Math 661 Notes

Based on *Topology: A Geometric Approach* by Terry Lawson

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Contents

Chapter 1: Basic Point Set Topology

Topology of $\mathbb{R}^n$

Topology: an abstraction of continuity in calculus

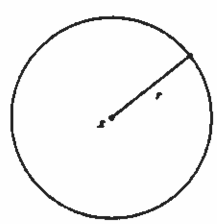
Recall from calculus:

$$\mathbb{R}^n = \{\vec{x} = (x_1, x_2, \ldots, x_n) | x \in \mathbb{R}\} \qquad d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

(Note, it is understood that for $x \in \mathbb{R}^n$, $\vec{x} = x$. Thus, $\vec{x}$ will be represented by x)

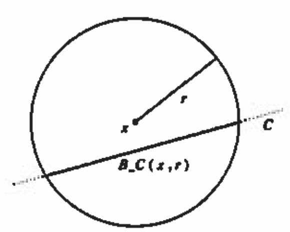
Definition. The (open) **ball of radius r around x**:

$$B(x, r) = \{y \in \mathbb{R}^n | d(x, y) < r\}$$



Definition. If $C \subset \mathbb{R}^n$ define:

$$B_C(x, r) = C \cap B(x, r) = \{y \in C | d(x, y) < r\}$$



Definition. (Version 1) Let X, Y be subsets of $\mathbb{R}^n$ and $f : X \to Y$ be a function. We say f is **continuous at $x \in X$** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y) < \varepsilon$.

Definition. (Version 2) Let X, Y be subsets of $\mathbb{R}^n$ and $f : X \to Y$ be a function. We say f is **continuous at** $x \in X$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $y \in B(x, \delta)$ then $f(y) \in B(f(x), \varepsilon)$ ($\iff$ $y \in f^{-1}(B(f(x), \varepsilon))$
Note: Since $y \in B_X(x, \delta) \Rightarrow y \in f^{-1}(B_Y(f(x), \varepsilon)$ then $B(x, \delta)) \subset f^{-1}(B_Y(f(x), \varepsilon))$

Definition. f is **continuous** if it is continuous at all $x \in X$.

Example. Define $f : \mathbb{R}^n \to R$ by $f(x) = d(0, x)$. Show f is continuous.
Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Suppose $y \in B(x, \delta)$. Then, using the triangle inequality:

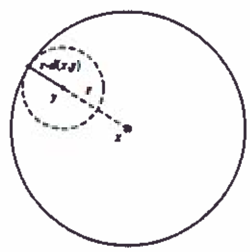
$$|f(x) - f(y)| = |d(x, 0) - d(y, 0)| \leqslant |d(x, 0)| + |d(y, 0)| \leqslant d(x, y) < \delta = \varepsilon$$

Definition. A set $U \subset \mathbb{R}^n$ is **open** if for each $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$.

Definition. If $C \subset \mathbb{R}^n$, $U \subset C$, we say U **is open in** C if for all $x \in U$ there exists $r > 0$ such that $B_C(x, r) \subset U$.

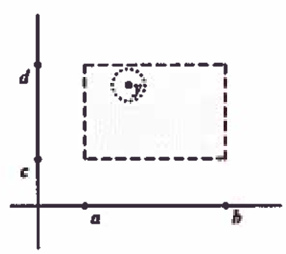
Examples. 1. $\varnothing$ is open.

2. $\mathbb{R}^n$ is open in $\mathbb{R}^n$ (in fact, any set is open in itself)
3. $B(x, r)$ is open in $\mathbb{R}^n$



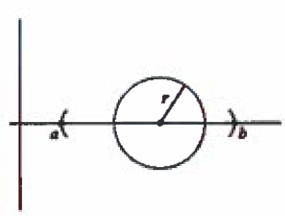
Proof. Let $y \in B(x, r)$. Then $B(y, r - d(x, y)) \subset B(x, r)$. □

4. $R = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$ is open in $\mathbb{R}^2$



Proof. Let $(x, y) \in R$. Then $B(\,(x, y)\,, \min\{b - x, x - a, d - y, y - c\}) \subset R$. □

5. (a, b) is open in $\mathbb{R}$. This is not the case if we look at the interval as part x-axis in $\mathbb{R}^2$, that is $A = \{(x, y) : x \in (a, b), y = 0\} \subset \mathbb{R}^2$. Since the positive y-coordinates are not in A, we cannot create a two dimensional ball contained in A.



Definition. (Version 3) $f : X \to Y$ is **continuous** (on all of X) if the inverse image of any open set in Y is open in X. That is, if $U \subset Y$ is open, then $f^{-1}(U) \subset X$ is open.

Theorem Definition Version 2 is equivalent to Definition Version 3.

Proof. ($\Rightarrow$) Assume for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^{-1}(B(f(X), \varepsilon))$. Let $U \subset Y$ be open. We want to show that $f^{-1}(U)$ is open in X. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there exists $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$. By version 2 there exists a $\delta > 0$ such that $B_X(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset f^{-1}(U)$. Therefore, $f^{-1}(U)$ is open.
($\Leftarrow$) Assume that if $U \subset Y$ is open, then $f^{-1}(U)$ is open. Let $x \in X$. Let $\varepsilon > 0$. Let $U = B(f(x), \varepsilon)$. Then $f^{-1}(U)$ is open by definition 3. So for all $y \in f^{-1}(U)$ there exists a $\delta > 0$ such that $B(y, \delta) \subset f^{-1}(U)$. Thus for all $y \in f^{-1}(U)$, $B(y, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Specifically, $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$ □

Moral: to define continuous functions, all we need is a concept of open sets.

Example. The composition of continuous functions is continuous.

Proof. Let $f : X \to Y$ and $g : Y \to Z$. Consider $g \circ f : X \to Z$. Let $U \subset Z$ be open. Let $V = g^{-1}(U)$. Since g is continuous, V is open Likewise, let $W = f^{-1}(V)$. Since f is continuous, W is open. Thus:

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V) = W$$

Therefore, $g \circ f$ is continuous. □

Topological Spaces

Definition. Let X be a set and let $\mathscr{T} = \{U_i \subseteq X : i \in I\}$ be a collection of subsets of X. Then $\mathscr{T}$ is called a **topology** on X, $(X, \mathscr{T})$ is called a **topological space**, and U_i are called **open sets** if:

1. $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$
2. For any $\mathscr{A} \subset I$, $\bigcup_{\alpha \in \mathscr{A}} U_\alpha \in \mathscr{T}$
3. For any finite $\mathscr{B} \subset I$, $\bigcap_{\beta \in \mathscr{B}} U_\beta \in \mathscr{T}$

That is to say, arbitrary unions of open sets are open and finite intersections of open sets are open.

Examples:

1. $X = \{a, b, c\}$
   (a) $\mathscr{T} = \{\varnothing, X\}$ is a topology
   (b) $\mathscr{T} = \{\varnothing, \{a\}, \{b\}, X\}$ is not a topology
   (it does not contain $\{a\} \cup \{b\} = \{a, b\}$)
   (c) $\mathscr{T} = \{\varnothing, \{a\}, \{a, b\}, X\}$ is a topology
2. Let X be any set. The **cofinite topology** $\mathscr{T} = \varnothing \cup \{U | X \backslash U \text{is finite}\}$ is a topology.

   Proof. Clearly $\varnothing, X \in \mathscr{T}$.
   Consider $U_\alpha \in \mathscr{T}$. Since $X \backslash (\bigcup_{\alpha \in \mathscr{A}} U_\alpha) = \bigcap_{\alpha \in \mathscr{A}} (X \backslash U_\alpha)$ is finite, $\bigcup_{\alpha \in \mathscr{A}} U_\alpha \in \mathscr{T}$.
   Consider $U_i \in \mathscr{T}$ for $i = 1, \ldots, n$. Since $X \backslash (\bigcap_{i=1}^n U_i) = \bigcup_{i=1}^n (X \backslash U_i)$ is finite union of finite sets and is thus finite, $\bigcap_{i=1}^n U_i \in \mathscr{T}$. $\square$

3. $\mathbb{R}^n$ is a topological space with open sets defined as in the previous section.

In fact: Let X be any set and $d : X \times X \to \mathbb{R}$ be a function, then d is called a **metric** if

- $d(x, y) > 0$, $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leqslant d(x, y) + d(y, z)$

A metric determines a topology by defining U to be open if for all $x \in U$ there exists and $r > 0$ such that $B(x, r) = \{y | d(x, y) < r\} \subset U$. Thus every metric space is a topology.

Example. Let X be any set. Define $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ Clearly d is a metric. Note that $B(x, \frac{1}{2})) = \{x\}$. Thus, every set is open. The topology on X is the power set of X. This is called the **Discrete Topology**.

Question. Does every topological space come from a metric?
No. Proof by "contradiction". Need to show that there exists an obstruction to get in the way of a topology being a metric.

Definition. A topological space $(\mathscr{T})$ is called **Hausdorff** if for all $x, y \in X$ there exists open sets U_x, U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. (i.e. you can separate points between open sets.)

Metric spaces are Hausdorff.

Proof. Let $x, y \in X$. $r = d(x,y)$ let $U_x = B\left(x, \frac{r}{2}\right)$ and $U_y = B\left(y, \frac{r}{2}\right)$ □

Exercise. Use the triangle inequality to show that $U_x \cap U_y = \varnothing$.

Exercise. Find an example of a non-metric space topology.

Example.
1. $\mathscr{T} = \{\varnothing, X\}$, $X \neq \varnothing$ is not Hausdorff, therefore no metric.
2. $\mathscr{T} = \{\varnothing, \{a\}, \{a,b\}, X\}$ not Hausdorff
3. cofinite topology when X is an infinite set. no metric.

Definition. Suppose $A \subseteq X$ is a topological space. We define the **subspace topology on** A by saying $U \subset A$ is open if there exists V open in X such that $U = A \cap V$.

Exercise. If $A \subseteq X$ has the subspace topology then the **inclusion map** $i : A \hookrightarrow X$ where $i(a) = a$ is continuous.

Definition. A set $C \subseteq X$ is **closed** if its complement is open.

Example. $[0,1]$ is closed. $[0,1)$ is not closed.

By DeMorgan's laws closed sets satisfy:

1. $\varnothing$, X are closed
2. The arbitrary intersections of closed sets are closed.
3. Finite union of closed sets are closed.

$f : X \to Y$ is continuous if and only if $f^{-1}(closedset)$ is closed.

Proof. ($\Leftarrow$) Suppose $f^{-1}(C)$ is closed for all $C \subseteq Y$ closed. Let $U \subset Y$ be open. Then $Y \backslash U$ is closed, so by assumption $f^{-1}(Y \backslash U) = f^{-1}(Y) \backslash f^{-1}(U) = X \backslash f^{-1}(U)$ is closed. Thus $f^{-1}(U)$ is open. Therefore f is continuous. □

Proposition. If $A \subseteq X$ has subspace topology, then $D \subset A$ is closed if and only if $D = A \cap E$ where E is closed in X.

Definition. The **closure** of a set $A \subset X$, $\overline{A}$ or $cl(A)$, is the intersection of all the closed sets which contain A.

Definition. The **interior** of a set $A \subset X$, $intA$ or $\mathring{A}$, is the union of all of the open sets contained in A.

Note: $Cl(A)$ is always closed and $int(A)$ is always open. Thus, $int(A)$ is the largest open set contained in A and $cl(A)$ is the smalled closed set containing A.

Exercise. $A = int(A)$ if and only if A is open. $A = cl(A)$ if and only if A is closed.

Definition. The **boundary** of A, $bd(A)$ is $\overline{A} \cap (\overline{X \backslash A})$.

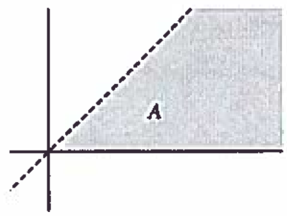
Exercise. $\overline{A} \cap \overline{B}$? $\overline{A \cap B}$

Example. $A = \{(x, y) | x > y > 0\}$, note A is open.

$int(A) = A$

$\overline{A} = \{(x, y) | x \geqslant y \geqslant 0\}$ (Note $x \in \overline{A}$ iff $\forall A \subset C$ closed iff every open set containing x intersects A. Further, $x \notin \overline{A}$ iff there exists open set disjoint from A that contains x.)

$bd(A) = \{(x, y) | x = y, y > 0\}$ (Note $x \in bd(A)$ iff every open set containing x intersects A and $X \backslash A$.)



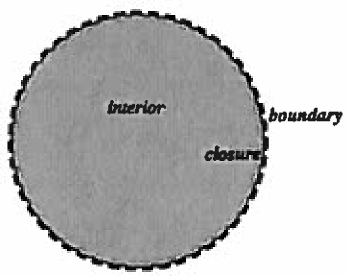
Topological Spaces

Last time:

- $int(A)$ = union of all open sets contained in A
- $\overline{A}$ = intersection of all closed sets containing A
- $int(A) \subseteq A \subseteq \overline{A}$
- $int(A)$ is the largest open set contained in A
- $\overline{A}$ is the smallest closed set containing A.
- $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Useful fact: $x \in \overline{A}$ if and only if every open set containing x intersects A nontrivially. That is, $U \cap A \neq \varnothing$ for all open $U \ni x$.

Examples. 1. If $A \subset \mathbb{R}^2$ then:



2. $B = \{(x, y) : x \in \mathbb{Q}, y > 0\}$: $int(B) = \varnothing$ since every open set in $\mathbb{R}^2$ contains a point with irrational coordinates. $\overline{A} = \{(x, y) : x \in \mathbb{R}, y \geqslant 0\}$. $bd(A) = \overline{A}$.

Useful Fact. $\overline{A} = int(A) \cup bd(A)$, $int(A) \cap bd(A) = \varnothing$, $bd(A) = \overline{A} \backslash int(A)$.

3. Let X an an infinite set with cofinite topology. Let $A \subset X$ be an infinite set with infinite compliment. $int(A) = \varnothing$, $bd(A) = X$, $\overline{A} = X$. That is, closed sets are finite or X.

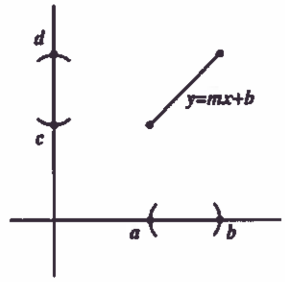
Planar Homology

Homeomorphism

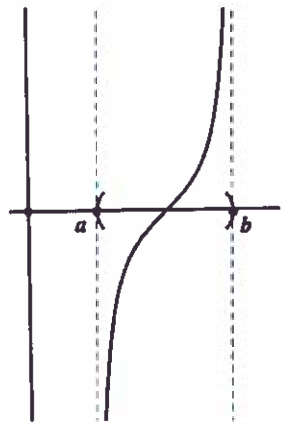
Definition. A map $f : X \to Y$ is called a **homeomorphism** if f is continuous, 1-1, onto and f^{-1} is continuous.

X being homeomorphic to Y forms an equivalence relation on the collection of topological spaces. We think of a homeomorphic meaning the spaces are topologically the same.

Examples. 1. (a, b) open interval in $\mathbb{R}$, $(c, d) \subset \mathbb{R}$ then $(a, b) \approx (c, d)$.



Moreover, $(a,b) \approx \mathbb{R}$



2. $f : [0, 2\pi) \to S^1$ via $f(t) = (\cos t, \sin t)$. Note, $S^1 = \{(x,y) : x^2 + y^2 = 1\}$. Note, f is continuous, 1-1, and onto. However, f^{-1} is not continuous: $(f^{-1})^{-1} = f([0,\pi]) = \{(x,y) : x^2 + y^2 = 1, y > 0 \cup (1,0)\}$

Definition. A **Planar homeomorphism** is a homeomorphism $f : A \to B$ for $A, B \subset \mathbb{R}^2$.

Example. An isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $d(x,y) = d(f(x), f(y))$ for all $x, y \in \mathbb{R}^2$.

(Homework: Show any isometry is continuous, use $d(x,y) = d(f(f-1(x)), f(f_1(y))) = d(f^{-1}(x), f^{-1}(y))$.

So if f is an isometry, so is f^{-1}. Thus f^{-1} is continuous. Therefore any isometry is a homeomorphism.

In fact, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of the form $f(x) = Ax + b$ where $b \in \mathbb{R}^2$, $A \in M_2$.

Definition. An **affine** map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a map of the form $f(x) = Ax + b$ where A is a 2×2 matrix and $b \in \mathbb{R}^2$.

Any affine map is continuous because it's a composition of a linear map and a translation.

Theorem. *A is invertible if and only if f is a homeomorphism.*

Proof. Set $f^{-1}(y) = A^{-1}(y) - A^{-1}b$ and check that $f \circ f^{-1} = I$ and $f^{-1} \circ f = id$. Recall if $f \circ g$ is 1-1 then g is 1-1. If $f \circ g$ is onto then f is onto. Note that f^{-1} is also affine. Therefore f is a homeomorphism. $\square$

Triangles

All triangles in $\mathbb{R}^2$ are homeomorphic.

Example. Consider the triangle $\Delta(e_0, e_1, e_2)$

$$e_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\Delta(e_0, e_1, e_2) = \{\lambda_1 e_1 + \lambda_2 e_2 | \lambda_1 > 0, 0 \leqslant \lambda_1 + \lambda_2 \leqslant 1\} = \{(\lambda_1, \lambda_2) | \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 \leqslant 1\}$$

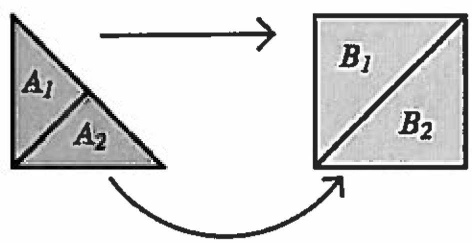
Definition. We say that vectors v_0, v_1, and v_2 are **affinely independent** if $v_1 - v_0$, $v_2 - v_0$ are linearly independent if and only if v_0, v_1, and v_2 form a triangle.

Proposition. $\Delta(e_0, e_1, e_2) \simeq \Delta(v_0, v_1, v_2)$.

Proof. Let $a_1 = v_1 - v_2$ and $a_1 = v_2 - v_0$. Let $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$. Suppose $f(x) = Ax + v_0$ such that $f(\Delta(e_0, e_1, e_2)) = \Delta(v_0, v_1, v_2)$. Thus $\Delta(e_0, e_1, e_2) \simeq \Delta(v_0, v_1, v_2)$. □

Question: Is a triangle always homeomorphic to a square? Yes, but not by affine map.

Pasting Lemma. Suppose $X = A \cup B$, $Y = C \cup D$, A, B, C, D are all closed. Suppose $f : A \to C$ and $g : B \to D$ are homeomorphisms that agree on $A \cap B$. Define $h : X \to Y$ such that $h|_A = f$ and $h|_B = g$. If h is a bijection then it is a homeomorphism.

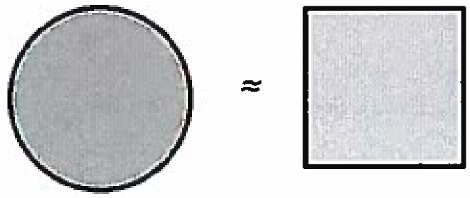


Example. By pasting lemma there exists a homeomorphism $h : A \to B$.

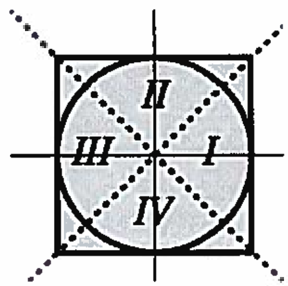
Definition. Let x and Y be a union of triangles along with edges and points. A **piecewise linear map** $f : X \to Y$ is a map such that the restriction to each triangle on f is affine map.

Planar Homology

Example. A circle is homeomorphic to a square.



Circle $= \{(x,y) : x^2 + y^2 \leqslant 1\}$ and square $= \{(x,y) : -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$



In polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$ where $0 \leqslant r \leqslant 1$,
$I : \frac{-\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$ $\quad II : \frac{\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{4}$ $\quad III : \frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4}$ $\quad IV : \frac{5\pi}{4} \leqslant \theta \leqslant \frac{7\pi}{4}$
I: $f_I(x,y) = t(x,y) = \frac{1}{\cos\theta}(x,y)$ $\quad t(\cos\theta, \sin\theta) = (1, ?) \Longrightarrow t\cos\theta = 1 \Rightarrow t = \frac{1}{\cos\theta}$
$f_I^{-1}(x,y) = \cos\theta(x,y)$
Polar coordinate: $f_I(x,y) = f_I(r\cos\theta, r\sin\theta) = \frac{1}{\cos\theta}(r\cos\theta, r\sin\theta) = (r, r\tan\theta)$
II: $f_{II}(x,y) = t(x,y) = \frac{1}{\sin\theta}(x,y) = (r\cot\theta, r)$
III: $f_{III}(x,y) = \frac{-1}{\cos\theta}(x,y) = (-r, -r\tan\theta)$
IV: $f_{IC}(x,y) = t(x,y) = \frac{-1}{\sin\theta}(x,y) = (-r\cot\theta, -r)$
These agree on intersection of regions. By piecing (or pasting) lemma we obtain a homeomorphism.

We will see that the annulus is not homeomorphic to a circle.

Compactness

Definition. Let X be a topological space, let $A \subseteq X$. An **open cover of** A is a collection of open sets $\{U_i\}_{i\in I}$ such that $A \subset \bigcup_{i\in I} U_i$.

Definition. A **subcover** of $\{U_i\}$ is a collection $\{V_j\}$ such that $\{V_j\} \subseteq \{U_i\}$ and $A \subset \bigcup_j V_j$.

Definition. A is **compact** if every open cover of A has a finite subcover.

Examples. 1. $\mathbb{R}$ is noncompact

Proof. $U_k = (-k, k)$, $k \in \mathbb{N}$ $\mathbb{R} = \bigcup_{k\in\mathbb{N}} U_k$. Note, any finite subset of $\{U_k\}$ will not be a cover. □

2. $[0,1)$ noncompact

Proof. $(-\frac{1}{2}, 1-\frac{1}{n}) = U_n$ is an open cover, but it has no finite subcover □

0,1 is compact. We will see this later.

Properties of Compactness

Theorem. Let $f : X \to Y$ be continuous and X is compact, then so is $f(X)$.

Proof. Let $\{U_i\}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_i)\}$ is an open cover of X. Since X is compact, there exists a finite subcover of X: $f^{-1}(U_1), \ldots, f^{-1}(U_n)$. That is $f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n) \supset X$. Further $f(f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)) = U_1 \cup \cdots \cup U_n \supset f(X)$. Therefore $U_1, \ldots, U_n$ is a finite subcover of $f(X)$. □

Note: $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) = f(A) \cap f(B)$.

Corollary. If $A \simeq B$ and A is compact, then B is compact.

Definition. A property that is invariant under homeomorphisms is called a **topological invariant**,

Definition. A subset A of a metric space X is **bounded** if it is contained in some ball $B(x,r)$.

Exercise. If $A \subset X$ is a compact subset of a metric space X then A is bounded. So not bounded $\Rightarrow$ not compact.

Recall: A space X is **Hausdorff** if for all $x, y \in X$ there exists open sets U_x and U_y such that $x \in U_x$, $y \in U_y$, $U_x \cap U_y = \varnothing$.

Proposition. In a Hausdorff space compact sets are also closed.

Proof. Let $A \subset X$ be a compact subset of Hausdorff space. Consider $X \backslash A$. Proceeding to show that it's open, let $x \in X\backslash A$. Then for all $y \in A$ there exists open sets U_y, V_y such that $y \in U_y$, $x \in V_y$, and $U_y \cap V_y = \varnothing$. Then $\{U_y : y \in A\}$ is an open cover of A. Since A is compact, there exists a finite subcover, $U_{y_1}, \ldots, U_{y_n}$. Define $V = \bigcap V_{y_n}$. Notice $(\bigcup_{i=1}^{n} U_{y_i}) \cap V = \varnothing$. But $A \subseteq \bigcup_{i=1}^{n} U_{y_n}$ so $A \cap V = \varnothing$. Thus $V \subset X\backslash A$. Thus $X\backslash A$ is open. Therefore A is closed. □

Corollary. If X is a metric space and $A \subset X$ is compact, then A is closed and bounded.

Question: does closed and bounded imply compact in a metric space?

Consider X to be an infinite set with the discrete metric. Every set is closed, every set is bounded, but only finite sets are compact.

Compactness

Properties.
- The continuous image of a compact set is compact
- In a Hausdorff space compact sets are closed
- In a metric space compact sets are closed and bounded

Proposition. Let X be compact, $A \subseteq X$ is closed, then A is compact.

Proof. Let $\{U_\alpha\}$ be an open cover of A. Since A is closed, $X\backslash A$ is open. Thus $\{U_\alpha\} \cup X\backslash A$ is an open cover of X. By compactness of X, there exists a finite subcover $U_1, U_2, \ldots, U_n, X\backslash A$. Thus $U_1, \ldots, U_n$ is an open cover of A. Therefore A has a finite subcover. ☐

Proposition. Let $f : X \to Y$ be a bijection. Suppose f is continuous, X is compact, and Y is Hausdorff. The f is a homeomorphism.

Proof. Need to show $f^{-1} : Y \to X$ is continuous. Let C be a closed subset of X. Then $(f^{-1})^{-1}(C) = C$. Thus C is compact as a closed subset of a compact set X. Then $f(C)$ is compact. Since Y is Hausdorff and $f(C)$ is compact, $f(C)$ must be closed. Therefore, f^{-1} is continuous. ☐

Recall that $\mathbb{R}$ has the least upper bound property: if $A \subset \mathbb{R}$ nonempty and bounded above, then A has a finite supremum.

Theorem. The closed interval $[a, b]$ is compact.

Proof. Let U_α be an open cover of $[a, b]$. Let $A = \{x \in [a, b] : [a, x] hasafinitesubcover\}$. A is nonempty since there exist a U_i such that $a \in U_i$. Since U_i is open, there exists an $\varepsilon > 0$ such that $[a, a + \varepsilon] \subseteq U_i$. Then $a + \varepsilon \in A$. Let $u = \sup(A)$.

Claim: $u = b$.

Proof. Suppose $u \neq b$. Then there exists some $U_{i(u)}$ such that $u \in U_{i(u)}$. Then there exists ε such that $(u - \varepsilon, u + \varepsilon) \subset U_{i(u)}$. Then there exists some $v \in A$ such that $u - \frac{\varepsilon}{2} < v < u$. So $[a, v]$ has a finite subcover $U_1, U_2, \ldots, U_n$. But $U_{i(u)}, U_1, \ldots, U_n$ is a finite subcover covering $[a, u + \frac{\varepsilon}{2}]$. Then $u + \frac{\varepsilon}{2} \in A$. However, $u = \sup A$. Thus, by contradiction, $u = b$. ☐

Since $u = b$, then there exists $U_{i(b)}$ such that $b \in U_{i(b)}$ such that for $\delta > 0$ $(b - \delta, b] \subset U_{i(b)}$ such that $w \in A$ such that $b - \frac{\delta}{2} < w < n$. There exists a finite subcover $U_1, \ldots, U_n$ covers $[a, w]$. Then $\{U_1, \ldots, U_n, U_{i(b)}\}$ is a finite subcover of $[a, b]$. ☐

Corollary. If $A \subset \mathbb{R}$ is closed and bounded then A is compact.

Proof. Since A is bounded, there exists $k \in \mathbb{R}$ such that $A \subset [-k, k]$. This is compact. Thus A is a closed subset of a compact set. Therefore A is compact. ☐

Proposition. A compact subset A of $\mathbb{R}$ has a largest and smallest element M.

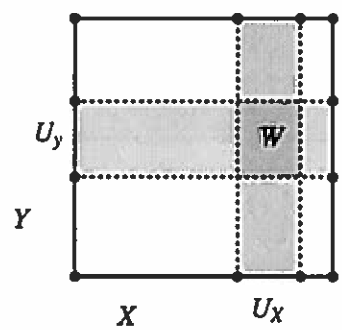
Proof. A compact $\Rightarrow$ A bounded $\Rightarrow$ A has a least upper bound U. Since A is closed, it contains its least upper bound. See text for proof. $\square$

Corollary. Let $f : X \to \mathbb{R}$ be continuous and X is compact. Then f achieves its maximum and minimum, i.e. there exists $x, y \in X$ such that $f(x) \leqslant f(z) \leqslant f(y)$ for all $z \in X$.

Proof. $f(x)$ is a compact set, thus it is closed and bounded. Therefore if has a max and a min. $\square$

The Product Topology

Definition. Suppose X and Y are topological spaces. Note $X \times Y = \{(x, y) | x \in X, y \in Y\}$. Define a topology on $X \times Y$ by saying that $W \subset X \times Y$ is open if for all $(x, y) \in W$ there exists open sets $U_x \subseteq X$ open and $U_y \subseteq Y$ open such that $(x, y) \in U_x \times U_y \subset X \times Y$. Call this the **Box topology**.



In fact, $U \times V$, U open in X and V open in Y form a basis for the topology.

Proposition. The topology coming from the standard metric on $\mathbb{R}^n$ is equivalent to product topology $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. That is, the open sets are the same.

Proof. Suppose U is open in $\mathbb{R}^n$ for all $x \in U$ there exists an $r > 0$ such that $B(x, r) \subseteq U$. Note that $x = (x_1, \ldots, x_n)$. For each x there exist $\varepsilon > 0$ such that $(x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \subseteq B_x, r$. Conversely, any open box contains an open ball so U open in $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. Therefore U is open in $\mathbb{R}^n$. $\square$

Lemma. If X, Y are topological spaces and $X \times Y$ is given the product topology. Then the inclusions, $y \in Y$ $i_y : X \to X \times Y$ such that $i_y(x) = (x, y)$ and $x \in X$ $i_x : Y \to X \times Y$ such that $i_x(y) = (x, y)$ are continuous. Furthermore, the projections, $p_x : X \times Y \to X$ such that $p(x, y) = x$ and $p_y : X \times Y \to Y$ such that $p(x, y) = y$ are continuous.

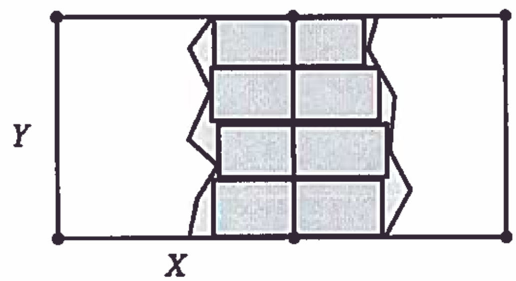
Theorem. (Tychanoff Theorem) Suppose X and Y are compact. Then $X \times Y$ is compact. (Common qual question)

The Product Topology

Theorem. (Heine-Borel) $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Lemma. (Tube Lemma) Let X, Y be compact. Let W be an open set in $X \times Y$ containing $\{x\} \times Y$ then there exists open set $U_x \subset X$ such that $\{x\} \times Y \subset U_x \times Y \subset W$. (Common qual question)

Proof. Since W is open, for all $y \in Y$ there exists $U_y \times V_y$ with U_y open in X and V_y open in Y such that $(x, y) \in U_y \times V_y \subset W$. Observe $U_y \times V_y$ is an open cover of $\{x\} \times Y$. Since i_x is continuous, Y is compact, and $i_x(Y) = \{x\} \times Y$, we know that $\{x\} \times Y$ is compact. Thus, there exists a finite subcover $\{U_{y_1} \times V_{y_1}, \ldots, U_{y_n} \times V_{y_n}\}$. Let $U_x = \bigcap_{i=1}^{n} U_{y_i}$. Clearly $\{x\} \times Y \subset U_x \times Y$ (since $x \in U_{y_i}$ for all $y_i \in Y$). Let $(x', y) \in U_x \times Y$, then $y \in V_{y_k}$ for some k. Thus $(x', y) \in U_{y_k} \times V_{y_k} \subset W$. Therefore $U_x \times Y \subset W$. □



Theorem. (Tychanoff Theorem) Suppose X and Y are compact. Then $X \times Y$ is compact. (Common qual question)

Proof. Let $\{W_i\}$ be an open cover of $X \times Y$. Let $x \in X$. There exists finitely many W_i that cover $\{x\} \times Y$, $\{W_{x(1)}, W_{x(2)}, \ldots, W_{x(n)}\}$. Let $W_x = \bigcup_{i=x(1)}^{x(n)} W_i$. BY the tube lemma there exists U_x such that $U_x \times Y \subset W_x$. Since X is compact there exists a finite subcover of $\{U_x | x \in X\}$, $\{U_{x_1}, U_{x_2}, \ldots, U_{x_r}\}$. Then $U_{x_1} \times Y, \ldots U_{x_r} \times Y$, is an open cover of $X \times Y$. So $W_{x_1}, \ldots, W_{x_r}$ is an open cover of $X \times Y$. Recall each W_{x_i} is a union of finitely many sets in original $\{W_i\}$. Taking all of these W_i together we get a finite subcover. □

Definition. A sequence $\{x_n\}$ **converges to** x in a topological space if given any open set U with $x \in U$ there exists N such that $x_n \in U$ for all $n \geqslant N$.

Example.
- X with discrete topology, eventually all elements in sequence are just X, it's eventually constant.
- X with $\mathscr{T} = \{\varnothing, X\}$, x_n converges to x if and only if every sequence converges to every point (limits are not unique)
- unique limits if and only if Hausdorff

Definition. X is called **sequentially compact** if every sequence has a convergent subsequence.

Theorem. In a metric space sequentially compact if and only if compact.

Proof. ($\Rightarrow$) Suppose (X, d) is sequentially compact. Let $\{U_i\}$ be an open cover. Then $\{U_i\}$ has a finite Lebesgue number, δ. Since (X, d) is totally bounded there exists a finite open cover of X by balls of radius $\frac{\delta}{3}$. Then $diam(B(x, \frac{\delta}{3})) \leqslant \frac{2\delta}{3} < \delta$. If $\{B(x, \frac{\delta}{3}) : i = 1, 2, \ldots, n\}$ is finite cover by δ-balls for each i. There exists U_i such that $B(x_i, \frac{\delta}{3}) \subset U_i$. Then U_i, $i = 1, \ldots, n$ is a finite subcover of $\{U_i\}$. □

Lemma. If $\{x_n\}$ is a sequence in a metric space such that there exists $x \in X$ such that any ball around x contains infinitely many elements of the sequence, then $\{x_n\}$ has a convergent subsequence.

Proof. Choose n_1 such that $x_{n_1} \in B(x, 1)$, n_2 such that $x_{n_2} \in B(x, \frac{1}{2})$, $\ldots$, n_j such that $x_{n_j} \in B(x, \frac{1}{j})$ where $n_j > n_{j-1}$. (Note, we can pick such n_j because there are infinitely many). Then $x_{n_k} \to x$.

Need to show that if X is compact then X is sequentially compact. Proceeding by contraposition, assume there exists x_n such that x_n does not have a convergent subsequence. Show X is not compact. By lemma, for all $x \in X$ there exists an open set U_x such that U_x contains only finitely many elements of $\{x_n\}$. So $\{U_x\}$ is an open cover of X. If it had a finite subcover, then there would be only finitely many distinct elements in $\{x_n\}$. This is not possible since $\{x_n\}$ does not have a convergent subsequence. □

Definition. Lebesgue Number Let $A \subset (X, d)$ be a metric space. Let $D_A = \{d(a_i, a_j) : a_i, a_j \in A\}$. If D_A bounded above, $diam(A) = \sup D_A$. In $\mathbb{R}^n$ $B(x, r) = A$, $diam A = 2r$.

Definition. A covering $\{U_i\}$ of a metric space (X, d) is said to have **Lebesgue number** $\delta > 0$ if for all $A \subset X$ of *diam* $> \delta$, A is contained in some element of the covering.

Note $\{B(x, 1) : x \in \mathbb{R}^n\}$ then has Lebesgue number $\delta < 2$.

Proposition. Let X be a metric space that is sequentially compact, then every covering has a finite Lebesgue number.

Proof. Contrapositive. Suppose there exists cover that does not have a Lebesgue number. Construct a sequence with no convergent subsequence. Let $\{U_i\}$ be a cover with no Lebesgue number. Then for all n there exists A_n such that $diam A_n < \frac{1}{n}$ which is not contained in a U_i. Let $x_n \in A_n$. Let $\{x_{n_k}\}$ be a subsequence that converges to x. Let $U_{x(i)}$ be an element of $\{U_i\}$ that contains x. For m large enough $B(x, \frac{1}{m}) \subset U_{x(i)}$. Choose $k \geqslant 2m$ such that $k \geqslant k_i$. Then $a_{n_k} \in B(x, \frac{1}{2m})$. Then for all $a \in A_{n_k}$, $d(a, x) \leqslant d(a, a_{n_K}) + d(a_{n_k}, x) < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}$. Then $A_{n_k} \subseteq B(x, \frac{1}{m}) \subset U_{x(i)}$. Therefore, there is no convergent subsequence. Thus, we've reached a contradiction. □

Definition. X is **totally bounded** if for all $\varepsilon > 0$ we can cover X by finitely many ε-balls.

Proposition. Sequentially compact implies totally bounded.

Proof. Suppose there exist $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Let $x_1 \in X$, $B(x_1, \varepsilon)$ is not a cover. Therefore, there exists $x_2 \in X$ $d(x_1, x_2) > \varepsilon$. Then $\{B(x_1, \varepsilon), B(x_2, \varepsilon)\}$ so there exists $x_3 \in X$. Then $d(x_1, x_3) \geqslant \varepsilon$, $d(x_2, x_3) \geqslant \varepsilon$, $d(x_1, x_2) \geqslant \varepsilon$. By induction there exists x_n such that $d(x_i, x_n) \geqslant \varepsilon$ for all $i < n$. Take $\{x_n\}$, $d(x_n, x_n) \geqslant \varepsilon$, $n \neq m$. Then there is no convergent subsequence. □

Connectedness

Definition. A topological space is called **separated** if it is the disjoint union of two nonempty open sets. ($A \subset X$ is separated if separated in subspace)

Definition. A set is called **connected** if is is not separated.

Exercise. X is connected if and only if only subsets of X that are both open and closed are $\varnothing$ and X.

Proposition. $A \subseteq X$ connected if and only if whenever there exists U, V open such that $U \cap V \cap A \neq \varnothing$ and $A \subseteq U \cup V$ then $A \subseteq U$ or $A \subseteq V$.

Proposition If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.

Proof. Assume $f(x)$ is separated, ten there exists U, V open, nonempty such that $U \cap V \cap f(X) = \varnothing$, $f(x) \subset U \cup V$. Then $f^{-1}(U)$, $f^{-1}(V)$ is a separation of X. □

Corollary. If $A \simeq B$, A is connected if and only if B is connected. (Example, $[0, 1]$ and $[1, 2] \cup [3, 4]$ cannot be homeomorphic.)

Lemma. If $A \subseteq \mathbb{R}$ is connected then for all $x, y \in A$, $[x, y] \subseteq A$.

Proof. Suppose $x, y \in A$ such that there exists $z \in [x, y]$ such that $z \notin A$. Let $U = (-\infty, z)$ and $V = (z, \infty)$. Then $A \cap U$ and $A \cap V$ forms a separation of A. □

Proposition. If $A \subseteq \mathbb{R}$ is connected, then A is an interval, ray, $\mathbb{R}$.

Proof. Suppose $A \subset \mathbb{R}$ is connected and for all $x, y \in A$, $[x, y] \subseteq A$. Consider the cases: A is bounded, A is bounded above but not below, A is bounded below but not above, and A is not bounded above or below.

If A is bounded, let $a = \inf(A)$, $b = \sup(A)$. Let $c \in (a, b)$, $c < b$ then there exists a $b' \in A$ such that $c < b' < b$ and $c > a$ so there exists $a' \in A$ such that $a < a' < c$. Thus $a' < c < b'$. By our initial suppositions, $c \in A$. thus $(a, b) \subset A \subset [a, b]$. This works similarly for $\pm\infty$. □

Proposition. $[a, b]$ is connected.

Proof. Suppose $[a, b]$ were separated. Then $[a, b] \subset U \cup V$ where $U \cap V \cap [a, b] = \varnothing$, then $U \cap [a, b] \neq \varnothing$ and $V \cap [a, b] \neq \varnothing$. Without loss of generality, let $a \in U$. Let $A = \{x \in [a, b] : [a, x] \subset U\}$. Then A has a supremum, $u = \sup A$. We want $u \in U$. Suppose $u \notin U$, then $u \in V$. Thus there exists $\varepsilon > 0$ such that $(u - \varepsilon, u + \varepsilon) \subset V$. This is a contradiction. Thus $u = \sup A$. Suppose $u \neq b$. Then, since $u \in U$ and U open, there exists $\varepsilon > 0$ such that $(u - \varepsilon, u + \varepsilon) \subseteq U$. Then $[a, u + \frac{\varepsilon}{2}] \subset U$ and $u + \frac{\varepsilon}{2} \in A$. However, this is not possible since u is the upper bound. Therefore $u = b$ and since $u \in A$ then $b \in A$. So $[a, b] \subset U$. Thus $V \cap [a, b] = \varnothing$. Thus, we've reached a contradiction. Therefore $[a, b]$ is connected □

Proposition. $\{A_i\}_{i \in I}$ is a collection of connected subsets of X and $\bigcap_{i \in I} A_i \neq \varnothing$ Thus $A = \bigcup_{i \in I} A_i$ is connected.

Proof. Suppose A were separated. $A \subseteq U \cup V$ such that $U \cap V \cap A = \emptyset$. Let $a \in U \cap A$, then there exists i such that $a \in A_i$. A_i is connected. Then there exists an i such that $a \in A_i$. Since A_i is connected then $A_i \subseteq U$. Thus, $x \in \bigcap_{i \in I} A_i \subset U$ Then $A_i \subset U$ for all i. Since $x \in A_i$ for all i, $A \subset U$. Thus $V \cap A = \emptyset$. ☐

Examples. All intervals are connected

- $(a, b] = \bigcup_{i=i_0}^{\infty} [a + \frac{1}{i}, b]$ choose i_0 such that $a + \frac{1}{i_0} < b$.
- $(a, b) = \bigcup_{i=i_0}^{\infty} [a + \frac{1}{i}, b - \frac{1}{i}]$.
- $(a, \infty) = \bigcup_{i=i_0}^{\infty} [a + \frac{1}{i}, i]$.

Proposition. If X and Y are connected, then $X \times Y$ is connected.

Proof. Let $A_x = (\{x\} \times Y) \cup (X \times \{y_0\})$. Then A_x is the union of two connected sets who's intersection if $\{(x, y_0)\}$. Therefore A_x is connected. Consider $\{A_x\}_{x \in X}$. Then $\bigcup A_x = X \times Y$, $\bigcap A_x = X \times \{y_0\}$. Therefore, by proposition, $X \times Y$ is connected. ☐

(Some more here, but just scratch level work)

Connectedness

Theorem. (Intermediate Value Theorem) Suppose $f : X \to \mathbb{R}$ is continuous and X is connected. Then for all $v \in \mathbb{R}$ such that there exists $x, y \in X$ such that $f(x) \leqslant v \leqslant f(y)$, there exists a $c \in X$ such that $f(c) = v$.

Proof. Homework, use the face that a connected set in $\mathbb{R}$ is an interval. ☐

Definition. A space is called **path connected** if for all $x, y \in X$ there exists $p : [0,1] \to X$ continuous such that $p(0) = x$, $p(1) = y$.

Proposition. The continuous image of a path connected set is path connected.

Proof. Let $f : X \to Y$ be continuous. Let $f(x), f(y) \in f(X)$. Since X is path connected there exists $p : [0,1] \to X$ such that $p(0) = x$, $p(1) = y$. So $f \circ p$ is a path from $f(x)$ to $f(y)$ in $f(X)$. ☐

Proposition. If X is path connected, then X is connected.

Proof. Assume X is path connected. Let $x_0 \in X$, For all $y \in X$ there exists p_y such that $p_y(0) = x_0$ and $p_y(1) = y$. Let $A_y = p_y([0,1])$. A_y are connected. Since $\bigcup A_y = X$ and $\bigcap A_y = x_o$, X is also connected. ☐

Note, the converse doesn't hold. Consider the topologist sin curve is $A \cup B$ for: $A = \{(x, \sin \frac{1}{x}) | x \in \mathbb{R}\}$ $B = \{(0,y) | -1 \leqslant y \leqslant 1\}$.

It is useful to define equivalence relations related to connectedness/ path connectedness.

Definition. $\sim$ is an **equivalence relation** if:

- $x \sim x$
- $x \sim y \Rightarrow y \sim x$
- $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

we say $x \sim' y$ if and only if there exists a connected set A, $A \subseteq X$ such that $x, y \in A$. Note that $\sim'$ is indeed an equivalence relation:

- $x \sim' x$ because $\{x\}$ is connected
- $x \sim' y \Rightarrow y \sim' x$ by definition
- $x \sim' y$ and $y \sim' z$ then $x \sim' z$ because $x \sim' y \Rightarrow x, y \in A_1$, $y \sim' z \Rightarrow y, z \in A_2$. Note $y \in A_1 \cap A_2$, so intersection is nonempty. Thus $A_1 \cup A_2$ is connected. Thus $x \sim' z$.

Definition. These equivalence classes are called the **connected components of** X.

Consider the equivalence relation $x \sim y$ if there exists $p : [0,1] \to X$ such that $p(0) = x$, $p(1) = y$. The equivalence classes are the same as path components.

Example. $\mathbb{Q} \subset \mathbb{R}$. The connected components are the singletons, $\{x\}$. These are also the path components.

Proposition. The connected components are connected, disjoint subsets of X whose union is all of X and each connected subset of X intersects only one component.

Proof. Let C be a component of X. Let $x \in C$. Then $C = [x] = \{y : x \sim' y\}$. Then for all $y \in C$ there exists a connected set A_y such that $x, y \in A_y$, $x \in \bigcap_y A_y$, $\bigcup A_y = C$ is connected. Therefore C is connected. If $A \subset X$ is connected, suppose there exists C_1, C_2 are two components and $x \in A \cap C_1$ and $y \in A \cap C_2$. Then $x \sim' y$. Thus, we've reached a contradiction. $\square$

Definition. A space X is **locally connected** at x if for all open sets U such that $x \in U$ contains an open set V containing x such that V is connected.

Examples.
1. (a, b) is connected and locally connected
2. $(0,1) \cup (1,2)$ is not connected but is locally connected
3. Topologists sine curve, connected but not locally connected at any point on B.
4. $\mathbb{Q}$ is not connected not locally connected

Proposition. A space is locally connected if and only if for all open set $U \subset X$ the connected component of Y are open. (Note, components are also closed if locally connected.)

Proof. Suppose X is locally connected. Let $U \subset X$ open. Let C be a component of U. Let $x \in C$. Since X is locally connected there exists some U_x that is connected such that $x \in U_x$, $U_x \subset U$, so $U_x \subset C$. Then $C = \bigcup U_x$. Then then connected components are open.

Suppose every component of U is open. Let $x \in U$, take C_x a component of U containing x. C_x is open and connected. Thus X is locally connected. $\square$

Similarly, define locally path connected. Propositions 1 and 2 work the same exact way for path component.

Theorem. If X is locally path connected, then path components and connected components are the same.

Corollary. If X is connected and locally path connected then X is path connected.

Connectedness

Path connected $\Rightarrow$ connected, but connected $\not\Rightarrow$ path connected

BUT: A space is locally connected at x if for all open U such that $x \in U$ there exists V open such that $x \in V$, $V \subseteq U$ and U is path connected.

Theorem. If X is locally path connected, then the components and path components are the same. In particular, connected implies path connected when X is locally path connected.

Proof. Let $x \in X$. Let C be a component of X containing x. Let P be the path component of X containing x. Because path connected implies connected, $P \subset C$ always. Suppose $P \neq C$. Since X locally path connected so P is open. Let $Q = \bigcup_{y \notin P} P_y$. Then P and Q are both open, $P \cup Q = X$ and $P \cap Q = \varnothing$. If $P \neq C$ then P, Q separates C, a contradiction since components are always connected. □

Quotient Topology

Definition. Suppose X is a topological space and $q : X \to Y$ is surjective, the **Quotient Topology** is the topology on Y given by U is open is Y if and only if $q^{-1}(U)$ is open in X.

Y is called the quotient space of X, q is called the quotient map.

if Y is already a topological space we say q is a quotient map and Y has the quotient topology.

Equivalent to $Y = X/\sim$, $q : X \to X/\sim$ via $q(x) = [x] = \{y | x \sim y\}$.

For any $q : X \to Y$ can define $\sim$ on X from $x_1 \sim x_2$ if and only if $q(x_1) = q(x_2)$. Thus equivalence classed are $q^{-1}(a)$.

Examples. 1. $X = [0, 1]$, $0 \sim 1$ (Identify 0 and 1) so $q : X \to X/\sim \approx S^1$.

2. $X = \mathbb{R}$, $x \sim y$ if there exists $n \in \mathbb{Z}$ such that $y = x + n$.

Definition. Open sets of form $q^{-1}(U)$ where U is open in Y are called **saturated** open sets if and only if the open sets which contain all equivalence classes that they intersect

Examples. 1. Saturated sets of $[0, 1]$ are open subsets of $(0, 1)$ and the open sets that contain 0 and 1. So $[0, 1) \cup (b, 1]$ is saturated. $[0, a)$ is not saturated but is open in $[0, 1]$.

2. Saturated sets of $\mathbb{R}$ are of form $\bigcup_{n \in \mathbb{Z}} (a + n, b + n)$.

Proposition. Suppose $q : X \to Y$ is a quotient map. Then $g : Y \to Z$ is continuous if and only if $g \circ q : X \to Z$ is continuous.

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "g\circ q"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Proof. If g is continuous then $g \circ q$ is continuous.

Assume $g \circ q$ is continuous. Let U be open. Then $(g \circ q)^{-1}(U)$ is open iff $q^{-1}(g^{-1}(U)$ open iff $g^{-1}(U)$ is open. Thus g is continuous. □

Suppose $q : X \to Y$, $Y = X/\sim$. Let $f : X \to Z$ be a function such that $x_1 \sim x_2$ then $f(x_1) = f(x_2)$. Define $\overline{f} : Y \to Z$, $\overline{f}([x]) = f(x)$ if and only if well defined. Then $\overline{f}(q(x)) = \overline{f}([x]) = f(x)$. Thus $\overline{f} \circ g = f$.

Suppose there are two quotient spaces, $q : X \to Y$ and $q' : X' \to Y'$. Then there exists $f : X \to X'$ and $\overline{f} : Y \to Y'$. such that $\overline{f}q = q'f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & X' \arrow[d, "q'"] \\ Y \arrow[r, "\overline{f}"'] & Y' \end{tikzcd}$$

Then $\overline{f}$ is continuous if f is continuous. Why? Because $\overline{f}$ is continuous if $\overline{f}q$ is continuous if and only if $q'f$ is continuous.

Proposition. Suppose $f : X \to Y$ is a surjective continuous map with X compact and Y Hausdorff. Define $u, v \in X$ $u \sim v$ if and only if $f(u) = f(v)$. Then the induced map $\overline{f} : X/\sim \to Y$ is a homeomorphism.

Proof. Note f is continuous, bijection $\overline{f}([x_1]) = \overline{f}([x_2])$ then $f(x_1) = f(x_2)$ so $x_1 \sim x_2$ thus $[x_1] = [x_2]$. □

Example. Consider $f : [0,1] \to S^1$ via $f(x) = (\cos(2\pi x), \sin(2\pi x))$. By proposition $[0,1]/\sim \approx S^1$.

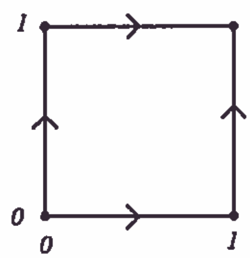
$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "q"] & S^1 \\ {[0,1]/\sim} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Quotient Topology

$f : X \to Y$ is surjective and continuous. Define $\sim$ on X via $u \sim v$ if and only if $f(u) \sim f(v)$ Then $\overline{f} : X/\sim \to Y$ via $\overline{f}([x]) = f(x)$. Then $\overline{f}$ is a continuous bijection. If Y is Hausdorff and $X/\sim$ is compact, then $\overline{f}$ is a homeomorphism. In general $\overline{f}$ is a homeomorphism if and only if f takes saturated open sets of X to open sets of Y.

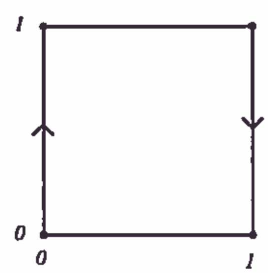
Examples. 1. Real line with two origins. $X = \mathbb{R} \times \{0,1\}$ $(x,1) \sim (x,0)$ if $x \neq 0$
If U is open around 0, then $(U \backslash \{0\}) \cup (\{p\})$ is also open in $X/\sim$. Can show $X/\sim$ is connected. d

2. Torus: $[0,1] \times [0,1]$ with $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$.



Note $f : [0,1] \times [0,1] \to S^1 \times S^1$ via $f(s,t) = (\cos(2\pi s), \sin(2\pi s), \cos(2\pi t), \sin(2\pi t))$. Then $f(0,t) = f(1,t)$ and $f(s,0) = f(s,1)$. Therefore $\overline{f}$ induced is a homeomorphism since $[0,1]$ is compact and $S^1 \times S^1$ is Hausdorff.

3. Mobius Band: $[0,1] \times [0,1]/\sim$ with $(0,t) \sim (1,-1-t)$.



4. Real Projective Space: Let $X = \mathbb{R}^{n+1} \backslash \{0\}$. Then $x \sim y$ if there exists $t \in \mathbb{R}$ and $t \neq 0$, $y = tx$. Then $x \sim y$ if and only if x, y lie on some line through origin. Then $X/\sim = \mathbb{R}P^n = P^n$ is real projective space, the space of lines through origin.

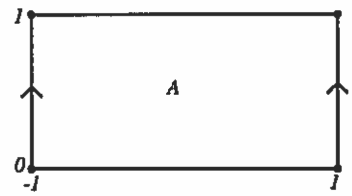
5. $X + \mathbb{C}^{n+1} \backslash \{0\}$. Then $w \sim z$ if there exists $\alpha \in \mathbb{C}$ and $\alpha \neq 0$ such that $w \neq \alpha z$. This is called complex projective space of "Complex lines" through origin in $\mathbb{C}^{n+1}$.

Another common way to construct quotient spaces is to take the disjoint union and glue along a closed set.

Let A, B be two topological spaces. Then $A \sqcup B$ as a set is disjoint union of A and B. Open sets are just union of open set in A and open sets in B. Let K be a closed subset of B and f a homeomorphism from K to $f(K) \subset A$. Then $A \cup_f B = A \sqcup B/\sim$ where $y \sim x$ if and only if $f(x) = y$ where $y \in f(K)$ for $x \in K$.

Examples. 1. Line with two origins: $\mathbb{R} \cup_{id} \mathbb{R}$ via $id : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$.

2. Annulus: $A = \{(x, y) | 1 \leqslant x^2 + y^2 \leqslant 2\}$. Consider $f : [-1, 1] \times [1, 2] \to A$ via $f(s, t) = (t \cos(\pi s), t \sin(\pi s))$. Note $A \approx [-1, 1] \times [1, 2]/\sim$ where $(1, t) \sim (-1, t)$.



Now, consider $B = [-1, 0] \times [1, 2]$ and $C = [0, 1] \times [1, 2]$. Consider $B \sqcup C$. $K = \{0\} \times [1, 2] \cup \{1\} \times [1, 2]$. Then $f(0, t) = f(0, t)$ and $f(1, t) = f(-1, t)$. Then $A = B \cup_f C$. Alternatively $[-1, 0] \times [1, 2] \cup_g [0, 1] \times [1, 2]$ where $g(0, t) = (-1, t)$ and $g(1, t) = (0, t)$.

Quotient Topology

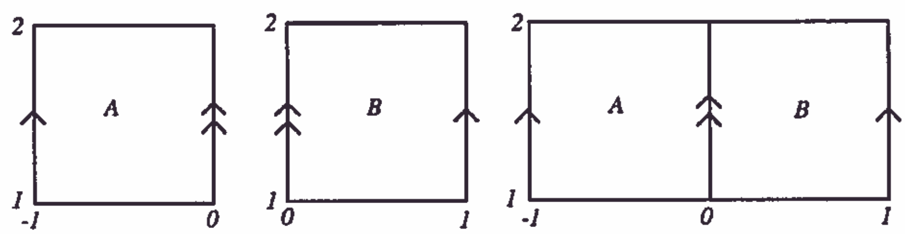
Porposition. (1.7.3 / 1.7.4) Let $f : X \to Y$ be a continuous surjection. Define $\sim$ such that $u \sim v$ if and only if $f(u) = f(v)$. Then the induced map is:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & Y \\ X/\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

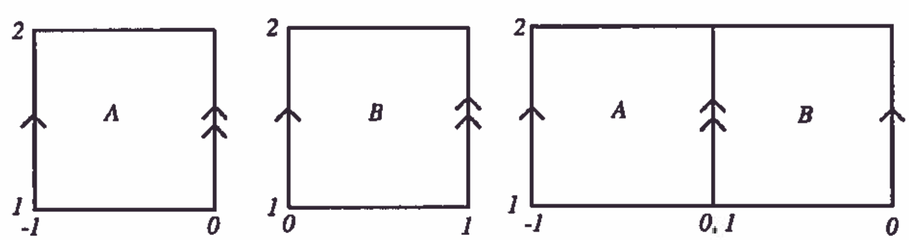
where $\overline{f}([x]) = f(x)$. If X is compact and Y is Hausdorff the $\overline{f}$ is a homomorphism.

Examples. Annulus: $A = \{(x,y) | 1 \leqslant x^2 + y^2 \leqslant 2\}$.

1. Consider $B = [-1,0] \times [1,2]$ and $C = [0,1] \times [1,2]$. Consider $B \sqcup C$. Then $g(0,t) = g(0,t)$ and $g(1,t) = g(-1,t)$. Then $A = B \cup_g C$.



2. Alternatively $[-1,0] \times [1,2] \cup_g [0,1] \times [1,2]$ where $g(0,t) = (-1,t)$ and $g(1,t) = (0,t)$.



Note that $1 \approx 2$.

Proof. First note that $A = [-1,1] \times [1,2]/\sim$ where $(1,t) \sim (-1,t)$. Let $f : [-1,1] \times [1,2] \to A \cup_g B$ via $f(s,t) = [s,t]$. Then $u \sim v$ if and only if $f(u) = f(v)$. Thus $(1,t) \sim (-1,t)$ implies that $[(1,t)] = [(-1,t)]$ by definition at g. If $[(s_1,t_1)] = [(s_2,t_2)]$ then either $(s_i,t_i) = (1,t), (-1,t)$ or $s_1 = s_2$, $t_1 = t_2$. Thus $[-1,1] \times [1,2]/\sim \approx A \cup_g B$. We want to show that $A \cup_g B \approx A \cup_h B$. We know:

$$\begin{tikzcd} A \sqcup B \arrow[r, "F_A \sqcup F_B"] \arrow[d, "g"] & A \sqcup B \arrow[d, "h"] \\ A \cup_g B & A \cup_h B \end{tikzcd}$$

where $F_A : A \to A$ and $F_B : B \to B$ are homeomorphisms. Want $f : A \sqcup B \to A \cup_h B$ where $f(a) = [F_A(a)]$ and $f(b) = [F_B(b)]$. We want $u \sim v$ if and only if $f(u) = f(v)$. Consider $a \in A$, $b \in B$ such that $g(b) = a$. Then $a \sim b$ if and only if $[F_A(a)] = [F_B(b)]$ if and only if $F_A(a) = h(F_B(b))$ if and only if $F_A(g(b)) = h(F_B(b))$. We see that $u \sim v$ if and only if $f(u) = f(v)$ is satisfied if and only if $F_B(k_1) = k_1$) and $F_A g = hF_B$. First, proposition 1.7.4 implies that there exists a continuous bijection such that $A \cup_g B \to A \cup_h B$. Since F_A, F_B, g, h are all homeomorphisms we can assume all arrows, get inverse map. So $A \cup_g B \to A \cup_h B$ is a homeomorphism. Using proposition 1.7.6 we can see that $F_A g = hF_B$:

$$F_B(s,t) = (1-s,t) \qquad F_A(s,t) = (s,t)$$

$$F_A g(0,t) = F_A(0,t) = (0,t) \qquad hF_B(0,t) = h(1,t) = (0,t)$$

$$F_A g(1,t) = F_A(-1,t) = (-1,t) \qquad hF_B(1,t) = h(0,t) = (-1,t)$$

Therefore, $A \cup_g B \approx A \cup_h B$

□

Exercise 1.7.8 Mobius band is homeomorphic to $D^1 \times D^1 \cup_f D^1 \times D^1$. Here $D^1 = [-1,1]$. $f(-1,t) = (-1,t)$ and $f(1,t) = (1,-t)$. Then Mobius band is $D^1 \times D^1/\sim$ where $(-1,t) \sim (1,-t)$. BY same argument as annulus, $D^1 \times D^1/\sim = [-1,0] \times D^1 \cup_g [0,1] \times D^1$ where $g(0,t) = (0,t)$ and $g(1,t) = (-1,-t)/$ Then $F_B(s,t) = (2s-1,t)$ and $F_A(s,t) = (-1-2s,t)$. Check that $fF_B = F_A g$.

Chapter 2: Classification of Surfaces

Definitions and Construction of Models

Definition. A **surface** is a space the locally looks like a plane. That is, every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^2$.

Definition. A **manifold** is a space such that every point has a neighborhood homeomorphic to $\mathbb{R}^n$. (A surface is a 2-manifold)

Definition. A map is an **embedding** if it is a homeomorphism onto its image with subspace topology.

Definition. A topological space is an n-**manifold** if:

1. there is an embedding of M into $\mathbb{R}^N$
2. given $x \in M$ there exists open set U such that $x \in U$ and U is homeomorphic to an open subset of $\mathbb{R}^n$

Note: 1 is a technical assumption. Consider:

1'. M is Hausdorff and topology for M has countable basis.

Observe, 1 implies 1' always. Thus 1 and 2 implies 1' and 2.

Classification Problem: We want to find a list M_i of all n-manifolds such that $M_i \not\approx M_j$. Restricting our scope to compact, connected n-manifolds:

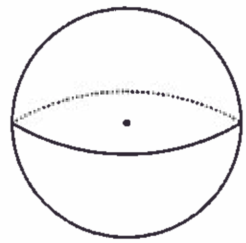
- 1 manifolds (in HW) just a circle
- 2 manifolds – what we will do
- 3 manifolds –open problem, lots of recent problems
- 4 manifolds and higher– unsolvable.

Examples. 1. Simplest example of a manifold is the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$:

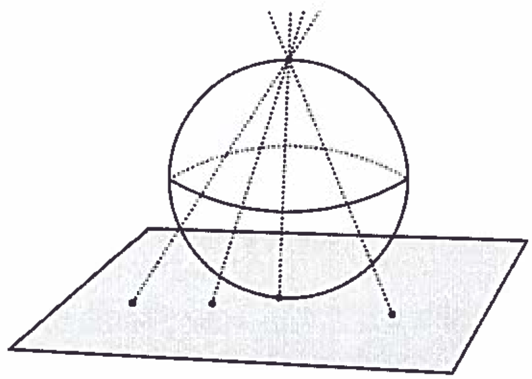
$$graph(f) = \{(x, f(x)) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{n+1}$$

or the projection $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ via $p(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$.

2. $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$ (For $n = 2$ this is the normal sphere).



Note $S^n\backslash\{x\}$ can be projected onto $\mathbb{R}^n$ via the stereographic projection.



Question. "Invariance of Domain" $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ $n \neq m$. Is it possible for $U \approx V$?

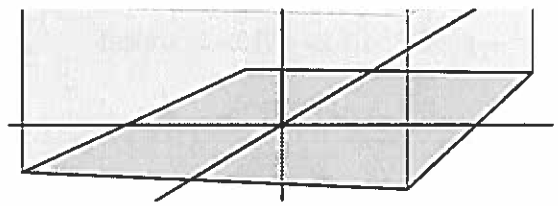
No. This is hard to show. The proof uses algebraic topology and the notion of homology. It is, however easy enough to show for $n = 1$, $m = 2$. More specifically, removing a point from a 1-manifold leaves a disconnected manifold. Removing a point from a 2-manifold leaves a connected manifold. Thus they cannot be homeomorphic.

Definitions and Construction of Models

Last time: A topological n-manifold is a topological space M such that:

1. M can be embedding in $\mathbb{R}^N$
2. for all $x \in M$ there exists open neighborhood U of x such that U is homeomorphic to open subset of $\mathbb{R}^n$.

$\mathbb{H}^n$ is the upper half space $= \{(x_1, \dots, x_n) | x_1 \geqslant 0\}$



Definition. A **topological n-manifold with boundary**, M, is a topological space such that

1. M can be embedded in $\mathbb{R}^N$.
2. for all $x \in M$ there exists open neighborhood U of x such that U is homeomorphic to open subset of $\mathbb{H}^n$.

There are two kinds of points in $\mathbb{H}^n$. The interior points $x_i > 0$ and boundary points, $x_n = 0$.

Fact. An open neighborhood of a boundary point is never homeomorphic to an open neighborhood of an interior point (by algebraic topology).

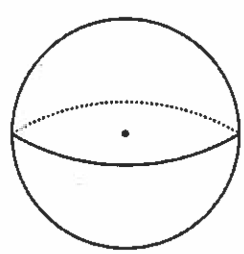
Definition. A point $x \in M$ is called an **interior point** if an open neighborhood of x is homeomorphic to an open neighborhood of an interior point in $\mathbb{H}^n$.

Definition. A point $x \in M$ is called an **boundary point** if an open neighborhood of x is homeomorphic to an open neighborhood of a boundary point in $\mathbb{H}^n$.

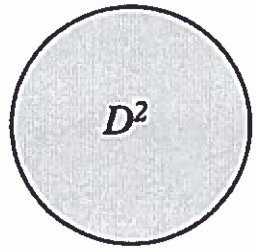
Fact. Interior and boundary are well defined.

Write $int(M)$ for interior points (This is an n-manifold with no boundary) and ∂M for boundary points (this is an $n-1$-manifold with no boundary).

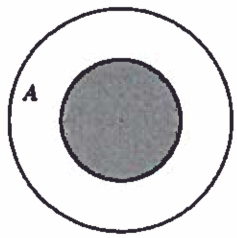
Examples. 1. $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$



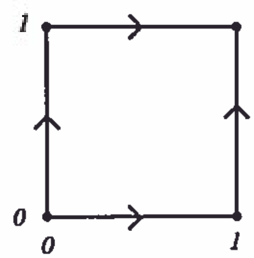
2. $D^2 = \{(x, y) : x^2 + y^2 \leqslant 1\}$ (Note, $\partial D^2 = S^1$)



3. $A^2 = \{(x, y) : 1 \leqslant x^2 + y^2 \leqslant 2\}$ (Note ∂A is 2 circles)

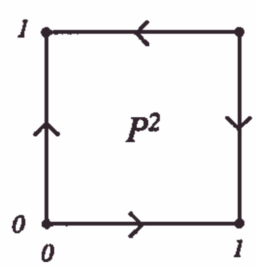


4. $T^2 = S^1 \times S^1$ this is the surface.

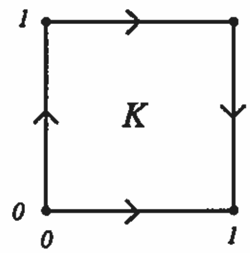


5. M- Mobius band is a surface with boundary

6. P^2 - Projective plane, a surface without boundary



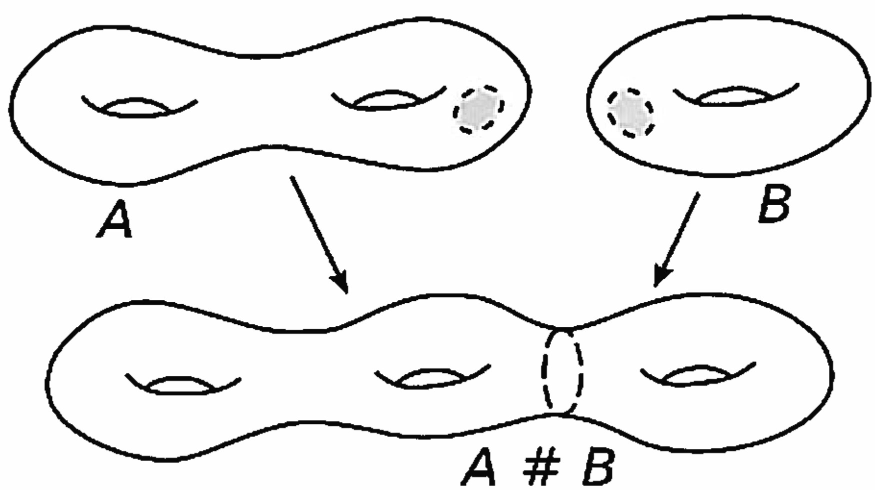
7. K^2 - Klein Bottle



8. Take a surface and remove an open disk. The New surface with boundary and boundary of removed disk is a new boundary.

Handle-bodies

Connected Sums



Handle Decomposition: 2 dimensional i-Handle $D^1 \times D^j$ for $i + j = 2$.

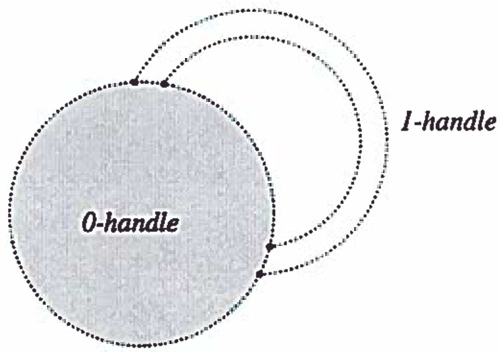
$D^0 = \{0\}$; $D^1 = [-1, 1]$ and $D^2 = \{(x, y) | x^2 + y^2 \leqslant 1\}$

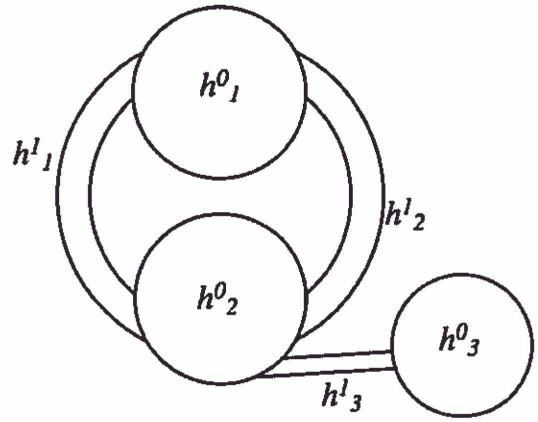
0-Handle: $D^0 \times D^2 = h^0$; 1-Handle: $D^1 \times D^1 = h^1$; 2-Handle $D^2 \times D^0 = h^2$

Handle-Body: $h_1^0 \cup h_2^0 \cup \cdots \cup h_{k_0}^0 \cup h_1^1 \cup h_2^1 \cup \cdots \cup h_{k_1}^1 \cup h_1^2 \cup h_2^2 \cup \cdots \cup h_{k_2}^2$

Where $k_0 \geqslant 1$, $\partial D^i \times D^j$ is attached to the preceding union by a homeomorphism to the boundary of the preceding union.

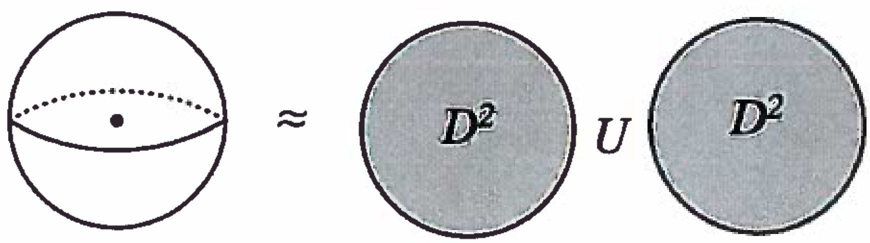
$\partial D^0 = \varnothing$; $\partial D^1 \times D^1 = \{-1\} \times D^1 \sqcup D^1 \times \{1\}$; $\partial D^2 \times \{0\} = S^1$.



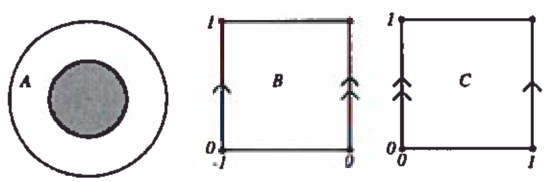


A handle body is always a surface with boundary. Conversely, we will assume that any surface with boundary can be written as a handlebody, that is that it has a handle body decomposition.

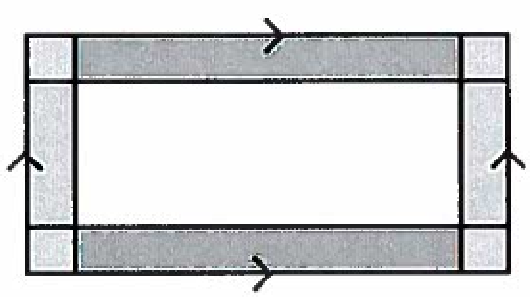
Examples. 1. $S^2 \approx D^2 \cup_{id} D^2$



2. $A^2 \approx (D^1 \times D^1) \cup_h (D^1 \times D^1)$



3. $M^2 \approx (D^1 \times D^1) \cup_f (D^1 \times D^1)$
4. T^2 is a one h^0, two h^1, one h^2

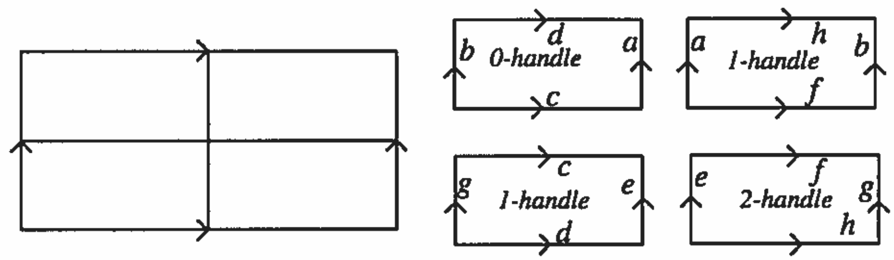


Handle-bodies

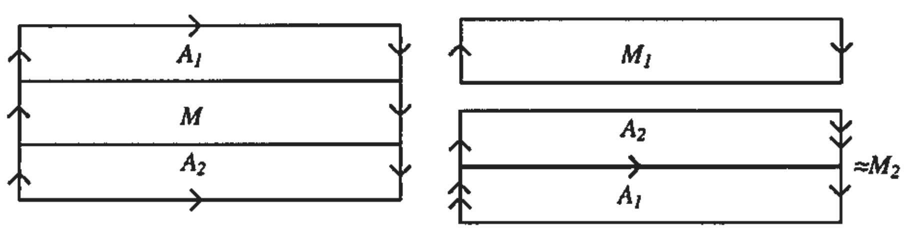
Handle-body: Take some number of 0-handles, attach 1-handles, then attach 2-handles along $\partial D^1 \times \{0\}$.

Examples. Last time: wrote Mobius band, annulus, 2-space, and torus as handlebodies.

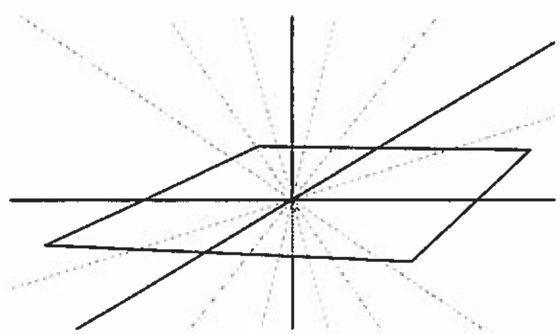
1. Write T^2 as a handle body



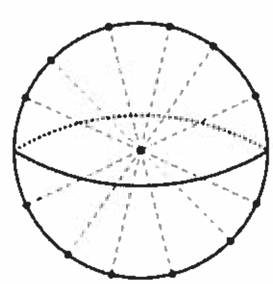
2. Klein bottle



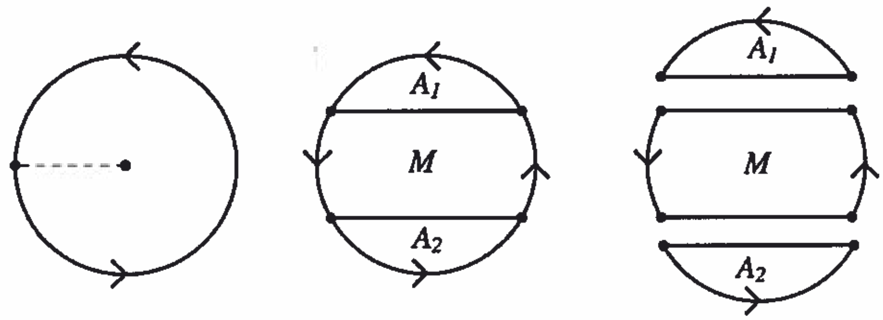
3. Projective plane. Recall $P^2 = \mathbb{R}^3 \backslash \{0\} / \sim^1$ where $x \sim^1 y$ if $y = tx$ for $t \in \mathbb{R}$, $t \neq 0$. That is, it is the space of lines through the origin in $\mathbb{R}^3$.

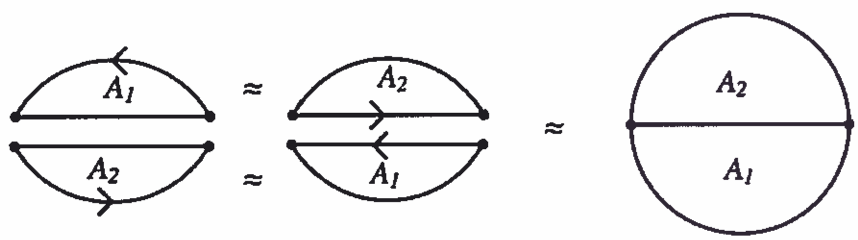


Furthermore, $P^2 = S^2 / \sim^2$ where $p \sim' q$ is $q = -p$.



Lastly $\S^2/\sim^2 \approx D^2/\sim^3$ where $x \sim^3 y$ if and only if $y = -1$ for $x, y \in S^1$.





Thus P^2 is homeomorphic to Mobius band with disc attached along ∂M giving the handle body decomposition: one 0-handle, one 1-handle, one 2-handle.

Isotopy and Attaching Handles

Assume we have a handle body with only 0 and 1 handles, H. You can attach 1-handle in two ways: $H \cup_f h^1$ and $H \cup_g h^1$.

Question: When are these two homeomorphic?

Answer: Depends on f and g $\partial D^1 \times D^1 \to \partial H = \{-1\} \times I \sqcup \{1\} \times I$. embedding image $\subseteq \partial H$.

Temporary inductive assumption is that $\partial H \approx$ disjoint union of circles S^1 and ∂H has a collar neighborhood. (i.e. there exists U closed such that $U \approx \partial H \times I$ such that $\partial H \times \{0\} = \partial H$.)

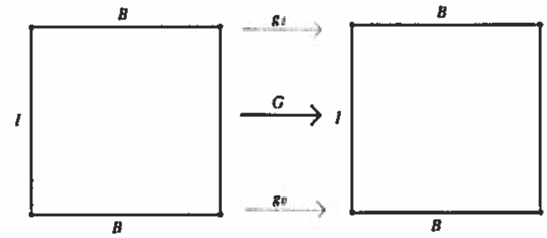
Remark. Isotopy and ambient isotopy are equivalent on space of homeomorphisms/ embeddings respectively.

Image $f, g \subset$ one of the circles by connectedness.

Understand. Embeddings of intervals into circles.

Note: To understand attaching 2-handles we need to understand $f : S^1 \to S^1$ is a homeomorphism, more specifically and embedding.

Definition. We say homeomorphisms $g_0, g_1 : B \to B$ are **isotopic** if there exists a homeomorphism $G : B \times I \to B \times I$, $I = [0,1]$, such that $G(b, 0) = g_0$ and $G(b, 1) = g_1$.



Definition. We say embeddings $f_0, f_1 : A \to B$ are **ambient isotopic** if there exists isotopy $G : B \times I \to B \times I$ so that $G(b,0) = G_0 = b$ for all b and $G(f_0(b), 1) = G_1 = f_1(b)$ for all b.

Isotopy and Attaching Handles

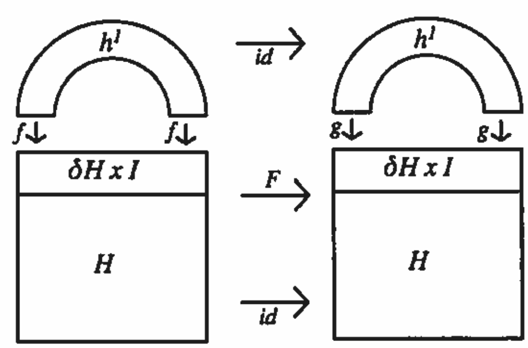
Exercises. 1. $G : B \times I \to B \times I$ is an isotopy if and only if $G_t : B \to B$ is a homeomorphism for all $t \in I$.

2. Isotopy is an equivalence relation.

3. Ambient isotopy is an equivalence relation.

Lemma. $H \cup_f h^1$, $H \cup_g h^1$ are handle bodies such that ∂H = disjoint union of circles and ∂H has a collar neighborhood. If f and g are ambient isotopic, then $H \cup_f h^1 \approx H \cup_g h^1$.

Proof. Suppose f, g are ambient isotopic. There exists $F : \partial H \times I \to \partial H \times I$ such that $F_0 f = g$ (which tells me that they agree on the boundary), and $F_1 = id$.



□

Let $f : [a,b] \to [c,d]$ be a homeomorphism.

Proposition. f is either strictly increasing ($x < y \Rightarrow f(x) < f(y)$), which would mean it is orientation preserving. OR f is strictly decreasing ($x < y \Rightarrow f(x) > f(y)$), which would mean it is orientation reversing.

Proof. $\overline{f} : [a,b] \times [a,b] \to \mathbb{R}$ where $\overline{f}(x,y) = f(x) - f(y)$. Let $A = \{(x,y) | (x,y) \in [a,b] \times [a,b], x < y\}$. Note, $\overline{f}(A)$ is connected. If f is not strictly monotone, there exists some (x_1, y_1), $x_1 < y_1$, where $f(x_1) < f(y_1)$ and (x_2, y_2), $x_2 < y_2$, where $f(x_2) > f(y_2)$. Then $\overline{f}(x_1, y_1) < 0$, $\overline{f}(x_2, y_2) > 0$ there exists some (x_0, y_0) with $\overline{f}(x_0, y_0) = 0$ so f is not strictly monotone. □

Lemma. 1. An orientation preserving homeomorphism $f : I \to I$ is isotopic to the identity

2. An orientation reversing homeomorphism $g : I \to I$ is isotopic to $r(t) = 1 - t$

3. $F : I|timesI \to I$ is an isotopy and F_0 is orientation preserving, then F_t is orientation preserving for all $t \in I$.

Proof. (3) $A = \{(x, y, t) : (x, y, t) \in I \times I \times I, x < y\}$

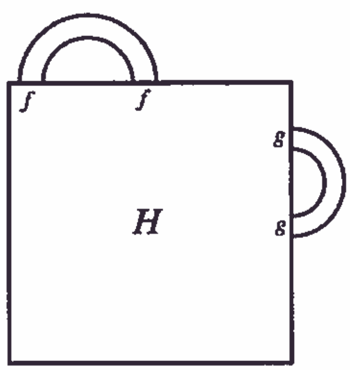
(1) $F(s,t) = ((1-t)f(s) + ts, t)$ rewritten as $F_t(s) = (1-t)f(s) + ts$. Then $F_0(s) = f(s)$, $F_1(s) = s$. Then For $0 \leqslant s_1 \leqslant s_2 \leqslant 1$: $F_t(s_1) = (1-t)f(s_1) + ts_1 < x(1-t)f(s_2) + ts_2 = F_t(s_2)$. Thus $F_t : I \to I$ is orientation preserving for all t, F_t is a homeomorphism for all t.

(2) Homework problem □

Defintion. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphisms, also always orientation preserving or orientation reversing. Then f is called **periodic and orientation preserving** if $f(x+1) = f(x) + 1$. Further f is called **periodic and orientation reversing** if $f(x+1) = f(x) - 1$.

Exercise. Any such f induces a homeomorphism $\overline{f} : S^1 \to S^1$ via $\overline{f}([x]) = [f(x)]$ where $S^1 = \mathbb{R}/\sim$, $x \sim x + n$ and any homeomorphism arises in this way.

Issue 1: $H \cup_f h^1$, $H \cup_g h'$, $Image(f)$, $Image(g)$ are contained in same circle in ∂H. However, $image(f) \neq image(g)$



Lemma. Let I_1, I_2 and I'_1, I'_2 be disjoint arcs on a circle, there is a homeomorphism $f : S^1 \to S^1$ isotopic to the identity such that $f(I_1) = I'_1$ and $f(I_2) = I'_2$.

Proof. Let I, I' be disjoint arcs on S^1. Let $p(t) = (\cos(2\pi t), \sin(2\pi t))$. There exists $J, J' \subset \mathbb{R}$ such that $p(J) = I$, $p(J') = I'$. There exists a homeomorphism $f : J \to J'$. Then $p \circ f \circ p^{-1}$ is a homeomorphism from I to I'. Then $p^{-1} : I_1, I_2 \to J_1 = [v_1, v_2], J_2 = [v_3, v_4] \in \mathbb{R}$ where $v_1 < v_2 < v_3 < v_4 < v_1 + 1$. Then $p^{-1} : I'_1, I'_2 \to [v'_1, v'_2], [v'_3, v'_4]$. $\overline{f}$ is a homeomorphism from $S^1 \to S^1$. Since f is increasing, isotopy of f to $\overline{f}$ identifies on $\mathbb{R}$ induces isotopy on $0 \cap S^1$ from $\overline{f}$ to identity. □

Proposition Consider embedding $D^1 \to S^1$. Any two OP are ambient isotopic. Any two OR are ambient isotopic. However OP and OR are not ambient isotopic.

Proof. Let $I_1 = f_1(D^1)$, $I_2 = f_2(D^2)$. By lemma there exists isotopy $F : S^1 \times I \to S^1 \times I$ such that $F_0 = id$, $F_1(I_1) = I_2$.

Consider $g_1 : D^1 \to S^1$ such that $g_1 = F_1 f_1$, Then g_1 is ambient isotopic to f_1 and $g_1(D^1) = F_1(f_1(D^1)) = I_2$. Therefore f_2, g_1 have the same image. Then $f_2 : D^1 \to I(D^1)$ and $g_1^{-1} : I(D^1) \to D^1$.

By construction F_t is OP for all t. Therefore $g_1^{-1} \circ f_2 : D^1 \to D^1$ is order preserving. There exists isotopy $G : D^1 \times I \to D^1 \times I$ where $G_0(x) = x$, $G_1(x) = g_1^{-1} \circ f_2(x)$.

Let $H : I_2 \times I \to I_2 \times I$ by $H_t = g_1 G_t g_1^{-1}$. Then $H_0 = id$, $H_1 = g_1 G_1 g_1^{-1} = g_1 g_1^{-1} f_2 g_1^{-1} = f_2 g_1^{-1}$. So $H_1 g_1 = f_2$. So g_1 and f_2 are ambient isotopic in I_2. Extend H_t to the identity on $S^1 \backslash I_2$. So $g_1 \sim f_2$, $f_1 \sim g_1$, and finally $f_1 \sim f_2$. □

Isotopy and Attaching Handles

Attaching 1-handles:

Let H be a handle body, with 0- and 1-handles

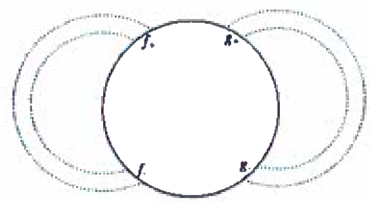
Assume ∂H is the disjoint union of circles and ∂H has a collar neighborhood

To attach a 1-handle to H specify an embedding: $f : \{\pm 1\} \times D^1 \to \partial H$ where $f_+ : \{+1\} \times D^1 \to \partial H$ and $f_- : \{-1\} \times D^1 \to \partial H$

Label circles that make up ∂H as $C_1, C_2, \ldots, C_k$, $image(f_+) \subset C_+$ and $image(f_-) \subset C_-$.

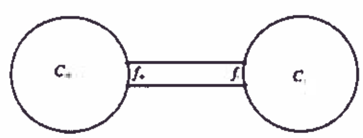
Two cases $C_+ = C_-$ or $C_+ \neq C_-$

If $C_+ = C_-$ four cases: f_+ OP, f_- OP; f_+ OP, f_- OR; f_+ OR, f_- OP; f_+ OR, f_- OR.



Then $r : D^1 \times D^1 \to D^1 \times D^1$ via $r(x, y) = (x, -y)$ induces homeomorphism: $H \cup_f h_1^1 to H \cup_f h_4^1$ and $H \cup_f h_2^1 to H \cup_f h_2^1$.

If $C_+ \neq C_-$ then when you attach you attach with both order similar (both OP/ both OR) or order different (OP,OR or OR,OP).



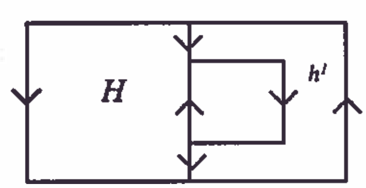
Theorem. Let H be a handlebody such that ∂H is a disjoint union of circles with collar neighborhood. The homeomorphism type of $H \cup_f h^1$ depends only on which circle f attaches to and whether the two attaching maps are order similar or order distinct.

Theorem. Let H be a handle body with handle decomposition with only 0- and 1-handles. Then the ∂H is disjoint union of circles and ∂H has a collar neighborhood.

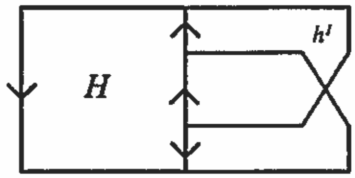
Proof. (induction on the number of 1-handles) Suppose there are no 1-handles. H is the disjoint union of D^2. Then $\partial D^2 = S^1$. has collar neighborhood $|x| > \frac{1}{2}$.

Assume that this is true for k 1-handles. If we have $k+1$ 1-handles then it is $H \cup_f h^1$ where H has k 1-handles. Then if $C_+ = C_-$:

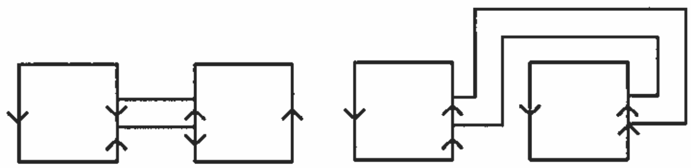
They can be order distinct:



They can be order similar:



We can prove this with drawing because the only possibilities are ones we can draw something homeomorphic to a circle is a boundary. This is always the case for $C_+ \neq C_-$.

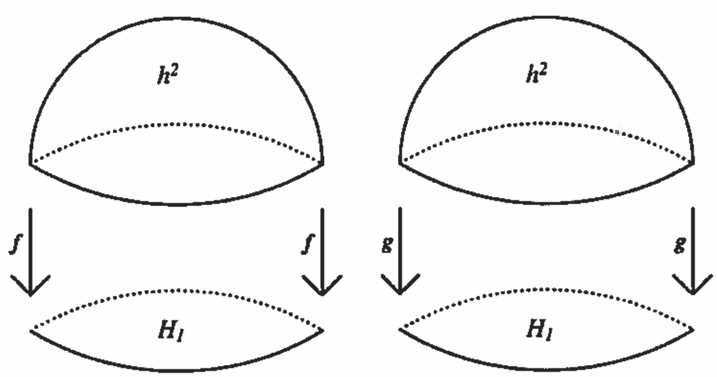


$\square$

Corollary. Cases (1), (2), and (3) are not homeomorphic because different number of circles.

Attaching 2-handles: $H = H' \cup_f h^2$ (recall $h^2 = D^2 \times \{pt\}$) where $f : S^1 \to \partial H'$ is the attaching map. Note $image(f) \subseteq C$ (where C is one of the circles in $\partial H'$. Note S^1 is not homeomorphic to $[a, b]$. So $image(f) = S^1$. Thus $f : S^1 \to S^1$ is a homeomorphism. Further $F : D^2 \to D^2$ where $F(0) = 0$ and, for $y \in D^2$, $y = rx$ $0 < r < 1$, $x \in S$, $F(y) = F(rx) = rF(x)$.

Proposition. If $f, g : S^1 \to \partial H'$ are attaching maps for 2-handle which map to the same boundary circle, then they're homeomorphic. Then $H' \cup_f h^2 \approx H' \cup_g h^2$

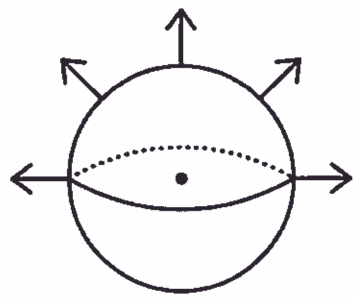


$H_l \cup_f h^2$ $H_l \cup_g h^2$

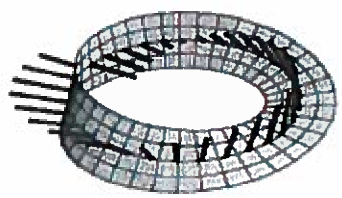
Proof. $G = g^{-1} \circ f : S^1 \to S^1$ extends to $G : D^2 \to D^2$ so $gG = g$. $\square$

Orientation

Examples. 1. Consider S^2, which is orientable, because there is a well-defined global vector fields, $\langle e_1, e_2 \rangle = N^{\perp}$, which is found by the right hand rule. Since there is a continuous choice of basis, we are able to define the vector field globally.



2. Consider M which is non-orientable. This does not have a well-defined global vector field.



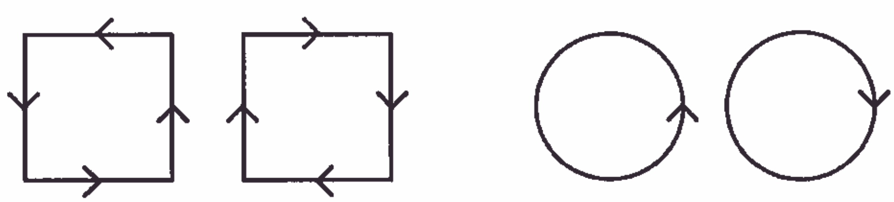
Definition. (Version 1) A surface with boundary is **orientable** if it does not contain an embedded mobius band.

Definition. (Version 2) A surface S is call **disc oriented** if there is more that one equivalence class of embeddings of the disc into surface $f : D^2 \to S$ under ambient isotopy.

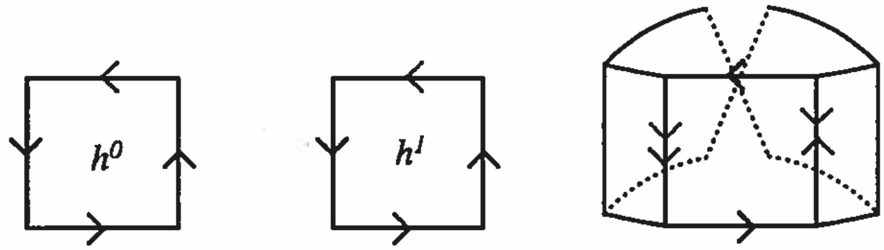
(Can see definition of orientable in terms of homology in 761. See 88-93 of Lawson for discussion.)

Orientation for Handle-bodies:

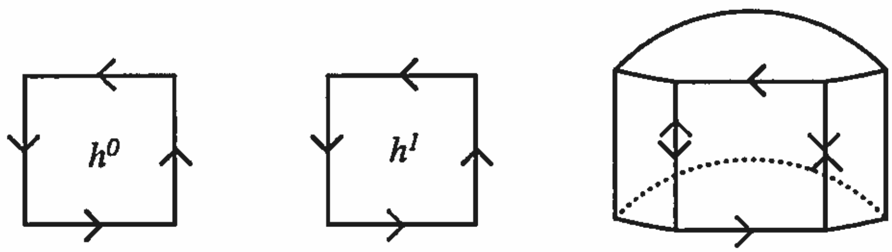
An orientation of a handle is just an orientation of its boundary circle:



Examples. 1. Mobius band: one 0-handle one 1-handle. Note that on the attaching region one disagrees and one agrees.

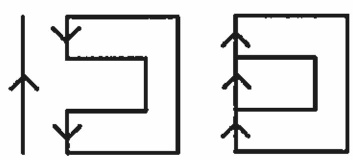


2. Annulus: one 0-handle one 1-handle. Note that on the attaching region one disagrees and one agrees. Note that they either both agree or both disagree.



Definition. A handle body is **orientable** if we can orient the handles so that all orientations disagree on all identifications. Otherwise, call non-orientable.

Remark. If we do it so all orientations disagree when we add a 1-handle, then new handle body $H \cup h^1$ has an orientation of ∂H automatically:



Remark. Definition is inductive, so at each step we obtain H (sub-handle body of H' which is orientable.

Remark. Nothing to prove when adding 0- and 2-handles.

Examples. 1. M, P and K are all non-orientable.

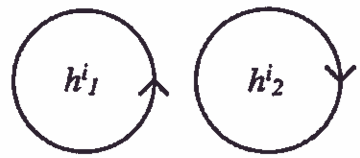
2. T^2 is orientable.

An orientation of a handle body is a choice of orientation of handle that all disagree. It follows that every oriented handlebody has at least 2 orientations. (reverse all orientations of all handles)

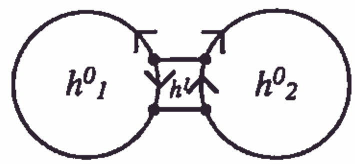
Proposition. If H is a connected orientable handle body then H has two-orientations.

Proof. 2.4.5 any 2 handles in a connected handle body can be connected by a chain of handles (via induction). $h_a - h_1 - h_2 - \cdots - h_k - h_b$. □

Example. Not connected: 2 0-handles, 4 orientations: ++, +-, -+, --.



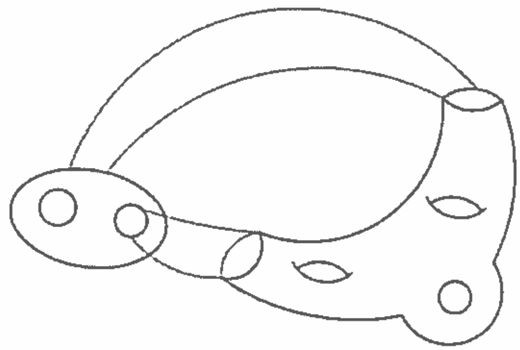
If you connect the two 0-handles with a 1-handle, then the two 0-handles can't have opposite orientations. If we change the orientation so that the two are different then it works.



Remark. If H is not connected, may have to go back and change orientations on H to show $H \cup_f h'$ is orientable.

Connected Sums

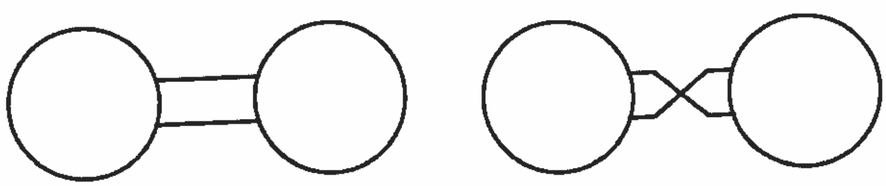
Definition. Let M, N be two surfaces (or handle bodies) with nonempty boundary. Identify and arc in ∂M with an arc in ∂N by choosing an embedding $f = \{\pm 1\} \times D^1$, Then $M \sqcup N$ whose image is the arcs. The **boundary connected sum** $M \natural N$ is $M \cup_f (D^1 \times D^1) \cup_f N \approx (M \sqcup N) \cup_f h^1$.



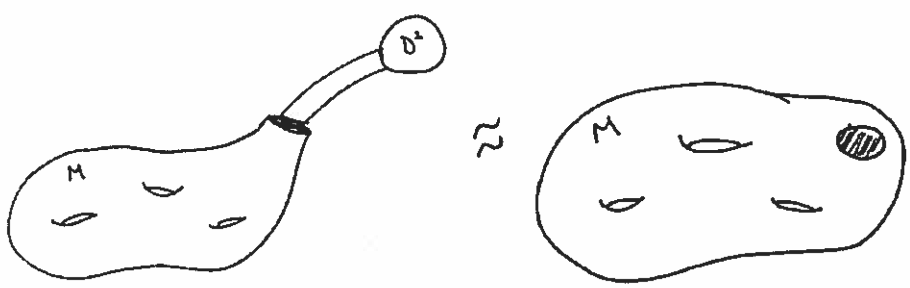
Moreover if M, N are orientable, choose f such that orientations disagree on identification such that $M \natural N$ orientable.

This is well defined. If you make different choices, you still get homeomorphic surfaces. This follows from Theorem three from last time in orientable case. Identifications don't if 1 is non-orientable.

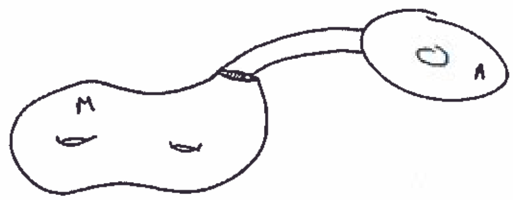
Examples. 1. $D^2 \natural D^2 \approx D^2$



2. $M \natural D^2 \approx M$



3. $M \natural A$



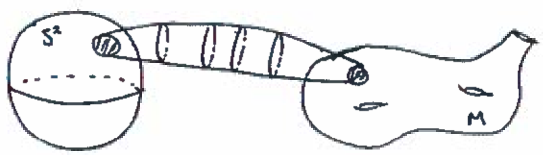
$M_{(1)}$, which is simply M with a disc removed. In general $M_{(k)}$ is M with k discs removed.

4. Problem 5 Show $D^2 \cup_{f^-} (D^1 \times D^1) \approx D^2$. This uses fats from Lemma 2.3.2. Isotopy implies homeomorphism.

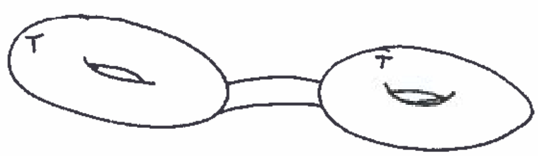
Definition. Let M and N be surfaces (connected, compact), the **connected sum** $M\#N$ is: $M\#N = (M_{(1)} \natural N_{(1)}) \cup h^2$ (note, h^2 makes a tub.

Examples. 1. $(M\#N)_{(1)} \approx M_{(1)} \natural N_{(1)}$

2. $M\#S^2 \approx M_{(1)} \natural S^2_{(1)} \approx (M_{(1)} \natural D^2) \cup h^2 \approx (M_{(1)}) \cup h^2 \approx M$



3. $T\#T$



$\approx$

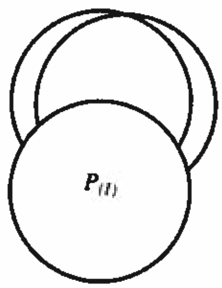
4. $P\#P \to M \cup_f M \Rightarrow K$

Lemma. $M\#N \approx M\backslash int(D_1^2) \cup_f N\backslash int(D_2^2)$ where $f : \partial D^2 \to \partial D^2$. This is called surgery, no tube.

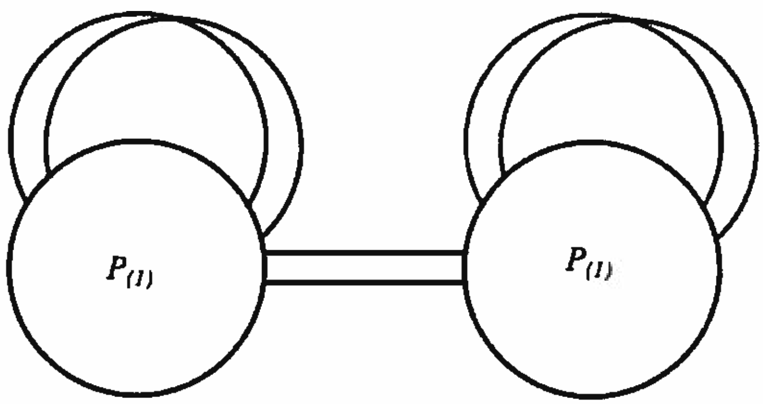
Connected Sums

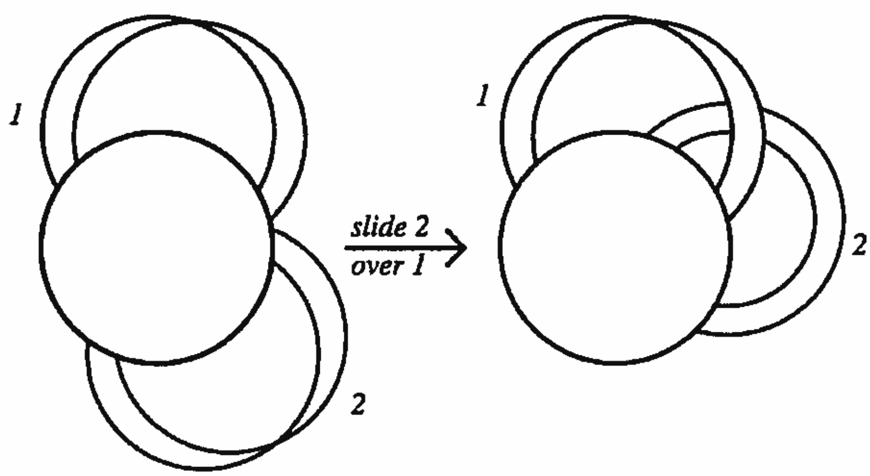
Last Time. $P\#P \approx K$ is the Mobius band glued along boundary. Another proof using handle slides: $P\#P \approx P_{(1)} \natural P_{(1)} \cup h^2$. Want to show $P_{(1)} \cup P_{(1)} \approx K$

$P_{(1)}$:



$P_{(1)} \natural P_{(1)}$:

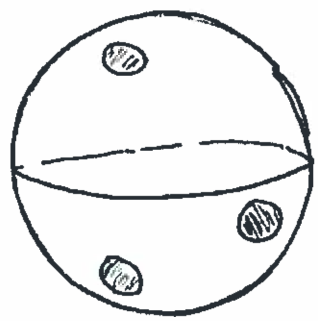




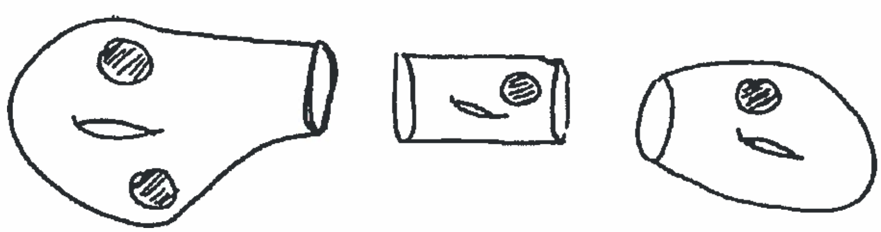
Classification Theorem

Let S be a connected, compact surface, then S is homeomorphic to $S^2_{(p)}$, $T^{(n)}_{(p)}$, or $P^{(n)}_{(p)}$, where $M_{(p)}$ is M with p discs removed. Further, $M^{(n)} = M\#M\#\dots\#M$. (Remember $A\#B \approx B\#A$ and $(A\#B)\#C \approx A\#(B\#C)$).

e.g. $S^2_{(3)}$

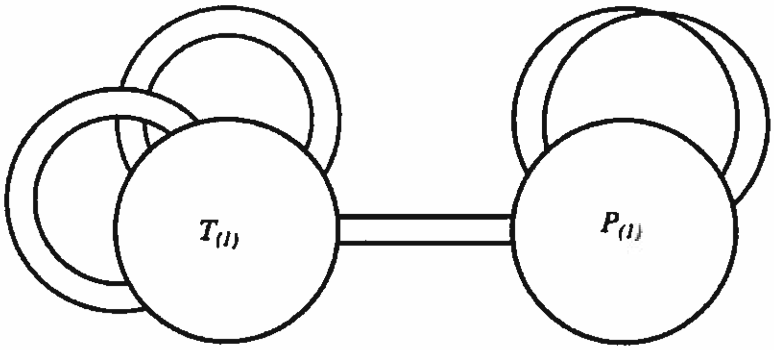


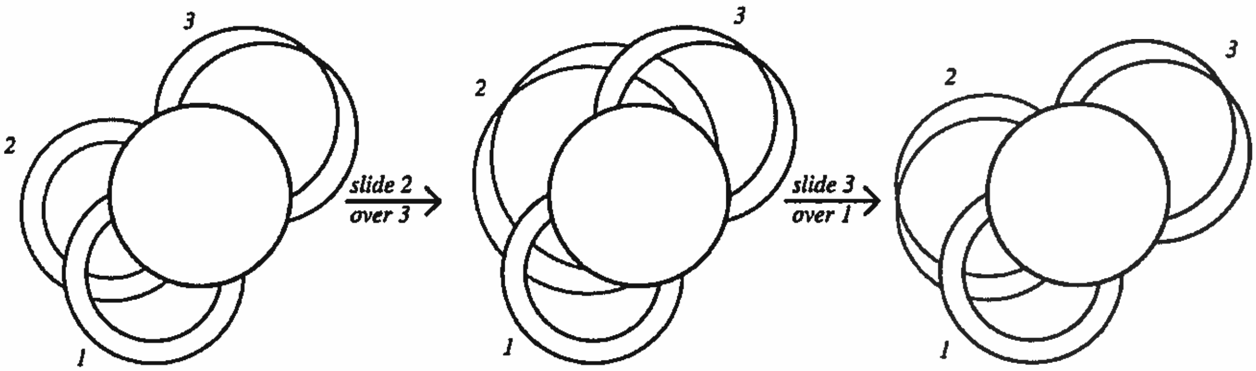
$T^{(3)}_{(4)}$



Examples. $P\#P \approx K$ $\quad M \approx P_{(1)}$ $\quad D^2 \approx S^2_{(1)}$ $\quad A \approx S^2_{(2)}$ $\quad N\#S^2 \approx N$ $\quad N\#S^2_{(p)} = N_{(p)}$

Fundamental Lemma of Surface Theory: $T\#P \approx K\#P \approx P\#P\#P$.





Proof. Handle slides $T\#P \approx (T_{(1)} \natural P_{(1)}) \cup h^2$. Thus $T\#P_{(1)} \approx T_{(1)} \natural P_{(1)}$. □

Outline of proof of classification

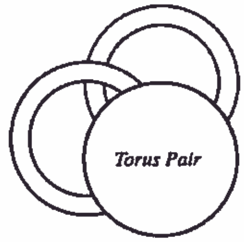
1. Every surface has a handlebody structure (Rado 1925)
2. Any connected handlebody is homeomorphic to one with only one 0-handle
3. If the classification is true for handle bodies with no 2-handles with no two handles, then true for all handle bodies
4. Work orientable and non orientable cases separately and use induction on 1-handles and handle slides.

Remark. By handle slides all 1-handles are attached disjointly to the 0-handles.

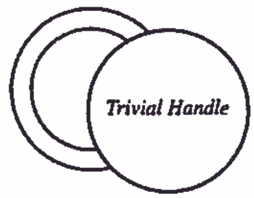
Lemma. If H is a connected handlebody with only 0- and 1-handles, then H is homeomorphic to H^1 has only one 0-handle.

Proof. By induction one the number of 0-handles. If there is only one 0-handle, this is trivially true. Assume this is true for all connected handle bodies with k 0-handles with $k > 1$. Since H is connected there exists $h_1^0 \cup h_2^0 \cup h_1^1$ where h_1^1 is attached to h_1^0 and h_2^0. Then $h_1^0 \cup h_2^0 \cup h_1^1 \approx h_3^0$. □

Orientable handle bodies with one zero handle no 2-handles : Torus pair



Trivial handle

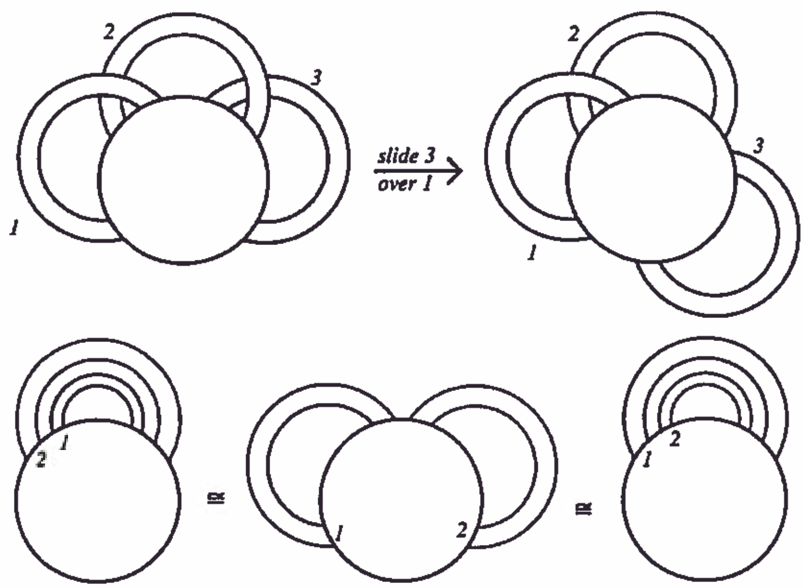


Normal Form: If all 1-handles occur in torus pairs or are trivial

Claim. All handle bodies can be put in normal form

Proof. By induction of the number of 1-handles. If only one 1-handle then this is trivial. Assume any with k 1-handles can be put in normal form. Let H be a handle body with $k+1$ one handles. Then $H = H' \cup h^1$. Since H' has k 1-handles, it can be put in normal form. So now we must examine how one can attach another 1-handle.

Case 1: one attaching region for h^1 is interior to a torus pair. Using handle slides we are able to get this into normal form. Then H' is orientable.



Case 2: At least one of the handles, h^1, is twisted. Then H' is non-orientable, by induction hypothesis H' can be put in normal form. ☐

Identifying Surfaces

Invariants: orientability, number of boundary circles, Euler characteristic of a handle body ($\chi(H) = \#h^0 - \#h^1 + \#h^2$).

Examples.

$$\chi(S^2) = 2 \qquad \chi(D^2) = 1 \qquad \chi(A) = 0 = 1{-}1 = \chi(M) \qquad \chi(T) = 1{-}2{+}1 = 0 = \chi(K)$$

$$\chi(P) = 1 - 1 + 1 = 1 \qquad \chi(S^2_{(p)}) = 1 - (p-1) = 2 - p \qquad \chi(A \natural B) = \chi(A) + \chi(B) - 1$$

$$\chi(A_{(1)} \natural B_{(1)}) + 1 = \chi(A_{(1)}) + \chi(B_{(1)}) - 1 + 1 = \chi(A_{(1)}) + \chi(B_{(1)}) = \chi(A) + \chi(B) - 2$$

$$\chi(A \# B) = \chi(A_{(1)} \natural B_{(1)} \cup h^2)$$

$$\chi(S^2_{(p)}) = 2 - p \qquad \chi(T^{(g)}_{(p)}) = 2 - 2g - p \qquad \chi(P^{(n)}_{(p)}) = 2 - n - p$$

Our proof of the classification shows that if H is a handlebody, $H \approx$ one of the models above, by our proof $\chi(H) = \chi(model)$

It is true that if $N \approx M$ then $\chi(N) = \chi(M)$, but proof uses Algebraic Topology.

The only "moves" we did in the proof are showing that two 0-handles with a 1-handle is homeomorphic to a single 0-handle. These two have the same Euler characteristic.

The handle slides which don't change the number of handles

Example. A connected compact surface is orientable has four boundary circles and $\chi = -6$, what is the surface. $0 \Rightarrow S_p$ or T^g_p, $p = 4$ $\chi = -6$. Then this is T^g_p with $g = 2$ since $2 - 4 - 4 = -6$

Chapter 3: The Fundamental Group and its Applications

Fundamental Group

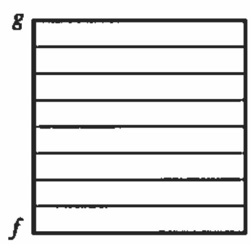
Idea: Associate to X a topological space, an algebraic object such that $f : X \to Y$ is continuous if and only if $f_* : A_x \to A_y$ where f_* is a "morphism" in algebraic context.

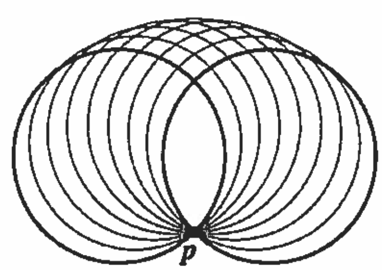
Recall: A group, G, is a set with binary operation $*$ such that

1. there exists $e \in G$ such that $g * e = e * g = g$ for all $g \in G$
2. Inverses: for all $g \in G$ there exists g' such that $g * g' = g' * g = e$
3. Associativity: for all $g_1, g_2, g_3 \in G$ $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$

Definition. Let X be a topological space, $x \in X$. $f : [0,1] \to X$ such that $f(0) = f(1) = x$. Then $\pi_1(X, x) = \{f : 0 \to X | f(0) = f(1) = x, f \text{ is continuous } \}/\sim$ is the **fundamental group**

Definition. $\sim$ is a **homotopy**, define $f \sim g$ if there exists $F : I \times I \to X$ such that F is continuous and $F(s,0) = f(s)$ and $F(s,1) = g(s)$. We say f is **homotopic** to g.

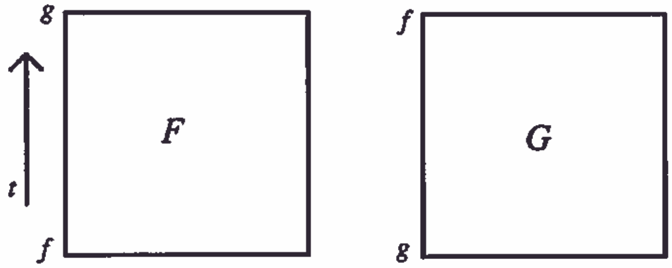




Note $F(s,t) = F_t(s) =$ the loop for each t.

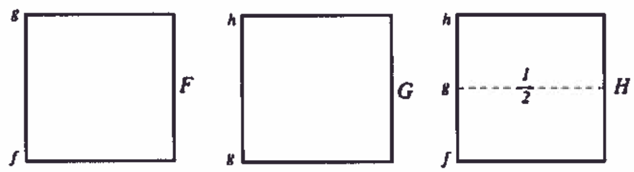
Lemma. Homotopy is an equivalence relation.

1. $f \sim f$ since $F(s,t) = f(s)$ for all t.
2. If $f \sim g$ then $g \sim f$. There exists $F : I \times I \to X$ such that $F(s,0) = f(s)$, $F(s,1) = g(s)$. Consider $G(s,t) = F(s, 1-t)$, then $G(s,0) = g(s)$ and $G(s,1) = f(s)$.



3. If $f \sim g$ and $g \sim h$ then $f \sim h$. There exists $F : I \times I \to X$ such that $F(s,0) = f(s)$, $F(s,1) = g(s)$ and $G : I \times I \to X$ such that $G(s,0) = g(s)$, $G(s,1) = h(s)$. Consider

$$H(s,t) = \begin{cases} F(s,2t) & t \leqslant \frac{1}{2} \\ G(s,2t-1) & t \geqslant \frac{1}{2} \end{cases}$$



Then $[f] \in \pi_1(X,x)$

Examples. 1. D^2: $\pi_1(D^2,x) = \{e\}$ (homotopy maps to a point)

2. S^2: $\pi_1(S^2,x) = \{e\}$
3. $\mathbb{R}^n$: $\pi_1(\mathbb{R}^n,x) = \{e\}$
4. Annulus: $\pi(A,x) = \mathbb{Z}$
5. S^1: $\pi(S^1,x) = \mathbb{Z}$
6. T: $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ (Recall $T = S^1 \times S^1$)
7. P: $\pi_1(P) = \mathbb{Z}_2$

Let $f, g : [0,1] \to X$ such that $f(0) = f(1) = g(0) = g(1) = x$. Then

$$f * g(s) = \begin{cases} f(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

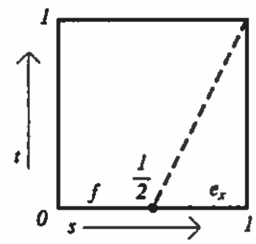
We want to show that $*$ defines a group.

<u>Identity:</u>

Suppose $f_0 * g_0 \sim f_1 * g_1$. Then

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0,\frac{1}{2}] \\ G(2s-1,t) & s \in [\frac{1}{2},1] \end{cases}$$

Show the group operations: Let $e_x : [0,1] \to X$ via $e_x(s) = x$ for all s. Want to show $e_x * f \sim f \sim f * e_s$.



This is left as an exercise. To do so, consider

$$F(s,t) = \begin{cases} f(\frac{2s}{1+t}) & s \in [0, \frac{1+t}{2}] \\ x & s \in [\frac{1+t}{2}, 1] \end{cases}$$

.

<u>Inverses:</u>

$\overline{f} : [0,1] \to X$ where $\overline{f}(0) = \overline{f}(1) = x$. Define $\overline{f}(s) = f(1-s)$ (same loop but opposite orientation.

Show that $f * \overline{f} \sim e_x$.

$$F(s,t) = \begin{cases} f(2st) & s \in [0, \frac{1}{2}] \\ \overline{f}(2t(s-1)+1) & s \in [\frac{1}{2}, 1] \end{cases}$$

$F(s,0) = e_x$ and $F(s,1) = f * \overline{f}$ since $\overline{\overline{f}} = f$ so $\overline{f} \times \overline{f} \sim e_x$.

<u>Associativity</u>

$$(f * g) * h = \begin{cases} (f * g)(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ h(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$
$$= \begin{cases} f(4s) & 0 \leqslant s \leqslant \frac{1}{4} \\ g(4s-1) & \frac{1}{4} \leqslant s \leqslant \frac{1}{2} \\ h(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

$$f * (g * h) = \begin{cases} f(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ g(4s-2) & \frac{1}{2} \leqslant s \leqslant \frac{3}{4} \\ h(4s-3) & \frac{3}{4} \leqslant s \leqslant 1 \end{cases}$$

Fundamental Group

Last time: Fundamental Group, X is a topological space, $x \in X$. Then $\pi_1(X,x) = \{f : [0,1] \to X : f \text{ is continuous }, f(0) = f(1) = x\}/\sim$ where $\sim$ is a homotopy.

$$[f] * [g] = \begin{cases} f(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Then $(\pi_1(X,x), *)$ is a group

Remark. The definition of a homotopy makes sense, homotopy is an equivalence relation on paths $f(0) = x$, $f(1) = y$) where $\overline{f}(s) = f(1-s)$ is the inverse.

Recall. Let $(G, \cdot)$ and $(H, *)$ be groups. Then $\phi : G \to H$ is a **group homomorphism** if $\phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2)$. Further ϕ is a **group isomorphism** if it is a bijective group homomorphism. $G \approx H$ if there exists a group isomorphism from G to H.

Definition. Let $f : X \to Y$ be continuous. The **induced homomorphism** on the fundamental group is $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ where $f_*([p]) = [f \circ p]$

Note, f_* is well defined because if $p \sim_F q$ then $F : I \times I \to X$ such that $F(s,0) = p(s)$, $F(s,1) = q(s)$, $F(0,t) = F(1,t) = x$

$$[f_*(p * q) = \begin{cases} f(p(2s)) & 0 \leqslant s \leqslant \frac{1}{2} \\ f(q((2s-1)) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} = (f \circ p) * (f \circ q) = f_*(p) * f_*(q)$$

Thus, f is a group homomorphism.

Corollary. Suppose X, Y are path connected. If X is homeomorphic to Y, then $\pi_1(X,x) \approx \pi_1(Y,y)$.

Proof. $f : X \to Y$ is a homeomorphism so $f^{-1} : Y \to X$ is continuous. The induced homomorphism is $f_* : \pi_1(X,x) \to \pi_1(Y,y)$. Consider $(f^{-1})_* : \pi_1(Y,y) \to \pi_1(X,x)$. to show this is also a group homomorphism note that $(f^{-1})_*(f_*)([p]) = (f^{-1})_*([f \circ p]) = [f^{-1} \circ f \circ p] = [p]$. Therefore $(f^{-1})_* = (f_*)^{-1}$. Therefore f_* is a group isomorphism. □

Dependence on Base Point

Suppose there exists a path P from x_1 to x_2 in X. $\pi_1(X,x_1) \approx \pi_1(X,x_2)$. Let f be a loop at x_2, $\phi : f \to p * f * \overline{p}$. (Showing that ϕ is a group isomorphism has been left as an exercise. Showing that ϕ is a homomorphism goes as follows: $\phi(f * g) = [p * f * g * \overline{p}] = [p * f * \overline{p} * p * g * \overline{p}] = \phi(f) * \phi(g)$

Proposition. $\pi_1(\mathbb{R}^n, x) \approx \{e\}$ and $\pi_1(D^n, x) \approx \{e\}$.

Proof. Let $f : [0,1] \to D^n$ such that $f(0) = f(1) = x$. Then $F(s,t) - (1-t)f(s) + tx$ so that $F(s,0) = f(s)$, $F(s,1) = x$, $F(0,t) = (1-t)x + tx = x$ and $F(1,t) = 1x = x$. □

The Fundamental Group of the Circle

Theorem. $\pi_1(S^1) \approx \mathbb{Z}$.

Proof. $\Phi : \mathbb{Z} \to \pi_1(S^1, (1,0))$ where $\Phi(n) = [\omega n]$ where $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$. For $n > 0$ go counter clockwise n-times and for $n < 0$ go clockwise n-times. want to show that Φ is a group isomorphism. Let $p : \mathbb{R} \to S^1$ where $p(s) = (\cos(2\pi s), \sin(2\pi s))$. This is a homeomorphism.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "\omega_n"'] & S^1 \end{tikzcd}$$

Definition. $\tilde{f}$ is a **lift** of ω_n

Let $\tilde{h}$ be another path from $[0,1] \to \mathbb{R}$ such that $\tilde{h}(0) = 0$ and $\tilde{h}(1) = n$. Then $\tilde{h} \sim_F \tilde{f}$ where $F(s,t) = t\tilde{h}(s) + (1-t)\tilde{f}(s)$. Note $F(s,0) = \tilde{f}s$, $F(s,1) = \tilde{h}$, $F(0,t) = F(1,t)$. Then $p \circ F$ is a homotopy from $p \circ \tilde{f} = \omega n$ to $p \circ \tilde{h}$. So $\omega_n \sim p \circ \tilde{h}$. Then $\Phi(n+m) = \omega_{n+m} = p\tilde{f}$ where $\tilde{f}(s) = (n+m)s$. Notice

$$\Phi(n)*\Phi(m) = \begin{cases} \omega_n(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ \omega_m(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} = p\tilde{h} \quad \text{where} \quad \tilde{h} = \begin{cases} 2ns & 0 \leqslant s \leqslant \frac{1}{2} \\ (2s-1)m + n & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} = p\tilde{h}$$

Note $\tilde{h}(1) = n + m$. $F : I \to S^1$ is continuous with $F(0) = x_0$ and $p(\tilde{x}_0) = x_0$. Then there is a unique $\tilde{f} : I \to \mathbb{R}$ such that $p\tilde{f} = f$ and $\tilde{f}(x_0) = \tilde{x}_0$ (this is called the path lifting property). □

Theorem. Homotopy Lifting Property. Suppose $F : I \times I \to S^1$ is continuous. Then there is a continuous lifting $\tilde{F} : T \to \mathbb{R}$ such that $p\tilde{F} = F$. Moreover it is unique if we require $\tilde{F}(0,0) = \tilde{x}_0$, $p(\tilde{x}) = F(0,0)$.

Proof. Proof that Φ is onto: Let $f : I \to S^1$ be a loop with $f(0) = f(1) = (1,0)$. We know there exists a lift $\tilde{f}$ such that $p\tilde{f} = f$ starting at $\tilde{x}_0 = 0$. Note $f(1) = (1,0)$ so $p(\tilde{f}(1)) = (1,0)$, then $\tilde{f}(1) =$ integer. Then $p\tilde{f} \sim \omega_n$. Therefore $f \sim \omega_n = \Phi(n)$. Therefore Φ is onto.

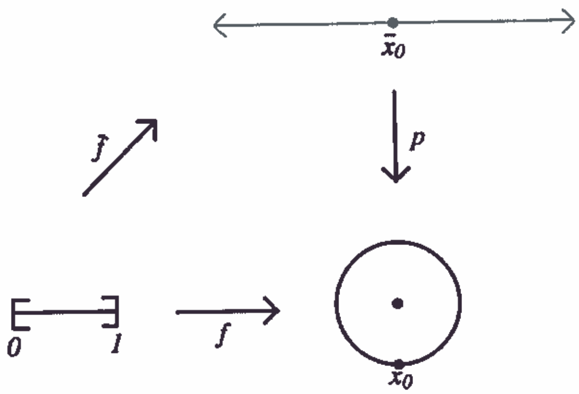
Proof that Φ is one-to-one: Suppose $\Phi(n) = \Phi(m)$ then $\omega_n \sim_F \omega_m$. By Theorem above there exists $\tilde{F} : I \times I \to \mathbb{R}$ such that $p\tilde{F} = F$, then $\tilde{F}(0,0) = 0$. By uniqueness of lift of path $\tilde{F}(s,0) = ns$, and $\tilde{F}(s,1) = ms$. Then $(1,0) = F(1,t)$ for all t, $\tilde{F}(1,t)$ is an integer for all t. $\tilde{F}(1,1) = m$, $F(1,0) = n$, $p\tilde{F} = F$ so $F(1,t) = p^{-1}(1,0) = \mathbb{Z}$. □

The Fundamental Group of the Circle

Last Time: $\pi_1(S^1) \approx \mathbb{Z}$. Where $p : \mathbb{R} \to S^1$ where $p(s) = (\cos(2\pi s), \sin(2\pi s))$.

Theorem. Unique Path Lifting Property:

If $f : I \to S^1$ is continuous, $f(0) = f(1) = x_0$, $p(\tilde{x}_0) = x_0$ then there is a unique lift $\tilde{f} : I \to \mathbb{R}$ such that $p\tilde{f} = f$, $\tilde{f}(0) = \tilde{x}_0$.



Proof. $S^1 = A_1 \cup A_2$ where A_i are open intervals. Then $p^{-1}(A_i)$ = Disjoint union of open intervals $= \bigsqcup_\alpha I_{\alpha_i}$. Then $p|_{I_{\alpha_i}}$ is a homeomorphism. Consider $f : I \to S^1$ such that $f(0) = f(1) = x_0$. Let $\tilde{x}_0 = p^{-1}(\{x_0\})$. For each $s \in I$, there exists interval $J_x \subset I$ such that $f(J_s) \subset A_i$ for $i = 1, 2$. Then J_s for $s \in I$ is an open cover of I. Let s be the Lebesgue number of the cover. That is, there exist $s\delta >>)$ such that if $|s_1 - s_2| < \delta$ then there exists s_3 such that $s_1, s_2 \in J_{s_3}$. Let $K \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Then $0 = x_0 < x_1 < x_2 < \cdots < x_k = 1$ is a partition of $[0, 1]$ where $x_1 = \frac{1}{k}$. Then $x_0 \in A_1$, $[x_0, x_1] \in J_s$ so $f([x_0, x_1]) \in A_i$. Let I_{α_i} be a component of $p^{-1}(A_i)$ with $\tilde{x}_0 \in I_{\alpha_i}$. Define $\tilde{f}(s) = \left(p|_{I_{\alpha_i}}\right)^{-1}(f(s))$. Then $\tilde{f}$ is a lift of $f[0, x_1]$. Assume, for induction, we've extended $f|_{[0,x_n]}$ to $\tilde{f}$. Similar argument shows/ allows to extend to $[0, x_{n+1}]$.

Let $f : I \to S^1$. Let $\tilde{f}$, $\tilde{f}'$ be two lifts of f such that $\tilde{f}(0) = \tilde{f}'(0)$. Assume for induction that $\tilde{f} = \tilde{f}'$ on $[0, x_j]$. Note $[x_j, x_{j+1}]$ is connected. Therefore $\tilde{f}([x_j, x_{j+1}])$ and $\tilde{f}'([x_j, x_{j+1}])$ have a point in common. So $\tilde{f}(x_j) = \tilde{f}'(x_j)$, so they are in the same connected component of $p^{-1}(A_i)$, I_{α_i}. Since p is 1-1 on I_{α_i}. Then $\tilde{f}$, $\tilde{f}'$ are lifts such that $p\tilde{f}(s) = f = p\tilde{f}'(s)$. Thus $\tilde{f}(s) = \tilde{f}'(s)$ since p is 1-1. □

Theorem. Homotopy Lifting:

Suppose $F : I \times I \to S^1$ is continuous, then there is a continuous lifting $\tilde{F} : I \times I \to \mathbb{R}$ such that $p\tilde{F} = F$. Further $\tilde{F}$ is unique if we fix $\tilde{F}(0, 0) = \tilde{x}_0$, $\tilde{x}_0 \in p^{-1}(\{x_0\})$.

Proof. Let A_1, A_2 be as before, $f^{-1}(A_1)$, $f^{-1}(A_2)$. Consider $F : I \times I \to S^1$. Let $(s, t) \in I \times I$ such that open set $U_{(s,t)} \subset I \times I$.Fix s_0, for each (s_0, t) there exists $I_t \times (a_t, b_t)$ such that $s_0 \in I_t$, $t \in (a_t, b_t)$. So there exists a finite subcover $I \times (a_{t_j}, b_{t_j})$ where $j = 1, \ldots, k$. Let $N = \bigcap_{j=1}^k I_{t_j}$. Then $f(N \times (a_{t_j}, b_{t_j})) \subset A_i$, $i = 1, 2$. Using a similar argument with Lebesgue numbers we can lift f to $\tilde{f}$ on $N \times T$. For all $s \in I$ there exists $N_s \times I$ where we can lift F to $\tilde{F}$. But $F_s(t)$ is a path interval, so it is the unique lift once we specify $F(s, 0)$.

Define $\tilde{F}$ on $I \times I$ as $\tilde{F}(s,t) =$ the lift of F or $N_s \times I$. Uniqueness also follows from uniqueness of cross sections $F_s(t)$. $\square$

Applications of the Fundamental Group

Definition. Let $A \subset X$ where X is a topological space, and let $i : A \to X$ be the inclusion. A continuous map $r : X \to A$ is called a **retraction** if $r \circ i(a) = a$ for all $\cancel{x} a \in A$.

Example. Projection of $r : \mathbb{R}^{n+k} \to \mathbb{R}^n$ where $r(x_1, \ldots, x_{n+k}) = (x_1, \ldots, x_n)$. Let $i : A \hookrightarrow X$ be the inclusion map. Then $r(a) = a$, so $r \circ i = id_A$. Thus $(r \circ i)_* = (id_A)_*$, a bijection. Recall $f_* : \pi_1(X) \to \pi_1(Y)$ where $f_*(f * [p]) = [f \circ p]$.

Lemma. Suppose $r : X \to A$ is a retraction, then $r_* : \pi_1(X) \to \pi_1(A)$ is onto and $i_* : \pi_1(A) \to \pi_1(X)$ is one-to-one.

Corollary. $D^2 = \{(x,y) | x^2 + y^2 \leqslant 1\}$. There is no retraction mapping from D^2 to S^1.

Proof. $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(D^2) = \{e\}$ and there is no onto map from $\pi_1(D^2)$ to $\pi_1(S^1)$. $\square$

Theorem. (Brower Fixed Point Theorem) Consider a continuous function $f : D^2 \to D^2$. Then f has a fixed point, i.e. there exists $x \in D^2$ such that $f(x) = x$.

Proof. Suppose not. Then $f(x) \neq x$ for all $x \in D^2$. Let $r(x)$ be a point in S^1 on ray starting at $f(x)$ and going through x. Note r is continuous since small perturbation of x causes small part of $f(x)$ causing small perturbation of ray, causes a small perturbation of $r(x)$. Note, r is a retraction since $r(x) = x$ for all x. Thus we've reached a contradiction. $\square$

Remark. Brower Fixed Point Theorem is true if $f : D^n \to D^n$ for all n. Proof for $n > 2$ uses homology.

Theorem. (Borsuk-Ulam Theorem) Suppose $f : S^2 \to \mathbb{R}^2$ is continuous. Then there exists a pair of points $x, -x \in S^2$ such that $f(x) = f(-x)$.

Proof. Assume not, so $f(x) \neq f(-x)$ for all x. $g = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. Then $g : S^2 \to S^1$. Consider $\alpha(s) = (\cos(2\pi s), \sin(2\pi s), 0)$. Then $h = g \circ \alpha$ is a loop in S^1. Note $g(-x) = -g(x)$ so $h(s + \frac{1}{2}) = -h(s)$. Lifting this we see that $p(\tilde{h}(s + \frac{1}{2})) = -p(\tilde{h}(s))\frac{q}{2} + \tilde{h}(s) = \tilde{h}(s + \frac{1}{2})$. Then $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$. Thus $\tilde{h}(1) - \tilde{h}(0) = q$. Thus $[h] =$ some odd integer in $\mathbb{Z}$. On the other hand there exists some homotopy H in S^2 taking α to a point, so$[h] = 0$. Thus we've reached a contradiction. $\square$

Applications of the Fundamental Group

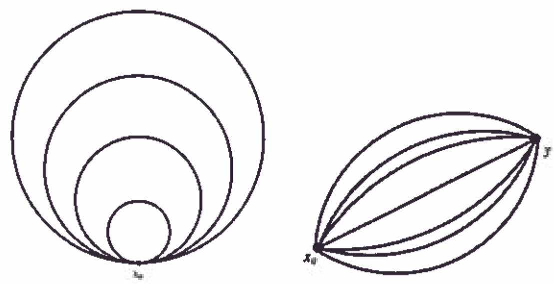
Definition. A topological space X is called **simply connected** if X is path connected and $\pi_1(X) = \{e\}$.

Examples. $\mathbb{R}^n$ and D^n are simply connected

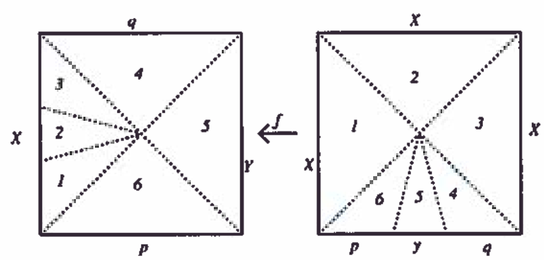
S^1 is not: it is path connected, but $\pi_1(S^1) = \mathbb{Z}$.

Proposition. $\pi_1(X) = \{e\}$ if and only if for all paths p, q such that $p(0) = q(0) = x_0$, $p(1) = q(1) = y$ then $p \sim q$ via a homotopy fixing endpoints.

$\pi_1(X, x_0) = \{e\}$



Proof. Let p, q be such that $p(0) = q(0) = x_0$, $p(1) = q(1) = y$. Let $F : I \times I \to X$ be a homotopy with $F(s, 0) = p(s)$, $F(s, 1) = q(s)$, $F(0, t) = x$, and $F(1, t) = y$.



There exists f via the piecing lemma homeomorphism of $[0,1] \times [0,1]$ to itself, preserves the identifications of the boundary. Then $F \circ f$ s a homotopy of loops taking $p * e_y * q$ to e_x. Shrink e_y to get a homotopy $p * q$ to e_x.

□

Exercise. Check: $G \circ f$ and $H \circ f^{-1}$ give desired homotopies

Proposition. $\pi_1(S^n) = \{e\}$ for $n \neq 1$ where $S^n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}$

Proof. Suppose $n = 2$. Let $H =$ the northern hemisphere (which is closed). Then $S^2 \backslash H = D^2$. Note D^2 is simply connected. Let $p : [0,1] \to S^1$ where $p(0) = p(1) =$ *south pole*. Observe $p^{-1}(H)$ is a closed subset of I, and is thus compact. So $p^{-1}(H)$ has finitely many components $J_1, \ldots, J_k$. Note $p|_{J_i}$ is a path in H connecting points on ∂H. Since H is simply connected, $\pi_1(H) = \{e\}$ by proposition. $p|_{J_i}$ is homotopic to a curve in ∂H (equator). Since there are finitely any J_i, can inductively homotop loop contained in the southern hemisphere. Then, since $\pi_1(\textit{southern hemisphere}) = \{e\}$. Further, homotop p to be e_s. □

Proposition. $\pi_1(X \times Y, (x, y)) \approx \pi_1(X, x) \oplus \pi_1(Y, y)$.

Proof. Let $p_1 : X \times Y \to X$ via $p(x, y) = x$ be the projection onto x and $p_2 : X \times Y \to X$ via $p(x, y) = y$ be the projection onto y. Then $(p_1)_*, (p_2)_* L\pi_1(X \times Y) \to X$ or Y. Define $P : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \oplus \pi_1(Y, y)$. So $P = (p_1)_* \oplus (p_2)_*$ is a homomorphism. Likewise, Define $Q : \pi_1(X, x) \oplus \pi_1(Y, y) \to \pi_1(X \times Y, (x, y))$ via $Q([f] \oplus [g]) \to [(f(s), g(s))]$. Then show that Q is a well-defined homomorphism and $Q = P^{-1}$. □

Corollary. $\pi_1(T) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$.

Corollary. $\mathbb{R}^n$, $n \neq 2$ and $\mathbb{R}^2$ are not homeomorphic.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. Then remove $(0,0)$ from $\mathbb{R}^2$. Note $f|_{\mathbb{R}^2 \setminus \{(0,0)\}}$ is homeomorphic to $\mathbb{R}^n \setminus \{f(0,0)\}$. Observe $\mathbb{R}^2 \setminus \{(0,0\} \approx S^1 \times \mathbb{R}$ and $\mathbb{R}^n \setminus \{(0,0\} \approx S^{n-1} \times \mathbb{R}$. However, $\pi_1(S^1 \times \mathbb{R}) = \mathbb{Z}$ and $\pi_1(S^{n-1} \times \mathbb{R}) \approx \{e\}$. □

Corollary. In $\mathbb{H}^2$ an open neighborhood of a boundary point is never homeomorphic to an open neighborhood of an interior point.

Proof. Recall $\mathbb{H}^2 = \{(x, y) : y \geqslant 0\}$. Consider $U \setminus \{x\} \approx [0, 1] \times (0, \varepsilon) \approx S^1 \times (0, 1)$

□

Homotopy Equivalence

If $X \approx Y$ (homeomorphic and path connected) then $\pi_1(X) \approx \pi_1(Y)$.

We need a less restrictive notion to compute $\pi_1(x)$.

Definition. Let f_0, f_1 be continuous maps where $f_{0,1} : X \to Y$ are **homotopic** if there's a continuous map $F : X \times I \to Y$ such that $F_0(x) = f_0(x)$ and $F_1(x) = f_1(x)$ where $f_t(x) = F(x, t)$.

A continuous map $g : X \to Y$ is called a **homotopy equivalence** if there exists $h : Y \to X$ continuous such that $h \circ g \sim id_X$ and $g \circ h \sim id_Y$. Further, X and Y are called **homotopy equivalent.**

Exercise. Homotopy of continuous maps and homotopy equivalence are equivalence relations.

Proposition. If X and Y are homotopy equivalent then $\pi_1(X) \approx \pi(Y)$.

Proof. Consider $g : X \to Y$ and $h : Y \to X$. Let p be a loop in X. Let $F : X \times I \to Y$ be a homotopy such that $h \circ g \sim id_x$. Note that $p : I \to X$. Thus $h(g(p(s)))$ is a loop in X. Let $G_t = F_t \circ p$, then G_t is a homotopy such that $G_0(p) = h(g(p(s)))$. Since $G(s) = p(s)$ we see that $h \circ g \circ p \sim p$. Examining the induced homomorphism we see that $(h \circ g)_*([p]) = h_* \circ g_*([p]) = [p]$. Thus $h_* \circ g_* = id_{\pi_1(x)}$. Similarly $g_* \circ h_* = id_{\pi_1(x)}$. Thus h_*, g_* are group isomorphisms. Therefore $\pi_1(X) = \pi_1(Y)$. □

Recall. A retraction mapping is a map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ let $i : A \hookrightarrow X$ be the inclusion map.

Definition. A retraction r is called a **deformation retraction** if $i \circ r \sim id_X$ relative to A. (i.e. there exists a homotopy $F : X \times I \to X$ such that $F_0(x) = i \circ r$, $F_1(x) = id_x$ and $F_t|_A = id_A$ for all t.

Note a deformation retraction is a homotopy equivalence because $r \circ i(a) = r(a) = a$ for all a. Thus $r \circ i = id_A$. Then r_* is an isomorphism of $\pi_1(X)$ to $\pi_1(A)$.

Examples. $\mathbb{R}^n$, $F_t(x) = tx + (1-t)x + 0$ is a deformation retraction of $\mathbb{R}^n$ onto $\{x_0\}$

Annulus deformation retracts to S^1.

Homotopy Equivalence

Examples. 1. D^n deformation retracts onto $\{x_0\}$ for $x_0 \in D^n$. Consider $F : X \times I \to X$ where $F_t(x) = (1-t)x + tx_0$. Then $F_0(x) = x$, $F_1(x) = x_0$, $F_t(x_0) = x_0$.

Definition. In general, if X is homotopy equivalent to a point, then X is called **contractible.**

It is possible to be contractible but not deformation retract to any point (e.g. comb space on pg 207)

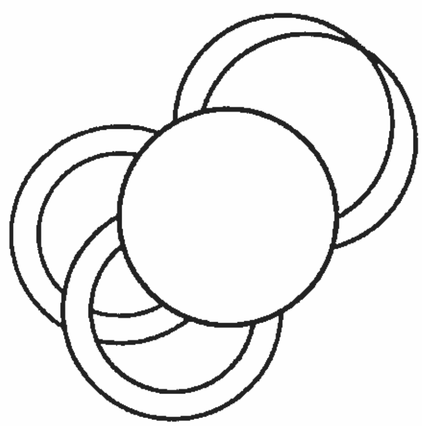
Contractible implies $\pi_1(x) = \{e\}$ but not the other way (e.g. S^2).

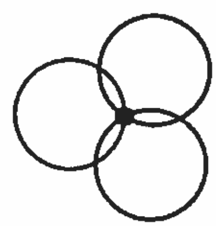
2. Annulus deformation retracts onto core circle. Consider $F_t(x, y) = (x, (1-t)y)$, $F_0 = id$, $F_1(x, y) = (x, 0)$, $F_t(x, 0) = (x, 0)$. Note that $\pi_1(A) \approx \pi_1(S^1) \approx \mathbb{Z}$.

3. The Mobius band deformation retracts onto core circle. Consider $F_t(x, y) = (x, (1-t)y)$, $F_0 = id$, $F_1(x, y) = (x, 0)$, $F_t(-1, y) = (1, -y)$. (This is the same as for the annulus) Note that $\pi_1(M) \approx \pi_1(S^1) \approx \mathbb{Z}$.

Note, S^1, Annulus, and Mobius band are all homotopy equivalent to each other but not homeomorphic.

More generally, let H be a connected handlebody. Assume it has 1 0-handle, 3 1-handles (see below), and no 2-handles. Then H deformation retracts to A which is $S^1 \vee S^1 \vee S^1$.





Definition. Let X and Y be topological spaces. Then the **wedge** of X and Y is $X \vee Y = X \cup_f Y$ where $f : \{y_0\} \to \{x_0\}$. More specifically $X \vee Y = X \sqcup Y/\sim$ where $y_0 \sim x_0$. Let $w_k =$ the wedge of k circles.

So if H has 1 0-handle, k 1-handles, and no 2-handles, then H deformation retracts onto w_k. So $\pi_1(H) \approx \pi_1(w_k)$

Question: What is $\pi_1(S^1 \vee S^1)$?

Answer: $\mathbb{Z} * \mathbb{Z}$ (the free group with 2 variables)

In general, for groups G and H, $G * H$ is a group of reduced words in G an H.

Theorem. (Seifert Van Kampen Theorem) Let X be a path connected space $x_0 \in X$. Suppose $X = A \cup B$, where A, B, $A \cap B$ are open, path connected. Let $x_0 \in A \cap B$. Then

$$\pi_1(X, x_0) \approx \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{N}$$

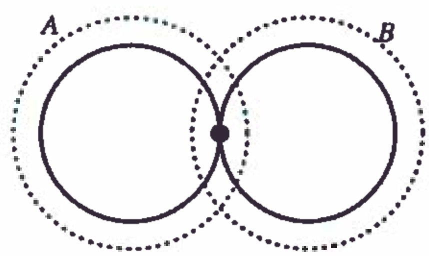
where N is a normal subgroup of $\pi_1(A, x_0) * \pi_1(B, x_0)$. More specifically, N is a normal subgroup generated by elements of the form $\psi_1(k)\psi_2(k)^{-1}$ where $\psi_1 : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ and $\psi_2 : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ are induced maps of the inclusion.

Amalgamated free product of $\pi_1(A)$ and $\pi_1(B)$ along k.

Note, $A \cap B$ must be path connected. Let $A, B \subset S^1$ such that $A \cap B = \varnothing$ Then $\pi_1(A) = \{e\}$, $\pi_1(B) = \{e\}$, $\pi_1(A) * \pi_1(B) = \{E\}$. However, $\pi_1(S^1) = \mathbb{Z}$.

Special Cases. If $\pi_1(A \cap B, x_0) = \{e\}$ then $\pi_1(X, x_0) = \pi_1(A, x_0) * \pi_1(B, x_0)$

If $\pi_1(B, x_0) = \{e\}$ then $\pi_1(X, x_0) = \pi_1(A, x_0)/N$ where N is the normal subgroup guaranteed by $\psi_1 : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$



Example. $S^1 \vee S^1$. Let $A = U_1 \cap X$, $B = U_2 \cap X$. Note $A \cap B = X$. Note $\pi_1(A \cap B) = \{e\}$ where $\pi_1(A) = \mathbb{Z}$, $\pi_1(B) = \mathbb{Z}$, thus $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$.

(Can prove by induction that $\pi_1(w_k) = F_k$)

Theorem. $\pi_1(S^2_{(p)}, x) = F_{p-1}$ for $p \geqslant 1$.

$\pi_1(T^{(g)}_{(p)}, x) = F_{2g-p-1}$

$\pi_1(P^{(h)}_{(p)}, x) = F_{h+p-1}$

Proof. The handlebody decomposition with no 2-handles deformation retracts to w_k where k is the number of 1-handles. □

What about surfaces with no boundary? $\pi_1(\mathbb{R}^n) = \{e\}$, $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$

Consider projective space $P = M \cup D^2$ Let A be an ε-neighborhood of M and B be an ε-neighborhood of D^2. $\pi_1(A) = \mathbb{Z}$ and $\pi_1(B) = \{e\}$. Thus, $\psi_1 : \pi_1(A \cap B) \to \pi_1(A)$ via $\psi_1(a) = 2a$.

Thus $\pi_1(P) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

Homotopy Equivalence

Theorem. (SVK)

Proof. Want $\Phi_{AB} : \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ surjective. Want to show that this is indeed a group homomorphism. Consider free groups G_1, G_2, and H. Using the universal mapping property of free products we see that:

$$\begin{tikzcd}
G_1 \arrow[d, "i_1"] \arrow[dr, "\phi_1"] & \\
G_1 * G_2 \arrow[r, "\phi"] & H \\
G_2 \arrow[u, "i_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Consider $\Phi_A : \pi_1(A, x_0) \to \pi_1(X, x_0)$ and $\Phi_B : \pi_1(B, x_0) \to \pi_1(X, x_0)$ which are induced homomorphisms from the inclusion maps. Then Φ_{AB} is a homomorphism. Let $a_i \in \pi_1(A, x_0)$ and $b_i \in \pi_1(B, x_0)$. Then

$$\Phi_{AB}(a_1 b_1 a_2 b_2 \dots a_n b_n) = \Phi_A(a_1) * \Phi_B(b_1) * \dots * \Phi_A(a_n) * \Phi_B(b_n).$$

Show Φ_{AB} is onto: Let $f \in \pi_1(X, x_0)$. Show $f = a_1 * b_1 * a_2 * b_2 * \dots * a_n * b_n$. Then $im(a_i) \subset A$, $im(b_i) \subset B$ where a_i, b_i are loops based at x_0. The $f^{-1}(A)$, $f^{-1}(B)$ form an open cover of I. There exists Lebesgue number δ such that if $\frac{1}{n} < \delta$ write $I = \bigcup_{i=1}^{n} J_i$ where $f(J_i) \subset A$ or B.

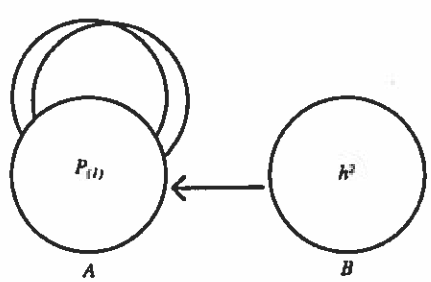
Let $f_i = f|_{J_i}$, f_i is not a loop, is a path. $f = f_1 * f_2 * \dots * f_k$.

If $f(J_i) \subset A$ there exists path p_{i1} in A from x_0 to $f($ left endpoint of $J_i)$. There exists path $p_{i2} \in A$ from x_0 to $f($ right endpoint of $J_i)$. Let $\alpha_i = p_{i1} * f * \overline{p}_{i2}$ be a loop contained in A. Choose $p_{i2} = p_{(i+1)1}$ can do if we always take p_i contained in $A \cap B$ when possible. Then $f \sim (\alpha_1 * \alpha_2) * \alpha_3 * \dots$ so $f_1 * \overline{p}_{12} * p_{12} f_2 * \overline{p}_{21} * \dots$. We can reduce words if α_i, α_{i+1} both in A or B. Want to show $\ker \Phi_{AB} = N$, so $a_1 * b_1 * a_2 * b_2 * \dots * a_k * b_k \sim e$. □

Group Presentations:

Start with F_k, then write the generators as $x_1, x_2 \dots, x_k$. Let $r_1, \dots, r_m$ be m-words in $x_1, \dots, x_k$. Let $N(r_1, \dots, r_m)$ be a normal subgroup generated by $r_1, \dots, r_m$. All finite products of the from $g r_j g^{-1}$, $g \in F_k$. Quotient $F_k/N(r_1, \dots, r_m) = \langle x_1, \dots, x_k | r_1, \dots, r_m \rangle$ (all these relations become the identity. That is, we want all words, but cancel out words that look like $r_1, \dots, r_m$. Finitely presented group with generators $x_1, \dots, x_k$ and relations $r_1, \dots, r_m$.

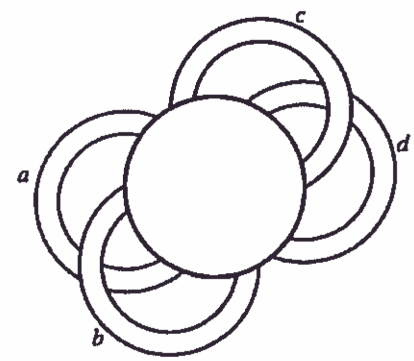
Examples. First consider the handle body diagram.



Let A be the ε-neighborhood of $h^0 \cup h^1$ and B be the ε-neighborhood of h^2. Note that $A \cap B$ is an annulus. A is homotopy equivalent to a S^1. Thus $\pi_1(A, x_0) = \mathbb{Z}$ and $\pi_1(B, x_0) = \{e\}$. By SVK $\pi_1(P) = \mathbb{Z}/N$ where N is the image of the induced homomorphism of the inclusion map $i_* : \pi_1(A \cap B) \to \pi_1(A)$. Thus $\pi_1(P) = \langle a | a^2 \rangle = \mathbb{Z}_2 = \{-1, 1\}$.

$T^{(2)} = T\#T$

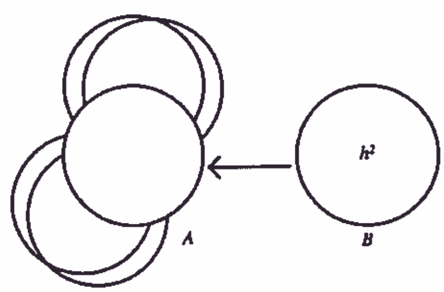
First consider the handle body diagram.



Let $A = \langle a, b, c, d \rangle$ (one generator for each handle) and B be the ε-neighborhood of h^2. Following the handles we see that $aba^{-1}b^{-1}cd^{-1}c^{-1}d = e$. Thus $\pi_k(T\#T)\langle a, b, c, d | aba^{-1}b^{-1}cd^{-1}c^{-1}d$

2. $P^{(2)} = P\#P$

First consider the handle body diagram.



Following the handles we see that $a^2b^2 = e$. Thus $\pi_1(P\#P)\langle a, b | a^2b^2 \rangle$.

In general Theorem 3.9.5

$$\pi_1(T^{(g)}, x) = \cancel{\langle} a_1 b_1 a_2 b_2 \ldots a_g b_g | a_1 b_1 a_1^{-1} b_1 - 1 \ldots a_g b_g a_g^{-1} b_g - 1 \cancel{\rangle}$$

$$\pi_1(P^{(h)}, x) = \cancel{\langle} a_1 a_2 \ldots a_h | a_1^2 a_2^2 \ldots a_h^2 \cancel{\rangle}$$

In general, infinite nonabelian groups.

Homotopy Equivalence

$\pi_1^{ab}(X) = \frac{\pi_1(X)}{N}$ where N is the commutator subgroup (the normal subgroup generated by elements of form $xyx^{-1}y^{-1}$)

$\pi_1^{ab}(T^{(g)}, x) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (2g copies) Therefore $T^{(g)}$ is not homeomorphic to $T^{(g')}$ for $g \neq g'$. Note $\chi(T^{(g)}) = 2 - 2g$. Thus $\pi_1^{ab}(T^{(g)})$ detects $\chi(T^{(g)})$.

$P^{(h)}$ Se $g_1 = a_1 a_2 \ldots a_h$ then $\langle g_1, a_2, \ldots, a_h | g_1^2 = e \rangle$. Then $\pi_1^{ab}(P^{(h)}) = \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (h-1 copies).

Lecture Notes on CW complexes have been omitted and left for 761.

Math 761 Notes

Based on *Algebraic Topology* by Allen Hatcher

Spring 2018

Erin Griffin

July 24, 2018

Contents

Review

Topological space X is a set of points equipped with a topology

subset $A \subset X$, we have the closure $\overline{A}$ and the interior $\operatorname{int} A$

So $x \in \overline{A}$ means x is in A or at least infinitely close to A.

Continuous, preserves notion of infinite closeness

- maps, homeomorphisms (invertible continuous maps with continuous inverse)
- homotopy equivalent (every homeomorphism is homotopic, but not the other way around)

Compactness, connectedness are preserved under continuous maps, especially homeomorphisms. These are called topological properties

X is a space and $x_0 \in X$, then $\pi_1(X, x_0)$ (a group generated by loops based at x_0) is called the fundamental group.

Example: $\pi_1(S^1, x_0) \approx \mathbb{Z}$ but $\pi_1(S^n, x_0) \approx \{e\}$ for $n > 1$.

Using fundamental group we can find the degree of a map:

$f : S^1 \to S^1$ implies that $\deg(f) = [f] \in \pi_1(S^1, x_0) \approx \mathbb{Z}$

Equivalently, to define the degree of f, choose a continuous map $\tilde{F} : I \to \mathbb{R}$ such that

$$\begin{tikzcd}
I \arrow[r, "\tilde{f}"] \arrow[d, "t \to e^{2\pi i t}"'] & R \arrow[d, "t \to e^{2\pi i t}"] \\
S^1 \arrow[r, "f"'] & S^1
\end{tikzcd}$$

(This is the path lifting property)

Then $\deg(f) := \tilde{f}(1) - \tilde{f}(0)$

Theorem. Fundamental Theorem of Algebra: Every complex polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $n > 0$ has a root. That is, there is a $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

Proof. Suppose $f(z)$ is never 0. Then we can view f as a map $f : \mathbb{C} \to \mathbb{C}\setminus\{0\}$. Restrict f to rS^1, a circle of radius r around 0, then $f|_{rS^1} : rS^1 \to \mathbb{C}\setminus\{0\}$. For $r >> 0$, $|z^n| > |a_{n-1}z^{n-1} + \cdots + a_0|$ for all $z \in rS^1$. So, for such r, the map $f|_{rS^1}$ is homotopic to $z^n|_{rS^1} : rS^1 \to \mathbb{C}\setminus\{0\}$ via the the homotopy $f_t(z) = z^n + (1-t)[a_{n-1}z^{n-1} + \cdots + a_0]$. Therefore, the map $\deg(f|_{rS^1}) = \deg(z^n|_{rS^1}) = n > 0$. but $\deg(f|_{rS^1}) = 0$ because $f|_{rS^1}$ extends to D^2. Thus we've reached a contradiction. $\square$

Theorem (Borsuk-Ulam) There exists no continuous map $f : S^2 \to S^1$ such that $f(-x) = -f(x)$ for all $x \in S^1$.

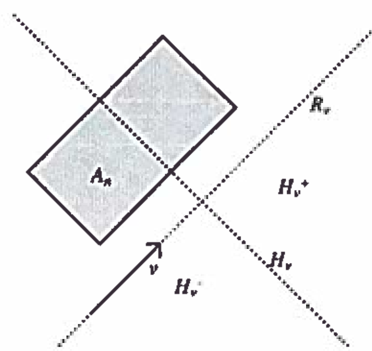
Proof. (general idea) $f|_{\text{equator of } S^2}$ would have odd degree because f is an odd function. On the other hand $\deg(f|_{\text{equator of } S^2}) = 0$ because $f_{\text{equator of } S^2}$ extends to f, specifically the upper hemisphere. This is a contradiction. ☐

In higher dimensions there exists no continuous map $f : S^{n+1} \to S^n$ such that $f(-x) = -f(x)$ for all $x \in S^{n+1}$. This proof of this uses homology or cohomology.

Theorem (Higher Dimensional ham sandwich Theorem) If $A_1, \ldots, A_n \subset \mathbb{R}^n$ are bounded and measurable sets, there exists an affine hyperplane $H \subset \mathbb{R}^n$ which cuts each A_i into 2 halves of equal volume.

Consider the three dimensional case: $A_1, A_2, A_3 \subset \mathbb{R}^3$.

Proof. Suppose there does not exists such an H. Define $f : S^{n-1} \to S^{n-2}$ as follows. Let $v \in S^{n-1} \subset \mathbb{R}^n$, let $H_v :=$ any affine hyperplane in $\mathbb{R}^n$ which is perpendicular to $\mathbb{R}^v$. Let H_v^+, H_v^- denote the two half spaces of $\mathbb{R}^n \backslash H_v$.



Choose H_v such that it cutes A_n in 2 halves of equal volume. Define real numbers $r_1(v), \ldots, r_{n-1}(v) \in \mathbb{R}$ by $r_i(v) = vol(H_v^+ \cap A_i) - vol(H_v^- \cap A_i)$. Note $r_i(v) = 0$ if and only if $vol(H_v^+ \cap A_i) = vol(H_v^- \cap A_i)$. By assumption there exists at least one i such that $r_i(v) = 0$. Define

$$f(v) := \frac{(r_1(v), \ldots, r_{n-1}(v))}{\sqrt{r_1(v)^2 + \cdots + r_{n-1}(v)^2}} \in S^{n-2} \subset \mathbb{R}^{n-1}$$

Thus, we have a map $f : S^{n-1} \to S^{n-2}$. One can show that this map is continuous. Note

$$r_i(-v) = vol(H_{-v}^+ \cap A_i) - vol(H_{-v}^- \cap A_i) = vol(H_v^- \cap A_i) - vol(H_v^+ \cap A_i) = -r_i(v)$$

Because $H_v = H_{-v}$ we see that $H_{-v}^+ = H_v^-$ and $H_{-v}^- = H_v^+$. Then $f(-v) = -f(v)$ But this contradicts the Borsuk-Ulam Theorem. ☐

Some invariant in algebraic topology:

- $(X, x_0) \longrightarrow$ fundamental group $\pi_1(X, x_0)$
- $X \longrightarrow$ homology $H_n(X)$
- $X \longrightarrow$ cohomology $H^n(X)$
- $(X, x_0) \longrightarrow$ higher homotopy groups $\pi_n(X, x_0)$

Chapter 0: Some Underlying Geometric Notes

Homotopy Equivalence

Definition. If $f, g : X \to Y$ are continuous then a **homotopy** $f \simeq g$ is a continuous family of continuous maps $f_t : X \to Y$, $t \in I$ such that $f_0 = f$, $f_1 = g$. SO $F(x,t) = f_t(x)$ is continuous as a map $X \times I \to Y$.

[3.5in]102.png

If $A \subset X$ and if $f_t|_A$ is independent of t, then f_t is called a **homotopy relative to** A.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Not g is called the **homotopy inverse** of f.

In this case X and Y are said to be **homotopy equivalent** or of the same **homotopy type**. (Notation $X \simeq Y$)

Definition. X is **contractible** if $X \simeq a\ point$.

Definition. If $A \subseteq X$, then the **deformation retraction** is a homotopy $f_t : X \to X$, $t \in I$, such that $f_0 = 1_X$, $f_1(X) = A$, $f_t|_A = 1_A$.

In this case A is called a **deformation retract** of X.

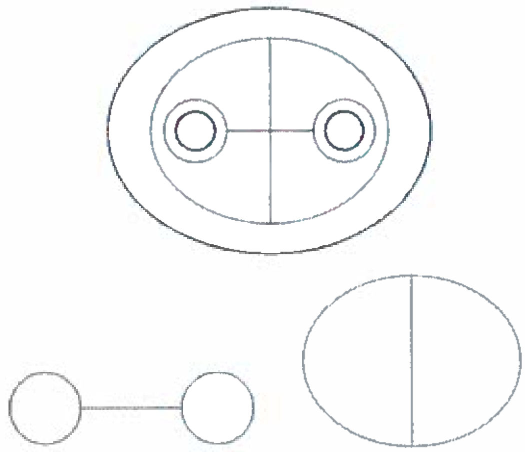
Note, if f_t is a deformation retraction, then f_1 can be views as a retraction $r : X \to A$. That is r is a continuous map such that $r|_A = i_A$.

Moreover $r : X \to A$ is a homotopy equivalence with homotopy inverse the inclusion $i : A \hookrightarrow X$. $r \circ i = 1_A$, $i \circ r = f_1 \simeq f_0 \simeq 1_X$.

If A is a deformation retract of $X \to A$ is homotopy equivalent to X.

Homotopy Equivalence

Example. Consider the following graphs:



Question. If two spaces are homotopy equivalent, is it always true that there exists a space X such that X and Y are both homeomorphic to deformation retractions of Z?

Proposition If (X, A) is a CW pair and A is contractible then the quotient map $X \to X/A$ is a homotopy equivalence.

Definition. We say X is a **CW complex** is a space built recursively by attaching cells. Further, $A \subset X$ is a **CW subcomplex** if A is itself a CW complex built from cells of X with cell structure. Thus (X, A) is a **CW pair**.

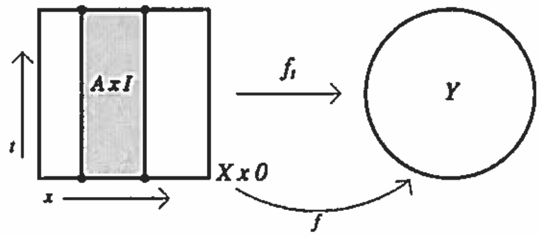
Example.

Homotopy Extension Property

Definition. Let (X, A) be a pair, X a space, $A \subset X$ a subspace. We say (X, A) has the **homotopy extension property** (HEP) if the following is given:

- a homotopy $f_t : A \to Y$, $t \in I$
- a continuous map $f : X \to Y$ such that $f|_A = f_0$

then there is a homotopy $\tilde{f}_t : X \to Y$ such that $\tilde{f}_t|_A = f_t$ and $\tilde{f}_0 = f$.



We're given a map $F : (X, \{0\}) \cup (A \times I) \to Y$. If

$$F(x,t) := \begin{cases} f_t(x) & (x,t) \in A \times I \\ f(x) & (x,t) \int X \times \{0\} \end{cases}$$

The continuity condition above means that F is continuous.

In this situation, the homotopy extension property says that there exists a continuous map $\tilde{F} : X \times I \to Y$ such that $\tilde{F}((X \times \{0\}) \cup (A \times I)) = F$.

Notation. Let $P(X, A) = (X \times \{0\}) \cup (A \times I)$.

Definition. (Equivalent to HEP) Pair (X, A) has the HEP if every continuous map $F : P(X, A) \to Y$ extends to a continuous map $\tilde{F} : X|timesI \to Y$ for all spaces Y. So:

$$\begin{tikzcd} P(X,A) \arrow[r, "F"] \arrow[d, shift right, "i"'] & Y \\ X \times I \arrow[u, shift right, "r"'] \arrow[ur, "\exists\, continuous\ \tilde{F}"'] & \end{tikzcd}$$

Lemma. (X, A) has the HEP if and only if $P(X, A)$ is a retract of $X \times I$.

Proof. Suppose first that there exists a retraction r. Then, given any continuous maps $F : P(X, A) \to Y$ we can define an extension $\tilde{F} : X \times I \to Y$ by $\tilde{F} = F \circ r$ so $\tilde{F}|_{P(X,a)} = F$.

$$\begin{tikzcd} P(X,A) \arrow[r, "F"] \arrow[d, shift right, "i"'] & Y \\ X \times I \arrow[u, shift right, "r"'] \arrow[ur, "\exists\, continuous\ \tilde{F}"'] & \end{tikzcd}$$

So (X, A) has the HEP.

Conversely, suppose (X, A) has the HEP. Then we can define a retraction $r : X \times I \to P(X, a)$ by extending the identity map $1_{P(X,A)} : P(X, A) \to P(X, A)$. □

Corollary. If (X, A) has the HEP, then so does $(X \times Z, A \times Z)$ for any space Z.

Proof. Suppose (X, A) has the HEP, the there exists a retractor $r : X \times I \to P(X, A)$. Consider the map

$$\begin{array}{rcl} r \times 1_Z = (X \times I) \times Z & \to & P(X, A) \times Z \\ \approx & & = \\ (X \times Z) \times I & & [(X \times \{0\}) \cup (A \times I)] \times Z \\ & & \approx \\ & & ((x \times Z) \times \{0\}) \cup ((A \times Z) \times I) \\ & & = \\ & & P(X \times Z, A \times Z) \end{array}$$

Thus, we can view the map $r \times 1_Z$ as a retraction $(X \times Z) \times I \to P(X \times Z, A \times Z)$. By Lemma, $(X \times Z, A \times Z)$ has HEP. □

Corollary. If (X, A) has the HEP, then $P(X, A)$ is a deformation retraction of $X \times I$.

Proof. Suppose $(X \times A)$ has the HEP. Then there exists a retraction $r : X \times I \to P(X, A) = (X \times \{0\}) \cup (A \times I)$. Define a family of maps $r_s : X \times I \to P(X, A) \subset X \times I$ for $s \in I$ as follows for $(x, t) \in X \times I$:

$$r_s(x, t) := (p_1(r(x, st)), (1 - s)t + sp_2(r(x, t))) \in X \times I$$

Then:

- $r_s(x, t)$ is continuous in (x, t, s)
- $r_0(x, t) = (p_1(r(x, 0)), (t + 0)) = (p_1(x, 0), t) = (x, t)$
- $r_1(x, t) = (p_1(r(x, t)), p_2(r(x, t))) = r(x, t)$
- Suppose $(x, t) \in P(X, A)$ then $r(x, t) = (x, t)$, so $(x, st) \in P(X, A)$ then $r(x, st) = (x, st)$. Thus $r_s(x, t) = (p_1(x, st), (1 - s)t + sp_2(x, t)) = (x, t - st + st) = (x, t)$. Then $r_s|_{P(X,A)} = 1_{P(X,A)}$.

□

Summary. (X, A) has the homotopy extension property if and only if $X \times I$ retracts to $P(X, A)$ if and only if $X \times I$ deformation retracts onto $P(x, A)$. Further $(X \times Z, A \times Z)$ has the HEP for all spaces Z.

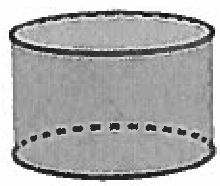
Homotopy Extension Property

Definition. A **deformation retraction** from X to $A \subset X$ is a homotopy $f_t : X \to X$, $t \in I$, such that $f_0 = 1_X$, $f_1(X) = A \subset X$, and $f_t|_A$ is the inclusion $i : A \hookrightarrow X$.

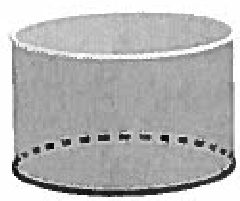
Example. $(X, A) = (D^2, \partial D^2)$ where $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 1\}$.

Claim: $(D^2, \partial D^2)$ has the HEP.

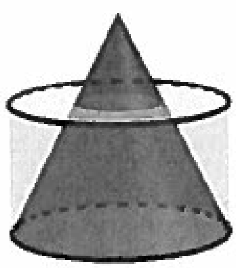
Proof. $D^2 \times I$:

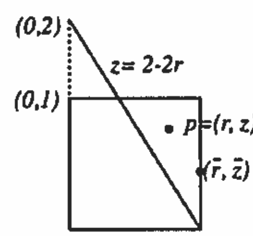


$P(D^2, \partial D^2) = (D^2 \times \{0\}) \cup ((\partial D^2) \times I)$



Regard $D^2 \times I$ as a subspace of $\mathbb{R}^3$. Let $Q := (0, 0, 2)$. Define $r : D^2 \times I \to P(D^2, \partial D^2)$ by stereographic projection from Q. (To find the explicit formula for r Consider the cone with Q as its vertex and D^2 as its base.)





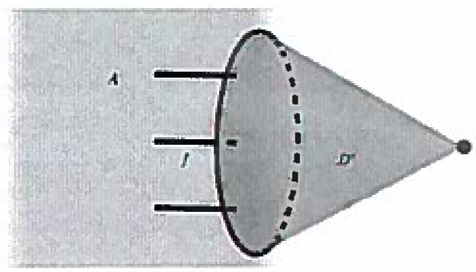
Summary, $r : D^2 \times I \to P(D^2, \partial D^2)$ is given by

$$r(r, \varphi, z) = \begin{cases} \left(\frac{2r}{2-z}, \varphi, 0\right) & z \leqslant 2 - 2r \\ \left(1, \varphi, 2 - \frac{2-z}{r}\right) & z \geqslant 2 - 2r \end{cases}$$

Thus r is continuous and is indeed a retraction from $D^2 \times I$ to $P(D^2, \partial D^2)$. Thus $(D^2, \partial D^2)$ has HEP.

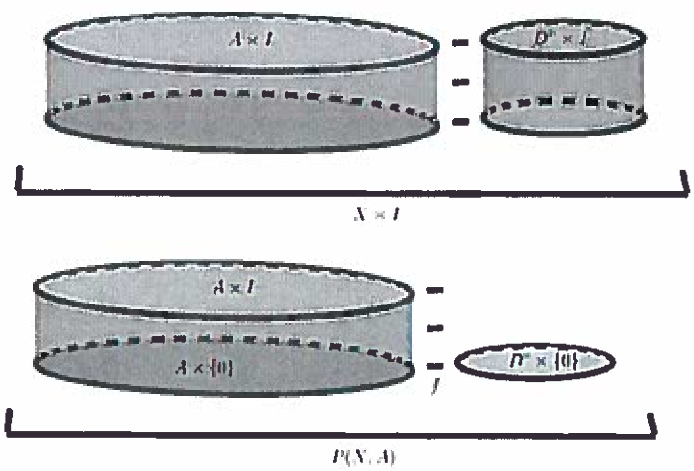
Similarly $(D^n, \partial D^n)$ has the HEP for all $n \geqslant 0$.

Consequentially if A is any space, $f : \partial D^n \to A$ is continuous. Let $X = A \sqcup_f D^n = \frac{A \cup D^n}{f(x) \sim x \ \forall x \in \partial D^n}$ with quotient topology. Then X is A with D^n attaching along f:



Claim: (X, A) has the HEP. To figure this out we will glue the regions $(\partial D^n \times I)$

$$X \times I = (A \sqcup_f D^n) \times I = (A \times I) \sqcup_{f \times 1_I} (D^n \times I)$$



$X \times I = P(X, A) \sqcup_g (D^n \times I)$ attached along $P(D^n, \partial D^n) \subseteq D^n \times I$. We can get a retraction from $X \times I$ to $P(X, A)$. Thus (X, A) has the HEP.

Note, the claim remains true if X is obtained from A by attaching any number of n-disks. $\square$

CW Complex

A CW complex is a space built inductively by attaching cells.

More formally, $X = \bigcup_{n \geq 0} x^n$, $X^0 \subset X^1 \subset X^2 \subset \ldots$ where

- X^0 is a discrete set of points
- X^n is obtained from X^{n-1} by attaching n-dimensional disks, D^n, using continuous maps, $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$. So for all $\alpha \in I_n$: $X^n = X^{n-1} \sqcup \coprod_{\alpha \in I_n} D^n_\alpha$ where $\varphi_\alpha(x) \sim x$ for all $x \in \partial D^n_\alpha$, for all $\alpha \in I_n$.

The topology on $X = \bigcup_{n\geqslant 0} X^n$ is assumed to be such that if a set $U \subseteq X$ is open in X if and only if $U \cap X_n$ is open in X^n for all $n \geqslant 0$. Call this the **weak topology**.

Terminology:

$$(D^n_\alpha \backslash \partial D^n_\alpha \subseteq)\ D^n_\alpha \hookrightarrow X^{n-1} \cup \coprod_{\alpha\in I} D^n_\alpha \twoheadrightarrow X^n \hookrightarrow X \supseteq \Phi_\alpha(D^n_\alpha \backslash \partial D^n_\alpha)$$

- Then $I_\alpha(D^n_\alpha \backslash \partial D^n_\alpha)$ is denoted $e^n + \alpha$ and called an n-**cell**.
- Φ_α is called the **characteristic map** of e^n_α
- Φ_α sends $D^n_\alpha \backslash \partial D^n_\alpha$ homeomorphically to e^n_α
- $\Phi_\alpha / \partial D^n_\alpha$ is given by $\partial D^n_\alpha \to \varphi_\alpha X^{n-1} \hookrightarrow X$.
- X^n is called the **n-skeleton** of X.

Definition. A CW complex is (at most) **n-dimensional** if it has no cells of dimension greater than n. In this case $X = X^n$ (and $X^n = X^{n+1} = X^{n+2} = \ldots$)

Definition. A subcomplex of a CW-complex X is a subspace $A \subseteq X$ which is a union of cells of X, such that the closure of each cell in A is also in A. (So A is itself a CW complex).

Facts:

1. If $A \subset X$ is a subcomplex then A is closed in X
2. IN a CW complex, the closure of each cell has nonempty intersection with only finitely many other cells. This is called **closure finiteness**.
3. Every CW pair has HEP.

CW Complex

Proposition. If (X, A) is a CW-pair then (X, A) has the HEP.

Proof. $A \subset X = \bigcup_{n\geqslant 0} X^n = \bigcup_{n\geqslant 0}(X^n \cup A)$.
$P(X, X^n \cup A) = (X^n \cup A) = (X \times \{0\}) \cup [(X^n \cup A) \times I]$.
Then $X \times I = \bigcup_{n\geqslant -1} P(X, X^n, A)$ where $P(X, A) \subseteq P(X, X^0 \cup A) \subseteq P(X, X^1 \cup A) \subseteq \ldots$.
Note: $X^n \cup A$ is obtained from $X^{n-1} \cup A$ by attaching n-dimensional disks D^n along ∂D^n. So $P(X, X^n \cup A)$ is obtained from $P(X, X^{n-1} \cup A)$ by attaching copies of $D^n \times I$ along the region $P(D^n, \partial D^n)$ (as in previous proof).

Thus, there exists a retraction $r_n : P(X, X^n \cup A) \to P(X, X^{n-1} \cup A)$ given by retracting each $D^n \times I$ to $P(D^n, \partial D^n)$. Define a retraction $r : X \times I \to P(X, A)$ as follows:

$$r|_{P(X, X^n \cup A)} = r_0 \circ r_1 \circ \cdots \circ r_n$$

We can check that r is well defined and continuous. (This is not obvious in general). Then we get a retraction $r : X \times I \to P(X, A)$, so $P(X, A)$ has the HEP. ☐

Proposition. If (X, A) has the HEP and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence (so $X \approx X/A$).

Proof. Suppose A is contractible. Then $1_A \approx$ constant map at a_0 for some $a_0 \in A$. Therefore, there exists a homotopy $f_t : A \to A$, $t \in I$. such that $f_0 = 1_A$ and $f_1(A0 = \{a_0\}$. Let $i : A \to X$ be the inclusion and consider $i \circ f_t : A \to X$. Since (X, A) has the HEP, there exists an extension $\tilde{f}_t : X \to X$, $t \in I$ such that $\tilde{f}_0 = 1_X$ and $\tilde{f}_t|_A = i \circ f_t$. Now note that $\tilde{f}_t(A) = (i \circ f_t)(A) \subset A$, so $\tilde{f}_t$ sends A to A.

Consider $q \circ \tilde{f}_t : X \to X/A$ where $q \circ \tilde{f}_t(A) = A/A = \{a_0\}$. So $q \circ \tilde{f}_t$ factors through a map $\overline{f}_t : X/A \to X/A$ so we have a commutative diagram for all $t \in I$.

$$\begin{tikzcd}
A \subset X \arrow[r, "\tilde{f}_t"] \arrow[d, "q"] & X \supset A \arrow[d, "q"] \\
X/A \arrow[r, "\overline{f}_t"'] & X/A
\end{tikzcd}$$

Setting $t = 1$: let

$$\square := \begin{tikzcd}
X \arrow[r, "\tilde{f}_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\
X/A \arrow[r, "\overline{f}_1"'] & X/A
\end{tikzcd}$$

Note that $\tilde{f}_1(A) = (i \circ f_1)(A) = i(\{a_0\}) = \{a_0\}$, so $\tilde{f}_1$ factors through a map $g : X/A \to X$. Hence we have another commutative diagram:

$$\Delta := \begin{tikzcd} A \subset X \arrow[r, "\tilde{f}_1"] \arrow[d, "q"'] & X \\ X/A \arrow[ur, "g"'] & \end{tikzcd}$$

Thus $g \circ q = \tilde{f}_1 \simeq \tilde{f}_0 = 1_X$.

Then $X, \sim$ for any continuous map $f : X \to Y$ such that $f(x) = f(x')$ whenever $x \sim x'$, there is a unique continuous map $g : X/\sim \to Y$ such that

$$\begin{tikzcd} A \subset X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/A \arrow[ur, "g"'] & \end{tikzcd}$$

Since $\square$, Δ commute and since q is surjective, the following commutes:

$$\begin{tikzcd} & X \arrow[d, "q"] \\ X/A \arrow[ur, "g"] \arrow[r, "\bar{f}_1"'] & X/A \end{tikzcd}$$

So in summary, $g \circ q \simeq 1_X$ and $q \circ g \simeq 1_{X/A}$, so g is a homotopy inverse for q. Thus $q : X \to X/A$ is a homotopy equivalence. $\square$

Corollary. If (X, A) is a CW-pair and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence (so $X \simeq X/A$)

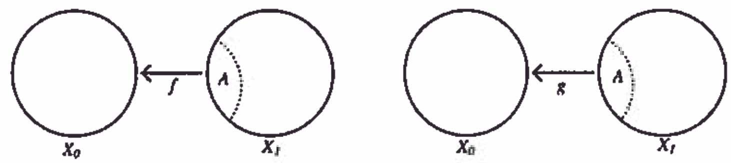
Example:

Definition. For spaces X, Y such that $X \cap Y \neq \varnothing$ with a common subspace A. We say that $X \simeq Y \operatorname{rel} A$ if there exists continuous maps f and g $f : X \to Y$, $g : Y \to X$. Such that $f|_A = g|_A = 1_A$ and $f \circ g \simeq 1_Y$, $g \circ f \simeq 1_X$ via homotopies which restrict to the identity map of A at all times.

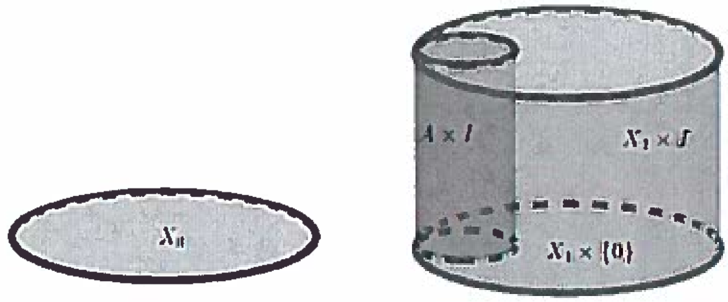
Proposition. Let x_0 be any space and (X_1, A) be a pair which has the HEP. Then, if $f, g : A \to X_0$ are homotopic continuous maps, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \operatorname{rel} X_0$.

$$X_0 \sqcup_f X_1 = \frac{X_0 \sqcup X_1}{f(a) \sim a \ \forall a \in A} \qquad X_0 \sqcup_g X_1 \frac{X_0 \sqcup X_1}{g(a) \sim a \ \forall a \in A}$$

Note, $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ contain X_0 as a subspace.



Proof. Let $F : A \times I \to X_0$ be a homotopy between f and g, and $Z := X_0 \sqcup (X \times I)$



Z

Since (X_1, A) has the HEP, there exists a deformation retraction $r_s : X_1 \times I \to P(X_1, A) = (X_1 \times \{0\}) \cup (A \times I)$ for $s \in I$. Because r_s is a deformation retraction, it restricts to the identity on $P(X_1, A)$ and hence on $A \times I \subset P(X_1, A)$ for all $s \in I$.

So r_s extends to deformation retraction, $r : X_0 \sqcup_f (X_1 \times I) = Z \to X_0 \sqcup_f P(X_1, A)$. Then:

$$X_0 \sqcup_f P(X_1, A) = X_0 \cup_f ((X_1 \times \{0\}) \cup (A \times I)) = X_0 \sqcup (X_1 \times \{0\}) = X_0 \sqcup_f X_1$$

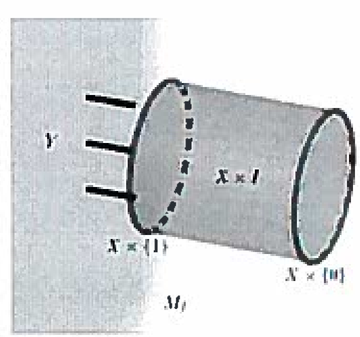
So Z deformation retracts onto $X_0 \sqcup_f X_1$. Similarly Z deformation retracts onto $X_0 \sqcup_g X_1$. Thus $X_0 \sqcup_f X_1 \simeq Z \simeq X_0 \sqcup_g X_1 \mathrm{rel} X_0$. ☐

Question: If $X \simeq Y$ is there a Z such that X and Y are both homeomorphic to deformation retracts of Z.

Mapping Cylinders

Definition. For a continuous map $f : X \to Y$, the **mapping cylinder** of f is the following space:

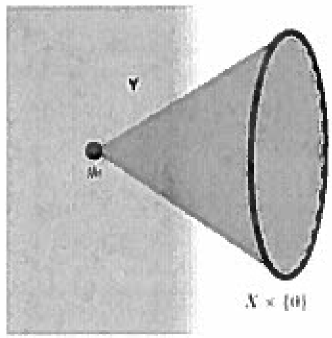
$$M_f := \frac{Y \sqcup (X \times I)}{f(x) \sim (x, 1)\ \forall x \in I} = Y \sqcup_g (X \times I) \qquad g : X \times \{1\} \to Y \text{ via } g(x, 1) = f(x)$$



Mapping Cylinders

Note: f doesn't have to be injective or surjective.

Example. Suppose f is a constant map, $f(x) = \{y_0\}$, $y_0 \in Y$:



Observations: 1. There exists inclusions $i : X \hookrightarrow M_f$, $j : Y \hookrightarrow M_f$ where $i(x) = (x, 0) \in X \times \{0\} \subseteq M_f$, $j(y) = y \in Y \subseteq M_f$.
One can check that i and j are homeomorphisms onto their images.

2. $X \times I$ deformation retracts onto $X \times \{1\}$, so M_f deformation retracts onto $Y \subseteq M_f$. Thus $Y \simeq M_f$ and $j : Y \hookrightarrow M_f$ is a homotopy equivalence with homotopy inverse given by the retraction $r : M_f \to Y$:

$$\begin{cases} r(y) = y & \forall y \in Y \subseteq M_f \\ r(x,t) = f(x) & \forall (x,t) \in X \times I \subseteq M_f \end{cases}.$$

$$\begin{tikzcd} & M_f \arrow[dl, shift right, "q"'] & \\ B \arrow[ur, shift right, "j"'] & & C \arrow[ul, "i"'] \arrow[ll, "f"] \end{tikzcd}$$

So j and r are homotopy inverses of each other. So $r \circ j \simeq 1_Y$ and $j \circ r \simeq 1_{M_f}$.
Claim: $r \circ i = f$ and $j \circ f \simeq i$ (showing this would add an arrow to f)

Proof. Let $x \in X$, $(r \circ i)(x) = r(x, 0) = f(x)$. So $r \circ i = f$. Further, $j \circ f = j \circ r \circ i \simeq 1_{M_f} \circ i = i$. Thus $j \circ f \simeq i$. □

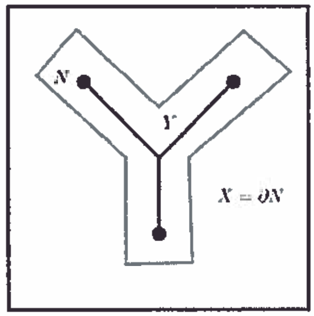
Proposition. $f : X \to Y$ is a homotopy equivalence if and only if $i : X \hookrightarrow M_f$, where $i(x) = (x, 0)$, is a homotopy equivalence.

Proof. By claim $r \circ i = f$, $j \circ i \simeq i$. Thus r and j are homotopy equivalences. Suppose i is a homotopy equivalence, so $r \circ i$ is a homotopy equivalence, and thus f is a homotopy equivalence. Suppose f is a homotopy equivalence, so $j \circ f$ is a homotopy equivalence, and thus i is a homotopy equivalence. □

Mapping Cylinder Neighborhoods

Z is a space, $Y \subseteq Z$ subspace, N is a **closed neighborhood** of Y (closure of an open subset of Z that contains Y). Define $X = \partial N$.

Example. $Z = \mathbb{R}^2$, $Y =$ the union of the segments from $(0,0)$ to the points $(1,1), (-1,1), (0,-1)$.



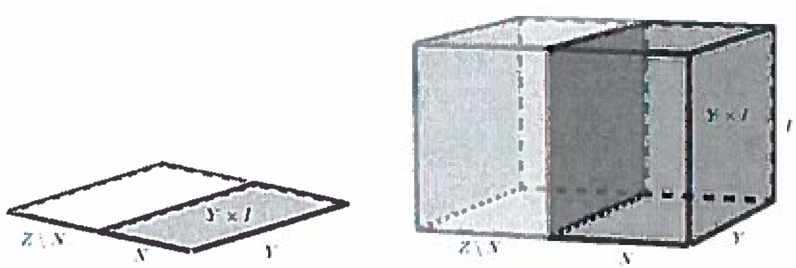
Definition. N is a **mapping cylinder neighborhood** of Y if there exists a continuous map $f : X \to Y$ and a homeomorphism $h : N \to M_f$ such that

$$h|_X = i : X \hookrightarrow X \times \{0\} \subseteq M_f \qquad h|_Y = j : Y \hookrightarrow Y \subseteq M_f$$

Claim: Picture, X is mapping cylinder of Y. Define a map $f : X \to Y$ by sending outer endpoint of each segment to the inner endpoint, then $N \cong M_f$.

Note: if N is a mapping cylinder neighborhood of Y, then N deformation retracts to Y.

Proposition. If $Y \subseteq Z$ has a mapping cylinder neighborhood then (Z, Y) has the HEP.



Proof. We get a retraction from $Z \times I$ to $P(Z, Y)$. □

Examples. $f : X \to Y$ continuous, $Z :=<_f$

Y has a mapping cylinder neighborhood $N := Y \cup [X \times [\frac{1}{2}, 1)] \subseteq M_f$

$X \times \{0\}$ also has a mapping cylinder neighborhood $N := X \times [0, \frac{1}{2}] \cong X \times I \cong M_{1_X}$

Corollary. (M_f, Y) and $(M_f, X \times \{0\})$ have HEP

Theorem. If (X, A) has the HEP and the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retraction of X.

Proof. Later $\square$

Proposition. $f : X \to Y$ is a homotopy equivalence if and only if $X \times \{0\}$ is a deformation retraction of the mapping cylinder.

Proof. Since $(M_f, X \times \{0\})$ has the HEP, the theorem tells us that $X \times \{0\}$ is a deformation retraction of M_f if and only if $i : X \times \{0\} \hookrightarrow M_f$ is a homotopy equivalence if and only if $f : X \to Y$ is a homotopy equivalence. $\square$

Corollary. If X and Y are homotopy equivalent then there exists Z such that X, Y are both homeomorphic to deformation retracts of Z.

Proof. Let $f : X \to Y$ be a homotopy equivalence and take $Z := M_f$. Then Z deformation retracts to $X \times \{0\} \cong X$ because f is a homotopy equivalence. Further Z deformation retracts to Y because M_f always deformation retracts to Y. $\square$

Mapping Cylinder Neighborhoods

Theorem. If (X, A) has the HEP and the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retraction of X.

Proof. Let $g : X \to A$ be a homotopy inverse of the inclusion $i : A \hookrightarrow X$. Then $g \circ i \simeq 1_A$ and $i \circ g \simeq 1_X$. We will prove this theorem in two steps. First, we will construct a retraction $r : X \to A$ such that $i \circ r \simeq 1_X$. Then we will show that $i \circ r \simeq 1_X \operatorname{rel} A$ and thus the homotopy is time independent on A.(that is $f_t|_A = i : A \hookrightarrow X$ for all $t \in I$.

Let $h_t : A \to A$ be a homotopy from $h_0 = g \circ i$ to $h_1 = 1_A$. Let $g_t : X \to A$ be an extension (allowed by HEP) such that $g_0 = g$ and $g_t|_A = h_t$. Define $r := g_1 : X \to A$ so $r \circ i = g_1 \circ i = g_1|_A = h_1 = 1_A$. So r is a retraction from X to A. Thus $i \circ r = i \circ g_1 \simeq i \circ g_0 i \circ g \simeq 1_X$. Therefore $i \circ r \simeq 1_X$ so we've constructed the desired retraction.

Using this r, let $k_t : X \to X$ be a homotopy where $k_0 = i \circ r$ and $k_1 = 1_X$. The only potential problem with this k is that k_t might not be rel A, i.e., $k_t|_A$ might not be equal to $i : A \hookrightarrow X$. However, we know that $k_0|_A = i \circ r|_A$, $k_1|_A = 1_X|_A = i$. So at times $t = 0$ and $t = 1$, $k_t|_A = i$. Proceeding, define a new homotopy l_t from $i \circ r$ to 1_X as follows:

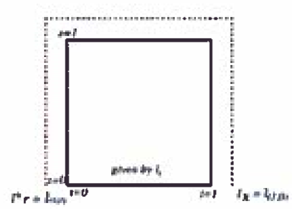
$$l_t := \begin{cases} k_{1-2t}|_A \circ r & 0 \leqslant t \leqslant \frac{1}{2} \text{ which is a homotopy from } i \circ r \text{ to } i \circ r \\ k_{2t-1}|_A & \frac{1}{2} \leqslant t \leqslant 1 \text{ which is a homotopy } k_t \text{ at twice the original speed} \end{cases}$$

Note that l_t is better than k_t because $l_t|_A$ is symmetric around $t = \frac{1}{2}$ so $l_t|_A = l_{1-t}|_A$.

Idea is to get a homotopy which fixes a pointwise at all times t, collapse each of the paths $l_t(x)$, $x \in A$, $t \in I$ to the constant path at x.

Formally: Define a family of maps $l_{(t,s)} : A \to X$, $(t, s) \in I \times I$ as follows. Define $l_{(t,0)} = l_t|_A$. Define $l_{(t,s)}$ to be constant on each boundary. (This definition is unambiguous because $l_t|_A = l_{1-t}|_A$.) Note $l_{(t,0)} = l_t|_A$ by definition. $l_{(t,1)} = l(0, 0) = l_0|_A$. We can view $l_{(t,s)}$ for $s \in I$ as a homotopy between the following homotopies. So when $s = 0$ we look at $l_t|_A$ and when $s = 1$ we look at the constant homotopy which is i for all t. These are both maps defined on $A \times I$. Since (X, A) has the HEP, the pair $(X \times I, A \times I)$ also has the HEP. So we can extend $l_{(t,s)} : A \to X$ to a family $\tilde{l}_{(t,s)}$, $(t, s) \in I \times I$, such that $\tilde{l}_{(t,0)} = l_t$.

How should we think of $\tilde{l}_{(t,s)}$?



Then $\tilde{l}_{(t,s)}|_{\sqcap}$ is a homotopy from $i \circ r$ to 1_X.

On A, this homotopy is given by the inclusion $i : A \hookrightarrow X$ at all times because $\tilde{l}_{(t,s)}|_A = l_{(t,s)}$ and on $\sqcap$ $l_{(t,s)}$ is constant and is equal to $i : A \hookrightarrow X$. Then $\tilde{l}_{(t,s)}|_{\sqcap}$ is a homotopy from $i \circ r$ to $1_X \operatorname{rel} A$. $\square$

Review of Metric Spaces

Definition. A **metric space** is a set X with a function $d : X \times X \to [0, \infty)$ such that:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leqslant d(x, y) + d(y, z)$ for all x, y, z.

Definition. Let $x \in X$, $\varepsilon > 0$, then the **open ε-ball centered at** x or **ε-neighborhood** is:

$$B(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\}$$

Definition. We say that a set U is **open** if for all $x \in U$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$.

Facts:

- This definition turns X into a normal Hausdorff space
- $B(x, \varepsilon)$ is open with respect to this topology
- $d(x, y) \to [0, \infty)$ is continuous

Distance from a set:

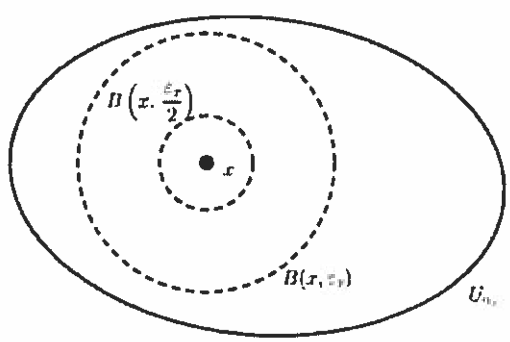
- $x \in X$, $A \subset X$, $A \neq \varnothing$: $D(x, A) = \inf\{d(x, a) | a \in A\}$
- $d(x, A)$ is continuous in X if A is fixed.
- if A is closed then $d(x, A) = 0$ if and only if $x \in A$. This characterizes closed sets completely.

Definition. If $A \subset X$, X is a metric space, $A \neq \varnothing$ then the **diameter of** A is:

$$\operatorname{diam}(A) = \sup\{d(x, y) | x, y \in A\} \in [0, \infty) \cup \{\infty\}$$

Lemma. (Lebesgue number lemma) Suppose (X, d) is a compact metric space and $\{U_\alpha\}$ is an open cover X. Then there exists a $\delta > 0$ such that every $A \subseteq X$ with $\operatorname{diam}(A) < \delta$ is contained in one of the U_α. We call δ a **Lebesgue number of** $\{U_\alpha\}$.

Proof. Since $\{U_\alpha\}$ is a cover of X_i, every $x \in X$ is in a U_{α_x}, and since U_{α_x} is open there exists an $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subseteq U_{\alpha_x}$.



Consider the following sets $B\left(x, \frac{\varepsilon_x}{2}\right)$ for $x \in X$. These sets form an open cover of X, so by compactness of x there exists finitely many $x_1, \ldots, x_n \in X$ such that

$$B\left(x_1, \frac{\varepsilon_{x_1}}{2}\right) \cup \cdots \cup B\left(x_n, \frac{\varepsilon_{x_n}}{2}\right) = X$$

Define $\delta := \min\{\frac{\varepsilon_{x_i}}{2} | i = 1, \ldots n\} > 0$ and let $A \subseteq X$ be such that $\operatorname{diam}(A) < \delta$. Choose $a \in A$ and let i be such that $a \in B\left(x_i, \frac{\varepsilon_{x_i}}{2}\right)$. Then $A \subset B\left(x_i, \frac{\varepsilon_{x_i}}{2} + \delta\right) \subset B\left(x_i, \varepsilon_{x_i}\right) \subset U_{\alpha_x}$ $\square$

Review of Metric Spaces

Theorem. (Nagata-Smirna) A topological space X is metrizable if and only it is a normal Hausdorff space and has a basis B that is countable locally finite.

Proof. ($\Leftarrow$) Suppose that X is a normal Hausdorff space and has a basis B that is countable locally finite. Choose any countably finite basis $B = \bigcup_{n\in\mathbb{N}} B_n$ and let $\mathscr{I} := \{(n, B) | n \in \mathbb{N}, B \in B_n\}$. Then show that there exists a topological embedding

$$\varphi : X \hookrightarrow [0, 1]^{\mathscr{I}} := \{(r_\alpha)_{\alpha\in\mathscr{I}} | r_\alpha \in [0, 1]\}$$

where topology in $[0, 1]^{\mathscr{I}}$ is given by the uniform metric $d((r_\alpha), (s_\alpha)) := \sup\{|r_\alpha - s_\alpha| \,|\alpha \in \mathscr{I}\}$. Then $X \simeq \varphi(X) \subseteq [0, 1]^{\mathscr{I}}$ which is a metric space because $\varphi(X)$ is a metric space and thus so is X. Therefore X is metrizable. □

To define φ: For each $\alpha = (n, B) \in \mathscr{I}$, construct a function $\varphi_\alpha : X \to [0, \frac{1}{n}] \subseteq [0, 1]$ such that $\varphi_\alpha(x) > 0$ if $x \in B$ and $\varphi_\alpha(x) = 0$ if $x \notin B$. Define $\varphi : X \to [0, 1]^{\mathscr{I}}$ by $\varphi = (\varphi_\alpha)_{\alpha\in\mathscr{I}}$.

Definition. A topological space X is **metrizable** if there exists a metric on X inducing a topology

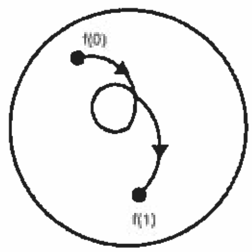
Definition. A basis, B, is **countably locally finite** if B c an be written as $B = \bigcup_{n\in N} B_n$ where each B_n is locally finite, or every point $x \in x$ is contained in at most finitely many elements of the collection of B_n.

Corollary. X is metrizable if and only if X is homeomorphic to a subspace of $[0, 1]^J$ equipped with the uniform metric for some set J.

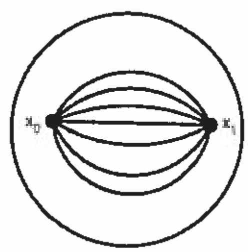
Chapter 1: The Fundamental Group

Basic Constructions

Definition. Let X be a topological Space. A **path** in X is a continuous map $f : I \to X$.



Definition. Two paths f, g are path homotopic if $f \simeq g \operatorname{rel} \partial I$.



Notation: $f \simeq_p g$

$[f] :=$ the path homotopy class of f.

Definition. A **loop** in X based at x_0 is a path $f : I \to X$ such that $f(0) = f(1) = x_0$. (Note, if f is a loop based at x_0 and $g \simeq_p f$ then g is also a loop based at x_0.)

Definition. A reparametrization of a path $f : I \to X$ is a composition $f \circ \varphi$ where $\varphi : I \to I$ is a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

Note: $f \circ Y \simeq_p f$ because $\varphi \simeq 1_I \operatorname{rel} \partial I$ via the following homotopy: $\varphi_t : I \to I$, $t \in I$ via $\varphi_t(s) = (1-t)\varphi(s) + ts$.

Composition/ Product: $f, g : I \to X$ paths such that $f(1) = g(1)$ then

$$f \circ g(s) = f \cdot g(s) = \begin{cases} f(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Note: If $f \simeq_p f'$ and $g \simeq_p g'$ then $f \cdot g \simeq_p f' \cdot g' \simeq_p f' \cdot g \simeq_p f \cdot g'$. So it makes sense to define the following: $[f][g] = [f \cdot g]$.

Definition. $\pi_1(X, x_0) = \{[f] | f$ is a loop in X based at $x_0 \in X\}$.

Definition. $\pi_1(X, x_0)$ is a group with respect to the above product, and is called the **fundamental group** of X at x_0.

Identity: $e = [c_{x_0}]$, the constant path at x_0, so $c_{x_0}(s) = x_0$ for all $s \in I$.

Inverse: $[f]^{-1} = [\overline{f}]$ where $\overline{f}$ is the inverse path, $\overline{f}(s) = f(1-s)$ for all $s \in I$.

Change of Base Point Homomorphism: $x_0, x_1 \in X$, $h : I \to X$ is a path with $h(0) = x_0$, $h(1) = x_1$. Define $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ via $\beta_h([f]) = [h \cdot f \cdot \overline{h}]$.

Note: β_h depends only on $[h]$. Further, if h, k are paths in X with $h(1) = k(0)$ then $\beta_{h \cdot k} = B_h \circ B_k$.

Proposition. β_h is a group isomorphism with inverse.

Proof. We see that B_h is a homomorphism since

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{[h]} \cdot h \cdot g \cdot \overline{h}] = \beta_h[f]\beta_h[g].$$

Further, β_h is an isomorphism with inverse $\beta_{\overline{h}}$ since

$$\beta_h \beta_{\overline{h}}[f] = \beta_h[\overline{h} \cdot f \cdot h] = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f] = \beta_{\overline{h}} B_h.$$

□

Proposition. If X is path-connected, then (up to isomorphism), $\pi_1(X, x_0)$ is independent of x_0.

Definition. X is **simply connected** if it is path connected and $\pi_1(X, x_0) = 0$ for all $x_0 \in X$. e.g. every convex subspace of $\mathbb{R}^n$ is simply connected.

Proposition. A space X is simply connected if and only if there exists a unique path homotopy class of paths connecting any two points in X.

Proof. Suppose X is simply connected. Thus $\pi_1(X) = 0$. If f and g are two paths from x_0 to x_1 , then $f \simeq_p f \cdot \overline{g} \cdot g \simeq_p g$ since $\overline{g} \cdot g$ and $f \cdot \overline{g}$ are loops that are homotopic to constant loops. If there is a unique path homotopy class of paths connecting a base point x_0 to itself, then all loops x_0 are homotopic to the constant loop and $\pi_1(X, x_0) = 0$. □

If $\varphi : X \to Y$ is a map such that $\varphi(x_0) = y_0$, then the **induced homeomorphism** is defined by $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ via $\varphi_*([f]) = [\varphi \circ (f)]$ for loops f bases at x_0.

Properties:

- φ_* is a group homomorphism
- $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$
- $(1_X)_* = 1_{\pi_1(X, x_0)}$

Consequences: If φ is a homomorphism, then φ_* is a group isomorphism. (This is still true is φ is just a homotopy equivalence.

Lemma. If $\varphi_t : X \to Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a base point $x_0 \in X$, then the three maps in the diagram below satisfy $\varphi_{0_*} = B_h \varphi_{1_*}$.

$$\begin{tikzcd} & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \\ & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

Examples 1. Every contractible space has trivial fundamental group. That is, $\varphi_1(X, x_0) = 0$ for all contractible X.

2. $\pi_1(S^1) = \mathbb{Z}$
3. $\pi_1(S^n) = 0$ for $n \geqslant 2$.
4. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$
   So, $\pi_1(T) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$

Van Kampen's Theorem

Free Product:

Let G_1, G_2 be groups. Then $G_1 * G_2 = \{$reduced words in G_1 and $G_2\}$ (Here "reduced' means no letter is the identity element, consecutive letters belong to different groups and "words" are finite (possibly empty) sequences of letters in G_1 or G_2.

Multiplication is induced by the concatenation of words.

Note, there exists inclusions $i_j : G_j \hookrightarrow G_1 * G_2$ where $i_j(g) = g$ for $g \neq e$ and $i_j(e) =$ empty word.

Amalgated Free Product:

If K is a group, $\psi_j : K \to G_j$ are homomorphisms, $j = 1, 2$

Define $N :=$ the normal subgroup of $G_1 * G_2$ generated by $\{\psi_1(k)\psi_2(k)^{-1} | k \in K\}$

Then $G_1 *_K G_2 := \frac{G_1 * G_2}{N}$ is the **Amalgated Free Products.**

Note: i_j indeces maps $\overline{i}_j : G_j \to G_1 *_K G_2$, so $\overline{i}_1 \circ \psi_1 = \overline{i_2} \circ \psi_2$.

Universal Property:

If $\varphi_1 : G_1 \to H$ and $\varphi_2 : G_2 \to H$ are any homomorphisms such that $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$ then there exists a unique homomorphism $\varphi : G_1 *_K G_2 \to H$ such that $\varphi \circ \overline{i_j} = \varphi_j$. (this follows from definitions.

$$\begin{tikzcd}
K \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & G_1 \arrow[d, "\overline{i}_1"'] \arrow[dr, dashed, "\varphi_1"] & \\
G_2 \arrow[r, "\overline{i}_2"'] \arrow[rr, dashed, bend right, "\varphi_1"'] & G_1 *_K G_2 \arrow[r, dashed, "\exists! \varphi"'] & H
\end{tikzcd}$$

In terms of generators and relations:

$$G_1 = \langle g_\alpha | r_\beta \rangle \qquad G_2 = \langle g'_{\alpha'} | r'_{\beta'} \rangle \qquad K = \langle g''_{\alpha''} | r''_{\beta''} \rangle$$

$$G_1 *_K G_2 = \langle g_\alpha, \ g'_{\alpha'} | r_\beta, r'_{\beta'}, \ \psi_1(g''_{\alpha''})\psi_2(g''_{\alpha''})^{-1} \rangle$$

Seifert-Van Kampen:

X is a space, $U, V \subseteq X$ open such that $U \cup V = X$, $x_0 \in U \cap V$. Then

$$\begin{tikzcd}
U \cap V \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook] \\
V \arrow[r, hook] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r, "\psi_1"] \arrow[d, "\psi_2"] & \pi_1(U, x_0) \arrow[d, "\varphi_1"] \\
\pi_1(V, x_0) \arrow[r, "\varphi_2"] & \pi_1(X, x_0)
\end{tikzcd}$$

Theorem. (SVK) If $U \cap V$ is path connected, then

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \frac{\pi_1(U) * \pi_1(V)}{\text{Normal subgroup generated by } \psi_1(k)\psi_2(k)^{-1} \text{ where } k \in \pi_1(U \cap V)}$$

Note, the theorem generalized to the case where $X = \bigcup_{\alpha \in \mathscr{I}} U_\alpha$ provided the U_α are open and contain an x_0 in the intersection.

Corollary. If $X = U \cup V$ where U, V are simply connected open subsets of X are such that $U \cap V$ is path connected, then $pi_1(X) = 0$.

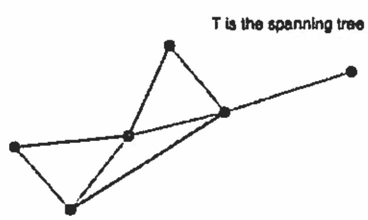
Proof. $0 * 0 = \pi_1(U) * \pi_1(V) = \pi_1(U \cap V) = 0$ because U, V are simply connected. □

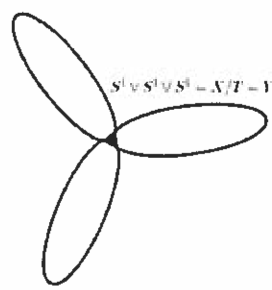
Example. Let $X = S^n$, $n > 1$. Let N, S be the north and south poles respectively. Let

$$U = S^n \backslash \{N\} \cong \text{int } D^n \qquad V = S^n \backslash \{S\} \cong \text{int } D^n \qquad U \cap V = S^n \backslash \{N, S\} \cong S^{n-1} \times (-1, 1) \cong S^{n-1}$$

Note S^{n-1} is path connected for $n > 1$. Thus, $\pi_1(S^n) = 0$ for all $n > 1$. However $\pi_1(S^1) = \mathbb{Z}$.

Example. Let $X =$ a finite connected graph (1-dimensional CW complex. Let T be the spanning tree of X. Let $n =$ the number of edges not in T. Then $Y = X/T = S^1 \vee S^1 \vee \dots S^1$ (repeated n times).

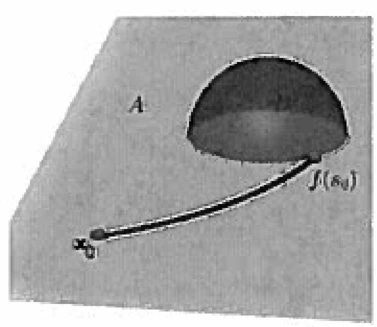




Let $U_i = Y \backslash \{x_1, \dots, \hat{x}_i, \dots, x_n\}$. That is, we cut open all of the loops excepts the ith one. Then $U_1 \approx S^1$. Thus $\pi_1(U_i) \cong \mathbb{Z}$.

Further, for $i \neq j$: $U_1 \cap U_j = Y \backslash \{x_1, \dots, x_n\} \approx$ one point. Thus, $\pi_1(U_i \cap U_j) = 0$. Note that $U_i \cap U_j$ is path connected and $U_i \cap U_j \cap U_k$ is path connected. Thus $\pi_1(X) \cong \pi_1(Y) = \frac{\pi_1(U_1) * \dots * \pi_1(U_k)}{0} = \pi_1(U_1) * \dots * \pi_1(U_k) \cong \mathbb{Z} * \dots \mathbb{Z} =$ a free group on n generators.

Example. (Attaching 2-cells) $X = A \cup_f D^2$, $f : \partial D^2 \to A$ is continuous.



Can regard f as a loop in A based at point $f(s_0)$. Let $N :=$ the normal subgroup of $\pi_1(A, x_0)$ generated by $[\gamma \cdot f \cdot \overline{\gamma}]$. Then $\pi_1(X, x_0) \cong \frac{\pi_1(A, x_0)}{N}$.

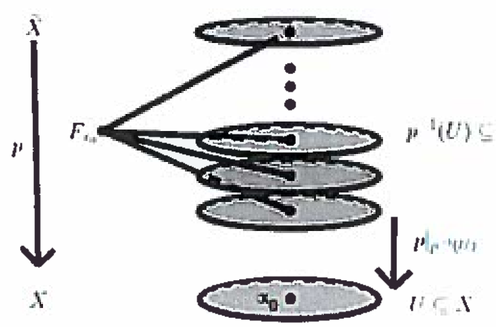
To see this, use SVK Theorem. Replace X by a homotopy equivalent space Z. Then let $U := Z\backslash\{p\} \simeq A$, $V := Z\backslash A \simeq \text{int}(D^2) \simeq \{p\}$. Then $U \cap V \simeq \text{int}(D^2)\backslash\{p\} \simeq S^1 \times (0, 1) \simeq S^1$. Thus $\pi_1(X) \cong \pi_1(Z) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \pi_1(A)/N$.

Remarks: Attaching n-cells for $n > 2$ doesn't change π_1 because $\partial D^n = S^{n-1}$ and $\pi_1(S^{n-1}) = 0$ for all $n > 0$.

If X is a CW complex then $\pi_1(X) = \pi_1(X^2)$

Covering Spaces

Definition. A **covering space** of X is a space $\tilde{X}$ together with a continuous map $p : \tilde{X} \to X$ satisfying that for all $a \in A$ there exists an open neighborhood $U \ni x$ such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically to h by p.



Note that $p^{-1}(U) = \coprod_j \tilde{U}_j$ is open in $\tilde{X}$ and $p|_{\tilde{U}_j} : \tilde{U}_j \to U$ is a homeomorphism.

In this situation:

- U is called an evenly covered neighborhood
- p is called a **covering map** or **covering projection**
- $F_x := p^{-1}(x)$ is called the fiber above x.

Covering Spaces

(continuing from previous page) $\tilde{U}_j \cap F_x$ contains one point. Since F_x is a discrete space, we see that 1-points in F_x are open.

Since $p^{-1}(U) \to U \times F_x$ is an isomorphism, the following diagram commutes (where $\{\tilde{x}_j\} = \tilde{U}_j \cap F_x$

$$\begin{tikzcd} \tilde{U}_j \arrow[rr, "p|_{U_j}"] \arrow[dr, "p|_{p^{-1}(U)}"'] & & U \times \{\tilde{x}_j\} \arrow[dl, "\text{projection onto first factor}"] \\ & U & \end{tikzcd}$$

Definition. For $x \in X$, $n(x) := |F_x|$. Note $n(x)$ could be infinite. Further, $n(x)$ is locally constant. Lastly, if X is connected then $n(x)$ is constant.

Thus, the **n-sheeted covering** is a covering $p : \tilde{X} \to X$ for which $|F_x| = n$ for all $x \in X$.

Remark: If $U \subseteq X$ is evenly covered open neighborhood, then every open $V \subseteq U$ is also evenly covered.

Reason: because $p^{-1}(V) = \coprod_j \tilde{V}_j$ where $\tilde{V}_j := \tilde{U}_j \cap p^{-1}(V)$.

Consequences. If $W \subseteq X$ is open and $x \in W$, then there exists an evenly covered neighborhood V of x such that $V \subseteq W$.

Reason: Take $V := W \cap U$ where U is an evenly covered neighborhood of X.

Corollary. Evenly covered neighborhoods form a basis for the topology of the base space X.

Example. Let X be any space, F a discrete space. $\tilde{X} := X \times F$, $p : X \times F \to X$ where $p(x, f) = x$. Then X (and hence every open subset of X) is evenly covered. Thus $\tilde{X}$ is a covering space of X.

Definition. A map $p : \tilde{X} \to X$ is called a **trivial** covering map if X is evenly covered.

Example. $p : \tilde{X} \to X$ is a 2-sheeted covering (so $|p^{-1}(x)| = 1$ for all $x \in X$ where P is a homeomorphism.

So 1-sheeted covering maps (the trivial covering map) are homeomorphisms.

Example. $X = S^1$, $\tilde{X} = \mathbb{R}$. Consider $p : \mathbb{R} \to S^1 \subset \mathbb{R}^2$ where $p(s) = (\cos(2\pi s), \sin(2\pi s))$

Let $U = S^1 \backslash \{(1, 0)\}$. Then $p^{-1}(U) = R \backslash \mathbb{Z} = \coprod_{n \in \mathbb{Z}} (n, n+1)$. Note, U is evenly covered.

Let $V = S^1 \backslash \{(-1, 0)\}$ Then $p^{-1}(V) = R \backslash \mathbb{Z} = \coprod_{n \in \mathbb{Z}} (n - \frac{1}{2}, n + \frac{1}{2})$. Note, V is evenly covered.

Consider $U \cup V = S^1$ every point of S^1 is contained in an evenly covered open set. Thus $p : \mathbb{R} \to S^1$ is a covering map.

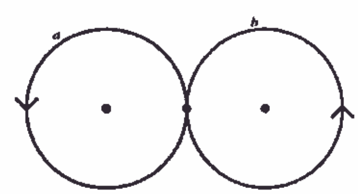
But $p^{-1}(S^1 = \mathbb{R} \cong \coprod S^1$ Thus $X = S^1$ is not an evenly covering, to $p : \mathbb{R} \to S^1$ is non-trivial.

Consider $i : \mathbb{R} \hookrightarrow \mathbb{R}^3$ via $i(s) = (\cos(2\pi s), \sin(2\pi s), s)$ $q : \mathbb{R}^3 \to \mathbb{R}^2$ via $q(x, y, z) = (x, y)$. Then $(q \circ i)(s) = (\cos(2\pi s), \sin(2\pi s))$, so $q \circ i = p$.

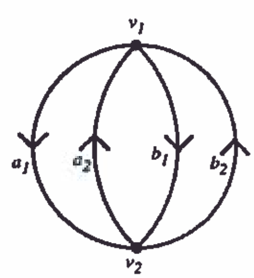
Example. Let $X = S^1 \subset \mathbb{C} \approx \mathbb{R}^2$, so $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. Then $\tilde{X} = S^1 \subset \mathbb{C}$. Consider $q : \mathbb{C} \to \mathbb{C}$ where $q(z) = z^n$, $n \in \mathbb{Z}$, $n \neq 0$.

Define $p := q|_{S^1} : S^1 \to S^1$ (wrap S^1 n-times around itself. Then $p : S^1 \to S^1$ is an n-sheeted covering map.

Example. $X = S^1 \vee S^1$



Consider $\tilde{X} =$

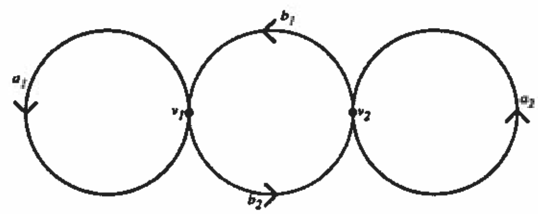


Define a continuous map $p : \tilde{X} \to X$ by $V_i \to V$, $a_i \to a$, $b_i \to b$, for $i = 1, 2$.

Then p is a covering map (near v_1 and v_2 $\tilde{X}$ looks that same as X.

Example. $X = S^1 \vee S^1$ (as above)

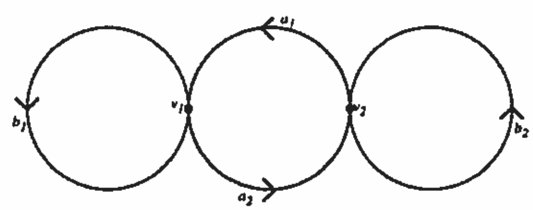
Consider $\tilde{X}_1 =$



Define a map p_1 as before. This is a 2-sheeted covering map. Note that $\tilde{X} \not\cong \tilde{X}_1$ since $\tilde{X}$ can be disconnected by removing a single point, but the same is not true for $\tilde{X}$.

Example. $X = S^1 \vee S^1$ (as above)

Consider $\tilde{X}_2 =$



Define a map p_1 as before. This is a 2-sheeted covering map. Note that $\tilde{X}_2 \cong \tilde{X}_1$ as spaces but there does not exists a homeomorphism from $\tilde{X}_2 \to \tilde{X}_2$ compatible with p.

Isomorphism of Covering Spaces

Let X be a space, $p_1 : \tilde{X}_1 \to X$, $p_2 : \tilde{X}_2 \to X$ is a homeomorphism.

Definition. An isomorphism between p_1 and p_2 is a homeomorphism $\varphi : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ \varphi = p_1$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Isomorphism of Covering Spaces

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$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} \quad \text{equivalently} \quad \begin{tikzcd} \tilde{X}_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] & \tilde{X}_2 \arrow[d, "p_2"] \\ X \arrow[r, equal, "1_X"'] & X \end{tikzcd}$$

Definition. An isomorphism from $p : \tilde{X} \to X$ to itself is called a **covering transformation** (or **deck transformation**) of $p : \tilde{X} \to X$.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\tau"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

This is the case if and only if τ is fiber preserving, i.e. $\tau(p^{-1}(x) = p^{-1}(x)$ for all $x \in X$.

Note: if τ_1 and τ_2 are both deck transformations of $p : \tilde{X} \to X$, then $\tau_1 \circ \tau_2$ and τ_i^{-1} are deck transformations of p. Thus deck transformations form a group.

Notation: $G(\tilde{X}) = G(\tilde{X}, p) = \{\text{deck transformations of } p : \tilde{X} \to X\}$.

Example. Let $p : \mathbb{R} \to S^1$, $p(s) = (\cos(2\pi s), \sin(2\pi s))$. Let $n \in \mathbb{Z}$. Then $\tau_n : \mathbb{R} \to \mathbb{R}$ via $\tau_n(s) := s + n$. Then $p(\tau_n(s)) = p(s+n) = p(s)$. Thus τ_n is a deck transformation of $p : \mathbb{R} \to S^1$.

Note that τ_n preserves the fibers set-wise. Further τ_n is a covering/ deck transformation of p.

Lifts of Continuous Maps

Let $p : \tilde{X} \to X$, $f : Y \to X$ be continuous maps.

Definition. A **lift** (or lifting) of f to $\tilde{X}$ is a continuous map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Example. Suppose $\varphi : \tilde{X}_1 \to \tilde{X}_2$ is an isomorphism between two covering spaces. Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$. Then

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Then φ can be viewed as a lift of p_1 to $\tilde{X}_2$ and φ^{-1} can be viewed as a lift of p_2 to $\tilde{X}_1$. Note $G(\tilde{X}_1) \to G(\tilde{X}_2)$ via $\tau \to \varphi \circ \tau \circ \varphi^{-1}$.

Assume:

- $f(Y) \subset U$, U is an evenly covered neighborhood, $p^{-1}(U) = \coprod \tilde{U}_j$
- $\tilde{f}(Y) \subseteq \tilde{U}_j$ for some j

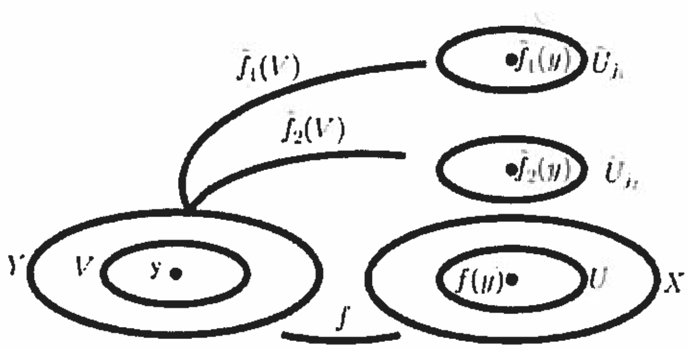
$$\begin{tikzcd} & \tilde{U}_j \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & U \end{tikzcd}$$

Then, since $p \circ \tilde{f} = f$, $(p|_{\tilde{U}_j}) \circ \tilde{f} = f$ implying $\tilde{f} = (p|_{\tilde{U}_j})^{-1} \circ f$

Consequently, suppose p, f are as about and $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ are lifts of f which map Y to the same $\tilde{U}_j \subset p^{-1}(U)$. Then $\tilde{f}_1 = (p|_{\tilde{U}_j})^{-1} \circ f = \tilde{f}_2$, so $\tilde{f}_1 = \tilde{f}_2$.

Proposition. (Unique Lifting Property) Given a covering space $p : \tilde{X} \to X$ and a continuous map $f : Y \to X$ with lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ that agree at one point of Y, then if Y is connected, these lifts must agree on all of Y.

Proof. Fix $y \in Y$ and let $U \subseteq X$ be an evenly covered neighborhood of $f(y)$, so $p^{-1}(U) = \coprod_j \tilde{U}_j$. Consider $\tilde{X}_2 =$



Then $\tilde{f}_1(y) \in U_{j_1}$, $\tilde{f}_2(y) \in U_{j_2}$, for some j_1, j_2. Now let $V := \tilde{f}_1^{-1}(\tilde{U}_{j_1}) \cap \tilde{f}_2^{-1}(\tilde{U}_{j_2})$. Then V is an open neighborhood of $y \in Y$ and $\tilde{f}_1(V) \subset \tilde{U}_{j_1}$, $\tilde{f}_2(V) \subset \tilde{U}_{j_2}$.

We have two cases:

Case 1: If $\tilde{f}_1(y) = \tilde{f}_2(y)$ then $\tilde{U}_{j_1} = \tilde{U}_{j_2}$, so $\tilde{f}_1(V), \tilde{f}_2(V) \subset \tilde{U}_{j_1} = \tilde{U}_{j_2}$, thus $\tilde{f}_1|_V = \tilde{f}_1|_V$ by the preceding discussion.

Case 2: If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then $\tilde{U}_{j_1} \cap \tilde{U}_{j_2} = \varnothing$. Thus $\tilde{f}_1|_V \neq \tilde{f}_2|_V$ for all $y \in V$.

This implies that $\{y \in Y | \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is both open and closed. Therefore, if Y is connected and $\{y \in Y | \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \varnothing$ it must be all of Y. $\square$

Lifts of Continuous Maps

Corollary If $\tilde{X}$ is a connected covering space of X and $\tau_1, \tau_2 \in G(\tilde{X})$ agree at one point of $\tilde{X}$, then $\tau_1 = \tau_2$.

This follows because τ_1, τ_2 can be viewed as lifts:

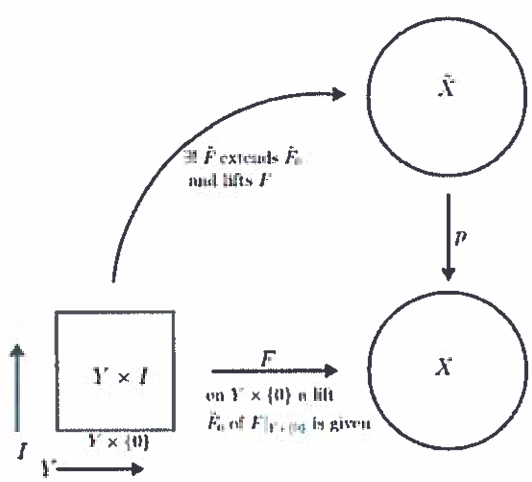
$$\begin{tikzcd} \tilde{X} \arrow[rr, shift left, "\tau_1"] \arrow[rr, shift right, "\tau_2"'] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

<u>Homotopy Lifting Property:</u>

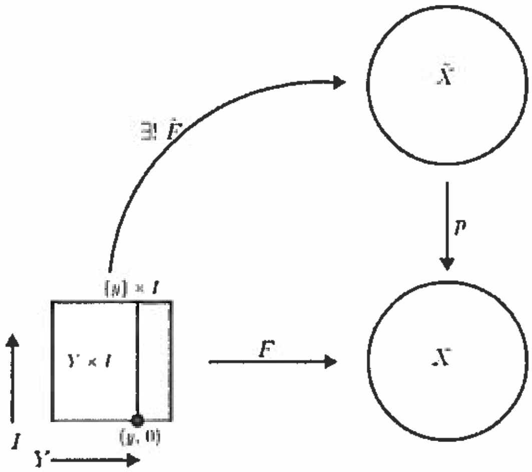
Let $p : \tilde{X} \to X$ be continuous and Y a space. (Y, p) has the **UHLP** if for all commutative diagram of continuous maps:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{F}_0"] \arrow[d, hook, "i"'] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

there exists a unique continuous map $\tilde{F} : Y \times I \to \tilde{X}$ such that the diagram with $\tilde{F}$ filled in still commutes. That is, both triangles commute. (i.e. $\tilde{F}$ extends $\tilde{F}_0$ and lifts F.)



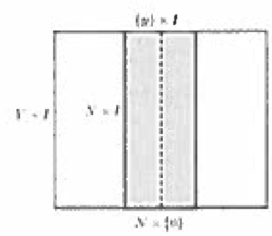
Proposition. If $p : \tilde{X} \to X$ is a covering space, then (Y, p) has the UHLP for all spaces Y.



Proof. Must show that there exists a unique lift $\tilde{F}$ of F extending $\tilde{F}_0$.

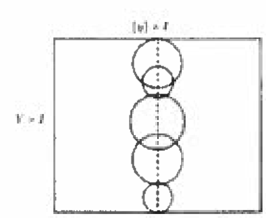
Uniqueness: Suppose $\tilde{F}_1$ and $\tilde{F}_2$ are lifts of F extending $\tilde{F}_0$. Then for each $y \in Y$, $\tilde{F}_1|_{\{y\}\times I}$ and $\tilde{F}_2|_{\{y\}\times I}$ are lifts of $F|_{\{y\}\times I}$ which agree at $(y, 0)$. But $\{y\} \times I$ is connected so $\tilde{F}_1|_{\{y\}\times I} = \tilde{F}_2|_{\{y\}\times I}$ for all $y \in Y$. Because Y is the union of y's, $Y \times I = \bigcup \{y\} \times I$. SO $\tilde{F}_1 = \tilde{F}_2$ on all of $Y \times I$.

Existence: For each $y \in Y$ we will construct an open neighborhood $N \subset Y$ of y and a lift $\tilde{F}_N$ of $F|_{N\times I}$ extending $F|_{N\times\{0\}}$.

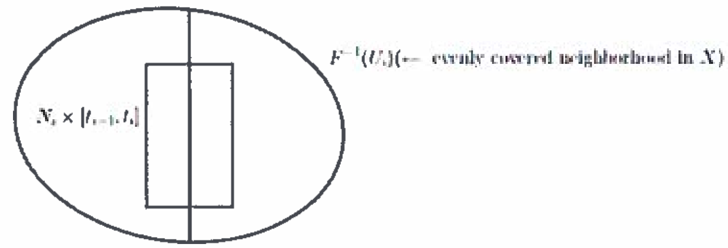


Then, since the lifts $\tilde{F}_N$ constructed on sets of the form $N \times I$ are unique when restricted to each $\{y\} \times I$, they must agree whenever the two such sets overlap. SO by the gluing lemma, we obtain a well-0defined continuous lift $\tilde{F}_1$ defined on all of $Y \times I$, and extending $\tilde{F}_0$.

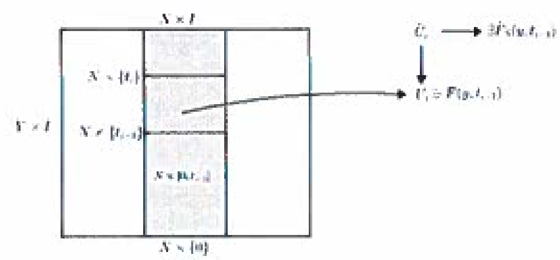
Construction of $\tilde{F}_N$: Fix $y \in Y$. Then the pre-images under F of all evenly covered neighborhoods in X form an open cover of $\{y\} \times I$:



Since $\{y\} \times I$ is a compact metric space, there exists a Lebesgue number $\delta > 0$ for this cover, so if $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ is a sufficiently fine subdivision of $I = [0, 1]$, then for each i, $\{y\} \times [t_{i-1}, t_i] \subset F^{-1}(U_i)$ for some evenly covered neighborhood $U_i \subseteq X$.



Moreover, by Homework 1 Problem 3, there exists an open neighborhood $N_i \subseteq U$ of Y such that $N_i \times [t_{i-1}, t_i] \subseteq F^{-1}(U)$. Now let $N := N_1 \cap \cdots \cap N_n \subseteq Y$. Then N is an open neighborhood of y and $F(N \times [t_{i-1}, t_i]) \subseteq F(N_i \times [t_{i-1}, t_i]) \subseteq U_i$.



Assume by induction that a life $\tilde{F}_N$ of F has been constructed on $N \times [0, t_{i-1}]$. Let $\tilde{U}_i \subseteq p^{-1}(U_i)$ be an open set which gets mapped homeomorphically to U_i, and such that $\tilde{F}_N(y, t_{i-1}) \subset \tilde{U}_i$. We can assume without loss of generality that $\tilde{F}_N(N \times \{t_{i-1}\}) \subseteq \tilde{U}_i$ (if not, replace N by $N \cap \left(\tilde{F}_N|_{N \times \{t_{i-1}\}}\right)^{-1}(\tilde{U}_i)$.)

Define $\tilde{F}_N|_{N \times [t_{i-1}, t_i]} := (p|_{\tilde{U}_i})^{-1} \circ F|_{N \times [t_{i-1}, t_i]}$. Then $\tilde{F}_N|_{N \times [t_{i-1}, t_i]}$ is a lift of $F|_{N \times [t_{i-1}, t_i]}$ and it agrees with $\tilde{F}_N|_{N \times [0, t_{i-1}]}$ on $N \times \{t_{i-1}\}$ because both lifts send $N \times \{t_{i-1}\}$ to the same sheet $\tilde{U}_i$. So by gluing lemma, for continuous functions, we get a lift $\tilde{F}_N$ of F on $N \times [0, t_i]$. ☐

Corollary (Unique Path Lifting Property) Suppose $p : \tilde{X} \to X$ is a covering space and $f : I \to X$ is a path starting at $x_0 \in X$. Then for every $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{f} : I \to \tilde{X}$ of f starting at $\tilde{x}_0$.

Proof. We can view $f : I \to X$ as a homotopy $f : Y \times I \to X$ where $Y = \{y\}$ is a 1-point space. So that UPLP follows from the UHLP. ☐

Warning: In general, if $f : I \to X$ i a loop the $\tilde{f} : I \to \tilde{X}$ will not be a loop.

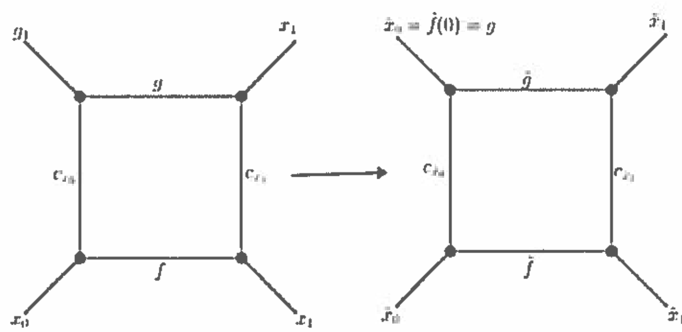
For example, consider $p : \mathbb{R} \to S^1$ via $p(s) = (\cos(2\pi s), \sin(2\pi s))$. The lift of the loop that is S^1 is no longer a loop.

Corollary. Suppose $p : \tilde{X} \to X$ is a covering space and suppose $f, g : I \to X$ are paths such that $f \simeq_p g$. If $\tilde{f}$, $\tilde{g}$ are lifts of f, g such that $\tilde{f}(0) = \tilde{g}(0)$ then $\tilde{f} \simeq_p \tilde{g}$. In particular $\tilde{f}(1) = \tilde{g}(1)$.

Proof. Let $F : I \times I \to X$ be a path homotopy from f to g. Then by UHLP, there exists a lift $\tilde{F} : I \times I \to \tilde{X}$ such that $\tilde{F}|_{I \times \{0\}} = \tilde{f}$.

We must show that $\tilde{F}$ is a path homotopy from $\tilde{f}$ to $\tilde{g}$:

$$x_0 := f(0) = g(0) \qquad x_1 := f(1) = g(1) \qquad \tilde{x}_0 := \tilde{f}(0) = \tilde{g}(0) \qquad \tilde{x}_1 := \tilde{f}(1) = \tilde{g}(1)$$



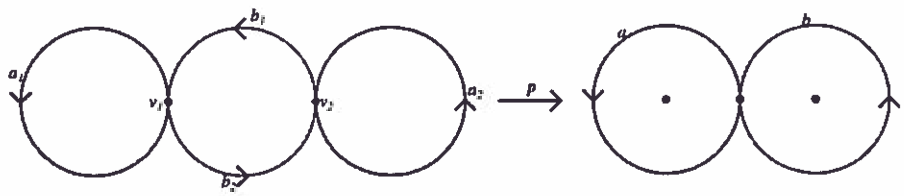
□

Lifts of Continuous Maps

Let $p : \tilde{X} \to X$ be a covering space, $x_0 \sin X$, $\tilde{x}_0 \in p^{-1}(x_0)$. We have the following facts:

1. If $f : I \to X$ is a path starting at x_0, then there exists a unique lift $\tilde{f} : I \to \tilde{X}$ of f starting at $\tilde{x}_0$.
2. Suppose $f, g : I \to X$ are paths such that $f \simeq g$, and suppose $\tilde{f}, \tilde{g} : I \to \tilde{X}$ are lifts of f, g such that $\tilde{f}(0) = \tilde{g}(0)$. Then $\tilde{f} \simeq_p \tilde{g}$. In particular $\tilde{f}(1) = \tilde{g}(1)$.

Example.



Example. Applications. Let $\omega_n : I \to S^1$ the path given by $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$. Let $x_0 : \omega_n(0) = (1, 0) \in S^1$.

Claim: The map $\Phi : \mathbb{Z} \to \pi_1(S^1, x_0)$ via $\Phi(n) = [\omega_n]$ is an isomorphisms.

Proof. Let $p : \mathbb{R} \to S^1$ the covering $p(s) := (\cos(2\pi s), \sin(2\pi s))$. Given a loop $f : I \to S^1$ based at $x_0 = (1, 0) \in S^1$, let $\tilde{f} : I \to \mathbb{R}$ denote the lift starting at $\tilde{x}_0 := 0 \in \mathbb{R}$. In particular $\tilde{\omega}_n(s) = ns$. Then $\tau_n : \mathbb{R} \to \mathbb{R}$ is the deck transformation from $s \to sn$, $n \in \mathbb{Z}$.

Φ is injective: Suppose $[\omega_n] = [\omega_m]$. Then $\omega_n \simeq_p \omega_m$. Then $n = \tilde{\omega}_n(1) = \tilde{\omega}_m(1) = m$. Thus $n = m$.

Φ is surjective: Let $[f] \in \pi_1(S^1, x_0)$. Then $p(\tilde{f}(1)) = f(1) = x_0 = (1, 0) \in S^1$. Then $\tilde{f}(1) z p^{-1}(1, 0) = \mathbb{Z}$. Thus $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. Thus $\tilde{f}$ has the endpoints as $\tilde{\omega}_n$ and so $\tilde{f} \simeq_p \tilde{\omega}$ so $p \circ \tilde{f} \simeq p \circ \tilde{\omega}_n$, so $f \simeq \omega_n$. Then $[f] = [\omega_n] = \Phi(n)$. SO $[f] \in \operatorname{im}(\Phi)$. Therefore Φ is surjective.

Φ is a group homomorphism: Let $n, m \in \mathbb{Z}$, $\tilde{\omega}_n \cdot (\tau \circ \tilde{\omega}_m)$ is a lift of $\omega_n \cdot \omega_m$ with endpoints 0 and $n + m$. Then $\tilde{\omega}_n \cdot (\tau_n \circ \omega_m) \simeq_p \tilde{\omega}_{n+m}$ because both paths have the same endpoints and $\mathbb{R}$ is simply connected. Thus $\omega_n \cdot \omega_m \simeq \omega_{n+m}$. So $[\omega_n][\omega_m] = [\omega_{n+m}]$. Therefore $\Phi(n)\Phi(m) = \Phi(n + m)$. □

Proposition. Let $p : \tilde{X} \to X$ be a covering space, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$. Then map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective with the image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ is equal to:

$$H := \{[f] \in \pi_1(X, x_0) | \text{ the lift of } f \text{ to } \tilde{X} \text{ starting at } \tilde{x}_0 \text{ is a loop}\}$$

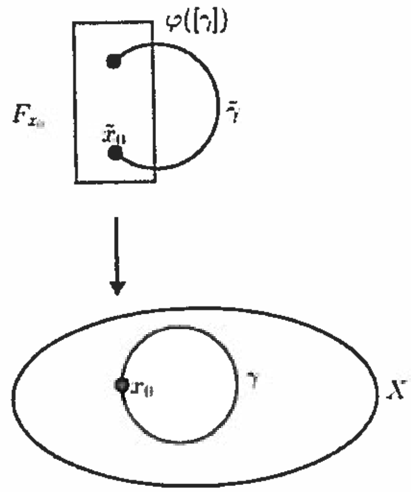
Proof. p_* is injective: Suppose $[\tilde{f}], [\tilde{g}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ satisfy $p_*[\tilde{f}] = p_*[\tilde{g}]$. Then $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$. So by fact 2 (above), $\tilde{f} \simeq_p \tilde{g}$ and hence $[\tilde{f}] = [\tilde{g}]$.

$p_*\pi_1(\tilde{X}, \tilde{x}_0) \subseteq H$: Let $[f] \in p_*\pi_1(\tilde{X}, \tilde{x}_0)$ and let $\tilde{f}$ be the lift of f starting at $\tilde{x}_0$. Then $[f] = p_*[\tilde{g}]$ for a $[\tilde{g}] \in \pi_1(\tilde{X}, \tilde{x}_0)$. So $f \simeq_p p \circ \tilde{g}$, and, by fact 2, $\tilde{f} \simeq_p \tilde{g}$. So $\tilde{f}$ is a loop because $\tilde{g}$ is a loop. Thus $[f] \in H$.

$H \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$: Let $[f] \in H$ and let $\tilde{f}$ be the lift of f starting at $\tilde{x}_0$. Then $\tilde{f}$ is a loop by definition of H_1 so $[\tilde{f}] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then $f = p \circ \tilde{f}$. Then $[f] = p_*[\tilde{f}]$. Thus $[f] \in p_*\pi_1(\tilde{X}, \tilde{x}_0)$. □

Remark: $p : \tilde{X} \to X$ covering space, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) = F_{x_0}$. $H := p_*\pi_1(\tilde{X}, x_0) < \pi_1(X, x_0)$. We can define a map $\varphi : \pi_1(X, x_0) \to F_{x_0}$ by $\varphi([\gamma]) := \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is a lift of γ starting at x_0

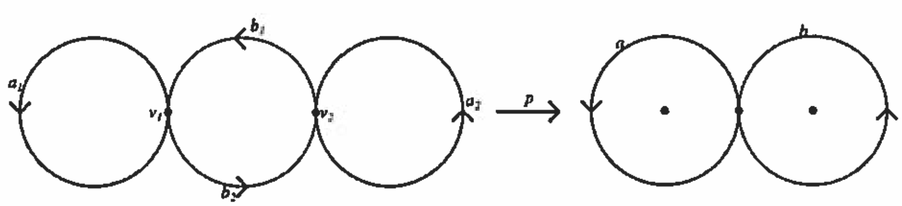
It is easy to see that φ is well defined and that If $\tilde{X}$ is path connected, φ descends to a bijection from $H\backslash\pi_1(X, x_0) \leftrightarrow F_{x_0}$.



Where $H\backslash\pi_1(X, x_0)$ is that right cosets of H. That is, $\pi_1(X, x_0)/\sim$ where $g \sim g'$ if and only if $g(g')^{-1} \in H = p_*\pi_1(\tilde{X}, \tilde{x}_0)$. So if $\tilde{X}$ is path connected, then:

$$|F_{x_0}| = |H\backslash\pi_1(X, x_0)| = \text{ the index of } H \text{ in } \pi_1(X, x_0).$$

Corollary. If $p : \tilde{X} \to X$ is a path connected n-sheeted covering, then the subgroup $H := p_*\pi_1(\tilde{X}, \tilde{x}_0)$ has index n in $\pi_1(X, x_0)$.



Example. Observe $\pi_1(X, x_0) = \langle a, b \rangle$ = the free group generated by a, b viewed as elements of $\pi_1(X, x_0)$. Then $p_*\pi_1(\tilde{X}, \tilde{x}_0) = \langle a, b^2, bab^{-1} \rangle < \langle a, b \rangle$.

Note, Since $\tilde{X}$ is a graph, $\pi_1(\tilde{X}, \tilde{x}_0)$ is free and hence so is H. (A similar argument shows that every subgroup of a free group is free.)

Further, $H\backslash\pi_1(X, x_0) = H\backslash\langle a, b\rangle = \langle H, Hb\rangle$. The fact that there are 2 right cosets of H corresponds to that fact that $\tilde{X}$ is a 2-sheeted covering space.

It turns out that $H = \{$ reduced words in $a^{\pm 1}$, $b^{\pm 1}$ for which the exponent sum of the b's is even $\}$. Further $Hb = \{$ reduced words in $a^{\pm 1}$, $b^{\pm 1}$ for which the exponent sum of the b's is odd$\}$.

Lifts of Continuous Maps

Proposition. (Lifting Criterion) Let $p : \tilde{X} \to X$ be a covering space where $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$. Let Y be any space, $y_0 \in Y$. Let Ybe locally path connected and path-connected and let $f : Y \to X$ be a continuous map such that $f(y_0) = x_0$. Then there exists a lift $\tilde{f} : Y \to X$ of f such that $\tilde{f}(y_0) = \tilde{x}_0$ if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

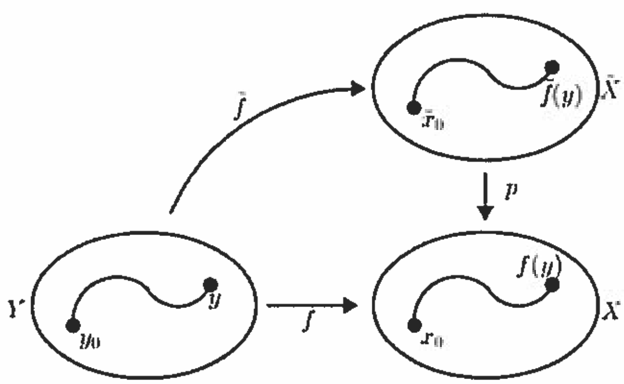
Proof. Suppose such a lift $\tilde{f}$ exists

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd} \implies \begin{tikzcd} & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

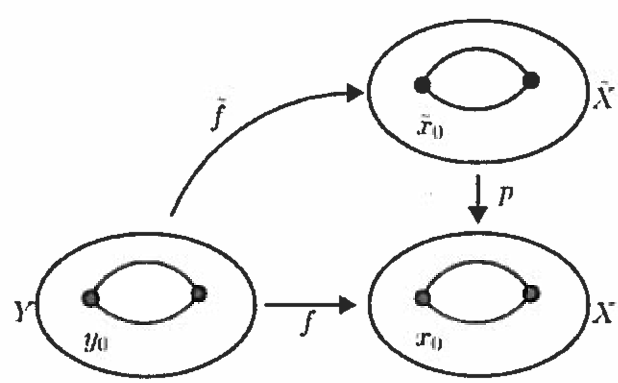
Thus $f_* = p_* \circ \tilde{f}_*$, so

$$f_*\pi_1(Y, y_0) = \operatorname{im}(f_*) = \operatorname{im}(p_* \circ \tilde{f}_*) \subseteq \operatorname{im}(p_*) = p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Suppose $f_*\pi_1(Y, y_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$. For $y \in Y$ choose a path $\gamma : I \to Y$ from y_0 to y. Let $\widetilde{f \circ \gamma} : I \to \tilde{X}$ be a lift of $f \circ \gamma$ starting at $\tilde{x}_0$. Define $\tilde{f}(y) = (\widetilde{f \circ \gamma})(1)$.



$\tilde{f}$ is well defined: Suppose γ' is another path from y_0 to y.



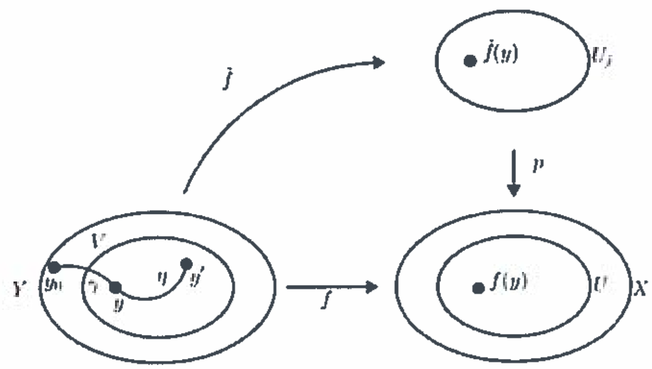
Then $\gamma' \cdot \overline{\gamma}$ is a loop in Y bases at y_0. So $f(\gamma' \cdot \overline{\gamma}) = (f \circ \gamma') \cdot (f \circ \overline{\gamma})$ is a loop in X based at x_0. Then $[f \circ (\gamma' \cdot \overline{\gamma}] \in f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0) = H$. The lift of $f \circ (\gamma' \cdot \overline{\gamma})$ starting at $\tilde{x}_0$ is a loop.

$$\tilde{h} := \widetilde{f \circ \gamma' \cdot \overline{\gamma}} = \widetilde{(f \circ \gamma') \cdot (f \circ \overline{\gamma})}$$

So $\widetilde{f \circ \gamma'} := \tilde{h}(\frac{s}{2})$, $s \in [0,1]$ is a lift of $f \circ \gamma'$ starting $\tilde{x}_0$. Likewise, $\widetilde{f \circ \gamma} := \tilde{h}(\frac{-s}{2})$, $s \in [0,1]$ is a lift of $f \circ \gamma$ starting $\tilde{x}_0$. Thus $\widetilde{f \circ \gamma'}(1) = \tilde{h}(\frac{1}{2} = \widetilde{f \circ \gamma}(1)$. Therefore $\tilde{f}(y)$ is well defined, so it is path independent.

$p \circ \tilde{f} = f$: Let $y \in Y$. Then $(p \circ \tilde{f})(y) = p(\tilde{f}(y)) = p(\widetilde{f \circ \gamma}(1) = (f \circ \gamma)(1) = f(\gamma(1)) = f(y)$. Therefore $p \circ \tilde{f} = f$.

$\tilde{f}$ is continuous: Assume y is path and locally path connected. Let $y \in Y$, γ a path in Y from y_0 to y. Let $U \subseteq X$ be an evenly covered neighborhood containing $f(y)$. Let $\tilde{U}_j \subset p^{-1}(U)$ be the sheet above U which contains $\tilde{f}(x)$. Let $V \subset f^{-1}(U)$ be a path connected open neighborhood of $y \in Y$.



Let $y' \in V$, and choose a path η in V from y to y'. Then $\gamma\eta$ is a path in Y from y_0 to y'. So

$$\tilde{f}(y') = \widetilde{f \circ (\gamma \circ \eta)}(1) = \widetilde{(f \circ \gamma)}\widetilde{(f \circ \eta)}(1) = \widetilde{f \circ \gamma} \cdot \widetilde{f \circ \eta}(1)$$

Note that $\widetilde{f \circ \gamma}$ starts at x_0, while $\widetilde{f \circ \eta}(1)$ starts $\tilde{f}(y)$. Thus $\tilde{f}(y') = \widetilde{f \circ \eta}(1)$. Further

$$\tilde{f}(y') = \widetilde{f \circ \eta}(1) = \left(p|_{\tilde{U}_j}\right)^{-1} \circ f \circ \eta)(1) = \left(\left(p|_{\tilde{U}_j}\right)^{-1} \circ f\right)(\eta(1)) = \left(\left(p|_{\tilde{U}_j}\right)^{-1} \circ f\right)(y')$$

So, $\tilde{f}|_v = ((p|_{\tilde{U}_j})^{-1} |_V \circ f|_V$, the right hand of which we know is continuous. Thus $\tilde{f}|_V$ is continuous. Thus $\tilde{f}$ is continuous on all of Y. $\square$

Lifts of Continuous Maps

There exists a lift $\tilde{f} : Y \to \tilde{X}$ of f such that $\tilde{f}(x_0) = \tilde{x}_0)$ if and only if $f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_0)$ (provided Y is path-connected and locally path-connected.)

Let X be a space, $p_i : \tilde{X}_i \to X$ be covering spaces, $i = 1, 2$. Let $H_i := (p_1)_* \pi_1(\tilde{X}_i, x_i) < \pi_1(X, x_0)$.

Proposition. If X is locally path connected and path connected, and if $\tilde{X}_1$ and $\tilde{X}_2$ are path connected. Then there exists an isomorphism of covering spaces $f : \tilde{X}_1 \to \tilde{X}_2$ such that $f(\tilde{X}_1) = \tilde{X}_2$ if and only if $H_1 = H_2$.

Proof. Suppose such an f exists. Then we have a commutative diagram:

By the lifting criterion $H_1 \subset H_2$, $H_2 \subset H_1$, thus $H_1 = H_2$.

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, shift left, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[ll, shift left, "f^{-1}"] \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

Note that f is a lift of p_1 and f^{-1} is a lift of p_2.

Conversely, suppose $H_1 = H_2$. Then by lifting criterion there exists lifts $f : \tilde{X}_1 \to \tilde{X}_2$ and $g : \tilde{X}_2 \to \tilde{X}_1$ of p_1 and p_2 respectively such that $f(\tilde{X}_1) = \tilde{X}_2$ and $g(\tilde{X}_2) = \tilde{X}_1$.

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl] \\
& (X, x_0) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\tilde{X}_2, \tilde{x}_2) \arrow[rr, "g"] \arrow[dr, "p_2"'] & & (\tilde{X}_1, \tilde{x}_1) \arrow[dl] \\
& (X, x_0) &
\end{tikzcd}$$

Consider $g \circ f$ and $f \circ g$ are lifts of p_1 and p_2 fixing $\tilde{x}_1$, $\tilde{x}_2$ respectively. Since $\tilde{X}_1$ and $\tilde{X}_2$ are connected, $g \circ f = 1_{\tilde{X}_1}$ and $f \circ g = 1_{\tilde{X}_2}$. Thus f and g are inverse isomorphisms. ☐

If X is path connected, locally path connected and $x_0 \in X$ then:

$$\{\text{based path connected covering spaces } p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)\} \hookrightarrow \{\text{ Subgroups of } \pi_1(X, x_0)\}$$
$$\text{based isomorphisms taking base point to base point} \to \{p_* \pi_1(\tilde{X}, \tilde{x}_0)\}$$

Can we get rid of the dependence on the base points?

Is there a condition on X under which the above inclusion is actually a bijection?

Dependence of $p_* \pi_1(\tilde{X}, \tilde{x}_0)$:

$p : \tilde{X} \to X$ be a covering space, $x_0 \in X$, $\tilde{x}_1, \tilde{x}_2 \in F_{x_0}$. Let $H_i := p_* \pi_1(\tilde{X}, \tilde{x}_i) < \pi_1(X, x_0)$ for $i = 1, 2$.

Claim: If $\tilde{X}$ is path-connected , then H_1 and H_2 are conjugate subgroups.

Proof. Let $\tilde{h}$ be a path in $\tilde{X}$ from $\tilde{x}_1$ to $\tilde{x}_2$, and let $h := p \circ \tilde{h}$. Then $[h] \in \pi_1(X < x_0)$ and we have a commutative diagram:

$$\begin{array}{ccc} \pi_1(\tilde{X}, \tilde{x}_2) & \xrightarrow{\beta_{\tilde{h}}} & \pi_1(\tilde{X}, \tilde{x}_1) \\ {\scriptstyle p_{*2}}\downarrow & & \downarrow{\scriptstyle p_{*1}} \\ \pi_1(X, x_0) & \xrightarrow[\beta_h]{} & \pi_1(X, x_0) \end{array}$$

Then:

$$\begin{aligned} & \beta_h \circ p_{*2} && = p_{*1} \circ \beta_{\tilde{h}} \\ \Longrightarrow \quad & \text{im}\,(\beta_h \circ p_{*2}) && = \text{im}(p_{*1} \circ \beta_{\tilde{h}}) \\ \Longrightarrow \quad & [h](\text{im}\,(p_{*2}))[\overline{h}] && = \text{im}\,(p_{*1}) \\ \Longrightarrow \quad & [h]H_2[\overline{h}] && = H_1 \end{aligned}$$

Therefore H_1 and H_2 are conjugate. ☐

Claim: For $p : \tilde{X} \to X$ covering space, $x_0 \in X$, $\tilde{x}_1 \in F_{x_0}$, if $H_1 = p_*\pi_1(\tilde{X}, \tilde{x}_1)$ and if $H_2 < \pi_1(X, x_0)$ is conjugate to H_1, then there exists $\tilde{x}_2 \in F_{x_0}$ such that $= p_*\pi_1(\tilde{X}, \tilde{x}_2) = H_2$

Proof. Follows essentially from proof of the previous claim. ☐

Corollary. (Easy Consequence) Let X be a space, $x_0 \in X$. If X is pat connected and locally path connected, then two path connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isometric if and only if $H_i := (p_i)_*\pi_1(\tilde{X}_i, \tilde{x}_i)$ are conjugate in $\pi_1(X, x_0)$ for any $\tilde{x}_i \in F_{x_0}$

So, if X is path connected and locally path connected, then

$$\frac{\{\text{path connected covering spaces} p : \tilde{X} \to X\}}{\text{isomorphism}} \hookrightarrow \{\text{ Conjugacy classes of subgroups of } \pi_1(X, x_0)\}$$
$$\tilde{X} \to \{p_*\pi_1(\tilde{X}, \tilde{x}_0)\}$$

Turns out this is a bijection if X is also semi-locally simply connected.

Universal Cover

Definition. X is **semi-locally simply connected** if for all $x \in X$ there exists $U \ni x$ such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion of $i : U \hookrightarrow X$, is trivial.

Proposition. If X is path connected, locally path connected, and semi-locally simply connected, then for all $H < \pi_1(X, x_0)$ there exists a path-connected covering space $p : \tilde{X}_H \to X$ and $\tilde{x}_0 \in F_{x_0}$ such that $p_*\pi_1(\tilde{X}_H, \tilde{x}_0) = H$.

Note: For given H, $\tilde{X}_H$ is unique up to isomorphism.

Definition. The covering space $\tilde{X}_H$ associated to $H = \{0\}$ is called the **universal cover** of X.

Note: $\pi_1(\tilde{X}_{\{0\}}) \cong p_*\pi_1(\tilde{X}_{\{0\}}) = \{0\}$

So the universal cover of X is the unique simply-connected nonempty covering space of X.

Example. $X = S^1$:

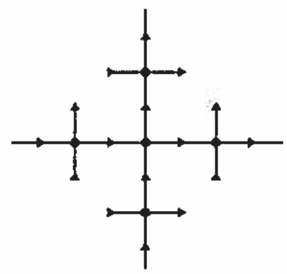
$p : \mathbb{R} \to S^1$ via $p(s) = (\cos(2\pi s), \sin(2\pi s))$. Recall that $\mathbb{R}$ is simply connected. Thus $\mathbb{R}$ is the universal cover of S^1.

Example. $X = S^1 \times I$.

$\tilde{X} = \mathbb{R} \times I$. The covering map is: $p : \mathbb{R} \times I \to S^1 \times$ via $p(s,t) = (\cos(2\pi s), \sin(2\pi s), t)$.

Thus $\mathbb{R} \times I$ is the universal cover of $S^1 \times I$.

Example. $X = S^1 \vee S^1$:



Let $\tilde{X}$ = the infinite graph. Consider $p : \tilde{X} \to X$ the map which sends each horizontal edge to a and vertical edge to b. One can check that p is a covering map and that $\tilde{X}$ s contractible. Thus $\tilde{X}$ is the universal cover of $S^1 \vee S^1$.

Note: In general $H \backslash \pi_1(X, x_0) \leftrightarrow F_{x_0}$ if $\tilde{X}$ is path connected.

For the universal cover: $\pi_1(X, x_0) \leftrightarrow F_{x_0}$

Note, we could define the "Cayley Graph" by $\tilde{X}$ = the 1-dimensional CW complex with $\tilde{X}_0 = \pi_1(X, x_0) = \langle a, b \rangle$. Then $\tilde{X}$ is obtained from $\tilde{X}_0$ by attaching an edge $g \to ga$ and and edge $g \to gb$ for all $g \in \pi_1(X, x_0) = \tilde{X}_0$.

Universal Cover

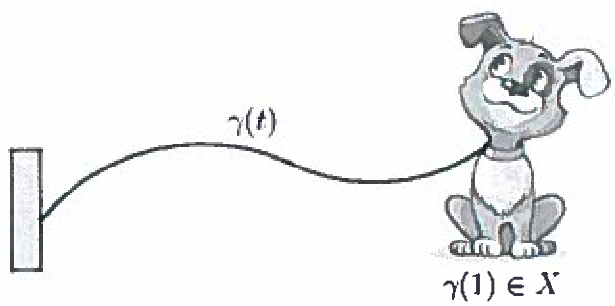
Summary:

- X is semi-locally simply connected if for all $x \in X$ there exists a neighborhood $U \ni x$ such that $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.
- If X is path connected, locally path connected, semi-locally simply connected then for all subgroups $H < \pi_1(X,x_0)$ there exists a unique path connected covering space $p : \tilde{X}_h \to X$ and an $\tilde{x}_0 \in F_{x_0}$ such that $p_*\pi_1(\tilde{X},\tilde{x}_0) = H$.
- The universal cover of X is the covering space associated to $H = \{0\}$. That is the unique simply connected covering space of X

Construction of the Universal Cover:

Let X be a path connected, locally path connected, semi-locally simply connected space such that $x_0 \in X$.

Idea: We want a point in $\tilde{X}$ records the position of the dog and the path homotopy class of the leash.



Goal: Construct the universal cover $\tilde{X}$ of X. As a set define:

$$\tilde{X} := \{[\gamma] | \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

Let $\tilde{x}_0 := [c_{x_0}]$ and define a map $p : \tilde{X} \to X$ by $p([\gamma]) := \gamma(1)$ (this will remember the position of the dog X and forget about the leash.)

Note: $p(\tilde{x}_0) = p([c_{x_0}]) = c_{x_0}(1) = x_0$.

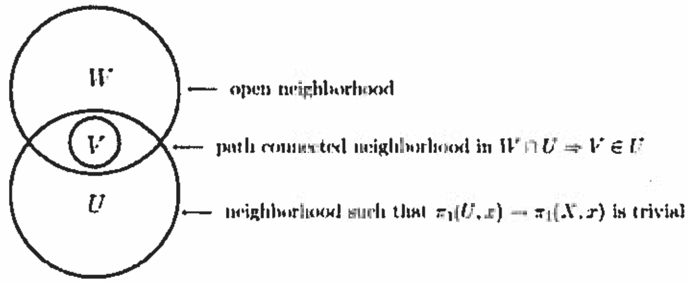
Have to define a topology on $\tilde{X}_0$. To do so let

$$U := \{ \text{ path connected open subsets } U \subseteq X \text{ such that } \pi_1(U) \to \pi_1(X) \text{ is trivial}\}$$

Observations:

1. If $\pi_1(U,x) \to \pi_1(X,x)$ is trivial and $V \subseteq U$ is a subset such that $x \in V$, then $\pi_1(V,x) \to \pi_1(X,x)$ is also trivial.
   (Reason: $\pi_1(V,x) \to \pi_1(X,x)$ factors through $\pi_1(U,x) \to \pi_1(X,x)$.)
2. If $U \in \mathscr{U}$ and if $V \subset U$ is open and path connected, then $V \in \mathscr{U}$.

3. $\mathscr{U}$ is a basis for the topology on X. (This follows from (1) because X s path connected and semi-locally simply connected.



Let $U \in U$ and let $[\gamma] \in \tilde{X}$ be such that $\gamma(1) \in U$. Define

$$U_{[\gamma]} := \{[\gamma \cdot \eta] | \eta \text{a path in } U \text{ starting at } \gamma(1)\} \subseteq \tilde{X}$$

$$\tilde{U} := \{U_{[\gamma]} | u \in U, \gamma \in \tilde{X}, \text{ such that } \gamma(1) \in U\}$$

Observations:

1. $[\gamma] \in U_{[\gamma]}$ because $[\gamma] = [\gamma \cdot \eta]$ for $\eta = c_{\gamma(1)}$
2. $p : \tilde{X} \to X$ via $p([\gamma]) = \gamma(1)$ implies that $p(U_{[\gamma]}) \subset U$ follows because $p([\gamma \cdot \eta]) = (\gamma \cdot \eta)(1) = \eta(1) \subset U$.
3. $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is surjective (because U is path connected.
4. $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is injective

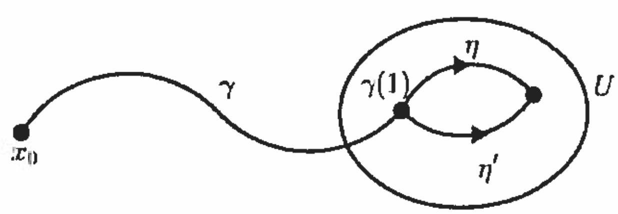
   Proof. Suppose $p([\gamma \cdot \eta]) = p([\gamma \cdot \eta'])$. Then:

   $$(\gamma \cdot \eta)(1) = (\gamma \cdot \eta')(1) \Rightarrow \eta(1) = \eta'(1) \Rightarrow \eta' \cdot \overline{\eta} \text{ is a loop in } U \text{ based at } \gamma(1)$$

   So $[\eta' \cdot \eta] \in \pi_1(X)$ is trivial (because $\pi_1(U) \to \pi_1(X)$ is trivial). Thus $[\gamma \cdot \eta] = [\gamma \cdot \eta' \cdot \overline{\eta} \cdot \eta] = [\gamma \cdot \eta']$. Therefore p is injective. □

5. $p^{-1}(U) = \bigcup_{\gamma(1) \in U} U[\gamma]$ (follows from 0 and 1)
6. If $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma]} = U_{[\gamma']}$
7. If $U_{[\gamma]} \cap U_{[\gamma']} \neq \varnothing$ then $U_{[\gamma]} = U_{[\gamma']}$
8. We can write $p^{-1}(U)$ as a disjoint union of sets of the form $U_{[\gamma]}$
9. $\tilde{\mathscr{U}}$ is a basis for a topology on $\tilde{X}$.

Equip X with the aforementioned topology.



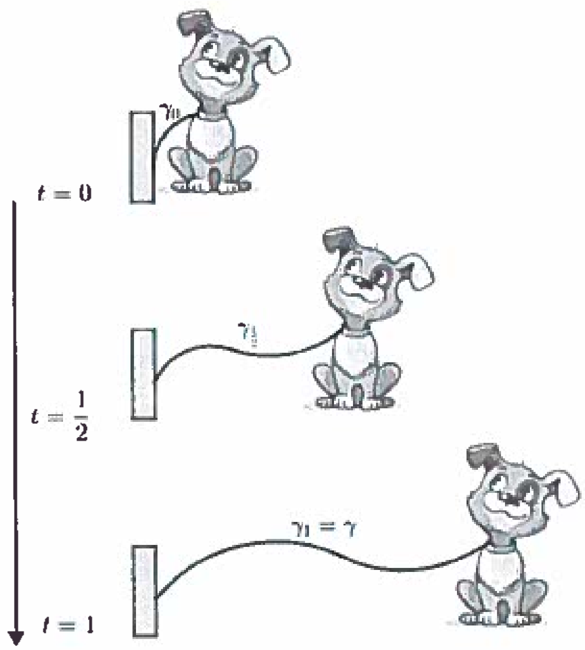
Claim: $p : \tilde{X} \to X$ is a covering map.

Proof. We already know that $p^{-1}(U) = \coprod($ sets of that form $U_{[\gamma]})$, which are open in $\tilde{X}$. Further $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is a bijection.

So P is continuous because $\mathscr{U}$ is a basis for X and $p^{-1}(U)$ is open for all $U \in \mathscr{U}$. Moreover, p is open because $\tilde{U}$ is a basis for $\tilde{X}$ and $p(U_{[\gamma]}) = U$ is open for all $U_{[\gamma]} \in \tilde{\mathscr{U}}$. So $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is a homeomorphism. Therefore p is a covering map. □

Claim: $\tilde{X}$ is path connected.

Proof. Let $[\gamma] \in \tilde{X}$. Then γ is a path in X starting at x_0. For $t \in$I , let $\gamma - t : I \to X$ be that path obtained by restricting γ to $[0, t]$ and identifying $[0, t]$ with I. More explicitly $\gamma_t(s) = \gamma(t, s)$, $s \in I$.



Define $\tilde{\gamma} : I \to \tilde{X}$ by $\tilde{\gamma}(t) := [\gamma_t] \in \tilde{X}$, $t \in I$. Can check that $\overline{\gamma}$ is continuous, so $\tilde{\gamma}$ is a path in $\tilde{X}$ from $\tilde{\gamma}(0) = [\gamma_0] = [c_{x_0}] = \tilde{x}_0$ to $\tilde{\gamma}(1) = [\gamma_1] = [\gamma]$.

So we've shown that for all $\gamma \in \tilde{X}$ there exists a path $\tilde{\gamma} \in \tilde{X}$ from $\tilde{x}_0$ to $[\gamma]$ so $\tilde{X}$ is path connected. □

Remark: For $\tilde{\gamma}$ as in proof, $p(\tilde{\gamma}(t)) = p([\gamma_t]) = \gamma_t(1) = \gamma(t \cdot 1) = \gamma(t)$. Thus $p \circ \tilde{\gamma} = \gamma$. Therefore $\tilde{\gamma}$ is a lift of γ starting at $\tilde{x}_0$.

Claim: $p_*\pi_1(\tilde{X}, \tilde{x}_0) = \{0\}$.

Proof. Let $[\gamma] \in p_*\pi_1(\tilde{X}, \tilde{x}_0)$. Then the lift of γ starting at $\tilde{x}_0$ is a loop. But by the remark, this lift is the path $\tilde{\gamma}$ given by $\tilde{\gamma}(t) = [\gamma_t]$. For this path to be a loop we must have $[\gamma] = [\gamma_1] = [\gamma_0] = [c_{x_0}] = 0 \in \pi_1(X, x_0)$. □

Universal Cover

Generalization:

Take X, x_0 as before. Suppose that $H < \pi_1(X, x_0)$ is any subgroup.

Goal: Construct a path connected covering space $\varphi : \tilde{X}_H \to X$ such that $p_*\pi_1(\tilde{X}_H, \tilde{x}_0) = H$ for a $\tilde{x}_0 \in p^{-1}(x_0)$.

For paths $\gamma, \gamma' \in X$ starting at x_0, define $\gamma \sim_H \gamma'$ if $\gamma(1) = \gamma'(1)$ and $[\gamma' \cdot \overline{\gamma}] \in H$.

Let $[\gamma]_H :=$ the H-equivalence class of γ.

Define $\tilde{X}_H := \{[\gamma]_H | \gamma \text{ is a path in } X \text{ starting at } x_0\}$.

Rest of the construction is the same s in the case of the universal cover.

In particular, $p([\gamma]_H) := \gamma(1)$.

Summary: If $X \neq \varnothing$, path connected, locally path connected, semi-locally simply connected, then

$$\frac{\{\text{nonempty path connected covering spaces of } X\}}{\text{isomorphism}} \leftrightarrow \{\text{conjugacy classes of subgroups of } \pi_1(X)\}$$

Deck Transformations and Group Actions

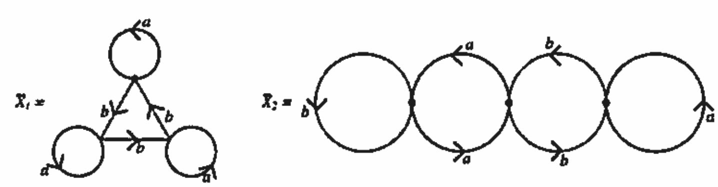
Let $G(\tilde{X})$ be the group of deck transformations or covering transformations, $p : \tilde{X} \to X$ be a covering space, $F_x = p^{-1}(x)$.

Definition. $\tilde{X}$ is **normal** if $G(\tilde{X})$ acts transitively on F_x for all $x \in X$.

Remark: If X is path connected and $G(\tilde{X})$ acts transitively on F_{x_0} for an $x_0 \in X$, then $G(\tilde{X})$ acts transitively on F_x for all $x \in X$.

Normal = $\tilde{X}$ has a "maximal subgroup"

Example. $X = S^1 \vee S^1$



$G(\tilde{X}_1) \cong \mathbb{Z}_3$ thus $\tilde{X}_1$ is normal.

$G(\tilde{X}_2) \cong \{0\}$ thus $\tilde{X}_2$ is not normal.

Definition. Let $p : \tilde{X} \to X$ be a covering space, $x_0 \in X$, $\tilde{x}_0 \in F_{x_0}$. Then the **normalizer** of H in fundamental group $\pi_1(X, x_0)$ is

$$N(H) := \{g \in \pi_1(X, x_0) | gHg^{-1} = H\}$$

Recall that $H \triangleleft N$, and $N(H)$ is the largest subgroup of $\pi_1(X, x_0)$ which contains H as a normal subgroup.

Proposition. If X is path connected and locally path connects and $\tilde{X}$ is path connected, then:

1. $\tilde{X}$ is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$, (equivalently, $N(H) = \pi_1(X, x_0)$)
2. $G(\tilde{X}) \cong N(H)/H$

In particular, if $\tilde{X}$ is normal, then $G(\tilde{X}) \cong \pi_1(X, x_0)/H$

Hence, for the universal cover $G(\tilde{X}) \cong \pi_1(X, x_0)$. (This can be used to show $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. Let $\tilde{x}_1 \in F_{x_0}$ and let $\tilde{\gamma}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Let $\gamma - p \circ \tilde{\gamma}$, so that $[\gamma] \in \pi_1(X, x_0)$. We've seen that there exists a deck transformation taking $(\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_1)$

$$\begin{aligned} &\iff p_*(\tilde{X}, \tilde{x}_0) = p_*(\tilde{X}, \tilde{x}_1) = [\overline{\gamma}] p_* \pi_1(\tilde{X}, \tilde{x}_0)[\gamma] \\ &\iff H = [\overline{\gamma}] H [\gamma] (\text{ since } H = p_* \pi_1(\tilde{X}, \tilde{x}_0)) \\ &\iff [\gamma] \in N(H) \end{aligned}$$

So we see that:

$$\begin{aligned} \tilde{X} \text{ is normal } &\iff G(\tilde{X}) \text{ acts transitively on } F_{x_0} \\ &\iff [\gamma] \in N(H)\ \forall\ [\gamma] \in \pi_1(X, x_0) \\ &\iff N(H) = \pi_1(X, x_0) \\ &\iff H \text{ is a normal subgroup of } \pi_1(X, x_0) \end{aligned}$$

To prove (b), define a map $\rho : N(H) \to G(\tilde{X})$. Let $[\gamma] \in N(H)$ and $\tilde{\gamma}$ be the lift of γ starting at $\tilde{x}_0$. Define $\tilde{x}_1 := \tilde{\gamma}(1)$. Since $[\gamma] \in N(H)$ there exists a deck transformation $\tau_\gamma : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_1)$. Moreover, τ_γ is unique because $\tilde{X}$ is path connected. Define $\rho([\gamma]) := \tau_\gamma$.

Claim: ρ is surjective, ρ is a group homomorphism, and $\ker \rho = h$ imply that $N(H)/H \cong N(H)/\ker \rho \cong G(\tilde{X})$.

We will only prove that ρ is a homomorphism. Let $[\gamma], [\eta] \in N(H)$ and $\tilde{\gamma}$, $e\tilde{t}a$ be the lifts of γ and η starting at $\tilde{x}_0$. Then

$$(\tau_\gamma \circ \tau_\eta)(\tilde{x}_0) = \tau_\gamma(\tau_\eta(\tilde{x}_0)) = \tau_\gamma(\tilde{\eta}(1))$$

On the other hand $\tilde{\gamma}(\tau_\gamma \circ \tilde{\eta})$ is a path in $\tilde{X}$ starting at $\tilde{x})$ and lifting $\gamma \cdot \eta$. So

$$\tau_{\gamma \cdot \eta}(\tilde{x}_0) = \tilde{\gamma}(\tau_\gamma \circ \tilde{\eta})(1) = (\tau_\gamma \circ \tilde{\eta})(1) = \tau_\gamma(\tilde{\eta}(1))$$

Thus $\tau_\gamma \circ \tau_\eta = \tau_{\gamma \cdot \eta}$, so $\rho([\gamma]) = \rho([\eta]) = \rho([\gamma][\eta])$. Therefore ρ is a homomorphism. ☐

Corollary. Let X be path connected, locally path connected, semi-locally simply connected, and let G be any group and $\rho : \pi_1(X, x_0) \to G$ be a surjective group homomorphism. Then the covering space $p : \tilde{X}_H \to X$ associated to $H = \ker(\rho)$ is normal and $G(\tilde{X}_H) \cong G$.

Proof. $\tilde{X}_H$ is normal because $H = \ker \rho$ is a normal subgroup of $\pi_1(X, x_0)$. Furthermore, since $\tilde{X}_H$ is normal $G(\tilde{X}_H) \cong \pi_1(X, x_0)/H \cong G$. ☐

Deck Transformations and Group Actions

Definition. Let Y be a space, $\text{Homeo}(Y) := \{\text{homeomorphisms } |Y \to Y\}$ is a group with respect to composition. Let G be any group. An **action** of G on Y by homeomorphisms is a group homomorphism $\rho : G \to \text{Homeo}(Y)$

Notation: For $g \in G$ we will write $\rho(g) : Y \to Y$ as $g : Y \to Y$.

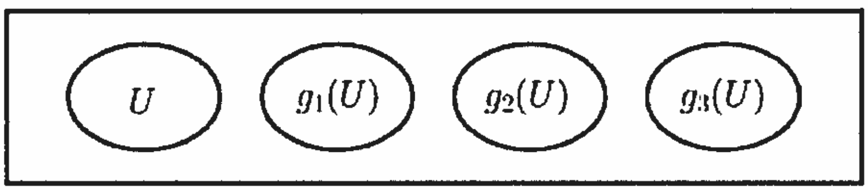
Definition. If $y \in Y$, then the **G-orbit** of Y is the set $G_y := \{g(y)|g \in G\}$.

Definition. The **orbit space** of ρ is the space:

$$Y/G := \frac{Y}{y \sim g(y)\,\forall y \in Y,\, g \in G} = \{G \text{ orbits in } Y\}$$

Question: When is the quotient map $p : Y \to Y/G$ such that $p(y) = G_y$ a covering map?

Definition. An action $\rho : G \to \text{Homeo}\,(Y)$ is called a **covering space action** if for all $y \in Y$ there exists an open neighborhood $U \ni x$ such that $g_1(U) \cap g_1(U) = \varnothing$ for all $g_1 \neq g_2$.



Note: If p is a covering space action, then $g_1(y) \neq g_2(y)$ for all $y \in Y$, for all $g_1 \neq g_2$

In particular $g(y) \neq y$ for all $y \in Y$, for all $g \neq e$. Then:

- ρ is free
- $\rho : G \to \text{Homeo}\,(Y)$ is injective
- $G \cong \rho(G) < \text{Homeo}(Y)$

Proposition. If $\rho : G \to \text{Homeo}(Y)$ is a covering space action, then:

1. The quotient map $p : Y \to Y/G$ (is a covering map) is a normal covering space (with fibers $p^{-1}(\rho(y)) = G_y$).
2. If Y is path connected, then $G \cong G(Y)$.
3. If Y is path connected and locally path connected, then

$$G \cong \frac{\pi_1(Y/G)}{p_*\pi_1(Y)}$$

In particular if Y is simply-connected and locally path connected then $G \cong \pi_1(Y/G)$

Proof. 1. If U is as in (*), then P identifies the disjoint copies $g(U)$ of U for $g \in G$ with each other. SO $g(U) \cong p(U)$ and $p^{-1}(\rho(U)) = \coprod_{g \in G} g(U)$. Then $\rho(U)$ is an evenly covered neighborhood of $\rho(Y)$. Thus p is a covering map.

Further, p is normal because $p^{-1}(\rho(y)) = G_y$ and G acts transitively on G_y

2. G acts by covering transformations. So $\rho(G) \subset G(Y)$. To prove the opposite inclusion, let $\tau \in G(Y)$ and let $y \in Y$. Then $\tau(Y) \in G(Y)$, and so $g \in G$ such that $\tau(Y) = \rho(g)(y)$. Since Y is path connected, this implies $\tau = \rho(y)$, so $\tau \in \rho(G)$, so $G(Y) \subseteq \rho(G)$. Thus $G(Y) = \rho(G) \cong G$.
3. $G \cong G(Y) \cong \frac{\pi_1(Y/G)}{p_*\pi_1(Y)}$

$\square$

Definition. If $\rho : G \to \text{Homeo}(Y)$ is a covering space action, then a fundamental domain for g is a subset $D \subset Y$ which contains exactly one point from each G-orbit in Y.

Remarks:

1. If D is a fundamental domain, then $p|_D : D \to Y/G$ is bijective (but in general not a homeomorphism.
2. Usually, one wants a fundamental domain to satisfy additional properties.

Example. $Y = \mathbb{R}$, $G = \mathbb{Z}$, action defined by translation. Then $\mathbb{R}/\mathbb{Z} \cong S^1$. So $\pi_1(S^1) \cong G \cong \mathbb{Z}$.

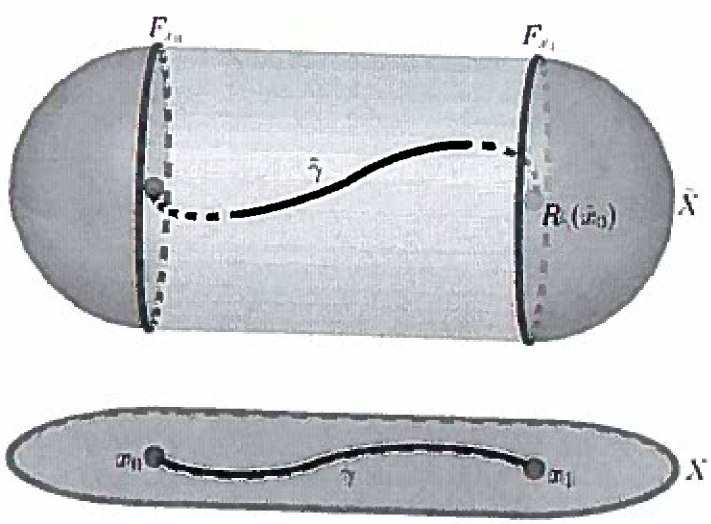
Fundamental Domain $D : [0, 1)$ or $D = [0, \frac{1}{2}] \cup [\frac{3}{2}, 2]$.

Example. $Y = \mathbb{R}^2$, $G = \mathbb{Z}^2$, action by translation. Then $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1 = T$. Then $\pi_1(T) \cong G \cong \mathbb{Z}^2$.

Fundamental Domain $D : [0, 1] \times [0, 1]$

Monodromy Action

Definition. $p : \tilde{X} \to X$ is a covering space, $x_0, x_1 \in X$. Consider $\gamma \in X$ a path from x_0 to x_1. Can now define a map $R_\gamma : F_{x_0} \to F_{x_1}$ by sending $\tilde{x} \in F_{x_0}$ to the endpoint of the unique lift $\tilde{\gamma}$ of γ starting at $\tilde{x}$.



Properties of R_γ:

1. R_γ only depends on the path homotopy class of γ.
2. $R_{c_{x_0}} = 1_{c_{x_0}}$
3. If $\gamma(1) = \eta(0)$, then $R_{\gamma\cdot\eta = R_\eta \circ R_\gamma}$
4. R_γ is a bijection with inverse $R_{\overline{\gamma}}$ (follows from 1,2,3)
5. If $x_0 = x_1$, then $[\gamma] \in \pi_1(X, x_0)$ and R_γ is a bijection $R_\gamma : F_{x_0} \to F_{x_0}$.
   Here the assignment $[\gamma] \to R_\gamma$ defines a right action.

Properties of this Action:

1. If $\tilde{x}_0 \in F_{x_0}$, then stabilizer, $\text{stab}(\tilde{x}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_0)$.
2. If $\tilde{X}$ is path connected, then the action of F_{x_0} is transitive (in this cases $F_{x_0} \leftrightarrow \text{stab}(\tilde{x}_0) \backslash \pi_1(X, x_1) = p_* \pi_1(\tilde{X}, \tilde{x}_0) \backslash \pi_1(X, x_0)$.
3. Could define left-action L_γ by $L_\gamma := R_\gamma^{-1} = R_{\overline{\gamma}}$

Proposition. (Connection with the action of $G(\tilde{X})$) Let $p : \tilde{X} \to X$ be a covering space, $x_0, x_1 \in X$, γ a path in X from x_0 to x_1, τ is a covering transformation, $\tau \in G(\tilde{X})$. Then $\tau|_{F_{x_1}} \circ R_\gamma = R_\gamma \circ \tau|_{F_{x_0}}$.

Note to Reader: There may be more notes on this including a proof of a proposition. Please refer to the text for such information.

Chapter 2: Homology

Motivation

Problems with $\pi_1(X)$:

- $\pi_1(X)$ depends on a base point,
- $\pi_1(X)$ doesn't detect higher dimensional structures (e.g. $\pi_1(S^n) = 0$ for all $n > 1$)
- If X is a CW complex then $\pi_1(X) = \pi_1(X^2)$.

Possible Solution: Introduce higher homotopy groups $\pi_n(X)$.

Let X be a topological space, $x_0 \in X$, $n \geqslant 1$. Then $I^n = I \times I \times \cdots \times I$

Definition. $\pi_n(X, x_0) = \{$ continuous maps $f : I^n \to X$ such that $f(\partial I^n) = \{x_0\}$, up to homotopy rel $\partial I^n\}$.

Note: such maps f descent to maps $\overline{f} : I^n/\partial I^n \to X$ (sub-note $I^n/\partial I^n \approx S^n$).

Turns out: $\pi_n(X, x_0)$ is a group for all $n \geqslant 1$, and Abelian for all $n \geqslant 2$.

Disadvantages of $\pi_n(X)$:

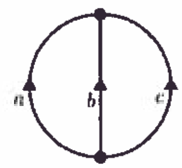
- $\pi_n(X)$ still depends on base point.
- $\pi_n(X)$ hard to compute (e.g. $\pi_3(S^2) \cong \mathbb{Z}$ so there exists a non-nullhomotopic map $S^3 \to S^2$)
- If p is a covering space ($p : \tilde{X} \to X$) then $\pi_n(\tilde{X}) \cong \pi_n(X)$ by lifting criterion.

Advantages of Singular Homology Groups $H_n(X)$:

- $H_n(X)$ is an Abelian Group
- $H_n(X)$ is defined for all spaces X, for all $n \geqslant 0$
- Does not require the choice of base point
- More computable than $|pi_n(X)$
- $H_n(S^n) \cong \mathbb{Z}$ for all $n > 0$
- If X is a CW complex then $H)n(X) = H_n(X^{n+1})$.

Idea of Homology:

Let X=



Then $\pi_1(X) = \langle ab^{-1}, bc^{-1}\rangle$ is a free group generated by the two loops, non-Abelian. (We could Abelianize by identifying $ab^{-1} \sim b^{-1}a$.) Then "based loops" become "unbased loops". Could further write composition additively (e.g. $ab^{-1} = a - b$). Thus, any loop in X can be written as $ka + fb + mc$ for $f, b, m \in \mathbb{Z}$.

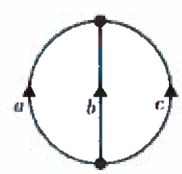
Definition. (Informal 1) An integer linear combination of edges of X is called a (cellular) **1-chain** in X.

Definition. (Informal 2) A 1-chain is called a 1-cycle if it can be "decomposed" into "loops" (e.g. $2a + 3b - 5c = 2(a - b) + 5(b - c)$, so it is a 1-cycle.)

Definition. (Informal) $H_1(X) = \{1 - cycles\} = \{m(a-b) + n(c-d) | m, n \in \mathbb{Z}\} \cong \mathbb{Z}^2 =$ Abelianization of $\langle ab^{-1}, bc^{-1}\rangle = \pi_1(X, x_0)$.

In general if $X \neq \varnothing$ is path connected, then $H_1(X) \cong$ abelianization of $\pi_1(X)$.

Example. Y=



As before

$$\{\text{(cellular) 1-cycles}\} = \{m(a-b) + n(b-c) | m, n \in \mathbb{Z}\},$$

but now $a - b$ bounds the 2-cell A.

Definition. (Informal) A multiple $m(a-b)$, $m \in \mathbb{Z}$ is called a **1-boundary** in Y .

Definition. (Informal)

$$H_1(Y) = \frac{\{\text{1-cycles}\}}{\{\text{1-boundaries}\}} = \frac{\{m(a-b) + n(c-d) | m, n \in \mathbb{Z}\}}{\{m(a-b) | m \in \mathbb{Z}\}} = \{n(c-d) | n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Definition. (Informal) In general X is a CW complex, a cellular n-change, or integer combination of n-dimensional cells in X then $H_n(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-boundaries}\}}$

For X an arbitrary space cellular n-chains imply singular n-chains.

Chain Complexes and Homology

Definition. A **chain complex** C is a sequence of Abelian group (or modules) C_n , $n \in \mathbb{Z}$, with homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial n = \partial_{n+1} = 0$

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \dots$$

where C_n = the n**th chain group**, ∂_n= **differential/ boundary map**, n =(homological) **degree**

Note: $\partial \circ \partial_{n+1} = 0$ if and only if $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$.

$$Z_n := \ker(\partial_n) = \{n\text{-cycles}\} \subseteq C_n \qquad B_n := \mathrm{im}(\partial_n) = \{n\text{-boundaries}\} \subseteq C_n$$

Definition. The n**th homology** of C is the quotient

$$H_n(C) := \frac{\ker(\partial_n)}{\mathrm{im}(\partial_n)} = \frac{Z_n}{B_n}$$

The elements of $H_n(C)$ are called the **homology classes.**

Notation: If $z \in Z_n$, $H_n(C) \ni [z]$:= the homotopy class represented by z.

Definition. $z, z' \in Z_n$ are called **homologous** if $[z] = [z']$ (if and only if $z - z' \in \partial_{n+1}c$ for all $c \in C_{n+1}$)

Definition. C is **acyclic** or **exact** if $H_n(C) = 0$ for all $n \in \mathbb{Z}$. (That is, $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$.)

Thus, homology measures the extent to which C fails to be exact.

Definition. C is **bounded** if $C_n = 0$ for all but finitely many $n \in \mathbb{Z}$

Definition. C is supported in non-negative degrees if $C_n = 0$ for all $n < 0$.

Chain Complexes and Homology

Definition. Let C, C' be chain complexes with differentials ∂_n and ∂'_n. A **chain map** $f : C \to C'$ is a family of homomorphisms $f_n : C_n \to C'_n$, $n \in \mathbb{Z}$ such that $f_{n-1} \circ \partial_n = \partial'_n \circ f_n$ for all $N \in Z$. f_n is called the **nth component** of f.

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Definition. A chain map $f : C \to C'$ is called a **chain isomorphism** if each $f_n : C_n \to C'_n$ is an isomorphism. In this case, there exists an **inverse chain map** $f^{-1} : C' \to C$ given by $(f^{-1})_n = (f_n)^{-1}$

Easy to see: If $f : C \to C'$ is a chain map then:

1. $f_n(Z_n) \subseteq Z'_n$
2. $f_n(B_n) \subseteq B'_n$

Thus f induces a map $f_* : H_n(C) \to H_n(C')$ via $f_*([z]) = [f_n(z)]$ for all $n \in Z$.

Properties 1. $(1_C)_* = 1_{H_n(C)}$

2. $(f \circ g)_* = f_* \circ g_*$

Thus, the nth homology is a functor from category of chain complexes and chain maps to the category of Abelian groups.

Definition. A chain map $f : C \to C'$ is called a **quasi-isomorphism** if the induced map $f_* : H_n(C) \to H_n(C')$ is an isomorphism for all $n \in \mathbb{Z}$.

Remark: Any sum of two chain maps is again a chain map, likewise for the composition of two chain maps.

$$(f+g)_n := f_n + g_n \qquad (cf)_n := cf_n \qquad (f \circ g)_n := f_n \circ g_n$$

Definition. Let $f, g : C \to C'$ be chain maps. A **chain homotopy** h between fundamental groups is a family of homomorphisms $h_n : C_n \to C_n$ such that

$$f_n - g_n = h_{n-1} \circ \partial_n + \partial'_{n+1} \circ h_n \quad \forall n \in Z$$

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, "f_n - g_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] & \cdots \\ \cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Note that this is not commutative. However: the following commutes:

$$\begin{tikzcd} C_n \arrow[d, "\begin{pmatrix} h_n \\ \partial_n \end{pmatrix}"'] \arrow[dr, "f_n - g_n"] & \\ C'_{n+1} \oplus C_{n-1} \arrow[r, "\begin{pmatrix} \partial'_{n+1} & h_{n-1} \end{pmatrix}"'] & C'_n \end{tikzcd}$$

Definition. If such an h exists, then f and g are called **homotopic**, $f \simeq g$.

Remarks.

1. homotopy is compatible with sums
2. $f, f', f'' : C \to C'$ are chain maps. h a homotopy $f \simeq f'$, h' a homotopy $f' \simeq f''$. Then:
$$\begin{aligned}(h + h')\partial + \partial'(h + h') &= h\partial + \partial' h + h'\partial + \partial' h' \\ &= f - f' + f' - f'' \\ &= f - f''\end{aligned}$$
3. Suppose $f - g = h\partial + \partial' h$ and let $z \in \mathbb{Z}_n = \ker(\partial_n)$. Recall that this means $h\partial_z = 0$ and that $\partial' h_z \in B'_h$. Thus,
$$f_*[z] - g_*[z] = [f(z)] - [g(z)] = [(f - g)(z)] = [h\partial_z + \partial' h_z] = [\partial' h_z] = 0.$$
So $f_* = g_*$. Thus, homotopic chain maps induce the same map in homology.

Definition. A chain map $f : C \to C'$ is called a **homology equivalence** if there exists a chain map $g : C' \to C$ such that $f \circ g \simeq 1$, $g \circ f \simeq 1_C$. (That is, g is a homotopy inverse of f.) In this case C, C' are called **homotopy equivalent** ($C \simeq C'$).

Remark.

1. f is homotopy equivalence implies that f is a quasi-isomorphism.
2. $C \simeq C'$ implies $H_n(C) \cong H_n(C')$ for all $n \in \mathbb{Z}$.
3. Any chain isomorphism $f : C \to C'$ is a homotopy equivalence with homotopy inverse given by f^{-1}.
So: {chain isomorphisms} $\subset$ {homotopy equivalence} $\subset$ {quasi-isomorphisms}

Definition. A chain map $f : C \to C'$ is called **nullhomotopic** if $f \simeq 0$.

Definition. C is **contractible** if $C \simeq 0$ (iff $1_C \simeq 0$)

Recall C is **acyclic** (or **exact**) if $H_n(C) = 0$ for all $n \in \mathbb{Z}$, By (2) contractible implies acyclic.

Further, if C is acyclic the $0 : 0 \to C$ is a quasi-isomorphism.

Fact: If $f : C \to C'$ is a chain map between bounded below chain complexes of free Abelian groups (or free modules), then f is a homotopy equivalence if and only if f is a quasi-isomorphism.

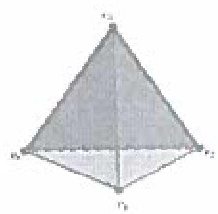
Simplices

Let $v_0, \ldots, v_n \in \mathbb{R}^m$ be $n+1$ points in $\mathbb{R}^m$ where $m \geqslant n$.

Assume $v_0, \ldots, v_n$ are in general position, i.e. not contained in any affine hyperplane of dim¡n, (iff pairwise linearly independent)

Definition. $[v_0, \ldots, v_n]$:= the convex hull of $\{v_1, \ldots, v_n\}$ = the smallest convex subset of $\mathbb{R}^m$ containing $v_0 \ldots v_n$ = the (affine) n-simplex spanned by $v_0, \ldots, v_n$

Example. Here $[v_0, v_1, v_2, v_3]$ create the 3-simplex spanned by $v_0, \ldots, v_3$



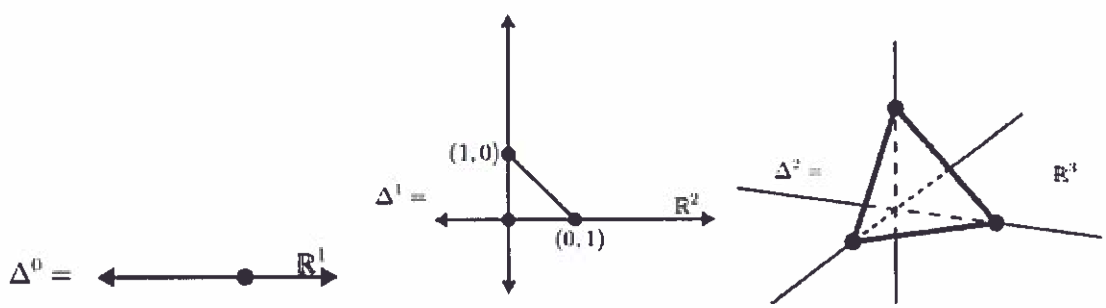
Convention: assume vertices $[v_0, \ldots, v_n]$ are equipped with the ordering $v_0 < v_1 < \cdots < v_n$. This ordering dictates the orientation of each edge $[v_i, v_j]$ (points towards the point with higher index). (See previous example)

Easy Fact: As a set

$$[v_0, \ldots, v_n] = \{\text{“convex combinations” of } v_0, \ldots, v_n\} = \{\sum_{i=0}^{n} t_i v_i | \sum_{i=1}^{n} t_i = 1, t_i \geqslant 0 \forall i\}$$

Definition. $t_0, \ldots, t_n$ are called the **barycentric coordinates** of the point $\sum_{i=0}^{n} t_i v_i \in [v_0, \ldots, v_n]$

Defintion $\Delta^n := [e_0, \ldots e_n] \subset \mathbb{R}^{n+1}$ form the **standard n-simplex.**



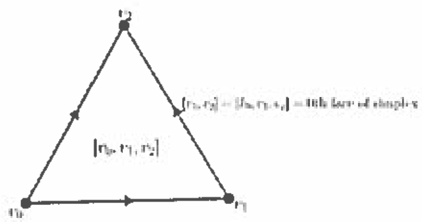
Simplices

For each affine n-simplex with vertices $[v_0, \dots, v_n]$, there exists a canonical affine homeomorphism $f_{[v_0,\dots,v_n]} : \Delta^n \to [v_0, \dots, v_n]$ via $f(t_0, \dots, t_n) = \sum_{i=0}^n t_i v_i$.

Definition. Let $[v_0, \dots, v_n]$ be an n-simplex. Then the **bith face** of $[v_0, \dots, v_n]$ is:

$$[v_0 \dots, \hat{v}_i, \dots, v_n] = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n] \subset [v_0, \dots, v_n]$$

Example. $n = 2$



Definition The **ith face map** $F_i^n : \Delta^{n-1} \to \Delta n$ is the composition: $\Delta^{n-1} \to [e_0, \dots, \hat{e}_i, \dots, e_n] \hookrightarrow \Delta^n$. That is:

$$F_i^n(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \in \Delta^n$$

Singular Simplices

Definition. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$.

Notation: $S_n(X) :=$ {singular n-simplex in X}

So: $S_0(X) = \{\sigma | \sigma : \Delta^0 \to X\} \leftrightarrow \{\text{points in } X\} = X$ $S_1(X) = \{\text{continuous maps that send } \Delta^1 \to X\}$

For $\sigma \in S_n(X)$, write $\sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} = \sigma \circ F_i^n \in S_{n-1}$

Singular Chain Complex

Suppose X is a topological space and $C(X)$ is its chain complex.

Definition (of $C(X)$) For $n < 0$ $C_n(X) := 0$

For $n > 0$:

$$\begin{aligned} C_n(X) :=& \text{ the free Abelian group generated by } S_n(X) \\ =& \text{ the free } \mathbb{Z}\text{-module spanned by } S_n(X) \\ =& \operatorname{span}_{\mathbb{Z}} S_n(X) \\ =& \bigoplus_{\sigma \in S_n(X)} \mathbb{Z}_\sigma \\ =& \{\text{formal finite sums } \textstyle\sum n_i \sigma_i | n_i \in \mathbb{Z}, \sigma_i \in S_n(X)\} \end{aligned}$$

Define the boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ as follows:

For $n \leqslant 0$, $\partial_n := 0$

For $n > 0$: ∂_n on $S_n(X)$ is:

$$\partial_n(\sigma) := \sum_{i=0}^{n}(-1)^i(\sigma \circ F^n) \in C_{n-1}(X)$$

Extend ∂_n to $C_n(X) = \mathrm{span}_{\mathbb{Z}} S_n(X)$ by linearity.

Claim: $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Proof. Let $n > 0$. Recall for $\sigma \in S_n(X)$, $\partial_n(\sigma) = \sum(-1)^i\sigma|_{[e_0,\dots,\hat{e_i},\dots,e_n]}$. Here i is the number of vertices to the left of the removed vertex and σ sums over all possible ways of removing a vertex.

Let $\sigma \in S_{n+1}(X)$, $(\partial_n \circ \partial_{n+1})$ = a sum over all possible ways of consecutively removing two vertices from Δ^{n+1}

Let e_i, e_j be the vertices of Δ^{n+1} that get removed first and second respectively.

If $j < i$: $(-1)^i(-1)^j\sigma|[e_0,\dots,\hat{e_j},\dots,\hat{e_i},\dots,e_{n+1}]$

If $i < j$: $(-1)^i(-1)^j\sigma|[e_0,\dots,\hat{e_i},\dots,\hat{e_j},\dots,e_{n+1}]$

$$\begin{aligned}\partial_n \circ \partial_{n+1}(\sigma) = &\sum_{0 \leqslant j < i \leqslant n+1} (-1)^i(-1)^j\sigma|[e_0,\dots,\hat{e_j},\dots,\hat{e_i},\dots,e_{n+1}] \\ &+ \sum_{0 \leqslant j < i \leqslant n+1} (-1)^i(-1)^j\sigma|[e_0,\dots,\hat{e_i},\dots,\hat{e_j},\dots,e_{n+1}]\end{aligned}$$

After renaming the indices $i \leftrightarrow j$ in the second sum, the two sums are the same up to sign. Thus they cancel. So $\partial_n \circ \partial_{n+1}(\sigma) = 0$. □

Definition. $\{C_n(X), \partial_n\}_{n\in\mathbb{Z}}$ is a chain complex called the **singular chain complex of** X.

Definition. The nth singular homology of X is the Abelian group $H_n(X) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}$

$Z_n(X) := \ker(\partial_n) = \{\text{singular } n\text{-cells in } X\}$

$B_n(X) := \mathrm{im}(\partial_{n+1}) = \{\text{singular } n\text{-boundaries in } X\}$

Note: for $n < 0$ $H_n(X) = 0$.

Example. $X = \varnothing$ Then there exists no maps $\sigma : \Delta^n \to X$, thus $S_n(X) = \varnothing$, $C_n(X) = 0$, and $C(X) = 0$. Therefore $H_n(\varnothing) = 0$

Example. $X = \{x_0\}$. Then there exists a unique map $\sigma_n : \Delta^n \to X$, namely the constant map at x_0. Then $C_n(X) = \mathbb{Z}_{\sigma(n)}$.

What is $\partial_n\sigma_n$? $\partial_n\sigma_n = \sum_{i=1}^{n}(-1)^i\sigma_n \circ F_i^n = \left(\sum_{i=1}^{n}(-1)^i\right)\sigma_{n-1}$. Thus

$$\partial_n\sigma_n = \begin{cases}\sigma & \text{if } n \text{ even and positive} \\ 0 & \text{other}\end{cases}$$

$C(\{x_0\})$: (Recal that ker(0) = everything

$$\cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} 0 \ldots$$

$$H_3 = 0 \qquad H_2 = 0 \qquad H_1 = 0 \qquad H_0 = \mathbb{Z} \qquad H_{-1} = 0$$

$$H_n(\{x_0\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Homology and Path Components

Proposition. If X is a space with path components X_α, then:

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha) \quad \forall n$$

Proof. $\sigma(\Delta^n)$ is path connected for all $\sigma \in S_n(X)$. It follows that:

$$\begin{aligned} &\Longrightarrow \sigma(\Delta^n) \text{ contained in one of the } X_\alpha \\ &\Longrightarrow S_n(X) = \coprod S_n(X_\alpha) \\ &\Longrightarrow C_n(X) = \bigoplus_\alpha C_n(X_\alpha) \end{aligned}$$

Moreover, $\operatorname{im}(\sigma \circ F_i^n) \subset \operatorname{im}(\sigma)$. Thus ∂_n maps $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$ to $C(X) = \bigoplus_\alpha C(X_\alpha)$ which implies the proposition. $\square$

Singular Homology

{singular n-simplices} = { continuous maps $\sigma : \Delta^n \to X$}

$C_n(X) = \text{span}_{\mathbb{Z}} S_n(X)$

For $r \in S_n(X)$, $\partial_n \sigma := \sim_{i=0}^{n} (-1)^i \sigma \circ F_i^n$

Def $H_n(X) = \frac{\ker(\partial_n)}{\text{im}(\partial_{n+1})}$.

Saw: If X has path components $X_2 \to H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$

Proposition. $H_0(X) \cong \text{span}_{\mathbb{Z}}\{$ path components of $X\} \cong \bigoplus_\alpha \mathbb{Z}$ (one copy of $\mathbb{Z}$ for each path component.

Proof. Recall $S_0(X) \leftrightarrow$ {points in X} and $S_1(X) \leftrightarrow$ {paths in X}. Note further, if $\sigma \in S_1(X)$ then

$$\partial_1 \sigma = \sigma|_{[\hat{e_0}, e_1]} - \sigma|_{[e_0, \hat{e_1}]} = \sigma|_{[e_1]} - \sigma|_{[e_0]} = \gamma(1) - \gamma(0).$$

$$\implies H_0(X) = \frac{\ker(\partial_0)}{\text{im}(\partial_1)} = \frac{c_0(X)}{\text{im}(\partial_1)} \cong \frac{\text{span}_{\mathbb{Z}}\{\text{points in } X\}}{\text{span}_{\mathbb{Z}}\{\gamma(1) - \gamma(0) | \gamma \text{ a path in } X\}}$$

$$\cong \text{span}_{\mathbb{Z}}\{\text{path components of } X\}$$

□

The proposition implies that if $X \neq \varnothing$, then rank $H_0(X) \geqslant 1$.

Defintion. If A is an Abelian group, then rank(A) := the cardinality of the maximal linearly independent subset.

Thus rank$(A) := \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$. Further, if A is a finitely generated Abelian group, $A \cong \mathbb{Z}^n \oplus$torsion, so rank$(A) = r$.

Reduced Singular Homology

Definition. Let X be a nonempty space. Define the **augmented map** to be the map $\varepsilon : C_0(X) \to \mathbb{Z}$ as follows:

$$\varepsilon\left(\sum n_i \sigma_i\right) := \sum n_i \quad (\sigma_i \in S_0(X))$$

Then ε is surjective is $X \neq \varnothing$.

If $\sigma \in S_1(X)$, then $\varepsilon(\partial, \sigma) = \varepsilon(\sigma|_{[e_1]} - \sigma|_{[e_0]}) = 1 + (-1) = 0$. Thus $\varepsilon \circ \partial_1 = 0$.

Definition. The **augmented singular chain complex** of X is the chain complex:

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

Note: ε is like ∂_0 of chain complex.

Definition. $\tilde{H}_n(X)$ = the nth homology of this chain complex, also called the **nth reduced singular homology.**

If $X \neq \varnothing$ then $\tilde{H}_n(X) = H_n(X)$ for all $n \neq 0$.

Remark: Can think of this extra term as being spanned by the unique map $\sigma : \varnothing \to X$ where $\varnothing$ behaves like a -1-dimensional simplex.

Recall $H_n(\{x_0\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$. Then $\tilde{H}_n(\{x_0\}) = 0$ for all n. Further the augmented complex, $\tilde{C}(\{x_0\})$, is contractible.

Consequence: there exists an exact sequence

$$
\begin{tikzcd}
 & \frac{\ker(i)}{\operatorname{im}\partial_1} \arrow[d, equal] & \frac{C_0(X)}{\operatorname{im}(\partial_1)} \arrow[d, equal] & & \\
0 \arrow[r] & \tilde{H}_0(X) \arrow[r, hook, "i"] & H_0(X) \arrow[r, "\varepsilon_*"] & \mathbb{Z} \arrow[r] \arrow[l, bend right, "j"'] & 0
\end{tikzcd}
$$

Define homomorphism j by sending 1 to any preimage of 1 under ε_*

Then $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$ and rank $\tilde{H}_0(X) =$ rank $H_0(X) - 1$.

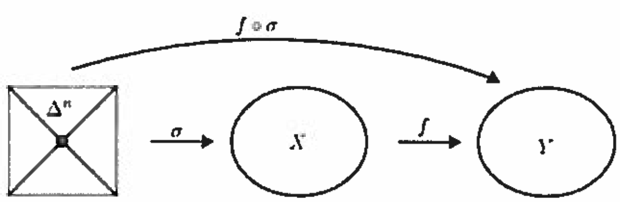
Induced Homomorphism and Homotopy Invariance

Definition. Let $f : X \to Y$ be continuous. Define the **induced homomorphism** $f_\# : C_n(X) \to C_n(Y)$ by $f_\#(\sigma) := f \cdot \sigma$.

Claim: $\partial \circ f_\# = f_\# \circ \partial$.

Proof. Let $\sigma \in S_n(X)$. Then

$$
\begin{aligned}
(\partial \circ f_\#)(\sigma) = \partial(f_\#(\sigma)) = \partial(f \circ \sigma) &= \sum_{i=0}^{n} (-1)^i (f \circ \sigma) \circ F_i^n \\
&= \sum_{i=0}^{n} (-1)^i f \circ (\sigma \circ F_i^n) = \sum_{i=0}^{n} (-1)^i f_\#(\sigma \circ F_i^n) = f_\# \left(\sum_{i=0}^{n} (-1)^i (\sigma \circ F_i^n) \right) \\
&= f_\#(\partial\sigma) = f_\# \circ \partial)(\sigma)
\end{aligned}
$$



□

This implies that $f_\#$ is a chain map $C(X) \to C(Y)$.

Further $f_\#$ induces maps $f_* : H_n(X) \to H_n(Y)$ via $f_*([z]) = [f_\#(z)]$.

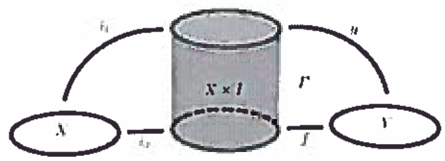
Properties of $f_\#$ and f_*:

1. $(1_X)_\# = 1_{C(X)}$; $(1_X)_* = 1_{H_n(X)}$
2. $(f \circ g)_\# = f_\# \circ g_\#$; $(f \circ g)_* = f_* \circ g_*$
3. Is f is a homomorphism, then $f_\#$ and f_* are isomorphisms.

Corollary. If $X \cong Y$, then $C(X) \cong C(Y)$ and $H_n(X) \cong H_n(Y)$ for all n.

Theorem. Let $f, g : X \to Y$ continuous. Then:

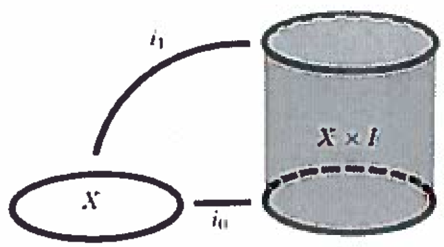
$$f \simeq g \Longrightarrow f_\# \simeq g_\# \Longrightarrow f_* \simeq g_*$$



Proof. Suppose $F : X \times I \to Y$ is a homotopy between two continuous maps f, g. Let $i_0, i_1 : X \to X \times I$ be the maps given by $i_0(x) := (x, 0)$, $i_1(x) := (x, 1)$. Then

$$F \circ i_0 = f \qquad F \circ i_1 = g \Longrightarrow \begin{cases} F_\# \circ (i_0)_\# = f_\# \\ F_\# \circ (i_1)_\# = g_\# \end{cases}$$

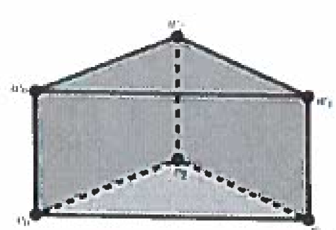
So it suffices to show that $(i_0)_\# \simeq (i_1)_\#$.



Let $v_0, \ldots, v_n$ be the vertices of $\Delta^n \times \{0\} \subset \Delta^n \times I$ and $w_0, \ldots, w_n$ be the vertices of $\Delta^n \times \{1\} \subset \Delta^n \times I$

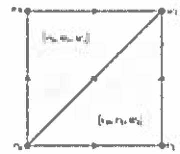
First a couple clarifying examples:

- $n = 2$



Note $\Delta^n \times I$ can be decomposed into $n + 1$ simplices $[v_0, \ldots, v_i, w_i, \ldots, w_n]$

- $n = 1$

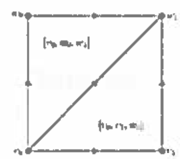


Let $f_{[v_0,\dots,v_i,w_i,\dots,w_n]} : \Delta^{n+1} \to [v_0, \dots, v_i, w_i, \dots, w_n] \subset \Delta^n \times I$ be the canonical homeomorphisms. For $\sigma \in S_n(X)$, let $\tilde{\sigma} : \Delta^n \times I \to X \times I$ be the map $\tilde{\sigma} : \sigma \times 1_I$. Define $P(\sigma) \in C_{n+1}(X \times I)$ as

$$P(\sigma) := \sum_{i=0}^{n} (-1)^i \tilde{\sigma} \circ f_{[v_0,\dots,v_i,w_i,\dots,w_n]}$$

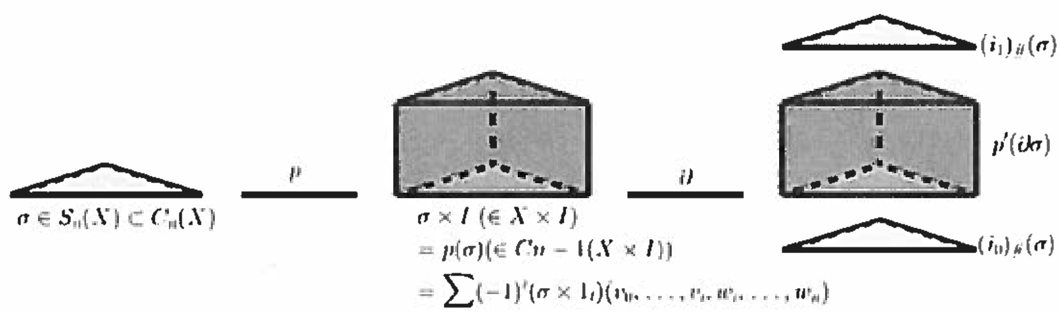
Clarifying this using our example above:

- $n = 1$



Then $P(\sigma) = \tilde{\sigma}|_{\text{upper } \Delta} - \tilde{\sigma}|_{\text{lower } \Delta}$ where $\tilde{\sigma} = \sigma \times I$.

Consider the Geometric interpretation of P:



Since $\partial_{n+1} P(\sigma) = (i_1)_\#(\sigma) - (i_0)_\#(\sigma) - P(\partial_n \sigma)$, $\partial P + P\partial = (i_1)_\# - (i_0)_\#$. (note, a brief justification was given using simplices). Thus P is a homotopy between $(i_0)_\#$ and $(i_1)_\#$. $\square$

Corollary. $X \simeq Y \implies C(X) \simeq C(Y) \implies H_n(X) \simeq H_n(Y)$

Exact Sequences

Definition. A sequence of Abelian groups or modules where f_n are homomorphisms

$$\dots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \dots$$

is called **exact at** A_n if $\operatorname{im}(f_{n+1}) = \ker(f_n)$.

It is called **exact** if it is exact at every A_n which is not an endpoint of the sequence.

Examples 1. $0 \xrightarrow{0} A \xrightarrow{f} B$ exact $\iff$ $\operatorname{im}(0) = \ker(f) = 0 \iff f$ is injective

2. $B \xrightarrow{g} C \xrightarrow{0} 0$ exact $\iff$ $\operatorname{im}(g) = \ker(0) = c \iff g$ is surjective

3. $0 \xrightarrow{0} B \xrightarrow{g} C \xrightarrow{0} 0$ exact $\iff$ g is injective and surjective $\iff$ g is isomorphism

4. $0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$ exact $\iff$ f is injective and g is surjective $\iff$ $\operatorname{im} f = \ker g$.

Definition. An exact sequence as in (4) is called a **short exact sequence**.

Definition. A **short exact sequence of chain complexes** is a sequence of chain complexes and chain maps:

$$0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$$

such that for each n

$$0 \xrightarrow{0} A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \xrightarrow{0} 0$$

.

More explicitly:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r, "0"] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial"'] & C_{n+1} \arrow[r, "0"] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r, "0"] & A_n \arrow[r, "f_n"] \arrow[d, "\partial"'] & B_n \arrow[r, "g_n"] \arrow[d, "\partial"'] & C_n \arrow[r, "0"] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r, "0"] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "0"] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Saying that this is a short exact sequence of chain complexes means that each row is exact.

Question: Suppose $A \longrightarrow B \longrightarrow C$ is an exact (at B_n for all n) sequence of chain complexes.

Does it follow that $H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C)$ is also exact (at $H_n(B)$)?

Answer: No. Consider the examples:

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & \\ A & & B & & C \end{array}$$

$$H_n = 0 \qquad H_n = \mathbb{Z} \qquad H_n = 0$$

All squares commute and each row is exact because $A \longrightarrow B \longrightarrow C$ is an exact sequence of chain complexes. However, if $A \longrightarrow B \longrightarrow C$ is exact, then $g \circ f = 0$ and hence $g_* \circ f_* = 0$ so $\operatorname{im} f_* \subset \ker g_*/$

Lemma. If $A \longrightarrow B \longrightarrow C$ is an exact sequence of chain complexes, and if for an $n \in \mathbb{Z}$, f_{n-1} is injective and g_{n+1} is surjective, then $H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C)$ is exact (at $H_n(B)$).

Proof.

$$\begin{tikzcd} & & B_{n+1}(\ni b') \arrow[r, two heads, "g_{n+1}"] \arrow[d, "\partial"] & C_{n+1}(\ni c) \arrow[r] \arrow[d, "\partial"] & 0 \\ & A_n \arrow[r, "f_n"] \arrow[d, "\partial"] & B_n(\ni b) \arrow[r, "g_n"] \arrow[d, "\partial"] & C_n(\ni g_n(b)) & \\ 0 \arrow[r] & A_{n-1} \arrow[r, hook, "f_{n-1}"] & B_{n-1} & & \end{tikzcd}$$

:

Need to show: $\ker(g_*) \subset \operatorname{im}(f_*)$

Let $b \in B_n$ be a cycle such that $[b] \in \ker(g_*)$. Then $g_*[b] = 0$, so $g_n(b) = \partial_c$ for a $c \in C_{n+1}$. Since g_{n+1} is surjective there exists $b' \in B_{n+1}$ such that $g_{n+1}(b') = c$. Note that:

$$g_n(\partial b') = \partial(g_{n+1}(b')) = \partial_c = g_n(b).$$

Si $g)n(b - \partial b') = g_n(b) - g_n(\partial b') = 0$. Hence by exactness at B_n, there exists $a \in A_n$ such that $f_n(a) = b - \partial b'$. Moreover, $f_{n-1}(\partial_n) = \partial(f_n(a)) = \partial(b - \partial b') = 0$. SO since f_{n-1} is injective, $\partial_n = 0$, and hence a is a cycle. Therefore

$$f_*[a] = [f_n(a)] = [b - \partial b'] = [b].$$

So $[b] \in \operatorname{im}(f_*)$, so $\ker(g_*) \subseteq \operatorname{im}(f_*)$. Thus $\ker(g_*) = \operatorname{im}(f_*)$. Therefore

$$H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C)$$

is exact. □

Theorem. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of chain complexes, then there exists natural homomorphisms $\delta_n : H_n(C) \to H_{n-1}(A$ such that the following sequence is exact:

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta_{n+1}"] \\
H_n(A) \arrow[r, "f_*"] & H_n(B) \arrow[r, "g_*"] & H_n(C) \arrow[dll, "\delta_n"] \\
H_{n-1}(A) \arrow[r, "f_*"] & H_{n-1}(B) \arrow[r, "g_*"] & H_{n-1}(C) \arrow[dll, "\delta_{n-1}"] \\
H_{n-2}(A) \arrow[r] & \dots &
\end{tikzcd}$$

This is called a **Long Exact Sequence** in homology.

Remarks:

1. Exactness at $H_n(B)$ follows from lemma for all n.
2. ∂_n is called the **connecting homomorphism** and is sometimes denoted ∂
3. "Natural" means: given a commutative diagram of chain complexes and chain maps:

$$\begin{tikzcd}
0 \arrow[r, "0"] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "0"] \arrow[d, "c"] & 0 \\
0 \arrow[r, "0"] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "0"] & 0
\end{tikzcd}$$

Then there exists an induced commutative diagram:

$$\begin{tikzcd}
H_n(C) \arrow[r, "\delta_n"] \arrow[d, "c_*"] & H_{n-1}(A) \arrow[d, "a_*"] \\
H_n(C') \arrow[r, "\delta_n'"] & H_{n-1}(A')
\end{tikzcd}$$

Proof. (Proof of Theorem) Definition: of $\delta_n : H_n(C) \to H_{n-1}(A)$

$$\begin{tikzcd}
 & & B_{n+1}(\ni b') \arrow[r, two heads, "g_{n+1}"] \arrow[d, "\partial"] & C_{n+1}(\ni c) \arrow[r] \arrow[d, "\partial"] \arrow[dll, bend right=40, "\delta_{n+1}"'] & 0 \\
 & A_n \arrow[r, "f_n"] \arrow[d, "\partial"] & B_n(\ni b) \arrow[r, "g_n"] \arrow[d, "\partial"] & C_n(\ni g_n(b)) & \\
0 \arrow[r] & A_{n-1} \arrow[r, hook, "f_{n-1}"] & B_{n-1} & & :
\end{tikzcd}$$

Let $c \in C_n$ be a cycle. Since g_n is surjective, there exists $b \in B_n$ such that $g_n(b) = c$. Let $b' := \partial$. Then:

$$g_{n-1}(b') = g_{n-1}(\partial b) = \partial(g_n(b)) = \partial(c) = 0.$$

So $b' \in \ker g_{n-1}$. By exactness at B_{n-1}, there exists $a \in A_{n-1}$ such that $f_{n-1}(a) = b'$:

$$f_{n-2}(\partial a) = \partial f_{n-1}(a) = \partial b' = \partial^2 b = 0$$

Since f_{n-2} injective, $\partial a = 0$, whence a is a cycle. Define $\delta_n([c]) = [a]$.

Remains to show:

- $\delta_n([c])$ is well defined (independent of choice of c, b)
- δ_n is a homomorphism
- Long exact sequence in homology is exact at $H_n(A), H_n(C)$ for all n.
- δ_n is natural

Proofs in Hatcher 116- 117. □

Exact Sequences

Short exact sequence of chain complexes $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ create long exact sequences in homology.

Corollary. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of chain complexes, then:

1. If A acyclic, then $H_n(B) \cong H_n(C)$ for all n.
2. If B acyclic, then $H_n(C) \cong H_{n-1}(A)$ for all n.
3. If C acyclic, then $H_n(A) \cong H_n(B)$ for all n.

Proof. Have exact sequence $H_n(A) \longrightarrow H_n(B) \hookrightarrow\!\!\!\!\!\rightarrow H_n(C) \longrightarrow H_{n-1}(A)$ because $H_n(A) = H_{n-1}(A) = 0$. (Note, $\hookrightarrow\!\!\!\!\!\rightarrow$ is an isomorphism.)

Proof of 2 and 3 are analogous. □

Definition. A **subcomplex** of (B, ∂) is a chain complex (A, ∂') such that

- $A_n \subseteq B_n$ (subgroup/ submodule) for all n
- $\partial A_n \subseteq A_{n-1}$
- $\partial'_n := \partial_n|_{A_n}$

Definition. If (A, ∂') is a subcomplex of (B, ∂) then the **quotient complex** $(B/A, \overline{\partial})$ is defined as follows:

$$(B/A)_n := B_n/A_n \qquad \overline{\partial} \text{ induced by } \partial$$

$(B/A, \overline{\partial})$ fits into short exact sequence:

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

Corollary. Let A be a subcomplex of B.

1. If A acyclic, then $H_n(B) \cong H_n(B/A)$ for all n.
2. If B acyclic, then $H_n(B/A) \cong H_{n-1}(A)$ for all n.
3. If B/A acyclic, then $H_n(A) \cong H_n(B)$ for all n.

Relative Singular Homology

Let (X, A) be a pair of spaces such that $A \subset X$ with subspace topology.

We can view every singular simplex in A as a singular simplex in X

Can view $C(A)$ as a subspace of $C(X)$.

Definition. The relative singular chain complex of (X, A) is the quotient complex $C(X, A) := C(X)/C(A)$.

So $C_n(X, A) := C_n(X)/C_n(A)$ and $\partial_n^{(X,A)}$ is induced by ∂_n^X.

Definition. The nth **relative singular homology** is

$$H_n(X, A) := \text{ the } n\text{th homology of } C(X, A)$$

Remarks:

1. $S_n(X) = S_n(X) \backslash S_n(A) \sqcup S_n(A)$. For the first component σ is not fully contained in A and for the second component σ is fully contained in A.

   $\Longrightarrow C_n(X) = \operatorname{span}_{\mathbb{Z}} S_n' \oplus C_n(A)$
   $\Longrightarrow C_n(X) \,/\, C_n(A) \cong \operatorname{span}_{\mathbb{Z}} S_n'$
   $\Longrightarrow C_n(X, A) =$ free Abelian group generated by all singular n-simplices in X that are not fully c
   $\Longrightarrow C_n(X) \cong C_n(X, A) \oplus C_n(A)$

   But differential in X (∂^X) does not preserve this decomposition (Reason: can happen that a simplex σ not fully contained in A has faces contained in A)

2. A $c \in C_n(X)$ represents a cycle in $C_n(A)$ if and only if $\partial_c \in C_{n-1}(A)$. Then:

$$H_n(X, A) \cong \frac{\partial_n^{-1}(C_{n-1}(A))}{\operatorname{im}(\partial_{n+1}) + C_n(A)}$$

$$\begin{tikzcd}
C_n(X) \arrow[d] \arrow[r, phantom, "="] & C_n(X, A) \arrow[d] \arrow[drr] & \oplus & C_n(A) \arrow[d, "\partial^A"] \\
C_{n-1}(X) \arrow[r, phantom, "="] & C_{n-1}(X, A) & \oplus & C_{n-1}(A)
\end{tikzcd}$$

3. Since $C(X, A) = C(X)/C(A)$ there exists a short exact sequence

$$0 \longrightarrow C(A) \longrightarrow C(B) \longrightarrow C(B/A) \longrightarrow 0$$

$$c \longmapsto c;\, c \longmapsto \overline{c}$$

   Thus there exists an induced long exact sequence:

$$\begin{tikzcd}
 & & H_{n+1}(X,A) \arrow[dll, "\delta_{n+1}"'] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[dll, "\delta_n"'] \\
H_{n-1}(A) \arrow[r] & \dots \arrow[r] & {}
\end{tikzcd}$$

Explicit Definition of δ_n:

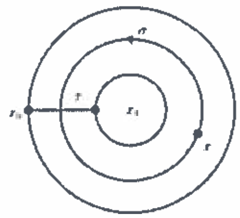
Suppose $c \in C_n(X)$ represents some $[\overline{c}] \in H_n(X, A)$. Then $\partial_c \in C_{n-1}(A)$. Define $\delta_n([\overline{c}]) := [\partial_c] \in H_n(A)$ (would be 0 in $H_{n-1}(X)$ because in X ∂_c is a boundary.

Example. $X = S^1 \times I$. $A := \partial^X$. Then

$$\sigma_1 \tau \in S_1(X) \qquad x, x_0, x_1 \in S_0(X)$$

Thus $\partial_\sigma = x - x = 0 \implies \partial$ is a cycle in $C(X)$

Further $\partial_\tau = x_1 - x_0 \in C_0(A)$ then τ represents a cycle in $C(X, A)$



Can show: $H_1(X) = \mathbb{Z}[\sigma]$, $H_1(X, A) = \mathbb{Z}[\tau]$.

Consider the induced long exact sequence in homology:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_1(X) & \xrightarrow{0} & H_1(X,A) & \xrightarrow{\begin{pmatrix}-1\\1\end{pmatrix}} & H_0(A) & \xrightarrow{\begin{pmatrix}-1 & 1\end{pmatrix}} & H_0(X) & \longrightarrow \cdots \\ & \| & & \| & & \| & & \| & \\ & \mathbb{Z}[\sigma] & & \mathbb{Z}[\tau] & & \mathbb{Z}[x_0] \oplus \mathbb{Z}[x_1] & & \mathbb{Z}[x] & \end{array}$$

Then $[\sigma] \mapsto 0$; $[\tau] \mapsto [x_1] - [x_0]$; $[x_i] \mapsto [x_i] = [x]$.

Does there exists a reduced version of $H_n(X, A)$?

$\tilde{C}(X)$, $\tilde{C}(A)$- augmented singular complex of X, A.

Could define:

$$\tilde{C}(X, A) = \frac{\tilde{C}(X)}{\tilde{C}(A)} = \frac{C(X)}{C(A)} = C(X, A).$$

Then $\tilde{H}_n(X, A) := H_n(X, A)$.

Also have short exact sequences:

$$0 \longrightarrow \tilde{C}(A) \hookrightarrow \tilde{C}(X) \twoheadrightarrow C(X, A) \longrightarrow 0$$

Thus, there exists an induced long exact sequence in homology. In particular, if $A = \{x_0\}$, then $\tilde{C}(A)$ is a cycle, so $\tilde{H}_n(X) \cong H_n(X, \{x_0\})$ for all n whenever $X \neq \varnothing$. (Note also that $H_n(X) \cong H_n(X, \varnothing)$.)

Definition. Let $f(X, A) \to (Y, B)$ be a continuous map of maps such that $f(A) \subset B$. Then $f_\# : C(X) \to C(Y)$ descends to a **chain map** $\overline{f}_\# : C(X, A) \to C(Y, B)$. Further, there exists an **induced map** $\overline{f}_* : H_n(X, A) \to H_n(Y, B)$ for all n. Thus the following commutes:

$$\begin{array}{ccccccccc} 0 \longrightarrow & C(A) & \longrightarrow & C(X) & \longrightarrow & C(X,A) & \longrightarrow 0 \\ & \downarrow{\scriptstyle f_\#|_A} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle \overline{f}_\#} & \\ 0 \longrightarrow & C(B) & \longrightarrow & C(Y) & \longrightarrow & C(Y,B) & \longrightarrow 0 \end{array}$$

Corresponding diagrams for LES commute.

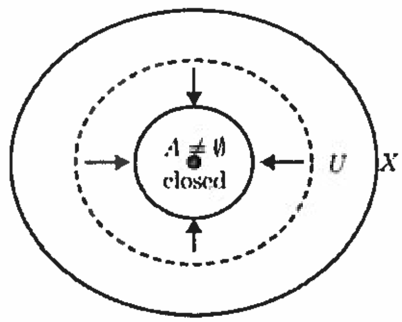
Not that $\overline{f}_\#$ and $\overline{f}_*$ satisfy the same properties as $f_\#$ and f_*.

Proposition. For $f, g : (X, A) \to (Y, B)$, via a homotopies of pairs such that $f_t(A) \subseteq B$ for all t:

$$f \simeq g \Longrightarrow \overline{f}_\# \simeq \overline{g}_\# \Longrightarrow \overline{f}_* = \overline{f}_*$$

Proof. Show that the homotopy $P : C_n(X) \to C_{n+1}(X \times I)$ descends to a homotopy $\overline{P} : C_n(X, A) \to C_{n+1}(X \times I, A \times I)$. ☐

Definition. A pair (X, A) is call **good** if $A \subset X$ is closed, nonempty, and there exists an open neighborhood $U \supset A$ which deformation retracts to A.



Example. If $A \subset X$ is nonempty and has a mapping cylinder neighborhood, then (X, A) is good.

Example. If (X, A) is a CW pair, $A \neq \varnothing$, then (X, A) is good.

Proposition. If (X, A) is good, then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism between:

$$H_n(X, A) \cong_{q_*} H_n(X/A, A/A) \cong \tilde{H}_n(X / A)$$

So $H_n(X, A) \cong \tilde{H}_n(X/A)$.

Relative Singular Homology

If (X, A) is good, then there exists a long exact sequence:

$$\begin{tikzcd} & H_n(A) \arrow[r] & H_n(X) \arrow[r] & \tilde{H}_n(X/A) \arrow[dll, "\delta_n"] \\ & H_{n-1}(A) \arrow[r] & \dots \arrow[r] & {} \end{tikzcd}$$

Moreover, there exists an analogous long exact sequence in which $H_n(A)$ and $H_n(X)$ are replaced by their reduced versions $\tilde{H}_n(A)$ and $\tilde{H}_n(X)$.

Example. (Homology of Spheres.)

Let $n > 0$. $X = D^n =$ closed unit ball in $\mathbb{R}^n$. Let $A = \partial D^n - S^{n-1}$. Then $(X, A) = (D^n, \partial D^n)$ is good because, for $n > 0$, $A = \partial D^n$ is closed and nonempty, $U : D^n \backslash \{0\}$ deformation retracts to the boundary ∂D^n. Thus there exists a long exact sequence:

$$\longrightarrow \underset{=0}{\tilde{H}_i(D^n)} \longrightarrow \underset{\cong S^n}{\tilde{H}_i(D^n/\partial D^n)} \xhookrightarrow{\cong} \underset{S^{n-1}}{\tilde{H}_{i-1}(\partial D^n)} \longrightarrow \underset{=0}{H_{i-1}(D^n)}$$

Thus $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$. (This is true for all $n > 0$.)

$\tilde{H}_3$	$\tilde{H}_2$	$\tilde{H}_1$	$\tilde{H}_0$	i/n
0	0	0	$\mathbb{Z}$	S^0
0	0	$\mathbb{Z}$	0	S^1
0	$\mathbb{Z}$	0	0	S^2
$\mathbb{Z}$	0	0	0	S^3

Thus $H_0(S^0) = \mathbb{Z}^2$, $\tilde{H}_0(S^0) = \mathbb{Z}$, $\tilde{H}_i(S^0) = 0$ for $i \neq 0$, $H_0(S^n) \cong \mathbb{Z}$ $n > 0$ thus $\tilde{H}_0(S^n) = 0$. $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}S^{n-1}$ is constant along diagonals. So

$$\tilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

Corollary. $S^m \not\cong S^n$ for $n \neq m$.

Corollary. $\mathbb{R}^m \not\cong \mathbb{R}^n$ for $n \neq m$. (these are homotopy equivalent, but not homeomorphic.)

Definition. Let (X, A, B) be a **triple of spaces** such that $B \subseteq A \subseteq X$.

There exists a short exact sequence of complexes:

$$0 \longrightarrow \underset{\frac{C(A)}{C(B)}}{\underset{\parallel}{C(A,B)}} \xhookrightarrow{i_\#} \underset{\frac{C(X)}{C(B)}}{\underset{\parallel}{C(X,B)}} \xrightarrow{q} \underset{\frac{C(X)}{C(A)} = \frac{C(X)/C(B)}{C(A)/C(B)}}{\underset{\parallel}{C(X,A)}} \longrightarrow 0$$

Where $i_\#$ is induced by the inclusion of A into X and q is the quotient map. Furthermore, there exists an induced long exact sequence in homology.

Δ- Complexes

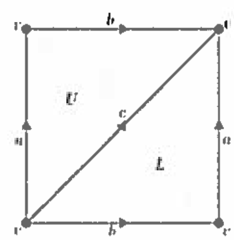
These are certain CW-complexes in which each n-cell is identified with the interior of the standard n-simplex: $\Delta^{\partial n} - \Delta^n \backslash \partial\Delta^n :=$ union of the faces of Δ^n

For $\sigma \in S_n(X)$ we write $n(\sigma) := n$

Definition. A Δ- complex is a topological space X together with a collection C of singular simplices in X such that:

- $\sigma|_{\mathring{\Delta}^{n(\sigma)}}$ is injective for all $\sigma \in C$
- As a set, X is the disjoint union of the images of the $\sigma|_{\mathring{\Delta}^{n(\sigma)}}$ for $\sigma \in C$.
- If $\sigma \in C$, $n(\sigma) \geqslant 1$, then $\sigma \circ F_i^n(\sigma) \in C$ for all $i = 0, \ldots, n(\sigma)$.
- $U \subset X$ is open in X if and only if $\sigma^{-1}(U) \subset \Delta^{n(\sigma)}$ is open for all $\sigma \in C$.

Example. Let $X = S^1 \times S^1$.



We turn this into a Δ complex by cutting it in half. Then $\Delta^0 = v$, $\Delta^1 = a, b, c$ (orientation preserving), $\Delta^2 = U, L$

Simplicial Homology

Definition. Let (X, C) be a Δ-complex. Define $S_n^{\Delta}(X) := S_n(C) \cap C$

Definition. $\Delta_n(X) := C_n^{\Delta}(X) := \operatorname{span}_{\mathbb{Z}} S_n\Delta(X) \subset C_n(X)$ (that is, all n chains that only involve simplices from X.

The differential in $C(X)$ maps $C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$. Thus $C_n^{\Delta}(X)$ form a subcomplex $C^{\Delta}(X) \subseteq C(X)$.

Definition. The **nth simplicial homology of** (X, C) $H_n^{\Delta}(X) :=$ the nth homology of $C^{\Delta}(X)$.

Proposition. If (X, C) is a Δ-complex, then the inclusion of $C^{\Delta}(X) \hookrightarrow C(X)$ induces an isomorphism between $H_n\Delta(X) \cong H_n(X)$ for all n. (That is, an isomorphism between the simplicial and singular homology.

Proof. Later $\square$

Example. $X = S_1 \times S^1$. (See picture above for break down into Δ-complex.)

Then $S_0^\Delta(X) - \{v\}$, $S_1^\Delta(X) - \{a,b,c\}$, $S_2^\Delta(X) - \{U,L\}$

Chain complex, compute boundaries:

- $\partial v = 0 \implies \partial_0 = 0$
- $\partial a = v - v = 0 = \partial b = \partial c \implies \partial_1 = 0$
- $\partial U = U|_{[\hat{e_0}, e_1, e_2]} - U|_{[e_0, \hat{e_1}, e_2]} + U|_{[e_0, e_1, \hat{e_2}]} = b - c + a$
- $\partial L = L|_{[\hat{e_0}, e_1, e_2]} - L|_{[e_0, \hat{e_1}, e_2]} + L|_{[e_0, e_1, \hat{e_2}]} = a - c + b = \partial U$

$C^\Delta(X)$ looks as follows:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow 0 & \xrightarrow{0} & \mathbb{Z}\langle U, L\rangle & \xrightarrow{\partial_2} & \mathbb{Z},\langle a,b,c\rangle & \xrightarrow{\partial_1 = 0} & \mathbb{Z}\langle v\rangle & \longrightarrow 0 \\ & & \| & & \| & & \| & \\ & & \mathbb{Z}^2 & & Z^3 & & \mathbb{Z} & \end{array}$$

$$\partial_2 : \begin{cases} U \to b - c + a \\ L \to b - c + a \end{cases} \implies \partial_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $\ker(\partial_2) = \mathbb{Z}\langle L - U\rangle$ and $\operatorname{im}(\partial_2) = \mathbb{Z}(a + b - c)$. So:

$$H_0^\Delta(X) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}\langle v\rangle}{0} = \mathbb{Z}\langle v\rangle \cong \mathbb{Z}$$
$$H_1^\Delta(X) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\langle a,b,c\rangle}{\mathbb{Z}\langle a+b-c\rangle} = \mathbb{Z}\langle b,c\rangle \cong \mathbb{Z}^2$$
$$H_2^\Delta(X) = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \frac{\mathbb{Z}\langle L - U\rangle}{0} = \mathbb{Z}\langle L - U\rangle \cong \mathbb{Z}$$

Remark $\pi_2(X) = 0$.

Some remaining notes on Simplicial Homology (specifically on the use of a basis to define ∂_i) from 4/19/2018 have been omitted due to a lack of clarity.

Subdivision

Definition. X is a topological space, $\mathscr{U}$ is a collection of subsets $A_i \subset X$ such that $\bigcup_{A_i \in \mathscr{U}} \text{int}(A_i) = X$. Define the **chain complex of the subdivision** as:

$$\begin{aligned} C_n^{\mathscr{U}}(X) &:= \text{span}_{\mathbb{Z}}\{\text{singular } n \text{ simplices in } X \text{ which are fully contained in one of the } A_i\} \\ &= \sum C_n(A_i) \subset C_n(X) \end{aligned}$$

Then the $C_n^{\mathscr{U}}(X)$ form a subcomplex $C^{\mathscr{U}}(X) \subseteq C(X)$.

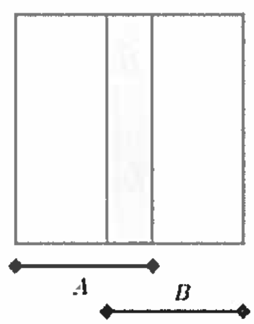
Definition. $H_n^{\mathscr{U}}(X) =$ the n**th homology of** $C^{\mathscr{U}}(X)$.

Proposition. The inclusion $i : C^{\mathscr{U}}(X) \hookrightarrow C(X)$ is a homotopy equivalence. Moreover, there exists a homotopy inverse $\rho : C(X) \to C^{\mathscr{U}}(X)$ such that $\rho \circ i = 1_{C^{\mathscr{U}}(X)}$ and $i \circ \rho = 1_{C(X)}$.

Corollary. $H_n^{\mathscr{U}}(X) \cong H_n(X)$ for all n.

Excision

Theorem. (Version 1 of Excision) Let X be a topological space and $A, B \subseteq X$. If $\text{int}(A) \cup \text{int}(B) = X$, then $H_n(X, A) \cong H_n(B, A \cap B)$ for all n (where the isomorphism is induced by the inclusion $(B, A \cap B) \hookrightarrow (X, A)$).



Proof. Let $\mathscr{U} = \{A, B\}$ and let $C^{\{A,B\}}(X) := C^{\mathscr{U}}(X) = C(A) + C(B) \subseteq C(X)$. The inclusion $C(B) \hookrightarrow C^{\{A,B\}}(X) = C(A) + C(B)$ descends to the isomorphism:

$$\frac{C(B)}{C(A \cap B)} \cong \frac{C^{\{A,B\}}(X)}{C(A)}$$

Examining $C(B) \hookrightarrow \frac{C^{\{A,B\}}(X)}{C(A)}$ we see that $\ker = C(A \cap B)$. Thus:

$$\frac{C(B)}{C(A \cap B)} \hookrightarrow \frac{C^{\{A,B\}}(X)}{C(A)}.$$

Further:

$$C(B, A \cap B) = \frac{C(B)}{C(A \cap B)} \cong \frac{C^{\{A,B\}}(X)}{C(A)} \simeq \frac{C(X)}{C(A)} = C(X, A).$$

Therefore $C(B, A \cap B) = C(X, A)$, so $H_n(B, A \cap B) \cong H_n(X, A)$. □

Theorem (Version 2) Let X be a topological space and $Z \subseteq A \subseteq X$. If $\overline{Z} \subset \operatorname{int}(A) \subset X$, then $H_n(X,A) \cong H_n(X-Z, A-Z)$ for all n (where the isomorphism is induced by the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$).

Proof. Let $B = X - Z$, so $A \cap B = A - Z$, and apply version 1. □

Corollary. If $x \in X$ is a closed point and $U \ni x$ is an open neighborhood of x, then $H_n(X, X - \{x\}) \cong H_n(U, U - \{x\})$ for all n.

Proof. Set $A := X - \{x\}$, $B := U$ and apply version 1 of excision. □

Definition. $H_n(X, X-\{x\})$ is called the n**th local homology group** of X at x.

Theorem. (Invariance of Dimension) If nonempty open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Proof. Let $f : U \to V$ be a homeomorphism and let $x \in U$ and $y := f(x) \in V$. Then, by corollary,

$$H_k(U, U-\{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{k-1}(\mathbb{R}^m - \{x\}) \cong \tilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z} & k = m \\ 0 & k \neq m \end{cases}$$

Analogously,

$$H_k(V, V-\{y\}) \cong \tilde{H}_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z} & k = m \\ 0 & k \neq m \end{cases}$$

(f induces a map, specifically an isomorphism, between homology groups. So f induces an isomorphism between $H_k(U, U-\{x\}) \cong H_k(V, V-\{y\})$. Therefore $m = n$. □

Recall. If (X,A) is good, then $H_n(X,A) \cong H_n(X/A, A/A)$ for all n (where the isomorphism is induced by the quotient map $q : X \to X/A$).

Proof. If (X,A) is good, then A is closed and nonempty. So there exists $U \supset A$ such that U deformation retracts to A. Thus:

$$\begin{tikzcd} H_n(X,A) \arrow[r, "i_*", "\cong"'] \arrow[d, "q_*"'] & H_n(X,U) \arrow[d, "q'_*"'] & H_n(X-A, U-A) \arrow[l, "j_*"', "\cong"] \arrow[d, "r_*"', "\cong"] \\ H_n(X/A, A/A) \arrow[r, "\cong", "i'_*"'] & H_n(U/A, A/A) & H_n(X/A - A/A, U/A - A/A) \arrow[l, "\cong"', "j'_*"] \end{tikzcd}$$

Note:

- i_*, i'_* induced by $1_X, 1_{X/A}$
- j_*, j'_* induced by inclusion
- q_*, q'_* induced by $q : X \to X/A$
- r_* induced by $q|_{X-A}$

Furthermore we know:

- j_*, j'_* are isomorphisms by excision
- i_* are isomorphisms because we have long exact sequences associated to (X, U, A):

$$(0 = H_n(A, A) \cong) H_n(U, A) \longrightarrow H_n(X, A) \xhookrightarrow{\cong} H_{n-1}(U, A) (\cong H_{n-1}(A, A) = 0)$$

- Analogously i'_* is an isomorphism.
- r_* is an isomorphism because $r : X - A \to X/A - A/A$ is bijective because it doesn't matter if one first removes A then collapses of vis versa. Thus r is continuous, bijective, and closed. Thus r is a homeomorphism. Therefore, the induced map r_* is an isomorphism.

□

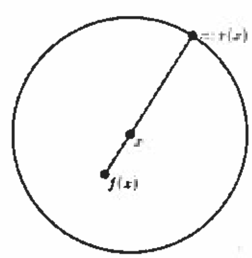
Consequences

Theorem. (No Retraction Theorem) There are no (continuous) retractions $r : D^n \to S^{n-1}(= \partial D^n)$.

Proof. Suppose such a retraction, r, exists. Then $i_* : \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(D^n)$ is injective. Contradiction. □

Theorem. (Brower Fixed point Theorem) Every continuous map $f : D^n \to D^n$ has a fixed point.

Proof. Suppose f has no fixed point. Define $r : D^n \to S^{n-1}(= \partial D^n)$ as follows:



Can check: r is a continuous retraction. This contradicts the no retraction theorem.

Explicit formula for r: We know $r(x) = x + t(x - f(x))$ for $t \geqslant 0$ and $||r(x)|| = 1$. Thus:

$$1 = ||r(x)||^2 = ||x||^2 + 2t\langle x, x - f(x)\rangle + t^2||x - f(x)||^2$$

$$t = \frac{-2\langle x, x - f(x)\rangle + \sqrt{(\langle x, x - f(x)\rangle)^2 - 4||x - f(x)||^2(||x||^2 - 1)}}{2||x - f(x)||^2}$$

Since the numerator is $\leqslant 0$ and the denominator $\neq 0$, $t = t(x)$ is well defined and continuous. So we get $r(x) = x + t(x)(x - f(x))$. □

Remarks:

1. There is an alternative proof of Brower fixed point theorem which does not use the no retraction theorem (Homework 8, #3)
2. Brower fixed point theorem is not true for continuous maps $f : D^n \backslash \partial D^n \to D^n \backslash \partial D^n$.

Theorem. (Excision) If $\operatorname{int}(A) \cup \operatorname{int}(B) = X$ then $H_n(X, A) \cong H_n(B, A \cap B)$.

Proposition. If (X, A) is good, then: $H_n(X, A) \cong H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$.

Corollary. If $A, B \subset X$ are subcomplexes of a CW complex X such that $A \cup B = X$, then $H_n(X, A) \cong H_n(B, A \cap B)$ for all n. (This is different because we don't assume that the interiors cover X.)

Proof. If $A \cap B \neq \varnothing$ then (X, A) and $(B, A \cap B)$ are good. So $H_n(X < A) \cong \tilde{H}(X/A)$ and $H_n(B, A \cap B) \cong \tilde{H}(B/(A \cap B))$. One can show $X/A \cong B/(A \cap B)$. □

Corollary. If X_α are spaces with base points $x_\alpha \in X_\alpha$ such that the pairs $(X_\alpha, \{x_\alpha\})$ are good, then the inclusion $X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ (formed at points $\{x_\alpha\}$) induce an isomorphism from $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha)$

Proof. Use that $\tilde{H}_n(\bigvee_\alpha X_\alpha) \cong H_n(\coprod X_\alpha, \coprod\{x_\alpha\})$. □

Mayer-Vietoris Sequence Let X be a space, $A, B \subset X$ such that $\operatorname{int}(A) \cup \operatorname{int}(B) = X$. Let $\mathscr{U} := \{A, B\}$. Define:

$$C^{\{A,B\}}(X) := C^{\mathscr{U}}(X) = C(A) + C(B) \subseteq C(X).$$

Then we have a short exact sequence:

$$0 \longrightarrow C(A \cap B) \xrightarrow{\varphi} C(A) \oplus C(B) \xrightarrow{\psi} C^{\{A,B\}}(X) \longrightarrow 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) := x + y$

Why is this exact?

- φ is injective, so both components are and ψ is surjective because we are going from a direct sum to a non direct sum
- $\operatorname{im}(\varphi) \subseteq \ker(\psi)$ because $\psi(\varphi(x)) = \psi(x, -x) = x + -x = 0$.
- $\ker(\psi) \subseteq \operatorname{im}(\varphi)$ because $\varphi(x, y) = 0 \Rightarrow x + y = 0 \Rightarrow x = -y \Rightarrow x$ a chain in $A \cap B \Rightarrow (x, y) = (x, -x) = \varphi(x)$. Thus $(x, y) \in \operatorname{im}(\varphi)$.

Thus, there exists an induced long exact sequence: (letting $\oplus$ represent $H_n(A) \oplus H_n(B)$)

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+1}"] & H_n(A \cap B) \arrow[r, "\varphi_*"] & \oplus \arrow[r, "\psi_*"] & H_n^{\mathscr{U}}(X) \arrow[r, "\delta_n"] & H_{n-1}(A \cap B) \arrow[r] & \cdots \arrow[r] & {}
\end{tikzcd}$$

This is called a Mayer- Vietoris sequence and exists whenever 2 subsets have interiors covering X.

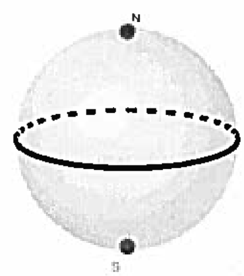
Explicit definition of δ_n:

Given $[z] \in H_n^{\mathcal{U}}(X)$, write z and $z = x + y = \psi(x, y)$ for an $x, y \in C_n(A) \oplus C_n(B)$ Then $\partial z - \partial_x + \partial_y = 0$ because z is a cycle. Further $\partial_x = -\partial_y$. Thus ∂_x is a chain in $A \cap B$ (in fact, a cycle). Define $\partial_n([z]) := [\partial_x] \in H_{n-1}(A \cap B) = [-\partial_y]$

Remarks:

1. MV sequence also exists for reduced homology
2. MV sequence also exists if $A \cup B \backslash X$ and if there exists open neighborhoods U, V of A, B respectively such that U, V deformation retract to A, B respectively. Then $U \cap V$ deformation retracts to $A \cap B$.
   See Hatcher pg 150, satisfied if X is a CW complex and A, B are subcomplexes such that $A \cup B = X$.
3. There exists a relative version of the MV sequence for pairs $(X, Y) = (A \cup B, C \cup D)$ where $Y \subset X$, $C \subset A$, $D \subset B$, and $X = \text{int}(A) \cup \text{int}(B)$, $Y = \text{int}(C) \cup \text{int}(D)$.

Example. Let $X = S^n$, N be the north pole and S the south pole.

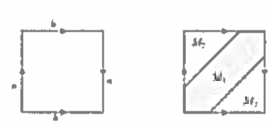


Let $A := S^n - \{S\}$ $B := S^n - \{N\}$. Then $A \cap B = S^n - \{N, S\} \cong S^{n-1} \times (-1, 1) \cong S^{n-1}$. The reduced MV Sequence is:

$$\tilde{H}_i(\text{pt}) \oplus \tilde{H}_i(\text{pt}) \longrightarrow \tilde{H}_i(S^n) \hookrightarrow\!\!\!\!\!\twoheadrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \tilde{H}_{i-1}(\text{pt}) \oplus \tilde{H}_{i-1}(\text{pt})$$

So $\tilde{H}_i(S^n) \cong \tilde{H}_i(S^{n-1})$ for all i, for all $n > 0$.

Example. X = Klein Bottle,



Let A, B = the two mobius bands (CW-subcomplexes). Then $A, B \cong S^1$, $A \cap B \cong S^1$. Then $H_2(A), H_2(B), \tilde{H}_0(A \cap B)$ are all 0. Thus:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \tilde{H}_2(X) & \longrightarrow & \tilde{H}_1(S^1) & \xrightarrow{\varphi_*} & \tilde{H}_1(S^1) \oplus \tilde{H}_1(S^1) & \xrightarrow{\psi_*} & \tilde{H}_1(X) & \longrightarrow & 0 \\ & & & & \| & & \| & & \| & & \\ & & & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

Can show $\varphi_*(n) = (2n, -2n)$.

Consequences

Recall $\tilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$. In particular $\tilde{H}_n(S^n) = \mathbb{Z}$.

Generators for $\tilde{H}_n(S^n) = \mathbb{Z}$

$i_n : \Delta^n \to \Delta^n$ is the identity map.

We can view i_n as a singular n-simplex in Δ^n or as an element of $C_n(\Delta^n)$.

Let $\bar{i}_n := i_n$ view as an element of $C(\Delta^n, \partial\Delta^n) = \frac{C_n(\Delta^n)}{\partial C_n(\Delta)}$. Note, $\bar{i}_n$ is a cycles in $\frac{C_n(\Delta^n)}{\partial C_n(\Delta)}$ because $i_n(\partial\Delta^n) = \partial(\Delta^n)$.

Claim: $[\bar{i}_n]$ generates

$$H_n(\Delta^n, \partial\Delta^n)(\cong \tilde{H}_n(\frac{\Delta^n}{\partial\Delta^n}) \cong \tilde{H}_n(S^n)) \cong \mathbb{Z}$$

Proof. (1) Use simplicial homology □

Proof. Induction on n (Details on Hatcher pg 135)

Idea: Identify Δ^{n-1} with the 0th face of Δ^n. Let $\Lambda = \bigcup$ (all other faces of the standard n-simplex Δ^n).

One can show that there exists isomorphisms:

$$([\bar{i}_n] \in) H_n(\Delta^n, \partial\Delta^n) \xrightarrow[\cong]{\delta} H_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow[\cong]{i_*} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})(\ni [\bar{i}_{n-1}])$$

Where δ comes from the long exact sequence of $(\Delta^n, \partial\Delta^n, \Lambda)$ and i_* s the inclusion induced by $i : \Delta^{n-1} \hookrightarrow \partial\Delta^{n-1}$.

By induction on n one can assume that $[\bar{i}_{n-1}]$ generates $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. Can check:

$$\partial[\bar{i}_n] = i_*[\bar{i}_{n-1}] \Longrightarrow [\bar{i}_n] = (\delta^{-1} \circ i_*)[\bar{i}_{n-1}] \Longrightarrow [\bar{i}_n] \text{ is a generator}$$

□

Example. Let $n > 0$. Let $q_n : \Delta^n \to \Delta^n/\partial\Delta^n$ be the quotient map. (e.g. for $n = 1$ $f_1 : \Delta^1 \to \Delta^1/\partial\Delta^1 \cong S^1$. Let $c_n : \Delta^n \to \Delta^n/\partial\Delta^n$ be the constant map at $\partial\Delta^n/\partial\Delta^n$. Note that we can view q_n, c_n as singular n=simplices in $\Delta^n/\partial\Delta^n$ or as elements of $C_n(\Delta^n/\partial\Delta^n)$.

Note: $q_n - c - n$ is a cycle in $C_n(\Delta^n/\partial\Delta^n)$ because $q_n|_{\partial\Delta^n} = c_n|_{\partial\Delta_n}$.

Claim: $[q_n - c_n]$ generates $H_n(\Delta^n/\partial\Delta^n) \cong H_n(S^n) \cong \mathbb{Z}$ for $n > 0$.

Proof. There exists isomorphisms:

$$(\mathbb{Z}\langle[\bar{i}_n]\rangle =) H_n(\Delta^n, \partial\Delta^n) \xrightarrow[\cong]{q_*} H_n(\tfrac{\Delta^n}{\partial\Delta^n}, \tfrac{\partial\Delta^n}{\partial\Delta^n}) \xleftarrow[\pi]{p_*} \tilde{H}_n(\tfrac{\Delta^n}{\partial\Delta^n})(\ni [q_n - c_n])$$

Can show: $(p_*{-1} \circ q_*)([\bar{i}_n]) = [q_n - c_n] \Longrightarrow [q_n - c_n]$ generates $\tilde{H}_n(\Delta^n/\partial\Delta^n)$. □

Remark 1: For n odd, q_n is a cycle and $c_n = \partial c_{n+1}$ implies that for n odd $[q_n - c_n] = [q_n]$
e.g. $H_1(S^1) \cong \mathbb{Z}$ is generated by $q_1 : \Delta^1 \to \Delta^1/\partial\Delta^1 \cong S^1$.

Remark 2: Alternatively, can construct generator of $\tilde{H}_n(S^n)$ as follows:

$$S^n_+ := \{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} | \sum t_i^2 = 1, t_{n+1} \geqslant 0\}$$

$$S^n_- := \{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} | \sum t_i^2 = 1, t_{n+1} \leqslant 0\}$$

Then $P_\pm : S^n_\pm \xrightarrow{\cong} D^n$ such that $p(t_1, \dots, t_{n+1}) = (t_1, \dots, t_n)$ are the projections. Fix a homeomorphism $h : \Delta^n \to D^n$ and define $\sigma_\pm := p_\pm^{-1} \circ h : \Delta^n \to S^n_\pm \subset S^n$. Note that we can view σ_+, σ_- as singular n-simplices in S^n and as elements of $C_n(S^n)$. Note $\sigma_+ - \sigma_-$ is a cycle in $C_n(S^n)$ because $\sigma_+|_{\partial\Delta^n} = \sigma_-|\partial\Delta^n$.

Claim: $[\sigma_+, \sigma_-]$ generates $H_n(S^n) \cong \mathbb{Z}$.

Proof. (Hatcher, pg 125) Consider

$$([\bar{i}_n] \in) H_n(\Delta^n, \partial\Delta^n) \xrightarrow{(\sigma_+)_*} H_n(S^n, S^n_-) \xleftarrow{p_*} \tilde{H}_n(S^n)(\ni [\sigma_+ - \sigma_-)$$

where p_* comes from the long exact sequence. □

Degree

Definition. Consider $f : S^n \to S^n$ a continuous map, $n > 0$. Then there exists an induced homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. Then f_* must be of the form $f_*(\alpha) = m\alpha$ for some $m \in \mathbb{Z}$. The integer $m \in \mathbb{Z}$ is called the **degree** of f, denoted $\deg(f)$.

Properties:

- $\deg(1_{S^n}) = 1$ because $(1_{S^n})_* = 1_{H_n(S^n)}$
- $\deg(f) = 0$ if f is not surjective.

  (If $x_0 \in S^n \backslash f(S^n)$, then f_* factors as $H_n(S^n) \to H_n(S^n \backslash \{x_0\}) \xrightarrow{i_*} H_n(S^n)$, where $H_n(S^n \backslash \{x_0\}) \overset{i_*}{=} 0$. Thus $f_* = 0$.
- If $f \simeq g$, then $\deg(f) = \deg(g)$ (because $f_* = g_*$)
- $\deg(f \circ g) = \deg(f)\deg(g)$. (because $(f \circ g)_* = f_* \circ g_*$
- If f is a homotopy equivalence, then $\deg(f) = \pm 1$ (in this cases f_* is an isomorphism)
- If $g : S^n \to S^n$ is a homeomorphism, then $\deg(g \circ f \circ g^{-1}) = \deg(f)$.
- If f is a restriction of a reflection $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of $\mathbb{R}^{n+1}$ about a hyperplane $W \subseteq \mathbb{R}^{n+1}$ through 0, then $\deg(f) = -1$.

  Proof. Can assume that W is a hyperplane. Consider $W = \{(t_1, \dots, t_{n+1}) \in S^n | t_{n+1=0}\}$. Can take

  $$[\sigma_+ - \sigma_-] = [f \circ \sigma_+ - f \circ \sigma_-] = [\sigma_- - \sigma_+] = -[\sigma_+ - \sigma_-].$$

  Therefore $\deg(f) = -1$. □

- If $f = -1_{S^n}$ is the antipodal map, then $\deg(f) = (-1)^{n+1}$. (because -1_{s^n} is the composition of $n+1$ reflections)
- If $f : S^n \to S^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$ (follows because if f has no fixed points then $f \simeq 1_{S^n}$ by homework 8 problem 3)

Corollary. If n is even, then $\mathbb{Z}_2$ is the only nontrivial group which can act (via homomorphisms) freely on S^n.

Proof. Suppose n is even and $\rho : G \to \text{Homeo}(S^n)$ is a free group action. Can define a group homomorphism $\varphi : G\ to \{\pm 1\} \cong \mathbb{Z}$ via $\varphi(g) = \deg(\rho(g))$.

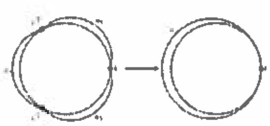
If $g \neq e$, then $\rho(g)$ has no fixed points so $\deg(\rho(g)) = (-1)^{n+1} = -1$. Thus $\ker(\varphi) = \{e\}$ so φ is injective. Therefore $G = \{e\}$ or $G \cong \mathbb{Z}_2$. ☐

Note: This is not true if n is odd (e.g. S^3 is a topological group by HW 4, and S^3 acts freely on itself by homeomorphisms.)

$$S^{2n-1} \subset \mathbb{R}^{2n} = \mathbb{C}^n \text{ (acts on itself by scalar multiplication)} \qquad S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$$

Only fixed point is $0 \in \mathbb{C}^n$ which is not in $\S^{2n-1}$.

Example. $f : S^1 \to S^1$ via $f(z) = z^n$, $n \in \mathbb{Z}$. Claim $\deg(f) = n$. Suppose that $n = 3$. Then



We know that $[a]$ generates $H - 1(S^1)$ if and only if $[a] \neq 0$. It is easy to see that $[a] = [a_1 + a_2 + a_3]$. Thus

$$f_*[a] = f_*[a_1 + a_2 + a_3] = [f \circ a_1 + f \circ a_2 + f \circ a + 3] = [a + a + a] = 3[a]$$

Therefore $\deg(f) = 3$.

Cellular Homology

Definition. Let X be a **CW Complex** with X^n **the** n**-skeleton**, where X^n is obtained from X^{n-1} by attaching n-dimensional D^n_α along ∂D^n_α via a continuous map $\varphi_\alpha : D^n_\alpha \to X^{n-1}$.

Definition. $C_n^{CW}(X) := \mathbb{Z}\langle n - cells \rangle$

Definition. $d_n : C_n^{CW} \to C_{n-1}^{CW}(X)$is defined as follows:

Let e_α be an n-cell ($= D^n_\alpha \backslash X_{n-1}$) and e_β be an $(n-1)$-cell.

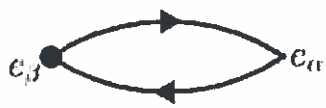
$$\begin{tikzcd}
\partial D^n_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, equal] & X^{n-1} \arrow[r, two heads] & \frac{X^{n-1}}{X^{n-1}\backslash e_\beta} \arrow[r, phantom, "\cong"] & \frac{D^{n-1}+\beta}{\partial D^{n-1}_\beta} \arrow[d, equal] \\
S^{n-1} \arrow[rrr, "f_{\alpha\beta}"] & & & S^{n-1}
\end{tikzcd}$$

Define $d_{\alpha\beta} := \deg(f_{\alpha\beta}) \in \mathbb{Z}$. Then $d_n(e_\alpha) := \sum_\beta \partial_{\alpha\beta} e_\beta$

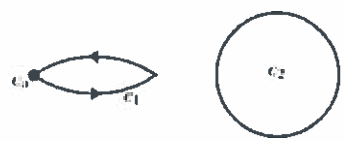
Examples. For $n = 1$:

$d_{\alpha\beta} = +1$

$d_{\alpha\beta} = -1$



$d_{\alpha\beta} = 0$
$\mathbb{R}P^2 =$



$$\mathbb{Z}\langle e^2\rangle \longrightarrow \mathbb{Z}\langle e_1\rangle \longrightarrow \mathbb{Z}\langle e_0\rangle$$

Cellular Homology

Definition. The **characteristic map**:

$$\Phi_\alpha^n : D_\alpha \int X^{n-1} \sqcup \coprod D_\alpha^n \twoheadrightarrow X^n \hookrightarrow X$$

Definition. The n**-cells** $e_\alpha :=$ the interior of $D_\alpha^n = \Phi_\alpha^n(D_\alpha^n \backslash \partial D_\alpha^n)$

Last Time:

$$C_n^{CW} := \mathbb{Z}\langle n\text{-cells}\rangle$$
$$d_n|_{e_\alpha} := \sum_\beta d_{\alpha\beta} e_\beta$$

Definition. $H_n^{CW}(X) :=$ the nth homology of $C^{CW}(X)$. (Claim $H_n^{CW}(X) \cong H_n(X)$)

Proposition. If X is a CW complex, then:

1. $H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbb{Z}\langle n\text{-cells}\rangle & k = n \\ 0 & k \neq n \end{cases}$
2. $H_k(X^n) = 0$ for $k > n$
3. $i : X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n) \cong H_k(X)$ if $k = n$

1. *Proof.* Note that

$$\begin{aligned} H_k(X^n, X^{n-1}) &\cong \tilde{H}_k(\frac{X^n}{X^{n-1}}) \\ &\cong \tilde{H}_k(\frac{\coprod D_\alpha^n}{\coprod \partial D_\alpha^n}) \quad \left(\Phi_\alpha^n : D_\alpha^n \to X^n \text{ induces a homeomorphism } \frac{\coprod D_\alpha^n}{\coprod \partial D_\alpha^n} \to \frac{X^n}{X^{n-1}}\right) \\ &\cong \tilde{H}_k(\bigvee_\alpha S^n) \quad \left(\text{choose homeomorphism } h_\alpha : \frac{D_\alpha^n}{\partial D_\alpha^n} \to S^n\right) \\ &\cong \bigoplus_\alpha \tilde{H}_n(S^n) \quad (\text{one summand for each } n\text{-cell}) \\ &\cong \begin{cases} \mathbb{Z}\langle n\text{-cell}\rangle & k = n \\ 0 & k \neq n \end{cases} \quad \left(\text{to make canonical choose a generator } [S^n] \in \tilde{H}_n(S^n)\right) \end{aligned}$$

Explicit isomorphism $\Phi : \mathbb{Z}\langle n\text{-cell}\rangle \to H_n(X^n, X^{n-1}$:
$[D_\alpha^n] :=$ the generator of $H_n(D_\alpha^n, \partial D_\alpha^n) \cong \tilde{H}_n(D^n/\partial D_\alpha^n)$ which corresponds to $[S^n]$ of $\tilde{H}_n(S^n$ under $(h_\alpha)_*$ Define:

$$\Phi : \mathbb{Z}\langle n\text{-cell}\rangle \to H_n(X^n, X^{n-1}) \Longrightarrow \Phi(e_\alpha) = (\Phi_\alpha^n)^*([D_\alpha^n])$$

$$\Phi_\alpha^n : (D_\alpha^n, \partial D_\alpha^n) \to (X^n, X^{n-1}) \Longrightarrow \Phi([D_\alpha^n]) = (\Phi_\alpha^n)_*([D_\alpha^n])$$

□

2. *Proof.* Consider the long exact sequence of (X^n, X^{n-1})

$$H_{n+1}(X^n, X^{n-1}) \longrightarrow H_n(X^{n-1}) H_n(X^n) \hookrightarrow\!\!\!\!\rightarrow H_n(X^n, X^{n-1})$$

Note the first and last term $= 0$ by (1) because $k \neq n$. Now suppose $k \neq n, n-1$ Then $H_k(X^n) \cong H_k(X^{n-1})$.
Suppose $k > n \geqslant 0$. Then:

$$H_k(X^n) \cong H_k(X^{n-1} \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) = 0$$

Therefore $H_k(X^n) = 0$ implying (2) □

3. *Proof.* Continuing from the proof of (2), suppose $k < n$. Then:

$$H_k(X^n) \cong H_k(X^{n+1}) \cong H_k(X^{n+2}) \cong \cdots \cong H_k(X^m)$$

Thus $H_k(X^n) \cong H_k(X^m)$ for all $m \geqslant n$. So if X is finite dimensional this proves (3) because $X = X^m$ for $m >> 0$.
Infinite dimensional case, Hatcher 138-139. □

Definition. (new definition of the cellular chain complex) Let X be a CW complex. Define

$$C_n^{CW}(X) := H_n(X^n, X^{n-1}) \cong \tilde{H}_n\left(\bigvee_\alpha S^n\right) \cong \mathbb{Z}\langle n\text{-cells}\rangle$$

Definition. Define $d_n : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ as the connecting homomorphism such that:

$$d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

in the long exact sequence of (X^n, X^{n-1}, X^{n-2}).

Explicit formula: $d_n([z]) := [\partial z]$ where $[z]$ is an n-cycle in $C(X^n, X^{n-1})$ and $[\partial_z]$ is an $(n-1)$ -cycle in $C(X^{n-1}, X^{n-2})$.

Claim: $d_n \circ d_{n+1} = 0$

Proof.

$$(d_n \circ d_{n+1})([z]) = [\partial^2 z] = 0$$

□

Definition. The n**-th cellular homology** of X is: $H_n^{CW}(X) :=$ the nth homology of $C^{CW}(X)$.

Theorem. $H_n^{CW}(X) \cong H_n(X)$ for all n.

Proof. Consider the following diagram:

$$
\begin{tikzcd}
 & & & & H_n(X^{n+1},X^n) \arrow[r, equal, "(1)"] & 0 \\
H_n(X^{n-1}) \arrow[drr] & & & H_n(X^{n+1}) \arrow[ur] \arrow[r, equal, "(1)"] & H_n(X) & \\
0 \arrow[rr, equal, "(2)"] & & H_n(X^n) \arrow[ur, two heads] \arrow[dr, hook, "j_n"] & & & \\
H_{n+1}(X^{n+1},X^n) \arrow[urr, "\partial_{n+1}"] \arrow[rrr, "d_{n+1}"] & & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2}) \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, hook, "j_{n-1}"'] & \\
 & & & H_{n-1}(X^{n-2}) \arrow[ur] \arrow[r, equal, "(2)"] & 0 &
\end{tikzcd}
$$

Note that the diagonals are exact. Further because j_n is injective. Further the triangles commute:

$$
\begin{aligned}
H_n^{CW}(X) &= \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})} \\
&= \frac{\ker(\partial_n)}{\operatorname{im}(j_n \circ \partial_{n+1})} \quad (j_n \text{ is injective and the triangles commute } \Rightarrow d_{n+1} = j_n \circ \partial_{n+1}) \\
&= \frac{\operatorname{im}(j_n)}{\operatorname{im}(j_n \circ \partial_{n+1})} \quad (\text{exactness at } H_n(X^n.X^{n-1})) \\
&= \frac{H_n(X^n)}{\operatorname{im}\partial_{n+1}} \cong H_n(X^{n+1}) \cong H_n(X)
\end{aligned}
$$

□

Last time we say that:

$$
\begin{tikzcd}
\partial D^n_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, equal] & X^{n-1} \arrow[r, two heads] & \frac{X^{n-1}}{X^{n-1}\setminus e_\beta} \cong \frac{D^{n-1}+\beta}{\partial D^{n-1}_\beta} \arrow[d, equal] \\
S^{n-1} \arrow[rr, "f_{\alpha\beta}"] & & S^{n-1}
\end{tikzcd}
$$

Want to see that our definitions do in fact align:

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[rrr, "d_n"] \arrow[dr, "\partial"] & & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[dl] \arrow[dd, shift left, "=:p_\beta"] & \\
& H_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[urr] & \tilde{H}_{n-1}\left(\frac{X^{n-1}}{X^{n-1}-e^\beta}\right) & & \\
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r] \arrow[uu, "(\Phi^n_\alpha)_*"] \arrow[d, equal] \arrow[dr] & H_{n-1}(\partial D^n_\alpha) \arrow[u, "(\varphi_\alpha)_*"] \arrow[d, equal] & H_{n-1}\left(\frac{D^{n-1}_\beta}{\partial D^{n-1}_\beta}\right) \arrow[u, "\cong"] \arrow[d, "\cong"] & H_{n-1}(D^{n-1}_\beta, \partial D^{n-1}_\beta) \arrow[l, "\cong"'] \arrow[uu, shift left, "(\Phi^{n-1}_\beta)_*"] \arrow[r, equal] & \mathbb{Z}[D^{n-1}_\beta] \\
\mathbb{Z}[D^n_\alpha] & H_{n-1}(S^{n-1}) \arrow[r, "(f_{\alpha\beta})_*"] \arrow[d, equal] & \tilde{H}_{n-1}(\partial^{n-1}) \arrow[ur] \arrow[d, equal] & & \\
& \mathbb{Z}[S^{n-1}] & \mathbb{Z}[S^{n-1}] & &
\end{tikzcd}$$

Then $(p_\beta \circ \partial_n \circ (\Phi^n_\alpha)_*)([D^n_\alpha]) = \deg(f_{\alpha\beta})[D^{n-1}_\beta]$. Thus, it is east to see that:

$$d_n(\Phi^n_\alpha)_*[D^n_\alpha]) = \sum_\beta \deg(f_{\alpha\beta})(\Phi^{n-1}_\beta)_*([D^{n-1}_\beta])$$

Betti Numbers and Euler Characteristic

Definition. Let X be a space. The **nth Betti Number** of X is $b_n(X) := \text{rank}(H_n(X))$ (where $\text{rank}(A) :=$ the cardinality of the maximal independent subset)

Note: $b_n(X)$is an invariant of the boundary type of X.

(*) Suppose thank $\text{rank}(H_n(X) < \infty$ for all n and $\neq 0$ for only finitely many n

Definition. The **Euler Characteristic of** X is:

$$\chi(X) := \sum_n (-1)^n b_n(X) \quad \in \mathbb{Z}$$

Remarks

1. $\chi(X)$ is an invariant of the homotopy type of X.
2. If X is a finite CW complex then $\chi(X) = \sum(-1)^n c_n(X)$ where $c_n(X)$ is the $rank(C_n^{CW}(X))$ (Note, $c_n(X)$ is not a topological invariant of X.)
3. If $A, B \subseteq X$ are subcomplexes of a finite CW complex X such that $A \cup B = X$, then $c_n(X) = c_n(A) + c_n(B) - c_n(A \cap B)$ so $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$ (this also holds if A, B, X are spaces that satusty (*) and $A, B \subset X$, $\text{int}(A) \cup \int(B) = X$ yielding a M-V sequence).
4. If X is a finite CW complex then $b_n(X) = \text{rank}(\ker(d_n)) = \text{rank}(\text{im}(d_{n+1})$

Topology Qualifying Exam   August 2017

Justify your answers with the necessary proofs, unless otherwise noted. While you should attempt every problem, try your best to answer questions fully. Therefore it might be best to do first the problems you know how to do.

Unless otherwise stated, assume that $\mathbb{R}$, $\mathbb{R}^2$, S^1, and $[a, b)$ are equipped with their standard topology and that if $A \subset X$ for a topological space X then A is given the subspace topology.

1. (a) Suppose that X is compact and Y is Hausdorff. Let $f : X \to Y$ be continuous. Show that $f(X)$ is closed.
   (b) Suppose that X is compact and non-empty and Y is Hausdorff and connected. Show that every continuous open map $f : X \to Y$ is onto.

2. Suppose that X is compact and $X \times \{y\} \subset U$ where $U \subset X \times Y$ is open. Prove that there exists an open set $Z \subset Y$ such that
$$X \times \{y\} \subset X \times Z \subset U.$$

3. Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if x and y are both rational (and otherwise $x \sim x$). Let $Y = \mathbb{R}/\sim$ with the quotient topology.
   (a) Describe the open sets in Y.
   (b) What are the continuous functions from Y to $\mathbb{R}$?
   (c) Show that Y is not Hausdorff.

4. Let U be an open, connected subset of $\mathbb{R}^n$.
   (a) Show that U is path connected.
   (b) For $p, q \in U$, define $\operatorname{dist}(p, q)$ to be the infimum of the length of all paths connecting p and q whose image lies in U. Show that dist defines a metric on U. (You may assume in this problem that the infimum is finite).
   (c) Show that dist induces the same topology on U as the standard Euclidean distance between points in U. Give an example of a connected, open set $U \subset \mathbb{R}^n$ such that dist is not equal to the standard Euclidean distance.

5. (a) Show how a Klein Bottle (KB) is obtained from a square with certain identifications.
   (b) Show that a KB can be obtained from two Möbius bands by identifying their boundaries.
   (c) Can a torus be obtained from two KBs by identifying certain edges?

6. Let X be the quotient space of S^2 obtained by identifying its north and south poles.
   (a) Give X a cell complex structure and use it to compute $\pi_1(X)$.
   (b) Let Y be the union $S^2 \cup D$ where D is the diameter (on the z-axis in $\mathbb{R}^3$ connecting the north and south poles of S^2. Show that X and Y are homotopy equivalent. If M is a meridian (half of a great circle) on S^2 connecting the north and south poles, show that the quotient space Y/M obtained from Y by collapsing M to a point is homotopy equivalent to the one-point union $S^2 \vee S^1$.
   (c) Assume the truth of (b) and use it to again compute $\pi_1(X)$.

7. Let X be obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus. Compute $\pi_1(X)$. Describe it in terms of generators and relations. Also compute the singular homology groups $H_i(X)$.

8. Let $S^1 \vee S^1$ be the one-point union of two circles with union point as basepoint $\{x_0\}$. Call the counterclockwise path around the left circle and the counterclockwise path around the right circle. Let E be a circle union an equilateral triangle inscribed in the circle. Choose the basepoint $\{e_0\}$ of E to be one of the vertices of the triangle. Describe a covering projection $p : (E, e_0) \to (S^1 \vee S^1, x_0)$ by labeling the arcs and edges of E and describe the corresponding subgroup $p_*(\pi_1(X))$ in terms of a and b.

9. For a space X the suspension SX of X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Here $I = [0, 1]$. The cone CX of X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point. Then there is an isomorphism $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ of reduced singular homology groups. The abbreviated proof of this statement goes as follows.

   (a) SX is homeomorphic to the union of two cones CX with their bases identified.

   (b) $\widetilde{H}_{n+1}(SX) = \widetilde{H}_{n+1}(CX \cup CX) \cong \widetilde{H}_{n+1}(CX, X) \cong \widetilde{H}_n(X)$.

   Fill in the details (reasoning) in each of these two steps. In particular, state which theorems are being applied to explain the two isomorphisms in step (b).

1a. Suppose X is compact and Y is Hausdorff. Let $f: X \to Y$ be continuous. Then $f(X) \subset Y$ is a compact subset of a Hausdorff set. Thus, $f(X)$ is closed.

b. Suppose X is a compact, nonempty set and Y is Hausdorff and connected. It follows then that the only sets in Y that are both open and closed are $\emptyset$ and Y. Let $f: X \to Y$ be an arbitrary open, continuous map. By definition, for any open $U \subseteq X$ $f(U)$ is open. To show that f is onto we need to show that $f(X) = Y$. Since X is itself open and closed we see that $f(X)$ is both open (by def) and closed. However, the only subsets of Y that are both open and closed are $\emptyset$ and Y. Since $X \neq \emptyset$, $f(X) \neq \emptyset$. Thus $f(X) = Y$. Therefore f is onto ☑

2. Suppose X is compact and $X \times \{y\} \subset U$ where $U \subset X \times Y$ is open. Since U is open $\forall\ (x,y) \in U\ \exists\ V_x, W_x$ open in X, Y resp. such that $(x,y) \in V_x \times W_x \subset U$. Consider the map $i_y : X \longrightarrow X \times \{y\}$, which is cts and onto (by definition of the product topology). Since i_y is continuous and X is compact, $i_y(X) = X \times \{y\}$ is compact. Thus for any open cover, like $\{V_x \times W_x\}_{x \in X}$ there is a finite subcover $\{V_{x_1} \times W_{x_1}, \ldots, V_{x_n} \times W_{x_n}\}$. Let $Z = \bigcap_{i=1}^{n} W_{x_i}$. Note $X \times \{y\} \subset X \times Z$. because $y \in W_{x_i}\ \forall\ i = 1, \ldots, n$.

Lastly, to show $X \times Z \subset U$, consider $(x_0, y_0) \in X \times Z$. Then $x_0 \in V_{x_k}$ and $y_0 \in W_{x_k}$ for some $1 \leq k \leq n$ (since $y_0 \in Z = \bigcap_{i=1}^{n} W_{x_i}$, $y_0 \in W_{x_i}\ \forall\ i = 1, \ldots, n$). Thus $(x_0, y_0) \in V_{x_k} \times W_{x_k} \subset U$. Therefore $X \times Z \subset U$. $\square$

3. Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if x and y are both rational. Let $Y = \mathbb{R}/\sim$ with the quotient topology.

a. Note that By definition $\exists$ a continuous map $q : \mathbb{R} \longrightarrow Y$, the quotient map. So $U \subseteq Y$ is open iff $q^{-1}(U)$ is open.

- If $U \subseteq \mathbb{R}\setminus\mathbb{Q}$ then U is not open because $q^{-1}(U)$ is not open.

- If $[\mathbb{Q}] \in U$, then $q^{-1}(U)$ is open if it is an open subset of $\mathbb{R}$ that contains $\mathbb{Q}$ (e.g. $\mathbb{R}\setminus\sqrt{2}$)

So, the open sets U of Y are sets s.t. $q^{-1}(U)$ is an open subset of $\mathbb{R}$ that contains $\mathbb{Q}$.

b The continuous functions from Y to $\mathbb{R}$ are constant functions.

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d, "q"'] & \mathbb{R} \\ Y \arrow[ur, "f"'] & \end{tikzcd}$$

pf. Let f be a cts fn. Then f is cts iff $f \circ q$ is continuous. Then $\forall\, x, y \in \mathbb{Q}$ $q(x) = q(y)$, $f(q(x)) = f(q(y))$ $\therefore$ $f \circ q$ $\mathbb{R} \to \mathbb{R}$ is constant on the rationals Thus $f \circ q$ is constant on $\mathbb{R}$ (b/c of the density of $\mathbb{Q}$ in $\mathbb{R}$. Therefore f is constant. ☑

c. Suppose Y is Hausdorff Then $\forall\, x, y \in Y$ $\exists\, U_x, U_y$ s.t. $U_x \cap U_y = \phi$

However, consider $[0]$, $[\sqrt{2}]$. By part (a) we know that any

contains all $\mathbb{Q}$

open set in Y contains all rationals. Thus $\forall$ $V \supseteq [\sqrt{2}]$ $\mathbb{Q} \subseteq V$

$\therefore$ $U \cap V \neq \phi$. ☑

4. Let U be an open connected subset of $\mathbb{R}^n$.

a. First note that we are given that U is connected. Thus, it suffices to show that U is locally connected. Since open sets in $\mathbb{R}^n$ look like balls, which are path connected (for any $x, y \in B(x_0, \varepsilon)$ can connect x to y via $p(0) = x$ $p(1/2) = x_0$ $p(1) = y$, where $p([0,1/2])$ is a path by def and $p([1/2, 1])$ is a path.) Thus U is connected and locally path connected. Therefore U is path connected $\blacksquare$

b. For $p, q \in U$ define $dist(p,q)$ to be the infimum of the length of all paths connecting p and q whose image lies in U.

Before showing that dist is a metric, we will note that

$$dist(p,q) \geq d(p,q) \quad \text{where } d \text{ is the Euclidean metric}$$

because the Euclidean metric looks at the infimum of all paths connecting p, q, not just the paths whose image is in U.

Thus:

- $dist(p,q) \geq d(p,q) \geq 0 \quad \forall p,q$

  if $p = q \implies dist(p,p) = \inf\{\ell(\rho) \mid \rho(0) = \rho(1) = p, \rho([0,1]) \subset U\} = 0$

  if $dist(p,q) = 0 \overset{?}{\longleftrightarrow} 0 \geq d(p,q) \geq 0 \overset{?}{\longrightarrow} d(p,q) = 0 \longleftrightarrow p = q$.

- $dist(p,q) = dist(q,p)$ because the paths that connect q to p are the reversal of those connecting p to q & thus have the same inf.

- $dist(x,y) + dist(y,z) = \inf\{\ell(f) \mid f(0) = x, f(1) = y\} + \inf\{\ell(g) \mid g(0) = y, g(1) = z\} \geq \inf\{\ell(h) \mid h(0) = x, h(1/2) = y, h(1) = z\}$

could be more precise.

4c. Note that in order to show the topologies are the same one must simply show that the open sets are the same. Let

$B^d(x,\varepsilon)$ represent balls centered around x w/ radius ε measured in terms of the aforementioned metric defined by dist.(p,q).

Let $B^e(x,\varepsilon)$ ' ' ' ' ' ' ' ' ' ' ' ' ' ' Euclidean metric.

NTS that if something is open in one ~~metric~~ topology it is also open in the other.

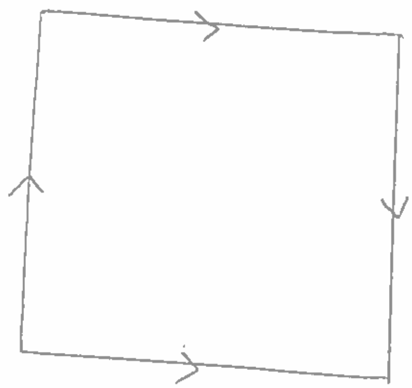
Let $V \subseteq U$ be open in the subspace topology. We WTS V is open in dist-topology. We know that $\forall x \in V\ \exists \varepsilon > 0$ s.t. $B^e(x,\varepsilon) \subseteq V \subseteq U$

Then let $B^d(x,\varepsilon) = B^e(x,\varepsilon)$ to see that $B^d(x,\varepsilon) \subseteq V$ so. V is open in distance topology.

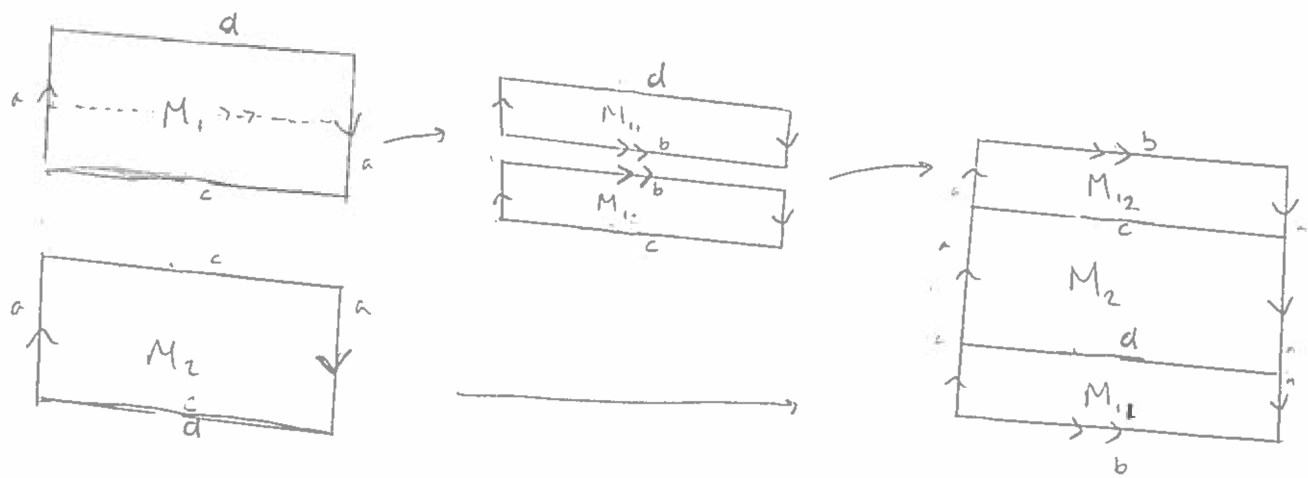
Similarly, assume $V \subseteq U$ is open in the distance topology Then $\forall x \in V\ \exists\ \delta > 0$ s.t. $B^d(x,\delta) \subseteq V \subseteq U$ But since $\text{dist}(x,y) \geq d(x,y)\ \forall\ x,y \in U$ we know $\exists\ \delta' \leq \delta$ s.t. $B^e(x,\delta') \subseteq B^d(x,\delta) \subseteq V$ Thus V is also open with regard to the Euclidean topology. Therefore the topologies are the same. $\boxtimes$

e. Consider U:

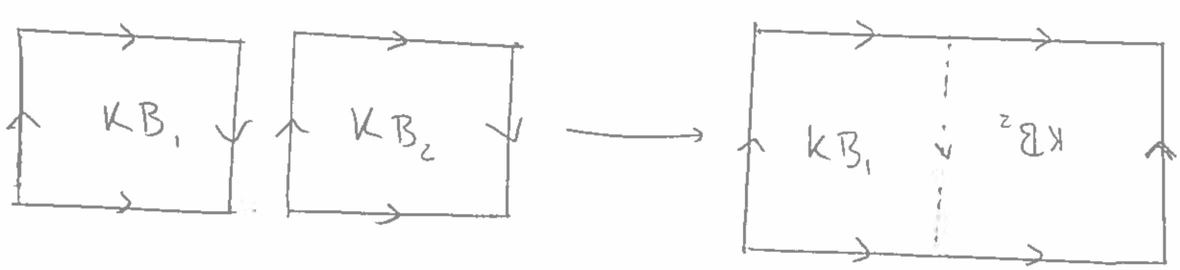
5a. Klein Bottle



b.

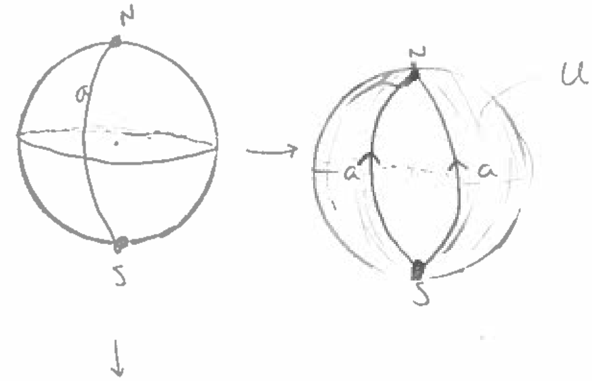


c. Torus out of 2 KBs:



6. Let X be the quotient space of S^2 obtained by identifying its north + South poles.

a. Give X a cell complex



0 - cells: $[N] = [S]$

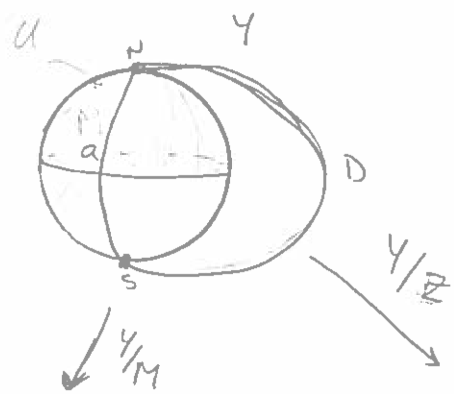
1 - cells: a

2 - cells: $u = aa^{-1}$

By Van Kampens Theorem since X has 1 0-cell we know that $\pi_1(X)$ is generated by the 1-cells with relations given by the 2-cells. Thus $\pi_1(X) = \langle a \mid aa^{-1} = 1 \rangle = \langle a \mid a = a \rangle \cong \mathbb{Z}$

b. Let $Y = S^2 \cup D$

Note the cell structure of Y is:

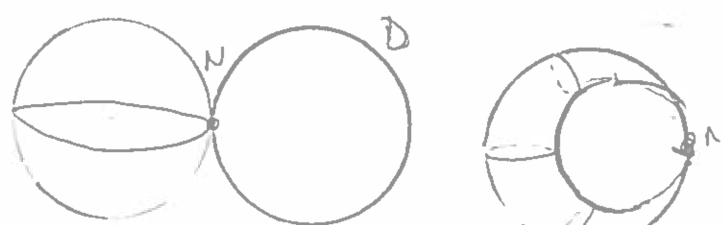


0-cells $[N] = [S]$

1-cells: D, a

2-cells: u

Consider the subcomplex Z consisting of $[N]$, D, $[S]$



Note, Z is contractible to a point.

Thus, by Theorem Y/Z is homotopy equiv to Y

Clearly $Y/Z = X = S^2/\sim$ where $N \sim S$.

Similarly consider the subcomplex M consisting of $[N]$, a, $[S]$ Thus $Y/M \simeq Y \simeq X$

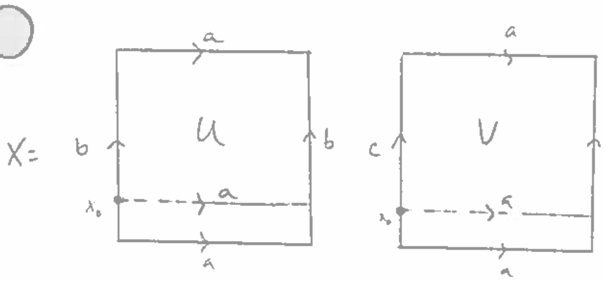
Further Y/M breaks down to 1 0-cell, $[N] = [S]$ connected to a 2-cell, u and connected to a 1-cell, D. This cell complex is homeomorphic to $S^2 \vee S^1$!

6c: Since $X \simeq Y \simeq Y/M \simeq S^2 \vee S^1$

$$\pi_1(X) \cong \pi_1(S^2 \vee S^1) = \pi_1(S^2) * \pi_1(S^1) = \{e\} * \mathbb{Z} = \mathbb{Z} \quad \square$$

(b/c intersection is a pt

7. Let X be obtained from 2 tori, $S^1 \times S^1$, by identifying a circle in one tori w/ corresponding circle in the other



0 - cells: x_0

1 - cells: a, b, c

2 - cells: u, v

By previous Thm

$$\pi_1(X) = \langle a, b, c \mid aba^{-1}b^{-1} = 1,\ aca^{-1}c^{-1} = 1 \rangle$$

$H_0(A) = \mathbb{Z}^k$ where k is the # of path components of A Thus $H_0(X) = \mathbb{Z}$ because X is path connected

Because X can be broken down into a CW complex we can look at the Cellular homology:

$$\longrightarrow H_3(X^3, X^2) \xrightarrow{d_3} H_2(X^2, X^1) \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0, X^{-1})$$

C

$$0 \xrightarrow{d_3} \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z}$$

Should be chain complex

$d_1 = 0$ because there is 1 0-cell

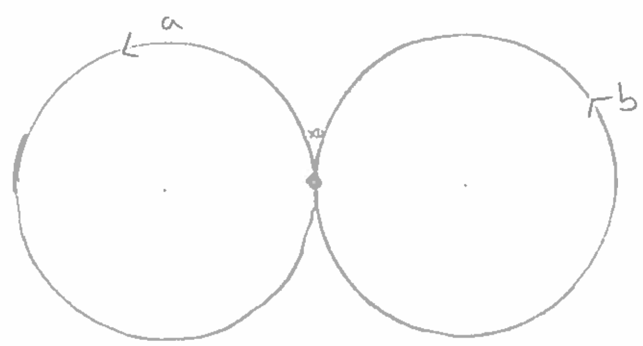
$d_2 = 0$ because $\deg(aba^{-1}b^{-1}) = 0$

Since $H_1(X) = \dfrac{\ker d_1}{\operatorname{im} d_2} = \mathbb{Z}^3$ b/c d_1 is the zero map so $\ker d_1 = H_1(X^1, X^0)$ and d_2 is the zero map so $\operatorname{im} d_2 = 0$

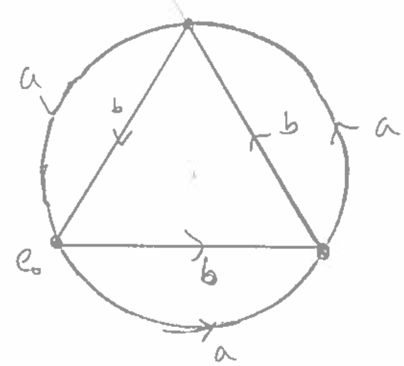
Likewise $H_2(X) = \dfrac{\ker d_2}{\operatorname{im} d_3} = \mathbb{Z}^2$ b/c d_2 is the zero map so the $\ker d_2 = H_2(X^2, X^1)$ and $\operatorname{im} d_3$ is 0 b/c $H_3(X^3, X^2) = 0$

Lastly $H_i(X) = 0$ for $i \geq 3$

8 Consider $S^1 \vee S^1$ and S^1 with E inscribed in it:



$S^1 \vee S^1$



E

Consider $p(E, e_0) \longrightarrow (S^1 \vee S^1, x_0)$ described by the labels above

$P_*(\pi_1(E)) \longrightarrow \pi_1(S^1 \vee S^1)$

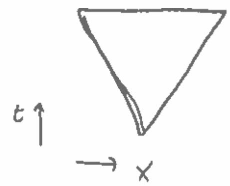
$P_*(\pi_1(E)) = \langle a, b \mid a^3, b^3, ab^{-1}, ba^{-1} \rangle$ note, this is $\underline{\text{not}}$ $\pi_1(E)$

is this what the question is asking for? yes

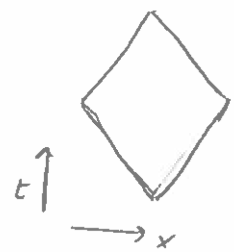
b/c $P_*(\pi(X))$ is not a thing b/c X is not defined

9a. Consider

$$CX = X \times I \Big/ (x,0) \sim (x',0) \quad =$$



$$SX = X \times I \Big/ \begin{matrix} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{matrix} \quad =$$



Then:

$$CX_1 \cup CX_2 \approx \frac{[0, 1/2]}{(x,0) \sim (x',0)} \underset{\substack{(x_1, 1/2) \sim (x_2, 1/2) \\ x_1 \in CX_1 \\ x_2 \in CX_2}}{\cup} \frac{[1/2, 1]}{(x,1) \sim (x',1)}$$

$$\approx [0,1] \Big/ \begin{matrix} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{matrix}$$

☑

9b. WTS: $\tilde{H}_{n+1}(SX) = \tilde{H}_{n+1}(CX_1 \cup CX_2) \cong \tilde{H}_{n+1}(CX, X) \cong \tilde{H}_n(X)$

- $\tilde{H}_{n+1}(SX) = \tilde{H}_{n+1}(CX_1 \cup CX_2)$ because $SX \simeq CX_1 \cup CX_2$ by (a)

- $\tilde{H}_{n+1}(CX_1 \cup CX_2) \cong \tilde{H}_{n+1}(CX_1, X)$ (note the cones have been #'d wlog)

Consider $\tilde{H}_{n+1}(CX_1 \cup CX_2, CX_2)$. Since this pair is good we get the LES:

$$\tilde{H}_{n+1}(CX_2) \longrightarrow \tilde{H}_{n+1}(CX_1 \cup CX_2) \longrightarrow \tilde{H}_{n+1}(CX_1 \cup CX_2, CX_2) \longrightarrow \tilde{H}_n(CX_2)$$

Since CX_2 is contractible $\tilde{H}_{n+1}(CX_2) \cong \tilde{H}_n(CX_2) \cong 0$, so we get the SES.

$$0 \longrightarrow \tilde{H}_{n+1}(CX_1 \cup CX_2) \longrightarrow \tilde{H}_{n+1}(CX_1 \cup CX_2, CX_2) \longrightarrow 0$$

Thus $\tilde{H}_{n+1}(CX_1 \cup CX_2) \cong \tilde{H}_{n+1}(CX_1 \cup CX_2, CX_2)$.

Since $CX_1 \cup CX_2 = \mathring{CX_2} \cup \mathring{CX_1}$ we can apply the excision theorem:

$$\tilde{H}_{n+1}(CX_1 \cup CX_2, CX_2) \cong \tilde{H}_{n+1}(CX_1, CX_1 \cap CX_2) = \tilde{H}_{n+1}(CX_1, X)$$

Therefore $\tilde{H}_{n+1}(CX_1 \cup CX_2) \cong \tilde{H}_{n+1}(CX_1 \cup CX_2, CX_2) \cong \tilde{H}_{n+1}(CX_1, X)$

- $\tilde{H}_{n+1}(CX, X) \cong \tilde{H}_n(X)$. (As above, note $\tilde{H}_n(X) = H_n(X)$ for $n>0$)

Since (CX, X) is good, we get the following LES.

$$\tilde{H}_{n+1}(CX) \longrightarrow \tilde{H}_{n+1}(CX, X) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(CX)$$

Again using the fact that CX is contractible so $\tilde{H}_n(CX) \cong 0$ we get the SES:

$$0 \longrightarrow \tilde{H}_{n+1}(CX, X) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

Therefore $\tilde{H}_{n+1}(CX, X) \cong \tilde{H}_n(X)$

Therefore $\tilde{H}_{n+1}(SX) = \tilde{H}_{n+1}(CX \cup CX) \cong \tilde{H}_{n+1}(CX, X) \cong \tilde{H}_n(X)$ ☑

Topology Qualifying Exam: Fall 2016

Your full name: ______________________________

Instructions. There are 8 questions on 2 pages. Do all questions, and justify your answers with the necessary proofs.

1. Prove the following statements:

(a) The open interval $(0, 1)$ is homeomorphic to the real line $\mathbb{R}$.

(b) The two dimensional sphere $S^2 \subset \mathbb{R}^3$ with the north pole removed is homeomorphic to $\mathbb{R}^2$.

2. Prove the following statements:

(a) Let X, Y be topological spaces and $A \subset X, B \subset Y$ be closed sets respectively. Then $A \times B$ is a closed set in $X \times Y$.

(b) Let $f : X \to Y$ be a continuous bijection between two topological spaces X and Y. Assume that X is compact and Y is Hausdorff. Then f is a homeomorphism.

3. Show that $(\mathbb{R}^2, d)$ is a metric space, where

$$d\left((x, y), (x', y')\right) = \begin{cases} |y| + |y'| + |x - x'| & \text{if } x \neq x' \\ |y - y'| & \text{if } x = x' \end{cases}.$$

Illustrate by diagrams in the real plane $\mathbb{R}^2$ what the open balls of this metric space are.

4. Let $X = A \cup B$ be a subset in $\mathbb{R}^2$, where

$$A = \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0, 1) \right\}, B = \{(0, y) : y \in [-1, 1]\}.$$

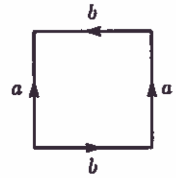
Show that X is connected but not path connected.

5. Let $\gamma \subset \mathbb{RP}^2$ be a simple closed curve representing a generator of $\pi_1(\mathbb{RP}^2)$, and let X be the space obtained from $\mathbb{RP}^2$ by attaching a Möbius band via a homeomorphism from the boundary of the Möbius band to γ.

(a) Compute $\pi_1(X)$.

(b) Determine the number of connected covering spaces of X up to equivalence.

6. Let S be the surface obtained from a square by identifying edges as shown below.



Prove or disprove:

(a) a is a retract of S.

(b) b is a retract of S.

(c) a is a deformation retract of S.

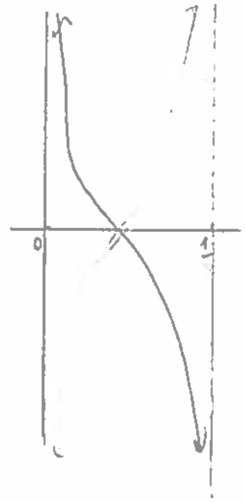
7. For $(m, n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, let $X_{m,n}$ be the CW complex obtained from S^1 with its standard cell structure by attaching two 2-cells by maps of degrees m and n, respectively.

(a) Compute the cellular homology groups of $X_{m,n}$.

(b) Give a necessary and sufficient condition under which $X_{m,n}$ and $X_{m',n'}$ are homotopy equivalent.

8. Show the following: If $Y = U \cup V$ is a union of two open subsets U and V such that $H_k(U \cap V)$ contains a nonzero homology class which is zero in both $H_k(U)$ and $H_k(V)$, then $H_{k+1}(Y) \neq 0$.

1 a. Consider $(0,1)$. Consider the function $f(x) = \cot \pi x$. Note



$f: (0,1) \rightarrow \mathbb{R}$

Clearly, f is continuous on $(0,1)$

$f^{-1}(x) = \frac{1}{\pi} \cot^{-1}(x)$

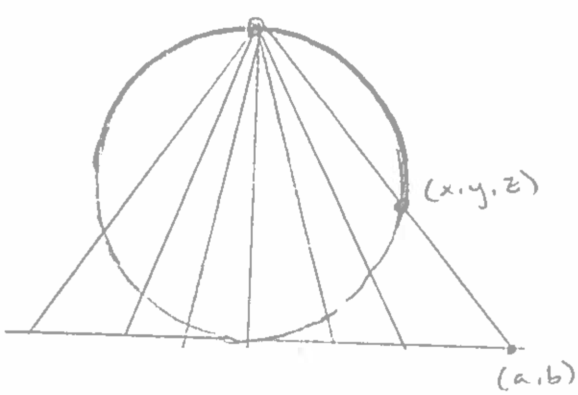
Note, $f^{-1}(x): \mathbb{R} \rightarrow (0,1)$ is also continuous

$\therefore$ f is a homeomorphism.

b. Consider $S^2 \setminus N$ where S^2 is the 2-dimensional sphere and N is

e North pole. Consider $f: S^2 \setminus N \longrightarrow \mathbb{R}^2$ where f is the

Stereographic projection. That is:



$$f(x,y,z) = \frac{1}{1-z}(x,y)$$

$$f^{-1}(a,b) = \frac{1}{1+a^2+b^2}\left(2a, 2b, a^2+b^2-1\right)$$

Since we are examining $S^2 \setminus N$ we know $z \neq 1$. Thus it is

clear that f, f^{-1} are continuous $\therefore$ f is a homeomorphism.

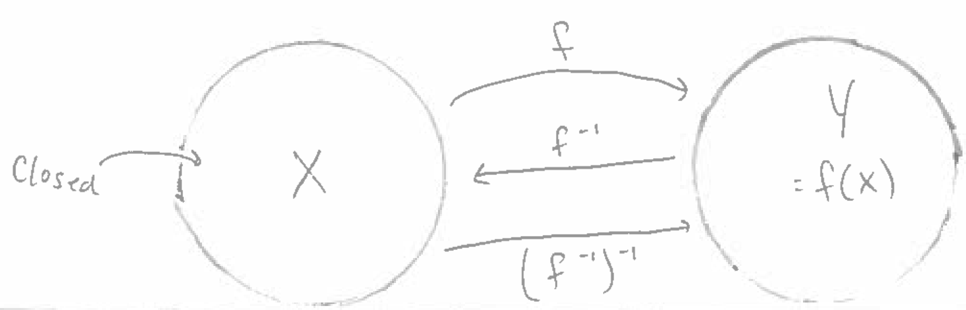
2.a. Let X, Y be topological spaces and $A \subset X$, $B \subset Y$ be closed sets respectively. Then $X \setminus A$, $Y \setminus B$ are open sets.

Since X, Y are both open (and closed) it follows that $X \setminus A \times Y$ and $X \times Y \setminus B$ are open by definition of the product topology. Furthermore $(X \setminus A \times Y) \cap (X \times Y \setminus B) = X \setminus A \times Y \setminus B$ is open. Therefore, it's complement $(X \setminus A \times Y \setminus B)^c = (A \times B)$ is closed. ☑

b. Let $f: X \to Y$ be a continuous bijection between 2 topological spaces X and Y. Assume X is compact and Y is Hausdorff. Since f is a bijection, $f^{-1}(f(X)) = X$. Since X is compact and f is continuous, $f(X)$ is compact. Since Y is Hausdorff, $f(X)$ is closed. Since f is continuous $f^{-1}(f(X)) = X$ is closed.

We want to show that $(f^{-1})^{-1}$ takes closed sets to closed sets.

Observe $(f^{-1})^{-1}(X) = (f^{-1})^{-1}(f^{-1}(f(X))) = f(X)$, which is closed. Thus f^{-1} is continuous. Therefore f is a homeomorphism.



3. Consider $(\mathbb{R}^2, d)$ where $d = \begin{cases} |y| + |y'| + |x - x'| & \text{if } x \neq x' \\ |y - y'| & \text{if } x = x' \end{cases}$

Observe:

- $\underline{d((x,y),(x',y')) \geq 0}$

  - $|y| + |y'| + |x - x'| > 0 \quad \forall \ (x,y),(x',y')$ s.t. $x \neq x'$

  - $|y - y'| > 0 \quad \forall \ (x,y),(x',y')$ s.t. $x = x', \ y \neq y'$

  - $d(x,y) = 0$ iff $x = x'$ (by first assertion), $y = y'$ (by second assertion)

    More explicitly: $|y| + |y'| + |x - x'| = 0$ iff $y = y' = 0 \rightarrow |x - x'| = 0 \rightarrow x = x'$

    this forces us into the second case, so $|y - y'| = 0$ iff $y = y'$

    Therefore $d(x,y) = 0$ iff $(x,y) = (x',y')$

- $\underline{d((x,y),(x',y')) = d((x',y'),(x,y))}$:

$$d((x,y),(x',y')) = \begin{cases} |y| + |y'| + |x - x'| & x \neq x' \\ |y - y'| & x = x' \end{cases} = \begin{cases} |y'| + |y| + |x' - x| & x' \neq x \\ |y' - y| & x' = x \end{cases} = d((x',y'),(x,y))$$

- $\underline{\text{Triangle inequality}}$:

$$d((x,y),(x',y')) + d((x',y'),(x'',y'')) = \begin{cases} |y| + |y'| + |x - x'| + |y'| + |y''| + |x' - x''| & x \neq x' \neq x'' \\ |y - y'| + |y'| + |y''| + |x' - x''| & x = x' \ x' \neq x'' \\ |y| + |y'| + |x - x'| + |y' - y''| & x \neq x' \ x' = x'' \\ |y - y'| + |y' - y''| & x = x' = x'' \end{cases}$$

using Δ ineq; reverse Δ ineq, and stated equalities we see that:

$$\geq \begin{cases} |y| + 2|y'| + |y''| + |x - x' + x' - x''| & x \neq x' \neq x'' \\ |y| - |y'| + |y'| + |y''| + |x - x''| & x = x' \ x' \neq x'' \\ |y| + |y'| + |x - x''| + |y''| - |y'| & x \neq x' \ x' = x'' \\ |y - y' + y' - y''| & x = x' = x'' \end{cases} = \begin{cases} |y| + |y''| + |x - x''| & x \neq x'' \\ |y - y''| & x = x'' \end{cases} = d((x,y),(x'',y''))$$

Therefore d is a metric, and $(\mathbb{R}^2, d)$ is a metric space. $\square$

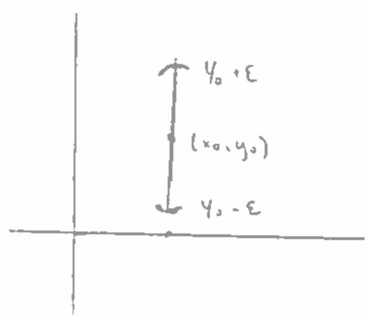
Open balls in this metric space can take on two forms depending on the chosen ε. Let $\varepsilon > 0$ be given. Then $B((x_0,y_0),\varepsilon) = \{(x,y) \mid d((x_0,y_0),(x,y)) < \varepsilon\}$.

So for $x \neq x_0$ we want all points (x,y) such that $|y_0| + |y| + |x - x_0| < \varepsilon$.

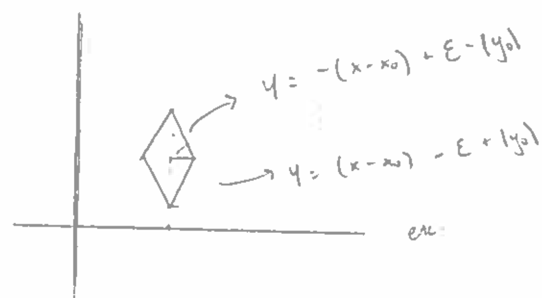
Note, if $|y_0| \geq \varepsilon$ then there are no such (x,y) (as it would already fail the desired inequality.) If $|y_0| < \varepsilon$ then we see that we get a diamond shape defined by: $|y| < -|x - x_0| + (\varepsilon - |y_0|)$. Lastly, if $x = x_0$ then we simply get an interval, $\{(x_0, y) \mid y_0 - \varepsilon < y < y_0 + \varepsilon\}$.

Thus, open balls look like:

$|y_0| \geq \varepsilon$



$|y_0| < \varepsilon$



4. Let $X = A \cup B$ be a subset of $\mathbb{R}^2$ where

$A = \{ (x, \sin 1/x) \mid x \in (0,1) \}$ $B = \{(0,y) : y \in [-1,1]\}$

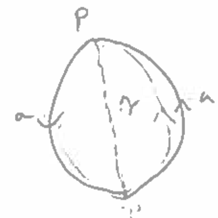
That is, X is the topologists sine curve.

To show X is connected, first note that A is connected because $\sin 1/x$ is a continuous map on a connected set in $(0,1)$. So its image is connected, and thus its graph, A, is connected. Further, B is connected because it is a vertical interval in $\mathbb{R}^2$. Observe, $B \subset Bd(A)$ because $\forall (0,y) \in B$ every open set $U_y \ni (0,y)$ contains infinitely many points of A (because $\lim_{x \to 0} (x, \sin 1/x) = B$) and $U_y \cap \mathbb{R}^2 \setminus A \neq \emptyset$ ($\exists\ \varepsilon > 0$ s.t. $B((0,y),\varepsilon) \subset U_y$ by def $\rightarrow (-\varepsilon/4, y) \in \mathbb{R}^2 \setminus A$). Thus $B \subset bd\, A$ Therefore $A \subset A \cup B \subset \bar{A}$. Suppose $\exists$ a separation in $A \cup B$, so $A \cup B = C \cup D$. Since A is connected A must lie entirely in C or in D. WLOG, $A \subset C$, then $\bar{A} \subset \bar{C}$. Since $\bar{C} \cap D = \emptyset$ we know for $A \cup B \subset \bar{A} \subset \bar{C}$, $(A \cup B) \cap D = \emptyset$. Thus, D is not a nonempty subset of $A \cup B$. Therefore $A \cup B$ is connected. E

Path connected part could use work.

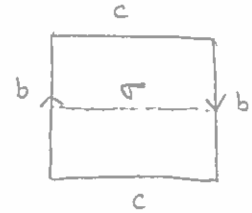
To show X is not connected consider $a \in A$, $b \in B$. T a graph of a continuous function we know A is path connec... is an interval and is therefore path connected. But $\nexists$ a continuous $p : [0,1] \to X$ s.t. $p(0) = a$, $p(1) = b$ because $\lim_{x \to 0} (x, \sin 1/x)$ is not a point. Essentially p would need to go through said point to connect to any points in B

5. Let $\gamma \subset \mathbb{R}P^2$ be a simple closed curve representing the generator of $\pi_1(\mathbb{R}P^2)$



Note $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 = \langle \gamma \mid \gamma^2 = 1 \rangle$

a. Consider the mobius band with generator σ:



Attaching the boundary of the Mobius band to γ to get X, recall that the boundary is itself a simple closed curve (because of the identifications. Thus $\sigma^2 = \gamma$. Therefore we see that

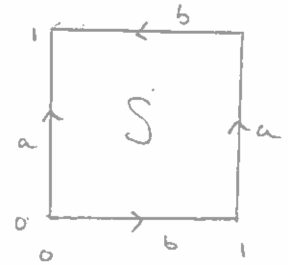
$$\pi_1(X) = \langle \gamma, \sigma \mid \gamma^2 = 1, \sigma^2 = \gamma \rangle = \langle \sigma \mid \sigma^4 = 1 \rangle = \mathbb{Z}_4$$

b. The number of covering spaces of X is equal to the number of subgroups of $\pi_1(X)$. The subgroups of $\mathbb{Z}_4$ are:

$\langle 4 \rangle = \langle 0 \rangle \qquad \langle 2 \rangle \qquad \mathbb{Z}_4 = \langle 1 \rangle = \langle 3 \rangle$

Thus $\mathbb{Z}_4$ has 3 distinct subgroups. Therefore X has 3 covering spaces

6. Consider the surface S obtained from $I \times I$ by identifying the edges as shown below.



Note that S is the Klein bottle, and $\pi_1(S) = \langle a, b \mid a b a^{-1} b = 1 \rangle$ by Van Kampen's Theorem.

a. a is a retract of S

pf. Consider the map $r : S \longrightarrow a'$ via $r(x,y) = (0,y)$. That is, r is a projection onto the left-most copy of a. Further, note that this copy of $a = \{(x,y) \mid x = 0,\ y \in [-1,1]\}$. Observe $r(0,y) = (0,y)$. That is $r((x,y)) = (x,y)$ $\forall (x,y) \in a$. Let $i : a \hookrightarrow S$ be the inclusion map. Then $r \circ i \,|_{a} = id$. Therefore r is a retraction, so a is a retract of S.

b. b is not a retract of S.

pf. Note that $b \approx S^1$, so $\pi_1(b) \cong \pi_1(S^1) \cong \mathbb{Z}$. Suppose a retraction $r : S \longrightarrow b$ existed. Then $r_* : \pi_1(S) \longrightarrow \pi_1(b)$ is injective. However $r_* : \langle a, b \mid a b a^{-1} b = 1 \rangle \longrightarrow \mathbb{Z}$ implies that $a b = b^{-1} a$

$b \longmapsto 1$

That is $b = b^{-1}$. However b generates $\mathbb{Z}$ and that cannot be the additive identity. Σ.

Therefore S does not retract to b.

c. a is not a deformation retract of S.

pf. Suppose a were a deformation retract of S. Note, $a \approx S^1$.

Then $\pi_1(S) \cong \pi_1(a) \cong \pi_1(S^1) \cong \mathbb{Z}$. However

$\langle a, b \mid aba^{-1}b = 1 \rangle \not\cong \mathbb{Z}$ because it has two generators.

Therefore a is not a deformation retract of S. $\square$

7.a. For $(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$, let $X_{m,n}$ be the CW complex obtained from S^1 with its standard cell complex by attaching two 2-cells by maps of degrees m and n, respectively.

Note that $X_{m,n}$ has:

$$1 \quad 0\text{-cell} \qquad 1 \quad 1\text{-cell} \qquad 2 \quad 2\text{-cells}$$

Thus, the cellular chain complex.

$d_2 = [m, n]$

$d_1 = [0]$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z} & \xrightarrow[0]{d_1} & \mathbb{Z} & \xrightarrow{0} & 0 \\ & & \langle p, q \rangle & \longrightarrow & \langle x \rangle & \longrightarrow & \langle y \rangle & & \end{array}$$

$H_0 = \mathbb{Z}$

$H_k = \ker d_k / \operatorname{Im} d_{k+1} \longrightarrow H_1 = \ker d_1 / \operatorname{Im} d_2 = \mathbb{Z} / \gcd(m,n)\mathbb{Z}$

$\longrightarrow H_2 = \ker d_2 / \operatorname{Im} d_3 = \ker d_2 = \mathbb{Z}$

$H_n = 0 \quad \forall \quad n \geq 3$

b. Give a necessary and sufficient condition und ... X ... and $X_{m',n'}$ are homotopy equivalent.

Needs work

$\gcd(m,n) = \gcd(m',n')$ is at the very ... ssary.

Not sure if it is sufficient

8. Let $Y = U \cup V$ be the union of two open subsets U and V such that $H_k(U \cap V)$ contains a nonzero homology class which is zero in both $H_k(U)$ and $H_k(V)$. Since U, V are open $\operatorname{int} U = U$, $\operatorname{int} V = V$. Thus Y satisfies the conditions necessary to assert the existence of a Mayer-Vietoris sequence:

$$\cdots \longrightarrow H_{n+1}(U) \oplus H_{n+1}(V) \longrightarrow H_{n+1}(Y) \longrightarrow$$
$$H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow \cdots$$

By assumption, since $H_k(U \cap V)$ contains a nonzero homology class which is zero in both $H_k(U)$ and $H_k(V)$, we get the short exact sequence:

$$0 \longrightarrow H_{n+1}(Y) \longrightarrow H_n(U \cap V) \longrightarrow 0$$

Therefore $H_{n+1}(Y) = H_{n+1}(U \cup V) \cong H_n(U \cap V)$. Since $H_n(U \cap V)$ contains a nonzero homology class, so does $H_{n+1}(Y)$. Therefore $H_{n+1}(Y) \neq 0$. ☑

Topology Qualifying Exam August 2015

Justify your answers with the necessary proofs, unless otherwise noted. While you should attempt every problem, try your best to answer questions fully. Therefore it might be best to do first the problems you know how to do.
Unless otherwise stated, assume that $\mathbb{R}$, $\mathbb{R}^2$, S^1, and $[a,b)$ are equipped with their standard topology.

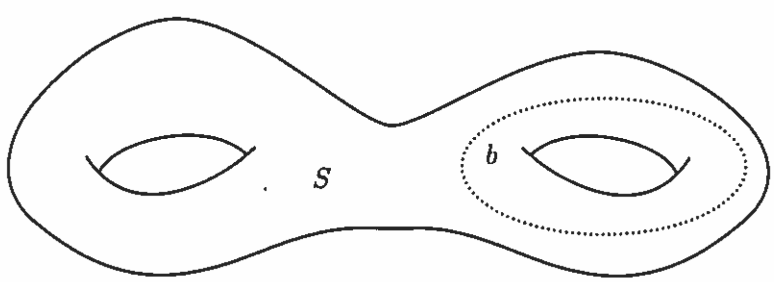
1. (a) Show that a product of two regular spaces is again regular.
   (b) State the definition of a quotient map.
   (c) Show that if X is regular and $A \subset X$ is closed, then X/A is Hausdorff.

2. Let X be a space. Define an equivalence relation on X by setting $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$. The equivalence classes of X with respect to $\sim$ are called the quasicomponents of X.
   (a) Show that each connected component of X lies in a quasicomponent of X.
   (b) State the definition of local connectedness and show that if X is locally connected, then the connected components and the quasicomponents of X are the same.
   (c) Determine the connected components and the quasicomponents of the following subspace of $\mathbb{R}^2$:

$$X = (\mathbb{R} \times \{-1, 1\}) \cup \{(x,y) \mid x^2 + y^2 = (1 - 1/n)^2 \text{ for an integer } n > 1\}$$

3. Recall that a retraction of a space X onto a subspace $A \subset X$ is a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. In the following, let $S^1 \vee S^1 = (S^1 \times \{\theta_1\}) \cup (\{\theta_1\} \times S^1)$ and $p = (\theta_2, \theta_2)$ for $\theta_1 \neq \theta_2 \in S^1$.
   (a) Show that if X is Hausdorff and $f\colon X \to X$ is continuous, then the set of fixed points of f is closed in X.
   (b) Is there a retraction from $[0,2)$ to $[0,1)$?
   (c) Is there a retraction from $S^1 \times S^1$ to $S^1 \vee S^1$?
   (d) Is there a retraction from $(S^1 \times S^1) - \{p\}$ to $S^1 \vee S^1$?

4. Let S be the closed, orientable surface of genus 2, and b the depicted curve. Attach a D^2 to S by gluing its boundary to b by a degree 5 map. Compute the homology groups of the resulting space.



5. Recall the standard CW structure for $\mathbb{RP}^3 = e^0 \cup e^1 \cup e^2 \cup e^3$ where each e^i is glued to the $(i-1)$-skeleton via the antipodal map. Recall that $\mathbb{RP}^2$ is the 2-skeleton of this CW complex. Let A and B be copies of this CW complex, and let $X = A \cup_{\mathbf{RP}^2} B$.
   (a) Find a CW structure for X with 5 total cells.
   (b) Calculate $\pi_1(X, x)$ where the basepoint x is one of the 0-cells in the CW structure.
   (c) Calculate the homology groups of X.
   (d) Calculate the homology groups of (X, A).

6. This question has parts.
   (a) Let X and Y be path-connected, locally path-connected, semilocally simply-connected spaces. Show that if X and Y are homeomorphic, then their universal covering spaces $\widetilde{X}$ and $\widetilde{Y}$ are homeomorphic.
   (b) Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$. Show X and Y have isomorphic homology groups but their universal covers do not.
   (c) Use (a) and (b) to conclude that X and Y are not homeomorphic.

1 a Consider regular spaces X, Y. We know that

Since X, Y are Hausdorff $X \times Y$ is also Hausdorff (Munkres).

Thus, single point sets in $X \times Y$ are closed in $X \times Y$.

Consider $(x,y) \in X \times Y$. Let U be a neighborhood of (x,y).

Choose $U_1 \times U_2$ a basis element contained in U. Choose neighborhoods

V_1, V_2 of x, y respectively such that $\overline{V_1} \subset U_1$, $\overline{V_2} \subset U_2$.

(Note, we can do this because X, Y are regular) ~~clo~~ the case

that $U_1 = X$ & $U_2 = Y \longrightarrow$ let $V_1 = X$, $V_2 = Y$. Let

$V = V_1 \times V_2$, which is a neighborhood of (x,y) in $X \times Y$.

Since $\overline{V} = \overline{V_1} \times \overline{V_2}$ it follows that

$$\overline{V} \subset U_1 \times U_2 \subset U$$

Therefore, by Munkres Lemma 31.1, $X \times Y$ is regular.

b. A quotient map is a surjective map $q: X \to Y$ such that $U \subset Y$ is open iff $q^{-1}(U) \subset X$ is open.

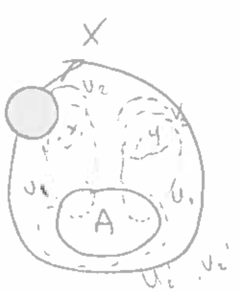
c. Suppose X is regular and $A \subset X$ is closed.

To show X/A is Hausdorff, first let $q: X \to X/A$ be the quotient map that sends all of A to a single point a. We want to show that $\forall\ x, y \in X/A$ $\exists$ open sets $U \ni x$, $V \ni y$ s.t. $U \cap V = \emptyset$. Note, since q is a quotient map, U, V are open iff $q^{-1}(U), q^{-1}(V)$ are open. There are two nontrivial cases:

1. $x, y \neq a$; so $q^{-1}(x), q^{-1}(y) \in X \setminus A$
2. $x = a$, $y \neq a$, so $q^{-1}(x) \in A$, $q^{-1}(y) \in X \setminus A$

1. Since X is Hausdorff we know $\exists\ U_1, V_1$ s.t. $x \in U_1$, $y \in V_1$, $U_1 \cap V_1 = \phi$

Since X is regular and A is closed, $\exists\ U_2, U_2'$ s.t. $x \in U_2$ $A \subset U_2'$ $U_2 \cap U_2' = \phi$ and V_2, V_2' s.t. $y \in V_2$, $A \subset V_2'$, $V_2 \cap V_2' = \phi$. Consider $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Since q is 1-1 on $X \setminus A$ (and onto by def of quotient map) and since $U, V \subset X \setminus A$ we know that $q(U), q(V)$ are open. furthermore $q(U) \cap q(V) = \phi$ because $U \cap V = \emptyset$.



2. Suppose $x = a$, $y \neq a$. Then $q^{-1}(x) \in A$. $q^{-1}(y) \in X \setminus A$.

Again appealing to the regularity of X we know $\exists$ U, V s.t. $q^{-1}(y) \in U$, $A \subset V$ & $U \cap V = \emptyset$. Since q is 1-1 on $X \setminus A$ and $U \subset X \setminus A$ we know that $q(U)$ is open and contains y. Furthermore $V \setminus A$ is 1-1, and $q(V \setminus A) \cup a = q(V)$.

Therefore $q(V)$ is open, contains $x = a$. Lastly, since $U \cap V = \emptyset$ we know that $q(U) \cap q(V) = \emptyset$.

Thus for any $x, y \in X/A$ $\exists$ open $q(U) \ni x$, $q(V) \ni y$ s.t. $q(U) \cap q(V) = \emptyset$. Therfore X/A is Hausdorff

2a. Let X be a space and let Q be a quasicompant of X. That is, $Q = [x]$ for some $x \in X$ s.t. $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets s.t. $x \in A$ & $y \in B$

Let C be a connected component of X. We want to show that $C \subseteq Q$ for $x \in C$. (That is, consider the connected component of X containing x). Clearly $x \in C \longrightarrow x \in [x]$. We want to show that $\forall y \in C$, $y \sim x \Longrightarrow y \in Q$.

Because C is a connected component we know that there is no separation of C. That is, suppose $\exists$ open $A, B \subset C$ s.t. $x \in A$, $y \in B$, $A \cap B = \emptyset$. However this contradicts the connectedness of C.

Therefore, $x \sim y$. Thus $y \in Q$ and $C \subset Q$.

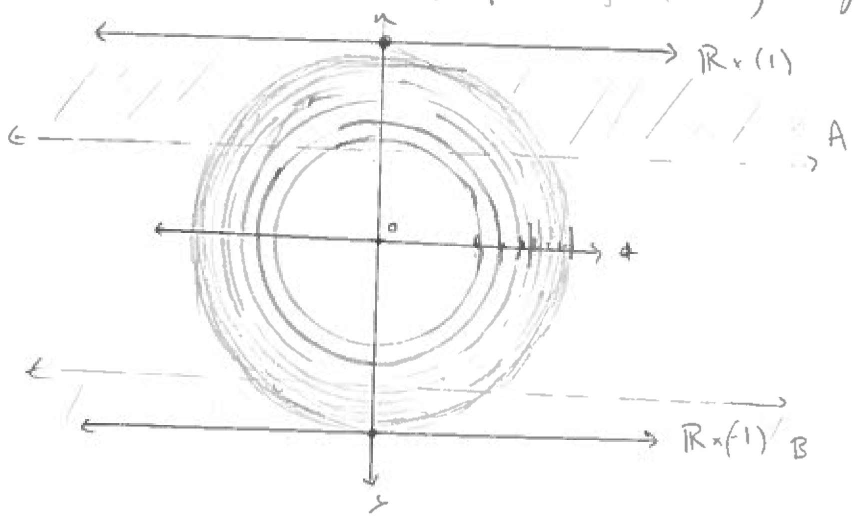
Therefore each connected component of X lies in a quasicomponent of X

b. A space X is locally connected if $\forall$ $x \in X$, $\forall$ open sets U s.t. $x \in U$ $\exists$ a connected $V \ni x$ s.t. $V \subset U$

pf. Suppose X is locally connected. Let Q be a quasicomponent and C be a component s.t $C \subseteq Q$. We want to show that $C = Q$. Suppose, to the contrary, that $C \subsetneq Q$. Recall that components of locally connected sets are open. Thus C is open. Since $Q \setminus C$ is composed of components, i.e. $Q \setminus C = \bigcup_{\alpha \in A} C_\alpha$, then $Q \setminus C$ must also be open (b/c unions of open sets are open). Lastly $Q \setminus C \cap C = \phi$ by definition. Suppose then that $x \in C$, $y \in Q \setminus C$. Since $x, y \in Q$, $x \sim y$. However there is a separation $Q = C \cup Q \setminus C$ of Q into disjoint open sets, which contradicts the fact that $x \sim y$. Thus, by contradiction, $C = Q$.

c. Consider the following subspace of $\mathbb{R}^2$:

$$X = \{\mathbb{R} \times (-1,1)\} \cup \{(x,y) \mid x^2+y^2 = (1-1/n)^2 \text{ for } n>1,\ n \in \mathbb{Z}\}$$



Connected components are:

- $\mathbb{R} \times (1)$
- $\mathbb{R} \times (-1)$
- $x^2+y^2 = (1-1/n)^2$ for $n = 1, 2, 3, \dots$

Quasicomponents

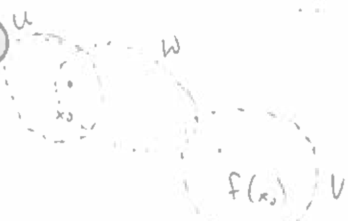
- $x^2+y^2 = (1-1/n)^2$ (Clear separation between these, for $0 < \varepsilon < \frac{1}{n(n-1)}$

  Consider $A = B(0, 1-\frac{1}{n}+\varepsilon)$ $B = D^2 \setminus B(0, 1-\frac{1}{n+1}-\varepsilon)$) ← unit disk

- $\mathbb{R} \times \{-1, 1\}$: Clearly $[(0,1)]_c = \mathbb{R} \times (1)$, $[(0,-1)]_c = \mathbb{R} \times (-1)$ so it suffices to show that $(0,1) \sim (0,-1)$. Suppose this was not the case. Then $\exists$ a separation $A \cup B$ s.t. A, B are open. WLOG let $(0,1) \in A$. Since A is open A contains infinitly many points of circles $x^2+y^2 = (1-1/n)^2$. But since those circles are connected A must contain the entire circle. But then $\nexists$ open B s.t. $[(0,-1)]_c \subset B$, $B \cap A = \emptyset$. Thus $[(0,1)]_Q = [(0,-1)]_Q \longrightarrow \mathbb{R} \times \{-1, 1\}$ is a quasi component

3a Suppose X is Hausdorff and f is continuous. Let $A = \{x \mid x = f(x)\}$ be the set of fixed points. Then $X \setminus A = \{x \mid x \neq f(x)\}$. It suffices to show that $X \setminus A$ is open. Let $x_0 \in X \setminus A$. Then $f(x_0) \in X \setminus A$.

Since X is Hausdorff $\exists\ U \ni x_0,\ V \ni f(x_0)$ s.t. $U \cap V = \emptyset$. Since f is continuous it pulls ~~back~~ open sets onto open sets. Let $f^{-1}(V) = W$. Note $x_0 \in W$. Further $U \cap W$ is open and contains x_0. We must show that $U \cap W \subset X \setminus A$. Consider some $x \neq x_0 \in U \cap W$. Then $f(x) \in V$. Thus $f(x) \notin U$ $f(x) \notin U \cap W$. Thus $X \setminus A$ is open.



Therefore A, the set of fixed points is closed. ∎

b. No.

pf. Suppose there were a retraction $r: [0,2) \longrightarrow [0,1)$. By definition r is continuous. Consider x_n that converge upward to 1, $\{x_n\} \nearrow 1$. Then $\{r(x_n)\} \nearrow r(1)$, by continuity. But $r(x_n) = x_n$ b/c $x_n \in [0,1)$ and r is a retraction onto $[0,1)$. Thus $\{x_n\} \nearrow r(1)$. By uniqueness of limits $r(1) = 1$. However $1 \notin [0,1)$ Σ

Thus $[0,2]$ doesn't retract onto $[0,1)$. ∎

C. No.

pf. Suppose there is a retraction $r: S^1 \times S^1 \longrightarrow S^1 \vee S^1$.

Then the induced homomorphism: $r_*: \pi_1(S^1 \times S^1) \longrightarrow \pi_1(S^1 \vee S^1)$ is surjective. Note $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$ which has generators a, b. Further $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. = The free group on 2 elts. Note, $\mathbb{Z}^2$ is abelian, so $ab = ba$.

Since $r_*: \pi_1(S^1 \times S^1) \longrightarrow \pi_1(S^1 \vee S^1)$, $r_*(a) = c =$ a generator of $\pi_1(S^1 \vee S^1)$. Likewise $r_*(b) = d =$ a second generator of $\langle c, d \rangle$

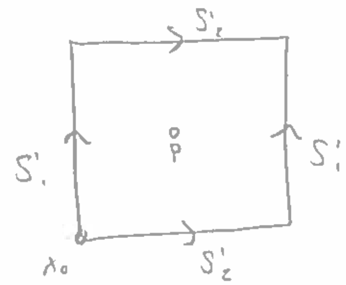
Furthermore since $aba^{-1}b^{-1} = id_{\mathbb{Z}^2}$ $r_*(aba^{-1}b^{-1}) = id_{\mathbb{Z} * \mathbb{Z}}$

However $r_*(aba^{-1}b^{-1}) = r_*(a)\, r_*(b)\, (r_*(a))^{-1} (r_*(b))^{-1} = id_{\mathbb{Z} * \mathbb{Z}}$.

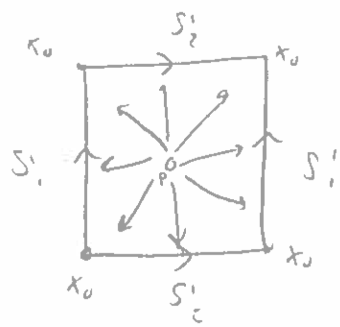
which would imply that $\mathbb{Z} * \mathbb{Z}$ is commutative, which it is not. Therefore, $S^1 \times S^1$ doesn't retract onto $S^1 \vee S^1$.

3d Yes

Consider $S^1 \times S^1 - \{p\}$ ~~given~~ shown by the following:



Note this deformation retracts to its boundary:



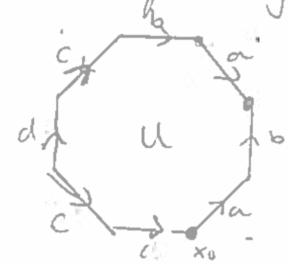
This boundary can be represented by:

Therefore $S^1 \times S^1 \setminus \{p\}$ deformation retracts (and thus retracts) to

$S^1 \vee S^1$ ☑

4. Let S be the closed orientable surface of genus 2, and b the depicted curve.

Note, S has the following cell complex.



0-cells : x_0

1-cells : a, b, c, d

2-cells : u.

Note that our 1-cell b corresponds to the curve b above.
Attaching a disk D^2 to S by gluing its boundary to b by a deg 5 map, we simply get an additional 2-cell, yielding the cell complex of X

0-cells: x_0 1-cell: a, b, c, d. 2-cells: u, D^2.

Thus the cellular chain complex is.

ker $d_0 = \mathbb{Z}$

$$0 \xrightarrow{d_3} \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since D^2 is attached with a 5-degree map and u is attached with a 0-degree map we see that $d_2 : \mathbb{Z}^2 \to \mathbb{Z}^4$ is:

$$d_2 = \begin{array}{c} \\ a \\ b \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cc} u & D^2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \longrightarrow \begin{array}{l} \text{im} = 5\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 5b \\ \text{ker} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \mathbb{Z} \end{array}$$

$d_1 = [0\ 0\ 0\ 0]$ ker $d_1 = \mathbb{Z}^4$ im $d_1 = 0$

↳ zero map b/c 1 zero cell

Likewise since all points of

Using these we see that

$$H_0(X) = \mathbb{Z} \quad \text{(b/c } X \text{ is one connected component)}$$

$$H_1(X) = \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}^4 / 5\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{\left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\rangle}{5\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} = \mathbb{Z}^3 \oplus \mathbb{Z}/5\mathbb{Z}$$

$$H_2(X) = \ker d_2 / \operatorname{im} d_3 = \mathbb{Z}/0 = \mathbb{Z}$$

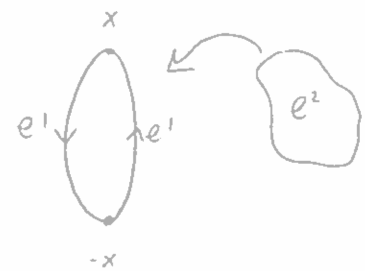
$$H_k(X) = 0 \quad \forall \ k \geq 3.$$

5 a. $A = e^0 \cup e^1 \cup e^2 \cup e^3$ $\quad B = f^0 \cup f^1 \cup f^2 \cup f^3$

$X = A \cup_{\mathbb{RP}^2} B = e^0 \cup e^1 \cup e^2 \cup e^3 \cup f^3$

$\mathbb{RP}^2 = e^0 \cup e^1 \cup e^2 = f^0 \cup f^1 \cup f^2$

b. $\pi_1(X, x) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

$X =$ 

$+ \; e^3, f^3 = \mathbb{RP}^2 \cup e^3 \cup f^3$

adding any number of 3 cells does not effect fund gp.
↳ by Van Kampen adding any cell ≥ 2 doesn't do anything to fund gp

c. $H_0(X) = \mathbb{Z}$ because X is path-connected

$H_1(X) = \mathbb{Z}_2$, the abelianization of $\pi_1(X) = \mathbb{Z}_2$

$H_2(X) = {}^{\ker d_2}/_{\operatorname{im} d_3} = {}^{0}/_{0} = 0$

$H_3(X) = {}^{\ker d_3}/_{\operatorname{im} d_4} = {}^{\mathbb{Z}^2}/_{0} = \mathbb{Z}^2$

Cellular chain

$$\xrightarrow{d_4} \mathbb{Z}^2 \xrightarrow[{[0,0]}]{d_3} \mathbb{Z} \xrightarrow[{[2]}]{d_2} \mathbb{Z} \xrightarrow[{[0]}]{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

b/c we are gluing using the antipodal map w/ deg = 0 for odd cells
deg = 2 for even cells.

5c. Defining A, B as above we know we satisfy the requirements of excision. Thus

$$H_n(X,A) \cong H_n(X/A) \cong \underbrace{H_n\left(\overset{\text{3-cell}}{B}/\underset{\text{2-skeleton}}{A\cap B}\right)}_{\text{3-cell attached at a pt}} \cong H_n(S^3)$$

$$H_n(S^3) = 0 \quad \forall n \neq 3$$

$$H_3(S^3) = \mathbb{Z}$$

6a. Let X, Y be path-connected, locally path-connected, semilocally simply-connected spaces. Given this, we know that the universal covers $\tilde{X}, \tilde{Y}$ are simply connected. Suppose X and Y are homeomorphic where f is said homeomorphism

$$
\begin{tikzcd}
\tilde{X} \arrow[d, "p_1"'] & \tilde{Y} \arrow[d, "p_2"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

We will first show that $(f \circ p_1)$ is a covering space. For each point $y \in Y$ there is an open neighborhood U of y in Y such that $(f \circ p_1)^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by $f \circ p_1$. (Some work to be done here.)

~~Thus $(f \circ p_1)_* = (p_2)_*$~~. Since $\tilde{X}, \tilde{Y}$ are simply connected $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = \{e\}$. Thus $(f \circ p_1)_* \cong (p_{2*})$. Since the covering spaces have isomorphic fundamental groups

$\exists$ homeomorphism $h : \tilde{X} \longrightarrow \tilde{Y}$. $\square$

(Not best quality).

b. Let $X = S^1 \times S^1 = T^2$ and $Y = S^1 \vee S^1 \vee S^2$

$$\left.\begin{aligned} H_0(T) &= \mathbb{Z} = H_0(S^1 \vee S^1 \vee S^2) \\ H_1(T) &= \mathbb{Z} \times \mathbb{Z} = H_1(S^1 \vee S^1 \vee S^2) \\ H_2(T) &= \mathbb{Z} = H_2(S^1 \vee S^1 \vee S^2). \end{aligned}\right\}$$ Needs more justification

Note that $\mathbb{R}^2$ is the universal cover of T. Recall that $\mathbb{R}^2$ contracts down to a point so

$$H_*(\tilde{T}) = 0$$

However, the universal cover of $S^1 \vee S^1 \vee S^2$ is the infinite graph with a sphere at every intersection, which contracts down to $(S^2 \vee S^2 \vee \ldots)$, or an infinit bouquet of S^2's. Therefore

$$H_2(\widetilde{S^1 \vee S^1 \vee S^2}) = \mathbb{Z}^\infty \neq 0 = H_2(\tilde{T}).$$

Therefore the homology groups of the universal covers are not isomorphic

c. If there was such a homeomorphism between X & Y then there would be a homeomorphism between $\tilde{X}$ and $\tilde{Y}$. However since the homology groups of $\tilde{X}, \tilde{Y}$ are not isomorphic they are not homeomorphic. Therefore X and Y are not homeomorphic

January 2015 Topology Qualifying Exam

Please read the following instructions carefully.

Vocabulary: "map"= continuous function; "space"= topological space; "R"= real numbers.
Reasoning must be given for *ALL* answers.
There are six problems.

1 For any subset C of a metric space (X, d) define the distance from $x \in X$ to C to be

$$d(x, C) = \inf\{d(x, y) | y \in C\}$$

a) Show that $\{x \in X | d(x, C) = 0\} = \bar{C}$, the closure of C.
b) Assume $d(x, C) \leq d(x, y) + d(y, C)$ for all $x, y \in X$. Show that the function $X \xrightarrow{f} R$ given by $f(x) = d(x, C)$ is continuous.
c) Prove the inequality given in b).

2 Let X be a space, Y a set, and $X \xrightarrow{f} Y$ a function. Recall that the collection of all subsets U of Y such that its inverse image in X is open, is a topology on Y (called the quotient topology induced by the quotient map f).

a) Now assume Y is a space and $X \xrightarrow{f} Y$ an onto map. Show that f is a quotient map if there exists a map $Y \xrightarrow{g} X$ such that the composition $Y \xrightarrow{fg} Y$ is the identity map.
b) Assuming a) is true, show that the projection map $X_1 \times X_2 \xrightarrow{h} X_1, h(x_1, x_2) = x_1$, is a quotient map.
c) Show that there is a quotient map $(0,1) \xrightarrow{f} [0,1]$, and that there is no quotient map $[0,1] \xrightarrow{f} (0,1)$.

3 Let X be a space whose one-point subsets are closed. Recall X is regular if for each $x \in X$ and each closed subset $C \sqsubset X$ there are disjoint open sets U, V with $x \in U$ and $C \sqsubset V$. Further X is normal if whenever C, D are disjoint closed sets there are disjoint open sets U, V with $C \sqsubset U$ and $D \sqsubset V$.

a) Show that a compact, Hausdorff space Y is regular.
b) Show that a compact, Hausdorff space Y is normal.

4 The 2-dimensional torus $S^1 \times S^1$ can be viewed as the quotient space of $S^1 \times D^1$ via the identification $(z, -1) \sim (z, 1)$ for all $z \in S^1$ and the solid 2-dimensional torus $D^2 \times S^1$ can be similarly viewed as the quotient space of $D^2 \times D^1$ via the identification $(z, -1) \sim (z, 1)$ for all $z \in D^2 = \{z \in C | |z| \leq 1\}$. The Möbius band can be viewed as the quotient space of $D^1 \times D^1$ via the identification $(z, -1) \sim (-z, 1)$

for all $z \in D^1 = [-1, 1]$. Give an embedding of the Möbius band in the solid torus such that the boundary circle of the Möbius band is embedded in the boundary of the solid torus (which is the torus!). An acceptable answer would have the form of a verbal description with a picture of the spaces involved.

5 a) Show that the fundamental group of the figure-eight (the one-point union of two circles) is non-abelian by finding a suitable covering space and using lifting.
b) The double torus is the surface obtained by taking two copies of the torus, deleting a small open disc from each of them, and pasting the remaining pieces together along their edges. Describe a retraction of the double torus onto the figure-eight. You may use pictures. Briefly explain why it is a retraction.
c) Use a) and b) to show that the double torus and the torus are not homeomorphic.

6 a) What is the fundamental group of real projective n-space RP^n for $n > 1$? Give a brief discussion why.
b) Show that every continuous map $RP^n \xrightarrow{f} S^1$ must be nullhomotopic if $n > 1$.

Jan 2015

1. Let $C \subset (X, d)$ and define $d(x, C) = \inf\{d(x,y) \mid y \in C\}$

a. To show that $\{x \in X \mid d(x,C) = 0\} = \bar{C}$, let us first investigate $\bar{C}$. By def. $x \in \bar{C}$ iff $B(x, r) \cap A \neq \emptyset \ \forall \ r > 0$ (Theorem 17.5, Munkres). Proceeding, note $d(x, C) = 0$ iff $\inf\{d(x,y) \mid y \in C\} = 0$ iff $\forall \ \varepsilon > 0 \ B(x, \varepsilon) \cap C \neq \emptyset$ iff $x \in \bar{C}$. Thus $d(x, C) = 0$ iff $x \in \bar{C}$. Therefore

$$\{x \in X \mid d(x.C) = 0\} = \bar{C}. \ \square$$

b. Assume $d(x, C) \leq d(x,y) + d(y, C) \ \forall \ x, y \in X$. Let $f : X \to Y$ s.t. $f(x) = d(x, C)$

Let $\varepsilon > 0$, $\delta > 0$ be given. Let $\delta = \varepsilon$. Then for x, y such that $d(x,y) < \delta = \varepsilon$:

dist. is positive definite.

$$|f(x) - f(y)| = |d(x, C) - d(y, C)| \leq |d(x, C)| - |d(y, C)| = d(x, C) - d(y, C)$$

$$\leq d(x,y) < \varepsilon$$

Therefore $f : X \to Y$ via $f(x) = d(x, C)$ is continuous. $\square$

c. Note that since X is a metric space, $\forall\ x, y, z \in X$

$$d(x,z) \leq d(x,y) + d(y,z)$$

Suppose $z \in C \subset X$. It follows that, $\forall\ z \in C$,

$$d(x,C) \leq d(x,z) \leq d(x,y) + d(y,z)$$

Then, consider $z_n \in C$, $\forall\ z_n \in C$:

$$d(x,C) \leq d(x,y) + d(y,z_n) \longrightarrow \lim_{n\to\infty} d(x,C) \leq \lim_{n\to\infty} \left(d(x,y) + d(y,z_n)\right)$$

$$= d(x,y) + d(y,C)$$

Therefore $d(x,C) \leq d(x,y) + d(y,C)$. ☑

2a. Let X be a space, Y a space and $f: X \longrightarrow Y$ an onto map.

Suppose $\exists$ $g: Y \to X$ a continuous function s.t. $fg: Y \longrightarrow Y$ is the identity map. We want to show that f is a quotient map

So we must show that $U \subset Y$ is open iff $f^{-1}(U) \subset X$ is open

($\Longrightarrow$) Suppose $U \subset Y$ is open. Since f is continuous $f^{-1}(U)$ is also open.

($\Longleftarrow$) Suppose $f^{-1}(U) \subset X$ is open. Since g is continuous, $g^{-1}(f^{-1}(U)) \subset Y$ is open. However $g^{-1} f^{-1} = (fg)^{-1} = (id)^{-1} = id$. Therefore $g^{-1}(f^{-1}(U)) = U \subset Y$ is open.

Thus $U \subset Y$ is open iff $f^{-1}(U)$ is open. Therefore f is a quotient map. $\square$

b. Consider the projection map $h: X_1 \times X_2 \longrightarrow X_1$ via $h(x_1, x_2) = x_1$. Clearly h is an onto map. Consider the inclusion $i_{x_2}: X_1 \hookrightarrow X_1 \times X_2$ via $i_{x_2}(x_1) = (x_1, x_2)$. Note $\forall x_2 \in X_2$ $h i_{x_2}: Y \longrightarrow Y$ s.t. $h(i_{x_2}(x_1)) = h(x_1, x_2) = x_1$

Thus $h \circ i_{x_2} = id$. Therefore h is a quotient map.

2c. Consider $f: (0,1) \longrightarrow [0,1]$ via

$$f(x) = \begin{cases} 0 & 0 < x < 1/3 \\ 3x-1 & 1/3 \le x \le 2/3 \\ 1 & \frac{2}{3} < x < 1 \end{cases}$$

Clearly f is continuous. $f(0,1) = [0,1]$ so f is surjective.

Lastly $U \subset [0,1]$ is open iff $f^{-1}(U)$ is open.

(Note $(\Longrightarrow)$ is obvious by continuity One should also note that any open set gets mapped to an open set.).

Now suppose there were some onto map

$$h: [0,1] \longrightarrow (0,1)$$

Hower this cannot be the case because $h([0,1])$ is compact and $(0,1)$ is not compact. So $h([0,1]) \subsetneq (0,1)$

Therefore h cannot be surjective. $\boxed{\checkmark}$

3a. Let Y be a space whose one-point subsets are closed. Further, let Y be a compact Hausdorff space. Because Y is Hausdorff, $\forall\, x, y \in Y$ $\exists\, U_x, U_y$ s.t. $U_x \cap U_y = \emptyset$.

Let $C \subset Y$ be closed. Then for $x \notin C$, $\forall\, y \in C$ $\exists$ U_{x_y}, U_y s.t. $U_{x_y} \cap U_y = \emptyset$. Since Y is compact x, C are closed, then x, C are compact. So x, C have a finite subcover. Let $V_{x_y} = \{U^i_{x_y}\}_{i=1}^n$ where y's are s.t.: $V_y = \{U^i_y\}_{i=1}^n$. Then $V_x = \bigcap_{i=1}^n U^i_{x_y}$ is open because finite intersections are open. Further, $V_x \cap U_y = \emptyset$ $\forall y$.

That is, once we've chosen a finite subcover for C, let the corresponding U^i_y form a finite subcover of the pt. x. Intersecting those we get the smallest open set around x possible. Further, $V_x \cap \bigcup_{i=1}^n U^i_y = \emptyset$.

Thus, $x \in V_y$ open, $C \subset \bigcup_{i=1}^n U^i_y$ open, $V_y \cap \bigcup_{i=1}^n U^i_y = \emptyset$. Thus Y is regular. $\square$

3b. Let Y be a compact Hausdorff space. Let C, D be closed subsets of Y. Note, C, D are compact because they are closed subsets of a compact set. Since Y is Hausdorff

$\forall x \in C, \forall y \in D \quad \exists \quad U_x \ni x, \quad U_y \ni D$ s.t. $U_x \cap U_y = \emptyset$

Note $\{U_x\}_{x \in C}$ and $\{U_y\}_{y \in D}$ form open covers of C, D

Since C, D are compact Let $\{U_x^i\}_{i=1}^n$, $\{U_y^j\}_{j=1}^m$ be finite subcovers of C, D.

* Need some way to guarantee the following don't intersect*

Then $V_x = \bigcup_{i=1}^n U_x^i$ and $V_y = \bigcup_{j=1}^m U_y^j$ Then

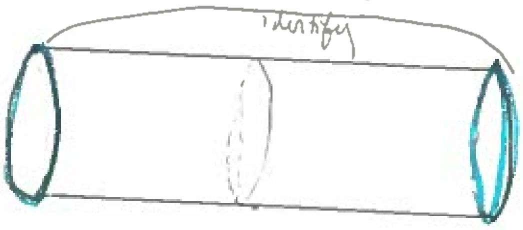
$C \subset V_x$, $D \subset V_y$, $V_x \cap V_y = \emptyset$ because

$U_x^i \cap U_y^j = \emptyset \quad \forall \; i, j$.

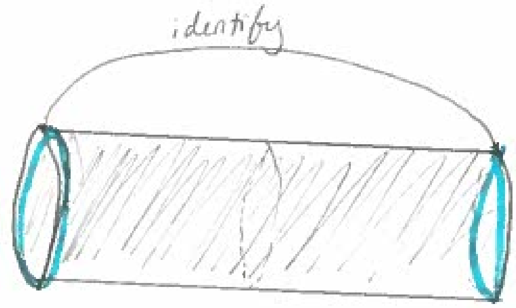
→ Technical point to work on later.

idea: ensure sets that form open cover don't intersect any other sets among intersections.

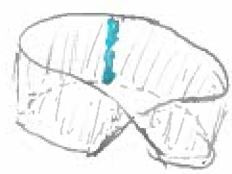
4. An embedding of the Mobius band into the solid Torus is as follows:



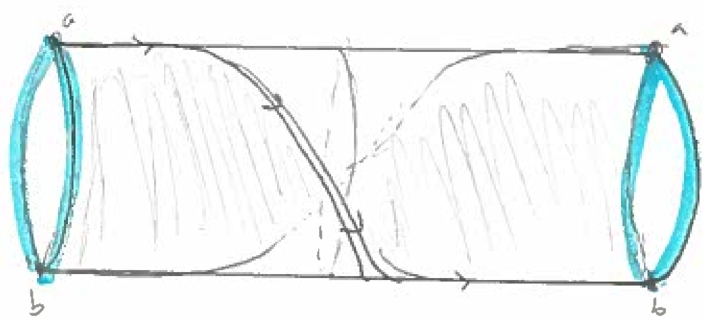
$T = S^1 \times S^1$



$ST = D^2 \times S^1$

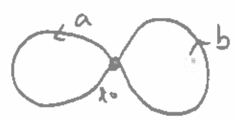


$M = D^1 \times D^1$



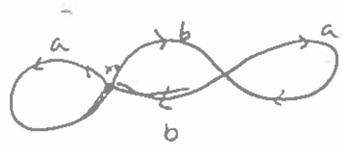
Starting at the top left we take our circle across the top and down the front of the cylinder, across the bottom. Then b gets sent back around to the other side via identification. Then its brought across the back of the cylinder to the top. Lastly unification brings us back to the start.

5a. Consider $S^1 \vee S^1 = X$



$\pi_1(X) = \langle a, b \rangle$

Consider the cover



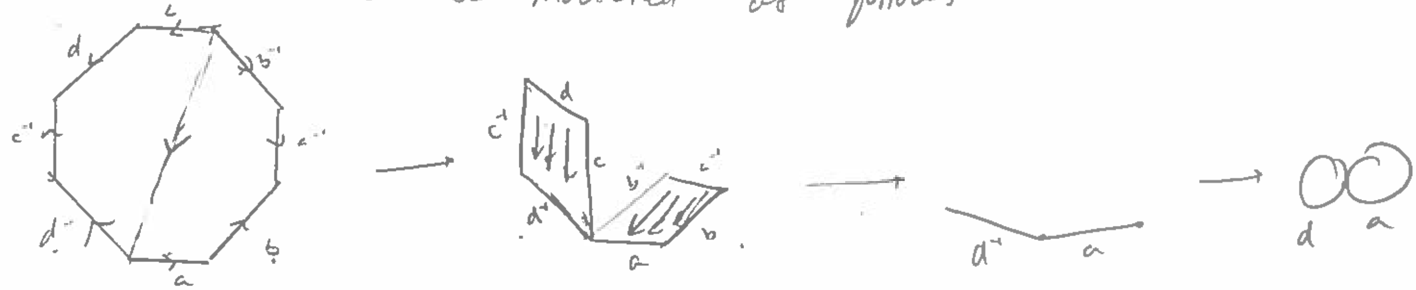
$\langle a, b^2, bab^{-1} \rangle$

Note this representation is not abelian (if it was it'd be $= \pi_1(X)$)

$\pi_1(\tilde{X})$ is not abelian $\underline{\text{so}}$ $\pi_1(X)$ is not abelian.

vague

b. The double Torus can be modeled as follows



c. Suppose T was homeomorphic to $T^{(2)}$, the double Torus.

Recall $\pi_1(T) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$

This group is abelian.

Since $T^{(2)}$ retracts to $S^1 \vee S^1$ via r,

$$r_* : \pi_1(T^{(2)}) \longrightarrow \pi_1(S^1 \vee S^1) \text{ is surjective.}$$

Therefore $\pi_1(T^{(2)})$ is not abelian.

Thus $\pi_1(T) \not\cong \pi_1(T^{(2)})$

Therefor $T \approx T^{(2)}$. $\square$

6a. $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n>1$.

The universal cover of $\mathbb{R}P^n$ is S^n (we get $\mathbb{R}P^n$ using the antipodal map.) The universal cover is two sheeted.

Therefore $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$

6b Suppose $f: \mathbb{R}P^n \longrightarrow S^1$ is a cts map. Then the induced homomorphism

$$\begin{array}{ccc} f_* : \pi_1(\mathbb{R}P^n) & \longrightarrow & \pi_1(S^1) \\ \cong & & \cong \\ \mathbb{Z}_2 & & \mathbb{Z} \end{array}$$

must be trivial. By the Lifting Criterion. f lifts to $\tilde{f} : \mathbb{R}P^2 \longrightarrow \mathbb{R}$ s.t. $p \circ \tilde{f} = f$, where $p : \mathbb{R} \longrightarrow S^1$ is the standard universal covering

Since $\mathbb{R}$ is contractible $\tilde{f}$ must be homotopic (via H) to a constant map. Hence $p \circ H$ is a homotopy from f to a constant map.

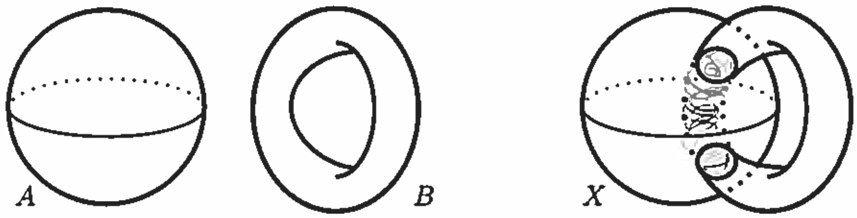
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{R}P^n \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Incomplete / Scratch quality work

Topology Qualifying Exam January 2014

> Justify your answers with the necessary proofs, unless otherwise noted. While you should attempt every problem, try your best to answer questions fully. Therefore it might be best to do first the problems you know how to do.

1. Let $A \cong S^2$ and $B \cong T^2$, and form $X = A \cup B$ as shown:



   (a) Why doesn't Seifert-van Kampen apply to the decomposition $X = A \cup B$?
   (b) Pick a basepoint and *carefully* calculate $\pi_1(X)$.
   (c) Calculate $H_*(X)$.

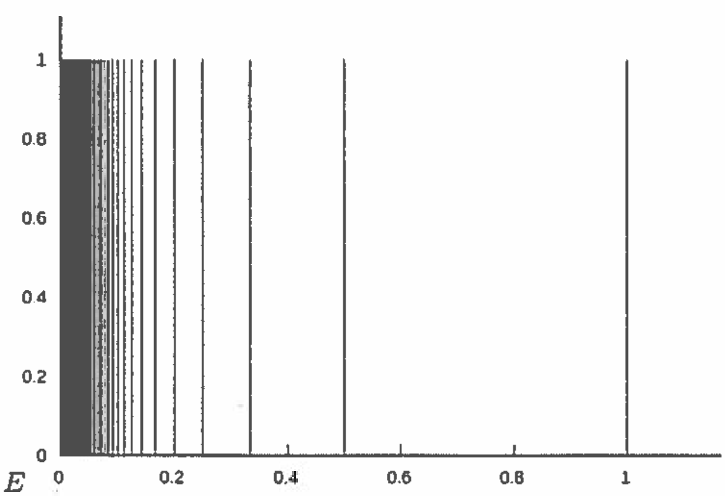
2. Let Y denote the CW complex gotten by attaching to a circle S^1 two discs, with attaching maps of degree 6 and degree 8, respectively.
   (a) Calculate $H_*(Y, S^1)$.
   (b) Calculate $H_*(Y; \mathbb{Z}_3)$. → ? pg 74 in notes

3. Let C denote the 3-fold dunce cap (a circle with a two-cell attached by a degree 3 map), and let $D = S^1 \times C$. Find all covering spaces of D up to equivalence. How many are there? Which ones are regular? Justify why you found all of them. → Find gp, sbgp ⊲ subgp ↳ all of them.

4. Let $K = \{1, 1/2, 1/3, \ldots\}$ and E denote the infinite comb

$$E = [0,1] \times 0 \cup K \times [0,1] \cup 0 \times [0,1]$$

topologized as a subspace of $\mathbb{R}^2$.



E

(a) Show E is a connected but not locally connected subspace of $\mathbb{R}^2$.

(b) Let $F = 0 \times [0,1] \subset E$. Show the map $f : E \to F$ given by $f(x,y) = (0,y)$ is a retract of E onto F.

(c) Briefly argue why F is not a deformation retract of E.

5. Consider the orthogonal group $O(n)$ consisting of all $n \times n$ orthogonal matrices, i.e., $n \times n$ matrices whose columns $v_1, \ldots, v_n$ form an orthonormal basis for $\mathbb{R}^n$. Give the set S of all $n \times n$ real matrices the topology induced by a bijection of it onto $\mathbb{R}^{n^2}$ that identifies the n^2 entries of a matrix with a point in $\mathbb{R}^{n^2}$. Then give $O(n)$ the subspace topology from S.

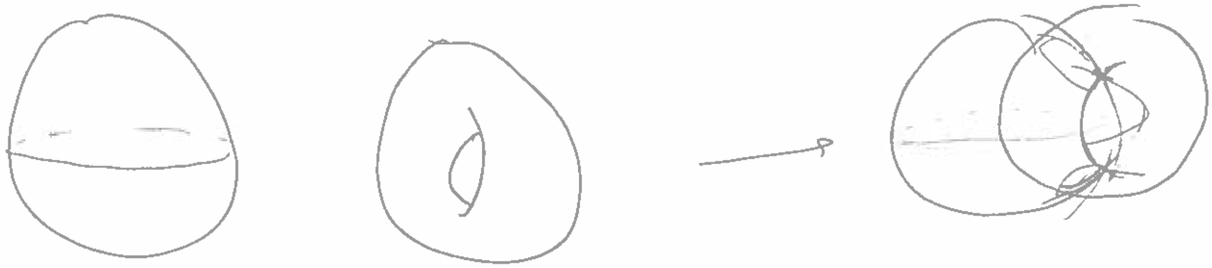
   (a) Show that $O(n)$ is a closed subset of S by considering the dot product $v_i \cdot v_j$ of the columns of matrices in $O(n)$ as functions $S \to \mathbb{R}$.

   (b) Show $O(n)$ is compact.

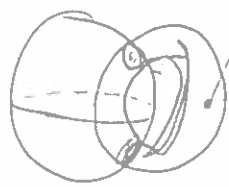
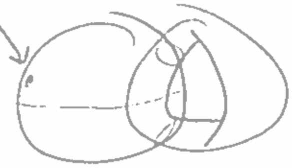
6. Let G be the subset of $\mathbb{R}^2$ which is the union of the line segments L_n from $(0,0)$ to $(1,1/n)$ for $n = 1, 2, \ldots$, together with the limiting segment L_∞ from $(0,0)$ to $(1,0)$. Define a topology $\mathcal{T}$ on G by saying a set $O \subset G$ is open if $O \cap L_n$ is open in L_n for all $n \leq \infty$. Here L_n is given the subspace topology from $\mathbb{R}^2$). Show this topology on G is normal but is not defined by any metric. Hint: Given a metric on G, find a sequence of points $x_n \in L_n$, $n = 1, 2, \ldots$, converging to $(0,0)$ in the metric topology but not in $\mathcal{T}$.

January 2014

1. $A \cong S^2$, $B \cong T^2$ $X = A \cup B$



a. SVK doesn't ~~work~~ apply to decomp $X = A \cup B$ because $A \cap B$ is not path connected

b. Let $U =$  remove a pt. $V =$ 

Then U def. retracts to A , V def retracts to B

$\pi_1(U) \cong \pi_1(A) \cong \pi_1(S^2) \cong 0$ $\quad\quad$ $\pi_1(V) \cong \pi_1(B) \cong \pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

Note $U \cap V$ deformation retracts to a cylinder, which is the same as an annulus

$\pi_1(U \cap V) \cong \pi_1(\text{Annulus}) \cong \pi_1(S^1) \cong \mathbb{Z}$

$$\pi_1(X) = \frac{\pi_1(U) * \pi_1(V)}{\boxed{\pi_1(U \cap V)}} = \frac{0 * \mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}} \overset{?}{=} \mathbb{Z}$$

Normal subgp gen by $\psi_1(k)\psi_1(k)^{-1}$ $k \in \pi_1(U \cap V)$ Commutator

$$= \frac{\pi_1(U) * \pi_1(V)}{N}$$

$N =$ normal subgroup generated by $i_1(\omega)\, i_2(\omega)^{-1}$ for $\omega \in \pi_1(U \cap V)$.

$i_2 : \pi_1(U \cap V) \hookrightarrow \pi_1(V)$

$i_1 : \pi_1(U \cap V) \hookrightarrow \pi_1(U)$

1. Calculate $H_*(X)$

Note $\text{int}(A) = A$, $\text{int}(B) = B$

$X = A \cup B = \text{int}\, A \cup \text{int}\, B \longrightarrow$ can use Mayer Vietoris

$\exists$ LES

$$\begin{tikzcd} & & H_{n+1}(X) \arrow[dll, bend right] \\ H_n(A \cap B) \arrow[r] & H_n(A) \oplus H_n(B) \arrow[r] & H_n(X) \arrow[dll, bend right] \\ H_{n-1}(A \cap B) \arrow[r] & \cdots & \end{tikzcd}$$

$A = S^2 \longrightarrow H_0(S^2) = 0 \quad H_1(S^2) = 0 \quad H_2(S^2) = \mathbb{Z} \quad H_3(S^2) = 0$

$B = T^2 \longrightarrow H_0(T) = \mathbb{Z} \quad H_1(T) = \mathbb{Z}^2 \quad H_2(T) = \mathbb{Z} \quad H_3(T) = 0$

$A \cap B =$ Annulus $\underset{\text{retract}}{\overset{\text{def}}{\rightsquigarrow}} S^1 \longrightarrow H_0(S^1) = 0 \quad H_1(S^1) = \mathbb{Z} \quad H_2(S^1) = 0 \quad H_3(S^1) = 0$

$$\begin{tikzcd} & H_3(S^2) \oplus H_3(T) \arrow[r] & H_3(X) \arrow[dll, bend right] \\ H_2(S^1) \arrow[r] & H_2(S^2) \oplus H_2(T) \arrow[r] & H_2(X) \arrow[dll, bend right] \\ H_1(S^1) \arrow[r] & H_1(S^2) \oplus H_1(T) \arrow[r] & H_1(X) \arrow[dll, bend right] \\ H_0(S^1) \arrow[r] & H_0(S^2) \oplus H_0(T) \arrow[r] & H_0(X) \arrow[r] & 0. \end{tikzcd}$$

$$\begin{tikzcd} & 0 \oplus 0 \arrow[r] & H_3(X) \arrow[dll, bend right] \\ 0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f"] & H_2(X) \arrow[dll, bend right, "g"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r, "d_1"] & H_1(X) \arrow[dll, bend right, "\delta_1"] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & H_0(X) \arrow[r] & 0 \end{tikzcd}$$

$\rightarrow$ SES $\quad H_3(X) = 0$

f injective $\rightarrow H_2(X) \supseteq f(\mathbb{Z} \oplus \mathbb{Z})$

g surjective $\rightarrow g(H_2(X)) = \mathbb{Z}$

$\ker g = \text{im} f = \mathbb{Z}^2$

$\text{im}\, d_1 = \ker \delta_1$

$H_3(X) = 0 \qquad H_2(X) = \mathbb{Z}^3 \qquad H_1(X) = \mathbb{Z}^2 \qquad H_0(X) = \mathbb{Z}$

$\hookrightarrow \text{im} f = \mathbb{Z}^2 = \ker g$

$\curvearrowleft$ Abelianization of π_1

2: a. $H_*(Y, S) \cong H_*(Y/S^1) \cong H_*(S^2 \vee S^2) \cong H_*(S^2) \oplus H_*(S^2)$

good pair

$H_i(Y, S) = 0 \qquad i \neq 2$

$H_2(Y, S) = \mathbb{Z} \oplus \mathbb{Z}$

b $H_*(Y, \mathbb{Z}_3)$

"mod"

$[\,6|_3 3,\ 8|3\,]$

$0 \to \mathbb{Z}_3^2 \xrightarrow{[0,2]} \mathbb{Z}_3 \xrightarrow{0} \mathbb{Z}_3 \xrightarrow{0} 0$

$H_0 = \mathbb{Z}_3$

$H_1 = \dfrac{\ker d_1}{\operatorname{im} d_2} = \mathbb{Z}_3 / \mathbb{Z}_3 = 0$

$H_2 = \dfrac{\ker d_2}{\operatorname{im} d_3} = \mathbb{Z}^3 / 0 = \mathbb{Z}^3$

Jan 14 26

$$\mathbb{Z}_3^2 \xrightarrow[d_2]{\begin{array}{cc} 0 & 2 \\ \left[6\right. & \left.8\right] \end{array}} \mathbb{Z}_3 \xrightarrow[d_1]{0} \mathbb{Z}_3$$

$$\begin{array}{l} \mathbb{Z}_3 \xrightarrow{0} \mathbb{Z}_3 \xrightarrow{0} \mathbb{Z}_3 \\ \oplus \\ \mathbb{Z}_3 \xrightarrow{2} \end{array}$$

$\ker(d_2) = \mathbb{Z}_3 \oplus 0 = H_2(Y;\mathbb{Z}_3)$

$\operatorname{im}(d_1) = \mathbb{Z}_3$

$$H_1(Y;\mathbb{Z}_3) = \frac{\ker(d_1)}{\operatorname{im}(d_2)} = 0$$

$H_0(Y;\mathbb{Z}_3) = \mathbb{Z}_3$

$$\mathbb{Z}^2 \xrightarrow[d_2]{[6\ 8]} \mathbb{Z} \xrightarrow[d_1]{0} \mathbb{Z}$$

$H_2(Y) = \mathbb{Z}$ $\quad \ker(d_2) = \{(m,n) \mid 6m + 8n = 0\}$

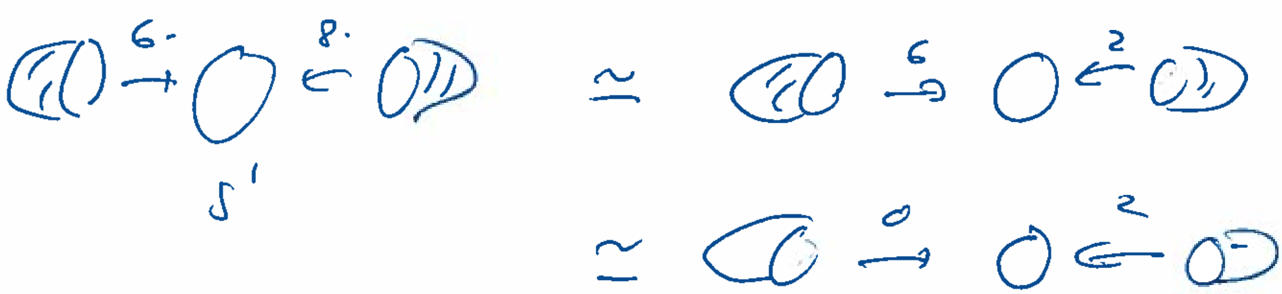
$= \{(m,n) \mid 3m = -4n\}$

$= \mathbb{Z}\begin{pmatrix} 4 \\ -3 \end{pmatrix} \cong \mathbb{Z}$

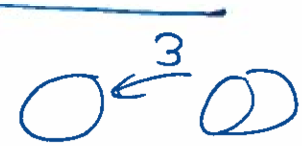
Jan 14 26

$$H_1(Y) = \cancel{\ker}\ \frac{\mathbb{Z}}{\operatorname{im}(d_2)} = \frac{\mathbb{Z}}{6\mathbb{Z}+8\mathbb{Z}}$$

$$= \frac{\mathbb{Z}}{\gcd(6,8)\mathbb{Z}} = \frac{\mathbb{Z}}{2\mathbb{Z}}$$



$\simeq \quad S^2 \vee \mathbb{R}P^2$

Jan 14 #3 $\quad C =$ 

Find covering spaces of $S^1 \times C$.

Examples:

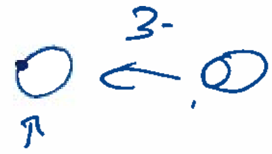
- $\mathbb{R} \times C \xrightarrow{\text{"}e^{2\pi i t} \times id\text{"}} S^1 \times \mathbb{C}$
- $S^1 \times C \xrightarrow{\text{"}z^n \times id\text{"}} S^1 \times \mathbb{C}$
-

3. "Too hard of a problem for a qual" - Wehrli

Take aways:

- Correspondence btwn # of covers and # of subgroups of $\pi_1(X, x_0)$.

- Better example is 2016 #5.

Covering spaces of $C =$ 

circle

$\pi_1(C) = \mathbb{Z}/3\mathbb{Z}$

$p_*\pi_1 = H = \mathbb{Z}/3\mathbb{Z}$: $C \xrightarrow{id} C$

$H = \{0\}$: universal cover $\tilde{C}$

equator

$z \mapsto z^3$

circle

3 disks glued along boundary

$\downarrow p$

C

<u>Example:</u> $S^1 \times \tilde{C} \xrightarrow[id \times p]{} S^1 \times C$

<u>Example:</u> e.g. $\mathbb{R} \times \tilde{C} \longrightarrow S^1 \times C$

$\pi_1(S^1 \times C) = \mathbb{Z} \times \mathbb{Z}_3 = \mathbb{Z} \oplus \mathbb{Z}_3$

obvious: $n\mathbb{Z} \oplus 0$ etc.

non obvious $\mathbb{Z}(1,1) \in \mathbb{Z} \oplus \mathbb{Z}_3$

$$p : \tilde{X} \longrightarrow X \quad \rightsquigarrow \quad H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$$

$$= \left\{ \begin{array}{l} \text{loops in } X \text{ based at } x_0 \\ \text{whose lift starting at } \tilde{x}_0 \\ \quad \text{is a loop} \end{array} \right\}$$

<u>Example</u> : $D = S^1 \times C$ $\qquad S^1 = I/\partial I$

$I \times \tilde{C} \leftarrow$ universal cover

Let $\tau : \tilde{C} \to \tilde{C}$ be a fixed covering transf.

$$\rightsquigarrow \quad \frac{I \times \tilde{C}}{(0,x) \sim (1, \tau(x))} \quad \longrightarrow \text{[illegible]}$$

---

4. $K = \{1/n\}_{n=1}^{\infty}$ $E = ([0,1] \times 0) \cup (K \times [0,1]) \cup (0 \times [0,1])$

a. E is connected but not locally connected.

- each component is connected to $([0,1] \times 0)$

- path-connected $\longrightarrow$ connected

$\hookrightarrow$ for any $(x_0,y_0) \in E$ $\exists$ $p: [0,1] \to E$ $p(0) = (x_0,y_0)$ $p(1) = (x,0)$

- For any (x_0, y_0) (x_2, y_2) $P_1 : [0,1] \to E$ & P_2 $[0,1] \to E$

$p(0) = (x_0, y_0)$ $\xrightarrow{p(1)}$ $(x_1, 0)$ $P_2(0) = (x_1, 0)$

$P_2(1) = (x_2, y_2)$

$\therefore$ E is path connected $\therefore$ connected

Not locally connected:

Consider $U = B((\tfrac{1}{2}, \tfrac{1}{2}), 1/4)$ Ball centered @ $(\tfrac{1}{2}, 1/2)$ w/ radius $1/4$.

$\nexists$ subset $V \subset U$ s.t. V is connected. (separate each line

b. $F = 0 \times [0,1] \subset E$ $f: E \longrightarrow F$ is a retract of E to F

$(x,y) \longrightarrow (0,y)$

$i: F \hookrightarrow E$

$(0,y) \longleftarrow (0,y)$

$r \circ i = id$ $r \circ i(0,y) = r(0,y) = (0,y)$

$\therefore r \circ i = id$

c. F is not a deformation retract of E b/c when collapsing all of the vertical lines to $0 \times [0,1]$ one would have to make a "jump" so to speak. misses continuity.

You couldn't find a homotopy $F_t(s)$ s.t. $F_0(s) = id$ $F_1(s) = i \circ r$

$F_t|_A = F_t$.

5 $O(n)$ all $n\times n$ orthog matrices.

S all $n\times n$ real matrices topg induced by on $\mathbb{R}^{n^2}$

a. Show $O(n)$ is a closed subset of S by considering $v_i \cdot v_j$ of columns

$f_{ij}: S \to \mathbb{R}$ $f_{ij}: \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$

$v_i \cdot v_j \in A \in S \longrightarrow v_i, v_j$ $(\;\; v_i \;\; v_j \;\;) \longrightarrow v_{i1}v_{j1} + \dots + v_{in}v_{jn}$

$\forall \; A \in O(n) \;\; \forall \; v_i, v_j \in A \quad v_i \cdot v_j = \delta_{ij} = \begin{cases} 0 & i\neq j \\ 1 & i=j \end{cases}$

This is cts   this is closed

pulls back to a closed set.

Need to figure out a more precise map to include $\forall$ S

look at S as $\mathbb{R}^{n^2}$. send other vectors to $0 \to$ just look at prod of 2 vectors

b Show $O(n)$ is compact $\longrightarrow$ ~~can show~~ ~~S is Hausdorff~~

Examine

~~Show S is hausdorff $\longrightarrow$ Subset of $\mathbb{R}^{n^2} \longrightarrow$ Hausdorff~~

$\therefore$ ~~$O(n)$ is closed~~

- Show $O(n)$ as subset of $\mathbb{R}^{n^2}$ is bounded

$\hookrightarrow \forall \; v_i \in M \in O(n)$ matrix $|v_i| = 1 \longrightarrow$

$O(n) \subset S^{n^2}(1) \subset \mathbb{R}^{n^2}$

n^2-ball of rad 1

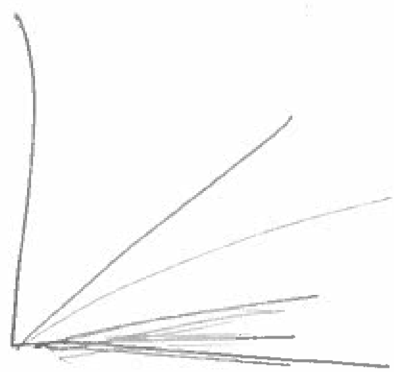
$\therefore$ $O(n)$ is closed & bdd $\therefore$ cmpct.

6. $G \subset \mathbb{R}^2$ $\quad G = \bigcup L_n \cup L_\infty$.

$L_n : (0,0) \to (1, 1/n)$

G is ∞ broom.

$O \subset G$ is open if $O \cap L_n$ is open in L_n $\forall\, n \leq \infty$



Closed set $\subset$ open sets.

Topology is normal but not defined on any metric.

WTS $\forall$ Closed $D \to G \setminus D$ is open.

$G \setminus D \cap L_n$ is open $\forall\, n \leq \infty$

D closed, E closed.

$\uparrow$

U open set containing D

idea: Pick 2 closed sets, look @ where it hits L_n

- Show each L_n is normal
- Take open sets U_n, V_n for each L_n enclosing closed sets
- Union these open sets $U = \bigcup U_n$ $V = \bigcup V_n$ to get normal

---

Sequence, $(1/n, 1/n) \to$ Open set of ~~rad~~ rad $1/n^2 \cap L_n$

$\hookrightarrow B(r, 1/n^2)$ doesn't do this

(low quality)

$\square$ = come back to if time

Ultimately a very tricky exam, not rep. of 2018 hopefully

TOPOLOGY QUALIFYING EXAM: FALL 2013

Instructions. *There are 8 questions worth 200 points. Justify your answers with the necessary proofs, unless otherwise noted. While you should attempt every problem, try your best to answer questions fully. (Therefore it might be best to do first the problems you know how to do). Unless otherwise stated, assume that $\mathbb{R}$ and $\mathbb{C} = \mathbb{R}^2$ are equipped with their standard topology/metric.*

1. (25 points)
(a) Let $f: X \to Y$ be a continuous bijection. Show that if Y is Hausdorff, then X is also Hausdorff.
(b) Let $f: X \to Y$ be a quotient map. Show that if Y is connected and if $f^{-1}(y)$ is connected for each $y \in Y$, then X is also connected.
(c) Does (b) remain true if f is only a continuous surjection?

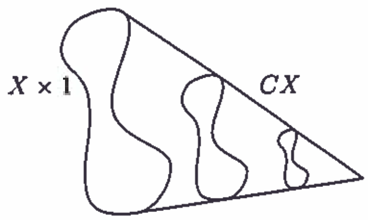
2. (25 points) Let $\mathbf{S}$ be the Sorgenfrey plane (i.e., the plane $\mathbb{R}^2$ equipped with the topology generated by the basis consisting of all rectangles of the form $[x, x+\delta) \times [y, y+\epsilon)$, for all $x, y \in \mathbb{R}$ and all $\delta, \epsilon > 0$). Determine which of the following are true:
(a) $\mathbf{S}$ is first countable.
(b) $\mathbf{S}$ is Lindelöf.
(c) $\mathbf{S}$ is connected.
(d) $\mathbf{S}$ is regular.
(e) $\mathbb{Q}^2 \subset \mathbf{S}$ equipped with the subspace topology is metrizable.

3. (25 points) Let $C(\mathbb{R}, \mathbb{R})$ denote the set of continuous functions $f: \mathbb{R} \to \mathbb{R}$.
(a) How does the topology of pointwise convergence on $C(\mathbb{R}, \mathbb{R})$ compare to the uniform topology (coarser/finer)?
(b) Show that the set $\mathcal{B}(\mathbb{R}, \mathbb{R}) \subset C(\mathbb{R}, \mathbb{R})$ of bounded functions $f: \mathbb{R} \to \mathbb{R}$ is closed and open in the uniform topology.
(c) Show that $C(\mathbb{R}, \mathbb{R})$ has uncountably many connected components with respect to the uniform topology. (Hint: Use part (b) to show that, for $r \neq s$, the functions $f_r(x) := rx$ and $f_s(x) := sx$ belong to different connected components).

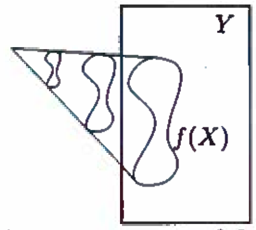
4. (25 points) Let $S^1 := \{u \in \mathbb{C} \mid |u| = 1\}$ and $S^3 := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$.
(a) State the Urysohn Lemma.
(b) Let $L \subset S^3$ be a 2-component link (i.e., a subspace of S^3 consisting of two connected components $L_0, L_1 \subset S^3$, each of which is homeomorphic to S^1). Show that there is a continuous injection $f: S^3 \to S^3 \times [0,1]$ such that $f(L_0) \subset S^3 \times \{0\}$ and $f(L_1) \subset S^3 \times \{1\}$.
? (c) Let $K \subset S^3$ be the knot $K := \{(z, w) \in S^3 \mid z^2 + \sqrt{2}w^3 = 0\}$. Show that the map $f: S^1 \to S^3 - K$ given by $f(u) := (u, 0)$ is not nullhomotopic. (Hint: Compose f with the map $g: S^3 - K \to S^1$ given by $g(z, w) := (z^2 + \sqrt{2}w^3)/|z^2 + \sqrt{2}w^3|$).

W **5.** (25 points) Let X be the space obtained by attaching an 8-cell to $\mathbb{R}P^7$ where the composition of the quotient map and attaching map $\partial e^8 \cong S^7 \to \mathbb{R}P^7 \twoheadrightarrow \overline{e^7}/\partial\overline{e^7} \cong S^7$ has degree 17. Since $\mathbb{R}P^3$ is a subspace of $\mathbb{R}P^7$, we have $\mathbb{R}P^3$ is a subspace of X.
(a) Compute the homology groups of $X/\mathbb{R}P^3$ with $\mathbb{Z}$-coefficients.
(b) Compute the homology groups of $X/\mathbb{R}P^3$ with $\mathbb{Z}_2$-coefficients.

6. (25 points) Recall the cone CX of a space X is the quotient of $X \times I$ by the relation $x \times 0 \sim x' \times 0$ for all $x, x' \in X$. Given a map $f : X \to Y$, the mapping cone Cf is obtained by identifying CX to Y according to $x \times 1 \sim f(x)$ for all $x \in X$.

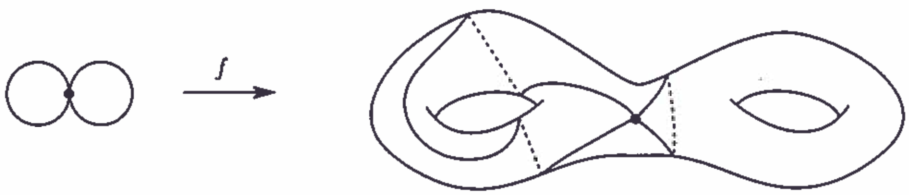


The cone of X



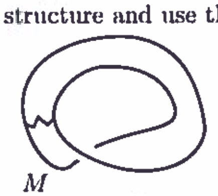
The mapping cone of $f : X \to Y$

Let $f : S^1 \vee S^1 \to \Sigma_2$ be the injection pictured below:



(a) Show that CX (for any space X) is contractible.
(b) Calculate $\pi_1(Cf)$ with basepoint at the image of the wedge point. Carefully describe generators and relators.
(c) Calculate $H_*(Cf)$. Carefully describe generators.

7. (25 points)
(a) Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.
(b) Let $(M, \partial M)$ be the Möbius band rel boundary. Show the transversal arc (the zigzag in the figure below) generates $H_1(M, \partial M)$ (hint: give M a CW structure and use the definition of relative homology).



(c) Calculate $H_1(\mathbb{R}, \mathbb{Q})$ and find a basis (hint: recall that $\mathbb{Q}$'s path components are precisely the singletons).

8. (25 points) Consider a group presentation $G = \langle g_\alpha | r_\beta \rangle$. We describe a graph $\widetilde{X}_G$. Let the vertices of $\widetilde{X}_G$ be the elements of G, and at each vertex g, put an edge to $g \cdot g_\alpha$ for each generator g_α of G. We call $\widetilde{X}_G$ the **Caley graph** of G with respect to the generators g_α. Each edge inherits the orientation $g \mapsto g \cdot g_\alpha$ from the generators g_α, so some might call the Caley graph a directed graph.
(a) Show that the Caley graph of G is connected.
(b) Fix a vertex g in the Caley graph. Show that each relator r_β determines a loop in the Caley graph based at g. (In fact, relators and their consequences are the *only* way to produce loops in the Caley graph).
(c) Draw the Caley graph of $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b | a^2, b^2 \rangle$.
(d) The group G acts on its Caley graph $\widetilde{X}_G$ by multiplying on the left: an element $g \in G$ sends a vertex $g' \in G$ to the vertex $gg' \in G$, and sends the edge from g' to $g'g_\alpha$ to the edge from gg' to $gg'g_\alpha$. This action is a covering space action (no justification required). Show that the *regular* covering spaces of $S^1 \vee S^1$ are precisely the Caley graphs of (presentations of) groups with two generators.

Fall 2013

1a. $f: X \to Y$ be a ~~cts~~ bij. Suppose Y is Hausdorff $\to$

$\forall\, y_1, y_2 \in Y \quad \exists^{\text{open}}\, U, V$ s.t. $y_1 \in U$, $y_2 \in V$, $U \cap V = \emptyset$

Consider arbitrary $x_1, x_2 \in X$. $\exists$ open U, V s.t. $f(x_1) \in U$, $f(x_2) \in V$ s.t. $U \cap V = \emptyset$

~~Since xf~~ Since f is bijection $x_1 \in f^{-1}(U)$, $x_2 \in f^{-1}(V)$

$f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset = f^{-1}(U) \cap f^{-1}(V)$

Thus X is Hausdorff

$f(X) = Y$.

b) $f: X \to Y$ is a quotient map. So f is a cts surjection st

U is open in Y iff $f^{-1}(U)$ is open in X.

Suppose Y is connected and $f^{-1}(y)$ is connected for each $y \in Y$

~~First note that $f^{-1}(Y) = X$~~

Suppose X was not connected. Then $\exists$ 2 open sets U, V s.t. $U \cap V = \emptyset$.

$U \cup V = X$. $U, V \subset X$ are open iff $f(U), f(V)$

$U = f^{-1}(W)$ $V = f^{-1}(Z)$ $\iff$ W, Z are open

f is surj. $\to f(U \cap V) = f(\emptyset) = \emptyset = f(U) \cap f(V) \to$ separates Y Σ.

careful w/ pull backs & push forwards

* mention saturation

maybe

ii b. $\ddot{f}: X \longrightarrow Y$ is surjective map s.t. $U \subset Y$ is open iff $f^{-1}(U)$ is open.

Suppose Y is connected. Suppose X was not connected. Then

$\exists\ \phi \neq A \subset X$ s.t. A is both open & closed. $\longrightarrow$ A open and $X \setminus A$

is open. $f(A) = U \subseteq Y$ $\longrightarrow$ $f^{-1}(U) = A$ is open iff U is open.

$X \setminus A = f^{-1}(Y \setminus U)$ open iff $Y \setminus U$ open

Can I do this.

7c. No Consider the constant map.

$$f: X \longrightarrow c$$

Where X is not a connected set.

e.g. $X = [0,1] \cup [2,3]$

$$f: X \longrightarrow \{1\}$$

$\{1\}$ is connected b/c only sets that are open + closed are $\{1\}$, $\emptyset$

Clearly X is not connected

2. $\mathbb{S}$ is plane $\mathbb{R}^2$ ~~topo~~ generated w/ rectangles of form $[x, x+\delta) \times [y, y+\varepsilon)$

$\forall\, x, y \in \mathbb{R}$, $\delta, \varepsilon > 0$ ⟶ need better understanding of this.

True b/c boxes

False if boxes can intersect

b Lindeloff: Every open covering contains a countable subcovering

Yes b/c basis elmts are countable

c

3. $C(\mathbb{R}, \mathbb{R}) = \{f \mid f \text{ cts}, f: \mathbb{R} \to \mathbb{R}\}$.

a. Coarser / finer?

- Find what open sets look like.
  - which one have "more" open sets

no ~~defn's~~

come back to if time

b. Show $\mathcal{B}$ set of

4. $S^1 = \{u \in \mathbb{C} \mid |u| = 1\}$ $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$

a. X is a normal space, A, B are disjoint closed subsets of X. Let $[a,b]$ be a closed : $\exists$ cts map $f: X \to [a,b]$ s.t. $f(A) = a$ $f(B) = b$.

b. $L \subset S3$ be a 2-component ~~link~~ of

show $\to$ $\exists$ a cts injection $f: S^3 \to S^3 \times [0,1]$ s.t. $f(L_0) \subset S^3 \times \{0\}$ & $f(L_1) \subset S^3 \times \{1\}$.

$\hookrightarrow$ use Urysohn's

c. $K \subset S^3$ $K = \{(z,w) \in S^3 \mid z^2 + \sqrt{2}\, w^2 = 0\}$ Show $f\ S^1 \to S^3 \setminus K$ via $f(u) = (u,0)$ is not null homotop...

~~$\hookrightarrow f \circ g =$~~ $g \circ f(u) = g(u,0) = \dfrac{u^2 + \sqrt{2}(0)}{|u^2|} = \dfrac{u^2}{|u^2|}$ = normalize u^2?

why $\to$

5. "X be a space obtained by attaching an 8-cell to $\mathbb{R}P^7$

↳ what does composition of quotient & attaching map has degree 17.

a. Compute homology gps of $X / \mathbb{R}P^7$ w/ $\underline{\mathbb{Z}\text{-coefficients}}$
?

Fall 2013 5

$\mathbb{RP}^7$ e_3 e_2 e_1 e_0

$C(X)$:

$$\begin{array}{l} \mathbb{Z} \\ \downarrow 17 \\ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \underbrace{\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}}_{C(\mathbb{RP}^3)} \end{array}$$

$C(X/\mathbb{RP}^3)$

$$\begin{array}{l} \mathbb{Z} \\ 17 \downarrow \\ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to 0 \to 0 \to \mathbb{Z} \end{array}$$

$$\begin{array}{l} \mathbb{Z}_2 \\ 1 \downarrow \\ \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0 \to 0 \to 0 \to \mathbb{Z}_2 \end{array}$$

$\uparrow$

$C(X/\mathbb{R}^3; \mathbb{Z}_2)$

4. Show that CX $\forall$ X is contractible.

$H : (X \times [0,1]) \times I \longrightarrow X \times [0,1]$

$H((x,t),s) = x, (1-s)t)$

H is cts, $H((x,0),s) = (x,0)$ $\forall x \in X$ & $s \in [0,1]$.

H induces a cts map

$\hat{H} : CX \times [0,1] \longrightarrow CX$ s.t.

$\hat{H}([(x,t)],s) = [(x,(1-s)t)]$.

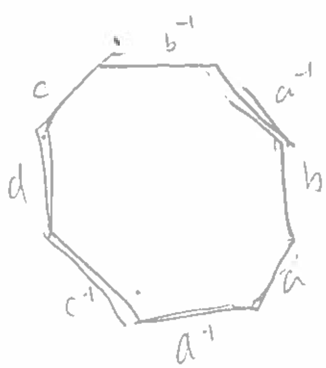
Stack exchange

$\hookrightarrow$ $\hat{H}$ is a homotopy in CX from id on CX to a pt on $X \times \{0\}$

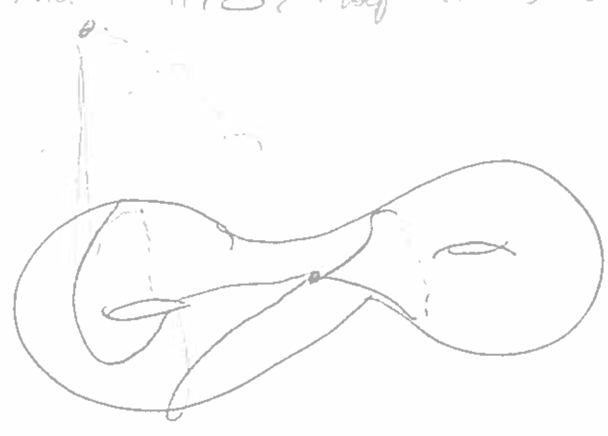
$\Longrightarrow$ CX is contractible

6. $Cf = C(S^1 \vee S^1)$ identified to Σ_2 via $x \times 1 \sim f(x)$ $\forall x \in$

8 $\longrightarrow$ does not help that $\pi_1 \Sigma_2$ ~~that~~ retr $S^1 \vee S^1$



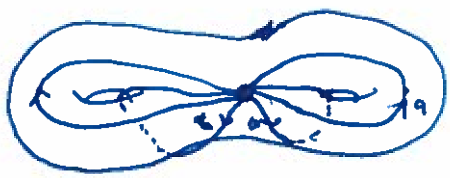
?

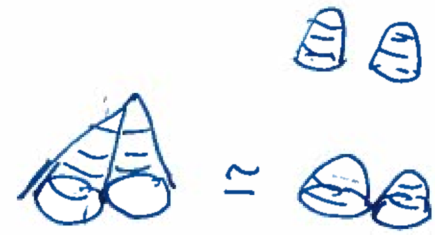


~~Fall~~ Fall 2013 6

(a) $CX = \dfrac{X \times I}{(x,0)\sim(x',0)} \overset{\text{retract } X \times \{0\}}{\rightsquigarrow} \dfrac{}{(x,0)\sim(x',0)} = $ a point

(b)

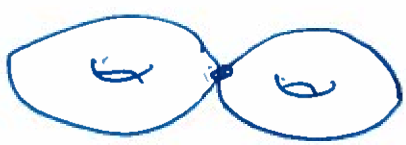




$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] \rangle$$

$$\pi_1(\underbrace{\Sigma_2 \cup \text{right disk}}) = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1]=0,\ [a_2,b_2]=0 \rangle$$

$$= \mathbb{Z}^2 * \mathbb{Z}^2$$

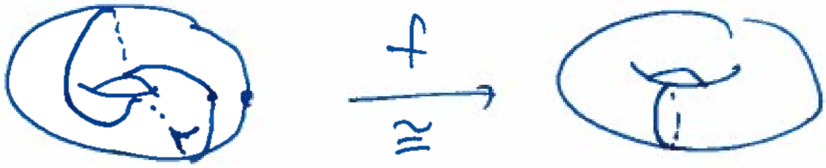


$$\pi_1(Cf) = \frac{\pi_1(\ \ )}{a_1 b_1^2 = 0}$$

$$= \frac{\mathbb{Z}^2}{\mathbb{Z}(1,2)} * \mathbb{Z}^2 \cong \mathbb{Z} * \mathbb{Z}^2$$

$$a_1^p b_1^q \triangleq (p,q)$$

Fall 2013 6



$\downarrow$ cut one

$\cong$ 

$H_1(\text{torus}) = \mathbb{Z}^2 \circlearrowleft f_*$

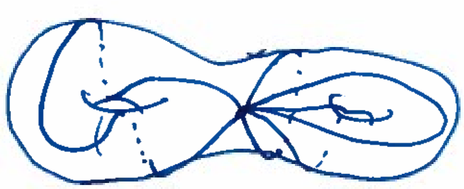
$\circlearrowleft f$

$(p,q) \rightsquigarrow (1,0)$

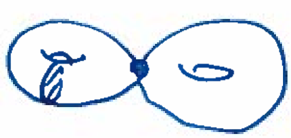
$\gcd(p,q) = 1$

$H_*(X)$

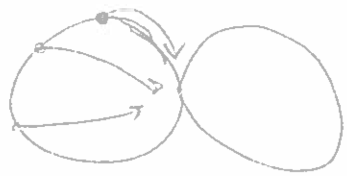
$n=2$	$n=1$	$n=0$
$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}$



etc...

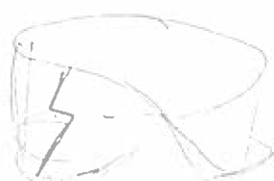
$X \simeq$ 

"7a"



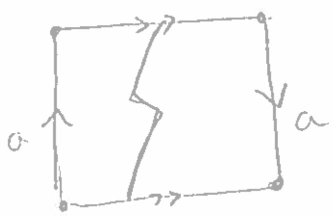
Math 761 HW ?

b $(M, \partial M)$ rec



$\pi_1(M) = \pi_1(S^1) = \mathbb{Z}$

$\pi_1(\partial M) = \pi_1(S^1) = \mathbb{Z}$



identify these two a point.

τ transversal arc

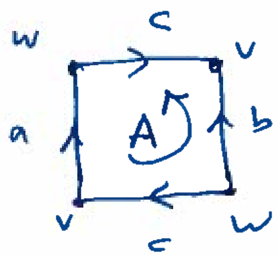
H_1 is abelianization of $\pi_1(M/\partial M)$.

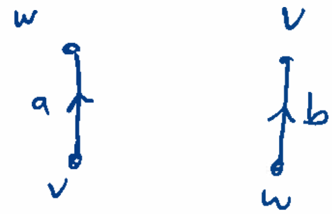
$H_1(M, \partial M) = H_1(M/\partial M)$

? excision version?..

c. $H_1(\mathbb{R}, \mathbb{Q}) = H_1(\mathbb{R} \setminus \mathbb{Q})$

Fall 2013   7

$M =$ 

∂M 

$C(M)$

$$\mathbb{Z} \xrightarrow{\begin{bmatrix} -1 \\ +1 \\ -2 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{array}{c|l} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} & \begin{array}{l} v \\ w \end{array} \end{array}} \mathbb{Z}^2$$

$$\downarrow$$

$$C(M,\partial M) = \frac{C(M)}{C(\partial M)} \Bigg\} \begin{array}{l} \text{set} \\ a, b = 0 \\ v, w = 0 \end{array}$$

$$\mathbb{Z} \xrightarrow{-2} \mathbb{Z} \xrightarrow{0} 0$$

$\{ H(M,\partial M)$

$$0 \qquad \mathbb{Z}/2\mathbb{Z} \qquad 0$$

generated by c

Alternatively:

$$M/\partial M \cong \mathbb{RP}^2$$

$$\Rightarrow H(M,\partial M) \cong \tilde{H}(\mathbb{RP}^2)$$

$X \neq \emptyset$

$\tilde{H}_i(X) = H_i(X) \quad i > 0$

$H_i(X) \cong \tilde{H}_i(X) \oplus \mathbb{Z}$

8. $G = \langle g_\alpha \mid r_\beta \rangle$

$\tilde{X}_G$. Let verts of $\tilde{X}_G$ be elts of G @ each vertex g, put

$\tilde{X}_G$ is the Cayley graph of G.

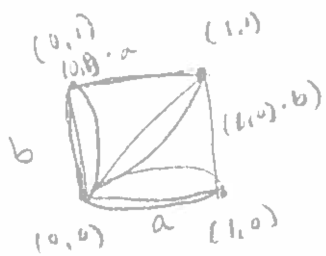
a. Show Cayley graph is connected

$\longrightarrow$ not separated / only gone that is open / closed is $\tilde{X}$, $\emptyset$

b. each relater r_β determines a loop

$\hookrightarrow$ makes sense, $r_\beta = g_\alpha^1 g_\alpha^2 \cdot \quad g_\alpha^{?} = 1$

c. $\mathbb{Z}_2 \times \mathbb{Z}_2$



SU Topology Qualifying Examination, January 2013

Instructions. *There are 4 questions worth the total of 100 points. Do all questions, and justify your answers with the necessary proofs. Unless otherwise stated, assume that $\mathbb{R}$ is equipped with its standard topology/standard metric.*

1. (25 points) Let $\mathbb{R}^\infty \subset \mathbb{R}^{\mathbb{Z}_+}$ be the subset $\mathbb{R}^\infty := \{f\colon f(n) = 0 \text{ for all but finitely many } n\}$.

(a) State the definitions of the product topology and of the uniform topology on $\mathbb{R}^{\mathbb{Z}_+}$.

(b) Show that the closure of $\mathbb{R}^\infty \subset \mathbb{R}^{\mathbb{Z}_+}$ in the uniform topology consists of all functions $f \in \mathbb{R}^{\mathbb{Z}_+}$ such that $f(n) \to 0$ as $n \to \infty$.

(c) What is the closure of $\mathbb{R}^\infty \subset \mathbb{R}^{\mathbb{Z}_+}$ in the product topology?

2. (25 points) Let X be a locally connected space.

(a) Show that the connected components of every open subset of X are open.

(b) Show that if $f\colon X \to Y$ is a quotient map, then Y is also locally connected.

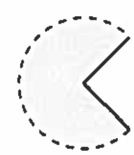
(c) Show that if a totally ordered set Z contains a subset which has an upper bound but no least upper bound in Z, then Z is disconnected with respect to the order topology.

Hint: In (b), start with an open subset $V \subset Y$ and define an equivalence relation on V such that the preimage of each equivalence class is a union of connected components of $f^{-1}(V)$.

3. (25 points) One can define a topology on $\mathbb{R}^2$ by declaring a set $U \subset \mathbb{R}^2$ to be open if for every $(x_0, y_0) \in U$ there exist real numbers $\epsilon > 0$ and $m > 0$ such that the set

$$P_{\epsilon,m}(x_0, y_0) := \{(x, y) \in \mathbb{R}^2 \colon (x - x_0)^2 + (y - y_0)^2 < \epsilon^2 \text{ and } |y - y_0| \geq m(x - x_0)\}$$

is contained in U. (See the figure on the right for an example of $P_{\epsilon,m}(x_0, y_0)$).



(a) How does this topology compare to the standard topology on $\mathbb{R}^2$?

(b) Is it regular?

(c) Is it first countable?

(d) Is it second countable?

(e) Show that the subspace $\mathbb{Q}^2 \subset \mathbb{R}^2$ is metrizable with respect to this topology.

4. (25 points)

(a) Show that every continuous map from a space Z to a convex subspace X of $\mathbb{R}^n$ is homotopic to a constant map.

(b) If X is a convex subspace of $\mathbb{R}^n$ and Y is a subspace of X homeomorphic to a standard circle, can there be a retraction $f\colon X \to Y$?

(c) Sketch a proof for <u>one</u> of the following two theorems: Brouwer's fixed point theorem in dimension 2 and the fundamental theorem of algebra.

Topology Part of the Qualifying Examination, January 2012

Instructions. *There are 4 questions worth the total of 100 points. Do all questions, and justify your answers with the necessary proofs.*

1. (30 points) Recall that the lower limit topology on $\mathbb{Q}$ is generated by the basis sets $[a, b) \cap \mathbb{Q}$ for all pairs of rational numbers a, b with $a < b$.

(a) Use the Urysohn metrization theorem to show that $\mathbb{Q}$ with the lower limit topology is metrizable.

(b) Let $\{X_\alpha\}$ be a collection of topological spaces. State the definitions of the product topology and the box topology on $\prod X_\alpha$.

(c) Show that if $\{X_\alpha\}$ is a countable collection of second countable spaces, then $\prod X_\alpha$ with the product topology is second countable.

(d) Show that a metric space containing a countable dense subset is second countable.

2. (20 points) Let $f\colon X \to Y$ be continuous.

(a) Suppose Y is Hausdorff. Show that the set $\{(x, x') \mid f(x) = f(x')\}$ is closed in $X \times X$.

(b) Suppose that f is a quotient map and that the subspace $f^{-1}(y) \subset X$ is connected for every $y \in Y$. Show that if Y is connected, then so is X.

3. (25 points) Let $\mathcal{C}(X, Y)$ be the set of all continuous functions from a space X to a metric space Y, and let $C \subset X$ be a compact subspace and $U \subset Y$ an open subset.

(a) Show that the set $S(C, U) := \{f \in \mathcal{C}(X, Y) \mid f(C) \subset U\}$ is open in $\mathcal{C}(X, Y)$ in the topology of uniform convergence.

(b) Let $f_n\colon [0, 1] \to \mathbb{R}$ ($\mathbb{R}$ = the real numbers with the usual metric) be defined by

$$f_n(x) := \min\{|nx - 1|, 1\}$$

Show that $\{f_n\}$ converges pointwise but not uniformly to the constant function $f(x) = 1$.

(c) Give an example to show that $S(C, U)$ need not be open in $\mathcal{C}(X, Y)$ in the topology of pointwise convergence.

4. (25 points) Let X, Y be spaces with basepoints $x \in X$, $y \in Y$.

(a) Show that $\pi_1(X \times Y, x \times y)$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

(b) Show that there is no retraction of $S^1 \times B^2$ onto $S^1 \times S^1$ where S^1 is the unit circle in $\mathbb{R}^2$ and B^2 the closed unit disk in $\mathbb{R}^2$.

(c) Let $M_n(\mathbb{C})$ be the set of all complex $n \times n$ matrices with the topology induced from the standard topology on $\mathbb{C}^{n^2}$ via the bijection $A = (a_{ij}) \mapsto (a_{11}, a_{12}, \ldots, a_{21}, \ldots, a_{nn})$, and let $GL_n(\mathbb{C})$ be the subspace of all matrices satisfying $\det(A) \neq 0$. Show that $GL_n(\mathbb{C})$ is not simply connected.

Topology Qualifying Exam
January 2011

Do all of the following problems.

1. Let X be a topological space and $A \subset X$ be a subset.

   (a) (5 points) Give the definition of the set ClA, the *closure* of A

   (b) (10 points) Let $x \in X$. Show that $x \in ClA$ if and only if for every open set U with $x \in U$, $U \cap A \neq \emptyset$.

2. Let X be a space.

   (a) (5 points) Define what it means to say X is *normal.*

   (b) (10 points) Show that if X is a metric space, then X is normal.

3. Let X be a space.

   (a) (5 points) Let $D \subset X$ be a subset. Define what it means to say D is *dense* in X.

   (b) (10 points) Let $\mathbb{R}^\omega$ be the product of countably many copies of the real line $\mathbb{R}$ with the usual topology. Show that $\mathbb{R}^\omega$ contains a countable dense set. (Hint: Recall the proof that the product of arbitrarily many connected spaces is connected.)

4. Let X be a space.

   (a) (5 points) Define what it means to say the collection $\mathcal{B}$ of open sets in X is a *basis for the topology* on X.

   (b) (10 points) Let $\mathbb{R}^\omega$ be the product of countably many copies of the real line $\mathbb{R}$ with the usual topology. Show that $\mathbb{R}^\omega$ has a countable basis.

5. Let X be a space and $A \subset X$ be a subspace of X.

   (a) (5 points) Define what it means to say A is a *retract* of X.

   (b) (10 points) Suppose A is a retract of X and that $a \in A$. Show that

   $$i_* : \pi_1(A, a) \to \pi_1(X, a)$$

   is one-to-one where $i : A \to X$ is the inclusion.

6. Let $X = \mathbb{R}^2 - \{(0,0)\}$ and $A = \mathbb{R}^2 - \{(0,0),(2,0)\} = X - \{(2,0)\}$ be topologized as subspaces of $\mathbb{R}^2$.

(a) (5 points) Is A a retract of X?

(b) (10 points) Show your answer to part (a) is correct.

7. (10 points) Let (X, x_0) and (Y, y_0) be spaces with base point and

$$f, g : (X, x_0) \to (Y, y_0)$$

be base point preserving maps. Suppose f is homotopic to g relative to x_0; that is, suppose there is a function

$$H : X \times [0,1] \to Y$$

with $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H(x_0, t) = y_0$ for all $t \in [0,1]$ Show that

$$f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

August 2011

Qualifying Examination
Topology

1 If Y is a topological space and $I = [0, 1]$ is the unit interval, let $C(Y) = {Y \times I}/{Y \times \{1\}}$, the quotient space which collapses the subspace $Y \times \{1\}$ to a point.

a) Prove that $C(Y)$ is contractible.

b) If Y is Hausdorff, prove $C(Y)$ is also Hausdorff.

2 a) If X is regular and $A \subset X$ is closed, prove X/A is Hausdorff.

b) Prove or disprove: all quotient spaces of Hausdorff spaces are Hausdorff.

3 If $f : X \to Y$ is a continuous map, let $C(f)$ be the quotient space obtained from the disjoint union of $X \times I$ and Y by identifying $(x, 0)$ with $f(x)$ for each $x \in X$. If $z \in X \times I$ or $z \in Y$, let $[z]$ denote its equivalence class in $C(Y)$. We have continuous maps $i : X \to C(f)$ and $\rho : C(f) \to Y$ given by $i(x) = [(x, 1)]$ and $\rho[(x,t)] = f(x)$, $\rho[y] = y$.

a) Show that ρ is a homotopy equivalence by showing that $j : Y \to C(f)$, $j(y) = [y]$ is a homotopy inverse of ρ.

b) If the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\downarrow & & \downarrow\psi \\ Z & \xrightarrow{g} & W \end{array}$$

commutes ($\psi \circ f = g \circ \varphi$), show that there exists a continuous map $C(f) \xrightarrow{\lambda} C(g)$ such that the following diagram commutes in each square:

$$\begin{array}{ccccc} X & \xrightarrow{i} & C(f) & \xrightarrow{\rho} & Y \\ \phi\downarrow & & \downarrow\lambda & & \downarrow\psi \\ Z & \xrightarrow{i} & C(g) & \xrightarrow{\rho} & W \end{array}$$

4 a) If $X \xrightarrow{f} Y \xrightarrow{g} X$ are continuous with $gf = id_X$, and Y is Hausdorff, then X is also Hausdorff and $f(X)$ is closed in Y.

b) If X is regular, show that every pair of points in X have neighborhoods whose closures are disjoint.

5 Recall that there is a simple argument that shows the n-sphere S^n and $R^{n+1} - \{one\ point\}$ have the same homotopy type. Let X be the complement of a single point in $S^1 \times S^1$, Y the complement of two points in $S^1 \times S^1$, and Z the complement of a single point in the real projective plane RP^2. Give three familiar spaces which have the same homotopy types as X, Y and Z (and give the corresponding simple arguments that prove this).

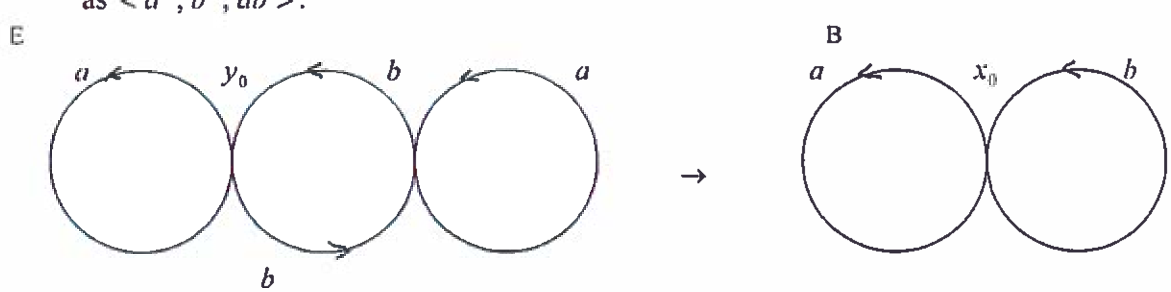
6 a) If $S^1 \vee S^1$ denotes the one-point union of two one-spheres, give two *nonhomotopic* retractions $S^1 \vee S^1 \to S^1$ where the range is say the left S^1 (and give the proof that they are nonhomotopic).

b) Prove or disprove: there exists a retraction of the two-sphere S^2 onto one of its equatorial one-spheres S^1.

7 See the diagram below for graphs of (1) the based figure-eight B, x_0 with loops a, b based at x_0 and oriented counterclockwise, and (2) the based three-loop figure E, y_0 with loops and arcs similarly labelled a, b. The labels a, b in E define a covering projection $p: E, y_0 \to B, x_0$. For example each semicircular arc b in E is mapped to its namesake loop in B, x_0 as indicated by the assigned orientations. [fyi: (1) the fundamental group $\pi_1(B, x_0)$ of B, x_0 is the free group $< a, b >$ on the two generators a, b (here we write a or b both for the loop and its class in the fundamental group); (2) any (nontrivial) subgroup of a free group is again a free group.]

a) Determine the subgroup $p_*\pi_1(E, y_0)$ in the free group $\pi_1(B, x_0)$.

b) With B, x_0 as in a) find a covering space $p': E', y_0' \to B, x_0$ such that $p'_*\pi_1(E', y_0')$ is the free subgroup of $< a, b >$ generated by a^2, b^2, ab, which we can thus describe as $< a^2, b^2, ab >$.



Topology Qualifying Exam
August 23, 2010

Do all of the following problems.

1. Let X be a topological space.

   (a) (5 points) Define what it means to say that X is *regular*.

   (b) (10 points) Give a complete proof that a product of two regular spaces is regular.

2. Let X be a space.

   (a) (5 points) Define what it means to say that X is *normal*.
   It is easy to show that a normal space is regular. On the other hand the space $\mathbb{R}_l^2$, the so-called *Sorgenfrey plane*, is a space that is regular, but not normal. Here $\mathbb{R}_l$ is the real line with the lower limit topology.

   (b) (5 points) Prove that $\mathbb{R}_l^2$ is regular.

   (c) (10 points) Prove that $\mathbb{R}_l^2$ is not normal.

3. Let X be a topological space.

   (a) (5 points) Define what it means to say X is *metrizable*.

   (b) (10 points) Let $\mathbb{R}$ be the real line with the usual topology. Let $\mathbb{R}^\omega$ be the product of countably many copies $\mathbb{R}$ and give $\mathbb{R}^\omega$ the product topology. Show that $\mathbb{R}^\omega$ is metrizable.

4. Let X be a topological space.

   (a) (10 points) Give two necessary conditions, other than separation properties, for X to be metrizable.

   (b) (10 points) Let I be an uncountable set and $\mathbb{R}^I$ be the product of $|I|$ copies of $\mathbb{R}$, where $|I|$ means the cardinality of I. Show that $\mathbb{R}^I$ with the product topology is not metrizable.

5. Let X be a space and $A \subset X$ be a subspace of X.

   (a) (5 points) Define what it means to say A is a *retract* of X.

   (b) (5 points) Define what it means to say A is a *deformation retract* of X.

6. Let $T = \{(r,\theta) \,|\, 0 \leq r \leq 1, \theta = 0, 2\pi/3, 4\pi/3\}$ where points in $\mathbb{R}^2$ are given polar coordinates (r,θ). Thus T consists of the three line segments on length 1 that start at the origin and run to the points $(1,0), (1,2\pi/3)$, and $(1,4\pi/3)$.

   Let X be the quotient space obtained from $T \times [0,1]$ by identifying $((r,\theta),0)$ with $((r,\theta+2\pi/3),1)$. Thus X is obtained from $T \times [0,1]$ by identifying $T \times 0$ with $T \times 1$ after a rotation of $2\pi/3$ radians. Let

$$q : T \times [0,1] \to X$$

   be the quotient map.

   (a) (10 points) Let $A = q(\{(0,0)\} \times [0,1]) \subset X$. Note that A is a circle in X. Show that A is a deformation retract of X.

   (b) (10 points) Let $B = q(\{(1,0),(1,2\pi/3),(1,4\pi/3)\} \times [0,1]) \subset X$. Note that B is also a circle in X. Is B a retract of X? Show why or why not.

TOPOLOGY QUALIFYING EXAM
AUGUST 2009

There are six problems. Begin your answer to any problem on a new page in your blue book(s). Make a space between answers to separate parts of a question to facilitate grading. All answers must be justified with proofs. The relative point values are indicated.

1 (12 pts) Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

a) Show that the set $\{x \mid f(x) \geq g(x)\}$ is closed in X.

b) Let $h : X \to Y$ be the function
$$h(x) = \max\{f(x),\ g(x)\}.$$
Show that h is continuous.

2 (18 pts) a) Show if Y is Hausdorff, then Y^X (= the space of all functions from X to Y) with the compact open topology is also Hausdorff.

b) Let $f_n : R \to R$ (R = the real numbers with the usual topology) be defined by
$$f_n(x) = \begin{cases} \frac{1}{n}\sqrt{n^2 - x^2} & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n \end{cases}$$
Show that $\{f_n\}$ does not converge to the constant function $g(x) = 1$ in the uniform topology, but that it does converge to this constant function in the compact convergence topology.

c) Let $\{f_n\}$ be a sequence in $C(X, Y)$ (= the space of all continuous functions) where Y is metric with metric d. Show if $\{f_n\}$ converges to a function g in the compact convergence topology, then it converges to this g in the compact open topology. [Note the converse is also true, but the problem does not ask for its proof.]

3 (10 pts) a) Is the real line R with the usual topology second countable? Reasoning?

b) Prove or disprove the assertion "the real line with any metric topology is second countable."

4 (20 pts) a) Let $p : X \to Y$ be a closed map. Given any subset $S \subset Y$ and any open U containing $p^{-1}(S)$, show there exists an open $V \supset S$ such that $p^{-1}(V) \subset U$.

b) Let Y be normal and $p : Y \to Z$ be a closed, continuous surjection. Show that Z is normal.

c)) Let Y be Hausdorff and $p : Y \to Z$ be a closed, continuous surjection such that $p^{-1}(y)$ is compact for each ~~$y \in Y$~~. Show that Z is Hausdorff.

z $\qquad$ $z \in Z$.

5 (20 pts) Let S^2 and T^2 be the 2-sphere and the torus. Recall that the space $S^2 - p$, where p is a point in S^2, has the homotopy type of a point. Reasoning: it is homeomorphic to an open 2-dimensional disk which can be contracted to its center along its radii. The following spaces also have the homotopy type of fairly simple spaces. Identify these simple spaces and give the reasoning why they have the same homotopy type of the given spaces.

a) $S^2 - p - q$, where p and q are two distinct points in S^2; and
$S^2 - p - q - r$, where p, q and r are three distinct points in S^2.

b) $T^2 - p$, where p is a point in T^2; and
$T^2 - p - q$, where p and q are two distinct points in T^2.

6 (20 pts) a) Show that there exists no map $f : S^1 \times S^1 \to \textit{figure} - \textit{eight}$ such that the induced homomorphism of fundamental groups is an epimorphism. You may assume that the fundamental group of the $\textit{figure} - \textit{eight}$ is the free group on two generators. [Hint: you do not need a covering space of the $\textit{figure} - \textit{eight}$ to answer this question.]

b) Give an example of an surjective map $g : X \to Y$ (X, Y are suitable spaces) whose induced homomorphism on fundamental groups is NOT an epimorphism.

c) Give an example of an injective map $h : X \to Y$ (X, Y are suitable spaces) whose induced homomorphism on fundamental groups is NOT a monomorphism.

Topology Qualifying Exam
January 9, 2009

Do all of the following problems each of which is worth 20 points.

1. Let X be a topological space.

   (a) Define what it means to say that X is *compact.*

   (b) State the Tube Lemma.

   (c) Prove the Tube Lemma.

2. Let X and Y be topological spaces and $f : X \to Y$ be a continuous function.

   (a) Define what it means to say that the continuous function $f : X \to Y$ is *closed.*

   (b) Let $\pi_1 : X \times Y \to X$ be projection on the first factor. If Y is compact, show that π_1 is closed.

   (c) Is the statement in part (b) true if Y is not compact? Show your answer is correct.

3. Let X be a T_1-space and $\mathcal{F} = \{f_j : X \to \mathbb{R} \,|\, j \in J\}$ be a family of continuous real valued functions that separates points and closed sets. Define $F : X \to \mathbb{R}^J$ by $F(x) = (f_j(x))_{j \in J}$. Show that F is an embedding of X into $\mathbb{R}^J$.

   Hint: Recall that a family $\mathcal{F} = \{f_j : X \to \mathbb{R} \,|\, j \in J\}$ of continuous real valued functions defined on the space X is said to *separate points and closed sets* if, for every point $x \in X$ and every closed set $A \subset X$ with $x \notin A$, there is a function $f_j \in \mathcal{F}$ with $f_j(x) > 0$ and $f_j(a) = 0$ for all $a \in A$

   In the next two problems you will need to know that the Möbius band M is the quotient space obtained from $[0,1] \times [-1,1]$ by identifying $(0,t)$ with $(1,-t)$. Let

   $$q : [0,1] \times [-1,1] \to M$$

   be the quotient map.

4. (a) Define what it means to say that the subspace A is a *deformation retract* of the space X.

   (b) Notice that the image $q([0,1] \times 0)$ is a circle C_1 in M. Show C_1 a deformation retract of M.

5. (a) Define what it means to say that the subspace A is a *retract* of the space X.

   (b) Notice that the image $q([0,1] \times \{1,-1\})$ is also a circle C_2 in M. Is C_2 a retract of M? Prove your answer is correct.

August 2008

Qualifying Examination
Topology

Problems 1-4 consist of true or false statements. Each statement is to be proved or disproved with brief but complete reasoning. Provide definitions of all underlined, italicized words and phrases. On page 2 find definitions and notations of some items appearing in the problems. There are 5 problems in total. Spacing: begin each problem on a new page.

1 a) Any infinite set with the *finite complement topology* is connected.
b) The subspace X of R^2, where $X = \{(x, y) \mid y = e^x \text{ or } y = 0\}$ is connected.
c) The real line with the *lower limit topology* is connected.
d) The complement of the "equatorial S^1" in the Mobius band is connected.

2 a) The set of integers Z with the topology generated by the *basis* $\{(-n, n) \mid n \geq 1\}$ is compact.
b) The space Z defined in 2 a) is *limit point compact*.
c) The set of all $n \times n$ real invertible matrices is compact.
d) The set of all $n \times n$ real orthogonal matrices is compact.

3 a) Define a function p from R onto a three point set $\{a, b, c\}$ by

$$p(x) = a \text{ if } x < 0; \quad p(x) = b \text{ if } x = 0; \quad p(x) = c \text{ if } x > 0.$$

With the resulting *quotient topology* induced by p, $\{a, b, c\}$ is Hausdorff.
b) For any map $f : S^2 \to R$, there does not exist a point $c \in S^2$ such that $f(c) = f(-c)$.
c) There exists a compact Hausdorff space which is not *regular*.
d) The unit circle $S^1 \in R^3$ is a *retract* of $R^3 - \{(0, 0, 0)\}$.

4 For $n \geq 2$ any map $RP^n \to (S^1)^n$ is *nullhomotopic*.

5 a) Define a *compactification* of a space X, and define when *two compactifications* of a space X are *equivalent*.
b) Consider $X = \{(x,y) \in R^2 \mid y = x \text{ or } y = -x \text{ for } -1 < x < 1\}$ with the subspace topology from the standard topology on R^2. Describe an imbedding $h : X \to R^2$ whose associated compactification is equivalent to the 2-point compactification of X. (You may use arcs and arrows for the description.)
c) For the space X given in b) describe an imbedding $h : X \to R^3$ whose associated compactification is equivalent to the 1-point compactification of X.
d) For the space X given in b) describe an imbedding (if it exists) $h : X \to R^2$ whose associated compactification is equivalent to the 1-point compactification of X.

For your information:

R, R^n : real line, real n – space, with standard topology unless otherwise indicated

S^n : unit sphere in R^{n+1} sometimes viewed in a larger R^{n+k}

RP^n : real projective n – space

Z : the set of all integers

$(X)^n$: n – fold Cartesian product

Topology Qualifying Exam
August 20, 2007

Do all of the following problems.

1. (20 points) Let X be a set.

   (a) Define what it means to say that the collection $\mathcal{B}$ of subsets of X is a *basis* for a topology on X.

   (b) Suppose now that X is endowed with a topology $\mathcal{T}$. Define what it means to say that the subset C of X is *compact*

   (c) Let R be the real line. Give a basis for the lower limit topology on R. Let R_l denote R with this topology.

   (d) Show that if $C \subset R_l$ is compact, then C is closed and bounded.

   (e) Show that the converse of (d) is false.

2. (20 points) Let X be a topological space with topology $\mathcal{T}$.

   (a) Define what it means to say that X is *connected.*

   (b) Let D be a subset of X. Define what it means to say D is *dense* in X.

   (c) Suppose X is connected and that $D \subset X$ is dense. Let $\mathcal{T}'$ be the topology on X with subbasis $\mathcal{T} \cup \{D\}$. Show that $(X, \mathcal{T}')$ is connected.

3. (20 points) Recall that a topological space X is said to be *completely regular* if one-point sets are closed in X and if for all $x_0 \in X$ and all closed sets C in X with $x_0 \notin C$, there is a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(C) = 0$.

   (a) Suppose X is a normal space in which one-point sets are closed. Show that X is completely regular.

   (b) Show that if X is completely regular, then there is an embedding $F : X \to R^A$ for some index set A. (Hint: Recall the proof of the Urysohn Metrization Theorem.)

Questions 4 and 5 refer to the same space N called the *necklace with two hollow beads.* N is the subspace of R^3 defined as follows: Let

$$A_1 = \{(x, y, z) \in R^3 | x^2 + (y - 1/2)^2 = 1, y \geq 1/2, z = 0\}$$

$$A_2 = \{(x, y, z) \in R^3 | x^2 + (y + 1/2)^2 = 1, y \leq -1/2, z = 0\}$$

$$B_1 = \{(x, y, z) \in R^3 | (x-1)^2 + y^2 + z^2 = 1/4\}$$

$$B_2 = \{(x, y, z) \in R^3 | (x+1)^2 + y^2 + z^2 = 1/4\}$$

Then

$$N = A_1 \cup A_2 \cup B_1 \cup B_2$$

You should draw a sketch this space.

4. (20 points) Let N be the space above and $n_0 = (0, 3/2, 0)$. Use the Seifert - van Kampen Theorem to find $\pi_1(N, n_0)$. In particular,
   (a) State the Seifert - van Kampen Theorem.
   (b) Show how you will apply the Theorem.
   (c) Carry out the necessary calculations giving full details.
5. Let $p : E \to B$ be a map.
   (a) (20 points) Define what it means to say p is a *covering map* and a *universal covering map*.
   (b) Construct a universal covering space E of the space N above. You may describe E as a labelled picture and describe p on the various parts of E.

January 10, 2007

NAME: ________________

Topology Qualifying Examination

Every answer must be supported with reasoning (proof!). Please start each of the five problems on a new page in the blue book.

1. (20 points) Let X be the set of $n \times n$ real matrices with the Euclidean topology induced from R^{n^2} via the bijection $A = (a_{ij}) \mapsto (a_{11}, a_{12}, ..., a_{21}, ..., a_{nn})$, and let Y be the subspace of all matrices A satisfying $A^T A = I$, $A^T = transpose(A)$, $I = identity\ matrix$.
a) Is Y compact?
b) Is Y normal?
c) If S^{n-1} is the unit sphere in R^n, is the function $f: S^{n-1} \times Y \rightarrow S^{n-1}, f(x, A) = A(x)$, continuous?

2. (20 points) a) Define a *locally compact space.*
b) Show that the rationals are not locally compact.
c) Assume as known that a finite Cartesian product of compact spaces is itself compact. Let K_i be a compact subspace of the space X_i, $i = 1,2,\ldots,n$. Show that the subspace which is the product of the K_i in the product of the X_i is a compact subspace.
d) If each X_i is a locally compact space $i = 1,2,\ldots,n$, show that the product of the X_i is also locally compact.

3. (20 points) a) Define and compare the product and box topologies on R^ω, the countable product of real lines.
b) Let X be the subset of all sequences $(x_1, x_2, ...)$ with $x_i \neq 0$ for only finitely many i. Describe the closure of X in each topology.

4. (20 points) Let $p: E \rightarrow B$ be a covering map. If U is evenly covered and $\{V_\alpha\}$ is a partition of $p^{-1}U$ into slices and C is a closed set of B such that $C \subset U$, then show $(p^{-1}C) \cap V_\alpha$ is a closed subset of E.

5. (20 points) a) Define a *covering space* and a *universal covering space.*
b) Construct a universal covering space $p: E \rightarrow B$, where $B = S^2 \cup D$ is the union of the unit sphere S^2 in 3-space and D the diameter of S^2 with endpoints $(-1, 0, 0)$ and $(1, 0, 0)$. You may describe E as a labeled picture and describe p on each part of E.
c) Show that the two paths $\alpha(s) = (2s-1, 0, 0)$ and $\beta(s) = (\cos(s+1)\pi, \sin(s+1)\pi, 0)$ in B are not path homotopic.

d) Let $X = \{x \in R^3 \mid \|x\| \leq 1\}$ be the unit ball in 3-space and let C be the circle centered at the origin of radius $\frac{1}{2}$ in the plane $x_1 = 0$. Describe a deformation retraction of the complement $X - C$ onto the space B described in b). (Hint: describe the deformation in each 2-dim plane that contains the subset D described in b).)

Topology Part.

Do all five problems as directed. All answers are to be supported by proofs and/or reasoning.

1. Do any two of (a), (b) or (c).
   - (a) Let $f_i\colon X_i \to Y_1$, $i = 1, 2$ be continuous maps. With respect to the product topologies, show that $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$, $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is continuous.
   - (b) Let $\prod_\alpha X_\alpha$ denote the product space with the product topology, and let $A_\alpha \subset X_\alpha$ for all $\alpha \in J$. Show that $\prod_\alpha \overline{A}_\alpha = \overline{\prod_\alpha A_\alpha}$
   - (c) Show that the diagonal function $\Delta\colon \mathbb{R} \to \mathbb{R}^\omega$ is continuous with respect to the product topology, and not continuous with respect to the box topology, into the countable product of the real line.

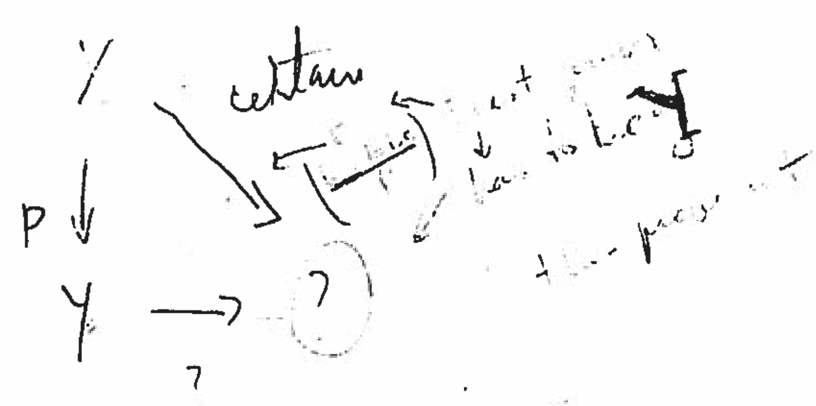
2. Do all three parts.
   - (a) Show that a continuous surjection $p\colon X \to Y$ is a quotient map if for every space Z and each function $g\colon Y \to Z$, g is continuous whenever the composition $g \circ p$ is continuous.
   - (b) Is the projection of the xy-plane onto the x-axis a quotient map?
   - (c) Recall that there is the known criterion for asserting when a function $Z \to \prod_\alpha X_\alpha$ into a product space is continuous. State and prove a corresponding assertion for (certain) functions out of a quotient space Y ($p\colon X \to Y$ a quotient map) to be continuous.

3. Recall that a space X is *locally compact at x* if there is a compact subspace C of X that contains a neighborhood (i.e. an open set) of x. Show for a Hausdorff space that is locally compact at x, that for each neighborhood U of x there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

4. Do either (a) or (b).
   - (a) Show if X is normal, every pair of disjoint closed sets has neighborhoods whose closures are disjoint.
   - (b) Show if X has the order topology, X is regular.

5. Do either (a) or (b).
   - (a) Let X be compactly generated space and let (Y, d) be a metric space. Show that the space of continuous functions $C(X, Y)$ is closed in Y^X in the topology of compact convergence. (Recall that a function f in this topology is continuous if its restriction to each compact set C is continuous.)
   - (b) Let X be locally compact Hausdorff and let the space of continuous functions $C(X, Y)$ from X to Y have the compact-open topology. Show the function $e\colon X \times C(X, Y) \to Y$, $e(x, f) = f(x)$ is continuous. (Suggestion: begin with any open set in Y and use that f is continuous and X is locally compact Hausdorff.)

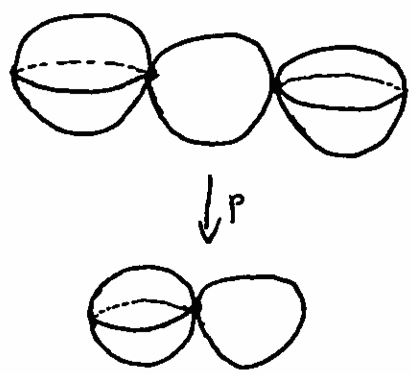
Solution to 2) c)



TOPOLOGY (661, 664)
Fall 2002

Examiner: Ucci
Observer: Anderson
(both asked questions)

1. Define regular, normal, locally compact.
2. Prove that a locally compact Hausdorff space is regular.
3. Name a topological property that is not generally inherited by subspaces.
4. Define retraction, identification. Prove that every retraction is an identification.
5. What is a covering space?
6. What is the universal covering space of S^1?
7. If the following p is a covering map, what is the map p?



$S^2 \vee S^1 \vee S^2$

$S^2 \vee S^1$

What is the fundamental group of $S^2 \vee S^1$? What is p_*?
What is the universal covering space of $S^2 \vee S^1$?

8. What is the Möbius band? Is the boundary of the Möbius band M a retract of M? What about the centerline of M?

9. Define exact sequence (of groups), chain complex. Is every chain complex exact?

10. Define chain map. If f is a chain map, what is f_*? Show that f_* is well-defined.

11. Define the connecting homomorphism of a pair (X, A). Show that it is well-defined.

12. What is the Mayer-Vietoris sequence? What is the map $H_n(X_1 \cap X_2) \longrightarrow H_n(X_1) \oplus H_n(X_2)$?

13. What lemma is used to prove the exitence of the Mayer-Vietoris sequence. Can you prove it?

Topology (L.G. Lewis - examiner, J. Ucci - observer)

1) Let $T = S^1 \times S^1$ the torus. What is $H_n(T)$ $\forall n \geq 0$? Calculate this.

2) What is $H_n(S^1 \vee S^1 \vee S^2)$ $\forall n \geq 0$? Is $S^1 \vee S^1 \vee S^2$ homeomorphic to T? Are they homotopy equivalent?

3) Suppose $f:[0,1] \to [3,5]$ is a homeomorphism. What can you say about $f(0)$? Prove it.

4) Suppose (E,p) is a covering space of S^3. What can you say about E?

5) Suppose X is a space with $\pi_1(X,x_0) = \mathbf{Z}_6$. What can you say about the covering spaces of X? What "nice" properties of X do you need for this?

6) Let X be a compact Hausdorff space. What properties does X have? Prove X is regular.

7) Consider $\mathfrak{R}^\omega$. Let $\mathfrak{R}^\infty = \{\{x_n\} \in \mathfrak{R}^\omega \mid x_n = 0$ for all but finitely many $n\} \subseteq \mathfrak{R}^\omega$. What is the closure of $\mathfrak{R}^\infty$ in
   (a) the product topology?
   (b) the box topology?

8) Calculate $H_n(X,A)$

Homework Notes

HW1

- basis elements
  - every $x \in B_i$ for some i
  - $x \in B_i \cap B_j \implies \exists\ B_x$ s.t. $B_x \subset B_i \cap B_j$

eg balls is basis for topo of a metric space X.

HW2

- $\overline{A} = \text{int}(A) \cup \text{bd}(A)$, $\text{int}(A) \cap \text{bd}(A) = \emptyset$
- $f: \mathbb{R}^2 \to \mathbb{R}^2$ dist. preserving is cts.
- use pasting lemma 
- limit point: every open set $U \ni x \to U \cap A \setminus \{x\} \neq \emptyset$ A'
- closed iff contains limit pts

  $\hookrightarrow$ $\overline{A} = A \cup A'$

HW3

- bdd = contained in ball
- cts & cmpct $\to$ uniformly cts
- limit pt cmpct $\longrightarrow$ every ∞ set has a limit pt.
  - cmpct is limit pt cmpct
  - metric space: lim pt cmpct $\longleftrightarrow$ seq. cmpct $\longleftrightarrow$ cmpct.

Hdorff:

- regular: $x \in X$, closed C w/ $x \notin C \to \exists$ open U, V disjoint w/ $x \in U$, $C \subset V$ 
- normal. C, D disjoint closed sets $\longrightarrow \exists\ U, V$ disjoint open $C \subset U$, $D \subset V$ 

HW4

- X is connected iff only subsets of X that are both open + closed are ϕ, X
- $A \subset X$ is sep. iff $\exists$ 2 open sets $U, V \subset X$ s.t. $A \subset U \cup V$ $\Longrightarrow$ $U \cap V \cap A = \emptyset$, $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$
- $A \subset X$ connected iff $\exists$ 2 open set $U, V \subset X$ s.t. $U \cap V \cap A = \phi$, $A \subset U \cup V \Longrightarrow A \subset U$ or $A \subset V$
- I.V.T.
- union of path connected sets + a pt. in common is path connected

---

HW5

- quotient topo $\longrightarrow$ open iff open + closed iff closed
- f is surjective, f is 1-1 except when ($f = \cup$) $\longrightarrow$ identification
- locally compact @ x $\longrightarrow$ open U, compact C $x \in U \subset C$
- $\mathbb{R}^\infty \longrightarrow$ info
- Annulus $\longrightarrow$ T^2

Exam 1.

$\not{\beta}$ - want Ex of non Hausdorff set

$\not{\beta}$ go through later

Homework 6

- homeo $g: [0,1) \longrightarrow [0, \infty)$ $g = \frac{x}{1-x^2}$
- homeo $B(y, r)$ onto $B(0,1)$.
- homeo $B(0,1) \longrightarrow \mathbb{R}^n$
- $h: M \to N$ is homeo $h|_{\partial M} \cdot \partial N$ is homeo
- $\mathbb{H}^2 \approx H^2$.
- K is quotient space of $\mathbb{R}^2$
- S^1 - compact connected, covered w/ 2 open sets $U \cap V \approx$ 2 open rays.

$$\begin{array}{ccc} [0,1] & \xrightarrow{f} & \\ \downarrow q & & S^1 \\ [0,1]/\sim & \xrightarrow{\bar{f}} & \end{array}$$

- $M \longrightarrow$ Cylinder

HW 7

- $D^2/\sim \approx S^2$
- or. / o.p.
- O.R. homeo isotopic $= id$? $r(x,y)$.
- $T \to K$
- $T \to S^1_a \times \{p\} \cup \{p\} \times S^1_b$

HW 8

- handle bodies
- 1 0-handle = connected.
- Specific examples of attaching things.

HW9

- $N_{(p)} \sqcup S_{(q)} \simeq N_{(p+q-1)}$
- $A \# B \# C \simeq A_{(1)} \sqcup B_{(1)} \sqcup C_{(1)} \cup h^2$
- handle slides
- χ from orientable & slides

HW10

- $\chi(T \# T)$.
- surfaces
- homotopy: $\bar{f} * f \simeq e$, $e_x * f \to f$.

HW11 $\longrightarrow$ Good Algebra Review

- $(\mathbb{R}, +)$, $(S^1, \overset{\text{Com mult}}{\cdot})$, $(GL(2,\mathbb{R})$ are topo gps
- subgroups
- homotopy & gps
- mx examples.

HW12

- retractions
- mx examples
- $M \not\cong A$ $\to$ not using orientation
- gp presentation using SVK. $\hookrightarrow T$, $\hookrightarrow K$.
- $K \approx P^{(2)} = P \# P$

· HW 1

- $f_t: X \to X$ def ret. $g_t: A \to A$ def ret. $\to g \circ f = h_t$ d.r.
- inclusion is null homotopic $\to$ $A \subset$ path component
- P+ set problem

- X def retr. to $x \in X$ $\to$ $\forall\, U \ni x\ \exists\, V \subset U$ s.t. $V \hookrightarrow U$ null homotopic.
- null homotopic : homotopic to constant. (2, 4)
- + def. retract.

Map Extend

HW2 → H Equiv

- def ret through $M \cup A$ $\to$ $M \#_s A$
- PROBLEM 2 → CW pair has HEP
- HEP. → Wedges.
  → investigate connection btwn π_1 & homotopy.

HW3

- UPLP
- P1 → good example of being specific
- Covering spaces, deck transformations

HW 4

Vectors

HW5

$f: S^1 \longrightarrow S^1$ $f \simeq c$, $g: Y \longrightarrow S^1$ $g \simeq c$

Y path-con loc. path-con , finite fund gp.

Problem 1 pf

· Cell make up of S^2, $\mathbb{RP}^2$

↳ · Connecting covers & local homeos.

$$\begin{tikzcd} & \tilde{X} \supset \tilde{U} \arrow[d, "p"] \\ U \arrow[ur, "\tilde{i}"] \arrow[r, hook] & X \supset U \end{tikzcd}$$

$$\begin{tikzcd} \tilde{U} \arrow[r] \arrow[d, "p|_{\tilde{U}}"'] & \tilde{X} \arrow[d, "p"] \\ U \arrow[r, hook] \arrow[ur] & X \end{tikzcd}$$

↳ evenly covered nbhd.

⟶ $\pi_1(\tilde{U}) \longrightarrow \pi_1(\tilde{X})$ is trivial

- $p_1: \tilde{X} \to X$ $p_2: \tilde{\tilde{X}} \longrightarrow \tilde{X}$ p_1 is finite sheeted $p_1 \circ p_2$ is covering

↳ also works if X is lpc & slsc.

HW6.

- 3 sheeted cover of K.
- X "nice" → $\tilde{X}$ path con n-sheeted coveringspace for n prime → $G(\tilde{X}) = 0$ or $\tilde{X}$ is normal + $G(\tilde{X}) \cong \mathbb{Z}_n$

↳ # of sheets is index of sbgp $p_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$

good explanation of Wythes covering explanation for $\mathbb{RP}^n$

3: Covering / group actions

4. Chain Complexes

HW7

- For def. retr. F show reduced singular chain complex of X is contractible. → good to look over if time, not super helpful.
  - → simplex
  - → use cellular normally.

- A is retract of X → $H_n(A) \xrightarrow{i_*} H_n(X)$

  i_* is injective.

- <u>Split</u> A chain complex C w/ differential ∂ is split if $\exists$ family of homomorphisms $S_n : C_n \to C_{n+1}$ s.t. $\partial = \partial \circ s \circ \partial$.
  - → Z_n & B_n $\not\subset$ → important role → $Z_n / B_n \cong H_n$
  - $\not\subset$ → for vector spaces

- compact simplicial chain complex & <u>simplicial hom.</u>

Homology of universal cover of $S^1 \vee S^1 \vee S^2$ is not the same as homology of $\mathbb{R}^2$

I want to show, that although the homology groups of $X := S^1 \vee S^1 \vee S^2$ and the torus T^2 are isomorphic, the homology groups of their universal covers are not. Let U_X be the universal cover of X

The first part was easy since $\tilde{H}_k(Y_1 \vee Y_2) = \tilde{H}_k(Y_1) \oplus \tilde{H}_k(Y_2)$. The universal cover of T^2 is $\mathbb{R}^2$, since $\mathbb{R}$ is the universal cover of S^1. Therefore $H_k(\mathbb{R}^2) = 0$ if $k > 0$ and $H_0(\mathbb{R}^2) = \mathbb{Z}$. Of course computing $H_k(U_X)$ is a way to solve this problem, but I don't know how compute U_X. Also my tutor told me, that this is not necessary and the properties of the universal cover are sufficient to solve this.

my thoughts so far

I think, that showing $H_2(U_X) \neq 0$ is now the best way to proof the statement, because $H_2(S^2) = \mathbb{Z}$ and S^2 is the universal cover of S^2 and has to be "contained" in U_X somehow...

Let $p\colon U_X \to X$ the covering map. I tried to show that the induced map $p_*\colon H_2(U_X) \to H_2(X) \cong \mathbb{Z}$ has non-trivial image (or even is surjective) by looking at the pair sequences of (U_X, A) and (X, B), where $A \subset U_X$ is one of the disjoint subsets of $p^{-1}(B)$, but that did not work out.

Thanks in advance for any help :)

PS: I already read this, but it didn't help me for the reasons mentioned above.

(algebraic-topology)

edited Apr 13 '17 at 12:21
Community ♦
1

asked Dec 17 '14 at 22:54
JBantje
198 1 9

2 It's not too hard to see what U_X looks like. The universal cover of $S^1 \vee S^1$ is the Cayley graph on the free group of two letters, and the universal cover of X is simply that tree with a S^2 attached at each vertex. As your link mentions, this deformation retracts onto an infinite wedge sum of spheres. Alternatively, you could argue that $H_2(U_X) \cong \pi_2(U_X) \cong \pi_2(X)$, and show that this last group receives an injection from $\pi_2(S^2) \cong \mathbb{Z}$, but this seems complicated. – JHF Dec 18 '14 at 17:40

Answer

The universal cover of T is contractible hence homology vanishes. Now consider $\hat{X}$ the universal cover. You know that $H_2(X) \neq 0$ and you also know that you have the inclusion $i : S^2 \to X$ and that $H_2(i) : H_2(S^2) \to H_2(X)$ is non-trivial. But S^2 is simply connected, hence there is a lift $\bar{i} : S^2 \to \hat{X}$ and since $H_2(i) = H_2(p\bar{i}) = H_2(p)H_2(\bar{i})$ is non trivial as mentioned before, so is $H_2(\bar{i})$. In particular $\hat{X}$ has non-trivial homology.

answered Dec 18 '14 at 22:43
Daniel Valenzuela
5,324 7 18

Prob. 10 (b), Sec. 25 in Munkres' TOPOLOGY, 2nd ed: How to show that components and quasicomponents are the same for locally connected spaces?

Let X be a topological space; let us define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$.

This relation is an equivalence relation; the equivalence classes are called the quasicomponents of X. This I can show.

Moreover, each connected component of X lies in a quasicomponent of X. This too have I managed to show.

How to show that the components and quasicomponents of X are the same if X is locally connected?

My effort:

Suppose that X is locally connected.

Let C be a component of X and let Q be the quasicomponent containing C. Suppose that $C \subset\neq Q$.

Let D be the union of all the components of X different from C that intersect Q. Then $Q = C \cup D$.

And, $C \cap D = \emptyset$ because distinct components, being distinct equivalence classes of a certain equivalence relation, are non-empty and disjoint.

Now since X is locally connected, components of open sets in X are open.

Since X itself is open, C is open and D is a union open sets and therefore open.

Thus we have written Q as a union of two disjoint non-empty sets each of which is open in X.

Now let $x \in C$ and $y \in D$. Then can we obtain a separation $X = A \cup B$ of X into disjoint non-empty open sets A and B such that $x \in A$ and $y \in B$?

Or, can we obtain a contradiction some other way?

Following the hint given in Stefan Hamcke's answer:

Suppose that X is locally connected and suppose also that the component C is properly contained in the quasicomponent Q.

Components of open sets are open in any locally connected topological space, and components of any topological space are closed.

Since X is open, therefore C is both open and closed in X.

Now suppose that $x \in C$ and $y \in Q - C$.

Then $X = C \cup (X - C)$ is a separation of X into disjoint non-empty open subsets C and $X - C$ such that $x \in C$ and $y \in X - C$. Then $x \nsim y$, which contradicts the assumption that $x, y \in Q$ and Q is a quasicomponent and hence $x \sim y$.

Thus our sopposition that C is properly contained in Q is wrong. So $C = Q$.

Is this proof correct now?

(general-topology) (algebraic-topology) (connectedness)

edited May 10 '15 at 18:06

asked May 10 '15 at 16:02
Saaqib Mahmood
7,095 4 21 69

1 Answer

You correctly noted that components in a locally connected space are open. Can you say something about closedness of components (maybe even for general topological spaces)?

answered May 10 '15 at 16:18
Stefan Hamcke
20.9k 4 24 75

yes the components of any topological space are closed. – Saaqib Mahmood May 10 '15 at 17:49

@SaaqibMahmuud: Your proof is correct now. – Stefan Hamcke May 10 '15 at 18:17

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Computing the homology of groups: The geometric way☆

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ABSTRACT

In this paper, we present several algorithms related with the computation of the homology of groups, from a geometric perspective (that is to say, carrying out the calculations by means of simplicial sets and using techniques of Algebraic Topology). More concretely, we have developed some algorithms which, making use of the *effective homology* method, construct the homology groups of Eilenberg–MacLane spaces $K(G, 1)$ for different groups G, allowing one in particular to determine the homology groups of G.

Our algorithms have been programmed as new modules for the Kenzo system, enhancing it with the following new functionalities:

- construction of the effective homology of $K(G, 1)$ from a given finite type free resolution of the group G;
- construction of the effective homology of $K(A, 1)$ for every finitely generated Abelian group A (as a consequence, the effective homology of $K(A, n)$ is also available in Kenzo, for all $n \in \mathbb{N}$);
- computation of homology groups of some 2-types;
- construction of the effective homology for central extensions.

In addition, an *inverse* problem is also approached in this work: given a group G such that $K(G, 1)$ has effective homology, can a finite type free resolution of the group G be obtained? We provide some algorithms to solve this problem, based on a notion of *norm* of a group, allowing us to control the convergence of the process when building such a resolution.

1. Introduction

When using homological algebra techniques to study group theory, two different (but related) alternatives are possible (see Brown (1982) for details on the following discussion). One is algebraic

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and is based on the notion of *resolution* (replacing the group under study by an acyclic object in a suitable category of modules). The other alternative is geometric in nature. It consists in finding a contractible topological space with a free action of a group G. Then the space of orbits of the action can be endowed with a convenient quotient topology, in such a way that we obtain an *aspherical* space (that is to say, a space whose only non-null homotopy group is the first, fundamental one). The homology of this space is, by definition, the homology of G, and it does not depend on the choosing of the contractible space or of the action. Each aspherical space (unique up to homotopy type) is a particular *Eilenberg–MacLane space* for G, and is generically denoted by $K(G, 1)$.

If we move to *computational* mathematics, the preferred via chosen was the algebraic one, as exemplified by the package HAP (Ellis, 2009) of the computer algebra system GAP (The GAP Group, 2008), which contains an impressive number of algorithms dealing with resolutions. The geometric way has been up to now neglected from the algorithmic point of view. The reason is that the contractible spaces to be constructed are very frequently of infinite type (even in cases where the group G is not too complicated), apparently closing the possibility of a computational treatment.

This view changed drastically when Sergeraert introduced at the end of the 1980's his theory of *effective homology* (Sergeraert, 1994). His methods allow the programmer to deal with spaces of infinite dimension, encoded in a lazy functional programming style, producing a complete revision of Algebraic Topology from a constructive point of view (see Rubio and Sergeraert (2006) for recent developments of this theory). Perhaps more important from a practical point of view was Sergeraert's construction of the *Kenzo* system, a Common Lisp program implementing the effective homology methods (Dousson et al., 1999). Since then, the programmer can work on a computer with simplicial sets, loop spaces, fibrations, classifying spaces and many other Algebraic Topology constructions, computing, at the end, homology groups of complicated spaces (under the combinatorial form of *simplicial sets*).

Taking into account this new situation, this paper represents a first step to take up again the geometrical way of approaching *group homology*, by means of techniques from effective homology and using, and extending, Kenzo as a computing platform.

Our proposal is not opposed to the algebraic view. Our aim is rather to take the best of both worlds. Therefore, and as a first module, we programmed, in collaboration with Graham Ellis (see Romero et al., 2009), an OpenMath link between HAP and Kenzo, allowing Kenzo to import from HAP resolutions of groups. Once a resolution of a group G is internally stored in Kenzo, an algorithm allows us to construct the Eilenberg–MacLane space $K(G, 1)$, with *effective homology*. This provides not only access to some homology groups of G, but also makes it possible to apply on the space $K(G, 1)$ all the powerful tools available in Kenzo, and construct in this way further spaces.

This via is explored in this paper. We show two applications in Algebraic Topology, and another one in Homological Algebra. As a first application, we develop a Kenzo package to compute, as *objects with effective homology*, the generalized Eilenberg–MacLane spaces $K(G, n)$ for any finitely generated Abelian group G and for all $n \in \mathbb{N}$. These objects are very important in Algebraic Topology, to study and compute homotopy groups, through Whitehead and Postnikov towers (see May (1967) and Rubio and Sergeraert (2006)).

As a second topological application, we compute mechanically (for the first time, up to our knowledge) some homology groups of 2-types, the second step (the first one consists of Eilenberg–MacLane spaces) towards the difficult problem of characterizing homotopy types.

Our last application provides programs to deal with the effective homology of central extensions of groups. The theoretical algorithms were known some time ago (see Rubio, 1997), but only now the technological tools explained before allow us to tackle the problem of programming them. Let us observe that this *algebraic* application has also positive consequences on topological problems, since it enlarges the field of application of our 2-types package: we can also compute with 2-types whose fundamental group is a central extension.

Finally, we have also approached an *inverse* problem: how to obtain a resolution of a group G from the knowledge of an effective homology of $K(G, 1)$. The results in this area are still partial, and more research will be needed to get fully satisfactory algorithms, and to proceed to implement them as Kenzo modules.

The organization of the paper is as follows. The next section is devoted to preliminaries. Section 3 contains our main algorithm, which constructs the effective homology of $K(G, 1)$ from a finite type resolution of a group G, and then Section 4 collects some interesting fields of application of this result. Section 5 explains how our algorithms have been translated to Common Lisp and comments on experimental results. In Section 6 an inverse problem is considered: given a group G with effective homology, it is (sometimes) possible to determine a resolution for G. The paper ends with conclusions, open problems and the bibliography.

2. Definitions and preliminaries

2.1. *Some fundamental notions about homology of groups*

The following definitions and important results about homology of groups can be found in MacLane (1963) and Brown (1982).

Definition 1. Given a ring R, a *chain complex* of R-modules is a pair of sequences $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ where, for each degree $n \in \mathbb{Z}$, C_n is an R-module and $d_n : C_n \to C_{n-1}$ (*the differential map*) is an R-module morphism such that $d_{n-1} \circ d_n = 0$ for all n.

Definition 2. Let $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ be a chain complex of R-modules, with R a general ring. For each degree $n \in \mathbb{Z}$, the *nth homology module* of C_* is defined to be the quotient module $H_n(C_*) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. A chain complex C_* is *acyclic* if $H_n(C_*) = 0$ for all n.

Definition 3. Let G be a group and $\mathbb{Z}G$ the free $\mathbb{Z}$-module generated by the elements of G. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}G$ which makes $\mathbb{Z}G$ a ring. This is called the *integral group ring* of G.

Definition 4. A *resolution* F_* for a group G is an acyclic chain complex of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} = \mathbb{Z} \longrightarrow 0$$

where $F_{-1} = \mathbb{Z}$ is considered a $\mathbb{Z}G$-module with the trivial action and $F_i = 0$ for $i < -1$. The map $\varepsilon : F_0 \to F_{-1} = \mathbb{Z}$ is called *augmentation*. If F_i is free for each $i \geq 0$, then F_* is said to be a *free resolution*.

Very frequently, resolutions come equipped with a *contracting homotopy* h, which is a set of Abelian group morphisms $h_n : F_n \to F_{n+1}$ for each $n \geq -1$ (in general not compatible with the G-action), such that

$$\begin{aligned}
\varepsilon h_{-1} &= \mathrm{Id}_{\mathbb{Z}} \\
h_{-1}\varepsilon + d_1 h_0 &= \mathrm{Id}_{F_0} \\
h_{i-1} d_i + d_{i+1} h_i &= \mathrm{Id}_{F_i}, \quad i > 0.
\end{aligned}$$

The existence of the contracting homotopy for F_* assures in particular the exactness of the resolution.

Given a free resolution F_*, one can consider the chain complex of $\mathbb{Z}$-modules (that is to say, Abelian groups) $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ defined by

$$C_n = (\mathbb{Z} \otimes_{\mathbb{Z}G} F_*)_n, \quad n \geq 0$$

(where $\mathbb{Z} \equiv C_*(\mathbb{Z}, 0)$ is the chain complex with only one non-null $\mathbb{Z}G$-module in dimension 0, $C_0 = \mathbb{Z}$) with differential maps $d_{C_n} : C_n \to C_{n-1}$ induced by $d_n : F_n \to F_{n-1}$.

Let us emphasize the difference between the chain complexes F_* and $C_* = \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$. The elements of F_n ($n \geq 0$) can be seen as *words* $\sum \lambda_i(g_i, z_i)$ where $\lambda_i \in \mathbb{Z}$, $g_i \in G$ and z_i is a generator of F_n (which is a *free* $\mathbb{Z}G$-module). On the other hand, the associated chain complex $C_* = \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$ of Abelian groups has elements in degree n of the form $\sum \lambda_i z_i$ with $\lambda_i \in \mathbb{Z}$ and z_i a generator of the free $\mathbb{Z}$-module C_n.

Although the chain complex of $\mathbb{Z}G$-modules F_* is acyclic, $C_* = \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$ is in general not exact and its homology groups are thus not null. An important result in homology of groups claims that these homology groups are independent of the chosen resolution for G.

Theorem 5 (Brown, 1982). *Let G be a group and F_*, F'_* two free resolutions of G. Then*

$$H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F'_*) \quad \text{for all } n \in \mathbb{N}.$$

The hypothesis that F_* and F'_* are free can in fact be relaxed; it suffices for the modules F_* and F'_* to be *projective*. This theorem leads to the following definition.

Definition 6. Given a group G, the *homology groups* $H_n(G)$ are defined as

$$H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*), \quad n \in \mathbb{N}$$

where F_* is any free (or projective) resolution for G.

Let G be a group, how can we determine a free resolution F_*? One approach is to consider the *Bar resolution* $B_* = \mathrm{Bar}_*(G)$ (explained, for instance, in MacLane (1963)) whose associated chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ can be viewed as the chain complex of the Eilenberg–MacLane space $K(G, 1)$ (see Brown (1982), for details). The homology groups of $K(G, 1)$ are those of the group G and this space has a big structural richness. But it has a serious drawback: its size. If $n > 1$, then $K(G, 1)_n = G^n$. In particular, if $G = \mathbb{Z}$, the space $K(G, 1)$ is of infinite type in each dimension. This fact is an important obstacle to using $K(G, 1)$ as a means for computing the homology groups of G. It would be therefore convenient to construct *smaller* resolutions.

For some particular cases, small (or minimal) resolutions can be directly constructed. For instance, let G be the cyclic group of order m with generator t, $G = C_m$. The resolution F_* for G

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where N denotes the *norm element* $1 + t + \cdots + t^{m-1}$ of $\mathbb{Z}G$, produces the chain complex of Abelian groups

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

and therefore

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/m\mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even and } i > 0. \end{cases}$$

But in general it is not so easy to obtain a resolution for a group G, and in fact this problem provides an interesting research field where many papers and works have appeared trying to determine resolutions for different kinds of groups. As we will see later, the GAP package HAP has been designed as a tool for constructing resolutions for a wide variety of groups. On the other hand, our work shows that the *effective homology* method, introduced in the following section, could also be helpful in order to compute the homology of some groups.

2.2. *Effective homology*

We now present the general ideas of the effective homology method, devoted to the computation of homology groups of *spaces*. See Rubio and Sergeraert (2002) and Rubio and Sergeraert (2006) for more details.

Definition 7. A *reduction* ρ between two chain complexes $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$ (which is denoted $\rho : C_* \Rrightarrow D_*$) is a triple (f, g, h) where: (a) the components f and g are chain complex morphisms $f : C_* \to D_*$ and $g : D_* \to C_*$; (b) the component h is a homotopy operator $h : C_* \to C_{*+1}$ (a graded group morphism of degree $+1$); (c) the following relations are satisfied: $fg = \mathrm{Id}_D$; $d_C h + h d_C = \mathrm{Id}_C - gf$; $fh = 0$; $hg = 0$; $hh = 0$.

These properties express that C_* is the direct sum of D_* and an acyclic complex. This decomposition is simply $C_* = \mathrm{Ker} f \oplus \mathrm{Im}\, g$, with $\mathrm{Im}\, g \cong D_*$ and $H_*(\mathrm{Ker} f) = 0$. In particular, this implies that the graded homology groups $H_*(C_*)$ and $H_*(D_*)$ are canonically isomorphic.

Remark 8. A reduction is in fact a particular case of chain equivalence in the classical sense (see MacLane (1963), page 40), where the homotopy operator on the small chain complex D_* is the null map.

Definition 9. A *(strong chain) equivalence* ε between two chain complexes C_* and D_*, denoted by $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple (B_*, ρ_1, ρ_2) where B_* is a chain complex, and ρ_1 and ρ_2 are reductions $\rho_1 : B_* \Rrightarrow C_*$ and $\rho_2 : B_* \Rrightarrow D_*$.

Remark 10. We need the notion of *effective* chain complex: it is essentially a free chain complex C_* where each group C_n is finitely generated, and a provided algorithm returns a (distinguished) $\mathbb{Z}$-basis in each degree n; in particular, its homology groups are elementarily computable (for details, see Rubio and Sergeraert (2002)).

Definition 11. An *object with effective homology* X is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where $C_*(X)$ is a chain complex canonically associated with X (which allows us to study the homological nature of X), HC_* is an effective chain complex, and ε is an equivalence $\varepsilon : C_*(X) \Lleftarrow\!\!\Rrightarrow HC_*$.

It is important to understand that in general the HC_* component of an object with effective homology is *not* made of the homology groups of X; this component HC_* is a free $\mathbb{Z}$-chain complex of finite type, in general with a non-null differential, whose homology groups $H_*(HC_*)$ can be determined by means of an elementary algorithm. From the equivalence ε one can deduce the isomorphism $H_*(X) := H_*(C_*(X)) \cong H_*(HC_*)$, which allows one to compute the homology groups of the initial space X. In this way, the notion of object with effective homology provides a method to compute homology groups of complicated spaces by means of homology groups of effective complexes.

The effective homology technique is based on the following idea: given some topological spaces $X_1, \dots, X_n$, a topological constructor Φ produces a new topological space X. If effective homology versions of the spaces $X_1, \dots, X_n$ are known, then one should be able to build an effective homology version of the space X, and this version would allow us to compute the homology groups of X. A typical example of this kind of situation is the loop space constructor. Given a 1-reduced simplicial set X with effective homology, it is possible to determine the effective homology of the loop space $\Omega(X)$, which in particular allows one to compute the homology groups $H_*(\Omega(X))$. Moreover, if X is m-reduced, this process may be iterated m times, producing an effective homology version of $\Omega^k(X)$, for $k \le m$. Effective homology versions are also known for classifying spaces or total spaces of fibrations, see Rubio and Sergeraert (2006) for more information.

All these constructions have been implemented in the Kenzo system (Dousson et al., 1999), a Common Lisp program which makes use of the effective homology method to determine homology groups of complicated spaces; it has obtained some results (for example homology groups of iterated loop spaces of a loop space modified by a cell attachment, components of complex Postnikov towers, etc.) which had never been determined before. Furthermore, Kenzo implements Eilenberg–MacLane spaces $K(G, n)$ for every n but only for $G = \mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z}$ (these spaces appear in different constructions of Algebraic Topology), although in principle it is not designed to determine the homology of groups and it does not know how to work with resolutions.

These ideas suggest that the effective homology technique and the Kenzo program should have a role in the computation of the homology of a group G. To this end, we can consider the Eilenberg–MacLane space $K(G, 1)$, whose homology groups coincide with those of G. The size of this space makes it difficult to calculate the groups in a direct way, but it is possible to operate with this simplicial set making use of the *effective homology* technique: if we construct the effective homology of $K(G, 1)$ then we would be able to *efficiently* compute the homology groups of $K(G, 1)$, which are those of G. Furthermore, it should be possible to extend many group theoretic constructions to effective homology constructions of Eilenberg–MacLane spaces. We thus introduce the following definition.

Definition 12. A group G is a *group with effective homology* if $K(G, 1)$ is a simplicial set with effective homology.

The problem is, given a group G, how can we determine the effective homology of $K(G, 1)$? If the group G is finite, the simplicial set $K(G, 1)$ is effective too, so that it can be considered with effective homology in a trivial way. However, the enormous size of this space makes it difficult to obtain real calculations, and therefore even in this case we will try to obtain an equivalence $C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow E_*$ where E_* is effective and (much) smaller than the initial complex. Section 3 of this paper presents an algorithm that computes this desired equivalence provided that the group G is endowed with a finite type resolution.

3. Effective homology of a group from a resolution

This section is devoted to an algorithm computing the effective homology of a group G given a (small) free $\mathbb{Z}G$-resolution. This algorithm was the main theoretical result included in the work Romero et al. (2009) and has been implemented in Common Lisp enhancing the Kenzo system. We will see some examples of use of these new programs in Section 5. A brief description of the construction of the algorithm is included in the following paragraphs. For more details, see Romero et al. (2009).

Let G be a group and F_* a free (augmented) finite type resolution for G with a contracting homotopy h. We want to construct the effective homology of the space $K(G,1)$, that is to say, a (strong chain) equivalence $C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow E_*$ where E_* is an effective chain complex.

We begin by considering the Bar resolution $B_* = \mathrm{Bar}_*(G)$ for G, with augmentation ε' and contracting homotopy h' (the definition of these maps can be found in Brown (1982)). As B_* and the given resolution F_* are free resolutions for G, it is well known (see Brown, 1982) that one can explicitly construct morphisms of chain complexes of $\mathbb{Z}G$-modules $f : B_* \to F_*$ and $g : F_* \to B_*$ which are homotopy equivalences. Moreover, one can construct graded morphisms of $\mathbb{Z}G$-modules

$$k : F_* \to F_{*+1}, \qquad k' : B_* \to B_{*+1}$$

such that $d_F k + k d_F = \mathrm{Id}_F - fg$ and $d_B k' + k' d_B = \mathrm{Id}_B - gf$.

We have therefore a homotopy equivalence (in the classical sense):

$$\begin{tikzcd} B_* \arrow[loop left, "k'"] \arrow[r, bend left, "f"] & F_* \arrow[loop right, "k"] \arrow[l, bend left, "g"] \end{tikzcd}$$

in which the four components f, g, k and k' are compatible with the action of the group G.

If we now apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$, which is additive, we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$\begin{tikzcd} \mathbb{Z} \otimes_{\mathbb{Z}G} B_* \arrow[loop above, "k'"] \arrow[r, bend left, "f"] & \mathbb{Z} \otimes_{\mathbb{Z}G} F_* \arrow[loop above, "k"] \arrow[l, bend left, "g"] \end{tikzcd}$$

where both chain complexes provide us the homology of the initial group G, that is,

$$H_*(\mathbb{Z} \otimes_{\mathbb{Z}G} B_*) \cong H_*(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \equiv H_*(G).$$

In order to obtain a strong chain equivalence (in other words, a pair of reductions, following the framework of effective homology), we make use of the mapping cylinder construction (see Weibel, 1994). This allows one to produce a (strong chain) equivalence

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \overset{\rho'}{\Lleftarrow} \mathrm{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*.$$

The definitions of the different components of both reductions are included in Romero et al. (2009).

Now we recall that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(K(G,1))$. On the other hand, if we suppose that the initial resolution F_* is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_* \equiv E_*$ is effective (and small too), so that we have obtained the desired effective homology of $K(G,1)$,

$$C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow E_*.$$

We have constructed in this way an algorithm computing the effective homology of a group, formally described in Algorithm 1.

Algorithm 1 Computation of the effective homology of a group

Require: a group G and a free (augmented) finite type resolution F for G with a contracting homotopy h.
Ensure: a (strong chain) equivalence $\varepsilon : C_*(K(G,1)) \Longleftrightarrow E$ where E is an effective chain complex.

1: $B = BarResolution(G)$

[Compute the Bar resolution of the group G and store it in the variable B]

2: $f = 2ResolutionsRightZGMorphism(B, F)$

[$f : B \to F$ is the morphism of chain complexes of $\mathbb{Z}G$-modules between both resolutions]

3: $g = 2ResolutionsLeftZGMorphism(B, F)$

[$g : F \to B$ is the morphism of chain complexes of $\mathbb{Z}G$-modules between both resolutions]

4: $k = 2ResolutionsRightHomotopy(B, F)$

[$k : F_* \to F_{*+1}$ is the graded morphism of chain complexes of $\mathbb{Z}G$-modules such that $d_F k + k d_F = \mathrm{Id}_F - fg$]

5: $k' = 2ResolutionsLeftHomotopy(B, F)$

[$k' : B_* \to B_{*+1}$ is the graded morphism of chain complexes of $\mathbb{Z}G$-modules such that $d_B k' + k' d_B = \mathrm{Id}_B - gf$]

6: $\rho = CylinderRightReduction(f)$

[Computes a reduction $\rho : \mathrm{Cylinder}(f)_* \Rightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$. Only the parameter f is necessary]

7: $\rho' = CylinderLeftReduction(f, g, k', k)$

[Computes a reduction $\rho' : \mathrm{Cylinder}(f)_* \Rightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} B_*$]

8: $\varepsilon = BuildHomotopyEquivalence(\rho', \rho)$

[Constructs a (strong chain) equivalence from two reductions. The result is a reduction $C_*(K(G,1)) \Longleftrightarrow E_*$]

All the functions included in Algorithm 1 are new functions which have been implemented in Common Lisp enhancing the Kenzo system, with the exception of the function `BuildHomotopyEquivalence` which was already included in Kenzo. As we will see in Section 5 the names of our real Lisp functions are not exactly the same as those included in this algorithm. Our implementation follows in most cases Kenzo's habit of using only the first consonants involved in the description of a function; for the general algorithm we have preferred to include a complete name for being more intuitive. The same will be done for all algorithms in this paper.

The strong chain equivalence determined by Algorithm 1 makes it possible to determine the homology groups of G, and, what is more useful, once we have $K(G, 1)$ with its effective homology we could apply different constructors and obtain the effective homology of the results. This could allow one, for instance, to determine the homology of some groups (obtained from other initial groups with effective homology) without constructing a resolution for them. Some fields of application of our algorithm are introduced in the following section.

4. Applications

4.1. 2-types

Let A be an Abelian group and G a group acting on A; a *2-type* for G and A is a (topological) space with $\pi_1(X) = G$, $\pi_2(X) = A$, and $\pi_n(X) = 0$ for all $n \geq 3$; the computation of the homology groups of these spaces is a difficult problem in the field of group homology (Ellis, 1992). It is well known that a 2-type X for G and A corresponds to a cohomology class $[f]$ in $H^3(G, A)$, and there exists a fibration

$$K(A, 2) \hookrightarrow X \to K(G, 1).$$

The theoretical existence of this fibration can be made *constructive* as follows, when the action of G on A is trivial. A cohomology class $[f]$ is given by a 3-cocycle f, which is a map $f : K(G, 1)_3 \to A$ (satisfying some properties). This map induces a simplicial morphism $f : K(G, 1) \to K(A, 3)$, which can be composed with the universal fibration (see May, 1967) $K(A, 2) \hookrightarrow E \to K(A, 3)$ in order to construct the desired fibration. In this way, we obtain a twisting operator $\tau_f : K(G, 1)_* \to K(A, 2)_{*-1}$ which allows one to express the total space X as a twisted Cartesian product

$$X = K(A, 2) \times_f K(G, 1).$$

Supposing now that the group G is given with a finite type free resolution, our Algorithm 1 can be applied in order to produce the effective homology of $K(G, 1)$. Analogously, provided a finite type resolution for A, we can determine the effective homology of $K(A, 1)$. Since $K(A, 1)$ is a simplicial Abelian group one can apply the classifying space constructor B that gives us $B(K(A, 1)) = K(A, 2)$, which is also a simplicial Abelian group. Furthermore, the effective homology version of the classifying space constructor B (see Rubio and Sergeraert (2006), for details) provides us the effective homology of the space $K(A, 2)$ from the effective homology of $K(A, 1)$ (iterating the process, $K(A, n) = B(K(A, n-1))$ has effective homology for every $n \in \mathbb{N}$). In this way, both spaces $K(A, 2)$ and $K(G, 1)$ are *objects with effective homology*. Finally, the effective homology version for a fibration (described also in Rubio and Sergeraert (2006)), makes use of the effective homologies of $K(A, 2)$ and $K(G, 1)$ and of the twisting operator $\tau_f : K(G, 1)_* \to K(A, 2)_{*-1}$ and gives us the effective homology of the total space $X = K(A, 2) \times_f K(G, 1)$. In particular, this leads to the desired homology groups of the 2-type X. The process is formally described in our Algorithm 2.

Algorithm 2 Computation of the effective homology of a 2-type

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Require: an Abelian group A with a free resolution F_A of finite type (with a contracting homotopy); a group G (acting trivially on A) with a free resolution F_G of finite type (with a contracting homotopy); a cohomology class $[f] \in H^3(G, A)$ given by a 3-cocycle $f : K(G, 1)_3 \to A$.

Ensure: $\varepsilon : C_*(X) \Longleftarrow\!\!\!\!\Longrightarrow E_*$ where E_* is an effective chain complex and $X = K(A, 2) \times_f K(G, 1)$.

1: $X = 2Type(A, G, f)$

[New function which computes the 2-type associated to the groups A and G and the 3-cocycle f, that is, $X = K(A, 2) \times_f K(G, 1)$; the implementation follows the ideas explained at the beginning of this subsection]

2: $efhmKG1 = Algorithm1(G, F_G)$

[Apply Algorithm 1 to the group G with the finite type resolution F_G; the effective homology of $K(G, 1)$ is obtained]

3: $efhmKA2 = ClassifyingSpaceEfhm(K(A, 1), Algorithm1(A, F_A))$

[Apply Algorithm 1 to compute the effective homology of $K(A, 1)$; then we use the Kenzo function computing the effective homology of the classifying space of a simplicial Abelian group from its effective homology, which in this case produces the effective homology of $B(K(A, 1)) = K(A, 2)$]

4: $\varepsilon = FibrationEfhm(X, efhmKA2, efhmKG1)$

[Use the Kenzo function which computes the effective homology of a fibration from the effective homologies of the two factors, in this case $K(A, 2)$ and $K(G, 1)$]

This algorithm has been implemented in Common Lisp as part of our new module for the Kenzo system dealing with homology of groups. See Section 5.3 for some examples of calculations.

If the group G acts non-trivially on A, an action $K(G, 0) \times K(A, 2) \to K(A, 2)$ must also be considered in the fibration $K(A, 2) \hookrightarrow X \to K(G, 1)$. The explicit construction of the twisting operator which describes the fibration cannot be obtained as easily as in the previous case, and a more deep study of the fibration is necessary. It should be done as a further work.

4.2. Central extensions

Let $0 \to A \to E \to G \to 1$ be a central extension of groups (that is, A is an Abelian group and G acts on A in a trivial way). Then, it is well-known (see Brown, 1982) there exists a set-theoretic map $f : G \times G \to A$ which satisfies:

(1) $f(g, 1) = 0 = f(1, g)$
(2) $f(gh, k) = f(h, k) - f(g, h) + f(g, hk)$

In addition, the initial extension is equivalent to another extension

$$0 \to A \to A \times_f G \to G \to 1$$

where the elements of $A \times_f G$ are pairs (a, g) with $a \in A$ and $g \in G$, and the group law is defined by

$$(a_1, g_1)(a_2, g_2) \equiv (a_1 + a_2 + f(g_1, g_2), g_1g_2).$$

The set-theoretic map f is called the *2-cocycle* of the extension, since it corresponds to a map $f : K(G, 1)_2 \to A$ in $H^2(G, A)$.

Very frequently, the groups G and A are not complicated and their homology groups are known. On the contrary, the homology groups of $E \cong A \times_f G$ are not always easy to obtain. The effective homology technique and our Algorithm 3 will provide a method computing the desired homology groups of E from finite type resolutions of G and A. In this way, it will not be necessary to determine a finite type resolution for E.

As explained in a previous work of the second author of this paper (see Rubio, 1997), given a 2-cocycle f defining a central extension of a group G by an Abelian group A, one can (explicitly) construct a fibration

$$K(A, 1) \hookrightarrow X \to K(G, 1)$$

where the total space X can be seen as a twisted Cartesian product $K(A, 1) \times_\tau K(G, 1)$. Furthermore, it can be proved that this space is in fact isomorphic to the Eilenberg–MacLane space $K(A \times_f G, 1)$, whose homology groups are those of the group $A \times_f G \cong E$. The simplicial morphisms $\Phi : K(A \times_f G, 1) \to K(A, 1) \times_\tau K(G, 1)$ and $\Phi^{-1} : K(A, 1) \times_\tau K(G, 1) \to K(A \times_f G, 1)$ can be found in Rubio (1997).

On the other hand, in the case where both the fiber and base spaces of the fibration, $K(A, 1)$ and $K(G, 1)$, are objects with effective homology, the effective homology version of a fibration (see Rubio and Sergeraert, 2006) provides the effective homology of the total space $K(A, 1) \times_\tau K(G, 1)$, which in particular will make it possible to obtain the homology groups of E. Finally, if the groups G and A are given with finite type (small) resolutions, our Algorithm 1 provides the necessary effective homologies of $K(G, 1)$ and $K(A, 1)$. We obtain therefore the following algorithm.

This algorithm has also been implemented in Common Lisp and in particular it allows us to determine the homology groups of central extensions of finitely generated Abelian groups. In Section 5.4 we include some examples of calculations.

5. New modules for Kenzo and experimental results

As already mentioned in Section 2.2, Kenzo (Dousson et al., 1999) is a Common Lisp program devoted to Symbolic Computation in Algebraic Topology, developed by Francis Sergeraert and some co-workers. This system makes use of the effective homology method to determine homology groups of complicated spaces and has obtained some results which had never been determined before. In principle Kenzo was not intended to compute homology of groups but we have enhanced this system with a new module dealing with groups, resolutions, and Eilenberg–MacLane spaces (which were already implemented in Kenzo for the particular cases $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$) and we have written

Algorithm 3 Computation of the effective homology of a central extension

Require: groups G and A (A is Abelian) and for both of them free resolutions of finite type F_G and F_A with the corresponding contracting homotopies (or more generally, G and A with effective homology); a 2-cocycle f defining an extension of G by A.

Ensure: $\varepsilon : C_*(K(A \times_f G, 1)) \Lleftarrow\!\!\Rrightarrow E_*$ where E_* is an effective chain complex.

1: $X = \mathit{CentralExtensionFibration}(A, G, f)$

[Compute the fibration associated with a central extension; in other words, $X = K(A, 1) \times_\tau K(G, 1)$. The implementation follows the construction of Rubio (1997)]

2: $\mathit{efhmKG1} = \mathit{Algorithm1}(G, F_G)$

[Apply Algorithm 1 to the group G with the finite type resolution F_G; the effective homology of $K(G, 1)$ is obtained]

3: $\mathit{efhmKA1} = \mathit{Algorithm1}(A, F_A)$

[Apply Algorithm 1 to the group A with the finite type resolution F_A; the effective homology of $K(A, 1)$ is obtained]

4: $\varepsilon' = \mathit{FibrationEfhm}(X, \mathit{efhmKA1}, \mathit{efhmKG1})$

[Use the Kenzo function which computes the effective homology of a fibration from the effective homologies of the two factors, in this case $K(A, 1)$ and $K(G, 1)$]

5: $\Phi = \mathit{CentralExtensionLeftIsomorphism}(A, G, f)$

[Produce the map $\Phi : K(A \times_f G, 1) \rightarrow K(A, 1) \times_\tau K(G, 1)$ as explained in Rubio (1997)]

6: $\Phi' = \mathit{CentralExtensionRightIsomorphism}(A, G, f)$

[Produce the map $\Phi^{-1} : K(A, 1) \times_\tau K(G, 1) \rightarrow K(A \times_f G, 1)$ as explained in Rubio (1997)]

7: $\varepsilon = \mathit{Composition}(\varepsilon', \Phi, \Phi')$

[Compose the equivalence $\varepsilon' : C_*(X) \Lleftarrow\!\!\Rrightarrow E_*$ with the isomorphism $K(A \times_f G, 1) \cong K(A, 1) \times_\tau K(G, 1) = X$ given by the maps Φ and Φ'. We obtain an equivalence $C_*(K(A \times_f G, 1)) \Lleftarrow\!\!\Rrightarrow E_*$.]

the corresponding programs implementing Algorithm 1, which produces the effective homology of the space $K(G, 1)$ given a finite type resolution for the group G. Since the construction of a finite type resolution for a group is not always an easy task, we have allowed Kenzo to connect with the GAP package HAP and obtain a resolution from it. Furthermore, as already announced, we provide programs which determine the homology groups of some 2-types and central extensions. All the programs presented in this section can be found in http://www.unirioja.es/cu/anromero/research2.html.

5.1. *Interoperating with GAP*

GAP (The GAP Group, 2008) is a system for computational discrete algebra with particular emphasis on Computational Group Theory. In our work we consider the HAP homological algebra library (Ellis, 2009) for use with GAP; it was written by Graham Ellis and is still under development. The initial focus of HAP is on computations related to the cohomology of groups. A range of finite and infinite groups are handled, with particular emphasis on integral coefficients. It also contains some functions for the integral (co)homology of: Lie rings, Leibniz rings, cat-1-groups and digital topological spaces. And in particular, HAP allows one to obtain (small) resolutions of many different groups, although it does not implement the Bar resolution nor Eilenberg–MacLane spaces $K(G, 1)$.

As presented in Romero et al. (2009), in a joint work with Graham Ellis, we have developed a new module making it possible to export resolutions from HAP and import them into Kenzo. As interchange language we have used OpenMath (The OpenMath Society, 2004), an XML standard for representing mathematical objects. There exist OpenMath translators from several Computer Algebra systems, and in particular GAP includes a package (Solomon and Costantini, 2009) which produces OpenMath code from some GAP elements (lists, groups...). We have extended this package in order to represent resolutions, including a new GAP function which provides the OpenMath code of these elements. In Romero et al. (2009) a detailed description of our OpenMath representation for resolutions can be found.

The communication between HAP and Kenzo is done as follows: given a group G, the system HAP produces a $\mathbb{Z}G$-resolution (including the homotopy operator). This resolution can be automatically translated to OpenMath code thanks to our new function added to the OpenMath package for GAP, and this code is written in a text file. Then Kenzo imports the file (and translates the OpenMath code into Kenzo elements thanks to the corresponding parser) so that one can use the resolution directly in Kenzo without the need of programming it in Common Lisp. Once the resolution is defined in Kenzo, we can use it to determine the effective homology of $K(G, 1)$ as explained in Section 3. In this way, if the construction of a resolution for a group G is complicated, we can avoid programming it by hand; it will be automatically implemented in Kenzo by obtaining it from HAP.

5.2. Computations with $K(G, n)$'s

Let G be a group and let us suppose that Kenzo knows a resolution for it (for some particular groups the system can construct it directly; for others, it could obtain it from HAP). Making use of our main Algorithm 1 one can determine the effective homology of the simplicial Abelian group $K(G, 1)$, and in particular, compute its homology groups.

Let us consider, for instance, $G = C_5$, the cyclic group of order 5. As already seen in Section 2.1, in this case it is not difficult to construct a small resolution F_* of G. The group can be built in Kenzo with the function `cyclicGroup`; the program computes automatically the well-known resolution for G (coded as a reduction $F_* \Rightarrow\!\!\!\Rightarrow \mathbb{Z}$) and stores it in the slot `resolution` of the group.

```
> (setf C5 (cyclicGroup 5))
[K1 Abelian-Group]
> (resolution C5)
[K10 Reduction K2 => K5]
```

This resolution is then used by our programs, following Algorithm 1, to determine the effective homology of $K(G, 1)$. The corresponding homotopy equivalence is spontaneously computed and stored in the slot `efhm`. In this way one can obtain the homology groups of this Eilenberg–MacLane space.

```
> (setf KC51 (K-G-1 C5))
[K11 Abelian-Simplicial-Group]
> (efhm KC51)
[K50 Homotopy-Equivalence K11 <= K40 => K31]
> (homology KC51 0 5)
Homology in dimension 0 :
Component Z
---done---
Homology in dimension 1 :
Component Z/5Z
---done---
Homology in dimension 2 :
---done---
Homology in dimension 3 :
Component Z/5Z
---done---
Homology in dimension 4 :
---done---
```

Moreover, since $G = C_5$ is Abelian, $K(G, 1)$ is a simplicial Abelian group, and we can apply the classifying space constructor B (already implemented in Kenzo) which gives us $B(K(G, 1)) = K(G, 2)$, a new simplicial Abelian group *with effective homology*.

```
> (setf KC52 (classifying-space KC51))
[K51 Abelian-Simplicial-Group]
> (efhm KC52)
[K190 Homotopy-Equivalence K51 <= K180 => K176]
> (homology KC52 3 6)
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/5Z
---done---
Homology in dimension 5 :
---done---
```

Iterating the process, $K(G, n) = B(K(G, n-1))$ has effective homology for every $n \in \mathbb{N}$. Our new Kenzo function `K-Cm-n` allows us to directly construct $K(C_m, n)$; we observe that the slot `efhm` is automatically created.

```
> (setf KC42 (K-Cm-n 4 2))
[K204 Abelian-Simplicial-Group]
> (efhm KC42)
[K378 Homotopy-Equivalence K204 <= K368 => K364]
> (homology KC42 4)
Homology in dimension 4 :
Component Z/8Z
---done---
```

The construction of Eilenberg–MacLane spaces $K(G, n)$ for every cyclic group $G = C_m$ (with the corresponding effective homology) is an important enhancing of the Kenzo system, which previously was only able to deal with cases $G = \mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z} = C_2$. The homology groups obtained for some of these new spaces have been tested comparing them with the results shown in Alain Clément's thesis (Clément, 2002). It is important to stress that Clément's tables, computed by using a direct algorithm created by Henry Cartan (see Cartan, 1954–1955), contain much more groups than those that can be computed with our programs in its current state. Nevertheless, Clément's tables give only the homology groups of the spaces, while our approach provides the *effective homology*. Our information is much more complete, giving access to geometrical generators of the homology and, in fact, fully solving the *homological problem* of these groups (see Rubio and Sergeraert, 2006). And, perhaps more important, our programs allow us to continue working with the corresponding $K(G, n)$, to produce new interesting topological spaces and to determine their homology groups. The information computed by Clément is not enough to carry out this further work.

The same technique explained for cyclic groups can be used to compute the effective homology of spaces $K(G, n)$, where G is a finitely generated Abelian group. In this case, the homology of $K(G, n)$ is one of the main ingredients to compute homotopy groups of spaces (see Rubio and Sergeraert (2002) and Rubio and Sergeraert (2006) for details).

5.3. *An example of homology of a* 2*-type*

Let us consider now $G = C_3$ the cyclic group of order 3. Let $A = \mathbb{Z}/3\mathbb{Z}$ be the Abelian group of three elements with trivial G-action (the groups G and A are in fact isomorphic; different notations are used to distinguish multiplicative and additive operations). Then the third cohomology group of G with coefficients in A is

$$H^3(G, A) = \mathbb{Z}/3\mathbb{Z}.$$

The classes $[f]$ of this cohomology group correspond to 2-types with $\pi_1 = G, \pi_2 = A$, and one such 2-type X can be seen as a twisted Cartesian product $X = K(A, 2) \times_f K(G, 1)$. It can be constructed by Kenzo in the following way:

```
> (setf KC31 (K-Cm-n 3 1))
[K380 Abelian-Simplicial-Group]
> (setf chml-clss (chml-clss KC31 3))
[K427 Cohomology-Class on K407 of degree 3]
> (setf tau (zp-whitehead 3 KC31 chml-clss))
```

```
[K442 Fibration K380 -> K428]
> (setf X (fibration-total tau))
[K448 Kan-Simplicial-Set]
```

As seen in the previous section, $K(A, 2)$ and $K(G, 1)$ are objects with effective homology. From the two equivalences $C_*(K(A, 2)) \Lleftarrow\!\Rrightarrow E_*$ and $C_*(K(G, 1)) \Lleftarrow\!\Rrightarrow E'_*$, Kenzo knows how to construct the effective homology of the twisted Cartesian product $X = K(A, 2) \times_f K(G, 1)$, which allows one to determine its homology groups.

```
> (efhm X)
[K660 Homotopy-Equivalence K448 <= K650 => K646]
> (homology X 5)
Homology in dimension 5 :
component Z/3Z
---done---
```

In the same way, the homology groups of $X = K(A, 2) \times_f K(G, 1)$ can be determined for all groups A and G with given (small) resolutions and cohomology classes $[f]$ in $H^3(G, A)$. Up to now, only the homology of finitely presented groups has been considered, restricting the kind of 2-types that can be studied with our methods, since only spaces with Abelian fundamental group would be in its scope. The range of groups which can be considered is considerably enlarged with the central extension constructions, as explained in the following subsection.

5.4. Central extensions

Let us introduce an interesting example of central extension extracted from Leary (1991). Let E be the group defined by the following presentation:

$$E = \langle x, y, z | x^p = y^p = z^{p^{n-2}} = [x, z] = [y, z] = 1; [x, y] = z^{p^{n-3}} \rangle.$$

This group can be seen as a central extension of the groups

$$A = \langle z | z^{p^{n-2}} = 1 \rangle,$$

isomorphic to the cyclic group with p^{n-2} elements, and

$$G = \langle x, y | x^p = y^p = [x, y] = 1 \rangle,$$

which is the direct sum of two cyclic groups of cardinality p. A 2-cocycle of the extension is defined by

$$f(x^{p_1}y^{q_1}, x^{p_2}y^{q_2}) = z^{q_1 p_2 (p-1) p^{n-3}}.$$

As already explained, the group $A \cong C_{p^{n-2}}$ has effective homology. On the other hand, the effective homology of $G \cong C_p \oplus C_p$ can be easily obtained from the effective homology of the cyclic group C_p (a direct sum of two groups can in fact be considered as a particular case of central extension, where the 2-cocycle is trivial, so that its effective homology can be computed given the effective homologies of the two factors). In this way, Algorithm 3 can be applied to obtain the effective homology of E and then compute its homology groups.

Let us consider, for instance, $p = 3$ and $n = 4$. The following Kenzo instructions construct the group E.

```
> (progn
    (setf p 3 n 4)
    (setf A (cyclicGroup (expt  p (- n 2))))
    (setf G (gr-crts-prdc (cyclicGroup p) (cyclicGroup p)))
    (setf cocycle #'(lambda (crpr1 crpr2)
                (with-grcrpr (x1 y1) crpr1
                  (with-grcrpr (x2 y2) crpr2
                    (mod (* y1 x2 (1- p) (expt p (- n 3))) (expt p (- n 2)))))))
    (setf E (gr-cntr-extn A G cocycle)))
[K663 Group]
```

The spaces $K(A, 1)$ and $K(G, 1)$ can be constructed with the function K-G-1; both of them are Abelian simplicial groups with effective homology.

```
> (setf KA1 (K-G-1 A))
[K664 Abelian-Simplicial-Group]
> (efhm KA1)
[K710 Homotopy-Equivalence K664 <= K700 => K691]
> (setf KG1 (K-G-1 G))
[K711 Abelian-Simplicial-Group]
> (efhm KG1)
[K775 Homotopy-Equivalence K711 <= K765 => K745]
```

Given the effective homologies of $K(A, 1)$ and $K(G, 1)$, our Algorithm 3 returns the effective homology of $K(E, 1)$, which is then stored in the corresponding slot efhm.

```
> (setf KE1 (K-G-1 E))
[K776 Simplicial-Group]
> (efhm KE1)
[K884 Homotopy-Equivalence K776 <= K870 => K866]
> (homology KE1 0 5)
Homology in dimension 0 :
Component Z
---done---
Homology in dimension 1 :
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
Homology in dimension 2 :
Component Z/3Z
Component Z/3Z
---done---
Homology in dimension 3 :
Component Z/9Z
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
Homology in dimension 4 :
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
```

In this way, one can determine the homology groups of the central extension E. The computations obtained by our programs have been compared with Leary's theoretical results for different values of p and n; the same groups have been obtained by both methods. We can repeat here the discussion made at the end of Section 5.2 with respect to Clément's computations for $H_*(K(G, n))$: Leary's methods give more groups than our techniques, but with less information. In particular, our results allow us to compute the homology of 2-types whose fundamental groups are central extensions, while Leary's groups are not enough for this task.

6. The inverse problem: recovering a resolution from the effective homology of a group

In Section 3 we have presented an algorithm which, given a group G with a free finite type resolution F_*, constructs the effective homology of the simplicial Abelian group $K(G, 1)$. This effective homology allows one to determine the homology groups of G and, as seen in Sections 4 and 5, makes it possible to use the space $K(G, 1)$ as initial data for some constructions in Algebraic Topology, computing in this way homology groups of other interesting objects.

We consider now the inverse problem: let G be a group such that an equivalence

$$C_*(K(G, 1)) \Lleftarrow\!\!\Rrightarrow E_*$$

is given, E_* being a finite type chain complex of Abelian groups. Is it possible to obtain a finite type free resolution for the group G? It seems, in principle, that the answer should be negative in the general case; since no condition is imposed on the arrows, they surely do not respect the G-action and, thus, it would not be possible to build a $\mathbb{Z}G$-resolution. We have proved, however, that supposing that some additional conditions for the given chain equivalence are satisfied, one can construct the desired resolution with the corresponding contracting homotopy.

The algorithm we have developed makes use of the *Basic Perturbation Lemma* (BPL), one of the fundamental results in Constructive Algebraic Topology. The general idea of this theorem is that given a reduction $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\Rightarrow D_*$, if we modify the initial differential d_C of the *big* complex C_* by adding some *perturbation*, then it is possible to perturb the differential d_D in the *small* chain complex D_* so that we obtain a new reduction between the perturbed complexes. But the result is not always true, a necessary condition must be satisfied: the composite function $h \circ d_C$ must be *locally nilpotent*. An endomorphism $\alpha : C_* \to C_*$ is locally nilpotent if for every $x \in C_*$ there exists $m \in \mathbb{N}$ such that $\alpha^m(x) = 0$. The condition of local nilpotency ensures the convergence of a formal series used in the BPL to build the perturbed differential on the small complex (and, in fact, to construct also all the arrows defining the new reduction between the perturbed complexes). The Basic Perturbation Lemma was discovered by Shih (1962), and then generalized by Brown (1967). In its modern form it was formulated by Gugenheim (1972) and its essential use in Kenzo has been documented in Rubio and Sergeraert (2006).

Let us suppose that G is a group and we have an equivalence $C_*(K(G,1)) \overset{\rho_1}{\Leftarrow\!\!\!\Leftarrow} D_* \overset{\rho_2}{\Rightarrow\!\!\!\Rightarrow} E_*$, where $\rho_1 = (f_1, g_1, h_1)$, $\rho_2 = (f_2, g_2, h_2)$, E_* is an effective chain complex, and the composition $h_2 g_1 \partial_n f_1$ is locally nilpotent (∂_n is the face of index n over the elements of $K(G,1)_n$, which can be extended to $C_n(K(G,1))$). We want to construct a free resolution F_* for G of finite type with a contracting homotopy h.

Let us start by considering the universal fibration $K(G,0) \to K(G,0) \times_\tau K(G,1) \to K(G,1)$ (see May, 1967). The total space $K(G,0) \times_\tau K(G,1)$ is acyclic and one can construct a reduction

$$C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\Rightarrow \mathbb{Z}$$

where $\mathbb{Z}$ represents the chain complex (of Abelian groups) $C_*(\mathbb{Z}, 0)$ with a unique non-null component $\mathbb{Z}$ in dimension 0.

On the other hand, one can consider the Eilenberg–Zilber theorem (Eilenberg and Zilber, 1953), which relates the chain complex of a Cartesian product with the tensor product of the chain complexes of the two components, and allows one to build a reduction

$$C_*(K(G,0) \times K(G,1)) \Rightarrow\!\!\!\Rightarrow C_*(K(G,0)) \otimes C_*(K(G,1)).$$

Applying the BPL (it can be proved that the nilpotence condition is satisfied) we obtain a *perturbed* reduction (this is in fact the *twisted* Eilenberg–Zilber theorem, see May (1967))

$$C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\Rightarrow C_*(K(G,0)) \otimes_t C_*(K(G,1))$$

where $C_*(K(G,0)) \otimes_t C_*(K(G,1))$ is a chain complex with the same underlying graded module as the tensor product $C_*(K(G,0)) \otimes C_*(K(G,1))$, but its differential is modified to take account of the twisting operator τ.

Now, from the given equivalence $C_*(K(G,1)) \Leftarrow\!\!\!\Leftarrow D_* \Rightarrow\!\!\!\Rightarrow E_*$, it is not difficult to construct a new equivalence

$$C_*(K(G,0)) \otimes C_*(K(G,1)) \Leftarrow\!\!\!\Leftarrow C_*(K(G,0)) \otimes D_* \Rightarrow\!\!\!\Rightarrow C_*(K(G,0)) \otimes E_*$$

and applying again the BPL, provided that $h_2 g_1 \partial_n f_1$ is locally nilpotent, we obtain

$$C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Leftarrow\!\!\!\Leftarrow C_*(K(G,0)) \otimes_t D_* \Rightarrow\!\!\!\Rightarrow C_*(K(G,0)) \otimes_t E_*.$$

Finally, one can observe that $C_*(K(G,0)) \equiv \mathbb{Z}G$ and composing the reductions $C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\Rightarrow \mathbb{Z}$ and $C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\Rightarrow C_*(K(G,0)) \otimes_t C_*(K(G,1))$ with the last equivalence, we get a contracting homotopy on $\mathbb{Z}G \otimes_t E_*$ which is a resolution for G.

This construction can be formalized by means of our Algorithm 4.

Algorithm 4 *Inverse* algorithm

Require: a group G and a (strong) chain equivalence $\varepsilon : C_*(K(G,1)) \overset{\rho_1}{\Leftarrow\!\!\Leftarrow} D \overset{\rho_2}{\Rightarrow\!\!\Rightarrow} E$, where $\rho_1 = (f_1, g_1, h_1)$, $\rho_2 = (f_2, g_2, h_2)$, E is an effective chain complex, and the composition $h_2 g_1 \partial_n f_1$ is locally nilpotent.

Ensure: F is a free resolution for G of finite type with a contracting homotopy h.

1: $X = \mathit{UniversalFibration}(K(G,0))$

[Construct the acyclic space $K(G,0) \times_\tau K(G,1)$ (May, 1967). This is a new function]

2: $\tau = \mathit{UniversalFibrationPerturbation}(K(G,0))$

[Compute the perturbation τ in $K(G,0) \times_\tau K(G,1)$]

3: $\rho 1 = \mathit{UniversalFibrationReduction}(K(G,0))$

[Construct the reduction $K(G,0) \times_\tau K(G,1) \Rightarrow\!\!\Rightarrow \mathbb{Z}$. New function]

4: $\rho 2 = \mathit{TwistedEilenbergZilber}(K(G,0), K(G,1), \tau)$

[Kenzo function which computes the reduction $C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\Rightarrow C_*(K(G,0)) \otimes_t C_*(K(G,1))$. It makes use of the Basic Perturbation Lemma as explained before]

5: $C = \mathit{BottomChainComplex}(\rho 2)$

[Kenzo returns the bottom chain complex in a reduction: in our case it is the space $C_*(K(G,0)) \otimes_t C_*(K(G,1))$]

6: $\rho 3 = \mathit{CompositionAsReduction}(\rho 2, \rho 1)$

[Since the bottom chain complex in the reduction $\rho 1$ is $\mathbb{Z}$, one can compute a new reduction $\rho 3 : C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Rightarrow\!\!\Rightarrow \mathbb{Z}$. This is a new function not included in Kenzo]

7: $\varepsilon 1 = \mathit{TwistedTensorProductEfhm}(C, \mathit{TrivialEfhm}(K(G,0)), \varepsilon)$

[Kenzo knows how to determine the effective homology of a (twisted) tensor product from the effective homologies of the two components. In our case, $K(G,0)$ has trivial effective homology and the effective homology of $K(G,1)$ is the given equivalence ε. In this way the new object $\varepsilon 1$ is an equivalence $\varepsilon 1 : C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Leftarrow\!\!\Leftarrow\!\!\Rightarrow\!\!\Rightarrow C_*(K(G,0)) \otimes_t E$]

8: $\rho 4 = \mathit{CompositionAsReduction}(\rho 3, \varepsilon 1)$

[The composition of the reduction $\rho 3 : C_*(K(G,0) \otimes_t K(G,1)) \Rightarrow\!\!\Rightarrow \mathbb{Z}$ with the equivalence $\varepsilon 1 : C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Leftarrow\!\!\Leftarrow\!\!\Rightarrow\!\!\Rightarrow C_*(K(G,0)) \otimes_t E$ leads to a reduction $\rho 4 : C_*(K(G,0)) \otimes_t E \Rightarrow\!\!\Rightarrow \mathbb{Z}$]

9: $F = \mathit{TopChainComplex}(\rho 4)$

[The top chain complex in the reduction $\rho 4$ is $C_*(K(G,0)) \otimes_t E \equiv \mathbb{Z}G \otimes_t E$ which can be seen as a chain complex of $\mathbb{Z}G$-modules]

10: $h = \mathit{hMorphism}(\rho 4)$

[The component h in the reduction $\rho 4$ is a contracting homotopy for F]

As a first possible application of this algorithm, one can consider the integer group $G = \mathbb{Z}$ and the well known effective homology of $K(\mathbb{Z}, 1)$, given by a reduction $C_*(K(\mathbb{Z},1)) \Rightarrow\!\!\Rightarrow C_*(S^1)$, where S^1 denotes a simplicial model for the sphere of dimension 1. In this case it is not difficult to prove the desired condition, $h_2 g_1 \partial_n f_1$ is locally nilpotent, and therefore one can construct a finite type resolution for $G = \mathbb{Z}$, as a reduction $\mathbb{Z}G \otimes_t C_*(S^1) \Rightarrow\!\!\Rightarrow \mathbb{Z}$.

A natural question which appears in this context is whether, given a group G and an equivalence $C_*(K(G,1)) \Leftarrow\!\!\Leftarrow\!\!\Rightarrow\!\!\Rightarrow E_*$ which has been obtained by means of our Algorithm 1 from a finite type resolution F_*, the necessary condition of $h_2 g_1 \partial_n f_1$ being locally nilpotent is satisfied or not. The answer is positive if the group G and the resolution F_* satisfy some particular properties. More concretely, we suppose that a *norm* is defined on G and it can be extended to F_* in the following *natural* way.

Definition 13. Let G be a group. A *norm* for G is a map $\|.\| : G \to \mathbb{N}$ such that

- $\|g\| > 0$ for each $g \in G$ and $\|g\| = 0$ if and only if $g = 1$;
- $\|g_1 g_2\| \leq \|g_1\| + \|g_2\|$ for all $g_1, g_2 \in G$.

We suppose that the resolution is reduced ($F_0 = \mathbb{Z}G$) and define $\|.\| : F_0 = \mathbb{Z}G \to \mathbb{N}$ as $\|\sum \lambda_i g_i\| = \max\{\|g_i\|\}$. We say that the norm is *compatible* with the resolution F_* if for each $n \geq 1$ we can also define $\|.\| : F_n \to \mathbb{N}$ such that

- $\|(g, z)\| = \|g\| + \|z\|$ for all $g \in G$ and z a generator of F_n;
- there exists $i_n \in \mathbb{Z}$ such that $\|h_n(x)\| \leq \|x\| - i_n$ and $\|d_{n+1}(x')\| \leq \|x'\| + i_n$ for all $x \in F_n$, $x' \in F_{n+1}$.

The last condition introduces a control measure on the contracting homotopy h, with respect to the structure of the group, allowing us (as shown in the following result) to ensure in this case the convergence of the Basic Perturbation Lemma. Examples of resolutions with this kind of norm are the Bar resolution, the canonical small resolution for $G = \mathbb{Z}$ and, for instance, the small resolutions for cyclic groups introduced in Section 2.1.

Theorem 14. *Let G be a group and F_* a free resolution for G with contracting homotopy h. Let us suppose that G is provided with a norm $\|.\| : G \to \mathbb{N}$ which is compatible with the resolution. Then the effective homology of $K(G, 1)$ obtained from F_* by our Algorithm 1 satisfies the necessary condition of $h_2 g_1 \partial_n f_1$ being locally nilpotent, and therefore it is possible to construct a (new) free finite type resolution for G.*

Proof. Let us recall that the effective homology of $K(G, 1)$ given by our Algorithm 1 is given by an equivalence:

$$C_*(K(G,1)) \overset{\rho'}{\Lleftarrow} \mathrm{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} E_*$$

obtained from an equivalence in the classical sense:

$$\begin{tikzcd} C_*(K(G,1)) \arrow[loop above, "k'"] \arrow[r, shift left, "f"] & E_* \arrow[l, shift left, "g"] \arrow[loop above, "k"] \end{tikzcd}$$

Taking into account the definition of the different components of the reductions ρ and ρ' (included in Romero et al. (2009)), one can observe that in this case the composition $f_1 h_2 g_1$ is in fact the morphism $k' : C_*(K(G, 1)) \to C_{*+1}(K(G, 1))$, and therefore the condition of $h_2 g_1 \partial_n f_1$ being locally nilpotent is equivalent to $\partial_n k'$ being locally nilpotent.

We recall too that the morphism k' is obtained by tensorizing the map $k' : \mathrm{Bar}_*(G) \to \mathrm{Bar}_{*+1}(G)$, which, as explained in Romero et al. (2009), is defined over the generators u_α^n of $B_n \equiv \mathrm{Bar}_n(G)$ as

$$k'_0(u_\alpha^0) = h'(u_\alpha^0) - h'gf(u_\alpha^0)$$
$$k'_n(u_\alpha^n) = h'(\mathrm{Id} - gf - k'_{n-1}d(u_\alpha^n))$$

where h' is the contracting homotopy of the Bar resolution $\mathrm{Bar}_*(G) \equiv B_*$.

The norm $\|.\| : G \to \mathbb{N}$ can be extended to B_* as follows: $\|.\| : B_* \to \mathbb{N}$ given by $\|g.[g_1|\cdots|g_n]\| = \|g\| + \sum_j \|g_j\|$ and $\|\sum_i \lambda_i(g^i.[g_1^i|\cdots|g_n^i])\| = \max_i\{\|g^i\| + \sum_j \|g_j^i\|$. From the definitions of the differential map d in B_* and the contracting homotopy h' (Brown, 1982) one can easily observe that both maps preserve the norm $\|.\|$. Furthermore, one can prove in a recursive way that the composition gf preserves the norm $\|.\|$ too, so that using an inductive reasoning one has that k' preserves $\|.\|$ too. Finally it is not difficult to observe that ∂_n decreases $\|.\|$ at least in one unit, and then the composition $\partial_n k'$ is locally nilpotent, as desired. $\square$

The new resolution F'_* given by Algorithm 4 has in this case the same structural components as the initial resolution F_*; in other words, $F_n = F'_n$ for all $n \in \mathbb{N}$. However, the differential and contracting homotopy maps could be different.

A final example of application of Algorithm 4 and Theorem 14 is the following one.

Theorem 15. *Let G, G' be groups with free resolutions F_* and F'_* (with contracting homotopies h and h' respectively). Let us suppose that there exists norms on G and G' which are compatible with the corresponding resolutions. Then the effective homology of $G \oplus G'$ (obtained from those of G and G' as a particular case of central extension) satisfies that $h_2 g_1 \partial_n f_1$ is locally nilpotent, so that it is also possible to determine a resolution for the direct sum $G \oplus G'$.*

Again, we know the graded part of the output resolution, but it is still unknown if the differential and contracting homotopy constructed have some good geometrical behavior.

7. Conclusions and further work

In this paper we have defended this proposal: the *geometric way* for computing group homology can be sensible and fruitful. To this aim, we have worked inside Sergeraert's *effective homology*, and added packages devoted to group homology in Sergeraert's Kenzo system.

In their current state our methods have a performance penalty when compared with the more standard algebraic approach (based on *resolutions*). Nevertheless, this claim is only true for computations reachable by previous means. Furthermore, what is more important, to get available the homology of a group G through an Eilenberg–MacLane space $K(G, 1)$ with *effective homology* allows us to use that space for further topological constructions. The poorer performance is therefore balanced with the richer information we get.

The paper illustrates our approach with concrete computer experiments for general Eilenberg–MacLane spaces $K(G, n)$, for central extensions of groups and for 2-types. In the first two applications, the computer results have been compared with previously published works. In the case of the homology groups of 2-types computed with Kenzo, no comparison is possible, because no other source of results is known by us.

Furthermore we have explored the problem of computing a resolution of G from the effective homology of $K(G, 1)$, obtaining some partial algorithmic results which have not yet been implemented.

Some of the lines opened in this paper have not been completely closed, signaling clear lines of further work. Starting from the end, the scope of the methods to compute resolutions from effective homologies should be enlarged, and more examples should be worked out. In particular, a comparison between the initial resolutions and the ones constructed in the case of *normed* groups should be undertaken, trying to elucidate if our output resolution is better in some geometrical sense.

In the area of 2-types, the more important task would be to extend our approach to 2-types with non-trivial action of the fundamental group. The main obstacle here is to obtain a fibration expressed as a twisted Cartesian product, in order to be able to apply the previous Kenzo infrastructure.

Another challenge consists in trying to get better algorithms from the efficiency point of view, in such a way that our programs can compete with other approaches. In particular, we should improve the algorithm to construct the effective homology from a resolution, at least in certain cases, to obtain execution times closer to those of the source system, HAP. For finitely generated Abelian groups (which are the building blocks to start many of our constructions) a more direct approach, much more efficient, could be extracted from the original papers by Eilenberg and MacLane (1953, 1954a,b).

Finally, the application of our methods for wider classes of groups (for instance, extensions beyond the central extensions dealt with in this paper) is likely possible and surely an interesting research topic.

Acknowledgement

Thanks are due to Graham Ellis who collaborated with us in the connection of Kenzo and GAP and in work dealing with 2-types, as documented in Romero et al. (2009).

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Computing the homology groups.

Let X be space obtained by first removing the the interior of two disjoint closed disks from the unit closed disk in $\mathbb{R}^2$ and then identifying their boundaries clockwise. Compute the homology of this space.

My idea is to do this using cellular homology: We can have cell complex structure on X: one 0-cell, one 1-cell and one 2-cell. Attaching the 2-cell to the 1-skeleton by first diving the S^1 into 3 parts, then mapping these parts to the 1-skeleton in the same direction.

Thus the cellular boundary map d_2 will be multiplication by 3 and we have the homology groups $H_0(X) = \mathbb{Z}$ and $H_1(X) = \mathbb{Z}_3$ and $H_i(X) = 0$, otherwise.

Please check the calculations and share some ideas for such questions. Thanks in advance!

(algebraic-topology)

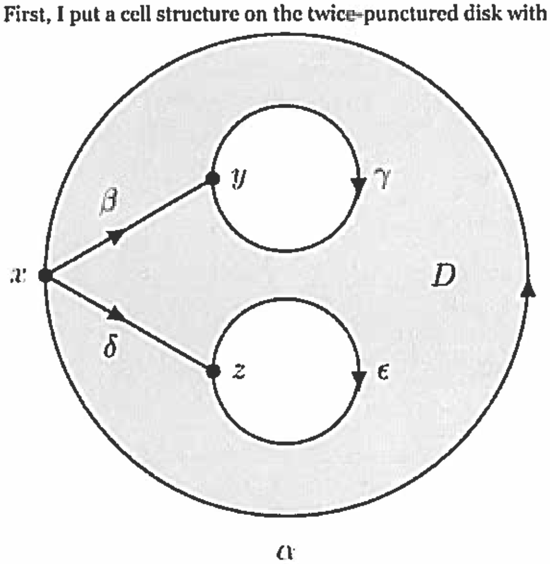
edited May 1 '15 at 4:22 guest196883 4,820 11 39

asked Dec 19 '12 at 13:38 user51266

2 Answers

Many thanks to Steve D, user17786, and Dave Hartman for their helpful corrections.

First, I put a cell structure on the twice-punctured disk with 3 0-cells, 5 1-cells, and 1 2-cell:



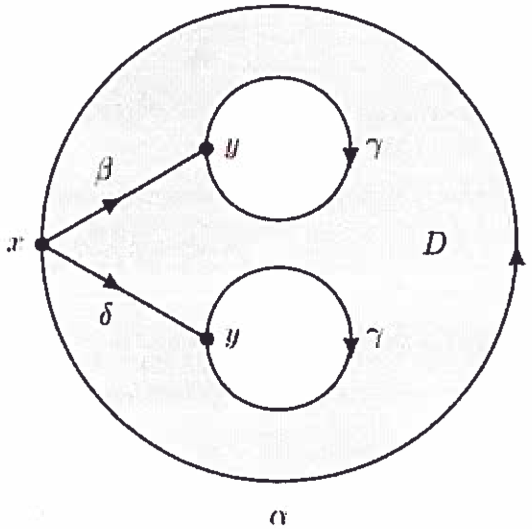
Note that the boundary of the 2-cell D is

$$d_2D = \alpha + \beta + \gamma - \beta + \delta + \epsilon - \delta = \alpha + \gamma + \epsilon,$$

and that the boundaries of the 1-cells are

$$\begin{aligned} d_1\alpha &= 0 \\ d_1\beta &= y - x \\ d_1\gamma &= 0 \\ d_1\delta &= z - x \\ d_1\epsilon &= 0 \end{aligned}$$

Now, we identify y with z, and γ with ϵ, to produce a cell structure on X:



For X, the chain groups are

$$C_0(X) = \langle x, y\rangle$$
$$C_1(X) = \langle \alpha, \beta, \gamma, \delta\rangle$$
$$C_2(X) = \langle D\rangle$$

where D is our 2-cell, and we have

$$d_1\alpha = 0$$
$$d_1\beta = y - x$$
$$d_1\gamma = 0$$
$$d_1\delta = y - x$$

$$d_2D = \alpha + \beta + \gamma - \beta + \delta + \gamma - \delta = \alpha + 2\gamma.$$

Thus,

$$H_0(X) = \ker(d_0)/\operatorname{im}(d_1) = \langle x, y\rangle/\langle y - x\rangle = \langle \overline{x}\rangle \cong \mathbb{Z}$$

$$H_1(X) = \ker(d_1)/\operatorname{im}(d_2) = \langle \alpha, \gamma, \beta - \delta\rangle/\langle \alpha + 2\gamma\rangle = \left\langle \overline{\gamma}, \overline{\beta - \delta}\right\rangle \cong \mathbb{Z}^2$$

$$H_2(X) = \ker(d_2)/\operatorname{im}(d_3) = 0/0 \cong 0.$$

edited Dec 19 '12 at 18:05 answered Dec 19 '12 at 15:22
Zev Chonoles
108k 16 217 408

Shouldn't the first homology be $\mathbb{Z} \times \mathbb{Z}$? I'm thinking about this by taking a cylinder, and gluing the boundary as decribed in the question, then puncturing the result. – user641 Dec 19 '12 at 16:38

When we construct the space in the way you describe, we just get a punctured torus, which would indeed have $H_1 \cong \mathbb{Z}^2$, but I think the difference is explained by the outer edge of the disk having the opposite orientation of the inner circles, so that it ends up mattering which pair of circles you connect. Here is a pair of pants (original image from Wikipedia) with the edges oriented so that the "waist" is the outer edge of the disk, and the "pants cuffs" are the punctures. – Zev Chonoles Dec 19 '12 at 16:58

Your construction attaches a pant cuff to the waist, and my construction attaches the pants cuffs together. However, I'm no longer sure if my construction is the one intended in the problem, or possibly if what I'm saying is making sense. What do you think? – Zev Chonoles Dec 19 '12 at 17:00

You should not identify δ with β, only γ and ϵ. The result cannot be drawn as something planar without identifications. – user17786 Dec 19 '12 at 17:01

1 The punctured Klein bottle has no torsion in its homology. An easy way to see this is to compute the fundamental group, which is free on two generators, then abelianize. – user641 Dec 19 '12 at 17:56

X as mentioned earlier is a punctured Klein Bottle, hence deformation retracts onto wedge of two circles. So $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$.

Homology of surface of genus g

This is a homework question given to me by someone of the community here and it's a generalisation of this. I was wondering if you could have a look and tell me if it's right. Thanks for your help!

Task: Compute the homology of a surface of genus g, Σ_g.

My calculations:

(i) The cell decomposition:

- 1 two-cell e^2 (a $4g$-gon)
- $2g$ one-cells e^1_i
- 1 zero-cell e^0

(ii) The attaching map of e^2:

- $f_2 = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$

  The attaching map of e^1:

- $f_1 = e^0$

(iii) The chain groups:

- $C_0(\Sigma_g) = \mathbb{Z}$
- $C_1(\Sigma_g) = \mathbb{Z}^{2g}$
- $C_2(\Sigma_g) = \mathbb{Z}$
- $C_k(\Sigma_g) = 0, k > 2$

(iv) The boundary homomorphisms:

$$\dots \xrightarrow{d_3} C_2(\Sigma_g) \xrightarrow{d_2} C_1(\Sigma_g) \xrightarrow{d_1} C_0(\Sigma_g) \xrightarrow{d_0} 0$$

- $d_0 = 0$
- $d_1 = 0$, because f_1 has degree 0
- $d_2(e^2) = 0$, because each coefficient is 0

(v) The homology groups:

- $H_0(\Sigma_g) = ker d_0 / im d_1 = \mathbb{Z}/0 = \mathbb{Z}$
- $H_1(\Sigma_g) = ker d_1 / im d_2 = \mathbb{Z}^{2g}/0 = \mathbb{Z}^{2g}$
- $H_2(\Sigma_g) = ker d_2 / im d_3 = \mathbb{Z}/0 = \mathbb{Z}$

(algebraic-topology)

edited Apr 13 '17 at 12:21 Community ♦ 1

asked Jun 20 '11 at 10:06 Rudy the Reindeer 25.6k 13 83 226

2 You say you're using g one-cells, but your attaching map for e^2 involves $2g$ of them. – Chris Eagle Jun 20 '11 at 10:09

There is something wrong. Thank you! – Rudy the Reindeer Jun 20 '11 at 10:13

1 It's a $4g$-gon, not a $2g$-gon! Thanks!! – Rudy the Reindeer Jun 20 '11 at 10:17

Ok, maybe someone could write something as an answer, then I can accept it and this question is resolved. Thanks! – Rudy the Reindeer Jun 20 '11 at 10:52

1 Answer

You can get the genus g-surface by doing the connected sum of g tori $T = S^1 \times S^1$, i.e., If you know the homology of T, and how to find that of the connected sum, done.

If not, or if you prefer a different approach, you can: i) Find the homology of S^1, then ii) Find the homology of the product $S^1 \times S^1$, and iii) Find the homology of the connected sum of g copies in ii):

1. $H_1(S^1) = \mathbb{Z}$
2. How to find the homology of a product space, (e.g., Künneth's formula) it is $\mathbb{Z}^2$
3. Finding the homology of connected sums; it is the direct sum of the respective homologies; the basis curves are pairwise disjoint, so the homology is the direct sum (what happens in one Torus, stays in that Torus) You ultimately get:

$$H_1(S_g) = \mathbb{Z}^{2g}$$

you are done.

edited Jun 20 '11 at 17:42
t.b.
61k 7 202 279

answered Jun 20 '11 at 14:36
gary
3,381 1 10 19

I don't know if this is the approach you wanted, but I think it may give you a kind of quick-and-dirty way of getting the homology without having to do the whole thing ground-up all the time (tho it is a good idea to do ground-up a few times) – gary Jun 20 '11 at 14:42

1 I tried to fix your formatting and TeX. Note: the sign $\times$ is obtained by using `\times` . If you want more than one letter as superscript use curly braces: $\mathbb{Z}^{2g}$ is obtained by `\mathbb{Z}^{2g}` . You can see what I did by clicking on the 'edited xx time' ago over my name. – t.b. Jun 20 '11 at 14:52

1 @Matt: they are a meridian and a parallel; two representatives of non-trivial cycles that do not bound. The class (m,n) then goes m times around a meridian and n times around a parallel. – gary Jun 20 '11 at 14:58

1 @gary: Okay, I see. By the way: add an @ -sign in front of your comments, then the corresponding user gets notified. – t.b. Jun 20 '11 at 15:01

1 @Matt, another comment: the choice of the two curves has to see with the fact that if we were to cut or remove the curves, the remaining space would be connected; in that sense, they do not bound, so they are non-trivial curves. Hope this is not too trivial of a comment; it helped me when I learnt about it. – gary Jun 20 '11 at 15:02

SPACES THAT ARE CONNECTED BUT NOT PATH-CONNECTED

KEITH CONRAD

1. Introduction

A topological space X is called *connected* if it's impossible to write X as a union of two nonempty disjoint open subsets: if $X = U \cup V$ where U and V are open subsets of X and $U \cap V = \emptyset$ then one of U or V is empty. Intuitively, this means X consists of one piece. A subset of a topological space is called connected if it is connected in the subspace topology. The most fundamental example of a connected set is the interval $[0,1]$, or more generally any closed or open interval in $\mathbf{R}$.

Most reasonable-looking spaces that appear to be connected can be proved to be connected using properties of connected sets like the following [2, pp. 149–151]:

- if $f\colon X \to Y$ is continuous and X is connected then $f(X)$ is connected,
- if C is a connected subset of X then $\overline{C}$ is connected and every set between C and $\overline{C}$ is connected,
- if C_i are connected subsets of X and $\bigcap_i C_i \neq \emptyset$ then $\bigcup_i C_i$ is connected,
- a direct product of connected sets is connected.

Proving complicated fractal-like sets are connected can be a hard theorem, such as connectedness of the Mandelbrot set [1].

We call a topological space X *path-connected* if, for every pair of points x and x' in X, there is a path in X from x to x': there's a continuous function $p\colon [0,1] \to X$ such that $p(0) = x$ and $p(1) = x'$. Since $q(t) = p(1-t)$ is also continuous with $q(0) = p(1) = x'$ and $q(1) = p(0) = x$, we can think of a path going in either direction, x to x' or x' to x. A subset $Y \subset X$ is called path-connected if any two points in Y can be linked by a path taking values entirely inside Y.

Path-connectedness shares some properties of connectedness:

- if $f\colon X \to Y$ is continuous and X is path-connected then $f(X)$ is path-connected,
- if C_i are path-connected subsets of X and $\bigcap_i C_i \neq \emptyset$ then $\bigcup_i C_i$ is path-connected,
- a direct product of path-connected sets is path-connected.

Compared to the list of properties of connectedness, we see one analogue is missing: every set lying between a path-connected subset and its closure is path-connected. In fact that property is not true in general.

For reasonable-looking subsets of Euclidean space, connectedness and path-connectedness are the same thing: one property holds if and only if it the other property does. But the properties are not always the same. We will set out here the precise logical connection (pun intended): path-connectedness implies connectedness, but the converse direction is false and we'll give three explicit examples of a connected set that is not path-connected. The first two will use objects you can find around your house: a broom and a comb. Well, not quite. The examples will be figures made up of carefully arranged line segments in the plane, together with one extra point, that are infinite versions of a broom and a comb. All three examples will be path-connected subsets together with one limit point, and including the limit point will wreck path-connectedness.

2. Path-connectedness implies connectedness

Theorem 2.1. *Every path-connected space is connected.*

Proof. Let X be path-connected. We will use paths in X to show that if X is not connected then $[0,1]$ is not connected, which of course is a contradiction, so X has to be connected.

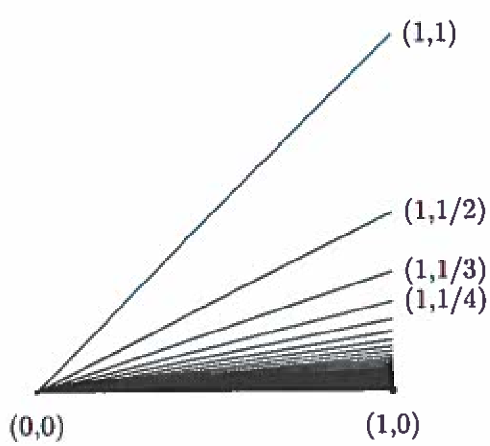
Suppose X is not connected, so we can write $X = U \cup V$ where U and V are nonempty disjoint open subsets. Pick $x \in U$ and $y \in V$. There is a path $p\colon [0,1] \to X$ where $p(0) = x$ and $p(1) = y$. The partition of X into U and V leads via this path to a partition of $[0,1]$: $[0,1] = A \cup B$ where $A = p^{-1}(U)$ and $B = p^{-1}(V)$.

Note $0 \in A$ and $1 \in B$, so A and B are nonempty subsets of $[0,1]$. Obviously A and B are disjoint, since no point in $[0,1]$ can have its p-value in both U and V. Since p is continuous and U and V are both open in X, A and B are both open in $[0,1]$. Thus the equation $[0,1] = A \cup B$ exhibits $[0,1]$ as a disjoint union of two nonempty open subsets, which contradicts the connectedness of $[0,1]$. □

Remark 2.2. A second proof of Theorem 2.1 is based on a property of connectedness listed earlier: if C_i are connected subsets and $\bigcap_i C_i \neq \emptyset$, then $\bigcup_i C_i$ is a connected subset. If X is path-connected and we fix a point $x \in X$ then for each $y \in X$ there's a path p_y in X from x to y, so we can cover X by the images of these paths: $X = \bigcup_{y \in X} p_y([0,1])$. Each $p_y([0,1])$ is connected since the image of a connected set under a continuous function is connected, and since $x = p_y(0)$ for all $y \in X$, the different subsets $p_y([0,1])$ have a nonempty intersection. Thus X is connected. B. Conrad noted that this proof can be condensed to a sentence: "All roads lead to Rome" (or equivalently, all roads lead from Rome).

3. Connectedness does not imply path-connectedness

Examples of connected sets that are not path-connected all look weird in some way. We will describe two examples that are subsets of $\mathbf{R}^2$. The first one is called the deleted infinite broom. It is pictured below and consists of the closed line segments L_n from $(0,0)$ to $(1,1/n)$ as n runs over the positive integers together with the (red) point $(1,0)$. The x-axis strictly between 0 and 1 is not part of this.



Theorem 3.1. *The deleted infinite broom is connected.*

Proof. Each point on L_n can be linked to $(0,0)$ by a path along L_n. By concatenating such paths, points on L_m and L_n can be linked by a path via $(0,0)$ if $m \neq n$, so the union $\bigcup_{n\geq 1} L_n$ is path-connected and therefore is connected (Theorem 2.1). The point $(1,0)$ is a limit point of $\bigcup_{n\geq 1} L_n$, so the deleted infinite broom lies between $\bigcup_{n\geq 1} L_n$ and its closure in $\mathbf{R}^2$. Therefore by the second property of connectedness in the introduction, the deleted infinite broom is connected. □

Remark 3.2. The closure of $\bigcup_{n\geq 1} L_n$ is obtained by adjoining to this union the segment L_∞ from $(0,0)$ to $(1,0)$, and the closure is called the infinite broom, which is why the space we care about is called the deleted infinite broom. The infinite broom is path-connected.

It makes sense intuitively that the deleted infinite broom is not path-connected: if a path starts at $(1,0)$ and stays within the deleted infinite broom it is hard to imagine how the path could "make the leap" to the rest of the space. In other words, you should have a feeling that any path in the deleted infinite broom that starts at $(1,0)$ has to be constant.

To prove that path property, we will first look at the endpoints of the segments L_n that lie on the line $x = 1$ together with $(1,0)$. The line $x = 1$ is homeomorphic to the real line, and rotating it by 90 degrees makes those endpoints and $(1,0)$ look like the figure below, which is the number 0 and $1/n$ for all $n \in \mathbf{Z}^+$.

Lemma 3.3. *The set* $\{0\} \cup \{1/n : n \in \mathbf{Z}^+\}$ *with its subspace topology in* $\mathbf{R}$ *has one-element subsets as its only nonempty connected subsets.*

Proof. Let C be a nonempty connected subset of $\{0\} \cup \{1/n : n \in \mathbf{Z}^+\}$. Assume C contains some $1/n$. Since $\{1/n\}$ is both closed and open in this set, writing $C = \{1/n\} \cup (C - \{1/n\})$ expresses C as a union of disjoint open subsets, so one of the subsets is empty. Thus $C - \{1/n\}$ is empty, so $C = \{1/n\}$. If C does not contain any $1/n$ then the only choice is $C = \{0\}$. □

Remark 3.4. A topological space whose only nonempty connected subsets are one-element subsets is called *totally disconnected*, so the set in Lemma 3.3 is totally disconnected. Other examples include $\mathbf{Q}$ with its standard topology as a subset of $\mathbf{R}$, and $\prod_{n\geq 1}\{1,-1\}$ with the product topology.

Lemma 3.3 is the key technical idea for proving the deleted infinite broom is not path-connected.

Theorem 3.5. *The deleted infinite broom is not path-connected.*

Proof. Denote the deleted infinite broom as B and let $p\colon [0,1] \to B$ be a path such that $p(0) = (1,0)$. We will prove $p(t) = (1,0)$ for all $t \in [0,1]$, so no path in B links $(1,0)$ to any other point of B.

Let

$$A = \{t \in [0,1] : p(t) = (1,0)\}$$

This is a nonempty subset of $[0,1]$ since it contains 0. Our goal is to show $A = [0,1]$.

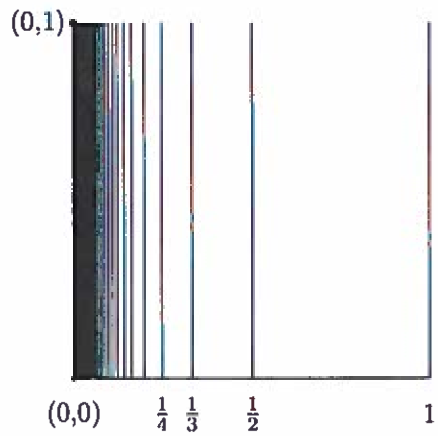
The set A is closed in $[0,1]$ since it is $p^{-1}((1,0))$ and p is continuous.

Next we show A is open in $[0,1]$. This will require a lot more work than showing it is closed. For $t_0 \in A$ we want to find an open interval around t_0 in $[0,1]$ that is also in A. By continuity of p at t_0 there's a $\delta > 0$ such that if $t \in [0,1]$ satisfies $|t - t_0| < \delta$ then $||p(t) - p(t_0)|| < 1/2$, where $||\cdot||$ is the length of a vector in $\mathbf{R}^2$.[1] Then $p(t) \neq (0,0)$ since $||p(t_0)|| = ||(1,0)|| = 1 > 1/2$, so $p(t)$ has a positive x-coordinate for all $t \in [0,1]$ satisfying $|t - t_0| < \delta$.

Consider the slope function $m\colon \{(x,y) \in \mathbf{R}^2 : x > 0\} \to \mathbf{R}$ defined by $m(x,y) = y/x$. This is the slope of the line connecting (x,y) to $(0,0)$ and it is clearly continuous. (We'd run into a problem if we tried to extend m to the y-axis.) Since $p(t)$ has positive x-coordinate for all $t \in [0,1]$ satisfying $|t - t_0| < \delta$, we can compose p with m to get the continuous function $t \mapsto m(p(t))$ mapping the interval $I := (t_0 - \delta, t_0 + \delta) \cap [0,1]$ to $\mathbf{R}$. Since the values of p on I are in the deleted infinite broom without the origin, we get $m(p(I)) \subset \{0\} \cup \{1/n : n \in \mathbf{Z}^+\}$. The set $m(p(I))$ is connected since this is the image of a connected set I under a continuous function. Therefore by Lemma 3.3, $m(p(I))$ is a single point. Since $t_0 \in I$ and $m(p(t_0)) = m(1,0) = 0$, we get $m(p(I)) = 0$, so I is an open set around t_0 in $[0,1]$ that is contained in A. Thus A is open in $[0,1]$.

The only nonempty open and closed subset of $[0,1]$ is $[0,1]$, since $[0,1]$ is connected. Therefore $A = [0,1]$, which means $p(t) = (1,0)$ for all $t \in [0,1]$. □

To understand the ideas in this argument, we apply them to a second subset of $\mathbf{R}^2$ that is connected but not path-connected, called the deleted comb space D. It is pictured below.



By definition, D is the union of the interval $[0,1]$ along the x-axis together with vertical line segments connecting $(1/n, 0)$ to $(1/n, 1)$ for $n \in \mathbf{Z}^+$ and the single (red) point $(0,1)$:

$$D = ([0,1] \times \{0\}) \cup \bigcup_{n \geq 1} (\{1/n\} \times [0,1]) \cup (0,1).$$

The y-axis strictly between 0 and 1 is not part of this.

Theorem 3.6. *The deleted comb space is connected but not path-connected.*

Proof. The set $D' = D - \{(0,1)\}$ is obviously path-connected: there's a path in D' linking any point in a bristle to the point on the x-axis at the end of that bristle, and any two points in D' on the x-axis can obviously be linked by a path in D' on the x-axis. Concatenating

[1] We're using here the ε-δ definition of continuity of $p\colon [0,1] \to B$ at t_0 with $\varepsilon = 1/2$.

these constructions proves D' is path-connected, and thus connected. Since $(0,1)$ is a limit point of D', D lies between D' and its closure, so D is connected for the same reason the deleted infinite broom is connected. (The closure of D' in $\mathbf{R}^2$ is D together with the y-axis from 0 to 1, and it is path-connected.)

To prove D is not path-connected we'll show no path in D links $(0,1)$ to any other point: if $p\colon [0,1] \to D$ has $p(0) = (0,1)$ then $p(t) = (0,1)$ for all t.

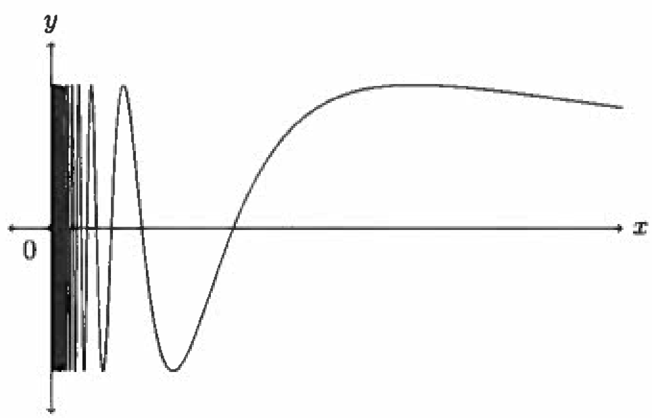
Let
$$A = \{t \in [0,1] : p(t) = (0,1)\}$$
Since $0 \in A$, this is a nonempty subset of $[0,1]$. We will show $A = [0,1]$ by showing A is open and closed in $[0,1]$.

The set A is closed since $A = p^{-1}((0,1))$ and p is continuous.

To show A is open, choose $t_0 \in A$. From continuity of p, there's a $\delta > 0$ such that if $t \in [0,1]$ satisfies $|t - t_0| < \delta$ then $||p(t) - p(t_0)|| < 1/2$, so $||p(t) - (0,1)|| < 1/2$. No point on the x-axis is within $1/2$ of $(0,1)$, so $p(t)$ is not on the x-axis when $t \in [0,1]$ satisfies $|t - t_0| < \delta$.

In place of the slope function m from the previous proof we will use the x-coordinate function. For points in D that are not on the x-axis, their x-coordinate is 0 or of the form $1/n$ for a positive integer n. The x-coordinate function $x\colon \mathbf{R}^2 \to \mathbf{R}$ is continuous and we can define a function $f\colon (t_0 - \delta, t_0 + \delta) \cap [0,1] \to \mathbf{R}$ by $f(t) = x(p(t))$, which is continuous since it's the composition of continuous functions. Set $I := (t_0 - \delta, t_0 + \delta) \cap [0,1]$, which is an open interval of $[0,1]$ and thus is connected. Therefore $f(I)$ is connected and it belongs to $\{0\} \cup \{1/n : n \in \mathbf{Z}^+\}$, so $f(I)$ is a single point by Lemma 3.3. Since $t_0 \in I$ and $f(t_0) = x(p(t_0)) = x((0,1)) = 0$ we get $f(I) = \{0\}$, so $I \subset A$. Therefore A is open (for each $t_0 \in A$ some open interval around t_0 in $[0,1]$ is also in A.) $\square$

Our third example of a topological space that is connected but not path-connected is the topologist's sine curve, pictured below, which is the union of the graph of $y = \sin(1/x)$ for $x > 0$ and the (red) point $(0,0)$. (We stretch the graph horizontally to make its shape clearer, which doesn't affect the topological features.)



Theorem 3.7. *The topologist's sine curve is connected but not path-connected.*

Proof. The graph of $y = \sin(1/x)$ for $x > 0$, like any graph of a function, is path-connected and therefore is connected. Since $(0,0)$ is a limit point of this graph, adjoining it to the graph gives us a connected set for the same reason the deleted infinite broom and deleted comb space are connected.

Let S denote the topologist's sine curve. To show S is not path-connected, we'll show no path in S links $(0,0)$ to any other point in S. At first it might seem we could argue as in the first two examples, using the points in S along the x-axis as a totally disconnected set analogous to the one in Lemma 3.3, but it does not seem to work; try it!

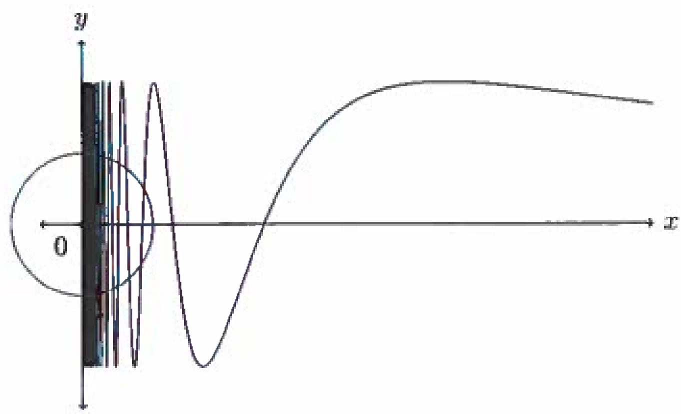
Suppose there is a path p in S from $(0,0)$ to a point on the graph of $y = \sin(1/x)$ with $x > 0$. Let $x\colon \mathbf{R}^2 \to \mathbf{R}$ be the x-coordinate function, which is continuous. The path p starts off on the y-axis and at some point has to "jump" onto the graph of $\sin(1/x)$, which is the points in S with positive x-coordinate. Let t_0 be the time this happens; precisely, set

$$t_0 = \inf\{t \in [0,1] : x(p(t)) > 0\}. \tag{3.1}$$

For $t < t_0$, $x(p(t)) = 0$. By continuity of $x \circ p$ at t_0, $x(p(t_0)) = \lim_{t \to t_0^-} x(p(t)) = 0$, so $p(t_0) = (0,0)$. By continuity of p at t_0, there is a $\delta > 0$ such that

$$t_0 \leq t < t_0 + \delta \Rightarrow \|p(t)\| < \frac{1}{2}. \tag{3.2}$$

We try to convey this visually in the picture below, where the red circle around $(0,0) = p(t_0)$ has radius $1/2$.



By the definition of t_0 as an infimum, for this same δ there is a t_1 with $t_0 < t_1 < t_0 + \delta$ such that $a := x(p(t_1)) > 0$. The image $x(p([t_0, t_1]))$ is connected and contains $0 = x(p(t_0))$ and $a = x(p(t_1))$, and every connected subset of $\mathbf{R}$ is an interval, so

$$[0, a] \subset x(p([t_0, t_1])). \tag{3.3}$$

This contradicts continuity of $t \mapsto x(p(t))$ at t_0 by the picture above, because the graph of $\sin(1/x)$ is oscillating in and out of the red circle, so the x-values on S inside the circle do *not* contain a whole interval like $[0, a]$. To turn this visual idea into a strict logical argument we look at where the peaks and troughs occur in S.

Since $\sin(\theta) = 1$ if and only if $\theta = (4k+1)\frac{\pi}{2}$ and $\sin(\theta) = -1$ if and only if $\theta = (4k-1)\frac{\pi}{2}$, where $k \in \mathbf{Z}$, we have $(x, \sin(1/x)) = (x, 1)$ if $x = 2/((4k+1)\pi)$ and $(x, \sin(1/x)) = (x, -1)$ if $x = 2/((4k-1)\pi)$ for $k \in \mathbf{Z}$. Such x-values get arbitrarily close to 0 for large k, so there are such x-values of both kinds in $[0, a]$. Therefore by (3.3) we get $p(t') = (*, 1)$ and $p(t'') = (*, -1)$ for some t' and t'' in $[t_0, t_1] \subset [t_0, t_0 + \delta)$. But $\|p(t')\| = \|(*, 1)\| > 1/2$ and $\|p(t'')\| = \|(*, -1)\| > 1/2$, which both contradict (3.2). $\square$

The closures of the deleted infinite broom and deleted comb space are path-connected since all points in the closure are linked to $(0,0)$ by a path in the closure, but the closure of the topologist's sine curve, which is obtained by adjoining the whole interval $\{0\} \times [-1,1]$ on the y-axis to the graph, is *not* path-connected.

Corollary 3.8. *The closure of the toplogist's sine curve is not path-connected.*

Proof. We modify the previous proof to show there is no path starting at a point in $\{0\} \times [-1,1]$ and ending at a point on the graph of $y = \sin(1/x)$. Assuming there is such path, p, we have $x(p(0)) = 0$ and $x(p(1)) > 0$, so we can define t_0 as in (3.1) and $x(p(t_0)) = 0$. (It may not be that $p(t_0)$ is $(0,0)$ anymore, but $p(t_0)$ does lie on the y-axis.) Choose δ so that

$$t_0 \leq t < t_0 + \delta \Rightarrow ||p(t) - p(t_0)|| < \frac{1}{2}.$$

Once again there's a $t_1 \in (t_0, t_0+\delta)$ such that $x(p(t_1)) > 0$, so (3.3) holds where $a = x(p(t_1))$.

For some large k we have $2/((4k \pm 1)\pi) \in [0,a]$ for both signs, so these are x-coordinates of $p(t')$ and $p(t'')$ for some t' and t'' in $[t_0, t_1] \subset [t_0, t_0+\delta)$: $p(t') = (*, 1)$ and $p(t'') = (*, -1)$. Since $||p(t') - p(t_0)|| < 1/2$ and $||p(t'') - p(t_0)|| < 1/2$, we get $||p(t') - p(t'')|| < 1$, but $||p(t') - p(t'')|| = ||(*,1) - (*,-1)|| \geq \sqrt{(1 - -1)^2} = 2 > 1$, a contradiction. □

Depending where you read about it, the term "topologist's sine curve" could mean the closure of what we call the topologist's sine curve.

Appendix A. A partial converse to Theorem 2.1

There is an important case where a converse to Theorem 2.1 holds: open subsets of $\mathbf{R}^n$.

Theorem A.1. *If a nonempty open subset of* $\mathbf{R}^n$ *is connected then it is path-connected.*

Proof. Let X be a nonempty open subset of $\mathbf{R}^n$ and pick $x \in X$. We want to show there is a path in X from x to every point in X. The special property of Euclidean space that we're going to use in the proof is that *balls in* $\mathbf{R}^n$ *are path-connected.*

Set

$$U = \{x' \in X : \text{there is a path in } X \text{ from } x \text{ to } x'\}.$$

This is a nonempty subset of X since $x \in U$ (use the constant path $p\colon [0,1] \to X$ where $p(t) = x$ for all t). We will show next that U is open. Suppose $x' \in U$. Since X is open in $\mathbf{R}^n$, there's an open ball B in $\mathbf{R}^n$ such that $x' \in B \subset X$. For every point $b \in B$ there is a (straight line) path p_1 from x' to b that doesn't leave B (so it doesn't leave X either), and since $x' \in U$ there is a path p_2 in X from x to x'. Concatenating these paths together gives us a path in X from x to b. Strictly speaking, since paths are only defined with domain $[0,1]$ we get a path from x to b by spending the first half of our time going from x to x' and the second half going from x' to b: the function $p\colon [0,1] \to X$ defined by

$$p(t) = \begin{cases} p_1(2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ p_2(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

is continuous (since $p_1(1) = x' = p_2(0)$) and $p(0) = p_1(0) = x$, $p(1) = p_2(1) = b$. We have shown every $b \in B$ is in U, so to each $x' \in U$ there's an open ball around x' that is entirely inside U as well. Thus U is open in X.

The complement of U in X is

$$V = \{x' \in X : \text{there is no path in } X \text{ from } x \text{ to } x'\}.$$

We want to prove $V = \emptyset$. If V were nonempty and x' is an element of V, then there is an open ball B in $\mathbf{R}^n$ such that $x' \in B \subset X$. Since all points in B can be linked to x' by a path in B, if any point in B were in U then it could be linked by a path to x in X and we'd then be able to link x and x' by a path in X, which contradicts what it means for x' to lie in V. Thus B is disjoint from U, so $B \subset V$. Therefore V is open in X (every point in V is contained in an open ball of $\mathbf{R}^n$ that's a subset of V).

The equation $X = U \cup V$ exhibits X as a disjoint union of nonempty open subsets if $V \neq \emptyset$, which is a contradiction of the connectedness of X, so $V = \emptyset$. That means $U = X$, so *every* point in X can be linked to x by a path in X. Since this holds for all $x \in X$, X is path-connected. $\square$

References

[1] A. Douady and J. Hubbard, Itération des polynômes quadratiques complexes, *Comptes Rendus, Académie des Sciences, Paris* **294** (1982), 123–126.

[2] J. Munkres, "Topology, A First Course," Prentice-Hall, 1975.

CHAPTER 17

Introduction to Homological Algebra and Group Cohomology

Let R be a ring with 1. In Section 10.5 we saw that a short exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \qquad (17.1)$$

of R-modules gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \qquad (17.2)$$

for any R-module D and that the homomorphism ψ' is in general not surjective so this sequence cannot always be extended to a short exact sequence. Equivalently, homomorphisms from L to D cannot in general be lifted to homomorphisms from M into D. In this chapter we introduce some of the techniques of "homological algebra," which provide a method of extending some exact sequences in a natural way. For the situation above one obtains an infinite exact sequence involving the "cohomology groups" $\mathrm{Ext}^n_R(\_\_, D)$ (cf. Theorem 8), and these groups provide a measure of the set of homomorphisms from L into D that cannot be extended to M. We then consider the analogous questions for the other two functors considered in Section 10.5, namely taking homomorphisms *from* D into the terms of the sequence (1) and tensoring the sequence (1) with D.

In the subsequent sections we concentrate on an important special case of this general type of homological construction—the "cohomology of finite groups." We make explicit the computations in this case and indicate some applications of these techniques to establish some new results in group theory. In this sense, Sections 2–4 may be considered as an explicit "example" illustrating some uses of the general theory in Section 1.

Cohomology and homology groups occur in many areas of mathematics. The formal notions of homology and cohomology groups and the general area of homological algebra arose from algebraic topology around the middle of the 20^{th} century in the study of the relation between the higher homotopy groups and the fundamental group of a topological space, although the study of certain specific cohomology groups, such as Schur's work on group extensions (described in Section 4), predates this by half a century. As with much of algebra, the ideas common to a number of different areas were abstracted into general theories. Much of the language of homology and cohomology reflects its topological origins: homology groups, chains, cycles, boundaries, etc.

17.1 INTRODUCTION TO HOMOLOGICAL ALGEBRA—EXT AND TOR

In this section we describe some general terminology and results in homological algebra leading to the so called Long Exact Sequence in Cohomology. We then define certain (cohomology) groups associated to the sequence (2) and apply the general homological results to obtain a long exact sequence extending this sequence at the right end. We then indicate the corresponding development for sequences obtained by taking homomorphisms from D to the terms in (1) or by tensoring the terms with D.

We begin with a generalization of the notion of an exact sequence, namely a sequence of abelian group homomorphisms where successive maps compose to zero (i.e., the image of one map is contained in the kernel of the next):

Definition. Let C be a sequence of abelian group homomorphisms:

$$0 \longrightarrow C^0 \xrightarrow{d_1} C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \cdots. \qquad (17.3)$$

(1) The sequence C is called a *cochain complex* if the composition of any two successive maps is zero: $d_{n+1} \circ d_n = 0$ for all n.
(2) If C is a cochain complex, its n^{th} *cohomology group* is the quotient group $\ker d_{n+1}/\,\mathrm{image}\, d_n$, and is denoted by $H^n(C)$.

There is a completely analogous "dual" version in which the homomorphisms are between groups in *decreasing* order, in which case the sequence corresponding to (3) is written $\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} C_0 \to 0$. Then if the composition of any two successive homomorphisms is zero, the complex is called a *chain complex*, and its *homology groups* are defined as $H_n(C) = \ker d_n/\,\mathrm{image}\, d_{n+1}$. For chain complexes the notation is often chosen so that the indices appear as subscripts and are decreasing, whereas for cochain complexes the indices are superscripts and are increasing. We shall instead use a uniform notation for the maps on both, since it will be clear from the context whether we are dealing with a chain or a cochain complex.

Chain complexes were the first to arise in topological settings, with cochain complexes soon following. With our applications in Section 2 in mind, we shall concentrate on cochains and cohomology, although all of the general results in this section have similar statements for chains and homology. We shall also be interested in the situation where each C^n is an R-module and the homomorphisms d_n are R-module homomorphisms (referred to simply as a *complex of R-modules*), in which case the groups $H^n(C)$ are also R-modules.

Note that if C is a cochain (respectively, chain) complex then C is an exact sequence if and only if all its cohomology (respectively, homology) groups are zero. Thus the n^{th} cohomology (respectively, homology) group measures the failure of exactness of a complex at the n^{th} stage.

Definition. Let $\mathcal{A} = \{A^n\}$ and $\mathcal{B} = \{B^n\}$ be cochain complexes. A *homomorphism of complexes* $\alpha : \mathcal{A} \to \mathcal{B}$ is a set of homomorphisms $\alpha_n : A^n \to B^n$ such that for every n the following diagram commutes:

$$
\begin{tikzcd}
\cdots \arrow[r] & A^n \arrow[r] \arrow[d, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & \cdots \\
\cdots \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}
\tag{17.4}
$$

Proposition 1. A homomorphism $\alpha : A \to B$ of cochain complexes induces group homomorphisms from $H^n(A)$ to $H^n(B)$ for $n \geq 0$ on their respective cohomology groups.

Proof: It is an easy exercise to show that the commutativity of (4) implies that the images and kernels at each stage of the maps in the first row are mapped to the corresponding images and kernels for the maps in the second row, thus giving a well defined map on the respective quotient (cohomology) groups.

Definition. Let $A = \{A^n\}$, $B = \{B^n\}$ and $C = \{C^n\}$ be cochain complexes. A *short exact sequence* of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a sequence of homomorphisms of complexes such that $0 \to A^n \xrightarrow{\alpha_n} B^n \xrightarrow{\beta_n} C^n \to 0$ is short exact for every n.

One of the main features of cochain complexes is that they lead to long exact sequences in cohomology, which is our first main result:

Theorem 2. *(The Long Exact Sequence in Cohomology)* Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of cochain complexes. Then there is a long exact sequence of cohomology groups:

$$
\begin{aligned}
0 \to H^0(A) \to H^0(B) \to H^0(C) &\xrightarrow{\delta_0} H^1(A) \\
\to H^1(B) \to H^1(C) &\xrightarrow{\delta_1} H^2(A) \to \cdots
\end{aligned}
\tag{17.5}
$$

where the maps between cohomology groups at each level are those in Proposition 1. The maps δ_n are called *connecting homomorphisms*.

Proof: The details of this proof are somewhat lengthy. For each n the verification that the sequence $H^n(A) \to H^n(B) \to H^n(C)$ is exact is a straightforward check of the definition of exactness of each map, similar to the proof of Theorem 33 in Section 10.5. The construction of a connecting homomorphism δ_n is outlined in Exercise 2. Some work is then needed to show that δ_n is a homomorphism, and that the sequence is exact at δ_n.

One immediate consequence of the existence of the long exact sequence in Theorem 2 is the fact that if any two of the cochain complexes A, B, C are exact, then so is the third (cf. Exercise 6).

Homomorphisms and the Groups $Ext^n_R(A,B)$

To apply Theorem 2 to analyze the sequence (2), we try to produce a cochain complex whose first few cohomology groups in the long exact sequence (5) agree with the terms in (2). To do this we introduce the notion of a "resolution" of an R-module:

Definition. Let A be any R-module. A *projective resolution* of A is an exact sequence

$$
\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0 \tag{17.6}
$$

such that each P_i is a projective R-module.

Every R-module has a projective resolution: Let P_0 be any free (hence projective) R-module on a set of generators of A and define an R-module homomorphism ϵ from P_0 onto A by Theorem 6 in Chapter 10. This begins the resolution $\epsilon : P_0 \to A \to 0$. The surjectivity of ϵ ensures that this sequence is exact. Next let $K_0 = \ker \epsilon$ and let P_1 be any free module mapping onto the submodule K_0 of P_0; this gives the second stage $P_1 \to P_0 \to A$ which, by construction, is also exact. We can continue this way, taking at the n^{th} stage a free R-module P_{n+1} that maps surjectively onto the submodule $\ker d_n$ of P_n, obtaining in fact a *free* resolution of A.

One of the reasons that *projective* modules are used in the resolution of A is that this makes it possible to lift various maps (cf. the proof of Proposition 4 following, for instance).

In general a projective resolution is infinite in length, but if A is itself projective, then it has a very simple projective resolution of finite length, namely $0 \to A \xrightarrow{1} A \to 0$ given by the identity map from A to itself.

Given the projective resolution (6), we may form a related sequence by taking homomorphisms of each of the terms into D, keeping in mind that this reverses the direction of the homomorphisms. This yields the sequence

$$
\begin{aligned}
0 \longrightarrow \mathrm{Hom}_R(A, D) &\xrightarrow{\epsilon} \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1} \mathrm{Hom}_R(P_1, D) \xrightarrow{d_2} \cdots \\
\cdots &\xrightarrow{d_{n-1}} \mathrm{Hom}_R(P_{n-1}, D) \xrightarrow{d_n} \mathrm{Hom}_R(P_n, D) \xrightarrow{d_{n+1}} \cdots
\end{aligned}
\tag{17.7}
$$

where to simplify notation we have denoted the induced maps from $\mathrm{Hom}_R(P_{n-1}, D)$ to $\mathrm{Hom}_R(P_n, D)$ for $n \geq 1$ again by d_n and similarly for the map induced by ϵ (cf. Section 10.5). This sequence is not necessarily exact, however it *is* a cochain complex (this is part of the proof of Theorem 33 in Section 10.5). The corresponding cohomology groups have a special name.

Definition. Let A and D be a R-modules. For any projective resolution of A as in (6) let $d_n : \mathrm{Hom}_R(P_{n-1}, D) \to \mathrm{Hom}_R(P_n, D)$ for all $n \geq 1$ as in (7). Define

$$
\mathrm{Ext}^n_R(A, D) = \ker d_{n+1} / \operatorname{image} d_n
$$

where $\mathrm{Ext}^0_R(A, D) = \ker d_1$. The group $\mathrm{Ext}^n_R(A, D)$ is called the n^{th} *cohomology group derived from the functor* $\mathrm{Hom}_R(\_\_, D)$. When $R = \mathbb{Z}$ the group $\mathrm{Ext}^n_{\mathbb{Z}}(A, D)$ is also denoted simply $\mathrm{Ext}^n(A, D)$.

Note that the groups $\mathrm{Ext}^n_R(A, D)$ are also the cohomology groups of the cochain complex obtained from (7) by replacing the term $\mathrm{Hom}_R(A, D)$ with zero (which does not effect the cochain property), i.e., they are the cohomology groups of the cochain complex $0 \to \mathrm{Hom}_R(P_0, D) \to \cdots$.

We shall show below that these cohomology groups do not depend on the choice of projective resolution of A. Before doing so we identify the 0^{th} cohomology group and give some examples.

Proposition 3. For any R-module A we have $\mathrm{Ext}^0_R(A, D) \cong \mathrm{Hom}_R(A, D)$.

Proof: Since the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ is exact, it follows that the corresponding sequence $0 \to \mathrm{Hom}_R(A, D) \xrightarrow{\epsilon} \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1} \mathrm{Hom}_R(P_1, D)$ is also exact by Theorem 33 in Section 10.5 (noting the first comment in the proof). Hence $\mathrm{Ext}^0_R(A, D) = \ker d_1 = \mathrm{image}\,\epsilon \cong \mathrm{Hom}_R(A, D)$, as claimed.

Examples

(1) Let $R = \mathbb{Z}$ and let $A = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the proposition we have $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D)$, and it follows that $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong {}_mD$, where ${}_mD = \{d \in D \mid md = 0\}$ are the elements of D that have order dividing m. For the higher cohomology groups, we use the simple projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for A given by multiplication by m on $\mathbb{Z}$. Taking homomorphisms into a fixed $\mathbb{Z}$-module D gives the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \xrightarrow{m} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \longrightarrow 0 \longrightarrow \cdots.$$

We have $D \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, D)$ (cf. Example 4 following Corollary 32 in Section 10.5) and under this isomorphism we have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$ for any abelian group D. It follows immediately from the definition and the cochain complex above that $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0$ for all $n \geq 2$ and any abelian group D, which we summarize as

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong {}_mD$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$$
$$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0, \quad \text{for all } n \geq 2.$$

(2) The same abelian groups may be modules over several different rings R and the Ext_R cohomology groups depend on R. For example, suppose $R = \mathbb{Z}/m\mathbb{Z}$ for some integer $m \geq 1$. An R-module D is the same as an abelian group D with exponent dividing m, i.e., $mD = 0$. In particular, for any divisor d of m, the group $\mathbb{Z}/d\mathbb{Z}$ is an R-module, and

$$\cdots \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

is a projective (in fact, free) resolution of $\mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}/m\mathbb{Z}$-module, where the final map is the natural projection mapping $x \bmod m$ to $x \bmod d$. Taking homomorphisms into the $\mathbb{Z}/m\mathbb{Z}$-module D, using the isomorphism $\mathrm{Hom}_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D$, and removing the first term gives the cochain complex

$$0 \longrightarrow D \xrightarrow{d} D \xrightarrow{m/d} D \xrightarrow{d} D \xrightarrow{m/d} \cdots.$$

Hence

$$\mathrm{Ext}^0_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D) \cong {}_dD,$$
$$\mathrm{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D) \cong {}_{(m/d)}D/dD, \quad n \text{ odd}, n \geq 1,$$
$$\mathrm{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D) \cong {}_dD/(m/d)D, \quad n \text{ even}, n \geq 2,$$

where ${}_kD = \{d \in D \mid kd = 0\}$ denotes the set of elements of D killed by k. In particular, $\mathrm{Ext}^n_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ for all $n \geq 0$, whereas, for example, $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $n \geq 2$.

In order to show that the cohomology groups $\mathrm{Ext}^n_R(A, D)$ are independent of the choice of projective resolution of A we shall need to be able to "compare" resolutions. The next proposition shows that an R-module homomorphism from A to B lifts to a homomorphism from a projective resolution of A to a projective resolution of B — this lifting property is one instance where the projectivity of the modules in the resolution is important.

Proposition 4. Let $f : A \to A'$ be any homomorphism of R-modules and take projective resolutions of A and A', respectively. Then for each $n \geq 0$ there is a lift f_n of f such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d_2'"] & P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] & 0
\end{tikzcd} \tag{17.8}$$

where the rows are the projective resolutions of A and A', respectively.

Proof: Given the two rows and map f in (8), then since P_0 is projective we may lift the map $f\epsilon : P_0 \to A'$ to a map $f_0 : P_0 \to P_0'$ in such a way that $\epsilon' f_0 = f\epsilon$ (Proposition 30(2) in Section 10.5). This gives the first lift of f. Proceeding inductively in this fashion, assume f_n has been defined to make the diagram commutative to that point. Thus image $f_n d_{n+1} \subseteq \ker d_n'$. The projectivity of P_{n+1} implies that we may lift the map $f_n d_{n+1} : P_{n+1} \to P_n'$ to a map $f_{n+1} : P_{n+1} \to P_{n+1}'$ to make the diagram commute at the next stage. This completes the proof.

The commutative diagram in Proposition 4 implies that the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(A, D) \arrow[r] & \mathrm{Hom}_R(P_0, D) \arrow[r] & \mathrm{Hom}_R(P_1, D) \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(A', D) \arrow[r] \arrow[u, "f"] & \mathrm{Hom}_R(P_0', D) \arrow[r] \arrow[u, "f_0"] & \mathrm{Hom}_R(P_1', D) \arrow[r] \arrow[u, "f_1"] & \cdots
\end{tikzcd} \tag{17.9}$$

is also commutative. The two rows of this diagram are cochain complexes, and this commutative diagram depicts a homomorphism of these cochain complexes. By Proposition 1 we have an induced map on their cohomology groups:

Proposition 5. Let $f : A \to A'$ be a homomorphism of R-modules and take projective resolutions of A and A' as in Proposition 4. Then for every n there is an induced group homomorphism $\varphi_n : \text{Ext}^n_R(A', D) \to \text{Ext}^n_R(A, D)$ on the cohomology groups obtained via these resolutions, and the maps φ_n depend only on f, not on the choice of lifts f_n in Proposition 4.

Proof: The existence of the map on the cohomology groups Ext^n_R follows from Proposition 1 applied to the homomorphism of cochain complexes (9). The more difficult part is showing these maps do not depend on the choice of lifts f_n in Proposition 4. This is easily seen to be equivalent to showing that if f is the zero map, then the induced maps on cohomology groups are also all zero. Assume then that $f = 0$. By the projectivity of the modules P_i one may inductively define R-module homomorphisms $s_n : P_n \to P'_{n+1}$ with the property that for all n,

$$f_n = d'_{n+1}s_n + s_{n-1}d_n \tag{17.10}$$

so the maps s_n give reverse downward diagonal arrows across the squares in (8). (The collection of maps $\{s_n\}$ is called a *chain homotopy* between the chain homomorphism given by the f_n and the zero chain homomorphism, cf. Exercise 4.) Taking homomorphisms into D gives diagram (9) with additional upward diagonal arrows from the homomorphisms induced by the s_n, and these induced homomorphisms satisfy the relations in (10) (i.e., they form a homotopy between cochain complex homomorphisms). It is now an easy exercise using the diagonal maps added to (9) to see that any element in $\text{Hom}_R(P'_n, D)$ representing a coset in $\text{Ext}^n_R(A', D)$ maps to the zero coset in $\text{Ext}^n_R(A, D)$ (cf. Exercise 4). This completes the argument.

One may also check that the homomorphism $\varphi_0 : \text{Ext}^0_R(A', D) \to \text{Ext}^0_R(A, D)$ in Proposition 5 is the same as the map $f : \text{Hom}_R(A', D) \to \text{Hom}_R(A, D)$ defined in Section 10.5 once the corresponding groups have been identified via the isomorphism in Proposition 3.

Theorem 6. The groups $\text{Ext}^n_R(A, D)$ depend only on A and D, i.e., they are independent of the choice of projective resolution of A.

Proof: In the notation of Proposition 4 let $A' = A$, let $f : A \to A'$ be the identity map and let the two rows of (8) be two projective resolutions of A. For any choice of lifts of the identity map, the resulting homomorphisms on cohomology groups $\varphi_n : \text{Ext}^n_R(A', D) \to \text{Ext}^n_R(A, D)$ are seen to be isomorphisms as follows. Add a third row to the diagram (8) by copying the projective resolution in the top row below the second row. Let g be the identity map from A' to A and lift g to maps $g_n : P'_n \to P_n$ by Proposition 4. Let $\psi_n : \text{Ext}^n_R(A, D) \to \text{Ext}^n_R(A', D)$ be the resulting map on cohomology groups. The maps $g_n \circ f_n : P_n \to P_n$ are now a lift of the identity map $g \circ f$, and they are seen to induce the homomorphisms $\varphi_n \circ \psi_n$ on the cohomology groups. However, since the first and third rows are identical, taking the identity map from P_n to itself for all n is a particular lift of $g \circ f$, and this choice clearly induces the identity map on cohomology groups. The last assertion of Proposition 5 then implies that $\varphi_n \circ \psi_n$ is also the identity on $\text{Ext}^n_R(A, D)$. By a symmetric argument $\psi_n \circ \varphi_n$ is the

identity on $\text{Ext}^n_R(A', D)$. This shows the maps φ_n and ψ_n are isomorphisms, as needed to complete the proof.

For a fixed R-module D and fixed integer $n \geq 0$, Proposition 5 and Theorem 6 show that $\text{Ext}^n_R(\_\_, D)$ defines a (contravariant) functor from the category of R-modules to the category of abelian groups.

The next result shows that projective resolutions for a submodule and corresponding quotient module of an R-module M can be fit together to give a projective resolution of M.

Proposition 7. *(Simultaneous Resolution)* Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules, let $L = A$ have a projective resolution as in (6) above, and let N have a similar projective resolution where the projective modules are denoted by $\overline{P}_n$. Then there is a resolution of M by the projective modules $P_n \oplus \overline{P}_n$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus \overline{P}_1 & \longrightarrow & \overline{P}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus \overline{P}_0 & \longrightarrow & \overline{P}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{17.11}$$

Moreover, the rows and columns of this diagram are exact and the rows are split.

Proof: The left and right nonzero columns of (11) are exact by hypothesis. The modules in the middle column are projective (cf. Exercise 3, Section 10.5) and the row maps are the obvious ones to make each row a split exact sequence. It remains then to define the vertical maps in the middle column in such a way as to make the diagram commute. This is accomplished in a straightforward manner, working inductively from the bottom upward — the first step in this process is outlined in Exercise 5.

Theorem 2 and Proposition 7 now yield the long exact sequence for Ext_R that extends the exact sequence (2).

Theorem 8. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \to \operatorname{Hom}_R(N, D) \to \operatorname{Hom}_R(M, D) \to \operatorname{Hom}_R(L, D) \xrightarrow{\delta_0} \operatorname{Ext}^1_R(N, D) \\ \to \operatorname{Ext}^1_R(M, D) \to \operatorname{Ext}^1_R(L, D) \xrightarrow{\delta_1} \operatorname{Ext}^2_R(N, D) \to \cdots \end{aligned} \tag{17.12}$$

where the maps between groups at the same level n are as in Proposition 5 and the connecting homomorphisms δ_n are given by Theorem 2.

Proof: Take a simultaneous projective resolution of the short exact sequence as in Proposition 7 and take homomorphisms into D. To obtain the cohomology groups Ext^n_R from the resulting diagram, as noted in the discussion preceding Proposition 3 we replace the lowest nonzero row in the transformed diagram with a row of zeros to get the following commutative diagram:

$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & \operatorname{Hom}_R(\overline{P}_1, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_1 \oplus \overline{P}_1, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_1, D) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(\overline{P}_0, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_0 \oplus \overline{P}_0, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_0, D) \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{17.13}$$

The columns of (13) are cochain complexes, and the rows are split by Proposition 29(2) of Section 10.5 and the discussion following it. Thus (13) is a short exact sequence of cochain complexes. Theorem 2 then gives a long exact sequence of cohomology groups whose terms are, by definition, the groups $\operatorname{Ext}^n_R(\_\_, D)$, for $n \geq 0$. The 0^{th} order terms are identified by Proposition 3, completing the proof.

Theorem 8 shows how the exact sequence (2) can be extended in a natural way and shows that the group $\operatorname{Ext}^1_R(N, D)$ is the first measure of the failure of (2) to be exact on the right — in fact (2) can be extended to a short exact sequence on the right if and only if the connecting homomorphism δ_0 in (12) is the zero homomorphism. In particular, if $\operatorname{Ext}^1_R(N, D) = 0$ for all R-modules N, then (2) will be exact on the right for *every* exact sequence (1). We have already seen (Corollary 35 in Section 10.5) that this implies the R-module D is injective. Part of the next result shows that the converse is also true and characterizes injective modules in terms of Ext_R groups.

Proposition 9. For an R-module Q the following are equivalent:
(1) Q is injective,
(2) $\operatorname{Ext}^1_R(A, Q) = 0$ for all R-modules A, and
(3) $\operatorname{Ext}^n_R(A, Q) = 0$ for all R-modules A and all $n \geq 1$.

Proof: We showed (2) implies (1) above, and (3) implies (2) is trivial, so it remains to show that if Q is injective then $\operatorname{Ext}^n_R(A, Q) = 0$ for all R-modules A and all $n \geq 1$. Take a projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

for A. Since Q is injective, the sequence

$$0 \to \operatorname{Hom}_R(A, Q) \to \operatorname{Hom}_R(P_0, Q) \to \cdots \to \operatorname{Hom}_R(P_{n-1}, Q) \to \operatorname{Hom}_R(P_n, Q) \to \cdots$$

is still exact (Corollary 35 in Section 10.5), so all of the cohomology groups for this cochain complex are 0. In particular, the groups $\operatorname{Ext}^n_R(A, Q)$ for $n \geq 1$ are all trivial, which is (3).

For a fixed R-module D, the result in Theorem 8 can be viewed as explaining what happens to the short exact sequence $0 \to L \to M \to N \to 0$ on the right after applying the left exact functor $\operatorname{Hom}_R(\_\_, D)$. This is why the (contravariant) functors $\operatorname{Ext}^n_R(\_\_, D)$ are called the *right derived functors* for the functor $\operatorname{Hom}_R(\_\_, D)$.

One can also consider the effect of applying the left exact functor $\operatorname{Hom}_R(D, \_\_)$, i.e., by taking homomorphisms *from* D rather than *into* D. The next theorem shows that in fact the same Ext_R groups define the (covariant) right derived functors for $\operatorname{Hom}_R(D, \_\_)$ as well.

Theorem 10. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \to \operatorname{Hom}_R(D, L) \to \operatorname{Hom}_R(D, M) \to \operatorname{Hom}_R(D, N) \xrightarrow{\gamma_0} \operatorname{Ext}^1_R(D, L) \\ \to \operatorname{Ext}^1_R(D, M) \to \operatorname{Ext}^1_R(D, N) \xrightarrow{\gamma_1} \operatorname{Ext}^2_R(D, L) \to \cdots. \end{aligned} \tag{17.14}$$

Proof: Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. By taking a projective resolution of D and then applying $\operatorname{Hom}_R(\_\_, L)$, $\operatorname{Hom}_R(\_\_, M)$ and $\operatorname{Hom}_R(\_\_, N)$ to this resolution one obtains the columns in a commutative diagram similar to (13), but with L, M and N in the second positions rather than the first. Applying the Long Exact Sequence Theorem to this array gives (14).

Theorem 10 shows that the group $\operatorname{Ext}^1_R(D, L)$ measures whether the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \longrightarrow \operatorname{Hom}_R(D, M) \longrightarrow \operatorname{Hom}_R(D, N)$$

can be extended to a short exact sequence — it can be extended if and only if γ_0 is the zero homomorphism. In particular, this will always be the case if the module D has the property that $\operatorname{Ext}^1_R(D, B) = 0$ for all R-modules B; in this case it follows by Corollary 32 in Section 10.5 that D is a projective R-module. As in the situation of injective R-modules in Proposition 9, the vanishing of these cohomology groups in fact characterizes projective R-modules:

Proposition 11. For an R-module P the following are equivalent:

(1) P is projective,
(2) $\text{Ext}^1_R(P, B) = 0$ for all R-modules B, and
(3) $\text{Ext}^n_R(P, B) = 0$ for all R-modules B and all $n \geq 1$.

Proof: We proved (2) implies (1) above, and (3) implies (2) is trivial, so it remains to prove that (1) implies (3). If P is a projective R-module, then the simple exact sequence

$$0 \longrightarrow P \xrightarrow{1} P \longrightarrow 0$$

given by the identity map on P is a projective resolution of P. Taking homomorphisms into B gives the simple cochain complex

$$0 \to \text{Hom}_R(P, B) \xrightarrow{1} \text{Hom}_R(P, B) \to 0 \to \cdots \to 0 \to \cdots$$

from which it follows by definition that $\text{Ext}^n_R(P, B) = 0$ for all $n \geq 1$, which gives (3).

Examples

(1) Since $\mathbb{Z}^m$ is a free, hence projective, $\mathbb{Z}$-module, it follows from Proposition 11 that

$$\text{Ext}^n_{\mathbb{Z}}(\mathbb{Z}^m, B) = 0$$

for all abelian groups B, all $m \geq 1$, and all $n \geq 1$.

(2) It is not difficult to show that $\text{Ext}^n_R(A_1 \oplus A_2, B) \cong \text{Ext}^n_R(A_1, B) \oplus \text{Ext}^n_R(A_2, B)$ for all $n \geq 0$ (cf. Exercise 10), so the previous example together with the example following Proposition 3 determines $\text{Ext}^n_{\mathbb{Z}}(A, B)$ for all finitely generated abelian groups A. In particular, $\text{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all finitely generated groups A, all abelian groups B, and all $n \geq 2$.

We have chosen to define the cohomology group $\text{Ext}^n_R(A, B)$ using a projective resolution of A. There is a parallel development using an *injective resolution* of B:

$$0 \to B \to Q_0 \to Q_1 \to \cdots$$

where each Q_i is injective. In this situation one defines $\text{Ext}^n_R(A, B)$ as the n^{th} cohomology group of the cochain sequence obtained by applying $\text{Hom}_R(A, \_)$ to the resolution for B. The theory proceeds in a manner analogous to the development of this section. Ultimately one shows that there is a natural isomorphism between the groups $\text{Ext}^n_R(A, B)$ constructed using both methods.

Examples

(1) Suppose $R = \mathbb{Z}$ and A and B are $\mathbb{Z}$-modules, i.e., are abelian groups. Recall that a $\mathbb{Z}$-module is injective if and only if it is divisible (Proposition 36 in Section 10.5). The group B can be embedded in an injective $\mathbb{Z}$-module Q_0 (Corollary 37 in Section 10.5) and the quotient, Q_1, of Q_0 by the image of B is again injective. Hence we have an injective resolution

$$0 \longrightarrow B \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow 0$$

of B. Applying $\text{Hom}_{\mathbb{Z}}(A, \_)$ to this sequence gives the cochain complex

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(A, B) \longrightarrow \text{Hom}_{\mathbb{Z}}(A, Q_0) \longrightarrow \text{Hom}_{\mathbb{Z}}(A, Q_1) \longrightarrow 0 \longrightarrow \cdots$$

from which it follows immediately that

$$\text{Ext}^n_{\mathbb{Z}}(A, B) = 0$$

for all abelian groups A and B and all $n \geq 2$, showing that the result of the previous example holds also when A is not finitely generated.

(2) Suppose A is a torsion abelian group. Then we have $\text{Ext}^0(A, \mathbb{Z}) \cong \text{Hom}(A, \mathbb{Z}) = 0$ since $\mathbb{Z}$ is torsion free. The sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives an injective resolution of $\mathbb{Z}$. Applying $\text{Hom}(A, \_)$ gives the cochain complex

$$0 \longrightarrow \text{Hom}(A, \mathbb{Z}) \longrightarrow \text{Hom}(A, \mathbb{Q}) \longrightarrow \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \longrightarrow \cdots$$

and since $\mathbb{Q}$ is also torsion free, this shows that

$$\text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

The group $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is called the *Pontriagin dual group* to A. If A is a finite abelian group the Pontriagin dual of A is isomorphic to A (cf. Exercise 14, Section 5.2). In particular, $\text{Ext}^1(A, \mathbb{Z}) \cong A$ is nonzero for all nonzero finite abelian groups A. We have $\text{Ext}^n(A, \mathbb{Z}) = 0$ for all $n \geq 2$ by the previous example.

We record an important property of Ext^1_R, which helps to explain the name for these cohomology groups. Recall that equivalent extensions were defined at the beginning of Section 10.5.

Theorem 12. For any R-modules N and L there is a bijection between $\text{Ext}^1_R(N, L)$ and the set of equivalence classes of extensions of N by L.

Although we shall not prove this result, in Section 4 we establish a similar bijection between equivalence classes of group extensions of G by A and elements of a certain cohomology group, where G is any finite group and A is any $\mathbb{Z}G$-module.

Example

Suppose $R = \mathbb{Z}$ and $A = B = \mathbb{Z}/p\mathbb{Z}$. We showed above that $\text{Ext}^1_R(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, so by Theorem 12 there are precisely p equivalence classes of extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$. These are given by the direct sum $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (which corresponds to the trivial class in $\text{Ext}^1_R(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$) and the $p - 1$ extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{i} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

defined by the map $i(x) = ix \bmod p$ for $i = 1, 2, \ldots, p - 1$. Note that while these are inequivalent as extensions, they all determine the same group $\mathbb{Z}/p^2\mathbb{Z}$.

Tensor Products and the Groups $Tor_n^R(A,B)$

The cohomology groups $\mathrm{Ext}_R^n(A, B)$ determine what happens to short exact sequences on the right after applying the left exact functors $\mathrm{Hom}_R(D, \_\_)$ and $\mathrm{Hom}_R(\_\_, D)$. One may similarly ask for the behavior of short exact sequences on the left after applying the right exact functor $D \otimes_R \_\_$ or the right exact functor $\_\_ \otimes_R D$. This leads to the Tor (homology) groups (whose name derives from their relation to torsion submodules), and we now briefly outline the development of these left derived functors. In some respects this theory is "dual" to the theory for Ext_R. We concentrate on the situation for $D \otimes_R \_\_$ when D is a right R-module. When D is a left R-module there is a completely symmetric theory for $\_\_ \otimes_R D$; when R is commutative and all R-modules have the same left and right R action the homology groups resulting from both developments are isomorphic.

Suppose then that D is a right R-module. Then for every left R-module B the tensor product $D \otimes_R B$ is an abelian group and the functor $D \otimes \_\_$ is covariant and right exact, i.e., for any short exact sequence (1) of left R-modules,

$$D \otimes L \longrightarrow D \otimes M \longrightarrow D \otimes N \longrightarrow 0$$

is an exact sequence of abelian groups. This sequence may be extended at the left end to a long exact sequence as follows. Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$$

be a projective resolution of B, and take tensor products with D to obtain

$$\cdots \longrightarrow D \otimes P_n \xrightarrow{1 \otimes d_n} D \otimes P_{n-1} \longrightarrow \cdots \xrightarrow{1 \otimes d_1} D \otimes P_0 \xrightarrow{1 \otimes \epsilon} D \otimes B \longrightarrow 0. \quad (17.15)$$

It follows from the argument in Theorem 39 of Section 10.5 that (15) is a chain complex — the composition of any two successive maps is zero — so we may form its homology groups.

Definition. Let D be a right R-module and let B be a left R-module. For any projective resolution of B by left R-modules as above let $1 \otimes d_n : D \otimes P_n \to D \otimes P_{n-1}$ for all $n \geq 1$ as in (15). Then

$$\mathrm{Tor}_n^R(D, B) = \ker(1 \otimes d_n)/\operatorname{image}(1 \otimes d_{n+1})$$

where $\mathrm{Tor}_0^R(D, B) = (D \otimes P_0)/\operatorname{image}(1 \otimes d_1)$. The group $\mathrm{Tor}_n^R(D, B)$ is called the n^{th} *homology group derived from the functor* $D \otimes \_\_$. When $R = \mathbb{Z}$ the group $\mathrm{Tor}_n^{\mathbb{Z}}(D, B)$ is also denoted simply $\mathrm{Tor}_n(D, B)$.

Thus $\mathrm{Tor}_n^R(D, B)$ is the n^{th} homology group of the chain complex obtained from (15) by removing the term $D \otimes B$.

A completely analogous proof to Proposition 3 (but relying on Theorem 39 in Section 10.5) implies the following:

Proposition 13. For any left R-module B we have $\mathrm{Tor}_0^R(D, B) \cong D \otimes B$.

Example

Let $R = \mathbb{Z}$ and let $B = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the proposition, $\mathrm{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z})$ is isomorphic to $D \otimes \mathbb{Z}/m\mathbb{Z}$, so we have $\mathrm{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong D/mD$ (Example 8 following Corollary 12 in Section 10.4). For the higher groups we apply $D \otimes \_\_$ to the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

of B and use the isomorphisms $D \otimes \mathbb{Z} \cong D$ and $D \otimes \mathbb{Z}/m\mathbb{Z} \cong D/mD$. This gives the chain complex

$$\cdots \longrightarrow 0 \longrightarrow D \xrightarrow{m} D \longrightarrow D/mD \longrightarrow 0.$$

It follows that $\mathrm{Tor}_1^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong {}_mD$ is the subgroup of D annihilated by m and that $\mathrm{Tor}_n^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) = 0$ for all $n \geq 2$, which we summarize as

$$\begin{aligned} &\mathrm{Tor}_0(D, \mathbb{Z}/m\mathbb{Z}) \cong D/mD, \\ &\mathrm{Tor}_1(D, \mathbb{Z}/m\mathbb{Z}) \cong {}_mD, \\ &\mathrm{Tor}_n(D, \mathbb{Z}/m\mathbb{Z}) = 0, \quad \text{for all } n \geq 2. \end{aligned}$$

As for Ext, the Tor groups depend on the ring R (cf. Exercise 20).

Following a similar development to that for Ext_R, one shows:

Proposition 14.

(1) The homology groups $\mathrm{Tor}_n^R(D, B)$ are independent of the choice of projective resolution of B, and

(2) for every R-module homomorphism $f : B \to B'$ there are induced maps $\psi_n : \mathrm{Tor}_n^R(D, B) \to \mathrm{Tor}_n^R(D, B')$ on homology groups (depending only on f).

There is a Long Exact Sequence in Homology analogous to Theorem 2, except that all the arrows are reversed, whose proof follows mutatis mutandis from the argument for cohomology. This together with Simultaneous Resolution gives:

Theorem 15. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of left R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} \cdots \to \mathrm{Tor}_2^R(D, N) &\xrightarrow{\delta_1} \mathrm{Tor}_1^R(D, L) \to \mathrm{Tor}_1^R(D, M) \to \\ &\mathrm{Tor}_1^R(D, N) \xrightarrow{\delta_0} D \otimes L \to D \otimes M \to D \otimes N \to 0 \end{aligned}$$

where the maps between groups at the same level n are as in Proposition 14 (and the maps δ_n are called connecting homomorphisms).

There is a characterization of flat modules corresponding to Propositions 9 and 11 whose proof is very similar and is left as an exercise.

Proposition 16. For a right R-module D the following are equivalent:
(1) D is a flat R-module,
(2) $\operatorname{Tor}_1^R(D, B) = 0$ for all left R-modules B, and
(3) $\operatorname{Tor}_n^R(D, B) = 0$ for all left R-modules B and all $n \geq 1$.

We have defined $\operatorname{Tor}_n^R(A, B)$ as the homology of the chain complex obtained by tensoring a projective resolution of B on the left with A. The same groups are obtained by taking the homology of the chain complex obtained by tensoring a projective resolution of A on the right by B. Put another way, the $\operatorname{Tor}_n^R(A, B)$ groups define the (covariant) left derived functors for both of the right exact functors $A \otimes_R \_$ and $\_ \otimes_R B$: if D is a left R-module, then the short exact sequence $0 \to L \to M \to N \to 0$ of right R-modules gives rise to the long exact sequence

$$\cdots \to \operatorname{Tor}_2^R(N, D) \xrightarrow{\gamma_1} \operatorname{Tor}_1^R(L, D) \to \operatorname{Tor}_1^R(M, D) \to$$
$$\operatorname{Tor}_1^R(N, D) \xrightarrow{\gamma_0} L \otimes_R D \to M \otimes_R D \to N \otimes_R D \to 0$$

of abelian groups. In particular, the left R-module D is flat if and only if $\operatorname{Tor}_1^R(A, D) = 0$ for all right R-modules A.

When R is commutative, $A \otimes_R B \cong B \otimes_R A$ (Proposition 20 in Section 10.4) for any two R-modules A and B with the standard R-module structures, and it follows that $\operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^R(B, A)$ as R-modules. When R is commutative the Tor long exact sequences are exact sequences of R-modules.

Examples

(1) If $R = \mathbb{Z}$, then since $\mathbb{Z}^m$ is free, hence flat (Corollary 42, Section 10.5), we have $\operatorname{Tor}_n(A, \mathbb{Z}^m) = 0$ for all $n \geq 1$ and all abelian groups A.
(2) Since $\operatorname{Tor}_n^R(A, B_1 \oplus B_2) \cong \operatorname{Tor}_n^R(A, B_1) \oplus \operatorname{Tor}_n^R(A, B_2)$ (cf. Exercise 10), the previous two examples together determine $\operatorname{Tor}_n^R(A, B)$ for all abelian groups A and all finitely generated abelian groups B.
(3) As a particular case of the previous example, $\operatorname{Tor}_1(A, B)$ is a torsion group and $\operatorname{Tor}_n(A, B) = 0$ for every abelian group A, every finitely generated abelian group B, and all $n \geq 2$. In fact these results hold without the condition that B be finitely generated.
(4) The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives the long exact sequence

$$\cdots \to \operatorname{Tor}_1(D, \mathbb{Q}) \to \operatorname{Tor}_1(D, \mathbb{Q}/\mathbb{Z}) \to D \otimes \mathbb{Z} \to D \otimes \mathbb{Q} \to D \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (Example 2 following Corollary 42 in Section 10.5), the proposition shows that we have an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1(D, \mathbb{Q}/\mathbb{Z}) \longrightarrow D \longrightarrow D \otimes \mathbb{Q}$$

and so $\operatorname{Tor}_1(D, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the kernel of the natural map from D into $D \otimes \mathbb{Q}$, which is the torsion subgroup of D (cf. Exercise 9 in Section 10.4).

The following results show that, for $R = \mathbb{Z}$, the Tor groups are closely related to torsion subgroups. The Tor groups first arose in applications of torsion abelian groups in topological settings, which helps explain the terminology.

Proposition 17. Let A and B be $\mathbb{Z}$-modules and let $t(A)$ and $t(B)$ denote their respective torsion submodules. Then $\operatorname{Tor}_1(A, B) \cong \operatorname{Tor}_1(t(A), t(B))$.

Proof: In the case where A and B are finitely generated abelian groups this follows by Examples 3 and 4 above. For the general case, cf. Exercise 16.

Corollary 18. If A is an abelian group then A is torsion free if and only if $\operatorname{Tor}_1(A, B) = 0$ for every abelian group B (in which case A is flat as a $\mathbb{Z}$-module).

Proof: By the proposition, if A has no elements of finite order then we have $\operatorname{Tor}_1(A, B) = \operatorname{Tor}_1(t(A), B) = \operatorname{Tor}_1(0, B) = 0$ for every abelian group B. Conversely, if $\operatorname{Tor}_1(A, B) = 0$ for all B, then in particular $\operatorname{Tor}_1(A, \mathbb{Q}/\mathbb{Z}) = 0$, and this group is isomorphic to the torsion subgroup of A by the example above.

The results of Proposition 17 and Corollary 18 hold for any P.I.D. R in place of $\mathbb{Z}$ (cf. Exercise 26 in Section 10.5 and Exercise 16).

Finally, we mention that the cohomology and homology theories we have described may be developed in a vastly more general setting by axiomatizing the essential properties of R-modules and the Hom_R and tensor product functors. This leads to the general notions of *abelian categories* and *additive functors*. In the case of the abelian category of R-modules, any additive functor $\mathcal{F}$ to the category of abelian groups gives rise to a set of *derived functors*, $\mathcal{F}_n$, also from R-modules to abelian groups, for all $n \geq 0$. Then for each short exact sequence $0 \to L \to M \to N \to 0$ of R-modules there is a long exact sequence of (cohomology or homology) groups whose terms are $\mathcal{F}_n(L)$, $\mathcal{F}_n(M)$ and $\mathcal{F}_n(N)$, and these long exact sequences reflect the exactness properties of the functor $\mathcal{F}$. If $\mathcal{F}$ is left or right exact then the 0^{th} derived functor $\mathcal{F}_0$ is naturally equivalent to $\mathcal{F}$ (hence the 0^{th} degree groups $\mathcal{F}_0(X)$ are isomorphic to $\mathcal{F}(X)$), and if $\mathcal{F}$ is an exact functor then $\mathcal{F}_n(X) = 0$ for all $n \geq 1$ and all R-modules X.

EXERCISES

1. Give the details of the proof of Proposition 1.
2. This exercise defines the connecting map δ_n in the Long Exact Sequence of Theorem 2 and proves it is a homomorphism. In the notation of Theorem 2 let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes, where for simplicity the cochain maps for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are all denoted by the same d.
   (a) If $c \in C^n$ represents the class $x \in H^n(\mathcal{C})$ show that there is some $b \in B^n$ with $\beta_n(b) = c$.
   (b) Show that $d_{n+1}(b) \in \ker \beta_{n+1}$ and conclude that there is a unique $a \in A^{n+1}$ such that $\alpha_{n+1}(a) = d_{n+1}(b)$. [Use $c \in \ker d_{n+1}$ and the commutativity of the diagram.]
   (c) Show that $d_{n+2}(a) = 0$ and conclude that a defines a class $\bar{a}$ in the quotient group $H^{n+1}(\mathcal{A})$. [Use the fact that α_{n+2} is injective.]
   (d) Prove that $\bar{a}$ is independent of the choice of b, i.e., if b' is another choice and a' is its unique preimage in A^{n+1} then $\bar{a} = \overline{a'}$, and that $\bar{a}$ is also independent of the choice of c representing the class x.
   (e) Define $\delta_n(x) = \bar{a}$ and prove that δ_n is a group homomorphism from $H^n(\mathcal{C})$ to $H^{n+1}(\mathcal{A})$. [Use the fact that $\delta_n(x)$ is independent of the choices of c and b to compute $\delta_n(x_1 + x_2)$.]

3. Suppose

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' & \end{tikzcd}$$

is a commutative diagram of R-modules with exact rows.

(a) If $c \in \ker h$ and $\beta(b) = c$ prove that $g(b) \in \ker \beta'$ and conclude that $g(b) = \alpha'(a')$ for some $a' \in A'$. [Use the commutativity of the diagram.]

(b) Show that $\delta(c) = a'$ mod image f is a well defined R-module homomorphism from $\ker h$ to the quotient $A'/\text{image } f$.

(c) *(The Snake Lemma)* Prove there is an exact sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

where $\operatorname{coker} f$ (the *cokernel* of f) is $A'/(\text{image } f)$ and similarly for $\operatorname{coker} g$ and $\operatorname{coker} h$.

(d) Show that if α is injective and β' is surjective (i.e., the two rows in the commutative diagram above can be extended to short exact sequences) then the exact sequence in (c) can be extended to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0$$

4. Let $\mathcal{A} = \{A^n\}$ and $\mathcal{B} = \{B^n\}$ be cochain complexes, where the maps $A^n \to A^{n+1}$ and $B^n \to B^{n+1}$ in both complexes are denoted by d_{n+1} for all n. Cochain complex homomorphisms α and β from $\mathcal{A}$ to $\mathcal{B}$ are said to be *homotopic* if for all n there are module homomorphisms $s_n : A^{n+1} \to B^n$ such that the maps $\alpha_n - \beta_n$ from A^n to B^n satisfy

$$\alpha_n - \beta_n = d_n s_{n-1} + s_n d_{n+1}.$$

The collection of maps $\{s_n\}$ is called a *cochain homotopy* from α to β. One may similarly define chain homotopies between chain complexes.

(a) Prove that homotopic maps of cochain complexes induce the same maps on cohomology, i.e., if α and β are homotopic homomorphisms of cochain complexes then the induced group homomorphisms from $H^n(\mathcal{A})$ to $H^n(\mathcal{B})$ are equal for every $n \geq 0$. (Thus "homotopy" gives a sufficient condition for two maps of complexes to induce the same maps on cohomology or homology; this condition is not in general necessary.) [Use the definition of homotopy to show $(\alpha_n - \beta_n)(z) \in \text{image } d_n$ for every $z \in \ker d_{n+1}$.]

(b) Prove that the relation $\alpha \sim \beta$ if α and β are homotopic is an equivalence relation on any set of cochain complex homomorphisms.

5. Establish the first step in the Simultaneous Resolution result of Proposition 7 as follows: assume the first two nonzero rows in diagram (11) are given, except for the map from $P_0 \oplus \overline{P}_0$ to M (where the maps along the row of projective modules are the obvious injection and projection for this split exact sequence). Let $\mu : \overline{P}_0 \to M$ be a lifting to $\overline{P}_0$ of the map $\overline{P}_0 \to N$ (which exists because $\overline{P}_0$ is projective). Let λ be the composition $P_0 \to L \to M$ in the diagram. Define

$$\pi : P_0 \oplus \overline{P}_0 \to M \qquad \text{by} \qquad \pi(x, y) = \lambda(x) + \mu(y).$$

Show that with this definition the first two nonzero rows of (11) form a commutative diagram.

6. Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes. Prove that if any two of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are exact, then so is the third. [Use Theorem 2.]

7. Prove that a finitely generated abelian group A is free if and only if $\operatorname{Ext}^1(A, \mathbb{Z}) = 0$.

8. Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ the sequence $0 \to \operatorname{Ext}^n_R(N, D) \to \operatorname{Ext}^n_R(M, D) \to \operatorname{Ext}^n_R(L, D) \to 0$ is also short exact and split. [Use a splitting homomorphism and Proposition 5.]

9. Show that

$$0 \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \cdots$$

is an injective resolution of $\mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}/m\mathbb{Z}$-module. [Use Proposition 36 in Section 10.5.] Use this to compute the groups $\operatorname{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z})$ in terms of the dual group $\operatorname{Hom}_{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/m\mathbb{Z})$. In particular, if $m = p^2$ and $d = p$, give another derivation of the result $\operatorname{Ext}^n_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.

10. (a) Prove that an arbitrary direct sum $\oplus_{i \in I} P_i$ of projective modules P_i is projective and that an arbitrary direct product $\prod_{j \in J} Q_j$ of injective modules Q_j is injective.

(b) Prove that an arbitrary direct sum of projective resolutions is again projective and use this to show $\operatorname{Ext}^n_R(\oplus_{i \in I} A_i, B) \cong \prod_{i \in I} \operatorname{Ext}^n_R(A_i, B)$ for any collection of R-modules A_i $(i \in I)$. [cf. Exercise 12 in Section 10.5.]

(c) Prove that an arbitrary direct product of injective resolutions is an injective resolution and use this to show $\operatorname{Ext}^n_R(A, \prod_{j \in J} B_j) \cong \prod_{j \in J} \operatorname{Ext}^n_R(A, B_j)$ for any collection of R-modules B_j $(j \in J)$. [cf. Exercise 12 in Section 10.5.]

(d) Prove that $\operatorname{Tor}^R_n(A, \oplus_{j \in J} B_j) \cong \oplus_{j \in J} \operatorname{Tor}^R_n(A, B_j)$ for any collection of R-modules B_j $(j \in J)$.

11. *(Bass' Characterization of Noetherian Rings)* Suppose R is a commutative ring.

(a) If R is Noetherian, and I is any nonzero ideal in R show that the image of any R-module homomorphism $f : I \to \oplus_{j \in \mathcal{J}} Q_j$ from I into a direct sum of injective R-modules Q_j $(j \in \mathcal{J})$ is contained in some finite direct sum of the Q_j.

(b) If R is Noetherian, prove that an arbitrary direct sum $\oplus_{j \in \mathcal{J}} Q_j$ of injective R-modules is again injective. [Use Baer's Criterion (Proposition 36) and Exercise 4 in Section 10.5 together with (a).]

(c) Let $I_1 \subseteq I_2 \subseteq \ldots$ be an ascending chain of ideals of R with union I and let $I/I_i \to Q_i$ for $i = 1, 2, \ldots$ be an injection of the quotient I/I_i into an injective R-module Q_i (by Theorem 38 in Section 10.5). Prove that the composition of these injections with the product of the canonical projection maps $I \to I_i$ gives an R-module homomorphism $f : I \to \oplus_{i=1,2,\ldots} Q_i$.

(d) Prove the converse of (b): if an arbitrary direct sum $\oplus_{j \in \mathcal{J}} Q_j$ of injective R-modules is again injective then R is Noetherian. [If the direct sum in (c) is injective, use Baer's Criterion to lift f to a homomorphism $F : R \to \oplus_{i=1,2,\ldots} Q_i$. If the component of $F(1)$ in Q_i is 0 for $i \geq n$ prove that $I = I_n$ and the ascending chain of ideals is finite.]

12. Prove Proposition 13: $\operatorname{Tor}^R_0(D, A) \cong D \otimes_R A$. [Follow the proof of Proposition 3.]

13. Prove Proposition 16 characterizing flat modules.

14. Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules. Prove that if C is a flat R-module, then A is flat if and only if B is also flat. [Use the Tor long exact sequence.] Give an example to show that if A and B are flat then C need not be flat.

15. (a) If I is an ideal in R and M is an R-module, prove that $\operatorname{Tor}_1^R(M, R/I)$ is isomorphic to the kernel of the map $M \otimes_R I \to M$ that maps $m \otimes i$ to mi for $i \in I$ and $m \in M$. [Use the Tor long exact sequence associated to $0 \to I \to R \to R/I \to 0$ noting that R is flat.]

(b) (*A Flatness Criterion using* Tor) Prove that the R-module M is flat if and only if $\operatorname{Tor}_1^R(M, R/I) = 0$ for every finitely generated ideal I of R. [Use Exercise 25 in Section 10.5.]

16. Suppose R is a P.I.D. and A and B are R-modules. If $t(B)$ denotes the torsion submodule of B show that $\operatorname{Tor}_1^R(A, t(B)) \cong \operatorname{Tor}_1^R(A, B)$ and deduce that $\operatorname{Tor}_1^R(A, B)$ is isomorphic to $\operatorname{Tor}_1^R(t(A), t(B))$. [Use Exercise 26 in Section 10.5 to show that $B/t(B)$ is flat over R, then use the Tor long exact sequence with $D = A$ applied to the short exact sequence $0 \to t(B) \to B \to B/t(B) \to 0$ and the remarks following Proposition 16.]

17. Let $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \cdots$. Prove that $\operatorname{Ext}^1(A, B) \cong (B/2B) \times (B/3B) \times (B/4B) \times \cdots$ for any abelian group A. [Use Exercise 10.] Prove that $\operatorname{Ext}^1(A, B) = 0$ if and only if B is divisible.

18. Prove that $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module and deduce that $\operatorname{Tor}_1^{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$.

19. Suppose $r \neq 0$ is not a zero divisor in the commutative ring R.

(a) Prove that multiplication by r gives a free resolution $0 \to R \xrightarrow{r} R \to R/rR \to 0$ of the quotient R/rR.

(b) Prove that $\operatorname{Ext}_R^0(R/rR, B) = {}_rB$ is the set of elements $b \in B$ with $rb = 0$, that $\operatorname{Ext}_R^1(R/rR, B) \cong B/rB$, and that $\operatorname{Ext}_R^n(R/rR, B) = 0$ for $n \geq 2$ for every R-module B.

(c) Prove that $\operatorname{Tor}_0^R(A, R/rR) = A/rA$, that $\operatorname{Tor}_1^R(A, R/rR) = {}_rA$ is the set of elements $a \in A$ with $ra = 0$, and that $\operatorname{Tor}_n^R(A, R/rR) = 0$ for $n \geq 2$ for every R-module A.

20. Prove that $\operatorname{Tor}_0^{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \cong A/dA$, that $\operatorname{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \cong {}_dA/(m/d)A$ for n odd, $n \geq 1$, and that $\operatorname{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \cong {}_{(m/d)}A/dA$ for n even, $n \geq 2$. [Use the projective resolution in Example 2 following Proposition 3.]

21. Let $R = k[x, y]$ where k is a field, and let I be the ideal (x, y) in R.

(a) Let $\alpha : R \to R^2$ be the map $\alpha(r) = (yr, -xr)$ and let $\beta : R^2 \to R$ be the map $\beta((r_1, r_2)) = r_1x + r_2y$. Show that

$$0 \longrightarrow R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \longrightarrow k \longrightarrow 0$$

where the map $R \to R/I = k$ is the canonical projection, gives a free resolution of k as an R-module.

(b) Use the resolution in (a) to show that $\operatorname{Tor}_2^R(k, k) \cong k$.

(c) Prove that $\operatorname{Tor}_1^R(k, I) \cong k$. [Use the long exact sequence corresponding to the short exact sequence $0 \to I \to R \to k \to 0$ and (b).]

(d) Conclude from (c) that the torsion free R-module I is not flat (compare to Exercise 26 in Section 10.5).

22. (*Flat Base Change for* Tor) Suppose R and S are commutative rings and $f : R \to S$ is a ring homomorphism making S into an R-module as in Example 6 following Corollary 12 in Section 10.4. Prove that if S is flat as an R-module, then $\operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^S(S \otimes_R A, B)$ for all R-modules A and all S-modules B. [Show that since S is flat, tensoring an R-module projective resolution for A with S gives an S-module projective resolution of $S \otimes_R A$.]

23. (*Localization and* Tor) Let $D^{-1}R$ be the localization of the commutative ring R with respect to the multiplicative subset D of R. Prove that localization commutes with Tor, i.e., $D^{-1}\operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{D^{-1}R}(D^{-1}A, D^{-1}B)$ for all R-modules A and B and all $n \geq 0$. [Use the previous exercise and the fact that $D^{-1}R$ is flat over R, cf. Proposition 42(6) in Section 15.4.]

24. (*Flatness is local*) Suppose R is a commutative ring. Prove that an R-module M is flat if and only if every localization M_P is a flat R_P-module for every maximal (hence also for every prime) ideal in R. [Use the previous exercise together with the characterization of flatness in terms of Tor.]

25. If R is an integral domain with field of fractions F, prove that $\operatorname{Tor}_1^R(F/R, B) \cong t(B)$ for any R-module B, where $t(B)$ denotes the R-torsion submodule of B.

An R-module M is said to be *finitely presented* if there is an exact sequence

$$R^s \longrightarrow R^t \longrightarrow M \longrightarrow 0$$

of R-modules for some integers s and t. Equivalently, M is finitely generated by t elements and the kernel of the corresponding R-module homomorphism $R^t \to M$ can be generated by s elements.

26. (a) Prove that every finitely generated module over a Noetherian ring R is finitely presented. [Use Exercise 8 in Section 15.1.]

(b) Prove that an R-module M is finitely presented and projective if and only if M is a direct summand of R^n for some integer $n \geq 1$.

27. Suppose that M is a finitely presented R-module and that $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} M \to 0$ is an exact sequence of R-modules. This exercise proves that if B is a finitely generated R-module then A is also a finitely generated R-module.

(a) Suppose $R^s \xrightarrow{\psi} R^t \xrightarrow{\varphi} M \to 0$ and $e_1, \ldots, e_t$ is an R-module basis for R^t. Show that there exist $b_1, \ldots, b_t \in B$ so that $\beta(b_i) = \varphi(e_i)$ for $i = 1, \ldots, t$.

(b) If f is the R-module homomorphism from R^t to B defined by $f(e_i) = b_i$ for $i = 1, \ldots, t$, show that $f(\psi(R^s)) \subseteq \ker \beta$. [Use $\varphi \circ \psi = 0$.] Conclude that there is a commutative diagram

$$\begin{tikzcd}
& R^s \arrow[r, "\psi"] \arrow[d, "g"] & R^t \arrow[r, "\varphi"] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

of R-modules with exact rows.

(c) Prove that $A/\operatorname{image} g \cong B/\operatorname{image} f$ and use this to prove that A is finitely generated. [For the isomorphism, use the Snake Lemma in Exercise 3. Then show that image g and $A/$ image g are both finitely generated and apply Exercise 7 of Section 10.3.]

(d) If I is an ideal of R conclude that R/I is a finitely presented R-module if and only if I is a finitely generated ideal.

28. Suppose R is a local ring with unique maximal ideal $\mathfrak{m}$ and M is a finitely presented R-module. Suppose $m_1, \ldots, m_s$ are elements in M whose images in $M/\mathfrak{m}M$ form a basis for $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$.

(a) Prove that $m_1, \ldots, m_s$ generate M as an R-module. [Use Nakayama's Lemma.]

(b) Conclude from (a) that there is an exact sequence $0 \to \ker\varphi \to R^s \xrightarrow{\varphi} M \to 0$ that maps a set of free generators of R^s to the elements $m_1, \ldots, m_s$. Deduce that there is an exact sequence

$$\operatorname{Tor}_1^R(M, R/\mathfrak{m}) \longrightarrow (\ker\varphi)/\mathfrak{m}(\ker\varphi) \longrightarrow 0.$$

[Use the Tor long exact sequence with respect to tensoring with $R/\mathfrak{m}$, using the fact that $N \otimes R/\mathfrak{m} \cong N/\mathfrak{m}N$ for any R-module N (Example 8 following Corollary 12 in Section 10.4] and the fact that $\varphi : (R/\mathfrak{m})^s \cong M/\mathfrak{m}M$ is an isomorphism by the choice of $m_1, \dots, m_s$.]

(c) Prove that if $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$ then $m_1, \dots, m_s$ are a set of *free* R-module generators for M. [Use the previous exercise and Nakayama's Lemma to show that $\ker\varphi = 0$.]

29. Suppose R is a local ring with unique maximal ideal $\mathfrak{m}$. This exercise proves that a finitely generated R-module is flat if and only if it is free.

(a) Prove that $M = F/K$ is the quotient of a finitely generated free module F by a submodule K with $K \subseteq \mathfrak{m}F$. [Let F be a free module with $F/\mathfrak{m}F \cong M/\mathfrak{m}M$.]

(b) Suppose $x \in K$ and write $x = a_1e_1 + \cdots + a_ne_n$ where $e_1, \dots, e_n$ are an R-basis for F. Let $I = (a_1, \dots, a_n)$ be the ideal of R generated by $a_1, \dots, a_n$). Prove that if M is flat, then $I = \mathfrak{m}I$ and deduce that $K = 0$, so M is free. [Use Exercise 25(d) of Section 10.5 to see that $x \in IK \subseteq \mathfrak{m}IF$ and conclude that $I \subseteq \mathfrak{m}I$. Then apply Nakayama's Lemma to the finitely generated ideal I.]

30. Suppose R is a local ring with unique maximal ideal $\mathfrak{m}$, M is an R-module, and consider the following statements:

(i) M is a free R-module,
(ii) M is a projective R-module,
(iii) M is a flat R-module, and
(iv) $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$.

(a) Prove that (i) implies (ii) implies (iii) implies (iv).

(b) Prove that (i), (ii), and (iii) are equivalent if M is finitely generated. (Exercise 34 below shows (iii) need not imply (i) or (ii) if M is finitely generated but R is not local.) [Use the previous exercise.]

(c) Prove that (i), (ii), (iii), and (iv) are equivalent if M is finitely presented. (Exercise 35 below shows that (iv) need not imply (i), (ii) or (iii) if M is finitely generated but not finitely presented.) [Use Exercise 28.]

Remark: It is a theorem of Kaplansky (cf. *Projective Modules*, Annals of Mathematics, 68(1958), pp. 372-377) that (i) and (ii) are equivalent without the condition that M be finitely generated.

31. *(Localization and* Hom *for Finitely Presented Modules)* Suppose $D^{-1}R$ is the localization of the commutative ring R with respect to the multiplicative subset D of R, and let M be a finitely presented R-module.

(a) For any R-modules A and B prove there is a unique $D^{-1}R$-module homomorphism from $D^{-1}\operatorname{Hom}_R(A, B)$ to $\operatorname{Hom}_{D^{-1}R}(D^{-1}A, D^{-1}B)$ that maps $\varphi \in \operatorname{Hom}_R(A, B)$ to the homomorphism from $D^{-1}A$ to $D^{-1}B$ induced by φ.

(b) For any R-module N and any $m \geq 1$ show that $\operatorname{Hom}_R(R^m, N) \cong N^m$ as R-modules and deduce that $D^{-1}\operatorname{Hom}_R(R^m, N) \cong (D^{-1}N)^m$ as $D^{-1}R$-modules.

(c) Suppose $R^s \longrightarrow R^t \longrightarrow M \longrightarrow 0$ is exact. Prove there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D^{-1}\operatorname{Hom}_R(M, N) \arrow[r] \arrow[d] & D^{-1}\operatorname{Hom}_R(R^t, N) \arrow[r] \arrow[d] & D^{-1}\operatorname{Hom}_R(R^s, N) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{D^{-1}R}(D^{-1}M, D^{-1}N) \arrow[r] & \operatorname{Hom}_{D^{-1}R}((D^{-1}R)^t, D^{-1}N) \arrow[r] & \operatorname{Hom}_{D^{-1}R}((D^{-1}R)^s, D^{-1}N)
\end{tikzcd}$$

of $D^{-1}R$-modules with exact rows. [For the first row first take R-module homomorphisms from the terms in the presentation for M into N using Theorem 33 of Section 10.5 (noting the first comment in the proof) and then tensor with the flat R-module $D^{-1}R$, cf. Propositions 41 and 42(6) in Section 15.4. For the second row first tensor the presentation with $D^{-1}R$ and then take $D^{-1}R$-module homomorphisms into $D^{-1}N$.]

(d) Use (b) to prove that localization commutes with taking homomorphisms when M is finitely presented, i.e., $D^{-1}\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{D^{-1}R}(D^{-1}M, D^{-1}N)$ as $D^{-1}R$-modules. [Show the second two vertical maps in the diagram above are isomorphisms and deduce that the left vertical map is also an isomorphism.] (This result is not true in general if M is not finitely presented.)

32. *(Localization and* Ext *for Finitely Presented Modules)* Suppose $D^{-1}R$ is the localization of the commutative ring R with respect to the multiplicative subset D of R. Prove that if M is a finitely presented R-module then $D^{-1}\operatorname{Ext}_R^n(M, N) \cong \operatorname{Ext}_{D^{-1}R}^n(D^{-1}M, D^{-1}N)$ as $D^{-1}R$-modules for every R-module N and every $n \geq 0$. [Use a projective resolution of N and the previous exercise, noting that tensoring the resolution with $D^{-1}R$ gives a projective resolution for the $D^{-1}R$-module $D^{-1}N$.]

33. Suppose R is a commutative ring and M is a finitely presented R-module (for example a finitely generated module over a Noetherian ring, or a quotient, R/I, of R by a finitely generated ideal I, cf. Exercises 26 and 27). Prove that the following are equivalent:

(a) M is a projective R-module,
(b) M is a flat R-module,
(c) M is locally free, i.e., each localization M_P is a free R_P-module for every maximal (hence also for every prime) ideal P of R.

In particular show that finitely generated projective modules are the same as finitely presented flat modules. [Exercises 24 and 30 show that (b) is equivalent to (c). Use the Ext criterion for projectivity and Exercises 30 and 32 to see that (a) is equivalent to (c).]

34. (a) Prove that *every* R-module for the commutative ring R is flat if and only if every finitely generated ideal I of R is a direct summand of R, in which case every finitely generated ideal of R is principal and projective (such a ring is said to be *absolutely flat*). [Use Exercise 15, the previous exercise applied to the finitely presented R-module R/I, and the remarks following Proposition 16.]

(b) Prove that every Boolean ring is absolutely flat. [Use Exercise 24 in Section 7.4, noting that if $I = Rx$ then x is an idempotent so $R = Rx \oplus R(1 - x)$.]

(c) Let R be the direct product and I the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$. Prove that I is an ideal of the Boolean ring R that is not finitely generated and that the cyclic R-module $M = R/I$ is flat but not projective (so finitely generated flat modules need not be projective).

35. Let R be the local ring obtained by localizing the ring of C^∞ functions on the open interval $(-1, 1)$ at the maximal ideal of functions that are 0 at $x = 0$ (cf. Exercise 45 of Section 15.2), let $\mathfrak{m} = (x)$ be the unique maximal ideal of R and let P be the prime ideal $\cap_{n\geq 1}\mathfrak{m}^n$. Set $M = R/P$.

(a) Prove that $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$. [Use Exercise 19 applied with $r = x$, noting that R/P is an integral domain.]

(b) Prove that M is not flat (hence not projective). [Let F be as in Exercise 45 of Section 15.2. Show that the sequence $0 \to R \to R \to R/(F) \to 0$ induced by multiplication by F is exact, but is not exact after tensoring with M.]

17.2 THE COHOMOLOGY OF GROUPS

In this section we consider the application of the general techniques of the previous section in an important special case.

Let G be a group.

Definition. An abelian group A on which G acts (on the left) as automorphisms is called a *G-module*.

Note that a G-module is the same as an abelian group A and a homomorphism $\varphi : G \to \mathrm{Aut}(A)$ of G into the group of automorphisms of A. Since an abelian group is the same as a module over $\mathbb{Z}$, it is also easy to see that a G-module A is the same as a module over the integral group ring, $\mathbb{Z}G$, of G with coefficients in $\mathbb{Z}$. When G is an infinite group the ring $\mathbb{Z}G$ consists of all the finite formal sums of elements of G with coefficients in $\mathbb{Z}$.

As usual we shall often use multiplicative notation and write ga in place of $g \cdot a$ for the action of the element $g \in G$ on the element $a \in A$.

Definition. If A is a G-module, let $A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}$ be the elements of A fixed by all the elements of G.

Examples

(1) If $ga = a$ for all $a \in A$ and $g \in G$ then G is said to act *trivially* on A. In this case $A^G = A$. The abelian group $\mathbb{Z}$ will always be assumed to have trivial G-action for any group G unless otherwise stated.

(2) For any G-module A the fixed points A^G of A under the action of G is clearly a $\mathbb{Z}G$-submodule of A on which G acts trivially.

(3) If V is a vector space over the field F of dimension n and $G = GL_n(F)$ then V is naturally a G-module. In this case $V^G = \{0\}$ since any nonzero element in V can be taken to any other nonzero element in V by some linear transformation.

(4) A semidirect product $E = A \rtimes G$ as in Section 5.5 in the case where A is an abelian normal subgroup gives a G-module A where the action of G is given by the homomorphism $\varphi : G \to \mathrm{Aut}(A)$. The subgroup A^G consists of the elements of A lying in the center of E. More generally, if A is any abelian normal subgroup of a group E, then E acts on A by conjugation and this makes A into a E-module and also an E/A-module. In this case $A^E = A^{E/A}$ also consists of the elements of A lying in the center of E.

(5) If K/F is an extension of fields that is Galois with Galois group G then the additive group K is naturally a G-module, with $K^G = F$. Similarly, the multiplicative group $K^\times$ of nonzero elements in K is a G-module, with fixed points $(K^\times)^G = F^\times$.

The fixed point subgroups in this last example played a central role in Galois Theory in Chapter 14. In general, it is easy to see that a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules induces an exact sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \tag{17.15}$$

that in general cannot be extended to a short exact sequence (in general a coset in the quotient C that is fixed by G need not be represented by an *element* in B fixed by G). One way to see that (15) is exact is to observe that A^G can be related to a Hom group:

Lemma 19. Suppose A is a G-module and $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ is the group of all $\mathbb{Z}G$-module homomorphisms from $\mathbb{Z}$ (with trivial G-action) to A. Then $A^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$.

Proof: Any G-module homomorphism α from $\mathbb{Z}$ to A is uniquely determined by its value on 1. Let α_a denote the G-module homomorphism with $\alpha(1) = a$. Since α_a is a G-module homomorphism, $a = \alpha_a(1) = \alpha_a(g \cdot 1) = g \cdot \alpha_a(1) = g \cdot a$ for all $g \in G$, so that a must lie in A^G. Likewise, for any $a \in A^G$ it is easy to check that the map $\alpha_a \mapsto a$ gives an isomorphism from $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ to A^G.

Combined with the results of the previous section, the lemma not only shows that the sequence (15) is exact, it shows that any projective resolution of $\mathbb{Z}$ considered as a $\mathbb{Z}G$-module will give a long exact sequence extending (15). One such projective resolution is the *standard resolution* or *bar resolution* of $\mathbb{Z}$:

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \to \cdots \xrightarrow{d_1} F_0 \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0. \tag{17.16}$$

Here $F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$ (where there are $n+1$ factors) for $n \geq 0$, which is a G-module under the action defined on simple tensors by $g \cdot (g_0 \otimes g_1 \otimes \cdots \otimes g_n) = (gg_0) \otimes g_1 \otimes \cdots \otimes g_n$. It is not difficult to see that F_n is a free $\mathbb{Z}G$-module of rank $|G|^n$ with $\mathbb{Z}G$ basis given by the elements $1 \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$, where $g_i \in G$. The map $\text{aug} : F_0 \to \mathbb{Z}$ is the *augmentation map* $\text{aug}(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g$, and the map d_1 is given by $d_1(1 \otimes g) = g - 1$. The maps d_n for $n \geq 2$ are more complicated and their definition, together with a proof that (16) is a projective (in fact, free) resolution can be found in Exercises 1–3.

Applying ($\mathbb{Z}G$-module) homomorphisms from the terms in (16) to the G-module A (replacing the first term by 0) as in the previous section, we obtain the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(F_0, A) \xrightarrow{d_1} \mathrm{Hom}_{\mathbb{Z}G}(F_1, A) \xrightarrow{d_2} \mathrm{Hom}_{\mathbb{Z}G}(F_2, A) \xrightarrow{d_3} \cdots, \tag{17.17}$$

the cohomology groups of which are, by definition, the groups $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. Then, as in Theorem 8, the short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of G-modules gives rise to a long exact sequence whose first terms are given by (15) and whose higher terms are the cohomology groups $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$.

To make this more explicit, we can reinterpret the terms in this cochain complex without explicit reference to the standard resolution of $\mathbb{Z}$, as follows. The elements of $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ are uniquely determined by their values on the $\mathbb{Z}G$ basis elements of F_n, which may be identified with the n-tuples $(g_1, g_2, \ldots, g_n)$ of elements g_i of G. It follows for $n \geq 1$ that the group $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ may be identified with the set of functions from $G \times \cdots \times G$ (n copies) to A. For $n = 0$ we identify $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)$ with A.

Definition. If G is a finite group and A is a G-module, define $C^0(G, A) = A$ and for $n \geq 1$ define $C^n(G, A)$ to be the collection of all maps from $G^n = G \times \cdots \times G$ (n copies) to A. The elements of $C^n(G, A)$ are called *n-cochains* (*of G with values in A*).

Each $C^n(G, A)$ is an additive abelian group: for $C^0(G, A) = A$ given by the group structure on A; for $n \geq 1$ given by the usual pointwise addition of functions: $(f_1 + f_2)(g_1, g_2, \ldots, g_n) = f_1(g_1, g_2, \ldots, g_n) + f_2(g_1, g_2, \ldots, g_n)$. Under the identification of $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ with $C^n(G, A)$ the cochain maps d_n in (17) can be given very explicitly (cf. also Exercise 3 and the following comment):

Definition. For $n \geq 0$, define the n^{th} *coboundary* homomorphism from $C^n(G, A)$ to $C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n) \qquad (17.18) \end{aligned}$$

where the product $g_i g_{i+1}$ occupying the i^{th} position of f is taken in the group G.

It is immediate from the definition that the maps d_n are group homomorphisms. It follows from the fact that (17) is a projective resolution that $d_n \circ d_{n-1} = 0$ for $n \geq 1$ (a self contained direct proof just from the definition of d_n above can also be given, but is tedious).

Definition.

(1) Let $Z^n(G, A) = \ker d_n$ for $n \geq 0$. The elements of $Z^n(G, A)$ are called *n-cocycles*.

(2) Let $B^n(G, A) = \text{image}\, d_{n-1}$ for $n \geq 1$ and let $B^0(G, A) = 1$. The elements of $B^n(G, A)$ are called *n-coboundaries*.

Since $d_n \circ d_{n-1} = 0$ for $n \geq 1$ we have image $d_{n-1} \subseteq \ker d_n$, so that $B^n(G, A)$ is always a subgroup of $Z^n(G, A)$.

Definition. For any G-module A the quotient group $Z^n(G, A)/B^n(G, A)$ is called the n^{th} *cohomology group of G with coefficients in A* and is denoted by $H^n(G, A)$, $n \geq 0$.

The definition of the cohomology group $H^n(G, A)$ in terms of cochains will be particularly useful in the following two sections when we examine the low dimensional groups $H^1(G, A)$ and $H^2(G, A)$ and their application in a variety of settings. It should be remembered, however, that $H^n(G, A) \cong \mathrm{Ext}^n(\mathbb{Z}, A)$ for all $n \geq 0$. In particular, these groups can be computed using *any* projective resolution of $\mathbb{Z}$.

Examples

(1) For $f = a \in C^0(G, A)$ we have $d_0(f)(g) = g \cdot a - a$ and so $\ker d_0$ is the set $\{a \in A \mid g \cdot a = a \text{ for all } g \in G\}$, i.e., $Z^0(G, A) = A^G$ and so

$$H^0(G, A) = A^G,$$

for any group G and G-module A.

(2) Suppose $G = 1$ is the trivial group. Then $G^n = \{(1, 1, \ldots, 1)\}$ is also the trivial group, so $f \in C^n(G, A)$ is completely determined by $f(1, 1, \ldots, 1) = a \in A$. Identifying $f = a$ we obtain $C^n(G, A) = A$ for all $n \geq 0$. Then, if $f = a \in A$,

$$d_n(f)(1, 1, \ldots, 1) = a + \sum_{i=1}^{n} (-1)^i a + (-1)^{n+1} a = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases},$$

so $d_n = 0$ if n is even and $d_n = 1$ is the identity if n is odd. Hence

$$H^0(1, A) = A^G = A$$
$$H^n(1, A) = 0 \text{ for all } n \geq 1.$$

Example: (Cohomology of a Finite Cyclic Group)

Suppose G is cyclic of order m with generator σ. Let $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1} \in \mathbb{Z}G$. Then $N(\sigma - 1) = (\sigma - 1)N = \sigma^m - 1 = 0$, and so we have a particularly simple free resolution

$$\cdots \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0$$

where aug denotes the augmentation map (cf. Exercise 8). Taking $\mathbb{Z}G$-module homomorphisms from the terms of this resolution to A (replacing the first term by 0) and using the identification $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) = A$ gives the chain complex

$$0 \longrightarrow A \xrightarrow{\sigma-1} A \xrightarrow{N} A \xrightarrow{\sigma-1} A \xrightarrow{N} \cdots$$

whose cohomology computes the groups $H^n(G, A)$:

$$H^0(G, A) = A^G, \text{ and } H^n(G, A) = \begin{cases} A^G/NA & \text{if } n \text{ is even}, n \geq 2 \\ {}_NA/(\sigma - 1)A & \text{if } n \text{ is odd}, n \geq 1 \end{cases}$$

where ${}_NA = \{a \in A \mid Na = 0\}$ is the subgroup of A annihilated by N, since the kernel of multiplication by $\sigma - 1$ is A^G.

If in particular $G = \langle \sigma \rangle$ acts trivially on A, then $N \cdot a = ma$, so that in this case $H^0(G, A) = A$, with $H^n(G, A) = A/mA$ for even $n \geq 2$, and $H^n(G, A) = {}_mA$, the elements of A of order dividing m, for odd $n \geq 1$. Specializing even further to $m = 1$ gives Example 2 previously.

Proposition 20. Suppose $mA = 0$ for some integer $m \geq 1$ (i.e., the G-module A has exponent dividing m as an abelian group). Then

$$mZ^n(G, A) = mB^n(G, A) = mH^n(G, A) = 0 \quad \text{for all } n \geq 0.$$

In particular, if A has exponent p for some prime p then the abelian groups $Z^n(G, A)$, $B^n(G, A)$ and $H^n(G, A)$ have exponent dividing p and so these groups are all vector spaces over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Proof: If $f \in C^n(G, A)$ is an n-cochain then $f \in A$ (if $n = 0$), in which case $mf = 0$, or f is a function from G^n to A (if $n \geq 1$), in which case mf is a function from G^n to $mA = 0$, so again $mf = 0$. Hence $mZ^n(G, A) = mB^n(G, A) = 0$ since these are subgroups of $C^n(G, A)$. Then $mH^n(G, A) = 0$ since $mZ^n(G, A) = 0$, and the remaining statements in the proposition are immediate.

By Example 1, the long exact sequence in Theorem 10 written in terms of the cohomology groups $H^n(G, A)$ becomes

Theorem 21. *(Long Exact Sequence in Group Cohomology)* Suppose

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of G-modules. Then there is a long exact sequence:

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta_0} H^1(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C) \xrightarrow{\delta_1} \cdots$$
$$\cdots \xrightarrow{\delta_{n-1}} H^n(G,A) \longrightarrow H^n(G,B) \longrightarrow H^n(G,C) \xrightarrow{\delta_n} H^{n+1}(G,A) \longrightarrow \cdots$$

of abelian groups.

Among many other uses of the long exact sequence in Theorem 21 is a technique called *dimension shifting* which makes it possible to analyze the cohomology group $H^{n+1}(G,A)$ of dimension $n+1$ for A by instead considering a cohomology group of dimension n for a different G-module. The technique is based on finding a G-module almost all of whose cohomology groups are zero. Such modules are given a name:

Definition. A G-module M is called *cohomologically trivial for* G if $H^n(G,M) = 0$ for all $n \geq 1$.

Corollary 22. *(Dimension Shifting)* Suppose $0 \to A \to M \to C \to 0$ is a short exact sequence of G-modules and that M is cohomologically trivial for G. Then there is an exact sequence

$$0 \longrightarrow A^G \longrightarrow M^G \longrightarrow C^G \longrightarrow H^1(G,A) \longrightarrow 0$$

and

$$H^{n+1}(G,A) \cong H^n(G,C) \text{ for all } n \geq 1.$$

Proof: Since M is cohomologically trivial for G, the portion

$$H^n(G,M) \longrightarrow H^n(G,C) \longrightarrow H^{n+1}(G,A) \longrightarrow H^{n+1}(G,M)$$

of the long exact sequence in Theorem 21 reduces to

$$0 \longrightarrow H^n(G,C) \longrightarrow H^{n+1}(G,A) \longrightarrow 0$$

which shows that $H^n(G,C) \cong H^{n+1}(G,A)$ for $n \geq 1$. Similarly, the first portion of the long exact sequence in Theorem 21 gives the first statement in the corollary.

We now indicate a natural construction that produces a G-module given a module over a subgroup H of G. When $H = 1$ is the trivial group this construction produces a cohomologically trivial module M and an exact sequence as in Corollary 22 for any G-module A.

Definition. If H is a subgroup of G and A is an H-module, define the *induced G-module* $M_H^G(A)$ to be $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$. In other words, $M_H^G(A)$ is the set of maps f from G to A satisfying $f(hx) = hf(x)$ for every $x \in G$ and $h \in H$.

The action of an element $g \in G$ on $f \in M_H^G(A)$ is given by $(g\cdot f)(x) = f(xg)$ for $x \in G$ (cf. Exercise 10 in Section 10.5).

Recall that if H is a subgroup of G and A is an H-module, then the module $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ obtained by extension of scalars from $\mathbb{Z}H$ to $\mathbb{Z}G$ is a G-module. For a finite group G, or more generally if H has finite index in G, we have $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ (cf. Exercise 10). When G is infinite this need no longer be the case (cf. Exercise 11). The module $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ is sometimes called the *induced G-module* and the module $M_H^G(A)$ is sometimes referred to as the *coinduced G-module*. For finite groups, associativity of the tensor product shows that $M_H^G(M_K^H(A)) = M_K^G(A)$ for subgroups $K \leq H \leq G$, and the same result holds in general (this follows from the definition using Exercise 7).

Examples

(1) If H is a subgroup of G and $0 \to A \to B \to C \to 0$ is a short exact sequence of H-modules then $0 \to M_H^G(A) \to M_H^G(B) \to M_H^G(C) \to 0$ is a short exact sequence of G-modules, since $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ and $\mathbb{Z}G$ is free, hence flat, over $\mathbb{Z}H$.

(2) When G is finite and A is the trivial H-module $\mathbb{Z}$, the module $M_H^G(\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $m = |G : H|$. There is a basis $b_1, \dots, b_m$ such that G permutes these basis elements in the same way it permutes the left cosets of H in G by left multiplication, i.e., if we let $b_i \leftrightarrow g_i H$ then $gb_i = b_j$ if and only if $gg_i H = g_j H$. The module $M_H^G(\mathbb{Z})$ is the *permutation module* over $\mathbb{Z}$ for G with stabilizer H. A special case of interest is when $G = S_m$ and $H = S_{m-1}$ where S_m permutes $\{1, 2, \dots, m\}$ as usual. Permutation modules and induced modules over fields are studied in Part VI.

(3) Any abelian group A is an H-module when $H = 1$ is the trivial group. The corresponding induced G-module $M_1^G(A)$ is just the collection of all maps f from G into A. For $g \in G$ the map $g\cdot f \in M_1^G(A)$ satisfies $(g\cdot f)(x) = f(xg)$ for $x \in G$.

(4) Suppose A is a G-module. Then there is a natural map

$$\varphi : A \longrightarrow M_1^G(A)$$

from A into the induced G-module $M_1^G(A)$ in the previous example defined by mapping $a \in A$ to the function f_a with $f_a(x) = xa$ for all $x \in G$. It is clear that φ is a group homomorphism, and $f_{ga}(x) = x(ga) = (xg)a = f_a(xg) = (g\cdot f_a)(x)$ shows that φ is a G-module homomorphism as well. Since $f_a(1) = a$, it follows that f_a is the zero function on G if and only if $a = 0$ in A, so that φ is an injection. Hence we may identify A as a G-submodule of the induced module $M_1^G(A)$.

(5) More generally, if A is a G-module and H is any subgroup of G then the function $f_a(x)$ in the previous example is an element in the subgroup $M_H^G(A)$ since we have $f_a(hx) = (hx)(a) = h(xa) = hf_a(x)$ for all $h \in H$. The associated map from A to $M_H^G(A)$ is an injective G-module homomorphism.

(6) The fixed points $(M_H^G(A))^G$ are maps f from G to A with $gf = f$ for all $g \in G$, i.e., with $(gf)(x) = f(x)$ for all $g, x \in G$. By definition of the G-action on $M_H^G(A)$, this is the equation $f(xg) = f(x)$ for all $g, x \in G$. Taking $x = 1$ shows that f is constant on all of G: $f(g) = f(1) = a \in A$. The constant function $f = a$ is an element of $M_H^G(A)$ if and only if $a = f(hx) = hf(x) = ha$ for all $h \in H$, so $(M_H^G(A))^G \cong A^H$.

An element $f_a(x)$ in the previous example is contained in the subgroup $(M_H^G(A))^G$ if and only if xa is constant for $x \in G$, i.e., if and only if $a \in A^G$.

One of the important properties of the G-module $M_H^G(A)$ induced from the H-module A is that its cohomology with respect to G is the same as the cohomology of A with respect to H:

Proposition 23. *(Shapiro's Lemma)* For any subgroup H of G and any H-module A we have $H^n(G, M_H^G(A)) \cong H^n(H, A)$ for $n \geq 0$.

Proof: Let $\cdots \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0$ be a resolution of $\mathbb{Z}$ by projective G-modules (for example, the standard resolution). The cohomology groups $H^n(G, M_H^G(A))$ are computed by taking homomorphisms from this resolution into $M_H^G(A) = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$. Since $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module it follows that this G-module resolution is also a resolution of $\mathbb{Z}$ by projective H-modules, hence by taking homomorphisms into A the same resolution may be used to compute the cohomology groups $H^n(H, A)$. To see that these two collections of cohomology groups are isomorphic, we use the natural isomorphism of abelian groups

$$\Phi : \mathrm{Hom}_{\mathbb{Z}G}(P_n, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)) \cong \mathrm{Hom}_{\mathbb{Z}H}(P_n, A)$$

given by $\Phi(f)(p) = f(p)(1)$, for all $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A))$ and $p \in P_n$. The inverse isomorphism is defined by taking $\Psi(f')(p)$ to be the map from $\mathbb{Z}G$ to A that takes $g \in G$ to the element $f'(gp)$ in A for all $f' \in \mathrm{Hom}_{\mathbb{Z}H}(P_n, A)$ and $p \in P_n$, i.e., $(\Psi(f')(p))(g) = f'(gp)$. Note this is well defined because P_n is a G-module. (These maps are a special case of an Adjoint Associativity Theorem, cf. Exercise 7.) Since these isomorphisms commute with the cochain maps, they induce isomorphisms on the corresponding cohomology groups, i.e., $H^n(G, M_H^G(A)) \cong H^n(H, A)$, as required.

Corollary 24. For any G-module A the module $M_1^G(A)$ is cohomologically trivial for G, i.e., $H^n(G, M_1^G(A)) = 0$ for all $n \geq 1$.

Proof: This follows immediately from the proposition applied with $H = 1$ together with the computation of the cohomology of the trivial group in Example 2 preceding Proposition 20.

By the corollary, the fourth example above gives us a short exact sequence of G-modules

$$0 \longrightarrow A \xrightarrow{\varphi} M \longrightarrow C \longrightarrow 0$$

where $M = M_1^G(A)$ is cohomologically trivial for G and where C is the quotient of $M_1^G(A)$ by the image of A. The dimension shifting result in Corollary 22 then becomes:

Corollary 25. For any G-module A we have $H^{n+1}(G, A) \cong H^n(G, M_1^G(A)/A)$ for all $n \geq 1$.

We next consider several important maps relating various cohomology groups. Some applications of the use of these homomorphisms appear in the following two sections.

In general, suppose we have two groups G and G' and that A is a G-module and A' is a G'-module. If $\varphi : G' \to G$ is a group homomorphism then A becomes a G'-module by defining $g' \cdot a = \varphi(g')a$ for $g' \in G'$ and $a \in A$. If now $\psi : A \to A'$ is a homomorphism of abelian groups then we consider whether ψ is a G'-module homomorphism:

Definition. Suppose A is a G-module and A' is a G'-module. The group homomorphisms $\varphi : G' \to G$ and $\psi : A \to A'$ are said to be *compatible* if ψ is a G'-module homomorphism when A is made into a G'-module by means of φ, i.e., if $\psi(\varphi(g')a) = g'\psi(a)$ for all $g' \in G'$ and $a \in A$.

The point of compatible homomorphisms is that they induce group homomorphisms on associated cohomology groups, as follows.

If $\varphi : G' \to G$ and $\psi : A \to A'$ are homomorphisms, then φ induces a homomorphism $\varphi^n : (G')^n \to G^n$, and so a homomorphism from $C^n(G, A)$ to $C^n(G', A)$ that maps f to $f \circ \varphi^n$. The map ψ induces a homomorphism from $C^n(G', A)$ to $C^n(G', A')$ that maps f to $\psi \circ f$. Taken together we obtain an induced homomorphism

$$\lambda_n : C^n(G, A) \longrightarrow C^n(G', A')$$
$$f \longmapsto \psi \circ f \circ \varphi^n.$$

If in addition φ and ψ are *compatible* homomorphisms, then it is easy to check that the induced maps λ_n commute with the coboundary operator:

$$\lambda_{n+1} \circ d_n = d_n \circ \lambda_n$$

for all $n \geq 0$. It follows that λ_n maps cocycles to cocycles and coboundaries to coboundaries, hence induces a group homomorphism on cohomology:

$$\lambda_n : H^n(G, A) \longrightarrow H^n(G', A')$$

for $n \geq 0$.

We consider several instances of such maps:

Examples

(1) Suppose $G = G'$ and φ is the identity map. Then to say that the group homomorphism $\psi : A \to A'$ is compatible with φ is simply the statement that ψ is a G-module homomorphism. Hence any G-module homomorphism from A to A' induces a group homomorphism

$$H^n(G, A) \longrightarrow H^n(G, A') \quad \text{for } n \geq 0.$$

In particular, if $0 \to A \to B \to C \to 0$ is a short exact sequence of G-modules we obtain induced homomorphisms from $H^n(G, A)$ to $H^n(G, B)$ and from $H^n(G, B)$ to $H^n(G, C)$ for $n \geq 0$. These are simply the homomorphisms in the long exact sequence of Theorem 21.

(2) *(The Restriction Homomorphism)* If A is a G-module, then A is also an H-module for any subgroup H of G. The inclusion map $\varphi : H \to G$ of H into G and the identity

map $\psi : A \to A$ are compatible homomorphisms. The corresponding induced group homomorphism on cohomology is called the *restriction homomorphism*:

$$\text{Res} : H^n(G, A) \longrightarrow H^n(H, A), \quad n \geq 0.$$

The terminology comes from the fact that the map on cochains from $C^n(G, A)$ to $C^n(H, A)$ is simply restricting a map f from G^n to A to the subgroup H^n of G^n.

(3) *(The Inflation Homomorphism)* Suppose H is a normal subgroup of G and A is a G-module. The elements A^H of A that are fixed by H are naturally a module for the quotient group G/H under the action defined by $(gH) \cdot a = g \cdot a$. It is then immediate that the projection $\varphi : G \to G/H$ and the inclusion $\psi : A^H \to A$ are compatible homomorphisms. The corresponding induced group homomorphism on cohomology is called the *inflation homomorphism*:

$$\text{Inf} : H^n(G/H, A^H) \longrightarrow H^n(G, A), \quad n \geq 0.$$

(4) *(The Corestriction Homomorphism)* Suppose that H is a subgroup of G of index m and that A is a G-module. Let $g_1, \dots, g_m$ be representatives for the left cosets of H in G. Define a map

$$\psi : M_H^G(A) \longrightarrow A \qquad \text{by} \qquad f \longmapsto \sum_{i=1}^{m} g_i \cdot f(g_i^{-1}).$$

Note that if we change any coset representative g_i by $g_i h$, then $(g_i h) f((g_i h)^{-1}) = g_i h f(h^{-1} g_i^{-1}) = g_i h h^{-1} f(g_i^{-1}) = g_i f(g_i^{-1})$ so the map ψ is independent of the choice of coset representatives. It is easy to see that ψ is a G-module homomorphism (and even that it is surjective), so we obtain a group homomorphism from $H^n(G, M_H^G(A))$ to $H^n(G, A)$, for all $n \geq 0$. Since A is also an H-module, by Shapiro's Lemma we have an isomorphism $H^n(G, M_H^G(A)) \cong H^n(H, A)$. The composition of these two homomorphisms is called the *corestriction homomorphism*:

$$\text{Cor} : H^n(H, A) \longrightarrow H^n(G, A), \quad n \geq 0.$$

This homomorphism can be computed explicitly by composing the isomorphism Ψ in the proof of Shapiro's Lemma for any resolution of $\mathbb{Z}$ by projective G-modules P_n (note these are G-modules and not simply H-modules) with the map ψ, as follows. For a cocycle $f \in \text{Hom}_{\mathbb{Z}H}(P_n, A)$ representing a cohomology class $c \in H^n(H, A)$, a cocycle $\text{Cor}(f) \in \text{Hom}_{\mathbb{Z}G}(P_n, A)$ representing $\text{Cor}(c) \in H^n(G, A)$ is given by

$$\text{Cor}(f)(p) = \sum_{i=1}^{m} g_i \cdot \Psi(f)(p)(g_i^{-1}) = \sum_{i=1}^{m} g_i f(g_i^{-1} p).$$

for $p \in P_n$. When $n = 0$ this is particularly simple since we can take $P_0 = \mathbb{Z}G$. In this case $f \in \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A) = M_H^G(A)$ is a cocycle if $f = a$ is constant for some $a \in A^H$ and then $\text{Cor}(f)$ is the constant function with value $\sum_{i=1}^{m} g_i \cdot a \in A^G$:

$$\text{Cor} : H^0(H, A) = A^H \longrightarrow A^G = H^0(G, A)$$
$$a \longmapsto \sum_{i=1}^{m} g_i \cdot a.$$

The next result establishes a fundamental relation between the restriction and corestriction homomorphisms.

Proposition 26. Suppose H is a subgroup of G of index m. Then $\text{Cor} \circ \text{Res} = m$, i.e., if c is a cohomology class in $H^n(G, A)$ for some G-module A, then

$$\text{Cor}(\text{Res}(c)) = mc \in H^n(G, A) \quad \text{for all } n \geq 0.$$

Proof: This follows from the explicit formula for corestriction in Example 4 above, as follows. If $f \in \text{Hom}_{\mathbb{Z}H}(P_n, A)$ were in $\text{Hom}_{\mathbb{Z}G}(P_n, A)$, i.e., if f were also a G-module homomorphism, then $g_i f(g_i^{-1} p) = g_i g_i^{-1} f(p) = f(p)$, for $1 \leq i \leq m$. Since restriction is the induced map on cohomology of the natural inclusion of $\text{Hom}_{\mathbb{Z}G}(P_n, A)$ into $\text{Hom}_{\mathbb{Z}H}(P_n, A)$, for such an f we obtain

$$\text{Hom}_{\mathbb{Z}G}(P_n, A) \xrightarrow{\text{Res}} \text{Hom}_{\mathbb{Z}H}(P_n, A) \xrightarrow{\text{Cor}} \text{Hom}_{\mathbb{Z}G}(P_n, A)$$
$$f \longmapsto f \longmapsto mf.$$

It follows that $\text{Res} \circ \text{Cor}$ is multiplication by m on the cohomology groups as well.

Corollary 27. Suppose the finite group G has order m. Then $mH^n(G, A) = 0$ for all $n \geq 1$ and any G-module A.

Proof: Let $H = 1$, so that $[G : H] = m$, in Proposition 26. Then for any class $c \in H^n(G, A)$ we have $mc = \text{Cor}(\text{Res}(c))$. Since $\text{Res}(c) \in H^n(H, A) = H^n(1, A)$, we have $\text{Res}(c) = 0$ for all $n \geq 1$ by the second example preceding Proposition 20. Hence $mc = 0$ for all $n \geq 1$, which is the corollary.

Corollary 28. If G is a finite group then $H^n(G, A)$ is a torsion abelian group for all $n \geq 1$ and all G-modules A.

Proof: This is immediate from the previous corollary.

Corollary 29. Suppose G is a finite group whose order is relatively prime to the exponent of the G-module A. Then $H^n(G, A) = 0$ for all $n \geq 1$. In particular, if A is a finite abelian group with $(|G|, |A|) = 1$ then $H^n(G, A) = 0$ for all $n \geq 1$.

Proof: This follows since the abelian group $H^n(G, A)$ is annihilated by $|G|$ by the previous corollary and is annihilated by the exponent of A by Proposition 20.

Note that the statements in the preceding corollaries are not in general true for $n = 0$, since then $H^0(G, A) = A^G$, which need not even be torsion.

We mention without proof the following result. Suppose that H is a normal subgroup of G and A is a G-module. The cohomology groups $H^n(H, A)$ can be given the structure of G/H-modules (cf. Exercise 17). It can be shown that there is an exact sequence

$$0 \to H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(H, A)^{G/H} \xrightarrow{\text{Tra}} H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A)$$

where $H^1(H, A)^{G/H}$ denotes the fixed points of $H^1(H, A)$ under the action of G/H and Tra is the so-called *transgression homomorphism*. This exact sequence relates the cohomology groups for G to the cohomology groups for the normal subgroup H and for the quotient group G/H. Put another way, the cohomology for G is related to the cohomology for the factors in the filtration $1 \leq H \leq G$ for G. More generally, one could try to relate the cohomology for G to the cohomology for the factors in a longer filtration for G. This is the theory of *spectral sequences* and is an important tool in homological algebra.

Galois Cohomology and Profinite Groups

One important application of group cohomology occurs when the group G is the Galois group of a field extension K/F. In this case there are many groups of interest on which G acts, for example the additive group of K, the multiplicative group $K^\times$, etc. The Galois group $G = \mathrm{Gal}(K/F)$ is the inverse limit $\varprojlim \mathrm{Gal}(L/F)$ of the Galois groups of the finite extensions L of F contained in K and is a compact topological group with respect to its Krull topology (i.e., the group operations on G are continuous with respect to the topology defined by the subgroups $\mathrm{Gal}(K/L)$ of G of finite index), cf. Section 14.9. In this situation it is useful (and often essential) to take advantage of the additional topological structure of G. For example the subfields of K containing F correspond bijectively with the *closed* subgroups of $G = \mathrm{Gal}(K/F)$, and the example of the composite of the quadratic extensions of $\mathbb{Q}$ discussed in Section 14.9 shows that in general there are many subgroups of G that are not closed. Fortunately, the modifications necessary to define the cohomology groups in this context are relatively minor and apply to arbitrary inverse limits of finite groups (the *profinite* groups). If G is a profinite group then $G = \varprojlim G/N$ where the inverse limit is taken over the open normal subgroups N of G (cf. Exercise 23).

Definition. If G is a profinite group then a *discrete G-module* A is a G-module A with the discrete topology such that the action of G on A is continuous, i.e., the map $G \times A \to A$ mapping (g, a) to $g \cdot a$ is continuous.

Since A is given the discrete topology, every subset of A is open, and in particular every element $a \in A$ is open. The continuity of the action of G on A is then equivalent to the statement that the stabilizer G_a of a in G is an open subgroup of G, hence is of finite index since G is compact (cf. Exercise 22). This in turn is equivalent to the statement that $A = \cup A^H$ where the union is over the open subgroups H of G.

Some care must be taken in defining the cohomology groups $H^n(G, A)$ of a profinite group G acting on a discrete G-module A since there are not enough projectives in this category. For example, when G is infinite, the free G-module $\mathbb{Z}G$ is not a discrete G-module (G does not act continuously, cf. Exercise 25). Nevertheless, the explicit description of $H^n(G, A)$ given in this section (occasionally referred to as the *discrete* cohomology groups) can be easily modified — it is only necessary to require the cochains $C^n(G, A)$ to be *continuous* maps from G^n to A. The definition of the coboundary maps d_n in equation (18) is precisely the same, as is the definition of the groups of cocycles, coboundaries, and the corresponding cohomology groups. It is customary not to introduce a separate notation for these cohomology groups, but to specify which cohomology is meant in the terminology.

Definition. If G is a profinite group and A is a discrete G-module, the cohomology groups $H^n(G, A)$ computed using continuous cochains are called the *profinite* or *continuous* cohomology groups. When $G = \mathrm{Gal}(K/F)$ is the Galois group of a field extension K/F then the *Galois cohomology groups* $H^n(G, A)$ will always mean the cohomology groups computed using continuous cochains.

When G is a finite group, every G-module is a discrete G-module so the discrete and continuous cohomology groups of G are the same. When G is infinite, this need not be the case as shown by the example mentioned previously of the free G-module $\mathbb{Z}G$ when G is an infinite profinite group. All the major results in this section remain valid for the continuous cohomology groups when "G-module" is replaced by "discrete G-module" and "subgroup" is replaced by "closed subgroup." For example, the Long Exact Sequence in Group Cohomology remains true as stated, the restriction homomorphism requires the subgroup H of G to be a closed subgroup (so that the restriction of a continuous map on G^n to H^n remains continuous), Proposition 26 requires H to be closed, etc.

We can write $G = \varprojlim (G/N)$ and $A = \cup A^N$ where N runs over the open normal subgroups of G (necessarily of finite index in G since G is compact). Then A^N is a discrete G/N-module and it is not difficult to show that

$$H^n(G, A) = \varinjlim_{N} H^n(G/N, A^N) \tag{17.19}$$

where the cohomology groups are continuous cohomology and the direct limit is taken over the collection of all open normal subgroups N of G (cf. Exercise 24). Since G/N is a finite group, the continuous cohomology groups $H^n(G/N, A^N)$ in this direct limit are just the (discrete) cohomology groups considered earlier in this section. The computation of the continuous cohomology for a profinite group G can therefore always be reduced to the consideration of finite group cohomology where there is no distinction between the continuous and discrete theories.

EXERCISES

1. Let $F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$ ($n+1$ factors) for $n \geq 0$ with G-action defined on simple tensors by $g \cdot (g_0 \otimes g_1 \otimes \cdots \otimes g_n) = (gg_0) \otimes g_1 \otimes \cdots \otimes g_n$.
   (a) Prove that F_n is a free $\mathbb{Z}G$-module of rank $|G|^n$ with $\mathbb{Z}G$ basis $1 \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$ with $g_i \in G$.

   Denote the basis element $1 \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$ in (a) by $(g_1, g_2, \ldots, g_n)$ and define the G-module homomorphisms d_n for $n \geq 1$ on these basis elements by $d_1(g_1) = g_1 - 1$ and

   $$\begin{aligned} d_n(g_1, \ldots, g_n) = g_1 \cdot (g_2, \ldots, g_n) &+ \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ &+ (-1)^n (g_1, \ldots, g_{n-1}), \end{aligned}$$

   for $n \geq 2$. Define the $\mathbb{Z}$-module *contracting homomorphisms*

   $$\mathbb{Z} \xrightarrow{s_{-1}} F_0 \xrightarrow{s_0} F_1 \xrightarrow{s_1} F_2 \xrightarrow{s_2} \cdots$$

   on a $\mathbb{Z}$ basis by $s_{-1}(1) = 1$ and $s_n(g_0 \otimes \cdots \otimes g_n) = 1 \otimes g_0 \otimes \ldots \otimes g_n$.

(b) Prove that

$$\epsilon s_{-1} = 1, \qquad d_1 s_0 + s_{-1}\epsilon = 1, \qquad d_{n+1}s_n + s_{n-1}d_n = 1, \text{ for all } n \geq 1$$

where the map aug : $F_0 \to \mathbb{Z}$ is the augmentation map $\text{aug}(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g$.

(c) Prove that the maps s_n are a chain homotopy (cf. Exercise 4 in Section 1) between the identity (chain) map and the zero (chain) map from the chain

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0 \qquad (*)$$

of $\mathbb{Z}$-modules to itself.

(d) Deduce from (c) that all $\mathbb{Z}$-module homology groups of $(*)$ are zero, i.e., $(*)$ is an exact sequence of $\mathbb{Z}$-modules. Conclude that $(*)$ is a projective G-module resolution of $\mathbb{Z}$.

2. Let P_n denote the free $\mathbb{Z}$-module with basis $(g_0, g_1, g_2, \ldots, g_n)$ with $g_i \in G$ and define an action of G on P_n by $g\cdot(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n)$. For $n \geq 1$ define

$$d_n(g_0, g_1, g_2, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n),$$

where $(g_0, \ldots, \hat{g}_i, \ldots, g_n)$ denotes the term $(g_0, g_1, g_2, \ldots, g_n)$ with g_i deleted.

(a) Prove that P_n is a free $\mathbb{Z}G$-module with basis $(1, g_1, g_2, \ldots, g_n)$ where $g_i \in G$.

(b) Prove that $d_{n-1} \circ d_n = 0$ for $n \geq 1$. [Show that the term $(g_0, \ldots, \hat{g}_j, \ldots, \hat{g}_k, \ldots, g_n)$ missing the entries g_j and g_k occurs twice in $d_{n-1} \circ d_n(g_0, g_1, g_2, \ldots, g_n)$, with opposite signs.]

(c) Prove that $\varphi : P_n \to F_n$ defined by

$$\varphi((g_0, g_1, g_2, \ldots, g_n)) = g_0 \otimes (g_0^{-1}g_1) \otimes (g_1^{-1}g_2) \ldots \otimes (g_{n-1}^{-1}g_n)$$

is a G-module isomorphism with inverse $\psi : P_n \to F_n$ given by

$$\psi(g_0 \otimes g_1 \otimes \ldots \otimes g_n) = (g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1g_2\cdots g_n).$$

(d) Prove that if $\epsilon(g_0) = 1$ for all $g_0 \in G$ then

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad (**)$$

is a free G-module resolution of $\mathbb{Z}$. [Show that the isomorphisms in (c) take the G-module resolutions $(**)$ and $(*)$ of the previous exercise into each other.]

3. Let F_n and P_n be as in the previous two exercises and let A be a G-module.

(a) Prove that $\text{Hom}_{\mathbb{Z}G}(F_n, A)$ can be identified with the collection $C^n(G, A)$ of maps from $G \times G \times \cdots \times G$ (n copies) to A and that under this identification the associated coboundary maps from $C^n(G, A)$ to $C^{n+1}(G, A)$ are given by equation (18).

(b) Prove that $\text{Hom}_{\mathbb{Z}G}(P_n, A)$ can be identified with the collection of maps f from $n+1$ copies $G \times G \times \cdots \times G$ to A that satisfy $f(gg_0, gg_1, \ldots, gg_n) = gf(g_0, g_1, \ldots, g_n)$.

The group $C^n(G, A)$ is sometimes called the group of *inhomogeneous n-cochains of G in A*, and the group in (b) of the previous exercise is called the group of *homogeneous n-cochains of G in A*. The inhomogeneous cochains are easier to describe since there is no restriction on the maps from G^n to A, but the coboundary map d_n on homogeneous cochains is less complicated (and more naturally suggested in topological contexts) than the coboundary map on inhomogeneous cochains. The results of the previous exercises show that the cohomology groups $H^n(G, A)$ defined using either homogeneous or inhomogeneous cochains are the same and indicate the origin of the coboundary maps d_n used in the text. Historically, $H^n(G, A)$ was originally defined using homogeneous cochains.

4. Suppose H is a normal subgroup of the group G and A is a G-module. For every $g \in G$ prove that the map $f(a) = ga$ for $a \in A^H$ defines an automorphism of the subgroup A^H.

5. Suppose the G-module A decomposes as a direct sum $A = A_1 \oplus A_2$ of G-submodules. Prove that for all $n \geq 0$, $H^n(G, A) \cong H^n(G, A_1) \oplus H^n(G, A_2)$.

6. Suppose $0 \to A \to M_1 \to M_2 \to \cdots \to M_k \to C \to 0$ is an exact sequence of G-modules where $M_1, M_2, \ldots, M_k$ are cohomologically trivial. Prove that $H^{n+k}(G, A) \cong H^n(G, C)$ for all $n \geq 1$. [Decompose the exact sequence into a succession of short exact sequences and use Corollary 22. For example, if $0 \to A \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2 \xrightarrow{\gamma} C \to 0$ is exact, show that $0 \to A \to M_1 \to B \to 0$ and $0 \to B \to M_2 \to C \to 0$ are both exact, where $B = M_1/\text{image}\,\alpha = M_1/\ker\beta \cong \text{image}\,\beta = \ker\gamma$.]

7. *(Adjoint Associativity)* Let R, S and T be rings with 1, let P be a left S-module, let N be a (T, S)-bimodule, and let A be a left T-module. Prove that

$$\Phi : \text{Hom}_S(P, \text{Hom}_T(N, A)) \longrightarrow \text{Hom}_T(N \otimes_S P, A)$$

defined by $\Phi(f)(n \otimes p) = f(p)(n)$ is an isomorphism of abelian groups. (See also Theorem 43 in Section 10.5).

8. Suppose G is cyclic of order m with generator σ and let $N = 1+\sigma+\sigma^2+\cdots+\sigma^{m-1} \in \mathbb{Z}G$.

(a) Prove that the *augmentation* map $\text{aug}(\sum_{i=0}^{m-1} a_i\sigma^i) = \sum_{i=0}^{m-1} a_i$ is a G-module homomorphism from $\mathbb{Z}G$ to $\mathbb{Z}$.

(b) Prove that multiplication by N and by $\sigma - 1$ in $\mathbb{Z}G$ define a free G-module resolution of $\mathbb{Z}$: $\ldots \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \ldots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0.$

9. Suppose G is an infinite cyclic group with generator σ.

(a) Prove that multiplication by $\sigma - 1 \in \mathbb{Z}G$ defines a free G-module resolution of $\mathbb{Z}$: $0 \longrightarrow \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$.

(b) Show that $H^0(G, A) \cong A^G$, that $H^1(G, A) \cong A/(\sigma-1)A$, and that $H^n(G, A) = 0$ for all $n \geq 2$. Deduce that $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$ (so free modules need not be cohomologically trivial).

10. Suppose H is a subgroup of finite index m in the group G and A is an H-module. Let $x_1, \ldots, x_m$ be a set of left coset representatives for H in G: $G = x_1H \cup \cdots \cup x_mH$.

(a) Prove that $\mathbb{Z}G = \bigoplus_{i=1}^m x_i\mathbb{Z}H = \bigoplus_{i=1}^m \mathbb{Z}Hx_i^{-1}$ and $\mathbb{Z}G \otimes_{\mathbb{Z}H} A = \bigoplus_{i=1}^m (x_i \otimes A)$ as abelian groups.

(b) Let $f_{i,a}$ be the function from $\mathbb{Z}G$ to A defined by

$$f_{i,a}(x) = \begin{cases} ha & \text{if } x = hx_i^{-1} \text{ with } h \in H \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $f_{i,a} \in M_H^G(A) = \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$, i.e., $f_{i,a}(h'x) = h'f_{i,a}(x)$ for $h' \in H$.

(c) Prove that the map $\varphi(f) = \sum_{i=1}^m x_i \otimes f(x_i^{-1})$ from $M_H^G(A)$ to $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ is a G-module homomorphism. [Write $x_i^{-1}g = h_ix_{i'}^{-1}$ for $i = 1, \ldots, m$ and observe that $x_i \otimes f(x_i^{-1}g) = x_i \otimes h_if(x_{i'}^{-1}) = x_ih_i \otimes f(x_{i'}^{-1}) = gx_{i'} \otimes f(x_{i'}^{-1})$.]

(d) Prove that φ gives a G-module isomorphism $\varphi : M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$. [For the injectivity observe that an H-module homomorphism is 0 if and only if $f(x_i^{-1}) = 0$ for $i = 1, \ldots, m$. For the surjectivity prove that $\varphi(f_{i,a}) = x_i \otimes a$.]

11. Prove that the isomorphism $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ in (d) of the previous exercise need not hold if H is not of finite index in G. [If G is an infinite cyclic group show that Shapiro's Lemma implies $H^1(G, M_1^G(\mathbb{Z})) = 0$ while $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$ by Exercise 9.]

12. If H is a subgroup of G and A is an abelian group let $M_{G/H}(A)$ denote the abelian group of all maps from the left cosets gH of H in G to A.
(a) Prove that $M_1^G(A) \cong M_1^H(M_{G/H}(A))$ as H-modules. [If $\{g_i\}_{i\in\mathcal{I}}$ is a choice of left coset representatives of H in G define the correspondence between $f \in M_1^G(A)$ and $F : H \to M_{G/H}(A)$ by $F(h)(g_iH) = f(g_ih)$, and check that this is an isomorphism of H-modules.]
(b) A G-module A such that $H^n(H, A) = 0$ for all $n \geq 1$ and all subgroups H of G is called *cohomologically trivial*. Prove that $M_1^G(A)$ is a cohomologically trivial for any abelian group A.
(c) If G is finite, prove that $\mathbb{Z}G \otimes_{\mathbb{Z}} A$ is cohomologically trivial for all abelian groups A.

13. Suppose A is a G-module and H is a subgroup of G. Prove that the group homomorphism from $H^n(G, A)$ to $H^n(G, M_H^G(A))$ for all $n \geq 0$ induced from the G-module homomorphism from A to $M_H^G(A)$ in Example 3 following Corollary 22 composed with the isomorphism $H^n(G, M_H^G(A)) \cong H^n(H, A)$ of Shapiro's Lemma is the restriction homomorphism from $H^n(G, A)$ to $H^n(H, A)$.

14. Suppose $\varphi : H \to G$ is the inclusion map of the subgroup H of G into G. If A is an H-module and $M_H^G(A)$ the associated induced G-module, define the group homomorphism $\psi : M_H^G(A) \to A$ by mapping f to its value at 1: $\psi(f) = f(1)$.
(a) Prove that φ and ψ are compatible homomorphisms.
(b) Prove that the induced group homomorphism from $H^n(G, M_H^G(A))$ to $H^n(H, A)$ for $n \geq 0$ is the isomorphism in Shapiro's Lemma.

15. Suppose H is a normal subgroup of G and A is a G-module. For fixed $g \in G$, let $\psi(a) = ga$ and $\varphi(h) = g^{-1}hg$ for $h \in H$.
(a) Prove that φ and ψ are compatible homomorphisms.
(b) For each $n \geq 0$, prove that the homomorphism θ_g from $H^n(H, A)$ to $H^n(H, A)$ induced by the compatible homomorphisms φ and ψ is an automorphism of $H^n(H, A)$. [Observe that both φ and ψ have inverses.]
(c) Show that θ_g acting on $H^0(H, A)$ is the automorphism in Exercise 4.

16. Let A be a G-module and for $g \in G$ let θ_g denote the automorphism of $H^n(G, A)$ defined in the previous exercise.
(a) Prove that θ_g acting on $H^0(G, A) = A^G$ is the identity map.
(b) Prove that θ_g acting on $H^n(G, A)$ is the identity map for $n \geq 1$. [By induction on n and dimension shifting. For $n = 1$, use the exact sequence in Corollary 22, together with (a) applied to θ_g on C^G. For $n \geq 2$ use the isomorphism $H^{n+1}(G, A) \cong H^n(G, C)$ in Corollary 22.]

17. Suppose that H is a normal subgroup of G and A is a G-module. For $n \geq 0$ prove that $H^n(H, A)$ is a G/H-module where gH acts by the automorphism θ_g induced by conjugation by g on H and the natural action of g on A as in Exercise 15. [Use the previous exercise to show this action of a coset is well defined.]

18. Suppose that G is cyclic of order m, that H is a subgroup of G of index d, and that $\mathbb{Z}$ is a trivial G-module. Use the projective G-module resolution in Exercise 8 to prove
(a) that Cor : $H^n(H, \mathbb{Z}) \to H^n(G, \mathbb{Z})$ is multiplication by d from $\mathbb{Z}$ to $\mathbb{Z}$ for $n = 0$, from $\mathbb{Z}/(m/d)\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ if n is odd, and from 0 to 0 if n is even, $n \geq 2$, and
(b) that Res : $H^n(G, \mathbb{Z}) \to H^n(H, \mathbb{Z})$ is the identity map from $\mathbb{Z}$ to $\mathbb{Z}$ for $n = 0$, and is the natural projection map from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/(m/d)\mathbb{Z}$ or from 0 to 0, depending on the parity of $n \geq 1$.

19. Let p be a prime and let P be a Sylow p-subgroup of the finite group G. Show that for any G-module A and all $n \geq 0$ the map Res : $H^n(G, A) \to H^n(P, A)$ is injective on the p-primary component of $H^1(G, A)$. Deduce that if $|A| = p^a$ then the restriction map is injective on $H^n(G, A)$. [Use Proposition 26.]

20. Let p be a prime, let $G = \langle \sigma \rangle$ be cyclic of order p^m and let W be a vector space of dimension $d > 0$ over $\mathbb{F}_p$ on which σ acts as a linear transformation. Assume W has a basis such that the matrix of σ is a $d \times d$ elementary Jordan block with eigenvalue 1.
(a) Prove that $d \leq p^m$. [Use facts about the minimal polynomial of an elementary Jordan block.]
(b) Prove that $\dim_{\mathbb{F}_p} W^G = 1$.
(c) Prove that $\dim_{\mathbb{F}_p} (\sigma - 1)W = d - 1$.
(d) If $N = 1 + \sigma + \cdots + \sigma^{p^m - 1}$ is the usual norm element, prove that NW is of dimension 1 if $d = p^m$ (respectively, of dimension 0 if $d < p^m$) and that the dimension of ${}_NW$ is $d - 1$ (respectively, d). [Let R be the group ring $\mathbb{F}_pG$, and show that every nonzero R-submodule of R contains N. Note that W is a cyclic R-module and let $\varphi : R \to W$ be a surjective homomorphism. Conclude that if φ is not an isomorphism then $N \in \ker \varphi$.]
(e) Deduce that if $d = p^m$ then $H^n(G, W) = 0$, and if $d < p^m$ then $H^n(G, W)$ has order p, for all $n \geq 1$ (i.e., these cohomology groups are zero if and only if W is a free $\mathbb{F}_pG$-module).

21. Let p be a prime, let $G = \langle \sigma \rangle$ be cyclic of order p^m and let V be a G-module of exponent p. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be a decomposition of V giving the Jordan Canonical Form of σ, where each V_i is σ-invariant and a matrix of σ on V_i is an $d_i \times d_i$ elementary Jordan block with eigenvalue 1, $d_i \geq 1$ (cf. Section 12.3). Prove that $|V^G| = p^k$ and $|H^n(G, V)| = p^s$ where s is the number of V_i of dimension less than p^m over $\mathbb{F}_p$, for all $n \geq 1$. [Use the preceding exercise and Exercise 5.]

22. Suppose G is a topological group, i.e., there is a topology on G such that the maps $G \times G \to G$ defined by $(g_1, g_2) \mapsto g_1g_2$ and $G \to G$ defined by $g \mapsto g^{-1}$ are continuous.
(a) If H is an open subgroup of G and $g \in G$, prove that the cosets gH and Hg and the subgroup $g^{-1}Hg$ are also open.
(b) Prove that any open subgroup is also closed. [The complement is the union of cosets as in (a).]
(c) Prove that a closed subgroup of finite index is open.
(d) If G is compact prove that every open subgroup H is of finite index.

23. Suppose G is a compact topological group. Prove the following are equivalent:
(i) G is profinite, i.e., $G = \varprojlim G_i$ is the inverse limit of finite groups G_i.
(ii) There exists a family $\{N_i\}$ $(i \in \mathcal{I})$ of open normal subgroups N_i in G such that $\cap_i N_i = 1$ and in this case $G \cong \varprojlim (G/N_i)$.
(iii) There exists a family $\{H_j\}$ $(j \in \mathcal{J})$ of open subgroups H_j in G such that $\cap_j H_j = 1$.

[To show (iii) implies (ii), let H be open in G and use (d) of the previous exercise to show that $N = \cap_{g\in G} g^{-1}Hg$ is a finite intersection and conclude that $N \subseteq H \subseteq G$ and N is open and normal in G.]

24. Suppose N and N' are open normal subgroups of the profinite group G and $N' \subseteq N$. Prove that the projection homomorphism $\varphi : G/N' \to G/N$ and the injection $\psi : A^N \to A^{N'}$ are compatible homomorphisms and deduce there is an induced homomorphism from $H^n(G/N, A^N)$ to $H^n(G/N', A^{N'})$.

25. If G is an infinite profinite group show that G does not act continuously on $A = \mathbb{Z}G$. [Show that the stabilizer of $a \in A$ is not always of finite index in G.]

17.3 CROSSED HOMOMORPHISMS AND $H^1(G,A)$

In this section we consider in greater detail the cohomology group $H^1(G, A)$ where G is a group and A is a G-module. From the definition of the coboundary map d_1 in equation (18), if $f \in C^1(G, A)$ then

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

Thus any function $f : G \to A$ is a 1-cocycle if and only if it satisfies the identity

$$f(gh) = f(g) + gf(h) \qquad \text{for all } g, h \in G. \tag{17.20}$$

Equivalently, a 1-cocycle is determined by a collection $\{a_g\}_{g \in G}$ of elements in A satisfying $a_{gh} = a_g + ga_h$ for $g, h \in G$ (and then the 1-cocycle f is the function sending g to a_g). Note that if 1 denotes the identity of G, then $f(1) = f(1^2) = f(1) + 1 \cdot f(1) = 2f(1)$, so $f(1) = 0$ is the identity in A. Thus 1-cocycles are necessarily "normalized" at the identity. It then follows from the cocycle condition that $f(g^{-1}) = -g^{-1}f(g)$ for all $g \in G$.

If A is a G-module on which G acts trivially, then the cocycle condition (20) is simply $f(gh) = f(g) + f(h)$, i.e., f is simply a *homomorphism* from the multiplicative group G to the additive group A. Because of this the functions from G to A satisfying (20) are called *crossed homomorphisms*.

A 1-cochain f is a 1-coboundary if there is some $a \in A$ such that

$$f(g) = g \cdot a - a \qquad \text{for all } g \in G, \tag{17.21}$$

(equivalently, $a_g = ga - a$ in the notation above). Note that since $-a \in A$, the coboundary condition in (21) can also be phrased as $f(g) = a - g \cdot a$ for some fixed $a \in A$ and all $g \in G$. The 1-coboundaries are called *principal crossed homomorphisms*. With this terminology the cohomology group $H^1(G, A)$ is the group of crossed homomorphisms modulo the subgroup of principal crossed homomorphisms.

Example: (Hilbert's Theorem 90)

Suppose $G = \text{Gal}(K/F)$ is the Galois group of a finite Galois extension K/F of fields. Then the multiplicative group $K^\times$ is a G-module and $H^1(G, K^\times) = 0$. To see this, let $\{\alpha_\sigma\}$ be the values $f(\sigma)$ of a 1-cocycle f, so that $\alpha_\sigma \in K^\times$ and $\alpha_{\sigma\tau} = \alpha_\sigma \sigma(\alpha_\tau)$ (the cocycle condition written multiplicatively for the group $K^\times$). By the linear independence of automorphisms (Corollary 8 in Section 14.2), there is an element $\gamma \in K$ such that

$$\beta = \sum_{\tau \in G} \alpha_\tau \tau(\gamma)$$

is nonzero, i.e., $\beta \in K^\times$. Then for any $\sigma \in G$ we have

$$\sigma(\beta) = \sum_{\tau \in G} \sigma(\alpha_\tau)\, \sigma\tau(\gamma) = \alpha_\sigma^{-1} \sum_{\tau \in G} \alpha_{\sigma\tau}\, \sigma\tau(\gamma) = \alpha_\sigma^{-1}\beta$$

where the second equality comes from the cocycle condition. Hence $\alpha_\sigma = \beta/\sigma(\beta)$, which is the multiplicative form of the coboundary condition (21) (for the element $a = \beta^{-1}$). Since every 1-cocycle is a 1-coboundary, we have $H^1(G, K^\times) = 0$. The same result holds for infinite Galois extensions by equation (19) in the previous section since $H^1(G, K^\times)$ is the direct limit of trivial groups.

As a special case, suppose K/F is a Galois extension with cyclic Galois group G having generator σ. The cohomology groups for G were computed explicitly in the previous section, and in particular, $H^1(G, A) = {}_N A/(\sigma - 1)A$ for any G-module A (written additively). Since this group is trivial in the present context, we see that an element α in K is in the kernel of the norm map, i.e., $N_{K/F}(\alpha) = 1$ if and only if $\alpha = \sigma(\beta)/\beta$ for some $\beta \in K$. (For a direct proof of this result in the cyclic case, cf. Exercise 23 in Section 14.2.)

This famous result for cyclic extensions was first proved by Hilbert and appears as "Theorem 90" in his book (known as the "*Zahlbericht*") on number theory in 1897. As a result, the more general result $H^1(G, K^\times) = 0$ is referred to in the literature as "Hilbert's Theorem 90." In general, the higher dimensional cohomology groups $H^n(G, K^\times)$ for $n \geq 2$ can be nontrivial (cf. Exercise 13).

Example

Suppose $G = \text{Gal}(K/F)$ is the Galois group of a finite Galois extension K/F of fields as in the previous example. Then the additive group K is also a G-module and $H^n(G, K) = 0$ for all $n \geq 2$. The proof of this in general uses the fact that there is a *normal basis* for K over F, i.e., there is an element $\alpha \in K$ whose Galois conjugates give a basis for K as a vector space over F, or, equivalently, $K \cong \mathbb{Z}G \otimes_{\mathbb{Z}} F$ as G-modules. The latter isomorphism shows that K is induced as a G-module, and then $H^n(G, K) = 0$ follows from Corollary 24 in Section 2. For a direct proof in the case where G is cyclic, cf. Exercise 26 in Section 14.2.

If G acts trivially on A, then $g \cdot a - a = 0$, so 0 is the only principal crossed homomorphism, i.e., $B^1(G, A) = 0$. This proves the following result:

Proposition 30. If A is a G-module on which G acts trivially then $H^1(G, A) = \text{Hom}(G, A)$, the group of all group homomorphisms from G to H.

If G is a profinite group, then the same result holds for the continuous cohomology group $H^1(G, A)$ provided one takes the group of continuous homomorphisms from G into A.

Examples

(1) If G acts trivially on A then $H^1(G, A) = H^1(G/[G, G], A)$ since any group homomorphism from G to the abelian group A factors through the commutator subgroup $[G, G]$ (cf. Proposition 7(5) in Section 5.4), so computing H^1 for trivial G-action reduces to computing H^1 for some abelian group.

(2) If G is a finite group acting trivially on $\mathbb{Z}$, then $H^1(G, \mathbb{Z}) = 0$ because $\mathbb{Z}$ has no nonzero elements of finite order so there is no nonzero group homomorphism from G to $\mathbb{Z}$.

(3) If A is cyclic of prime order p and G is a p-group then G must act trivially on A (since the automorphism group of A has order $p-1$), so in this case one always has $H^1(G, A) = \text{Hom}(G, A)$.

(4) If G is a finite group that acts trivially on $\mathbb{Q}/\mathbb{Z}$ then $H^1(G, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \hat{G}$ is the *dual group* of G (cf. Exercise 14 in Section 5.2.). Since $\mathbb{Q}/\mathbb{Z}$ is abelian, any homomorphism of G into $\mathbb{Q}/\mathbb{Z}$ factors through the commutator quotient $G^{ab} = G/[G, G]$ of G, so $\text{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G^{ab}, \mathbb{Q}/\mathbb{Z})$. It follows that $\text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \cong \widehat{G^{ab}}$ (which by cf. Exercise 14 again is noncanonically isomorphic to G^{ab}).

If $0 \to A \to B \to C \to 0$ is a short exact sequence of G-modules then the long exact sequence in group cohomology in Theorem 21 of the previous section begins with terms

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta_0} H^1(G, A) \longrightarrow \cdots.$$

The connecting homomorphism δ_0 is given explicitly as follows: if $c \in C^G$ then there is an element $b \in B$ mapping to c and then $\delta_0(c)$ is the class in $H^1(G, A)$ of the 1-cocycle given by

$$\begin{aligned} \delta_0(c) : G &\longrightarrow A \\ g &\longmapsto g \cdot b - b. \end{aligned}$$

Note that $g \cdot b - b$ is (the image in B of) an element of A for all $g \in G$ since $c \in C^G$. To verify directly that $f = \delta_0(c)$ satisfies the cocycle condition in (20), we compute

$$f(gh) = gh \cdot b - b = (g \cdot b - b) + g \cdot (h \cdot b - b) = f(g) + gf(h).$$

From the explicit expression $f = g \cdot b - b$ it is also clear that $\delta_0(c) \in H^1(G, A)$ maps to 0 in the next term $H^1(G, B)$ of the long exact sequence above since f is the coboundary for the element $b \in B$.

Example: (Kummer Theory)

Suppose that F is a field of characteristic 0 containing the group μ_n of all n^{th} roots of unity for some $n \geq 1$. Let K be an algebraic closure of F and let $G = \mathrm{Gal}(K/F)$. The group G acts trivially on μ_n since $\mu_n \subset F$ by assumption, i.e., $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ as G-modules. Hence the Galois cohomology group $H^1(G, \mu_n)$ is the group $\mathrm{Hom}_c(G, \mathbb{Z}/n\mathbb{Z})$ of continuous homomorphisms of G into $\mathbb{Z}/n\mathbb{Z}$. If χ is such a continuous homomorphism, then $\ker \chi \subseteq G$ is a closed normal subgroup of G, hence corresponds by Galois theory to a Galois extension L_χ/F. Then $\mathrm{Gal}(L_\chi/F) \cong \mathrm{image}\,\chi$, so L_χ is a cyclic extension of F of degree dividing n. Conversely, every such cyclic extension of F defines an element in $\mathrm{Hom}_c(G, \mathbb{Z}/n\mathbb{Z})$, so there is a bijection between the elements of the Galois cohomology group $H^1(G, \mu_n)$ and the cyclic extensions of F of degree dividing n.

The homomorphism of raising to the n^{th} power is surjective on $K^\times$ (since we can always extract n^{th} roots in K) and has kernel μ_n. Hence the sequence

$$1 \longrightarrow \mu_n \longrightarrow K^\times \xrightarrow{n} K^\times \longrightarrow 1$$

is an exact sequence of discrete G-modules. The associated long exact sequence in Galois cohomology gives

$$1 \longrightarrow \mu_n^G \longrightarrow (K^\times)^G \xrightarrow{n} (K^\times)^G \longrightarrow H^1(G, \mu_n) \longrightarrow H^1(G, K^\times) \longrightarrow \cdots.$$

We have $\mu_n^G = \mu_n$ and $(K^\times)^G = F^\times$ by Galois theory, and $H^1(G, K^\times) = 0$ by Hilbert's Theorem 90, so this exact sequence becomes

$$1 \longrightarrow \mu_n \longrightarrow F^\times \xrightarrow{n} F^\times \longrightarrow H^1(G, \mu_n) \longrightarrow 0,$$

which in turn is equivalent to the isomorphism

$$H^1(G, \mu_n) \cong F^\times / F^{\times n}$$

where $F^{\times n}$ denotes the group of n^{th} powers of elements of $F^\times$. This isomorphism is made explicit using the explicit form for the connecting homomorphism given above: for every $\alpha \in F^\times$ and $\sigma \in G$, the element $\sqrt[n]{\alpha}$ in $K^\times$ maps to α in the exact sequence and

$$\chi(\sigma) = \frac{\sigma(\sqrt[n]{\alpha})}{\sqrt[n]{\alpha}}$$

defines an element in $H^1(G, \mu_n)$ (cf. Exercise 11). The kernel of this homomorphism χ is the field $F(\sqrt[n]{\alpha})$. By the results of the previous paragraph, when F contains the n^{th} roots of unity an extension L/F is Galois with cyclic Galois group of order dividing n if and only if $L = F(\sqrt[n]{\alpha})$ for some $\alpha \in F^\times$. Furthermore, the class of α in $F^\times/F^{\times n}$ is unique, i.e., α is unique up to an n^{th} power of an element in F. Such an extension is called a *Kummer extension*, cf. Section 14.7 and Exercise 12.

If the characteristic of F is a prime p, the same argument applies when n is not divisible by p, replacing the algebraic closure of F with the separable closure of F (the largest separable algebraic extension of F).

Example: (The Transfer Homomorphism)

Suppose G is a finite group and H is a subgroup. The corestriction defines a homomorphism from $H^1(H, \mathbb{Q}/\mathbb{Z})$ to $H^1(G, \mathbb{Q}/\mathbb{Z})$, which by Example 4 above gives a homomorphism from $\hat{H}^{\text{ab}}$ to $\hat{G}^{\text{ab}}$. This gives a homomorphism

$$\mathrm{Ver} : G^{\text{ab}} \longrightarrow H^{\text{ab}}$$

called the *transfer* (or *Verlagerungen*) homomorphism (cf. Exercise 14). To make this homomorphism explicit, consider the exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow M_1^G(\mathbb{Q}/\mathbb{Z}) \longrightarrow C \longrightarrow 0 \tag{17.22}$$

defined by the homomorphism mapping $a \in \mathbb{Q}/\mathbb{Z}$ to $f_a \in M_1^G(\mathbb{Q}/\mathbb{Z})$ in Example 4 preceding Proposition 23 in the previous section (so $f_a(g) = g \cdot a$ for $g \in G$). This is a short exact sequence of G-modules and hence also of H-modules. The first portions of the associated long exact sequences for the cohomology with respect to H and then G give the rows in the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & C^H \arrow[r, "\delta_0"] \arrow[d, "\mathrm{Cor}"] & H^1(H, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Cor}"] & 0 \\ \cdots \arrow[r] & C^G \arrow[r, "\delta_0"] & H^1(G, \mathbb{Q}/\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

since $H^1(H, M_1^G(\mathbb{Q}/\mathbb{Z})) = H^1(G, M_1^G(\mathbb{Q}/\mathbb{Z})) = 0$ (cf. Exercise 12 in Section 2). Let $\chi \in H^1(H, \mathbb{Q}/\mathbb{Z})$ and suppose that $c \in C^H$ is an element mapping to χ by the surjective connecting homomorphism δ_0 in the first row of the diagram above. By the commutativity, $\chi' = \mathrm{Cor}(\chi)$ is the image under the connecting homomorphism δ_0 of $c' = \mathrm{Cor}(c) \in C^G$ in the second row of the diagram. By our explicit formula for the coboundary map δ_0, if $F \in M_1^G(\mathbb{Q}/\mathbb{Z})$ is any element mapping to c' in (22) then $g \cdot F - F = f_{a'}$ for a unique $a' \in \mathbb{Q}/\mathbb{Z}$, and we have $\chi'(g) = \delta_0(c')(g) = a'$ for $g \in G$. Since $f_{a'}(x) = x \cdot a' = a'$ for any $x \in G$ because G acts trivially on $\mathbb{Q}/\mathbb{Z}$, the function $g \cdot F - F$ in fact has the constant value a', and so can be evaluated at any $x \in G$ to determine the value of $\chi'(g)$.

Since $c' = \sum_{i=1}^{m} g_i \cdot c \in C^G$ where $g_1, \dots, g_m$ are representatives of the left cosets of H in G (cf. Example 4 preceding Proposition 26), such an element F is given by

$$F = \sum_{i=1}^{m} g_i \cdot f,$$

where $f \in M_1^G(\mathbb{Q}/\mathbb{Z})$ is any element mapping to c in (22). This f can be used to compute the explicit coboundary of c as before: $h \cdot f - f = f_a$ for a unique $a \in \mathbb{Q}/\mathbb{Z}$ and $\chi(h) = a$ for $h \in H$. As before, the function $h \cdot f - f = f_a$ has the constant value a and so can be evaluated at any element x of G to determine the value of $\chi(h)$.

Computing $g \cdot F - F$ on the element $1 \in G$ it follows that

$$\chi'(g) = \sum_{i=1}^{m} f(gg_i) - \sum_{i=1}^{m} f(g_i).$$

For $i = 1, \dots, m$, write

$$gg_i = g_j h(g, g_i) \qquad \text{with } h(g, g_i) \in H, \tag{17.23}$$

noting that the resulting set of g_j is some permutation of $\{g_1, \dots, g_m\}$. Then

$$\sum_{i=1}^{m} f(gg_i) - \sum_{i=1}^{m} f(g_i) = \sum_{i=1}^{m} [f(g_j h(g, g_i)) - f(g_j)] = \sum_{i=1}^{m} \chi(h(g, g_i))$$

since as noted above, $\chi(h) = f(xh) - f(x)$ for any $x \in G$. Hence

$$\chi'(g) = \chi\Big(\prod_{i=1}^{m} h(g, g_i)\Big)$$

and so the transfer homomorphism is given by the formula

$$\operatorname{Ver}(g) = \prod_{i=1}^{m} h(g, g_i) \tag{17.24}$$

with the elements $h(g, g_i) \in H$ defined by equation (23). Note that this proves in particular that the map defined in (24) is a homomorphism from G^{ab} to H^{ab} that is independent of the choice of representatives g_i for H in G in (23). Proving that this map is a homomorphism directly is not completely trivial. The same formula also defines the transfer homomorphism when G is infinite and H is a subgroup of finite index in G.

As an example of the transfer, suppose $H = n\mathbb{Z}$ and $G = \mathbb{Z}$ and choose $0, 1, 2, \dots, n-1$ as coset representatives for H in G. If $g = 1$, then all the elements $h(g, g_i)$ are 0 for $i = 1, 2, \dots, n-1$ and $h(1, n-1) = n$. Hence the transfer map from $\mathbb{Z}$ to $n\mathbb{Z}$ maps 1 to n, so is simply multiplication by the index. Similarly, the transfer map from any cyclic group G to a subgroup H of index n is the n^{th} power map. See also Exercise 8.

For the cyclic group $\mathbb{F}_p^\times$ for an odd prime p and subgroup $\{\pm 1\}$, it follows that the transfer map is the homomorphism $\operatorname{Ver} : \mathbb{F}_p^\times \to \{\pm 1\}$ given by

$$\operatorname{Ver}(a) = a^{(p-1)/2} = \left(\frac{a}{p}\right) = \begin{cases} +1 & \text{if } a \text{ is a square} \\ -1 & \text{if } a \text{ is not a square} \end{cases}$$

(the symbol $(\frac{a}{p})$ is called the *Legendre symbol* or the *quadratic residue symbol*). If instead we take the elements $1, 2, \dots, (p-1)/2$ as coset representatives for $\{\pm 1\}$ in $\mathbb{F}_p^\times$ we see that

$$\left(\frac{a}{p}\right) = (-1)^{m(a)}$$

where $m(a)$ is the number of elements among $a, 2a, \dots, (p-1)a/2$ whose least positive remainder modulo p is greater than $(p-1)/2$ (in which case the element differs by -1 from one of our chosen coset representatives and contributes one factor of -1 to the product in (24)). This result is known as *Gauss' Lemma* in elementary number theory and can be used to prove Gauss' celebrated Quadratic Reciprocity Law (cf. also Exercise 15).

Next we give two important interpretations of $H^1(G, A)$ in terms of semidirect products. If A is a G-module, let E be the semidirect product $E = A \rtimes G$, where A is normal in E and the action of G (viewed as a subgroup of E) on A by conjugation is the same as its G-module action: $gag^{-1} = g \cdot a$. In the notation of Section 5.5, $E = A \rtimes_\varphi G$, where φ is the homomorphism of G into $\operatorname{Aut}(A)$ given by the G-module action. In particular, E will be the direct product of A and G if and only if G acts trivially on A. As in Section 5.5, we shall write the elements of E as (a, g) where $a \in A$ and $g \in G$, with group operation

$$(a_1, g_1)(a_2, g_2) = (a_1 + g_1 \cdot a_2, g_1 g_2).$$

Note that A is written additively, while G and E are written multiplicatively.

Definition. Let X be any group and let Y be a normal subgroup of X. The *stability group* of the series $1 \trianglelefteq Y \trianglelefteq X$ is the group of all automorphisms of X that map Y to itself and act as the identity on both of the factors Y and X/Y, i.e.,

$$\operatorname{Stab}(1 \trianglelefteq Y \trianglelefteq X) = \{\sigma \in \operatorname{Aut}(X) \mid \sigma(y) = y \text{ for all } y \in Y, \\ \text{and } \sigma(x) \equiv x \bmod Y \text{ for all } x \in X\}.$$

In the special case where Y is an *abelian* normal subgroup of X, conjugation by elements of Y induce (inner) automorphisms of X that stabilize the series $1 \trianglelefteq Y \trianglelefteq X$, and in this case $Y/C_Y(X)$ is isomorphic to a subgroup of $\operatorname{Stab}(1 \trianglelefteq Y \trianglelefteq X)$ (where $C_Y(X)$ is the elements of Y in the center of X).

Proposition 31. Let A be a G-module and let E be the semidirect product $A \rtimes G$. For each cocycle $f \in Z^1(G, A)$ define $\sigma_f : E \to E$ by

$$\sigma_f((a, g)) = (a + f(g), g).$$

Then the map $f \to \sigma_f$ is a group isomorphism from $Z^1(G, A)$ onto $\operatorname{Stab}(1 \trianglelefteq A \trianglelefteq E)$. Under this isomorphism the subgroup $B^1(G, A)$ of coboundaries maps onto the subgroup $A/C_A(E)$ of the stability group.

Proof: It is an exercise to see that the cocycle condition implies σ_f is an automorphism of E that stabilizes the chain $1 \trianglelefteq A \trianglelefteq E$. Likewise one checks directly that $\sigma_{f_1+f_2} = \sigma_{f_1} \circ \sigma_{f_2}$, so the map $f \mapsto \sigma_f$ is a group homomorphism. By definition of σ_f this map is injective. Conversely, let $\sigma \in \operatorname{Stab}(1 \trianglelefteq A \trianglelefteq E)$. Since σ acts trivially on E/A, each element $(0, g)$ in this semidirect product maps under σ to another element (a, g) in the same coset of A; define $f_\sigma : G \to A$ by letting $f_\sigma(g) = a$. If we identify A with the elements of the form $(a, 1)$ in E, then the group operation in E shows that

$$f_\sigma(g) = \sigma((0, g))(0, g)^{-1}.$$

Because σ is a stability automorphism of E, it is easy to check that f_σ satisfies the cocycle condition. It follows immediately from the definitions that $f_{\sigma_f} = f$, so the map $f \mapsto \sigma_f$ is an isomorphism.

Now f is a coboundary if and only if there is some $x \in A$ such that $f(g) = x - g\cdot x$ for all $g \in G$. Thus f is a coboundary if and only if $\sigma_f((a, g)) = (a+x-g\cdot x, g)$. But conjugation in E by the element $(x, 1)$ maps (a, g) to the same element $(a+x-g\cdot x, g)$, so the automorphism σ_f is conjugation by $(x, 1)$. This proves the remaining assertion of the proposition.

Corollary 32. In the notation of Proposition 31 let φ_a denote the automorphism of E given by conjugation by a for any $a \in A$. Then the cocycles f_1 and f_2 are in the same cohomology class in $H^1(G, A)$ if and only if $\sigma_{f_1} = \varphi_a \circ \sigma_{f_2}$, for some $a \in A$.

The proposition and corollary show that 1-cocycles may be computed by finding automorphisms of E that stabilize the series $1 \trianglelefteq A \trianglelefteq E$, and vice versa. The first cohomology group is then given by taking these automorphisms modulo inner automorphisms, i.e., is the group of "outer stability automorphisms" of this series.

Example

Let $G = Z_2$ act by inversion on $A = \mathbb{Z}/4\mathbb{Z}$. The corresponding semidirect product $E = A \rtimes G$ is the dihedral group of order 8, which has automorphism group isomorphic to D_8; viewing E as a normal (index 2) subgroup of D_{16}, conjugation in the latter group restricted to E exhibits 8 distinct automorphisms of E (cf. Proposition 17 in Section 4.4). The subgroup A of E is characteristic in E, hence every automorphism of E sends A to itself, and therefore also acts on E/A (necessarily trivially since $|E/A| = 2$). Half the automorphisms of E invert A and half centralize A; in fact, the cyclic subgroup of order 8 in D_{16} (which contains A) maps to a cyclic group of order 4 of automorphisms centralizing A. Thus $\text{Stab}(1 \trianglelefteq A \trianglelefteq E) \cong Z_4 \cong Z^1(G, A)$. Since the center of E is a subgroup of A of order 2, $|A/Z(E)| = 2 = |B^1(G, A)|$. This proves $|H^1(G, A)| = 2$.

In the semidirect product E the subgroup G is a complement to A, i.e., $E = AG$ and $A \cap G = 1$; moreover, every E-conjugate of G is also a complement to A. But A may have complements in E that are not conjugate to G in E. Our second interpretation of $H^1(G, A)$ shows that this cohomology group characterizes the E-conjugacy classes of complements of A in E.

Proposition 33. Let A be a G-module and let E be the semidirect product $A \rtimes G$. For each 1-cocycle f let

$$G_f = \{(f(g), g) \mid g \in G\}.$$

Then G_f is a subgroup complement to A in E. The map $f \mapsto G_f$ is a bijection from $Z^1(G, A)$ to the set of complements to A in E. Two complements are conjugate in E if and only if their corresponding 1-cocycles are in the same cohomology class in $H^1(G, A)$, so there is a bijection between $H^1(G, A)$ and the set of E-conjugacy classes of complements to A.

Proof: By the cocycle condition,

$$(f(g), g)(f(h), h) = (f(g)+gf(h)g^{-1}, gh) = (f(g)+g\cdot f(h), gh) = (f(gh), gh),$$

and it follows that G_f is closed under the group operation in E. As observed earlier, each cocycle necessarily has $f(1) = 0$, so G_f contains the identity $(0, 1)$ of E. The inverse to $(f(g), g)$ in E is $(f(g^{-1}), g^{-1})$, so G_f is closed under inverses. This proves G_f is a subgroup of E. Since the distinct elements of G_f represent the distinct cosets of A in E, G_f is a complement to A in E. Distinct cocycles give different coset representatives, hence they determine different complements.

Conversely, if C is any complement to A in G, then C contains a unique coset representative $a_g g$ of Ag for each $g \in G$. Since C is closed under the group operation the element $(a_g g)(a_h h) = (a_g g a_h g^{-1}) gh$ represents the coset Agh, and so a_{gh} is $a_g g a_h g^{-1} = a_g(g\cdot a_h)$ (written additively in A this becomes $a_{gh} = a_g + (g\cdot a_h)$). This shows that the map $f : G \to A$ given by $f(g) = a_g$ is a cocycle, and so $C = G_f$. Hence there is a bijection between 1-cocycles and complements to A in E.

Since $\text{Stab}(1 \trianglelefteq A \trianglelefteq E)$ normalizes A it permutes the complements to A in E. In the notation of Proposition 31, for 1-cocycles f_1 and f_2 it follows immediately from the definition that $\sigma_{f_1}(G_{f_2}) = G_{f_1+f_2}$. This shows that the permutation action of $\text{Stab}(1 \trianglelefteq A \trianglelefteq E)$ on the set of complements to A in E is the (left) regular representation of this group. Furthermore, if $a \in A$ and φ_a is the stability automorphism conjugation by a, then

$$aG_f a^{-1} = \varphi_a(G_f) = G_{f+\beta_a} \tag{17.25}$$

where β_a is the 1-coboundary $\beta_a : g \mapsto a - g\cdot a$. Since G_f is a complement to A, any $e \in E$ may be written as ag for some $a \in A$ and $g \in G_f$. Then $eG_f e^{-1} = aG_f a^{-1}$, i.e., the E-conjugates of G_f are the just the A-conjugates of G_f. Now the complements G_{f_1} and G_{f_2} are conjugate in E if and only if $G_{f_2} = aG_{f_1}a^{-1} = G_{f_1+\beta_a}$ for some $a \in A$ by (25). This shows two complements are conjugate in E if and only if their corresponding cocycles differ by a coboundary, i.e., represent the same cohomology class in $H^1(G, A)$, which completes the proof.

Corollary 34. Under the notation of Proposition 33, all complements to A are conjugate in E if and only if $H^1(G, A) = 0$.

Corollary 35. If A is a finite abelian group whose order is relatively prime to $|G|$ then all complements to A in any semidirect product $E = A \rtimes G$ are conjugate in E.

Examples

(1) Let $A = \langle a \rangle$ and $G = \langle g \rangle$ both be cyclic of order 2. The group G must act trivially on A, hence $A \rtimes G = A \times G$ is a Klein 4-group. Here $A \rtimes G$ is abelian, so every subgroup is conjugate only to itself, and since $H^1(G, A) = \text{Hom}(Z_2, \mathbb{Z}/2\mathbb{Z})$ has order 2, there are precisely two complements to A in E, namely $\langle g \rangle$ and $\langle ag \rangle$.

(2) If $A = \langle a \rangle$ is cyclic of order 2 and $G = \langle x \rangle \times \langle y \rangle$ is a Klein 4-group, then as before G must act trivially on A, so $H^1(G, A) = \text{Hom}(Z_2 \times Z_2, \mathbb{Z}/2\mathbb{Z})$ has order 4. The four complements to A in $A \times G$ are G, $\langle ax, y \rangle$, $\langle x, ay \rangle$ and $\langle ax, ay \rangle$.

(3) Proposition 33 can also be used to compute $H^1(G, A)$. Let $A = \langle r \rangle$ be cyclic of order 4 and let $G = \langle s \rangle$ be cyclic of order 2 acting on A by inversion: $srs^{-1} = r^{-1}$ as in the Example following Corollary 32. Then $A \rtimes G$ is the dihedral group D_8 of order 8. The subgroup A has four complements in D_8, namely the groups generated

by each of the four elements of order 2 not in A: $\langle s\rangle$, $\langle r^2s\rangle$, $\langle rs\rangle$ and $\langle r^3s\rangle$. The former pair and the latter pair are conjugate in D_8 (in both cases via r), but $\langle s\rangle$ is not conjugate to $\langle rs\rangle$. Thus A has 2 conjugacy classes of complements in $A\rtimes G$ and hence $H^1(Z_2, \mathbb{Z}/4\mathbb{Z})$ has order 2. This also follows from the computation of the cohomology of cyclic groups in Section 2.

EXERCISES

1. Let G be the cyclic group of order 2 and let A be a G-module. Compute the isomorphism types of $Z^1(G, A)$, $B^1(G, A)$ and $H^1(G, A)$ for each of the following:
   (a) $A = \mathbb{Z}/4\mathbb{Z}$ (trivial action),
   (b) $A = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ (trivial action),
   (c) $A = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ (any nontrivial action).
2. Let p be a prime and let P be a p-group.
   (a) Show that $H^1(P, \mathbb{F}_p)\cong P/\Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of P (cf. the exercises in Section 6.1).
   (b) Deduce that the dimension of $H^1(P, \mathbb{F}_p)$ as a vector space over $\mathbb{F}_p$ equals the minimum number of generators of P. [Use Exercise 26(c), Section 6.1.]
3. If G is the cyclic group of order 2 acting by inversion on $\mathbb{Z}$ show that $|H^1(G, \mathbb{Z})| = 2$. [Show that in $E = \mathbb{Z}\rtimes G$ every element of $E - \mathbb{Z}$ has order 2, and there are two conjugacy classes in this coset.]
4. Let A be the Klein 4-group and let $G = \mathrm{Aut}(A)\cong S_3$ act on A in the natural fashion. Prove that $H^1(G, A) = 0$. [Show that in the semidirect product $E = A\rtimes G$, G is the normalizer of a Sylow 3-subgroup of E. Apply Sylow's Theorem to show all complements to A in E are conjugate.]
5. Let G be the cyclic group of order 2 acting on an elementary abelian 2-group A of order 2^n. Show that $H^1(G, A) = 0$ if and only if $n = 2k$ and $|A^G| = 2^k$. [In $E = A\rtimes G$ show that (a, x) is an element of order 2 if and only if $a\in A^G$, where $G = \langle x\rangle$. Then compare the number of complements to A with the number of E-conjugates of x.]
6. *(Thompson Transfer Lemma)* Let G be a finite group of even order, let T be a Sylow 2-subgroup of G, let $M\leq T$ with $|T : M| = 2$, and let x be an element of order 2 in G. Show that if G has no subgroup of index 2 then M contains some G-conjugate of x as follows:
   (a) Let $\mathrm{Ver} : G/[G, G]\to T/[T, T]$ be the transfer homomorphism. Show that
   $$\mathrm{Ver}(x) = \prod_g g^{-1}xg \bmod [T, T]$$
   where the product is over representatives of the cosets gT that are fixed under left multiplication by x.
   (b) Show that under left multiplication x fixes an odd number of left cosets of T in G.
   (c) Show that if G has no subgroup of index 2 then $\mathrm{Ver}(x)\in M/[T, T]$. Deduce that for some $g\in G$ we must have $g^{-1}xg\in M$. [Consider the product $\mathrm{Ver}(x)$ in the group T/M of order 2.]
7. Let H be a subgroup of G and let $x\in G$. The transfer $\mathrm{Ver} : G/[G, G]\to H/[H, H]$ may be computed as follows: let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$ be the distinct orbits of x acting by left multiplication on the left cosets of H in G, let $\mathcal{O}_i$ have length n_i and let g_iH be any representative of $\mathcal{O}_i$.

   (a) Show that $\mathcal{O}_i = \{g_iH, xg_iH, x^2g_iH, \dots, x^{n_i-1}g_iH\}$ and that $g_i^{-1}x^{n_i}g_i\in H$.
   (b) Show that $\mathrm{Ver}(x) = \prod_{i=1}^k g_i^{-1}x^{n_i}g_i \bmod [H, H]$.
8. Assume the center, $Z(G)$, of G is of index m. Prove that $\mathrm{Ver}(x) = x^m$, for all $x\in G$, where Ver is the transfer homomorphism from $G/[G, G]$ to $Z(G)$. [Use the preceding exercise.]
9. Let p be a prime, let $n\geq 3$, and let V be an n-dimensional vector space over $\mathbb{F}_p$ with basis $v_1, v_2, \dots, v_n$. Let V be a module for the symmetric group S_n, where each $\pi\in S_n$ permutes the basis in the natural way: $\pi(v_i) = v_{\pi(i)}$.
   (a) Show that $|H^1(S_n, V)| = \begin{cases} 0, & \text{if } p\neq 2\\ 2, & \text{if } p = 2\end{cases}$. [Use Shapiro's Lemma.]
   (b) Show that $H^1(A_n, V) = 0$ for all primes p.
10. Let V be the natural permutation module for S_n over $\mathbb{F}_2$, $n\geq 3$, as described in the preceding exercise, and let $W = \{a_1v_1 + \cdots + a_nv_n \mid a_1 + \cdots + a_n = 0\}$ (the "trace zero" submodule of V). Show that if n is even then $H^1(A_n, W)\neq 0$. [Show that in the semidirect product $V\rtimes A_n$ the element v_1 induces a nontrivial outer automorphism on $E = W\rtimes A_n$ that stabilizes the series $1\trianglelefteq W\trianglelefteq E$.]
11. Let F be a field of characteristic not dividing n and let a be any nonzero element in F. Let K be a Galois extension of F containing the splitting field of $x^n - a$, and let $\sqrt[n]{a}$ be a fixed n^{th} root of a in K.
    (a) Prove that $\sigma(\sqrt[n]{a})/\sqrt[n]{a}$ is an n^{th} root of unity.
    (b) Prove that the function $f(\sigma) = \sigma(\sqrt[n]{a})/\sqrt[n]{a}$ is a 1-cocycle of G with values in the group μ_n of n^{th} roots of unity in K (note μ_n is not assumed to be contained in F).
    (c) Prove that the 1-cocycle obtained by a different choice of n^{th} root of a in K differs from the 1-cocycle in (b) by a 1-coboundary.
12. Let F be a field of characteristic not dividing n that contains the n^{th} roots of unity, and suppose L/F is a Galois extension with abelian Galois group of exponent dividing n. Prove that L is the composite of cyclic extensions of F whose degrees are divisors of n and use this to prove that there is a bijection between the subgroups of the multiplicative group $F^\times/F^{\times n}$ and such extensions L.
13. The Galois group of the extension $\mathbb{C}/\mathbb{R}$ is the cyclic group $G = \langle\tau\rangle$ of order 2 generated by complex conjugation τ. Prove that $H^2(G, \mathbb{C}^\times)\cong\mathbb{R}^\times/\mathbb{R}^+\cong\mathbb{Z}/2\mathbb{Z}$ where $\mathbb{R}^+$ denotes the positive real numbers.
14. For any group G let $\hat{G} = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ denote its dual group.
    (a) If $\varphi : G_1\to G_2$ is a group homomorphism prove that composition with φ induces a homomorphism $\hat{\varphi} : \hat{G}_2\to\hat{G}_1$ on their dual groups.
    (b) For any fixed g in G, show that evaluation at g gives a homomorphism φ_g from $\hat{G}$ to $\mathbb{Q}/\mathbb{Z}$.
    (c) Prove that the map taking $g\in G$ to φ_g in (b) defines a homomorphism from G to its *double dual* $\widehat{(\hat{G})}$.
    (d) Prove that if G is a finite abelian group then the homomorphism in (c) is an isomorphism of G with its double dual. (By Exercise 14 in Section 5.2 the group G is (noncanonically) isomorphic to its dual $\hat{G}$. This shows that G is *canonically* isomorphic to its double dual — the isomorphism is independent of any choice of generators for G.)
    (e) If $\psi : \hat{G}_2\to\hat{G}_1$ is a homomorphism where G_1 and G_2 are finite abelian groups, then by (a) and (d) there is an induced homomorphism $\varphi : G_1\to G_2$. Prove that

$\varphi(g_1) = g_2$ if $\chi(g_2) = \chi'(g_1)$ for $\chi' = \psi(\chi)$.

15. Use Gauss' Lemma in the computation of the transfer map for $\mathbb{F}_p^\times$ to $\{\pm 1\}$ to prove that 2 is a square modulo the odd prime p if and only if $p \equiv \pm 1 \bmod 8$. [Count how many elements in $2, 4, \dots, p-1$ are greater than $(p-1)/2$.]

17.4 GROUP EXTENSIONS, FACTOR SETS AND $H^2(G,A)$

If A is a G-module then from the definition of the coboundary map d_2 in equation (18) a function f from $G \times G$ to A is a 2-cocycle if it satisfies the identity

$$f(g,h) + f(gh,k) = g \cdot f(h,k) + f(g,hk) \qquad \text{for all } g,h,k \in G. \tag{17.26}$$

Equivalently, a 2-cocycle is determined by a collection of elements $\{a_{g,h}\}_{g,h\in G}$ of elements in A satisfying $a_{g,h} + a_{gh,k} = g \cdot a_{h,k} + a_{g,hk}$ for $g,h,k \in G$ (and then the 2-cocycle f is the function sending (g,h) to $a_{g,h}$).

A 2-cochain f is a coboundary if there is a function $f_1 : G \to A$ such that

$$f(g,h) = gf_1(h) - f_1(gh) + f_1(g), \qquad \text{for all } g,h \in G \tag{17.27}$$

i.e., f is the image under d_1 of the 1-cochain f_1.

One of the main results of this section is to make a connection between the 2-cocycles $Z^2(G,A)$ and the *factor sets* associated to a group extension of G by A, which arise when considering the effect of choosing different coset representatives in defining the multiplication in the extension. In particular, we shall show that there is a bijection between equivalence classes of group extensions of G by A (with the action of G on A fixed) and the elements of $H^2(G,A)$.

We first observe some basic facts about extensions. Let E be any group extension of G by A,

$$1 \longrightarrow A \xrightarrow{\ \iota\ } E \xrightarrow{\ \pi\ } G \longrightarrow 1. \tag{17.28}$$

The extension (28) determines an action of G on A, as follows. For each $g \in G$ let e_g be an element of E mapping onto g by π (the choice of such a set of representatives for G in E is called a set-theoretic *section* of π). The element e_g acts by conjugation on the normal subgroup $\iota(A)$ of E, mapping $\iota(a)$ to $e_g\iota(a)e_g^{-1}$. Any other element in E that maps to g is of the form $e_g\iota(a_1)$ for some $a_1 \in A$, and since $\iota(A)$ is abelian, conjugation by this element on $\iota(A)$ is the same as conjugation by e_g, so is independent of the choice of representative for g. Hence G acts on $\iota(A)$, and so also on A since ι is injective. Since conjugation is an automorphism, the extension (28) defines A as a G-module.

Recall from Section 10.5 that two extensions $1 \to A \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} G \to 1$ and $1 \to A \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} G \to 1$ are *equivalent* if there is a group isomorphism $\beta : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\iota_1"] \arrow[d, "\mathrm{id}"] & E_1 \arrow[r, "\pi_1"] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & A \arrow[r, "\iota_2"] & E_2 \arrow[r, "\pi_2"] & G \arrow[r] & 1. \end{tikzcd} \tag{17.29}$$

In this case we simply say β is the equivalence between the two extensions. As noted in Section 10.5, equivalence of extensions is reflexive, symmetric and transitive. We also observe that

equivalent extensions define the same G-module structure on A.

To see this assume (29) is an equivalence, let g be any element of G and let e_g be any element of E_1 mapping onto g by π_1. The action of g on A given by conjugation in E_1 maps each a to $\iota_1^{-1}(e_g\iota_1(a)e_g^{-1})$. Let $e'_g = \beta(e_g)$. Since the diagram commutes, $\pi_2(e'_g) = g$, so the action of g on A in the second extension is given by conjugation by e'_g. This conjugation maps a to $\iota_2^{-1}(e'_g\iota_2(a)e'^{-1}_g)$. Since ι_1, ι_2 and β are injective, the two actions of g on a are equal if and only if they result in the same image in E_2, i.e., $\beta \circ \iota_1(\iota_1^{-1}(e_g\iota_1(a)e_g^{-1})) = e'_g\iota_2(a)e'^{-1}_g$. This equality is now immediate from the definition of e'_g and the commutativity of the diagram.

We next see how an extension as in (28) defines a 2-cocycle in $Z^2(G,A)$. For simplicity we identify A as a subgroup of E via ι and we identify G as E/A via π.

Definition. A map $\mu : G \to E$ with $\pi \circ \mu(g) = g$ and $\mu(1) = 0$, i.e., so that for each $g \in G$, $\mu(g)$ is a representative of the coset Ag of E and the identity of E (which is the zero of A) represents the identity coset, is called a *normalized section* of π.

Fix a section μ of π in (28). Each element of E may be written uniquely in the form $a\mu(g)$, where $a \in A$ and $g \in G$. For $g,h \in G$ the product $\mu(g)\mu(h)$ in E lies in the coset Agh, so there is a unique element $f(g,h)$ in A such that

$$\mu(g)\mu(h) = f(g,h)\mu(gh) \qquad \text{for all } g,h \in G. \tag{17.30}$$

If in addition μ is normalized at the identity we also have

$$f(g,1) = 0 = f(1,g) \qquad \text{for all } g \in G. \tag{17.31}$$

Definition. The function f defined by equation (30) is called the *factor set* for the extension E associated to the section μ. If f also satisfies (31) then f is called a *normalized* factor set.

We shall see in the examples following that it is possible for different sections μ to give the same factor set f.

We now verify that the factor set f is in fact a 2-cocycle. First note that the group operation in E may be written

$$\begin{aligned}(a_1\mu(g))(a_2\mu(h)) &= (a_1 + \mu(g)a_2\mu(g)^{-1})\mu(g)\mu(h) \\ &= (a_1 + g \cdot a_2)(\mu(g)\mu(h)) \\ &= (a_1 + g \cdot a_2 + f(g,h))\mu(gh)\end{aligned} \tag{17.32}$$

where $g \cdot a_2$ denotes the G-module action of g on a_2 given by conjugation in E. Now use (32) and the associative law in E to compute the product $\mu(g)\mu(h)\mu(k)$ in two different ways:

$$\begin{aligned}(\mu(g)\mu(h))\mu(k) &= (f(g,h) + f(gh,k))\mu(ghk) \\ \mu(g)(\mu(h)\mu(k)) &= (gf(h,k) + f(g,hk))\mu(ghk).\end{aligned} \tag{17.33}$$

It follows that the factors in A of the two right hand sides in (33) are equal for every $g, h, k \in G$, and this is precisely the 2-cocycle condition (26) for f. This shows that the factor set associated to the extension E and any choice of section μ is an element in $Z^2(G, A)$.

We next see how the factor set f depends on the choice of section μ. Suppose μ' is another section for the same extension E in (28), and let f' be its associated factor set. Then for all $g \in G$ both $\mu(g)$ and $\mu'(g)$ lie in the same coset Ag, so there is a function $f_1 : G \to A$ such that $\mu'(g) = f_1(g)\mu(g)$ for all g. Then

$$\mu'(g)\mu'(h) = f'(g,h)\mu'(gh) = (f'(g,h) + f_1(gh))\mu(gh).$$

We also have

$$\begin{aligned}\mu'(g)\mu'(h) &= (f_1(g)\mu(g))(f_1(h)\mu(h)) = (f_1(g) + g \cdot f_1(h))(\mu(g)\mu(h))\\ &= (f_1(g) + g \cdot f_1(h) + f(g,h))\mu(gh).\end{aligned}$$

Equating the factors in A in these two expressions for $\mu'(g)\mu'(h)$ shows that

$$f'(g,h) = f(g,h) + (gf_1(h) - f_1(gh) + f_1(g)) \qquad \text{for all } g, h \in G,$$

in other words f and f' differ by the 2-coboundary of f_1 as in (27).

We have shown that the factor sets associated to the extension E corresponding to different choices of sections give 2-cocycles in $Z^2(G, A)$ that differ by a coboundary in $B^2(G, A)$. Hence associated to the extension E is a well defined cohomology class in $H^2(G, A)$ determined by the factor set in (30) for any choice of section μ.

If the extension E of G by A is a *split* extension (which is to say that $E = A \rtimes G$ is the semidirect product of G by A with the given conjugation action of G on A), then there is a section μ of G that is a *homomorphism* from G to E. In this case the factor set f in (30) is identically 0: $f(g,h) = 0$ for all $g, h \in G$. Hence the cohomology class in $H^2(G, A)$ defined by a split extension is the trivial class.

Suppose now that β is an equivalence between the extension in (28) and an extension E':

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & A & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & G & \longrightarrow & 1. \end{array}$$

If μ is a section of π, then $\mu' = \beta \circ \mu$ is a section of π', so what we have just proved can be used to determine the cohomology class in $H^2(G, A)$ corresponding to E'. Applying the homomorphism β to equation (30) gives

$$\beta(\mu(g))\beta(\mu(h)) = \beta(f(g,h))\beta(\mu(gh)) \qquad \text{for all } g, h \in G.$$

Since β restricts to the identity map on A, this is

$$\mu'(g)\mu'(h) = f(g,h)\mu'(gh) \qquad \text{for all } g, h \in G,$$

which shows that the factor set for E' associated to μ' is the same as the factor set for E associated to μ. This proves that equivalent extensions define the same cohomology class in $H^2(G, A)$.

We next show how this procedure may be reversed: Given a class in $H^2(G, A)$ we construct an extension E_f whose corresponding factor set is in the given class in $H^2(G, A)$. The process generalizes the semidirect product construction of Section 5.5 (which is the special case when f is the zero cocycle representing the trivial class).

Note first that any 2-cocycle arising from the factor set of an extension as above where the section μ is normalized satisfies the condition in (31).

Definition. A 2-cocycle f such that $f(g, 1) = 0 = f(1, g)$ for all $g \in G$ is called a *normalized* 2-cocycle.

The construction of E_f is a little simpler when f is a normalized cocycle and for simplicity we indicate the construction in this case (the minor modifications necessary when f is not normalized are indicated in Exercise 4).

We first see that any 2-cocycle f lies in the same cohomology class as a normalized 2-cocycle. Let $d_1 f_1$ be the 2-coboundary of the constant function f_1 on G whose value is $f(1,1)$. Then $f(1,1) = d_1 f_1(1,1)$, and one easily checks from the 2-cocycle condition that $f - d_1 f_1$ is normalized.

We may therefore assume that our cohomology class in $H^2(G, A)$ is represented by the normalized 2-cocycle f. Let E_f be the set $A \times G$, and define a binary operation on E_f by

$$(a_1, g)(a_2, h) = (a_1 + g \cdot a_2 + f(g,h), gh) \tag{17.34}$$

where, as usual, $g \cdot a_2$ denotes the module action of G on A. It is straightforward to check that the group axioms hold: Since f is normalized, the identity element is (0,1) and inverses are given by

$$(a, g)^{-1} = (-g^{-1} \cdot a - f(g^{-1}, g), g^{-1}). \tag{17.35}$$

The cocycle condition implies the associative law by calculations similar to (32) and (33) earlier — the details are left as exercises.

Since f is a normalized 2-cocycle, $A^* = \{(a, 1) \mid a \in A\}$ is a subgroup of E_f, and the map $\iota^* : a \mapsto (a, 1)$ is an isomorphism from A to A^*. Moreover, from (34) and (35) it follows that

$$(0, g)(a, 1)(0, g)^{-1} = (g \cdot a, 1) \qquad \text{for all } g \in G \text{ and all } a \in A. \tag{17.36}$$

Since E_f is generated by A^* together with the set of elements $(0, g)$ for $g \in G$, (36) implies that A^* is a normal subgroup of E_f. Furthermore, it is immediate from (34) that the map $\pi^* : (a, g) \mapsto g$ is a surjective homomorphism from E_f to G with kernel A^*, i.e., $E_f / A^* \cong G$. Thus

$$1 \longrightarrow A \xrightarrow{\iota^*} E_f \xrightarrow{\pi^*} G \longrightarrow 1 \tag{17.37}$$

is a specific extension of G by A, where (36) ensures also that the action of G on A by conjugation in this extension is the module action specified in determining the 2-cocycle f in $H^2(G, A)$. The extension sequence (37) shows that this extension has the normalized section $\mu(g) = (0, g)$ whose corresponding normalized factor set is f. Note that this proves not only that every cohomology class in $H^2(G, A)$ arises from some extension E, but that every normalized 2-cocycle arises as the normalized factor set of some extension.

Finally, suppose f' is another normalized 2-cocycle in the same cohomology class in $H^2(G, A)$ as f and let $E_{f'}$ be the corresponding extension. If f and f' differ by the coboundary of $f_1 : G \to A$ then $f(g,h) - f'(g,h) = gf_1(h) - f_1(gh) + f_1(g)$ for all $g, h \in G$. Setting $g = h = 1$ shows that $f_1(1) = 0$. Define

$$\beta : E_f \longrightarrow E_{f'} \qquad \text{by} \qquad \beta((a,g)) = (a + f_1(g), g).$$

It is immediate that β is a bijection, and

$$\begin{aligned}\beta((a_1,g)(a_2,h)) &= \beta((a_1 + g\cdot a_2 + f(g,h), gh))\\ &= (a_1 + g\cdot a_2 + f(g,h) + f_1(gh), gh))\\ &= (a_1 + f_1(g) + g\cdot(a_2 + f_1(h)) + f'(g,h), gh)\\ &= (a_1 + f_1(g), g)(a_2 + f_1(h), h) = \beta((a_1,g))\beta((a_2,h))\end{aligned}$$

shows that β is an isomorphism from E_f to $E_{f'}$.

The restriction of β to A is given by $\beta((a,1)) = (a + f_1(1), 1) = (a, 1)$, so β is the identity map on A. Similarly β is the identity map on the second component of (a, g), so β induces the identity map on the quotient G. It follows that β defines an equivalence between the extensions E_f and $E_{f'}$. This shows that the equivalence class of the extension E_f depends only on the cohomology class of f in $H^2(G, A)$.

We summarize this discussion in the following theorem.

Theorem 36. Let A be a G-module. Then

(1) A function $f : G \times G \to A$ is a normalized factor set of some extension E of G by A (with conjugation given by the G-module action on A) if and only if f is a normalized 2-cocycle in $Z^2(G, A)$.

(2) There is a bijection between the equivalence classes of extensions E as in (1) and the cohomology classes in $H^2(G, A)$. The bijection takes an extension E into the class of a normalized factor set f for E associated to any normalized section μ of G into E, and takes a cohomology class c in $H^2(G, A)$ to the extension E_f defined by the extension (37) for any normalized cocycle f in the class c.

(3) Under the bijection in (2), split extensions correspond to the trivial cohomology class.

Corollary 37. Every extension of G by the abelian group A splits if and only if $H^2(G, A) = 0$.

Corollary 38. If A is a finite abelian group and $(|A|, |G|) = 1$ then every extension of G by A splits.

Proof: This follows immediately from Corollary 29 in Section 2.

We can use Corollary 38 to prove the same result without the restriction that A be an abelian group.

Theorem 39. (*Schur's Theorem*) If E is any finite group containing a normal subgroup N whose order and index are relatively prime, then N has a complement in E.

Remark: Recall that a subgroup whose order and index are relatively prime is called a *Hall subgroup*, so Schur's Theorem says that every normal Hall subgroup has a complement that splits the group as a semidirect product.

Proof: We use induction on the order of E. Since we may assume $N \neq 1$, let p be a prime dividing $|N|$ and let P be a Sylow p-subgroup of N. Let E_0 be the normalizer in E of P and let $N_0 = N \cap E_0$. By Frattini's Argument (Proposition 6 in Section 6.1) $E = E_0N$. It follows from the Second Isomorphism Theorem that N_0 is a (normal) Hall subgroup of E_0 and $|E_0 : N_0| = |E : N|$ (cf. Exercise 10 of Section 3.3).

If $E_0 < E$, then by induction applied to N_0 in E_0 we obtain that E_0 contains a complement K to N_0. Since $|K| = |E_0 : N_0|$, K is also a complement to N in E, as needed. Thus we may assume $E_0 = E$, i.e., P is normal in E.

Since the center of P, $Z(P)$, is characteristic in P, it is normal in E (cf. Section 4.4). If $Z(P) = N$, then N is abelian and the theorem follows from Corollary 38. Thus we may assume $Z(P) \neq N$. Let bars denote passage to the quotient group $E/Z(P)$. Then $\overline{N}$ is a normal Hall subgroup of $\overline{E}$. By induction it has a complement $\overline{K}$ in $\overline{E}$. Let E_1 be the complete preimage of $\overline{K}$ in E. Then $|E_1| = |\overline{K}||Z(P)| = |E/N||Z(P)|$, so $Z(P)$ is a normal Hall subgroup of E_1. By induction $Z(P)$ has a complement in E_1 which is seen by order considerations to also be a complement to N in E. This completes the proof.

Examples

(1) If $G = Z_2$ and $A = \mathbb{Z}/2\mathbb{Z}$ then G acts trivially on A and so $H^2(G, A) = A^G/NA = \mathbb{Z}/2\mathbb{Z}$ by the computation of the cohomology of cyclic groups in Section 2, so by Theorem 36 there are precisely two inequivalent extensions of G by A. These are the cyclic group of order 4 and the Klein 4-group, the latter being split and hence corresponding to the trivial class in H^2.

(2) If $G = \langle g \rangle \cong Z_2$ and $A = \langle a \rangle \cong \mathbb{Z}/4\mathbb{Z}$ is a group of order 4 on which G acts trivially, then $H^2(G, A) = A/2A \cong \mathbb{Z}/2\mathbb{Z}$ by the computation of the cohomology of cyclic groups. As in the previous example there are two inequivalent extensions of G by A; evidently these are the groups Z_8 and $Z_4 \times Z_2$, the latter split extension corresponding to the trivial cohomology class.

If $E = \langle r \rangle \times \langle s \rangle$ denotes the split extension of G by A, where $|r| = 4$ and $|s| = 2$, then $\mu_i(g) = r^i s$ for $i = 0, \dots, 3$ give the four normalized sections of G in E. The sections μ_0, μ_2 both give the zero factor set f. The sections μ_1, μ_3 both give the factor set f' with $f'(g, g) = a^2 \in A$. Both f and f' give normalized 2-cocycles lying in the trivial cohomology class of $H^2(G, A)$. The extension E_f corresponding to the zero 2-cocycle f is the group with the elements $(a, 1)$ and $(1, g)$ as the usual generators (of orders 4 and 2, respectively) for $Z_4 \times Z_2$. In $E_{f'}$, however, $(a, 1)$ has order 4 but so does $(1, g)$ since $(1, g)^2 = (f'(g, g), g^2) = (a^2, 1)$. The 2-cocycles f and f' differ by the coboundary f_1 with $f_1(1) = 1$ and $f_1(g) = r$. The isomorphism $\beta(a, g) = (a + f_1(g), g)$ from E_f to $E_{f'}$ maps the generators $(a, 1)$ and $(1, g)$ of E_f to the generators $(a, 1)$ and (a, g) of $E_{f'}$ and gives the explicit equivalence of these two extensions.

The situation where G acts on A by inversion is handled in Exercise 3.

(3) Suppose $G = Z_2$ and A is the Klein 4-group. If G acts nontrivially on A then G interchanges two of the nonidentity elements, say a and b, of A and fixes the third nonidentity element c. Then $A^G = NA = \{1, c\}$ and so $H^2(G, A) = 0$, and so every extension E of G by A splits. This can be seen directly, as follows. Since the action is nontrivial, such a group must be nonabelian, hence must be D_8. From the lattice of D_8 in Section 2.5 one sees that for each Klein 4-group there is a subgroup of order 2 in D_8 not contained in the 4-group and that subgroup splits the extension.

If G acts trivially on A then $H^2(G, A) = A/2A \cong A$, so there are 4 inequivalent extensions of G by A in this case. These are considered in Exercise 1.

Example: (Groups of Order 8 and $H^2(Z_2 \times Z_2, \mathbb{Z}/2\mathbb{Z})$)

Let $G = \{1, a, b, c\}$ be the Klein 4-group and let $A = \mathbb{Z}/2\mathbb{Z}$. The 2-group G must act trivially on A. The elements of $H^2(G, A)$ classify extensions E of order 8 which has a quotient group by some Z_2 subgroup that is isomorphic to the Klein 4-group. Although there are, up to group isomorphism, only four such groups, we shall see that there are *eight* inequivalent extensions.

Since $G \times G$ has 16 elements, we have $|C^2(G, A)| = 2^{16}$. The cocycle condition (26) here reduces to

$$f(g, h) + f(gh, k) = f(h, k) + f(g, hk) \qquad \text{for all } g, h, k \in G. \tag{17.38}$$

The following relations hold for the subgroup $Z^2(G, A)$ of cocycles:

(1) $f(g, 1) = f(1, g) = f(1, 1)$, for all $g \in G$

(2) $f(g, 1) + f(g, a) + f(g, b) + f(g, c) = 0$, for all $g \in G$

(3) $f(1, h) + f(a, h) + f(b, h) + f(c, h) = 0$, for all $h \in G$.

The first of these come from (38) by setting $h = k = 1$ and by setting $g = h = 1$. The other two relations come from (38) by setting $g = h$ and $h = k$, respectively, using relations (1) and (2). It follows that every 2-cocycle f can be represented by a vector $(\alpha, \beta, \gamma, \delta, \epsilon)$ in $\mathbb{F}_2$ where

$$\alpha = f(1, g) = f(g, 1), \ \text{for all } g \in G,$$
$$\beta = f(a, a), \quad \gamma = f(a, b), \quad \delta = f(b, a), \quad \epsilon = f(b, b)$$

because the relations above then determine the remaining values of f:

$$f(a, c) = \alpha + \beta + \gamma \qquad f(b, c) = \alpha + \delta + \epsilon \qquad f(c, a) = \alpha + \beta + \delta$$
$$f(c, b) = \alpha + \gamma + \epsilon \qquad f(c, c) = \alpha + \beta + \gamma + \epsilon.$$

It follows that $|Z^2(G, A)| \leq 2^5$. Although one could eventually show that every function satisfying these relations is a 2-cocycle (hence the order is exactly 32), this will follow from other considerations below.

A cocycle f is a coboundary if there is a function $f_1 : G \to A$ such that

$$f(g, h) = f_1(h) - f_1(gh) + f_1(g), \qquad \text{for all } g, h \in G.$$

This coboundary condition is easily seen to be equivalent to the conditions:

(i) $f(g, 1) = f(1, g) = f(g, g)$ for all $g \in G$, and

(ii) $f(g, h) = f(g', h')$ whenever g, h are distinct nonidentity elements and so are g', h'.

These relations are equivalent to $\alpha = \beta = \epsilon$ and $\gamma = \delta$. Thus $B^2(G, A)$ consists of the vectors $(\alpha, \alpha, \gamma, \gamma, \alpha)$, and so $H^2(G, A)$ has dimension at most 3 (i.e., order at most $2^3 = 8$). It is easy to see that $\{(0, \beta, \gamma, 0, \epsilon)\}$ with β, γ, and ϵ in $\mathbb{F}_2$ gives a set of representatives for $Z^2(G, A)/B^2(G, A)$, and each of these representative cocycles is normalized. We now prove $|H^2(G, A)| = 8$ (and also that $|Z^2(G, A)| = 2^5$) by explicitly exhibiting eight inequivalent group extensions.

Suppose E is an extension of G by A, where for simplicity we assume $A \leq E$. If $\mu : G \to E$ is a section, the factor set for E associated to μ satisfies

$$\mu(g)\mu(h) = f(g, h)\mu(gh).$$

The group E is generated by $\mu(a)$, $\mu(b)$ and A, and A is contained in the center of E since G acts trivially on A. Hence E is abelian if and only if $\mu(a)\mu(b) = \mu(b)\mu(a)$, which by the relation above occurs if and only if $f(a, b) = f(b, a)$. If g is a nonidentity element in G, we also see from the relation above that $\mu(g)$ is an element of order 2 in E if and only if $f(g, g) = 0$. Because A is contained in the center of E, both elements in any nonidentity coset $A\mu(g)$ have the same order (either 2 or 4).

There are four groups of order 8 containing a normal subgroup of order 2 with quotient group isomorphic to the Klein 4-group: $Z_2 \times Z_2 \times Z_2$, $Z_4 \times Z_2$, D_8, and Q_8.

The group $E \cong Z_2 \times Z_2 \times Z_2$ is the split extension of G by A and has $f = 0$ as factor set.

When $E \cong Q_8$, in the usual notation for the quaternion group $A = \langle -1 \rangle$. In this (nonabelian) group every nonidentity coset consists of elements of order 4, and this property is unique to Q_8, so the resulting factor set f satisfies $f(g, g) \neq 0$ for all nonidentity elements in G.

When $E \cong Z_4 \times Z_2 = \langle x \rangle \times \langle y \rangle$ we must have $A = \langle x^2 \rangle$. The cosets Ax and Axy both consist of elements of order 4, and the coset Ay consists of elements of order 2, so exactly one of $\mu(a)$, $\mu(b)$ or $\mu(c)$ is an element of order 2 and the other two must be of order 4. This suggests three homomorphisms from E to G, defined on generators by

$$\begin{aligned} \pi_1(y) &= a & \pi_1(x) &= b \\ \pi_2(y) &= b & \pi_2(x) &= a, \\ \pi_3(y) &= c & \pi_3(x) &= a \end{aligned}$$

Each of these homomorphisms maps surjectively onto G, has A as kernel, and has $\mu(a)$ (respectively, $\mu(b)$, $\mu(c)$) an element of order 2 in E. Any isomorphism of E with itself that is the identity on A must take the unique nonidentity coset Ay of A consisting of elements of order 2 to itself. Hence any extension equivalent to the extension E_1 defined by π_1 also maps y to a (since the equivalence is the identity on G). It follows that the three extensions defined by π_1, π_2 and π_3 are inequivalent.

The situation when $E \cong D_8 = \langle r, s \rangle$ is similar. In this case $A = \langle r^2 \rangle$, the cosets As and Asr consist of elements of order 2, and the coset Ar consists of elements of order 4. In this case exactly one of $\mu(a)$, $\mu(b)$ or $\mu(c)$ is an element of order 4 and the other two are of order 2, suggesting the three homomorphisms defined on generators by

$$\begin{aligned} \pi_1(r) &= a & \pi_1(s) &= b \\ \pi_2(r) &= b & \pi_2(s) &= a, \\ \pi_3(r) &= c & \pi_3(s) &= a \end{aligned}$$

As before, the corresponding extensions are inequivalent.

The existence of 8 inequivalent extensions of G by A proves that $|H^2(G, A)| = 8$, and hence that these are a complete list of all the inequivalent extensions. In particular, the extension $E_1' \cong Z_4 \times Z_2$ defined by the homomorphism π_1' mapping y to a and x to c must be equivalent to the extension E_1 above (and similarly for the other two extensions isomorphic to $Z_4 \times Z_2$ and the three extensions for D_8). This proves the existence of certain outer automorphisms for these groups, cf. Exercise 9.

Remark: For any prime p the cohomology groups of the elementary abelian group E_{p^m} with coefficients in the finite field $\mathbb{F}_p$ may be determined by relating them to the cohomology groups of the factors in the direct product as mentioned at the end of Section 2. In general, $H^2(E_{p^m}, \mathbb{F}_p)$ is a vector space over $\mathbb{F}_p$ of dimension $\frac{1}{2}m(m+1)$. When $p = 2$ and $m = 2$ this is the result $H^2(Z_2 \times Z_2, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3$ above.

Crossed Product Algebras and the Brauer Group

Suppose F is a field. Recall that an F-algebra B is a ring containing the field F in its center and the identity of B is the identity of F, cf. Section 10.1.

Definition. An F-algebra A is said to be *simple* if A contains no nontrivial proper (two sided) ideals. A *central simple F-algebra* A is a simple F-algebra whose center is F.

Among the easiest central simple F-algebras are the matrix algebras $M_n(F)$ of $n \times n$ matrices with coefficients in F.

If K/F is a finite Galois extension of fields with Galois group $G = \mathrm{Gal}(K/F)$, then we can use the normalized 2-cocycles in $Z^2(G, K^\times)$ to construct certain central simple K-algebras. The construction of these algebras from 2-cocycles and their classification in terms of $H^2(G, K^\times)$ (cf. Theorem 42 below) are important applications of cohomological methods in number theory. Their construction in the case when G is cyclic was one of the precursors leading to the development of abstract cohomology.

Suppose $f = \{a_{\sigma,\tau}\}_{\sigma,\tau \in G}$ is a normalized 2-cocycle in $Z^2(G, K^\times)$. Let B_f be the vector space over L having basis u_σ for $\sigma \in G$:

$$B_f = \left\{ \sum_{\sigma \in G} \alpha_\sigma u_\sigma \mid \alpha_\sigma \in K \right\}. \tag{17.39}$$

Define a multiplication on B_f by

$$u_\sigma \alpha = \sigma(\alpha) u_\sigma \qquad u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau} \tag{17.40}$$

for $\alpha \in L$ and $\sigma, \tau \in G$. The second equation shows that the $a_{\sigma,\tau}$ give a "factor set" for the elements u_σ in B_f and is one reason this terminology is used. Using this multiplication we find

$$(u_\sigma u_\tau)u_\rho = a_{\sigma,\tau} a_{\sigma\tau,\rho} u_{\sigma\tau\rho} \qquad \text{and} \qquad u_\sigma (u_\tau u_\rho) = \sigma(a_{\tau,\rho}) a_{\sigma,\tau\rho} u_{\sigma\tau\rho}.$$

Since $a_{\sigma,\tau} a_{\sigma\tau,\rho} = \sigma(a_{\tau,\rho}) a_{\sigma,\tau\rho}$ is the multiplicative form of the cocycle condition (26), it follows that the multiplication defined in (40) is associative.

Since the cocycle is normalized we have $a_{1,\sigma} = a_{\sigma,1} = 1$ for all $\sigma \in G$ and it follows from (40) that the element u_1 is an identity in B_f. Identifying K with the elements αu_1 in B_f, we see that B_f is an F-algebra containing the field K and having dimension n^2 over F if $n = [K : F] = |G|$.

Proposition 40. The F-algebra B_f with K-vector space basis u_σ in (39) and multiplication defined by (40) is a central simple F-algebra.

Proof: It remains to show that the center of B_f is F and that B_f contains no nonzero proper ideals. Suppose $x = \sum_{\sigma \in G} \alpha_\sigma u_\sigma$ is an element in the center of B_f. Then $x\beta = \beta x$ for $\beta \in K$ shows that $\sigma(\beta) = \beta$ if $\alpha_\sigma \neq 0$. Since there is an element $\beta \in K$ not fixed by σ for any $\sigma \neq 1$, this shows that $\alpha_\sigma = 0$ for all $\sigma \neq 1$, so $x = \alpha_1 u_1$. Then $x u_\tau = u_\tau x$ if and only if $\tau(\alpha_1) = \alpha_1$, so if this is true for all τ then we must have $\alpha_1 = a \in K$. Hence $x = a u_1$ and the center of B_f is F.

To show that B_f is simple, suppose I is a nonzero ideal in B_f and let

$$x = \alpha_{\sigma_1} u_{\sigma_1} + \cdots + \alpha_{\sigma_m} u_{\sigma_m}$$

be a nonzero element of I with the minimal number m of nonzero terms. If $m > 1$ there is an element $\beta \in K^\times$ with $\sigma_m(\beta) \neq \sigma_{m-1}(\beta)$. Then the element $x - \sigma_m(\beta)\, x\, \beta^{-1}$ would be an element of the ideal I with the nonzero element $(1 - \sigma_m(\beta)\, \sigma_{m-1}(\beta)^{-1}) \alpha_{\sigma_{m-1}}$ as coefficient of $u_{\sigma_{m-1}}$, and would have fewer nonzero terms than x since the coefficient of u_{σ_m} is 0. It follows that $m = 1$ and $x = \alpha u_\sigma$ for some $\alpha \in K$ and some σ. This element is a unit, with inverse $\sigma^{-1}(\alpha^{-1}) u_{\sigma^{-1}}$, so $I = B_f$, completing the proof.

Definition. The central simple F-algebra B_f defined by (39) and (40) is called the *crossed product algebra* for the factor set $\{a_{\sigma,\tau}\}$.

If $f' = a'_{\sigma,\tau}$ is a normalized cocycle in the same cohomology class in $H^2(G, K^\times)$ as $a_{\sigma,\tau}$ then there are elements $b_\sigma \in K^\times$ with

$$a'_{\sigma,\tau} = a_{\sigma,\tau}(\sigma(b_\tau) b_{\sigma\tau}^{-1} b_\sigma)$$

(the multiplicative form of the coboundary condition (27)). If $B_{f'}$ is the F-algebra with K-basis v_σ defined from this cocycle as in (39) and (40), then the K-vector space homomorphism φ defined by mapping u'_σ to $b_\sigma u_\sigma$ satisfies

$$\begin{aligned}\varphi(u'_\sigma u'_\tau) &= \varphi(a'_{\sigma,\tau} u'_{\sigma\tau}) = a'_{\sigma,\tau} b_{\sigma\tau}\, u_{\sigma\tau} = b_\sigma\, \sigma(b_\tau)\, u_\sigma u_\tau \\ &= (b_\sigma u_\sigma)(b_\tau u_\tau) = \varphi(u'_\sigma)\varphi(u'_\tau).\end{aligned}$$

It follows that φ is an F-algebra isomorphism from $B_{f'}$ to B_f.

We have shown that every cohomology class c in $H^2(G, K^\times)$ defines an isomorphism class of central simple F-algebras, namely the isomorphism class of any crossed product algebra for a normalized cocycle $\{a_{\sigma,\tau}\}$ representing the class c. The next result shows that the trivial cohomology class corresponds to the isomorphism class containing $M_n(F)$.

Proposition 41. The crossed product algebra for the trivial cohomology class in $H^2(G, K^\times)$ is isomorphic to the matrix algebra $M_n(F)$ where $n = [K : F]$.

Proof: If $\alpha \in K$ then multiplication by α defines a linear transformation T_α of K viewed as an n-dimensional vector space over F. Similarly, every automorphism $\sigma \in G$ defines an F-linear transformation T_σ of K, and we may view both T_α and T_σ as elements of $M_n(F)$ by choosing a basis for K over F. If B_0 denotes the crossed product algebra for the trivial factor set ($a_{\sigma,\tau} = 1$ for all $\sigma, \tau \in G$), consider the additive map $\varphi : B_0 \to M_n(F)$ defined by $\varphi(\alpha u_\sigma) = T_\alpha T_\sigma$. Since $T_{a\alpha} = aT_\alpha$ for $a \in F$, the map φ is an F-vector space homomorphism. If $x \in K$, we have

$$T_\sigma T_\alpha(x) = T_\sigma(\alpha x) = \sigma(\alpha x) = \sigma(\alpha)\sigma(x) = T_{\sigma(\alpha)}T_\sigma,$$

so $T_\sigma T_\alpha = T_{\sigma(\alpha)}T_\sigma$ as linear transformations on K. It then follows from $u_\sigma u_\tau = u_{\sigma\tau}$ that

$$\varphi((\alpha u_\sigma)(\beta u_\tau)) = \varphi(\alpha\sigma(\beta)\,u_{\sigma\tau}) = T_{\alpha\sigma(\beta)}T_{\sigma\tau} = T_\alpha T_{\sigma(\beta)}T_\sigma T_\tau$$
$$= T_\alpha T_\sigma T_\beta T_\tau = \varphi(\alpha u_\sigma)\,\varphi(\beta u_\tau)$$

which shows that φ is an F-algebra homomorphism from B_0 to $M_n(F)$. Since $\ker\varphi$ is an ideal in B_0 and $\varphi \neq 0$, it follows from Proposition 40 that $\ker\varphi = 0$ and φ is an injection. Since both B_0 and $M_n(F)$ have dimension n^2 as vector spaces over F, it follows that φ is an F-algebra isomorphism, proving the proposition.

Example

If $K = \mathbb{C}$ and $F = \mathbb{R}$, then $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is of order 2 and generated by complex conjugation τ. We have $|H^2(G, \mathbb{C}^\times)| = 2$. The central simple $\mathbb{R}$-algebra B_0 corresponding to the trivial class is $\mathbb{C}u_1 \oplus \mathbb{C}u_\tau$ with $u_\tau(a+bi) = (a-bi)u_\tau$ and $u_\tau^2 = u_1$. This is isomorphic to the matrix algebra $M_2(\mathbb{R})$ under the map

$$\varphi((a+bi)u_1 + (c+di)u_\tau) = aI + bT_i + cT_\tau + dT_iT_\tau = \begin{pmatrix} a+c & -b+d \\ b+d & a-c \end{pmatrix}.$$

A normalized cocycle f representing the nontrivial cohomology class is defined by the values $a_{1,1} = a_{1,\tau} = a_{\tau,1} = 1$ and $a_{\tau,\tau} = -1$. The corresponding central simple $\mathbb{R}$-algebra B_f is given by $\mathbb{C}v_1 \oplus \mathbb{C}v_\tau$. The element v_1 is the identity of B_f, and we have the relations $v_\tau(a+bi) = (a-bi)v_\tau$ and $v_\tau^2 = -v_1$. Letting $v_1 = 1$ and $v_\tau = j$ we see that B_f is isomorphic as an $\mathbb{R}$-algebra to the real Hamilton Quaternions $\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.

There is a rich theory of simple algebras and we mention without proof the following results. Let A be a central simple F-algebra of finite dimension over F.

I. If $F \subseteq B \subseteq A$ where B is a simple F-algebra define the *centralizer* B^c of B in A to be the elements of A that commute with all the elements of B. Define the *opposite algebra* B^{opp} to be the set B with opposite multiplication, i.e., the product b_1b_2 in B^{opp} is given by the product b_2b_1 in B. Both B^c and B^{opp} are simple F-algebras and we have
   a. $(\dim_F B)(\dim_F B^c) = \dim_F A$
   b. $A \otimes_F B^{opp} \cong M_r(B^c)$ as F-algebras, where $r = \dim_F B$
   c. $B \otimes_F B^c \cong A$ if B is a central simple F-algebra.

II. If A' is an Artinian (satisfies D.C.C. on left ideals) simple F-algebra, then $A \otimes_F A'$ is an Artinian simple F-algebra with center $(A')^c$.

III. We have $A \cong M_r(\Delta)$ for some division ring Δ whose center is F and some integer $r \geq 1$. The division ring Δ and r are uniquely determined by A. The same statement holds for any Artinian simple F-algebra.

The last result is part of Wedderburn's Theorem described in greater detail in the following chapter.

Definition. If A is a central simple F-algebra then a field L containing F is said to *split* A if $A \otimes_F L \cong M_m(L)$ for some $m \geq 1$.

It follows from (II) that every maximal commutative subalgebra of Δ is a field E with $E = E^c = E^{opp}$; if $[E : F] = m$ we obtain $\dim_F \Delta = m^2$. Applying (II) to $A = \Delta$ and $B = E$ we also see that $\Delta \otimes_F E \cong M_m(E)$. It can also be shown that a maximal subfield E of the central simple F-algebra A also satisfies $E = E^c = E^{opp}$ and so again by (II) it follows that $A \otimes_F E \cong M_r(E)$ ($r^2 = \dim_F A$).

If $A = M_r(\Delta)$ then the field L splits A if and only if L splits Δ, as follows. If $\Delta \otimes_F L \cong M_n(L)$ then

$$A \otimes_F L \cong M_r(\Delta) \otimes_F L \cong M_r(\Delta \otimes_F L) \cong M_r(M_n(L)) \cong M_{rn}(L).$$

Conversely if $A \otimes_F L \cong M_n(L)$ then

$$M_n(L) \cong M_r(\Delta) \otimes_F L \cong M_r(\Delta \otimes_F L).$$

By (II) and (III), $\Delta \otimes_F L \cong M_s(\Delta')$ for some division ring Δ'. Together with the previous isomorphism, the uniqueness statement in (III) shows that $\Delta' \cong L$ and then the isomorphism $\Delta \otimes_F L \cong M_s(L)$ shows that L splits Δ.

We see from the discussion above that a maximal commutative subfield of Δ splits both Δ and $A \cong M_r(\Delta)$ for any $r \geq 1$. It is not too difficult to show from this that every central simple F-algebra of finite dimension over F can be split by a finite Galois extension of F.

Applying (I) by taking A to be the crossed product algebra B_f and taking $B = K$ shows that $K = K^c = K^{opp}$ and $B_f \otimes_F K \cong M_n(K)$. In particular, the crossed product algebras B_f are always split by K.

Example

In the example of the Hamilton Quaternions above we have $B_f \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. We have $B_f \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} + \mathbb{C}i + \mathbb{C}j + \mathbb{C}k$ and an explicit isomorphism φ to $M_2(\mathbb{C})$ is given by

$$\varphi(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \qquad \varphi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and extending $\mathbb{C}$-linearly.

By (III) every central simple F-algebra A is isomorphic as an F-algebra to $M_r(\Delta)$ for some division ring Δ uniquely determined up to F-isomorphism, called the *division ring part* of A.

Definition. Two central simple F-algebras A and B are *similar* if $A \cong M_r(\Delta)$ and $B \cong M_s(\Delta)$ for the same division ring Δ, i.e., if A and B have the same division ring parts.

Let $[A]$ denote the similarity class of A. By (II), if A and B are central simple F-algebras then $A \otimes_F B$ is again a central simple F-algebra, so we may define a multiplication on similarity classes by $[A][B] = [A \otimes_F B]$. The class $[F]$ is an identity for this multiplication and associativity of the tensor product shows that the multiplication is associative. By (Ib) applied with $B = A$ (so then $B^c = F$ since A is central) we have $[A][A^{opp}] = [F]$, so inverses exist with this multiplication.

Definition. The group of similarity classes of central simple F-algebras with multiplication $[A][B] = [A \otimes_F B]$ is called the *Brauer group* of F and is denoted $Br(F)$.

If L is any extension field of F then by (II) the algebra $A \otimes_F L$ is a central simple L-algebra. It is easy to check that the map $[A] \to [A \otimes_F L]$ is a well defined homomorphism from $Br(F)$ to $Br(L)$. The kernel of this homomorphism consists of the classes of the algebras A with $A \otimes_F L \cong M_m(L)$ for some $m \geq 1$, i.e., the algebras A that are split by L.

Definition. If L/F is a field extension then the *relative Brauer group* $Br(L/F)$ is the group of similarity classes of central simple F-algebras that are split by L. Equivalently, $Br(L/F)$ is the kernel of the homomorphism $[A] \to [A \otimes_F L]$ from $Br(F)$ to $Br(L)$.

The following theorem summarizes some major results in this area and shows the fundamental connection between Brauer groups and the crossed product algebras constructed above.

Theorem 42. Suppose K/F is a Galois extension of degree n with $G = \mathrm{Gal}(K/F)$.

(1) The central simple F-algebra A with $\dim_F A = n^2$ is split by K if and only if $A \otimes_F K \cong M_n(K)$ if and only if A is isomorphic to a crossed product algebra B_f as in (39) and (40).

(2) There is a bijection between the F-isomorphism classes of central simple F-algebras A with $A \otimes_F K \cong M_n(K)$ and the elements of $H^2(G, K^\times)$. Under this bijection the class $c \in H^2(G, K^\times)$ containing the normalized cocycle f corresponds to the isomorphism class of the crossed product algebra B_f defined in (39) and (40), and the trivial cohomology class corresponds to $M_n(F)$.

(3) Every central simple F-algebra of finite dimension over F and split by K is similar to one of dimension n^2 split by K. The bijection in (2) also establishes a bijection between $Br(K/F)$ and $H^2(G, K^\times)$ which is also an isomorphism of groups.

(4) There is a bijection between the collection of F-isomorphism classes of central simple division algebras over F that are split by K and $H^2(G, K^\times)$.

As previously mentioned, every central simple F-algebra of finite dimension over F can be split by some finite Galois extension of F, and it follows that

$$Br(F) = \bigcup_K Br(K/F)$$

where the union is over all finite Galois extensions of F. It follows that there is a bijection between $Br(F)$ and $H^2(\mathrm{Gal}(F^s/F), (F^s)^\times)$ where F^s denotes a separable algebraic closure of F. Here $\mathrm{Gal}(F^s/F)$ is considered as a profinite group and the cohomology group refers to continuous Galois cohomology.

One consequence of this result and Theorem 42 is that a full set of representatives for the F-isomorphism classes of central simple division algebras Δ over F can be obtained from the division algebra parts of the crossed product algebras for finite Galois extensions of F. Those division algebras that are split over K occur for the crossed product algebras for K/F.

Example

Since $H^2(\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q), \mathbb{F}_{q^d}^\times) = 0$ (cf. Exercise 10), we have $Br(\mathbb{F}_{q^d}/\mathbb{F}_q) = 0$ and hence also $Br(\mathbb{F}_q) = 0$. As a consequence, every finite division algebra is a field (cf. Exercise 13 in Section 13.6 for a direct proof), and every finite central simple algebra $\mathbb{F}_q$-algebra is isomorphic to a full matrix ring $M_r(\mathbb{F}_q)$.

EXERCISES

1. Let $A = \{1, a, b, c\}$ be the Klein 4-group and let $G = \langle g \rangle$ be the cyclic group of order 2 acting trivially on A.
   (a) Prove that $|C^2(G, A)| = 2^8$.
   (b) Show that coboundaries are constant functions, and deduce that $|B^2(G, A)| = 4$.
   (c) Use the cocycle condition to show that $|Z^2(G, A)| \leq 2^4$.
   (d) If $E = Z_4 \times Z_2 = \langle x \rangle \times \langle y \rangle$, prove that the extensions $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ defined by $\pi(x) = g$, $\pi(y) = 1$ and $\iota_1(a) = x^2$, $\iota_1(b) = y$ (respectively, $\iota_2(b) = x^2$, $\iota_2(a) = y$, and $\iota_3(c) = x^2$, $\iota_3(a) = y$), together with the split extension $Z_2 \times Z_2 \times Z_2$ give 4 inequivalent extensions of Z_2 by the Klein 4-group. Deduce that $H^2(G, A)$ has order 4 by explicitly exhibiting the corresponding cocycles.
2. Let $A = \mathbb{Z}/4\mathbb{Z}$ and let G be the cyclic group of order 2 acting trivially on A.
   (a) Prove that $|C^2(G, A)| = 2^8$.
   (b) Use the coboundary condition to show that $|B^2(G, A)| = 2^3$.
   (c) Use the cocycle condition to show that $|Z^2(G, A)| \leq 2^4$.
   (d) Show that $|H^2(G, A)| = 2$ by exhibiting two inequivalent extensions of G by A and their corresponding cocycles.
3. Let $A = \mathbb{Z}/4\mathbb{Z}$ and let G be the cyclic group of order 2 acting by inversion on A.
   (a) Show that there are four coboundaries and that only the zero coboundary is normalized.
   (b) Prove by a direct computation of cocycle and coboundary groups that $|H^2(G, A)| = 2$.
   (c) Exhibit two distinct cohomology classes and their corresponding extension groups.
   (d) Show that for a given extension of G by A with extension group isomorphic to D_8 there are four normalized sections, all of which have the zero 2-cocycle as their factor set.
   (e) Show that for a given extension of G by A with extension group isomorphic to Q_8 there are sixteen sections, four of which are normalized, and all of the latter have the same factor set.
4. For a *non*-normalized 2-cocycle f one defines the extension group E_f on the set $A \times G$ by the same binary operation in equation (34). Verify two of the group axioms in this case by showing that identity is now $(-f(1, 1), 1)$ and inverses are given by
$$(a, x)^{-1} = (-x^{-1} \cdot a - f(x^{-1}, x) - f(1, 1), x^{-1}).$$
(Verification of the associative law is essentially the same as for normalized 2-cocycles.) Prove also that the set $A^{**} = \{(a - f(1, 1), 1) \mid a \in A\}$ is a subgroup of E_f and the map $\iota^{**} : a \mapsto (a - f(1, 1), 1)$ is an isomorphism from A to A^{**}. Show that this extension E_f, with the injection ι^{**} and the usual projection map π^* onto G, is equivalent to an extension derived from a normalized cocycle in the same class as f.
5. Show that the set of equivalences of a given extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ with itself form a group under composition, and that this group is isomorphic to the stability group

$\mathrm{Stab}(1 \trianglelefteq \iota(A) \trianglelefteq E)$. (Thus Proposition 31 implies $Z^1(G, A)$ is the group of equivalences of the extension with itself).

6. *(Gaschütz's Theorem)* Let p be a prime, let A be an abelian normal p-subgroup of a finite group G, and let P be a Sylow p-subgroup of G. Prove that G is a split extension of G/A by A if and only if P is a split extension of P/A by A. (Note that $A \leq P$ by Exercise 37 in Section 4.5). [Use Sylow's Theorem to show if G splits over A then so too does P. Conversely, show that a normalized 2-cocycle associated to the extension of P/A by A via Theorem 36 is the image of a normalized 2-cocycle in $H^2(G/A, A)$ under the restriction homomorphism $\mathrm{Res} : H^2(G/A, A) \to H^2(P/A, A)$. Then use Proposition 26 and the fact that multiplication by $|G : P|$ is an automorphism of A.]

7. (a) Prove that $H^2(A_4, \mathbb{Z}/2\mathbb{Z}) \neq 0$ by exhibiting a nonsplit extension of A_4 by a cyclic group of order 2. [See Exercise 11, Section 4.5.]
   (b) Prove that $H^2(A_5, \mathbb{Z}/2\mathbb{Z}) \neq 0$ by showing that $SL_2(\mathbb{F}_5)$ is a nonsplit extension of A_5 by a cyclic group of order 2. [Use Propositions 21 and 23 in Section 4.5.]

8. The *Schur multiplier* of a finite group G is defined as the group $H^2(G, \mathbb{C}^\times)$, where the multiplicative group $\mathbb{C}^\times$ of complex numbers is a trivial G-module. Prove that the Schur multiplier is a finite group. [Show that every cohomology class contains a cocycle whose values lie in the n^{th} roots of unity, where $n = |G|$, as follows: If f is any cocycle then by Corollary 27, $f^n \in B^2(G, \mathbb{C}^\times)$. Define $k \in C^2(G, \mathbb{C}^\times)$ by $k(g_1, g_2) = f(g_1, g_2)^{1/n}$ (take any n^{th} roots). Show that $k \in B^2(G, \mathbb{C}^\times)$ and fk^{-1} takes values in the group of n^{th} roots of 1.]

9. Use the classification of the extensions of the Klein 4-group by Z_2 in the example following Theorem 39 to prove the following (in the notation of that example):
   (a) There is an (outer) automorphism of $Z_4 \times Z_2$ which interchanges the cosets Ax and Axy and fixes the coset Ay.
   (b) There is an outer automorphism of D_8 which interchanges the cosets As and Asr and fixes the coset Ar.

10. Suppose $\mathbb{F}_q$ is a finite field with $G = \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q) = \langle \sigma_q \rangle$ where σ_q is the Frobenius automorphism, and let N be the usual norm element for the cyclic group G.
    (a) Use Hilbert's Theorem 90 to prove that $|N(\mathbb{F}_{q^d}^\times)| = (q^d - 1)/(q - 1)$, and deduce that the norm map from $\mathbb{F}_{q^d}$ to $\mathbb{F}_q$ is surjective.
    (b) Prove that $H^n(G, \mathbb{F}_{q^d}^\times) = 0$ for all $n \geq 1$.

§71 The Fundamental Group of a Wedge of Circles

In this section, we define what we mean by a *wedge of circles*, and we compute its fundamental group.

Definition. Let X be a Hausdorff space that is the union of the subspaces $S_1, \dots, S_n$, each of which is homeomorphic to the unit circle S^1. Assume that there is a point p of X such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$. Then X is called the ***wedge of the circles*** $S_1, \dots, S_n$.

Note that each space S_i, being compact, is closed in X. Note also that X can be imbedded in the plane; if C_i denotes the circle of radius i in $\mathbb{R}^2$ with center at $(i, 0)$, then X is homeomorphic to $C_1 \cup \dots \cup C_n$.

Theorem 71.1. *Let X be the wedge of the circles $S_1, \dots, S_n$; let p be the common point of these circles. Then $\pi_1(X, p)$ is a free group. If f_i is a loop in S_i that represents a generator of $\pi_1(S_i, p)$, then the loops $f_1, \dots, f_n$ represent a system of free generators for $\pi_1(X, p)$.*

Proof. The result is immediate if $n = 1$. We proceed by induction on n. The proof is similar to the one given in Example 1 of the preceding section.

Let X be the wedge of the circles $S_1, \dots, S_n$, with p the common point of these circles. Choose a point q_i of S_i different from p, for each i. Set $W_i = S_i - q_i$, and let

$$U = S_1 \cup W_2 \cup \dots \cup W_n \quad \text{and} \quad V = W_1 \cup S_2 \cup \dots \cup S_n.$$

Then $U \cap V = W_1 \cup \dots \cup W_n$. See Figure 71.1. Each of the spaces U, V, and $U \cap V$ is path connected, being the union of path-connected spaces having a point in common.

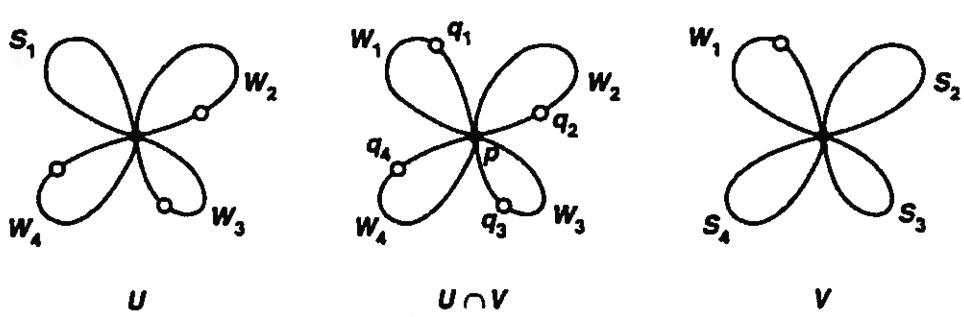


Figure 71.1

The space W_i is homeomorphic to an open interval, so it has the point p as a deformation retract; let $F_i : W_i \times I \to W_i$ be the deformation retraction. The maps F_i fit together to define a map $F : (U \cap V) \times I \to U \cap V$ that is a deformation retraction of $U \cap V$ onto p. (To show that F is continuous, we note that because S_i is a closed subspace of X, the space $W_i = S_i - q_i$ is a closed subspace of $U \cap V$, so that $W_i \times I$

is a closed subspace of $(U \cap V) \times I$. Then the pasting lemma applies.) It follows that $U \cap V$ is simply connected, so that $\pi_1(X, p)$ is the free product of the groups $\pi_1(U, p)$ and $\pi_1(V, p)$, relative to the monomorphisms induced by inclusion.

A similar argument shows that S_1 is a deformation retract of U and $S_2 \cup \cdots \cup S_n$ is a deformation retract of V. It follows that $\pi_1(U, p)$ is infinite cyclic, and the loop f_1 represents a generator. It also follows, using the induction hypothesis, that $\pi_1(V, p)$ is a free group, with the loops $f_2, \ldots, f_n$ representing a system of free generators. Our theorem now follows from Theorem 69.2. ∎

?

We generalize this result to a space X that is the union of *infinitely* many circles having a point in common. Here we must be careful about the topology of X.

Definition. Let X be a space that is the union of the subspaces X_α, for $\alpha \in J$. The topology of X is said to be ***coherent*** with the subspaces X_α provided a subset C of X is closed in X if $C \cap X_\alpha$ is closed in X_α for each α. An equivalent condition is that a set be open in X if its intersection with each X_α is open in X_α.

If X is the union of finitely many closed subspaces $X_1, \ldots, X_n$, then the topology of X is automatically coherent with these subspaces, since if $C \cap X_i$ is closed in X_i, it is closed in X, and C is the finite union of the sets $C \cap X_i$.

Definition. Let X be a space that is the union of the subspaces S_α, for $\alpha \in J$, each of which is homeomorphic to the unit circle. Assume there is a point p of X such that $S_\alpha \cap S_\beta = \{p\}$ whenever $\alpha \neq \beta$. If the topology of X is coherent with the subspaces S_α, then X is called the ***wedge of the circles*** S_α.

In the finite case, the definition involved the Hausdorff condition instead of the coherence condition; in that case the coherence condition followed. In the infinite case, this would no longer be true, so we included the coherence condition as part of the definition. We would include the Hausdorff condition as well, but that is no longer necessary, for it follows from the coherence condition:

Lemma 71.2. *Let X be the wedge of the circles S_α, for $\alpha \in J$. Then X is normal. Furthermore, any compact subspace of X is contained in the union of finitely many circles S_α.*

Proof. It is clear that one-point sets are closed in X. Let A and B be disjoint closed subsets of X; assume that B does not contain p. Choose disjoint subsets U_α and V_α of S_α that are open in S_α and contain $\{p\} \cup (A \cap S_\alpha)$ and $B \cap S_\alpha$, respectively. Let $U = \bigcup U_\alpha$ and $V = \bigcup V_\alpha$; then U and V are disjoint. Now $U \cap S_\alpha = U_\alpha$ because all the sets U_α contain p, and $V \cap S_\alpha = V_\alpha$ because no set V_α contains p. Hence U and V are open in X, as desired. Thus X is normal.

Now let C be a compact subspace of X. For each α for which it is possible, choose a point x_α of $C \cap (S_\alpha - p)$. The set $D = \{x_\alpha\}$ is closed in X, because its intersection with each space S_α is a one-point set or is empty. For the same reason, each *subset* of D is closed in X. Thus D is a closed discrete subspace of X contained in C; since C is limit point compact, D must be finite. ∎

Theorem 71.3. *Let X be the wedge of the circles S_α, for $\alpha \in J$; let p be the common point of these circles. Then $\pi_1(X, p)$ is a free group. If f_α is a loop in S_α representing a generator of $\pi_1(S_\alpha, p)$, then the loops $\{f_\alpha\}$ represent a system of free generators for $\pi_1(X, p)$.*

Proof. Let $i_\alpha : \pi_1(S_\alpha, p) \to \pi_1(X, p)$ be the homomorphism induced by inclusion; let G_α be the image of i_α.

Note that if f is any loop in X based at p, then the image set of f is compact, so that f lies in some finite union of subspaces S_α. Furthermore, if f and g are two loops that are path homotopic in X, then they are actually path homotopic in some finite union of the subspaces S_α.

It follows that the groups $\{G_\alpha\}$ generate $\pi_1(X, p)$. For if f is a loop in X, then f lies in $S_{\alpha_1} \cup \cdots \cup S_{\alpha_n}$ for some finite set of indices; then Theorem 71.1 implies that $[f]$ is a product of elements of the groups $G_{\alpha_1}, \dots, G_{\alpha_n}$. Similarly, it follows that i_β is a monomorphism. For if f is a loop in S_β that is path homotopic in X to a constant, then f is path homotopic to a constant in some finite union of spaces S_α, so that Theorem 71.1 implies that f is path homotopic to a constant in S_β.

Finally, suppose there is a reduced nonempty word

$$w = (g_{\alpha_1}, \dots, g_{\alpha_n})$$

in the elements of the groups G_α that represents the identity element of $\pi_1(X, p)$. Let f be a loop in X whose path-homotopy class is represented by w. Then f is path homotopic to a constant in X, so it is path homotopic to a constant in some finite union of subspaces S_α. This contradicts Theorem 71.1. ∎

The preceding theorem depended on the fact that the topology of X was coherent with the subspaces S_α. Consider the following example:

EXAMPLE 1. Let C_n be the circle of radius $1/n$ in $\mathbb{R}^2$ with center at the point $(1/n, 0)$. Let X be the subspace of $\mathbb{R}^2$ that is the union of these circles; then X is the union of a countably infinite collection of circles, each pair of which intersect in the origin p. However, X is *not* the wedge of the circles C_n; we call X (for convenience) the ***infinite earring***.

One can verify directly that X does not have the topology coherent with the subspaces C_n; the intersection of the positive x-axis with X contains exactly one point from each circle C_n, but it is not closed in X. Alternatively, for each n, let f_n be a loop in C_n that represents a generator of $\pi_1(C_n, p)$; we show that $\pi_1(X, p)$ is *not* a free group with $\{[f_n]\}$ as a system of free generators. Indeed, we show the elements $[f_i]$ do not even *generate* the group $\pi_1(X, p)$.

Consider the loop g in X defined as follows: For each n, define g on the interval $[1/(n+1), 1/n]$ to be the positive linear map of this interval onto $[0, 1]$ followed by f_n. This specifies g on $(0, 1]$; define $g(0) = p$. Because X has the subspace topology derived from $\mathbb{R}^2$, it is easy to see that g is continuous. See Figure 71.2. We show that given n, the element $[g]$ does not belong to the subgroup G_n of $\pi_1(X, p)$ generated by $[f_1], \dots, [f_n]$.

Choose $N > n$, and consider the map $h : X \to C_N$ defined by setting $h(x) = x$ for $x \in C_N$ and $h(x) = p$ otherwise. Then h is continuous, and the induced homomorphism $h_* : \pi_1(X, p) \to \pi_1(C_N, p)$ carries each element of G_n to the identity element. On the other hand, $h \circ g$ is the loop in C_N that is constant outside $[1/(N+1), 1/N]$ and on this interval equals the positive linear map of this interval onto $[0, 1]$ followed by f_N. Therefore, $h_*([g]) = [f_N]$, which *generates* $\pi_1(C_N, p)$! Thus $[g] \notin G_n$.

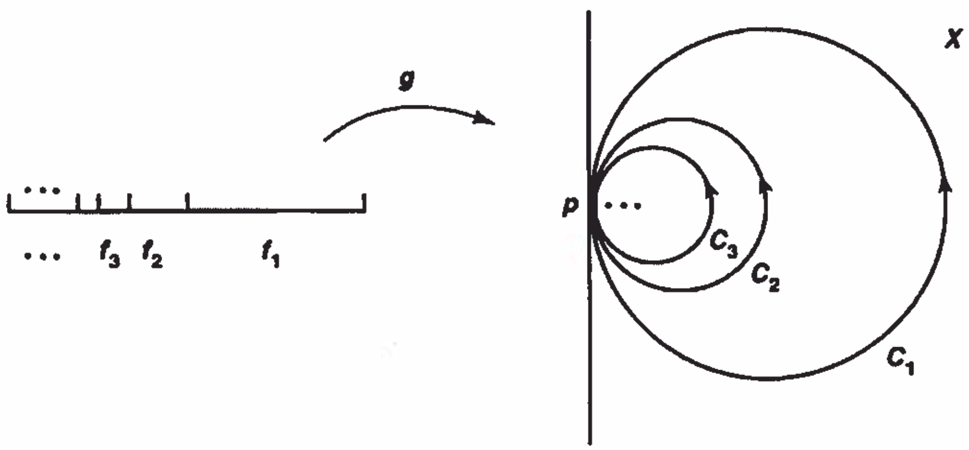


Figure 71.2

In the preceding theorem, we calculated the fundamental group of a space that is an infinite wedge of circles. For later use, we now show that such spaces do exist! (We shall use this result in Chapter 14.)

***Lemma 71.4.** *Given an index set J, there exists a space X that is a wedge of circles S_α for $\alpha \in J$.*

Proof. Give the set J the discrete topology, and let E be the product space $S^1 \times J$. Choose a point $b_0 \in S^1$, and let X be the quotient space obtained from E by collapsing the closed set $P = b_0 \times J$ to a point p. Let $\pi : E \to X$ be the quotient map; let $S_\alpha = \pi(S^1 \times \alpha)$. We show that each S_α is homeomorphic to S^1 and X is the wedge of the circles S_α.

Note that if C is closed in $S^1 \times \alpha$, then $\pi(C)$ is closed in X. For $\pi^{-1}\pi(C) = C$ if the point $b_0 \times \alpha$ is not in C, and $\pi^{-1}\pi(C) = C \cup P$ otherwise. In either case, $\pi^{-1}\pi(C)$ is closed in $S^1 \times J$, so that $\pi(C)$ is closed in X.

It follows that S_α is itself closed in X, since $S^1 \times \alpha$ is closed in $S^1 \times J$, and that π maps $S^1 \times \alpha$ homeomorphically onto S_α. Let π_α be this homeomorphism.

To show that X has the topology coherent with the subspaces S_α, let $D \subset X$ and suppose that $D \cap S_\alpha$ is closed in S_α for each α. Now

$$\pi^{-1}(D) \cap (S^1 \times \alpha) = \pi_\alpha^{-1}(D \cap S_\alpha);$$

the latter set is closed in $S^1 \times \alpha$ because π_α is continuous. Then $\pi^{-1}(D)$ is closed in $S^1 \times J$, so that D is closed in X by definition of the quotient topology. ∎

Exercises

1. Let X be a space that is the union of subspaces $S_1, \dots, S_n$, each of which is homeomorphic to the unit circle. Assume there is a point p of X such that $S_i \cap S_j = \{p\}$ for $i \neq j$.
   (a) Show that X is Hausdorff if and only if each space S_i is closed in X.
   (b) Show that X is Hausdorff if and only if the topology of X is coherent with the subspaces S_i.
   (c) Give an example to show that X need not be Hausdorff. [*Hint:* See Exercises 5 of §36.]
2. Suppose X is a space that is the union of the closed subspaces $X_1, \dots, X_n$; assume there is a point p of X such that $X_i \cap X_j = \{p\}$ for $i \neq j$. Then we call X the ***wedge*** of the spaces $X_1, \dots, X_n$, and write $X = X_1 \vee \cdots \vee X_n$. Show that if for each i, the point p is a deformation retract of an open set W_i of X_i, then $\pi_1(X, p)$ is the external free product of the groups $\pi_1(X_i, p)$ relative to the monomorphisms induced by inclusion.
3. What can you say about the fundamental group of $X \vee Y$ if X is homeomorphic to S^1 and Y is homeomorphic to S^2?
4. Show that if X is an infinite wedge of circles, then X does not satisfy the first countability axiom.
5. Let S_n be the circle of radius n in $\mathbb{R}^2$ whose center is at the point $(n, 0)$. Let Y be the subspace of $\mathbb{R}^2$ that is the union of these circles; let p be their common point.
   (a) Show that Y is not homeomorphic to a countably infinite wedge X of circles, nor to the space of Example 1.
   (b) Show, however, that $\pi_1(Y, p)$ is a free group with $\{[f_n]\}$ as a system of free generators, where f_n is a loop representing a generator of $\pi_1(S_n, p)$.

§72 Adjoining a Two-cell

We have computed the fundamental group of the torus $T = S^1 \times S^1$ in two ways. One involved considering the standard covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ and using the lifting correspondence. Another involved a basic theorem about the fundamental group of a product space. Now we compute the fundamental group of the torus in yet another way.

If one restricts the covering map $p \times p$ to the unit square, one obtains a quotient map $\pi : I^2 \to T$. It maps Bd I^2 onto the subspace $A = (S^1 \times b_0) \cup (b_0 \times S^1)$, which is the wedge of two circles, and it maps the rest of I^2 bijectively onto $T - A$. Thus, T can be thought of as the space obtained by pasting the edges of the square I^2 onto the space A.