h(t)}{t-z}}{r}$

$= \lim_{r\to 0} \int_0^1 \frac{h(t)\,[t-z-t+z+r]}{(t-z-r)(t-z)r}$

$= \lim_{r\to 0} \int_0^1 \frac{h(t)}{(t-z-r)(t-z)}$

$= \int_0^1 \frac{-h(t)}{(t-z)^2}$ which exists $\forall z$

$\Rightarrow H$ is analytic and $H'(z) = \int_0^1 \frac{-h(t)}{(t-z)^2}$

$\square$

2.3.3 Show if $f, \bar{f}$ are analytic on a domain D then f is constant.

Pf Let $f = u + iv$, $\bar{f} = u - iv$

f analytic $\Rightarrow u_x = v_y \quad u_y = -v_x$

$\bar{f}$ analytic $\Rightarrow u_x = -v_y \quad u_y = v_x$

CRE $u_x = v_y \quad u_y = -v_x$

$\therefore v_y = -v_y$ and $v_x = -v_x$

So $v_y = v_x = 0 \Rightarrow u_x = u_y = 0$

$\Rightarrow f$ is analytic with $f' = 0$.

$\Rightarrow f$ is constant on D. $\square$

2.3.4 Show if f is analytic on D and if $|f|$ is constant then f is constant.

Pf case 1 $\exists z_0$ s.t. $f(z_0) = 0$

$\Rightarrow |f(z_0)| = 0$

$\Rightarrow |f| = 0 \quad \forall z$

$\Rightarrow f \equiv 0$ ie constant

if f has root we only care $|f|$ is constant

case 2 $f(z) \neq 0 \ \forall z$.

$\Rightarrow \bar{f} = \dfrac{|f|^2}{f}$ is anaylic since $|f|$ & $\frac{1}{f}$ are.

↳ since $f \neq 0$

$\Rightarrow f$ constant since f and $\bar{f}$ are analytic. $\square$

if f has no root use f analytic and $|f|$ analytic.

2.3.5 If $f = u + iv$ is analytic then $|\nabla u| = |\nabla v| = |f'|$

Pf $|f'|^2 = |u_x + iv_x|^2 = u_x^2 + v_x^2$

$|\nabla u|^2 = u_x^2 + u_y^2 = u_x^2 + (-v_x)^2 = u_x^2 + v_x^2$

$|\nabla v|^2 = v_x^2 + v_y^2 = v_x^2 + u_x^2$

$u_x = v_y$
$u_y = -v_x$

By CRE

$\Rightarrow |f'| = |\nabla u| = |\nabla v|$.

$\square$

2.3.6 If $f = u + iv$ is analytic on D then ∇v is obtained by rotating ∇u by $90°$.
Show ∇u and ∇v are orthogonal

Pf Let $\theta \in \mathbb{R}$.
$\Rightarrow$ rotation of $(x, y) \in \mathbb{C}$ by $\measuredangle \theta$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow A_\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ if $\theta = 90$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -y \\ x \end{pmatrix}$

Now $\nabla v = (v_x, v_y) = (-u_y, u_x) = A_\theta(\nabla u)$

$\Rightarrow \nabla v$ is obtained by rotating ∇u $90°$

$\Rightarrow \nabla v$ and ∇u are orthogonal.

2.5.1 Show $xy + 3x^2y - y^3$ is harmonic + find harm conj.

Pf $u_{xx} + u_{yy} = (y + 6xy)_x + (x + 3x^2 - 3y^2)_y = 6y - 6y = 0$ ✓

$\Rightarrow$ harmonic.

$u_x = y + 6xy = V_y \Rightarrow V = \frac{y^2}{2} + 3y^2x + h(x)$

$u_y = x + 3x^2 - 3y^2 = -V_x \Rightarrow V = \frac{-x^2}{2} - x^3 + 3y^2x + g(y)$

$\Rightarrow V = \frac{y^2}{2} - \frac{x^2}{2} - x^3 + 3y^2x$ is the harmonic conj.

∎

2.5.5 Show Laplaces equation in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Pf Let $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow x_r = \cos\theta, \quad y_r = \sin\theta, \quad x_\theta = -r\sin\theta, \quad y_\theta = r\cos\theta$

$u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \cdot \sin\theta$

$u_\theta = u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + y_r r\cos\theta$

$u_{rr} =$

X

2.7.1a Compute fractional linear trans $(1+i, 2, 0) \to (0, \infty, i-1)$

Pf $f(z) = \alpha \frac{z - i - 1}{z - 2}$

$f(0) = \alpha \frac{i+1}{2}$

$\Rightarrow i - 1 = \alpha \frac{i+1}{2}$

$\Rightarrow 2i - 2 = \alpha(i+1)$

$\Rightarrow \frac{2i-2}{i+1} = \alpha$

$\Rightarrow 2i = \alpha$

$$\Rightarrow \boxed{f(z) = \frac{2i(z - (i+1))}{z - 2}}$$

$\frac{w - w_0}{w - w_2}\frac{w_1 - w_2}{w_1 - w_0} = \frac{z - z_0}{z - z_2}\frac{z_1 - z_2}{z_1 - z_0}$

General formula

2.7.2 Find image of circle $\{|z-1|=1\}$, $\mathbb{R}$, $\{|z-1|<1\}$ using above

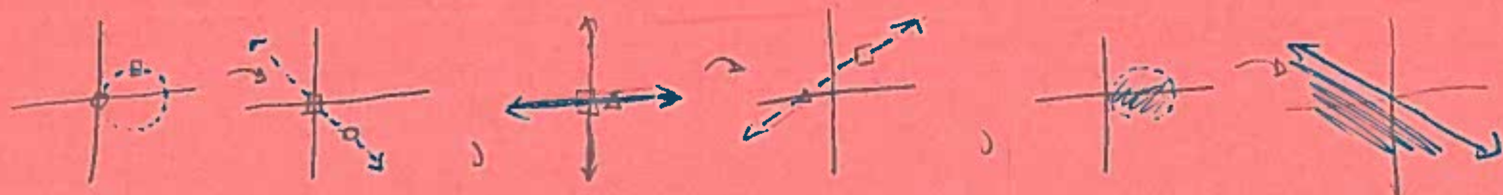
Pf From above $f(z) = \frac{2i(z-(i+1))}{z-2}$

Pole is at $z_0 = 2$

$\{|z-1|=1\}$ contains pole so is mapped to a line.

$f(0) = i-1$, $f(1+i) = 0$

$f(1) = \frac{2i(-i)}{-1} = -2$



2.7.7 Show $f(z) = \frac{az+b}{cz+d} = z \Leftrightarrow b=c=0 \quad a=d\neq 0$

Pf $\frac{az+b}{cz+d} = z \Leftrightarrow az+b = cz^2+dz \Leftrightarrow cz^2+(d-a)z-b=0$

$z=1 \Rightarrow c+d-a-b=0$
$z=0 \Rightarrow -b=0$
$z=-1 \Rightarrow c+a-d-b=0$

$\Rightarrow c=a-d=d-a \Rightarrow c=0 \Rightarrow d=a\neq 0$

$[c=0 \Rightarrow d\neq 0]$

$\square$

2.7.3. $1 \mapsto i$, $0 \mapsto 1+i$, $-1 \mapsto 1$ Determine image of $\{|z|=1\}$ $\{|z|<1\}$ and $[1,1] \subset \mathbb{R}$

Pf $\frac{w-w_0}{w-w_2}\frac{w_1-w_2}{w_1-w_0} = \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0} \Rightarrow \frac{w-i}{w-1}\cdot\frac{i}{1} = \frac{z-1}{z+1}\cdot\frac{-1}{+1} \Rightarrow w = i\left(\frac{z^2+z}{1+z}\right)$

$z=-1$ is pole so $\{|z|=1\}$ is sent to pole.

$f(0) = i$

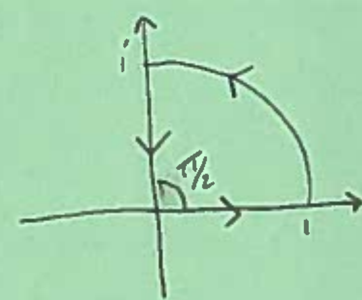
Gamelin Chapter 3

3.1. # 3, 5, 8
3.2 # 1, 2
3.3 # 2, 4?
3.4 # 1
3.5 # 1, 2, 3, 4, 5, 6, 7

<u>3.1.3</u> Evaluate $\int_{\partial D} x^2 dy$ directly and w/ Green's thm D=

Pf Green's Thm:

$$\int_{\partial D} P dx + Q dy = \iint_D (Q_x - P_y)\, dx\, dy$$
$$= \iint_D 2x - 0\, dx\, dy$$
$$= \int_0^1 \int_0^{\pi/2} 2r\cos\theta\, r\, d\theta\, dr$$
$$= \int_0^1 2r^2 \sin\theta \Big|_0^{\pi/2} dr$$
$$= \int_0^1 2r^2\, dr$$
$$= \frac{2r^3}{3}\Big|_0^1 = \boxed{2/3}$$



$x = r\cos\theta$

Directly:

$$\int_{\partial D} x^2 dy = \int_0^{\pi/2} \cos^2\theta \cos\theta\, d\theta \qquad \partial D = \{(\cos\theta, \sin\theta) \mid \theta \in (0, \pi/2)\}$$
$$= \int_0^{\pi/2} (1 - \sin^2\theta)\cos\theta\, d\theta$$
$$= \int_0^{\pi/2} \cos\theta\, d\theta - \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$$
$$= \sin\theta \Big|_0^{\pi/2} - u^3/3 \Big|_0^1$$
$$= 1 - 1/3$$
$$= \boxed{2/3}$$

$x = \cos\theta$
$y = \sin\theta$
$dy = \cos\theta\, d\theta$

Definition of line Integral

$$\int_a^b P(x(t), y(t))\, x'(t)\, dt$$

3.1.5 Show $\int_{\partial D} x\,dy = \text{area } D$ and $\int_{\partial D} y\,dx = -\text{area } D$

Pf By Green's thm $\int_{\partial D} x\,dy = \iint_D 1\, dx\,dy = \text{area } D$

$\int_{\partial D} y\,dx = \iint_D -1\, dx\,dy = -\text{area } D$

$\square$

3.2.1 Determine if independent of path.
If so find h s.t. $dh = P\,dx + Q\,dy$.
If not find closed form around which integral is 0

Pf a) $x\,dx + y\,dy = dh$

$\Rightarrow h = \frac{x^2}{2} + \frac{y^2}{2}$

b) $x^2 dx + y^5 dy = dh$

$\Rightarrow h = \frac{x^3}{3} + \frac{y^6}{6}$.

independent $\Rightarrow \int \neq 0$

closed $\Rightarrow \partial P/\partial y = \partial Q/\partial x$

c) $y\,dx + x\,dy = dh$

$\Rightarrow h = xy$

d) $y\,dx - x\,dy$

Not independent since not closed.

$$\frac{d}{dy} y \neq \frac{d}{dx}(-x)$$

$$\int_{|z|=1} y\,dx - x\,dy = \iint_{|z|<1} -2\,dx\,dy = -2\pi \neq 0.$$

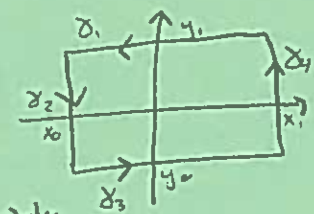
exact = $p\,dx + q\,dy = dh$ for some h

exact $\Leftrightarrow$ independent of path

3.1.8 Prove Green's thm for $x_0 < x < x_1$ $y_0 < y < y_1$

(a) directly
(b) Using result for triangles

Pf
(a) $\int_{\partial D} P\,dx + Q\,dy = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}\right)(P\,dx + Q\,dy)$



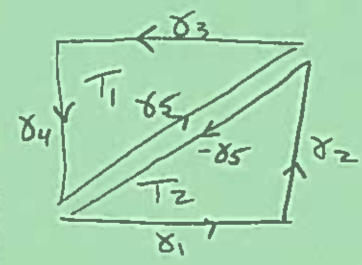
$$= \int_{x_0}^{x_1} P(x,y_0)\,dx - \int_{x_0}^{x_1} P(x,y_1)\,dx + \int_{y_0}^{y_1} Q(x_1,y)\,dy - \int_{y_0}^{y_1} Q(x_0,y)\,dy$$

$$= \int_{x_0}^{x_1} P(x,y_0) - P(x,y_1)\,dx - \int_{y_0}^{y_1} Q(x_0,y) - Q(x_0,y)\,dy$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{dP}{dy}(x,y)\,dy\,dx - \int_{y_0}^{y_1}\int_{x_0}^{x_1} \frac{dQ}{dx}(x,y)\,dx\,dy$$

$$= \iint_D \frac{dP}{dy} - \frac{dQ}{dy}\,dx\,dy \qquad \checkmark$$

(b) Let $\omega = P\,dx + Q\,dy$.



$$\int_{\partial T_1} \omega = \int_{\gamma_3}\omega + \int_{\gamma_4}\omega + \int_{\gamma_5}\omega = \iint_{T_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\int_{\partial T_2} \omega = \int_{\gamma_1}\omega + \int_{\gamma_2}\omega + \int_{\gamma_5}\omega = \iint_{T_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \iint_{T_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} + \iint_{T_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$= \int_{\partial T_1}\omega + \int_{\partial T_2}\omega$$

holds on Δ's so apply it

$$= \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5} + \int_{\gamma_1} + \int_{\gamma_2} - \int_{\gamma_5}$$

$$= \int_{\gamma_3}\omega + \int_{\gamma_4}\omega + \int_{\gamma_1}\omega + \int_{\gamma_2}\omega$$

diagonals going opposite directions so they cancel out.

$$= \int_{\partial D}\omega$$

$$= \int_{\partial D} P\,dx + Q\,dy. \qquad \square$$

3.2.2 Show $f = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ is closed but not independent of path on annulus centered at 0

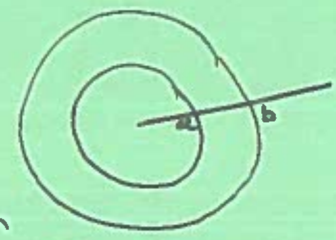
Pf closed $\Rightarrow \dfrac{dP}{dy} = \dfrac{dQ}{dx}$

$$P = \frac{-y}{x^2+y^2} \Rightarrow \frac{dP}{dy} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$Q = \frac{x}{x^2+y^2} \Rightarrow \frac{dQ}{dx} = \frac{(x^2+y^2)-(2x)(x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore$ f is closed

Consider annulus $a \leq x \leq b = A$



Let $x = r\cos\theta$, $y = r\sin\theta$
for some $r \in (a,b)$ we integrate on γ a circle of radius r where $\theta \in [0, 2\pi]$

$$\int_{|z|=r} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{-r^2\sin^2\theta - r^2\cos^2\theta}{r^2}\,d\theta$$

$$= \int_0^{2\pi} (-1)\,d\theta$$

$$= -2\pi \neq 0$$

So f is not path independent

$\square$

3.3.2 Show $h(z)$ is harmonic on ☆ shaped domain. $h(z) = f(z) + \overline{g(z)}$ where f and g are analytic

Pf Let $h = u + iv$

h harmonic on ☆ shaped domain

$\Rightarrow u, v$ have harmonic conjugates

$\Rightarrow u = \text{re}\,\varphi$ for some analytic φ

$v = \text{im}\,\psi$ for some analytic ψ

$\Rightarrow h = \text{re}\,\varphi + i\,\text{im}\,\psi$

$$\Rightarrow h = \frac{\varphi + \overline{\varphi}}{2} + i\,\frac{\psi - \overline{\psi}}{2i}$$

$$\Rightarrow h = \frac{\varphi}{2} + \frac{\psi}{2} + \frac{\overline{\varphi}}{2} - \frac{\overline{\psi}}{2}$$

$$\Rightarrow h = \underbrace{\left(\frac{\varphi + \psi}{2}\right)}_{f} + \underbrace{\left(\overline{\frac{\varphi - \psi}{2}}\right)}_{g}$$

Let $h = f(z) + \overline{g(z)}$

$$\Rightarrow \Delta h = \frac{\partial^2}{\partial z \partial \bar{z}} h$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}} f\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z} \bar{g}\right)$$

$$= 0 + 0$$

$$= 0$$

$\Rightarrow h$ is harmonic $\square$

3.3.4 Let $u(z)$ be harmonic on $\{a<|z|<b\}$. Show $\exists$ a constant C s.t. $u(z)-C\log|z|$ has a harmonic conjugate on the annulus. Show

$$C=\frac{r}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial r}(re^{i\theta})\,d\theta.$$

Pf u harmonic $\Rightarrow u_{xx}+u_{yy}=0$

$\Rightarrow u$ has harmonic conjugate v.

3.5.1 Let D be bdd domain and let u be a real valued harmonic function on D that extends continuously to boundary ∂D.
Show if $a \le u \le b$ on ∂D then $a \le u \le b$ on D

Pf $D \cup \partial D$ is compact
$\Rightarrow u$ attains both max and min on $D \cup \partial D$

if bigger then max on bdry must be constant on interior.

Assume $\exists z_0 \in D$ s.t. $u(z_0) = c > b$.
$\Rightarrow u = c$ on D by max principal
This contradicts since u extends continuously to ∂D

for other bd just flip everything by considering $-u$

To show $u \geq a$ on D consider $-u$
$\Rightarrow -u \le -a$ on ∂D then by above $-u \le -a$ on D
$\Rightarrow a \le u$ on D

$\square$

3.5.2 Fix $n \geq 1$, $r > 0$ and $\lambda = \rho e^{i\varphi}$. What is max modulus of $z^n + \lambda$ over disk $\{|z| \le r\}$? Where does $z^n + \lambda$ attain max modulus over disk?

Pf By the max principle we know max is attained on $|z| = r$.

$$|z| = r \Rightarrow |z^n + \lambda| \le |z|^n + |\lambda|$$
$$= r^n + |\lambda|$$
$$= |(r^n + |\lambda|)e^{i\theta}|$$
$$= |(re^{i\theta})^n + \lambda| \quad \text{ask}$$

$\square$

3.4.1 Let f be continuous on domain D.
Show if f has MVP wrt circles then f has MVP wrt disks.
(i.e. if $z_0 \in D$ and D_0 is disk centered at z_0 with area A in D then $f(z_0) = \frac{1}{A}\iint_{D_0} f(z)\,dx\,dy$)

Pf Assume f has MVP wrt circles

$\Rightarrow f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})\,d\theta \quad 0 < r < \varepsilon$

Let $D_0 = \{z \in \mathbb{C} \mid |z - z_0| \leq R\} \subset D$

$$\frac{1}{A}\iint f(z)\,dx\,dy = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^{R} f(z_0 + re^{i\theta})\,r\,dr\,d\theta$$

$$= \frac{1}{\pi R^2}\int_0^{R}\int_0^{2\pi} f(z_0 + re^{i\theta})\,r\,d\theta\,dr$$

$$= \frac{1}{\pi R^2}\int_0^{R} 2\pi f(z_0)\,r\,dr$$

$$= \frac{2\pi f(z_0)}{\pi R^2}\frac{R^2}{2}$$

$$= f(z_0) \qquad \square$$

3.5.5 Suppose f is bdd analytic on right half plane. Suppose f extends continuously to imaginary axis and $|f(iy)| \le M \ \forall iy \in i\mathbb{R}$.
Show $|f(z)| \le M \ \forall z$ in RHP

Pf Let $\varepsilon > 0$, $g(z) = \dfrac{f(z)}{1+\varepsilon z}$

g bdd by M on bdry

f bdd $\Rightarrow f(z) < C$ (C a constant)

$\Rightarrow |g(z)| \le \dfrac{C}{1+\varepsilon R} \quad \forall |z|\ R \quad \text{Re}\, z > 0.$

$\Rightarrow |g(z)| \le M$ for R sufficiently large.

$|f(iy)| \le M \Rightarrow |g(iy)| \le M$

⇒ bdd on inside

$\Rightarrow |g(z)| \le M \quad \forall z \le R \quad \text{Re}\, z > 0$ by max as $\varepsilon \to 0$.

$\Rightarrow f(z) \le M \quad \forall z$ in RHP

$\square$

3.5.7 f bdd analytic on $\mathbb{D}$. $\exists$ finite # pts on bdry st $f(z)$ extends continuously to arcs of $\partial\mathbb{D}$, seperating pts and $|f(e^{i\theta})| \le M$. Show $|f| \le M$ on $\mathbb{D}$

Pf Let the points be $a_1, \ldots, a_n$.

$\Rightarrow (z-a_j)^\varepsilon \quad (1 \le j \le n)$ is analytic

$|z-a_j|^\varepsilon \to 0$ as $z \to a_j$

$\Rightarrow |\prod (z-a_j)^\varepsilon|\, f(z)| = f_\varepsilon(z)$ is cont on $\mathbb{D} \cup \partial\mathbb{D}$

$\Rightarrow |f_\varepsilon(z)| \le M$ on $\partial\mathbb{D}$

$\Rightarrow |f_\varepsilon(z)| \le M$ on $\mathbb{D}$

$\Rightarrow |f(z)| \le M$ on $\mathbb{D}$ as $\varepsilon \to 0$

3.5.3 Use max principle to prove fundamental thm of algebra (any polynomial $p(z)$ of degree $n \geq 1$ has a zero)

Pf Assume BWOC $p(z)$ has no zeros
$\Rightarrow \frac{1}{p(z)}$ is entire

Let $m(R)$ be max of $\frac{1}{p(z)}$ on $\{|z|=R\}$.
$\Rightarrow m(R) \to 0$ as $R \to \infty$ (unless p is constant)
Since p is a polynomial.
$\Rightarrow |1/p(z)| \leq m(R)$ when $|z| \leq R$ by max princ.
$\Rightarrow \left|\frac{1}{p(z)}\right| = 0$ Since $m(R) \to 0$
$\Rightarrow$ This contradicts
$\Rightarrow p(z)$ has a zero. $\square$

3.5.4 Let f be analytic on D w/ no zeros
(a) Show $|f(z)|$ attains min on D then f constant.
(b) Show D bdd and extents continuously to ∂D
then $f(z)$ attains min on ∂D

Pf (a). $|f|$ attains min
$\Rightarrow |1/f|$ attains max on D
$\Rightarrow f$ Constant

(b) $D \cup \partial D$ compact & f continuous
$\Rightarrow |f(z)|$ attains max and min on $D \cup \partial D$
Assume $|f|$ doesn't attain min on ∂D
$\Rightarrow |f|$ attains min on D
$\Rightarrow f$ is constant
$\Rightarrow |f|$ attans min on ∂D. $\square$

Gamelin Chapter 4

4.1 # 2, 4, 5

4.2 # 3

4.3 # 3, 6

4.4 # 1aefh, $\boxed{2}$

4.5 # 1, 2, 3, 4

4.6 # 1, 2, $\boxed{3}$

4.8 # 1, 2, 3, $\boxed{4}$ 5

radios
behind the moon

4.1.2 Let $\gamma = \{|z|=1\}$ evaluate the following.

(a) $\int_\gamma z^m dz$ (b) $\int_\gamma \bar{z}^m dz$ (c) $\int_\gamma z^m |dz|$

Pf Since integrating around a circle we use the substitution $z = e^{i\theta}$ $0 \le \theta < 2\pi$, $dz = ie^{i\theta}d\theta$

$$\int_{|z|=1} z^m dz = \int_0^{2\pi} e^{i\theta m} ie^{i\theta} d\theta$$

$$= \int_0^{2\pi} i e^{i\theta(m+1)} d\theta$$

$$= \frac{ie^{i\theta(m+1)}}{i(m+1)}\Big|_0^{2\pi} = \begin{cases} 2\pi i & m=-1 \\ 0 & m\neq -1 \end{cases}$$

$$\int_{|z|=1} \bar{z}^m dz = \int_0^{2\pi} e^{-i\theta m} ie^{i\theta} d\theta$$

$$= \int_0^{2\pi} ie^{i\theta(1-m)} d\theta$$

$$= \frac{ie^{i\theta(1-m)}}{i(1-m)}\Big|_0^{2\pi}$$

$$= \frac{1}{1-m}\left(e^{2\pi i(1-m)} - 1\right) = \begin{cases} 2\pi i & m=1 \\ 0 & m\neq 1 \end{cases}$$

integrating around circle let $z = e^{i\theta}$ $dz = ie^{i\theta}d\theta$

$$\int_{|z|=1} z^m |dz| = \int_0^{2\pi} e^{i\theta m} d\theta$$

$$= \frac{e^{i\theta m}}{im}\Big|_0^{2\pi}$$

$$= \frac{1}{im}\left(e^{2\pi i m} - 1\right) = \begin{cases} 2\pi & m=0 \\ 0 & m\neq 0 \end{cases}$$

4.1.4 Show if D bdd domain with smooth bdry

$$\int_{\partial D} \bar{z}\,dz = 2i \text{ Area } D$$

Pf Let $z = x+iy$

$$\int_{\partial D} \bar{z}\,dz = \int_{\partial D} (x-iy)(dx+idy)$$

$$= \int_{\partial D} (x-iy)dx + (xi+y)dy$$

$$= \iint_D i-(-i)\,dxdy \quad \text{by Greens thm}$$

$$= \iint_D 2i\,dxdy$$

$$= 2i \text{ Area } D \qquad \square$$

4.1.5 Show $\left| \oint_{|z-1|=1} \frac{e^z}{z+1} dz \right| \le 2\pi e^2$

Pf $\left| \int_{|z-1|=1} \frac{e^z}{z+1} dz \right| \le \int_{|z-1|=1} \left| \frac{e^z}{z+1} \right| |dz|$

Now $\left| \frac{e^z}{z+1} \right| \le \frac{|e^z|}{|z-1+2|}$ $|e^z| \le e^{re z}$

$$\le \frac{e^{re z}}{|z-1+2|}$$

$$= \frac{e^{re z}}{2-|z-1|}$$

$$= \frac{e^2}{2-1} = e^2$$

By ML estimate $\left| \int_{|z-1|=1} \frac{e^z}{z+1} dz \right| \le e^2\, 2\pi$

length of γ

$\square$

<u>4.2.3</u> Show if $m \neq -1$ then z^m has primitive on $\mathbb{C} \setminus \{0\}$

<u>Pf</u> $\frac{z^{m+1}}{m+1}$ is a primitive for z^m given $m \neq -1$.

<u>4.3.3</u> Let $f(z) = c_0 + c_1 z + \dots + c_n z^n$,

(a) If $c_k \in \mathbb{R}$ show $\int_{-1}^{1} f(x)^2 \, dx \leq \pi \int_0^{2\pi} |f(e^{i\theta})| \frac{d\theta}{2\pi} = \pi \sum c_k^2$

(b) If $c_k \in \mathbb{R}$ Show $\int_{-1}^{1} |f(x)|^2 \, dx \leq \pi \int_0^{2\pi} |f(e^{i\theta})| \frac{d\theta}{2\pi} = \pi \sum_0^n |c_k|^2$

(c) establish $\left| \sum \frac{c_j c_k}{j+k+1} \right| \leq \pi \sum_0^n |c_k|^2$ strict unless $c_j = 0 \ \forall j$.

4.3.6 Suppose $f(x)$ continuous in closed disk $\{|z| \le R\}$ and analytic on $\{|z|<R\}$ Show $\int_{|z|=R} f(z)dz = 0$

Pf Let $\varepsilon > 0$.

f continuous on $\{|z| \le R\}$

$\Rightarrow f$ continuous on compact set

$\Rightarrow f$ uniformly continuous on $\{|z| \le R\}$

$\Rightarrow \exists \delta > 0$ s.t. $|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \frac{\varepsilon}{2\pi R}$ $z_1, z_2 \in \{|z| \le R$

Let $r \in (0,1)$ and $f_r : \{|z| < R/r\} \to \mathbb{C}$ s.t. $f_r(z) = f(rz)$

$\Rightarrow f_r(z)$ is analytic on domain containing $\{|z| \le R\}$

$\Rightarrow \oint_{|z|=R} f_r(z) = 0$ by Cauchys thm

Choose $r \in (0,1)$ s.t. $0 < 1 - \frac{\delta}{R} < r < 1$

$\Rightarrow \forall z$ s.t. $|z| = R$ we have $|z - rz| = R(1-r) < \delta$

$\Rightarrow |f(z) - f(rz)| < \varepsilon/2\pi R$ by uniform Continuity.

$$\Rightarrow \left|\int_{|z|=R} f(z)dz\right| = \left|\int_{|z|=R} f(z)dz - \int_{|z|=R} f_r(z)dz\right|$$

$$= \left|\int_{|z|=R} (f(z) - f_r(z))dz\right|$$

$$\le \int_{|z|=R} |f(z) - f_r(z)|\, dz$$

$$= \int_{|z|=R} |f(z) - f(rz)|\, dz$$

$$< \frac{\varepsilon}{2\pi R} \cdot 2\pi R = \varepsilon \text{ by ML estimate}$$

$\therefore \left|\int_{|z|=R} f(z)dz\right| = 0$ as $\varepsilon \to 0$. $\square$

4.4.1 (a) $\int_{|z|=2} \frac{z^n}{z-1} dz$ (e) $\int_{|z|=1} \frac{e^z}{z^m} dz$

(f) $\int_{|z-1-i|=5/4} \frac{\text{Log} z}{(z-1)^2}$ (h) $\int_{|z-1|=2} \frac{dz}{z^2(z^2-4)e^z}$

Pf Note Cauchy's integral formula gives us

$$\frac{2\pi i\, f^{(m)}(z)}{m!} = \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw$$

(a) $\int_{|z|=2} \frac{z^n}{z-1} dz = \frac{2\pi i\, w^n\big|_{w=1}}{0!} = 2\pi i$ $\boxed{\begin{array}{c} m=0 \\ f(w)=w^n \\ z=1 \end{array}}$

(e) $\int_{|z|=1} \frac{e^z}{z^m} dz = \frac{2\pi i\, e^0}{(m-1)!} = \begin{cases} \frac{2\pi i}{(m-1)!} & m\geq 1 \\ 0 & m\leq 0 \end{cases}$ $\boxed{\begin{array}{c} m=m-1 \\ z=0 \\ f(w)=e^z \end{array}}$

(f) $\int_{|z-1-i|=5/4} \frac{\text{Log} z}{(z-1)^2} = \frac{2\pi i\, \text{Log}'(z)}{1!} = \frac{2\pi i}{z}\Big|_{z=1} = 2\pi i$ $\boxed{\begin{array}{c} m=1 \\ z=1 \\ f(w)=\text{Log}(w) \end{array}}$

(h) $\int_{|z-1|=2} \frac{dz}{z^2(z^2-4)e^z}$ $\boxed{\begin{array}{c} m=1 \\ z=0 \\ f(w)=\frac{e^{-2w}}{w^2-4} \end{array}}$ $\boxed{\begin{array}{c} m=0 \\ z=2 \\ f(w)=\frac{e^{-z}}{z^2(z+2)} \end{array}}$

$$= \int_{|z|=\varepsilon} \frac{e^{-z}dz}{z^2(z^2-4)} + \int_{|z-2|=\varepsilon} \frac{e^{-z}}{z^2(z^2-4)}$$

Singularities are $0, \pm 2$
-2 is not in $|z-1|\leq 2$ so we ignore it

$$= 2\pi i \frac{d}{dz}\left(\frac{e^{-w}}{w^2-4}\right)\Big|_{w=0} + 2\pi i \frac{e^{-z}}{z^2(z+2)}\Big|_{z=4}$$

$$+ \frac{\pi i}{8e^2}$$

finish

4.4.2 Show harmonic functions have partial derivatives of all orders

Pf Let $f = u + iv$ be a harmonic function.

$\Rightarrow f' = u_x + iv_x = v_y - iu_y \qquad (u_x = v_y \ \& \ u_y = -v_x)$

$\Rightarrow \mathrm{Re} f' = u_x = u_y$

4.5.1 Show if u is harmonic on $\mathbb{C}$ & bdd above then u is constat.

Pf u harmonic

$\Rightarrow u = \mathrm{re} f$ for some analytic $f = u + iv$.

Let $g = e^{u+iv}$

$\Rightarrow g$ is entire function

$\Rightarrow |g| = |e^u||e^{iv}| = e^u$

$\Rightarrow g$ bdd above since u is

$\Rightarrow g$ bdd and entire

$\Rightarrow g$ constant by Louiville

$\Rightarrow f$ is constant

$\Rightarrow u$ is constant. $\square$

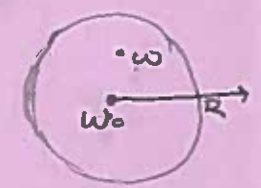
if dont know anything about 0's of f consider e^f

4.5.2 Show if $f(z)$ is entire and there is a nonempty disk s.t. $f(z)$ doesn't attain any values in disk then f is constant.

Pf Let f attain no values in $A = \{|w_0 - w| < R \quad w \in \mathbb{C},\ R > 0\}$

$\Rightarrow f(z) - w \neq 0$ for any $w \in A$ and $\forall z \in \mathbb{C}$

$\Rightarrow \dfrac{1}{f(z)-w} = g$ is entire



$\Rightarrow |g| = \dfrac{1}{|f(z)-w|}$ is bdd

$\Rightarrow g$ entire and bounded

$\Rightarrow g$ constant by Louiville

$\Rightarrow f$ is constant

$\square$

4.5.3 Prove the only doubly periodic functions are constant

Pf Let f be doubly periodic

$\Rightarrow \exists w_0 \neq w_1$ s.t. $f(z+w_0) = f(z+w_1) = f(z) \quad \forall z \in \mathbb{C}$

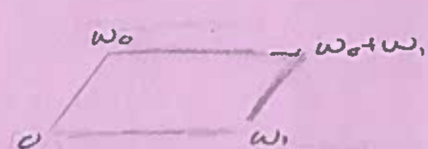
Let P = parallelogram w/ vertices $0, w_0, w_1, w_0+w_1$

$\Rightarrow \forall z \in \mathbb{C}, \quad z = z_0 + m w_0 + n w_1 \quad z_0 \in P,\ m, n \in \mathbb{N}.$

$\Rightarrow |f(z)| = |f(z_0)| \leq M$ since $z_0 \in P$ for some M

$\Rightarrow f$ is bounded and entire

$\Rightarrow f$ is constant by Louiville



4.5.4 Let f be an entire function. f/z^n bounded for $\{|z|<R\}$. Show f is is a poly of degree at most n. What if f/z^n is bdd on entire pa

Pf f/z^n bounded for $\{|z|<R\}$

$\Rightarrow |f/z^n| < M$

$\Rightarrow |f(z)| < M|z|^n$

Fix $z \in \mathbb{C}$. Consider $|\xi - z| = \rho$

$$\Rightarrow \rho - |z| = |\xi - z| - |z| \le |\xi - z + z| = |\xi| = |\xi - z + z| \le |\xi - z| + |z| = \rho + |z|$$

$f^{(n)}(z) \le \frac{n!\, M}{\rho}$ where $f(z) \le M$ on $|z| < \rho$

$\Rightarrow |f(\xi)| \le M|\xi|^n \le M(\rho + |z|)^n$

$\Rightarrow |f^{(n+1)}(\xi)| \le \frac{(n+1)!\, M(\rho+|z|)^n}{\rho^{n+1}} \to 0$ as $\rho \to \infty$

$\Rightarrow f^{(n+1)} \equiv 0$

$\Rightarrow f$ is a poly of degree at most n.

f/z^n bdd on $\mathbb{C}$

$\Rightarrow f = a_0 + a_1 z + \dots + a_n z^n$

$\Rightarrow |a_0 + a_1 z + \dots + a_n z^n| \le M|z|^n$

$\Rightarrow a_0 = 0$ if we let $z \to 0$

$\Rightarrow |a_1 + \dots + a_n z^{n-1}| \le M|z|^{n-1}$ $\forall z \ne 0$

$\Rightarrow a_1 = 0$ etc

$\vdots$

$\Rightarrow a_0 = a_1 = \dots = a_{n-1} = 0$

$\Rightarrow f = a_n z^n$. $\square$

4.6.1 Let L be line in complex plane. Suppose $f \in \mathbb{C}$ continuous on D analytic on $D \setminus L$. Show f analytic on D.

Pf Simply rotate f s.t. $L = \mathbb{R}$. call f_θ

$\Rightarrow f_\theta$ is analytic on $D \setminus \mathbb{R}$

$\Rightarrow f_\theta$ is analytic on D

rotate back

$\Rightarrow f$ analytic on D $\square$

4.6.2 Let h be continuous on $[a,b]$. Show $H(z) = \int_a^b h(t) e^{-itz} dt$ is entire and $|H(z)| \leq C e^{A|y|}$ for some A, C.

Pf h continuous on $[a,b]$

$\Rightarrow h(t) e^{-itz}$ continuous on $\mathbb{C}$.

$\Rightarrow H$ entire on $\mathbb{R}$.

h continuous on $[a,b]$.

$\Rightarrow |h| \leq M$ for some M.

Let $C = M(b-a)$, $A = \max\{|a|, |b|\}$,

$$
\begin{aligned}
\Rightarrow |H(z)| &= \left| \int_a^b h(t) e^{-itz} dt \right| \\
&\leq \int_a^b |h(t)| \, |e^{-itz}| \, |dt| \\
&\leq \int_a^b M e^{|yt|} dt \\
&\leq \int_a^b M e^{A|y|} dt \\
&= M(b-a) e^{A|y|}. \\
&= C e^{A|y|}
\end{aligned}
$$

$\square$

4.6.3 Show $H(z)$ (as in 4.6.2) is bdd in lower half plane if h is cont. on $[a,b] \subset [0,A]$

Pf $H(z) = \int_a^b h(t) e^{-itz} dt$

Assume h is continuous on $[a,b] \subset [0,\infty]$

$|H(z)| \leq C e^{A|y|}$

In LHP

4.8.1 Show $\frac{\partial}{\partial z} z = 1$, $\frac{\partial}{\partial \bar{z}} z = 0$ $\frac{\partial}{\partial z} \bar{z} = 0$ $\frac{\partial}{\partial \bar{z}} \bar{z} = 1$

Pf $\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right]$ $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right]$

$\frac{\partial}{\partial z}(x+iy) = \frac{1}{2}(1 - ii) = 1 \Rightarrow \frac{\partial}{\partial z} z = 1$

$\frac{\partial}{\partial \bar{z}}(x+iy) = \frac{1}{2}(1 + ii) = 0 \Rightarrow \frac{\partial}{\partial \bar{z}} z = 0$

$\frac{\partial}{\partial z}(x-iy) = \frac{1}{2}(1 - i(-i)) = 0 \Rightarrow \frac{\partial}{\partial z} \bar{z} = 0$

$\frac{\partial}{\partial \bar{z}}(x-iy) = \frac{1}{2}(1 + i(-i)) = 1 \Rightarrow \frac{\partial}{\partial \bar{z}} \bar{z} = 1$

$\square$

4.8.2 Compute $\frac{\partial}{\partial \bar{z}}(az + bz\bar{z} + c\bar{z}^2)$
Use to find where its complex differentiable and where analytic

Pf $\frac{\partial}{\partial \bar{z}}(az+bz\bar{z}+c\bar{z}^2) = bz + 2c\bar{z}$

Complex differentiable if $bz+2c\bar{z}=0$

$bz+2c\bar{z}=0 \Rightarrow b=c=0 \Rightarrow f$ entire.

if $b \neq 0 \Rightarrow f$ analytic at $z = \frac{-2c\bar{z}}{b}$. $\square$

4.8.4 Show $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Then show h harmonic $\overset{(a)}{\Leftrightarrow} \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0$

$\overset{(b)}{\Leftrightarrow} \frac{\partial h}{\partial z}$ is analytic

$\overset{(c)}{\Leftrightarrow} \frac{\partial h}{\partial \bar{z}} = 0$

$\overset{(d)}{\Rightarrow}$ m^{th} order partial is linear comb of $\frac{\partial^m h}{\partial z^m}$ and $\frac{\partial^m h}{\partial \bar{z}^m}$

Pf $4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

(a) clear
(b) h harmonic

4.8.3. Show Jacobian on smooth f is

$$\det J_f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2$$

Pf $J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ for $f = u + iv$

$$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right| = \frac{1}{4}|f_x - i f_y|^2 - \frac{1}{4}|f_x + i f_y|^2$$

4.8.5 Show $df = \frac{df}{dz}dz + \frac{df}{d\bar{z}}d\bar{z}$

Pf
$$\begin{aligned}
\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} &= \frac{1}{2}(f_x - i f_y)(dx + i dy) + \frac{1}{2}(f_x + i f_y)(dx - i dy) \\
&= \frac{1}{2}f_x dx - \frac{i}{2}\cancel{f_y}dx + \frac{i}{2}f_x dy + \frac{1}{2}f_y dy \\
&\quad + \frac{1}{2}f_x dx + \frac{i}{2}f_y \cancel{d}y - \frac{i}{2}\cancel{f_x}dy + \frac{1}{2}f_y dy \\
&= f_x dx + f_y dy \\
&= df.
\end{aligned}$$

□

Gamelin Chapter 5

5.3 # 4

5.4 # 1ab, 2, 12, 14

5.5 # 1a 2

5.7 # 1abcd, 2, 3, 6, 9, 11

5.3.4 Show that $f(z) = \sum z^{n!}$ is analytic on the open disk $\{|z|<1\}$. Show $|f(r\lambda)| \to \infty$ as $r \to 1$ whenever λ is a root of unity

Pf Let $f(z) = \sum z^{n!} = \sum a_n z^n$ where $a_n = \begin{cases} 1 & n = k! \text{ some } k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$

radius of Convergence $= R$

$$= \frac{1}{\limsup \sqrt[n]{a_n}}$$

$$= \frac{1}{1} = 1$$

$\Rightarrow f$ is analytic on $\{|z|<1\}$

Let λ be a root of unity

$\Rightarrow \lambda^k = 1$ for some $k \neq 0$

$$\Rightarrow |f(r\lambda)| = \left|\sum (r\lambda)^{n!}\right|$$

$$\geq \left|\sum_k r^{n!}(\lambda^k)^{n!/k}\right| - \left|\sum_0^{k-1} \underbrace{r^{n!}\lambda^{n!}}_{\leq 1}\right|$$

$$\geq \left|\sum_k^\infty r^{n!}\right| - \left|\sum_0^{k-1} 1\right|$$

$$\geq \sum_k^{k+N} r^{n!} - k \qquad (r \geq 0) \ \forall N \in \mathbb{N}$$

$$\Rightarrow \lim_{r\to 1} |f(r\lambda)| \geq \lim \left|\sum_k^{k+N} r^{n!} - k\right| = N - k \to \infty \text{ as } N \to \infty$$

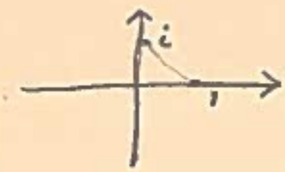
$$\Rightarrow \lim_{r \to 1} |f(r\lambda)| = \infty$$

$\square$.

♡

5.4.1 Find RoC of $\frac{1}{z-1}$ about i and $\frac{1}{\cos z}$ about 0

Pf Consider $\frac{1}{z-1} = f(z)$.



$\Rightarrow f$ has singularity at $z=1$

$\Rightarrow d(1,i) = \sqrt{1^2 - i^2} = \sqrt{2}$

$\Rightarrow$ RoC $= \sqrt{2}$

Consider $\frac{1}{\cos z} = g(z)$

$\Rightarrow g$ has singularities at $z = \frac{(2k+1)\pi}{2}$

$\Rightarrow$ distance from 0 to closest singularity is $\frac{\pi}{2}$

$\Rightarrow$ RoC $= \frac{\pi}{2}$

$\square$

5.4.2 Show RoC of $f(z) = \frac{z^2-1}{z^3-1}$ about $z=2$ is $\sqrt{7}$.

Pf $f = \frac{z^2-1}{z^3-1} \Rightarrow f = \frac{(z-1)(z+1)}{(z-1)(z^2+z+1)} = \frac{(z-1)(z+1)}{(z-1)(x+\frac{1}{2}-\frac{\sqrt{3}}{2})(x+\frac{1+\sqrt{3}i}{2})}$

$\Rightarrow d(2, \frac{-1}{2}+\frac{\sqrt{3}}{2}i) = \sqrt{(2+1/2)^2+(\frac{\sqrt{3}}{2})^2} = \sqrt{7}$.

$\Rightarrow$ RoC $= \sqrt{7}$

$\square$

5.4.12 Suppose $f = \sum a_n z^n$ is analytic. Show f even $\Rightarrow a_n = 0$ for odd n and f odd $\Rightarrow a_n = 0$ for even n.

Pf $f = \sum a_n z^n$ analytic $\Rightarrow f(-z) = \sum (-1)^n a_n z^n$ is analytic

f even $\Rightarrow f(z) - f(-z) = 0$

$\Rightarrow \sum a_n z^n (1-(-1)^n) = 0$

$\Rightarrow a_n = 0$ if n is odd

f odd $\Rightarrow f(z) + f(-z) = 0$

$\Rightarrow \sum a_n z^n (1+(-1)^n) = 0$

$\Rightarrow a_n = 0$ if n even.

$\square$

<u>6.4.14</u> Let f be a cont function on $\mathbb{T} = \{|z|=1\}$. Show f can be approximated uniformly on $\mathbb{T}$ by a sequence of polynomials in z $\iff$ f has extension F that is cont. on $\{|z| \le 1\}$ and analytic on interior.

<u>Pf</u> ($\Rightarrow$) Let $\{p_n\} \xrightarrow{u} f$ on $\mathbb{T}$

$\Rightarrow p_n$ cont on $\{|z| \le 1\}$ and analytic on $\{|z|<1\}$.

$\Rightarrow F$ is similary continuous and analytic.

WTS $p_n \xrightarrow{u} f$ on $\{|z|<1\}$

$\sup_{z \in \{|z| \le 1\}} |p_n - p_m| \le \sup_{z \in \mathbb{T}} |p_n - p_m|$ by Max principle.

$\Rightarrow \sup_{z \in \mathbb{T}} |p_n - p_m| < \varepsilon$ for any ε m,n large enough. since p_n converges uniformly on $\mathbb{T}$

$\Rightarrow F(z) = \lim p_n(z)$ is well defined. on $\{|z| \le 1\}$ and continuous $\{|z| \le 1\}$ and analytic on interior

($\Leftarrow$) Let F be extension as claimed.

Let $0<r<1$ and $F_r(z) = F(rz)$ analytic on $\{|z|<1\}$

Since F u.c. (cont on compact set.)

$\Rightarrow \forall \varepsilon > 0$ $\exists r \in (0,1)$ s.t. $|F_r(z) - f(z)| = |F(rz) - f(z)| < \varepsilon/2$

$\Rightarrow \exists N$ s.t. $|F_r(z) - \sum_1^N a_n r^n z^n| < \varepsilon/2$ since $F_r(z)$ analytic

$\Rightarrow \forall z \in \mathbb{T}$ $\sup|f(z) - \sum_1^N a_n r^n z^n| \le \sup|f(z) - F_r(z)| + \sup|F_r(z) - \sum a_n r^n z^n|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\Rightarrow f$ can be uniformly approximated by the polynomials $F_r(z) = \sum_1^n a_n r^n z^n$.

$\square$

5.5.1a Expand $\frac{1}{z^2+1}$ in power series about ∞

Pf $\frac{1}{z^2+1} = \frac{1}{z^2}\left(\frac{1}{1+\frac{1}{z^2}}\right)$

$= \frac{1}{z^2} \sum \left(\frac{-1}{z^2}\right)^n$

$= \sum (-1)^n z^{-(2n+2)}$

$\square$

5.5.2 Suppose f analytic at ∞, $f(z) = \sum \frac{b_k}{z^k}$ with $f(\infty) = b_0$ and $f'(\infty) = b_1$. Show $f'(\infty) = \lim z|f(z) - f(\infty)|$

Pf $f(z) = \sum \frac{b_k}{z^k} = f(\infty) + \frac{f'(\infty)}{z} + \sum_2^\infty \frac{b_k}{z^k}$

$\Rightarrow z|f(z) - f(\infty)| = z\left|\frac{f'(\infty)}{z} + \sum_2^\infty \frac{b_k}{z^k}\right|$

$= \left|f'(\infty) + \sum_2^\infty \frac{b_k}{z^{k-1}}\right|$

$= f'(\infty) + \sum_1^\infty \frac{b_{k+1}}{z^k}$

$\to f'(\infty)$ as $z \to \infty$

$\square$

5.7.1 Find zeros and orders of zeros of

(a) $\frac{z^2+1}{z^2-1}$ (b) $\frac{1}{z}+\frac{1}{z^5}$ (c) $z^2 \sin z$ (d) $\cos z - 1$

Pf (a) $\frac{z^2+1}{z^2-1} = \frac{(z+i)(z-i)}{(z+1)(z-1)}$ so has simple poles at $z=\pm 1$

analytic at ∞

(b) $\frac{1}{z}+\frac{1}{z^5} = \frac{1}{z^5}(z^4+1) = \frac{1}{z^5}(z^2-i)(z^2+i)$

$$\left.\begin{array}{l} z^2-i=0 \Rightarrow z^2=e^{i\pi/2} \Rightarrow z=\pm e^{i\pi/4} \\ z^2+i=0 \Rightarrow z^2=e^{3\pi i/2} \Rightarrow z=\pm e^{3\pi i/4} \end{array}\right\} \text{simple zeros.}$$

$\frac{1}{z^5}=0 \Rightarrow z=\infty$ is a simple zero.

(c) $z^2 \sin z$

$z=0$ is a triple 0

$z=n\pi$ $n=\pm 1, \pm 2, \pm 3 \dots$ are simple zeros

not analytic at ∞.

(d) $\cos z - 1$

$\Rightarrow \cos z = 1$ at $z=n\pi$ for $n \in \mathbb{N}$

$\Rightarrow \frac{d}{dz}\cos z - 1 = -\sin z = 0$ at $z=n\pi$ $n \in \mathbb{N}$

$\frac{d^2}{dz^2}\cos z - 1 = -\cos z \neq 0$ at $z=n\pi$ $n \in \mathbb{N}$

$\Rightarrow z=n\pi$ are double zeros.

$\square$

5.7.3 Show all zeros of $\sin z$ and $\tan z$ are simple.

Pf $\sin z = 0 \Rightarrow z = n\pi$ for $n \in \mathbb{N}$

$\frac{d}{dz}\sin z\Big|_{n\pi} = \cos(n\pi) \neq 0$

So $\sin z$ has simple zeros.

Similarly zeros of $\tan z$ are simple. $\square$

5.7.6 Suppose f analytic on D and $z_0 \in D$.
Show if $f^{(m)}(z_0) = 0$ for $m \geq 1$, then f is constant on D

Pf f analytic $\Rightarrow f(z) = \sum a_n(z-z_0)^n$ on some $\Delta(z_0, r)$ $r>0$.

$f^{(m)}(z_0) = 0 \Rightarrow a_n = \dfrac{f^{(n)}(z_0)}{n!} = 0$

$\Rightarrow f(z_0) = a_0(z-z_0)^0 + \sum 0\,(z-z_0)^n = a_0$

$\Rightarrow f(z_0) = a_0 \quad \forall z \in \Delta(z_0, r)$

$\Rightarrow f(z_0) = a_0$ on D by uniqueness.

$\square$

5.7.9 Show an analytic f has zero of order N at z_0 $\Rightarrow f(z) = g(z)^N$ for some analytic g near z_0 with $g'(z) \neq 0$

Pf f has zero of order N at z_0

$\Rightarrow f = (z-z_0)^N h(z)$ where $h(z_0) \neq 0$

equivalently $f^{(N)}(z_0) \neq 0$

f analytic

$\Rightarrow f = \sum_0^\infty a_n (z-z_0)^n$

$\Rightarrow f^{(N)} = \sum_0^\infty n \cdots (n-N+1)\, a_n (z-z_0)^{n-N}$ $f'(z) = N g(z)^{N-1} g'(z)$

5.7.11 Show if $f(z)$ is nonconstant analytic on a domain D then the image under $f(z)$ of any open set is open.

Gamelin Chapter 6

6.1 # 1ab, 4.5

6.2 # 1abef, 2ac, 5, 6, 7, 12

6.3 # 2, 3.

6.1.1 Find Laurent expansion centered at 0 of

a) $\frac{1}{z^2-z}$ b) $\frac{z-1}{z+1}$

Pf a) $\frac{1}{z^2-z} = \frac{1}{z(z-1)}$

On I : $0<|z|<1$

$$\frac{1}{z(z-1)} = \frac{-1}{z}\left(\frac{1}{1-z}\right)$$
$$= \frac{-1}{z}\sum_0^\infty z^k$$
$$= \sum_0^\infty -z^{k-1}$$
$$= \sum_1^\infty -z^k$$

on II: $|z|>1$

$$\frac{1}{z^2-z} = \frac{1}{z^2}\,\frac{1}{1-\frac{1}{z}}$$
$$= \frac{1}{z^2}\sum\left(\frac{1}{z}\right)^k$$
$$= \sum_{k=0}^\infty \frac{1}{z^{k+2}}$$
$$= \sum_{k=-2}^\infty \frac{1}{z^k}$$
$$= \sum_{-\infty}^{-2} z^k$$

b) $\frac{z-1}{z+1} = \frac{(z+1)-z}{z+1} = 1 - \frac{z}{z+1}$

On I: $0<|z|<1$

$$1-\frac{2}{z+1} = 1-\sum 2(-z)^n$$
$$= 1-\sum_0^\infty 2(-1)^n z^n$$
$$= 1-2\sum_0^\infty (-1)^n z^n$$

On II $|z|>1$

$$1-\frac{2}{z+1} = 1-\frac{1}{z}\,\frac{2}{1+\frac{1}{z}}$$
$$= 1-\frac{2}{z}\sum\left(-\frac{1}{z}\right)^n$$
$$= 1-2\sum_0^\infty (-1)^n z^{-n-1}$$
$$= 1-2\sum_{-\infty}^{-1} (-1)^n z^n$$

$\square$

6.1.4 Suppose $f(z) = f_0(z) + f_1(z)$ is Laurent Decomp of analytic $f(z)$ on $\{A < |z| < B\}$ Show if $f(z)$ is an even function then $f_0(z)$ and $f_1(z)$ are even function and Laurent series of f has only even powers of z. Similarly for odd fcns

Pf Let f be an analytic even function.

$\Rightarrow f(z) = f(-z) = \sum a_n (-1)^n z^n$

$\Rightarrow a_n = a_n(-1)^n$ by uniqueness of expansion

$\Rightarrow a_n = 0$ if n is odd

$\Rightarrow f_0(z) + f_1(z) = \sum a_n z^n$ where $a_n = 0$ if n is odd,

$\Rightarrow f_0(z) + f_1(-z) = \sum a_n z^n$

$\Rightarrow \sum (-1)^k b_k z^k + \sum (-1)^k c_k z^k = \sum a_n z^n$

$= \sum (-1)^k (b_k + c_k) z^k = \sum a_n z^n$

$\Rightarrow (-1)^k (b_k + c_k) = a_n = b_k + c_k$

$\Rightarrow (-1)^k = 0$ if k odd

$\Rightarrow f_0, f_1$ even.

Similarly for odd f

6.1.5 Suppose f analytic on $D = \mathbb{C}\setminus\{0\}$. Show if $\exists c$ s.t. $f(z) - c/z$ has a primitive in D. Give formula for c in terms of integral of f

Pf f analytic $\Rightarrow f(z) = \sum a_k z^k$

$$\Rightarrow a_k = \frac{1}{2\pi i}\int_{B_r(0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}}\, d\xi$$

Let $c = a_{-1} = \frac{1}{2\pi i}\int_{B_r(0)} f(\xi)\, d\xi$

$$\Rightarrow f(z) - c/z = \sum_{k\neq -1} a_k z^k$$

having a primitive means $\int = 0$ on any smooth closed path

WTS $f(z) - c/z$ has a primitive,

ie WTS $\int_\gamma f(z) - c/z = 0$ $\forall$ piecewise smooth closed $\gamma \in D$

Let γ be such a path.

$\Rightarrow \gamma \subset \{z : r \le |z| \le s\}$ for some $0 < r < s$ contained in some annulus since γ on $\mathbb{C}\setminus\{0\}$

$\sum_{k\neq -1} a_k z^k$ converges uniformly on $\{z : r \le |z| \le s\}$ since compact

$\Rightarrow \sum_{k\neq -1} a_k z^k$ converges uniformly on γ.

$$\Rightarrow \int_\gamma f(z) - c/z\, dz = \int_\gamma \sum_{k\neq -1} a_k z^k$$

$= \sum_{k\neq -1} \int_\gamma z^k dz$ since converges uniformly

$= \sum_{k\neq -1} a_k\, 0$ since z^k has primitive if $k\neq -1$

$= 0$

$\Rightarrow f(z) - c/z$ has primitive in D

$\square$

6.2.1 Find isolated singularities, type, order, principal

a) $\dfrac{z}{(z^2-1)^2}$ b) $\dfrac{ze^z}{z^2-1}$ c) $z^2\sin\left(\frac{1}{z}\right)$ d) $\dfrac{\cos z}{z^2-\pi^2/4}$

Pf a) $\dfrac{z}{(z^2-1)^2} = \dfrac{z}{(z+1)^2(z-1)^2} = f(z)$

$z=-1$ is a pole of order 2

$\left(f(z) = \dfrac{g(z)}{(z+1)^2} \right.$ where g analytic at $z=1)$

$z=1$ is also pole of order 2.

b) $\dfrac{ze^z}{z^2-1} = \dfrac{ze^z}{(z+1)(z-1)}$

$z=\pm 1$ are poles of order 1.

c) $z^2 \sin\left(\frac{1}{z}\right)$

There are no singularities away from 0

$z^2\sin\left(\frac{1}{z}\right) = z^2 \sum \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1}$

$= \sum \frac{(-1)^k}{(2k+1)!} z^{1-2k}$

There are infinitely many negative terms

$\Rightarrow$ $z=0$ is an essential singularity.

d) $\dfrac{\cos z}{z^2-\pi^2/4} = \dfrac{\cos z}{(z-\pi/2)(z+\pi/2)}$

$\Rightarrow$ f has isolated singularities at $z=\pm\pi/2$.

Since $\lim_{z\to\pm\pi/2} f(z) = -1/\pi$

$\Rightarrow$ $z=\pm\pi/2$ are isolated singularities

$\square$

6.2.2 Find RoC of $\frac{z-1}{z^4-1}$ about $3+i$ and $\frac{z}{\sin z}$ about πi.

Pf Consider $\frac{z-1}{z^4-1} = \frac{z-1}{(z^2-1)(z^2+1)} = \frac{z-1}{(z-1)(z+1)(z+i)(z-i)}$

$\Rightarrow z=1$ is a removable singularity

$\Rightarrow$ isolated singularities at $z=-1, z=\pm i$

$\text{RoC} = \min\{d(3+i,-1), d(3+i,i), d(3+i,-i)\}$.

$d(3+i,-1) = \sqrt{(3+1)^2+1^2} = \sqrt{17}$

$d(3+i,i) = \sqrt{3^2+0^2} = \sqrt{9} = 3$

$d(3+i,-i) = \sqrt{3^2+2^2} = \sqrt{13}$

$\text{RoC} = 3$

Consider $\frac{z}{\sin z}$

$\Rightarrow z=n\pi$ are singularities.

$\Rightarrow z=0$ is removable.

$\text{RoC} = \min\{d(\pi i, n\pi)\ n\neq 0\}$

$d(\pi i, \pi) = \sqrt{(n\pi)^2+\pi^2} = \pi\sqrt{n+1}$

$\Rightarrow$ RoC is $\pi\sqrt{2}$.

$\square$

6.2.5 By estimating the coefficients of the Laurent series prove if z_0 is an isolated singularity of f and if $(z-z_0)f(z) \to 0$ as $z \to z_0$ then z_0 is removable.

Pf Let f be analytic

$$\Rightarrow f = \sum_{-\infty}^{\infty} a_k z^k$$

$$\Rightarrow a_k = \frac{1}{2\pi i} \int_{|z-z_0|} \frac{f(\xi)}{(\xi - z)^{k+1}} \, d\xi$$

6.2.7 Prove if z_0 is an isolated singularity of f, and if $(z-z_0)^N f(z)$ is bounded near z_0 then z_0 is removable or a pole of at most N.

Pf Let z_0 be an isolated singularty of f.

$\Rightarrow (z-z_0)^N f(z)$ is bounded near z_0 by assumption

$\Rightarrow \exists r$ s.t. $(z-z_0)^N f(z) = \sum_0^\infty a_k (z-z_0)^k \quad \forall\, 0<|z-z_0|<r$

by Riemanns Thm

$\Rightarrow f(z) = \sum_0^\infty a_k (z-z_0)^{k-N}$

$= \sum_0^\infty a_{j+N} (z-z_0)^j$

$\Rightarrow z_0$ is a pole of order at most n

unless $a_k = 0$, $k = 0, 1, \dots, N-1$

$\Rightarrow z_0$ is a removable singularity

$\square$

6.2.12 Show if z_0 is an isolated singularity of $f(z)$ not removable then z_0 is an essential singularity for $e^{f(z)}$.

Pf Let z_0 be an isolated singularity of $f(z)$

Assume BWOC z_0 is a pole of $e^{f(z)}$

$\Rightarrow z_0$ a removable singularity of $e^{-f(z)}$

$\Rightarrow z_0$ is a removable singularity of $-f$

$\Rightarrow z_0$ is a removable singularity of f

Assume BWOC z_0 is a removable singularity of e^{f}

$\Rightarrow |e^{f(z)}| < K$ in some punctured nbhd of z_0, $D_\varepsilon(z_0)$

$\Rightarrow e^{\operatorname{Re} f} < K$

$\Rightarrow \operatorname{Re} f < \log K$ in $D_\varepsilon(z_0)$

$\Rightarrow z_0$ is removable singularity of f.

$\therefore$ z_0 is an essential singularity of $e^{f(z)}$

$\square$

6.3.2 Suppose f entire function not a polynomial. What kind of singularity can f have at ∞.

Pf f entire $\Rightarrow \sum_0^\infty a_n z^n$ $\forall n\ \exists N>n$ s.t $a_N \neq 0$ since f not a poly

$\Rightarrow g(w) = f(\frac{1}{w}) = \sum_{-\infty}^{0} a_n w^n$

$\Rightarrow g$ has essential singularity at 0

$\Rightarrow f$ has essential singularity at ∞

$\square$

6.3.3 If $f(z)$ is nonconstant entire function then $e^{f(z)}$ has an essential singularity at ∞.

Pf Assume $e^{f(z)}$ has a pole at ∞

$\Rightarrow e^{-f(z)}$ has a zero at ∞

$\Rightarrow e^{-f(z)}$ is constant since bdd & entire

$\Rightarrow f$ is constant which contradicts.

Assume $e^{f(z)}$ has removable singularity at ∞

$\Rightarrow e^{f(z)} = \sum_{-\infty}^{0} b_k z^k$

$\Rightarrow e^{f(z)}$ is bounded near ∞.

$\Rightarrow f$ entire since $e^{f(z)}$ is bounded everywhere

$\Rightarrow e^{f(z)}$ is constant

$\Rightarrow f(z)$ is constant which contradicts

$\therefore e^{f(z)}$ has essential singularity at ∞

$\square$

Gamelin Chapter 7

7.1 # 1acde, 2a, 3ab
7.2 # 1, 2, 4, 5, 7 8
7.3 # 1, 2, 4
7.4 # 1, 2, 3
7.5 # 1, 2, 4
7.6 # 1, ~~3~~
7.7 # 12, 3
7.8 # 1abce.

7.1.1 Evaluate the following Residues
a) $\mathrm{Res}\left(\frac{1}{z^2+4}, 2i\right)$ c) $\mathrm{Res}\left(\frac{1}{z^5-1}, 1\right)$ d) $\mathrm{Res}\left(\frac{\sin z}{z^2}, 0\right)$ e) $\mathrm{Res}\left(\frac{\cos z}{z^2}, 0\right)$

Pf a) $\mathrm{Res}\left(\frac{1}{z^2+4}, 2i\right)$

$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$ $\Rightarrow$ $2i$ is a simple pole

$\Rightarrow \mathrm{Res}\left(\frac{1}{z^2+4}, 2i\right) = \frac{1}{z+2i}\Big|_{2i} = \frac{1}{4i}$

c) $\mathrm{Res}\left(\frac{1}{z^5-1}, 1\right)$

$\frac{1}{z^5-1} = \frac{1}{(z-1)(z^4+z^3+z^2+z+1)}$ $\Rightarrow$ 1 is a simple pole

$\Rightarrow \mathrm{Res}\left(\frac{1}{z^5-1}, 1\right) = \frac{1}{z^4+z^3+z^2+z+1}\Big|_{z=1} = \frac{1}{5}$

d) $\mathrm{Res}\left(\frac{\sin z}{z^2}, 0\right)$

$\frac{\sin z}{z^2} = \frac{1}{z^2}\sum \frac{(-1)^{2n+1}}{(2n+1)!} z^{2n+1} = \sum \frac{(-1)^n}{(2n+1)!} z^{2n-1}$

$\Rightarrow \mathrm{Res}\left(\frac{\sin z}{z^2}, 0\right) = a_{-1} = \frac{(-1)^0}{1!} = 1$

e) $\mathrm{Res}\left(\frac{\cos z}{z^2}, 0\right)$

$\frac{\cos z}{z^2} = \frac{1}{z^2}\sum \frac{(-1)^n}{(2n)!} z^{2n} = \sum \frac{(-1)^n}{(2n)!} z^{2n-2}$

$2n-2 \neq 0 \ \forall n$ so $\mathrm{Res}\left(\frac{\cos z}{z^2}, 0\right) = 0$

or $\mathrm{Res}\left(\frac{\cos z}{z^2}, 0\right) = \frac{d}{dz}(z^2\cos z/z^2) = -\sin z\big|_{z=0} = 0.$

D

7.1.2a Calculate the residue at each singularity of $e^{1/z}$

Pf $z=0$ is only singularity of $e^{1/z}$

$$e^{1/z} = \sum \frac{z^{-k}}{k!}$$

$$\text{Res}(e^{1/z}, 0) = a_{-1} = \frac{1}{k!}\Big|_{k=1} = 1.$$

$\square$

7.1.3 Evaluate using Residue Thm a) $\int_{|z|=1} \frac{\sin z}{z^2}\,dz$. b) $\int_{|z|=2} \frac{z}{\cos z}\,dz$

Pf a) $\frac{\sin z}{z^2}$ has singularity at 0 and $\text{Res}\left(\frac{\sin z}{z^2}, 0\right) = 1$

$$\Rightarrow \int_{|z|=1} \frac{\sin z}{z^2}\,dz = 2\pi i \,\text{Res}\left(\frac{\sin z}{z^2}, 0\right) = 2\pi i$$

b) $\frac{z}{\cos z}$ has singularities at $\frac{(2k+1)\pi}{2}$

In $|z|=2$ the singularities are $\pm\frac{\pi}{2}$

$$\text{Res}\left(\frac{z}{\cos z}, \pm\pi/2\right) = \frac{z}{-\sin z}\Big|_{\pm\pi/2} = \pi/2$$

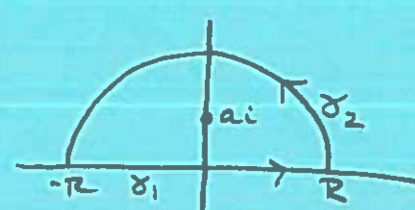
$$\Rightarrow \int_{|z|=2} \frac{z}{\cos z}\,dz = 2\pi i\left(\pi/2 + \pi/2\right) = 2\pi^2 i$$

$\square$

7.2.1 Show using Residue Theory that $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \pi/a$

Pf $\frac{1}{x^2+a^2} = f$ has singularities at $\pm ai$

Integrate f along γ_1, γ_2.



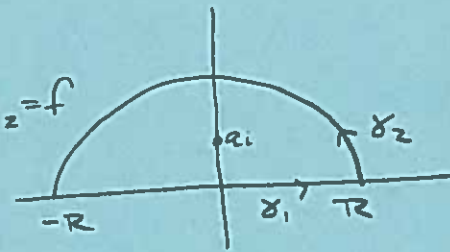
- $\int_{\gamma_1} \frac{dx}{x^2+a^2} = \int_{-R}^{R} \frac{dx}{x^2+a^2} \to \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = I$ as $R \to \infty$
- $\left|\int_{\gamma_2} \frac{dx}{x^2+a^2}\right| \le \frac{1}{R^2-a^2}\pi R \to 0$ as $R \to \infty$
- $\text{Res}\left(\frac{1}{x^2+a^2}, ai\right) = \lim_{x\to ai} \frac{x-ai}{x^2+a^2} = \frac{1}{x+ai}\Big|_{ai} = \frac{1}{2ai}$

$$2\pi i\,\text{Res}\left(\frac{1}{x^2+a^2}, ai\right) = \left(\int_{\gamma_2} + \int_{\gamma_1}\right) \frac{dx}{x^2+a^2} \to I$$

$$\therefore I = 2\pi i \cdot \frac{1}{2ai} = \pi/a$$

$\square$

7.2.2 $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$



Pf ai is a double pole of $\frac{1}{(z^2+a^2)^2} = f$

- $\displaystyle\int_{\gamma_1} f\,dx = \int_{-R}^{R} f\,dx \to \int_{-\infty}^{\infty} f\,dx$
- $\displaystyle\left|\int_{\gamma_2} f\,dx\right| \le \frac{1}{(R^2-a^2)^2}\pi R \to 0$ as $R \to \infty$.

$$\left(\int_{\gamma_1} + \int_{\gamma_2}\right) f\,dx = 2\pi i \cdot \operatorname{Res}(f, ai)$$
$$= 2\pi i \frac{d}{dz}\left(\frac{1}{(z+ia)^2}\right)\Big|_{ai}$$
$$= 2\pi i \; -2(z+ia)^{-3}\big|_{ai}$$
$$= -4\pi i\,(2ai)^{-3}$$
$$= \frac{-4\pi i}{8a^3 i^3} = \frac{\pi}{2a^3}$$

$\therefore$ as $R \to \infty$ we get $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$ $\square$

7.2.4 $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$

Pf $z^4+1=0 \Rightarrow z^4=-1 \Rightarrow z = e^{3\pi i/4}, e^{\pi i/4}$ are singularities of $f = \frac{1}{z^4+1}$ in upper half plane.

Now $\displaystyle 2\pi i\left[\operatorname{Res}(f, e^{i\pi/4}) + \operatorname{Res}(f, e^{3\pi i/4})\right] = \int_{-R}^{R}\frac{dx}{x^4+1} + \int_{\Gamma_R}\frac{dz}{z^4+1} \to \int_{-\infty}^{\infty}\frac{dx}{x^4+1}$

$z^4=-1$
$\Rightarrow \frac{1}{z^4+1}$

Since $\displaystyle\left|\int_{\Gamma_R}\frac{dz}{z^4+1}\right| \le \pi R\frac{1}{R^4-1} \to 0$ as $R \to \infty$

$$2\pi i\left[\operatorname{Res}(f, e^{i\pi/4}) + \operatorname{Res}(f, e^{3\pi i/4})\right] = 2\pi i\left[\frac{1}{4z^3}\Big|_{e^{3\pi i/4}} + \frac{1}{4z^3}\Big|_{e^{\pi i/4}}\right]$$
$$= 2\pi i\left[\frac{-z}{4}\Big|_{e^{3\pi i/4}} + \frac{-z}{4}\Big|_{e^{\pi i/4}}\right]$$
$$= -\pi i\,\frac{e^{3\pi i/4} + e^{\pi i/4}}{2}$$
$$= -\pi i\,\frac{e^{3\pi i/4} + e^{-3\pi i/4}}{2}$$
$$= -\pi i \cos 3\pi/4$$

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+a^2} = \frac{-i\pi}{\sqrt{2}}$$ $\square$

7.2.5 $\displaystyle\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$

Pf Note if $I = \int_0^\infty \frac{x^2}{x^4+1}dx$ then $2I = \int_{-\infty}^\infty \frac{x^2}{x^4+1}dx$.
As in 7.2.4, $e^{3\pi i/4}$ and $e^{\pi i/4}$ are singularities in $\mathbb{H}$

$$\Rightarrow 2I = 2\pi i\left[\operatorname{Res}(f, e^{i\pi/4}) + \operatorname{Res}(f, e^{3\pi i/4})\right] \quad \text{where } f = \frac{z^2}{z^4+1}$$

$$= 2\pi i\left[\frac{z^2}{4z^3}\Big|_{e^{i\pi/4}} + \frac{z^2}{4z^3}\Big|_{e^{3\pi i/4}}\right]$$

$$= 2\pi i\left[\frac{1}{4z}\Big|_{e^{i\pi/4}} + \frac{1}{4z}\Big|_{e^{3\pi i/4}}\right]$$

$$= \frac{\pi i}{2}\left[-z^3\Big|_{e^{i\pi/4}} - z^3\Big|_{e^{\pi i/4}}\right]$$

$$= \frac{-\pi i}{2}\; e^{3i\pi/4} + e^{9\pi i/4}$$

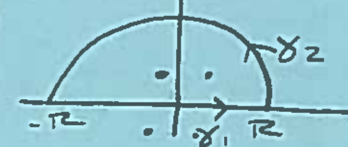
$$= \frac{-\pi i}{2}\; e^{-i\pi/4} + e^{\pi i/4}$$

$$= -\pi i \cos \pi/4 = \frac{-\pi i}{\sqrt{2}}$$

$$\Rightarrow I = \frac{-\pi i}{2\sqrt{2}}.$$

7.2.7 Show $\int_{-\infty}^{\infty} \frac{\cos ax}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}} e^{-a/\sqrt{2}}\left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}}\right)$

Pf Set $I = \int_{-\infty}^{\infty} \frac{\cos ax}{x^4+1}$ and $f(z) = \frac{e^{iaz}}{z^4+1}$



poles at $z^4+1=0 \Rightarrow z = \frac{\pm 1 \pm i}{\sqrt{2}}$

$$\text{Res}\left[f(z), \frac{1+i}{\sqrt{2}}\right] = \left.\frac{e^{iaz}}{4z^3}\right|_{z=\frac{1+i}{\sqrt{2}}}$$

$$= -\frac{\sqrt{2}e^{-a/\sqrt{2}}\cos(a/\sqrt{2})}{8} - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin(a/\sqrt{2})i}{8}$$

$$-\frac{\sqrt{2}e^{-a/\sqrt{2}}\cos(a/\sqrt{2})}{8}i + \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin(a/\sqrt{2})}{8} \quad *$$

$$\text{Res}\left[f, \frac{-1+i}{\sqrt{2}}\right] = \frac{\sqrt{2}e^{-a/\sqrt{2}}\cos(a/\sqrt{2})}{8} - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin(a/\sqrt{2})i}{8}$$

$$-\frac{\sqrt{2}e^{-a/\sqrt{2}}\cos(a/\sqrt{2})i}{8} - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin(a/\sqrt{2})}{8} \quad **$$

$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{1}{R^4-1}\pi R \to 0$ as $R\to\infty$.

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ax}}{x^4+1}\,dx$$

$$\to \int_{-\infty}^{\infty} \frac{\cos ax}{x^4+1} + i\int_{-\infty}^{\infty} \frac{\sin ax}{x^4+1} = I + i\int_{-\infty}^{\infty}\frac{\sin ax}{x^4+1}$$

By residue thm w/ $R\to\infty$

$$\Rightarrow I + i\int_{-\infty}^{\infty}\frac{\sin ax}{x^4+1} = 2\pi i(* + **)$$

$$\Rightarrow I = 2\pi i\left[-\frac{\sqrt{2}e^{-a/\sqrt{2}}\sin(a/\sqrt{2})}{4}i - \frac{\sqrt{2}e^{-a/\sqrt{2}}\cos(a/\sqrt{2})}{4}i\right]$$

$$= \frac{\pi e^{-a/\sqrt{2}}}{\sqrt{2}}\left(\cos\left(\frac{a}{\sqrt{2}}\right) + \sin\left(\frac{a}{\sqrt{2}}\right)\right)$$

$\square$

7.2.8 $\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\,dx = \pi/e$

Pf As in preceding exercises it suffices to find $2\pi i \sum \text{Res}(f, z_i)$ where $f = \frac{\cos z}{(1+z^2)^2}$

$z = i$ is only singularity in $\mathbb{H}$

$$2\pi i \,\text{Res}(f, i) = 2\pi i \frac{d}{dz} \frac{\cos z}{(i+z)^2}\Big|_{z=i}$$

$$= 2\pi i \frac{-(i+z)^2 \sin z - 2\cos z\,(i+z)}{(i+z)^4}\Big|_{z=i}$$

$$= 2\pi i \left(\frac{-(2i)^2 \sin i - 2\cos i\,(2i)}{(2i)^4} \right)$$

$$= \frac{2\pi i}{16}\; 4\sin i - 4i\cos i$$

$$= \frac{\pi i}{8}\; 4i\left(\frac{e}{2} - \frac{1}{2e}\right) - 4i\left(\frac{1}{2e} + \frac{e}{2}\right)$$

$$= \frac{-\pi}{2}\left(-\frac{1}{e}\right) = \frac{\pi}{2e} \qquad \square$$

7.3.1 Show $\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\,d\theta = 2\pi\left(1 - \frac{2}{\sqrt{3}}\right)$

Pf Let $z = e^{i\theta}$ $dz = iz\,d\theta \Rightarrow \frac{1}{2}(z + 1/z) = \cos\theta$

$$\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\,d\theta = \int_{|z|=1} \frac{\frac{1}{2}(z + 1/z)}{2 + 1/2\,(z + 1/z)} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{z^2+1}{4z + z^2 + 1} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_{|z|=1} \frac{z^2+1}{z^3 + 4z^2 + z}\,dz.$$

$$\text{Res}\left(\frac{z^2+1}{z^3+4z^2+z}, 0\right) = \frac{z^2+1}{3z^2+8z+1}\Big|_{z=0} = 1$$

$$\text{Res}\left(\frac{z^2+1}{z^3+4z^2+z}, -2+\sqrt{3}\right) = \frac{z^2+1}{3z^2+8z+1}\Big|_{-2+\sqrt{3}} = \frac{-2\sqrt{3}}{3}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\,d\theta = 2\pi i\left(1 - \frac{2}{\sqrt{3}}\right)$$

$\square$

7.3.2 Show $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2+b^2}}$

Pf $z=e^{i\theta}$ $dz = iz\,d\theta$ $\Rightarrow \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_{|z|=1} \frac{1}{a+b\frac{1}{2i}(z-1/z)} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{2}{2aiz+bz^2-b}\,dz$$

$$= \int_{|z|=1} \frac{2}{bz^2+2aiz-b}\,dz$$

Singularity at

$$z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2+b^2}}{b} i$$

Let $z_1 = \frac{-a+\sqrt{a^2+b^2}}{b} i$

$$\mathrm{Res}\left(\frac{2}{bz^2+2aiz-b}, z_1\right) = \frac{2}{2bz+2ai}\Big|_{z_1} = \frac{-i}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2+b^2}}$$

$\square$

7.3.4 Show $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \pi\sqrt{2}$

Pf First let $\theta = \pi - t$ $\Rightarrow \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_0^{2\pi} \frac{dt}{1+\sin^2 t}$

Now let $z = e^{it}$ $dz = iz\,dt$.

$$\Rightarrow \int_0^{2\pi} \frac{dt}{1+\sin^2 t} = \int_{|z|=1} \frac{1}{1+(\frac{1}{2i}(z+\frac{1}{z}))^2} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{-4z\,dz}{i(z^4-6z^2+1)}$$

$\Rightarrow$ Singularities at $z^4-6z^2+1=0$

$\Rightarrow$ singularities at $z = \pm 1 \pm \sqrt{2}$

$\Rightarrow$ Singularities in upper unit circle are $\pm(\sqrt{2}-1)$

$$\mathrm{Res}\left(\frac{z}{z^4-6z^2+1}, \pm(\sqrt{2}-1)\right) = \frac{z}{4z^3-12z}\Big|_{\pm(\sqrt{2}-1)} = \frac{-1}{8\sqrt{2}}$$

$$\Rightarrow \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = 2\pi i \left(\frac{-4}{i}\right)\left(\frac{-1}{8\sqrt{2}} + \frac{-1}{8\sqrt{2}}\right)$$

$$= \frac{8\pi}{4\sqrt{2}}$$

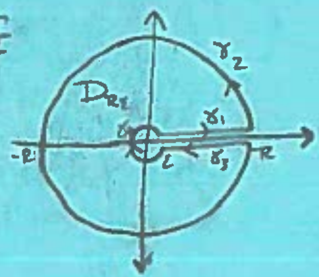
$$= \frac{2\pi}{\sqrt{2}}$$

$$= \sqrt{2}\pi.$$

$\square$

7.4.1 Integrate around keyhole contour to show $\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin \pi a}$.

Pf



$z=-1$ is a simple pole of $f(z) = \frac{z^{-a}}{1+z}$

$$\int_{\partial D_{R\varepsilon}} f(z)\,dz = 2\pi i \,\mathrm{Res}\left(\frac{z^{-a}}{1+z}, -1\right)$$
$$= 2\pi i \frac{z^{-a}}{1}\Big|_{z=-1}$$
$$= 2\pi i (-1)^{-a} \quad = 2\pi i e^{-a\pi i}$$

?

Alternatively, $\int_{\partial D_{R\varepsilon}} f(z)\,dz = \int_\varepsilon^R f(x) + \int_{|z|=\varepsilon} f(z) - \underbrace{\int_\varepsilon^R f(x) e^{-2\pi i a}}_{\text{why this term}} + \int_{|z|=R} f(z)$

$\left|\int_{\Gamma_R} \frac{z^{-a}}{1+z}\,dz\right| \le \int_{\Gamma_R} \left|\frac{z^{-a}}{1+z}\right| |dz| \le 2\pi R \frac{R^{-a}}{R-1} \to 0$ as $R \to \infty$

$\left|\int_{\gamma_\varepsilon} \frac{z^{-a}}{1+z}\,dz\right| \le \int_{\gamma_\varepsilon} \left|\frac{z^{-a}}{1+z}\right| |dz| \le 2\pi\varepsilon \frac{\varepsilon^{-a}}{1-\varepsilon} \to 0$ as $\varepsilon \to 0$

Let $R \to \infty$, $\varepsilon \to 0$

$\Rightarrow 2\pi i e^{-a\pi i} = \int_0^\infty \frac{x^{-a}}{1+x}\,dx - \int_0^\infty \frac{x^{-a} e^{-\pi i a}}{1+x}\,dx$

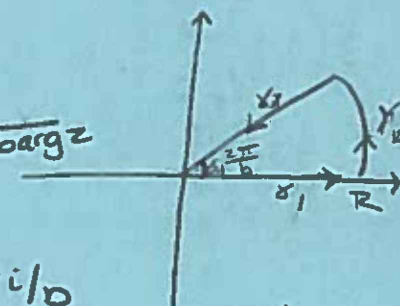
$\Rightarrow \int_0^\infty \frac{x^{-a}}{1+x} = \frac{2\pi i e^{-a\pi i}}{1-e^{-2\pi i a}} = \frac{2\pi i}{e^{ai\pi} - e^{\pi i a}} = \frac{2\pi i}{2i\sin \pi a} = \frac{\pi}{\sin \pi a}$

$\square$

7.4.2 By integrating around the bdry of a pie-slice domain of aperture $2\pi/b$ show that $\int_0^\infty \frac{dx}{1+x^b} = \frac{\pi}{b\sin \pi/b}$

Pf Set $I = \int_0^\infty \frac{dx}{1+x^b}$, $f(z) = \frac{1}{1+z^b}$

$$= \frac{1}{1+|z|^b e^{ib\arg z}}$$



$-\pi/2 < \arg z < 3\pi/2$

Residue at simple pole $z_1 = e^{\pi i/b}$

$$\mathrm{Res}(f, e^{\pi i/b}) = \frac{1}{bz^{b-1}}\Big|_{e^{\pi i/b}} = -\frac{1}{b} e^{\pi i/b}$$

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_1} \frac{1}{1+|z|^b e^{ib\arg z}} = \int_\varepsilon^R \frac{1}{1+x^b} dx$$

$$\to \int_0^\infty \frac{1}{1+x^b} dx = I$$

$$\left|\int_{\gamma_2} f(z)dz\right| \le \frac{1}{R^b - 1} \cdot \frac{2\pi R}{b} \to 0 \text{ as } R \to \infty$$

$$\int_{\gamma_3} f(z)dz = \int_R^\varepsilon \frac{1}{1+x^b} e^{2\pi i/b} dx$$

$$\to \int_\infty^0 \frac{1}{1+x^b} e^{2\pi i/b} = -e^{2\pi i/b} \int_0^\infty \frac{1}{1+x^b} dx = e^{\frac{2\pi i}{b}} I$$

$$\left|\int_{\gamma_4} f(z)dz\right| \le \frac{1}{1-\varepsilon^b} \frac{2\pi\varepsilon}{b} \to 0 \text{ as } \varepsilon \to 0$$

By residue thm w/ $R \to \infty$ $\varepsilon \to 0$. we get

$$I + 0 - e^{2\pi i/b} I + 0 = 2\pi i \frac{-1}{b} e^{\pi i b}$$

$$\Rightarrow I(e^{-\pi i/b} - e^{\pi i/b}) = \frac{-2\pi i}{b}$$

$$\Rightarrow I = \frac{\pi}{b} \frac{+2i}{e^{\pi i/b} - e^{-\pi i/b}}$$

$$= \frac{\pi}{b} \cdot \frac{1}{\sin \pi/b} \quad b > 1$$

7.4.3 By integrating around keyhole contour show

$$\int_0^\infty \frac{\log x}{x^a(x+1)}\,dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}$$

7.5.1 Use keyhole contour indented on lower edge of axis at $x=1$ to show

$$\int_0^\infty \frac{\log x}{x^a(x-1)}dx = \frac{2\pi^2}{1-\cos(2\pi a)}$$

7.5.2 Show using residue theory that $\int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2+1)} dx = \pi(1-e^{-a})$

7.5.4 Show using residue theory that $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi/2$

7.6.1 Integrate $\frac{1}{1-x^2}$ directly using partial fractions and show $PV\int_0^\infty \frac{dx}{1-x^2} = 0$

Show $\int_0^1 \frac{dx}{1-x^2} = \infty$ $\int_1^\infty \frac{dx}{1-x^2} = -\infty$.

Pf $\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

$\Rightarrow 1 = A(1+x) + B(1-x)$

$\Rightarrow A = 1/2, \quad B = -1/2$

$\Rightarrow \frac{1}{(1-x)(1+x)} = \frac{-1}{2(1+x)} + \frac{1}{2(1-x)}$

$$\int \frac{1}{(1-x)(1+x)} dx = \int \frac{1}{2(1+x)} - \frac{1}{2(1-x)} dx = \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(1+x)$$

$$PV\int_0^\infty \frac{dx}{1-x^2} = \lim_{\varepsilon\to 0} \int_0^{1-\varepsilon} \frac{dx}{1-x^2} + \int_{1+\varepsilon}^\infty \frac{dx}{1-x^2}$$

$$= \lim_{\varepsilon\to 0} 1/2\ln(1-x) - 1/2\ln(1+x)\Big|_0^{1-\varepsilon} + 1/2\ln(1-x) - 1/2\ln(1+x)\Big|_{1+\varepsilon}^\infty$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2}\ln\frac{\varepsilon}{2-\varepsilon} - \frac{1}{2}\ln(1) + \frac{1}{2}\ln\left(\frac{1-\infty}{1+\infty}\right) - \frac{1}{2}\ln\left(\frac{-\varepsilon}{2+\varepsilon}\right)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2}\cdot\ln\left(\frac{2-\varepsilon}{2+\varepsilon}\right) = 0$$

$$\int_0^1 \frac{dx}{1-x^2} = \int_0^{1-\varepsilon} \frac{dx}{1-x^2}$$

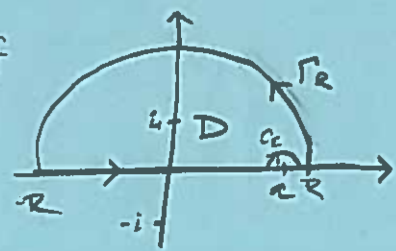
$$= \lim 1/2\left[\ln\left(\frac{1+x}{1-x}\right)\right]\Big|_0^{1-\varepsilon}$$

$$= \lim 1/2 \ln\frac{\varepsilon}{2-\varepsilon}$$

$=$

7.6.3 By integrating around bdry of indented half disk in $\mathbb{H}$ show $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}\,dx = -\frac{\pi a}{a^2+1}$

Pf



$PV \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{a-\varepsilon} + \int_{a+\varepsilon}^{\infty} \right) f(x)\,dx$

Let $f(z) = \frac{1}{(z^2+1)(z-a)}$

Let ε be small and R large consider figure above (D) with $|z|<R$ and $|z-a|<\varepsilon$. f has one pole, $z=i$, in D

$\text{Res}\left(\frac{1}{(z^2+1)(z-a)}, z=i\right) = \frac{1}{(z+i)(z-a)}\Big|_{z=i} = \frac{1}{2i(i-a)} = \frac{-1}{2+2ai}$

$\Rightarrow \int_{\partial D} f(z)\,dz = 2\pi i \frac{-1}{2+2ai} = \frac{\pi}{i-a}$.

alternatively, $\int_{\partial D} f\,dz = \left(\int_{\Gamma_R} + \int_{-R}^{a-\varepsilon} + \int_{C_\varepsilon} + \int_{a+\varepsilon}^{R} \right) f\,dx$

- By ML estimate $\left| \int_{\Gamma_R} f(z)\,dz \right| \le \frac{1}{(R^2+1)(R-a)} \pi R \to 0$ as $R \to \infty$
- $\text{Res}\left[\frac{1}{(z^2+1)(z-a)}, a\right] = \frac{1}{z^2+1}\Big|_a = \frac{1}{a^2+1}$

$\Rightarrow \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{1}{(z^2+1)(z-a)}\,dz = \frac{-\pi i}{a^2+1}$ by fractional Residue thm with angle $-\pi$

Let $\varepsilon \to 0$, $R \to \infty$

$\Rightarrow \int_{\partial D} f(z)\,dz = PV \int_{-\infty}^{\infty} f(x)\,dx - \frac{\pi i}{a^2+1} = \frac{\pi}{i-a}$

$$\Rightarrow PV \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{i-a} + \frac{\pi i}{a^2+1}$$

$$= \frac{\pi(a^2 \cancel{+1} - 1 - ia)}{(i-a)(a^2+1)}$$

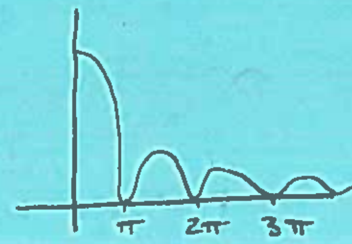
$$= \frac{\pi a(a-i)}{(i-a)(a^2+1)}$$

$$= \frac{-\pi a}{a^2+1}$$

$\square$

7.7.1 Show $\int_0^\infty \frac{|\sin x|}{x}dx = \infty$

Pf Consider m^{th} arch of $\frac{|\sin x|}{x}$.



For $x \in [(m-1)\pi + \pi/4, m\pi - \pi/4]$

$\Rightarrow |\sin x| \geq \sin \pi/4 = \frac{1}{\sqrt{2}}$

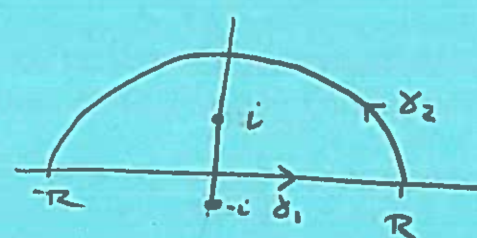
$\Rightarrow \frac{|\sin x|}{x} \geq \frac{1}{\sqrt{2}} \frac{1}{m\pi}$

$\Rightarrow \int_{(m-1)\pi+\pi/4}^{m\pi-\pi/4} \frac{|\sin x|}{x} \geq \frac{1}{\sqrt{2}}\frac{1}{m\pi}\pi = \frac{C}{m}$

$\Rightarrow \int_0^{m\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{m} \int_{(k-1)\pi+\pi/4}^{k\pi-\pi/4} \frac{|\sin x|}{x}dx = C\left(1+1/2+\dots+1/m\right) \to \infty$

$\square$

7.7.2 Show $\int_{-R}^{R} \frac{x^3\sin(x)}{(x^2+1)^2}dx = \frac{\pi}{2e}.$



Pf double poles at $\pm i$

$f(z) = \frac{z^3e^{iz}}{(z^2+1)^2}$ and $I = \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}$

$\text{Res}(f(z), i) = \frac{d}{dz}\frac{z^3e^{iz}}{(z+i)^2} = \frac{(z+i)^2(3z^2e^{iz}+z^3ie^{iz}) - z^3e^{iz}2(z+i)}{(z+i)^4} = \frac{1}{4e}$

$\int_{\gamma_1} \frac{z^3e^{iz}}{(z^2+1)^2}dz = \int_{-R}^{R}\frac{x^3e^{ix}}{(x^2+1)^2}$

$\to \int_{-\infty}^{\infty}\frac{x^2e^{ix}}{(x^2+1)^2} = \int_{-\infty}^{\infty}\frac{x^3\cos x}{(x^2+1)^2}dx + i\int_{-\infty}^{\infty}\frac{x^3\sin x}{(x^2+1)^2}dx$

$\left|\int_{\gamma_2} f(z)dz\right| \leq \frac{R^3}{(R^2-1)^2}\int_{\gamma_2}|e^{iz}|\,|dz| \overset{\text{Jordan}}{\leq} \frac{\pi R^3}{(R^2-1)^2} \to 0$ as $R \to \infty$.

$\Rightarrow$ By residue thm w/ $R \to \infty$

$\int_{-\infty}^{\infty}\frac{x^3\cos x}{(x^2+1)^2} + iI = 2\pi i\left(\frac{1}{4e}\right)$

$\Rightarrow I = \frac{\pi}{2e}.$

$\square$

7.7.3 Evaluate the limits $\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(ax)}{x^2+1}\,dx \quad -\infty < a < \infty$

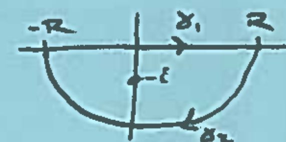
Show they do not depend continuously on a.

Pf Case $a=0$

$$\lim \int_{-R}^{R} \frac{x\sin ax}{x^2+1}\,dx = \lim \int_{-R}^{R} \frac{0}{x^2+1} = 0$$

Case $a<0$

Set $I = \int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+1}\,dx$ and $f(z) = \frac{ze^{iaz}}{z^2+1}$



f has simple pole at $z=-i$

$$\Rightarrow \operatorname{Res}(f,-i) = \frac{ze^{iaz}}{2z}\Big|_{z=-i} = \frac{-ie^{a}}{-2i} = \frac{e^a}{2}$$

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} \frac{ze^{iaz}}{z^2+1} = \int_{-R}^{R} \frac{xe^{iax}}{x^2+1}\,dx$$

$$\rightarrow \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{x\cos ax}{x^2+1} + i\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+1}$$

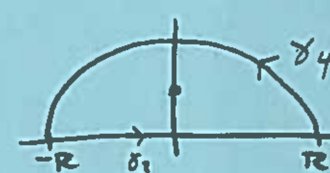
$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{R}{R^2-1}\int_{\gamma_2} |e^{iza}|\,|dz| \overset{\text{Jordan}}{<} \frac{\pi R}{R^2-1} \longrightarrow 0 \quad \text{as } R\to\infty$$

By Residue Thm:

$$\int_{-\infty}^{\infty} \frac{x\cos ax}{x^2+1} + iI = -2\pi i\frac{e^a}{2} \qquad \Rightarrow I = -\pi e^a$$

Case $a>0$

f has simple pole at $z=i$



$$\Rightarrow \operatorname{Res}(f,i) = \frac{ze^{iaz}}{2z}\Big|_{z=i} = \frac{e^{-a}}{2}$$

$$\int_{\gamma_3} f(z)\,dz = \int_{-\infty}^{\infty} \frac{x\cos ax}{x^2+1} + i\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+1} \quad \text{as above.}$$

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{R}{R^2-1}\int_{\gamma_4} |e^{iza}|\,|dz| < \frac{\pi R}{R^2-1} \rightarrow 0 \quad \text{as } R\to\infty$$

$\Rightarrow$ By residue thm

$$\int_{-\infty}^{\infty} \frac{x\cos xa}{x^2+1} + iI = 2\pi i\frac{e^{-a}}{2} \qquad \Rightarrow I = \pi e^{-a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+1} = \begin{cases} -\pi e^{a} & a<0 \\ 0 & a=0 \\ \pi e^{-a} & a>0 \end{cases}$$

Not cont at $a=0$ since $\sin 0 = 0$

7.8.1 Evaluate residue at ∞ of

a) $\frac{z}{z^2-1}$ b) $\frac{1}{(z^2+1)^2}$ c) $\frac{z^3+1}{z^2-1}$ e) $z^n e^{1/z}$ $n \in \mathbb{Z}$

Pf $\operatorname{Res}\left(\frac{z}{z^2-1}, \infty\right) = \operatorname{Res}\left(\frac{-1}{w^2} \frac{\frac{1}{w}}{\frac{1}{w^2}-1}, 0\right)$

$$= \operatorname{Res}\left(\frac{-1}{w^3} \frac{w^2}{1-w^2}, 0\right)$$

$$= \operatorname{Res}\left(\frac{-1}{w-w^3}, 0\right)$$

$$= \frac{-1}{1-3w^2}\Big|_{w=0} = -1$$

$$\operatorname{Res}\left(\frac{1}{(z^2+1)^2}, \infty\right) = \operatorname{Res}\left(\frac{-1}{w^2} \frac{1}{(\frac{1}{w^2}+1)^2}, 0\right)$$

$$= \operatorname{Res}\left(\frac{-w^2}{(1+w^2)^2}, 0\right)$$

$$= \frac{-w^2}{2(1+w^2)2w}\Big|_{w=0} = 0$$

$$\operatorname{Res}\left(\frac{z^3+1}{z^2-1}, \infty\right) = \operatorname{Res}\left(\frac{-1}{w^2} \frac{\frac{1}{w^3}+1}{\frac{1}{w^2}-1}, 0\right)$$

$$= \operatorname{Res}\left(\frac{-w^{-5}-w^{-2}}{w^{-2}-1}, 0\right)$$

$$= \operatorname{Res}\left(\frac{-1-w^3}{w^3-w^5}, 0\right)$$

$$= \frac{-1-w^3}{3w^2-5w^4}\Big|_{w=0} = -1$$

$$\operatorname{Res}(z^n e^{1/z}, \infty) = \operatorname{Res}\left(\frac{-1}{w^2} \frac{1}{w^n} e^w, 0\right)$$ $w=0$ is pole of order $n+2$

$$= \operatorname{Res}\left(\frac{-e^w}{w^{2+n}}, 0\right)$$

$$= \lim_{w \to 0} \frac{d^{n+1}}{dw^{n+1}} \frac{-w^{n+2} e^w}{w^{n+2}} \cdot \frac{1}{(n+1)!}$$

$$= \lim_{w \to 0} \frac{1}{(n+1)!} \frac{d^{n+1}}{dw^{n+1}} -e^w$$

$$= \frac{-1}{(n+1)!}$$

□

Gamelin Chapter 8

8.2 # 1, 3, 4, 5, 7

8.3 # 2

8.4 # 1, 3, 6, 8

8.2.1 Show $2z^5+6z-1$ has 1 root and 4 in the annulus $\{1<|z|<2\}$

Pf Let $h = 2z^5$ and $g = 6z-1$

$\Rightarrow |h| = 2\cdot 2^5 = 64$ and $|g| \leq 6(2)-1 = 11$ on $|z|=2$

$\Rightarrow |g| \leq |h|$ on $|z|=2$ and h has 5 roots.

$\Rightarrow p = g+h$ has 5 roots on $|z|=2$.

Now let $h = 6z$ and $g = 2z^5-1$.

$\Rightarrow |h| = 6$ on $|z|=1$, $|g| = |2z^5-1| = 3$ on $|z|=1$

$\Rightarrow |h| \geq |g|$ on $|z|=1$ and h has 1 root

$\Rightarrow p = g+h$ has one root on $|z|=1$

$\Rightarrow p$ has 4 roots on $\{1 \leq |z| \leq 2\}$.

$p(0) = 2(0)^5 + 6(0) - 1 = -1$
$p(1) = 2(1)^5 + 6(1) - 1 = 7$ $\Rightarrow p$ has one root on $0 \leq x \leq 1$

$\square$

8.2.3 Show if $m, n \in \mathbb{N}$ then $p(z) = 1 + z + \frac{z^2}{2} + \dots + \frac{z^m}{m!} + 3z^n$ has n zeros in unit disc.

Pf Let $p = f + h$ where $f = 3z^n$, $h = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^m}{m!}$

$\Rightarrow |f| = |3z^n| = 3$, $|h| = |1 + z + \dots + \frac{z^m}{m!}| \leq \sum \frac{z^k}{k!} \leq e.$

$\Rightarrow |f| \geq |h|$ on $|z| = 1$ and f has n roots.

$\Rightarrow f + h = p$ has exactly n roots.

□

8.2.4 Fix $|\lambda| < 1$ for $n \geq 1$. Show $(z-1)^n e^z - \lambda$ has n zeros with $|z-1| < 1$ and no others in right half plane. Determine multiplicity of zeros.

Pf Let $p = f + h$ where $f = (z-1)^n e^z$ $h = -\lambda$.

$|f| = |(z-1)^n e^z| = |z-1|^n |e^z| = |e^z| = e^x > 1$ on $|z-1| = 1$.

$|h| = |\lambda| < 1$

$\Rightarrow |f| > |h|$ on $|z-1| = 1$ and f has n roots

$\Rightarrow p = f + h$ has n roots

$f'(z) = n(z-1)^{n-1} e^z + (z-1)^n e^z = (n + z - 1)(z-1)^{n-1} e^z$

$\Rightarrow f'(z) = 0 \Rightarrow z = 1$ or $z = 1 - n$

$\Rightarrow z = 1$ is only zero in right half plane.

$f(1) = -\lambda \neq 0$ so $z = 1$ is a simple 0

unless $\lambda = 0$ then $z = 1$ is a zero of order n

□

8.2.5 Let λ be fixed, $|\lambda|<1$

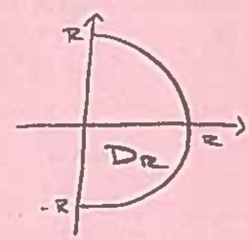
Show $(z-1)^n e^z + \lambda(z+1)^n$ has n zeros in right half plane, all simple if $\lambda \neq 0$.

Pf $f_n(z) = (z-1)^n e^z + \lambda(z+1)^n = 0$

$$\Leftrightarrow \frac{(z-1)^n}{(z+1)^n} e^z + \lambda = 0$$

$$\Leftrightarrow \underbrace{\left(\frac{z-1}{z+1}\right)^n}_{f} + \underbrace{\lambda e^{-z}}_{h} = 0$$

since $z+1 \neq 0 \neq e^{-z}$ on D_R



$|h| = |\lambda| e^{-|x|} < |\lambda| < 1$ on $\overline{D_R}$

$$|f| = \begin{cases} 1 & \text{if } \operatorname{Re} z = 0 \\ M_n & \text{if } |z| = R \end{cases}$$

on D_R $|f(z)| \geq \left(\frac{R-1}{R+1}\right)^n > |\lambda|$ if $R \geq R_0$ for some big R_0

So by Rouches Thm $\left(\frac{z-1}{z+1}\right)^n + \lambda e^{-z}$ has exactly n roots in D_R for R large.

$\Rightarrow f_n(z)$ has exactly n roots in RHP

Now assume $\lambda \neq 0$ and show all zeros are simple

i.e $f_n(z) = 0 \Rightarrow f_n'(z) \neq 0$. in RHP

$f_n(z) = 0 \Rightarrow (z-1)^n e^z + \lambda(z+1)^n = 0$

$f_n'(z) = 0 \Rightarrow n(z-1)^{n-1} e^z + (z-1)^n e^z + n\lambda(z+1)^{n-1} = 0$

$\Rightarrow (z-1)^{n-1}(n+z-1)e^z + n\lambda(z+1)^{n-1} = 0$

$\Rightarrow (z-\cancel{1})^{n-1}(n+z-1)\left(\frac{-\cancel{\lambda}(z+1)^{\cancel{n}}}{(z-1)^{\cancel{n}}}\right) + n\cancel{\lambda}(z+\cancel{1})^{n-1} = 0$

since not 0

$$\Rightarrow -\frac{(n+z-1)(z+1)}{z-1} + n = 0$$

$\Rightarrow (n+z-1)(z+1) = nz - n$

$\Rightarrow n\cancel{z} + z^2 - \cancel{z} + n + \cancel{z} - 1 = n\cancel{z} - n$

$\Rightarrow z^2 = -2n+1$

$\Rightarrow z = \sqrt{2n-1}\, i \notin$ RHP

$\therefore f_n(z) = 0 \Rightarrow f_n'(z) \neq 0 \Rightarrow$ not simple 0's.

8.3.2. Let S be a family of univalent functions $f(z)$ defined on $\mathbb{D}$. that satisfies $f(0)=0$ and $f'(0)=1$. Show S is closed under normal convergence.

Pf Let $\{f_k\} \in S$ be s.t. $f_k \to f$

$\Rightarrow f$ is constant or univalent

univalent functions converge to constants or other univalent fcns.

WTS $f \in S$.

$f(0)=0$ $f_k \rightrightarrows f$ on any closed disk

$\Rightarrow f_k \to f$ uniformly

$\Rightarrow f_k \to f$ pointwise

$\Rightarrow f(0)=0$ since $f_k(0)=0 \ \forall k$ ✓

$f'(0)=1$ $f_k' \rightrightarrows f'$ since $\{f_k\}$ holomorphic on bdd domain

normal convergence of analytic fcns $\Rightarrow$ derivatives converge normally too.

$\Rightarrow |f_k'(0) - f'(0)| < \varepsilon$

$\Rightarrow |1 - f'(0)| < \varepsilon$

$\Rightarrow f'(0)=1$ ✓

f univalent $f'(0)=1$

$\Rightarrow f$ not constant

$\Rightarrow f$ univalent ✓

$\therefore f \in S$ and so S is closed under normal convergence

univalent means conformal (analytic and 1-1)

<u>8.27</u> f, g analytic on bdd D that extends continuously to ∂D and satisfies $|f(z)+g(z)| < |f|+|g|$

Show f & g have same # of zeros.

<u>Pf</u> $|f(z)+g(z)| < |f(z)| + |g(z)|$

$\Rightarrow f(z), g(z) \neq 0$ on ∂D

Let $N_0(f) =$ # zeros of $f = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz$

$N_0(g) =$ # zeros of $g = \frac{1}{2\pi i} \int_{\partial D} \frac{g'(z)}{g(z)} dz$

note f, g have no poles since they're analytic on D

<u>Claim</u> $N_0(f) = N_0(g)$

Let $h = f/g$

$\Rightarrow f = h \cdot g$

$\Rightarrow f'/f = \frac{h'g + g'h}{hg}$

$\Rightarrow f'/f = h'/h + g'/g$

$\Rightarrow \int_{\partial D} \frac{h'(z)}{h(z)} dz = \int_{\Gamma} \frac{dw}{w} = 0$ $\quad w = h(z)$, $\Gamma = h(\partial D)$ $\quad \Gamma \subset \mathbb{C} \setminus [0, \infty)$ $\Rightarrow 1/w$ has primitive here and Γ closed curves

* on ∂D: $|h(z)+1| \leq |h(z)| + 1$

$\Leftrightarrow h(z) \in \mathbb{C} \setminus [0, \infty)$

$\Rightarrow N_0 f = N_0 g$

$\square$

8.4.1 D bdd w/ piecewise smooth bdry
Let $f(z)$ be meromorphic and $g(z)$ analytic on D
Suppose f, g extend analytically across ∂D
$f(z) \neq 0$ on ∂D Show $\frac{1}{2\pi i}\int_{\partial D} g(z) \frac{f'(z)}{f(z)} dz = \sum m_j g(z_j)$
z_j zeros and poles of f, m_j order of z_j

Pf $\frac{1}{2\pi i}\int_{\partial D} g(z)\frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_{f(\partial D)} g\frac{(f^{-1}(w))}{w} dw$ $w = f(z)$ $dw = f'(z)dz$ $z = f^{-1}(w)$

$= \operatorname{Res}(\frac{1}{w} g(f^{-1}(w)), 0)$
$= g(f^{-1}(0))$
$= \sum m_j g(z_j)$

$\square$

8.4.3 $\{f_k(z)\}$ analytic on D converges normally to f
$f_k(z)$ attains each w at most m times in D
Show either f constant of f attains w at most m times in D.

Pf Assume f is not constant (example $\frac{z^n}{n}$)
$\Rightarrow f$ attains w N times in D
Say f attains w_0 at z_j, n_j times $\Rightarrow \sum n_j = N$
By Hurwitz thm f_n attains w_0, n_j times in a nbhd of z_j.
?
Make nbhd's disjoint
$\Rightarrow f_n$ attains w_0 at least n_j times in D
$\Rightarrow \sum n_j \leq m$
$\Rightarrow f$ attains w_0 at most n times

$\square$

8.4.6 Let f be meromorphic on $\mathbb{C}$.
Suppose $\exists m \in \mathbb{Z}$ s.t. $f^{-1}(w)$ has at most m points for all $w \in \mathbb{C}$. Show f rational.

Pf Let w_0 be s.t. $f^{-1}(w)$ has max # of points

$\Rightarrow f$ attains values close to w_0 only close to those finite # of $z \in f^{-1}(w)$

$\Rightarrow \dfrac{1}{f(z)-w_0}$ is bdd at ∞

↳ since its quotient of meromorphic fcn's it is meromorphic on $\mathbb{C}^*$

$\Rightarrow \dfrac{1}{f-w_0}$ is rational

$\Rightarrow f$ is rational

□

8.4.8 Let D be bdd domain and $f(z)$ cont. function on $D \cup \partial D$ analytic on D.
Show $\partial(f(D)) \subseteq f(\partial D)$

Pf $\partial(f(D)) \subset \overline{f(D)} \subset f(\overline{D}) = f(D) \cup f(\partial D)$

Since compact hence closed

$f(D)$ is open

$\Rightarrow \partial f(D) \cap f(D) = \emptyset$ □

$\Rightarrow \partial f(D) \subset f(\partial D)$

Gamelin Chapter 9

9.1 # 1, 2, 4, 6, 8

9.2 # 1, 3, 5, 7, 13

9.1.1 f analytic, $|f(z)| \le M$ for $|z-z_0| < R$.
Show if f has zero of order m at z_0 then $|f(z)| \le \frac{M}{R^m}|z-z_0|^m$. Show equality at $z \ne z_0$ if f is a constant multiple of $|z-z_0|^m$

Pf f has a zero of order m at z_0

$\Rightarrow f(z) = (z-z_0)^m g(z)$

$\Rightarrow |g(z)| = \frac{|f(z)|}{|z-z_0|^m} = \frac{|f(z)|}{r^m} \le \frac{M}{r^m}$ for $|z-z_0| = r < R$

$\Rightarrow |f(z)| \le \frac{M}{R^m}|z-z_0|^m$ if $r \to R$

Assume equality holds.

$\Rightarrow \exists z' \ne z_0$ s.t. $|f(z')| = \frac{M}{R^m}|z'-z_0|^m$

$\Rightarrow |g(z')| = \frac{M}{R^m}$

$\Rightarrow g(z) = \lambda \frac{M}{r^m}$ by the strict max principle

$\Rightarrow f(z) = \frac{\lambda M}{r^m}(z-z_0)^m$

$\square$

9.1.2 Suppose f analytic and $|f(z)| \leq 1$ for $|z|<1$
Show if f has zero of order m at z_0
then $|z_0|^m \geq |f(0)|$

Pf Assume f has a zero of order m at z_0.
If $|z_0|>1$ then $|z_0|^m > 1$ and $|f(0)|<1$
$\Rightarrow$ claim holds

If $|z_0| \leq 1$ consider function $\psi(z) = \dfrac{z-z_0}{1-\bar{z}_0 z}$
$\Rightarrow \psi : \mathbb{D} \to \mathbb{D}$
$\Rightarrow f \circ \psi : \mathbb{D} \to \mathbb{D}$ and $f(-\psi(0)) = f(z_0) = 0$ order m.
$\Rightarrow f(-\psi(z)) \leq |z|^m$ by Schwartz Lemma
$\Rightarrow f(-\psi(z_0)) \leq |z_0|^m$ by Schwartz Lemma
$\Rightarrow f(0) \leq |z_0|^m$ since $-\psi(z_0) = 0$.
$\square$

9.1.4 Suppose f analytic for $|z|<1$ and $f(0)=0$ + $\operatorname{Re} f(z) < 1$
(a) Show $|f(z)| \leq \dfrac{2|z|}{1-|z|}$
(b) Show $|f'(0)| \leq 2$
(c) Fixed z_0, $0<|z_0|<1$ determine equality functions (a)
(d) Determine where equality for (b)
(e) obtain sharp estimates for $|g(z)| + |g'(0)|$
for g analytic for $|z|<R$ w/ $g(0)=0$ $\operatorname{Re} g < C$

Pf (a) Let $\varphi(z) = \dfrac{z}{2-z}$. $\varphi : \{\operatorname{Re} w < 1\} \to \mathbb{D}$.
$\Rightarrow |\varphi \circ f| \leq 1$ and $\varphi \circ f(0) = \varphi(0) = 0$
$\Rightarrow |\varphi \circ f(z)| \leq |z|$ by Schwartz Lemma
$\Rightarrow \dfrac{|f(z)|}{|2-f(z)|} \leq |z|$
$\Rightarrow |f(z)| < \dfrac{2|z|}{1-|z|}$

(b) $|f'(0)| = \left| \lim \frac{f(z)-f(0)}{z-0} \right|$

$= \lim |f(z)| / |z|$

$\leq \lim \frac{2|z|}{(1-|z|)|z|}$

$= 2$ as $z \to 0$

(c) Assume equality holds in (a).

$\Rightarrow |f(z_0)| = \frac{2|z_0|}{1-|z_0|}$ for $0 < |z_0| < 1$

$\Rightarrow \varphi \circ f(z) = \lambda z$ by schwartz

$\Rightarrow \frac{f(z)}{2-f(z)} = \lambda z$

$= f(z) = 2\lambda z - f(z)\lambda z$

$\Rightarrow f(z) = \frac{2\lambda z}{1+\lambda z}$

(d) $|f'(0)| = 2$

$\Rightarrow (\varphi \circ f)' = \varphi'(f(0)) \cdot f'(0) \leq 1$

$\Rightarrow \varphi'(\ 0\ ) \leq 1/2$

$\varphi' = \frac{2-z+z}{(2-z)^2} = \frac{2}{(2-z)^2} = 1$

=

9.1.6 Let $f: \mathbb{D} \to \mathbb{D}$ be conformal. Show $\text{dist}(f(0), \partial D) \leq |f'(0)|$

Pf Let $d = \text{dist}(f(0), \partial D)$

$\Rightarrow f^{-1}(w)$ is analytic in $I = \{w : |f(0) - w| < d\}$ by IFT

$\Rightarrow f^{-1}(I) \subset \mathbb{D}$

$\Rightarrow |f^{-1}(w)| < 1 \ \forall w \in I$ and $f^{-1}(f(0)) = 0$

$\Rightarrow |(f^{-1})'(f(0))| \leq 1/d$

$\Rightarrow f^{-1}(f(x)) = x$

$\Rightarrow (f^{-1})'(f(x)) \cdot f'(x) = 1$

$\Rightarrow (f^{-1})'(f(x)) = 1/f'(x)$

$\Rightarrow (f^{-1})'(f(0)) = 1/f'(0)$

$\Rightarrow \left|\frac{1}{f'(0)}\right| \leq 1/d$

$\Rightarrow d \leq |f'(0)|$ $\square$

9.1.8 f analytic on $|z|<1$ and $|f|<1$, $f(0)=0$, $|f'(0)|<1$
Let $r<1$, Show $\exists c>1$ s.t. $|f(z)| < c|z|$ for $|z|<r$
Show $f_n(z) = f \circ f \circ \dots \circ f(z)$ satisfies $|f_n(z)| \leq c^n |z|$ $|z| \leq r$
Deduce $f_n \rightrightarrows 0$.

Pf

9.2.1 A finite Blasche Product is a rational fcn.

$B(z) = e^{i\varphi}\left(\frac{z-a_1}{1-\bar{a}_1 z}\right)\cdots\left(\frac{z-a_n}{1-\bar{a}_n z}\right)$ $a_i \in \mathbb{D}$ $0 \le \varphi < 2\pi$.

Show if f is cont. for $|z| \le 1$ and analytic for $|z|<1$ and if $f(z)=1$ for $|z|=1$ then f is finite Blasche

Pf First note $f \not\equiv 0$ since $|f(z)|=1$ and its cont.

$\Rightarrow f$ has finitely many 0's in $\mathbb{D}$ say $a_1, \ldots, a_n$.

9.2.3 Suppose f analytic for $|z|<3$. If $|f(z)| \le 1$ and $f(\pm i) = f(\pm 1) = 0$. What is max of $|f(0)|$? When is max attained?

Pf Consider $f_3 : \mathbb{D} \to \mathbb{D}$ s.t. $f_3(x) = f(3x)$

Let B be finite Blasche product w/ $a_1 = \frac{1}{3i}$, $a_2 = \frac{-1}{3i}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{-1}{3}$

$\Rightarrow \left|\frac{f_3}{B}\right| \le 1$ on $|z|=1$

$\Rightarrow |f_3(z)| \le B(z)$ on $|z|=1$.

$\Rightarrow f = Bg$ for some analytic g.

$|f(0)| = |B(0)|\underbrace{|g(0)|}_{\le 1 \text{ by max princ}} = \left|\frac{-1/3}{1} \cdot 1/3 \cdot 1/3 \cdot 1/3\right||g(0)| = \frac{1}{81}$

9.2.5 Show any conformal self-map of $\mathbb{H}$ has form

$f(z) = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$ $ad-bc=1$ when do 2

Choices of coefficient determine same conformal self-map of $\mathbb{H}$

Pf Let f be a conformal self map of $\mathbb{H}$.

Let $g: \mathbb{H} \to \mathbb{D}$ s.t. $g(z) = \frac{z-i}{z+i}$ a linear fractional transformation.

$\Rightarrow g \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ is a conformal self map of $\mathbb{D}$.

$\Rightarrow g \circ f \circ g^{-1}$ is a fractional linear transformation.

$\Rightarrow f$ is a fractional linear transformation.

$\Rightarrow \exists a,b,c,d \in \mathbb{C}$ w/ $ad-bc \neq 0$ s.t. $f(z) = \frac{az+b}{cz+d}$

Any nonzero scalar multiple of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will determine same transformation.

WLOG assume $ad-bc=1$

Finally $f: \mathbb{H} \to \mathbb{H}$

$\Rightarrow f(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$

$\Rightarrow \exists$ distinct $z_1, z_2, z_3 \in \mathbb{R}$ s.t. $f(z_1), f(z_2), f(z_3) \in \mathbb{R}$

$\Rightarrow f$ determined by cross ratios via

$$[z, z_1, z_2, z_3] = [w, f(z_1), f(z_2), f(z_3)]$$

$\Rightarrow$ we can choose $a,b,c,d \in \mathbb{R}$

$\square$

9.2.7 Show every conformal self-map of $\mathbb{C}$ has form $f(z) = az+b$.

Pf f conformal

$\Rightarrow f$ 1-1

$\Rightarrow f^{-1}(w)$ is a single point $\forall w \in \mathbb{C}$.

$\Rightarrow f$ is rational by 8.4.6.

$\Rightarrow f$ a polynomial since f analytic + rational

f, f^{-1} have same degree

$\Rightarrow f$ is a polynomial of degree 1

$\Rightarrow f = az+b$.

$a \neq 0$ since a constant function is not 1-1.

$\square$

9.2.13 f analytic, $f : \mathbb{D} \to \mathbb{D}$ (not the identity)

Show f has at most 1 fixed point

Pf Let z_0 be a fixed point of f.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be the conformal selfmap sending $0 \mapsto z_0$,

$\Rightarrow f(z_0) = z_0$ and $\varphi(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$

$\Rightarrow \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D}$ and $\varphi^{-1} \circ f \circ \varphi(0) = 0$

$\Rightarrow |\varphi^{-1} \circ f \circ \varphi| \leq |z|$ by schwartz lemma.

Assume BWOC w is another fixed point of f, $\varphi(z_1) = w$

$\Rightarrow \varphi^{-1} \circ f \circ \varphi(z_1) = \varphi^{-1}(f(w)) = \varphi^{-1}(w) = z_1$

$\Rightarrow \varphi^{-1} \circ f \circ \varphi(z_1) = \lambda z_1$ by schwartz.

$\Rightarrow \varphi^{-1}(f(w)) = \lambda(\varphi^{-1}(w))$

$\Rightarrow f(w) = \varphi(\lambda(\varphi^{-1}(w)))$

$\Rightarrow w = f(w) = \varphi(\lambda(\varphi^{-1}(w)))$

$\Rightarrow \lambda = 1$

$\Rightarrow f(w) = w \quad \forall w$

which contradicts b/c f is not the identity

or $\dfrac{|f(z) - f(w)|}{|1 - \overline{f(w)} f(z)|} \leq \left|\dfrac{z - w}{1 - \bar{w} z}\right|$ and $|f'(z)| \leq \dfrac{1 - |f(z)|^2}{1 - |z|^2}$ $\square$

w/ equality $\Leftrightarrow f \in \mathrm{Aut}\,\mathbb{D}$.

If $f(z_1) = z_1$ and $f(z_0) = z_0$ where $z_1 = z_0$

$\Rightarrow z = f(z) = e^{i\theta} \dfrac{z - 1}{1 - \bar{a} z}$ at most one fixed pts

$\Rightarrow z - \bar{a} z^2 = e^{i\theta}(z - a)$

$\Rightarrow \bar{a} z^2 + (e^{i\theta} - 1) z - a e^{i\theta} = 0 \qquad a = 0$

$\Rightarrow z^2 + (e^{i\theta} - 1) z - \dfrac{a}{\bar{a}} e^{i\theta} = 0$

product equals this

$\Rightarrow |\cdot| = 1$ which cant happen inside $\mathbb{D}$ $\square$

Gamelin Chapter 10

10.1 # 2, 3, 4

10.2 # 2

10.3 # 6, 7, 8

10.1.2 Let $R>0$, $h(Re^{i\theta})$ a continuous fcn on $\{|z|=R\}$. Show $h(z) = \int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)} h(Re^{i\varphi}) \frac{d\varphi}{2\pi}$ is harmonic on disk $\{|z|<R\}$ and has bdry values $h(Re^{i\theta})$ on bdry circle.

10.1.3 Suppose $f(z) = u(z) + iv(z)$ is analytic for $|z|<1$ and $u(z)$ extends to be continuous on closed disk $\{|z| \le 1\}$. Show $f(z) = \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \frac{d\varphi}{2\pi} + i\,v(0)$.

10.1.4 Let $\{f_n(z) = u_n(z) + iv_n(z)\}$ be a sequence of analytic functions on $\mathbb{D}$ s.t. $u_n(z)$ extends continuously to $\partial\mathbb{D}$, $u_n(e^{i\varphi})$ converges uniformly on $\partial\mathbb{D}$ to $u(e^{i\varphi})$ and $v_n(0)$ converges. Show $f_n(z)$ converges normally on $\mathbb{D}$ to an analytic fcn $f(z)$ whose real part is $\tilde{u}(z)$

10.2.2 Assume $u(x,y)$ is twice continuously differentiable fcn on D

(a) For $(x_0, y_0) \in D$ Let $A_\varepsilon(x_0, y_0)$ be average of $u(x,y)$ on circle centered at (x_0, y_0) of radius ε. Show

$$\lim_{\varepsilon \to 0} \frac{A_\varepsilon(x_0,y_0) - u(x_0,y_0)}{\varepsilon^2} = \frac{1}{4} \Delta u(x_0,y_0)$$

(b). Let $B_\varepsilon(x_0, y_0)$ be area average of $u(x,y)$ on disk centered at (x_0, y_0) of radius ε. Show

$$\lim_{\varepsilon \to 0} \frac{B_\varepsilon(x_0,y_0) - u(x_0,y_0)}{\varepsilon^2} = \frac{1}{8} \Delta u(x_0,y_0)$$

10.3.6 Let $f(z)$ be entire fcn whose modulus is constant on some circle. Show $f(z) = C(z - z_0)^n$ for some $n \geq 0$ and some Constant C, (z_0 center)

<u>10.3.7</u> Show if f is meromorphic for $|z|<1$ and if $|f(z)| \to 1$ as $|z| \to 1$ then $f(z)$ is a rational fcn. Show further that $f(z)$ is quotient of 2 finite Blaschke products

10.3.8 The modulous of an annulus $\{a < |z-z_0| < b\}$ is defined to be $1/2\pi \log(b/a)$.

(a) Show any conformal map fromo one annulus centered on origin to another extends to a conformal self map of punctured $\mathbb{C}$

(b). Show $\exists$ a conformal map of one annulus onto another $\Leftrightarrow$ the annuli have same mod.

(c) Show any conformal self map of the annulus $\{a < |z| < b\}$ is either a rotation $z \mapsto e^{i\varphi} z$ or a rotation then an inversion $z \mapsto ab/z$.

Gamelin Chapter 11

11.1 # 2, 3, 5, 7, 11

11.2 # 1, 2

11.5 # 1, 2, 6, 7

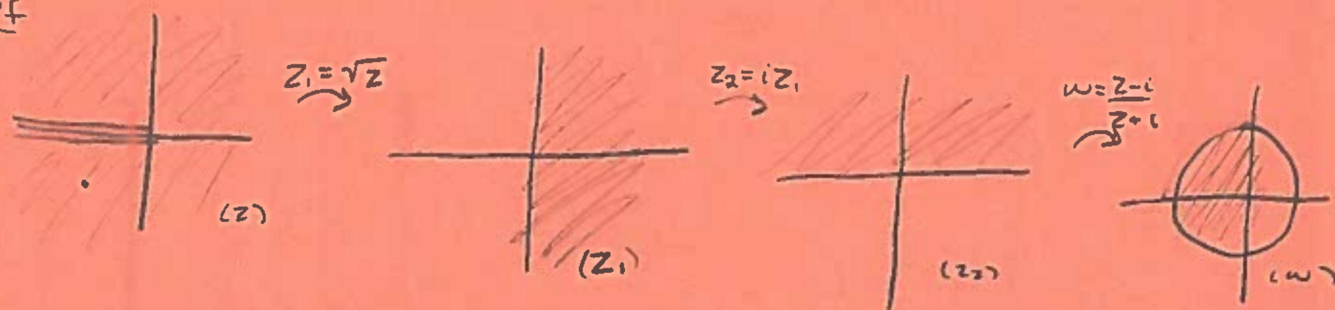
11.6 # 2

11.1.2 Find conformal map of $\mathbb{C} \setminus (-\infty, 0]$ onto $\mathbb{D}$

w/ $w(0)=i$, $w(-1+0i)=1$, $w(-1-0i)=-1$.

what is image of circle centered at 0 under map?

Pf



$\Rightarrow w = \lambda \dfrac{i\sqrt{z} - i}{i\sqrt{z} + i} = \lambda \dfrac{\sqrt{z} - 1}{\sqrt{z} + 1}$

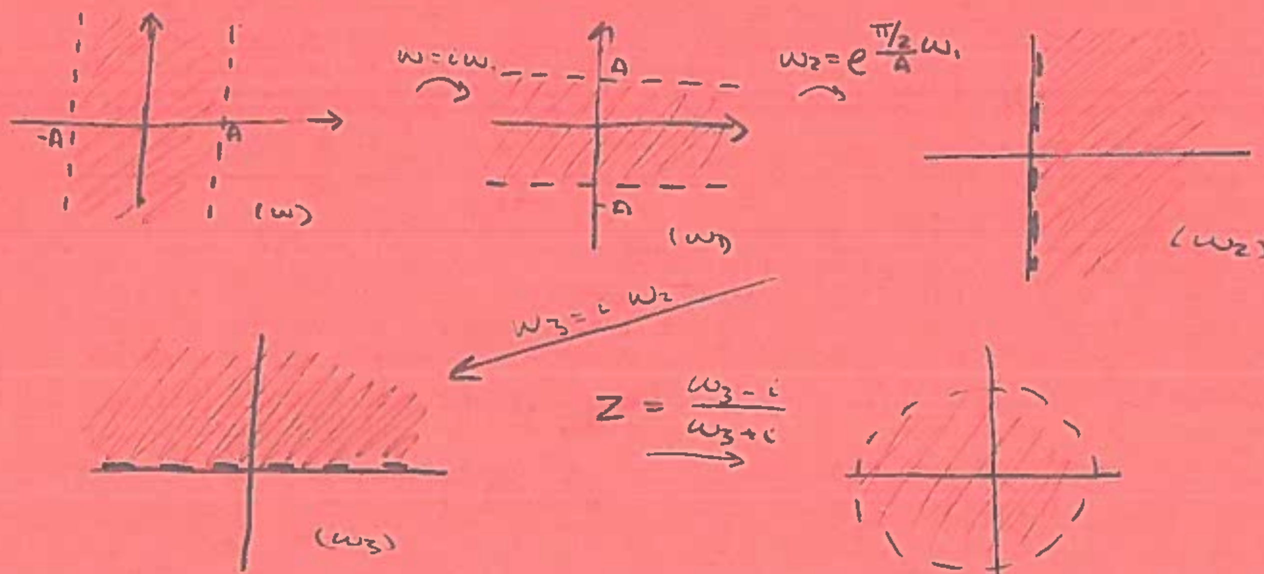
$w(0) = i \Rightarrow i = -\lambda \Rightarrow \lambda = -i$

$$\Rightarrow \boxed{w = -i\left(\frac{\sqrt{z} - 1}{\sqrt{z} + 1}\right)}$$

$w(|z| = a) =$

11.1.3 For fixed $A>0$ find conformal map $w(z)$ of the open unit disk $\{|z|<1\}$ onto vertical strip $\{-A<\operatorname{Re} w<A\}$ that satisfies $w(0)=0$ and $w'(0)>0$. Sketch curves in disk that correspond to vertical and horizontal lines in strip

Pf First lets find map from strip to disk.



$$\Rightarrow z=\lambda\frac{ie^{\pi/2A\, iw}-i}{ie^{\pi/2A\, iw}+i}=\lambda\frac{e^{\frac{\pi}{2A}iw}-1}{e^{\frac{\pi}{2A}iw}+1}$$

$w(0)=0 \Rightarrow \lambda=1$

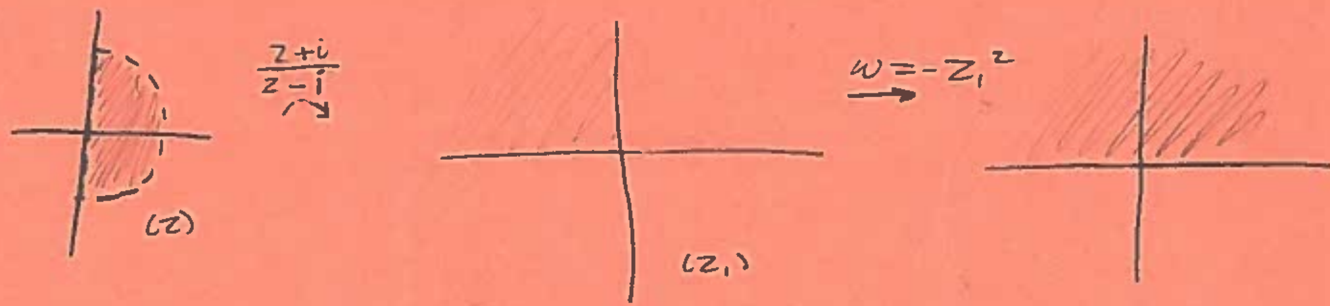
$$\Rightarrow z e^{\pi/2A\, iw}+z=e^{\frac{\pi}{2A}iw}-1$$

$$\Rightarrow \boxed{w=\frac{2A}{\pi i}\log\left(\frac{1+z}{1-z}\right)}$$

□

11.1.5 Find conformal map $w(z)$ of $\{\operatorname{Re} z > 0, |z| < 1\}$ onto $\mathbb{H}$ s.t. $-i \mapsto 0$, $i \mapsto \infty$, $0 \mapsto -1$, what is $w(1)$

Pf



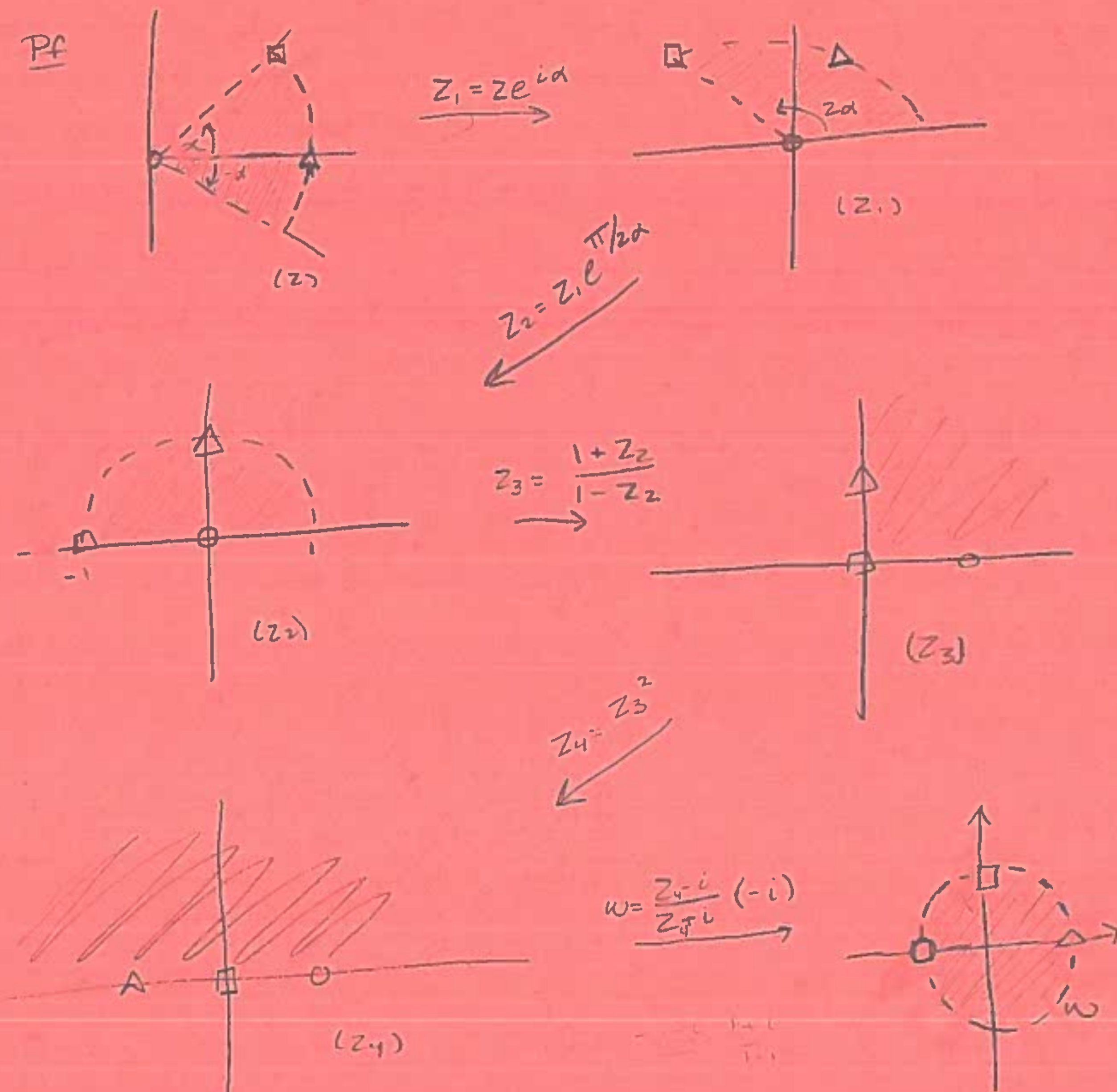
$$\Rightarrow \boxed{w = -\left(\frac{z+i}{z-i}\right)^2}$$

$$w(1) = -\left(\frac{1+i}{1-i}\right)^2 = -\frac{1+2i-1}{1-2i-1} = 1$$

$\square$

11.1.7 Find conformal map of pie slice $\{|\arg z| < \alpha,\ |z| < 1\}$ onto $\mathbb{D}$ s.t. $w(0) = -1$, $w(1) = 1$ and $w(e^{i\alpha}) = i$.

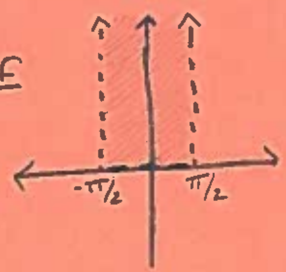
Pf



$$f(z) = \frac{\left(\frac{1+ze^{\pi/2\alpha+i\alpha}}{1-ze^{\pi/2\alpha+i\alpha}}\right)^2 - i}{\left(\frac{1+ze^{\pi/2\alpha+i\alpha}}{1-ze^{\pi/2\alpha+i\alpha}}\right)^2 + i} \cdot (-i)$$

11.1.11 Show half-strip $\{-\pi/2 < \operatorname{Re} z < \pi/2,\ \operatorname{Im} z > 0\}$ is mapped conformally by $w = \sin z$ onto $\mathbb{H}$

Pf

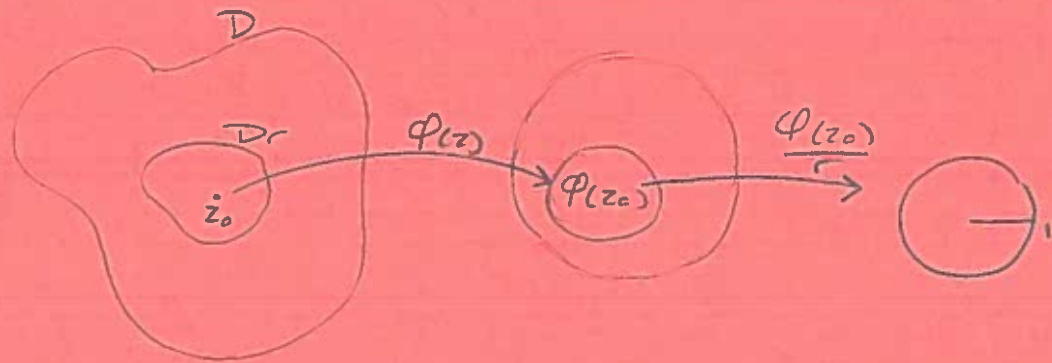


Let $w = \sin z$.

$w(\pi/2 + iy) = \sin(\pi/2 + iy) =$

11.2.2 Let $\varphi(z)$ be a conformal map from domain D onto the open unit disk $\mathbb{D}$. For $0<r<1$ let D_r be the set of $z\in D$ s.t. $|\varphi(z)|<r$. Find conformal map of D_r onto $\mathbb{D}$.

Pf



Let $D_r = \{z\in D \,|\, |\varphi(z)|<r\}$

and $\frac{\varphi(z)}{r} : D_r \to \mathbb{D}$

$\varphi : D_r \to r\mathbb{D}$ is onto.

□

11.2.1 Show no 2 of $\mathbb{C}, \mathbb{C}^*, \mathbb{D}$ are conformally equivalent.

Pf If $\varphi : \mathbb{C}^* \to \mathbb{D}$ is conformal then φ is entire and $|\varphi|\le 1$

$\Rightarrow \varphi$ is constant which contradicts.

Similarly for $\varphi : \mathbb{C} \to \mathbb{D}$

If $\varphi : \mathbb{C}^* \to \mathbb{C}$ is conformal.
However this is impossible since $\mathbb{C}^*$ is compact and $\mathbb{C}$ is not.

□

11.5.1 Let $\{f_n(z)\}$ be a uniformly bounded seq. of analytic functions on a domain D and let $z_0 \in D$. Suppose that $\forall m \geq 0$ $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$. Show $f_n(z) \rightrightarrows 0$ on D.

Pf Assume BWOC $f_n(z)$ does not converge normally

$\Rightarrow \exists K \subset D$ s.t. $f_n \not\to 0$ uniformly on K

$\Rightarrow \exists \varepsilon > 0$ s.t. $n_k \nearrow \infty$, $z_{n_k} \in K$ s.t. $|f_{n_k}(z_{n_k})| \geq \varepsilon$

$\Rightarrow \exists$ a normally convergent subseq. $f_{n_k} \to f$ on D

$\Rightarrow f_{n_{k_j}}^{(m)}(z_0) \to f^{(m)}(z_0)$ $\forall m \geq 0$ since f analytic

$\Rightarrow f^{(m)}(z_0) = 0$ for $m \geq 0$ since $f = \sum a_n (z-z_0)^n$

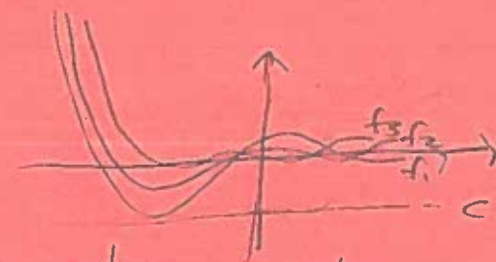
$\Rightarrow f \equiv 0$ by identity

$\Rightarrow f_{n_{k_j}} \to 0$ which contradicts

$\square$

11.5.2 Let $\{f_n(z)\}$ be a seq of analytic functions on domain D. Let $z_0 \in D$. Suppose $\operatorname{Re} f_n(z) \geq -C$ $\forall z \in D$ and $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$ $\forall m \geq 0$.

Show $f_n(z) \rightrightarrows 0$



Pf Let $\tilde{f}_n = \left(\frac{1}{1+C+f_n} \right)_{n \in \mathbb{N}}$

$\Rightarrow \tilde{f}_n$ is uniformly bdd

Since $\left| \frac{1}{1+C+f_n} \right| = \frac{1}{|1+C+f_n|} \geq \frac{1}{1+|C|+|f_n|} \geq \frac{1}{1+2|C|}$

$\Rightarrow \exists \tilde{f}_{n_k} \to \tilde{f}$

$\Rightarrow \frac{1}{1+C+f_{n_k}} \to \tilde{f}$

$\Rightarrow f_{n_k}$ converges either to some f or ∞

$\Rightarrow f_{n_k}^{(m)}(z_0) \to 0$ since

11.5.6 Let D be a bdd domain and let $f(z)$ be an analytic fcn from D onto D. Denote by $f_n(z)$ the n^{th} iterate of $f(z)$. Suppose that z_0 is an attracting fixed point for $f(z)$ so $f(z_0) = z_0$ and $|f'(z_0)| < 1$. Show $f_n(z)$ converges uniformly on compact subsets of D to z_0.

11.5.7 Let D be a bdd domain and let $f(z)$ be an analytic fcn from $D \to D$. Show if $z_0 \in D$ is a fixed point for $f(z)$ then $|f'(z_0)| \leq 1$

Pf Let $f_n = f \circ f \circ \ldots \circ f$ n times.

f analytic $\Rightarrow$ f_n analytic

$f(z_0) = z_0 \Rightarrow f_n(z_0) = z_0$

$$f_n'(z_0) = f'(f_{n-1}(z_0)) f'(f_{n-2}(z_0)) \cdots f'(z_0)$$
$$= f'(z_0) \cdots f'(z_0)$$
$$= (f'(z_0))^n$$

Since D is bdd, f_n is uniformly bdd on D.

$\Rightarrow f_n'$ is uniformly bdd on each compact subset

$\Rightarrow |f_n'| \leq M$ for some $M > 0$

$\Rightarrow |f_n'(z_0)| = |f'(z_0)|^n \leq M$

$\Rightarrow |f'(z_0)| \leq 1$ otherwise $|f'(z_0)|^n \to \infty$ as $n \to \infty$

$\square$

11.6.2 Let $\varphi(z)$ be the Riemann map of a simply connected domain D, onto $\mathbb{D}$ normalized by $\varphi(z_0)=0$ and $\varphi'(z_0)>0$. Show if $f(z)$ is any analytic fcn on D s.t. $|f(z)|\le 1$ for $z\in D$ then $|f'(z_0)|\le \varphi'(z_0)$ w/ equality only when $f(z)=\lambda\varphi(z)$.

Pf Let f, φ be as above.

Let $f\circ\varphi^{-1}: \mathbb{D}\to\overline{\mathbb{D}}$

$\Rightarrow f\circ\varphi^{-1}$ is well defined since φ is bijective

$\Rightarrow f\circ\varphi^{-1}(0)=f(z_0)$

Let $\psi: \mathbb{D}\to\mathbb{D}$ be conformal selfmap s.t. $\psi\circ f\circ\varphi^{-1}(0)=0$

$\Rightarrow \psi(z)=\dfrac{z-a}{1-\bar{a}z}$ where $\psi(f(z_0))=0$

$\Rightarrow \psi(z)=\dfrac{z-f(z_0)}{1-\overline{f(z_0)}z}$

$\Rightarrow |(\psi\circ f\circ\varphi^{-1})'(0)|\le 1$ by schwartz.

(equality if $\psi\circ f\circ\varphi^{-1}=\lambda z$ for $|\lambda|=1$

Notice: $(\psi\circ f\circ\varphi^{-1})'(0)=\psi'(f\circ\varphi^{-1}(0))\cdot f'(\varphi^{-1}(0))(\varphi^{-1})'(0)$

$$=\frac{\psi(f(z_0))\cdot f'(z_0)}{\varphi'(z_0)}$$

$$\Rightarrow \psi'(z)=\frac{(1-\overline{f(z_0)}z)(1)-(z-f(z_0))(-\overline{f(z_0)})}{(1-\overline{f(z_0)}z)^2}$$

$$=\frac{1-|f(z_0)|^2}{(1-\overline{f(z_0)}z)^2}$$

$$\Rightarrow \psi'(f(z_0)) = \frac{1-|f(z_0)|^2}{(1-|f(z_0)|^2)^2} = \frac{1}{1-|f(z_0)|^2}$$

$$\Rightarrow |(\psi \circ f \circ \varphi^{-1})'(0)| \leq 1$$

$$\Rightarrow \left| \frac{1}{1-|f(z_0)|^2} \cdot f'(z_0) \circ \frac{1}{\varphi'(z_0)} \right| \leq 1$$

Notice $|f'(z_0)| \leq (1-|f(z_0)|^2)\varphi'(z_0) \leq \varphi'(z_0)$ since $|f| \leq 1$

Now if equality holds

$\Rightarrow |f'(z_0)| = \varphi'(z_0)$

$\Rightarrow 1-|f(z_0)|^2 = 1$

$\Rightarrow |f(z_0)| = 0$

$\Rightarrow f(z_0) = 0$

$\Rightarrow \psi$ is the identity

$\Rightarrow (f \circ \varphi^{-1})(w) = \lambda w$ by schwartz w/ $|\lambda| = 1$

$\Rightarrow f(z) = \lambda \varphi(z)$

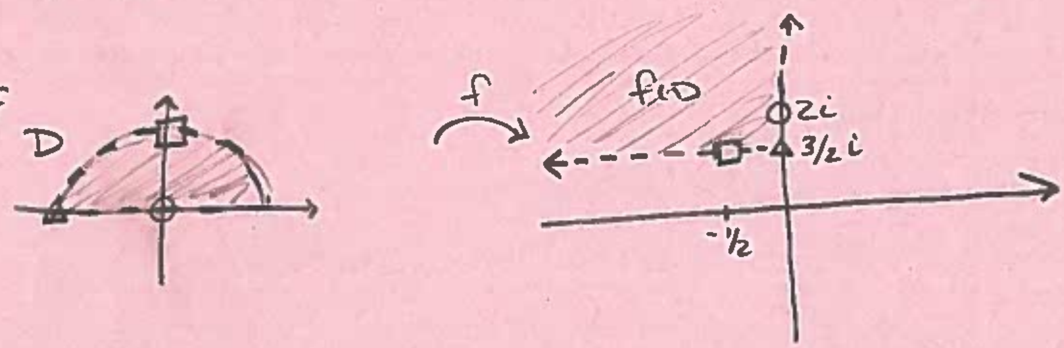
$\square$

Complex Analysis Exams

Midterm 2014

1. If $D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$ $f(z) = i\frac{z-2}{z-1}$, sketch D, find $f(D)$.

Pf



$f(z)$ has pole at $z=1$

$\Rightarrow$ unit circle is mapped to a line.

$f(-1) = i\left(\frac{-1-2}{-1-1}\right) = \frac{3}{2}i$

$f(i) = i\left(\frac{i-2}{i-1}\right) = -\frac{1}{2} + \frac{3}{2}i$

$\Rightarrow$ image of D is $y = \frac{3}{2}i$

$f(0) = i\frac{-2}{-1} = 2i$

$f(\frac{1}{2}i) = -\frac{2}{5} + \frac{9}{5}i$

$\therefore f(D) = \{z \in \mathbb{C} : \operatorname{Im} z > \frac{3}{2}, \operatorname{Re} z < 0.\}$

$\square$

2. Suppose $g: \mathbb{R} \to \mathbb{R}$ is a fcn of class C^2 s.t. $g''(x) > 0$ $\forall x \in \mathbb{R}$. $u: \mathbb{C} \to \mathbb{R}$ harmonic, $h = g \circ u$ is harmonic

Prove u is constant

<u>Pf</u> h harmonic

$$\Rightarrow h_{xx} + h_{yy} = 0$$

$$\Rightarrow (g \circ u)_{xx} + (g \circ u)_{yy} = 0$$

$$\Rightarrow ((g' \circ u) u_x)_x + ((g' \circ u) u_y)_y = 0$$

$$\Rightarrow (g' \circ u)_x u_x + (g' \circ u) u_{xx} + (g' \circ u)_y u_y + (g' \circ u) u_{yy} = 0$$

$$\Rightarrow (g'' \circ u) u_x^2 + (g' \circ u) u_{xx} + (g'' \circ u) \cdot u_y^2 + (g' \circ u) u_{yy} = 0$$

$$\Rightarrow (g'' \circ u) u_x^2 + (g'' \circ u) u_y^2 + (g'(u))(\underbrace{u_{xx} + u_{yy}}_{0}) = 0$$

$$\Rightarrow \underbrace{g''(u)}_{>0} (u_x^2 + u_y^2) = 0$$

$$\Rightarrow u_x^2 + u_y^2 = 0$$

$$\Rightarrow u_x = u_y = 0 \qquad \square$$

$$\Rightarrow u \text{ constant}$$

3. Assume f is entire s.t. $\int_{|z|=1} \frac{f(z)}{(nz-1)^2} dz = 0$ $\forall n \in \mathbb{Z}$ $n \geq 2$. Prove f constant.

Pf Note Cauchy Integral formula gives

$$f'(\omega) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{(z-\omega)^2} dz = 0$$

$$\Rightarrow 0 = \int_{|z|=1} \frac{f(z)}{(nz-1)^2} dz = \int_{|z|=1} \frac{f(z)}{n^2(z-1/n)^2} dz$$

$$= \frac{1}{n^2} \int_{|z|=1} \frac{f(z)}{(z-1/n)^2} dz$$

$$= \frac{1}{n^2} 2\pi i \, f'(1/n)$$

$\Rightarrow 0 = \frac{2\pi i}{n^2} f'(1/n)$

$\Rightarrow f'(1/n) = 0$ $\forall n \geq 2$

$\Rightarrow f'(z) = 0$ by identity principle.

$\Rightarrow f$ is constant. $\square$

4. Find and classify all singular points of

$$f(z) = \frac{z+\pi i}{e^z+1} + \cos 1/z$$

Pf Singular points:

1) $z_0 = 0$

2) $z_k = \pi i(1+2k)$ since $e^{z_k} = -1 \Rightarrow z_k = \log|-1| + i(\pi + 2k\pi)$
$= \log 1 + i(\pi + 2k\pi)$

3) $z_\infty = \infty$

1) $\cos 1/z = \sum_0^\infty \frac{(-1)^n}{(2n)!} z^{-2n}$

$\Rightarrow \exists$ infinitely many terms in Laurant expansion of f at z_0

$\Rightarrow z_0$ is an essential singularity of f.

2) $\lim_{z \to z_{-1}} \frac{z+\pi i}{e^z+1} = \lim_{z \to z_{-1}} \frac{1}{e^z} = -1 < \infty$

$\Rightarrow z_{-1} = -\pi i$ is a removable singularity.

for $k \neq -1$

$\lim_{z \to z_k} \frac{z+\pi i}{e^z+1} = \infty$

$\Rightarrow z_k$ is a pole of $f(z)$

$\frac{d}{dz}(e^z+1)\Big|_{z=z_k} = e^{i\pi(1+2k)} \neq 0$

$\Rightarrow z_k$ are simple zeros of e^z+1

$\Rightarrow z_k$ are simple poles of $f(z)$

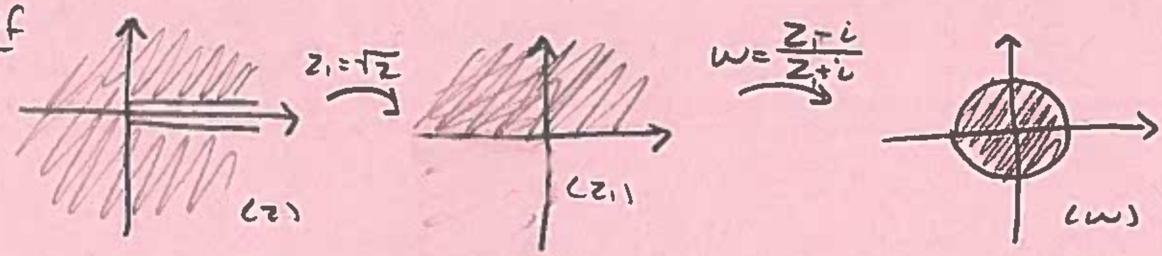
3) $\lim_{z \to z_\infty} z_k = \infty$

$\Rightarrow z_\infty$ is not isolated $\square$

Final Exam 2014

1) Find conformal map f from $\mathbb{C} \setminus [0,\infty) \to \mathbb{D}$ s.t. $f(-1) = c$

Pf



$$w = \frac{\sqrt{z} - i}{\sqrt{z} + i} \qquad w(-1) = 0 \quad \checkmark$$

2) Find all functions $g: \mathbb{R} \to \mathbb{R}$ of class C^2 s.t. $f(z) = x^2 - y^2 + ig(xy)$ $z = x + iy$ is entire.

Pf f entire $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\Rightarrow 2x = g'(x,y) x \quad -2y = -g'(x,y) y$$

$$\Rightarrow 2 = g'(xy) \qquad 2 = g'(xy).$$

$$\Rightarrow 2t + c = g(t) \quad \text{for } c \in \mathbb{R}$$

$\square$

3) Assume f is holomorphic on $D = \{|z| > 1\}$. $\lim_{z \to \infty} f = 1$ $r \in (0,1)$
Find $\int_{|z|=r} f(1/z) \, dz/z$.

Pf $\lim_{z \to \infty} f = 1 \Rightarrow f = 1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$

$\Rightarrow f$ converges uniformly in $\{|z| \geq R\}$.

$$|z| = r \Rightarrow \frac{1}{|z|} = \frac{1}{r} > 1$$

$$\Rightarrow f(1/z) 1/z = 1/z \, (1 + b_1 z + b_2 z^2 + \dots)$$
$$= 1/z + b_1 + b_2 z + \dots)$$

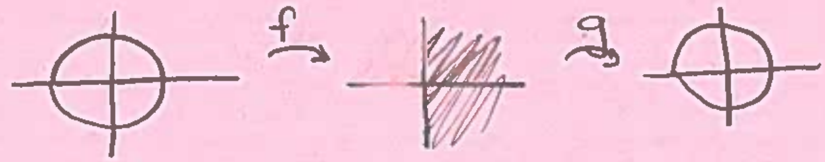
$$\Rightarrow \int_{|z|=r} f(1/z) 1/z = \int \frac{1}{z} + \sum b_n \underbrace{\int z^{n-1}}_{=0} = 2\pi i$$

$\square$

4) Let $\Delta = \{|z|<1\}$. $P = \{z \in \mathbb{C} \; \mathrm{Re}\, z > 0\}$. $f : \Delta \to P$ holomorphic with $f(0)=1$. Show $|f'(0)| \le 2$ and find all f s.t. $|f'(0)| = 2$.

Pf Let $g(w) = \frac{1-w}{1+w}$

$\Rightarrow g(P) = \Delta$



Let $h : \Delta \to \Delta$ s.t. $h(z) = g \circ f(z)$

$h(0) = g \circ f(0) = g(1) = 0$

$\Rightarrow |h'(0)| \le 1$ by schwartz

$\Rightarrow |g'(f(0)) f'(0)| \le 1$

$\Rightarrow |g'(1) f'(0)| \le 1$

$\Rightarrow \frac{1}{2} |f'(0)| \le 1$

$\Rightarrow |f'(0)| \le 2$

$|g'(w)| = \left| \frac{(1+w)(-1) - (1-w)(1)}{(1+w)^2} \right|$

$= \left| \frac{-2}{(1+w)^2} \right|$

Equality if $g \circ f(z) = \lambda z$ for $|\lambda| = 1$

$\Leftrightarrow \frac{1 - f(z)}{1 + f(z)} = \lambda z$

$\Leftrightarrow 1 - f(z) = \lambda z + \lambda z f(z)$

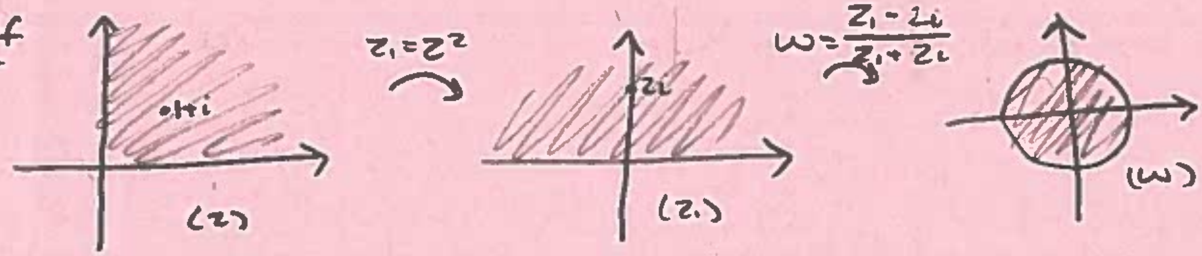
$\Leftrightarrow f(z) = \frac{1 - \lambda z}{1 + \lambda z}$ for $|\lambda| = 1$

$\square$

Final Exam May 2012

1) Find conformal map f from $D=\{z\in\mathbb{C} : \operatorname{Re} z>0, \operatorname{Im} z>0\}$ onto $\mathbb{D}$. $f(1+i)=0$

Pf

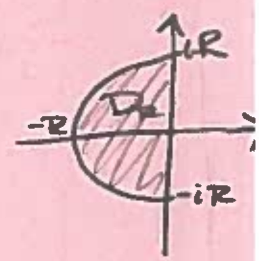


$$\Rightarrow \boxed{w=f(z)=\frac{z^2-2i}{z^2+2i}}$$

$\square$

2) How many zeros counted w/ multiplicity does $f(z)=z^4+e^z+z$ have in $D=\{z\in\mathbb{C} \ \operatorname{Re} z<0\}$.

Pf Consider $D_R=\{z\in\mathbb{C} \ \operatorname{Re} z<0 \ |z|<R\}$ $R>2$



if $z=iy$ then $|z^4+z|=y^4+z\geq 2 \ \forall y\in\mathbb{R}$

if $|z|=R$ then $|z^4+z|\geq R^4-2>14$ $(R>2)$

if $\operatorname{Re} z<0$ then $|e^z|=e^{\operatorname{Re} z}<1$

$\Rightarrow |z^4+z|>|e^z|$ for $z\in\partial D_R$

$\Rightarrow f$ has 2 zeros on D_R since z^4+z does.

$$z_1=2^{1/4}e^{3\pi i/4} \quad z_2=2^{1/4}e^{5\pi i/4}$$

$\Rightarrow f$ has 2 zeros in D.

$\square$

3) Find all entire fcns w/ $|f(z)| = 1$ on $|z|=1$.

Pf <u>Case 1</u> $f(z) \neq 0$ $\forall z \in \mathbb{C}$ w/ $|z|<1$

$\Rightarrow 1/f$ is bounded and entire since $|f| \leq 1$

$\Rightarrow 1/f$ is constant by Louiville

$\Rightarrow f$ is constant

$\Rightarrow f = \lambda$ where $|\lambda| = 1$

<u>Case 2</u> $z_1, \dots, z_k$ are zeros of f w/ order $n_1, \dots, n_k$.

Let $B = \sum_1^k \left(\frac{z - z_i}{1 - \bar{z}_i z}\right)^{n_i}$

$\Rightarrow f/B$ is holomorphic in nbhd of $\{|z| \leq 1\}$

$\Rightarrow |f/B| = 1$ if $z = 1$

f/B has no zeros $\Rightarrow f/B$ is co

$\Rightarrow f/B = \lambda$ w/ $|\lambda| = 1$ from above

$\Rightarrow f = \lambda \sum_1^k \left(\frac{z - z_i}{1 - \bar{z}_i z}\right)^{n_i}$

4. Find $\int_0^\infty \frac{\cos x}{1+x^4} dx$

Pf Let $f(z) = \frac{e^{iz}}{1+z^4}$ and $I = \int_{-\infty}^{\infty} f(z)$

$$\Rightarrow \int_0^\infty \frac{\cos x}{1+x^4}\, dx = \frac{1}{2} \operatorname{Re} I$$

By residue theory we know

$$I = 2\pi i \sum_{\operatorname{Im}\xi > 0} \operatorname{Res}\left(\frac{e^{iz}}{1+z^4}, \xi\right)$$

$f(z)$ has singularities at $e^{\pi i/4}, e^{2\pi i/4}, e^{3\pi i/4}, e^{\pi i}$ where $e^{\pi i/4}, e^{3\pi i/4}$ are in upper half plane

$$\operatorname{Res}(f(z), z_j) = \frac{e^{iz_j}}{4z_j^3} = \frac{-1}{4} z_j e^{iz_j} \qquad \text{Since } z_j^4 = -1$$

for $z_1 = e^{\pi i/4}$, $\frac{-1}{4} e^{\pi i/4} e^{ie^{\pi i/4}} = \frac{-1}{4} e^{\pi i/4} e^{i\frac{\sqrt{2}}{2}(1+i)}$

$$= \frac{-1}{4} e^{-\frac{\sqrt{2}}{2}} e^{i(\pi/4 + \frac{\sqrt{2}}{2})}$$

for $z_2 = e^{-3\pi i/4}$, $\frac{-1}{4} e^{-3\pi i/4} e^{i3\pi i/4} = \frac{1}{4} e^{-\frac{\sqrt{2}}{2}} e^{-i(\pi/4 + \frac{\sqrt{2}}{2})}$

So $I = 2\pi i \left(\frac{-1}{4} e^{-\sqrt{2}/2} \left(e^{i(\pi/4 + \sqrt{2}/2)} - e^{-i(\pi/4 + \sqrt{2}/2)}\right)\right)$

$$= 2\pi i \, \frac{-i}{2} e^{-\sqrt{2}/2} \sin(\pi/4 + \sqrt{2}/2)$$

$$= \pi e^{-\sqrt{2}/2} \sin(\pi/4 + \sqrt{2}/2)$$

$$\Rightarrow \int_0^\infty \frac{\cos x}{1+x^4} = \frac{\pi}{2} e^{-\frac{\sqrt{2}}{2}} \sin(\pi/4 + \sqrt{2}/2)$$

$\square$

Complex Exam Unknown Year

1. Find $\int_{|z|=2013} \frac{z^n}{z^{2013}-1}\,dz$ where $n \geq 0$ an integer.

Pf For $|z|>1$ $f(z)=\frac{z^n}{z^{2013}-1}$

$$\Rightarrow f(z)=\frac{z^n}{z^{2013}(1-\frac{1}{z^{2013}})} = z^{n-2013}\sum_0^{\infty}(1/z)^{2013k} = \sum_0^{\infty} z^{n-2013(1-k)}$$

By residue theory $\int_{|z|=2013} f(z)dz = -2\pi i\, \mathrm{Res}(f(z),\infty)$

Case 1 n is s.t. $\exists k$ s.t. $n-2013(1-k)=-1$ $\left(k=1-\frac{n+1}{2013}\right)$

$\Rightarrow \mathrm{Res}(f(z),\infty)=-1$ since $a_{-1}=1$

$\Rightarrow \int f(z)\,dz = 2\pi i$

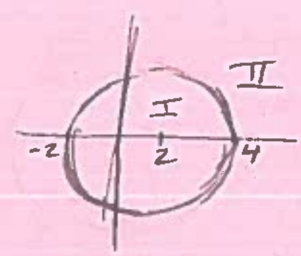
Case 2 there is no such k.

$\Rightarrow \int f(z)dz = 0.$

2. Find all Laurent Expansions centered at 2 of $\frac{z^2}{z+2}$

Pf $f(z)=\frac{z^2}{z+2} = \frac{z^2-4}{z+2}+\frac{4}{z-2} = z-2+\frac{4}{z+2}$

In I $|z-2|<4$



$$\frac{4}{z+2} = \frac{4}{z-2+4} = \frac{4}{4(\frac{z-2}{4}+1)} = \frac{1}{1-\left(-\left(\frac{z-2}{4}\right)\right)}$$

$$= \sum \left(\frac{-1}{4}\right)^k (z-2)^k$$

$$\Rightarrow \boxed{f(z) = z-2+\sum_0^{\infty} (-1/4)^k (z-2)^k}$$

In II $|z-2|>4$

$$f(z) = z-2+\frac{4}{z+2} = z-2+\frac{4}{z-2+4} = z-2+\frac{4}{z-2}\,\frac{1}{1+\frac{4}{z-2}}$$

$$\Rightarrow f(z) = z-2+\frac{4}{z-2}\left(\frac{1}{1+\frac{4}{z-2}}\right)$$

$$= z-2+\frac{4}{z-2}\sum (-4)^k (z-2)^{-k}$$

$$\boxed{f(z) = z-2+\sum_0^{\infty} (-1)^k 4^{k+1} (z-2)^{-k-1}}$$

3. Suppose f, g holomorphic w/ isolated singularity at 0
f has an essential one, g a pole. Show
$h = f/g$ has isolated singularity at 0. What type?

Pf g has a pole at 0

$\Rightarrow \lim_{z\to 0} g(z) = \infty$

$\Rightarrow \exists r>0$ s.t. f, g analytic and $|g(z)|>1$ for $0<|z|<r$

$\Rightarrow \frac{1}{g}$ is analytic and bdd in $0<|z|<r$

$\Rightarrow f \cdot 1/g$ is analytic for $0<|z|<r$

$\Rightarrow h$ has isolated singularity at 0

Now Suppose 0 is removable, or pole of h

$\Rightarrow h(z) = z^n H(z)$ where $H(0) \neq 0$, H analytic near 0

$\Rightarrow g(z) = z^{-m} G(z)$ where $G(0) \neq 0$, G analytic near 0

$\Rightarrow f(z) = z^{n-m} H(z) G(z)$ where $H(0)G(0) \neq 0$ HG analytic

$\Rightarrow$ f has removable singularity if $n \geq m$ and pole if $n<m$
which contradicts f has an essential singularity

$\Rightarrow$ 0 is an essential singularity of h.

$\square$

4. Let f be holomorphic on $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ s.t. $\operatorname{Im} f(z) \to 0$ as $z \in H \to x \in \mathbb{R}$ $\forall x \in \mathbb{R}$ and s.t. $|f(z)| \geq 1$ $\forall z \in H$. Prove $f(z) = c$ $\forall z \in \mathbb{C}$.

Pf By reflection across the real line we get f extends to an entire fcn, h, where $h(z) = \overline{f(\bar{z})}$ $\forall z \in \mathbb{C}$

By continuity $|h(z)| \geq 1$ $\forall z \in \mathbb{R}$.

$\operatorname{Im} z < 0 \Rightarrow |h(z)| = |\overline{f(\bar{z})}| = |f(\bar{z})| \geq 1$ since $\bar{z} \in H$

$\Rightarrow |h(z)| \geq 1$ $\forall z \in \mathbb{C}$.

Let $g(z) = 1/h(z)$

$\Rightarrow g$ is entire and bdd since h entire, $|h(z)| \geq 1$

$\Rightarrow g(z) = d$ for some $d \in \mathbb{C}$ by Louiville.

$\Rightarrow h(z) = c = 1/d$

$\operatorname{Im} c = 0$ on $\mathbb{R}$ $\Rightarrow$ $c \in \mathbb{R}$

$\Rightarrow h(z) = c$ $\quad c \in \mathbb{R}$

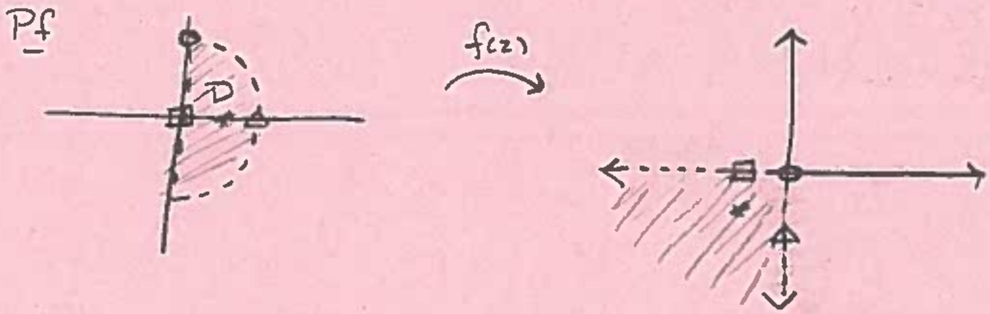
$\Rightarrow f(z) = c$ $\quad c \in \mathbb{R}$.

$\square$

Midterm Exam 2013

1. Find $f(D)$ wher $D = \{z \in \mathbb{C} : |z| < 2,\ \operatorname{Re} z > 0\}$ $f(z) = \dfrac{z-2i}{z+2i}$.

Pf



$-2i$ is pole of f $\Rightarrow$ $\{|z| = 2\}$ is sent to a line

$\Rightarrow \operatorname{Im} z$ is also sent to a line.

$f(2i) = 0$

$$f(2) = \frac{2-2i}{2+2i} \cdot \frac{2-2i}{2-2i} = \frac{4-8i-4}{8} = -i$$

$$f(0) = \frac{-2i}{2i} = -1$$

$$f(1) = \frac{1-2i}{1+2i} \cdot \frac{1-2i}{1+2i} = \frac{1-4i-4}{1+4} = \frac{-3}{5} - \frac{4}{5}i$$

$f(D) = \{z \in \mathbb{C} : \operatorname{Im} z < 0,\ \operatorname{Re} z < 0\}$ $\square$

2) If $u : D \to \mathbb{C}$ is C^2 show $\Delta u = 4 \dfrac{\partial^2 u}{\partial z \partial \bar{z}}$

Pf
$$4\frac{\partial^2 u}{\partial z \partial \bar{z}} = 4\frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right)(u)$$
$$= 2\left(\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u)\right.$$
$$= \left(\frac{\partial}{\partial^2 x} + \frac{\partial}{\partial^2 y}\right)(u)$$
$$= \Delta u$$

$\square$

3. If $f: D \to \mathbb{C}$ holomorphic on D s.t. $f(z) \neq c$ $\forall z \in D$
Show $\log|f|$ is harmonic.

Pf WTS $4 \dfrac{\partial^2 \log|f|}{\partial z \partial \bar{z}} = 0$

Let $u = \log|f| = \frac{1}{2}\log|f|^2 = \frac{1}{2}\log f\bar{f}$

$$4\frac{\partial^2 u}{\partial z \partial \bar{z}} = 4\frac{\partial}{\partial z}\left(\frac{1}{2}\frac{1}{f\bar{f}} f \frac{\partial \bar{f}}{\partial \bar{z}}\right)$$
$$= 4\frac{\partial}{\partial z}\left(\frac{1}{2}\overline{(f'/f)}\right)$$
$$= \frac{\partial}{\partial z}\overline{(f'/f)}$$
$$= \overline{\frac{\partial}{\partial \bar{z}}(f'/f)}$$

$= 0$ since f'/f is holomorphic on D.

$\Rightarrow \log|f|$ is harmonic. $\square$

4. Let L be line in $\mathbb{C}$ a det. D_1, D_2 connected Components of $\mathbb{C}/L$ Suppose u is cont. on $\mathbb{C}$ s.t u is harmonic on $D_1 \cup D_2$. Prove $\exists$ a harmonic v on $\mathbb{C}$ s.t. v is hamonic conjugate of u on $D_1 \cup D_2$ then u is hamonic on $\mathbb{C}$.

Pf Let $f = u + iv$

$\Rightarrow f$ is continuous on $D_1 \cup D_2$.

$\Rightarrow f$ is holomorphic on $\mathbb{C}$.

$\Rightarrow u = \text{Re} f$ is harmonic on $\mathbb{C}$. $\square$