January 2023

1. Let R be a ring.
(a) (3 points) Define what it means for a short exact sequence of left R-modules to be split. Do not give several equivalent conditions. Pick your favorite one.

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split if and only if $\exists\, h: B \to A$ such that $hf = 1_A$.

In parts (b) and (c) below, use your definition from part (a) to prove whether the given short exact sequence is split. Prove directly from your minimal definition. Any theorems about split sequences you use must be proven as part of your answer. We denote the quotient groups Z/nZ by Zn and consider them as Z-modules.

(b) (3.5 points) $0 \to \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_6 \xrightarrow{g} \mathbb{Z}_3 \to 0$ with the first map being $n + 2\mathbb{Z} \mapsto 3n + 6\mathbb{Z}$ and the second map being $n + 6\mathbb{Z} \mapsto n + 3\mathbb{Z}$.

This sequence splits. Using above def, consider $h: \mathbb{Z}_6 \to \mathbb{Z}_2$ given by $h(1+6\mathbb{Z}) = 1+2\mathbb{Z}$, and extending linearly. Then $h(3+6\mathbb{Z}) = 3h(1+6\mathbb{Z}) = 3+2\mathbb{Z} = 1+2\mathbb{Z}$. So $h \circ f(1+2\mathbb{Z}) = 1+2\mathbb{Z}$. Hence $h \circ f = 1_A$ and seq. splits.

(c) (3.5 points) $0 \to \mathbb{Z}_6 \xrightarrow{f} \mathbb{Z}_{12} \xrightarrow{g} \mathbb{Z}_2 \to 0$ with the first map being $n + 6\mathbb{Z} \mapsto 2n + 12\mathbb{Z}$ and the second map being $n + 12\mathbb{Z} \mapsto n + 2\mathbb{Z}$.

This seq. does not split. Using above def; Suppose $h: \mathbb{Z}_{12} \to \mathbb{Z}_6$ satisfies $h \circ f = 1_A$. Then $h(2+12\mathbb{Z}) = 1+6\mathbb{Z}$. But there is no such homomorphism. Indeed any hom. is determined by image of $1+12\mathbb{Z}$. There are 6 possible images:

$h(1+12\mathbb{Z}) = 1+6\mathbb{Z}$ $\qquad$ $h(1+12\mathbb{Z}) = 4+6\mathbb{Z}$

$h(1+12\mathbb{Z}) = 2+6\mathbb{Z}$ $\qquad$ $h(1+12\mathbb{Z}) = 5+6\mathbb{Z}$

$h(1+12\mathbb{Z}) = 3+6\mathbb{Z}$ $\qquad$ $h(1+12\mathbb{Z}) = 0+6\mathbb{Z}$

None satisfy $h(2+12\mathbb{Z}) = 1$, so there is no such homomorphism.

2. Let D be an integral domain such that D as a module over itself is injective. Prove that D is a field.

pf: Note integral domain $\Rightarrow$ commutative, so we just need to show every nonzero element has an inverse.

Let $\lambda \neq 0 \in D$. Then $\lambda x = 0 \Rightarrow x = 0$ so mult. by λ is an injective homomorphism $\lambda : D \to D$. So by the definition of injective module we may extend 1_D as in following:

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r, "\lambda"] \arrow[d, "1_D"'] & D \arrow[dl, dashed, "\exists h"] \\ & D & \end{tikzcd}$$

i.e $\exists h : D \to D$ s.t $h \circ \lambda = 1_D$. But $\mathrm{Hom}_D(D, D) \cong D$ so h is given by mult. by some element $x \in D$. That is $x \lambda = 1_D$. So λ is invertible and D is a field.

3. (a) (6 points) Let R be a commutative ring and let M and N be R-modules. Prove that if M and N are flat R-modules, then $M \otimes_R N$ is a flat R-module.

pf: Let $f: A \to B$ be an injective hom. of R-modules.

M flat $\Rightarrow$ $0 \to A \otimes M \xrightarrow{f \otimes M} B \otimes M$ is exact.

N flat $\Rightarrow$ $0 \to (A \otimes M) \otimes N \xrightarrow{(f \otimes M) \otimes N} (B \otimes M) \otimes N$ exact

So by associativity of tensor product:

$$0 \to A \otimes (M \otimes N) \xrightarrow{f \otimes (M \otimes N)} B \otimes (M \otimes N) \text{ exact}$$

$\Rightarrow$ $M \otimes N$ is flat $\square$

(b) (4 points) Describe problems you run into if you try to generalize this to non-commutative rings. Is there any way around them? Explain.

We have to be careful with left/right modules. But there are ways around it if we are careful. Here is one possible fix: For M an R-R bimodule which is flat as a left R-mod and N a left R-module which is flat, then $M \otimes_R N$ is a flat left R-module. The same proof above holds.

4. Let R be a commutative ring, M a R-module, N and N' submodules of M with N' finitely generated, and I an ideal of R contained in the Jacobson radical of R. Assume that $M = N + IN'$. Prove that M=N.

pf: Note $N + IN' \subseteq N + IM \subseteq N + JM$. Thus

$M \subseteq N + JM \subseteq M \Rightarrow M = N + JM$, where J is Jacobson radical.

Hence $M/N = \dfrac{N+JM}{N} \cong \dfrac{JM}{N \cap JM}$ by 2^{nd} iso thm

$\cong J(M/N)$

$\ast$ Hence $M/N \cong 0$ by Nakayama's Lemma So $M = N$.

(Note: I believe $\ast$ requires M/N to be f.g. so we'd still need to show N' f.g $\Rightarrow M/N$ is f.g)

$$M/N = \frac{N + IN'}{N} \cong \frac{IN'}{N \cap IN'}$$

N' f.g $\Rightarrow IN'$ f.g $\Rightarrow M/N$ f.g (quotients of f.g modules are f.g)

Same generators work!

5. (a) Prove the following: If R is a division ring, then the ring of $n \times n$ matrices over R under matrix addition and matrix multiplication is a simple ring.

(b) Give an example of a semisimple ring that is not a simple ring.

pf:

(a) Let J be a nonzero two-sided ideal of $\mathrm{Mat}_n(R)$. Let $X = (x_{ij})$ be a nonzero element. So $x_{ab} \neq 0$ for some $1 \leq a \leq n$, $1 \leq b \leq n$. We'll show $J = \mathrm{Mat}_n(R)$.

Let E_{ij} denote the $n \times n$ matrix with a 1 in position (i,j) and zeros elsewhere. Then notice for any (i,j)

$$E_{ia} X E_{bj} = \begin{bmatrix} \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & \\ \vdots & & \\ & x_{ab} & \end{bmatrix} \begin{array}{c} \text{col } j \\ \downarrow \\ \begin{bmatrix} & 0 & \\ \cdots 0 & 1 & 0 \cdots \\ & 0 & \\ & \vdots & \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow \text{row } b \\ \\ \end{array}$$

$$= \begin{array}{r} \\ \xrightarrow{\text{row } i} \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} & 1 & \end{bmatrix} \\ \uparrow \\ \text{col } a \end{array} \begin{array}{c} \begin{bmatrix} 0 & \begin{matrix} x_{1b} \\ \vdots \\ x_{nb} \end{matrix} & 0 \end{bmatrix} \\ \uparrow \\ \text{col } j \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} & x_{ab} & \end{bmatrix} \\ \uparrow \text{col } j \end{array} \leftarrow \text{row } i = x_{ab} E_{ij}$$

Multiplying by $\frac{1}{x_{ab}}$ gives $\quad E_{ij} = \frac{1}{x_{ab}} E_{ia} X E_{bj} \in J$

But every element in $\mathrm{Mat}_n(R)$ is a linear combination of E_{ij}.

Hence $J = \mathrm{Mat}_n(R)$, so $\mathrm{Mat}_n(R)$ is a simple ring $\square$

b) Let R be a division ring. Then the ring $S = R \times R$ is semisimple (By Artin-Wederburn) but not simple since $J = R \times 0$ is a nontrivial two-sided ideal.

6. Let R be a ring.

(a) Given an exact sequence of left R-modules $0 \to M \to N \to K \to 0$, show that there exists a commutative diagram of left R-modules

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "i_1"] \arrow[d, "\pi_1"'] & P \oplus Q \arrow[r, "p_2"] \arrow[d, "\pi_2"'] & Q \arrow[r] \arrow[d, "\pi_3"'] & 0 \\ 0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r, "g"] & K \arrow[r] & 0 \end{tikzcd}$$

such that the rows are exact, the vertical maps are surjective, and the map i_1 is the injection into the first factor and the map p_2 is the projection to the second factor.

(b) Explain why you can conclude that the kernels of the vertical maps form again a short exact sequence

$$0 \to \ker \pi_1 \to \ker \pi_2 \to \ker \pi_3 \to 0.$$

pf: Recall every R-module is isomorphic to a quotient of a projective (or even free) module. So there exist epis $P \xrightarrow{\pi_1} M \to 0$ and $Q \xrightarrow{\pi_3} K \to 0$ with P, Q projective.

Using lifting property, $\exists\ \sigma : Q \to N$ s.t $\pi_3 = g\sigma$

$$\begin{tikzcd} & Q \arrow[d, "\pi_3"] \arrow[dl, dotted, "\sigma"'] \\ N \arrow[r, "g"'] & K \end{tikzcd}$$

Now we get the following diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & P \oplus Q \arrow[r, "{[0\ 1]}"] \arrow[d, "{\pi_2 = [f\pi_1\ \ \sigma]}"] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[r] \arrow[d, "\pi_3"] & 0 \\ 0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r, "g"] & K \arrow[r] & 0 \end{tikzcd}$$

$gf = 0 \Rightarrow$ both squares commute. We show π_2 is surjective.

Let $n \in N$. Then $g(n) = \pi_3(q) = g\sigma(q)$

$\Rightarrow g(n - \sigma(q)) = 0 \Rightarrow n - \sigma(q) = f(m)$

for some m. π_1 surj $\Rightarrow m = \pi_1(p)$ for some p. So

$$n = f\pi_1(p) + \sigma(q) = \pi_2\left(\begin{bmatrix} p \\ q \end{bmatrix}\right).$$

(For an easier proof just use $P = M$, $Q = N$ with $\pi_1 = 1$, $\pi_3 = g$, $\pi_2 = [f\ g]$.)

b) $\operatorname{coker} \pi_1 = 0$, so this follows from Snake Lemma.

7. Find the colimit (also called the direct limit) of the following diagram.

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

Be sure to give a full proof that the universal property holds.

The direct limit is isomorphic to $\mathbb{Z}$ with inclusions as shown

$$\begin{tikzcd}
M_1 = \mathbb{Z} \arrow[d, "\phi^1_2 = 2"'] \arrow[dr, "\alpha_1 = 4"] \arrow[d, phantom, shift left=6ex, "\circlearrowright"] & \\
M_2 = \mathbb{Z} \arrow[r, "\alpha_2 = 2"] \arrow[d, "\phi^2_3 = 2"'] \arrow[d, phantom, shift left=6ex, "\circlearrowright"] & \varinjlim M_i = \mathbb{Z} \\
M_3 = \mathbb{Z} \arrow[ur, "\alpha_3 = 1"'] &
\end{tikzcd}$$

It is clear that

$\alpha_1 = \alpha_2 \circ \phi^1_2$ and

$\alpha_2 = \alpha_3 \circ \phi^2_3$

Now we show the universal property holds:

Suppose we have $f_i : \mathbb{Z} \longrightarrow X$, $i=1,2,3$ such that $f_1 = f_2 \phi^1_2$ and $f_2 = f_3 \phi^2_3$. (i.e $f_1 = f_2 \circ 2$; $f_2 = f_3 \circ 2$). Then we claim $\Theta = f_3$ is the unique map $\Theta : \mathbb{Z} \to X$ satisfying $\Theta \alpha_i = f_i$ for $i = 1,2,3$. Indeed $\alpha_3 = 1 \Rightarrow \Theta = f_3$; so our choice is unique. For existence, simply note

$f_2 = f_3 \circ 2 \Rightarrow f_2 = \Theta \circ \alpha_2$ and

$f_1 = f_2 \circ 2 = (f_3 \circ 2) \circ 2 = f_3 \circ 4 = \Theta \circ \alpha_1$.

Thus our diagram satisfies the desired universal property.

8. Let R be a commutative ring, and let I be an ideal and M an R-module.

(a) (7 points) Prove that there is an isomorphism of R-modules $R/I \otimes_R M \cong M/IM$.

(b) (7 points) Prove that $\mathrm{Tor}_1^R(R/I, R/I) \cong I/I^2$.

(c) (6 points) Compute $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$ for all $n \geq 0$.

a) Consider $\phi : R/I \times M \longrightarrow M/IM$

$(\bar{r}, m) \longmapsto rm + IM$.

Easy to check this is R-bilinear, so we get $\hat{\phi} : R/I \otimes M \to M/IM$ sending $\bar{r} \otimes m \mapsto \overline{rm}$. (Also check ϕ is well-defined: $r - s \in I \Rightarrow rm - sm \in IM$).

Consider $\psi : M/IM \to R/I \otimes M$

$\bar{m} \longmapsto \bar{1} \otimes m$.

Again easy to check ψ is an R-map and well defined.
We'll show W.D: $m - n \in IM \Rightarrow m - n = \sum r_i m_i$ with $r_i \in I$.

So $\bar{1} \otimes (m-n) = \bar{1} \otimes \sum r_i m_i = \sum (\bar{1} \otimes r_i m_i) = \sum \bar{r_i} \otimes m_i = \sum 0 = 0$

So $\bar{1} \otimes m = \bar{1} \otimes n \Rightarrow \psi$ is well-defined

Now it is easy to check $\psi \circ \hat{\phi} = 1$ and $\hat{\phi} \circ \psi = 1$

Suffices to check $\psi \circ \hat{\phi}$ on simple tensors:

$\psi \circ \hat{\phi}(\bar{r} \otimes m) = \psi(\overline{rm}) = \bar{1} \otimes rm = (\bar{1})r \otimes m = \bar{r} \otimes m$

and $\hat{\phi} \circ \psi(\bar{m}) = \hat{\phi}(\bar{1} \otimes m) = \overline{1m} = \bar{m}$. So $\hat{\phi}$ is an iso.

8b) $\mathrm{Tor}_1^R(R/I, R/I) \cong I/I^2$.

pf: Consider SES $0 \to I \xrightarrow{i} R \xrightarrow{\pi} R/I \to 0$. Use LES in $\mathrm{Tor}(R/I, -)$:

Get exact seq:

$$\mathrm{Tor}_1(R/I, R) \longrightarrow \mathrm{Tor}_1(R/I, R/I) \to R/I \otimes_R I \to R/I \otimes_R R \to R/I \otimes_R R/I \to 0$$

- Note $\mathrm{Tor}_1(R/I, R) = 0$ since R is proj. hence flat.
- $R/I \otimes_R R \cong R/I$
- $R/I \otimes_R R/I \cong R/I$
- $R/I \otimes I \cong I/I^2$

$\Big\}$ follow from $R/I \otimes_R M \cong M/IM$

So we have an exact seq:

$$0 \longrightarrow \mathrm{Tor}_1(R/I, R/I) \xrightarrow{f} I/I^2 \xrightarrow{g} R/I \xrightarrow{h} R/I \longrightarrow 0$$

Note h is simply the identity*. So $\ker h = \mathrm{Im}\, g = 0$.

Hence $g = 0$ and $\mathrm{Im}\, f = \ker g = I/I^2$.

So f is an isomorphism $\square$

*More details I guess:

$$\begin{tikzcd}
R/I \otimes R \arrow[r, "R/I \otimes \pi", "\beta"'] & R/I \otimes R/I \arrow[d, "\gamma", "\cong"'] \\
R/I \arrow[u, "\alpha"] \arrow[r, "h"] & R/I
\end{tikzcd}$$

$\alpha: \bar{r} \mapsto \bar{r} \otimes 1$, $\qquad \gamma: \bar{r} \otimes \bar{s} \mapsto \overline{rs}$

$$h(\bar{r}) = \gamma\beta\alpha(\bar{r}) = \gamma\beta(\bar{r} \otimes 1) = \gamma(\bar{r} \otimes \bar{1}) = \overline{r \cdot 1} = \bar{r}.$$

8c) $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \begin{cases} 0 & \text{for } n \geq 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n=0,1 \end{cases}$

pf: Start with a proj resolution of $\mathbb{Z}/2\mathbb{Z}$:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

The map $\mathbb{Z} \to \mathbb{Z}$ is mult. by 2. Since $pd(\mathbb{Z}/2\mathbb{Z}) = 1$
$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = 0$ for $n \geq 2$.

Now take $\mathrm{Hom}(-, \mathbb{Z}/6\mathbb{Z})$ of deleted resolution:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) & \xrightarrow{\mathrm{Hom}(2, \mathbb{Z}/6\mathbb{Z})} & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & \\ C^\bullet \quad 0 & \to & \mathbb{Z}/6\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 0 \end{array}$$

$$\mathrm{Ext}^0 = H^0(C^\bullet) = \frac{\ker(2)}{0} \cong 3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$$

$$\mathrm{Ext}^1 = H^1(C^\bullet) = \frac{\mathbb{Z}/6\mathbb{Z}}{\mathrm{Im}(2)} \cong \frac{\mathbb{Z}/6\mathbb{Z}}{2\mathbb{Z}/6\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$$

9. Let R be a ring, and let X and Y be complexes of left R-modules.

(a) Working from the definitions, show that if a chain map $f\colon X \to Y$ is nullhomotopic ($f \simeq 0$), then the induced map on homology

$$\mathrm{H}_n(f)\colon\ \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

vanishes for each n.

(b) Let $f\colon X \to Y$ be a chain map that is nullhomotopic. Suppose each homology module $\mathrm{H}_n(X)$ and $\mathrm{H}_n(Y)$ has finite length.

Find the lengths of the homology modules $\mathrm{H}_n(C(f))$ for the mapping cone $C(f)$ of the map f.

a) We'll use cochains, but proof is identical. Suppose $f\colon X' \longrightarrow Y'$ is null homotopic. So $\exists\ s\colon Y' \longrightarrow X'$ as shown

$$\begin{tikzcd}
X'\colon \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "f"] & X^{n} \arrow[r] \arrow[d, "f"] \arrow[dl, "s"'] & X^{n+1} \arrow[r, "d_X"] \arrow[d, "f"] \arrow[dl, "s"'] & X^{n+2} \arrow[r] \arrow[d, "f"] \arrow[dl, "s"'] & \cdots \\
& Y^{n-1} \arrow[r] & Y^{n} \arrow[r] & Y^{n+1} \arrow[r, "d_Y"'] & Y^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

(1) $$f^n = f^n - 0 = s\, d_X^n - d_Y^{n-1} s$$

(2)
$$\begin{tikzcd}
\ker(d_X^n) \arrow[r, "j_X^n"] \arrow[d, dashed, "\exists\, \bar{f}^n"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] \arrow[dr, phantom, "\circlearrowleft"] & X^{n+1} \arrow[d, "f^{n+1}"] \\
\ker(d_Y^n) \arrow[r, "j_Y^n"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1}
\end{tikzcd}$$
From U.P Kernel

and then $H^n(f)$ is given as in

(3)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(d_X^{n-1}) \arrow[r] \arrow[d] & \ker(d_X^n) \arrow[r, "\pi_X^n"] \arrow[d, "\bar{f}^n"] & H^n(X) \arrow[r] \arrow[d, dashed, "H^n(f)"] & 0 \\
0 \arrow[r] & \mathrm{Im}(d_Y^{n-1}) \arrow[r] & \ker(d_Y^n) \arrow[r, "\pi_Y^n"'] & H^n(Y) \arrow[r] & 0
\end{tikzcd}$$
U.P Cok.

To show $H^n(f) = 0$, it suffices to show $\pi_Y^n \circ \bar{f}^n = 0$ i.e that $\mathrm{Im}(\bar{f}) \subseteq \mathrm{Im}(d_Y^{n-1})$. But from (1)

$$j_Y^n \bar{f} = f^n j_X^n = (s\, d_X^n - d_Y^{n-1} s) \circ j_X^n = d_Y^{n-1} \circ s \circ j_X^n \quad (\text{since } d_X^n \circ j_X^n = 0)$$

$$\Rightarrow \mathrm{Im}(\bar{f}) \subseteq \mathrm{Im}(d_Y^{n-1}) \Rightarrow H^n(f) = 0.$$

b) Recall we have a SES

$$0 \longrightarrow Y^{\cdot} \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} C(f) \xrightarrow{[1\ 0]} X[1] \longrightarrow 0$$

with long exact seq in homology given by

$$H^n(X) \xrightarrow{\delta = H^n(f)} H^n(Y) \longrightarrow H^n(C(f)) \longrightarrow H^{n+1}(X) \rightarrow H^{n+1}(Y) \rightarrow \cdots$$

where $\delta = H^n(f)$ is the connecting homomorphism.

If f is null-homotopic, $H^n(f) = 0$ by part a). So we have an exact sequence

$$\rightarrow \cdots H^n(X) \xrightarrow{0} H^n(Y) \longrightarrow H^n(C(f)) \longrightarrow H^{n+1}(X) \xrightarrow{0} H^{n+1}(Y) \rightarrow \cdots$$

which gives short exact sequences

$$0 \rightarrow H^n(Y) \rightarrow H^n(C(f)) \rightarrow H^{n+1}(X) \rightarrow 0$$

So $\ell(H^n(C(f))) = \ell(H^{n+1}(X)) + \ell(H^n(Y))$ for all n. $\square$