Topology Qualifying Examination (661/761) Study Material

Henry Potts-Rubin

Contents

1 Topics & Overview

1.1 List of Topics in Introductory Topology

- Topological spaces: axioms of a topology, open and closed sets, basis for a topology, continuous maps, homeomorphisms
- Examples of topological spaces: subspaces, products, quotients, CW complexes, metric spaces
- Topological properties: connectedness, compactness, path-connectedness (and their local variants), Hausdorff, separation axioms
- Classification of surfaces: 2-dimensional manifolds, orientability, examples, construction, the classification theorem

1.2 List of Topics in Algebraic Topology

- Fundamental group: the fundamental group, induced homomorphisms, simple connectedness, the fundamental group of the circle, applications
- Seifert-van Kampen: group presentations, Seifert-van Kampen theorem, computations of the fundamental group for many examples, applications
- Covering spaces: covering spaces, examples, covering transformations, equivalence of covering spaces, the universal covering space, classification of covering spaces, applications
- Homology theory: (singular, simplicial, and cellular), homology of spaces, homology of pairs, exact sequences, chain complexes, excision, Mayer-Vietoris theorem, homology with coefficients, degree, computations, applications

1.3 Main & Additional References

- Munkres (main): 2-4, 9-14
- Lawson (main): 1-3, 5
- Hatcher (main): 0-2
- Massey (additional): 2-7
- Rotman (additional): 3-6, 8-10

1.4 Courses Taken

- 661 Topology; Professor - Wylie; Text - Lawson; Fall 2020
- 761 Introduction to Algebraic Topology; Professor - Wehrli; Text - Hatcher; Spring 2021

1.5 Notes on Notation

The symbol $\simeq$ means either "is homeomorphic to" or "is homotopy equivalent to", although more likely the former. Hopefully, the context will be clear. The symbol $\cong$ means "is isomorphic to". The symbol $\simeq_p$ means "is path homotopic to", whereas if the p is replace by a symbol for a map, say H, then the symbol $\simeq_H$ means "is homotopic to via H".

2 PREFACE

Take all of this with a grain of salt - if you think something is incorrect, it very well may be. In particular, you will notice in my solutions that for much of my studying, I was convinced $H_n(X/A) \cong \tilde{H}_n(X, A)$ for good pairs, when in fact $\tilde{H}_n(X/A) \cong H_n(X, A)$. This shows up in several "solved" problems.

3 Topics in Introductory Topology

3.1 Point Set Topology

3.1.1 Piecing Lemmas

Lemma 3.1.1 (Piecing Lemma for Continuous Functions)**.** Let $X = A \cup B$, where A and B are closed in X. Let $F : X \to Y$ be a function such that $f|_A$ and $f|_B$ are continuous (i.e., the compositions of f with the inclusions of A and B into X are continuous). Then, f is continuous.

Lemma 3.1.2 (Piecing Lemma for Homeomorphisms)**.** Let $X = A \cup B$ and $Y = C \cup D$, where A and B are closed in X and C and D are closed in Y. Let $f : A \to C$ and $g : B \to D$ be homeomorphisms, and suppose that $f|_{A\cap B} = g|_{A\cap B}$. Define $h : X \to y$ by $h|_A = f$ and $h|_B = g$. If H is a bijection (i.e., the only points that are in the image of both f and g are the points in the image of $A \cap B$), then h is a homeomorphism.

3.1.2 Compactness

Theorem 3.1.1. The continuous image of a compact set is compact.

Corollary 3.1.1. Compactness is a topological property.

Theorem 3.1.2. In a *metric space*, compact sets are closed.

Theorem 3.1.3. Let X be compact, and let A be closed in X. Then, A is compact.

Theorem 3.1.4. Let $f : X \to Y$ be a bijection. Suppose that f is continuous, X is compact, and Y is Hausdorff. Then, f is a homeomorphism.

Theorem 3.1.5. Suppose X and Y are topological spaces, and let $X \times Y$ have the product topology. Then, the inclusions $i_x : Y \to X \times Y$ given by $i_x(y) = (x, y)$ and $i_y : X \to X \times Y$ given by $i_y(X) = (x, y)$ are continuous. Furthermore, each projection $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ is continuous. In particular, the maps $X \to X \times \{y\}$ given from i_y by restricting the ranges and $Y \to \{x\} \times Y$ given from i_x by restricting the range are homeomorphisms.

Theorem 3.1.6. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Lemma 3.1.3 (Tube Lemma)**.** Let X be any space and x be point in X. Suppose Y is compact. Then, for each open set $N \subseteq X \times Y$ containing $\{x\} \times Y$, there is an open neighborhood $V \subseteq X$ of x such that $V \times Y \subseteq N$.

Theorem 3.1.7 (Tychanoff)**.** The product of compact sets is compact.

3.1.3 Connectedness

Definition 3.1.1. A topological space X is called **separated** if $X = A \cup B$, where $A \cap B = \emptyset$, and A and B are both nonempty and open. A topological space is called **connected** if it is not separated.

Theorem 3.1.8. Suppose that $\{A_i\}$ is a collection of connected subsets of a topological space X such that the A_i all have at least one point in common. Then, the union $\bigcup_i A_i$ is connected.

Definition 3.1.2. A **path** in a topological space X is a continuous map $f : [0,1] \to X$. We say that X is **path-connected** if given $x, y \in X$, there exists a path from x to y, i.e., if the is some such f with $f(0) = x$ and $f(1) = y$.

Theorem 3.1.9. Suppose that $\{A_i\}$ is a collection of path-connected subsets of a topological space X such that the A_i all have at least one point in common. Then, the union $\bigcup_i A_i$ is path-connected.

Theorem 3.1.10. The continuous image of a path-connected space is path-connected.

Corollary 3.1.2. Path-connectedness is a topological property.

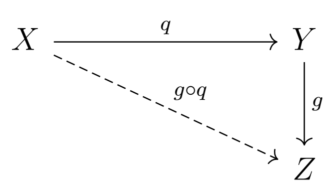
Theorem 3.1.11. Let X be a topological space. If X is path-connected, then X is connected.

Note: The converse of the previous theorem does not hold in general. For an example of why, look at the topologist's sine curve.

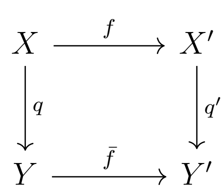
Theorem 3.1.12. Let $A \subset \mathbb{R}^n$ be open. If A is connected, then A is path-connected.

3.1.4 Quotient Spaces

Theorem 3.1.13. Let Y be a quotient space of X with quotinet map $q : X \to Y$. Let $g : Y \to Z$ be any map. Then, g is continuous if and only if $g \circ q$ is continuous. The following commutative diagram describes the situation:



Theorem 3.1.14. Suppose $q : X \to Y$ and $q' : X' \to Y'$ are quotient maps. Let $f : X \to X'$ and $\bar{f} : Y \to Y'$ be maps such that $q' \circ f = \bar{f} \circ q$. That is, the diagram



commutes. Then, $\bar{f}$ is continuous if f is continuous.

Theorem 3.1.15. Suppose $f : X \to Y$ is a surjective continuous map, where X is compact and Y is Hausdorff. Define an equivalence relation on X by $u \sim v$ if and only if $f(u) = f(v)$. The equivalence classes are the inverse images $f^{-1}(y)$. Then, the induced map $\bar{f} : X/\sim \to Y$ is a homeomorphism.

Theorem 3.1.16. Suppose $f : X \to Y$ is a surjective continuous map. Define an equivalence relation on X by $u \sim v$ if and only if $f(u) = f(v)$. The equivalence classes are the inverse images $f^{-1}(y)$. Then, the induced map $\bar{f} : X/\sim \to Y$ is a homeomorphism exactly when f sends saturated open sets $q^{-1}(U)$ to open sets. In particular, $\bar{f}$ is a homeomorphism if f is an open map.

3.2 The Classification of Surfaces

3.2.1 (Boundary) Connected Sums

Definition 3.2.1. Let M and N be two surfaces, each with nonempty boundary. Identify arcs in ∂M and ∂N. Choose an embedding

$$f : \{\pm 1\} \times D^2 \to M \sqcup N$$

to the arcs. The **boundary connected sum** $M \natural N$ is

$$M \natural N = M \cup_{f_-} (D^1 \times D^1) \cup_{f_+} N.$$

Equivalently, $M \natural N = (M \sqcup N) \cup_f h^1$, where h^1 is a 1-handle connecting M and N.

Definition 3.2.2. The **connected sum** of surfaces M and N is

$$M \# N = (M_{(1)} \natural N_{(1)}) \cup h^2,$$

where h^2 is a 2-handle turning the "bridge" between the surfaces into a tube. So, $(M \# N)_{(1)} \simeq M_{(1)} \natural N_{(1)}$.

3.2.2 The Fundamental Lemma of Surface Theory

Lemma 3.2.1 (Fundamental Lemma of Surface Theory)**.** We have that

$$T_{(1)} \sqcup P_{(1)} \simeq P_{(1)} \sqcup P_{(1)} \sqcup P_{(1)} \simeq K_{(1)} \sqcup P_{(1)},$$

and

$$T \# P \simeq P \# P \# P \simeq K \# P.$$

Note that $P \# P \simeq K$, but $P \# P \not\simeq T \not\simeq K$.

3.2.3 Euler Characteristic & Identifying Surfaces

Definition 3.2.3. Given a CW complex X with exactly n_i i-cells, the **Euler characteristic** of X is

$$\chi(X) = \sum_{i=0}^{k} (-1)^i n_i.$$

In particular, for a 2-dimensional CW complex X,

$$\chi(X) = n_0 - n_1 + n_2.$$

Definition 3.2.4. Given a handlebody H with a handle decomposition consisting of exactly n_i i-handles, the **Euler charactersitic** of H is

$$\chi(H) = \sum_{i=0}^{k} (-1)^i n_i.$$

In particular, for a handlebody H with only 0-, 1-, and 2-handles,

$$\chi(H) = n_0 - n_1 + n_2.$$

Theorem 3.2.1. Let H be a connected handlebody with a handle decomposition with only 0-handles and 1-handles, Euler characteristic χ, and p boundary circles. Then,

(i) If H is orientable, then H is homeomorphic to $S_{(p)}$ or $T^{(n)}_{(p)}$, where $\chi = 2 - 2n - p$. Note that $S_{(p)}$ is just alternative notation for $T^{(0)}_{(p)}$.

(ii) If H is nonorientable, then H is homeomorphic to $P^{(n)}_{(p)}$, where $\chi = 2 - n - p$.

Upshot: If we know orientability, the number of boundary components, and χ, then we know the surface:

Orientable $\longrightarrow \chi = 2 - 2n - p \longrightarrow$ **We know p and χ, so we know n**

Nonorientable $\longrightarrow \chi = 2 - n - p \longrightarrow$ **We know p and χ, so we know n**

3.3 Homotopy Theory

3.3.1 (Deformation) Retractions & Homotopy (Equivalence)

Definition 3.3.1. A **retraction** of a space X onto $A \subseteq X$ is a map $r : X \to X$ such that $r(X) = A$ and $r|_A =$ id. Equivalently, r satisfies $r^2 = r$, as this says that r is the identity on its image.

Definition 3.3.2. A **deformation retraction** of a space X onto $A \subseteq X$ is a family of maps $f_t : X \to X$, $t \in I = [0,1]$, such that $f_0 = \text{id}$, $f_1(X) = A$, and $f_t|_A = \text{id}$ for all t. The family of f_t should be continuous in the sense that the associated map $X \times I \to X$ given by $(x,t) \mapsto f_t(x)$ is continuous.

Example: A space X always retracts onto any point $x_0 \in X$ via the constant map. However, if X is not path-connected, then this map is not a deformation retraction.

Example: The annulus deformation retracts to onto its center circle.

Example: The Möbius band deformation retracts onto its center circle.

Example: The handle body $h^0 \cup h^1_1 \cup h^1_2$ deformation retracts onto W_2, the wedge of 2 circles.

Definition 3.3.3. A **homotopy** is a family of maps $f_t : X \to Y$, $t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. Two maps $g, h : X \to Y$ are **homotopic** if there exists a homotopy f_t connecting them.

Remark: In homotopy terms, a deformation retraction of X onto $A \subseteq X$ is a homotopy form id_X to a retraction of X onto A.

Definition 3.3.4. A homotopy $f_t : X \to Y$ whose restriction to a subspace A of X is independent of t is called a **homotopy rel** A. So, a deformation retraction of X onto A is a homotopy rel A from the identity map of X to a retraction of X onto A.

Theorem 3.3.1. If X deformation retracts onto $A \subseteq X$ via $f_t : X \to X$, if $r : X \to A$ is the resulting retraction, and if $i : A \hookrightarrow X$ is the inculsion, then we have that $r \circ i = \text{id}$ and $i \circ r \simeq \text{id}$, the latter homotopy given by f_t.

Definition 3.3.5. A map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $f \circ g \simeq \text{id}$ and $g \circ f \simeq \text{id}$.

Note: In the previous theorem, r and i are homotopy inverses.

Theorem 3.3.2. If $X \simeq Y$, then there exists Z such that X and Y are homeomorphic to deformation retracts of Z.

Definition 3.3.6. A space is called contractible if it is homotopy equivalent to a point.

Definition 3.3.7. A map is called **nullhomotopic** if it is homotopic to a constant map.

Theorem 3.3.3. A space is contractible if and only if its identity map is nullhomotopic.

Theorem 3.3.4. If (X,A) is a CW pair and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

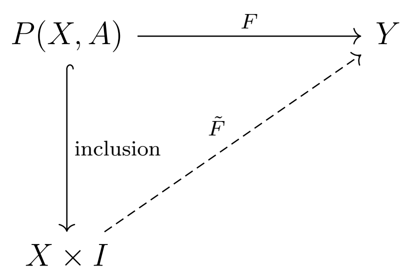
3.3.2 The Homotopy Extension Property

Definition 3.3.8. Let (X, A) be a pair, i.e., X is a space and A is a subspace of X. Define

$$P(X, A) = (X \times \{0\}) \cup (A \times I).$$

Figure 1 describes $P(X, A)$.

The following commutative diagram describes the situation:



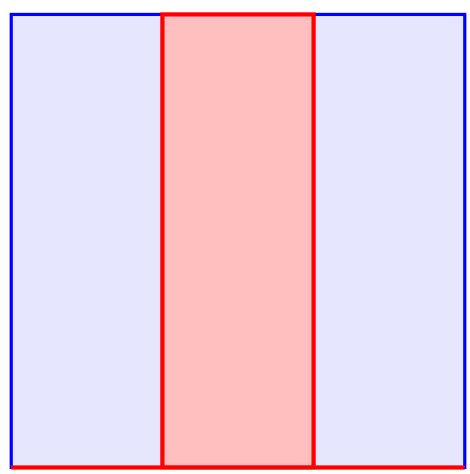


Figure 1: The blue is $X \times I$, with X horizontal and I vertical. The red is $P(X, A) = (X \times \{0\}) \cup (A \times I)$, where A is at the bottom of the red box (not at the bottom of the blue box).

Definition 3.3.9. A pair (X, A) has the **Homotopy Extension Property**, or **HEP**, if *every* continuous map $F : P(X, A) \to Y$ extends to a continuous map $\tilde{F} : X \times I \to Y$ *for all* Y.

Remark: In the situation of the HEP, we are given a homotopy $f_t : A \to Y$, $t \in I$, and an extension $f : X \to Y$ of f_0 such that $f : P(X, A) \to Y$ is continuous. Then, the HEP says that we can extend f_t to a homotopy $\tilde{f}_t : X \to Y$ such that $\tilde{f}_0 = f$.

Lemma 3.3.1. A pair (X, A) has the HEP if and only if $P(X, A)$ is a retract of $X \times I$.

Corollary 3.3.1. A pair (X, A) has the HEP if and only if $(X \times Z, A \times Z)$ for all Z.

Note: The "$\Leftarrow$" of the above corollary can be seen from the fact that if Z is just a point, then $(X \times Z, A \times Z) \simeq (X, A)$.

Corollary 3.3.2. A pair (X, A) has the HEP if and only if $X \times I$ deformation retracts to $P(X, A)$.

Theorem 3.3.5. Let A be any space, and let $f : \partial D^n \to A$ be a continuous map. Let $X = A \sqcup_f D^n = (A \sqcup D^n)/\sim_f$, where $f(x) \sim_f x$ for all $x \in \partial D^n$. Then, (X, A) has the HEP.

Theorem 3.3.6. If (X, A) is a CW pair, then (X, A) has the HEP.

Theorem 3.3.7. If $X(, A)$ has the HEP and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

Definition 3.3.10. Let X and Y be topological spaces with a common subspace A. We say that X and Y are **homotopy equivalent rel** A if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f|_A = \mathrm{id}_A = g|_A$ and $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$ via homotopies that restrict to id_A at all times t.

Theorem 3.3.8. Suppose we have a topological space X_0, and let (X_1, A) be a pair with the HEP. If $f, g : A \to X_0$ are continuous, homotopic maps, then

$$(X_0 \sqcup_f X_1) \simeq (X_0 \sqcup_g X_1) \text{ rel } X_0.$$

3.3.3 Mapping Cylinders & Mapping Cylinder Neighborhoods

Definition 3.3.11. Let $f : X \to Y$ be continuous. The **mapping cylinder** of f is the space

$$\begin{aligned} M_f &= \frac{Y \sqcup (X \times I)}{f(x) \sim (x, 1)\ \forall x \in X} \\ &= Y \sqcup_g (X \times I), \end{aligned}$$

where $g : X \times I \to Y$ is given by $(x, 1) \mapsto f(x)$. Here, M_f is equipped with the quotient topology.

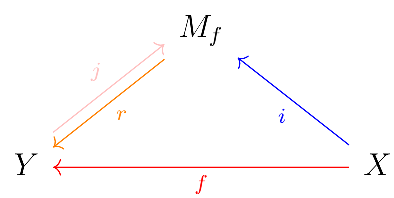
Remark: Let $f : X \to Y$ be continuous. The mapping cylinder M_f always contains Y, and we have that $X \times \{0\} \simeq X$. So, there exists inclusion $i : X \hookrightarrow M_f$ given by $x \mapsto (x, 0) \in X \times \{0\} \subseteq M_f$ and $j : Y \hookrightarrow M_f$ given by $y \mapsto y \in Y \subseteq M_f$. In fact, i and j are *homeomorphisms* onto their images.

Note: Since $X \times I$ deformation retracts to $X \times \{1\}$, we have that M_f deformation retracts to Y. So, $M_f \simeq Y$, and $j : Y \hookrightarrow M_f$ is a homotopy equivalence with homotopy inverse given by the retraction

$r : M_f \to Y$ defined by

$$r = \begin{cases} r(y) = y, & \forall y \in Y \subseteq M_f, \\ r(x,t) = f(x), & \forall (x,t) \in X \times [0,1) \subseteq M_f. \end{cases}$$

In summary, we have started with a continuous map $f : X \to Y$, we have the following diagram that *commutes up to homotopy*:



Theorem 3.3.9. A map $f : X \to Y$ is a homotopy equivalence if and only if $i : X \hookrightarrow M_f$ is a homotopy equivalence.

Definition 3.3.12. A closed neighborhood N of Y is called a **mapping cylinder neighborhood** if there exists a continuous function $f : \partial N \to Y$ and a homeomorphism $h : N \to M_f$ such that

(i) $h|_{\partial N} = i : \partial N \hookrightarrow \partial N \times \{0\} \subseteq M_f$ and

(ii) $h|_Y = j : Y \hookrightarrow Y \subseteq M_f$.

Note that any maping cylinder neighborhood N of Y deformation retracts onto Y, because M_f deformation retracts onto Y.

Theorem 3.3.10. If $Y \subseteq Z$ and Y has a mapping cylinder neighborhood, then (Z, Y) has the HEP.

Corollary 3.3.3. Let $f : X \to Y$ be continuous. Then, (M_f, Y) and $(M_f, X \times \{0\})$ each have the HEP.

Theorem 3.3.11. If (X, A) has the HEP and the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X. The converse is also true, i.e., if (X, A) has the HEP, then $i : A \hookrightarrow X$ is a homotopy equivalence if and only if A is a deformation retract of X.

Theorem 3.3.12. Let $f : X \to Y$ be continuous. Then, f is a homotopy equivalence if and only if $X \times \{0\}$ is a deformation retraction of M_f.

Note: The above theorem gives us the previous result that if X and Y are homotopy equivalent, then there exists Z such that X and Y are each homeomorphic to deformation retractions of Z. Simply take $Z = M_f$, where $f : X \to Y$ is any homotopy equivalence.

3.4 Lebesgue Number

Note: Metric spaces are normal Hausdorff spaces, i.e., singletons are closed (Hausdorff) and any two disjoint closed sets are contained in disjoint open neighborhoods (normal). The converse (that normal and Hausdorff implies metric) is not necessarily true.

Definition 3.4.1. Let X be a metric space and $A \subseteq X$ be nonempty. Let $x \in X$. The **distance from** x **to** A is

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Note: For a fixed set A, $d(-, A)$ is a continuous function $X \to \mathbb{R}$.

Note: If $A \subseteq X$ is closed, then $d(x, A) = 0$ if and only if $x \in A$.

Definition 3.4.2. Let X be a metric space and $A \subseteq X$ be nonempty. Define the **diameter** of A to be

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Lemma 3.4.1 (Lebesgue Number Lemma)**.** Let (X, d) be a *compact* metric space, and suppose $\{U_\alpha\}$ is an open cover of X. Then, there exists a $\delta > 0$ such that every $A \subseteq X$ with $\operatorname{diam}(A) < \delta$ is contained in a single U_α.

Here, δ is called a **Lebesgue number** for the cover $\{U_\alpha\}$.

3.5 Metrizability

Definition 3.5.1. A topological space X is called **metrizable** is there exists a metric on X inducing the given topology on X. Equivalently, X is metrizable if it is homeomorphic to a metric space.

Definition 3.5.2. A basis $\mathcal{B}$ for a topological space X is called **countably locally finite** if $\mathcal{B}$ can be written as

$$\mathcal{B} = \bigcup_{n=1}^{\infty} B_n,$$

where each B_n is locally finite, i.e., each $x \in X$ is contained in at most finitely many $B \in B_n$.

Note: Second countable, i.e., there exists a countable basis, implies countably locally finite.

Theorem 3.5.1 (Nagata-Smirnov Metrization Theorem)**.** A topological space X is metrizable if and only if it is a normal Hausdorff space which has a basis that is countably locally finite.

4 Topics in Algebraic Topology

4.1 The Fundamental Group

4.1.1 Induced Maps

Theorem 4.1.1. Let $\varphi : X \to Y$ be continuous. Let $x_0 \in X$, and set $y_0 = \varphi(x_0)$. Define

$$\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by

$$\varphi_*([f]) = [\varphi \circ f].$$

Then, φ_* is a group homomorphism. If φ is a homeomorphism, then φ_* is an isomorphism with $(\varphi_*)^{-1} = (\varphi^{-1})_*$. It follows that path-connected homeomorphic spaces have isomorphic fundamental groups. Also, $(\varphi \circ \phi)_* = \varphi_* \circ \phi_*$ for φ and another continuous map $\phi : Y \to Z$.

Note: $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$.

Theorem 4.1.2. Homotopy equivalent spaces have isomorphic fundamental groups.

Lemma 4.1.1. If $\varphi_t : X \to Y$ is a homotopy and $x_0 \in X$, then

$$(\varphi_0)_* = \beta_h \circ (\varphi_1)_*,$$

where β_h is the change-of-base-point map $\pi_1(X, \varphi_0(x_0)) \to \pi_1(X, \varphi_1(x_0))$ given by $\beta_h([f]) = [h \cdot f \cdot \bar{h}]$, and h is a path from $\varphi(x_0)$ to $\varphi_1(x_1)$ given by $h(s) = \varphi_s(x_0)$.

4.1.2 A Bunch of Fundamental Groups

Some Fundamental Groups

Space	Fundamental Group
S^1	$\mathbb{Z}$
S^2	$\{e\}$
$S_{(p)}$	$F_{p-1} \quad (p > 1)$
W_{p-1}	$F_{p-1} \quad (p > 1)$
T	$\mathbb{Z} \oplus \mathbb{Z}$
$T^{(n)}_{(p)}$	$F_{2n+p-1} \quad (p > 0)$
W_{2n+p-1}	$F_{2n+p-1} \quad (p > 0)$
P	$\mathbb{Z}_2$
$P^{(n)}_{(p)}$	$F_{n+p-1} \quad (p > 0)$
W_{n+p-1}	$F_{n+p-1} \quad (p > 0)$
$T^{(n)}$	$\langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \ldots a_nb_na_n^{-1}b_n^{-1} \rangle$
$P^{(n)}$	$\langle a_1, a_2, \ldots, a_n \mid a_1^2a_2^2 \ldots a_n^2 \rangle$

Some Abelianizations

Space	Abelianization of Fundamental Group
$T^{(n)}$	$\mathbb{Z}^{2n}$
$P^{(n)}$	$\mathbb{Z}_2 \oplus \mathbb{Z}^{n-1}$

Note: Abelianizations determine surfaces, since if $G_1 \cong G_2$, then $G_1/G_1' \cong G_2/G_2'$.

Note: If M is *orientable*, then $\pi_1^{ab}(M)$ is torsion-free. If M is *nonorientable*, then $\pi_1^{ab}(M)$ has torsion elements.

4.1.3 Free Products (With Amalgamation)

Definition 4.1.1. Let G and H be groups with generating sets S_G and S_H and relation sets R_G and R_H, respectively. The **free product** of G and H is the group denoted $G * H$ given by generating set $S_G \cup S_H$ and relation set $R_G \cup R_H$.

Definition 4.1.2. Let F, G, and H be groups. Let $\varphi : F \to G$ and $\phi : F \to H$ be injective homomorphisms. Let $N = \langle \varphi(f)\phi^{-1}(f) \mid f \in F \rangle$. Then, N is a normal subgroup of $G * H$. The **free product with amalgamation** of G and H along F is the group

$$G *_F H = (G * H)/N.$$

4.1.4 The Seifert-van Kampen Theorem

Theorem 4.1.3 (Seifert-van Kampen)**.** Let X be a path-connected space with basepoint x_0. Suppose $X = A \cup B$, where A and B are both open, path-connected subsets of X. Furthermore, assume $x_0 \in A \cap B$, and assume $A \cap B$ is path-connected. Let $\varphi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ and $\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ be the homomorphisms induced by inclusion. Then, $\pi_1(X, x_0)$ is isomorphic to the amalgamated free product

$$\pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0).$$

Corollary 4.1.1. If $\pi_1(A \cap B)$ is trivial, the $\pi_1(X, x_0)$ is the free product $\pi_1(A, x_0) * \pi_1(B, x_0)$.

Corollary 4.1.2. If $\pi_1(B, x_0)$ is trivial, then $\pi_1(X, x_0) \cong \pi_1(A, x_0)/N$, where N is the normal subgroup of $\pi_1(A, x_0)$ generated by the image of $\pi_1(A \cap B, x_0) \hookrightarrow \pi_1(A, x_0)$.

Note: Attaching n-cells where $n > 2$ to a space X does not change $\pi_1(X)$. This is because both $\partial D^n = S^{n-1}$ and D^n are simply connected. In particular, if X is a CW complex, then $\pi_1(X) \cong \pi_1(X^2)$. The isomorphism $\pi_1(X^2) \to \pi_1(X)$ is induced by the inclusion $X^2 \hookrightarrow X$.

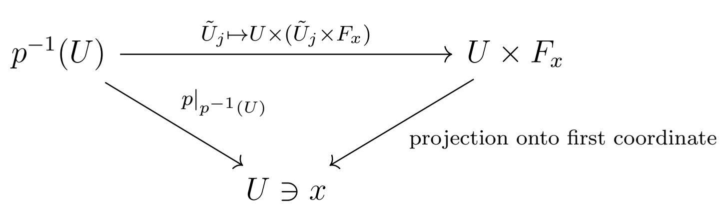
4.2 Covering Space Theory

4.2.1 Covering Spaces, Isomorphisms, Deck Transformations

Definition 4.2.1. A **covering space** of a space X is a space $\tilde{X}$ together with a continuous map $p : \tilde{X} \to X$ such that for all $x \in X$, there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically (by p) to U. In this situation, we call p a **covering map**, and such a set U is call **evenly covered**.

Definition 4.2.2. Denote by F_x the preimage $p^{-1}(x)$ for $x \in X$, p a covering map. Call F_x the **fiber above** x. If $|F_x| = n$ for all n, then we say that p is an n**-sheeted covering**.

The following commutative diagram describes covering spaces:



Here, $\tilde{U}_j$ is one of the sheets in $p^{-1}(U)$, i.e., one of the opens sets in the disjoint union that maps homeomorphically to U.

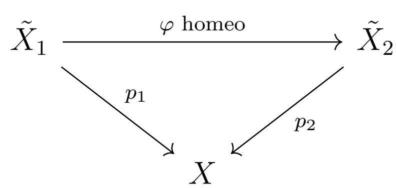
Note: If $p : \tilde{X} \to X$ is a covering map, then evenly covered neighborhoods in X form a basis for *the* topology of X.

Note: If $p : \tilde{X} \to X$ is a 1-sheeted covering, the p is a homeomorphism. The converse is also true, i.e., 1-sheeted covering maps and homeomorphisms are equivalent concepts.

Definition 4.2.3. If the entire space X is evenly covered, then we say that $\tilde{X}$ is **trivial**.

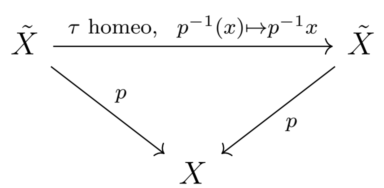
Definition 4.2.4. An **isomorphism** $\varphi : \tilde{X}_1 \to \tilde{X}_2$ of covering spaces $p_1 \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ is a homeomorphism such that $p_2 \circ \varphi = p_1$. Equivalently, φ maps $p_1^{-1}(x)$ to $p_2^{-1}(x)$ for all $x \in X$ and is a homeomorphism.

The following commutative diagram describes isomorphisms of covering spaces:



Definition 4.2.5. An automorphism τ of a covering space is called a **deck transformation**.

The following commutative diagram describes deck transformations:



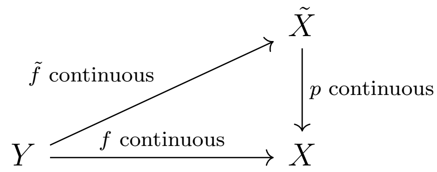
Note: The set of all deck transformations of a covering space $(\tilde{X}, p)$ is a group under function composition. This group is denoted $G(\tilde{X}, p)$, or, if the context is right, simply $G(\tilde{X})$.

Theorem 4.2.1. If $\tilde{X}$ is a nonempty connected 2-sheeted covering space, then the group of deck transformations of $\tilde{X}$ is isomorphic to $\mathbb{Z}_2$.

4.2.2 Lifting Properties, Lifting Criterion

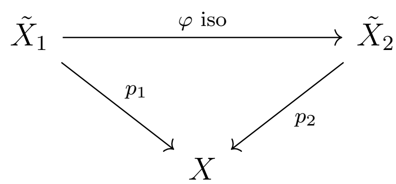
Definition 4.2.6. Let $p : \tilde{X} \to X$ and $f : Y \to X$ be continuous maps, where p is not necessarily a covering map. A **lift** of f to $\tilde{X}$ is a continuous map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

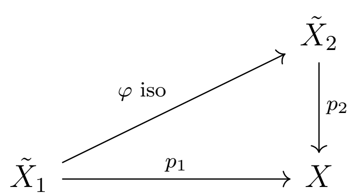
The following commutative diagram describes lifts:

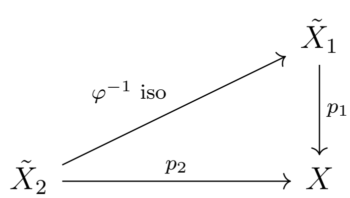


Note: Isomorphisms of covering spaces may be viewed as lifts. For $\varphi : \tilde{X}_1 \to \tilde{X}_2$ an isomorphism of covering spaces p_1 and p_2, we can view φ as a lift of p_1 and φ^{-1} as a lift of p_2. In particular, deck transformations can be viewed a lifts.

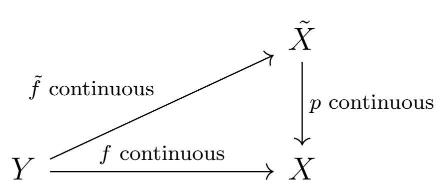
The following commutative diagrams describe viewing isomorphisms as lifts:







Remark: Given a lift $\tilde{f}$ of f:



if we have that

(i) $f(Y) \subseteq U$, where U is an evenly covered neighborhood,

(ii) $\tilde{f}(Y) \subseteq \tilde{U}_j$ for some j,

then, since $p \circ \tilde{f} = f$, we get that $p|_{\tilde{U}_j} \circ \tilde{f} = f$ and so $\tilde{f} = \left(p|_{\tilde{U}_j}\right)^{-1} \circ f$.

Upshot: If $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ are lifts of f that send Y to the same $\tilde{U}_j$, then

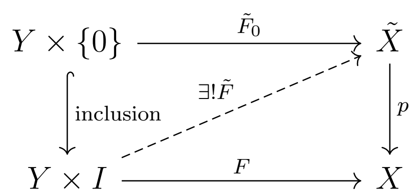
$$\tilde{f}_1 = \left(p|_{\tilde{U}_j}\right)^{-1} \circ f = \tilde{f}_2.$$

Theorem 4.2.2 (Unique Lifting Property (ULP))**.** Let $p : \tilde{X} \to X$ be a covering space, and let $f : Y \to X$ be a continuous map. Let $\tilde{f}_1$ and $\tilde{f}_2$ be lifts of f to $\tilde{X}$ that agree at *one point* of Y. If Y is connected, then $\tilde{f}_1$ and $\tilde{f}_2$ agree on *all* of Y.

Corollary 4.2.1. If $\tilde{X}_1$ and $\tilde{X}_2$ are connected covering spaces of X and $\varphi_1, \varphi_2 : \tilde{X}_1 \to \tilde{X}_2$ that agree at *one point*, then $\varphi_1 = \varphi_2$.

Corollary 4.2.2. If $\tilde{X}$ is a connected covering space of X and τ_1 and τ_2 are deck transformations of $\tilde{X}$ that agree at *one point*, then $\tau_1 = \tau_2$.

Definition 4.2.7 (Unique Homotopy Lifting Property (UHLP))**.** Let $p : \tilde{X} \to X$ be a continuous map, not necessarily a covering map. Let Y be a space. We say that (Y, p) has the **Unique Homotopy Lifting Property** if for all continuous maps $F : Y \times I \to X$ and for all lifts $\tilde{F}_0 : Y \times \{0\} \to \tilde{X}$ of $F_0 = F|_{Y \times \{0\}}$ we have that there exists a *unique* lift $\tilde{F} : Y \times I \to \tilde{X}$ of F such that $\tilde{F}|_{Y \times \{0\}} = F_0$. That is, the diagram



commutes.

Note: If $p : \tilde{X} \to X$ is a covering space, the (Y, p) has the UHLP for *all* spaces Y.

Theorem 4.2.3 (Unique Path Lifting Property (UPLP))**.** Let $p : \tilde{X} \to X$ be a covering space and $f : I \to X$ be a path starting at $x_0 \in X$. Then, for all $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a *unique* lift $\tilde{f} : I \to \tilde{X}$ of f starting at $\tilde{x}_0$.

Corollary 4.2.3. Let $p : \tilde{X} \to X$ be a covering space and $f, g : I \to X$ be paths such that $f \simeq_p g$. If $\tilde{f}, \tilde{g} : I \to \tilde{X}$ are lifts of f and g, respectively, such that $\tilde{f}(0) = \tilde{g}(0)$, then $\tilde{f} \simeq_p \tilde{g}$, and, in particular, $\tilde{f}(1) = \tilde{g}(1)$.

Note: Loops need not lift to loops.

Remark: The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective, and

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \left\{ [f] \in \pi_1(X, x_0) \mid \text{The lift of } f \text{ to } \tilde{X} \text{ starting at } \tilde{x}_0 \text{ is a loop} \right\}.$$

If $\tilde{X}$ is path-connected, then

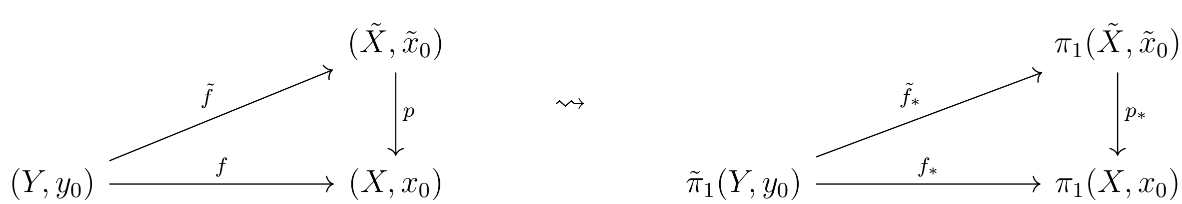
$$|F_{x_0}| = [\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))].$$

It follows that if $p : \tilde{X} \to X$ is a path-connected n-sheeted covering space, the $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has index n in $\pi_1(X, x_0)$.

Theorem 4.2.4 (Lifting Criterion)**.** Let $p : \tilde{X} \to X$ be a covering space, $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$, Y a space, and $y_0 \in Y$. If Y is path-connected and locally path-connected, and if $f : Y \to X$ is continuous such that $f(y_0) = x_0$, then there exists a lift $\tilde{f} : Y \to \tilde{X}$ of f such that $\tilde{f}(y_0) = \tilde{x}_0$ if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

The following commutative diagram describes the Lifting Criterion:



4.2.3 Covers Associated to Subgroups & The Universal Cover

Theorem 4.2.5. Let $p_i : \tilde{X}_i \to X$ be covering spaces for $i = 1, 2$. Let $\tilde{x}_i \in \tilde{X}_i$, and let $x_0 \in X$. Set

$$H_i = (p_i)_*(\pi_1(\tilde{X}_i, \tilde{x}_i)) \leq \pi_1(X, x_0).$$

If X is locally path-connected and path-connected, and if $\tilde{X}_i$ is path-connected fr $i = 1, 2$, then there exists an isomorphism $f : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ if and only if $H_1 = H_2$.

Note: For X a path-connected and locally path-connected space with $x_0 \in X$,

$$\left\{\text{based path-connected covering spaces } p : (\tilde{X}, \tilde{x}_0) \to (X, x_0),\ \tilde{x}_0 \mapsto x_0\right\} \overset{p \mapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))}{\hookrightarrow} \{\text{subgroups of } \pi_1(X, x_0)\}$$

Theorem 4.2.6. Let $p : \tilde{X} \to X$ be a covering space, $x_0 \in X$, $\tilde{x}_1, \tilde{x}_2 \in F_{x_0}$, $H_i = p_*(\pi_1(\tilde{X}, \tilde{x}_i)) \leq \pi_1(X, x_0)$ for $i = 1, 2,$. If $\tilde{X}$ is path-connected, then H_1 and H_2 are conjugate subgroups of $\pi_1(X, x_0)$.

Conversely, if $H_1 = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$ and $H_2 \leq \pi_1(X, x_0)$ where H_1 and H_2 are conjugate subgroups, say,

$$H_2 = [\bar{h}]H_1[h],$$

for some $[h] \in \pi_1(X, x_0)$, then there exists $\tilde{x}_2 \in F_{x_0}$ such that $H_2 = p_*(\pi_1(\tilde{X}, \tilde{x}_2))$.

Theorem 4.2.7. If $x_0 \in X$ and X is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 \tilde{X}_2 \to X$ are isomorphic if and only if the subgroups

$$H_i = (p_i)_*(\pi_1(\tilde{X}_i, \tilde{x}_i)) \qquad (i = 1, 2)$$

are conjugate in $\pi_1(X, x_0)$ for *any* choice of basepoints $\tilde{x}_i \in F_{i,x_0} = p_i^{-1}(x_0)$.

Remark: For $X \neq \emptyset$ a path-connected and locally path-connected space,

$$\left\{\text{path-connected covering spaces } p : \tilde{X} \to X,\ \tilde{X} \neq \emptyset\right\} \hookrightarrow \{\text{conjugacy classes of subgroups of } \pi_1(X, x_0)\},$$

where the above map is given by $p \mapsto$ "conjugacy class of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ for any $\tilde{x}_0 \in p^{-1}(x_0)$".

In fact, the above map is a bijection if we further require X to be **semi-locally simply connected**, meaning that for all $x \in X$, there exists an open neighborhood U of x such that the map $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ is the zero map.

Theorem 4.2.8. If X is path-connected, locally path-connected, and semi-locally simply connected, then for all $H \leq \pi_1(X, x_0)$, there exists a path-connected covering space $p : \tilde{X}_H \to X$ and a basepoint $\tilde{x}_0 \in p^{-1}(x_0)$ such that $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$. For a given $H \leq \pi_1(X, x_0)$, the space $(\tilde{X}_H, \tilde{x}_0)$ is *unique* up to based isomorphism.

Definition 4.2.8. The covering space $\tilde{X}_{\{0\}}$ associated to the trivial subgroup $\{0\}$ of $\pi_1(X, x_0)$ is called the **universal cover** of X.

Note: The universal cover of X is the *unique* path-connected simply connected covering space of X.

Example: Because $\mathbb{R}$ is a covering space of S^1 and $\mathbb{R}$ is both path-connected and simply connected, we have that $\mathbb{R}$ is the universal cover of S^1.

Note: The product of covering maps is a covering map.

4.2.4 Construction of the Universal Cover

For a nonempty, path-connected, locally path-connected, and semi-locally simply connected space X, take

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ a path in } X \text{ starting at } x_0\},$$

where $[\gamma]$ is the path homotopy class of γ. The map $p : \tilde{X} \to X$ given by $[\gamma] \mapsto \gamma(1)$ is a covering map. The space $\tilde{X}$ is the universal cover of X.

4.2.5 General Construction of the Cover Associated to a Subgroup of $\pi_1(X, x_0)$

For a nonempty, path-connected, locally path-connected, and semi-locally simply connected space X, and $x_0 \in X$, fix $H \leq \pi_1(X, x_0)$. For two paths γ and γ' in X each starting at x_0, define

$$\gamma \sim_H \gamma' \text{ if and only if } \gamma(1) = \gamma'(1) \text{ and } [\gamma' \cdot \bar{\gamma}] \in H.$$

Call $\sim_H$ H**-equivalence**. If H is trivial, then $\gamma \sim_H \gamma'$ if and only if $\gamma \simeq_p \gamma'$. The space

$$\tilde{X}_H = \{[\gamma]_H \mid \gamma \text{ a path in } X \text{ starting at } x_0\},$$

where $[\gamma]_H$ is the H-equivalence class of γ, is a path-connected covering space of X associated to H. If $H = \pi_1(X, x_0)$, then $\gamma \sim_H \gamma'$ if and only if $\gamma(1) = \gamma'(1)$. In this case, $\tilde{X}_H = X$.

4.2.6 Normal Coverings Spaces & Group Actions on Spaces

Definition 4.2.9. Recall that the set of all deck transformations of a covering space $\tilde{X}$ is a group under function composition denoted $G(\tilde{X})$. A covering space $\tilde{X}$ of X is called **normal**, or **regular**, if $G(\tilde{X})$ acts transitively on every fiber $F_x = p^{-1}(x)$, $x \in X$.

Note: If X is path-connected, then $G(\tilde{X})$ acts transitively of F_x for all $x \in X$ if and only if there exists $x_0 \in X$ such that $G(\tilde{X})$ acts transitively on F_{x_0}.

Upshot: If X is path-connected, we only need to know that $G(\tilde{X})$ acts transitively on a single fiber to know that $\tilde{X}$ is normal.

Definition 4.2.10. Recall from group theory that the **normalizer** of $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ is

$$N(H) = \left\{g \in \pi_1(X, x_0) \mid gHg^{-1} = H\right\}.$$

Theorem 4.2.9. If X is path-connected and locally path-connected, and if $\tilde{X}$ is path-connected, then

(i) $\tilde{X}$ is normal if and only if $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \lhd \pi_1(X, x_0)$;

(ii) $G(\tilde{X}) \cong N(H)/H$.

In particular, if $\tilde{X}$ is normal, then $G(\tilde{X}) \cong \pi_1(X, x_0)/H$.

Corollary 4.2.4. The universal cover is *always* normal, since the trivial subgroup is always normal. So, if $\tilde{X}$ is the universal cover of X, then $G(\tilde{X}) \cong \pi_1(X, x_0)$.

Theorem 4.2.10. If X is path-connected and locally path-connected, $x_0 \in X$, $\varphi : \pi_1(X, x_0) \to G$ a surjective group homomorphism, and $H = \ker \varphi$, then the covering space $\tilde{X}_H$ is normal, and

$$G(\tilde{X}) \cong \pi_1(X, x_0)/H \cong G$$

by the First Isomorphism Theorem. It follows that G acts on $\tilde{X}_H$ via covering transformations.

Definition 4.2.11. Let Y be a space, and define

$$\text{Homeo}(Y) = \{\text{homeomorphisms } Y \to Y\}.$$

The set $\text{Homeo}(Y)$ is a group under function composition.

Definition 4.2.12. Let G be a group. An **action** of G on a space Y by homeomorphisms is a group homomorphism $\varphi : G \to \text{Homeo}(Y)$. For $g \in G$, we write $g : Y \to Y$ to denote the homeomorphism $\varphi(g)$.

Definition 4.2.13. For $y \in Y$, the G**-orbit** of y is

$$G_y = \{g(y) \mid g \in G\} \subseteq Y.$$

The **orbit space** of the action φ is

$$Y/G = \frac{Y}{y \sim g(y), \forall y \in Y, \forall g \in G} = \{G\text{-orbits in } Y\}$$

equipped with the quotient topology. Note that "Y/G" is *not* a quotient of Y by G.

Definition 4.2.14. An action $\varphi : G \to \text{Homeo}(Y)$ is called a **covering spaces action** if for every $y \in Y$, there exists an open neighborhood U of y such that $g_1(U) \cap g_2(U) = \emptyset$ for all $g_1 \neq g_2 \in G$.

Theorem 4.2.11. If φ is a covering space action, then

(i) $g_1(y) \neq g_2(y)$ for all $y \in Y$ and for all $g_1 \neq g_2 \in G$;

(ii) In particular, $g(y) \neq y$ for all $y \in Y$ and for all $g \neq e$.

Thus, φ is a free action, and so it is faithful. It follows that $\varphi : G \to \text{Homeo}(Y)$ is injective. Furthermore, by the First Isomorphism Theorem,

$$G \cong \varphi(G) \leq \text{Homeo}(Y).$$

So, looking at actions φ is equivalent to looking at subgroups of $\text{Homeo}(Y)$.

Theorem 4.2.12. Let $\varphi : G \to \text{Homeo}(Y)$ be a covering space action. Then,

(i) The quotient map $p : Y \to Y/G$ is a normal covering space, adn the fibers of the map are the G-orbits of the action;

(ii) If Y is path-connected, then $G \cong G(Y)$;

(iii) If Y is path-connected and locally path-connected, then

$$G \cong \frac{\pi_1(Y/G)}{p_*(\pi_1(Y))}.$$

In particular, if Y is also simply connected, then $G \cong \pi_1(Y/G)$.

Definition 4.2.15. Let $\varphi : G \to \text{Homeo}(Y)$ be a covering space action. A **fundamental domain** for φ is a subset $D \subseteq Y$ which contains exactly one point from each G-orbit.

Note: If D is a fundamental domain for φ, then $p|_D \colon D \to Y/G$ is a continuous bijection. In general, $p|_D$ is *not* a homeomorphism, however.

4.2.7 Monodromy Action

4.3 Homology Theory

4.3.1 (Singular) Homology

Some properties of homology groups:

- Abelian
- Defined for all spaces and all $n > 0$
- Does not depend on a choice of base point
- Detect higher-dimensional structures
- $H_n(S^n) \cong \mathbb{Z}$ for all n
- For a CW complex X, $H_n(X)$ depends only on the $(n+1)$-skeleton of X
- Homology is preserved under homotopy equivalence

Definition 4.3.1. For a graph X, an integer linear combination of edges of X is called a **1-chain** in X. A 1-chain is called a **1-cycle** if it can be decomposed into loops.

Example: For the graph

PICTURE

$2a + 3b - 5c = 2(a - b) + 5(b - c)$ is a 1-cycle, whereas $6a$ is a 1-chain, but not a 1-cycle.

Definition 4.3.2. The **first homology group** $H_1(X)$ of X is the set of all 1-cycles in X. If X is nonempty and path-connected, then $H_1(X) \cong \pi_1^{ab}(X)$.

Example: For our graph above, $\pi_1(X) = F_2$, and so $H_1(X) \cong \mathbb{Z}^2$. This can also be seen in

$$H_1(X) = \{m(a - b) + n(b - c) \mid m, n \in \mathbb{Z}\} \cong \mathbb{Z}^2.$$

Definition 4.3.3. If we add in 2-cells to our graph, we can define 1-boundaries; an integer multiple $m(a - b)$, where $m \in \mathbb{Z}$ and $a - b$ is a loop, is called a **1-boundary** if $a - b$ bounded a 2-cell.

Definition 4.3.4. For a 2-dimensional CW complex X,

$$H_1(X) \cong \frac{\{\text{1-cycles}\}}{\{\text{1-boundaries}\}}.$$

Example: For X the 2-dimensional CW complex

PICTURE

$$H_1(X) \cong \frac{\{m(a - b) + n(b - c) \mid m, n \in \mathbb{Z}\}}{\{m(a - b) \mid m \in \mathbb{Z}\}} \cong \mathbb{Z}^2/\mathbb{Z} \cong \mathbb{Z}.$$

Definition 4.3.5. In general, for a CW complex X, an n-chain in X is an integer linear combination of n-cells in X. The n**th cellular homology** of X is

$$H_n^{\mathrm{CW}}(X) \cong \frac{\{n\text{-cycles in } X\}}{\{n\text{-boundaries in } X\}}.$$

4.3.2 Chain Complexes

Definition 4.3.6. A **chain complex** C is a sequence of Abelian groups C_n, $n \in \mathbb{Z}$, with homomorphisms $\partial_n : C_n \to C_{n-1}$ such that

$$\partial_n \circ \partial_{n+1} = 0.$$

A chain complex looks like

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

The group C_n is called the n**th chain group** of the chain complex C. The map ∂_n is called the **differential**. The integer n is called the **homological degree**. Elements of C_n are called n**-chains**.

Definition 4.3.7. Since $\partial_n \circ \partial_{n+1} = 0$, we have that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. We may write $Z_n = \ker \partial_n = \{n\text{-cycles}\} \subseteq C_n$ and $B_n = \operatorname{im} \partial_{n+1} = \{n\text{-boundaries}\} \subseteq C_n$. So, $B_n \subseteq Z_n$. We define the n**th homology group** of a chain complex C by

$$H_n(C) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{Z_n}{B_n}.$$

Elements of $H_n(C)$ are called n**th homology classes**. They are cosets of B_n in Z_n.

Definition 4.3.8. Two n-cycles z and z' are **homologous** if $[z] = [z']$, i.e., if $z - z'$ is a boundary, i.e., if $z - z' = \partial c$ for some $c \in C_{n+1}$.

Definition 4.3.9. A chain complex C is **acyclic**, or **exact**, if $H_n(C_n) = 0$ for all $n \in \mathbb{Z}$. This means that $\ker \partial_n = \operatorname{im} \partial_{n+1}$ for all n. So, every cycle is a boundary.

Note: In a sense, homology measures the extent to which a chain complex fails to be exact.

Definition 4.3.10. A chain complex C is **bounded** if $C_n = 0$ for all but finitely many n.

Definition 4.3.11. A chain complex C is **supported in nonnegative degree** if $C_n = 0$ for all $n < 0$.

4.3.3 Chain Maps

Definition 4.3.12. For C, C' chain complexes (of R-modules over the same ring R) with differentials ∂, ∂', respectively, a **chain map** $f : C \to C'$ is a family of homomorphisms $f_n : C_n \to C'_n$, $n \in \mathbb{Z}$, such that

$$f_{n-1} \circ \partial_n = \partial'_n \circ f_n$$

for all $n \in \mathbb{Z}$. A chain map looks like

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \xrightarrow{\partial'_{n-1}} & \dots \end{array}$$

where each square commutes.

A chain map $f : C \to C'$ is an **isomorphism** if f_n is an isomorphism for all n. In this case, $f^{-1} : C' \to C$ exists and is given by $(f^{-1})_n = (f_n)^{-1}$.

Remark: Any chain map $f : C \to C'$ induces a maps

$$f_* : H_n(C) \to H_n(C') \qquad (n \in \mathbb{Z})$$

given by $[z] \mapsto F_*([z]) = [f_n(z)]$, where $z \in \ker \partial_n$. In general,

(i) $(1_C)_* = 1_{H_n(C)}$ for all $n \in \mathbb{Z}$;

(ii) $(f \circ g)_n = f_n \circ g_n$;

(iii) $(f \circ g)_* = f_* \circ g_*$;

(iv) $(f + g)_* = f_* + g_*$.

Note: For each n, H_n is a functor $\mathsf{Ch} \to \mathsf{Ab}$.

Definition 4.3.13. A chain map $f : C \to C'$ is a **quasi-isomorphism** if $f_* : H_n(C) \to H_n(C')$ is an isomorphism for all $n \in \mathbb{Z}$.

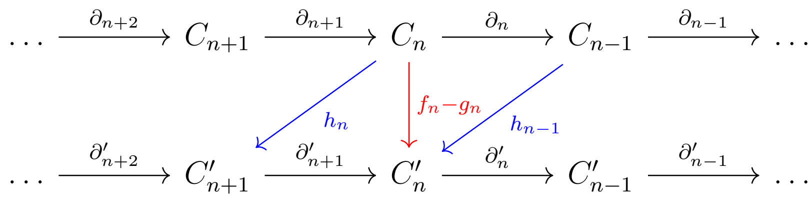
4.3.4 Chain Homotopies

Definition 4.3.14. Let $f, g : C \to C'$ be chain maps. A **chain homotopy** h between f and g is a family of homomorphisms $h_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = (h_{n-1} \circ \partial_n) + (\partial'_{n+1} \circ h_n)$$

for all $n \in \mathbb{Z}$.

A chain homotopy looks like



Theorem 4.3.1. We have that

(i) Homotopies are compatible with sums of chain maps, i.e., if $f \simeq_{h_1} f'$ and $g \simeq_{h_2} g'$, then $f + g \simeq_{h_1+h_2} f' + g'$.

(ii) If $f \simeq_h f'$ and $f' \simeq_{h'} f''$, then $f \simeq_{h+h'} f''$.

(iii) If $f \simeq_h g$, then $g \simeq_{-h} f$.

Theorem 4.3.2. Homotopic chain maps induce the same map in homology.

Definition 4.3.15. A chain map $f : C \to C'$ is a **homotopy equivalence** if there is a chain map $g : C' \to C$ such that $f \circ g \simeq 1_{C'}$ and $g \circ f \simeq 1_C$.

Note: Homotopy inverses are unique up to homotopy, if they exist. If f and g are homotopy inverses, then f_* and g_* are mutually inverse isomorphisms. So, any homotopy equivalence is a quasi-isomorphism. In general,

$$\text{chain isomorphism} \implies \text{homotopy equivalence} \implies \text{quasi-isomorphism},$$

and the first two are preserved under additive functors.

Definition 4.3.16. A chain map $f : C \to C'$ is **nullhomotopic** if $f \simeq 0$, the zero chain map.

Definition 4.3.17. A chain complex C is **contractible** is $C \simeq 0$, the zero chain complex.

Theorem 4.3.3. If a chain complex C is contractible, then it is acyclic.

Theorem 4.3.4. If a chain complex C is acyclic, then $0 : 0 \to C$ is a quasi-isomorphism.

Theorem 4.3.5. A chain complex C is acyclic if and only if C is quasi-isomorphic to the zero complex.

Theorem 4.3.6. If $f : C \to C'$ is a chain map between *bounded below* chain complexes of *free Abelian groups (or projective modules)*, then

$$f \text{ is a homotopy equivalence} \iff f \text{ is a quasi-isomorphism}.$$

4.3.5 Mapping Cones

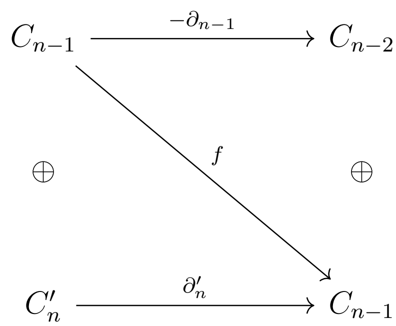
Definition 4.3.18. Let $f : C \to C'$ be a chain map. The **mapping cone** of f is the chain complex $C(f)$ given by $C(f)_n = C_{n-1} \oplus C'_n$ and with

$$\partial_{C(f),n} : C(f)_n \to C(f)_{n-1}$$

given by

$$\partial_{C(f),n} = \begin{pmatrix} -\partial_{n-1} & 0 \\ f & \partial'_n \end{pmatrix}.$$

As a picture, we have



Theorem 4.3.7. We have that

$$\begin{array}{ccc} f \text{ is a homotopy equivalence} & \iff & C(f) \text{ is contractible} \\ \Downarrow & & \Downarrow \\ f \text{ is a quasi-isomorphism} & \iff & C(f) \text{ is acyclic} \end{array}$$

4.3.6 Affine Simplices & Singular Simplicies

Definition 4.3.19. Let $v_0, v_1, \ldots, v_n$ be $n+1$ points in $\mathbb{R}^m$, where $m \geq n$. Assume $v_0, v_1, \ldots, v_n$ are in "general position", meaning that they are not contained in any affine hyperplane of dimension less than n. Equivalently, assume $v_1 - v_0$, $v_2 - v_0$, $\ldots$, $v_n - v_0$ are linearly independent. Denote by $[v_0, v_1, \ldots, v_n]$ the convex hull of $v_0, v_1, \ldots, v_n$. We call $[v_0, v_1, \ldots, v_n]$ the **affine n-simplex** spanned by $v_0, v_1, \ldots, v_n$, and we call the v_i **vertices** of the simplex.

Definition 4.3.20. The n-simplex $\Delta^n = [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$ is called the **standard n-simplex**. As a set, $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \; \forall i\}$.

Definition 4.3.21. For an n-simplex $[v_0, \dots, v_n]$, the i**th face** of $[v_0, \dots, v_n]$ is

$$[v_0, \dots, \hat{v}_i, \dots, v_n] = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n],$$

which is an $(n-1)$-simplex contained in $[v_0, \dots, v_n]$

Definition 4.3.22. For each affine n-simplex $[v_0, \dots, v_n]$, there exists a canonical homeomorphism

$$f_{[v_0, \dots, v_n]} : \Delta^n \to [v_0, \dots, v_n]$$

which sends e_i to v_i for all i. Explicitly,

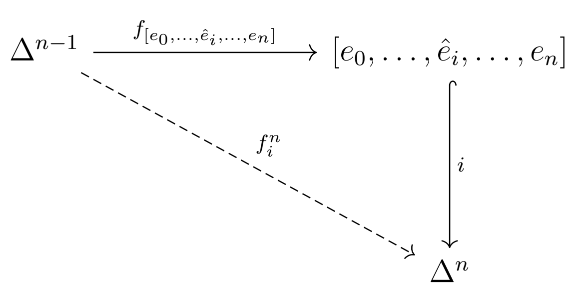
$$f_{[v_0, \dots, v_n]}(t_0, \dots, t_n) = \sum_{i=0}^{n} t_i v_i.$$

The i**th face map**

$$f_i^n : \Delta^{n-1} \to \Delta^n$$

is defined to be the composition $i \circ f_{[e_0, \dots, \hat{e}_i, \dots, e_n]}$, where i is the inclusion $i : [e_0, \dots, \hat{e}_i, \dots, e_n] \hookrightarrow \Delta^n$. In coordinates, $f_i^n(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$.

The following commutative diagram describes the ith face map:



Definition 4.3.23. A **singular n-simplex** in a topological space X is a continuous map

$$\sigma : \Delta^n \to X.$$

We denote by $S_n(X)$ the set of all singular n-simplices in X.

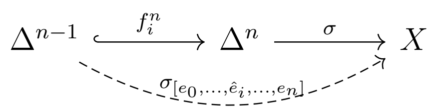
Note: The set of all singular 0-simplices in X is the set of all maps from a point into X. This set $S_0(X)$ is then in 1-1 correspondence with points in X.

Note: The set $S_1(X)$ is in 1-1 correspondence with the set of all paths in X, as the standard 1-simplex Δ^1 is homeomorphic to the closed interval I.

Definition 4.3.24. Given $\sigma \in S_n(X)$, write

$$\sigma_{[e_0,\ldots,\hat{e}_i,\ldots,e_n]} = \sigma \circ f_i^n \in S_{n-1}(X).$$

The following commutative diagram describes the map $\sigma_{[e_0,\ldots,\hat{e}_i,\ldots,e_n]}$:



4.3.7 Singular Chain Complexes

Definition 4.3.25. Given a topological space X we can define a chain complex $C(X)$ as follows:

- For $n < 0$, $C_n(X) = 0$.
- For $n \geq 0$,

$$\begin{aligned}
C_n(X) &= \text{the free Abelian group generated by } S_n(X) \\
&= \text{the free } \mathbb{Z}\text{-module spanned by } Sn(X) \\
&= \mathbb{Z}\text{-span}(S_n(X)) \\
&= \bigoplus_{\sigma \in S_n(X)} \mathbb{Z}\sigma \quad (\text{where } \mathbb{Z}\sigma \text{ is the free } \mathbb{Z}\text{-modules of rank 1 spanned by } \sigma) \\
&= \left\{ \text{formal finite sums } \sum n_i \sigma_i \mid n_i \in \mathbb{Z}, \sigma_i \in S_n(X) \right\}.
\end{aligned}$$

The formal finite sums $\sum n_i \sigma_i$, where $n_i \in \mathbb{Z}$ and $\sigma_i \in S_n(X)$, are called **singular n-chains** in X.

Define $\partial_n : C_n(X) \to C_{n-1}(X)$ by

- For $n \leq 0$, $\partial_n = 0$.
- For $n > 0$, define ∂_n on n-simplices by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \circ f_i^n) \in C_{n-1}(X)$$

and extended to n-chains by linearity:

$$\partial_n(\sum n_i\sigma_i) = \sum n_i\partial_n(\sigma_i).$$

In fact, ∂_n is a differential, and $C(X)$ with this differential is a chain complex called the **singular chain complex** of X.

Definition 4.3.26. The **nth singular homology** of X is

$$H_n(X) = \frac{\ker \partial_n}{\text{im } \partial_{n+1}}.$$

We may write $Z_n(X) = \ker \partial_n = \{\text{singular } n\text{-cycles in } X\}$ and $B_n = \text{ im } \partial_{n+1} = \{\text{singular } n\text{-boundaries in } X\}$.

4.3.8 Homology and Path Components

Theorem 4.3.8. Let X be a space with path components X_α. Then,

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$$

for all n.

Theorem 4.3.9. For a space X,

$$\begin{aligned} H_0(X) &\cong \mathbb{Z}\text{-span}\{\text{path components of } X\} \\ &= \bigoplus_\alpha \mathbb{Z}, \end{aligned}$$

where there is one copy of $\mathbb{Z}$ in the sum for each path component X_α of X. So, $H_0(X)$ is always a free Abelian group of rank equal to the number of path components of X. In particular, if $X \neq \emptyset$, then $\text{rank}(H_0(X)) \geq 1$.

4.3.9 Reduced Singular Homology

Definition 4.3.27. Let $X \neq \emptyset$ be any space. Define

$$\varepsilon : C_0(X) \to \mathbb{Z}$$

by

$$\sum n_i\sigma_i \mapsto \sum n_i \in \mathbb{Z}.$$

Since X is nonempty, ε is surjective. Note that $\varepsilon \circ \partial_1 = 0$.

The **augmented singular chain complex** of X is the chain complex

$$\ldots' \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \ldots$$

The **reduced singular homology** of X is the homology of this chain complex. Denote the nth reduced homology of X by $\tilde{H}_n(X)$.

Theorem 4.3.10. For $X \neq \emptyset$, $\tilde{H}_n(X) \cong H_n(X)$ for all $n \neq 0$. Furthermore, $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$, and so $\tilde{H}_0(X)$ is a free Abelian group of rank equal to $\text{rank}(H_0(X)) - 1$.

Example: The nth reduced homology of a point is zero for all n. So, the augmented singular chain complex of a point is acyclic.

4.3.10 Induced Maps & Homotopy Invariance

Definition 4.3.28. Let $f : X \to Y$ be a continuous map. Define

$$f_\# : C_n(X) \to C_n(Y)$$

by $\sigma \mapsto f \circ \sigma$, and denote $f \circ \sigma$ by $f_\#(\sigma)$. Extend by linearity to get that $\sum n_i\sigma_i \mapsto \sum n_i f_\#(\sigma_i)$.

Note that $\partial \circ f_\# = f_\# \circ \partial$. In fact, $f\#$ is a chain map, and it induces a map

$$f_* : H_n(X) \to H_n(Y)$$

for all n, given by $[z] \mapsto [f_\#(x)] = f_*([z])$, where z is an n-cycle.

Theorem 4.3.11. We have the following properties of $f_\#$:

(i) $(1_X)_\# = 1_{C(X)}$ and $(1_X)_{*,n} = 1_{H_n(X)}$;

(ii) $(f \circ g)_\# = f_\# \circ g_\#$ and $(f \circ g)_{*,n} = f_{*,n} \circ g_{*,n}$;

(iii) If f is a homoeomorphism, then both $f_\#$ and $f_{*,n}$ are isomorphisms.

Corollary 4.3.1. If X and Y are homeomorphic, then $C(X)$ and $C(Y)$ are homeomorphic and $H_n(X) \cong H_n(Y)$ for all n.

Corollary 4.3.2. For each n, H_n may be viewed as a functor $\mathsf{Top} \to \mathsf{Ab}$.

Theorem 4.3.12. Let $f, g : X \to Y$ be continuous. If $f \simeq g$, then $f_\# \simeq g_\#$, and so $f_{*,n} = g_{*,n}$ for all n.

Corollary 4.3.3. If $X \simeq Y$, i.e., if X and Y are homotopy equivalent, then $C(X) \simeq C(Y)$, and so $H_n(X) \cong H_n(Y)$ for all n.

4.3.11 Exact Sequences

Definition 4.3.29. A sequence

$$\ldots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \ldots$$

of Abelian groups (or modules) and homomorphisms is **exact at** A_n if $\operatorname{im} f_{n+1} = \ker f_n$. If the sequence is exact at A_n for all n such that A_n is not an endpoint of the sequence, then we call the entire sequence **exact**.

Theorem 4.3.13. We have that

(i) The sequence

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact if and only if f is injective;

(ii) The sequence

$$B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if g is surjective;

(iii) The sequence

$$0 \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if g is bijective if and only if g is an isomorphism;

(iv) The sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if f is injective, g is surjective, and $\operatorname{im} f = \ker g$. A sequence of this form (five terms, first and last terms equal to zero, exact) is called a **short exact sequence**, or **SES**.

Definition 4.3.30. An **SES of chain complexes** is a sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

of chain complexes and chain maps such that for all n,

$$0 \longrightarrow A_n \xrightarrow{\ f_n\ } B_n \xrightarrow{\ g_n\ } C_n \longrightarrow 0$$

is an SES of Abelian groups (or modules).

The following commutative diagram (in which each row is exact) describes short exact sequences of chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Note: Given an exact sequence of chain complexes

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

we can look at

$$H_n(A) \xrightarrow{\ f_{*,n}\ } H_n(B) \xrightarrow{\ g_{*,n}\ } H_n(C)$$

This sequence is not necessarily exact. However, it is true that $\operatorname{im} f_{*,n} \subseteq \ker g_{*,n}$ for all n.

Lemma 4.3.1. Suppose

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence of chain complexes such that for some n,

(i) f_{n-1} is injective, and

(ii) g_{n+1} is surjective.

Then,

$$H_n(A) \xrightarrow{f_{*,n}} H_n(B) \xrightarrow{g_{*,n}} H_n(C)$$

is exact.

Theorem 4.3.14. Suppose

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of chain complexes. Then, there exists a "natural" homomorphism

$$\delta_n : H_n(C) \to H_{n-1}(A)$$

such that the following sequence is exact:

$$\begin{array}{ccccc} & & \dots & \xrightarrow{g_{*,n+1}} & H_{n+1}(C) \\ & & & \swarrow_{\delta_{n+1}} & \\ H_n(A) & \xrightarrow{f_{*,n}} & H_n(B) & \xrightarrow{g_{*,n}} & H_n(C) \\ & & & \swarrow_{\delta_n} & \\ H_{n-1}(A) & \xrightarrow{f_{*,n-1}} & H_{n-1}(B) & \xrightarrow{g_{*,n-1}} & H_{n-1}(C) \\ & & & \swarrow_{\delta_{n-1}} & \\ H_{n-2}(A) & \xrightarrow{f_{*,n-2}} & \dots & & \end{array}$$

We call such a sequence a **long exact sequence**, or **LES**, and we call the δ_n **connecting homomorphisms**.

Note: "Natural", as used in the previous theorem, means that given a commutative diagram of chain complexes and chain maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

in which each row is exact, we have that the following diagram commutes:

$$\begin{array}{ccc} H_n(C) & \xrightarrow{\delta_n} & H_{n-1}(A) \\ \downarrow c_{*,n} & & \downarrow a_{*,n-1} \\ H_n(C') & \xrightarrow{\delta'_n} & H_{n-1}(A') \end{array}$$

Corollary 4.3.4. If

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an SES of chain complexes, then

(i) If A is acyclic, then $H_n(B) \cong H_n(C)$ for all n;

(ii) If B is acyclic, then $H_n(C) \cong H_{n-1}(A)$ for all n;

(iii) If C is acyclic, then $H_n(A) \cong H_n(B)$ for all n.

Remark: If C is a complex of free Abelian groups, then

(i) $f_n : A_n \to B_n$ has a left-inverse p_n,

(ii) $g_n : B_n \to C_n$ has a right inverse s_n,

and $\delta_n([c]) = [p_{n-1}(\partial(s_n(c)))]$.

4.3.12 Relative Singular Homology

Definition 4.3.31. Let (B, ∂) be a chain complex. A **subcomplex** of (B, ∂) is a chain complex (A, ∂') such that

(i) $A_n \leq B_n$ for all n,

(ii) $\partial_n(A_n) \subseteq A_{n-1}$ for all n, and

(iii) $\partial_n|_{A_n} = \partial'_n$ for all n.

Definition 4.3.32. Let (B, ∂) be a chain complex with (A, ∂') as a subcomplex. The **quotient complex** $(B/A, \bar{\partial})$ is defined by

- $(B/A)_n = B_n/A_n$ and
- $\bar{\partial}_n : B_n/A_n \to B_{n-1}/A_{n-1}$ induced by $\partial_n : B_n \to B_{n-1}$.

By construction,

$$0 \longrightarrow A \xhookrightarrow{\text{inclusion}} B \xrightarrow{\text{quotient map}} B/A \longrightarrow 0$$

is an SES.

Theorem 4.3.15. Let A be a subcomplex of a chain complex B. Then,

(i) If A is acyclic, then $H_n(B) \cong H_n(B/A)$ for all n;

(ii) If B is acyclic, then $H_n(B/A) \cong H_{n-1}(A)$ for all n;

(iii) If B/A is acyclic, then $H_n(A) \cong H_n(B)$ for all n.

Definition 4.3.33. Let (X, A) be a pair of spaces, by which we mean that X is a space and $A \subseteq X$. We can view singular simplices in A as singular simplices in X, and so we can view $C(A)$ as a subcomplex of $C(X)$. This allows us to form the quotient $C(X)/C(A)$, which we call the **relative singular chain complex** of (X, A), and we denote this quotient complex $C(X, A)$. Explicitly, $C_n(X, A) = C_n(X)/C_n(A)$ and $\partial_n^{(X,A)}$ is induced by ∂_n^X.

The nth homology of $C(X, A)$ is called the **nth relative homology of** (X, A), denoted $H_n(X, A)$.

Note: $C_n(X, A)$ is a free Abelian group spanned by all singular n-simplices in X that are *not fully contained in* A. Furthermore,

$$C_n(X) \cong C_n(X, A) \oplus C_n(A).$$

However, ∂^X does not preserve this decomposition.

Lemma 4.3.2 (Splitting Lemma)**.** For a short exact sequence

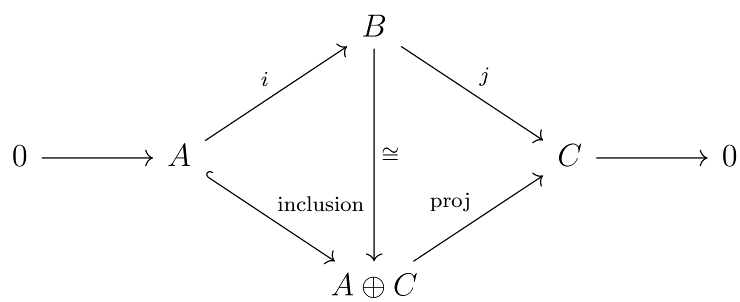
$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 0$$

of Abelian groups, the following are equivalent:

(i) There is a homomorphism $p : B \to A$ such that $p \circ i = \mathrm{id} : A \to A$;

(ii) There is a homomorphism $s : C \to B$ such that $j \circ s = \mathrm{id} : C \to C$;

(iii) There is an isomorphism $B \cong A \oplus C$ making the following diagram commute:



In particular, we get the following theorem.

Theorem 4.3.16. Let X be a space with a subspace A. If there is a retraction $X \to A$, then

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

Theorem 4.3.17. Let $c \in C_n(X)$. Then, c represents a cycle in $C(X, A)$ if and only if $\partial_n c \in C_{n-1}(A)$. Equivalently, c represents a cycle in $C(X, A)$ if and only if $c \in \partial_n^{-1}(C_{n-1}(A))$. So,

$$H_n(X, A) \cong \frac{\partial_n^{-1}(C_{n-1}(A))}{\mathrm{im}\ \partial_{n+1} + C_n(A)}.$$

Remark: Because $C(X, A)$ is a quotient,

$$0 \longrightarrow C(A) \xhookrightarrow{\text{inclusion}} C(X) \xrightarrow{\text{quotient map}} C(X, A) \longrightarrow 0$$

is an SES. Then, there exists an induced LES in homology:

$$
\begin{array}{ccccc}
 & & \dots & \longrightarrow & H_{n+1}(X,A) \\
 & & & \xrightarrow{\delta_{n+1}} & \\
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) \\
 & & & \xrightarrow{\delta_{n}} & \\
H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(X,A) \\
 & & & \xrightarrow{\delta_{n-1}} & \\
H_{n-2}(A) & \longrightarrow & \dots & &
\end{array}
$$

4.3.13 Reduced Relative Singular Homology

Definition 4.3.34. Let (X, A) be a pair. Let $\tilde{C}(X)$ and $\tilde{C}(A)$ be the augmented chain complexes of X and A, respectively. Define

$$\tilde{C}(X, A) = \tilde{C}(X)/\tilde{C}(A).$$

Note that the copies of $\mathbb{Z}$ in the numerator and denominator cancel, and so

$$\begin{aligned}
\tilde{C}(X, A) &= \tilde{C}(X)/\tilde{C}(A) \\
&= C(X)/C(A) \\
&= C(X, A).
\end{aligned}$$

It follows that there is no difference between $\tilde{H}_n(X, A)$ and $H_n(X, A)$ for all n.

Remark: The exists an SES

$$0 \longrightarrow \tilde{C}(A) \longrightarrow \tilde{C}(X) \longrightarrow \tilde{C}(X, A) = C(X, A) \longrightarrow 0$$

and an induced LES in homology:

$$
\begin{array}{ccccc}
 & & \dots & \longrightarrow & H_{n+1}(X,A) \\
 & & \xleftarrow{\delta_{n+1}} & & \\
\tilde{H}_n(A) & \longrightarrow & \tilde{H}_n(X) & \longrightarrow & H_n(X,A) \\
 & & \xleftarrow{\delta_{n}} & & \\
\tilde{H}_{n-1}(A) & \longrightarrow & \tilde{H}_{n-1}(X) & \longrightarrow & H_{n-1}(X,A) \\
 & & \xleftarrow{\delta_{n-1}} & & \\
\tilde{H}_{n-2}(A) & \longrightarrow & \dots & &
\end{array}
$$

Note: If A is a point, then $\tilde{H}_n(A) = 0$ for all n, and so $\tilde{H}_n(X) \cong H_n(X, A)$ for all n by exactness. This isomorphism is "natural" with respect to based maps.

Note: $H_n(X) \cong H_n(X, \emptyset)$ for all n.

4.3.14 Induced Maps in Relative Singular Homology

Theorem 4.3.18. Let $f : (X, A) \to (Y, B)$ be a continuous map of pairs. Then, the chain map $f_\# : C(X) \to C(Y)$ descends to a chain map $\bar{f}_\# : C(X, A) \to C(Y, B)$. Furthermore, there exists an induced map

$$\bar{f}_{*,n} : H_n(X, A) \to H_n(Y, B)$$

for all n, and we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C(A) & \longrightarrow & C(X) & \longrightarrow & C(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (f|_A)_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle \bar{f}_\#} & & \\
0 & \longrightarrow & C(B) & \longrightarrow & C(Y) & \longrightarrow & C(Y,B) & \longrightarrow & 0
\end{array}
$$

There also exists a corresponding commutative diagram for the induced long exact sequences for (X, A) and (Y, B).

Theorem 4.3.19. Let $f, g : (X, A) \to (Y, B)$ be continuous maps of pairs. If $f \simeq g$ rel A, then $\bar{f}_\# \simeq \bar{g}_\#$ and so $\bar{f}_{*,n} \simeq \bar{g}_{*,n}$ for all n.

In fact, this theorem holds if the assumption that "$f \simeq g$ rel A" is replaced by the assumption that each f_t in the homotopy between f and g maps A to B.

4.3.15 Good Pairs

Definition 4.3.35. A pair (X, A) is **good** if $A \subseteq X$ is closed, $A \neq \emptyset$, and there exists an open neighborhood U of A which deformation retracts to A.

Example: If A has a mapping cylinder neighborhood N, then (X, A) is good. The neighborhood U is simply $N \smallsetminus \partial N$.

Example: If (X, A) is a CW pair with A nonempty, then (X, A) is good.

Theorem 4.3.20. If (X, A) is good, then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism

$$\bar{q}_{*,n} : H_n(X, A) \to H_n(X/A, A/A).$$

Since A/A is just a point, we have that

$$\tilde{H}_n(X/A) \cong H_n(X/A, A/A) \cong H_n(X, A)$$

for all n. The isomorphism $\tilde{H}_n(X, A) \to H_n(X/A, A/A)$ is induced by the quotient map $\tilde{C}(X, A) \to \tilde{C}(X/A, A/A) = C(X/A, A/A)$.

Remark: If $f : (X, A) \to (Y, B)$ is a continuous map of good pairs, then the diagram

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\text{isomorphism}} & \tilde{H}_n(X/A) \\ \downarrow{\scriptstyle \bar{f}_{*,n}} & & \downarrow{\scriptstyle (\bar{f})_{*,n}} \\ H_n(Y, B) & \xrightarrow{\text{isomorphism}} & \tilde{H}_n(Y/B) \end{array}$$

commutes.

Corollary 4.3.5. If (X, A) is good, the there exists a natural LES

$$\begin{array}{ccccc} & & \dots & \longrightarrow & \tilde{H}_{n+1}(X/A) \\ & & \delta_{n+1} & & \\ H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & \tilde{H}_n(X/A) \\ & & \delta_n & & \\ H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \tilde{H}_{n-1}(X/A) \\ & & \delta_{n-1} & & \\ H_{n-2}(A) & \longrightarrow & \dots & & \end{array}$$

There exists an analogous LES with $H_n(A)$ and $H_n(X)$ replaced by $\tilde{H}_n(A)$ and $\tilde{H}_n(X)$, respectively, for all n.

4.3.16 Homology of Spheres

Remark: The pair $(D^n, \partial D^n = S^{n-1})$ is good for all $n > 0$.

Theorem 4.3.21. The homology of spheres is as follows:

$$\tilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases}$$

Corollary 4.3.6. We have that S^n and S^m are not homotopy equivalent for $n \neq m$, because their homologies differ.

Corollary 4.3.7. We have that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$. However, they are homotopy equivalent, since they are each homotopy equivalent to a point.

4.3.17 Simplicial Homology

Definition 4.3.36. A **Δ-complex** is a topological space X together with a collection C of singular simplices in X such that

(i) $\sigma|_{\overset{\circ}{\Delta^{n(\sigma)}}}$ is injective for all $\sigma \in C$, where "∘" means "interior" and $n(\sigma)$ signifies that $\sigma \in S_n(X)$;

(ii) As a set, X is a disjoint union of the $\sigma|_{\overset{\circ}{\Delta^{n(\sigma)}}}$ for all $\sigma \in C$;

(iii) If $\sigma \in C$ and $n(\sigma) > 0$, then $\sigma \circ f_i^{n(\sigma)} \in \mathbb{C}$ for all $i = 0, 1, \ldots, n(\sigma)$;

(iv) $U \subseteq X$ is open if and only if $\sigma^{-1}(U) \subseteq \Delta^{n(\sigma)}$ is open for all $\sigma \in C$.

Definition 4.3.37. Let (X, C) be a Δ-complex. Define $S_n^{\Delta} = S_n(X) \cap C$ and $C_n^{\Delta} = \mathbb{Z}\text{-span} S_n^{\Delta}(X) \subseteq C_n(X)$. Forming $C^{\Delta}(X)$ from the C_n^{Δ}, we get that $C^{\Delta}(X)$ is a subcomplex of $C(X)$. The nth homology of $C^{\Delta}(X)$, denoted $H_n^{\Delta}(X)$, is the n**th simplicial homology** of (X, C).

Theorem 4.3.22. If (X, C) is a Δ-complex, then $C^{\Delta}(X) \hookrightarrow C(X)$ induces an isomorphism $H_n^{\Delta}(X) \to H_n(X)$.

Corollary 4.3.8. The isomorphism type of $H_n^{\Delta}(X)$ is an invariant of the homotopy type of X (because the isomorphism type of $H_n(X)$ is).

4.3.18 Homology of the Torus

Theorem 4.3.23. The (simplicial) homology of the torus $S^1 \times S^1$ is

$$H_i(S^1 \times S^1) = \begin{cases} \mathbb{Z}, & i = 0, 2, \\ \mathbb{Z}^2, & i = 1, \\ 0, & i \neq 0, 1, 2. \end{cases}$$

4.3.19 Homology of the Projective Plane

Theorem 4.3.24. The (simplicial) homology of the projective plane P is

$$H_i(P) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}_2, & i = 1, \\ 0, & i \neq 0, 1. \end{cases}$$

Theorem 4.3.25. The (simplicial) homology of the n-dimensional projective plane $\mathbb{R}P^n$ is

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & i = 0, n \text{ odd}, \\ \mathbb{Z}_2, & i \text{ odd}, 0 < i < n, \\ 0, & \text{else}. \end{cases}$$

Note that the projective plane P is $\mathbb{R}P^2$.

4.3.20 Homology of the Klein Bottle

Theorem 4.3.26. The (simplicial) homology of the Klein bottle K is

$$H_i(K) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}, & i = 1, \\ 0, & \text{else}. \end{cases}$$

4.3.21 Homology of Genus g Orientable Surface

Theorem 4.3.27. The (simplicial) homology of the genus g orientable surface M_g is

$$H_i(M_g) = \begin{cases} \mathbb{Z}, & i = 0, 2, \\ \mathbb{Z}^{2g}, & i = 1, \\ 0, & i \neq 0, 1, 2. \end{cases}$$

Theorem 4.3.28. The orientable space of genus g with n boundary components deformation retracts onto the wedge of $2g - n + 1$ circles.

4.3.22 Subdivision ($C^{\mathcal{U}}(X)$)

Definition 4.3.38. Let X be a topological space, and let $\mathcal{U}$ be a collection of subsets A_i of X such that

$$\bigcup_{A_i \in \mathcal{U}} \text{int } A_i = X.$$

Define

$$\begin{aligned} C_n^{\mathcal{U}}(X) &= \mathbb{Z}\text{-span}\{\text{singular } n\text{-simplices in } X \text{ which are fully contained in one of the sets } A_i\} \\ &= \sum_{A_i \in \mathcal{U}} C_n(A_i) \\ &\subseteq C_n(X). \end{aligned}$$

So, the $C_n^{\mathcal{U}}$ form a subcomplex $C^{\mathcal{U}}(X) \subseteq C(X)$.

Definition 4.3.39. The n**th homology** of $C^{\mathcal{U}}(X)$ is $H_n^{\mathcal{U}}(X)$.

Theorem 4.3.29. The inclusion $i : C^{\mathcal{U}}(X) \hookrightarrow C(X)$ is a homotopy equivalence, and there exists a homotopy inverse $\zeta : C(X) \to C^{\mathcal{U}}(X)$ such that

- $\zeta \circ i = \text{id}_{C^{\mathcal{U}}(X)}$ and
- $i \circ \zeta = \text{id}_{C(X)} - (\partial \circ h) - (h \circ \partial)$ for a homotopy h with $h \circ i = 0$.

Note: The proof of the above theorem uses barycentric subdivision (hence the name of the section) and the Lebesgue Number Lemma.

Corollary 4.3.9. We have that $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.

4.3.23 Excision

Theorem 4.3.30 (Excision Theorem - Version 1)**.** Let X be a topological space, $A, B \subseteq X$, and $\text{int } A \cup \text{int } B = X$. Then, $H_n(X, A) \cong H_n(B, A \cap B)$ for all n. Furthermore, the isomorphism $H_n(B, A \cap B) \to H_n(X, A)$ is induced by the inclusion $(B, A \cap B) \hookrightarrow (X, A)$.

Theorem 4.3.31 (Excision Theorem - Version 2)**.** For X a topological space with $Z \subseteq A \subseteq X$, if we have that $\bar{Z} \subseteq \text{int } A$, then

$$H_n(X, A) \cong H_n(X \setminus Z, A \setminus Z)$$

for all n. Furthermore, the isomorphism $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ is induced by the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$.

Corollary 4.3.10. If $x \in X$, $\{x\}$ is closed in X, and $U \ni x$ is open, then

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$$

for all n.

Definition 4.3.40. We call $H_n(X, X \setminus \{x\})$ the n**th local homology** of X at x.

Theorem 4.3.32. If $U \subseteq \mathbb{R}^m$ and $V \subseteq R^n$ are nonempty open subsets and $m \neq n$, then U and V are not homeomorphic.

Theorem 4.3.33. If $A, B \subseteq X$ are subcomplexes of a CW complex X such that $A \cup B = X$, then $H_n(X, A) \cong H_n(B, A \cap B)$ for all n. The isomorphism $H_n(B, A \cap B) \to H_n(X, A)$ is induced by the inclusion $(B, A \cap B) \hookrightarrow (X, A)$.

Theorem 4.3.34. If X_α are spaces with base points $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good, then the inclusions

$$i_\alpha : X_\alpha \hookrightarrow \bigvee_\beta X_\beta,$$

where $\bigvee_\beta X_\beta$ is the wedge sum form by identifying the base points, induce an isomorphism

$$\bigoplus_n \tilde{H}_n(X_\alpha) \cong \tilde{H}_n\left(\bigvee_\alpha X_\alpha\right).$$

4.3.24 The Mayer-Vietoris Sequence

Theorem 4.3.35 (Mayer-Vietoris Sequence)**.** Let X be a space, $A, B \subseteq X$ such that $\text{int}A \cup \text{int}B = X$, and $\mathcal{U} = \{A, B\}$. Then, $C^{\mathcal{U}}(X) = C(A) + C(B) \subseteq C(X)$. Furthermore, there exists an SES

$$0 \longrightarrow C(A \cap B) \xrightarrow{\varphi} C(A) \oplus C(B) \xrightarrow{\psi} C^{\mathcal{U}}(X) \longrightarrow 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$. We then also have an induced LES in homology:

$$\begin{array}{ccccc}
 & & \dots & \longrightarrow & H_{n+1}^{\mathcal{U}}(X) \\
 & & \xrightarrow{\delta_{n+1}} & & \\
H_n(A \cap B) & \xrightarrow{\varphi_*} & H_n(A) \oplus H_n(B) & \xrightarrow{\psi_*} & H_n^{\mathcal{U}}(X) \\
 & & \xrightarrow{\delta_n} & & \\
H_{n-1}(A \cap B) & \xrightarrow{\varphi_*} & H_{n-1}(A) \oplus H_{n-1}(B) & \xrightarrow{\psi_*} & H_{n-1}^{\mathcal{U}}(X) \\
 & & \xrightarrow{\delta_{n-1}} & & \\
H_{n-2}(A \cap B) & \xrightarrow{\varphi_*} & \dots & &
\end{array}$$

This LES is called the **Mayer-Vietoris Sequence**. Recall that $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n, and so we may as well write the Mayer-Vietoris Sequence as

$$\begin{array}{ccccc}
 & & \dots & \longrightarrow & H_{n+1}(X) \\
 & & \xrightarrow{\delta_{n+1}} & & \\
H_n(A \cap B) & \xrightarrow{\varphi_*} & H_n(A) \oplus H_n(B) & \xrightarrow{\psi_*} & H_n(X) \\
 & & \xrightarrow{\delta_n} & & \\
H_{n-1}(A \cap B) & \xrightarrow{\varphi_*} & H_{n-1}(A) \oplus H_{n-1}(B) & \xrightarrow{\psi_*} & H_{n-1}(X) \\
 & & \xrightarrow{\delta_{n-1}} & & \\
H_{n-2}(A \cap B) & \xrightarrow{\varphi_*} & \dots & &
\end{array}$$

Remark: We can describe δ_n explicitly. If $[z] \in H_n^{\mathcal{U}}(X) \cong H_n(X)$, we can write $z = x + y \in C_n(A) + C_n(B)$. Then, $\partial z = \partial x + \partial y = 0$ because z is a cycle. This gives us that $\partial x = -\partial y$, and so ∂x is an $(n-1)$-chain in $A \cap B$. Note that ∂x is a cycle, since $\partial(\partial x) = \partial^2 x = 0$. We can then define $\delta_n([z]) = [\partial x] = -[\partial y]$.

Remark: A Mayer-Vietoris Sequence also exists if $X = A \cup B$ with $A, B \subseteq X$ such that there exists open neighborhood U and V of A and B, respectively, such that

(i) U deformation retracts to A,

(ii) V deformation retracts to B; and

(iii) $U \cap V$ deformation retracts to $A \cap B$.

For example, these conditions are satisfied if X is a CW complex and A and B are subcomplexes which together cover X.

Note: A Mayer-Vietoris Sequence also exists for reduced homology.

Note: The exists a relative version of the Mayer-Vietoris Sequence for pairs $(X, Y) = (A \cup B, C \cup D)$ where $\text{int } A \cup \text{int } B = X$ and $\text{int } C \cup \text{int } D = Y$.

4.3.25 Degree (Maps)

Definition 4.3.41. For $n > 0$, let $f : S^n \to S^n$ be a continuous map. Then, f induces a homomorphism

$$f : H_n(S^n) \cong \mathbb{Z} \to H_n(S^n) \cong \mathbb{Z}.$$

So, f is given by multiplication by some integer m, as these are the only group homomorphisms $\mathbb{Z} \to \mathbb{Z}$. Call this m the **degree** of the map f, and denote it $\deg(f)$.

Theorem 4.3.36. We have the following properties of the degree:

(i) $\deg(\text{id}_{S^n}) = 1$;

(ii) $\deg(f) = 0$ if f is *not* surjective;

(iii) $f \simeq g$ if and only if $\deg(f) = \deg(g)$; furthermore, if f and g induce the same map in homology, then $f \simeq g$;

(iv) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$, since $(f \circ g)_* = f_* \circ g_*$;

(v) $\deg(f) = -1$ if f is a reflection of S^n;

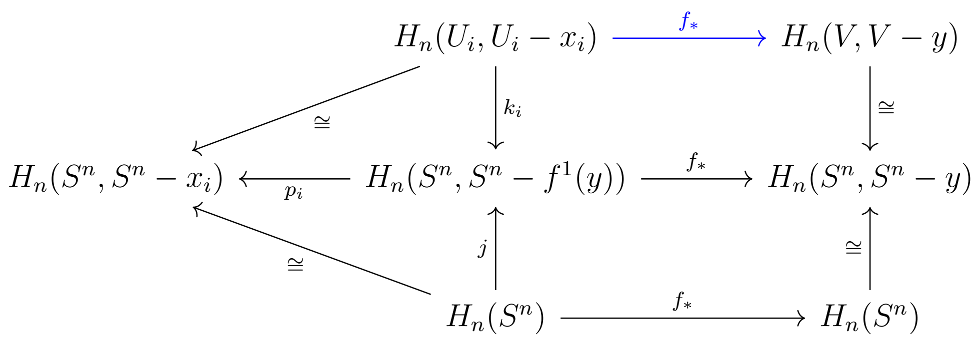
(vi) The antipodal map of S^n has degree $(-1)^{n+1}$, since it is the composition of $n+1$ reflections;

(vii) If $f : S^n \to S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$, because then f is homotopic to the antipodal map via
$$h_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

Example: The map $f : S^1 \to S^1$ given by $z \mapsto z^n$ has degree n.

Corollary 4.3.11. Recall that an action $G \curvearrowright X$ is **free** if given $e \neq g \in G$, then the action of g on X has no fixed points. If n is even, then $\mathbb{Z}_2$ is the only nontrivial group that acts *freely* on S^n. It does so via homeomorphisms. Note that this is not true if "freely" is replaced with "faithfully".

Definition 4.3.42. Let $f : S^n \to S^n$ be a continuous map, where $n > 0$, Assume there exists $y \in S^n$ such that $f^{-1}(y)$ is finite, say, $f^{-1}(y) = \{x_i\}_{i=1}^m$, $m \geq 0$. We may choose disjoint open neighborhoods U_i of x_i and an open neighborhood V of $f(U_1) \cup f(U_2) \cup \ldots \cup f(U_m)$, e.g. $V = S^n$. Then, $f(U_i - x_i) \subseteq V - y$, and for each i, we have the following commutative diagram:



in which k_i and p_i are induced by inclusions. Then, we can identify the top two groups in the diagram with $H_n(S^n)$ and view the top f_* has a map $H_n(S^n) \cong \mathbb{Z} \to H_n(S^n) \cong \mathbb{Z}$, which means it is given by multiplication by an integer z. Call this z the **local degree of f at x_i**, and denote it $\deg(f \mid x_i)$.

Theorem 4.3.37. If $f^{-1}(y)$ is finite, say, $f^{-1}(y) = \{x_i\}_{i=1}^m$, $m \geq 0$, then

$$\deg(f) = \sum_{i=1}^{m} \deg(f \mid x_i).$$

Remark: We can use degree to prove the Fundamental Theorem of Algebra. Let $f(x)$ be a complex polynomial of degree greater $n > 0$. Extend $f : \mathbb{C} \to \mathbb{C}$ to a map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by defining $f(\infty) = \infty$. As long as f is nonconstant, this extension is still continuous. We then have that

$$\deg(f) = \deg(f \mid \infty) = \deg(z^n \mid \infty) = n > 0.$$

Since $\deg(f) \neq 0$, we have that f is onto, and so there exists $z \in \mathbb{C}$ so that $f(z) = 0$, i.e., f has a complex root.

Theorem 4.3.38. The n-dimensional real projective plane $\mathbb{R}P^n$ is a CW complex with exactly one k-cell for each $0 \leq k \leq n$. The attaching maps are all antipodal maps. The degree of the attaching map

f attaching a k-cell to a $(k-1)$-cell is $1 + (-1)^k$, for the map is given by $z \mapsto z^k$. Thus, the degree of f is 2 if k is even and 0 if k is odd.

4.3.26 Cellular Homology

Theorem 4.3.39. Let X be a CW complex. Then,

$$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbb{Z}\langle n\text{-cells}\rangle, & \text{if } k = n, \\ 0, & \text{if } k \neq n, \end{cases}$$

and in particular, $H_k(X^n) = 0$ if $k > n$. This is important, for example, if X is of finite dimension n. Furthermore, the inclusion $X^n \hookrightarrow X$ induces an isomorphism $H_n(X^n) \hookrightarrow H_k(X)$ if $k < n$.

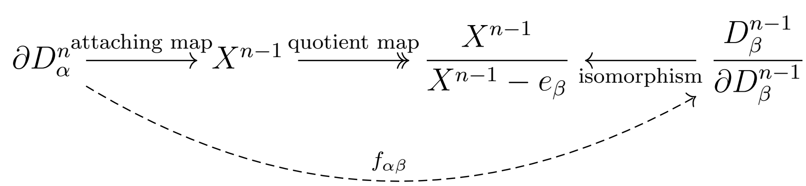
Definition 4.3.43. A CW complex X induces a **cellular chain complex** $C^{\mathrm{CW}}(X)$. This chain complex is given by $C_n^{\mathrm{CW}}(X) = 0$ if $n < 0$, and $C_n^{\mathrm{CW}}(X) = H_n(X^n, X^{n-1}) \cong \mathbb{Z}\langle n\text{-cells}\rangle$ if $n \geq 0$. Letting $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ be the connecting homomorphism in the LES for the triple (X^n, X^{n-1}, X^{n-2}), we we get that $d_n([z]) = [\partial n]$, where z is an n-cycle in $C(X^n, X^{n-1})$. We define

$$H_n^{CW} = \frac{\ker d_n}{\text{im } d_{n+1}},$$

and call H_n^{CW} the **cellular homology** of X.

Theorem 4.3.40. Cellular homology is isomorphic to singular homology.

Theorem 4.3.41 (Cellular Boundary Formula)**.** Let e_α be an n-cell and e_β be an $(n-1)$-cell. Consider



where the isomorphism is induced by Φ_β^{n-1}. Let $d_{\alpha\beta} = \deg(f_{\alpha\beta})$. We have the following formula:

$$d_n(e_\alpha) = \sum_\beta d_{\alpha\beta} e_\beta.$$

4.3.27 Homology With Coefficients

4.3.28 Cone and Suspension

Definition 4.3.44. Let X be a topological space. The **cone** of X, denoted CX, is the space $X \times [0,1]$ with $X \times \{0\}$ identified to a point.

Theorem 4.3.42. The cone CX of a topological space X is always contractible and thus simply connected.

Definition 4.3.45. Let X be a topological space. The **suspension** of X, denoted SX, is the space $X \times [0,1]$ with $X \times \{0\}$ identified to one point and $X \times \{1\}$ identified to another point.

Note: The suspension of X is the union of two cones of X identified at their bases.

Theorem 4.3.43. The suspension of S^n is S^{n+1}.

Theorem 4.3.44. For any topological space X, we have that

$$\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X).$$

Theorem 4.3.45. The suspension SX of a topological space X is always path-connected. If X is path-connected, then SX is simply connected.

5 RESTATEMENT OF PREFACE

Take all of this with a grain of salt - if you think something is incorrect, it very well may be. In particular, you will notice in my solutions that for much of my studying, I was convinced $H_n(X/A) \cong \tilde{H}_n(X,A)$ for good pairs, when in fact $\tilde{H}_n(X/A) \cong H_n(X,A)$. This shows up in several "solved" problems.

6 Worked Problems

6.1 Problems From 661

6.2 Problems From 761

6.2.1 Homework 1, Problem 1

Suppose there is a deformation retraction $f_t : X \to X$ from a space X to a subspace $A \subseteq X$ and a deformation retraction $g_t : A \to A$ form A to a subspace $B \subseteq A$. Show that there is a deformation retraction $h_t : X \to X$ from X to B.

Proof. Define

$$h_t(x) = \begin{cases} f_t(x), & t \in [0, \frac{1}{2}], \\ g_t(f_1(x)), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Certainly, $h_t : X \to X$, $h_0 = f_0 = \text{id}_X$, and $h_1(X) = g_1(f_1(X)) = g_1(A) = B$. Furthermore,

$$h_t|_B = \begin{cases} f_t|_B, & t \in [0, \frac{1}{2}], \\ g_t|_B, & t \in [\frac{1}{2}, 1], \end{cases} = \begin{cases} \text{id}_A, & t \in [0, \frac{1}{2}], \\ \text{id}_B, & t \in [\frac{1}{2}, 1], \end{cases} = \text{id}_B,$$

since $B \subseteq A$. Lastly, the associated map $H(x,t) : X \times I \to X$ is continuous, since

$$H(x,t) = \begin{cases} F(x, 2t), & t \in [0, \frac{1}{2}], \\ G(x, 2t-1), & t \in [\frac{1}{2}, 1], \end{cases}$$

is continuous by the Piecing Lemma, for at $t = \frac{1}{2}$, we have that $F(x,1) = G(x,0)$. Thus, h_t is a deformation retraction of X onto B. □

6.2.2 Homework 1, Problem 2

Show that if $A \subseteq Z$ is a subspace of a space Z such that the inclusion map $i : A \hookrightarrow Z$ is nullhomotopic, then A is contained in a single path component of Z.

Proof. Since the inclusion map $i : A \hookrightarrow Z$ is nullhomotopic, there is a homotopy h_t with $h_0 = i$ and $h_1(x) = q \in A$ for all $x \in X$. To see that A is contained in a single path component of Z, it suffices to show that between any two points in A there is a path in Z. But this is equivalent to showing that for any point p in A, there is a path from p to q.

Consider the associated map of h_t

$$H(x,t) : A \times I \to Z,$$

and define $g(t) = H(p,t)$. Then, g is continuous, because H is. Furthermore, $g(0) = H(p,0) = h_0(p) = i(p) = p$, and $g(1) = H(p,1) = h_1(p) = q$. Thus, g is a path in Z from p to q, and by our earlier discussion, we are done. □

6.2.3 Homework 1, Problem 3

Let X be any space and x be point in X. Show that if Y is compact, then for each open set $N \subseteq X \times Y$ containing $\{x\} \times Y$, there is an open neighborhood $V \subseteq X$ of x such that $V \times Y \subseteq N$.

Proof. The set $\{A \times B \mid A \text{ open in } X,\ B \text{ open in } Y\}$ is a basis for the product topology on $X \times Y$. Let $N \subseteq X \times Y$ be open and contain $\{x\} \times Y$. Since N is open, each point of $\{x\} \times Y$ is an interior point of N, and so we may find a basic open cover of $\{x\} \times Y$ completely contained in N. Since Y is compact and $\{x\} \times Y$ is homoeomorphic to Y, $\{x\} \times Y$ is compact. So, we need only finitely many of the basic open sets in our open cover to cover $\{x\} \times Y$. If these basic open sets are $A_i \times B_i$ for $i = 1, 2, \ldots, n$, then we may take

$$V = \bigcap_{i=1}^{n} A_i$$

to be our open neighborhood of x with $V \times Y \subseteq N$. □

6.2.4 Homework 1, Problem 4 FINISH

Show that if a space X deformation retracts to a point $x \in X$, then for each open neighborhood U of x there exists an open neighborhood $V \subseteq U$ of x such that the inclusion map $i : V \hookrightarrow U$ is nullhomotopic.

Proof. Note that the unit interval I is compact. By **Homework 1, Problem 3**, we know that for the open neighborhood $U \times I$ in $X \times I$, there is an open neighborhood $V \subseteq X$ such that $V \times I \subseteq U \times I$. Then, $V \subseteq U$.

To see that $i : V \hookrightarrow U$ is nullhomotopic, look at the deformation retraction $f_t : X \to \{x\}$. We have that $f_0 = \mathrm{id}_X$, $f_1(X) = \{x\}$, and $f_t|_{\{x\}} = \{x\}$ for all t. Furthermore, the associated map $F(s,t) : X \times I \to X$ is continuous. Consider f_t the restriction of f_t to V. Then, $f_t : V \to X$ and $F : V \times I \to X$ is continuous since the restriction of a continuous function is continuous. Furthermore, f_0 is the inclusion $i : V \hookrightarrow U$, as $V \subseteq U$ and f_0 is the identity map. Lastly, $f_1(V) = \{x\}$, i.e., f_1 is a constant map, since $F_1(X) = \{x\}$. □

6.2.5 Homework 1, Problem 5

Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0,1]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point in X.

Proof. We know that intervals deformation retract to any point they contain. So, each $\{r\} \times [0,1]$ deformation retracts to the r in the horizontal interval. This interval then deformation retracts to any point it contains. So, X deformation retracts to any point in $[0, 1] \times \{0\}$ by **Homework 1, Problem 1**.

Suppose $x \in X \smallsetminus ([0, 1] \times \{0\})$. If X deformation retracts to x, then **Homework 1, Problem 4** gives us that for each open neighborhood U of x, we have an open neighborhood $V \subseteq U$ of x such that $V \hookrightarrow U$ is nullhomotopic. **Homework 1 Problem 2** then yields that V is contained in a single path component of U. But the rationals are dense in $\mathbb{R}$. So there is a small open neighborhood of x consisting of infinitely many vertical line segments. This is not path-connected. Thus, if we take U to be this small, no open subset V of U can be path-connected, a contradiction. □

6.2.6 Homework 2, Problem 1

Construct a 2-dimensional cell complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

Proof. Let $f : S^1 \to \mathcal{M}$ be the embedding of S^1 onto the center circle of the Möbius band $\mathcal{M}$. The mapping cylinder M_f deformation retracts onto the target space, which in our case in $\mathcal{M}$. Since the Möbius band deformation retracts onto its center circle, M_f deformation retracts onto $S^1 \times I$, which we may then deformation retract to an annulus. That M_f is a 2-dimensional cell complex is clear, as all mapping cylinders of maps $g : S^1 \to S^1$ are cell complexes. More explicity, since the product of CW complexes is a CW complex, $S^1 \times I$ is a CW complex, and so M_f can be obtain by taking the disjoint union of cells in $S^1 \times I$ with those of $\mathcal{M}$ and attaching as the mapping cylinder indicates. □

6.2.7 Homework 2, Problem 2

Let $X := S^2 \sqcup_f I$ and $Y := S^2 \sqcup_g I$ where $f : \partial I \to S^2$ is the map that sends both endpoints of I to the north pole and $g : \partial I \to S^2$ sends one endpoint to the north pole and one endpoint to the south pole. Since $f \simeq g$ and since $(I, \partial I)$ has the homotopy extension property, it follows that $X \simeq Y$ rel S^2. Describe explicit homotopy inverse continuous maps $X \to Y$ and $Y \to X$ which restrict to the identity on S^2.

Proof. Pick an arc (a meridian) from the north pole to the south pole on S^2. Let $\varphi : X \to Y$ map $[f(\frac{1}{2}), f(1)]$ to the arc in a one-to-one fashion so that a space homeomorphic to Y is obtained. Let $\psi : Y \to X$ map $[g(\frac{1}{2}), g(1)]$, where $g(1)$ is the south pole, to the arc similarly. The maps are clearly homotopy inverses. □

6.2.8 Homework 2, Problem 3

Recall that the **wedge sum** of a collection of spaces X_α with basepoints $x_\alpha \in X_\alpha$ is the quotient space $\bigvee_\alpha X_\alpha$ obtained from the disjoint union $\bigsqcup_\alpha X_\alpha$ by identifying the points x_α. Show that the space obtained from S^2 by attaching n copies of D^2 along arbitrary continuous maps $\partial D^2 \to S^2$ is homotopy equivalent to a wedge sum of $n+1$ 2-spheres.

Proof. Let $f : \partial D^2 \to S^2$ be any continuous attaching map. Let $g : \partial D^2 \to S^2$ be the map that takes every point in ∂D^2 to a point $x_0 \in S^2$. We claim that $f \simeq g$.

The image of ∂D^2 under f is a loop γ on S^2. Since S^2 is path-connected, we may assume γ is based at x_0. Since S^2 is simply connected, γ contracts to x_0. Thus, $f \simeq g$, as x_0 is the image of ∂D^2 under g. Hence $S^2 \cup_f D^2$ and $S^2 \cup_g D^2$ are homotopy equivalent. It follows that $S^2 \cup_f D^2$ and $S^2 \vee S^2$ are homotopy equivalent, as the latter is clearly homeomorphic to $S^2 \cup_g D^2$.

Assume the statement holds when attaching $n-1$ disks. Attaching all disks at once or in succession does not matter, so we attach another disk, bring the total to n disks. Since S^2 with $n-1$ disks attached is homotopy equivalent to the wedge of n 2-spheres, the two spaces have the same fundamental group. The fundamental group of the wedge of 2-spheres is trivial, and this fact we will show later. Given this, however, we have that, as in the base case, the image of ∂D^2 contracts to a point and so S^2 with n disks attached is homotopy equivalent to the wedge of $n+1$ 2-spheres.

It remains to show that the wedge of k spheres has trivial fundamental group. We go by induction, the base case of one being obvious. Assume the wedge of $k-1$ 2-spheres has trivial fundamental group. Using the Seifert-van Kampen Theorem with an open neighborhood of the wedge of $k-1$ 2-spheres, an open neighborhood of one 2-sphere, and the intersection that is contractible to the wedge point, we get that the fundamental group of the wedge of k 2-spheres is indeed trivial.

This completes the proof. □

6.2.9 Homework 2, Problem 4

Show that any two retractions $S^1 \vee S^2 \to S^2$ are homotopic, but that there are infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$, where in each case the retraction is understood to be a retraction onto the second summand of the wedge sum.

Proof. The image of S^1 under any retraction $S^1 \vee S^2 \to S^2$ is a loop by continuity. Since S^2 is simply connected and path-connected, any such retraction is homotopy equivalent to the map that send each point of S^1 to some fixed $x_0 \in S^2$.

We claim that the retraction that wraps the first copy of S^1 about the second copy of S^1 a total of n times is different from the retraction that wraps the first copy of S^1 about the second copy of S^1 a total of m times for $n \neq m$. This is true because elements of $\pi_1(S^1)$ correspond to traversing the circle, and traversing n times and traversing m times are different elements for $n \neq m$, i.e., such loops are non-homotopic. □

6.2.10 Homework 3, Problem 1

Recall that if $\psi_1 : K \to G_1$ and $\psi_2 : K \to G_2$ are two group homomorphisms, then the amalgamated free product $G_1 *_K G_2$ is the quotient group obtained from $G_1 * G_2$ by imposing all relations of the form $\psi_1(k) = \psi_2(k)$ for $k \in K$.

(a) Compute $G_1 *_K G_2$ for $K = \langle a, b \mid aba^{-1}b^{-1} \rangle$, $G_1 = \langle c \rangle$, and $G_2 = \langle d \rangle$ if $\psi_1(a) = c^5$, $\psi_1(b) = c^6$, $\psi_2(a) = d$, and $\psi_2(b) = 1$, where 1 is the identity element of G_2.

Proof. We have, by definition, that

$$G_1 *_K G_2 \cong \langle c, d \mid c^5 = d,\ c^6 = 1 \rangle.$$

So, d is a redundant generator and the relation $c^5 = d$ is redundant. Thus, $G_1 *_K G_2 \cong \langle c \mid c^6 \rangle \cong \mathbb{Z}_6$. □

(b) Let U and V be two disjoint copies of the solid torus $S^1 \times D^2$, and let ∂U and ∂V be their boundaries, viewed as copies of $S^1 \times \partial D^2 \subseteq \mathbb{C} \times \mathbb{C}$. Compute the fundamental group of the glued space $X = U \sqcup_f V$ if $f : \partial V \to \partial U$ is the map $f(z, w) := (z^5 w^6, zw)$.

Proof. Regard ∂V as the intersection of U and V. We apply the Seifert-van Kampen Theorem to get that $\pi_1(X) \cong \pi_1(U) *_{\pi_1(\partial V)} \pi_1(V)$. It is clear that a solid torus deformation retracts to its central circle, and so $\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$. Thus, $\pi_1(X) \cong \langle c \rangle *_{\pi_1(\partial V)} \langle d \rangle$. Note that ∂V is a (non-solid) torus, and so $\pi_1(\partial V) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$. The map f induces two inclusions $\psi_1 : \partial V \hookrightarrow V$ and $\psi_2 : \partial V \hookrightarrow U$. Including ∂V into V maps any latitudinal loop to the identity and any longitudinal loop to itself, i.e., $\psi_1(a) = d$ and $\psi_1(b) = 1$. The map f tells us how to find what goes on the ψ_2. We get that $\psi_2(a) = c^5$ and $\psi_2(b) = c^6$. So, we are in the situation of part (a), albeit ψ_1 and ψ_2 are swapped. Thus, $\pi_1(X) \cong \mathbb{Z}_6$. □

6.2.11 Homework 3, Problem 2 FINISH

A continuous map $p : X \to Y$ is called a **local homeomorphism** if for every point $x \in X$, there is an open neighborhood U of x which is mapped homeomorphically to an open neighborhood V of $p(x)$.

(a) Give an example of a continuous map $p : X \to \mathbb{R}$ such that X is a nonempty Hausdorff space and $p^{-1}(s)$ is finite for each $s \in \mathbb{R}$, and such that p is a local homeomorphism but not a covering map.

Proof. □

(b) Show that the map p found in part (a) does not have the UPLP. That is, show that there exists path f in $\mathbb{R}$ and a point $x \in p^{-1}(f(0))$ such that the lift of f starting at x is either not unique or does not exist.

Proof. □

6.3 Problems From Previous Qualifying Exams

6.3.1 January 2021, Problem 1

Consider the integers $\mathbb{Z}$. We call a subset K of $\mathbb{Z}$ **balanced** if it is finite and if it has the properties that if $x \in K$ and x is even, then $x + 1 \in K$ and if $x \in K$ and x is odd, then $x - 1 \in K$. Let $\mathcal{T} = \{\mathbb{Z} \setminus K \mid K \text{ is balanced}\} \cup \{\emptyset\}$.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

Proof. We are given that $\emptyset \in \mathcal{T}$. To get that $\mathbb{Z} \in \mathcal{T}$, we need to show that $\emptyset$ is balanced, for $\mathbb{Z} \setminus \emptyset = \mathbb{Z}$. But this is vacuously true.

Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of elements of $\mathcal{T}$. Each U_α is of the form $\mathbb{Z} \setminus K_\alpha$, where K_α is balanced. Now,

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (\mathbb{Z} \setminus K_\alpha) = \mathbb{Z} \setminus \bigcup_{\alpha \in I} K_\alpha.$$

So, we would like to have that $\bigcup_{\alpha \in I} K_\alpha$ is balanced. If x is in the union, it is in one of the K_α, and since each of these are balanced, bases on the parity of x, either $x + 1$ or $x - 1$ is in the K_α and so in the union. That is, the union is balanced, and $\mathcal{T}$ is closed under arbitrary unions.

Let $\{V_i\}_{i=1}^n$ be a finite collection of elements of $\mathcal{T}$. Each V_i is of the form $\mathbb{Z} \setminus K_i$, where K_i is balanced. Now,

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (\mathbb{Z} \setminus K_i) = \mathbb{Z} \setminus \bigcap_{i=1}^{n} K_i.$$

We would like to have that $\bigcap_{i=1}^{n} K_i$ is balanced. If x is in the intersection, then x is in every K_i, and since each K_i is balanced, based on the parity of x, either $x+1$ or $x-1$ is in each K_i, as well. But then (again, based on the parity of x) either $x+1$ or $x-1$ is in the intersection. Thus, $\mathcal{T}$ is closed under finite intersections.

Thus, $\mathcal{T}$ is a topology on $\mathbb{Z}$. $\square$

(b) Is $(\mathbb{Z}, \mathcal{T})$ compact?

Proof. Let x be even. Then, $\mathbb{Z} \setminus \{x, x+1\}$ is balanced, for $x+2$ is even and $x-1$ is odd so that the properties of balanced sets do not require x nor $x+1$ to be in $\mathbb{Z} \setminus \{x, x+1\}$. By definition of $\mathcal{T}$, $\mathbb{Z} \setminus (\mathbb{Z} \setminus \{x, x+1\}) \in \mathcal{T}$. But this is set is just $\{x, x+1\}$. So, the collection $\{\{z, z+1\} \mid z \text{ is even}\}$ forms an open cover of $\mathbb{Z}$. But it is clear that this cover cannot admit a finite subcover, for removing even one set from the collection means that we are no longer covering $\mathbb{Z}$. $\square$

(c) Let $z \in \mathbb{Z}$. Find the closure of the set $\{z\}$ in $(\mathbb{Z}, \mathcal{T})$.

Proof. By definition of $\mathcal{T}$, the closed sets are the balanced sets. Note that $\{z, z+1\}$ is balanced if z is even and $\{z-1, z\}$ is balanced if z is odd. Since $\{z\}$ itself is not balanced, these are the possible closures of $\{z\}$. $\square$

6.3.2 January 2021, Problem 2 FINISH

Let X be a Hausdorff topological space. Suppose that $\{C_i\}_{i \in \mathbb{N}}$ is a collection of nonempty, nested, compact, connected subsets of X. Show that $\mathcal{C} = \bigcap_{i=1}^{\infty} C_i$ is nonempty, compact, and connected.

Proof. $\square$

6.3.3 January 2021, Problem 3

Determine whether each of the following statements are true. If the statement is true, give a short proof. If the statement is false, give a counterexample.

(a) Every compact subset of a topological space is closed.

Proof. This is false. Let $X = \{a, b, c\}$ have the trivial topology. Then, the only open cover of $\{a\}$, say, is $\{X\}$, and so $\{a\}$ is compact. However, $\{a\}$ is not closed, since the topology on X is trivial. $\square$

(b) The projective plane and the 2-sphere are not homeomorphic.

Proof. This is true. Regard each as handlebodies. The projective plane can be expressed as $h^0 \cup h^1 \cup h^2$ and the 2-sphere can be expressed as $h^0 \cup h^1 \cup h_1^2 \cup h_2^2$. Thus, the projective plane has Euler characteristic 1, but the 2-sphere has Euler characteristic 2. Thus, the two cannot be homeomorphic. □

(c) The continuous image of a Hausdorff space is Hausdorff.

Proof. This is false. Let $X = \{a, b, c\}$. Denote by X_D the space X equipped with the discrete topology and by X_T the space X equipped with the trivial topology. Let $f : X_D \to X_T$ be the identity map. Clearly, X_D is Haudorff, since singletons are open, and X_T is not Hausdorff, since the only nonempty open set is X itself. Since every set is open in X_D, we have that the preimage of any set (let alone the open ones) in X_T is open in X_D. So, f is continuous. This gives us our desired counterexample. □

(d) The two compact connected surfaces H_1 and H_2 represented by the handlebody diagrams in the original exam are homeomorphic.

Proof. This is true. To see why, we put H_1 into standard form via handle-sliding. Slide handle 3 over handle 1 (there is no twist), and then slide handle 1 along handle 2 (there is a twist). We then slide handle 2 out from under handle 1, twisting twice, and get precisely H_2. □

6.3.4 January 2021, Problem 4

Consider the torus T and the projective plane P as the square $I \times I$ with appropriate identifications, and let d be the diagonal of each square (see the original exam for a picture). Let $X = T \cup_d P$ be the space obtain by gluing T and P along d.

(a) Find $\pi_1(X)$.

Proof. We use the Seifert-van Kampen Theorem. First, we use the given pictures to describe the fundamental groups of T and P, or, rather, of small open neighborhoods of T and P that deformation retract onto T and P. Visualizing an open disk in the center of each square and overlapping open neighborhoods of the boundary, we use the Seifert-van Kampen Theorem to get that $\pi_1(T) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$ and $\pi_1(P) \cong \langle m, l \mid mlml \rangle$. The intersection of the open neighborhoods of T and P is clearly a circle, as the endpoints of d are glued together. So, $\pi_1(T \cap P) \cong \mathbb{Z}$. We then apply the Seifert-van Kampen Theorem again to find $\pi_1(X)$:

$$\pi_1(X) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle *_{\mathbb{Z}} \langle m, l \mid mlml \rangle.$$

The amalgamation yields the relations (via the inclusions) $abl^{-1}m^{-1} = baml = 1$. Then, $ab = ml$. But also $ml = l^{-1}m^{-1} = (ml)^{-1}$, and so we have that $ab = (ab)^{-1}$. Thus, $ab = 1$. So, $ml = 1$. That a and b commute yields that $ba = 1$. Since $mlml = 1$, we have that $lm = m^{-1}l^{-1} = (lm)^{-1}$, and so $lm = 1$. Thus,

$$\pi_1(X) \cong \langle a, b, m, l \mid ab = ba = lm = ml = 1 \rangle.$$

□

(b) Is X homeomorphic to a compact, connected surface? Justify your answer.

Proof. Indeed, X is homeomorphic to $T\#P$, which is a compact, connected surface by the classification of such surfaces. That X is in fact this connected sum is clear, considering d becomes circle (that does not bound an inner disk) upon gluing. □

6.3.5 January 2021, Problem 5 FINISH

Let $X = S^1 \vee S^1$, and let $\tilde{X}$ be the CW complex in the original exam. There is a 4-sheeted covering map $p : \tilde{X} \to X$ given by sending 1-cells labeled a to the first copy of S^1 and 1-cells labeled b to the second copy.

(a) Describe all covering transformations of $(\tilde{X}, p)$.

Proof. The covering transformations of $\tilde{X}$ correspond to automorphisms of the graph that is $\tilde{X}$. There is always the identity, and it is also clear that there is a 180° rotation. That there are no other automorphisms of the graph can be seen by the fact that other manipulations distort the directions of the arrows. □

(b) Find the generators for the subgroup $p_\#(\pi_1(\tilde{X}, v_1)) \leq \pi_1(X, p(v_1))$.

Proof. To do this, we trace out loops based at v_1 in $\tilde{X}$ that become wedged circles if we contract $\tilde{X}$ to a wedge sum of circles (after picking a^3 as our spanning tree). Doing so, we find that

$$p_\#(\pi_1(\tilde{X}, v_1)) = \langle aba^{-1}, a^2b, a^2b^{-1}, a^4, a^3ba \rangle.$$

□

(c) Find the generators for the normalizer of $p_{\#}(\pi_1(\tilde{X}, v_1))$ in $\pi_1(X, p(v_1))$.

Proof. □

6.3.6 January 2021, Problem 6 FINISH

Recall that a **retraction** from a space X to a subspace A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

(a) Prove that if there is a retraction from X to A, then $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.

Proof. Consider the LES in homology induced by $C(X, A)$:

$$\ldots \longrightarrow H_n(A) \xrightarrow{f} H_n(X) \xrightarrow{g} H_n(X, A) \longrightarrow \ldots$$

Since A is a retraction of X, f is injective and g is surjective. So, by exactness, we get the following SES:

$$0 \longrightarrow H_n(A) \xrightarrow{f} H_n(X) \xrightarrow{g} H_n(X, A) \longrightarrow 0$$

Note that $r_* \circ f = \mathrm{id} : A \to A$, where r_* is the map induced by the retraction $r : X \to A$, since f is induced by the inclusion $A \hookrightarrow X$. Then, by the Splitting Lemma, $H_n(X) \cong H_n(A) \oplus H_n(X, A)$, as desired. □

(b) Determine which of the following statements are true.

(1) There is a retraction from the Möbius band to its boundary.

Proof. This is a false statement. Let M be the Möbius band. Note that $M/\partial M$ is homeomorphic to the projective plane P (this can be perhaps most easily observed via the rectangular representation of M). By part (a), if we had such a retraction $M \to \partial M$, we would have that

$$H_1(M) \cong H_1(\partial M) \oplus H_1(P),$$

i.e., that

$$\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$

which is obviously false (by, for example, the Fundamental Theorem of Finitely-Generated Abelian Groups). $\square$

(2) There is a retraction from $S^1 \times S^1$ to $(S^1 \times S^1) \setminus \text{int } D^2$, where $D^2 \subseteq S^1 \times S^1$ is an embedded disk.

Proof. This is a false statement. We know the homology of the torus:

$$H_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z}, & n = 0, 2, \\ \mathbb{Z}^2, & n = 1, \\ 0, & n \neq 0, 1, 2. \end{cases}$$

We also know that the homology of int D^2 is trivial. If there were such a retraction, then we would have that, by part (a), $H_n(T) \cong H_n(\text{int } D^2) \oplus H_n(T \setminus \text{int } D^2)$. Again, the homology of int D^2 is trivial, and so we would have that

$$H_n(T) \cong H_n(T \setminus \text{int } D^2). \tag{$*$}$$

It remains to find $T \setminus \text{int } D^2$ (and its homology). Removal of the interior of a disk allows us to deform the new space into a loop with a tube attached, and this is clearly homotopy equivalent to $S^1 \vee S^1$. Thus, $H_n(T \setminus \text{int } D^2)$ is $\mathbb{Z}^2$ if $n = 0, 1$ and 0 otherwise. So, we have that $(*)$ is not true, meaning that the suggested retraction is impossible. $\square$

(3) If X is a nonempty space, then there is a retraction from X to any one of its points.

Proof. This is a true statement. Let $x_0 \in X$, and take the constant map $r(x) = x_0$ for all $x \in X$. The constant map is continuous, and $r(x_0) = x_0$. That is, r is a retraction. $\square$

6.3.7 January 2021, Problem 7 FINISH

Let $Z = \mathbb{C} \setminus \{\pm 1\}$ be the complex plane with the points ± 1 removed, and let X be the space obtained from Z by attaching three disks via the maps $f_i : \partial D^2 \to Z$, $i = 1, 2, 3$, given by $f_1(z) = z^4 + 1$, $f_2(z) = z^6 - 1$, and $f_3(z) = 2z$.

(a) Describe a CW complex which is homotopy equivalent to X and which has three 2-cells. [*Hint.* Use that Z deformation retracts to a wedge sum of two circles.]

Proof. Note that Z deformation retracts to $S^1 \vee S^1$, and we can view $S^1 \vee S^1$ as a 1-dimensional CW complex $e^0 \cup e^1_1 \cup e^1_2$. Attaching three 2-cells via the given maps, we obtain the desired CW structure for X. That is, attach one 2-cell by wrapping its boundary 4 times about one circle in the wedge sum, attach one 2-cell by wrapping its boundary 6 times about the other circle in the wedge sum, and attach the final 2-cell by wrapping its boundary once about each circle in the wedge sum. □

(b) Compute the cellular homology of the CW complex found in part (a).

Proof. Call the complex Y. We obtain the following chain complex, where the ranks of the groups are determined by the number of n-cells:

$$0 \xrightarrow{\partial_3} \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

Because Y is a 2-dimensional CW complex, $H_n(Y) \cong 0$ for all $n \neq 0, 1, 2$. Clearly, Y is path-connected, and so $H_0(Y) \cong \mathbb{Z}$.

For $H_1(Y)$, we have $H_1(Y) \cong \dfrac{\ker \partial_1}{\text{im } \partial_2}$. There is only one 0-cell, and so $\ker \partial_1 \cong \mathbb{Z}^2 \cong \langle a, b\rangle$, where a and b represent the 1-cells. The maps and part (a) tell us that $\text{im } \partial_2 \cong \langle 4a, 6b, a + b\rangle$. Thus, $H_1(Y) \cong \langle a, b \mid 4a, 6b, a + b\rangle$. Evaluating, we get that $a = -b$, and so $H_1(Y) \cong \langle a \mid 4a, -6a\rangle \cong \langle a \mid 2a\rangle \cong \mathbb{Z}_2$.

Finally, $H_2(Y) \cong \ker \partial_2 \cong \langle p, q, r \in \mathbb{Z} \mid p(4a) + q(6b) + r(a + b) = 0\rangle$. Evaluating, we get that $4p = 6q = -r$, and so choosing any one of the three variables determines the other two. Furthermore, we must have that $3 \mid p$, $2 \mid q$, and $\text{lcm}(4, 6) = 12 \mid r$. So, we could view the kernel of the second differential as any of $3\mathbb{Z}$, $2\mathbb{Z}$, $12\mathbb{Z}$. In any case, $H_2(Y)$ is seen to be infinite cyclic and thus isomorphic to $\mathbb{Z}$ ($3\mathbb{Z} \cong 2\mathbb{Z} \cong 12\mathbb{Z} \cong \mathbb{Z}$). □

6.3.8 August 2020, Problem 1

Let X be a set.

(a) Can we define a topology on X such that every real valued function on X is continuous?

Proof. Yes; take the discrete topology on X. Then, the preimage of any subset of $\mathbb{R}$ is open because every subset of X is open. □

(b) Can we define a topology on X such that no real valued function on X is continuous?

Proof. No matter the topology on X, the constant function $f(x) = 0$ is continuous. The preimage of any subset of $\mathbb{R}$ under f is either X or $\emptyset$, which must be open in X by the definition of "topology". □

6.3.9 August 2020, Problem 2 FINISH

Let $X_1 = \mathbb{R}^n$ be equipped with the Euclidean topology, and let $X_2 = \mathbb{R}^n$ be equipped with the Zariski topology. By definition, a set $S \subseteq \mathbb{R}^n$ is **closed** if there exist polynomials $p_1, \ldots, p_l$ such that $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid p_1(x_1, \ldots, x_n) = \ldots = p_n(x_1, \ldots, x_n) = 0\}$. We allow for $l = 0$ in this definition since we want $S = \mathbb{R}^n$ to be one of the closed sets.

(a) Describe the open sets in X_1.

Proof. The open sets in X_1 are arbitrary unions of open balls of any radius. □

(b) Prove that one of these topologies is contained in the other.

Proof. □

(c) Prove that these topologies are not homeomorphic.

Proof. □

6.3.10 August 2020, Problem 3

Let X be a Hausdorff space, and let S_1 and S_2 be compact subsets in X with $S_1 \cap S_2 = \emptyset$. Show that there exist open sets $U_1, U_2 \subseteq X$ such that

$$S_1 \subseteq U_1,\ S_2 \subseteq U_2, \text{ and } U_1 \cap U_2 = \emptyset.$$

Proof. For each $x \in S_1$, let K_x be an open set containing x, so that $\{K_x \mid x \in X\}$ is an open cover of S_1. By compactness, this open cover admits a finite subcover $\{K_{x_i} \mid i = 1, \ldots, n\}$.

Fix $x_i \in S_1$ (this x_i is one of the elements of S_1 referenced above in the finite subcover of S_1). For each $y \in S_2$, pick an open neighborhood $V_{y,i}$ of y and an open neighborhood V_{x_i} of x_i such that $V_{x_i} \cap V_{y,i} = \emptyset$. This is possible since X is Hausdorff and $x_i \neq y$ (because $S_1 \cap S_2 = \emptyset$). Then,

$$V_{x_i} \cap \left(\bigcup_{y \in Y} V_{y,i} \right) = \emptyset,$$

since V_{x_i} is disjoint from each $V_{y,i}$.

Set $U_1 = \bigcup_{i=1}^n K_{x_i}$ and $U_2 = \bigcap_{i=1}^n \left(\bigcup_{y \in Y} V_{y,i} \right)$. Then, U_1 and U_2 are open, $S_1 \subseteq U_1$, $S_2 \subseteq U_2$, and $U_1 \cap U_2 = \emptyset$. ☐

6.3.11 August 2020, Problem 5 FINISH

Let $X = \bigvee_{i=1}^3 S^1$ be the wedge sum of three circles.

(a) Describe all of the connected, two-sheeted covering maps $p : \tilde{X} \to X$.

Proof. Take two vertices v_1 and v_2. Put a path from v_1 to v_2 and a path from v_2 to v_1. We may put one or two more such pairs of paths and two or one (respectively) loops on each vertex. In each case, we can then label in six different ways (a, b, and c corresponding to the copies of S^1 in the wedge sum). ☐

(b) For each $\tilde{X}$ from part (a), describe the subgroup $H = p_*(\pi_1(\tilde{X})) \subseteq \pi_1(X)$.

Proof. For two vertices, three pairs of paths, we have $H = \langle ab, ab^{-1}, ac, ac^{-1} \rangle$, where a is the spanning tree. Permuting a, b, and c gives the options for H. With one pair of loops, $H = \langle aba^{-1}, b, ac, ac^{-1} \rangle$ and permutations (again, a is the spanning tree). With two pairs of loops, $H = \langle aba^{-1}, b, aca^{-1}, c \rangle$. ☐

(c) Each of the subgroups H from part (b) is normal. Prove this.

Proof. Index two. ☐

(d) Describe a connected, three-sheeted covering map $q : \tilde{X} \to X$ that is not normal.

Proof. Take three vertices v_1, v_2, and v_3. Make an oriented triangle with edges b, and again with c. Each of these are copies of S^1 is the wedge sum. For the last copy of S^1, take a path from v_1 to v_2 labeled a and a loop on v_3 labeled a. This is a three-sheeted covering of X. It is not normal because one lift of a is to a loop, and two are not. ☐

6.3.12 August 2020, Problem 6

Let X be the Δ-complex shown in the original exam.

(a) Using the definition, compute the singular homology of X.

Proof. From the diagram, we get the following sequence:

$$\dots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}\langle U, L\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v,w\rangle \xrightarrow{\partial_0} 0 \longrightarrow \dots$$

The diagram also gives us the kernels and images necessary to compute the homology of the space. It is clear that $\operatorname{im}\partial_3 = 0$. We claim that $\ker\partial_2 = 0$, as well. This is because $\partial U = a-b-c$ and $\partial L = b-a-c$ and no nontrivial linear combination of these is zero, for

$$m(a-b-c)+n(b-a-c)=0 \implies a(m+n)+b(n-m)+c(-m-n)=0 \implies m+n=n-m=0,$$

as a, b, and c are linearly independent. It follows that $m=n=-n$, and so $m=n=0$. Thus, $H_2^\Delta(X)\cong 0$.

We have that $\ker\partial_1 = \langle c, a-b\rangle$, for these are the paths of elements a, b, and c with endpoint the same as starting point. It is clear from the diagram that $\operatorname{im}\partial_2 = \langle a-b-c, b-a-c\rangle$. So,

$$H_1^\Delta(X) \cong \frac{\langle c, a-b\rangle}{\langle a-b-c, b-a-c\rangle}.$$

From the relations, we get that $a-b=c=b-a$. Since $a-b=-(b-a)$, we have that $c=-c$, i.e. $2c=0$. Since $c=a-b$, we can remove $a-b$ from the generators. What results is that $H_1^\Delta(X)\cong\langle c \mid 2c=0\rangle\cong\mathbb{Z}_2$.

Finally, it is clear that $\ker\partial_0=\langle v,w\rangle$. Since $\partial a=\partial b=w-v$ and $\partial c=0$, we have that $\operatorname{im}\partial_1=\langle w-v\rangle$. Thus,

$$H_0^\Delta \cong \frac{\langle v,w\rangle}{\langle w-v\rangle}\cong\mathbb{Z}^2/\mathbb{Z}\cong\mathbb{Z}.$$

All other homology groups are of course zero. $\square$

(b) Prove that X is homeomorphic to a compact surface M.

Proof. Part (a) suggests that X is homeomorphic to the projective plane P, as the two spaces have the same homology. Work in my notebook not replicated here shows that this is indeed the case. □

(c) Describe geometrically a homeomorphism of X that maps a onto b.

Proof. This can be seen by swapping the orientation of c and relabeling. □

(d) Compute the relative singular homology of the pairs (X, b) and (X, c), and use this information to prove that there is no homeomorphism that maps b onto c.

Proof. The pairs (X, b) and (X, c) are good, considering that they are CW pairs. So, $\tilde{H}_n(X, b) \cong H_n(X/b)$ and $\tilde{H}_n(X, c) \cong H_n(X/c)$. Collapsing b to a point identifies v and w and so yields an elliptical representation of the projective plane P, the homology of which is the same as that of X. Collapsing c to a point gives us a wedge sum with some edges identified. Performing one identification makes it clear that what we have is S^2 (again, we get an elliptical representation). In degree one, for example, reduced singular homology and singular homology coincide. Thus, $H_1(X, b) \cong \mathbb{Z}_2$, where $H_1(X, c) \cong 0$. Thus, there can be no homeomorphism mapping b onto c. □

6.3.13 August 2020, Problem 7 FINISH

Let ST denote the suspension of the torus T. Recall that $ST = (T \times [0, 1]/ \sim)$, where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in T$. Use the Seifert-van Kampen Theorem to compute the fundamental groups of T and ST.

Proof. The fundamental group of T is $\mathbb{Z} \oplus \mathbb{Z}$. Look at the rectangular representation.

We know that the suspension of T is the union of two cones CT with their bases identified. The cone CT is contractible and so it is simply connected. The Seifert-van Kampen Theorem applies the ST because T is path-connected so that $CT \cap CT$ is path-connected. As CT is simply connected, so is ST. □

6.3.14 August 2020, Problem 8 FINISH

Let T and ST be the torus and its suspension, as in the previous problem.

(a) Write down the homology of T.

Proof. For $n \neq 0, 1, 2$, $H_n(T) \cong 0$. Also, $H_1(T) \cong \mathbb{Z}^2$, and $H_0(T) \cong H_2(T) \cong \mathbb{Z}$. □

(b) Using properties of homology (e.g., long exact sequences), compute the homology of ST.

Proof. Because the cone of T is contractible, exactness yields that $H_{n+1}(ST) \cong H_n(T)$ for all $n > 1$. So, $H_3(ST) \cong \mathbb{Z}$, $H_2(ST) \cong \mathbb{Z}^2$, and $H_k(ST) \cong 0$ for all $k \neq 0, 1, 2, 3$. The Hurewicz Theorem yields that $H_1(ST) \cong 0$. Since ST is path-connected (because T is path-connected), we have that $H_0(ST) \cong \mathbb{Z}$. □

(c) Construct a map $ST \to S^3$ that induces an isomorphism on the third homology group.

Proof. □

6.3.15 August 2019, Problem 1

Assume X is a space that is *locally path-connected*, which means that for all $x \in X$ and all open neighborhoods V of x, there exists an open, path-connected set U with $x \in U \subseteq V$. Fix $x_0 \in X$ and consider the subset $X_0 \subseteq X$ consisting of points in X that are connected to x_0 by some continuous path $f : [0, 1] \to X$.

(a) Prove that X_0 is an open subset of X.

Proof. Let $y \in \overline{X \setminus X_0}$, the closure of $X \setminus X_0$. Every open set containing y intersects $X \setminus X_0$. Let V be an open set containing y. Since X is locally path-connected, there is an open, path-connected set U with $y \in U \subseteq V$. So, U intersects $X \setminus X_0$. No point in $U \cap (X \setminus X_0)$ is connected via a path to x_0, for any such point is in $X \setminus X_0$. If y is in the intersection, we are done, so assume $y \in U$ and $y \notin (X \setminus X_0)$. Then, $y \in X_0$, and so there is a path from y to x_0. But U is path-connected, and so there is a path from any point in U to x_0. But this means that U does not intersect $X \setminus X_0$, a contradiction. Thus, $y \in (X \setminus X_0)$, and so $X \setminus X_0$ is equal to its closure, i.e., $X \setminus X_0$ is closed. It follows that X_0 is open, as desired. □

(b) Prove that a connected and locally path-connected space is path-connected.

Proof. Further assume that X is connected. Since $x_0 \in X_0$ (there is always the constant path from a point to itself), we have that X_0 is nonempty. We saw in part (a) that X_0 is open in X. To show that X_0 is path-connected amounts to showing that $X_0 = X$, which will follow if we can show that X_0 is closed in X, for, since X is connected, any nonempty clopen set in X must be X itself.

Let $y \in \bar{X}_0$. Then, every open set containing y intersects X_0. Let V be an open set containing y. Since X is locally path-connected, there is an open, path-connected set U such that $y \in U \subseteq V$. Since U is open and contains y, U intersects X_0. But then $X_0 \cup U$ is path-connected, being the union of two nondisjoint path-connected sets. So, there is a path from y to x_0, i.e., $y \in X_0$. So, $X_0 = \bar{X}_0$, meaning that X_0 is closed in X. By our earlier discussion, we are done. □

6.3.16 August 2019, Problem 2 FINISH

Fix a non-compact Hausdorff space X. Define the one-point compactification $X^+ = X \cup \{\infty\}$, where ∞ denotes some point not in X. To define a topology, declare $U \subseteq X^+$ to be open if one of the following holds:

(i) $\infty \notin U$ and $U \subseteq X$ is open.

(ii) $\infty \in U$ and $X \setminus U$ is compact.

Prove the following:

(a) Show that the open sets defined above satisfy the properties of a topology.

Proof. Note that $\infty \notin \emptyset$ and $\emptyset$ is open in X, implying that $\emptyset$ is open in X^+. The empty set is compact, i.e., $X \setminus X^+$ is compact, and so X^+ is open in X^+, as well.

Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in X^+. If ∞ is not in any of the U_α, then there U_α's are all open in X, and so the union of all of them is open in X and does not contain ∞, meaning that the union is open in X^+. If ∞ is in some U_α, then ∞ is in the union of all the U_α's. So, we need that $X \setminus \bigcup_{\alpha \in I} U_\alpha$ is compact. □

(b) X^+ is compact. (It might be helpful to notice that, if $U \subseteq X^+$ is open and contains ∞, then $U \setminus \{\infty\}$ is open in X).

Proof. □

(c) If X is locally compact (i.e., for all $x \in X$, there exists an open set U and a compact set C such that $x \in U \subseteq C$), then X^+ is Hausdorff.

Proof. □

6.3.17 August 2019, Problem 3

Determine which of the following pairs of spaces are homeomorphic. If they are, describe the homeomorphism. If they are not homeomorphic, then identify a topological invariant or property that distinguishes them.

(a) $\mathbb{R}^2$ and S^2

Proof. Not homeomorphic; S^2 is a compact, but $\mathbb{R}^2$ is not compact. □

(b) $\mathbb{R}^2$ and $B^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

Proof. □

(c) $S^2 \setminus \{N, S\}$ (where N and S are the north and south poles, respectively) and $(S^1 \times S^1) \setminus (S^1 \times \{p\})$ (where p is a point)

Proof. Homeomorphic; S^2 remove the north and south poles can be deformed to an annulus, as can $(S^1 \times S^1) \setminus (S^1 \times \{p\})$, for this is a torus minus a longitudinal circle. □

(d) $(0, 1)$ and $[0, 1)$ with the induced topology

Proof. We assume "induced" refers to "subspace". These two spaces are not hoemomorphic. If there were a homeomorphism $f : [0, 1) \to (0, 1)$, then the restriction $f|_{(0,1)}$ would be a homeomorphism, as well. However, $(0, 1)$ is connected and $(0, 1) \setminus \{f(0)\}$ is not connected, and so such a homeomorphism f cannot exist. □

(e) The torus T and the Klein bottle K

Proof. Not homeomorphic; the torus is orientable, but the Klein bottle is non-orientable. □

(f) D^2 and the quotient space $D^2/\sim$, where $(x, y) \sim (-x, -y)$ for all $(x, y) \in D^2$

Proof. □

6.3.18 August 2019, Problem 4 FINISH

(a) Let T be the 2-torus, and let $Y = T \setminus \{p_1, p_2, \ldots, p_k\}$ for some nonempty, finite collection of points $p_1, p_2, \ldots, p_k \in T$. What is $\pi_1(Y)$?

Proof. The space Y deformation retracts to $\bigvee_{i=1}^{k+1} S^1$, meaning that its fundamental group is F_{k+1}, the free group of rank $k+1$. □

(b) Let S be any compact, oriented, connected surface with no boundary, and for some nonempty, finite collection of points $p_i \in S$, let $Y = S \setminus \{p_1, \ldots, p_k\}$. Show that $H_2(Y) = 0$.

Proof. Note that $(S, \{p_1, \ldots, p_k\})$ is a good pair, as we can view S as a CW complex and $\{p_1, \ldots, p_k\}$ as a subcomplex. So, we have an LES

$$\ldots \longrightarrow H_2(\{p_1, \ldots, p_k\}) \longrightarrow H_2(S) \longrightarrow H_2(Y) \longrightarrow H_1(\{p_1, \ldots, p_k\}) \longrightarrow \ldots$$

Both $H_2(\{p_1, \ldots, p_k\})$ and $H_1(\{p_1, \ldots, p_k\})$ are zero since $\{p_1, \ldots, p_k\}$ is a 0-dimensional CW complex. By exactness, $H_2(S) \cong H_2(Y)$. □

6.3.19 August 2019, Problem 5 FINISH

(a) Give an example of an injective continuous map $f : X \to Y$ such that the induced map $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is not injective.

Proof. □

(b) Give an example of a surjective continuous map $f : X \to Y$ such that the induced map $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is not surjective.

Proof. □

6.3.20 August 2019, Problem 6 FINISH

Let $\tilde{X}$ be the 3-sheeted covering space of the wedge of two circles drawn in the original exam where the covering map $p : \tilde{X} \to S^1 \vee S^1$ is given by identifying the edges labeled a and b in $\tilde{X}$ with the circles traversed counterclockwise by a and b in $S^1 \vee S^1$. Let $v \in S^1 \vee S^1$ be the intersection of the two circles and $p^{-1}(v) = \{v_1, v_2, v_3\}$.

(a) Is $p : \tilde{X} \to S^1 \vee S^1$ a normal covering space?

Proof. No; there are lifts of b to a loop (at v_1) and to a non-loop path (v_3 to v_2, for example). Thus, the group of covering transformations of $\tilde{X}$ cannot act transitively on $p^{-1}(v)$, i.e., $\tilde{X}$ is not normal. □

(b) Find all deck transformations of $p : \tilde{X} \to S^1 \vee S^1$.

Proof. Deck transformations of $\tilde{X}$ correspond to automorphisms of the graph that is $\tilde{X}$. The vertex v_1 must be fixed by any automorphism of $\tilde{X}$, since it is the only vertex at which a length-one loop is based. It is impossible to map v_2 to v_3, as if we did, then the direction of the arrow from v_2 to v_1 would be wrong. So, the autormorphism group of $\tilde{X}$ is trivial, i.e., the group of covering transformations of $\tilde{X}$ is trivial. □

(c) Draw two path-connected 3-sheeted covering of $S^1 \vee S^1$ which are not equivalent to $\tilde{X}$, one that is normal and one that is not normal.

Proof. □

6.3.21 August 2019, Problem 7 FINISH

Let X be the regular hexagon with sides identified as in the picture in the original exam.

(a) Give X a cell structure and use it to compute $\pi_1(X, v)$, $H_1(X)$, and $H_2(X)$.

Proof. The hexagon gives us a cell structure: $e^0_v \cup e^1_a \cup e^1_b \cup e^1_c \cup e^2_f$. Glue the cells together as the hexagon suggests.

We can find $\pi_1(X)$ using the Seifert-van Kampen Theorem. Set A equal to an open neighborhood of the boundary of the hexagon and B equal to an open disk in the center of the hexagon that overlaps A is an open annulus. The Seifert-van Kampen Theorem yields that $\pi_1(X) \cong \langle a, b, c \mid a^{-1}b^{-1}cab^{-1}c^{-1} \rangle$.

From the cell structure, we get the complex

$$0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$$

We get that $H_1(X) \cong \dfrac{\mathbb{Z}^3}{\langle 2b \rangle} \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2$ and $H_2(X) \cong 0$.

□

(b) Compute the relative homology groups $H_*(X, A)$ and $H_*(X, B)$, where $A = a \cup \{v\}$ and $B = b \cup \{v\}$.

Proof. Each pair in the problem is good, as they are each CW pairs. So, the relative homologies are just the reduced homologies of the quotients X/A and X/B. Contracting A to a point gives us a Klein bottle, as can be seen from the picture in the original exam. Contracting B yields a torus. So, $H_*(X, A)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}$ in degree one and zero in all other degrees, while $H_*(X, B)$ is $\mathbb{Z}$ in degree two, $\mathbb{Z}^2$ in degree one, and zero in all other degrees. □

(c) Is there a retraction mapping from X to A?

Proof. □

(d) Is there a retraction mapping from X to B?

Proof. Suppose there is such a retraction. Then,

$$H_1(X) \cong H_1(B) \oplus H_1(X, B).$$

That is, $\mathbb{Z}_2 \oplus \mathbb{Z}^2 \cong \mathbb{Z} \oplus \mathbb{Z}^2$, which is false by the Fundamental Theorem of Finitely Generated Abelian Groups. Thus, there is no such retraction □

6.3.22 August 2017, Problem 1 POSSIBLY OKAY

(a) Suppose that X is compact and Y is Hausdorff. Let $f : X \to Y$ be continuous. Show that $f(X)$ is closed.

Proof. The continuous image of a compact set is compact, and so $f(X)$ is compact. Compact subsets of Hausdorff spaces are closed, and we show this in what follows. Fix $y \in Y \setminus f(X)$. If no such y exists, that $f(X) = Y$ and we are done. For each $x \in f(X)$, there are disjoint open set U_x and V_x with $x \in U_x$ and $y \in V_x$. The collection of all U_x forms an open cover of $f(X)$, and by compactness, this collection admits a finite subcover $\{U_{x,i}\}_{i=1}^n$. Then, the intersection of the corresponding neighborhoods of y

$$\bigcap_{i=1}^{\infty} V_{x,i}$$

is the finite intersection of open sets and is thus open. It is disjoint from each element of the cover $\{U_{x,i}\}_{i=1}^n$ of $f(X)$, and so it is disjoint from $f(X)$. That is, $\bigcap_{i=1}^{\infty} V_{x,i}$ is an open neighborhood of

y contained completely within $Y \setminus f(X)$. Thus, $Y \setminus f(X)$ is open, meaning that $f(X)$ is closed, as desired. □

(b) Suppose that X is compact and nonempty and Y is Hausdorff and connected. Show that every continuous open map $f : X \to Y$ is onto.

Proof. Since X is nonempty, $f(X)$ is nonempty. Since X is open in X and f is an open map, $f(X)$ is open in Y. We saw in part (a) that $f(X)$ is also closed, and since Y is connected, any nonempty clopen subset of Y must be Y itself. Thus, $f(X) = Y$. □

6.3.23 August 2017, Problem 2 POSSIBLY OKAY

Suppose that X is compact and $X \times \{y\} \subseteq U$, where $U \subseteq X \times Y$ is open. Prove that there exists an open set $Z \subseteq Y$ such that

$$X \times \{y\} \subseteq X \times Z \subseteq U.$$

Proof. Since X and $X \times \{y\}$ are homeomorphic and X is compact, we have that $X \times \{y\}$ is compact.

The set $\{A \times B \mid A \text{ open in } X,\ B \text{ open in } Y\}$ forms a basis for the product topology on $X \times Y$. Since U is open, every point of U is an interior point. Then, for each $(x, y) \in U$, we have that there exists an open neighborhood of $A_x \times B_x \subseteq U$ of (x, y). The set of all such $A_x \times B_x$ forms an open cover of $X \times \{y\}$. By compactness, this cover admits a finite subcover $\{A_{x_i} \times B_{x_i}\}$. The intersection $Z = \bigcap_{i=1}^{n} B_{x_i}$ is the finite intersections of open sets in Y and so is open in Y. Then,

$$X \times \{y\} \subseteq X \times Z \subseteq U,$$

as desired. □

6.3.24 August 2017, Problem 3 FINISH

Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if x and y are both rational (and otherwise $x \sim x$). Let $Y = \mathbb{R}/\sim$ with the quotient topology.

(a) Describe the open sets in Y.

Proof. A subset V of Y is open by definition if its preimage under the quotient map q is open in $\mathbb{R}$. If V contains the image of a rational number, then $q^{-1}(V)$ contains all rationals. So, if $q^{-1}(V)$ is to be open, it must be all of $\mathbb{R}$, in which case $V = Y$. If V does not contain the image of a rational number, then $q^{-1}(V) = V \subseteq \mathbb{R}$. The only way $q^{-1}(V)$ is open in this case is if it is empty. So, the open sets of Y are the empty set and Y itself. □

(b) What are the continuous functions from Y to $\mathbb{R}$.

Proof. Let $f : Y \to \mathbb{R}$ be continuous. Because Y has the trivial topology, the preimage of any open set in $\mathbb{R}$ under f must be either Y or $\emptyset$. Because Y is nonempty, there is some open set V in $\mathbb{R}$ such that $f^{-1}(V) = Y$. So, there is some $v \in V$ such that $f^{-1}(v)$ exists. But then for any open neighborhood N_ε of v, we have that $f^{-1}(N_\varepsilon) = Y$. Thus, f must be constant and $f(y) = v$ for all $y \in Y$. That is, the continuous functions from Y to $\mathbb{R}$ are the constant functions. □

(c) Show that Y is not Hausdorff.

Proof. Let $x, y \in Y$. Since Y has the trivial topology, the only open set containing either is Y itself, meaning that Y cannot be Hausdorff. □

6.3.25 August 2017, Problem 4 POSSIBLY OKAY

Let U be an open, connected subset of $\mathbb{R}^n$.

(a) Show that U is path-connected.

Proof. Let $x \in U$, and let $V \subseteq U$ be the set of points v in U such that there is a path from x to v. Then, V is path-connected. Then, $U \setminus V$ is the set of points in U that cannot be connected to x via a path. We claim that both V and $U \setminus V$ are open. Clearly, they are disjoint and together cover U, and since $x \in V \neq \emptyset$, and because U is connected, we will have that $U = V$, meaning that U is path-connected because V is.

Let $v \in V$, and let γ be a path in U from v to x. Because U is open, there is an open neighborhood $B_\varepsilon \subseteq U$ of v. Open balls are path-connected, and so there is a path from v to any point of B_ε. But then there is a path from x to any point of B_ε. Thus, $B_\varepsilon \subseteq V$, and we get that V is open.

Let $y \in U \setminus V$. Since U is open, there is an open neighborhood $B_\delta \subseteq U$ of y. Because there is no path in U from x to y, there is no path in U from x to any other point of B_δ, since B_δ is path-connected, being an open ball. So, $B_\delta \subseteq U \setminus V$, meaning that $U \setminus V$ is open.

By our earlier discussion, we are done. □

(b) For $p, q \in U$, define $\text{dist}(p, q)$ to be the infimum of the length of all paths connecting p and q whose image lies in U. Show that dist defines a metric on U. (You may assume in this problem that the infimum is finite).

Proof. Clearly, $d(p, p) = 0$ and $d(p, q) = d(q, p)$. It remains to show the triangle inequality. But this too is rather obvious, for given path $p \to q$ and $q \to r$, the length of the path $p \to q \to r$ is the sum of the length of the summed paths. There may indeed be a shorter path, and so the triangle inequality holds. □

(c) Show that dist induces the same topology on U as the standard Euclidean distance between points in U. Give an example of a connected, open set $U \subseteq \mathbb{R}^n$ such that dist is not equal to the standard Euclidean distance.

Proof. Let V be an open subset of U with respect to the Euclidean distance, and let $v \in V$. Then, there is an open neighborhood $B_\varepsilon \subseteq V$ of v with respect to the Euclidean distance. Note that B_ε is open with respect to dist, as well, for the infimum of all paths between v and any in B_ε is found in the straight line segment between the two points. Thus, the two topologies are the same.

If $n = 1$, then dist and the Euclidean distance are the same. For larger n, take any concave open set U and pick points that are not connected by a straight line segment within U. For example, let U be $((0, 1) \times (0, 1)) \setminus ([1/4, 3/4] \times [1/4, 3/4])$ and look at the distance measures between $(1/10, 9/10)$ and $(9/10, 1/10)$. □

6.3.26 August 2017, Problem 5 POSSIBLY OKAY

(a) Show how a Klein bottle (KB) is obtained from a square with certain identifications.

Proof. Orient L and R same, and orient T and B opposite. □

(b) Show that a KB can be obtained from two Möbius bands by identifying their boundaries.

Proof. Clear. □

(c) Can a torus be obtained from two KBs by identifying certain edges?

Proof. Yes; proof by picture. □

6.3.27 August 2017, Problem 6 POSSIBLY OKAY

Let X be the quotient space of S^2 obtained by identifying its north and south poles.

(a) Give X a cell structure and use it to compute $\pi_1(X)$.

Proof. Take X to have one 0-cell, one 1-cell, and one 2-cell. We are using the oval representation of S^2, but with the vertices identified. The Seifert-van Kampen Theorem gives us that $\pi_1(X) \cong \langle a \rangle \cong \mathbb{Z}$. □

(b) Let Y be the union $S^2 \cup D$ where D is the diameter on the z-axis in $\mathbb{R}^3$ connecting the north and south poles of S^2. Show that X and Y are homotopy equivalent. If M is a meridian (half a great circle) on S^2 connecting the north and south poles, show that the quotient space Y/M obtained from Y by collapsing M to a point is homotopy equivalent to the one-point union $S^2 \vee S^1$.

Proof. Proof by picture. □

(c) Assume the truth of (b) and use it again to compute $\pi_1(X)$.

Proof. The Seifert van-Kampen Theorem on $S^2 \vee S^1$ yields the result (homotopy equivalent spaces have isomorphic fundamental groups). □

6.3.28 August 2017, Problem 7 POSSIBLY OKAY

Let X be obtain from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus. Compute $\pi_1(X)$. Describe it in terms of generators and relations. Also compute the singular homology groups $H_i(X)$.

Proof. The Seifert-van Kampen Theorem yields that $\pi_1(X) \cong \pi_1(S^1 \times S^1) *_{\pi_1(S^1)} \pi_1(S^1 \times S^1)$. Write this as

$$\pi_1(X) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle *_{\langle z \rangle} \langle c, d \mid cdc^{-1}d^{-1} \rangle.$$

The inclusions of S^1 into the two tori yield the relation $b = d$, and so $\pi_1(X) \cong \langle a, b, c \mid aba^{-1}b^{-1}, cbc^{-1}b^{-1} \rangle$.

If we identify the tori with their rectangular representations and identify b and d, we get two trapezoids $[a, b, a^{-1}, b^{-1}]$ and $[c, b, c^{-1}, b^{-1}]$, where the b in the first is glued to the b^{-1} in the second. We get the following chain complex, where the ranks are given by the number of n-cells:

$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$$

Both 2-cells map to 0, for one maps to $a+b-a-b$ and the other maps to $c+b-c-b$. So, $H_2(X) \cong \ker \partial_2 \cong \mathbb{Z}^2$. Clearly, X is path-connected, and so $H_0(X) \cong \mathbb{Z}$.

Finally, $H_1(X) \cong \dfrac{\ker \partial_1}{\text{im } \partial_2} \cong \ker \partial_1 \cong \langle a, b, c \rangle \cong \mathbb{Z}^3$. That the kernel of the first differential is everything follows from the fact that there is only one 0-cell.

□

6.3.29 August 2017, Problem 8 FINISH

Let $S^1 \vee S^1$ be the one-point union of two circles with union point as basepoint $\{x_0\}$. Call the counterclockwise path around the left circle a and the counterclockwise path around the right circle b. Let E be a circle union an equilateral triangle inscribed in the circle. Choose the basepoint $\{e_0\}$ of E to be one of the vertices of the triangle. Describe a covering projection $p : (E, e_0) \to (S^1 \vee S^1, x_0)$ by labeling arcs and edges of E and describe the corresponding subgroup $p_*(\pi_1(E))$ in terms of a and b.

Proof. Orient all arcs and edges of E counterclockwise. Label the arcs each by a and the edges each by b. The vertices are the intersection points of the circle and triangle.

Choose a spanning tree corresponding to a^2, based at e_0. Tracing out loops, we get that

$$p_*(\pi_1(E)) \cong \langle a^3, a^2b, a^2b^{-1}a^{-1}, a^2b^{-2} \rangle.$$

□

6.3.30 August 2017, Problem 9 FINISH

For a space X, the suspension SX of X us the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Here, $I = [0, 1]$. The cone CX of X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point. Then there is an isomorphism $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ of reduced singular homology groups. The abbreviated proof of this statement goes as follows.

(a) SX is homeomorphic to the union of two cones CX with their bases identified.

(b) $\tilde{H}_{n+1}(SX) \cong \tilde{H}_{n+1}(CX \cup CX) \cong \tilde{H}_{n+1}(CX, X) \cong \tilde{H}_n(X)$.

Fill in the details (reasoning) in each of these two steps. In particular, state which theorems are being applied to explain the two isomorphisms in step (b).

Proof. Part (a) is obvious – we essentially map each copy of the cone to either side of the suspension identically. The maps agree where they are glued, and the piecing lemma yields the result. The first isomorphism in part (b) follows immediately.

Note that X has an open neighborhood in CX that deformation retracts to X, and so we have that (CX, X) is a good pair. So, $\tilde{H}_{n+1}(CX, X) \cong \tilde{H}_n(CX/X)$. But collapsing X to a point just results in $SX = CX \cup CX$. Thus, $\tilde{H}_{n+1}(CX \cup CX) \cong \tilde{H}_{n+1}(CX, X)$.

The final isomorphism follows from the Mayer-Vietoris Sequence

$$\ldots \longrightarrow \tilde{H}_{n+1}(CX) \longrightarrow \tilde{H}_{n+1}(CX, X) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(CX) \longrightarrow \ldots$$

and the fact that CX is contractible (to the collapsed copy of X that is a point, for example). Because contractible spaces have zero homology (except for in degree zero), we get our result by exactness. □

6.3.31 August 2016, Problem 1 FINISH

Prove the following statements:

(a) The open interval $(0, 1)$ is homeomorphic to the real line $\mathbb{R}$.

Proof. Define $f : \mathbb{R} \to (0, 1)$ by $f(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}$. We know that this function is continuous, and its inverse $g(y) = \tan(\pi(x - \frac{1}{2}))$ is also continuous. Composition yields that these are indeed inverses, and so we have a bijection. Thus, f is a homeomorphism. □

(b) The two dimensional sphere $S^2 \subseteq \mathbb{R}^3$ with the north pole removed is homeomorphic to $\mathbb{R}^2$.

Proof. Let ∞ be a point at infinity in $\mathbb{R}^2$. Then, $\mathbb{R}^2 \cup \{\infty\}$ is homeomorphic, via f, say, to S^2. The restriction of a homeomorphism is still a homeomorphism, and so f restricted to $\mathbb{R}^2$ is a homeomorphism between $\mathbb{R}^2$ and S^2 minus a point. Since S^2 is path-connected, S^2 minus a point is homeomorphic to S^2 minus the north pole. Transitivity of "is homeomorphic to" yields the result. □

6.3.32 August 2016, Problem 2 POSSIBLY OKAY

Prove the following statements:

(a) Let X and Y be topological spaces and $A \subseteq X$ and $B \subseteq Y$ be closed sets, respectively. Then, $A \times B$ is a closed set in $X \times Y$.

Proof. We will show that $(X \times Y) \setminus (A \times B)$ is open in $X \times Y$. Let $(x, y) \in (X \times Y) \setminus (A \times B)$ So, either $x \notin A$ or $y \notin B$. Without loss of generality, assume the former. Since $X \setminus A$ is open in X, there is an open set U in X with $x \in U \subseteq X \setminus A$. We then have that $(x, y) \in U \times (Y \setminus B)$, which is open in $X \times Y$. We claim that $U \times (X \setminus B) \subseteq (X \times Y) \setminus (A \times B)$. Of course, $U \times (Y \setminus B) \subseteq (X \setminus A) \times (Y \setminus B)$. Let $(c, d) \in (X \setminus A) \times (Y \setminus B)$. Then, $c \notin A$ or $d \notin B$. So, $(c, d) \notin A \times B$, and we have that $(c, d) \in (X \times Y) \setminus (A \times B)$. Chaining together the subset relations, we have that $U \times (X \setminus B) \subseteq (X \times Y) \setminus (A \times B)$, and so $(X \times Y) \setminus (A \times B)$ is open, meaning that $A \times B$ is closed, as desired. □

(b) Let $f : X \to Y$ be a continuous bijection between two topological spaces X and Y. Assume that X is compact and Y is Hausdorff. Then, f is a homeomorphism.

Proof. It remains to show that f is an open map. Let U be open in X. Then, $X \setminus U$ is closed. Closed subsets of compact sets are compact, and so $X \setminus U$ is compact. The continuous image of a compact set is compact, and so $f(X \setminus U)$ is compact in Y. Compact subsets of Hausdorff spaces are closed, and so $f(X \setminus U)$ is closed in Y. But $f(X \setminus U) = f(X) \setminus f(U) = Y \setminus f(U)$ means that $f(U)$ is open in Y. Note that the previous equations follow from the bijectivity of f. □

6.3.33 August 2016, Problem 3 POSSIBLY OKAY

Show that $(\mathbb{R}^2, d)$ is a metric space, where

$$d((x, y), (x', y')) = \begin{cases} |y| + |y'| + |x - x'| & \text{if } x \neq x', \\ |y - y'| & \text{if } x = x'. \end{cases}$$

Illustrate by diagrams in the real plane $\mathbb{R}^2$ what open balls of this metric are.

Proof. If $(x, y) = (x', y')$, then $x = x'$ and so $d((x, y), (x', y')) = |y - y'| = 0$.

If $x = x'$, then

$$d((x, y), (x', y')) = |y - y'| = |y' - y| = d((x', y'), (x, y)),$$

and if $x \neq x'$, then

$$d((x, y), (x', y')) = |y| + |y'| + |x - x'| = |y'| + |y| + |x' - x| = d((x', y'), (x, y)).$$

It remains to show that the triangle inequality holds. Suppose $x_1 = x_2 = x_3$. Then,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) &= |y_1 - y_2| + |y_2 - y_3| \\ &\geq |y_1 - y_3| \\ &= d((x_1, y_1), (x_3, y_3)) \end{aligned}$$

by the usual triangle inequality.

Suppose x_1, x_2, and x_3 are distinct. Then,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) &= |y_1| + |y_2| + |x_1 - x_2| + |y_2| + |y_3| + |x_2 - x_3| \\ &\geq |y_1| + |y_3| + |x_1 - x_3| \\ &= d((x_1, y_1), (x_3, y_3)). \end{aligned}$$

Suppose $x_1 = x_2 \neq x_3$. Then,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) &= |y_1 - y_2| + |y_2| + |y_3| + |x_2 - x_3| \\ &\geq |y_1| - |y_2| + |y_2| + |y_3| + |x_2 - x_3| \\ &= |y_1| + |y_3| + |x_2 - x_3| \\ &= |y_1| + |y_3| + |x_1 - x_3| \\ &= d((x_1, y_1), (x_3, y_3)), \end{aligned}$$

because $x_1 = x_2$.

Suppose $x_1 \neq x_2 = x_3$. Then,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) &= |y_1| + |y_2| + |x_1 - x_2| + |y_2 - y_3| \\ &\geq |y_1| + |y_2 - y_3 - y_2| + |x_1 - x_2| \\ &= |y_1| + |y_3| + |x_1 - x_3| \\ &= d((x_1, y_1), (x_3, y_3)), \end{aligned}$$

because $x_2 = x_3$.

Finally, suppose $x_3 = x_1 \neq x_2$. Then,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) &= |y_1| + |y_2| + |x_1 - x_2| + |y_2| + |y_3| + |x_2 - x_3| \\ &\geq |x_1 - x_3| \\ &= d((x_1, y_1), (x_3, y_3)), \end{aligned}$$

by the usual triangle inequality.

Hence, $(\mathbb{R}^2, d)$ is a metric space.

The basic open sets (unit open balls) look like open square diamonds of side length 1 in the Euclidean norm (i.e., squares rotated 45 degrees). So, open sets look like unions of open square diamonds (of whatever varying side lengths). □

6.3.34 August 2016, Problem 4 FINISH

Let $X = A \cup B$ be a subset in $\mathbb{R}^2$, where

$$A = \left\{ \left(x, \sin \frac{1}{x} \right) \mid x \in (0, 1) \right\}, \; B = \{(0, y) \mid y \in [-1, 1]\}.$$

Show that X is connected but not path connected.

Proof. Note that B is connected. So, if $U \cap V$ is a separation of X, then, without loss of generality, $B \subseteq U$. Now, every open set containing B intersects U in infinitely many points, so there is no way that it does not intersect V, Similarly, A is connected, being the image of a connected set under a continuous function. So, it must be completely contained in U or V. If U, then there is actually no separation, as V is empty. If V, then U and V intersect, for any open neighborhood of B must intersect A in infinitely many points. Thus, this case, too, yields that there is no separation. Hence, X is connected. □

6.3.35 August 2016, Problem 5 POSSIBLY OKAY

Let $\gamma \subseteq \mathbb{R}P^2$ be a simple closed curve representing a generator of $\pi_1(\mathbb{R}P^2)$, and let X be the space obtained from $\mathbb{R}P^2$ by attaching a Möbius band via a homeomorphism from the boundary of the Möbius band to γ.

(a) Compute $\pi_1(X)$.

Proof. Choose A to be an open neighborhood of $\mathbb{R}P^2$ and B to be an open neighborhood of the Möbius band so that $A \cap B$ is an open neighborhood of γ that deformation retracts onto γ. By the

Seifert-van Kampen Theorem, $\pi_1(X) \cong \pi_1(\mathbb{R}P^2) *_{\pi_1(\gamma)} \pi_1(\mathcal{M})$, where $\mathcal{M}$ is the Möbius band. So,

$$\pi_1(X) \cong \mathbb{Z}_2 *_{\mathbb{Z}} \mathbb{Z}.$$

The inclusion $\gamma \hookrightarrow \mathbb{R}P^2$ is the identity, and the inclusion $\gamma \hookrightarrow \mathcal{M}$ is given by $z \mapsto z^2$. So,

$$\pi_1(X) \cong \langle a, b \mid a^2, ab^{-2} \rangle \cong \langle b \mid b^4 \rangle \cong \mathbb{Z}_4.$$

□

(b) Determine the number of connected covering spaces of X up to equivalence.

Proof. The set of connected covering spaces modulo isomorphism is in bijection with the set of subgroups of $\pi_1(X) \cong \mathbb{Z}_4$. There are three such subgroups: $\{\bar{0}\}$, $\mathbb{Z}_4$, and $\{\bar{0}, \bar{2}\}$. Thus, there are three connected covering spaces of X up to equivalence. □

6.3.36 August 2016, Problem 6 FINISH

Let S be the surface obtained from a square by identifying edges as shown in the original exam. Prove or disprove:

(a) a is a retract of S.

Proof. Identify the given square with $[0,1] \times [0,1]$ in the obvious way. Define $f : [0,1] \times [0,1] \to [0,1]/(0 \sim 1) = a$ by $f(x,y) = y$. The map is a projection, and thus it is continuous. Note that $f|_a = \text{id}$, and so f is a retraction $S \to a$. □

(b) b is a retract of S.

Proof. Assume such a retraction exists. Then, $H_1(S) \cong H_1(b) \oplus H_1(S, b)$. Since b is a subcomplex of S, (S, b) is a good pair, and so $H_1(S, b) \cong \tilde{H}_1(S/b) \cong H_1(S/b)$. Contracting b to a point yields a representation of a 2-sphere. Note that S is a Klein bottle and b is a circle. We then have that

$$\mathbb{Z} \oplus \mathbb{Z}_2 \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z},$$

which is false by the Fundamental Theorem of Finitely Generated Abelian Groups. Thus, no such retraction can exist. □

(c) a is a deformation retract of S.

Proof. If a were a deformation retract of S, the two spaces would be homotopy equivalent. In particular, they would have the same fundamental group. But $\pi_1(S) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $\pi_1(a) \cong \mathbb{Z}$. So, a is not a deformation retraction of S. □

6.3.37 August 2016, Problem 7 POSSIBLY OKAY

For $(m, n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, let $X_{m,n}$ be the CW complex obtained from S^1 with its standard cell structure by attaching two 2-cells by maps of degres m and n, respectively.

(a) Compute the cellular homology groups of $X_{m,n}$.

Proof. Because $X_{m,n}$ is a 2-dimension CW complex, $H_n(X_{m,n}) \cong 0$ for all $n \neq 0, 1, 2$. Because $X_{m,n}$ is path-connected, $H_0(X_{m,n}) \cong \mathbb{Z}$. From the cell structure on $X_{m,n}$, we have the following chain complex, where the ranks are determined by the number of k-cells:

$$\dots \longrightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z} \longrightarrow \dots$$

The degrees of the attaching maps tell us that f maps (a, b) to $ma + nb$. That is, $I :=$ im $f \cong \langle ma + nb \mid m, n \in \mathbb{Z}\rangle$. Set $d = \gcd(m, n)$. Then, $I = \langle d(pa + qb) \mid a, b \in \mathbb{Z},\ \gcd(p, q) = 1,\ dp = m,\ dq = n\rangle$. So, $I \cong d\mathbb{Z}$. That $X_{m,n}$ has one 0-cell means that $\ker g \cong \mathbb{Z}$. So,

$$H_1(X_{m,n}) \cong \frac{\mathbb{Z}}{d\mathbb{Z}} \cong \mathbb{Z}_d.$$

Note that $H_2(X_{m,n}) \cong \ker f \cong \langle (a, b) \mid ma+nb = 0\rangle$. So, the kernel is represented by $(m/d, -n/d)$. The entries are coprime because $d = \gcd(m, n)$, and so $\ker f \cong \mathbb{Z}$. Thus, $H_2(X_{m,n}) \cong \mathbb{Z}$. □

(b) Give a necessary and sufficient condition under which $X_{m,n}$ and $X_{m',n'}$ are homotopy equivalent.

Proof. If $X_{m,n}$ and $X_{m',n'}$ are homotopy equivalent, then their homologies must be isomorphic. In particular, we need $\mathbb{Z}_d \cong \mathbb{Z}_{d'}$, where $d = \gcd(m, n)$ and $d' = \gcd(m', n')$. This only holds when $d = d'$.

Suppose $d = \gcd(m, n) = \gcd(m', n') = d'$. Showing that $X_{m,n}$ and $X_{m',n'}$ are homotopy equivalent amounts to showing that the attaching maps are homotopic. This is true if the attaching maps induce the same map in homology. One attaching map induces "multiplication by d" and the other induces "multiplication by d'". Since $d = d'$, we have our result. □

6.3.38 August 2016, Problem 8 POSSIBLY OKAY

Show the following: If $Y = U \cup V$ is a union of two open sets U and V such that $H_k(U \cap V)$ contains a nonzero homology class which is zero in both $H_k(U)$ and $H_k(V)$, then $H_{k+1}(Y) \neq 0$.

Proof. We apply the following Mayer-Vietoris Sequence:

$$\ldots \longrightarrow H_{k+1}(Y) \xrightarrow{f} H_k(U \cap V) \xrightarrow{g} H_k(U) \oplus H_k(V) \longrightarrow \ldots$$

By hypothesis, the becomes

$$\ldots \longrightarrow H_{k+1}(Y) \xrightarrow{f} G \ncong 0 \xrightarrow{g} 0 \longrightarrow \ldots$$

The Mayer-Vietoris Sequence is exact, and so $\operatorname{im} f \cong \ker g$. Because g maps to the zero group, $\ker g \cong G \ncong 0$. Then, $\operatorname{im} f \cong G \ncong 0$. But then something nontrivial maps into G via f, i.e., $H_{k+1}(Y)$ is nonzero. □

6.3.39 August 2015, Problem 1 FINISH

(a) Show that the product of two regular spaces is again regular.

Proof. We claim that a space X is regular is if and only if given $x \in X$ and an open neighborhood V of x, there exists an open neighborhood U of x such that $x \in U \subseteq \bar{U} \subseteq V$.

($\Rightarrow$) Let X be a regular space. Let $x \in X$, and let V be an open neighborhood of x. Then, $X \setminus V$ is closed and does not contain X. So, we can separate x and $X \setminus V$ with disjoint open sets A and B. Then, $A \cap V$ is an open neighborhood of x. Furthermore, $\overline{A \cap B} \subseteq \bar{A}$. Since A and B are disjoint open sets, $\bar{A}$ and B are disjoint. So, $\bar{A} \subseteq V$. Thus,

$$x \in A \cap B \subseteq \overline{A \cap B} \subseteq \bar{A} \subseteq V.$$

($\Leftarrow$) Let X be a space. Suppose that given $x \in X$ and an open neighborhood V of x, there exists an open neighborhood U of x such that $x \in U \subseteq \bar{U} \subseteq V$. Let $x \in X$, and let $F \subseteq X$ be closed and not contain x. Then, $X \setminus F$ is an open set containing x, and by so by assumption there is an open set U such that $x \in U \subseteq \bar{U} \subseteq X \setminus F$. Since $\bar{U}$ is closed, $X \setminus \bar{U}$ is open. Furthermore, $X \setminus \bar{U}$

contains F. So, we have found disjoint open sets U and $X \setminus \bar{U}$ that contain x and F, respectively, i.e., X is regular.

Let X and Y be regular spaces. Let $(x, y) \in X \times Y$, and let N be an open neighborhood of (x, y) in $X \times Y$. Then, $N = \bigcup_{\alpha \in I} U_\alpha \times V_\alpha$, where U_α is open in X and V_α is open in Y. For some α, we have that $(x, y) \in U_\alpha \times V_\alpha$, and so $x \in U_\alpha$ and $y \in V_\alpha$. By the regularity of X and Y, there are open sets A and B with $x \in A \subseteq \bar{A} \subseteq U_\alpha$ and $y \in B \subseteq \bar{B} \subseteq V_\alpha$. Since the closure of the product of sets is the product of the closures of the sets, we have that

$$(x, y) \in A \times B \subseteq \bar{A} \times \bar{B} = \overline{A \times B} \subseteq U_\alpha \times V_\alpha \subseteq N.$$

Hence, $X \times Y$ is regular. □

(b) State the definition of a quotient map.

Proof. A quotient map $q : X \to Y$ is a surjective map defined so that $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open in X. □

(c) Show that if X is regular and $A \subseteq X$ is closed, then X/A is Hausdorff.

Proof. Let $q : X \to X/A$ be the quotient map. Let $x, y \in X/A$ be distinct points. Without loss of generality, let $y \in A$. Since x and y are distinct, $x \notin A$ and so we may find disjoint open neighborhoods U and V of x and A, respectively, in X. Since V contains A and U is disjoint from A, we have that U and V are saturated open sets and so $q(U)$ and $q(V)$ are disjoint open neighborhoods of x and y in X/A. Similarly, if neither x nor y is in A, we can find open neighborhoods M and N in X of x and y, respectively, that are each disjoint from A. Since these neighborhoods are disjoint from A, they are saturated, and so $q(M)$ and $q(N)$ are open in X/A. WHY DON'T THEY INTERSECT???!!!

REGULAR IMPLIES HAUSDORFF, INTERSECT SOME NBHDS □

6.3.40 August 2015, Problem 2 FINISH

Let X be a space. Define an equivalence relation on X by setting $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$. The equivalence classes of X with respect to $\sim$ are called the quasicomponents of X.

(a) Show that each connected component of X lies in a quasicomponent of X.

Proof. Let C be a connected component of X. Let $x, y \in C$. Suppose there is a separation $X = A \cup B$ of X into nonempty disjoint open sets such that $x \in A$ and $y \in B$. We claim that $C \cap A$ and $C \cap B$ separate C. Since A and B are disjoint, so are $C \cap A$ and $C \cap B$. Since every point of X is either in A or B, we have that $(C \cap A) \cup (C \cap B) = C$. Since $C \cap A$ and $C \cap B$ are open with respect to the subspace topology on C, we have a separation of C, which is supposedly connected. Thus, the original separation of X cannot exist, and we have that $x \sim y$, i.e., that C lies in a quasicomponent of X. ☐

(b) State the definition of local connectedness and show that if X is locally connected, then the connected components and the quasicomponents of X are the same.

Proof. A space X is locally connected if given $x \in X$ and an open neighborhood $U \subseteq X$ of x there exists a connected open set V such that $x \in V \subseteq U$.

We saw in part (a) that connected components lie within quasicomponents. So, it remains to show that quasicomponents lie within connected components. Let Q be a quasicomponent of X. Suppose $Q = A \cup B$ is a separation of Q, i.e., that A and B are nonempty, open in Q, and disjoint. Then, $A = X \cap G$ and $B = X \cap H$, where G and H are open in X. ☐

(c) Determine the connected components and the quasicomponents of the following subspace of $\mathbb{R}^2$:

$$X = (\mathbb{R} \times \{-1, 1\}) \cup \{(x, y) \mid x^2 + y^2 = (1 - 1/n)^2 \text{ for an integer } n > 1\}.$$

Proof. ☐

6.3.41 August 2015, Problem 3 POSSIBLY OKAY

Recall that a retraction of a space X onto a subspace $A \subseteq X$ is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. In the following, let $S^1 \vee S^1 = (S^1 \times \{\theta_1\}) \cup (\{\theta_1\} \times S^1)$ and $p = (\theta_2, \theta_2)$ for $\theta_1 \neq \theta_2 \in S^1$.

(a) Show that if X is Hausdorff and $f : X \to X$ is continuous, then the set of fixed points of f is closed in X.

Proof. Define $g : X \to X \times X$ by $g(x) = (x, f(x))$. Since X is Hausdorff, the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. The preimage of Δ under g is exactly the set $E = \{x \in X \mid f(x) = x\}$, which is the set of fixed points of f. Since f is continuous, the components of g are continuous, and so g is continuous. So, E is closed in X (the preimage of a closed set under a continuous function is closed). □

(b) Is there a retraction from $[0, 2)$ to $[0, 1)$?

Proof. Let $r : [0, 2) \to [0, 1)$ be a retraction. Then, $[0, 1)$ is the set of fixed points of r. Since $1 \notin [0, 1)$, we have that $1 \notin E$. Since $[0, 2)$ is Hausdorff and r is continuous, E must be closed in $[0, 2)$. But any open neighborhood about 1 intersects $[0, 1)$ and $1 \notin E$, meaning that E is not closed. So, by part (a), no such retraction can exist. □

(c) Is there a retraction $S^1 \times S^1$ to $S^1 \vee S^1$?

Proof. If there were such a retraction, we would have that

$$H_1(S^1 \times S^1) \cong H_1(S^1 \vee S^1) \oplus H_1(S^1 \times S^1, S^1 \vee S^1).$$

That is, we would have that $\mathbb{Z}^2 \cong F_2 \oplus H_1(S^1 \times S^1, S^1 \vee S^1)$. The left-hand side is Abelian, but the right-hand side has elements that do not commute. For example, $(a, 1)(b, 1) = (ab, 1) \neq (ba, 1) = (b, 1)(a, 1)$, where 1 is the identity element of $H_1(S^1 \times S^1, S^1 \vee S^1)$ and $F_2 = \langle a, b\rangle$. Thus, there is no such retraction. □

(d) Is there a retraction $(S^1 \times S^1) \smallsetminus \{p\}$ to $S^1 \vee S^1$?

Proof. Such a retraction exists, and we describe it geometrically. The removal of a point is equivalent to the removal of an arc, and then we can pull back the torus into a cylinder with a loop. The cylinder retracts to a circle, and so we are left with $S^1 \vee S^1$. □

6.3.42 August 2015, Problem 4 POSSIBLY OKAY

Let S be the closed orientable surface of genus 2 and b be the curve depicted in the original exam. Attach a D^2 to S by gluing its boundary to b by a degree 5 map. Compute the homology groups of the resulting space.

Proof. Let X denote the resulting space. View X as a CW complex, where S is given by the usual octagonal representation. Note that X has one 0-cell, four 1-cells, and two 2-cells. Thus, we get the complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$$

where the ranks are determined by the number of n-cells. Since X is a 2-dimensional cell complex, $H_n(X) \cong 0$ for $n \geq 3$. Since X is path-connected, $H_0(X) \cong \mathbb{Z}$. Since X has only one 0-cell, ∂_1 is the zero map, and so $H_1(X) \cong \dfrac{\mathbb{Z}^4}{\text{im } \partial_2}$. The image of one of the 2-cells is zero, but the other is $5b$. So, $H_1(X) \cong \dfrac{\mathbb{Z}^4}{5\mathbb{Z}} \cong \mathbb{Z}_5 \oplus \mathbb{Z}^3$.

Finally, $H_2(X) \cong \ker \partial_2$, since ∂_3 is the zero map. One of the 2-cells maps to zero, and so $H_2(X) \cong \ker \partial_2 \cong \mathbb{Z}$.

□

6.3.43 August 2015, Problem 5 POSSIBLY OKAY

Recall the standard CW structure for $\mathbb{R}P^3 = e^0 \cup e^1 \cup e^2 \cup e^3$ where each e^i is glued to the $(i-1)$-skeleton via the antipodal map. Recall that $\mathbb{R}P^2$ is the 2-skeleton of this CW complex. Let A and B be copies of of this CW complex, and let $X = A \cup_{\mathbb{R}P^2} B$.

(a) Find a CW structure for X with 5 total cells.

Proof. Take the usual CW structure for the intersection $\mathbb{R}P^2$: $e^0 \cup e^1 \cup e^2$. Attach via antipodal maps two 3-cells to get a CW structure for X. □

(b) Calculate $\pi_1(X, x)$ where the basepoint x is one of the 0-cells in the CW structure.

Proof. The fundamental group of a CW complex only depends on the 2-skeleton of the complex. So, the fundamental group of X is the same as the fundamental group of $\mathbb{R}P^2$, which is isomorphic to $\mathbb{Z}_2$. □

(c) Calculate the homology groups of X.

Proof. Since X is path-connected, $H_0(X) \cong \mathbb{Z}$. We use the following Mayer-Vietoris Sequence:

$$\ldots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \ldots$$

For $n \geq 4$, exactness yields that $H_n(X) \cong 0$. For $n = 3$, our sequence becomes

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_3(X) \longrightarrow 0$$

and so, by exactness, $H_3(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. The Hurewicz Theorem says that $H_1(X)$ is isomorphic to the Abelianization of $\pi_1(X) \cong \mathbb{Z}_2$, which is, of course, again $\mathbb{Z}_2$.

It remains to find $H_2(X)$. We examine the sequence

$$H_2(\mathbb{R}P^3) \oplus H_2(\mathbb{R}P^3) \longrightarrow H_2(X) \longrightarrow H_1(\mathbb{R}P^2)$$

Evaluating the known groups, we get

$$0 \longrightarrow H_2(X) \xrightarrow{f} \mathbb{Z}_2$$

The antipodal map is injective, and so $\ker f$ is trivial. Thus, $H_2(X) \cong 0$.

□

(d) Calculate the homology groups of (X, A).

Proof. Since A is a subcomplex of X, we have that (X, A) is a good pair. So, the homology of (X, A) is isomorphic to the reduced homology of X/A. Collapsing A to a point results in a 3-cell attached to a point, which is a CW structure for S^3. The reduced homology of S^3 is zero in all degrees other then three, in which it is $\mathbb{Z}$.

□

6.3.44 August 2015, Problem 6 FINISH

(a) Let X and Y be path-connected, locally path-connected, semilocally simply connected spaces. Show that if X and Y are homeomorphic, then their universal covering spaces $\tilde{X}$ and $\tilde{Y}$ are homeomorphic.

Proof. Let $f : X \to Y$ be a homeomorphism. The universal cover of X is $\tilde{X} = \{[\gamma] \mid \gamma \text{ a path starting at } x_0\}$, where $x_0 \in X$ and $[\gamma]$ is the path homotopy class of γ. Sending $[\gamma]$ to $\gamma(1)$ defines a covering map. The universal cover of Y is $\tilde{Y} = \{[\eta] \mid \gamma \text{ a path starting at } y_0\}$. Since $\tilde{X}$ and $\tilde{Y}$ are universal covers, they are path-connected, and so the base point for our paths may be any point. Furthermore, each η is $f(\gamma)$ for some γ. Thus, $\tilde{Y} = \{[f(\gamma)] \mid \gamma \text{ a path starting at } x_0\}$. As f is a homeomorphism, $\tilde{X} \cong \tilde{Y}$. □

(b) Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$. Show that X and Y have isomorphic homology groups, but that their universal covers do not.

Proof. We know that the homology of a wedge sum is the direct sum of the homologies of the components of the sum (in degree $n \neq 0$). Thus, $H_n(Y)$ is isomorphic to $\mathbb{Z}$ for $n = 2$, $\mathbb{Z} \oplus \mathbb{Z}$ for $n = 1$, and zero for $n > 2$. Since Y is path-connected, $H_0(Y) \cong \mathbb{Z}$. This all amounts to Y having the same homology as the torus, which is exactly what X is.

The product of covers is a cover of a product. So, $\mathbb{R}^2$ is a cover of $S^1 \times S^1$, as $\mathbb{R}$ is a cover (the universal cover, in fact) of S^1. Since $\mathbb{R}^2$ is path-connected and simply connected, we have that $\mathbb{R}^2$ is the universal cover of $S^1 \times S^1$. The homology of $\mathbb{R}^2$ is trivial in all degree other than zero, in which it is $\mathbb{Z}$.

The universal cover of Y is an infinite chain of 2-sphere with intervals connecting them, every other interval being identified with the same copy of S^1 in Y and the points where the intervals meet the sphere are all identified to a single point. Call this space $\tilde{Y}$. Clearly, $\tilde{Y}$ is a cover, is path-connected, and is simply connected, and so it is indeed the universal cover of Y. Also, note that $\tilde{Y}$ is homotopy equivalent to an infinite wedge sum of 2-sphere, and this space has nontrivial homology in degree 2, as $H_2(S^2) \cong \mathbb{Z}$ and the wedge sum of of spaces has homology the direct sum of homologies of wedged-spaces. Thus, the universal covers of X and Y do not have the same homology. □

(c) Use (a) and (b) to conclude that X and Y are not homeomorphic.

Proof. If X and Y were homeomorphic, then their universal covers $\tilde{X}$ and $\tilde{Y}$would be homeomorphic, by part (a). But by part (b), $\tilde{X}$ and $\tilde{Y}$ do not have the same homology groups, which homeomorphic spaces do. Thus, X and Y are not homeomorphic. □

6.3.45 January 2015, Problem 1 SOMETHING IS OFF IN PART (C), I THINK

For any subset C of a metric space (X, d) define the distance from $x \in X$ to C to be

$$d(x, C) = \inf\{d(x, y) \mid y \in C\}.$$

(a) Show that $\{x \in X \mid d(x, C) = 0\} = \bar{C}$, the closure of C.

Proof. Set $A = \{x \in X \mid d(x, C) = 0\}$.

Let $x \in \bar{C}$. Then, every open ball about x intersects C. That is, for all $\varepsilon > 0$, $N_\varepsilon(x) \cap C \neq \emptyset$. If $d(x, C) > 0$, then $N_{d(x,C)/2} \cap C = \emptyset$, giving us a contradiction. Thus, $d(x, C) = 0$, and we get that $\bar{C} \subseteq A$.

Let $x \in A$. Let $N_\varepsilon(x)$ be an open ball about x of radius $\varepsilon > 0$. Suppose $N_\varepsilon(x) \cap C = \emptyset$. Then, for all $y \in C$, $d(x, y) \geq \varepsilon > 0$. But then $\inf\{d(x, y) \mid y \in C\} = d(x, C) > 0$, a contradiction. So, $N_\varepsilon(x) \cap C \neq \emptyset$, giving us that $x \in \bar{C}$. □

(b) Assume $d(x, C) \leq d(x, y) + d(y, C)$ for all $x, y \in X$. Show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, C)$ is continuous.

Proof. Let $\varepsilon > 0$, and let $y \in X$. Consider

$$\begin{aligned} |f(x) - f(y)| &= |d(x, C) - d(y, C)| \\ &\leq |d(x, y) + d(y, C) - d(y, C)| \\ &= d(x, y). \end{aligned}$$

So, for $d(x, y) < \varepsilon$, we get that $|f(x) - f(y)| < \varepsilon$, giving us the continuity of f. □

(c) Prove the inequality given in (b).

Proof. Let $z \in C$. Since d is a metric, it obeys the triangle inequality. So,

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y).$$

Then,

$$\inf\{d(x, z) \mid z \in C\} - \inf\{d(y, z) \mid z \in C\} \leq \inf\{d(x, z) - d(y, z) \mid z \in C\} \leq d(x, y)$$

yields the result. □

6.3.46 January 2015, Problem 2

Let X be a space, Y a set, and $f : X \to Y$ a function. Recall that the collection of all subsets U of Y such that its inverse image is open in X is open is a topology on Y (called the quotient topology induced by the quotient map f).

(a) Now assume Y is a space and $f : X \to Y$ onto and continuous. Show that f is a quotient map if there exists a continuous function $g : Y \to X$ such that $f \circ g : Y \to Y$ is the identity map.

Proof. Since f is onto, the composition $f \circ g$ is well-defined.

If V is open in Y, then $f^{-1}(V)$ is open in X by the continuity of f.

Let $V \subseteq Y$ such that $f^{-1}(V)$ is open in X. We want to show that V is open in Y. Consider $(f \circ g)^{-1}(V) = V$, since $f \circ g$ is the identity. The preimage of V under $f \circ g$ is equal to the preimage under g of the preimage of V under f. That is,

$$V = (f \circ g)^{-1}(V) = g^{-1}(f^{-1}(V)).$$

By assumption, $f^{-1}(V)$ is open in X. By the continuity of g, we have that $V = g^{-1}(f^{-1}(V))$ is open in Y, as desired. □

(b) Assuming (a) is true, show that the projection map $X_1 \times X_2 \to X_1$ given by $h(x_1, x_2) = x_1$ is a quotient map.

Proof. Note that h is onto and continuous. Fix $y \in X_2$. Define $g : X_1 \to X_1 \times X_2$ by $g(x_1) = (x_1, y)$, and note that g is continuous. Furthermore, note that $h \circ g : X_1 \to X_1$ is the identity, since $(h \circ g)(x_1) = h(x_1, y) = x_1$. By part (a), h is a quotient map. □

(c) Show that there is a quotient map $f : (0,1) \twoheadrightarrow [0,1]$ and that there is no quotient map $g : [0,1] \twoheadrightarrow (0,1)$.

Proof. Quotient maps are continuous. But if g is continuous, then $g([0,1]) = (0,1)$ is compact (the continuous image of a compact set is compact), which is false. □

6.3.47 January 2015, Problem 6

(a) What is the fundamental group of $\mathbb{R}P^n$ for $n > 1$?

Proof. View $\mathbb{R}P^n$ as a CW complex with exactly one k-cell for each $0 \leq k \leq n$ with the attaching maps being antipodal maps. The fundamental group of a CW complex only depends on the 2-skeleton of the complex. So, $\pi_1(\mathbb{R}P^n) \cong \pi_1(\mathbb{R}P^2)$ for $n > 1$. The fundamental group of $\mathbb{R}P^2$ is $\mathbb{Z}_2$, and this can be seen via the oval representation of $\mathbb{R}P^2$ and using the Seifert-van Kampen Theorem. □

(b) Show that every continuous map $f : \mathbb{R}P^n \to S^1$ must be nullhomotopic if $n > 1$.

Proof. The induced map $f_* : \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2 \to \pi_1(S^1) \cong \mathbb{Z}$ must be trivial, since

$$f_*(1) + f_*(1) = f_*(1 + 1) = f_*(0) = 0 \implies f_*(1) = -f_*(1) \implies f_*(1) = 0.$$

So, we may apply the Lifting Criterion. We get that there is a lift $\tilde{f}$ of f to $\mathbb{R}$, the universal cover of S^1. If p is the covering map $\mathbb{R} \to S^1$, then $p \circ \tilde{f} = f$. Now, $\mathbb{R}$ is contractible, and so $\tilde{f}$ is nullhomotopic, i.e., there is a homotopy h from $\tilde{f}$ to a constant map. Then, $p \circ h$ is a homotopy of f and a constant map. □