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Syracuse University

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Contents

1. Funding

This conference is generously supported by the Mathematics Graduate Organization, Syracuse University Mathematics Department, American Mathematical Society, and the National Science Foundation.

2. Keynote Speakers

Ramon van Handel, Princeton University

Ramon received his PhD from Caltech in 2007, advised by Hideo Mabuchi. He is currently an associate professor at Princeton University. His research interests are in probability theory, analysis, geometry, and their intersections.

Title: Optimal Spectral Gaps

Abstract: The spectral gap of a geometric structure — think of a graph, a manifold, a group — measures how quickly a random walker explores the entire structure. It is a problem of great interest, both in pure and in applied mathematics, to construct geometric structures with the largest possible spectral gap. In many cases, it is a conjectured that optimal spectral gaps are ubiquitous, but this turns out to be remarkably difficult to prove. I will aim to introduce this general theme, explain some recent progress in this area, and highlight a few vexing open problems.

Álvaro Lozano-Robledo, University of Connecticut

Álvaro received his PhD from Boston University in 2004, advised by David Rohrlich. He subsequently held a position at Cornell University, and visiting professorships at BU and Universidad Autonoma de Madrid. He is currently a professor at University of Connecticut. His research interests are in Number Theory and Arithmetic Geometry.

Title: Mathematical Dragons

Abstract: In this talk, we explore “Mathematical Dragons”: objects with rare or pathological properties that defy intuition or narrowly evade the usual hypotheses of familiar identities or theorems. These dragons may be frustrating or fascinating, but constructing compelling counterexamples is an art that has often led to entirely new branches of mathematics. As a case study, we introduce a new family of functions — the *spiral trigonometric functions* — which include the exponential, sine, and cosine functions as special cases. We use these functions to produce a collection of surprisingly intricate (irritating?) integrals, illustrating how new dragons can arise from seemingly familiar ground.

3. **Talk Abstracts**

3.1. **9:45-10:15 Parallel Session 1.**

Carnegie 100: Sabrina Traver

Title: Global invertibility of Sobolev mappings with given homeomorphic boundary values

Abstract: The classical inverse function theorem guarantees local invertibility for continuously differentiable maps with a positive Jacobian determinant. In Nonlinear Elasticity, variational problems often lead to the question of whether a Sobolev mapping is invertible in an appropriate sense. This talk addresses this question and presents an analogue of the inverse function theorem for Sobolev mappings with an almost everywhere positive Jacobian that coincide with a homeomorphism (in the Sobolev sense) on the boundary. A key part of the result is monotonicity, a purely topological concept that has interesting implications for the analysis of these mappings.

Carnegie 114: Arya Dutta

Title: Investigating new resonance transitions in Oterma's orbit to explain the latest observations.

Abstract: The comet 39P/Oterma has been observed to make resonance transitions from a heliocentric orbit inside the Jupiter's orbit (3:2 resonance) to a heliocentric orbit outside that of Jupiter (2:3 resonance) and vice versa. However, in the last few decades, it was observed that Oterma no longer follows this pattern. Previous works on this, from the dynamical system's perspective, have explored the 3:2 to 2:3 resonance transition route by establishing the heteroclinic connection between two collinear Lagrange points L1 and L2, in the context of Planar Circular Restricted Three-Body Problem (PCRTBP). In this work, we investigate resonant orbits considering the Planar Elliptic Restricted Three-Body Problem (PERTBP) for the Sun-Jupiter-Oterma system. The PERTBP can be viewed as a perturbation of the PCRTBP, with the perturbation parameter being the eccentricity of Jupiter's orbit. We compute invariant manifolds associated to L1 and L2 in the PCRTBP and determine the effect of the perturbation on these manifolds and on resonance transitions. This work can explain the changes observed recently in the orbit of Oterma and help us find new resonance transitions.

Carnegie 115: Tony Wehbe

Title: Composition of Coupled Dynamical Systems

Abstract: The attractor lattice of a dynamical system encodes its global dynamics in an algebraic framework. For product systems, a natural question is how this lattice relates to those of the component systems. In this talk, I will present an algebraic characterization of the attractor lattice of the product system in terms of the component systems in the decoupled case. This motivates a sheaf-theoretic framework for studying attractor lattices in the coupled setting.

Carnegie 119: No Talk

Carnegie 120: Dion Mann

Title: Holonomic Gauge Theory and the Reconstruction Theorem

Abstract: We present a theorem originally due to S. Kobayashi in the 50s which is now called J. Barrett's reconstruction theorem. Roughly, it says that homomorphisms H from a loop-space of some manifold M into any Lie group G "reconstructs" a principal G-bundle over M equipped with a connection whose holonomy is H. We discuss a heuristic argument by I. Singer for the theorem. This presentation is designed for a general, mathematics graduate student audience and will avoid technicalities—all are welcome!

Carnegie 200: Mengchun Cai

Title: Mesoscopic Rates of Convergence for Classical Ensembles in Random Matrix Theory

Abstract: In this talk, we provide mesoscopic rates of convergence (ROC) with respect to the L^1-Wasserstein distance for the eigenvalue determinantal point processes (DPPs) from the classical ensembles, the Gaussian Unitary Ensemble (GUE), the Laguerre Unitary Ensemble (LUE), the Jacobi Unitary Ensemble (JUE) and the Circular Unitary Ensemble (CUE) to their limiting point processes. We prove ROCs for the bulk of the GUE and CUE spectrum, the left edge of the LUE spectrum under different assumptions and the right edge spectrum of the GUE, LUE and JUE. These results are called mesoscopic because we are able to directly compare the point counts between the converging and limit DPPs in a range of scales with a spatial parameter s. We are able to achieve these results by controlling the trace class norm of the integral operators determined by the DPP kernels.

Carnegie 219: Arturo Ortiz San Miguel

Title: Maximizing Subgraphs in Regular Graphs

Abstract: Given a graph H, we investigate the d-regular graphs G with the highest H-density. We reframe the problem as a continuous optimization problem on the eigenvalues of G by relating injective homomorphism numbers from H and homomorphism numbers from quotient graphs of H. For almost all H, this relation has non-spectral terms, which require bounding by spectral terms in a way that is sharp at the optimal graph.For bipartite H and d large enough, we show G consists of disjoint copies of $K_{d,d}$. For non-bipartite H and d sufficiently large, G is a collection of disjoint K_{d+1} graphs. For $H = C_5$ and $d = 3$, disjoint Petersen graphs emerge.

3.2. **10:20-10:50 Parallel Session 2.**

Carnegie 100: Austin Konkel

Title: When does a circle immersed generically into the plane bound an immersed disk?

Abstract: In his 1968 thesis, Sandy Blank formulated and proved a combinatorial if and only if condition which answers when a generically immersed circle in the plane bounds and immersed disk; Blank's solution also tells one how many suitably inequivalent such bounding disks there are. The goal of this expository talk will be to discuss Blank's result, and with time permitting, discuss the state of the art of generically immersed n-spheres in $(n+1)$-space alongside how such immersions relate to the smooth Poincaré conjecture in dimension n."

Carnegie 114: Ridvan Ozdemir

Title: A Rigorous Mathematical Theory for Topological Phases and Edge Modes in Spring-mass Mechanical Systems

Abstract: "In this work, we examine the topological phases of the spring-mass lattices when the spatial inversion symmetry of the system is broken and prove the existence of edge modes when two lattices with different topological phases are glued together. In particular, for the one-dimensional lattice consisting of an infinite array of masses connected by springs, we show that the Zak phase of the lattice is quantized, only taking the value 0 or π. We also prove the existence of an edge mode when two semi-infinite lattices with distinct Zak phases are connected. For the two-dimensional honeycomb lattice, we characterize the valley Chern numbers of the lattice when the masses on the lattice vertices are uneven. The existence of edge modes is proved for a joint honeycomb lattice formed by gluing two semi-infinite lattices with opposite valley Chern numbers together."

Carnegie 115: Jacob Brown

Title: A B-spline based finite element approach for estimating the Hausdorff dimensions of continued fraction iterated function systems

Abstract: In the fields of fractal geometry and dynamical systems, continued fraction iterated function systems produce fractals whose Hausdorff dimensions generally cannot be found exactly. In this talk, we discuss a B-spline based finite element approach for the estimation of these Hausdorff dimensions. The approach is based on theoretical results of a Perron-Frobenius operator and its B-spline approximation, which we are able to compute numerically. We discuss the advantages of using B-splines as basis functions, including higher-order convergence and extensions to higher dimensions. Finally, we share numerical results and discuss future work.

Carnegie 119: Arshiya Farhath Gulam Dasthagir

Title: Quasiconformal mappings and holomorphic motions

Abstract: We will discuss some basic properties of quasiconformal mappings and holomorphic motions. In particular, we will talk about the λ-lemma and Slodkowski's extension theorem. We will also discuss some explicit examples.

Carnegie 120: Joseph Beckmann

Title: The Excess Intersection Problem

Abstract: If you have algebraic curves C_1 and C_2 in $\mathbb{P}^2$ whose respective degrees are d_1 and d_2, then $C_1 \cap C_2$ contains exactly $d_1 d_2$ points, if you count multiplicities. This classic result is one of many iterations of Bézout's theorem, which can be extended to discuss any varieties X and Y in $\mathbb{P}^n$, as long as $\dim(X)+\dim(Y) \geq n$. But what if $\dim(X)+\dim(Y) < n$? For example, how many times can two curves in $\mathbb{P}^3$ intersect? What about a curve and a surface in $\mathbb{P}^9$? Two surfaces in $\mathbb{P}^{37}$? In this talk, I will discuss what was previously known about these excess intersection situations, present some original results using (Fulton-MacPherson) intersection theory, and propose a "correct way" to count multiplicities in this situation using Stückrad-Vogel intersection theory.

Carnegie 200: Samson Ajibade

Title: AN EFFICIENT INTEGRATOR FOR GENERALIZED MODELS USING RADIAL BASIS FUNCTIONS AND RATIONAL APPROXIMATION OF MITTAG-LEFFLER FUNCTIONS.

Abstract: Fractional reaction diffusion models play a crucial role in describing anomalous transport and memory effects arising in complex physical, biological and engineering systems. However, the nonlocal nature of fractional derivatives combined with nonlinear reaction terms makes the models computationally challenging and analytically unsolvable. In this work, we propose a novel numerical algorithm based on Exponential Time Differencing(ETD) coupled with Radial Basis Function (RBF) spatial approximation for the efficient solution of time fractional reaction diffusion equations. The proposed algorithm employs a rational approximation of the generalized Mittag-Leffler function (MLF) to accurately treat the fractional time derivative, while RBF interpolation is used to discretize the spatial operator without requiring structured meshes. The proposed algorithm was used to solve some examples and compared with existing methods.Keywords: Exponential Time Differencing, Fractional derivatives, Generalized Mittag-Leffler function, Radial Basis Function, Reaction Diffusion Model

Carnegie 219: Dalena Vien

Title: Algebraic Invariants of Edges Ideals Under Coning Constructions

Abstract: In this talk, we explore how algebraic invariants of edge ideals respond to coning constructions of the underlying graph. We begin by recalling the classical cone over a graph, an operation known to preserve the regularity of the edge ideal while increasing its projective dimension by one. Motivated by this phenomenon, we concentrate on coning over minimal vertex covers and maximal independent sets. For cones over minimal vertex covers of an arbitrary graph, we show that the familiar behavior of regularity and projective dimension persists. In contrast, this behavior does not hold in general for cones over maximal independent sets. To better understand this discrepancy, we restrict attention to paths and cycles, where we show that, in most cases, these invariants behave as they do under the classical cone construction. We conclude by discussing how the -invariant is affected by these coning operations. This is joint work with Selvi Kara.

3.3. **11:05-11:35 Parallel Session 3.**

Carnegie 100: Zhihe Li

Title: The Fourier Ratio: Uncertainty, Restriction, and Approximation for Compactly Supported Measures

Abstract: We introduce and study a continuous analogue of the Fourier ratio for compactly supported Borel measures. Given function f in L^2, Fourier ratio is defined to be the ratio of L^1 and L^2 norm of a regularized Fourier transform at scale R. This quantity, which interpolates between L^1 and L^2 Fourier information, serves as a fundamental parameter connecting uncertainty principles, Fourier restriction theory, and approximation by trigonometric polynomials.Using this framework, we establish sharp bounds for the Fourier ratio in terms of geometric features of the spatial and frequency supports. These bounds lead to a fractal uncertainty principle that limits how strongly a function can be localized in both space and frequency. This talk is based on joint work with Alex Iosevich, Eyvindur Ari Pálsson, and Alexia

Carnegie 114: Abdullah Al Rafi Mahmud

Title: Cavity Mediated Interactions in Spin Systems

Abstract: This research investigates the fundamental nature of cavity-mediated interactions in many-particle spin systems, utilizing the Tavis-Cummings model. Despite their significance in quantum electrodynamics, the exact characteristics and emergent collective behaviors of these interactions remain largely unresolved. A dual approach is employed: analytically deriving an effective Hamiltonian via the Schrieffer-Wolff transformation and validating these results through exact numerical quantum simulations on high-performance computing clusters. By benchmarking effective theories against full system dynamics, this study aims to establish computationally efficient models that capture essential physics. The findings will bridge fundamental cavity QED with practical applications in quantum information processing and polaritonic chemistry.

Carnegie 115: Hunter Young

Title: Fast and Accurate Surrogate Models: Comparing DeepONet, FNO, and PIKNO for Rapid Prediction of Laser-Tissue Interaction

Abstract: Our study explores the application of neural operators, particularly deep operator networks (DeepONets), Fourier neural operators (FNOs), and physics-informed kernel neural operators (PIKNOs) to solving the heat equation under a variety of conditions relevant to modeling laser heating of biological tissue. We constructed several test cases and applied the models to multi-dimensional time-dependent scenarios. The results demonstrated scalability and accuracy in FNOs over complex domains. We also compared FNO performance against corresponding PIKNOs to assess the benefits of incorporating physics directly into the models' architecture. We concluded that FNOs outperform DeepONet significantly in both training efficiency and predictive accuracy. The study provided valuable insights into the strengths and limitations of physics-informed and non physics-informed operator learning frameworks for modeling biologically relevant thermodynamics.

Carnegie 119: Jasmine Burns

Title: Oscillatory Prime Distributions and Thermodynamic Instability in Analytic Number Theory

Abstract: We develop a deterministic model for prime-gap oscillations based on a nonlinear analytic law governing residue densities. When expressed through an ergodic-theoretic and thermodynamic lens, the model predicts an error term that grows faster than classical bounds, revealing a structural instability in the heat-flow formulation of the zeta function. Numerical and analytic evidence show that the associated parameter controlling zero alignment must remain strictly positive. This framework unifies analytic number theory, dynamical systems, and statistical mechanics, offering a new perspective on the prime distribution's departure from equilibrium.

Carnegie 120: Ryan Gelnett

Title: On spaces of embeddings of circles in surfaces

Abstract: We consider the space of embeddings of finitely many circles that bound disks in non-positively curved surfaces. We index the connected components of this space with finite rooted trees and show that the connected components are classifying spaces of the "braided" automorphism groups of the associated trees. An intermediate step to proving these results is to construct a strong deformation retract onto the subspace of geometric circles; moreover, this strong deformation retraction is equivariant with respect to transformations of the surface.

Carnegie 200: Asir Intesar Tushar

Title: Markov Chain Monte Carlo Methods for Curve Reconstruction and Point Cloud Data Analysis

Abstract: Point cloud data have become increasingly vital due to their ability to capture detailed, multi-dimensional representations of physical objects and their surrounding environments. Point clouds are nowadays central to applications across industry and academia that range from robotics, navigation systems, and 3D printing to architecture, manufacturing, and agriculture. Despite its importance, the analysis of point cloud data faces several limitations, including high computational cost due to large data volume, contamination with localization noise and inaccuracies caused by sensor limitations, and missing points due to environmental factors or sensor positioning. Moreover, unavailable uncertainty quantification in reconstructed structures remains a critical gap in applications requiring reliable estimation. We present a fully Bayesian framework for point cloud data analysis and curve reconstruction. Our framework models a cloud's points as noisy perturbations of latent positions constrained to lie on closed polylines, jointly inferring polyline vertices and connectivity, latent coordinates, and noise characteristics. Posterior inference is performed via a specialized Markov chain Monte Carlo sampler tailored to point cloud data processing. Experiments on synthetic data demonstrate accurate curve reconstruction while providing uncertainty quantification.

Carnegie 219: Luna Perry

Title: A Diagramatic Approach to Coalgebraic Structures

Abstract: Classically in Algebra, one studies algebraic objects with property preserving morphisms between them: Modules, Algebras, Graded Algebras, DG Algebras... the list goes on. However, this is only half of the story: by Dualizing these structures (in the categorical sense) one gets coalgebraic objects: CoModules, CoAlgebras, etc., whose study and construction naturally complements their more familiar duals. In this talk, we will give the categorical definitions of (co)algebraic objects, motivating the constructions of some fundamental structures over a vector space such as the polynomial algebra and divided powers multiplication. Basic familiarity with Category Theory is encouraged, but not required.

3.4. **11:40-12:10 Parallel Session 4.**

Carnegie 100: Ke Yu

Title: The uncertainty principle and Orlicz spaces

Abstract: The uncertainty principle asserts that a function and its Fourier transform cannot both be sharply localized: the product of their support sizes must be large. Recently, Iosevich and Mayeli refined this principle for functions whose Fourier transform is supported on specific subsets of Z_N^d , using restriction theory in L_p spaces. In this talk, we will extend their ideas to Orlicz spaces, which generalize L_p spaces and allow finer control over growth conditions of functions. We will introduce (ϕ, ψ) restriction estimates and use them to establish sharper uncertainty principles and new results related to Bourgain's Λp sets. Finally, we will discuss an application of these results to the exact recovery problem, showing how the Orlicz space framework recovers and extends the classical results of Iosevich and Mayeli in the Lebesgue setting. This presentation is based on joint work with Alex Iosevich, Isaac Li, and Zhihe Li."

Carnegie 114: Ethan Shade

Title: Slip 'N Solve: Deriving the Sliding Catenary Curve Equation

Abstract: This talk presents the derivation of the ""Sliding Catenary"" curve-a hanging chain of fixed length whose endpoints are not fixed points, but are instead constrained to slide along frictionless boundary curves (specifically, a linear slant $y = mx$ and a vertical line $x = b$). We formulate the problem as minimizing the gravitational potential energy functional, $L[y] = \int y\sqrt{1 + (y')^2}dx$, subject to an isoperimetric constraint on the chain's total length. We utilize the method of Lagrange Multipliers combined with the Beltrami identity (a corollary of the Euler-Lagrange equation) to reduce the differential equation to a solvable first-order form. The core of the discussion focuses on the application of Transversality Conditions to determine the necessary boundary values for the free-moving endpoints. We further verify the sufficiency of the solution as a local minimum using the Legendre-Clebsch condition. Finally, the analytic solution is validated against a physical construction, demonstrating a near-perfect overlay between the derived curve and a real-world chain.

Carnegie 115: Steve Wainaina

Title: A Perturbed Physics-Informed Neural Network Surrogate Model for the One-Dimensional Heat Equation

Abstract: In the study of parameterized Partial Differential Equations (PDEs), we are often interested in obtaining the solution to a governing PDE under a variety of parameters representing specific physical properties of a system. In our work, we investigate perturbation and machine learning methods to surrogate the solution to the one-dimensional diffusion (heat) equations in terms of perturbations of the material's diffusivity coefficient. The perturbation approach leverages knowledge of one material's diffusivity to gain knowledge of another, using it to approximate a surrogate for the new parameter. This is a general perturbation theory approach to the one-dimensional diffusion PDE, but we investigate the particular advantages and disadvantages of using machine learning surrogates such as physics-informed neural networks (PINNs) and operator networks to represent these perturbation PDEs. Here, we mathematically derive the PINN and operator network representation of the perturbed PDE problem.

Carnegie 119: Xiaorun Wu

Title: Kolyvagin derivatives for higher rank Kolyvagin system construction

Abstract: This work generalizes the theory of Kolyvagin systems to the higher-rank setting, extending the foundational rank-one work of Mazur and Rubin. In this paper, we construct, under certain set of assumptions with the input Euler system following Burns and Sato, a system of Kolyvagin derivatives valued in the exterior powers of a-adic Galois representation, specifically addressing scenarios where the Selmer group has rank. By utilizing a modified transverse local condition at Kolyvagin primes, we establish the existence of derived cohomology classes that satisfy the rigorous local compatibility relations of an Euler system. We also verify the rigidity of this Kolyvagin systme constructed by seeing how these higher-rank classes could bound the dual Selmer group and determine the structure of the underlying Iwasawa module. Furthermore, we link these derived classes to the leading term of the associated-function, offering a strategy to potentially verify an analog of Iwasawa Main Conjectures in the Higher-rank contexts.

Carnegie 120: Courtney Hauf

Title: Equivariant parameterized cohomology

Abstract: In the 1950's Borel introduced equivariant cohomology, a theory graded on $\mathbb{Z}$ with group coefficients. In the 1960's Bredon introduced another flavor of equivariant cohomology, a theory that was at first still graded on the integers with local coefficient systems. With the introduction of Mackey Functors in 1971 and the desire for a finer cohomological view, in 1974 Bredon's cohomology was upgraded to a theory graded on the representation ring of G with Mackey Functor coefficients. Refining the view once more, in 2016 Constanoble and Waner introduced what they coined to be $\mathrm{RO}(\Pi_G B)$-graded cohomology, a theory graded on representations of the equivariant fundamental groupoid. This theory gives the equivariant cohomology of a G-space over some base space B, and hence is referred to in the title as "Equivariant Parameterized Cohomology".

Carnegie 200: Heena Shaikh

Title: Stability of Recurrent Neural Networks

Abstract: Stability is essential for the reliable use of recurrent neural networks (RNNs). Unlike feedforward networks, RNNs exhibit nonlinear behavior that limits the applicability of classical matrix-based stability methods. In this work, we study RNN stability using the method of Reduction of the Dissipativity Domain, which is computationally efficient and well suited for nonlinear neural systems. This approach reduces stability verification to the problem of minimization of nonlinear functions over polytopes. We show that this problem can be further reduced to finding local extrema of linear combinations of neuron transfer functions on affine subspaces. While uniqueness of minima is known for certain functions on hyperplanes, their behavior on higher-dimensional subspaces remains largely unexplored. For one-loop perceptrons, we demonstrate that along one-dimensional subspaces the number of local extrema can reach $\lfloor n/2 \rfloor$. We also present practical criteria for detecting and classifying these extrema and discuss ongoing work and open questions regarding the structure of maxima on two-dimensional hyperplanes.

Carnegie 219: Johnny Rivera

Title: Combinatorial proof of a permuted basement Macdonald polynomial identity

Abstract: A well-known and fundamental property of the Macdonald polynomials is their invariance under the transformation sending (q, t) to (q^{-1}, t^{-1}). Recently, Concha and Lapointe showed that this property extends in an interesting, nontrivial way to an identity for partially symmetric Macdonald polynomials. In this talk, we refine the Concha-Lapointe identity to a sub-family of Alexandersson's permuted basement Macdonald polynomials and give a combinatorial proof of the refined identity.

3.5. **2:50-3:20 Parallel Session 5.**

Carnegie 100: Obed Domson

Title: Sampling from Non-Smooth Composite Distributions

Abstract: Generating random samples from probability distributions is a fundamental problem in statistics and plays an essential role in Bayesian analysis. Classical methods, however, often become inefficient and unreliable in high dimensional settings. In this talk we explore sampling from non-smooth composite distributions, a setting where traditional methods face notable difficulties. I will also present a new method developed for this setting and illustrate its performance with numerical results.

Carnegie 114: Daniel Hobbs

Title: A Dominant Balance Series Solution to the Rayleigh Collapse

Abstract: For the classic Rayleigh Collapse problem for a single spherical vacuum in an infinite fluid, the exact analytic solution to the nonlinear ordinary differential equation that governs the bubble radius during the collapse is first obtained via a very slowly converging direct power series, and this includes a derivation of the formula of the series coefficients in general. 2 different asymptotically motivated factorizations are both considered including one previously discussed in the literature, and in both cases, this leads to an acceleration of the convergence of the series. To achieve machine precision, it is shown that the prior methods require an impractical number of terms varying from just over 80.9 million to just over $1.09 * 10^{37}$ series terms. Next, a dominant balance series solution, which is an exact analytic series solution as a function of a gauge variable that removes the singular behavior, was found as well, and a comparison of the estimated numerical errors of the different solution methods was considered. The error analysis shows that the dominant balance series achieves optimum computational efficiency by reaching machine precision with only 35 series terms.

Carnegie 115: Chidera Ugbonta

Title: Mathematical Modeling of Generalized Damped Nerve Equation and Its Analytical Solution

Abstract: Understanding how nerve signals propagate is vital for elucidating how information is conveyed within biological systems, a cornerstone of fields such as neuroscience, medicine, and bioengineering. Interruptions in this process are associated with neurological disorders, such as Parkinson's disease, multiple sclerosis, epilepsy, and neuropathic pain. Consequently, precise modeling of signal propagation is crucial for enhancing diagnostic methods, therapeutic approaches, and regenerative treatments, including spinal cord repair. Our study concentrates on the accurate modeling of nerve signal propagation, incorporating memory and damping effects, which are characteristic of real-world conditions. This is achieved through the application of fractional calculus. Moreover, a robust and precise analytical technique is utilized to derive solution profiles that accurately depict anomalies in signal propagation.

Carnegie 119: Andrew Pendleton

Title: Can you hear the aspect ratio of a drum?

Abstract: Eigenvalues of the Dirichlet Laplacian on rectangular domains can be interpreted physically as the overtones produced by an idealized rectangular drumhead with the same aspect ratio. The multiplicities of those eigenvalues therefore inform the timbre of their corresponding overtones. A recent conjecture proposed a strikingly simple criterion for when rectangular domains admit eigenvalues with an arbitrary multiplicity. We exploit some surprising connections to algebraic number theory and the study of quadratic forms to confirm this classification; thus showing that in some ways, you can really hear the aspect ratio of a drum. Finally, we explore a natural generalization to two-dimensional toroidal domains. This is a joint work with Siqi Fu of Rutgers University.

Carnegie 120: Juan Ricardo Rosas Mendoza

Title: Reconstruction Theorems and Non-commutative Algebraic Geometry

Abstract: If R is a commutative ring, then R can be recovered from the affine scheme $\mathrm{Spec}(R)$; in fact, R can also be recovered from its category of R-modules. Furthermore, if R is a Noetherian ring, Matlis' Theorem states that it is possible to recover $\mathrm{Spec}(R)$ from the set of isomorphism classes of indecomposable injective modules in R-mod. Similarly, R-mod can be recovered from the category of quasi-coherent modules over $\mathrm{Spec}(R)$. In this talk, we will discuss these results and a generalization for (quasi-separated) schemes given by Gabriel-Rosenberg.

Carnegie 200: Kody Angell

Title: Oh NAR! The Importance of Exogenous Inputs in Autoregressive Neural Networks

Abstract: Laser damage induced at blue wavelengths in pigmented tissues results in interactions between photothermal and photochemical mechanisms. Traditional modeling relies on the Arrhenius equation or zero-order kinetic models which fail to capture experimental data due to their inability to represent the nature of biological damage processes. These models can't account for complex interactions between multiple damage mechanisms occurring simultaneously. Here we apply a nonlinear autoregressive model with exogenous inputs to characterize photochemical & photothermal damage in tissues under light exposure and varying wavelengths. Different kernel types were explored and optimized. The NARX framework incorporates memory effects, nonlinear dynamics, and external input variables, enabling representation of complex biological responses to laser irradiation and captures immediate & delayed damage responses. This work enables spectral analysis of cellular damage mechanisms. This approach is adapted to complex differences between damage induced by different wavelengths, which offers a tool for evaluating phototherapy efficacy and implications in understanding interplay between damage types.

Carnegie 219: Kory Pollicove

Title: Hochschild Cohomology of Quasi-Complete Intersections in Positive Characteristic

Abstract: Quasi-complete intersections are ring maps that were initially studied for their connection to a conjecture of Quillen on the vanishing of Andre-Quillen cohomology. This class of morphisms strictly contains the class of complete intersections, while still retaining some nice smoothness properties. Hochschild Cohomology is another derived functor that relates to smoothness, and has been used to great success in studying complete intersections. In this talk, we will discuss recent work in computing Hochschild Cohomology for local quasi-complete intersections, with a focus on the case of positive characteristic residue field.

3.6. **3:25-3:55 Parallel Session 6.**

Carnegie 100: Justus Ademuyiwa

Title: Properties and Applications of the Cubic Transmuted Two-Parameter Lindley Distribution

Abstract: In probability theory, transmutation maps are used to extend classical distributions to obtain more flexible distributions. Several transmutation maps have been proposed and utilized to extend many classical continuous probability distributions. This study extends the two-parameter Lindley distribution using a cubic transmutation map to obtain a more flexible distribution. Different moment-based mathematical properties of the cubic transmuted distribution are obtained, and some descriptive statistics are simulated. The performance of the new extended distribution and other distributions related to the Lindley distribution is assessed on some lifetime datasets that are positively skewed and dispersed. Findings reveal that the transmuted distribution outperforms the baseline distributions.

Carnegie 114: Thomas Aldredge

Title: The Effect of Scaled Dispersion on Bourgain-type Bilinear Estimates for the Korteweg-de Vries Equation

Abstract: This work considers a famous problem first addressed by P. Lax and C. Levermore, whichlooks at the convergence of solutions to an ϵ-scaled Korteweg-de Vries (KdV) equation as ϵ tends to 0. Lax-Levermore proved that the corresponding solution converges weakly as the dispersion term vanishes. We reexamine this problem in the context of Bourgain spaces and discover an interesting phenomenon regarding the Bilinear estimate which obstructs the convergence in a stronger sense.

Carnegie 115: Keara Walsh

Title: Modeling the Threshold: A Mathematical Approach to the Thermal-Photochemical Transition in Retinal Damage

Abstract: Photochemical damage is a significant concern for retinal pigmented epithelial (RPE) cells, as it can lead to long-term or permanent cellular effects even at low temperatures and energy levels. We investigated the modeling of the effect of long-pulse, short-wavelength laser exposure to retinal cells, particularly to determine the time duration at which the main damage mechanism transitioned from photothermal to photochemical. Using the rate process model proposed by Clark et al. in 2011, we developed a MATLAB simulation to calculate photothermal and photochemical damage thresholds for a 413nm wavelength laser. By implementation of a numerical search algorithm and a binary search, the MATLAB simulation only required input of a time duration vector to compute both the photothermal and photochemical damage thresholds for each time. In this study, we also gained insight into the sensitivity of our parameters, the dependence on wavelength, and the limitations of the model.

Carnegie 119: Piyali Chakraborty

Title: Self adjoint extension of Moment operator, Unitary group and Spectral pair

Abstract: Self-adjoint operators and their extensions form a cornerstone of modern analysis, connecting operator theory, harmonic analysis, and spectral theory. A central theme is how symmetric operators can be extended to self-adjoint ones, and how these extensions determine both the spectral set and the associated one-parameter unitary group via Stone's theorem. Boundary conditions and moment operators play a decisive role in this process, shaping the spectrum and the integrability properties of the system. In this talk, I will explore the interplay between self-adjoint extensions of Moment operator, spectral sets, and unitary dynamics, highlighting the conditions under which these notions are equivalent. These ideas naturally link to recent developments involving spectral sets, Fourier bases, and moment problems.

Carnegie 120: Mithun Padinhare Veettil

Title: Locally Integer Polynomial Functions

Abstract: Locally Integer Polynomial (LIP) functions are $\mathbb{Z}$-valued functions on an infinite subset X of $\mathbb{Z}$ that are given by polynomials with integer coefficients in every finite subset of X. In this talk, we will explore the local properties of LIP functions on $\mathbb{N}$ with Kirch topology.

Carnegie 200: Abdulla Toleugazin

Title: Semi-supervised Classification in an Undirected Multilayer Networks with Groupwise Subspace Structure

Abstract: The paper studies a multilayer undirected network model where each layer network represents connections from the same set of nodes to possibly different sets of node. We assume that each of the layer network follow the Generalized Random Dot Product Graph model where layers can be partitioned into groups, with each group of layers sharing the same left latent subspace. Otherwise, all adjacency matrices can be different. Due to sparsity and the differences in layer-dependent network dimensions, those left latent subspaces may not be identifiable in some of the layer networks. The objective is to partition layers into groups with the same left latent subspace structure and recover those shared subspaces. The problem can also be seen as an extension of the classic setting in which one recovers a shared structure across an entire collection of matrices, extending it to the case where such structures are shared only within specific groups of matrices. Our paper introduces a novel two-stage algorithm.

Carnegie 219: Dorian Kalir

Title: Frobenius Endomorphisms for Koszul Complexes

Abstract: Recent work of Ballard, Iyengar, Lank, Mukhopadhyay, and Pollitz has shown that for an F-finite locally complete intersection ring, the first Frobenius pushforward of the ring is a strong generator for the bounded derived category. A key ingredient in their proof is that locally each residue field has finite level over the localized pushforward. In this talk, we will present an extension of these results to the dg setting that also refines the key ingredient even in the affine case.

3.7. **4:10-4:40 Parallel Session 7.**

Carnegie 100: Trevor Scheuing

Title: Measuring Brownian motion self-intersections

Abstract: Imagine a d-dimensional Brownian motion path B. How often will B intersect itself? Spoiler: Not at all if $d \geq 4$, and infinitely often if $d \leq 3$ - even just on a finite time interval. This suggests that, unlike the random walks that approximate it, the path of a Brownian motion is some kind of crazy fractal! The $d \leq 3$ case is still unsatisfactory. Can't we measure self-intersections in some way that isn't always infinite? Spoiler: Yes. To do so, we define a random measure on the time set $\{(s,t) : 0 \leq s < t\}$ that somehow characterizes how often $B(s) = B(t)$. Will this approach succeed, or will even this give an infinite answer? That I won't spoil. Tune in to find out!

Carnegie 114: No Talk

Carnegie 115: Obed Amo

Title: A Graph-Fused Lasso Regression Approach for Modeling Spatial Heterogeneity in Real Estate Markets

Abstract: Residential real estate prices are influenced not only by observable property characteristics but also by unobserved neighborhood heterogeneity. Traditional regression models, including penalized approaches such as Lasso, often fail to capture this type of spatial dependence. We introduce a graph-fused Lasso regression model that combines property-level covariates with the spatial structure represented through an adjacency graph. By penalizing the differences in estimated neighborhood effects, the method recovers piecewise-constant spatial patterns while preserving flexibility in feature contributions. Applied to housing transaction data from Hampton Roads, the model uncovers meaningful spatial clusters and delivers improved predictive accuracy. Beyond the real estate domain, this approach offers a general framework for modeling spatially structured heterogeneity, providing a useful tool for urban analysis, policy evaluation, and other domains where geographic context is of essence. (This is joint work with Michael Pokojovy, Simon Stevenson and Lei Zhang).

Carnegie 119: Gideon Mensah Banson

Title: Investigating Prospective Teachers' Reasoning About Multiplication of Rational Numbers: Insights from a Problem-Solving Context

Abstract: The purpose of the study was to examine prospective elementary teachers' (PTs') reasoning about the multiplication of a positive number and a rational number between 0 and 1. In this study we investigated 44 PTs' rational number reasoning in two sections of a mathematics course taught via problem solving. We adopted an interpretative paradigm and a qualitative technique where we collected data through a pre-unit assessment and a post-unit assessment on students' reasoning about the rational numbers. We found that students demonstrated conceptual understanding, procedural reliance, and some misconceptions about multiplying a positive number by a rational number between 0 and 1. This study extends existing research by showing how learning through problem-solving can reduce misconceptions while procedural dominance persists among PTs.

Carnegie 120: Gal Yehuda

Title: On the growth spectrum of hyperbolic groups

Abstract: The growth spectrum of a group G acting on a metric space X is the set of all exponential growth rates with respect to the metric of X achieved by the subgroups of G. We prove that there is no gap in the growth spectrum of free groups and hyperbolic groups acting geometrically on a metric space. We also prove that there is no gap in the lower half on the growth spectrum of any hyperbolic group. Based on a joint work with Mixalis Louvaris, Daniel Wise and Remi Coulon.

Carnegie 200: Hari Dahal

Title: Damped Proximal Augmented Lagrangian Method for Weakly Convex problems with Convex constraints

Abstract: In this talk, I will present a damped proximal augmented Lagrangian method (DPALM) for solving problems with a weakly-convex objective and convex linear/non-linear constraints. Instead of taking a full stepsize, DPALM adopts a (possibly) damped dual stepsize. It is known that DPALM can produce a (near) ϵ-KKT point within $O(\epsilon-2)$ outer iterations if each DPALM subproblem is solved to a proper accuracy. In addition, I will discuss overall oracle complexity (i.e., total number of evaluations of function values and (sub)gradients) of DPALM when the objective is either a regularized smooth function or in a regularized compositional form. For the former case, DPALM achieves a complexity of Oe ϵ -2.5 to produce an ϵ-KKT point by applying an accelerated proximal gradient (APG) method to each DPALM subproblem 1 . For the latter case, the complexity of DPALM

is Oe $\epsilon - 3$ to produce a near ϵ-KKT point by using an APG method to solve a Moreau-envelope smoothed version of each subproblem. The main result is that the outer iteration complexity and the overall complexity either generalize existing best ones from unconstrained or linear-constrained problems to convex-constrained ones, or improve over the best-known results on solving the same-structured problems. Furthermore, numerical experiments on linearly/quadratically constrained non-convex quadratic programs and linear constrained robust nonlinear least squares are conducted to demonstrate the numerical efficiency of DPALM.

Carnegie 219: Cory Eckert

Title: Formal Group Laws, the Lazard Ring and Complex Oriented Cohomology Theories

Abstract: Formal Group Laws (FGL's) historically come from an attempt to express group multiplication on Lie groups via power series expansions. For a commutative ring R, the set $\mathbf{FGL}(R)$ of all formal group laws over R defines a functor $\mathbf{CRing} \to \mathbf{Set}$. A theorem of Lazard shows that this functor is corepresentable and that there exists a universal ring L, generated by the coefficients of a universal formal group law modulo the ideal generated by a few constraints on these coefficients. In particular, L is isomorphic to a polynomial ring on countably many generators, each in even degree. We then turn to Quillen's work on Complex-Oriented Multiplicative Cohomology Theories, where a choice of orientation determines a FGL via the first Chern classes. Quillen's theorem identifies the Lazard ring with the coefficient ring of the cohomology theory MU of complex bordism. This reveals deep insight into the connective components of Homotopy Theory with that of Algebraic Geometry and even Number Theory.

3.8. **4:45-5:15 Parallel Session 8.**

Carnegie 100: Sucharitha Dodamgoda

Title: Maximum Likelihood Estimation for the Dirichlet Distribution

Abstract: The Dirichlet distribution is a multivariate generalization of the Beta distribution used to model compositional data in fields such as microbiome analysis, text classification, and market share modeling. Although the maximum likelihood estimator (MLE) is widely used for estimating Dirichlet parameters, its existence and uniqueness are often assumed without formal proof, limiting the reliability of bias and error assessments. We address this gap by providing an analytic proof of the existence and uniqueness of the MLE for the general Dirichlet distribution, using a specific representation of the digamma function. Our approach also resolves a conjecture left open by Ronning (1989) regarding MLE computation. We further consider the symmetric Dirichlet model, often arising in hypothesis testing where all components are equally represented on average, and we prove the existence and uniqueness of its single-parameter MLE, a result not previously established in the literature.

Carnegie 114: Amara Eze

Title: A SELF-ADAPTIVE INERTIAL MANN-TYPE ALGORITHM FOR COMPLEX OPTIMIZATION PROBLEMS WITH APPLICATIONS TO IMAGE RECONSTRUCTION

Abstract: Optimization problems arising in complex settings play a central role in modern mathematics and applied sciences, particularly in large-scale and ill-posed frameworks. Among these, variational inclusion problems (VIPs) form a broad class that models numerous phenomena in optimization theory and applied analysis. Developing algorithms that are both computationally efficient and adaptable to such problems remains an important and active area of research. In this research, we introduce a novel self-adaptive inertial iterative algorithm of Mann type for solving VIPs within complex optimization frameworks. The proposed method integrates inertial step technique with a self-adaptive step-size, leading to improved convergence behavior without imposing restrictive conditions on our control parameters. To demonstrate the effectiveness of the approach, we apply the algorithm to image reconstruction problems arising in areas such as medical imaging, astronomical observation, among others. Extensive numerical experiments indicate that the proposed scheme enhances reconstruction quality while reducing computational cost compared to several existing iterative methods.

Carnegie 115: Deepak Bastola

Title: Deep Learning Outperforms Traditional Machine Learning Methods in Predicting Childhood Malnutrition: Evidence from Survey Data

Abstract: Malnutrition among children is a critical global health challenge, particularly prevalent in developing and least developed countries such as Nepal. Early detection and prompt intervention are the secrets to reducing its negative impact, but traditional methods of identification of malnutrition are not always efficient or effective. To carry out this study, we applied a variety of machine learning and deep learning algorithms to predict the malnutrition status of children and identify the key drivers of malnutrition, based on data from the Nepal Multiple Indicator Cluster Survey 2019. We utilized typical metrics for both baseline and hyperparameter-tuned models for measuring model performance. Our results showed that support vector machines and artificial neural networks worked best from the baseline models, while better performance was achieved by TabNet after careful tuning of hyperparameters. Among the most important factors linked with malnutrition were child weight and age, household wealth index, mother's education, and location.

Carnegie 119: Anis Munfarikhatin

Title: Measurement Properties of a Likert-Scale Instrument Using the Rasch Partial Credit Model: A Pilot Study

Abstract: This study examines the measurement properties of a newly developed Likert-scale instrument using the Rasch Partial Credit Model (PCM). Although Likert responses are ordinal, the Rasch framework enables transformation into interval-level measures on a logit scale, while providing diagnostic tools to evaluate scale quality. Data from 42 respondents across 30 items were analyzed to assess model fit, reliability, separation, rating scale functioning, and targeting. Results indicate high person reliability and strong separation, suggesting effective differentiation along the latent trait continuum. Item reliability was moderate, and several items showed elevated fit statistics, indicating areas for refinement. Wright's map analysis revealed that item difficulty was generally lower than respondent ability. This pilot study demonstrates the application of the Rasch PCM in constructing interval measures and evaluating instrument performance in small samples.

Carnegie 120: Jon Kim

Title: Moduli of (b, c)-weighted stable marked cubic surfaces

Abstract: We will construct many KSBA compactifications of cubic surfaces with a marked line by computing the full wall-and-chamber decomposition for KSBA compactifications where we asymmetrically weight a marked line with weight b, and the other 26 lines uniformly with weight c. This generalizes the recent work done by Schock where the lines are weighted uniformly.

Carnegie 200: Shankhadeep Mondal

Title: Designing Optimal Duals for Reliable Reconstruction in Erasure-Prone Systems

Abstract: Frames provide redundancy essential for stable signal reconstruction under data loss. This talk presents recent advances on optimizing dual frames and dual frame pairs to minimize reconstruction error under probabilistic and structured erasures. We analyze optimality with respect to several operator-theoretic measures, including the operator norm, Frobenius norm, numerical radius, and spectral radius, and also consider ℓ^p-averaged error models. Conditions ensuring existence, optimality, and uniqueness of duals are established, with explicit results for uniform tight frames and K-frames. A graph-theoretic viewpoint is also explored, illustrating how frames generated by graphs behave under erasures. These results provide a unified framework for designing erasure-robust duals across different reconstruction settings.

Carnegie 219: Trang Quach

Title: Representations of braid groups

Abstract: We will be talking about the Burau representation and the Lawrence-Krammer representation of braid groups. Specifically, we'll be talking about the proof of Long-Paton and Moody on why the Burau representation is not faithful for $n > 5$. Then we'll talk about Krammer's algebraic proof for why the Lawrence-Krammer representation is faithful for all n.

3.9. **5:20-5:50 Parallel Session 9.**

Carnegie 100: Youssef Djellouli

Title: The Rigidity of Random Point Processes

Abstract: The rigidity of various random processes has been a very active area of research over the past 15 years. These processes can arise from a wide variety of areas such as physics, functional analysis, and random matrices. Point processes are called rigid if certain information/properties can be determined with certainty despite the random noise. In this talk, we give an introduction to some notions of rigidity, discuss some previous results and then go over some new findings that provide new tools for proving rigidity and improving some previous results. We will also discuss some potential applications of these new results to processes arising from physics and functional analysis.

Carnegie 114: Toluwani Okunola

Title: Efficient Edge Preservation in Large-Scale Linear and Nonlinear Inverse Problems

Abstract: Traditional image reconstruction algorithms assume a linear forward operator with known parameters. However, many practical applications involve uncertainty in acquisition parameters such as projection angles in computed tomography, or point spread function characteristics in image deblurring. These uncertainties introduce nonlinearity into the inverse problem, requiring simultaneous estimation of both the image and the uncertain model parameters. We develop a generalized Krylov subspace framework that extends the majorization-minimization generalized Krylov subspace method to handle nonlinear forward operators. Our approach incorporates recycling strategies to control memory usage and implements streaming variants for sequential data processing. We carry out rigorous numerical experiments in fan-beam computed tomography and photoacoustic tomography to demonstrate that our proposed methods achieve high-quality reconstructions with bounded memory requirements, making them suitable for large-scale dynamic imaging problems.

Carnegie 115: Randolph Russell

Title: Final hour price dynamics predict overnight return magnitude but not direction in U.S. equities

Abstract: We investigate the predictive power of the last-hour price dynamics of U.S. equities regarding the following overnight gap. Using 5-second data from the last hour of the regular trading day over 10 years across 9 large-cap S&P 500 stocks, we use multiple machine learning models to predict overnight gap direction and overnight gap magnitude. Using an expanding window and balanced accuracy as a primary metric, we find a statistically significant predictive signal in the final hour regarding the prediction of the subsequent overnight return magnitude. Specifically, models including last-hour information outperform models trained only using daily-level data, indicating strong incremental predictive power. Interestingly, when applied to the prediction of overnight gap direction, the model predictions are indistinguishable from chance. This suggests that while the final hour encodes information about overnight return magnitude, it contains little to no information regarding overnight return direction, both results confirmed by permutation testing.

Carnegie 119: Calistus Simiyu

Title: PROSPECTIVE TEACHERS' UNDERSTANDING OF DIVISION INVOLVING RATIONAL NUMBERS

Abstract: The purpose of this qualitative study was to explore prospective teachers' (PTs') understanding of why we multiply by the reciprocal when dividing by a rational number. As part of a larger project on PTs' rational number reasoning, this study involved analyzing a sample of 44 PTs' written responses on pre- and post-unit assessments, class homework, intern observation memos on PT tasks during class time, and end-of-semester exam questions that involved division by rational numbers. We coded data deductively and inductively to investigate PTs' conceptual and procedural understanding, and their misconceptions about rational number ideas. We found that PTs showed (a) continued reliance on procedures across the semester, (b) growth in conceptual understanding, and (c) a decrease in misconceptions about rational number division. We also discuss the implications of this research in mathematics teaching.

Carnegie 120: Naftoli Kolodny

Title: Intro to Topological Algebra via the Symmetric Group on the Natural Numbers

Abstract: We provide a simple introduction to the rich field of Topological Algebra by showing that the commutator length of the symmetric group on the naturals is 1. This is done primarily via topological methods, such as the Baire Category Theorem. In doing so, we introduce a natural way to topologize substructures of the full transformation monoid on arbitrary sets.

Carnegie 200: Odirachukwunma Ugwu

Title: Monotone Bilevel Variational Inequalities: Algorithm and Convergence Analysis with Applications

Abstract: Bilevel Variational Inequality Problems model complex hierarchical scenarios in which an upperlevel decision-maker interacts with a lowerlevel system that responds by solving a variational inequality capturing equilibrium behavior. These arise naturally in the optimization of transportation networks, smart city infrastructure,etc but their nested formulation makes them computationally demanding. Many existing projection-based algorithms rely on multiple projections per iteration, which increases computational cost and limits their practical scalability. Motivated by this challenge, we propose a computationally efficient algorithm that specifically minimizes projection complexity. The method incorporates an inertial extrapolation step and a self-adaptive stepsize to accelerate convergence, while requiring only a single projection onto a half-space. This design significantly reduces the computational cost, compared to projection-intensive schemes in the literature, without compromising theoretical convergence guarantees. Under mild assumptions on the control parameters, we establish strong convergence. Numerical experiments further illustrate the effectiveness of the method.

Carnegie 219: Alex LaJeunesse

Title: Homotopical Galois Theory

Abstract: Ever since Waldhausen coined the term "higher algebra" in the 1980's, algebraic topologists have found success in the field of stable homotopy theory by generalizing various algebraic constructions from the category of abelian groups to the category of spectra. The most important such generalization enlarges the category of commutative rings to a world where $\mathbb{Z}$ is no longer an initial object, that role instead being played by the sphere spectrum $\mathbb{S}$. There are many interesting examples of Galois extensions in this larger category, including all those of ordinary commutative rings but also more exotic examples like the natural map from real to complex K-theory. After providing some background on spectra and stable homotopy theory, we will define these generalized Galois extensions and provide several examples. We will then look at the homotopy-fixed-point spectral sequence — one of the fundamental calculational tools we can use to study them — and its application to the study of Picard groups.